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Andrée CHARLES EHRESMANN
Amiens, Juillet 1983

L'évangile selon Thomas (Laval théologique et philosophique)

Jésus a dit : Que celui qui cherche ne cesse pas de chercher jusqu'à ce qu'il trouve, et, quand il trouvera, il sera troublé, et, ayant été troublé, il sera émerveillé, et il dominera le Tout.

Citation copiée en 1962

Les articles originaux faisant l'objet de cette Partie IV-2 sont reproduits par procédé photographique dans les pages 407 à 774. Les numéros rajoutés dans les marges extérieures réfèrent aux Commentaires situés à la fin du livre.

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CATEGORIES OF SKETCHED STRUCTURES

by Andrée and Charles EHRESMANN

INTRODUCTION.

In the last decade algebraic structures have been defined on the objects of a category V:

1° A multiplication on an object e of V is a morphism k from a product $e \times e$ to e; monoids on e, groups on e, ... are obtained if further axioms are imposed on k by way of commuting diagrams ([Go], [EH]).

(The product may also be replaced by a «tensor product», but this point of view will not be considered here.)

2° The theory of fibre spaces and local structures led to p-structured categories (such as topological categories, differentiable categories, ordered categories, double categories) [E6] relative to a faithful functor p from V to the category of mappings (*), and more generally to categories in V (or category-objects in V).

Algebraic theories of Lawvere [Lw] (see also [B]) give an axiomatic way to define universal algebras; but they do not cover structures defined by partial laws, such as categories. However, all these structures may be defined by «sketches». Other examples of sketched structures are: categories equipped with a partial or a total choice of limits [Br], «discretely structured» categories [Bu], adjoint functors [L2], and also «less algebraic» structures, such as topologies [Br].

More precisely, let σ be a cone-bearing category, i.e. a category (or even a neocategory) Σ, equipped with a set of cones. A σ-structure in V is [E3] a functor from Σ to V, applying the distinguished cones on

(*) A category will be considered as the category of its morphisms and not, as usually, as the category of its objects.

limit-cones; a σ-morphism in V is a natural transformation between σ-structures in V. We denote by V^{σ} the category of σ-morphisms in V.

There are many cone-bearing categories σ' such that V^{σ} is equivalent to $V^{\sigma'}$; among them, we associate «universally» to σ:

- a limit-bearing category $\bar{\sigma}$ (i. e. the distinguished cones are limit-cones),

- a presketch $\bar{\bar{\sigma}}$ (i. e. a functor is at most the base of one distinguished cone),

- a prototype π (i. e. a presketch which is a limit-bearing category), and, if $\mathcal{I}$ is a set of categories containing the indexing-category of each distinguished cone of σ,

- a loose $\mathcal{I}$-type τ' (i. e. a limit-bearing category in which each functor indexed by an element of $\mathcal{I}$ is the base of at least one distinguished cone); for a universal algebra, τ' «is» its algebraic theory;

- a $\mathcal{I}$-type τ (i. e. a loose $\mathcal{I}$-type which is a presketch).

Moreover:

- $\bar{\sigma}$, $\bar{\bar{\sigma}}$, π and τ are defined up to an isomorphism,

- τ' is defined up to an equivalence,

- if σ is a sketch (i. e. if it is injectively immersed in π), then π and $\bar{\sigma}$ are isomorphic, while τ and τ' are equivalent.

The existence of π and τ was proved in [E4] and [E5] under the stronger assumption that σ were a presketch; this was necessary, the proof using the existence theorem for free structures whose hypotheses are not satisfied in the case of general cone-bearing categories. But subsequent works, in particular [Bu] and the (yet unpublished) paper of Lair on tensor products of sketches [L], showed that cone-bearing categories are often more convenient, and so they convinced us of the importance of immersing them in «universal» loose types.

We achieve this here by giving an explicit construction (by transfinite induction) of $\bar{\sigma}$, π, τ and τ'. These constructions are suggested by the explicit construction of the free $\mathcal{I}$-projective completion of a category in [E]. When applied to a prototype, the constructions of τ and τ' generalize theorems of [E] on completions of categories.

These results are proved in Part I in the case where the distinguished cones are projective, in Part II when there are both projective and inductive distinguished cones. They may also be expressed as adjunctions between the category $\mathcal{S}''$ of morphisms between cone-bearing categories, and some of its full subcategories. In fact, $\mathcal{S}''$ is the category of 1-morphisms of a representable and corepresentable 2-category, and these adjunctions extend into 2-adjunctions.

Part III is devoted to the problem:

(P) { If σ is a projective limit-bearing category on Σ and if V is underlying a symmetric monoidal closed category $\mathcal{V}$, does V^σ admit a symmetric monoidal closed structure?

We solve this problem in the case where σ is «cartesian», i.e. where the category $\mathcal{M}^\sigma$ of σ-morphisms in the category $\mathcal{M}$ of maps is cartesian closed (Proposition 20 is a characterisation of such a σ). More precisely, if σ is cartesian and if V admits «enough» limits, then V^σ is underlying a symmetric monoidal closed category as soon as either the tensor product of $\mathcal{V}$ commutes with the projective limits considered on σ, or the insertion functor from V^σ to V^Σ admits a left adjoint.

To prove this, we consider the symmetric monoidal closed category $\mathcal{V}^\Sigma$ defined by Day (Example 5-3 [D]) and we show that V^σ is closed for the closure functor (or Hom internal functor) of $\mathcal{V}^\Sigma$. The result is then deduced from a Proposition giving conditions under which a subcategory of a symmetric monoidal closed category admits such a structure (these conditions seem apparently slightly weaker than those we have just seen in a recent paper by Day [D1]). Notice that we use only a partial result of [D]; his general result is used in [FL] to get solutions of (P) under another kind of conditions (see Remark 2, page 82).

As an application, we deduce a symmetric monoidal closed structure on the category $\mathcal{F}(V)$ of functors in V (when σ is the prototype of categories), as was announced in [BE]. We finally show that the closure functor E on $\mathcal{F}(V)$ may also be constructed by a direct method (whose idea is to define the analogue of the «double category of quartets» generalizing the method used in a particular case in [BE]), which requires that V has only pullbacks and kernels (and not even finite sums, which have to be used in the first construction).

Sketched structures may be generalized in different ways: one of them (proposed two years ago by the first of the authors in a lecture) is to replace the cone-bearing categories by cone-bearing double categories (2-theories of [Du] and [G1] are examples of them). Another way consists in substituting «cylinders» to the cones, as is done in a just appeared paper by Freyd and Kelly [FK].

We use throughout the terminology of [E1], but we have tried to take lighter notations, nearer to those used in most papers on categorical Algebra. We stay in the frame of the Zermelo-Fraenkel set theory, with the supplementary axiom of universes: Any set belongs to a universe.

I. PROJECTIVE LIMIT-BEARING CATEGORIES

1. Neocategories and neofunctors.

Firstly, we recall the definition of a neocategory (or «graphe multiplicatif» [E1]). Graphs and also categories appear as «extreme» examples of neocategories.

A *neocategory* Σ is a couple formed by a set, denoted by $\underline{\Sigma}$, and a «partial law of composition» κ on $\underline{\Sigma}$ satisfying the following axioms:

1° κ is a mapping from a subset of $\underline{\Sigma}\times\underline{\Sigma}$ (denoted by $\Sigma * \Sigma$ and called *the set of composable couples*) into $\underline{\Sigma}$; instead of $\kappa(y, x)$, we write $y.x$ (or $y \circ x$, or yx, ...) and we call $y.x$ the *composite of* (y, x).

2° There exists a graph $(\underline{\Sigma}, \beta, \alpha)$ (i. e. α and β are retractions from $\underline{\Sigma}$ onto a subset of $\underline{\Sigma}$, denoted by Σ_0), such that:

a) For each element x of $\underline{\Sigma}$, the composites $x.\alpha(x)$ and $\beta(x).x$ are defined, and we have:

$$x.\alpha(x) = x = \beta(x).x \ ;$$

b) If the composite $y.x$ is defined, then:

$$\alpha(y) = \beta(x), \quad \alpha(y.x) = \alpha(x), \quad \beta(y.x) = \beta(y).$$

From the condition 2, it follows that the graph (Σ, β, α) is uniquely defined; moreover the set Σ_0 of its vertices (called *objects of* Σ) is the set of unit elements (i. e. identities) of Σ. We say that $\alpha(x)$ is the *source* of x, and that $\beta(x)$ is the *target* of x. The elements of $\underline{\Sigma}$ are called *morphisms of* Σ. We write

$$x \in \Sigma \quad \text{or} \quad x: e \to e' \text{ in } \Sigma$$

instead of: x is a morphism of Σ, with source e and target e'. If e and e' are two objects of Σ, the set of morphisms $f: e \to e'$ in Σ will be denoted by $e'.\Sigma.e$ or by $\Sigma(e', e)$ (and not $\Sigma(e, e')$ as usually).

EXAMPLES. 1° A graph $(\underline{\Sigma}, \beta, \alpha)$ may be identified with the neocategory Σ admitting $\underline{\Sigma}$ as its set of morphisms and in which the only composites are $x.\alpha(x)$ and $\beta(x).x$, for every element x of $\underline{\Sigma}$.

2° A category is a neocategory in which all the couples (y, x) where $\alpha(y) = \beta(x)$ are composable (so that $\Sigma * \Sigma$ is the pullback of (α, β)), the law of composition being furthermore associative.

Let Σ and Σ' be two neocategories. A *neofunctor* ϕ from Σ toward Σ' is a triple $(\Sigma', \underline{\phi}, \Sigma)$, where $\underline{\phi}$ is a mapping from $\underline{\Sigma}$ into $\underline{\Sigma}'$ such that $\phi(e) \in \Sigma'_0$ for each $e \in \Sigma_0$ and that:

If $y.x$ is defined in Σ, then $\underline{\phi}(y).\underline{\phi}(x)$ is defined in Σ', and

$$\underline{\phi}(y).\underline{\phi}(x) = \underline{\phi}(y.x).$$

We say also that $\phi: \Sigma \to \Sigma'$ is a neofunctor; we write $\phi(x)$ instead of $\underline{\phi}(x)$ and ϕ_0 denotes the restriction $\phi_0: \Sigma_0 \to \Sigma'_0$ of ϕ.

If $\phi: \Sigma \to \Sigma'$ and $\phi': \Sigma' \to \Sigma''$ are two neofunctors, we denote by $\phi' \cdot \phi$, or by $\phi' \phi$, the neofunctor from Σ to Σ'' assigning

$$\phi'(\phi(x)) \quad \text{to } x \text{ in } \Sigma.$$

Neofunctors between graphs reduce to morphisms between graphs (in the usual meaning [E1]) and neofunctors between categories are ordinary *functors*.

Let Σ and Σ' be neocategories. If ϕ and ψ are two neofunctors from Σ to Σ', a *natural transformation* τ *from* ϕ *to* ψ is defined as a triple (ψ, τ_0, ϕ), where τ_0 is a mapping associating to each object e of Σ a morphism $\tau_0(e): \phi(e) \to \psi(e)$ of Σ' (also denoted by $\tau(e)$), such that the composites $\psi(x).\tau(e)$ and $\tau(e').\phi(x)$ be defined and that

$$\psi(x).\tau(e) = \tau(e').\phi(x),$$

for each $x: e \to e'$ in Σ. We say also that $\tau: \phi \to \psi$ is a natural transformation (defined by τ_0).

EXAMPLES. 1° If u is an object of Σ', the constant mapping assigning u to each morphism x in Σ defines a neofunctor $u^\wedge: \Sigma \to \Sigma'$. If $z: u \to u'$ is a morphism in Σ', we denote by $z^\wedge$ the natural transformation (said constant on z) from $u^\wedge$ to $u'^\wedge$ such that $z^\wedge(e) = z$ for each $e \in \Sigma_0$.

2° A natural transformation from a constant neofunctor, i.e. a natural transformation $\gamma: u^\wedge \to \psi$, is called a *projective cone in* Σ', *indexed by*

Σ, *with base* $\psi : \Sigma \to \Sigma'$ *and vertex* u. Similarly, a natural transformation $\gamma' : \phi \to u^\wedge$ is called an *inductive cone*.

3° Let $\tau : \phi \to \psi$ be a natural transformation with $\phi : \Sigma \to \Sigma'$. If $\phi' : \Sigma' \to \Sigma''$ is a neofunctor, the mapping $\underline{\phi}'\tau_0$ defines a natural transformation denoted by $\phi'\tau : \phi'\phi \to \phi'\psi$; if τ is a projective (resp. inductive) cone, $\phi'\tau$ is also one. If $\phi'' : \Sigma'' \to \Sigma$ is a neofunctor, the mapping $\tau_0\,\phi''_0$ defines the natural transformation $\tau\phi'' : \phi\phi'' \to \psi\phi''$.

Let Σ be a neocategory and Σ' a category. If

$$\tau : \phi \to \psi \quad \text{and} \quad \tau' : \psi \to \psi'$$

are natural transformations, the mapping $\tau''_0 : \Sigma_0 \to \underline{\Sigma}'$ such that

$$\tau''_0(e) = \tau'(e).\tau(e) \quad \text{for each } e \in \Sigma_0$$

defines a natural transformation $\tau'' : \phi \to \psi'$, denoted by $\tau' \boxminus \tau$. (This is not true if Σ' is only a neocategory.) With this law of composition, the set of natural transformations between neofunctors from Σ to Σ' becomes a category, denoted by $\mathfrak{N}(\Sigma', \Sigma)$ or by Σ'^{Σ}.

EXAMPLES. 1° Let $z : u' \to u$ be a morphism of Σ'. If $\gamma : u^\wedge \to \psi$ is a projective cone in Σ', with vertex u, we denote by γz the projective cone $\gamma \boxminus z^\wedge : u'^\wedge \to \psi$. If $\gamma' : \phi \to u'^\wedge$ is an inductive cone, we define $z\gamma'$ as the inductive cone $z^\wedge \boxminus \gamma'$.

2° Suppose that Σ is the category $\mathbf{2}$, with only two objets 0 and 1, and one morphism $a = (0, 1)$ from 0 to 1. A functor $\phi : \mathbf{2} \to \Sigma'$ may be identified with the morphism $\phi(a)$ of the category Σ'; a natural transformation $\tau : \phi \to \phi'$ may be identified with the *quartet (commutative square)* $(\phi'(a), \tau(1), \tau(0), \phi(a))$. Then the category $\Sigma'^{\mathbf{2}}$ reduces to the *longitudinal category of quartets of* Σ' (often called category of pairs), denoted by $\boxminus\Sigma'$. By assigning (y', x', x, y) to the quartet (x', y', y, x), we define an isomorphism from $\boxminus\Sigma'$ onto a category $\boxminus\Sigma'$, called the *lateral category of quartets of* Σ'. The pair $(\boxminus\Sigma', \boxminus\Sigma')$ is a double category [E6], written $\square\Sigma'$.

A projective cone $\gamma : u^\wedge \to \phi$ in the category Σ' is called a *projective limit-cone* («limite projective naturalisée» in [E]) if, for any projective cone $\gamma' : u'^{\wedge} \to \phi$ in Σ admitting the same base than γ, there exists one and only one $z : u' \to u$ in Σ' such that $\gamma z = \gamma'$; in that case, z will be called *the factor of* γ' *through* γ, and denoted by $lim_{\gamma}\gamma'$. Dually, we define the notion of an *inductive limit-cone.*

If $\mathcal{J}$ is a set of categories, we say that thecategory Σ *admits* $\mathcal{J}$*-projective* (resp. $\mathcal{J}$*-inductive*) *limits* if each functor $\phi : K \to \Sigma$, where $K \in \mathcal{J}$, admits a projective (resp. an inductive) limit.

REMARK. Since we will essentially be concerned with projective cones or projective limits, we call them briefly cones or limits; but the dual notions will always be called explicitly inductive cone or inductive limit.

2. Cone-bearing neocategories.

By definition, a *cone-bearing neocategory* σ is a pair (Σ, Γ), where Σ is a neocategory and Γ a set of (projective) cones in Σ, said the *distinguished cones of* σ, indexed by categories. The set of the indexing-categories of all the distinguished cones is called *the set of indexing-categories of* σ.

If σ' is another cone-bearing neocategory (Σ', Γ'), a *morphism from* σ *to* σ' is defined as a triple $\overline{\psi} = (\sigma', \psi, \sigma)$, where $\psi : \Sigma \to \Sigma'$ is a neofunctor such that:

$$\psi\gamma \in \Gamma' \quad \text{for any} \quad \gamma \in \Gamma .$$

We say also that $\overline{\psi} : \sigma \to \sigma'$ is a morphism defined by ψ; we write:

$$\overline{\psi}(x) = \psi(x) \ \text{ if } \ x \in \Sigma, \quad \overline{\psi}\gamma = \psi\gamma \ \text{ if } \ \gamma \in \Gamma$$

or, more generally, if γ is a cone in Σ. Notice that the set of indexing-categories of σ must then be included in that of σ'.

If $\overline{\psi}' = (\sigma'', \psi', \sigma')$ is also a morphism from σ' to the cone-bearing neocategory σ'', then $\psi'\psi$ defines a morphism, denoted by

$$\overline{\psi}' \cdot \overline{\psi} : \sigma \to \sigma'' .$$

If ψ is an isomorphism and if its inverse defines also a morphism from σ' to σ, we say that $\overline{\psi}$ is an *isomorphism.*

Two cone-bearing neocategories σ and σ' are said *equivalent* if there exist morphisms

$$\overline{\psi} = (\sigma', \psi, \sigma) \quad \text{and} \quad \overline{\psi}' = (\sigma, \psi', \sigma')$$

such that $\psi\,\psi'$ and $\psi'\psi$ be equivalent to identities (which implies the equivalence of the underlying neocategories).

REMARK. Cone-bearing neocategories are sketches in the sense of [E3] (but the notions of a sketch considered in [E4] and [E5] are more strict, and here the word sketch will have the same meaning as in [E5]). They are used in [Bu] under the name «esquisse multiforme». Lair needs them in [L] to define tensor products of sketches. Morphisms between cone-bearing neocategories are called homomorphisms between sketches in [E3].

DEFINITION. A (projective) cone-bearing neocategory (Σ, Γ) is called a *limit-bearing category* if Σ is a category and if each distinguished cone $\gamma \in \Gamma$ is a (projective) limit-cone.

EXAMPLES. 1° Let Σ be a category and $\mathcal{I}$ a set of categories. Let Γ be the set of all limit-cones in Σ with indexing-categories in $\mathcal{I}$. Then (Σ, Γ) is a limit-bearing category, called the *full $\mathcal{I}$-limit bearing category on Σ.*

2° Let σ be a limit-bearing category (Σ, Γ) and K a category. Consider the category of natural transformations Σ^K; for each object i of K, denote by $\pi_i : \Sigma^K \to \Sigma$ the functor associating $\tau(i)$ to the natural transformation τ. Let $\overline{\Gamma}$ be the set of cones γ in Σ^K such that:

$$\pi_i \gamma \in \Gamma \quad \text{for any} \quad i \in K_0 .$$

Then $(\Sigma^K, \overline{\Gamma})$ is a limit-bearing category [E3], denoted by σ^K. In particular, if K is the category $\mathbf{2}$ and if Σ^2 is identified with the longitudinal category $\boxminus\Sigma$ of quartets of Σ (see example 2-1), we get the *longitudinal limit-bearing category of quartets of* σ, denoted by $\boxminus\sigma$. The canonical isomorphism from $\boxminus\Sigma$ to $\boxminus\Sigma$ defines an isomorphism from $\boxminus\sigma$ to the *lateral limit-bearing category of quartets of* σ, written $\boxminus\sigma$.

Let $\mathfrak{U}$ be a universe [AB]; an element of $\mathfrak{U}$ is called a $\mathfrak{U}$-*set* (or a small set). We denote by:

- $\mathcal{F}'_0$ (resp. $\mathcal{F}_0$) the set of neocategories (resp. of categories) Σ whose sets of morphisms are $\mathfrak{U}$-sets.

- $\mathcal{S}''_0$ the set of cone-bearing neocategories (Σ, Γ), where Γ and $\underline{\Sigma}$ are $\mathfrak{U}$-sets as well as $\underline{K}$, for each indexing-category K of $\sigma = (\Sigma, \Gamma)$.

- $\mathcal{P}'_0$ the set of limit-bearing categories belonging to $\mathcal{S}''_0$.

- $\mathfrak{M}$ the category of all mappings between $\mathfrak{U}$-sets (following our convention to name a category according to its morphisms).

- $\mathcal{F}'$ the category of all neofunctors $\phi: \Sigma \to \Sigma'$, where Σ and Σ' belong to $\mathcal{F}'_0$ (this category is denoted by $\mathfrak{N}'$ in [E1]).

- $p_{\mathcal{F}'}: \mathcal{F}' \to \mathfrak{M}$ the faithful functor which assigns the map

$$\underline{\phi}: \underline{\Sigma} \to \underline{\Sigma}' \quad \text{to} \quad \phi: \Sigma \to \Sigma'$$

and by $p'_{\mathcal{F}'}: \mathcal{F}' \to \mathfrak{M}$ the not-faithful functor associating

$$\phi_0 : \Sigma_0 \to \Sigma'_0 \quad \text{to} \quad \phi: \Sigma \to \Sigma' .$$

- $\mathcal{F}$ the full subcategory of $\mathcal{F}'$ formed by the functors and by $p_{\mathcal{F}}: \mathcal{F} \to \mathfrak{M}$ the faithful functor restriction of $p_{\mathcal{F}'}$.

The morphisms $\overline{\phi}: (\Sigma, \Gamma) \to (\Sigma', \Gamma')$ between cone-bearing neocategories (resp. between limit-bearing categories) belonging to $\mathcal{S}''_0$ form a category $\mathcal{S}''$ (resp. $\mathcal{P}'$). Assigning $\phi: \Sigma \to \Sigma'$ to $\overline{\phi}$, we define a faithful functor

$$q_{\mathcal{S}''}: \mathcal{S}'' \to \mathcal{F}' \quad (\text{resp. } q_{\mathcal{P}'}: \mathcal{P}' \to \mathcal{F}).$$

Let $p_{\mathcal{S}''}$ and $p_{\mathcal{P}'}$ be the composite functors:

$$p_{\mathcal{S}''} = p_{\mathcal{F}'}\, q_{\mathcal{S}''}: \mathcal{S}'' \to \mathfrak{M}, \quad p_{\mathcal{P}'} = p_{\mathcal{F}}\, q_{\mathcal{P}'}: \mathcal{P}' \to \mathfrak{M} .$$

The following elementary proposition will be used later on.

PROPOSITION 1. *$\mathcal{S}''$ admits $\mathcal{F}_0$-projective limits and $\mathcal{F}_0$-inductive limits; $q_{\mathcal{S}''}$ commutes with projective and inductive limits; $p_{\mathcal{S}''}$ commutes with projective limits and filtered inductive limits; $\mathcal{P}'$ is closed in $\mathcal{S}''$ for projective limits.* (See also [E4] and [L1].)

Δ. The proof is straightforward. Let $F: K \to \mathcal{S}''$ be a functor, where $\underline{K}$ is a $\mathfrak{U}$-set, and write

$$F(i) = (\Sigma_i, \Gamma_i) \quad \text{for any} \quad i \in K_0 .$$

1° Let Σ be a projective limit of the functor $q_{\mathcal{S}''}F$; then $\underline{\Sigma}$ is a projective limit of $p_{\mathcal{S}''}F$; denote by $\pi_i: \Sigma \to \Sigma_i$ the canonical projection and by Γ the set of cones γ in Σ such that

$$\pi_i \gamma \in \Gamma_i \quad \text{for any} \quad i \in K_0 .$$

Then (Σ, Γ) is a projective limit of F. If moreover F takes its values in $\mathcal{P}'$, we have also $(\Sigma, \Gamma) \in \mathcal{P}'_0$.

2° $q_{\mathcal{S}''} F$ admits [E1] an inductive limit Σ', with canonical injections $\nu_i : \Sigma_i \to \Sigma'$. Let Γ' be the set of all cones

$$\nu_i \gamma_i , \quad \text{where} \quad i \in K_0 \quad \text{and} \quad \gamma_i \in \Gamma_i .$$

Then (Σ', Γ') is an inductive limit σ' of F. If K is filtered, $\underline{\Sigma}'$ is [E1] an inductive limit of $p_{\mathcal{S}''} F$. ∇

Let σ be a cone-bearing neocategory (Σ, Γ) and $\mathcal{J}$ its set of indexing-categories.

DEFINITION. If σ' is a limit-bearing category (Σ', Γ'), we define a *σ-structure in σ'* as a neofunctor $\psi : \Sigma \to \Sigma'$ defining a morphism $\overline{\psi} : \sigma \to \sigma'$ (we also say [E5] that ψ is a realization of σ in σ'). If Σ' is a category and if σ' is the full $\mathcal{J}$-limit-bearing category on Σ' (example 1), a σ-structure in σ' is called a *σ-structure in Σ'*.

The set $\mathcal{S}(\sigma', \sigma)_0$ of σ-structures in the limit-bearing category $\sigma' = (\Sigma', \Gamma')$ is the set of objects of a full subcategory of Σ'^{Σ} denoted by $\mathcal{S}(\sigma', \sigma)$, and called *the category of morphisms between σ-structures in σ'*, or *category of σ-morphisms in σ'*.

If Σ' is a category, a σ-structure in Σ' is just a neofunctor ψ from Σ to Σ' such that $\psi\gamma$ is a limit-cone, for any $\gamma \in \Gamma$. We will denote by $\mathcal{S}(\Sigma', \sigma)$, or by Σ'^{σ}, the full subcategory of Σ'^{Σ} whose objects are the σ-structures in Σ'. Remark that Σ'^{σ} admits $\mathcal{S}(\sigma', \sigma)$ as a full subcategory, for any limit-bearing category σ' on Σ'.

PROPOSITION 2. *Let σ be a cone-bearing neocategory, σ' a limit-bearing category and $\boxminus\sigma'$ the lateral limit-bearing category of quartets of σ' (example 2). Then there exists a canonical bijection from the set of morphisms of $\mathcal{S}(\sigma', \sigma)$ to the set $\mathcal{S}(\boxminus\sigma', \sigma)_0$ of σ-structures in $\boxminus\sigma'$.*

Δ. To a natural transformation $\tau : \psi \to \psi'$, where $\psi : \Sigma \to \Sigma'$, there corresponds the neofunctor $T : \Sigma \to \boxminus \Sigma'$ which assigns the quartet

$$(\psi'(z), \tau(e'), \tau(e), \psi(z)) \text{ to } z : e \to e' \text{ in } \Sigma.$$

The map f associating T to τ is a bijection from $\underline{\Sigma'^{\Sigma}}$ to the set of neofunctors from Σ to $\boxminus \Sigma'$. (If we identify τ with a functor from **2** to Σ'^{Σ}, this bijection f is deduced from the canonical isomorphism:

$$(\Sigma'^{\Sigma})^{\mathbf{2}} \approx (\Sigma'^{\mathbf{2}})^{\Sigma} \approx (\boxminus \Sigma')^{\Sigma}.)$$

The natural transformation τ is a morphism between σ-structures iff(*) T is a σ-structure in $\boxminus \sigma'$. Therefore f induces a bijection

$$f' : \underline{\mathcal{S}}(\sigma', \sigma) \to \mathcal{S}(\boxminus \sigma', \sigma)_0 . \quad \nabla$$

3. Limit-bearing category generated by a cone-bearing neocategory.

The study of the category $\mathcal{S}(\sigma', \sigma)$ of morphisms between σ-structures in the limit-bearing category σ' will be much easier when the cone-bearing neocategory σ is itself a limit-bearing category. Hence the question: Does there exist a limit-bearing category $\bar{\sigma}$ such that $\mathcal{S}(\sigma', \sigma)$ and $\mathcal{S}(\sigma', \bar{\sigma})$ are isomorphic? The following proposition not only answers affirmatively this question, but it gives an explicit construction of a smallest $\bar{\sigma}$ of this kind.

PROPOSITION 3. *Let σ be a cone-bearing neocategory (Σ, Γ). There exists a limit-bearing category $\bar{\sigma} = (\bar{\Sigma}, \bar{\Gamma})$ and a morphism $\bar{\delta} : \sigma \to \bar{\sigma}$ satisfying the following conditions:*

1° $\bar{\Gamma} = \{ \bar{\delta}\gamma \mid \gamma \in \Gamma \}$.

2° If $\mathfrak{U}$ is a universe such that $\sigma \in \mathcal{S}''_0$, then $\bar{\sigma} \in \mathcal{P}'_0$.

3° $\bar{\sigma}$ is characterized, up to an isomorphism, by the universal property: If σ' is a limit-bearing category and $\bar{\psi} : \sigma \to \sigma'$ a morphism, then there
1 *exists one and only one morphism $\bar{\psi}' : \bar{\sigma} \to \sigma'$ such that $\bar{\psi}' \cdot \bar{\delta} = \bar{\psi}$.*

Δ. By transfinite induction, we shall construct a «tower» of cone-bearing neocategories σ_ξ such that σ_0 be σ and that $\sigma_{\xi+1}$ be deduced from σ_ξ by adding to σ_ξ «formal factors» through a distinguished cone γ for cones with the same base as γ. We will show that this tower ends for

(*) iff means if and only if.

some sufficiently big ordinal μ, and that σ_μ is the limit-bearing category $\bar{\sigma}$.

1° Let us decribe the step from σ_ξ to $\sigma_{\xi+1}$. Let σ_ξ be any cone-bearing neocategory (Σ_ξ, Γ_ξ).

a) If $\gamma \in \Gamma_\xi$ and if γ' is a cone in Σ_ξ with the same base as γ we consider the pair (γ, γ') (called the «formal factor» of γ' through γ). Let Ω be the set of all these pairs; let U be the sum («disjoint union») of $\underline{\Sigma}_\xi$ and Ω, with injections

$$v : \underline{\Sigma}_\xi \to U \quad \text{and} \quad v' : \Omega \to U.$$

We define a graph (U, β, α) in the following way:

- If $x : u \to u'$ in Σ_ξ, then

$$\alpha(v(x)) = v(u), \quad \beta(v(x)) = v(u').$$

- If $(\gamma, \gamma') \in \Omega$, where $\gamma : u^\wedge \to \phi$ and $\gamma' : u'^\wedge \to \phi$, then

$$\alpha(v'(\gamma, \gamma')) = v(u'), \quad \beta(v'(\gamma, \gamma')) = v(u).$$

Let L be the free category generated by (U, β, α); it is [E1] the «category of paths» on (U, β, α) and U is identified with paths of length 1. Consider the smallest equivalence relation r on $\underline{L}$ such that:

$$\text{(P)} \begin{cases} (v(x'), v(x)) \sim v(x'.x), \text{ if } x'.x \text{ is defined in } \Sigma_\xi, \\ (v(\gamma(i)), v'(\gamma, \gamma')) \sim v(\gamma'(i)), \text{ if } (\gamma, \gamma') \in \Omega \text{ and } i \in K_0, \\ v(z) \sim v'(\gamma, \gamma'), \text{ if } z \in \Sigma_\xi, \text{ if } (\gamma, \gamma') \in \Omega \text{ and if} \\ \qquad \gamma'(i) = \gamma(i).z \text{ for any } i \in K_0, \\ \text{where } K \text{ is the indexing-category of } \gamma. \end{cases}$$

There exists a quasi-quotient category [E1] of L by r, denoted by $\bar{\Sigma}_\xi$; since r identifies no objects, $\bar{\Sigma}_\xi$ is in fact the quotient category of L by the smallest equivalence relation compatible with the law of composition of L and containing r. Let ρ: $L \to \bar{\Sigma}_\xi$ be the canonical functor corresponding to r. The map $\rho\, v$ defines a neofunctor $\delta_\xi : \Sigma_\xi \to \bar{\Sigma}_\xi$ by the first condition imposed on r. Denote by $\bar{\Gamma}_\xi$ the set of all the cones $\delta_\xi \gamma$, where $\gamma \in \Gamma_\xi$. Then $(\bar{\Sigma}_\xi, \bar{\Gamma}_\xi)$ is a cone-bearing neocategory $\bar{\sigma}_\xi$ and δ_ξ defines a morphism $\bar{\delta}_\xi : \sigma_\xi \to \bar{\sigma}_\xi$. Moreover, for each formal factor $(\gamma, \gamma') \in \Omega$, we have

$$(\delta_\xi \gamma) z = \delta_\xi \gamma' , \text{ where } z = \underline{\rho}(v'(\gamma, \gamma')).$$

b) *Suppose that σ' is a limit-bearing category (Σ', Γ') and that ψ is a neofunctor defining a morphism $\overline{\psi} : \sigma_\xi \to \sigma'$. Then there exists a unique morphism*

$$\overline{\psi}' : \overline{\sigma}_\xi \to \sigma' \quad \text{such that} \quad \overline{\psi}' \cdot \overline{\delta}_\xi = \overline{\psi}.$$

Indeed, if $(\gamma, \gamma') \in \Omega$, where $\gamma : u^\wedge \to \phi$, the cone $\psi\gamma$ is a limit-cone with the same base as the cone $\psi\gamma'$; so there exists a unique y such that $(\psi\gamma) y = \psi\gamma'$, namely the factor of $\psi\gamma'$ through $\psi\gamma$. By assigning y to the formal factor (γ, γ'), we get a mapping $f : \Omega \to \underline{\Sigma}'$. The unique map $f' : U \to \underline{\Sigma}'$ such that

$$f' v = \underline{\psi} \quad \text{and} \quad f' v' = f$$

defines a neofunctor from (U, β, α) (considered as a neocategory) to Σ'. This neofunctor extends into a functor $F' : L \to \Sigma'$. Since σ' is a limit-bearing category, this F' is compatible with r, so that there exists one and only one functor

$$\psi' : \overline{\Sigma}_\xi \to \Sigma' \quad \text{such that} \quad \psi' \rho = F'.$$

This functor defines the unique morphism

$$\overline{\psi}' : \overline{\sigma}_\xi \to \sigma' \quad \text{such that} \quad \overline{\psi}' \cdot \overline{\delta}_\xi = \overline{\psi}.$$

c) If $\mathfrak{U}$ is a universe such that $\sigma_\xi \in \mathfrak{S}''_0$, then $\underline{K}$, for each indexing-category K of σ_ξ and Γ_ξ are $\mathfrak{U}$-sets; it results that the set Γ_K of cones in Σ_ξ indexed by K is also a $\mathfrak{U}$-set, as well as the set $\bigcup_{K \in \mathfrak{J}} \Gamma_K$, where $\mathfrak{J}$ is the set of indexing-categories of σ_ξ. From this we deduce succesively that Ω, U, $\underline{L}$ and $\underline{\overline{\Sigma}}_\xi$ are $\mathfrak{U}$-sets, and that $\overline{\sigma}_\xi$ belongs to $\mathfrak{S}''_0$.

2° We are now ready to construct the tower. Let $\mathfrak{J}$ be the set of indexing-categories of σ. If $K \in \mathfrak{J}$, we denote by $\overline{\overline{K}}$ the cardinal of $\underline{K}$. (An ordinal number ζ is considered as the set of ordinals ξ such that $\xi < \zeta$; the cardinal of a set E is identified with the initial ordinal equipotent to E). Let λ be the ordinal which is the upper bound of the ordinals $\overline{\overline{K}}$, where $K \in \mathfrak{J}$, and let μ be the least regular ordinal satisfying $\lambda < \mu$.

Accepting the «axiom of universes», there exists a universe $\mathfrak{U}$ to which belongs $\Gamma \cup \underline{\Sigma} \cup \bigcup_{K \in \mathcal{J}} \underline{K}$, i.e. such that σ is an object of the category $\mathcal{S}''$ corresponding to $\mathfrak{U}$. As $\underline{K}$ is a $\mathfrak{U}$-set, $\bar{\bar{K}}$ belongs to $\mathfrak{U}$ and, $\mathcal{J}$ being equipotent to a subset of the $\mathfrak{U}$-set Γ, the ordinal λ belongs also to $\mathfrak{U}$, as well as μ. (Here we use the fact that the upper bound of the ordinals which belong to a universe $\mathfrak{U}$ is an inaccessible ordinal [AB]).

For each ordinal ξ, let $<\xi>$ be the category defining the canonical order on ξ; its set of objects is ξ. (In particular, $\mathbf{2} = <2>$). By transfinite induction, we define a functor $\omega : <\mu + 1> \to \mathcal{S}''$:

- First, $\omega(0) = \sigma$.
- Let ζ be an ordinal, $\zeta \leqslant \mu$; suppose we have defined a functor

$$\omega_\zeta : <\zeta> \to \mathcal{S}'' \quad \text{such that} \quad \omega_\zeta(0) = \sigma,$$

and write

$$\omega_\zeta(\xi) = \sigma_\xi = (\Sigma_\xi, \Gamma_\xi) \quad \text{for any} \quad \xi < \zeta.$$

We extend ω_ζ into a functor $\omega_{\zeta+1} : <\zeta + 1> \to \mathcal{S}''$ in the following way:

If ζ is a limit ordinal, $\omega_{\zeta+1}(\zeta)$ is the canonical inductive limit, denoted by $\sigma_\zeta = (\Sigma_\zeta, \Gamma_\zeta)$, of the functor ω_ζ (which exists, Proposition 1) and $\omega_{\zeta+1}(\zeta, \xi) : \sigma_\xi \to \sigma_\zeta$ is the canonical injection, for any $\xi < \zeta$. We recall (Proposition 1 and [E1]) that $\underline{\Sigma}_\zeta$ is the canonical inductive limit of the functor $p_{\mathcal{S}''} \omega_\zeta$ from $<\zeta>$ to $\mathfrak{M}$ and that each composite $\overline{y'} . \overline{y}$ in Σ_ζ is of the form $\omega_{\zeta+1}(\zeta, \xi)(y' . y)$, for some $\xi < \zeta$, where $y' . y$ is a composite in Σ_ξ, and $\overline{y} = \omega_{\zeta+1}(\zeta, \xi)(y)$, $\overline{y'} = \omega_{\zeta+1}(\zeta, \xi)(y')$.

If ζ is the successor of ξ (that is: $\zeta = \xi + 1$), then $\omega_{\zeta+1}(\zeta)$ will be the cone-bearing neocategory $(\overline{\Sigma}_\xi, \overline{\Gamma}_\xi)$ associated to σ_ξ in Part 1, and $\omega_{\zeta+1}(\xi + 1, \xi) : \sigma_\xi \to \sigma_\zeta$ will be the morphism $\overline{\delta}_\xi$ constructed in Part 1. The induction hypothesis $\sigma_\xi \in \mathcal{S}''_0$ implies $\sigma_\zeta \in \mathcal{S}''_0$ (Part 1).

- Finally, we put

$$\omega = \omega_{\mu+1}, \quad \overline{\sigma} = (\overline{\Sigma}, \overline{\Gamma}) = \sigma_\mu, \quad \overline{\delta} = \omega(\mu, 0) = (\overline{\sigma}, \delta, \sigma).$$

By construction, $\overline{\sigma}$ is an object of $\mathcal{S}''$.

3° a) By transfinite induction, we prove that $\overline{\overline{\Sigma}}$ is a category. Indeed, suppose that ζ is an ordinal, $\zeta \leqslant \mu$, and that Σ_ξ is a category for any

ξ such that $0 \neq \xi < \zeta$. If $\zeta = \xi + 1$, then $\Sigma_\zeta = \overline{\Sigma}_\xi$ is a category, by construction. If ζ is a limit ordinal, Σ_ζ is the inductive limit of the functor $q_{\mathcal{S}''}\omega_\zeta$; since $<\zeta>$ is a filtered category and since the Σ_ξ, for $\xi < \zeta$ and $0 \neq \xi$, are categories, the neocategory Σ_ζ is a category. Hence Σ_μ is a category $\overline{\Sigma}$.

b) We are going to prove, by transfinite induction, that each cone $\overline{\gamma}$ of $\overline{\Gamma}$ is of the form $\delta\gamma$, for some $\gamma \in \Gamma$. Indeed, we have $\Gamma_0 = \Gamma$. Let ζ be an ordinal, $\zeta \leqslant \mu$, and suppose that, for any $\xi < \zeta$, we have:

$$\Gamma_\xi = \{ \omega(\xi, 0)\gamma \mid \gamma \in \Gamma \}.$$

- If ζ is a limit ordinal, Proposition 1 asserts that

$$\Gamma_\zeta = \{ \omega(\zeta, \xi)\gamma_\xi \mid \gamma_\xi \in \Gamma_\xi, \quad \xi < \zeta \}$$

and the induction hypothesis implies that

$$\gamma_\xi = \omega(\xi, 0)\gamma, \quad \text{for some} \quad \gamma \in \Gamma;$$

hence

$$\omega(\zeta, \xi)\gamma_\xi = \omega(\zeta, \xi)\cdot\omega(\xi, 0)\gamma = \omega(\zeta, 0)\gamma.$$

It follows that:

$$\Gamma_\zeta = \{ \omega(\zeta, 0)\gamma \mid \gamma \in \Gamma \}.$$

- If $\zeta = \xi + 1$, by Part 1-a, we have:

$$\Gamma_\zeta = \overline{\Gamma}_\xi = \{ \overline{\delta}_\xi \gamma_\xi \mid \gamma_\xi \in \Gamma_\xi \},$$

where

$$\overline{\delta}_\xi \gamma_\xi = \omega(\xi+1, \xi)\gamma_\xi = \omega(\xi+1, \xi)\cdot\omega(\xi, 0)\gamma = \omega(\zeta, 0)\gamma,$$

since $\gamma_\xi = \omega(\xi, 0)\gamma$, for some $\gamma \in \Gamma$. Therefore, in this case also,

$$\Gamma_\zeta = \{ \omega(\zeta, 0)\gamma \mid \gamma \in \Gamma \}.$$

c) The category $\overline{\Sigma}$ is determined independently of the universe $\mathfrak{U}$. For, let $\hat{\mathfrak{U}}$ be another universe such that

$$(\Gamma \cup \underline{\Sigma} \cup \bigcup_{K \in \mathcal{J}} \underline{K}) \in \hat{\mathfrak{U}},$$

and let $\hat{\mathcal{S}}''$ be the category of morphisms between cone-bearing neocategories corresponding to $\hat{\mathfrak{U}}$. If $F: C \to \mathcal{S}''$ and $\hat{F}: C \to \hat{\mathcal{S}}''$ are two functors taking the same values (i.e. $F(z) = \hat{F}(z)$ for any $z \in C$), then they have the same canonical inductive limit, according to the construction of this limit as a quotient of a sum. So the inductive limit σ_ζ of ω_ζ, for a limit ordinal ζ, does not depend on the choice of $\mathfrak{U}$. In particular, $\bar{\sigma}$ is independent of $\mathfrak{U}$.

d) $\bar{\delta}$ satisfies the condition 3 of the Proposition. Indeed, let $\bar{\psi}$ be a morphism (σ', ψ, σ) from σ to a limit-bearing category $\sigma' = (\Sigma', \Gamma')$. Part c above shows that we may suppose $\sigma' \in \mathcal{S}''_0$. Using the universal property of the inductive limit σ_ζ of ω_ζ and that of $\bar{\delta}_\xi : \sigma_\xi \to \bar{\sigma}_\xi = \sigma_{\xi+1}$ (Part 1-b), we construct by transfinite induction a sequence of morphisms $\bar{\psi}_\zeta : \sigma_\zeta \to \sigma'$, where $\zeta \leqslant \mu$, such that $\bar{\psi}_0 = \bar{\psi}$ and

$$\bar{\psi}_\zeta \cdot \omega(\zeta, \xi) = \bar{\psi}_\xi \quad \text{for any} \quad \xi < \zeta .$$

Then $\bar{\psi}_\mu$ is the unique morphism $\bar{\psi}' : \bar{\sigma} \to \sigma'$ satisfying $\bar{\psi}' \cdot \bar{\delta} = \bar{\psi}$.

(Notice that, up to now, we have not used the fact that μ is a given regular ordinal.)

4° To complete the proof, we have yet to show that $\bar{\sigma}$ is a limit-bearing category, i.e. that each cone $\bar{\gamma} \in \bar{\Gamma}$ is a limit-cone. This will imply that the tower ends with $\bar{\sigma}$ (this means that $\sigma_{\mu+1}$ is isomorphic to $\bar{\sigma}$). Suppose that $\bar{\gamma}$ is a distinguished cone of $\bar{\sigma}$; then there exists some cone $\gamma \in \Gamma$ such that $\bar{\gamma} = \delta\gamma$ (Part 3-b). Denote by ϕ the base of γ, by K its indexing-category, by γ_ξ the cone $\omega(\xi, 0)\gamma \in \Gamma_\xi$, for each $\xi < \mu$. Let $\bar{\gamma}' : u'^{\wedge} \to \phi$ be a cone in $\bar{\Sigma}$ with the same base as $\bar{\gamma}$.

a) We are going to prove the existence of an ordinal $\xi < \mu$ and of a cone γ' with the same base as the distinguished cone γ_ξ such that we have $\bar{\gamma}' = \omega(\mu, \xi)\gamma'$. Then the «formal factor» (γ_ξ, γ') determines a morphism z of $\bar{\Sigma}_\xi = \Sigma_{\xi+1}$ satisfying the equalities:

$$(\omega(\xi+1, \xi)\gamma_\xi) z = \omega(\xi+1, \xi)\gamma' ,$$

which gives, after transformation by $\omega(\mu, \xi+1)$:

$$(\delta\gamma)\bar{z} = \bar{\gamma}' , \quad \text{where} \quad \bar{z} = \omega(\mu, \xi+1)(z) .$$

Indeed, since $\bar{\Sigma}$ is the inductive limit of $p_{\mathcal{S}''}\omega_\mu : \langle \mu \rangle \to \mathfrak{M}$, for each object i of K there exists an ordinal $\xi_i < \mu$ and a $x_i \in \Sigma_{\xi_i}$ such that

$$\bar{\gamma}'(i) = \omega(\mu, \xi_i)(x_i).$$

Let $k: i \to i'$ be a morphism in K. By construction of the inductive limit $\bar{\sigma}$, the equality $\bar{\gamma}'(i') = \delta\phi(k).\bar{\gamma}'(i)$ means that there exists an ordinal ξ_k such that $\xi_i < \xi_k < \mu$, $\xi_{i'} < \xi_k < \mu$ and

(1) $\omega(\xi_k, \xi_{i'})(x_{i'}) = \omega(\xi_k, 0)(\phi(k)).\omega(\xi_k, \xi_i)(x_i)$.

μ being a regular ordinal such that

$$\bar{\bar{K}} < \mu \quad \text{and} \quad \xi_k < \mu \quad \text{for any } k \in K,$$

the upper bound ξ of the ξ_k, where $k \in K$, verifies $\xi < \mu$. For this ordinal ξ and for each $k: i \to i'$ in K, we get from (1):

$$\begin{aligned}\omega(\xi, \xi_{i'})(x_{i'}) &= \omega(\xi, \xi_k)\omega(\xi_k, \xi_{i'})(x_{i'}) \\ &= \omega(\xi, \xi_k)(\omega(\xi_k, 0)(\phi(k)).\omega(\xi_k, \xi_i)(x_i)) \\ &= \omega(\xi, 0)(\phi(k)).\omega(\xi, \xi_i)(x_i).\end{aligned}$$

This shows that the map

$$\gamma'_0 : K_0 \to \underline{\Sigma}_\xi \quad \text{such that} \quad \gamma'_0(i) = \omega(\xi, \xi_i)(x_i)$$

defines a cone γ' in Σ_ξ with the same base as $\gamma_\xi = \omega(\xi, 0)\gamma$. Moreover $\omega(\mu, \xi)\gamma' = \bar{\gamma}'$, since, for each object i of K, we have:

$$\omega(\mu, \xi)\gamma'(i) = \omega(\mu, \xi_i)(x_i) = \bar{\gamma}'(i).$$

b) We have found a $\bar{z}$ such that

$$\bar{\gamma}\bar{z} = \bar{\gamma}', \quad \text{namely} \quad \bar{z} = \omega(\mu, \xi+1)(z).$$

Suppose that $\bar{z}'$ is another morphism of $\bar{\Sigma}$ satisfying $\bar{\gamma}\bar{z}' = \bar{\gamma}'$; we show that $\bar{z} = \bar{z}'$. Indeed, there exists an ordinal $\zeta < \mu$ and a morphism z' in Σ_ζ with $\omega(\mu, \zeta)(z') = \bar{z}'$. We may suppose $\xi < \zeta$. For each $i \in K_0$, the equality $\bar{\gamma}'(i) = \bar{\gamma}(i).\bar{z}'$, which may also be written

$$\omega(\mu, \xi)\gamma'(i) = \omega(\mu, 0)\gamma(i).\omega(\mu, \zeta)(z')$$

implies the existence of an ordinal ζ_i such that $\zeta < \zeta_i < \mu$ and

$$\omega(\zeta_i, \xi)\gamma'(i) = \omega(\zeta_i, 0)\gamma(i).\omega(\zeta_i, \zeta)(z').$$

If ζ' is the upper bound of the ζ_i, for $i \in K_0$, we get as above $\zeta' < \mu$ and

$$\omega(\zeta', \xi)\gamma' = (\omega(\zeta', 0)\gamma)\omega(\zeta', \zeta)(z') = \gamma_{\zeta'}\hat{z}',$$

where $\hat{z}' = \omega(\zeta', \zeta)(z')$. From the equality

$$\omega(\xi+1, \xi)\gamma' = (\omega(\xi+1, \xi)\gamma_\xi)z,$$

it follows, by applying $\omega(\zeta', \xi+1)$:

$$\omega(\zeta', \xi)\gamma' = (\omega(\zeta', \xi)\gamma_\xi)\hat{z} = \gamma_{\zeta'}\hat{z},$$

where $\hat{z} = \omega(\zeta', \xi+1)(z)$. Hence $\hat{z}'$ and $\hat{z}$ are two morphisms such that $\gamma_{\zeta'}\hat{z} = \gamma_{\zeta'}\hat{z}'$, which implies

$$\omega(\zeta'+1, \zeta')(\hat{z}) = \omega(\zeta'+1, \zeta')(\hat{z}'),$$

by construction of $\Sigma_{\zeta'+1} = \overline{\Sigma}_{\zeta'}$ (Part 1). Finally, applying $\omega(\mu, \zeta'+1)$, we get $\overline{z} = \overline{z}'$. ∇

DEFINITION. With the hypotheses of Proposition 3, we call $\overline{\sigma}$ the *limit-bearing category generated by* σ.

COROLLARY 1. *The insertion functor* $I: \mathcal{P}' \to \mathcal{S}''$ *admits a (left) adjoint.* $\mathcal{P}'$ *admits* $\mathcal{F}_0$*-inductive limits, and there exist quasi-quotient limit-bearing categories.*

Δ. The first statement results from Proposition 3.

If $F: C \to \mathcal{P}'$ is a functor, where $\underline{C}$ is a $\mathcal{U}$-set, then $IF: C \to \mathcal{S}''$ admits an inductive limit σ (Proposition 1), and the limit-bearing category $\overline{\sigma}$ generated by σ is an inductive limit of F.

Let σ' be a cone-bearing neocategory (Σ', Γ') and ρ an equivalence relation on $\underline{\Sigma}'$. There exists a quasi-quotient cone-bearing neocategory σ of σ' by ρ (i.e. a quasi-quotient $p_{\mathcal{S}''}$-structure [E1]); namely, $\sigma = (\Sigma, \Gamma)$, where Σ is the neocategory quotient of Σ' by the smallest compatible equivalence relation on Σ' containing ρ and where

$$\Gamma = \{ \hat{\rho}\gamma' \mid \gamma' \in \Gamma' \}, \quad \text{if } \hat{\rho}: \Sigma' \to \Sigma$$

is the neofunctor corresponding to ρ. Hence the limit-bearing category $\overline{\sigma}$ generated by σ is the quasi-quotient limit-bearing category of σ' by ρ. If $\sigma' \in \mathcal{P}'_0$, then $\overline{\sigma}$ is a quasi-quotient $p_{\mathcal{P}'}$-structure of σ' by ρ. ∇

COROLLARY 2. *Let* σ *be a cone-bearing neocategory and* $\overline{\sigma}$ *the limit-bearing category generated by* σ. *If* σ' *is a limit-bearing category, then the* *categories* $\mathcal{S}(\sigma', \sigma)$ *and* $\mathcal{S}(\sigma', \bar{\sigma})$ *are isomorphic. In particular,* Σ'^{σ} *and* $\Sigma'^{\bar{\sigma}}$ *are isomorphic, for every category* Σ'.

Δ. Let $\boxminus\sigma'$ be the lateral limit-bearing category of quartets of σ' (Example 2-2). We have constructed, in Proposition 2, bijections

$$g: \underline{\mathcal{S}}(\sigma', \sigma) \to \mathcal{S}(\boxminus\sigma', \sigma)_0 \quad \text{and} \quad h: \underline{\mathcal{S}}(\sigma', \bar{\sigma}) \to \mathcal{S}(\boxminus\sigma', \bar{\sigma})_0 .$$

By Proposition 3, there is a canonical bijection

$$d: \mathcal{S}(\boxminus\sigma', \bar{\sigma})_0 \to \mathcal{S}(\boxminus\sigma', \sigma)_0 ,$$

assigning $\psi'\delta$ to the $\bar{\sigma}$-structure ψ', where $\bar{\delta} = (\bar{\sigma}, \delta, \sigma)$ is the canonical morphism. The bijection $g^{-1}dh$ defines the isomorphism from the category $\mathcal{S}(\sigma', \bar{\sigma})$ to $\mathcal{S}(\sigma', \sigma)$ assigning $\tau\delta$ to τ. ∇

REMARKS. 1° $\bar{\sigma}$ is «universal» relative to all σ-structures, and not only to those which are «small enough». The universe $\mathcal{U}$ is used as a tool in the proof of Proposition 3, and it does not appear in the conclusion (as we have shown in Part 3-c). We could have omitted $\mathcal{U}$ by considering «the category of morphisms between all cone-bearing neocategories» (i. e. by admitting a theory of sets and classes).

2° In [L], Corollary 1 of Proposition 3 is deduced from the general existence theorem for free structures of [E], the proof being identical with the argument used in [E5] to prove the existence of the prototype of σ. Above, we have not only shown the existence of $\bar{\sigma}$, but we have also given an explicit construction of it, from which many properties of $\bar{\sigma}$ may be deduced. This construction is suggested by the explicit construction of a free $\mathcal{I}$-projective completion of a category (Theorem 7 of [E]); the main difference, apart from adding «no objects», lies in the fact that the hypotheses of Theorem 7 of [E] (after adding «all formal cones») implied the injectivity of the functor $\delta_\xi : \Sigma_\xi \to \Sigma_{\xi+1}$, for any ordinal (which was difficult to prove and required a detailed description of the morphisms of $\Sigma_{\xi+1}$ as «reduced paths»); so, the category Σ_ζ, for a limit ordinal ζ, was just the union of the categories Σ_ξ, for $\xi < \zeta$. This is no more true here, and we have to define Σ_ζ, for a limit ordinal ζ, as the inductive limit of the functor $q\mathcal{S}''\omega_\zeta : \langle \zeta \rangle \to \mathcal{F}'$.

3° Proposition 3 may also be expressed as follows: Let σ be a cone-bearing neocategory. *There exists a limit-bearing category* $\bar{\sigma} = (\bar{\Sigma}, \bar{\Gamma})$, *characterized up to an isomorphism by the property:*

If $\mathfrak{U}$ *is a universe such that* $\sigma \in \mathcal{S}''_0$, *then* $\bar{\sigma}$ *is a free structure generated by* σ *relative to the insertion functor from* $\mathcal{P}'$ *to* $\mathcal{S}''$.

Intuitively, if $\sigma = (\Sigma, \Gamma)$, the set $\bar{\underline{\Sigma}}$ belongs to the smallest universe to which belongs $\underline{\Sigma}$, while $\bar{\sigma}$ solves the universal problem for *any* universe to which belongs $\underline{\Sigma}$.

4. Loose types.

Let σ be a cone-bearing neocategory and σ' a limit-bearing category. We have seen that there exist limit-bearing categories $\bar{\sigma}$ such that the category $\mathcal{S}(\sigma', \sigma)$ is isomorphic with $\mathcal{S}(\sigma', \bar{\sigma})$. In fact, we have constructed a $\bar{\sigma}$ which is minimal. Now the question arises: If any functor is the base of a distinguished cone in σ', does there exist a $\bar{\sigma}$ with the same property? We are going to solve this problem relative to a given set of categories.

We denote by $\mathcal{I}$ a set of categories.

DEFINITION. If σ is a cone-bearing neocategory (resp. a limit-bearing category) whose set of indexing-categories $\mathcal{I}_\sigma$ is a subset of $\mathcal{I}$, we also say that σ is a $\mathcal{I}$*-cone-bearing neocategory* (resp. a $\mathcal{I}$*-limit-bearing category).*

In particular, σ is a $\mathcal{I}_\sigma$-cone-bearing neocategory.

DEFINITION. Let σ be a limit-bearing category (Σ, Γ) and $\mathcal{I}$ its set of indexing-categories. We say that σ is *a loose type* (or, more precisely, a loose $\mathcal{I}$-type) if each functor $\phi: K \to \Sigma$, where $K \in \mathcal{I}$, is the base of at least one distinguished limit-cone $\gamma \in \Gamma$.

This condition implies that Σ admits $\mathcal{I}$-projective limits.

If $\mathfrak{U}$ is a universe such that $\mathcal{I}$ is a $\mathfrak{U}$-set, we denote by $\mathcal{S}''^{\mathcal{I}}$ (resp. $\mathcal{P}'^{\mathcal{I}}$, resp. $\mathcal{L}^{\mathcal{I}}$) the full subcategory of $\mathcal{S}''$ whose objects are the $\mathcal{I}$-cone-bearing neocategories (resp. the $\mathcal{I}$-limit-bearing categories, resp. the loose

$\mathcal{J}$-types) σ belonging to $\mathfrak{S}''_0$.

PROPOSITION 4. *Let σ be a $\dot{\mathcal{J}}$-cone-bearing neocategory (Σ, Γ). There exists a loose $\mathcal{J}$-type $\bar{\sigma}$ (unique up to an equivalence) and a morphism $\bar{\delta} = (\bar{\sigma}, \delta, \sigma)$ satisfying the following condition where σ' is a loose $\mathcal{J}'$-type for a set $\mathcal{J}'$ of categories containing $\mathcal{J}$:*

1° If $\bar{\psi} : \sigma \to \sigma'$ is a morphism, there exist morphisms $\bar{\psi}' : \bar{\sigma} \to \sigma'$ such that $\bar{\psi}' \cdot \bar{\delta} = \bar{\psi}$, and two such morphisms are equivalent.

2° If $\bar{\psi}' = (\sigma', \psi', \bar{\sigma})$ and $\bar{\psi}'' = (\sigma', \psi'', \bar{\sigma})$ are morphisms and if $\tau : \psi' \delta \to \psi'' \delta$ is a natural transformation (resp. an equivalence), there exists one and only one natural transformation (resp. equivalence)

$$\tau' : \psi' \to \psi'' \quad \textit{such that} \quad \tau' \delta = \tau .$$

Moreover, if $\mathfrak{U}$ is a universe such that $\mathcal{J}$ is a $\mathfrak{U}$-set and $\sigma \in \mathfrak{S}''_0$, then we
1 *have $\bar{\sigma} \in \mathfrak{L}_0^{\mathcal{J}}$.*

Δ. We will again construct, by transfinite induction, a tower of cone-bearing neocategories which stops («up to an equivalence») at the first regular ordinal μ greater than all the ordinals $\bar{\bar{K}}$, where $K \in \mathcal{J}$. The method is similar to that used in Proposition 3, but, in the «non-limit step» from σ_ξ to $\sigma_{\xi+1}$, we will add also «formal cones» for each neofunctor indexed by an element of $\mathcal{J}$.

1° Let us first describe this non-limit step. We suppose that σ_ξ is a cone-bearing neocategory (Σ_ξ, Γ_ξ).

a) Let us consider:

- the set Ω of pairs (γ, γ') (or «formal factors»), where $\gamma \in \Gamma_\xi$ and γ' is a cone in Σ_ξ with the same base as γ,
- the set M of neofunctors $\phi : K \to \Sigma_\xi$, where $K \in \mathcal{J}$, which are not the base of any distinguished cone $\gamma \in \Gamma_\xi$,
- the set M' of pairs (i, ϕ), where $\phi \in M$ and where i is an object of the indexing-category of ϕ,
- the sum («disjoint union») U of $\underline{\Sigma}_\xi$, Ω, M and M', with injections:

$$v : \underline{\Sigma}_\xi \to U, \quad v' : \Omega \to U, \quad w : M \to U, \quad w' : M' \to U .$$

We describe a graph (U, β, α) in the following way:

- if $x: u \to u'$ in Σ_ξ, then
$$\alpha(v(x)) = v(u), \quad \beta(v(x)) = v(u'),$$
- if $(\gamma, \gamma') \in \Omega$, with $\gamma : u^\wedge \to \phi$ and $\gamma' : u'^\wedge \to \phi$,
$$\alpha(v'(\gamma, \gamma')) = v(u'), \quad \beta(v'(\gamma, \gamma')) = v(u),$$
- $w(\phi)$ is a vertex, for each $\phi \in M$,
- if $(i, \phi) \in M'$, we have:
$$\alpha(w'(i, \phi)) = w(\phi), \quad \beta(w'(i, \phi)) = v(\phi(i)).$$

We denote by L the free category generated by (U, β, α) and by r the equivalence relation on $\underline{L}$ satisfying both the condition (P) of Part 1, Proposition 3 and the condition

$$(\text{P}') \begin{cases} (v(\phi(k)), w'(i, \phi)) \sim w'(i', \phi) \\ \text{if} \quad \phi \in M, \quad \phi: K \to \Sigma_\xi, \quad k: i \to i' \text{ in } K. \end{cases}$$

There exists a quotient category $\overline{\Sigma}_\xi$ of L by the smallest equivalence relation compatible on L and containing r. Let $\rho: L \to \overline{\Sigma}_\xi$ be the canonical functor corresponding to r; the map $\underline{\rho}\, v$ defines a neofunctor δ_ξ from Σ_ξ to $\overline{\Sigma}_\xi$.

If $\phi: K \to \Sigma_\xi$ belongs to M, let γ_ϕ be the cone in $\overline{\Sigma}_\xi$ with vertex $\rho(w(\phi))$ and base $\delta_\xi \phi$ such that
$$\gamma_\phi(i) = \rho(w'(i, \phi)) \quad \text{for any } i \in K_0$$
(it will be called «the formal cone associated to ϕ»). Put:
$$\overline{\Gamma}_\xi = \{ \delta_\xi \gamma \mid \gamma \in \Gamma_\xi \} \cup \{ \gamma_\phi \mid \phi \in M \}.$$
Then $(\overline{\Sigma}_\xi, \overline{\Gamma}_\xi)$ is a cone-bearing (neo)category $\overline{\sigma}_\xi$ and δ_ξ defines a morphism $\overline{\delta}_\xi : \sigma_\xi \to \overline{\sigma}_\xi$.

When $\mathfrak{U}$ is a universe such that $\mathcal{I}$ is a $\mathfrak{U}$-set and $\sigma_\xi \in \mathcal{S}''_0$, the set of neofunctors $\phi: K \to \Sigma_\xi$, where $K \in \mathcal{I}$, is a $\mathfrak{U}$-set, as well as the set of cones in Σ_ξ indexed by elements of $\mathcal{I}$. It follows that M, M' and Ω are $\mathfrak{U}$-sets. Hence $\overline{\sigma}_\xi \in \mathcal{S}''_0$.

b) Let $\mathcal{I}'$ be a set of categories containing $\mathcal{I}$ and σ' a loose $\mathcal{I}'$-type (Σ', Γ'). If $\overline{\psi} = (\sigma', \psi, \sigma_\xi)$ is a morphism, there exists at least one morphism $\overline{\psi}': \overline{\sigma}_\xi \to \sigma'$ such that $\overline{\psi}' \cdot \overline{\delta}_\xi = \overline{\psi}$.

If $\overline{\psi}'$ exists and if $\phi \in M$, then ψ' will transfer the formal cone γ_ϕ into a distinguished cone γ' of σ' admitting $\psi\phi$ as base; since there may be several cones of this kind, ψ' will be defined «up to a choice» of cones γ'.

Hence, for each element $\phi: K \to \Sigma_\xi$ of M, we choose a distinguished cone $\eta_\phi : e_\phi{}^\wedge \to \psi\phi$ of Γ' and we define mappings

- $g: M \to \underline{\Sigma}'$ by $g(\phi) = e_\phi$,
- $g': M' \to \underline{\Sigma}'$ by $g'(i, \phi) = \eta_\phi(i)$,
- $f: \Omega \to \underline{\Sigma}'$ by $f(\gamma, \gamma') = y$, where y is the unique morphism such that $(\psi\gamma)y = \psi\gamma'$.

As in Part 1, Proposition 3, there exists a unique functor $F': L \to \Sigma'$ «extending» ψ, f, g, g', and a unique functor:

$$\psi' : \overline{\Sigma}_\xi \to \Sigma' \quad \text{such that} \quad \psi'\rho = F'.$$

Moreover the equivalence relation r is such that ψ' is a $\overline{\sigma}_\xi$-structure in σ'. By construction, ψ' defines the unique morphism $\overline{\psi}': \overline{\sigma}_\xi \to \sigma'$ satisfying the conditions:

$$\overline{\psi}' \cdot \overline{\delta}_\xi = \overline{\psi} \quad \text{and} \quad \overline{\psi}'\gamma_\phi = \eta_\phi \quad \text{for any} \quad \phi \in M.$$

c) If σ' is a loose type, if $\overline{\psi}' = (\sigma', \psi', \overline{\sigma}_\xi)$ and $\overline{\psi}'' = (\sigma', \psi'', \overline{\sigma}_\xi)$ are morphisms and if $\tau: \psi'\delta_\xi \to \psi''\delta_\xi$ is a natural transformation, there exists a unique natural transformation $\tau': \psi' \to \psi''$ such that $\tau'\delta_\xi = \tau$.

Indeed, let us consider the lateral limit-bearing category $\boxminus\sigma' = (\boxminus\Sigma', \hat{\Gamma})$ of quartets of σ'. We identify the objects of $\boxminus\Sigma'$ with the morphisms of Σ'. Since σ' is a loose type, $\boxminus\sigma'$ is also one. Proposition 2 canonically associates to τ a neofunctor $T: \Sigma_\xi \to \boxminus\Sigma'$ defining a morphism $\overline{T}: \sigma_\xi \to \boxminus\sigma'$. If $\phi: K \to \Sigma_\xi$ belongs to M, the cones $\psi'\gamma_\phi$ and $\psi''\gamma_\phi$ are two distinguished limit-cones with bases $\psi'\delta_\xi\phi$ and $\psi''\delta_\xi\phi$. Since $\tau\phi: \psi'\delta_\xi\phi \to \psi''\delta_\xi\phi$ is a natural transformation, there exists a unique morphism x_ϕ in Σ' such that:

$$(\psi''\gamma_\phi)x_\phi = \tau\phi \,\square\, (\psi'\gamma_\phi).$$

By assigning to an object i of K the quartet

$$\hat{\eta}_\phi(i) = (\psi''\gamma_\phi(i), \tau\phi(i), x_\phi, \psi'\gamma_\phi(i)),$$

we define a cone $\hat{\eta}_\phi : x_\phi{}^\wedge \to T\phi$ in $\boxminus\Sigma'$ which belongs to $\hat{\Gamma}$ (by the definition of $\boxminus\sigma'$). Part b asserts the existence of a unique morphism

$$\overline{T}' = (\boxminus\sigma', T', \overline{\sigma}_\xi) \text{ such that } T'\delta_\xi = T \text{ and } T'\gamma_\phi = \hat{\eta}_\phi$$
$$\text{for any } \phi \in M.$$

Let $\tau' : \theta \to \theta'$ be the natural transformation to which T' is associated; the equality $T'\delta_\xi = T$ implies

$$\tau'\delta_\xi = \tau, \quad \theta\,\delta_\xi = \psi'\delta_\xi, \quad \theta'\delta_\xi = \psi''\delta_\xi$$

and, for each $\phi \in M$, from $T'\gamma_\phi = \hat{\eta}_\phi$, we deduce

$$\tau'(\rho(w(\phi))) = x_\phi, \quad \theta\gamma_\phi = \psi'\gamma_\phi, \quad \theta'\gamma_\phi = \psi''\gamma_\phi.$$

θ and θ' define morphisms from $\overline{\sigma}_\xi$ to σ'. Hence, using Part b, we get

$$\theta = \psi' \quad \text{and} \quad \theta' = \psi''.$$

Since $\overline{T}'$ and x_ϕ are determined in a unique way, τ' is the unique natural transformation from ψ' to ψ'' satisfying $\tau'\delta_\xi = \tau$. Moreover, if τ is an equivalence, x_ϕ is invertible for every $\phi \in M$, so that τ' is also an equivalence.

2° a) Let λ be the upper bound of the ordinals $\overline{\overline{K}}$, where $K \in \mathcal{J}$, and μ the least regular ordinal greater than λ. We can choose a universe $\mathcal{U}$, such that

$$(\mathcal{J} \cup \underline{\Sigma} \cup \Gamma \cup \bigcup_{K \in \mathcal{J}} \underline{K}) \in \mathcal{U};$$

then $\sigma \in \mathcal{S}''_0$. As in Part 2, Proposition 3, we see that μ is a $\mathcal{U}$-set and we define by transfinite induction a functor $\omega : \langle \mu + 1 \rangle \to \mathcal{S}''$ (whose values are independent of $\mathcal{U}$) satisfying the following conditions, where

$$\omega(\xi) = \sigma_\xi = (\Sigma_\xi, \Gamma_\xi) \quad \text{for any } \xi \leq \mu :$$

- $\omega(0) = \sigma$;
- for each limit ordinal ζ, with $\zeta \leq \mu$, we take for σ_ζ the canonical inductive limit of the functor $\omega_\zeta : \langle \zeta \rangle \to \mathcal{S}''$ restriction of ω, and for $\omega(\zeta, \xi)$ the injection from σ_ξ to σ_ζ, if $\xi < \zeta$.
- If $\zeta = \xi + 1$, where $\xi < \mu$, then σ_ζ is the cone-bearing (neo)cate-

gory $\overline{\sigma}_\xi$ associated to σ_ξ in Part 1 and $\omega(\zeta,\xi)$ is the morphism $\overline{\delta}_\xi$. We write

$$\begin{cases} \overline{\sigma} = \sigma_\mu, \quad \overline{\delta} = (\overline{\sigma}, \delta, \sigma) = \omega(\mu, 0) \\ \omega(\zeta,\xi) = (\sigma_\zeta, \omega_{\zeta\xi}, \sigma_\xi) \text{ when } \xi < \zeta \leqslant \mu . \end{cases}$$

b) Let σ' be a loose $\mathcal{J}'$-type, where $\mathcal{J}'$ contains $\mathcal{J}$, and $\overline{\psi}: \sigma \to \sigma'$ a morphism. Using Part 1-b, we construct by transfinite induction morphisms $\overline{\psi}_\zeta : \sigma_\zeta \to \sigma'$ for each $\zeta \leqslant \mu$, such that

$$\overline{\psi}_0 = \overline{\psi}, \quad \overline{\psi}_\zeta \cdot \omega(\zeta,\xi) = \overline{\psi}_\xi \quad \text{if } \xi < \zeta .$$

In particular $\overline{\psi}_\mu$ is a morphism $\overline{\psi}' : \overline{\sigma} \to \sigma'$ for which $\overline{\psi}' \cdot \overline{\delta} = \overline{\psi}$.

Now, let ψ' and ψ'' be two $\overline{\sigma}$-structures in σ' and $\tau: \psi'\delta \to \psi''\delta$ a natural transformation. Suppose that ζ is an ordinal, $\zeta \leqslant \mu$, and that, for each $\xi < \zeta$, there exists a natural transformation

$$\tau_\xi : \psi'\omega_{\mu\xi} \to \psi''\omega_{\mu\xi} \text{ such that } \tau_\xi\,\omega_{\xi 0} = \tau \text{ for any } \xi < \zeta .$$

- If $\zeta = \xi + 1$, Part 1-c shows the existence and the unicity of a natural transformation $\tau_\zeta : \psi'\omega_{\mu\zeta} \to \psi''\omega_{\mu\zeta}$ such that $\tau_\zeta\omega_{\zeta\xi} = \tau_\xi$, and so

$$\tau_\zeta\,\omega_{\zeta 0} = \tau_\xi\,\omega_{\xi 0} = \tau .$$

- If ζ is a limit ordinal and if $T_\xi : \Sigma_\xi \to \boxminus\Sigma'$ is the neofunctor associated to τ_ξ, for any $\xi < \zeta$, there exists a unique neofunctor

$$T_\zeta : \Sigma_\zeta \to \boxminus\Sigma' \text{ such that } T_\zeta\omega_{\zeta\xi} = T_\xi \text{ for any } \xi < \zeta ,$$

since Σ_ζ is the inductive limit of ω_ζ. Hence the natural transformation corresponding to T_ζ is the unique natural transformation

$$\tau_\zeta : \psi'\omega_{\mu\zeta} \to \psi''\omega_{\mu\zeta} \text{ satisfying } \tau_\zeta\omega_{\zeta\xi} = \tau_\xi \text{ for any } \xi < \zeta .$$

- By transfinite induction, we so define a natural transformation τ_μ, which is the unique natural transformation

$$\tau' : \psi' \to \psi'' \text{ such that } \tau'\delta = \tau .$$

3° We have yet to prove that $\overline{\sigma} = (\overline{\Sigma}, \overline{\Gamma})$ is a loose $\mathcal{J}$-type.

a) We see that $\overline{\Sigma}$ is a category as in Part 3-a Proposition 3. Suppose that $\overline{\gamma}$ is a distinguished cone of $\overline{\Gamma}$. By a method similar to that used in Part 4-a, Proposition 3, we get an ordinal $\zeta < \mu$ and a cone $\gamma \in \Gamma_\zeta$ such that $\overline{\gamma} = \omega_{\mu\zeta}\gamma$, and we still deduce similarly that $\overline{\gamma}$ is a limit-cone.

b) Let $\phi' : K \to \overline{\Sigma}$ be a functor, where $K \in \mathcal{I}$. There exists a cone $\overline{\gamma} \in \overline{\Gamma}$ with base ϕ'.

Indeed, for each $k \in K$, there exists an ordinal $\xi_k < \mu$ and a morphism x_k of Σ_{ξ_k} such that $\phi'(k) = \omega(\mu, \xi_k)(x_k)$. If the composite $k'.k$ is defined in K, the equality $\phi'(k').\phi'(k) = \phi'(k'.k)$ implies the existence of an ordinal $\xi_{k',k}$ greater than ξ_k, $\xi_{k'}$ and $\xi_{k'.k}$ such that $\xi_{k',k} < \mu$ and (1):

$$\omega(\xi_{k',k}, \xi_{k'})(x_{k'}).\omega(\xi_{k',k}, \xi_k)(x_k) = \omega(\xi_{k',k}, \xi_{k'.k})(x_{k'.k}).$$

We denote by ξ the ordinal upper bound of the family of the $\xi_{k',k}$, where (k', k) belongs to the set $K * K$ of composable couples. Since $\overline{\overline{K}} < \mu$, the cardinal of $K * K$ is strictly less than the regular ordinal μ, so that $\xi < \mu$. Put

$$\phi(k) = \omega(\xi, \xi_k)(x_k) \quad \text{for any } k \in K;$$

if the composite $k'.k$ is defined, we get

$$\phi(k').\phi(k) = \phi(k'.k)$$

(by applying $\omega(\xi, \xi_{k',k})$ to (1)); so, we have defined a functor

$$\phi : K \to \Sigma_\xi \quad \text{such that} \quad \omega_{\mu\xi}\phi = \phi'.$$

By construction of $\Sigma_{\xi+1} = \overline{\Sigma}_\xi$ (Part 1), there exists a distinguished cone $\gamma_\phi \in \Gamma_{\xi+1}$ with base $\omega_{\xi+1\,\xi}\phi$. Hence $\omega_{\mu\,\xi+1}\gamma_\phi$ is a cone of $\overline{\Gamma}$, admitting $\omega_{\mu\,\xi}\phi = \phi'$ as its base. ∇

DEFINITION. If $\overline{\sigma} = (\overline{\Sigma}, \overline{\Gamma})$ is a loose type satisfying the conditions of Proposition 4, we call $\overline{\sigma}$ a *loose $\mathcal{I}$-type generated by* σ (or of σ) and $\overline{\Sigma}$ *a loose $\mathcal{I}$-projective completion of* σ.

COROLLARY 1. *Let σ be a $\mathcal{I}$-cone-bearing neocategory and $\overline{\sigma}$ a loose $\mathcal{I}$-type generated by σ. If σ' is a loose $\mathcal{I}'$-type, where $\mathcal{I}'$ contains $\mathcal{I}$, the categories $\mathcal{S}(\sigma', \sigma)$ and $\mathcal{S}(\sigma', \overline{\sigma})$ are equivalent.*

Δ. Let $\overline{\delta} = (\overline{\sigma}, \delta, \sigma)$ be the canonical morphism. We have a functor $F : \mathcal{S}(\sigma', \overline{\sigma}) \to \mathcal{S}(\sigma', \sigma)$ assigning $\tau'\delta$ to the natural transformation τ' between $\overline{\sigma}$-structures in σ'. This functor defines an equivalence. Indeed, for each σ-structure ψ in σ', Proposition 4 asserts the existence of $\bar{\sigma}$-structures ψ' in σ' for which $\psi'\delta = \psi$; choosing one of them, we denote it by $G(\psi)$. If $\tau: \psi \to \theta$ is an element of $\mathcal{S}(\sigma',\sigma)$, there exists a unique natural transformation

$$G(\tau): G(\psi) \to G(\theta) \quad \text{such that} \quad G(\tau)\delta = \tau$$

(Proposition 4). From the unicity of $G(\tau)$, it results that we define in this way a functor $G: \mathcal{S}(\sigma',\sigma) \to \mathcal{S}(\sigma',\bar{\sigma})$. The equalities

$$FG(\tau) = G(\tau)\delta = \tau, \quad \text{for any} \quad \tau,$$

mean that FG is an identity.

On the other hand, for each $\bar{\sigma}$-structure ψ' in σ', we have

$$GF(\psi')\delta = F(\psi') = \psi'\delta,$$

so that there exists a unique equivalence $\eta(\psi'): \psi' \to GF(\psi')$ for which $\eta(\psi')\delta$ is an identity. If $\tau': \psi' \to \theta'$ is an element of $\mathcal{S}(\sigma',\bar{\sigma})$, we get

$$\eta(\theta') \boxdot \tau' = GF(\tau') \boxdot \eta(\psi'),$$

since

$$(\eta(\theta') \boxdot \tau')\delta = \eta(\theta')\delta \boxdot \tau'\delta = \tau'\delta = F(\tau')$$

and

$$(GF(\tau') \boxdot \eta(\psi'))\delta = GF(\tau')\delta \boxdot \eta(\psi')\delta = \\ = GF(\tau')\delta = F(\tau').$$

Hence we have defined an equivalence $\eta: Id_{\mathcal{S}(\sigma',\bar{\sigma})} \to GF$. ∇

COROLLARY 2. *Let σ be a $\mathcal{J}$-cone-bearing neocategory (Σ, Γ) and Σ' a category admitting $\mathcal{J}$-projective limits. Then the category Σ'^{σ} is equivalent to the full subcategory of $\Sigma'^{\bar{\Sigma}}$ whose objects are the functors from $\bar{\Sigma}$ to Σ' which commute with $\mathcal{J}$-projective limits, $\bar{\Sigma}$ denoting a loose $\mathcal{J}$-projective completion of σ.*

Δ. Let us denote by $\bar{\sigma} = (\bar{\Sigma}, \bar{\Gamma})$ a loose $\mathcal{J}$-type generated by σ and let σ' be the full $\mathcal{J}$-limit-bearing category on Σ'; since σ' is a loose type, the categories

$$\Sigma'^{\sigma} = \mathcal{S}(\sigma',\sigma) \quad \text{and} \quad \Sigma'^{\bar{\sigma}} = \mathcal{S}(\sigma',\bar{\sigma})$$

are equivalent, by Corollary 1. If $\psi': \bar{\Sigma} \to \Sigma'$ is a functor, it commutes with

$\mathcal{J}$-projective limits iff each functor from $K \in \mathcal{J}$ to $\overline{\overline{\Sigma}}$ is the base of a limit-cone γ, where $\psi'\gamma$ is a limit-cone. Hence ψ' is a $\overline{\sigma}$-structure in Σ' iff ψ' commutes with $\mathcal{J}$-projective limits. This means that $\Sigma'^{\overline{\sigma}}$ is the full subcategory of $\Sigma'^{\overline{\Sigma}}$ whose objects are functors commuting with $\mathcal{J}$-projective limits. ∇

COROLLARY 3. *A loose $\mathcal{J}$-projective completion $\overline{\overline{\Sigma}}$ of a $\mathcal{J}$-cone-bearing neocategory σ is characterized up to an equivalence by the conditions:*

1° $\overline{\overline{\Sigma}}$ admits $\mathcal{J}$-projective limits.

2° There exists a σ-structure δ in $\overline{\overline{\Sigma}}$ satisfying the universal property: If Σ' is a category admitting $\mathcal{J}$-projective limits and if ψ is a σ-structure in Σ', there exists a functor ψ', unique up to an equivalence, such that ψ' commutes with $\mathcal{J}$-projective limits and $\psi'\delta = \psi$.

Δ. Condition 2 results from Proposition 4, applied to the full $\mathcal{J}$-limit-bearing category σ' on Σ'. ∇

REMARKS. 1° The construction of the loose $\mathcal{J}$-type $\overline{\sigma} = (\overline{\overline{\Sigma}}, \overline{\Gamma})$, generated by $\sigma = (\Sigma, \Gamma)$, is yet suggested by the explicit construction of a free $\mathcal{J}$-projective completion of a category (Theorem 7 [E], in which Σ is a category and Γ is void); the difference is that we do not require that there exists only one cone of $\overline{\Gamma}$ with a given base (this problem will be studied in Paragraph 5). Notice that the general Proposition 4 and Corollaries cannot be immediatly deduced from the general existence theorem of free structures. Indeed, if σ' is a loose $\mathcal{J}$-type (Σ', Γ') and if A is a subset of $\underline{\Sigma}'$, there does not exist a «smallest» loose $\mathcal{J}$-type extracted from σ' and containing A.

2° The loose $\mathcal{J}$-type $\overline{\sigma}$ is defined up to an equivalence, and not up to an isomorphism (as the limit-bearing category generated by σ); so, Proposition 4 does not imply the existence of an adjoint for the insertion functor from $\mathcal{L}^{\mathcal{J}}$ to $\mathcal{S}''^{\mathcal{J}}$. In fact, we have proved the following result:

Let $\mathcal{L}^{\mathcal{J}}_{\sim}$ (resp. $\mathcal{S}''^{\mathcal{J}}_{\sim}$) be the quotient category of $\mathcal{L}^{\mathcal{J}}$ (resp. of $\mathcal{S}''^{\mathcal{J}}$) by the equivalence (generated by): $\overline{\psi}$ and $\overline{\psi}'$ are equivalent iff there exists an equivalence between the neofunctors defining them. This category has the same objects as $\mathcal{L}^{\mathcal{J}}$ (resp. as $\mathcal{S}''^{\mathcal{J}}$). From Proposition 4, it results:

COROLLARY 4. *Let* σ *be a* $\mathcal{J}$*-cone-bearing neocategory; there exists a loose* $\mathcal{J}$*-type* $\bar{\sigma}$ *satisfying the following condition:*

If $\mathfrak{U}$ *is a universe such that* $\mathcal{J}$ *is a* $\mathfrak{U}$*-set and* $\sigma \in \mathcal{S}''_0$*, then* $\bar{\sigma}$ *is a free structure generated by* σ *relative to the insertion functor* $\mathcal{L}^{\mathcal{J}}_{\sim} \hookrightarrow \mathcal{S}''^{\mathcal{J}}_{\sim}$.

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5. Presketches. Prototypes. Sketches.

These are special cone-bearing neocategories and limit-bearing categories. We are going to show that a cone-bearing neocategory generates a presketch and a prototype π. If σ is mapped injectively into π (we then call σ a sketch), the limit-bearing category generated by σ is itself a prototype, isomorphic with π.

DEFINITION. A cone-bearing neocategory (Σ, Γ) is called a (projective) *presketch* if there exists at most one distinguished cone $\gamma \in \Gamma$ with base a given neofunctor ϕ. A limit-bearing category which is a presketch is called a *prototype*.

The cone-bearing neocategory (Σ, Γ) is a presketch iff Γ is the image of a mapping assigning to some neofunctors $\phi: K \to \Sigma$ a cone in Σ with base ϕ. So, the notion of a presketch is equivalent to that used in [E5]. In particular, as in [E5] a prototype «is» a category equipped with a partial choice of projective limit-cones.

$\mathfrak{U}$ being a universe, we denote by $\mathcal{S}'$ (resp. by $\mathcal{P}$) the full subcategory of $\mathcal{S}''$ whose objects are the presketches (resp. the prototypes) belonging to $\mathcal{S}''$. It results from [E5] that $\mathcal{S}'$ and $\mathcal{P}$ are closed in $\mathcal{S}''$ for projective limits.

PROPOSITION 5. *Let* σ *be a cone-bearing neocategory* (Σ, Γ)*. There exists a presketch* $\bar{\sigma} = (\bar{\Sigma}, \bar{\Gamma})$*, determined up to an isomorphism by the following condition:*

If $\mathfrak{U}$ *is a universe such that* $\sigma \in \mathcal{S}''_0$*, then* $\bar{\sigma}$ *is a free structure generated by* σ *relative to the insertion functor from* $\mathcal{S}'$ *to* $\mathcal{S}''$.

Δ. We shall construct $\bar{\sigma}$ by transfinite induction, the idea being at each step to «identify» distinguished cones with the same base.

1° Let σ_ξ be a cone-bearing neocategory (Σ_ξ, Γ_ξ). We consider the smallest equivalence relation r on $\underline{\Sigma}_\xi$ such that:

$$(\mathrm{P''})\begin{cases}\gamma(i) \sim \gamma'(i), \text{ for each } i \in K_0, \text{ if } \gamma \text{ and } \gamma' \text{ are two cones of } \Gamma_\xi \\ \text{with the same base, indexed by } K.\end{cases}$$

There exists a canonical quasi-quotient neocategory $\overline{\Sigma}_\xi$ of Σ_ξ by r (it is [E1] the quotient neocategory of Σ_ξ by the smallest equivalence relation containing r and compatible with the law of composition and with the maps source and target of Σ_ξ). Let $\delta_\xi : \Sigma_\xi \to \overline{\Sigma}_\xi$ be the canonical neofunctor and put:

$$\overline{\Gamma}_\xi = \{ \delta_\xi \gamma \mid \gamma \in \Gamma_\xi \}, \quad \overline{\sigma}_\xi = (\overline{\Sigma}_\xi, \overline{\Gamma}_\xi).$$

Then $\overline{\sigma}_\xi$ is a cone-bearing neocategory and δ_ξ defines a morphism $\overline{\delta}_\xi$ from σ_ξ to $\overline{\sigma}_\xi$.

If $\mathfrak{U}$ is a universe, the quotient of a $\mathfrak{U}$-set is a $\mathfrak{U}$-set, so that $\overline{\underline{\Sigma}}_\xi$ is a $\mathfrak{U}$-set when $\underline{\Sigma}_\xi$ is a $\mathfrak{U}$-set; if Γ_ξ is also a $\mathfrak{U}$-set, $\overline{\Gamma}_\xi$ is a $\mathfrak{U}$-set.

If σ' is a presketch (Σ', Γ') and if $\overline{\psi} = (\sigma', \psi, \sigma)$ is a morphism, ψ is compatible with r, and the unique neofunctor

$$\psi' : \overline{\Sigma}_\xi \to \Sigma' \quad \text{such that} \quad \psi' \delta_\xi = \psi$$

defines the unique morphism

$$\overline{\psi}' : \overline{\sigma}_\xi \to \sigma' \quad \text{such that} \quad \overline{\psi}' \cdot \overline{\delta}_\xi = \overline{\psi}.$$

2° Let μ be the smallest regular ordinal such that $\overline{\overline{K}} < \mu$ for each indexing-category of σ. As in Proposition 3, by transfinite induction we construct a functor $\omega : \langle \mu \rangle \to \mathcal{S}''$ satisfying the following properties, where

$$\sigma_\xi = \omega(\xi) = (\Sigma_\xi, \Gamma_\xi) \quad \text{for any} \quad \xi \leqslant \mu :$$

- $\omega(0) = \sigma$;
- $\omega(\zeta)$, for any limit-ordinal $\zeta \leqslant \mu$, is the canonical inductive limit of the functor $\omega_\zeta : \langle \zeta \rangle \to \mathcal{S}''$, restriction of ω, and $\omega(\zeta, \xi) : \sigma_\xi \to \sigma_\zeta$ is the canonical injection;
- σ_ζ, for an ordinal $\zeta = \xi + 1 < \mu$, is the cone-bearing neocategory $\overline{\sigma}_\xi$ associated to σ_ξ in Part 1, and $\omega(\zeta, \xi)$ is the morphism $\overline{\delta}_\xi$ of Part 1.

We denote by $\overline{\sigma} = (\overline{\Sigma}, \overline{\Gamma})$ the neocategory σ_μ thus obtained, and by $\overline{\delta} = (\overline{\sigma}, \delta, \sigma)$ the morphism $\omega(\mu, 0)$. As in Part 3-b, Proposition

3, we see that

$$\Gamma_\xi = \{ \omega(\xi, 0)\gamma \mid \gamma \in \Gamma \} \text{ for any } \xi \leqslant \mu .$$

Let σ' be a presketch and $\overline{\psi}: \sigma \to \sigma'$ a morphism; the universal properties of the inductive limit and of δ_ξ (Part 1) permit to define by transfinite induction a unique sequence of morphisms $\overline{\psi}_\xi : \sigma_\xi \to \sigma'$, $\xi \leqslant \mu$, such that $\overline{\psi}_0 = \overline{\psi}$ and

$$\overline{\psi}_\zeta \cdot \omega(\zeta, \xi) = \overline{\psi}_\xi \quad \text{for} \quad \xi < \zeta \leqslant \mu .$$

3° It remains to prove that $\overline{\sigma}$ is a presketch. We suppose that $\overline{\gamma}$ and $\overline{\gamma}'$ are two distinguished cones of $\overline{\Gamma}$ with the same base. Then there exist cones $\gamma : u^\wedge \to \phi$ and $\gamma' : u'^\wedge \to \phi'$ of Γ such that

$$\overline{\gamma} = \delta\gamma \quad \text{and} \quad \overline{\gamma}' = \delta\gamma'.$$

Let K be the indexing-category of γ (and of γ'). For each morphism k of K, the equality $\delta\phi(k) = \delta\phi'(k)$ implies the existence of an ordinal ξ_k such that $\xi_k < \mu$ and

$$\omega(\xi_k, 0)(\phi(k)) = \omega(\xi_k, 0)(\phi'(k)) .$$

If ξ is the ordinal upper bound of the family of the ξ_k, for $k \in K$, we have $\xi < \mu$ (since μ is regular and $\overline{\overline{K}} < \mu$). By construction the cones

$$\omega(\xi, 0)\gamma \quad \text{and} \quad \omega(\xi, 0)\gamma'$$

are distinguished cones of σ_ξ with the same base. Hence they are identified in $\sigma_{\xi+1}$, i.e. we get

$$\omega(\xi+1, 0)\gamma = \omega(\xi+1, 0)\gamma' .$$

Applying $\omega(\mu, \xi+1)$, it follows $\overline{\gamma} = \overline{\gamma}'$. ∇

COROLLARY 1. *The insertion functor from* $\mathcal{S}'$ *to* $\mathcal{S}''$ *admits a left adjoint.*

DEFINITION. A presketch $\overline{\sigma}$ satisfying Proposition 5 is called a *presketch generated by* σ.

COROLLARY 2. *Let* σ *be a cone-bearing neocategory,* $\overline{\sigma}$ *a presketch generated by* σ *and* σ' *a prototype. Then the category* $\mathcal{S}(\sigma', \sigma)$ *is isomorphic with* $\mathcal{S}(\sigma', \overline{\sigma})$.

Δ. The Proof is similar to that of Corollary 2, Proposition 3. ∇

REMARK. In [F] Proposition 5 is proved more generally for **V** - categories, where **V** is a monoidal closed category.

PROPOSITION 6. *Let* σ *be a cone-bearing neocategory* (Σ, Γ). *There exists a prototype* $\bar{\sigma}$ *defined up to an isomorphism by the condition:*

If $\mathfrak{U}$ *is a universe such that* $\sigma \in \mathcal{S}''_0$, *then* $\bar{\sigma}$ *is a free structure generated by* σ *relative to the insertion functor from* $\mathcal{P}$ *to* $\mathcal{S}''$.

Δ. The prototype $\bar{\sigma}$ will be constructed by transfinite induction, as end of a «tower». The method is similar to that used in Proposition 3, the only difference being that the non-limit step has to be slightly modified in the following way:

Let us suppose that σ_ξ is a cone-bearing neocategory (Σ_ξ, Γ_ξ). As in Part 1, Proposition 3, we consider the set Ω of the «formal factors» (γ, γ'), where $\gamma \in \Gamma_\xi$ and γ' is a cone in Σ_ξ with the same base as γ, the same graph (U, β, α), on the sum U of $\underline{\Sigma}_\xi$ and Ω, and the free category L generated by it. Let r' be the smallest equivalence relation on $\underline{L}$ satisfying the condition $(\hat{\text{P}})$ formed by the condition (P) of Part 1, Proposition 3 and the condition

$$(v\text{P}'')\begin{cases} v(\gamma(i)) \sim v(\gamma'(i)) & \text{for any object } i \text{ of the indexing-category of } \gamma \\ & \text{when } (\gamma, \gamma') \in \Omega \text{ and } \gamma' \in \Gamma_\xi \end{cases}$$

(deduced from the condition (P") of Proposition 5), where $v : \underline{\Sigma}_\xi \to U$ still denotes the canonical injection.

Then $\bar{\Sigma}_\xi$ is the canonical quasi-quotient category of L by r' (we recall [E1] that $\bar{\Sigma}_\xi$ is defined as follows: let L' be the quotient neocategory of L by the smallest compatible equivalence relation containing r' and the free category L'' generated by the graph underlying L'; the category $\bar{\Sigma}_\xi$ is the quotient category of L'' by the smallest compatible equivalence relation such that

$$(x', x) \sim x'.x \quad \text{if } x'.x \text{ is defined in } L').$$

If $\underline{\Sigma}_\xi$ is a $\mathfrak{U}$-set, $\underline{\bar{\Sigma}}_\xi$ is also one.

Apart from this modification (i. e. r' satisfies both (P) and (vP"), not only (P)), the construction of $\bar{\sigma}$ and of the canonical morphism $\bar{\delta}$ from

σ to $\bar{\sigma}$ is essentially the same as done in Proposition 3; the proof of Proposition 3 may also be copied to prove that each morphism ψ from σ to a prototype is of the form $\bar{\psi}' \cdot \bar{\delta}$. Finally an argument similar to that of Proposition 3 shows that the distinguished cones of $\bar{\sigma}$ are limit-cones, and we prove as in Part 3, Proposition 5, that two distinguished cones admitting the same base are identical. Hence $\bar{\sigma}$ is a prototype. ∇

COROLLARY 1. *The insertion functors from $\mathcal{P}$ to $\mathcal{S}''$ and from $\mathcal{P}$ to $\mathcal{S}'$ admit (left) adjoints. The categories $\mathcal{S}'$ and $\mathcal{P}$ admit $\mathcal{F}_0$-inductive limits. If (Σ, Γ) is a cone-bearing neocategory, there exists a quasi-quotient prototype of it, by an equivalence relation on $\underline{\Sigma}$.*

Δ. The proof is similar to that of Corollary 1, Proposition 3. ∇

DEFINITION. A prototype $\bar{\sigma}$ satisfying Proposition 6 will be called a *prototype generated by* σ. If the canonical morphism $\bar{\delta} : \sigma \to \bar{\sigma}$ is injective, we say that σ is a *sketch*.

COROLLARY 2. *If σ is a cone-bearing neocategory and $\bar{\sigma}$ a prototype generated by σ, for every prototype σ', the category $\mathcal{S}(\sigma', \sigma)$ is isomorphic with $\mathcal{S}(\sigma', \bar{\sigma})$.*

Δ. The proof is similar to that of Corollary 2, Proposition 3. ∇

REMARK. The existence of an adjoint for the insertion functor from $\mathcal{P}$ to $\mathcal{S}'$ is deduced in [E5] from the general existence theorem for free structures. This fact is generalized in [F] for **V**-categories, where **V** is a monoidal closed category. Sketches are introduced in [E5]. Naturally each prototype is also a sketch, and every sketch σ generates a prototype of which σ is a subsketch.

PROPOSITION 7. *Let σ be a sketch, $\bar{\sigma}$ a limit-bearing category generated by σ and π a prototype generated by σ. Then $\bar{\sigma}$ and π are isomorphic.*

Δ. Let us denote by

$$\bar{\delta} = (\bar{\sigma}, \delta, \sigma) \quad \text{and} \quad \bar{\Pi} = (\pi, \Pi, \sigma)$$

the canonical morphisms. Since π is a fortiori a limit-bearing category, it exists a unique morphism

$$\overline{\overline{\Pi}}' = (\pi, \Pi', \overline{\sigma}) \text{ such that } \overline{\overline{\Pi}}' \cdot \overline{\delta} = \overline{\overline{\Pi}}$$

(this is valid even is σ is not a sketch). If $\overline{\sigma}$ is also a prototype, then there exists a unique morphism

$$\overline{\delta}' : \pi \to \overline{\sigma} \text{ such that } \overline{\delta}' \cdot \overline{\overline{\Pi}} = \overline{\delta},$$

and, from the equalities

$$\overline{\delta}' \cdot \overline{\overline{\Pi}}' \cdot \overline{\delta} = \overline{\delta} \quad \text{and} \quad \overline{\overline{\Pi}}' \cdot \overline{\delta}' \cdot \overline{\overline{\Pi}} = \overline{\overline{\Pi}},$$

we deduce that $\overline{\delta}'$ is an isomorphism, whose inverse is $\overline{\overline{\Pi}}'$. Hence Proposition 7 will be proved if we show that $\overline{\sigma} = (\overline{\Sigma}, \overline{\Gamma})$ is a prototype, when σ is a sketch.

Indeed, let $\overline{\gamma}$ and $\overline{\gamma}'$ be two distinguished cones of $\overline{\Gamma}$ with the same base ϕ'. Since

$$\overline{\Gamma} = \{ \delta\gamma \mid \gamma \in \Gamma \}$$

(Proposition 3), there exist cones γ and γ' of Γ such that $\overline{\gamma} = \delta\gamma$ and $\overline{\gamma}' = \delta\gamma'$. The cones $\overline{\Pi}\gamma$ and $\overline{\Pi}\gamma'$ are distinguished cones of the prototype π; as $\overline{\overline{\Pi}} = \overline{\overline{\Pi}}' \cdot \overline{\delta}$, we get

$$\overline{\Pi}\gamma = \overline{\Pi}'\overline{\gamma} \quad \text{and} \quad \overline{\Pi}\gamma' = \overline{\Pi}'\,\overline{\gamma}',$$

so that $\overline{\Pi}\gamma$ and $\overline{\Pi}\gamma'$ have the same base $\Pi'\phi'$. Hence $\overline{\Pi}\gamma = \overline{\Pi}\gamma'$. The injectivity of Π implies $\gamma = \gamma'$ and, therefore, $\overline{\gamma} = \overline{\gamma}'$. ∇

We denote by $\mathcal{S}$ the full subcategory of $\mathcal{S}'$ whose objects are the sketches $\sigma \in \mathcal{S}''_0$.

PROPOSITION 8. *Let σ be a cone-bearing neocategory (Σ, Γ) and let $\overline{\overline{\Pi}} = (\pi, \Pi, \sigma)$ be the canonical morphism from σ to a prototype $\pi = (\overline{\Sigma}, \overline{\Gamma})$ generated by σ. The presketch $\overline{\overline{\sigma}}$ image of σ by Π is a sketch, characterized up to an isomorphism by the condition:*

If $\mathcal{U}$ is a universe such that $\sigma \in \mathcal{S}''_0$, then $\overline{\overline{\sigma}}$ is a free structure generated by σ relative to the insertion functor from $\mathcal{S}$ to $\mathcal{S}''$.

Δ. We denote by $\overline{\overline{\Sigma}}$ the sub-neocategory of $\overline{\Sigma}$ defined by the set $\Pi(\underline{\Sigma})$ and by $\eta : \overline{\overline{\Sigma}} \to \overline{\Sigma}$ the insertion neofunctor. Let $\Pi' : \Sigma \to \overline{\overline{\Sigma}}$ be the neofunctor restriction of Π and $\overline{\overline{\Gamma}}$ the set of cones $\Pi'\gamma$, where $\gamma \in \Gamma$. Then, $\overline{\overline{\sigma}} = (\overline{\overline{\Sigma}}, \overline{\overline{\Gamma}})$ is a cone-bearing neocategory, Π' defines a morphism $\overline{\overline{\Pi}}'$

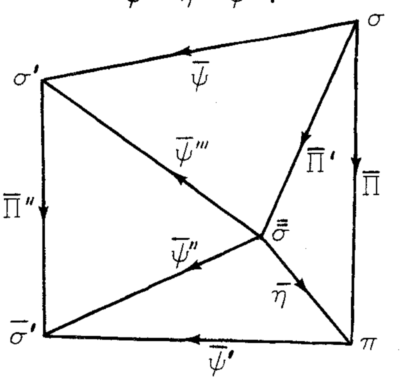
from σ to $\bar{\bar{\sigma}}$ and η defines a morphism $\bar{\eta}: \bar{\bar{\sigma}} \to \pi$. Moreover $\bar{\eta} \cdot \bar{\Pi}' = \bar{\Pi}$.

2° We are going to prove that π is also a prototype generated by $\bar{\bar{\sigma}}$, the canonical morphism being $\bar{\eta}$; it will follow that $\bar{\bar{\sigma}}$ is a sketch η being injective. Indeed, let $\bar{\sigma}'$ be a prototype and $\bar{\psi}'': \bar{\bar{\sigma}} \to \bar{\sigma}'$ a morphism. By definition of π, there exists a unique morphism

$$\bar{\psi}' : \pi \to \bar{\sigma}' \quad \text{such that} \quad \bar{\psi}' \cdot \bar{\Pi} = \bar{\psi}'' \cdot \bar{\Pi}' ;$$

this equality may also be written $\bar{\psi}' \cdot \bar{\eta} \cdot \bar{\Pi}' = \bar{\psi}'' \cdot \bar{\Pi}'$ and, $\bar{\Pi}'$ being surjective, it follows that $\bar{\psi}'$ is also the unique morphism satisfying

$$\bar{\psi}' \cdot \bar{\eta} = \bar{\psi}'' .$$



3° Let σ' be a sketch (Σ', Γ') and $\bar{\psi}: \sigma \to \sigma'$ a morphism. It remains to exhibit a morphism

$$\bar{\psi}''' : \bar{\bar{\sigma}} \to \sigma' \quad \text{such that} \quad \bar{\psi}''' \cdot \bar{\Pi}' = \bar{\psi};$$

the surjectivity of $\bar{\Pi}'$ will imply the unicity of such a morphism. Indeed, the canonical morphism $\bar{\Pi}'' = (\bar{\sigma}', \Pi'', \sigma')$ from σ' to a prototype $(\bar{\Sigma}', \bar{\Gamma}')$ generated by σ' is injective, σ' being a sketch. As π is a prototype generated by σ, there exists a unique morphism $\bar{\psi}' = (\bar{\sigma}', \psi', \pi)$ such that

$$\bar{\Pi}'' \cdot \bar{\psi} = \bar{\psi}' \cdot \bar{\Pi} = \bar{\psi}' \cdot \bar{\eta} \cdot \bar{\Pi}' .$$

As $\bar{\psi}' \bar{\eta}$ maps $\bar{\bar{\Sigma}} = \Pi'(\underline{\Sigma})$ into $\Pi''(\underline{\Sigma}')$ and as Π'' is injective, there is a unique neofunctor

$$\psi''' : \bar{\bar{\Sigma}} \to \Sigma' \quad \text{such that} \quad \Pi'' \psi''' = \psi' \eta ;$$

it satisfies $\psi''' \Pi' = \psi$, since Π'' is injective and

$$\Pi'' \psi''' \Pi' = \psi' \eta \Pi' = \psi' \Pi = \Pi'' \psi .$$

If $\bar{\bar{\gamma}} \in \bar{\bar{\Gamma}}$, we have $\bar{\bar{\gamma}} = \Pi' \gamma$ for some $\gamma \in \Gamma$; from the equality

$$\Pi'' \psi''' \bar{\bar{\gamma}} = \Pi'' \psi''' \Pi' \gamma = \Pi'' \psi \gamma ,$$

we deduce $\Pi'' \psi''' \bar{\bar{\gamma}} \in \bar{\Gamma}'$, the neofunctor $\Pi'' \psi$ defining a morphism. Now, Π'' is injective and $\bar{\Gamma}'$ is formed by the cones $\Pi'' \gamma'$, where $\gamma' \in \Gamma'$. Hence, $\psi''' \bar{\bar{\gamma}} \in \Gamma'$.

So, ψ''' defines the unique morphism

$$\bar{\psi}''' : \bar{\bar{\sigma}} \to \sigma' \quad \text{such that} \quad \bar{\psi}''' \cdot \bar{\Pi}' = \bar{\psi} . \quad \nabla$$

COROLLARY 1. *The insertion functors from $\mathcal{S}$ to $\mathcal{S}'$ and $\mathcal{S}''$ admit left adjoints; $\mathcal{S}$ admits $\mathcal{F}_0$-inductive limits.*

DEFINITION. A sketch $\bar{\bar{\sigma}}$ satisfying the condition of Proposition 8 is called a *sketch generated by* σ.

COROLLARY 2. *Let σ be a cone-bearing neocategory, $\bar{\bar{\sigma}}$ a sketch generated by σ and σ' a prototype. The categories $\mathcal{S}(\sigma', \sigma)$ and $\mathcal{S}(\sigma', \bar{\bar{\sigma}})$ are isomorphic.*

Δ. The proofs of these corollaries are similar to that of Corollaries 1 and 2, Proposition 3. ∇

6. Types.

A loose type which is a presketch will be called a type. We are going to show that each $\mathcal{J}$-cone-bearing neocategory σ generates a $\mathcal{J}$-type τ which is defined up to an isomorphism (and not only up to an equivalence, as the loose $\mathcal{J}$-type $\bar{\sigma}$ generated by σ). Moreover τ is equivalent to $\bar{\sigma}$, when σ is a sketch.

We still denote by $\mathcal{J}$ a given set of categories.

DEFINITION. A $\mathcal{J}$-cone-bearing neocategory which is a presketch (resp. a sketch, or a prototype) will be called a *$\mathcal{J}$-presketch* (resp. a *$\mathcal{J}$-sketch* or a *$\mathcal{J}$-prototype*). A loose $\mathcal{J}$-type which is a presketch is called a *$\mathcal{J}$-type*.

A $\mathcal{J}$-type $\sigma = (\Sigma, \Gamma)$ may be identified with a category Σ admitting $\mathcal{J}$-projective limits, equipped with a choice of a limit-cone with base ϕ for each functor $\phi: K \to \Sigma$, where $K \in \mathcal{J}$ (i.e. with a $\mathcal{J}$-type as defined in [E5]). If $\mathcal{J}$ is a $\mathfrak{U}$-set, denote by $\mathcal{S}'^{\mathcal{J}}$, $\mathcal{S}^{\mathcal{J}}$, $\mathcal{P}^{\mathcal{J}}$ and $\mathcal{F}^{\mathcal{J}}$ the full sub-categories of $\mathcal{S}''^{\mathcal{J}}$ whose objects are respectively the $\mathcal{J}$-presketches, the $\mathcal{J}$-sketches, the $\mathcal{J}$-prototypes and the $\mathcal{J}$-types belonging to $\mathcal{S}''_0$.

PROPOSITION 9. *Let σ be a $\mathcal{J}$-cone-bearing neocategory. There exists a $\mathcal{J}$-type $\bar{\sigma}$ characterized up to an isomorphism by the condition:*

If $\mathfrak{U}$ is a universe such that $\mathcal{J}$ is a $\mathfrak{U}$-set and $\sigma \in \mathcal{S}''_0$, then $\bar{\sigma}$ is a free structure generated by σ relative to the insertion functor $\mathcal{F}^{\mathcal{J}} \hookrightarrow \mathcal{S}''^{\mathcal{J}}$.

Δ. The construction of $\bar{\sigma}$ is obtained by modifying the construction of the loose $\mathcal{J}$-type generated by σ (Proposition 4) in a way similar to that used to deduce in Proposition 6 the construction of the prototype from Proposition 3. In fact, we have only to modify the transition from σ_ξ to $\sigma_{\xi+1}$ by also identifying two distinguished cones with the same base. More precisely:

1° If σ_ξ is a cone-bearing neocategory (Σ_ξ, Γ_ξ), we define as in Part 1, Proposition 4, the graph (U, β, α) and the free category L it generates. But now we denote by $\bar{\Sigma}_\xi$ the canonical quasi-quotient category of L by the equivalence relation satisfying not only conditions (P) and (P') as in Proposition 4, but also the condition (νP") of Proposition 6. After this modification,

$$\bar{\Gamma}_\xi\,,\quad \bar{\sigma}_\xi \quad \text{and} \quad \bar{\delta}_\xi = (\bar{\sigma}_\xi, \delta_\xi, \sigma_\xi)$$

are defined formally as in Part 1, Proposition 4.

Now, let $\mathcal{J}'$ be a set of categories containing $\mathcal{J}$, let σ' be a $\mathcal{J}'$-type and $\bar{\psi} = (\sigma', \psi, \sigma_\xi)$ be a morphism. For each functor $\phi: K \to \Sigma_\xi$, where $K \in \mathcal{J}$, there exists one and only one cone $\eta_\phi \in \Gamma'$ with base $\psi\phi$. Hence, by the method of Part 1-b, Proposition 4, we get one and only one morphism

$$\bar{\psi}': \bar{\sigma}_\xi \to \sigma' \quad \text{such that} \quad \bar{\psi}' \cdot \bar{\delta}_\xi = \bar{\psi}$$

(while in Proposition 4 the morphism $\bar{\psi}'$ was only defined up to an equivalence, the choice of η_ϕ being not unique).

2° By transfinite induction, exactly as in Proposition 4:

a) we construct a functor $\omega : < \mu + 1 > \to \mathcal{S}''$, where μ is yet the least regular ordinal such that $\bar{\bar{K}} < \mu$, for each $K \in \mathcal{J}$;

b) putting $\bar{\sigma} = \omega(\mu)$ and $\bar{\delta} = \omega(\mu, 0)$, we prove that $\bar{\sigma}$ is a loose $\mathcal{J}$-type;

c) using the last statement of Part 1, we show that, if $\bar{\psi} : \sigma \to \sigma'$ is a morphism from σ to a $\mathcal{J}'$-type, where $\mathcal{J}'$ contains $\mathcal{J}$, there exists a unique morphism

$$\bar{\psi}' : \bar{\sigma} \to \sigma' \quad \text{satisfying} \quad \bar{\psi}' \cdot \bar{\delta} = \bar{\psi} .$$

Finally, we see that $\bar{\sigma}$ is also a presketch (and therefore a $\mathcal{J}$-type), by an argument similar to that used in Part 3, Proposition 5. ∇

COROLLARY 1. *Let* $\mathcal{J}$ *be a* $\mathcal{U}$*-set; the insertion functors from* $\mathcal{F}^{\mathcal{J}}$ *to* $\mathcal{S}'^{\mathcal{J}}$, *to* $\mathcal{S}''^{\mathcal{J}}$, *to* $\mathcal{S}^{\mathcal{J}}$, *to* $\mathcal{P}^{\mathcal{J}}$ *and to* $\mathcal{L}^{\mathcal{J}}$ *admit left adjoints.* $\mathcal{F}^{\mathcal{J}}$ *admits* $\mathcal{F}_0$*-inductive limits. There exists a quasi-quotient* $\mathcal{J}$*-type of a* $\mathcal{J}$*-cone-bearing neocategory* (Σ, Γ) *by an equivalence relation on* $\underline{\Sigma}$.

△. The proof is similar to that of Corollary 1, Proposition 3. ∇

DEFINITION. A $\mathcal{J}$-type $\bar{\sigma}$ satisfying the condition of Proposition 9 is called a *$\mathcal{J}$-type generated by* σ .

COROLLARY 2. *Let* σ *be a* $\mathcal{J}$*-cone-bearing neocategory and* $\bar{\sigma}$ *a* $\mathcal{J}$*-type generated by* σ. *If* σ' *is* $\mathcal{J}'$*-type, where* $\mathcal{J}'$ *contains* $\mathcal{J}$, *the categories* $\mathcal{S}(\sigma', \sigma)$ *and* $\mathcal{S}(\sigma', \bar{\sigma})$ *are isomorphic.*

△. The proof is similar to that of Corollary 2, Proposition 3. ∇

REMARKS. In [E5] Proposition 9 is deduced from the existence theorem for free structures. The explicit construction of $\bar{\sigma}$ given here generalizes that of Theorem 7 [E] (where Γ is supposed void). Proposition 9 may be extended for **V**-categories (see [F]).

PROPOSITION 10. *Let* σ *be a* $\mathcal{J}$*-presketch* (Σ, Γ) ,

$$\bar{\delta} = (\bar{\sigma}, \delta, \sigma) \quad and \quad \bar{\theta} = (\tau, \theta, \sigma)$$

the canonical morphisms from σ *to a loose* $\mathcal{J}$*-type* $\bar{\sigma} = (\bar{\Sigma}, \bar{\Gamma})$ *and to a* $\mathcal{J}$*-type* τ *generated by* σ. *Then the following conditions are equivalent:*

1° σ is a sketch;

2° δ is injective;

3° θ is injective.

If they are satisfied, τ and $\bar{\sigma}$ are equivalent.

Δ. 1° Notice that, to prove the injectivity of δ, it is sufficient to exhibit an injective σ-structure ψ in a loose $\mathcal{J}$-type σ'; indeed, there exists then a neofunctor ψ' defining a morphism from $\bar{\sigma}$ to σ' which satisfies $\psi'\delta = \psi$; this equality implies the injectivity of δ when ψ is injective. In particular, if θ is injective, δ is also injective, for θ is a σ-structure in the (loose) $\mathcal{J}$-type τ. Similarly, θ will be injective as soon as there exists an injective σ-structure in a $\mathcal{J}$-type.

We denote by π a prototype $(\hat{\Sigma}, \hat{\Gamma})$ generated by σ and by $\bar{\Pi}$ the canonical morphism from σ to π.

a) If θ is injective, then σ is a sketch. Indeed, since τ is also a prototype, there exists a unique morphism

$$\bar{\Pi}' : \pi \to \tau \text{ such that } \bar{\Pi}' \cdot \bar{\Pi} = \bar{\theta}.$$

$\bar{\theta}$ being injective, $\bar{\Pi}$ is injective, i.e. σ is a sketch.

b) Supposing σ is a sketch, we now prove the injectivity of δ. Let $\mathcal{U}$ be a universe to which belong $\underline{K}$, for any $K \in \mathcal{J}$, and $u'.\hat{\Sigma}.u$, for any pair (u', u) of objects of $\hat{\Sigma}$. The category $\mathcal{M}$ of maps between $\mathcal{U}$-sets admits then $\mathcal{J}$-projective limits, so that the category $\Sigma' = \mathcal{M}^{\hat{\Sigma}^*}$ of natural transformations, where $\hat{\Sigma}^*$ is the dual of $\hat{\Sigma}$, admits $\mathcal{J}$-projective limits. Hence the full $\mathcal{J}$-limit-bearing category σ' on Σ' is a loose $\mathcal{J}$-type. If we consider the Yoneda immersion Y from $\hat{\Sigma}$ to Σ', it is injective and it commutes with projective limits; so Y defines a morphism $\bar{Y} : \pi \to \sigma'$. A fortiori, $\bar{Y} \cdot \bar{\Pi} : \sigma \to \sigma'$ is a morphism from σ to a loose $\mathcal{J}$-type and, $\bar{\Pi}$ being injective by definition of a sketch, $\bar{Y} \cdot \bar{\Pi}$ is injective. From the initial remark, we deduce that δ is also injective.

2° We have yet to show that, if δ is injective, θ is injective and τ is equivalent to $\bar{\sigma}$. For this, we will use the following result:

a) *Let σ' be a loose $\mathcal{J}$-type (Σ', Γ') and ψ an injective σ-structure* *in* σ'. *Then there exists a subset* Γ'' *of* Γ' *such that* (Σ', Γ'') *is a* $\mathcal{J}$-*type* σ'' *and that* ψ *defines also a morphism* $\overline{\psi}: \sigma \to \sigma''$.

Indeed, let $\phi' : K \to \Sigma'$ be a functor, where $K \in \mathcal{J}$. Since ψ is injective, there is at most one neofunctor

$$\phi : K \to \Sigma \quad \text{such that} \quad \psi\,\phi = \phi'$$

and, σ being a presketch, there exists at most one distinguished cone $\gamma \in \Gamma$ with ϕ as its base; hence there is at most one cone $\gamma \in \Gamma$ such that ϕ' is the base of $\psi\gamma \in \Gamma'$. If such a cone γ exists, we denote the cone $\psi\gamma$ by $\gamma_{\phi'}$; otherwise, we choose one cone $\gamma' \in \Gamma'$ with ϕ' as its base, and we denote it by $\gamma_{\phi'}$. The set Γ'' of cones

$$\gamma_{\phi'}\,, \quad \text{where} \quad \phi' : K \to \Sigma' \quad \text{and} \quad K \in \mathcal{J},$$

is a subset of Γ', and (Σ', Γ'') is a $\mathcal{J}$-type σ''; by construction, ψ defines a morphism from σ to σ''.

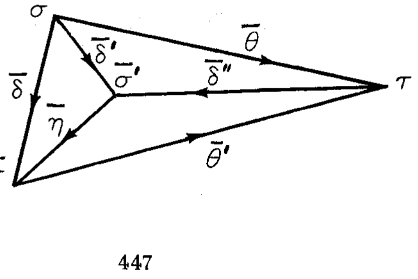
b) We suppose now that δ is injective. Part a applied to $\overline{\delta}: \sigma \to \overline{\sigma}$ asserts the existence of a $\mathcal{J}$-type $\overline{\sigma}' = (\overline{\Sigma}, \overline{\Gamma}')$ such that $\overline{\Gamma}'$ is a subset of $\overline{\Gamma}$ and that δ defines a morphism $\overline{\delta}' : \sigma \to \overline{\sigma}'$. By definition of the $\mathcal{J}$-type generated by σ, there exists a unique morphism $\overline{\delta}'' = (\overline{\sigma}', \delta'', \tau)$ satisfying $\overline{\delta}'' \cdot \overline{\theta} = \overline{\delta}'$. This implies the injectivity of θ.

The identity of $\overline{\Sigma}$ defines a morphism $\overline{\eta}: \overline{\sigma}' \to \overline{\sigma}$ and we have: $\overline{\eta} \cdot \overline{\delta}' = \overline{\delta}$. There exists a morphism $\overline{\theta}' = (\tau, \theta', \overline{\sigma})$ such that $\overline{\theta}' \cdot \overline{\delta} = \overline{\theta}$. From the equalities

$$\overline{\eta} \cdot \overline{\delta}'' \cdot \overline{\theta}' \cdot \overline{\delta} = \overline{\eta} \cdot \overline{\delta}'' \cdot \overline{\theta} = \overline{\eta} \cdot \overline{\delta}' = \overline{\delta}\,,$$

it follows (Proposition 4, condition 2) that the functor $\delta''\theta'$ which defines the morphism $\overline{\eta} \cdot \overline{\delta}'' \cdot \overline{\theta}' : \overline{\sigma} \to \overline{\sigma}$ is equivalent to the identity of $\overline{\Sigma}$. On the other hand, the equalities

$$\overline{\theta}' \cdot \overline{\eta} \cdot \overline{\delta}'' \cdot \overline{\theta} = \overline{\theta}' \cdot \overline{\eta} \cdot \overline{\delta}' = \overline{\theta}' \cdot \overline{\delta} = \overline{\theta}$$



imply that the functor $\theta' \delta''$ defining the morphism $\overline{\theta}' \cdot \overline{\eta} \cdot \overline{\delta}'' : \tau \to \tau$ is an identity. Hence, $\overline{\theta}'$ defines an equivalence from $\overline{\sigma}$ to τ. ∇

COROLLARY. *Let σ be a $\mathcal{I}$-prototype (Σ, Γ). The canonical morphism $\overline{\theta}: \sigma \to \tau$ from σ to a $\mathcal{I}$-type $\tau = (\overline{\Sigma}, \overline{\Gamma})$ generated by σ is injective. Moreover, $\overline{\Sigma}$ is a loose $\mathcal{I}$-projective completion of σ.* ∇

REMARK. The injectivity of $\overline{\theta}$ was shown in Theorem 6 of [E] .

II. MIXED LIMIT-BEARING CATEGORIES

7. Mixed sketches and mixed types.

Up to now, we have always considered neocategories equipped with projective cones. Dually, we could deduce similar results for neocategories Σ equipped with a set of inductive cones (since this is equivalent with equipping the dual of Σ with projective cones). In this paragraph, we will generalize all the preceding results to the case where the neocategory is equipped with both projective cones and inductive cones.

We denote by $\mathcal{I}$ and $\mathcal{J}$ two sets of categories.

DEFINITIONS. 1° *A mixed cone-bearing neocategory* (resp. *category*) is a triple (Σ, Γ, ∇), where Σ is a neocategory (resp. a category), Γ a set of projective cones in Σ indexed by categories and ∇ a set of inductive cones in Σ indexed by categories. We say more precisely that (Σ, Γ, ∇) is a $(\mathcal{I}, \mathcal{J})$-*cone-bearing neocategory* if the indexing-category of each γ of Γ belongs to $\mathcal{I}$ and that of each $\kappa \in \nabla$ belongs to $\mathcal{J}$.

2° If moreover Σ is a category, if Γ is a set of projective limit-cones and ∇ a set of inductive limit-cones, then (Σ, Γ, ∇) is called a *mixed limit-bearing category* (or, more precisely, a $(\mathcal{I}, \mathcal{J})$-*limit-bearing category)*.

3° A $(\mathcal{I}, \mathcal{J})$-limit-bearing category (Σ, Γ, ∇) is called a (mixed) *loose* $(\mathcal{I}, \mathcal{J})$-*type* if each functor $\phi: K \to \Sigma$, where $K \in \mathcal{I}$ (resp. where $K \in \mathcal{J}$) is the base of at least one cone $\gamma \in \Gamma$ (resp. of at least one cone $\kappa \in \nabla$).

4° A $(\mathcal{I}, \mathcal{J})$-cone-bearing neocategory (Σ, Γ, ∇) is called a (mixed)

$(\mathcal{I},\mathcal{J})$-*presketch* if two different cones of Γ (resp. of ∇) have different bases. A mixed presketch which is a mixed limit-bearing category (resp. a loose $(\mathcal{I},\mathcal{J})$-type) is called a *mixed prototype* (resp. a $(\mathcal{I},\mathcal{J})$-*type*).

5° A *morphism between mixed cone-bearing neocategories* is a triple (σ', ψ, σ), where

$$\sigma = (\Sigma, \Gamma, \nabla) \quad \text{and} \quad \sigma' = (\Sigma', \Gamma', \nabla')$$

are mixed cone-bearing neocategories and $\psi : \Sigma \to \Sigma'$ is a neofunctor such that

$$\psi\gamma \in \Gamma' \text{ for any } \gamma \in \Gamma, \quad \psi\kappa \in \nabla' \text{ for any } \kappa \in \nabla .$$

6° Let σ be a mixed cone-bearing neocategory and $\sigma' = (\Sigma', \Gamma', \nabla')$ a mixed cone-bearing category. A neofunctor ψ defining a morphism $\overline{\psi}$ from σ to σ' (still denoted by $\overline{\psi}: \sigma \to \sigma'$) is called a σ-*structure in* σ'. We denote by $\mathcal{S}(\sigma', \sigma)$ the full subcategory of Σ'^{Σ} formed by the natural transformations between σ-structures in σ'. 1+

EXAMPLES. 1° Let Σ' be a category. The *full* $(\mathcal{I},\mathcal{J})$-*limit-bearing category on* Σ' is the triple $(\Sigma', \Gamma', \nabla') = \sigma'$, where Γ' is the set of all the projective limit-cones in Σ' indexed by a category $K \in \mathcal{I}$ and ∇' the set of all the inductive limit-cones in Σ' indexed by a $K \in \mathcal{J}$. If σ is a mixed cone-bearing neocategory, a σ-structure ψ in σ' is called a σ-*structure in* Σ', and $\mathcal{S}(\sigma', \sigma)$ is then denoted by $\mathcal{S}(\Sigma', \sigma)$, or by Σ'^{σ}.

2° Let K be a category and $\sigma = (\Sigma, \Gamma, \nabla)$ a mixed cone-bearing category. We denote by σ^K the mixed cone-bearing category $(\Sigma^K, \overline{\Gamma}, \overline{\nabla})$, where $\overline{\Gamma}$ is defined as in Example 2-2 and $\overline{\nabla}$ is defined dually from ∇. When σ is a mixed limit-bearing category, such is σ^K. If K is the category $\mathbf{2}$, as in Example 2-2, we deduce from $\sigma^{\mathbf{2}}$ the *longitudinal mixed cone-bearing category* $\square\!\square\,\sigma$ *of quartets of* σ and the *lateral mixed cone-bearing category* $\boxminus\sigma$ *of quartets of* σ (they are mixed limit-bearing categories when such is σ).

PROPOSITION 11. *Let σ be a mixed cone-bearing neocategory and σ' a mixed cone-bearing category. There is a canonical bijection from the set of morphisms of the category $\mathcal{S}(\sigma', \sigma)$ onto $\mathcal{S}(\boxminus\sigma', \sigma)_0$.*

Δ. The proof is similar to that of Proposition 2. ∇

Let $\mathcal{U}$ be a universe. We denote by:

- $\mathcal{S}m''_0$ the set of mixed cone-bearing neocategories (Σ, Γ, ∇) such that $\underline{\Sigma}$, Γ, ∇ are $\mathcal{U}$-sets, as well as $\underline{K}$, for any indexing category K of a cone γ of Γ or ∇.
- $\mathcal{S}m''$ the category of morphisms between elements of $\mathcal{S}m''_0$.
- $q_{\mathcal{S}m''}: \mathcal{S}m'' \to \mathcal{F}'$ the functor associating ψ to (σ', ψ, σ).
- $\mathcal{P}m'$, $\mathcal{S}m'$, $\mathcal{P}m$ the full subcategories of $\mathcal{S}m''$ whose objects are those $\sigma \in \mathcal{S}m''_0$ which are respectively mixed limit-bearing categories, mixed pre-sketches and mixed prototypes.
- $\mathcal{S}''^{\mathcal{I}\mathcal{J}}$, $\mathcal{L}^{\mathcal{I}\mathcal{J}}$ and $\mathcal{F}^{\mathcal{I}\mathcal{J}}$, if $\mathcal{I}$ and $\mathcal{J}$ are $\mathcal{U}$-sets, the full subcategories of $\mathcal{S}m''$ whose objects are those $\sigma \in \mathcal{S}m''_0$ which are respectively $(\mathcal{I}, \mathcal{J})$-cone-bearing neocategories, loose $(\mathcal{I}, \mathcal{J})$-types and $(\mathcal{I}, \mathcal{J})$-types.
- $\mathcal{S}''^{\mathcal{I}\mathcal{J}}_{\sim}$ and $\mathcal{L}^{\mathcal{I}\mathcal{J}}_{\sim}$ the quotient categories of $\mathcal{S}''^{\mathcal{I}\mathcal{J}}$ and $\mathcal{L}^{\mathcal{I}\mathcal{J}}$ by the equivalence relation generated by:

$$(\sigma', \psi, \sigma) \sim (\sigma', \psi', \sigma) \text{ iff there exists an equivalence } \eta: \psi \to \psi'.$$

The category $\mathcal{S}''$ may be identified with the full subcategory of $\mathcal{S}m''$, whose objects are those $(\Sigma, \Gamma, \nabla) \in \mathcal{S}m''_0$ such that ∇ is void; similarly $\mathcal{P}'$, $\mathcal{S}'$ and $\mathcal{P}$ may be identified with subcategories of $\mathcal{P}m'$, $\mathcal{S}m'$ and $\mathcal{P}m$. The categories $\mathcal{L}^{\mathcal{I}}$ and $\mathcal{F}^{\mathcal{I}}$ will be identified with $\mathcal{L}^{\mathcal{I}\mathcal{J}}$ and $\mathcal{F}^{\mathcal{I}\mathcal{J}}$ corresponding to the case where the set $\mathcal{J}$ is void.

We also obtain the analogous categories of morphisms between inductive cone-bearing neocategories as subcategories of $\mathcal{S}m''$.

PROPOSITION 12. *$\mathcal{S}m''$ admits $\mathcal{F}_0$-projective limits and $\mathcal{F}_0$-inductive limits; $q_{\mathcal{S}m''}$ commutes with projective limits and with inductive limits. The categories $\mathcal{P}m'$, $\mathcal{S}m'$, $\mathcal{P}m$ are closed for projective limits in $\mathcal{S}m''$, as well as $\mathcal{F}^{\mathcal{I}\mathcal{J}}$, when $\mathcal{I}$ and $\mathcal{J}$ are $\mathcal{U}$-sets.*

Δ. The proof is similar to that of Proposition 1. The distinguished projective cones on the limit are defined as in Proposition 1, while the distinguished inductive cones are defined dually. ∇

PROPOSITION 13. *Let σ be a mixed cone-bearing neocategory. There exist:*

a mixed limit-bearing category $\bar{\sigma}$,

a mixed presketch π',

a mixed prototype π,

characterized up to an isomorphism by the condition:

If $\mathfrak{U}$ *is a universe such that* σ *belongs to* $\mathcal{S}m''_0$, *then* $\bar{\sigma}$, π' *and* π *are free structures generated by* σ *relative to the insertion functors toward* $\mathcal{S}m''$ *from respectively* $\mathcal{P}m'$, $\mathcal{S}m'$ *and* $\mathcal{P}m$.

Δ. Let μ be the least regular ordinal greater than $\bar{\bar{K}}$, for any category K indexing either a cone of Γ or a cone of ∇. We construct $\bar{\sigma}$ (resp. π', resp. π) by transfinite induction, as the end of a tower of mixed cone-bearing neocategories σ_ξ, for $\xi < \mu$, as in Proposition 3 (resp. 5, resp. 6), the only difference being in the non-limit step which we now describe.

We suppose for this that σ_ξ is any mixed cone-bearing neocategory $(\Sigma_\xi, \Gamma_\xi, \nabla_\xi)$.

1° In the construction of $\bar{\sigma}$, we associate to σ_ξ the mixed cone-bearing neocategory $\bar{\sigma}_\xi = (\bar{\Sigma}_\xi, \bar{\Gamma}_\xi, \bar{\nabla}_\xi)$ defined as follows. We denote by:

- Ω the set of pairs (γ, γ') (or «formal factors»), where $\gamma \in \Gamma_\xi$ and γ' is a projective cone in Σ_ξ with the same base as γ.
- $\hat{\Omega}$ the set of pairs (κ', κ) (or «formal cofactors»), where $\kappa \in \nabla_\xi$ and κ' is an inductive cone in Σ' with the same base as κ.
- U the sum of $\underline{\Sigma}_\xi$, Ω and $\hat{\Omega}$, with injections:

$$v : \underline{\Sigma}_\xi \to U, \quad v' : \Omega \to U, \quad \hat{v}' : \hat{\Omega} \to U.$$

- (U, β, α) the graph such that:

$$(G) \begin{cases} v(x) : v(u) \to v(u') & \text{if } x : u \to u' \text{ is in } \Sigma_\xi . \\ v'(\gamma, \gamma') : v(u') \to v(u) & \text{if } (\gamma, \gamma') \in \Omega \text{ and if } \gamma \text{ and } \gamma' \text{ have } u \text{ and } u' \text{ as vertices.} \\ \hat{v}'(\kappa', \kappa) : v(u) \to v(u') & \text{if } (\kappa', \kappa) \in \hat{\Omega} \text{ and if } \kappa \text{ and } \kappa' \text{ have } u \text{ and } u' \text{ as vertices.} \end{cases}$$

- L the free category generated by (U, β, α) and r the smallest equivalence relation on $\underline{L}$ satisfying the condition (Pm) obtained by adding

Condition (P), Part 1, Proposition 3, and

$$(\text{Pi}) \begin{cases} (\hat{v}(\kappa', \kappa), v(\kappa(i))) \sim v(\kappa'(i)), & \text{if } i \in K_0, \\ \hat{v}(\kappa', \kappa) \sim v(z), & \text{if } z \in \Sigma_\xi \text{ and } z.\kappa(i) = \kappa'(i) \text{ for any } i \in K_0, \\ \text{where } (\kappa', \kappa) \in \hat{\Omega} & \text{and } K \text{ is the indexing-category of } \kappa. \end{cases}$$

- $\overline{\Sigma}_\xi$ the quasi-quotient category of L by r and $\rho : L \to \overline{\Sigma}_\xi$ the canonical functor.

The map ρv defines a neofunctor $\delta_\xi : \Sigma_\xi \to \overline{\Sigma}_\xi$. The triple $\overline{\sigma}_\xi = (\overline{\Sigma}_\xi, \overline{\Gamma}_\xi, \overline{\nabla}_\xi)$, where

$$\overline{\Gamma}_\xi = \{ \delta_\xi \gamma \mid \gamma \in \Gamma_\xi \} \quad \text{and} \quad \overline{\nabla}_\xi = \{ \delta_\xi \kappa \mid \kappa \in \nabla_\xi \},$$

is a mixed cone-bearing neocategory and δ_ξ defines a morphism

$$\overline{\delta}_\xi : \sigma_\xi \to \overline{\sigma}_\xi .$$

2° In the construction of π', we associate to σ_ξ the following mixed cone-bearing neocategory $\overline{\sigma}_\xi$: Let r be the smallest equivalence relation on $\underline{\Sigma}_\xi$ satisfying the condition (P"m) obtained by adding to the condition (P") of Proposition 5 the condition:

$$(\text{P"i}) \begin{cases} \kappa(i) \sim \kappa'(i), \text{ for any } i \in K_0, \text{ if } \kappa \text{ and } \kappa' \text{ are two cones of } \nabla_\xi \\ \text{with the same base, indexed by } K. \end{cases}$$

We denote by $\overline{\Sigma}_\xi$ the quasi-quotient category of Σ_ξ by r and we define the canonical neofunctor $\delta_\xi : \Sigma_\xi \to \overline{\Sigma}_\xi$ and the sets $\overline{\Gamma}_\xi$ and $\overline{\nabla}_\xi$ formally as in Part 1. Then $\overline{\sigma}_\xi = (\overline{\Sigma}_\xi, \overline{\Gamma}_\xi, \overline{\nabla}_\xi)$ and $\overline{\delta}_\xi : \sigma_\xi \to \overline{\sigma}_\xi$ is defined by δ_ξ.

3° In order to get π, we construct $\overline{\sigma}_\xi$ as in Part 1, replacing only the condition (Pm) by the condition ($\hat{\text{P}}$m) deduced from the conditions (Pm) and (P"m) (as ($\hat{\text{P}}$) was deduced from (P) and (P") in Proposition 6).

4° To prove that $\overline{\sigma}$ (resp. π', resp. π) has the properties indicated in Proposition 13, we use the same arguments as in Proposition 3 (resp. 5, resp. 6) for the distinguished projective cones, and dual arguments for the distinguished inductive cones. (This is possible, since the parts of the constructions involving inductive cones are just deduced by duality from those involving projective cones.) ∇

DEFINITION. With the hypotheses of Proposition 13, we call $\overline{\sigma}$ (resp. π', resp. π) a *mixed limit-bearing category* (resp. a *mixed presketch*, resp. a *mixed prototype)* generated by σ. We say that σ is a *mixed sketch* if the canonical morphism from σ to π is injective.

We denote by $\mathcal{S}m$ the full subcategory of $\mathcal{S}m'$ whose objects are the mixed sketches $\sigma \in \mathcal{S}m'_0$.

PROPOSITION 14. *Let σ be a mixed cone-bearing neocategory. There exists a mixed sketch $\bar{\bar{\sigma}}$ defined up to an isomorphism by the condition:*

If $\mathcal{U}$ is a universe such that $\sigma \in \mathcal{S}m''_0$, then $\bar{\bar{\sigma}}$ is a free structure generated by σ relative to the insertion functor from $\mathcal{S}m$ to $\mathcal{S}m''$.

Δ. Let (π, Π, σ) be the canonical morphism from σ to a prototype generated by σ and $\bar{\bar{\sigma}}$ the mixed presketch image of σ by Π. Then a proof similar to that of Proposition 8 shows that $\bar{\bar{\sigma}}$ is a sketch satisfying the condition of Proposition 14. ∇

PROPOSITION 15. *Let σ be a $(\mathcal{I}, \mathcal{J})$-cone-bearing neocategory. There exist*

- *a loose $(\mathcal{I}, \mathcal{J})$-type $\bar{\sigma}$, defined up to an equivalence,*
- *a $(\mathcal{I}, \mathcal{J})$-type τ, defined up to an isomorphism,*

satisfying the condition:

Let $\mathcal{U}$ be a universe such that $\mathcal{I}$ and $\mathcal{J}$ are $\mathcal{U}$-sets and $\sigma \in \mathcal{S}m''_0$. Then $\bar{\sigma}$ and τ are free structures generated by σ relative to the insertion functors respectively from $\underset{\sim}{\mathcal{L}}^{\mathcal{I}\mathcal{J}}$ to $\underset{\sim}{\mathcal{S}}''^{\mathcal{I}\mathcal{J}}$ and from $\mathcal{F}^{\mathcal{I}\mathcal{J}}$ to $\mathcal{S}''^{\mathcal{I}\mathcal{J}}$.

Δ. The construction of $\bar{\sigma}$ (resp. of τ) is done by transfinite induction by a method similar to that used in Proposition 4 (resp. 9), the only modification occuring in the non-limit step, which we now describe.

Let σ_ξ be a $(\mathcal{I}, \mathcal{J})$-cone-bearing neocategory. We consider the sets

- $\underline{\Sigma}_\xi$, Ω, M and M', defined as in Part 1, Proposition 4,
- $\hat{\Omega}$, $\hat{M}$ and $\hat{M}'$ defined dually as follows:

$\hat{\Omega}$ is the set of pairs of cones (κ', κ), where $\kappa \in \nabla_\xi$ and κ' is an inductive cone in Σ_ξ with the same base as κ,
$\hat{M}$ is the set of neofunctors $\phi: K \to \Sigma_\xi$, where $K \in \mathcal{J}$, which are not the base of any inductive cone $\kappa \in \nabla_\xi$,
$\hat{M}'$ is the set of pairs (ϕ, i), where $\phi \in \hat{M}$ and $i \in K_0$.

We denote by U the sum of these seven sets, by

$$v,\ v',\ w,\ w',\ \hat{v}',\ \hat{w},\ \hat{w}'$$

the canonical injections into U. We get a graph (U, β, α) by imposing

$$\begin{cases} \text{condition (G) of Proposition 13 and} \\ w'(i,\phi): w(\phi) \to v(\phi(i)) \quad \text{if } (i,\phi) \in M', \\ \hat{w}'(\phi,i): v(\phi(i)) \to \hat{w}(\phi) \quad \text{if } (\phi,i) \in \hat{M}'. \end{cases}$$

Let L be the free category generated by this graph and r the smallest equivalence relation on $\underline{L}$ satisfying the condition (Pm) of Part 1 (resp. ($\hat{\text{P}}$m), of Part 3), Proposition 13, the condition (P') of Proposition 4 and

$$(\text{P}'\,\text{i})\begin{cases} (\hat{w}'(\phi,i), v(\phi(k))) \sim \hat{w}'(\phi,i'), \ \text{if } (\phi,i) \in \hat{M}',\ \phi: K \to \Sigma_\xi, \\ \qquad k: i' \to i \text{ in } K. \end{cases}$$

There exists a quasi-quotient category $\overline{\Sigma}_\xi$ of L by r and, if ρ is the canonical functor from L to $\overline{\Sigma}_\xi$, then $\underline{\rho}\, v$ defines a neofunctor δ_ξ, from Σ_ξ to $\overline{\Sigma}_\xi$.

Let ϕ: $K \to \Sigma_\xi$ be a functor. If $\phi \in M$, we define a projective cone $\gamma_\phi: \rho(w(\phi))^\wedge \to \delta_\xi \phi$, the «formal projective cone associated to ϕ», by

$$\gamma_\phi(i) = \rho(w'(i,\phi)), \quad \text{for any } i \in K_0.$$

If $\phi \in \hat{M}$, we define an inductive cone $\kappa_\phi: \delta_\xi \phi \to \rho(\hat{w}(\phi))^\wedge$, the «formal inductive cone associated to ϕ», by:

$$\kappa_\phi(i) = \rho(\hat{w}'(\phi,i)), \ \text{for any } i \in K_0.$$

We denote by

- $\overline{\Gamma}_\xi$ the set of cones $\delta_\xi \gamma$ where $\gamma \in \Gamma_\xi$, and γ_ϕ where $\phi \in M$,
- $\overline{\nabla}_\xi$ the set of cones $\delta_\xi \kappa$ where $\kappa \in \nabla_\xi$, and κ_ϕ where $\phi \in \hat{M}$,
- $\overline{\sigma}_\xi$ the mixed cone-bearing category $(\overline{\Sigma}_\xi, \overline{\Gamma}_\xi, \overline{\nabla}_\xi)$,
- $\overline{\delta}_\xi: \sigma_\xi \to \overline{\sigma}_\xi$ the morphism defined by δ_ξ.

If $\overline{\psi} = (\sigma', \psi, \sigma_\xi)$ is a morphism from σ_ξ to a loose $(\mathcal{I}, \mathcal{J})$-type σ', we can choose one (resp. to a $(\mathcal{I}, \mathcal{J})$-type σ', there exists one unique) distinguished projective cone η_ϕ in σ' with $\psi\phi$ as its base, for each $\phi \in M$, and one distinguished inductive cone $\hat{\eta}_{\phi'}$ with $\psi\phi'$ as its base, for each $\phi' \in \hat{M}$. As in Part 1 Proposition 4, we see there is a unique morphism $\overline{\psi}': \overline{\sigma}_\xi \to \sigma'$ such that

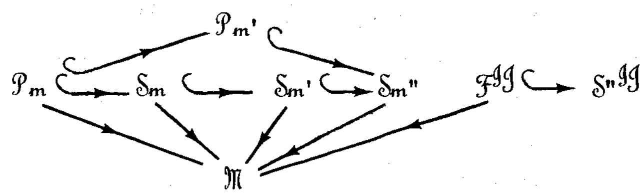
$$\overline{\psi}' \cdot \overline{\delta}_\xi = \overline{\psi}, \quad \overline{\psi}' \gamma_\phi = \eta_\phi \ \text{ if } \ \phi \in M,$$
$$\overline{\psi}' \kappa_{\phi'} = \hat{\eta}_{\phi'} \ \text{ if } \ \phi' \in \hat{M} .$$

The construction of $\overline{\sigma}$ (resp. of τ) is done just as in Proposition 4 (resp. 9), but with this modified definition of $\overline{\sigma}_\xi$. The proof of Proposition 15 is completed similarly. ∇

DEFINITION. With the hypotheses of Proposition 15, we call $\overline{\sigma}$ a *loose* $(\mathcal{I}, \mathcal{J})$*-type generated by* σ and τ a $(\mathcal{I}, \mathcal{J})$*-type generated by* σ. The category underlying $\overline{\sigma}$ is called a *loose* $(\mathcal{I}, \mathcal{J})$*-completion of* σ.

The preceding Propositions admit the following corollaries:

COROLLARY 1. *In the diagram*



the insertion functors admit left adjoints, all the categories admit $\mathcal{F}_0$*-inductive limits and the functors toward* $\mathfrak{M}$ *admit quasi-quotient structures.*

COROLLARY 2. *The corollaries of Propositions 3, 4, 5, 6, 8 and 9 are still valid when (projective) cone-bearing neocategories are replaced by mixed cone-bearing neocategories.* 1+

Let σ be a $(\mathcal{I}, \mathcal{J})$-cone-bearing neocategory. We will denote by:

- $\overline{\sigma}$ a mixed limit-bearing category,
- π a mixed prototype,
- τ' a loose $(\mathcal{I}, \mathcal{J})$-type,
- τ a $(\mathcal{I}, \mathcal{J})$-type,

generated by σ. From Corollary 2, we deduce:

COROLLARY 3. *1° If* σ' *is a* $(\mathcal{I}, \mathcal{J})$*-type, the categories*

$$\mathcal{S}(\sigma', \sigma), \quad \mathcal{S}(\sigma', \pi), \quad \mathcal{S}(\sigma', \overline{\sigma}) \ \textit{and} \ \mathcal{S}(\sigma', \tau)$$

are isomorphic, and they are equivalent to $\mathcal{S}(\sigma', \tau')$.

2° If Σ' is a category admitting $\mathcal{I}$-projective limits and $\mathcal{J}$-inductive limits, the categories Σ'^{σ} and $\Sigma'^{\overline{\sigma}}$ are isomorphic, and they are equivalent to the category $\Sigma'^{\tau'}$.

PROPOSITION 16. *Let σ be a $(\mathcal{I},\mathcal{J})$-cone-bearing neocategory. The following conditions are equivalent:*

1° σ is a mixed sketch.

2° The canonical morphism $\overline{\delta}: \sigma \to \tau'$ is injective.

3° The canonical morphism $\overline{\theta}: \sigma \to \tau$ is injective.

If they are satified, then

$$\begin{cases} \pi \text{ is isomorphic with } \overline{\sigma}, \\ \tau \text{ is equivalent to } \tau'. \end{cases}$$

Δ. The proof is just similar to that of Propositions 7 and 10, except that Part 1-b of Proposition 10 must be modified as follows.

We suppose that σ is a mixed sketch (Σ, Γ, ∇); we want to exhibit an injective σ-structure in a loose $(\mathcal{I},\mathcal{J})$-type. As in Proposition 10, we consider the canonical morphism $\overline{\Pi} = (\pi, \Pi, \sigma)$ from σ to a prototype $\pi = (\hat{\Sigma}, \hat{\Gamma}, \hat{\nabla})$ generated by σ, a universe $\mathcal{U}$ such that $\underline{K}$, for any category K belonging to $\mathcal{I}$ or $\mathcal{J}$, and $u'.\hat{\Sigma}.u$, for any pair (u', u) of objects of $\hat{\Sigma}$, are $\mathcal{U}$-sets, and the Yoneda immersion Y from $\hat{\Sigma}$ to $\mathcal{M}^{\hat{\Sigma}^*}$. But Y does not commute with inductive limits. So we take the full subcategory Σ'' of $\Sigma' = \mathcal{M}^{\hat{\Sigma}^*}$ whose objects are functors $F: \hat{\Sigma}^* \to \mathcal{M}$ commuting with $\mathcal{J}$-projective limits. It is known (see, for example, [J]) that Σ'' admits $\mathcal{F}_0$-projective and inductive limits. (In fact, Σ'' is closed for projective limits in Σ' and the insertion functor from Σ'' to Σ' admits a left adjoint). Moreover, there exists [Lb] a restriction

$$Y': \hat{\Sigma} \to \Sigma'' \quad \text{of} \quad Y: \hat{\Sigma} \to \Sigma',$$

which commutes with projective limits and with $\mathcal{J}$-inductive limits. It follows that the full $(\mathcal{I},\mathcal{J})$-limit-bearing category on Σ'' is a loose $(\mathcal{I},\mathcal{J})$-type σ'', and that Y' defines an injective morphism $\overline{Y'}: \pi \to \sigma''$. Hence $Y'\Pi$ is an injective σ-structure in the loose $(\mathcal{I},\mathcal{J})$-type σ''. ∇

REMARK. If σ is a mixed limit-bearing category, the «type part» of Proposition 15 and the injectivity of $\overline{\theta}$ are stated in Theorem 15 [E]. The explicit constructions of the generated loose $(\mathcal{I},\mathcal{J})$-type and $(\mathcal{I},\mathcal{J})$-type τ are yet suggested by that of Theorem 8 [E] (the construction of the type τ has also be done for **V**-categories [F]). Proposition 16 generalizes Theorem 14 of [E] (which corresponds to the case $\Gamma = \emptyset = \nabla$).

DEFINITION. Let σ be a mixed $(\mathcal{I},\mathcal{J})$-cone-bearing neocategory. If σ' is a mixed limit-bearing category $(\Sigma', \Gamma', \nabla')$, we say that σ is σ'*-regular* if each σ-structure in Σ' is equivalent to a σ-structure in σ'. If σ is σ'-regular for each $(\mathcal{I},\mathcal{J})$-type σ', we say that σ is *regular.* 1+

This definition means that the insertion functor from $\mathcal{S}(\sigma', \sigma)$ to Σ'^{σ} defines an equivalence between these two categories.

COROLLARY. *Let σ be a $(\mathcal{I},\mathcal{J})$-sketch and σ' a mixed prototype (resp. a $(\mathcal{I},\mathcal{J})$-type) $(\Sigma', \Gamma', \nabla')$. Then σ is σ'-regular iff a prototype (resp. a $(\mathcal{I},\mathcal{J})$-type) $\hat{\sigma}$ generated by σ is σ'-regular.*

Δ. We denote by $(\hat{\sigma}, \delta, \sigma)$ the canonical morphism, by F the functor from $\Sigma'^{\hat{\sigma}}$ to Σ'^{σ} assigning $\theta'\delta$ to θ'. By Proposition 16, $\hat{\sigma}$ is also a limit-bearing category (resp. a loose $(\mathcal{I},\mathcal{J})$-type) generated by σ. So, according to the proof of Corollary 2, Proposition 15 via Corollary 2, Proposition 3 (resp. via Corollary 1, Proposition 4), there exists a functor G from Σ'^{σ} to $\Sigma'^{\hat{\sigma}}$ such that G is an inverse of F (resp. such that FG is an identity and GF is equivalent to an identity).

1° If $\hat{\sigma}$ is σ'-regular and if μ is a σ-structure in Σ', there exists an equivalence η' from the $\hat{\sigma}$-structure $G(\mu)$ in Σ' to a $\hat{\sigma}$-structure ψ' in σ', and $\eta'\delta : G(\mu)\delta \to \psi'\delta$ is an equivalence from μ to the σ-structure $\psi'\delta$ in σ', since $G(\mu)\delta = FG(\mu) = \mu$. So σ is σ'-regular.

2° We suppose that σ is σ'-regular. Let ν be a $\hat{\sigma}$-structure in Σ'; there exists an equivalence ξ from $\nu\delta$ to a σ-structure ψ in σ', and $G(\xi)$ is an equivalence from $G(\nu\delta)$ to $G(\psi)$. By definition of $\hat{\sigma}$, there exists a $\hat{\sigma}$-structure ψ' in σ' satisfying $F(\psi') = \psi'\delta = \psi$. As ψ' is equivalent to $GF(\psi') = G(\psi)$ and ν to $G(\nu\delta) = GF(\nu)$, the functors ν and ψ' are equivalent. Hence, $\hat{\sigma}$ is σ'-regular. ∇

REMARK. Most usual sketches are regular. More generally, we say that

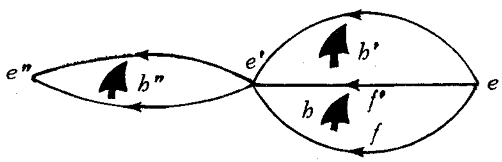
σ is loosely σ'-regular if the categories Σ'^{σ} and $\mathcal{S}(\sigma', \sigma)$ are equivalent. From Corollary 2, Proposition 15, we deduce at once that σ is loosely σ'-regular, where σ' is a mixed prototype (resp. a loose $(\mathcal{I}, \mathcal{J})$-type, resp. a $(\mathcal{I}, \mathcal{J})$-type) iff so is a prototype (resp. a loose $(\mathcal{I}, \mathcal{J})$-type, resp. a $(\mathcal{I}, \mathcal{J})$-type) generated by σ. In several papers, regular means loosely regular. In particular in [L], each mixed cone-bearing neocategory is universally immersed into a loosely regular one.

8. Corresponding 2-categories of bimorphisms.

In this paragraph, we give a reformulation of the preceding results in terms of 2-categories. The categories $\mathcal{P}_{m'}$, $\mathcal{P}_m$,... appear as the categories of 1-morphisms of representable and corepresentable 2-categories and the adjoint functors constructed above extend into 2-adjoints.

2-categories will be considered as those special double categories (or category-objects in $\mathcal{F}$) $(C^{\cdot}, C^{\perp})$ for which the objects of the category $C^{\cdot}$ are also objects of the category $C^{\perp}$ (they are often considered as $\mathcal{F}$-categories, relative to the closed cartesian category $\mathcal{F}$).

Let $\mathcal{C}$ be a 2-category $(C^{\cdot}, C^{\perp})$. The categories $C^{\cdot}$ and $C^{\perp}$ have the same set of morphisms, denoted by $\underline{\mathcal{C}}$, and whose elements are called *bimorphisms* (or 2-cells) of $\mathcal{C}$. The category $C^{\cdot}$ will be called the *category of bimorphisms of* $\mathcal{C}$ (or «strong category» [G]), and written $\mathcal{C}^{\cdot}$, while $C^{\perp}$, also denoted by $\mathcal{C}^{\perp}$, is called the *transverse category* (or «weak» category) of $\mathcal{C}$. We say that an object of $\mathcal{C}^{\cdot}$ is a *vertex of* $\mathcal{C}$, and that an object of $\mathcal{C}^{\perp}$ is a 1-*morphism* (or 1-cell) of $\mathcal{C}$. The set of 1-morphisms defines a subcategory of $\mathcal{C}^{\cdot}$, denoted by $|\mathcal{C}|$. If h is an element of $\underline{\mathcal{C}}$, it is both a morphism $h: f \to f'$ in $\mathcal{C}^{\perp}$ and a morphism in $\mathcal{C}^{\cdot}$, with source the source e of the 1-morphism f (or f') in $|\mathcal{C}|$ and with target the target e'



of f in $|\mathcal{C}|$; to «visualize» the two laws, we will write:

$$h : e \rightrightarrows e' \text{ in } \mathcal{C}$$

or, more precisely:

$$h : f \to f' : e \rightrightarrows e' \text{ in } \mathcal{C}.$$

The 2-category $\mathcal{C}$ is said *representable* (resp. *corepresentable*) [G1] if the insertion functor I from $|\mathcal{C}|$ to the category of bimorphisms $\mathcal{C}^{\cdot}$ admits a coadjoint (resp. a left adjoint) $\square$. A cofree (resp. free) structure generated by a vertex e of $\mathcal{C}$ is called a *representation* (resp. a *corepresentation*) *of* e. Hence $\square e$ is a representation of e iff there exists a bimorphism $\partial e : \square e \rightrightarrows e$ such that, for each bimorphism $h : e' \rightrightarrows e$, there is a unique 1-morphism

$$h' : e' \to \square e \quad \text{satisfying} \quad \partial e . h' = h.$$

If $\mathcal{C}$ is a representable 2-category, the triple on $|\mathcal{C}|$ associated to the pair $(I, \square)$ of adjoint functors admits the category of bimorphisms of $\mathcal{C}$ as its Kleisli category.

We still denote by $\mathcal{N}$ the 2-category of natural transformations associated to the universe $\mathcal{U}$ (we call a 2-category by its bimorphisms, and not by its vertices, as usual). Its category of 1-morphisms $|\mathcal{N}|$ is the category $\mathcal{F}$ of functors associated to $\mathcal{U}$. Its transverse category is the sum of the categories Σ'^{Σ}, where Σ and Σ' are categories whose sets of morphisms belong to $\mathcal{U}$. The law of its category of bimorphisms is the *lateral composition* of natural transformations: If

$$\tau : \phi \to \phi' : \Sigma \rightrightarrows \Sigma' \quad \text{and} \quad \tau' : \nu \to \nu' : \Sigma' \rightrightarrows \Sigma''$$

are natural transformations, their lateral composite, denoted by $\tau' \cdot \tau$ or by $\tau' \tau$ is the natural transformation:

$$\tau' \phi' \boxminus \nu \tau : \nu \phi \to \nu' \phi' : \Sigma \rightrightarrows \Sigma''.$$

$\mathcal{N}$ is representable and corepresentable, a representation of the category Σ being the lateral category $\boxminus \Sigma$ of quartets of Σ and a corepresentation of Σ being the product category $\Sigma \times \mathbf{2}$ (see [G1]).

Using $\mathcal{N}$, we are going to define a representable and corepresentable 2-category, whose category of 1-morphisms is the category of morphisms between mixed cone-bearing categories.

DEFINITION. A *bimorphism between mixed cone-bearing categories* is defined as a triple $\overline{\tau} = (\overline{\psi}', \tau, \overline{\psi})$, where

$$\overline{\psi} = (\sigma', \psi, \sigma) \quad \text{and} \quad \overline{\psi}' = (\sigma', \psi', \sigma)$$

are morphisms between mixed cone-bearing categories and $\tau: \psi \to \psi'$ is a natural transformation between the underlying functors.

We also say that $\overline{\tau}$ is a bimorphism from $\overline{\psi}$ to $\overline{\psi}'$ defined by τ, denoted by one of the following formulas:

$$\overline{\tau}: \overline{\psi} \to \overline{\psi}', \quad \overline{\tau}: \sigma \rightrightarrows \sigma', \quad \overline{\tau}: \overline{\psi} \to \overline{\psi}': \sigma \rightrightarrows \sigma'.$$

Let σ and σ' be two mixed cone-bearing categories. We define the longitudinal category $\overline{\mathcal{S}}(\sigma', \sigma)$ of bimorphisms between σ and σ' as the set of bimorphisms $\overline{\tau}: \sigma \rightrightarrows \sigma'$ equipped with the *longitudinal composition:* The longitudinal composite of $(\overline{\tau}', \overline{\tau})$ exists iff

$$\overline{\tau} = (\overline{\psi}', \tau, \overline{\psi}) \quad \text{and} \quad \overline{\tau}' = (\overline{\psi}'', \tau', \overline{\psi}'),$$

and it is then equal to the bimorphism, denoted by $\overline{\tau}' \boxminus \overline{\tau}$,

$$\overline{\tau}' \boxminus \overline{\tau}: \overline{\psi} \to \overline{\psi}'': \sigma \rightrightarrows \sigma',$$

defined by the natural transformation $\tau' \boxminus \tau$.

The category $\overline{\mathcal{S}}(\sigma', \sigma)$ is trivially isomorphic with $\mathcal{S}(\sigma', \sigma)$.

If $\overline{\tau}: \sigma \rightrightarrows \sigma'$ and $\overline{\tau}'': \sigma' \to \sigma''$ are bimorphisms, where

$$\overline{\tau} = (\overline{\psi}', \tau, \overline{\psi}) \quad \text{and} \quad \overline{\tau}'' = (\overline{\nu}', \tau'', \overline{\nu}),$$

the natural transformation $\tau'' \tau$ defines a bimorphism

$$\overline{\theta}: \overline{\nu}\overline{\psi} \to \overline{\nu}'\overline{\psi}': \sigma \rightrightarrows \sigma'';$$

we call $\overline{\theta}$ the *lateral composite of* $(\overline{\tau}'', \overline{\tau})$ and we denote it by $\overline{\tau}'' \cdot \overline{\tau}$.

We consider still the set $\mathcal{S}m_0''$ of mixed cone-bearing neocategories associated to the universe $\mathcal{U}$ and the corresponding category of morphisms $\mathcal{S}m''$. We denote by:

- $\mathcal{F}m_0''$ the subset of $\mathcal{S}m_0''$ formed by those σ whose underlying neocategory is a category,

- $\mathcal{F}m''$ the full subcategory of $\mathcal{S}m''$ of morphisms between mixed cone-bearing categories belonging to $\mathcal{F}m''_0$,

- $\mathcal{N}\mathcal{F}m''$ the 2-category of bimorphisms associated to $\mathcal{U}$: its category of bimorphisms is formed by the bimorphisms $\bar{\tau}: \sigma \rightrightarrows \sigma'$ such that σ and σ' belong to $\mathcal{F}m''_0$, the law of composition being the lateral composition; the law of its transverse category is the longitudinal composition (category sum of the categories $\bar{\mathcal{S}}(\sigma', \sigma)$). In particular, the category of 1-morphisms is $\mathcal{F}m''$.

- $\mathcal{N}\mathcal{F}m'$ and $\mathcal{N}\mathcal{F}m$ the 2-categories of bimorphisms between mixed pre-sketches and sketches on a category, i. e. the full sub-2-category of $\mathcal{N}\mathcal{F}m''$ whose sets of vertices are respectively

$$\mathcal{F}m'_0 = \mathcal{F}m''_0 \cap \mathcal{S}m'_0 \quad \text{and} \quad \mathcal{F}m_0 = \mathcal{F}m''_0 \cap \mathcal{S}m_0 .$$

- $\mathcal{N}\mathcal{P}m'$, $\mathcal{N}\mathcal{P}m$, $\mathcal{N}\mathcal{L}^{\mathcal{I}\mathcal{J}}$ and $\mathcal{N}\mathcal{F}^{\mathcal{I}\mathcal{J}}$, where $\mathcal{I}$ and $\mathcal{J}$ are $\mathcal{U}$-sets of categories, the 2-categories of bimorphisms between mixed limit-bearing categories, prototypes, loose $(\mathcal{I}, \mathcal{J})$-types and $(\mathcal{I}, \mathcal{J})$-types, i. e. the full sub-2-categories of $\mathcal{N}\mathcal{F}m''$ whose sets of vertices are respectively $\mathcal{P}m'_0$, $\mathcal{P}m_0$, $\mathcal{L}_0^{\mathcal{I}\mathcal{J}}$ and $\mathcal{F}_0^{\mathcal{I}\mathcal{J}}$.

All these 2-categories are canonically equipped with a faithful 2-functor toward $\mathcal{N}$.

PROPOSITION 17. *The 2-category* $\mathcal{N}\mathcal{F}m''$ *is representable and corepresentable.*

Δ. Let σ be a mixed cone-bearing category (Σ, Γ, ∇).

1° σ admits as a representation the lateral mixed cone-bearing category $\boxminus\sigma$ of quartets of σ, for any universe $\mathcal{U}$ such that $\sigma \in \mathcal{F}m''_0$.

Indeed, let a and b be the functors from $\boxminus\Sigma$ to Σ defined by the mappings source and target of the longitudinal category $\square\square\Sigma$. By definition (Example 2-7), $\boxminus\sigma$ is the category $\boxminus\Sigma$ equipped with the sets

- $\bar{\Gamma}$ of projective cones $\bar{\gamma}$ such that $a\bar{\gamma} \in \Gamma$ and $b\bar{\gamma} \in \Gamma$,
- $\bar{\nabla}$ of inductive cones $\bar{\kappa}$ such that $a\bar{\kappa} \in \nabla$ and $b\bar{\kappa} \in \nabla$.

In particular, a and b define morphisms

$$\bar{a}: \boxminus\sigma \to \sigma \quad \text{and} \quad \bar{b}: \boxminus\sigma \to \sigma .$$

To the identical morphism of $\boxminus\sigma$, Proposition 11 associates a bimorphism

$$\partial\sigma = (\overline{b}, j, \overline{a}) : \boxminus\sigma \rightrightarrows \sigma$$

(where j is the natural transformation from a to b assigning the morphism x of Σ to the object x of $\boxminus\Sigma$).

Let σ' be a mixed cone-bearing category $(\Sigma', \Gamma', \nabla')$ and

$$\overline{\tau} = (\overline{\psi}', \tau, \overline{\psi}) : \sigma' \rightrightarrows \sigma$$

a bimorphism. The unique functor $T : \Sigma' \to \boxminus\Sigma$ such that $jT = \tau$ defines a morphism from σ' to $\boxminus\sigma$ (Proposition 11), which is the unique morphism

$$\overline{T} : \sigma' \to \boxminus\sigma \quad \text{such that} \quad \partial\sigma \cdot \overline{T} = \overline{\tau}.$$

Hence $\mathcal{N}\mathcal{F}m''$ is representable, $\boxminus\sigma$ being a representation of σ.

2° We denote by:

- $\hat{\Sigma}$ the category $\Sigma \times \mathbf{2}$,
- ν and ν' the functors from Σ to $\hat{\Sigma}$ associating respectively $(x, 0)$ and $(x, 1)$ to the morphism x of Σ,
- $\hat{\Gamma}$ the set of cones $\nu\gamma$ and $\nu'\gamma$, where $\gamma \in \Gamma$,
- $\hat{\nabla}$ the set of cones $\nu\kappa$ and $\nu'\kappa$, where $\kappa \in \nabla$.

Then $(\hat{\Sigma}, \hat{\Gamma}, \hat{\nabla})$ is a cone-bearing category $\hat{\sigma}$ and ν and ν' define morphisms $\overline{\nu}$ and $\overline{\nu}'$ from σ to $\hat{\sigma}$. By assigning $(e, (1, 0))$ to an object e of Σ, we get a natural transformation $\theta : \nu \to \nu'$, and therefore a bimorphism $\overline{\theta} = (\overline{\nu}', \theta, \overline{\nu}) : \sigma \rightrightarrows \hat{\sigma}$.

$\hat{\sigma}$ is a corepresentation of σ in $\mathcal{N}\mathcal{F}m''$ for any universe $\mathcal{U}$ such that $\sigma \in \mathcal{F}m''_0$. Indeed, let

$$\overline{\tau}' = (\overline{\psi}', \tau', \overline{\psi}) : \sigma \rightrightarrows \sigma'$$

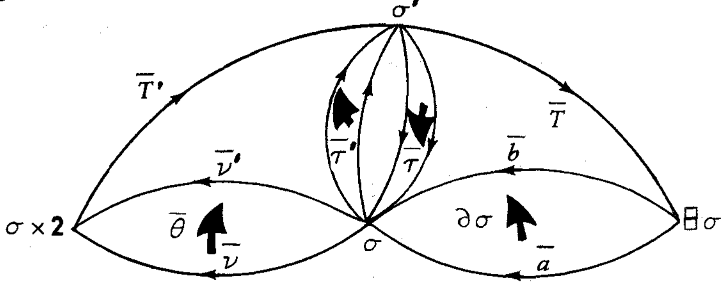
be a bimorphism, where σ' is a mixed cone-bearing category $(\Sigma', \Gamma', \nabla')$. As $\hat{\Sigma}$ is a corepresentation of Σ in $\mathcal{N}$, there exists a unique functor

$$T' : \hat{\Sigma} \to \Sigma' \quad \text{such that} \quad T'\theta = \tau' .$$

This functor defines a morphism $\overline{T}' : \hat{\sigma} \to \sigma'$, since

$$T'\nu\gamma = \overline{\psi}\gamma \quad \text{and} \quad T'\nu'\gamma = \overline{\psi}'\gamma,$$

for any distinguished cone γ in σ. Then $\overline{T}'$ is the unique morphism satisfying $\overline{T}' \cdot \overline{\theta} = \overline{\tau}'$. ∇



The cone-bearing category $\hat{\sigma}$ considered here above will be denoted by $\sigma \times \mathbf{2}$.

REMARK. $\sigma \times \mathbf{2}$ is not defined as the product of two cone-bearing categories. However, if $\mathcal{I}$ and $\mathcal{J}$ are given sets of categories, we can define a prototype $\hat{\mathbf{2}}$ by equipping $\mathbf{2}$ with the set of all «constant» projective cones in $\mathbf{2}$ indexed by a category $K \in \mathcal{I}$, and with the set of all constant inductive cones indexed by a category of $\mathcal{J}$. Then, for each $(\mathcal{I}, \mathcal{J})$-cone-bearing category σ , the product $\sigma \times \hat{\mathbf{2}}$ in $\mathcal{F}_{m''}$ is identical with $\sigma \times \mathbf{2}$.

PROPOSITION 18. *Let* $\mathcal{X}$ *denote anyone of the symbols*

$$\mathcal{F}_{m''}\,,\ \mathcal{F}_{m'}\,,\ \mathcal{F}_{m}\,,\ \mathcal{P}_{m'}\,,\ \mathcal{P}_{m}\,,\ \mathcal{L}^{\mathcal{I}\mathcal{J}}\,,\ \mathcal{F}^{\mathcal{I}\mathcal{J}}\,,$$

where $\mathcal{I}$ *and* $\mathcal{J}$ *are* $\mathcal{U}$*-sets of categories.*

1° If $\mathcal{X} \neq \mathcal{F}_{m'}$*, then* $\mathcal{N}\mathcal{X}$ *is representable, a representation of a vertex* σ *being* $\boxminus\sigma$*.*

2° If $\mathcal{X} \neq \mathcal{L}^{\mathcal{I}\mathcal{J}}$ *and* $\mathcal{X} \neq \mathcal{F}^{\mathcal{I}\mathcal{J}}$*, then* $\mathcal{N}\mathcal{X}$ *is corepresentable, a corepresentation of* σ *being* $\sigma \times \mathbf{2}$ *.*

3° $\mathcal{N}\mathcal{F}^{\mathcal{I}\mathcal{J}}$ *is corepresentable, a corepresentation of* σ *being a* $(\mathcal{I}, \mathcal{J})$*-type generated by* $\sigma \times \mathbf{2}$ *.*

Δ. 1° A full sub-2-category $\mathcal{N}\mathcal{X}$ of the representable (resp. corepresentable) 2-category $\mathcal{N}\mathcal{F}_{m''}$ to which belongs a representation (resp. a corepresentation) of each vertex σ of $\mathcal{N}\mathcal{X}$ is representable (resp. corepresentable). So assertions 1 and 2 result from the following facts.

a) If σ is a mixed limit-bearing category, so is $\boxminus\sigma$. Since a constant functor toward **2** admits its unique value both as a projective limit and as an inductive limit, $\sigma \times \mathbf{2}$ is also a limit-bearing category. Hence $\mathcal{N}\mathcal{P}_m'$ is representable and corepresentable.

b) If σ is a mixed presketch (resp. prototype), $\sigma \times \mathbf{2}$ is also one, so that $\mathcal{N}\mathcal{F}_m'$ and $\mathcal{N}\mathcal{P}_m$ are corepresentable.

c) Let σ be a mixed prototype (Σ, Γ, ∇). Then the mixed limit-bearing category $\boxminus\sigma$ is also a presketch, i. e. a prototype. Indeed, let $\bar{\gamma}$ be a distinguished projective cone in $\boxminus\sigma$, with base T and vertex x. Objects of $\boxminus\Sigma$ are identified with morphisms of Σ and we denote yet by a and b the functors from $\boxminus\Sigma$ to Σ determined by the mappings source and target of $\square\Sigma$. By construction, $a\bar{\gamma}$ and $b\bar{\gamma}$ belong to Γ, so that $a\bar{\gamma}$ and $b\bar{\gamma}$ are the only cones of Γ with bases aT and bT (for σ is a presketch). Moreover, $b\bar{\gamma}$ being a projective limit-cone, x is the unique morphism of Σ such that

$$(b\bar{\gamma})x = \theta \,\square\, a\bar{\gamma}, \quad \text{where} \quad \theta : aT \to bT$$

is the natural transformation canonically associated to the functor T toward $\boxminus\Sigma$. Hence $\bar{\gamma}$ is the unique distinguished projective cone in $\boxminus\sigma$, with base T. Similarly, there is at most one distinguished inductive cone of $\boxminus\sigma$ with a given base. This proves that $\boxminus\sigma$ is a mixed prototype. A fortiori, $\mathcal{N}\mathcal{P}_m$ is representable.

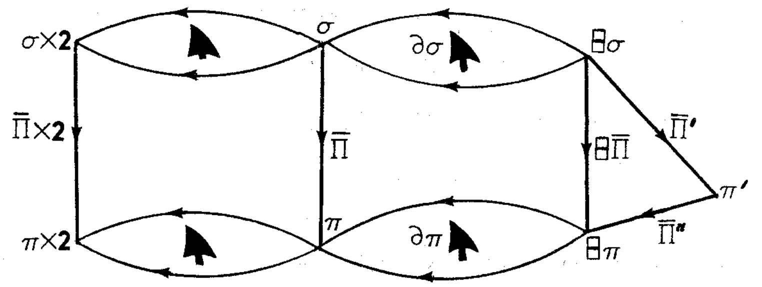
d) If σ is a $(\mathcal{I}, \mathcal{J})$-type (resp. a loose $(\mathcal{I}, \mathcal{J})$-type), so is $\boxminus\sigma$, which implies that $\mathcal{N}\mathcal{F}^{\mathcal{I}\mathcal{J}}$ and $\mathcal{N}\mathcal{L}^{\mathcal{I}\mathcal{J}}$ are representable.

e) Let σ be a mixed sketch (Σ, Γ, ∇), where Σ is a category; let $\bar{\Pi}$ be the canonical morphism (π, Π, σ) from σ to a prototype π generated by σ.

$\boxminus\sigma$ is a mixed sketch. Indeed, let $\bar{\Pi}'$ be the canonical morphism from $\boxminus\sigma$ to a mixed prototype it generates. From Part c, it follows that $\boxminus\pi$ is a prototype. The functor $\boxminus\Pi$ (assigning

$$(\Pi(y'), \Pi(x'), \Pi(y), \Pi(x)) \quad \text{to} \quad (y', x', x, y))$$

defines a morphism $\overline{\boxminus\Pi} : \boxminus\sigma \to \boxminus\pi$. So there exists a unique morphism $\bar{\Pi}''$ such that $\boxminus\bar{\Pi} = \bar{\Pi}'' \cdot \bar{\Pi}'$. Since Π is injective, $\boxminus\Pi$ is also injective



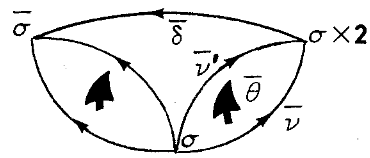
and the preceding equality implies the injectivity of $\overline{\Pi}'$. Therefore, $\boxminus\sigma$ is a mixed sketch, and $\mathfrak{N}\mathfrak{F}_m$ is representable.

Similarly, $\sigma\times 2$ is a mixed sketch, because the functor $\Pi\times 2$ defines an injective morphism $\overline{\Pi}\times 2$ from $\sigma\times 2$ to the prototype $\pi\times 2$ (Part b). So $\mathfrak{N}\mathfrak{F}_m$ is corepresentable.

2° Let σ be a $(\mathfrak{I},\mathfrak{J})$-type. Then $\sigma\times 2$ is not a $(\mathfrak{I},\mathfrak{J})$-type, but it generates a $(\mathfrak{I},\mathfrak{J})$-type $\overline{\sigma}$. By transitivity of free structures, $\overline{\sigma}$ is a free structure generated by σ relative to the composite insertion functor

$$\mathfrak{F}^{\mathfrak{I}\mathfrak{J}} \hookrightarrow \mathfrak{F}_m'' \hookrightarrow (\mathfrak{N}\mathfrak{F}_m'')^{\cdot}.$$

A fortiori $\overline{\sigma}$ is a corepresentation of σ in the full sub-2-category $\mathfrak{N}\mathfrak{F}^{\mathfrak{I}\mathfrak{J}}$ of $\mathfrak{N}\mathfrak{F}_m''$. ∇



COROLLARY. *$\mathfrak{N}\mathfrak{F}_m'$ is not representable.*

Δ. Let σ be a mixed presketch (Σ,Γ,∇), where Σ is a category.

1° $\boxminus\sigma$ may not be a presketch. Indeed, we still denote by a and b the functors from $\boxminus\Sigma$ to Σ determined by the mappings source and target of $\square\Sigma$. Let $T: K\to\boxminus\Sigma$ be a functor and $\tau: K \rightrightarrows \Sigma$ the corresponding natural transformation. If

$$\gamma: e^{\wedge}\to aT \quad\text{and}\quad \gamma': e'^{\wedge}\to bT$$

are cones of Γ, for any morphism $x: e\to e'$ in Σ such that $\gamma' x = \tau\square\gamma$, there exists a cone $\bar{\gamma}_x : x^{\hat{}} \to T$ in $\boxminus\Sigma$ such that

$$a\bar{\gamma}_x = \gamma \quad \text{and} \quad b\bar{\gamma}_x = \gamma' ,$$

and this cone is distinguished in $\boxminus\sigma$. As γ' is not necessarily a limit-cone, there may exist another

$$y \in \Sigma \quad \text{such that} \quad \gamma' y = \tau \boxdot \gamma ,$$

and so another distinguished cone $\bar{\gamma}_y$ in $\boxminus\sigma$ with base T. Then $\boxminus\sigma$ is not a presketch.

2° Let us suppose there exists a representation $\hat{\sigma}$ of σ in $\mathcal{N}\mathcal{F}_m'$ and denote by $\bar{\eta} = (\sigma, \eta, \hat{\sigma})$ the canonical bimorphism. Let

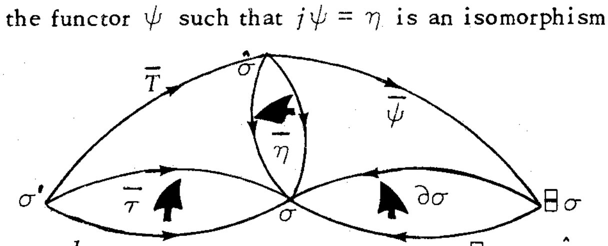
$$\partial\sigma = (\bar{b}, j, \bar{a}) : \boxminus\sigma \rightrightarrows \sigma$$

be the canonical bimorphism defining $\boxminus\sigma$ as a representation of σ in the 2-category $\mathcal{N}\mathcal{F}_m''$ (Proposition 17). There exists a unique morphism

$$\bar{\psi} = (\boxminus\sigma, \psi, \hat{\sigma}) \quad \text{such that} \quad \partial\sigma \cdot \bar{\psi} = \bar{\eta} .$$

We are going to show that $\bar{\psi}$ is an isomorphism, which is impossible in the case where $\boxminus\sigma$ is not a presketch.

a) ψ is an isomorphism. Indeed, let $\tau : \Sigma' \rightrightarrows \Sigma$ be any natural transformation. It defines a bimorphism $\bar{\tau} : \sigma' \rightrightarrows \sigma$, where σ' is the mixed presketch on Σ' without any distinguished cone. There exists a unique morphism $\bar{T} = (\hat{\sigma}, T, \sigma')$ such that $\bar{\eta} \cdot \bar{T} = \bar{\tau}$; this means that T is the unique functor satisfying $\eta T = \tau$. Hence η defines the underlying category of $\hat{\sigma}$ as a representation of Σ in $\mathcal{N}$. As j defines $\boxminus\Sigma$ as a representation of Σ in $\mathcal{N}$, the functor ψ such that $j\psi = \eta$ is an isomorphism.



b) The inverse ψ^{-1} of ψ defines a morphism from $\boxminus\sigma$ to $\hat{\sigma}$. Indeed, let $\bar{\gamma}$ be a distinguished cone of $\boxminus\sigma$. We get a mixed presketch $\bar{\bar{\sigma}}$ by equipping $\boxminus\Sigma$ with $\bar{\gamma}$ as its only distinguished cone; j defines a morphism $\bar{\bar{j}} : \bar{\bar{\sigma}} \to \sigma$. So there exists a unique morphism

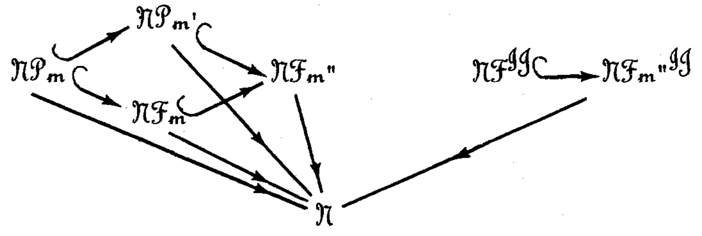
$$\overline{\psi}' = (\hat{\sigma}, \psi', \bar{\bar{\sigma}}) \quad \text{satisfying} \quad \overline{\eta} \cdot \overline{\psi}' = \bar{\bar{j}} \ ;$$

in other words, ψ' is a functor such that $\psi'\overline{\gamma}$ is a distinguished cone of $\hat{\sigma}$ and $\eta\psi' = j$. It follows $j\psi\psi' = j$, which implies that $\psi\psi'$ is an identity functor. ψ being an isomorphism, we have $\psi' = \psi^{-1}$. Hence, $(\mathrm{B}\sigma, \psi^{-1}, \hat{\sigma})$ is a morphism, inverse of $\overline{\psi}$. ∇

REMARK. The 2-category $\mathfrak{NL}^{\mathfrak{IJ}}$ is not corepresentable, but it is weakly corepresentable [G1], a vertex σ admitting as a weak corepresentation a loose $(\mathfrak{I}, \mathfrak{J})$-type $\overline{\sigma}$ generated by $\sigma \times \mathbf{2}$. More precisely, let $\overline{\theta}$ be the canonical bimorphism defining $\sigma \times \mathbf{2}$ as a corepresentation of σ in $\mathfrak{NF}_m''$ (Proposition 17, Part 2) and $\overline{\delta} : \sigma \times \mathbf{2} \to \overline{\sigma}$ the canonical morphism. If σ' is a loose $(\mathfrak{I}, \mathfrak{J})$-type, $\overline{\tau}' : \sigma \rightrightarrows \sigma'$ a bimorphism, there exists a morphism $\overline{T}'$, defined up to an *equivalence*, such that $\overline{T}' \cdot (\overline{\delta} \cdot \overline{\theta}) = \overline{\tau}'$.

If $\mathfrak{I}$ and $\mathfrak{J}$ are $\mathfrak{U}$-sets of categories, we denote by $\mathfrak{NF}_m''^{\mathfrak{IJ}}$ the full sub-2-category of $\mathfrak{NF}_m''$ whose vertices are those $(\mathfrak{I}, \mathfrak{J})$-cone-bearing categories belonging to $\mathfrak{S}_0''^{\mathfrak{IJ}}$.

PROPOSITION 19. *In the following diagram of 2-functors,*



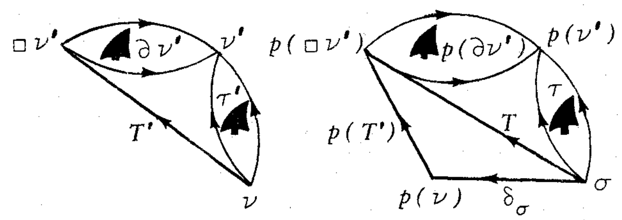
where the 2-functors toward $\mathfrak{N}$ assign to a bimorphism $(\sigma', \theta, \sigma)$ the natural transformation θ, all the 2-functors admit 2-adjoints.

Δ. All the 2-functors of the diagram are 2-functors between representable 2-categories, which commute with the representations by Proposition 18. Moreover their restrictions to the categories of 1-morphisms admit left adjoints. Indeed, this results from Corollary 1, Proposition 15, for the insertion 2-functors. Now let $p_{\mathfrak{X}} : \mathfrak{NX} \to \mathfrak{N}$ be one of the 2-functors toward $\mathfrak{N}$. Assigning to a category Σ the trivial mixed prototype $\bar{\bar{\Sigma}}$ on Σ (without any distinguished cone) and to a natural transformation $\tau : \Sigma \rightrightarrows \Sigma'$ the bimorphism $\overline{\tau}: \overline{\Sigma} \rightrightarrows \overline{\Sigma}'$ defined by τ, we get a 2-functor $J_{\mathfrak{X}}: \mathfrak{N} \to \mathfrak{N}\mathfrak{X}$; its restriction $|J_{\mathfrak{X}}|: \mathfrak{F} \to \mathfrak{X}$ is an adjoint of the restriction $|p_{\mathfrak{X}}|: \mathfrak{X} \to \mathfrak{F}$ of $p_{\mathfrak{X}}$. So Proposition 19 follows from the lemma:

LEMMA. *Let* $\mathcal{H}$ *be a representable 2-category and* $p: \mathcal{H} \to \mathcal{C}$ *a 2-functor satisfying the following conditions:*

- *For each vertex* ν *of* $\mathcal{H}$, *let* $\partial\nu: \square\nu \to \nu$ *be a bimorphism which defines* $\square\nu$ *as a representation of* ν; *then* $p(\partial\nu)$ *defines* $p(\square\nu)$ *as a representation of* $p(\nu)$ *in* $\mathcal{C}$.
- *The functor* $|p|: |\mathcal{H}| \to |\mathcal{C}|$ *restriction of* p *admits an adjoint* q. *Then* q *extends into a 2-adjoint of* p.

a) The functor $p^{\cdot}: \mathcal{H}^{\cdot} \to \mathcal{C}^{\cdot}$ underlying p admits an adjoint $Q^{\cdot}$ extending q. More precisely, for each vertex σ of $\mathcal{C}$, the canonical morphism $\delta_\sigma: \sigma \to p(q(\sigma))$ corresponding to the pair of adjoint functors $(|p|, q)$ defines also $q(\sigma)$ as a free structure generated by σ relative to $p^{\cdot}$. Indeed, let σ be a vertex of $\mathcal{C}$ and $\nu = q(\sigma)$. If ν' is a vertex of $\mathcal{H}$ and $\tau: \sigma \rightrightarrows p(\nu')$ a bimorphism in $\mathcal{C}$, there exists a unique 1-morphism $T: \sigma \to p(\square\nu')$ such that $p(\partial\nu') \cdot T = \tau$, since $p(\partial\nu')$ defines $p(\square\nu')$ as a representation of $p(\nu')$. To T is associated a uni-



que 1-morphism

$$T': \nu \to \square\nu' \quad \text{such that} \quad p(T') \cdot \delta_\sigma = T.$$

From the equalities

$$p(\partial\nu' \cdot T') \cdot \delta_\sigma = p(\partial\nu') \cdot p(T') \cdot \delta_\sigma = p(\partial\nu') \cdot T = \tau,$$

it follows that $\partial\nu' \cdot T'$ is the unique bimorphism

$$\tau': \nu \rightrightarrows \nu' \quad \text{such that} \quad p(\tau') \cdot \delta_\sigma = \tau.$$

Hence, ν is a free structure generated by σ relative to $p^{\cdot}$.

b) The map $\underset{\sim}{Q}$ underlying the adjoint functor $Q^{\cdot}$ of $p^{\cdot}$ also defines a 2-functor $Q: \mathcal{C} \to \mathcal{H}$, so that Q is a 2-adjoint [G] of p. Indeed, we denote by:

- $\mathcal{C}(\sigma', \sigma)$ the subcategory of the transverse category $\mathcal{C}^{\perp}$ of $\mathcal{C}$ formed by the bimorphisms $\theta: \sigma \rightrightarrows \sigma'$.
- $\mathcal{C}(\lambda', \lambda)$, if $\lambda: \rho \to \sigma$ and $\lambda': \sigma' \to \rho'$ are 1-morphisms of $\mathcal{C}$, the functor from $\mathcal{C}(\sigma', \sigma)$ to $\mathcal{C}(\rho', \rho)$ assigning the composite $\lambda' \cdot \theta \cdot \lambda$ to $\theta: \sigma \rightrightarrows \sigma'$.

Let σ and σ' be vertices of $\mathcal{C}$; we write

$$\nu = q(\sigma) \text{ and } \nu' = q(\sigma').$$

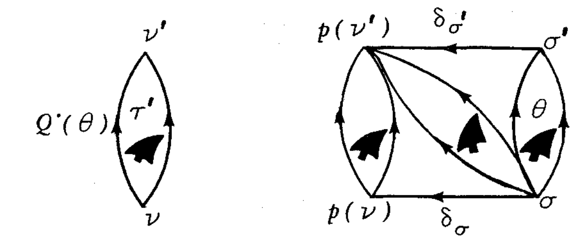
As p is a 2-functor, there exists a functor

$$p_{\nu'\nu}: \mathcal{H}(\nu', \nu) \to \mathcal{C}(p(\nu'), p(\nu))$$

defined by a restriction of p. The map g_σ assigning $p(\tau') \cdot \delta_\sigma$ to the bimorphism $\tau': \nu \rightrightarrows \nu'$ defines the functor

$$\hat{g}_\sigma = \mathcal{C}(p(\nu'), \delta_\sigma) p_{\nu'\nu}: \mathcal{H}(\nu', \nu) \to \mathcal{C}(p(\nu'), \sigma).$$

Part a proves that g_σ is a bijection; it follows that $\hat{g}_\sigma$ is an isomorphism.



The functor

$$\hat{g}_\sigma^{-1} \mathcal{C}(\delta_{\sigma'}, \sigma): \mathcal{C}(\sigma', \sigma) \to \mathcal{H}(\nu', \nu)$$

associates $Q^{\cdot}(\theta)$ to $\theta: \sigma \rightrightarrows \sigma'$, since $Q^{\cdot}(\theta)$ is the unique bimorphism

$$\theta': \nu \rightrightarrows \nu' \text{ such that } p(\theta') \cdot \delta_\sigma = \delta_{\sigma'} \cdot \theta.$$

The category $\mathcal{C}^{\perp}$ being a sum of the categories $\mathcal{C}(\sigma', \sigma)$, we deduce that $\underset{\sim}{Q}$ defines a functor from $\mathcal{C}^{\perp}$ to $\mathcal{H}^{\perp}$, and also a 2-functor Q. ∇

REMARK. Proposition 19 gives a more axiomatic proof of Corollary 2, Propositions 3 or 15.

III. MONOIDAL CLOSED CATEGORIES OF SKETCHED MORPHISMS

Let σ be a projective limit-bearing category and $\mathcal{V}$ a symmetric monoidal closed category [EK]. Under some conditions on σ, the category V^σ of σ-morphisms in the underlying category V of $\mathcal{V}$ admits a symmetric monoidal closed structure. This is applied to the category of functors (or «category of category objects») in V.

10. Cartesian closed structures on $\mathcal{M}^\sigma$.

After some notations, we give conditions on σ, insuring that $\mathcal{M}^\sigma$ admits a cartesian closed structure.

If $\theta: p \to p': L \rightrightarrows K$ is a natural transformation, for any $y: e \to e'$ in L, we denote by $\theta(y)$ the morphism $\theta(e').p(y) = p'(y).\theta(e)$.

Let $P: K' \times K \to C$ be a functor (of «two variables»). If $p: L \to K$ is a functor, we denote by $P(s, p\text{-})$ the functor from L to C assigning $P(s, p(y))$ to $y \in L$, for each object s of K; we denote by $P(x, p\text{-})$ (or by $P(x, p)$, if this does not lead to any confusion) the natural transformation from $P(s, p\text{-})$ to $P(s', p\text{-})$ such that $P(x, p\text{-})(y) = P(x, p(y))$, for any $y \in L$, if $x: s \to s'$ is a morphism in K'.

If $p = Id_K$, we write $P(x, \text{-})$ instead of $P(x, p\text{-})$. If p is the dual q^* of a functor q, we write also $P(x, q\text{-})$ instead of $P(x, q^*\text{-})$.

Similar notations are used relative to the other «variable», and for functors of «more than two variables».

Let K be a category. The functor $Hom_K: K \times K^* \to \mathcal{M}$ will often be denoted by $K(\text{-}, \text{-})$, so that the set of morphisms $x: e \to e'$ in K is written $K(e', e)$ (and not $K(e, e')$ as usual).

We say that K *admits a cartesian closed structure* if there exists a cartesian closed category $\mathbf{K}$, whose underlying category is K. This means that K admits finite products and that, for each object e of K, the partial product functor $\text{-} \times e: K \to K$ (corresponding to a choice of finite products on K) admits a right adjoint. Then we call *closure functor* on K a functor $D: K \times K^* \to K$ such that $D(\text{-}, e)$ is a right adjoint of $\text{-} \times e$, for any object e of K (such a functor is the internal Hom-functor for a closed category [EK] underlying a cartesian closed structure on K). The product functor $\times$ and D are defined up to an equivalence, so that **K** is determined up to an isomorphism of cartesian closed categories.

From now on, we denote by:

- σ a projective limit-bearing category (Σ, Γ),
- $\mathcal{J}$ the set of indexing categories of σ,
- σ^* the « dual of σ», which is the inductive limit-bearing category (Σ^*, Γ^*), whose distinguished inductive cones correspond by duality to projective cones $\gamma \in \Gamma$,
- $\mathfrak{U}$ a universe, to which belong $\underline{\Sigma}$ and $\underline{I}$, for each $I \in \mathcal{J}$,
- $\mathfrak{M}$ the category of maps between $\mathfrak{U}$-sets.

The functor $\Sigma(\cdot, u): \Sigma \to \mathfrak{M}$ commuting with projective limits, it is a σ-structure in $\mathfrak{M}$ for each object u of Σ. Hence the Yoneda immersion $\hat{Y}$ from Σ^* to $\mathfrak{M}^\Sigma$ takes its values in the category $\mathfrak{M}^\sigma$ of σ-morphisms (i. e. of morphisms between σ-structures) in $\mathfrak{M}$. We denote by Y the functor from Σ^* to $\mathfrak{M}^\sigma$, restriction of $\hat{Y}$.

This functor Y is in fact a σ^*-structure in $\mathfrak{M}^\sigma$, called the *Yoneda σ^*-structure*. (Indeed, this will result from Proposition 3-1 [Lb], if $\mathfrak{M}^\sigma(F, Y\text{-})$ is a σ-structure in $\mathfrak{M}$ for each object F of $\mathfrak{M}^\sigma$, i. e. for each σ-structure F; this holds since, by Yoneda Lemma, we get

$$\mathfrak{M}^\sigma(F, Y\text{-}) = \mathfrak{M}^\sigma(F, \text{-})\, Y^* = \mathfrak{M}^\Sigma(F, \text{-})\, \hat{Y}^* \approx F).$$

Let V be a category. The category V^σ of σ-morphisms in V is a full subcategory of V^Σ, closed for equivalences (i. e. a functor equivalent to a σ-structure in V is also one). If V admits projective limits indexed by a category K, the category V^σ admits also projective limits indexed by K, and the insertion functor from V^σ to V^Σ commutes with these limits [E4] (since in V^Σ these limits are computed evaluationwise and projective limit functors commute with projective limits of any kind). In other words, V^σ is closed in V^Σ for projective limits indexed by K.

Since $\mathfrak{M}$ admits $\mathcal{F}_0$-projective limits, where $\mathcal{F}_0$ is the set of all the categories whose sets of morphisms are $\mathfrak{U}$-sets, $\mathfrak{M}^\sigma$ admits also $\mathcal{F}_0$-projective limits. In particular, $\mathfrak{M}^\sigma$ admits finite products. From the canonical product functor on $\mathfrak{M}$, we deduce the product functors $\hat{\times}$ and $\times$ on $\mathfrak{M}^{\Sigma}$ and on $\mathfrak{M}^{\sigma}$. For each σ-structure F in $\mathfrak{M}$, we denote by $-\times F$ the canonical partial product functor from $\mathfrak{M}^{\sigma}$ to $\mathfrak{M}^{\sigma}$, which assigns to a σ-structure F' in $\mathfrak{M}$ the σ-structure $F'\times F$ in $\mathfrak{M}$ such that

$$(F'\times F)(x) = F'(x)\times F(x)$$

for each morphism $x: u \to u'$ in Σ. It is a restriction of $-\hat{\times} F: \mathfrak{M}^{\Sigma} \to \mathfrak{M}^{\Sigma}$.

In particular, for any object u of Σ, we have the partial product functor $-\times Y(u): \mathfrak{M}^{\sigma} \to \mathfrak{M}^{\sigma}$.

PROPOSITION 20. *$\mathfrak{M}^{\sigma}$ admits a cartesian closed structure iff the functor $-\times Y(u)$ commutes with $\mathfrak{I}$-inductive limits, for each object u of Σ. In this case:*

1° $\mathfrak{M}^{\sigma}$ admits a closure functor $\overline{M}$ assigning to the pair (F', F) of σ-structures in $\mathfrak{M}$ the functor $\mathfrak{M}^{\sigma}(F', F\times Y\text{-}) = \mathfrak{M}^{\sigma}(F', \text{-})(F\times \text{-})^{*}\, Y^{*}$.

2° For each σ-structure F in $\mathfrak{M}$, the functor $\overline{M}(F, Y\text{-}): \Sigma \to \mathfrak{M}^{\sigma}$ is a σ-structure in $\mathfrak{M}^{\sigma}$, and $\overline{M}(F', F) = \mathfrak{M}^{\sigma}(\overline{M}(F', Y\text{-}), F)$.

Δ. If $\mathfrak{M}^{\sigma}$ admits a cartesian closed structure, the partial product functor $-\times Y(u)$ admits a right adjoint, so that it commutes with inductive limits, for any object u of Σ.

We suppose now that $-\times Y(u)$ commutes with $\mathfrak{I}$-inductive limits, for each object u of Σ.

$\mathfrak{M}^{\Sigma}$ admits a cartesian closed structure whose closure functor $\hat{M}$ associates $\mathfrak{M}^{\Sigma}(\theta', \text{-})(\theta \hat{\times} \hat{Y}\text{-})^{*}$ to each pair (θ', θ) of morphisms of $\mathfrak{M}^{\Sigma}$ (see for example [GZ], Chapter 2-1). To show that $\mathfrak{M}^{\sigma}$ admits a cartesian closed structure, it is sufficient to prove that $\hat{M}(F', F)$ is a σ-structure when F and F' are σ-structures, for this implies the existence of a functor $\overline{M}: \mathfrak{M}^{\sigma}\times(\mathfrak{M}^{\sigma})^{*} \to \mathfrak{M}^{\sigma}$ restriction of $\hat{M}$, and $\overline{M}$ is a closure functor on $\mathfrak{M}^{\sigma}$. The proof will go in three steps.

1° Let F be a σ-structure in $\mathfrak{M}$ and u an object of Σ. Then, the functor $\hat{M}(F, Y(u))$ is a σ-structure in $\mathfrak{M}$. Indeed, by definition,

$$\hat{M}(F, Y(u)) = \mathfrak{M}^{\Sigma}(F, Y(u)\hat{\times}\hat{Y}\text{-}): \Sigma \to \mathfrak{M}.$$

As F and $Y(u)\times Y(u')$, for each object u' of Σ, are objects of the full subcategory $\mathfrak{M}^\sigma$ of $\mathfrak{M}^\Sigma$, we also have

$$\hat{M}(F, Y(u)) = \mathfrak{M}^\sigma(F, Y(u)\times Y\text{-}),$$

so that this functor is the dual of the composite functor G:

$$\Sigma^* \xrightarrow{Y} \mathfrak{M}^\sigma \xrightarrow{Y(u)\times -} \mathfrak{M}^\sigma \xrightarrow{\mathfrak{M}^\sigma(F,\,-)^*} \mathfrak{M}^*$$

where

- Y is a σ^*-structure,

- the functor $Y(u)\times$- commutes with $\mathcal{J}$-inductive limits, since it is equivalent to the functor $-\times Y(u)$ (the product functor being symmetrical) which commutes with $\mathcal{J}$-inductive limits according to the hypothesis,

- $\mathfrak{M}^\sigma(F, -)^*$ commutes with inductive limits

Hence G is a σ^*-structure in $\mathfrak{M}^*$ and its dual $\hat{M}(F, Y(u))$ is a σ-structure in $\mathfrak{M}$.

2° Let F be a σ-structure in $\mathfrak{M}$. From Part 1, it follows that the functor $\hat{M}(F, -)\hat{Y}^*: \Sigma \to \mathfrak{M}^\Sigma$ takes its values in $\mathfrak{M}^\sigma$. So it admits as a restriction a functor $L: \Sigma \to \mathfrak{M}^\sigma$. This functor L is a σ-structure in $\mathfrak{M}^\sigma$. Indeed let us denote by π_u for each object u of Σ the «projection functor» from $\mathfrak{M}^\sigma$ to $\mathfrak{M}$, which assigns $\theta(u)$ to the σ-morphism θ. Projective limits being computed evaluationwise in $\mathfrak{M}^\sigma$ (since the insertion functor from $\mathfrak{M}^\sigma$ to $\mathfrak{M}^\Sigma$ commutes with projective limits), L is a σ-structure in $\mathfrak{M}^\sigma$ iff $\pi_u L$ is a σ-structure in $\mathfrak{M}$ for each object u of Σ. As

$$\pi_u L(x) = \hat{M}(F, \hat{Y}(x))(u) = \mathfrak{M}^\sigma(F, Y(x)\times Y(u))$$

for each $x \in \Sigma$, we get

$$\pi_u L = \mathfrak{M}^\sigma(F, (Y\text{-})\times Y(u)).$$

The product being symmetrical, the functor $(Y\text{-})\times Y(u)$ is equivalent to $Y(u)\times(Y\text{-})$; a fortiori $\pi_u L$ is equivalent to $\mathfrak{M}^\sigma(F, Y(u)\times Y\text{-})$, which is identical to the σ-structure $\hat{M}(F, Y(u))$. So $\pi_u L$ is a σ-structure in $\mathfrak{M}$ for each u, and L is a σ-structure in $\mathfrak{M}^\sigma$ denoted by $\overline{M}(F, Y\text{-})$.

3° Let F and F' be σ-structures in $\mathfrak{M}$. Then $\hat{M}(F', F)$ is a σ-structure in $\mathfrak{M}$. Indeed, we have

$$\hat{M}(F', F) = \mathfrak{M}^\Sigma(F', F\hat{\times}\hat{Y}\text{-}).$$

As $\hat{M}$ is a closure functor on $\mathcal{M}^{\Sigma}$, the functor $\mathcal{M}^{\Sigma}(F', F\hat{\times}-)$ is equivalent to $\mathcal{M}^{\Sigma}(\hat{M}(F', -), F) = \mathcal{M}^{\Sigma}(-, F)\hat{M}(F', -)$. It follows that the functor

$$\hat{M}(F', F) = \mathcal{M}^{\Sigma}(F', F\hat{\times}\hat{Y}-) = \mathcal{M}^{\Sigma}(F', F\hat{\times}-)\hat{Y}^*$$

is equivalent to the functor

$$\mathcal{M}^{\Sigma}(\hat{M}(F', -), F)\hat{Y}^* = \mathcal{M}^{\Sigma}(\hat{M}(F', \hat{Y}-), F) = \mathcal{M}^{\sigma}(\overline{M}(F', Y-), F).$$

This last functor is a σ-structure in $\mathcal{M}$, since it is the composite of the σ-structure $\overline{M}(F', Y-)$ in $\mathcal{M}^{\sigma}$ with the functor $\mathcal{M}^{\sigma}(-, F)$ which commutes with projective limits. Hence $\hat{M}(F', F)$ is a σ-structure in $\mathcal{M}$, and there exists a functor $\overline{M} : \mathcal{M}^{\sigma} \times (\mathcal{M}^{\sigma})^* \to \mathcal{M}^{\sigma}$ restriction of $\hat{M}$. ∇

DEFINITION. With the hypothesis of Proposition 20, for each σ-structure F in $\mathcal{M}$ we call $\overline{M}(F, Y-)$ ($= \overline{M}(F, -)Y^*$) the *σ-structure in $\mathcal{M}^{\sigma}$ associated to F.*

COROLLARY. *If the insertion functor I from $\mathcal{M}^{\sigma}$ to $\mathcal{M}^{\Sigma}$ commutes with $\mathcal{J}$-inductive limits, then $\mathcal{M}^{\sigma}$ admits a cartesian closed structure.*

Δ. Let u be an object of Σ. The partial product functor $-\hat{\times} Y(u)$ from $\mathcal{M}^{\Sigma}$ to $\mathcal{M}^{\Sigma}$ commutes with $\mathcal{J}$-inductive limits, since it admits a right adjoint $\hat{M}(-, Y(u))$. It follows that the functor

$$P = (-\hat{\times} Y(u))I : \mathcal{M}^{\sigma} \to \mathcal{M}^{\Sigma}$$

also commutes with $\mathcal{J}$-inductive limits. As P takes its values in the full subcategory $\mathcal{M}^{\sigma}$ of $\mathcal{M}^{\Sigma}$, there exists a functor P' from $\mathcal{M}^{\sigma}$ to $\mathcal{M}^{\sigma}$ res-
1 triction of P, and P' commutes with $\mathcal{J}$-inductive limits. P' being the partial product functor $-\times Y(u)$ on $\mathcal{M}^{\sigma}$, the hypothesis of Proposition 20 is satisfied. So the Corollary results from this Proposition. ∇

REMARK. The insertion functor I from $\mathcal{M}^{\sigma}$ to $\mathcal{M}^{\Sigma}$ always admits a left adjoint and $\mathcal{M}^{\sigma}$ admits $\mathcal{F}_0$-inductive limits ([J] or [Br]). If I commutes with $\mathcal{F}_0$-inductive limits, it admits a right adjoint (Theorem 2-1 [GZ]). So the Corollary may then be deduced from the following result:

If V is a category admitting a cartesian closed structure and if V' is a full subcategory of V such that the insertion functor from V' to V admits both a left adjoint and a right adjoint, then V' admits a cartesian closed structure.

This last result proves also that, *if* σ' *is a mixed limit-bearing category* (Σ, Γ, ∇) *and if the insertion functor from* $\mathcal{M}^{\sigma'}$ *to* $\mathcal{M}^{\Sigma}$ *admits both a left adjoint and a right adjoint, then* $\mathcal{M}^{\sigma'}$ *admits a cartesian closed structure.* However this condition on σ' is very restrictive.

10. Monoidal closed categories.

A) The monoidal closed category $\mathcal{V}^{\Sigma}$.

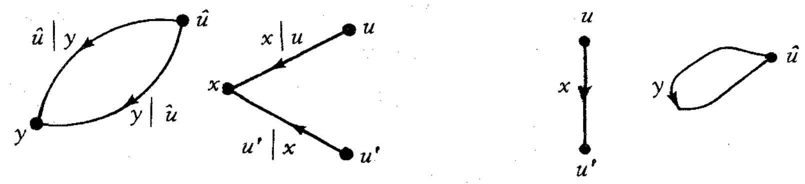
Let Σ be a category. We recall here the definition and some properties of the symmetric monoidal closed category $\mathcal{V}^{\Sigma}$ constructed by Day [D], where $\mathcal{V}$ is a symmetric monoidal closed category.

We denote by $\therefore\Sigma$ the *subdivision category of* Σ:

- its objects are the morphisms of Σ,
- for each morphism $x: u \to u'$ of Σ which does not belong to Σ_0, there are in $\therefore\Sigma$ two morphisms

$$x|u : u \to x \quad \text{and} \quad u'|x : u' \to x ,$$

- there are no other morphisms in $\therefore\Sigma$, and the only composites are those of a morphism with its source and its target.



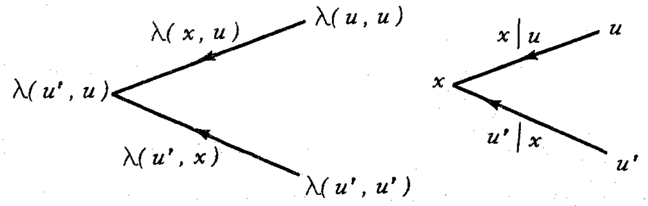
(Intuitively, x is replaced by «an abstract triangle» with vertex x). Naturally, $\therefore\Sigma$ depends on the graph underlying the category Σ and not on the law of composition of Σ.

Let V be a category. We define as follows a functor $\therefore$ from the category $V^{\Sigma \times \Sigma^*}$ to the category $V^{\therefore\Sigma}$:

If $\lambda: \Sigma \times \Sigma^* \to V$ is a functor, $\therefore(\lambda) : \therefore\Sigma \to V$ is the functor assigning $\lambda(u,u)$ to $u \in \Sigma_0$ and

$$\lambda(x,u) \text{ to } x|u, \quad \lambda(u',x) \text{ to } u'|x, \quad \lambda(u',u) \text{ to } x,$$

for each morphism $x: u \to u'$ in Σ.



- If $\theta : \lambda \to \lambda' : \Sigma \times \Sigma^* \rightrightarrows V$ is a natural transformation, $\therefore(\theta)$ is the natural transformation from $\therefore(\lambda)$ to $\therefore(\lambda')$ assigning $\theta(u', u)$ to the morphism $x : u \to u'$ in Σ.

If the functor $\therefore(\lambda)$ admits a projective limit s, then s is called *an end of* $\lambda : \Sigma \times \Sigma^* \to V$.

We say that V *admits* Σ*-ends* if V admits $\therefore\Sigma$-projective limits, i.e. if each functor $\lambda : \Sigma \times \Sigma^* \to V$ admits an end. In that case, if a choice of $\therefore\Sigma$-projective limits is done in V and if $L : V^{\therefore\Sigma} \to V$ is the corresponding canonical projective limit functor, we denote by $\int\theta$ the morphism $L(\therefore(\theta))$, for each $\theta \in V^{\Sigma\times\Sigma^*}$. We write also

$$\int_{x'x} \theta(x', x) \text{ instead of } \int\theta$$

(the usual notation, which does not seem explicit enough, is $\int_u \theta(u, u)$).

EXAMPLE. $\mathfrak{M}$ admits Σ-ends, when $\Sigma \in \mathcal{F}_0$. Let ψ and ψ' be two functors from Σ to $\Sigma' \in \mathcal{F}_0$ and consider the functor

$$\Sigma'(\psi'-, \psi-) : \Sigma \times \Sigma^* \to \mathfrak{M},$$

which assigns $\Sigma'(\psi'(x'), \psi(x))$ to the pair (x', x) of morphisms of Σ. The canonical end of this functor is the set $\Sigma'^{\Sigma}(\psi', \psi)$ of natural transformations from ψ to ψ'.

From now on, we denote by $\mathcal{V}$ a symmetric monoidal closed category $(V, \tau, i, a, b, c, m, D)$. In this notation:

- V is the underlying category,
- $\tau : V \times V \to V$ is the «tensor product functor» and we write

$$g \tau f \text{ instead of } \tau(g, f),$$

- i is the «unit» (up to an equivalence) of τ,
- the equivalences defining i as a unit of τ are

$$a: Id_V \to -\tau i \quad \text{and} \quad b: Id_V \to i\tau -,$$

- the equivalence defining the «associativity» of τ is

$$c: -\tau(-\tau-) \to (-\tau-)\tau- : V\times V\times V \rightrightarrows V,$$

- the equivalence defining the «symmetry» of τ is

$$m: (-\tau-) \to (-\tau-)\mu : V\times V \rightrightarrows V,$$

where μ is the symmetry functor from $V\times V$ to $V\times V$, assigning

$$(g,f) \quad \text{to} \quad (f,g) \in V\times V,$$

- $D: V\times V^* \to V$ is the closure functor, so that $D(-,s)$ is a right adjoint of $-\tau s$, for each object s of V.

We suppose that $V(s,i)$ belongs to the universe $\mathcal{U}$ for each object s of V, and that V admits sums indexed by $\mathcal{U}$-sets. Then the functor $V(-,i): V \to \mathfrak{M}$ admits a left adjoint, which we denote by q. If E is a $\mathcal{U}$-set, $q(E)$ is a sum $\coprod_E i$ in V (of the family

$$(s_z)_{z\in E} \quad \text{where} \quad s_z = i \quad \text{for each} \quad z\in E\ ).$$

In fact, q defines [K] a monoidal closed functor from the canonical cartesian closed category over $\mathfrak{M}$ to $\mathcal{V}$, so that the functors

$$q(-\times-) \quad \text{and} \quad \tau(q-,q-): \mathfrak{M}\times\mathfrak{M} \to V$$

are canonically equivalent.

Let Σ be a category such that V admits Σ-ends. Then Day ([D], example 5-3) has defined a symmetric monoidal closed category

$$\mathcal{V}^\Sigma = (V^\Sigma, \hat\tau, \hat i, \hat a, \hat b, \hat c, \hat m, \hat D)$$

as follows:

- If G and G' are functors from Σ to V, the functor $G'\hat\tau G: \Sigma \to V$ is the functor $\tau[G',G]$ which assigns $G'(x)\tau G(x)$ to $x\in\Sigma$. If

$$\theta: G \to F: \Sigma \rightrightarrows V \quad \text{and} \quad \theta': G' \to F'$$

are natural transformations, the natural transformation

$$\theta'\hat\tau\theta : G'\hat\tau G \to F'\hat\tau F : \Sigma \rightrightarrows V$$

assigns $\theta'(u)\tau\theta(u)$ to the object u of Σ.

- $i^\wedge$ is the constant functor from Σ to V, whose value is the unit i,

- the natural equivalences $\hat{a}$ and $\hat{b}$ assign the natural transformations aG and bG to the functor $G: \Sigma \to V$,

- the natural equivalence $\hat{c}$ assigns

$$c\,[\,G'', G', G\,] : \Sigma \rightrightarrows V \quad \text{to} \quad (G'', G', G),$$

where G'', G' and G are functors from Σ to V,

- the equivalence $\hat{m}$ assigns $m\,[\,G', G\,] : \Sigma \rightrightarrows V$ to the pair (G', G) of functors from Σ to V,

- if G' and G are functors from Σ to V, then $\hat{D}(G', G)$ is an end of the functor from $\Sigma \times \Sigma^*$ to V^Σ assigning the natural transformation

$$D(G'(x'), -)((G(x)\,\tau -)\,q\Sigma(x, -))^* : \Sigma \rightrightarrows V$$

to the pair (x', x) of morphisms of Σ. We will write:

$$\hat{D}(G', G) = \int_{x'x} D(G'(x'), G(x)\,\tau\, q\Sigma(x, -)).$$

In fact, Day proves a stronger result: $\mathcal{V}^\Sigma$ is a symmetric monoidal closed category over $\mathcal{V}$, which means that the functors and natural transformations in the construction above underly $\mathcal{V}$-functors or $\mathcal{V}$-natural transformations. From this, we will use only that, G and G' being functors from Σ to V, the functors

$$\int_{x'x} D(-, G(x))\,\hat{D}(G', -)(x') \quad \text{and} \quad \int_{x'x} D(G'(x'), -)(G\,\hat{\tau} -)(x)^*$$

from $(V^\Sigma)^*$ to V are equivalent. (This may be proved directly, using Fubini Theorem on ends [ML] and the $\mathcal{V}$-Yoneda Lemma [K] .)

B) Subcategories of a symmetric monoidal closed category.

We suppose here that $\mathcal{V}$ is a symmetric monoidal closed category

$$\mathcal{V} = (V, \tau, i, a, b, c, m, D),$$

and V' a full subcategory of V which is closed for D, i. e. such that it exists a functor $D' : V' \times V'^* \to V'$ restriction of D. Then, under some conditions, V' underlies a symmetric monoidal closed category having D' as its closure functor. This will be applied in the next Section to the subcategory V^σ of V^Σ.

If V' is also closed for τ, i. e. if it exists a functor

$$\tau' : V' \times V' \to V' \text{ restriction of } \tau,$$

and if i is an object of V', then the natural equivalences a, b, c and m admit restrictions a', b', c' and m' such that

$$(V', \tau', i, a', b', c', m', D')$$

is a symmetric monoidal closed subcategory of $\mathcal{V}$. More generally:

PROPOSITION 21. *We suppose that* V' *is a full subcategory of* V, *such that:*

1° there exists a functor $D' : V' \times V'^{*} \to V'$ *restriction of* D,

2° the insertion functor I *from* V' *to* V *admits a left adjoint* J.

Then there exists a symmetric monoidal closed category

$$(V', \tau', J(i), a', b', c', m', D'), \text{ where } f'\tau' f = J(f'\tau f). \qquad 1+$$

Δ. We denote by $\delta : Id_V \to I J$ the natural transformation defining J as an adjoint of I, by i' the object $J(i)$ of V' and by τ' the composite functor $J\,\tau(I-, I-)$:

$$V' \times V' \xrightarrow{I \times I} V \times V \xrightarrow{\tau} V \xrightarrow{J} V'$$

which assigns $J(f' \tau f)$ to the pair (f', f) of morphisms of V'.

1° Let s' be an object of V'. The functor $-\tau' s' : V' \to V'$ admits $D'(-, s')$ as a right adjoint. Indeed, as $D(-, s')$ is a right adjoint of $-\tau s'$, the functor $D(-, s')I$ is a right adjoint of $J(-\tau s')$. As V' is a full subcategory of V in which $D(-, s')I$ takes its values, the restriction

$$D'(-, s') : V' \to V' \text{ of } D(-, s')I$$

is also a right adjoint of the functor from V' to V' restriction of $J(-\tau s')$, i. e. of the functor $-\tau' s'$.

If τ' is a tensor-product functor on V' whose unit is i', Proposition 21 will result from Theorem II-5-8 of [EK].

2° We will establish some facts to be used afterwards.

a) Let s' and s'' be objects of V' and e of V. Then the maps

$$V(s'', \delta(e)\tau s') \text{ and } V(s'', s'\tau\delta(e))$$

are bijections. Indeed, we denote by:

- $p(s'', s') : D(s'', s')\tau s' \to s''$ the morphism defining $D(s'', s')$

as a cofree structure generated by s'' relative to the functor $-\tau s'$,

- $P(s'', s', e): V(D(s'', s'), e) \to V(s'', e\tau s')$ the bijection assigning $p(s'', s').(g\tau s')$ to $g: e \to D(s'', s')$.

Since V' is closed for D, the object $D(s'', s')$ belongs to V', and, from the adjunction between I and J, we deduce that

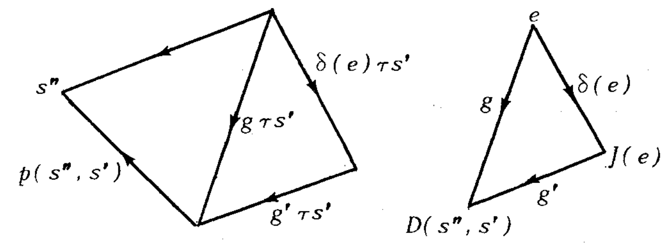
$$V(D(s'', s'), \delta(e)): V(D(s'', s'), J(e)) \to V(D(s'', s'), e)$$

is a bijection. The composite bijection

$$P(s'', s', e)\, V(D(s'', s'), \delta(e))\, P(s'', s', J(e))^{-1}$$

assigns $f'.(\delta(e)\tau s')$ to $f': J(e)\tau s' \to s''$; so it is the map

$$V(s'', \delta(e)\tau s'): V(s'', J(e)\tau s') \to V(s'', e\tau s').$$



The map $V(s'', s'\tau\delta(e))$ is also a bijection, the equality

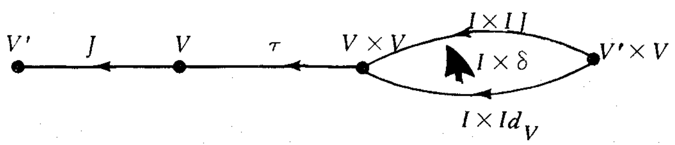
$$s'\tau\delta(e) = m(J(e), s').(\delta(e)\tau s').m(s', e)^{-1}.$$

implying that $V(s'', s'\tau\delta(e))$ is the composite bijection

$$V(s'', m(s', e)^{-1})\, V(s'', \delta(e)\tau s')\, V(s'', m(J(e), s')).$$

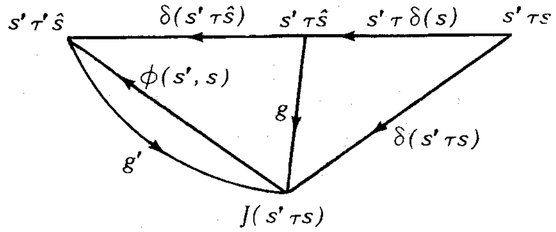
b) ϕ is the natural transformation

$$J\tau(I\times\delta): J\tau(I-,-) \to \tau'(-, J-): V'\times V \rightrightarrows V'.$$



It is an equivalence, i.e. $J(s'\tau J(s))$ is a free structure generated by $s'\tau s$ relative to I. Indeed, let s be an object of V and s' of V'; we write $\hat{s} = J(s)$; then $\phi(s', s) = J(s'\tau\delta(s))$ is the unique f such that

$$f.\delta(s'\tau s) = \delta(s'\tau\hat{s}).(s'\tau\delta(s)).$$



From Part a, it follows that $V(J(s'\tau s), s'\tau\delta(s))$ is a bijection, so that there exists a unique morphism

$$g : s'\tau\hat{s} \to J(s'\tau s) \quad \text{satisfying} \quad g.(s'\tau\delta(s)) = \delta(s'\tau s).$$

There also exists a unique morphism g' such that $g'.\delta(s'\tau\hat{s}) = g$. From the equalities

$$g'.f.\delta(s'\tau s) = g'.\delta(s'\tau\hat{s}).(s'\tau\delta(s)) = g.(s'\tau\delta(s)) = \delta(s'\tau s),$$

we get $g'.f = J(s'\tau s)$ and, from the equalities

$$f.g'.\delta(s'\tau\hat{s}).(s'\tau\delta(s)) = f.\delta(s'\tau s) = \delta(s'\tau\hat{s}).(s'\tau\delta(s)),$$

we deduce successively

$$f.g'.\delta(s'\tau\hat{s}) = \delta(s'\tau\hat{s}),$$

since $V(s'\tau'\hat{s}, s'\tau\delta(s))$ is a bijection and $f.g' = J(s'\tau\hat{s})$.

This proves that $\phi(s',s)$ admits g' as an inverse, and ϕ is an equivalence.

c) Similarly,

$$\phi' = J\tau(\delta \times I) : V \times V' \rightrightarrows V'$$

is an equivalence.

3° We are going to show that τ' is a tensor-product functor whose unit is $i' = J(i)$.

a) If s is an object of V', we denote by $a'(s)$ the morphism

$$\delta(s\tau i').(s\tau\delta(i)).a(s) : s \to s\tau' i'.$$

We so define a natural transformation $a' : Id_{V'} \to \cdot\,\tau' i'$ such that Ia' is the natural transformation

$$(\delta(\cdot\,\tau i') \square (\cdot\,\tau\delta(i)) \square a)I.$$

The morphism $a(s)$ being invertible and $\delta(s\tau i').(s\tau\delta(i))$ defining $s\tau' i'$ as a free structure generated by $s\tau i$ (Part 2-b), the morphism $a'(s)$ defines $s\tau' i'$ as a free structure generated by the object s of the full subcategory V' relative to the insertion functor I. Hence $a'(s)$ is invertible. So a' is an equivalence.

We define similarly the equivalence $b' : Id_{V'} \to i'\tau'\text{-}$, which assigns $\delta(i'\tau s).(\delta(i)\tau s).b(s)$ to the object s of V'.

b) $\mu : V\times V \to V\times V$ and $\mu' : V'\times V' \to V'\times V'$ being the «symmetry functors», we have $\mu(I\text{-}, I\text{-}) = (I\times I)\mu'$. The equivalence $m: \tau \to \tau\mu$ defining the symmetry of τ gives rise to the equivalence

$$m' = J\, m(I\text{-}, I\text{-}) : \tau' \to J\,\tau\mu(I\text{-}, I\text{-}),$$

which assigns the invertible morphism $J(m(s, s'))$ to the pair (s', s) of objects of V'. As

$$J\,\tau\mu(I\text{-}, I\text{-}) = J\,\tau(I\text{-}, I\text{-})\mu' = \tau'\mu',$$

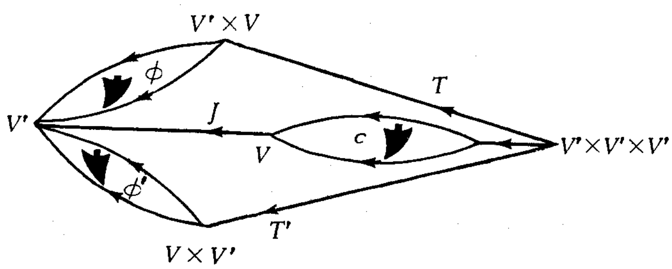
the equivalence m' is a symmetry of τ'.

c) We consider the functors

$$T : V'\times V'\times V' \to V'\times V \quad\text{and}\quad T' : V'\times V'\times V' \to V\times V'$$

assigning to (x'', x', x) respectively $(x'', x'\tau x)$ and $(x''\tau x', x)$. With the notations of Part 2, let c' be the natural transformation

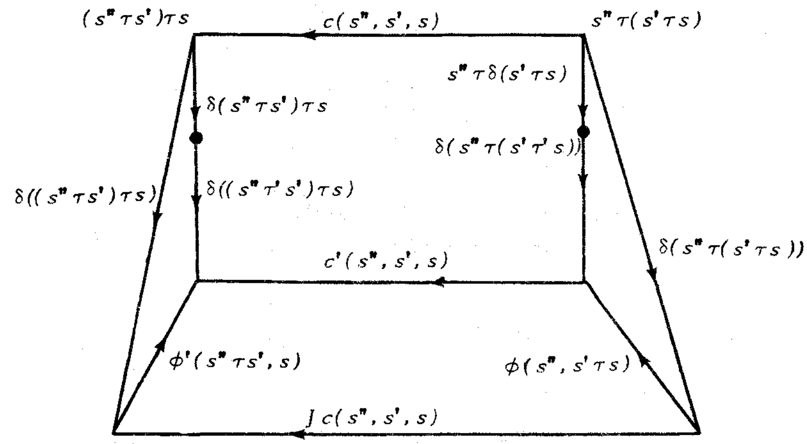
$$\phi' T' \square\square\, J\, c(I\times I\times I) \square\square (\phi T)^{-1} : \text{-}\,\tau'(\text{-}\,\tau'\,\text{-}) \to (\text{-}\,\tau'\,\text{-})\,\tau'\,\text{-},$$



which assigns

$$c'(s'', s', s) = \phi'(s''\tau s', s).\, J\, c(s'', s', s).\, \phi(s'', s'\tau s)^{-1}$$

to (s'', s', s), where s'', s' and s are objects of V'.



Since ϕ, Jc and ϕ' are equivalences, c' is also an equivalence. To prove that

$$(V', \tau', i', a', b', c', m', D')$$

is a symmetric monoidal closed category, we have yet to show that the three coherence axioms are satisfied.

d) The coherence axiom on units asserts that, if s and s' are objects of V', then

$$c'(s', i', s).(s'\tau' b'(s)) = a'(s')\tau' s.$$

Indeed, we have the following diagram, where

① is a quartet, c being a natural transformation,

② is a quartet, by definition of c',

③ is commutative, since we have

$$a'(s') = \delta(s'\tau i').(s'\tau\delta(i)).a(s'),$$

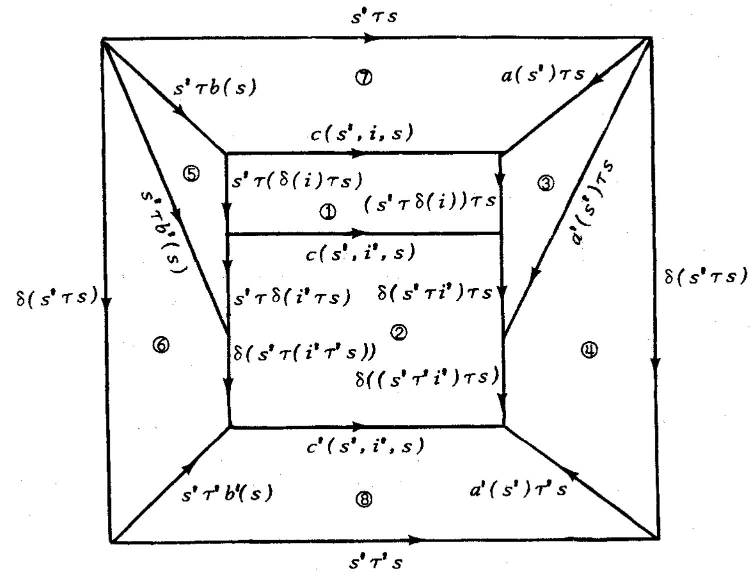
④ is commutative, as a consequence of the equality

$$a'(s')\tau' s = J(a'(s')\tau s),$$

⑤ is commutative, by definition of b',

⑥ is commutative, by definition of τ' (similarly to ④),

⑦ is commutative, the first coherence axiom being satisfied in the monoidal category (V, τ, i, a, b, c).



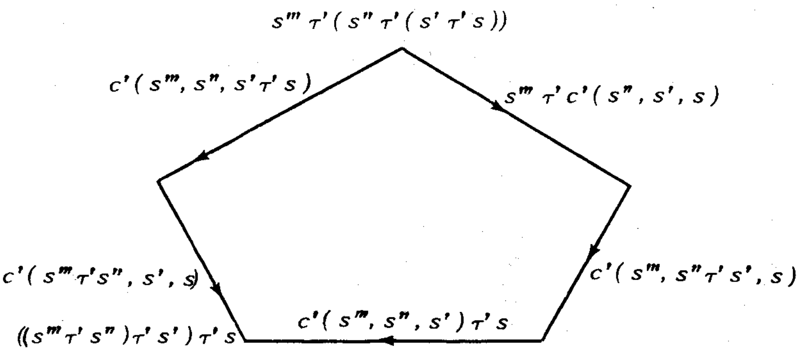
From this diagram we deduce

$$c'(s', i', s).(s'\tau' b'(s)).\,\delta(s'\tau s) = (a'(s')\tau' s).\delta(s'\tau s),$$

which implies, $\delta(s'\tau s)$ defining $s'\tau' s$ as a free structure generated by $s'\tau s$ relative to I,

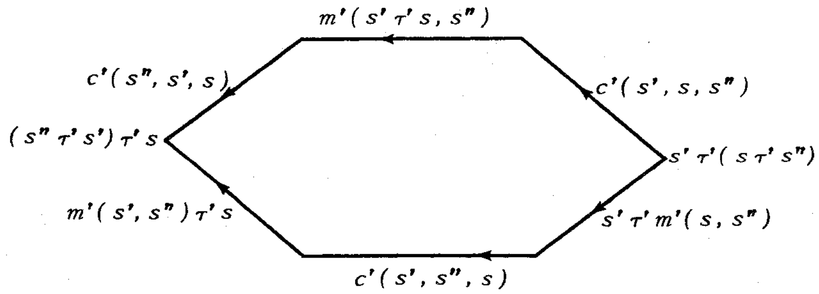
$$c'(s', i', s).(s'\tau' b'(s)) = a'(s')\tau' s.$$

e) We consider the second coherence axiom (on associativity), called axiom MC3 [EK], which says that, if s, s', s'' and s''' are objects of V', the following diagram commutes.



This may be proved directly from the axiom MC3 satisfied by τ and from the definition of c'. But we can also use Proposition II-2-1 [EK], since D' and τ' are in the «basic situation» of Chapter II-4 [EK]. So τ' satisfies the axiom MC3 iff D' satisfies the axiom CC3 of [EK] (associativity coherence axiom for closed categories). As D is a closure functor on V, it satisfies CC3 and, V' being a full subcategory of V, the restriction D' of D also verifies CC3 (which is independent of i and τ). Hence τ' is a tensor-product functor on V'.

f) The coherence axiom on symmetry asserts that, if s, s' and s'' are objects of V', the following diagram commutes:



This diagram is the exterior border of the following diagram, where:

①, ④ and ⑦ commute, by definition of c',

② commutes, m being a natural transformation,

③, ⑤ and ⑧ commute, since $m' = J\, m(I-, I-)$,

⑥ and ⑨ commute, as $\tau' = J\, \tau(I-, I-)$,

⑩ commutes, (V, τ, i, a, b, c, m) being a symmetric monoidal category, the mapping

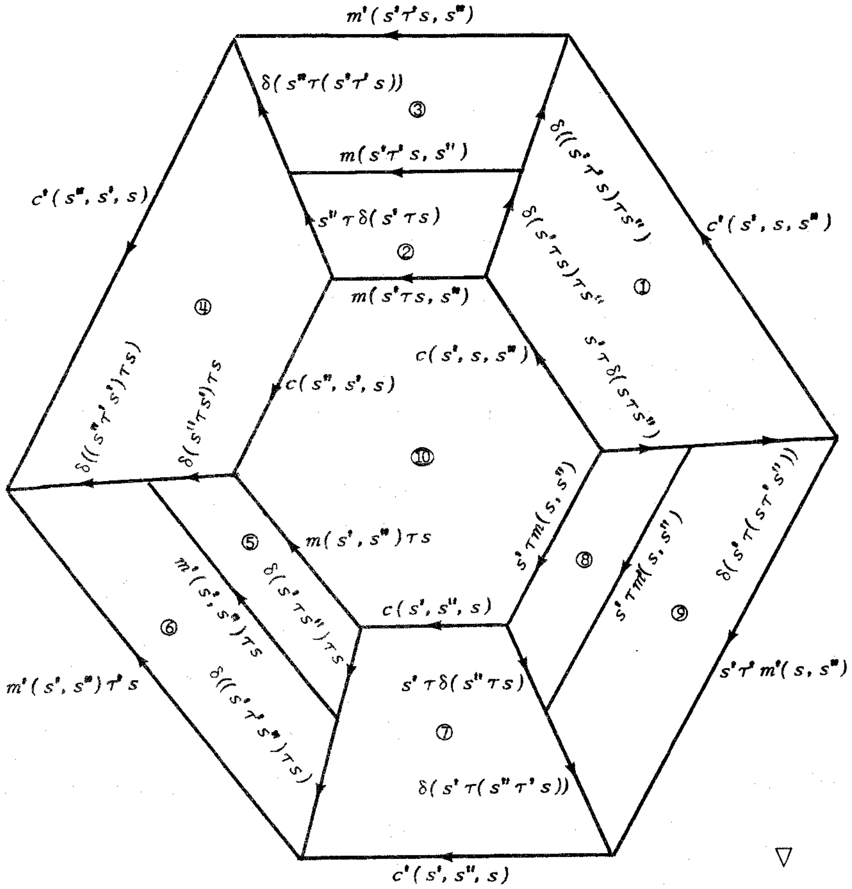
$$V((s''\tau' s')\tau' s,\ \delta(s'\tau(s\tau' s'')).(s'\tau\delta(s\tau s'')))$$

is a bijection (Part 2-b).

From all these properties, we deduce that the exterior border of this diagram commutes. Hence all the coherence axioms are satisfied, so that

$$(V', \tau', i', a', b', c', m', D')$$

is a symmetric monoidal closed category.



∇

COROLLARY. *With the hypotheses of Section A on* $\mathcal{V}$, *let* σ' *be a mixed limit-bearing category* (Σ, Γ, ∇) *such that:*

1° the insertion functor from $V^{\sigma'}$ *to* V^{Σ} *admits a left adjoint,*

2° $V^{\sigma'}$ *is closed for the closure functor* $\hat{D}$ *of the symmetric monoidal closed category* $\mathcal{V}^{\Sigma}$.

Then $V^{\sigma'}$ *is underlying a symmetric monoidal closed category whose closure functor is a restriction of* $\hat{D}$.

Δ. This results from Proposition 21 applied to $\mathcal{V}^{\Sigma}$ and $V^{\sigma'}$. ∇

PROPOSITION 22. *Let* V' *be a full subcategory of* V *such that:*

1° $i \in V'$ *and* τ *admits a restriction* $\tau' : V' \times V' \to V'$,

2° the insertion functor I from V' to V admits a right adjoint I'.
Then there exists a symmetric monoidal closed category

$$(V', \tau', i, a', b', c', m', D'), \quad \text{where} \quad D' = I'D(I-, I-).$$

Δ. The first condition implies that V' defines a symmetric monoidal subcategory $(V', \tau', i, a', b', c', m')$ of (V, τ, i, a, b, c, m).

Proposition 22 will result from Theorem II-5-8 [EK] if we prove that, for each object s' of V', the functor $-\tau' s' : V' \to V'$ admits $D'(-, s')$ as a right adjoint. Indeed, $I'D(-, s')$ is a right adjoint of $(-\tau s')I$. As V' is a full subcategory of V, it follows that the functor $D'(-, s') : V' \to V'$, restriction of $I'D(-, s')$, is a right adjoint of the functor $-\tau' s'$, restriction of $(-\tau s')I$. ∇

COROLLARY 1. *With the hypotheses of Section A on $\mathcal{V}$, let σ' be a mixed limit-bearing category (Σ, Γ, ∇) such that:*

1° the insertion functor from $V^{\sigma'}$ to V^{Σ} admits a right adjoint,

2° $V^{\sigma'}$ is closed for the tensor product $\hat{\tau}$ of $\mathcal{V}^{\Sigma}$, and $\hat{i}$ is a σ'-structure in V.

Then $V^{\sigma'}$ is underlying a symmetric monoidal closed category whose tensor product is a restriction of $\hat{\tau}$. 1

COROLLARY 2. *Let σ be a $\mathcal{J}$-limit-bearing category (Σ, Γ) and V a category admitting $\mathcal{J}$-projective limits, sums indexed by $\mathfrak{U}$-sets and Σ-ends. If V admits a cartesian closed structure, and if the insertion functor from V^{σ} to V^{Σ} admits a right adjoint, then V^{σ} admits a cartesian closed structure (deduced from that of V^{Σ}).*

Δ. V^{σ} being closed for finite products in V^{Σ}, this results from Corollary 1, applied to a symmetric cartesian closed category $\mathcal{V}$ over V. ∇

11. Symmetric monoidal closed category $\mathcal{V}^{\sigma}$.

If σ is «cartesian», V^{σ} is closed for the closure functor of $\mathcal{V}^{\Sigma}$ (section 10-A), so that the preceding corollaries give symmetric monoidal closed structures on V^{σ}.

As in the sections 10 and 11, we still denote by σ a projective limit-bearing category (Σ, Γ) whose set of morphisms is a $\mathfrak{U}$-set, by $\mathcal{J}$ its set of indexing-categories, by Y the Yoneda σ^*-structure in $\mathfrak{M}^\sigma$.

DEFINITION. We say that σ is *cartesian* if the functor $\cdot \times Y(u) : \mathfrak{M}^\sigma \to \mathfrak{M}^\sigma$ commutes with $\mathfrak{I}$-inductive limits, for each object u of Σ.

Proposition 20 says that σ is cartesian iff $\mathfrak{M}^\sigma$ admits a cartesian closed structure.

In all this Section, we will denote by V a category satisfying the following condition

(L) $V(s', s)$ is a $\mathfrak{U}$-set, for any pair (s', s) of objects of V.
V admits Σ-ends.
V admits sums indexed by $\mathfrak{U}$-sets (this property may be replaced by a weaker one, as is shown in Remark 2, after Proposition 23).

Finally, $\mathcal{V}$ denotes a symmetric monoidal closed category

$$(V, \tau, i, a, b, c, m, D),$$

$q : \mathfrak{M} \to V$ an adjoint of $V(-, i)$ and $\mathcal{V}^\Sigma$ the corresponding symmetric monoidal closed category constructed by Day (Section 10-A):

$$\mathcal{V}^\Sigma = (V^\Sigma, \hat{\tau}, i^\wedge, \hat{a}, \hat{b}, \hat{c}, \hat{m}, \hat{D}).$$

PROPOSITION 23. *We suppose σ is cartesian. Then: there exists a functor $\hat{D}' : V^\sigma \times (V^\sigma)^* \to V^\sigma$ restriction of $\hat{D}$; for each σ-structure G in V, the functor $\hat{D}'(G, qY\cdot)$ assigning $\hat{D}(G, qY(x))$ to $x \in \Sigma$ is a σ-structure in V^σ. Finally, if G and G' are σ-structures in V, we have*

$$\hat{D}'(G', G) \approx \int_{x' x} D(-, G(x)) \hat{D}'(G', qY(x')).$$

Δ. 1° Let u be an object of Σ and G a σ-structure in V. For each object s of V, let G_s be the functor $V(\hat{D}(G, qY(u)), s)$. We are going to prove that the functors G_s are σ-structures in $\mathfrak{M}$. This will imply [Lb] that $\hat{D}(G, qY(u))$ is a σ-structure in V, so that the functor $\hat{D}'(G, qY\cdot)$ from Σ to V^σ exists. Indeed, as $V(-, s)$ commutes with projective limits and

$$\hat{D}(G, qY(u)) = \int_{x' x} D(G(x'), qY(u)(x) \tau q\Sigma(x, \cdot)),$$

the functor G_s is an end of the functor $F : \Sigma \times \Sigma^* \to \mathfrak{M}^\Sigma$ assigning

$$V(\cdot, s) D(G(x'), -)(qY(u)(x) \tau q\Sigma(x, \cdot))^* \text{ to } (x', x).$$

The functors $V(D(G-, -), s)$ and $V(D(G-, s), -)$, from $\Sigma \times V^*$ to $\mathfrak{M}$

are equivalent, $\mathcal{V}$ being a symmetric monoidal closed category, as well as the functors

$$(q\text{-})\tau(q\text{-}) \quad \text{and} \quad q(\cdot\times\text{-}): \mathfrak{M}\times\mathfrak{M} \to V.$$

So F is equivalent to the functor F' assigning

$$V(D(G(x'), s), q\text{-})(Y(u)(x)\times\Sigma(x, \text{-}))^* \quad \text{to} \quad (x', x)$$

where x and x' are morphisms of Σ. Since q is adjoint to $V(\text{-}, i)$ and, by definition of a symmetric monoidal closed category, $V(D\text{-}, i)$ is equivalent to $V(\text{-}, \text{-})$, the functor F' is equivalent to the functor F'' from $\Sigma\times\Sigma^*$ to $\mathfrak{M}^{\Sigma}$ assigning to (x', x):

$$\mathfrak{M}(V(G(x'), s), \text{-})(Y(u)\times Y(\text{-}))(x)^*: \Sigma \rightrightarrows \mathfrak{M}.$$

It follows that G_s is also an end of F''.

As G is a σ-structure in V and $V(\text{-}, s)$ commutes with projective limits, the functor $\hat{G} = V(G\text{-}, s)$ is a σ-structure in $\mathfrak{M}$. We consider the σ-structure $\overline{M}(\hat{G}, Y\text{-})$ in $\mathfrak{M}^{\sigma}$ associated to $\hat{G}$ (Proposition 20); we have

$$\overline{M}(\hat{G}, Y(u)) = \mathfrak{M}^{\sigma}(\hat{G}, Y(u)\times Y\text{-}) \approx \int F'',$$

by definition of the set of natural transformations between two functors as an end. So G_s is equivalent to the σ-structure $\overline{M}(\hat{G}, Y(u))$ in $\mathfrak{M}$. A fortiori, G_s is a σ-structure in $\mathfrak{M}$, for any object s of V.

2° Let G be a σ-structure in V. Then $\hat{D}'(G, qY\text{-})$ is a σ-structure $\overline{G}$ in V^{σ}, equivalent to $\overline{\overline{G}}: \Sigma \to V^{\sigma}$, where $\overline{\overline{G}}(x'): \Sigma \rightrightarrows V$ is defined by $\overline{\overline{G}}(x')(y) = \overline{G}(y)(x)$, for $y \in \Sigma$. The proof is similar to Part 2, Prop. 20.

3° If G and G' are σ-structures in V, then $\hat{D}(G', G)$ is a σ-structure in V. Indeed, $\hat{D}(G', G)$ is an end of the functor $H: \Sigma\times\Sigma^* \to V^{\Sigma}$, assigning

$$D(G'(x'), \text{-})(G(x)\tau q\text{-})^*\Sigma(x, \text{-})^* = D(G'(x'), \text{-})(G\hat{\tau} qY(\text{-}))(x)^*$$

to (x', x). The functors

$$\int_{x'x} D(\text{-}, G(x))\hat{D}(G', \text{-})(x') \quad \text{and} \quad \int_{x'x} D(G'(x'), \text{-})(G\hat{\tau}\text{-})(x)^*$$

being equivalent (section 10-A), $\hat{D}(G', G)$ is also an end of the functor H' from $\Sigma\times\Sigma^*$ to V^{Σ} assigning

$$D(\text{-}, G(x))\hat{D}'(G', qY(x')) \quad \text{to} \quad (x', x).$$

If u and u' are objects of Σ, the functor $H'(u', u)$ is a σ-structure in V, for it is the composite functor

$$\Sigma \xrightarrow{\hat{D}(G', qY(u'))} V \xrightarrow{D(-, G(u))} V$$

where $\hat{D}(G', qY(u'))$ is a σ-structure in V (Part 1) and $D(-, G(u))$ commutes with projective limits. Hence, H' takes its values in V^σ. As V^σ is closed for Σ-ends in V^Σ (the category V admitting Σ-ends), it follows that the end $\hat{D}(G', G)$ of H' is a σ-structure in V.

This proves the existence of a functor $\hat{D}': V^\sigma \times (V^\sigma)^* \to V^\sigma$, restriction of $\hat{D}$. ∇

COROLLARY 1. *If σ is cartesian and if the insertion functor $\hat{I}$ from V^σ to V^Σ admits a left adjoint $\hat{J}$, it exists a symmetric monoidal closed category*

$$\mathcal{V}^\sigma = (V^\sigma, \hat{\tau}', \hat{i}', \hat{a}', \hat{b}', \hat{c}', \hat{m}', \hat{D}'),$$

where $\hat{D}'$ is a restriction of $\hat{D}$ and $\hat{\tau}'$ a restriction of $\hat{J}\,\hat{\tau}$.

Δ. By Proposition 23, V^σ is closed for $\hat{D}$. So this corollary results from the corollary of Proposition 21. ∇

COROLLARY 2. *Under the following conditions, V^σ defines a symmetric monoidal closed subcategory $\mathcal{V}^\sigma$ of $\mathcal{V}^\Sigma$:*

1° σ is cartesian,

2° τ commutes with $\mathcal{G}$-projective limits,

3° $i^\wedge$ is a σ-structure in V (for example, if all the indexing-categories of σ are connected or if i is a final object of V).

Δ. Proposition 23 asserts that V^σ is closed for $\hat{D}$.

If G and G' are σ-structures in V, the functor

$$G' \hat{\tau} G = \tau\,[G', G] : \Sigma \to V$$

is a σ-structure in V, since $[G', G]$ is a σ-structure in $V \times V$ and τ is commuting with $\mathcal{G}$-projective limits. Hence V^σ is also closed for $\hat{\tau}$. Since $i^\wedge$ belongs to V^σ (condition 3), V^σ defines a symmetric monoidal closed subcategory of $\mathcal{V}^\Sigma$. ∇

COROLLARY 3. *We suppose σ is cartesian and V is a category admitting* *a cartesian closed structure. If* V *satisfies condition* (L), *then* V^σ *admits a cartesian closed structure.* 1+

Δ. Let $\mathcal{V}$ be any symmetric cartesian closed category whose underlying category is V (it is defined up to an isomorphism). Its tensor-product functor τ is in fact a product functor, so that it commutes with $\mathcal{I}$-projective limits. As i is then a final object of V, the constant functor $i^\wedge$ commutes with projective limits; a fortiori, it is a σ-structure in V. Hence Corollary 2 asserts that V^σ defines a symmetric monoidal closed subcategory $\mathcal{V}^\sigma$ of $\mathcal{V}^\Sigma$. Since $\mathcal{V}^\Sigma$ is cartesian, $\hat{\tau}$ being a product functor, so is $\mathcal{V}^\sigma$. ∇

EXAMPLE. Let V be a category admitting $\mathcal{F}_0$-projective limits and $\mathcal{F}_0$-inductive limits; so it satisfies (L). If V satisfies also the condition:

(L') $\left\{\begin{array}{l}\text{There exists an } \mathfrak{U}\text{-ordinal } \xi \text{ such that } \mathcal{I}\text{-projective limits commu-}\\ \text{te with inductive limits indexed by } <\xi>, \text{ in } V,\end{array}\right.$

the insertion functor from V^σ to V^Σ admits a left adjoint [F1]. Then, by Corollary 1, V^σ underlies a symmetric monoidal closed category as soon as σ is cartesian. The condition (L') is verified, for instance, when V is locally ξ-presentable [GU], or when V is a fibred category over a category satisfying (L') (see [W]).

REMARKS. 1° The third property of Condition (L) may be replaced everywhere by the less restrictive condition:

$\left\{\begin{array}{l} V \text{ admits sums indexed by the sets } \Sigma(u', u), \text{ where } u \text{ and } u' \text{ are ob-}\\ \text{jects of } \Sigma.\end{array}\right.$

Indeed, in this case, let $\mathfrak{M}'$ be the full subcategory of $\mathfrak{M}$ whose objects are the $\mathfrak{U}$-sets E such that there exists in V a sum $\coprod_E i$ (of E exemplars of i), where i is still the unit of the symmetric monoidal closed category $\mathcal{V}$. Choosing such a sum $q'(E)$ for each object E of $\mathfrak{M}'$, we get a functor $q': \mathfrak{M}' \to V$, which is a «partial adjoint» of $V(-, i)$. The sets $\Sigma(u', u)$ are objects of $\mathfrak{M}'$. If E and E' are objects of $\mathfrak{M}'$, then $q'(E')\tau q'(E)$ is a sum of $E' \times E$ exemplars of i, since $-\tau q'(E)$, being a left adjoint, commutes with sums. Hence $\mathfrak{M}'$ is closed for finite products, and the functors

$$q'(-\times -): \mathfrak{M}' \times \mathfrak{M}' \to V \quad \text{and} \quad \tau(q'-, q'-)$$

are equivalent. The proofs of Proposition 23 and of its Corollaries using only the values of q on sets $E' \times E$, where E and E' are of the form $\Sigma(u', u)$, they are also valid if we replace q by q'.

2° We have not used the general result of Day [D], but a very special case of it (Example 5-3 of [D]). In fact, Day associates to any «premonoidal symmetric structure» $P : \Sigma^* \times \Sigma^* \times \Sigma \to V$ a symmetric monoidal closed category P^{Σ} whose underlying category is V^{Σ}. In a forthcoming paper [FL] Foltz and Lair prove that V^{σ} is also closed for the closure functor of P^{Σ} when P defines a double σ-costructure in V, i. e. when there exists a σ^*-structure p in $(V^{\sigma})^{\sigma^*}$ such that

$$p(y)(x')(x) = P(y, x', x), \text{ when } x,\ x' \text{ and } y \text{ belong to } \underline{\Sigma}.$$

So, in this case and if the insertion functor from V^{σ} to V^{Σ} admits a left ad-
1+ joint, V^{σ} is underlying a symmetric monoidal closed category P^{σ}. Notice that Proposition 23 and its corollaries cannot be deduced from this result of [FL]. Indeed, the category $\mathcal{V}^{\Sigma}$ used here is the category P^{Σ} associated to the premonoidal structure P such that

$$P(y, x', x) = (qY(y) \mathbin{\hat{\tau}} qY(x'))(x),$$

and P does not define a double σ-costructure, even if σ is cartesian.

Application.

We denote by:

- $\hat{\sigma}$ a sketch (definition p. 30) $(\hat{\Sigma}, \hat{\Gamma})$, where $\hat{\underline{\Sigma}}$ is a $\mathcal{U}$-set, $\mathcal{J}$ its set of indexing-categories and σ a prototype (Σ, Γ) generated by $\hat{\sigma}$,
- σ' a $\mathcal{J}$-type (V, Γ'), where V is a category satisfying Condition (L),
- $\mathcal{S}(\sigma', \hat{\sigma})$ the category of $\hat{\sigma}$-morphisms (i. e. of morphisms between $\hat{\sigma}$-structures) in σ',
- $\mathcal{S}(\sigma_{\mathcal{M}}, \hat{\sigma})$ the category of $\hat{\sigma}$-morphisms in the canonical $\mathcal{J}$-type $\sigma_{\mathcal{M}} = (\mathcal{M}, \Gamma_{\mathcal{M}})$ on $\mathcal{M}$.

PROPOSITION 24. *We suppose that $\hat{\sigma}$ is σ'-regular and $\sigma_{\mathcal{M}}$-regular (definition p. 47) and that $\mathcal{S}(\sigma_{\mathcal{M}}, \hat{\sigma})$ admits a cartesian closed structure. Then:*

1° $\mathcal{S}(\sigma', \hat{\sigma})$ is underlying a symmetric monoidal closed category if V underlies a symmetric monoidal closed category $\mathcal{V}$ and if one of the following *conditions is satisfied:*

a) There exists an ordinal ξ *in* $\mathfrak{U}$ *such that the* $\mathfrak{I}$*-projective limits in* V *commute with the inductive limits indexed by* $<\xi>$.

b) τ *commutes with* $\mathfrak{I}$*-projective limits and* $i^{\wedge}$ *is a* σ*-structure in* V.

2° $\mathcal{S}(\sigma', \mathfrak{S})$ *admits a cartesian closed structure if* V *admits one.*

Δ. 1° σ is cartesian. Indeed, $\mathfrak{S}$ being $\sigma_{\mathfrak{M}}$-regular, the categories $\mathfrak{M}^{\hat{\mathfrak{S}}}$ and $\mathcal{S}(\sigma_{\mathfrak{M}}, \mathfrak{S})$ are equivalent. By Proposition 7 (page 30), the prototype σ generated by $\mathfrak{S}$ is also a limit-bearing category generated by $\mathfrak{S}$, so that $\mathfrak{M}^{\sigma}$ is isomorphic to $\mathfrak{M}^{\hat{\mathfrak{S}}}$. Hence, $\mathfrak{M}^{\sigma}$ is equivalent to $\mathcal{S}(\sigma_{\mathfrak{M}}, \mathfrak{S})$ and, $\mathcal{S}(\sigma_{\mathfrak{M}}, \mathfrak{S})$ admitting a cartesian closed structure, $\mathfrak{M}^{\sigma}$ also admits one. It follows (Proposition 20) that σ is cartesian.

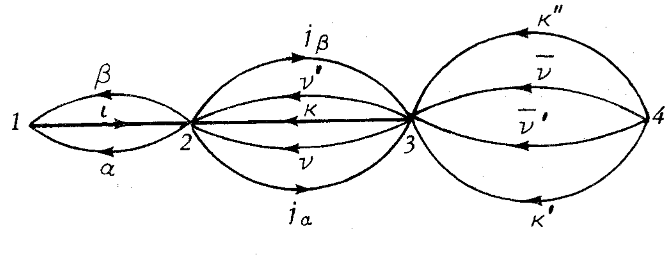
2° The categories $\mathcal{S}(\sigma', \mathfrak{S})$ and $\mathcal{S}(\sigma', \sigma)$ are isomorphic (Corollary 2, Proposition 6). As $\mathfrak{S}$ is σ'-regular, the Corollary of Proposition 16 asserts that the prototype σ generated by $\mathfrak{S}$ is also σ'-regular. This implies that the category V^{σ} is equivalent to $\mathcal{S}(\sigma', \sigma)$, and a fortiori to $\mathcal{S}(\sigma', \mathfrak{S})$. Hence $\mathcal{S}(\sigma', \mathfrak{S})$ underlies a symmetric monoidal closed category iff V^{σ} is underlying one; so the proposition results from the corollaries of Proposition 23 and from the Example. ∇

12. Application to categories of structured functors.

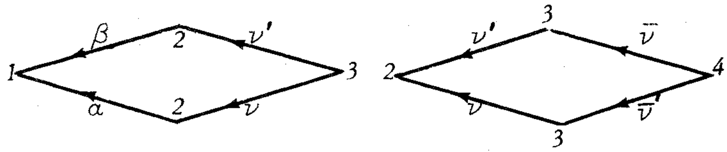
Applying the preceding results to the «sketch of categories», we deduce, from a monoidal closed structure on V, a similar one on the category of functors in V (or category of categories in V).

An integer n is considered as being the set $\{0, 1, \dots, n\text{-}1\}$ (i.e. as a finite ordinal); we denote by $\mathbf{n}$ the category $<n>$ defining the usual order on n.

Let Δ be the simplicial category: its objects are the integers, its morphisms are the monotone maps between integers equipped with their usual order. We denote by Σ the dual of the full subcategory of Δ whose objects are 1, 2, 3 and 4. A set of generators of Σ is formed by the morphisms drawn in the following diagram and by three other morphisms from 3 to 4. The denomination of the morphisms will result from the following properties:



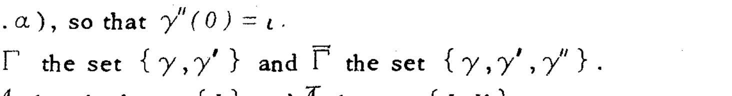
In Σ, we have the two pullbacks



and ι is a kernel of the pair $(2, \iota . \alpha)$.

We denote by:

- I the subdivision category of **2** and I' the category with two objects and two morphisms with the same source and the same target:



- γ and γ' the projective cones indexed by I defining respectively the pullbacks $((\alpha, \nu), (\beta, \nu'))$ and $((\nu, \bar\nu'), (\nu', \bar\nu))$, so that

$$\gamma(0) = \nu, \quad \gamma(1) = \nu', \quad \gamma'(0) = \bar\nu', \quad \gamma'(1) = \bar\nu .$$

- γ'' the projective cone indexed by I' and defining ι as a kernel of $(2, \iota . \alpha)$, so that $\gamma''(0) = \iota$.
- Γ the set $\{\gamma, \gamma'\}$ and $\bar\Gamma$ the set $\{\gamma, \gamma', \gamma''\}$.
- $\mathfrak{I}$ the singleton $\{I\}$ and $\bar{\mathfrak{I}}$ the set $\{I, I'\}$.
- σ and $\bar\sigma$ the pairs (Σ, Γ) and $(\Sigma, \bar\Gamma)$.
- $\sigma_{\mathfrak{M}}$ the canonical $\bar{\mathfrak{I}}$-type $(\mathfrak{M}, \Gamma_{\mathfrak{M}})$ on the category $\mathfrak{M}$.

PROPOSITION 25. *σ and $\bar\sigma$ are regular prototypes, which are cartesian. The category $\mathcal{S}(\sigma_{\mathfrak{M}}, \bar\sigma)$ is isomorphic to the category $\mathcal{F}$ of functors and $\mathfrak{M}^\sigma$ is equivalent to $\mathcal{F}$.*

Δ. 1° $\bar\sigma$ is a prototype, γ, γ' and γ'' being limit-cones. Let U be the subcategory of Σ generated by the set of morphisms drawn in the diagram above and $\sigma_{\mathcal{F}}$ the presketch obtained by equipping U with the cones (restrictions to U of) γ, γ' and γ''. Then $\sigma_{\mathcal{F}}$ is the sketch of categories considered in [E2], and $\mathcal{S}(\sigma_{\mathfrak{M}}, \sigma_{\mathcal{F}})$ is isomorphic to $\mathcal{F}$.

Each morphism y in Σ and not in U having the vertex 4 of γ' as its target, it is the factor of $\gamma' y$ through γ'; so the construction of the prototype generated by $\sigma_{\mathcal{F}}$ (Proposition 6) stops at the first step and gives $\bar{\sigma}$. Corollary 2, Proposition 6, asserts that $\mathcal{S}(\sigma_{\mathfrak{M}}, \bar{\sigma})$ is isomorphic to $\mathcal{S}(\sigma_{\mathfrak{M}}, \sigma_{\mathcal{F}})$. So there exists an isomorphism from $\mathcal{S}(\sigma_{\mathfrak{M}}, \bar{\sigma})$ to $\mathcal{F}$; it assigns to the $\bar{\sigma}$-structure F in $\sigma_{\mathfrak{M}}$ the category whose set of morphisms is $F(2)$, the law of composition being $F(\kappa)$, the maps source and target $F(\alpha)$ and $F(\beta)$.

As $\sigma_{\mathcal{F}}$ is a regular sketch (Propositions 4 and 5 of [E2]), its prototype $\bar{\sigma}$ is also regular (Corollary, Proposition 16). In particular, $\mathfrak{M}^{\bar{\sigma}}$ is equivalent to $\mathcal{F}$.

2° For each category V, the categories V^{σ} and $V^{\bar{\sigma}}$ are identical. Indeed, a $\bar{\sigma}$-structure in V is also a σ-structure in V. Now let F be a σ-structure in V. Since ι is a right inverse of α in Σ, the morphism $F(\iota)$ is a right inverse of $F(\alpha)$; this implies that $F(\iota)$ is a kernel of the pair $(F(2), F(\iota).F(\alpha))$ in V. Hence F is a $\bar{\sigma}$-structure in V.

It follows that $\mathfrak{M}^{\bar{\sigma}} = \mathfrak{M}^{\sigma}$ is equivalent to $\mathcal{F}$. Since $\mathcal{F}$ admits a cartesian closed structure, $\mathfrak{M}^{\sigma}$ also, i. e. σ and $\bar{\sigma}$ are cartesian (p. 78). ∇

REMARKS. 1° A σ-structure F in $\sigma_{\mathfrak{M}}$ corresponds to a category on $F(2)$ whose law of composition is $F(\kappa)$, equipped with an injection $F(\iota)$ defining $F(1)$ as a set of objects. So, $\mathcal{S}(\sigma_{\mathfrak{M}}, \sigma)$ is isomorphic to the category of functors between categories with a given set of objects.

2° In [E2] the sketch of categories was in fact defined as a «pointed sketch», i. e. the ι had to be mapped on a canonical injection. This condition is expressed here by asking $F(\iota)$ to be a «canonical» kernel, so that we have no need of pointed sketches.

DEFINITION. σ is called the *prototype of categories with objects* and $\bar{\sigma}$ the *prototype of categories.* If V is a category, we define a *category in* V as a σ-structure in V, a *functor in* V as a σ-morphism in V. If σ' is a

$\mathfrak{I}$-type, the $\bar{\sigma}$-structures and $\bar{\sigma}$-morphisms in σ' are called *categories in* σ' and *functors in* σ'.

The categories in V are called generalized structured categories in [E2], category-objects in V in most papers. We denote by:

- $\mathfrak{F}(V)$ the category V^{σ} of functors in V: in particular $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}^{\sigma}$.
- $\mathfrak{F}(\sigma')$ the category $\mathcal{S}(\sigma', \bar{\sigma})$ of functors in a $\mathfrak{I}$-type σ'.

PROPOSITION 26. *We suppose that V is a category which admits pullbacks, kernels and sums of pairs.*

1° If V admits a cartesian closed structure, $\mathfrak{F}(V)$ admits also one.

2° Let $\mathfrak{V} = (V, \tau, i, a, b, c, m, D)$ be a symmetric monoidal closed category.

a) If τ commutes with pullbacks, $\mathfrak{F}(V)$ defines a symmetric monoidal closed subcategory $\mathfrak{F}(\mathfrak{V})$ of $\mathfrak{V}^{\Sigma}$ (Section 10).

b) If the insertion functor from $\mathfrak{F}(V)$ to V^{Σ} admits a left adjoint J, there exists a symmetric monoidal closed category $\mathfrak{F}(\mathfrak{V})$ whose underlying category is $\mathfrak{F}(V)$ and whose tensor product assigns $J\tau[G', G]$ to the pair (G', G) of functors in V.

Δ. σ is cartesian (Proposition 25) and the only category I belonging to $\mathfrak{I}$ is connected. So Proposition 26 will result from the Corollaries of Proposition 23, if we prove that V satisfies the condition (L) of page 78 (modified according to Remark 1, page 81).

We may choose a universe $\mathfrak{U}$ to which belong the sets $V(s', s)$, where s and s' are objects of V (since we suppose the axiom of universes satisfied). As $\Sigma(u', u)$, where u and u' are equal to $1, 2, 3$ or 4, is a non void finite set, V admits sums indexed by $\Sigma(u', u)$. Finally, the subdivision category $\therefore\Sigma$ of Σ is a finite connected category, so that the existence of Σ-ends in V follows from the

LEMMA. *If V is a category admitting pullbacks and kernels of pairs, it admits projective limits indexed by any category generated by a sub-neocategory which is finite and connected.*

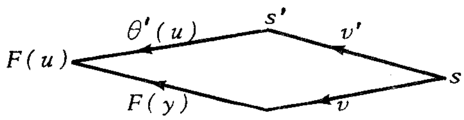
Δ. This (probably well-known) result is proved by induction on the number n of proper morphisms (i. e. different from an object) of the finite connected generating sub-neocategory. The assertion is evident if $n = 1$. We suppose it valid for $n = i$ and we take a functor $F: C \to V$, where C admits a generating sub-neocategory B which is finite, connected, and has $i+1$ proper morphisms. We can find a sub-neocategory B' of B which is connected and has i proper morphisms. We denote by:

- $y: e \to u$ the unique proper morphism of B not in B',
- C' the sub-category of C generated by B',
- $F': C' \to V$ the functor, restriction of F.

By the induction hypothesis, there exists a limit-cone $\theta': s'^{\wedge} \to F'$.

a) If u is not an object of B', then s' is also a projective limit of F.

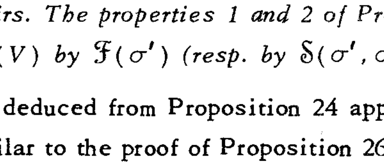
b) If u is an object of B' and e is not in B', there exists a pullback 1



and we get a limit-cone $\theta: s^{\wedge} \to F$ by defining

$$\theta(e) = v, \quad \theta(u') = \theta'(u').v' \text{ if } u' \in B'_0 .$$

c) If u and e are objects of B', let w be a kernel of the pair $(\theta'(u),$



$F(y).\theta'(e))$. Assigning $\theta(u') = \theta'(u').w$ to $u' \in B_0$ we define a limit-cone $\theta: \bar{s}^{\wedge} \to F$. This proves the Lemma by induction. ▽

COROLLARY. *Let σ' be a $\mathfrak{T}$-type (V, Γ'), where V is a category admitting sums of pairs. The properties 1 and 2 of Proposition 26 are also valid if we replace $\mathcal{F}(V)$ by $\mathcal{F}(\sigma')$ (resp. by $\mathcal{S}(\sigma', \sigma)$).*

△. This is deduced from Proposition 24 applied to $\bar{\sigma}$ (resp. to σ) by an argument similar to the proof of Proposition 26. ▽

EXAMPLE. Let p be a saturated homomorphism functor [E1], i.e. p is a faithful functor from V to the category $\mathcal{M}$ of maps and, if s is an object of V and f a bijection with source $p(s)$, there exists one and only one

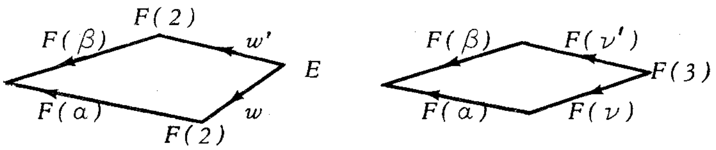
invertible morphism f' of V with source s satisfying $p(f') = f$. Let σ' be a $\overline{\mathfrak{J}}$-type (V, Γ') such that $p\hat{\gamma}$ is a canonical limit-cone in $\mathfrak{M}$ for any cone $\hat{\gamma}$ of Γ'. A category in σ' is called a *p-structured category* and $\mathcal{F}(\sigma')$ is identified with the category $\mathcal{F}(p)$ of p-structured functors [E2]. The Corollary gives conditions for a symmetric monoidal (resp. a cartesian) closed structure $\mathcal{V}$ on V to determine a similar structure on $\mathcal{F}(p)$. This statement generalizes Proposition 10 [BE], relative to the case where p is equivalent to the base functor $V(-, i)$ of $\mathcal{V}$ (this condition is very restrictive, since p is supposed faithful). It implies for instance, if p is the faithful functor $p_{\mathcal{F}}: \mathcal{F} \to \mathfrak{M}$, that the category $\mathcal{F}(p_{\mathcal{F}})$ of double func-
1 tors admits a cartesian closed structure, since $\mathcal{F}$ admits one (this does not result from [BE], the base functor $\mathcal{F}(-, 1)$ of $\mathcal{F}$ being equivalent to the not faithful functor $p'_{\mathcal{F}}$ which assigns to a functor ϕ its restriction ϕ_0).

13. Another construction of a closure functor on $\mathcal{F}(V)$.

We are going to give a direct construction of a closure functor of $\mathcal{F}(\mathcal{V})$; this construction proves that such a functor may be defined even if V admits pullbacks and kernels of pairs, but not sums of pairs.

A) Closure functor on $\mathcal{F}(\mathfrak{M})$.

Let $F: \Sigma \to \mathfrak{M}$ be an object of $\mathcal{F}(\mathfrak{M})$. In $\mathfrak{M}$, we have the pullbacks



where the first one is the canonical one, i. e. E is the set of pairs

$$(y, x) \in F(2) \times F(2) \text{ such that } F(\alpha)(y) = F(\beta)(x).$$

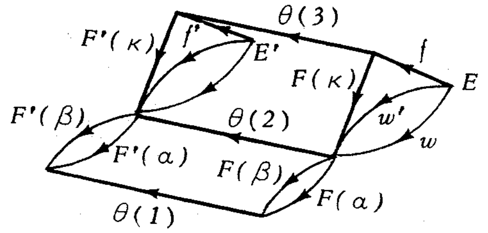
So there exists a unique bijection $f: E \to F(3)$ satisfying

$$F(\nu).f = w \text{ and } F(\nu').f = w'.$$

The map $F(\kappa).f$ is the law of composition of a category C whose set of morphisms is $F(2)$. We say that C is the category *determined by* F, and we denote it by $\eta(F)$.

We get an equivalence $\eta: \mathcal{F}(\mathfrak{M}) \to \mathcal{F}$ by assigning to a morphism

$\theta: F \to F'$ of $\mathcal{F}(\mathfrak{M})$ the functor $\eta(\theta): \eta(F) \to \eta(F')$ defined by the map $\theta(2)$.

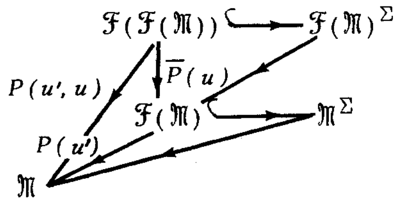


We consider the category $\mathcal{F}(\mathcal{F}(\mathfrak{M}))$ of functors in $\mathcal{F}(\mathfrak{M})$ and, for each object u of Σ, the «evaluation functors»

$$\overline{P}(u): \mathcal{F}(\mathcal{F}(\mathfrak{M})) \to \mathcal{F}(\mathfrak{M}) \quad \text{and} \quad P(u): \mathcal{F}(\mathfrak{M}) \to \mathfrak{M},$$

which assign $\theta(u)$ to θ. If u' is also an object of Σ, we write

$$P(u', u) = P(u')\overline{P}(u): \mathcal{F}(\mathcal{F}(\mathfrak{M})) \to \mathfrak{M}.$$



There exists an equivalence $\overline{\eta}$ from $\mathcal{F}(\mathcal{F}(\mathfrak{M}))$ to the category $\mathcal{F}(p\mathcal{F})$ of double functors, described as follows:

- Let G be an object of $\mathcal{F}(\mathcal{F}(\mathfrak{M}))$. Then $G(2) = \overline{P}(2)(G): \Sigma \to \mathfrak{M}$ determines a category $K^{\perp}$ and the functor

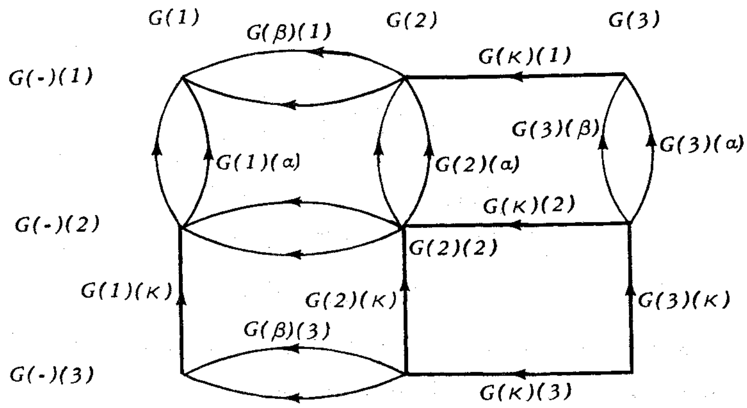
$$P(2)G = G(\cdot)(2): \Sigma \to \mathfrak{M} \quad \text{assigning} \quad G(x)(2) \quad \text{to } x \in \Sigma$$

determines a category $K^{\cdot}$, since $G(2)$ and $G(\cdot)(2)$ are objects of $\mathcal{F}(\mathfrak{M})$. The categories $K^{\cdot}$ and $K^{\perp}$ have $G(2)(2) = P(2,2)(G)$ as their sets of morphisms, and their laws of composition are, respectively, $G(\kappa)(2)$ and $G(2)(\kappa)$. The pair $(K^{\cdot}, K^{\perp})$ is a double category, called *the double category determined by* G. We denote it by $\overline{\eta}(G)$.

- If $\overline{\theta}: G \to G'$ is a morphism of $\mathcal{F}(\mathcal{F}(\mathfrak{M}))$, the map

$$P(2,2)(\overline{\theta}) = \overline{\theta}(2)(2): G(2)(2) \to G'(2)(2)$$

defines the double functor $\overline{\eta}(\overline{\theta})$ from $\overline{\eta}(G)$ to $\overline{\eta}(G')$.



PROPOSITION 27. *There exists a functor* $\partial : \mathcal{F}(\mathfrak{M}) \to \mathcal{F}(\mathcal{F}(\mathfrak{M}))$ *satisfying the following conditions:*

1° If C *is the category determined by an object* F *of* $\mathcal{F}(\mathfrak{M})$, *the double category determined by* $\partial(F)$ *is isomorphic to the double category of quartets of* C.

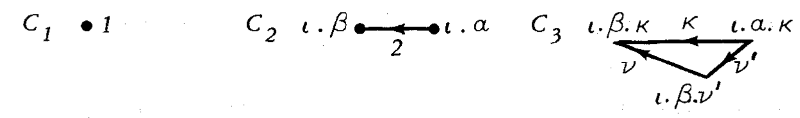
2° If u *and* u' *are objects of* Σ, *the functor* $P(u', u)\partial$ *is equivalent to* $P(u, u')\partial$, *and* $P(u, 1)\partial$ *is equivalent to* $P(u)$.

3° $\mathcal{F}(\mathfrak{M})$ *admits a closure functor* $\overline{M}$ *such that*

$$\overline{M}(F', F)(x) = \mathcal{F}(\mathfrak{M})(\partial(F')(x), F),$$

for a pair (F', F) *of objects of* $\mathcal{F}(\mathfrak{M})$ *and a morphism* x *of* Σ.

Δ. We denote by Y the Yoneda σ^*-structure in $\mathcal{F}(\mathfrak{M})$. For an object n of Σ, the category C_n determined by the object $Y(n)$ of $\mathcal{F}(\mathfrak{M})$ is isomorphic to the category **n**; in particular:



The image of Y is isomorphic to the full subcategory of $\mathcal{F}$ whose objects are the categories **1**, **2**, **3** and **4**. (It follows that a category K is isomorphic to the category determined by the object $\mathcal{F}(K, \eta Y\text{-})$ of $\mathcal{F}(\mathfrak{M})$.)

1° Let $\overline{M}$ be the closure functor on $\mathcal{F}(\mathfrak{M})$ constructed in Proposition 20. For an object F of $\mathcal{F}(\mathfrak{M})$, this proposition shows that $\overline{M}(F, Y\text{-})$ is the σ-structure in $\mathfrak{M}^\sigma$ (i.e. the category in $\mathfrak{F}(\mathfrak{M})$) assigning the natural transformation

$$\mathfrak{F}(\mathfrak{M})(F, Y(x)\times Y\text{-}) \quad \text{to} \quad x \in \Sigma.$$

Denoting $\overline{M}(F, Y\text{-})$ by $\partial(F)$, Proposition 20 also proves that $\overline{M}$ satisfies the third condition.

To the functor $\overline{M}(\text{-}, Y\text{-}): \mathfrak{F}(\mathfrak{M})\times\Sigma \to \mathfrak{F}(\mathfrak{M})$ is canonically associated the functor $M': \mathfrak{F}(\mathfrak{M}) \to \mathfrak{F}(\mathfrak{M})^\Sigma$ such that $M'(\theta) = \overline{M}(\theta, Y\text{-})$ for any θ in $\mathfrak{F}(\mathfrak{M})$. This functor takes its values in the full subcategory $\mathfrak{F}(\mathfrak{F}(\mathfrak{M}))$ of $\mathfrak{F}(\mathfrak{M})^\Sigma$, since $M'(F) = \partial(F)$ for each object F of $\mathfrak{F}(\mathfrak{M})$. Hence M' admits as a restriction a functor $\partial : \mathfrak{F}(\mathfrak{M}) \to \mathfrak{F}(\mathfrak{F}(\mathfrak{M}))$.

2° As $Y(1)$ is a final object of $\mathfrak{F}(\mathfrak{M})$, for each object u of Σ the functor

$$P(u, 1)\partial = \mathfrak{F}(\mathfrak{M})(\text{-}, Y(1)\times Y(u))$$

is equivalent to $\mathfrak{F}(\mathfrak{M})(\text{-}, Y(u))$, and therefore (by Yoneda Lemma) to $P(u)$. Let u and u' be objects of Σ. We have:

$$P(u', u)\partial = \mathfrak{F}(\mathfrak{M})(\text{-}, Y(u)\times Y(u')).$$

If we consider the «symmetry equivalence»

$$\pi(u', u): Y(u')\times Y(u) \to Y(u)\times Y(u')$$

(such that the isomorphism

$$\eta(\pi(u', u)): C_{u'}\times C_u \to C_u\times C_{u'}$$

assigns (y, x) to (x, y)), we get the equivalence

$$\mathfrak{F}(\mathfrak{M})(\text{-}, \pi(u', u)): P(u', u)\partial \to P(u, u')\partial.$$

3° Let F be an object of $\mathfrak{F}(\mathfrak{M})$. We denote by C the category determined by F.

a) $K^{\cdot}$ being the category determined by $\partial(F)(\text{-})(2)$, there exists an isomorphism $\phi(F)$ from $K^{\cdot}$ to the longitudinal category $\square C$ of quartets of C. Indeed, the functor

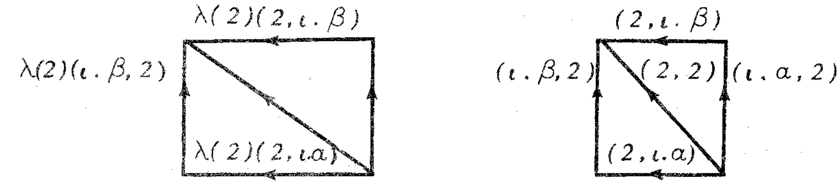
$$\partial(F)(\text{-})(2) = \mathfrak{F}(\mathfrak{M})(F, Y\text{-}\times Y(2))$$

is equivalent to the functor $\mathfrak{F}(C, \eta Y\text{-}\times C_2)$. So we get an isomorphism

$\phi(F): K^{\cdot} \to \square\!\square\, C$ assigning the quartet

$$(\lambda(2)(2, \iota.\beta), \lambda(2)(\iota.\beta, 2), \lambda(2)(\iota.\alpha, 2), \lambda(2)(2, \iota.\alpha))$$

to the natural transformation $\lambda: Y(2) \times Y(2) \to F$, element of K.

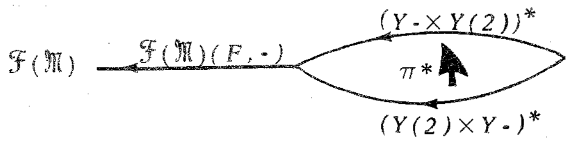


b) $\phi(F)$ defines a double functor from the double category $\overline{\eta}(\partial(F)) = (K^{\cdot}, K^{\perp})$ determined by $\partial(F)$ to the double category $(\square\!\square\, C, \boxminus C)$ of quartets. Indeed, $K^{\perp}$ is the category determined by

$$\partial(F)(2) = \mathfrak{F}(\mathfrak{M})(F, Y(2) \times Y\text{-}).$$

From the symmetry of the product on $\mathfrak{F}(\mathfrak{M})$ we deduce that the functors $\text{-}\times Y(2)$ and $Y(2)\times\text{-}$ from $\mathfrak{F}(\mathfrak{M})$ to $\mathfrak{F}(\mathfrak{M})$ are equivalent and that there exists an equivalence $\pi: Y\text{-}\times Y(2) \to Y(2)\times\text{-}$ where $\pi(u)$ is the equivalence $\pi(u, 2)$ considered in Part 2, for any object u of Σ. So, if π^* is the equivalence dual of π, we have the equivalence

$$\Pi = \mathfrak{F}(\mathfrak{M})(F, \text{-})\pi^* : \partial(F)(2) \to \partial(F)(\text{-})(2),$$



and $\Pi(2)$ assigns $\lambda \square\!\square\, \pi(2,2)$ to $\lambda: Y(2)\times Y(2) \to F$. The isomorphism $T = \eta(\Pi): K^{\perp} \to K^{\cdot}$ associates to λ the natural transformation $T(\lambda)$ such that, if x and y are morphisms of Σ,

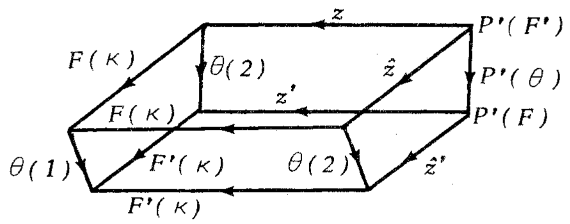
$$T(\lambda)(2)(y, x) = \lambda(2)(x, y).$$

If T' denotes the canonical isomorphism from $\square\!\square\, C$ to $\boxminus C$, it follows that the isomorphism $T'\phi(F)T$:

$$\boxminus C \xleftarrow{\;T'\;} \square\!\square\, C \xleftarrow{\;\phi(F)\;} K^{\cdot} \xleftarrow{\;T\;} K^{\perp}$$

is defined by the same map $\underline{\phi}(F)$ as the isomorphism $\phi(F)$. ∇

COROLLARY. *The functor* $P(2,2)\partial$ *is equivalent to the functor* P', *from* $\mathfrak{F}(\mathfrak{M})$ *to* $\mathfrak{M}$ *assigning to* $\theta : F \to F'$ *the canonical pullback* $P'(\theta)$, *defined by the following diagram, whose bases are canonical pullbacks in* $\mathfrak{M}$:



Δ. Let F be an object of $\mathfrak{F}(\mathfrak{M})$ and $\phi(F)$ the bijection considered in the preceding proof, from $K = P(2,2)\partial(F)$ to the set $\square C$ of quartets of the category C determined by F. There exists a bijection $\phi'(F)$ from $\square C$ to the canonical pullback $P'(F)$ of $(F(\kappa), F(\kappa))$, assigning

$$((y', x), (x', y)) \text{ to the quartet } (x', y', y, x).$$

If we associate to F the composite bijection

$$\psi(F) = \phi'(F)\,\phi(F) : K \to P'(F),$$

we get an equivalence $\psi : P(2,2)\partial \to P'$. ∇

B) Closure functors on $\mathfrak{F}(V)$.

PROPOSITION 28. *Let* V *be a category admitting pullbacks. There exists a functor* $\overline{\partial} : \mathfrak{F}(V) \to \mathfrak{F}(\mathfrak{F}(V))$ *such that, if* G *is a category in* V, *then* $\overline{\partial}(G)(2)$ *is equivalent to* $\overline{\partial}(G)(-)(2)$ *and, for* $s \in V_0$ *and* $x \in \Sigma$,

$$(A) \qquad V(-,s)\overline{\partial}(G)(x) \approx \partial(V(G-,s))(x).$$

Δ. We denote by $\mathfrak{L}$ the full subcategory of $\mathfrak{F}(\mathfrak{M})^{V^*}$ whose objects are the functors H such that the functor $P(2)H = H(-)(2) : V^* \to \mathfrak{M}$ is representable.

1° There exists an equivalence $d : \mathfrak{F}(V) \to \mathfrak{L}$. Indeed we have a func- 1
tor $d' : V^* \times \mathfrak{F}(V) \to \mathfrak{M}^{\Sigma}$ such that

$$d'(f, \theta) = V(-, f)\,\theta, \text{ for } f \in V, \ \theta \in \mathfrak{F}(V).$$

As $V(-,s)$ commutes with pullbacks, $d'(s, G) = V(G-, s)$ is an object of $\mathfrak{F}(\mathfrak{M})$ for each object (s, G); hence there exists a functor d'', from $V^* \times \mathfrak{F}(V)$ to $\mathfrak{F}(\mathfrak{M})$, restriction of d'. The functor $\hat{d}'' : \mathfrak{F}(V) \to \mathfrak{F}(\mathfrak{M})^{V^*}$

canonically associated to d'' is injective, and it takes its values in $\mathfrak{L}$ (since $P(u)\hat{d}''(G)$ is the representable functor $V(G(u),-)$ for any object u of Σ). So it admits as a restriction a functor d from $\mathcal{F}(V)$ to $\mathfrak{L}$; if $\theta \in \mathcal{F}(V)$, then $d(\theta): V^* \rightrightarrows \mathcal{F}(\mathfrak{M})$ is the natural transformation such that $d(\theta)(f) = V(-,f)\,\theta$, for any $f \in V$. It is known (see [Go] and [E3]) that d is an equivalence; $d^{-1}: \mathfrak{L} \to \mathcal{F}(V)$ will denote an equivalence.

2° We denote by:

- $Q'': V^* \times \mathcal{F}(V) \to \mathcal{F}(\mathfrak{M})^\Sigma$ the composite functor

$$V^* \times \mathcal{F}(V) \xrightarrow{d''} \mathcal{F}(\mathfrak{M}) \xrightarrow{\partial} \mathcal{F}(\mathcal{F}(\mathfrak{M})) \hookrightarrow \mathcal{F}(\mathfrak{M})^\Sigma$$

which assigns $\partial(V(-,f)\,\theta)$ to (f,θ).

- Q' the functor from $\Sigma \times \mathcal{F}(V)$ to $\mathcal{F}(\mathfrak{M})^{V^*}$ associated to Q''.

If $x: u \to u'$ is in V and $\theta: G \to G'$ in $\mathcal{F}(V)$, we have

$$Q'(u,G)(f) = \partial(V(G-,f))(u) = \overline{P}(u)\,\partial(V(G-,f)),$$

for any morphism f of V, and the natural transformation

$$Q'(x,\theta): Q'(u,G) \to Q'(u',G')$$

is such that, for any f in V, we have

$$Q'(x,\theta)(f) = \partial(V(-,f)\,\theta)(x) \in \mathcal{F}(\mathfrak{M}).$$

Let G be a category in V. We are going to show that the functor $Q'(-,G)$ takes its values in $\mathfrak{L}$. This will imply that Q' takes also its values in the category $\mathfrak{L}$.

a) $Q'(-,G)$ is a category in $\mathcal{F}(\mathfrak{M})^{V^*}$. Indeed, for each object s of V, the functor $Q'(-,G)(s): \Sigma \to \mathcal{F}(\mathfrak{M})$ is the object $\partial(V(G-,s))$ of $\mathcal{F}(\mathcal{F}(\mathfrak{M}))$. It follows that the cone $Q'(-,G)\hat{\gamma}$, whose components in $\mathcal{F}(\mathfrak{M})$ are the limit-cones $Q'(-,G)(s)\hat{\gamma}$, is a limit-cone in $\mathcal{F}(\mathfrak{M})^{V^*}$, if $\hat{\gamma}$ is equal to γ or to γ'. Hence, $Q'(-,G)$ is a category in $\mathcal{F}(\mathfrak{M})^{V^*}$.

b) We denote by R the functor from Σ to $\mathfrak{M}^{V^*}$ assigning

$$P(2)\,Q'(x,G): V^* \rightrightarrows \mathfrak{M} \quad \text{to any } x \in \Sigma.$$

The functor $Q'(-,G)$ will take its values in $\mathfrak{L}$ if we prove that $R(u)$ is representable for each object u of Σ. Indeed, for any f in V, we have

$$R(u)(f) = P(2)Q'(u,G)(f) = P(2)\overline{P}(u)\partial(V(G\text{-},f)) =$$
$$= P(2,u)\partial(V(G\text{-},f)).$$

- The functor $P(2,1)\partial$ being equivalent to $P(2)$ (Proposition 27), the functor $R(1)$ is equivalent to the functor assigning

$$P(2)(V(G\text{-},f)) = V(G(2),f) \quad \text{to} \quad f,$$

so that it is representable by $G(2)$.

- The functor $P(2,2)\partial$ being equivalent to the functor P' considered in the corollary of Proposition 27, the functor $R(2)$ is equivalent to the functor R' assigning $P'(V(G\text{-},f))$ to $f \in V$. By definition of P' and pullbacks being computed evaluationwise in $\mathcal{F}(\mathfrak{M})^{V^*}$, the functor R' is a pullback in $\mathfrak{M}^{V^*}$ of $(V(G(\kappa),\text{-}), V(G(\kappa),\text{-}))$. Such a pullback is equivalent to $V(S,\text{-})$, where S is a pullback of $(G(\kappa), G(\kappa))$ in V. So $R(2)$ is representable by S.

- $Q'(3,G)$ is a pullback of $(Q'(\alpha,G), Q'(\beta,G))$ (Part a) and $P(2)$ commutes with pullbacks, so that $R(3)$ is a pullback of $(R(\alpha), R(\beta))$ in $\mathfrak{M}^{V^*}$. We have just seen that $R(\alpha)$ and $R(\beta)$ are natural transformations between representable functors; hence $R(3)$ is representable.

- $Q'(4,G)$ being a pullback of $(Q'(\nu',G), Q'(\nu,G))$, we deduce similarly that $R(4)$ is representable, as a pullback of $(R(\nu'), R(\nu))$.

3° Q' taking its values in $\mathfrak{L}$, there exists a functor $Q: \Sigma\times\mathcal{F}(V) \to \mathfrak{L}$ restriction of Q'. We denote by $\overline{\partial}'$ the functor from $\mathcal{F}(V)$ to $\mathcal{F}(V)^{\Sigma}$ canonically associated to the composite functor $d^{-1}Q$:

$$\Sigma\times\mathcal{F}(V) \xrightarrow{\ Q\ } \mathfrak{L} \xrightarrow{\ d^{-1}\ } \mathcal{F}(V).$$

a) $\overline{\partial}'$ takes its values in $\mathcal{F}(\mathcal{F}(V))$. Indeed, if G is a category in V, we have $\overline{\partial}'(G) = d^{-1}Q(\text{-},G)$. As $\mathfrak{L}$ is closed for pullbacks in $\mathcal{F}(\mathfrak{M})^{V^*}$ the functor $Q(\text{-},G): \Sigma \to \mathfrak{L}$, restriction of the category $Q'(\text{-},G)$ in the category $\mathcal{F}(\mathfrak{M})^{V^*}$, is also a category in $\mathfrak{L}$. The equivalence d^{-1} commuting with pullbacks, $\overline{\partial}'(G)$ is a category in $\mathcal{F}(V)$. It follows that there exists a functor $\overline{\partial}: \mathcal{F}(V) \to \mathcal{F}(\mathcal{F}(V))$, restriction of $\overline{\partial}'$. If G is a category in V, if f is a morphism of V and if $x\in\Sigma$, we get

$$V(\text{-},f)\overline{\partial}(G)(x) = d(\overline{\partial}(G)(x))(f) \approx Q(x,G)(f) = \partial(V(G\text{-},f))(x).$$

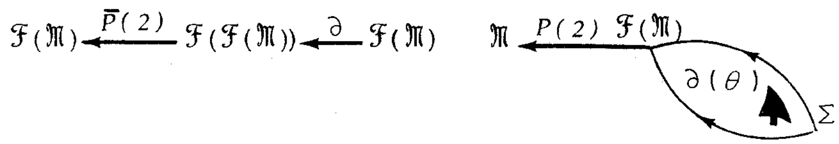
A fortiori $\overline{\partial}$ satisfies the condition (A) (we take $f = s$).

b) Let G be a category in V. It remains to show that $\overline{\partial}(G)(2)$ is equivalent to the functor $\overline{\partial}(G)(-)(2): \Sigma \to V$ assigning $\overline{\partial}(G)(x)(2)$ to $x \in \Sigma$. Indeed, if F is an object of $\mathfrak{F}(\mathfrak{M})$ and C the category $\eta(F)$ determined by F, there exists an equivalence

$$T(F) \text{ from } (\overline{P}(2)\partial)(F) = \partial(F)(2) \text{ to } P(2)\partial(F)$$

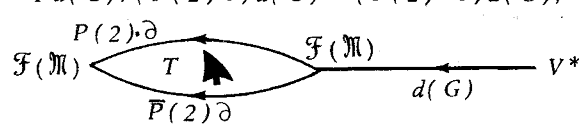
such that $\eta(T(F))$ is the canonical isomorphism from $\boxminus C$ to $\square\square C$. This defines an equivalence T from the functor $\overline{P}(2)\partial$ to the functor

$$P(2)\cdot\partial : \mathfrak{F}(\mathfrak{M}) \to \mathfrak{F}(\mathfrak{M}) \text{ assigning } P(2)\partial(\theta) \text{ to } \theta.$$



We have the equivalence

$$Td(G) : (\overline{P}(2)\partial)d(G) \to (P(2)\cdot\partial)d(G),$$



from A to A'. Since

$$A(f) = \overline{P}(2)\partial d(G)(f) = \partial(V(G-,f))(2)$$
$$\approx V(-,f)\overline{\partial}(G)(2) = d(\overline{\partial}(G)(2))(f),$$

for any f in V, it follows $A = d(\overline{\partial}(G)(2))$. On the other hand,

$$A'(f)(x) = P(2)\partial(d(G)(f))(x) = P(2)\partial(V(G\cdot,f))(x) =$$
$$= \partial(V(G\cdot,f))(x)(2) \approx V(\cdot,f)\overline{\partial}(G)(x)(2),$$

for any $x \in \Sigma$; so

$$A'(f) \approx V(-,f)\overline{\partial}(G)(-)(2) = d(\overline{\partial}(G(-)(2))(f)$$

for each f in V; this implies $A' \approx d(\overline{\partial}(G)(-)(2))$. Hence, $Td(G)$ belongs to $\mathfrak{L}$ and $d^{-1}(Td(G)): \overline{\partial}(G)(2) \to \overline{\partial}(G)(-)(2)$ is an equivalence.∇

DEFINITION. With the notations of Proposition 28, we call $\overline{\partial}(G)$ *the double category in* V *of quartets of* G, while $\overline{\partial}(G)(2)$ (resp. $\overline{\partial}(G)(\cdot)(2)$)
1 is called *the lateral* (resp. the *longitudinal*) category of quartets of G, and

denoted by $\boxminus G$ (resp. by $\boxbar G$).

The preceding proof shows that the categories determined by

$$V(-,s)\boxminus G \quad\text{and}\quad V(-,s)\boxbar G$$

are isomorphic to the lateral and to the longitudinal categories of quartets of the category determined by $V(G-,s)$, for any object s of V. Moreover,

- $\boxminus G$ and $\boxbar G$ are isomorphic,
- $\boxminus G(1)$ and $\boxbar G(1)$ are isomorphic to $G(2)$,
- $\boxminus G(2)$ and $\boxbar G(2)$ are pullbacks of $(G(\kappa), G(\kappa))$ in V.

REMARK. $\mathfrak{F}(V)$ is the category of 1-morphisms of the 2-category $\mathfrak{N}(V)$ of *natural transformations in* V: If $\theta: G \to G'$ and $\theta': G \to G'$ are functors in V, a natural transformation in V from θ to θ' is a functor Θ in V, from G to $\boxminus G'$, such that $\overline{\partial}(G')(\alpha)\boxbar\Theta = \theta$ and $\overline{\partial}(G')(\beta)\boxbar\Theta = \theta'$ (by construction, we may clearly identify $\overline{\partial}(G')(1)$ with G'). When V admits pullbacks, it is known [G1] that $\mathfrak{N}(V)$ is a representable 2-category, a representation of the category G in V being precisely the lateral category $\boxminus G$ in V of quartets of G.

PROPOSITION 29. *Let* $\mathcal{V} = (V, \tau, i, a, b, c, m, D)$ *be a symmetric monoidal closed category, where* V *admits pullbacks and kernels of pairs.*

1° There exists a functor $E: \mathfrak{F}(V)\times\mathfrak{F}(V)^* \to \mathfrak{F}(V)$ *such that, for a pair* (G', G) *of categories in* V, *we have:*

$$E(G', G) = \int_{x'x} D(-, G(x))\, \overline{\partial}(G')(x').$$

2° If the conditions of Proposition 26 are satisfied, E *is equivalent to the closure functor* $\hat{D}'$ *of* $\mathfrak{F}(\mathcal{V})$ *and* $\overline{\partial}(G)$ *is equivalent to the category* $\hat{D}'(G, qY-)$ *in* $\mathfrak{F}(V)$, *for any category* G *in* V. 1

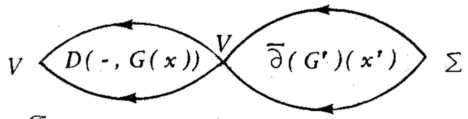
Δ. 1° The Lemma of Proposition 26 shows that the existence of pullbacks and kernels in V implies there exist Σ-ends in V. It follows that there exist also Σ-ends in $\mathfrak{F}(V)$, which are computed evaluationwise. We choose a Σ-end-functor $\int : \mathfrak{F}(V)^{\Sigma\times\Sigma^*} \to \mathfrak{F}(V)$.

a) Let G and G' be categories in V. There exists a functor A from $\Sigma\times\Sigma\times\Sigma^*$ to V which assigns

$$D(\overline{\partial}(G')(x')(y), G(x)) \quad\text{to}\quad (y, x', x).$$

The corresponding functor $A' : \Sigma \times \Sigma^* \to V^\Sigma$, which assigns

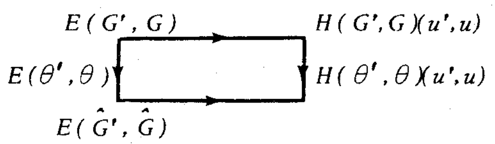
$$D(-, G(x))\, \overline{\partial}(G')(x') \text{ to } (x', x),$$

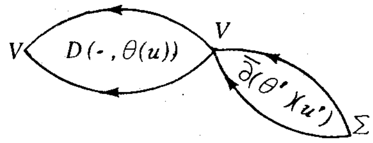


takes its values in $\mathfrak{F}(V)$, since $A'(u', u)$ is, for a pair (u', u) of objects of Σ, the composite of the category $\overline{\partial}(G')(u')$ in V with the functor $D(-, G(u))$ which commutes with pullbacks. So, there exists a functor $H(G', G) : \Sigma \times \Sigma^* \to \mathfrak{F}(V)$, restriction of A'. We denote by $E(G', G)$ the canonical end $\int H(G', G)$ in $\mathfrak{F}(V)$.

b) Let $\theta : \hat{G} \to G$ and $\theta' : G' \to \hat{G}'$ be functors in V. If u and u' are objects of Σ, we have the natural transformation

$$H(\theta', \theta)(u', u) = D(-, \theta(u))\, \overline{\partial}(\theta')(u'),$$





from $H(G', G)(u', u)$ to $H(\hat{G}', \hat{G})(u', u)$. Assigning this natural transformation to (u', u), we get a natural transformation

$$H(\theta', \theta) : H(G', G) \to H(\hat{G}', \hat{G}) : \Sigma \times \Sigma^* \rightrightarrows \mathfrak{F}(V).$$

We write $E(\theta', \theta) = \int H(\theta', \theta)$.

c) It is easily verified that we have so defined a functor

$$H : \mathfrak{F}(V) \times \mathfrak{F}(V)^* \to \mathfrak{F}(V)^{\Sigma \times \Sigma^*},$$

and a fortiori a functor

$$E = \int H : \mathfrak{F}(V) \times \mathfrak{F}(V)^* \to \mathfrak{F}(V).$$

2° We suppose moreover that the conditions of Proposition 26 are satisfied, i.e. V admits sums of pairs and also either τ commutes with pullbacks or the insertion functor from $\mathfrak{F}(V)$ to V^Σ admits a left adjoint. Then there exists a symmetric monoidal closed category $\mathfrak{F}(V)$ whose closure functor $\hat{D}'$ is defined by $\hat{D}'(G', G) = \int H'(G', G)$ (Proposition 23), the functor $H'(G', G) : \Sigma \times \Sigma^* \to \mathfrak{F}(V)$ assigning

$$D(-, G(x))\hat{D}'(G', qY-)(x') \text{ to } (x', x),$$

where q is a «partial adjoint» of $V(-, i)$.

For any category G in V, we denote the category $\hat{D}'(G, qY-)$ in $\mathfrak{F}(V)$ by $\delta(G)$.

a) Let G be a category in V. Then $\delta(G)$ is equivalent to $\overline{\partial}(G)$. Indeed, according to the proof of Proposition 23 (Part 1), for each object u of Σ, the category $\delta(G)$ is such that $V(-, s)\delta(G)(u)$ is canonically equivalent to $\partial(V(G-, s))(u)$, for any object s of V.

$$\partial(V(G-, s))(u) \approx V(-, s)\overline{\partial}(G)(u),$$

by Proposition 28. Hence, denoting yet by $d: \mathfrak{F}(V) \to \mathfrak{L}$ the isomorphism defined in Part 1, Proposition 28, we deduce that

$$d(\delta(G)(u)) \text{ and } d(\overline{\partial}(G)(u))$$

are equivalent; a fortiori there exists an equivalence $\xi(G)(u): \delta(G)(u) \to \overline{\partial}(G)(u)$. More precisely, we get an equivalence $\xi(G): \delta(G) \to \overline{\partial}(G)$.

b) Let G and G' be categories in V. We define an equivalence

$$X(G', G): H'(G', G) \to H(G', G): \Sigma \times \Sigma^* \rightrightarrows \mathfrak{F}(V)$$

assigning the equivalence

$$D(-, G(u'))\xi(G)(u) \text{ to } (u', u) \in \Sigma_0 \times \Sigma_0 .$$

Moreover, there exists

- a functor $H': \mathfrak{F}(V) \times \mathfrak{F}(V)^* \to \mathfrak{F}(V)^{\Sigma \times \Sigma^*}$, defined as in Part 1, such that $\hat{D}'(\theta', \theta)$ is an end of $H'(\theta', \theta)$, for each pair (θ', θ) of functors in V;

- an equivalence $X: H' \to H$ assigning $X(G', G)$ to (G', G).

Hence $\int X: \int H' \to \int H$ is an equivalence, and $\hat{D}'$ is equivalent to E. ∇

The construction of E does not depend upon the existence of sums in V. This suggests that E could always be a closure functor on $\mathfrak{F}(V)$. In fact, we have:

PROPOSITION 30. *Let $\mathcal{V}$ be a symmetric monoidal closed category*

$$(V, \tau, i, a, b, c, m, D).$$

If V admits pullbacks and kernels of pairs and if τ commutes with pullbacks, then there exists a symmetric monoidal closed category

$$(\mathfrak{F}(V),\ \hat{\tau}',\ i\hat{\ },\ \hat{a}',\ \hat{b}',\ \hat{c}',\ \hat{m}',\ E),$$

where E is the functor defined in Proposition 29 and where $\hat{\tau}'$ assigns the category $\tau[G', G]$ to the pair (G', G) of categories in V.

Δ. Let $\hat{\tau}$ be the tensor-product functor on V^{Σ} such that

$$(\theta' \hat{\tau} \theta)(x) = \theta'(x) \tau \theta(x), \quad \text{for any } x \in \Sigma,$$

if θ and θ' are natural transformations. As τ commutes with pullbacks, $G' \hat{\tau} G$ is a category in V when such are G and G'. So, there exists a functor $\hat{\tau}': \mathfrak{F}(V) \times \mathfrak{F}(V) \to \mathfrak{F}(V)$, restriction of $\hat{\tau}$ and, $\mathfrak{F}(V)$ being a full subcategory of V^{Σ}, the canonical symmetric monoidal category on V^{Σ}, whose tensor-product is $\hat{\tau}$, admits a symmetric monoidal subcategory

$$(\mathfrak{F}(V),\ \hat{\tau}',\ i\hat{\ },\ \hat{a}',\ \hat{b}',\ \hat{c}',\ \hat{m}')$$

since $i\hat{\ }$ is a category in V, the category I indexing pullbacks being connected. Hence Proposition 30 will result from Theorem II-5-8 [EK] if we know that $E(G', G)$ is a cofree structure generated by G' relative to the functor $-\hat{\tau}'G: \mathfrak{F}(V) \to \mathfrak{F}(V)$, for each pair (G', G) of categories in V. We will only sketch the proof of this assertion, omitting the purely technical computations.

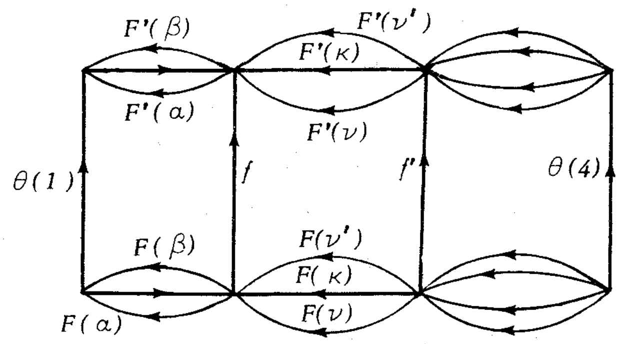
1° The following remarks will be useful:

a) Let F and F' be categories in V and $f: F(2) \to F'(2)$ a morphism of V. There exists a functor $\theta: F \to F'$ in V such that $\theta(2) = f$ iff f satisfies the equalities:

- $f.F(\iota.\alpha) = F'(\iota.\alpha).f, \quad f.F(\iota.\beta) = F'(\iota.\beta).f,$
- $f.F(\kappa) = F'(\kappa).f'$, where f' is the «pullback» morphism such that $F'(\nu).f' = f.F(\nu)$ and $F'(\nu').f' = f.F(\nu')$ (it exists, the two first equalities implying $F'(\alpha).f.F(\nu) = F'(\beta).f.F(\nu')$, since $F'(\iota)$ is a monomorphism and $\alpha.\nu = \beta.\nu'$).

In this case, we have

$$\theta(3) = f', \quad \theta(1) = F'(\alpha).f.F(\iota),$$



and $\theta(4)$ is defined by pullbacks. (The existence of θ means that σ admits an «idea» [E3], which is $(\kappa, \iota . \alpha, \iota . \beta)$.)

We will say that $\theta: F \to F'$ is the functor in V defined by f.

b) Let $B: \Sigma \times \Sigma^* \to V$ be a functor such that $B(\cdot, u)$ is a category in V for each object u of Σ, and S an end of B, with canonical projections $p(u): S \to B(u,u)$. If $g: s \to B(2,2)$ is a morphism in V, there exists a morphism $\hat{g}: s \to S$ such that $p(2).\hat{g} = g$ iff:

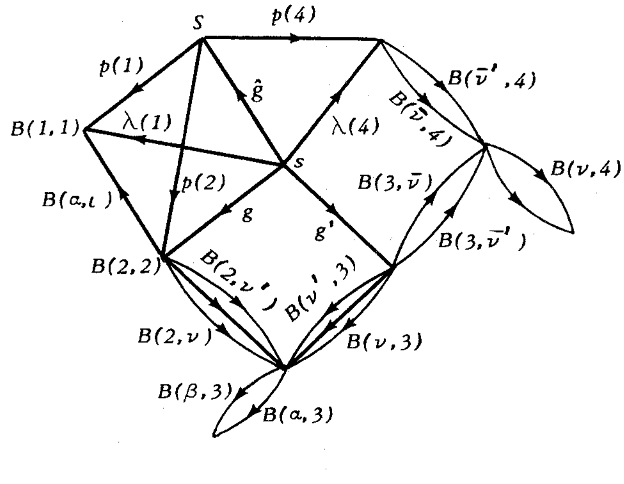
- $B(2, \iota . \alpha).g = B(\iota . \alpha, 2).g$, $B(2, \iota . \beta).g = B(\iota . \beta, 2).g$,
- $B(2, \kappa).g = B(\kappa, 3).g'$, where g' is the unique morphism such that

$$B(2, \nu).g = B(\nu, 3).g' \text{ and } B(2, \nu').g = B(\nu', 3).g'$$

(its existence follows from the fact that $B(\cdot, 3)$ is a category in V).

It is easily proved that there exists a cone $\lambda: s\hat{} \to \therefore B$, where

$$\lambda(2) = g, \quad \lambda(3) = g', \quad \lambda(1) = B(\alpha, \iota).g,$$



and $\lambda(4)$ is defined by pullback as being the morphism such that

$$B(3,\overline{\nu}).g' = B(\overline{\nu},4).\lambda(4),$$
$$B(3,\overline{\nu}').g' = B(\overline{\nu}',4).\lambda(4).$$

Then $\hat{g}$ is the factor of λ through the cone $p: S^{\wedge} \to \therefore B$ defining the end.

2° Let G and G' be categories in V.

a) We consider:

- the longitudinal category $\boxdot G' = \overline{\partial}(G')(\cdot)(2)$ of quartets of G', denoted by $\hat{G}'$ (definition page 97);
- the canonical projection $p(2)$ from the end

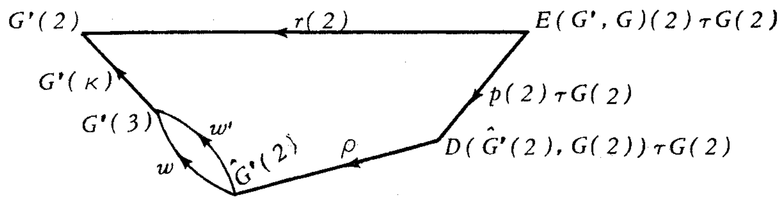
$$E(G',G)(2) = \int_{x'x} D(\hat{G}'(x'), G(x)) \text{ to } D(\hat{G}'(2), G(2));$$

- the morphism

$$\rho: D(\hat{G}'(2), G(2)) \tau G(2) \to \hat{G}'(2)$$

defining $D(\hat{G}'(2), G(2))$ as a cofree structure generated by $\hat{G}'(2)$ relative to the functor $-\tau G(2): V \to V$;
- the canonical projections $w: \hat{G}'(2) \to G'(3)$ and w' defining $\hat{G}'(2)$ as a pullback of $(G'(\kappa), G'(\kappa))$.

It may be shown that the composite morphism $r(2)$:



satisfies the hypothesis of Part 1-a, so it defines a functor:

$$r: E(G',G)\hat{\tau}'G \to G'.$$

b) r defines $E(G',G)$ as a cofree structure generated by G' relative to $-\hat{\tau}'G: \mathfrak{F}(V) \to \mathfrak{F}(V)$. Indeed, let $\theta: G''\hat{\tau}'G \to G'$ be a functor in V. To define the unique functor in V:

$$\theta': G'' \to E(G',G) \text{ such that } r \boxdot (\theta' \hat{\tau}' G) = \theta,$$

we are going to construct a morphism $g: G''(2) \to D(\hat{G}'(2), G(2))$ satisfying the hypothesis of Part 1-b, applied to the functor B assigning

$$D(\hat{G}'(x'), G(x)) \text{ to } (x', x).$$

Then there exists a morphism

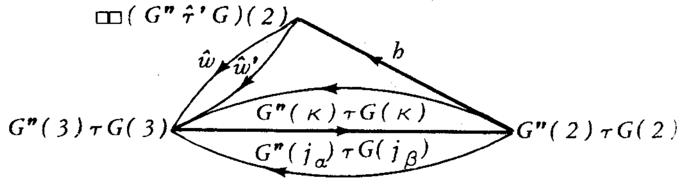
$$\hat{g}: G''(2) \to E(G', G)(2) \text{ such that } p(2).\hat{g} = g,$$

and a technical argument proves that $\hat{g}$ defines a functor θ' in V, from G'' to $E(G', G)$, satisfying the wanted property.

To construct g, we consider:

- the morphism $\overline{\partial}(\theta)(2)(2): \boxminus(G'' \hat{\tau}' G)(2) \to \hat{G}'(2)$,
- the morphisms $G''(j_\alpha)\tau G(j_\beta)$ and $G''(j_\beta)\tau G(j_\alpha)$ from $G''(2)\tau G(2)$ to $G''(3)\tau G(3)$,
- the projections $\hat{w}$ and $\hat{w}'$ of the pullback of

$$(G''(\kappa)\tau G(\kappa),\ G''(\kappa)\tau G(\kappa)) \text{ defining } \boxminus(G'' \hat{\tau}' G)(2).$$



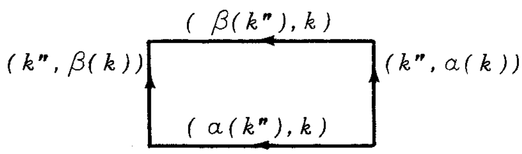
- the unique morphism $h: G''(2)\tau G(2) \to \boxminus(G'' \hat{\tau}' G)(2)$ such that

$$\hat{w}.h = G''(j_\beta)\tau G(j_\alpha), \quad \hat{w}'.h = G''(j_\alpha)\tau G(j_\beta);$$

it exists, since

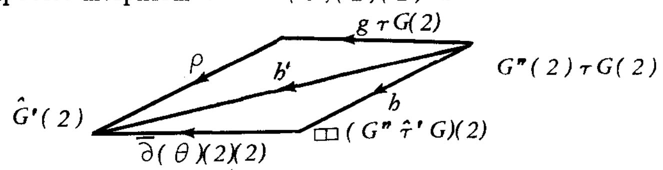
$$(G''(\kappa)\tau G(\kappa)).(G''(j_\beta)\tau G(j_\alpha)) = G''(\kappa.j_\beta)\tau G(\kappa.j_\alpha) =$$
$$G''(2)\tau G(2) = (G''(\kappa)\tau G(\kappa)).(G''(j_\alpha)\tau G(j_\beta)).$$

(For usual categories, this morphism corresponds to the map from the product category $C'' \times C$ to $\square(C'' \times C)$ assigning to (k'', k) the quartet



.)

- the composite morphism $h' = \overline{\partial}(\theta)(2)(2).h$.



Then g is the unique morphism $g: G''(2) \to D(\hat{G}'(2), G(2))$ such that

$$\rho.(g \tau G(2)) = h'. \quad \nabla$$

COROLLARY. *If V admits pullbacks, kernels of pairs and a cartesian closed structure, then $\mathfrak{F}(V)$ admits a cartesian closed structure.* ∇

REMARKS. 1° Proposition 30 (announced in [BE]) has been indicated by the first of the authors in 1971, in a lecture at the Séminaire Ehresmann (Paris). The proof given then was along the same arguments as above except that ends were not explicitely used (the authors did not know them) but constructed from kernels and pullbacks.

2° It may be asked whether Proposition 30 extends to more general cone-bearing categories. This does not seem true. Indeed, we denote now by σ any projective cone-bearing category (Σ, Γ). As in Part 1 Proposition 28, we prove that there exists an equivalence d from V^σ, for any category V, to the full subcategory $\mathfrak{L}'$ of $(\mathfrak{M}^\sigma)^{V^*}$ defined as follows:

the objects of $\mathfrak{L}'$ are those functors H such that the functor $H(-)(u)$ is representable, for any object u of Σ

(in the case where σ is the prototype of categories, $\mathfrak{L}'$ is identical with $\mathfrak{L}$, since *1*, *3* and *4* are constructed successively as projective limits).

But, even if σ is cartesian, there is no way to prove that, G being a σ-structure in V, the functor from Σ to $(\mathfrak{M}^\sigma)^{V^*}$ associated to

$$\overline{M}(d(G)-, Y-): V^* \times \Sigma \to \mathfrak{M}^\sigma$$

takes its values in $\mathfrak{L}'$. However, if such is the case, we may extend the construction of $\overline{\partial}(G)$, and then the construction of E.

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TENSOR PRODUCTS OF TOPOLOGICAL RINGOIDS

by Andrée and Charles EHRESMANN

INTRODUCTION.

A topological ringoid A is an Ab-category (category enriched in the category of abelian groups) A equipped with a topology such that the underlying category be a topological category (in the sense category internal to Top) and that the addition be also continuous. Topological ringoids arise in several problems of Differential Geometry: for instance the category of 1-jets from a differentiable manifold into itself «is» a topological ringoid; other topological ringoids are naturally associated to vector bundles. 1

If A and A' are topological ringoids and if σ is a «stable» set of subsets of A, we construct a topological ringoid $A' \otimes_\sigma A$ whose underlying Ab-category is the tensor product $A' \otimes A$ (it is known [10] that Ab-Cat admits a canonical monoidal closed structure). The continuous additive functors from $A' \otimes_\sigma A$ to a topological ringoid A'' are in 1-1 correspondence with the continuous additive functors from A' to the topological ringoid $\mathrm{Hom}_\sigma(A, A'')$ of continuous additive functors from A to A'', equipped with the σ-open topology. This answers a question unsolved in [17].

One of the main results gives weak enough conditions on the sets σ and σ' for the existence of an «associativity» morphism or equivalence $(-\otimes_{\sigma'} A') \otimes_\sigma A \to -\otimes_{\sigma' \otimes \sigma}(A' \otimes_{\sigma'} A)$. As a by-product, monoidal closed structures are defined on the category RdT of topological ringoids, on the subcategory of Hausdorff ringoids and on the category TAb-Cat (where TAb is the category of topological abelian groups).

Several authors [11, 12, 16] have given general existence Theorems for monoidal closed structures on a category. But these «global» structures 2 are rather scarce on categories related to Topology. So there is a need for «partial» tensor products, more adapted to a prescribed geometrical or topological situation; such problems were the motivation for this paper.

1. TENSOR PRODUCTS OF TOPOLOGIES.

The category *Top* of topological spaces is not cartesian closed. To remedy this hindrance several solutions have been proposed:

1° to extend *Top* into a cartesian closed category, e. g. the category of Choquet pseudo-topologies [7], the category of limit spaces [2,8], the category of Spanier quasi-topologies [21];

2° to restrict *Top*, e. g. by considering the category of Kelley spaces [13] which is cartesian closed but in which the product is different from
1+ the product in *Top*.

On *Top* itself, there are monoidal closed structures, associated to tensor product topologies defined on the product set. This is done in [1], from which we gather here some results used in the sequel.

A. σ-open topologies on functional spaces.

Let (E, T) be a topological space and σ a set of subsets Σ of E satisfying the axiom:

(a) Each point of E belongs to at least one $\Sigma \in \sigma$.

If (E', T') is a topological space, we denote by $C_\sigma(T, T')$ the set $C(T, T')$ of continuous maps $f: T \to T'$ from T to T', equipped with the *σ-open topology*, which is generated by all the sets

$$<\Sigma, U'> = \{ f: T \to T' \mid f(\Sigma) \subset U' \},$$

where $\Sigma \in \sigma$ and U' is open in T.

REMARK. In [1], $C_\sigma(T, T')$ is denoted by $C_\sigma(T', T)$; we come back here to the more usual notation.

There exists ([1], page 12) a functor $C_\sigma(T, -): Top \to Top$ associating to $g: T' \to T''$ the continuous map

$$C_\sigma(T, g): C_\sigma(T, T') \to C_\sigma(T, T'')$$

which sends $f: T \to T'$ to $g \circ f: T \xrightarrow{f} T' \xrightarrow{g} T''$.

B. σ-product of topologies ([1], page 23).

With the same hypotheses, we define on the product set $E' \times E$ a topology, called the *σ-product of (T', T)*, and denoted by $T' \times_\sigma T$ (instead of $T'\otimes_\sigma T$ in [1]). It is the finest topology $\hat{T}$ on $E'\times E$ such that:

1° For each x' in E' we have the continuous map

$$(x', -): T \to \hat{T}: x \mapsto (x', x).$$

2° For each $\Sigma \in \sigma$, the insertion from $E'\times\Sigma$ to $E'\times E$ is continuous from $T'\times(T/\Sigma)$ into $\hat{T}$ (where T/Σ is the topology induced by T on Σ).

The open sets of $T'\times_\sigma T$ are the subsets W of $E'\times E$ containing, for each point (x', x) of W:

1° a set $\{x'\}\times U$, where U is a neighborhood of x in T,

2° for each $\Sigma\in\sigma$ a set $V'\times V$, where V is a neighborhood of x in T/Σ and V' a neighborhood of x' in T'.

$T'\times_\sigma T$ has the following «universal property»: If (E'', T'') is a topological space, a map $f: E'\times E \to E''$ is continuous from $T'\times_\sigma T$ to T'' iff it satisfies the two conditions:

1° For each x' in E', we have the continuous map

$$f(x', -): T \to T'': x \mapsto f(x', x).$$

2° For each $\Sigma\in\sigma$, the restriction $f/_{E'\times\Sigma}: T'\times(T/\Sigma) \to T''$ is continuous.

In particular, $T'\times_\sigma T$ is finer than the product topology $T'\times T$, so that it is Hausdorff if so are T and T'.

EXAMPLES. 1° If σ is the set s of all the subsets with one element of E, then $T'\times_s T$ is the so-called *asterisk topology*, considered by several authors [5,6, 20], and which renders continuous the «separately continuous» maps. We get the same topology if we take for σ the set of all finite subsets of E.

2° If $E\in\sigma$, then $T'\times_\sigma T = T'\times T$.

3° If σ is the set c of all (Hausdorff) compact subspaces of T, we obtain the c-product $T'\times_c T$. When T is locally compact, we have: 1

$$T'\times_c T = T'\times T.$$

REMARK. In [22] other topologies are defined on $E'\times E$ by specifying not only a set σ of subsets of E but also a set σ' of subsets of E'. 2+

C. c-stable sets.

Let (E, T) be a topological space and σ a set of subsets of E. We say σ is *c-stable* ($c(T)$-stable in [1], page 14) if it satisfies the axiom (a) above and:

(b) for each $\Sigma \in \sigma$, the topology T/Σ is compact and each $x \in \Sigma$ admits a basis of neighborhoods in T/Σ formed by elements of σ.

For example, s and c are c-stable.

THEOREM 1 *([1], page 25-27). If σ is c-stable, the functor $C_\sigma(T, -)$ from Top to Top admits as a left adjoint the functor $-\times_\sigma T$: Top $\to$ Top, associating $g \times Id_T: T' \times_\sigma T \to T'' \times_\sigma T$ to $g: T' \to T''$.*

In other words, there exists a canonical equivalence

$$C(T', C_\sigma(T, -)) \to C(T' \times_\sigma T, -)$$

between functors from *Top* to *Set*. More precisely:

THEOREM 2 *([1], page 30). Suppose σ is c-stable and σ' is a c-stable set of subsets of the topological space (E', T'). Then*

$$\sigma' \times \sigma = \{ \Sigma' \times \Sigma \mid \Sigma' \in \sigma', \Sigma \in \sigma \}$$

is c-stable in $(E' \times E, T' \times_\sigma T)$ and the canonical equivalence above lifts into an equivalence

$$C_{\sigma'}(T', C_\sigma(T, -)) \to C_{\sigma' \times \sigma}(T' \times_\sigma T, -)$$

between functors from Top to Top.

Theorems 1 and 2 imply the following «associativity» result:

THEOREM 3 *([1], page 32). With the assumptions of Theorem 2 there exists a canonical equivalence between functors from Top to Top:*

$$(- \times_{\sigma'} T') \times_\sigma T \to - \times_{\sigma' \times \sigma} (T' \times_\sigma T).$$

COROLLARY. *There exist homeomorphisms:*

$$(T'' \times_s T') \times_s T \to T'' \times_s (T' \times_s T) \text{ and } (T'' \times_c T') \times_c T \to T'' \times_c (T' \times_c T)$$

defined by $((x'', x'), x) \mapsto (x'', (x', x))$ for any topological spaces (E, T), (E', T') and (E'', T'').

D. Monoidal closed structures on *Top* **and its subcategories.**

Given a topological space (E, T) and a c-stable set σ on it, we have constructed functors $-\times_\sigma T$ and $C_\sigma(T,-)$ from *Top* to *Top*. Is it possible to «glue together» such functors to obtain a monoidal closed structure on *Top* or on subcategories of *Top* ?

Suppose given a full subcategory S of *Top* containing at least a one-point topological space, and a map $\sigma(-)$ associating to each object (E, T) of S a c-stable set $\sigma(T)$ of subsets of E such that

(c) For each $f: T \to T'$ in S, we have $f(\Sigma) \in \sigma(T')$ for any $\Sigma \in \sigma(T)$.

EXAMPLES. 1° The map s associating to each topological space the set of its one-point subsets satisfies (c) with respect to *Top*.

2° The map c associating to any topological space the set of its compact subsets satisfies (c) with respect to the subcategory *HTop* of Hausdorff spaces, but not with respect to *Top* itself.

THEOREM 4. *If* $T' \times_{\sigma(T)} T$ *and* $C_{\sigma(T)}(T, T')$ *are in* S *for any objects* T *and* T' *of* S, *then* S *admits a non associative (in general) monoidal closed structure whose tensor product* $\times_{\sigma(-)}$ *extends the functors* $-\times_{\sigma(T)} T: S \to S$ *and whose internal Hom functor* $C_{\sigma(-)}$ *extends the functors*

$$C_{\sigma(T)}(T,-): S \to S.$$

The tensor product always admits as a unit the one-point topology.

COROLLARY 1. *Top is a symmetric monoidal closed category* Top_s *when equipped with the tensor product* $\times_s$ *and the internal Hom* C_s.

COROLLARY 2. *HTop becomes a monoidal closed category:*

1° $HTop_s$ *when equipped with* $-\times_s-$ *and* $C_s(-,-)$;

2° $HTop_c$ *when equipped with* $-\times_c-$ *and* $C_c(-,-)$.

The tensor product $-\times_c-$ on *HTop* is not symmetric, while $-\times_s-$ is.

Let S satisfy the assumptions of Theorem 4 and let S' be a full coreflective subcategory of S containing a one-point topological space.

COROLLARY 3. *If* $T' \times_{\sigma(T)} T$ *is in* S' *when* T *and* T' *are in* S', *then* S' *is a non associative monoidal closed category for the restriction of the ten-*

sor product $\times_{\sigma(-)}$ *and the internal Hom:* $S'\times S'^{*} \xrightarrow{C_{\sigma(-)}(-,.)} S \xrightarrow[k]{} S'$ *where* k *is the coreflector.*

As an application of this last corollary, we consider the full subcategory *Ke* of *HTop* whose objects are the Kelley spaces (also called compactly generated spaces) (see [13,15]).

THEOREM 5. *Ke is a cartesian closed category and the product of* (T', T) *in Ke is identical with* $T'\times_c T$.

PROOF. It is well-known that *Ke* is a coreflective subcategory of *HTop*, the coreflector being the Kelleyfication functor $K: HTop \to Ke$. If we prove that $T'\times_c T$ is a Kelley space for any Kelley spaces (E, T) and (E', T'), it will result from Corollaries 2 and 3 that *Ke* is a monoidal closed category for the tensor product $-\times_c-$ and the internal Hom $K \circ C_c$. In fact, we shall prove that $T'\times_c T$ is identical with the product $T' \circ T$ of (T', T) in *Ke*, so that *Ke* is cartesian closed (see also [13]).

- Indeed, a subspace W of $T'\times_c T$ is open iff:

$$W_{x'} = W \cap (\{x'\}\times E) \quad \text{and} \quad W_B = W \cap (E'\times B)$$

are open in the topology induced by the product topology $T'\times T$, for each point x' of E' and each compact B of T. Now, $\{x'\}\times T$ and $T'\times B$ are Kelley spaces [13] so that $W_{x'}$ and W_B are open iff their intersection with each compact of $\{x'\}\times T$ and of $T'\times B$ are open. Hence W is open in the topology $T'\times_c T$ iff its intersection with any $B'\times B$, where B' is a compact of T', is open. But this is exactly the definition of the open sets for the Kelley product $T' \circ T$. So $T'\times_c T = T' \circ T$.

2. TENSOR PRODUCTS OF TOPOLOGICAL RINGOIDS.

A. Monoidal closed structure on *AbCat*.

The category *Ab* of abelian groups has a well-known monoidal closed structure. The tensor product $G'\otimes G$ of the abelian groups G' and G is their tensor product as Z-modules.

From general results [10], it follows that the category *AbCat* of *Ab*-

categories admits a monoidal closed structure which we recollect briefly for later use.

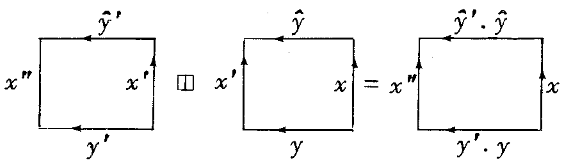
Ab-categories (i.e. categories enriched in *Ab*) are variously named; to keep the idea of «rings with several objects» [19] with a shorter name, we call them *ringoids* (annoïdes in French [3]) and we reserve the often used name «additive categories» for those ringoids admitting finite products (as in [3]). An *Ab*-category may be defined in several ways, the simplest one being probably the data A of a category $A^{\cdot}$ and of a lifting of its Hom functor $A^{*} \times A^{\cdot} \to Set$ into a functor

$$A(-,-): A^{*} \times A^{\cdot} \to Ab.$$

We denote by A_0 the set of objects of A, i.e. of $A^{\cdot}$, by A^{+} the groupoid coproduct (in *Cat*) of the abelian groups $A(e, e')$, for any objects e and e' of A, and by $0_{e e'}$ the zero of $A(e, e')$. The couple $(A^{\cdot}, A^{+})$ entirely determines the ringoid A.

We denote by Rd (shorter than *Ab-Cat*) the category of ringoids.

To the ringoid A is associated [3] the *horizontal ringoid* $\boxbar A$ *of commutative squares of* $A^{\cdot}$, whose multiplication is:

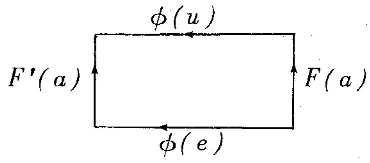


and the *vertical ringoid* $\boxminus A$; their couple $\square A$ is called *the double ringoid of squares of* A.

If A and A' are ringoids, we denote by $Hom(A, A')$ the ringoid of additive functors from A to A'. The morphisms of this ringoid, i.e. the natural transformations between additive functors from A to A', are identified [3] with additive functors from A to $\boxminus A$, by identifying

$$\phi: F \Longrightarrow F': A \to A'$$

with the additive functor $\Phi: A \to \boxminus A'$ which sends $a: e \to u$ in A onto the commutative square



This defines the «internal» *Hom* of the closed category *Rd*.

The tensor product in *Rd* associates to the ringoids A and A' the ringoid $A'\otimes A$ whose set of objects is $A'_0 \times A_0$, the abelian group from the object (e', e) to (u', u) being the tensor product group

$$A'(e', u') \otimes A(e, u).$$

The canonical bi-additive functor $J: (A', A) \to A'\otimes A$ is defined by

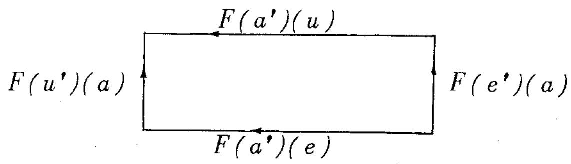
$J(a', a) = a'\otimes a$ for any morphisms a' of A' and a of A.

The image $J(A'\times A)$ «additively generates» the ringoid $A'\otimes A$.

The additive functors from $A'\otimes A$ to a ringoid A'' are in 1-1 correspondence with the bi-additive functors from (A', A) to A'', and also with the additive functors from A' to $Hom(A, A'')$. The canonical isomorphism

$$Hom(A', Hom(A, A'')) \to Hom(A'\otimes A, A'')$$

maps $F: A' \to Hom(A, A'')$ onto the additive functor sending $a'\otimes a$ onto the diagonal of the square $F(a')(a) =$



for $a: e \to u$ in A and $a': e' \to u'$ in A'.

B. Topological ringoids.

Ringoids may also be considered as sketched structures [4] : indeed there exists a projective cone-bearing category, the *sketch of ringoids*, whose realizations into *Set* «are» the ringoids [18] . The realizations of this sketch into *Top* are called topological ringoids.

A *topological ringoid* A is a couple (A, T) of a ringoid A and of a topology T on the set of morphisms of A, such that:

1° $(A^{\cdot}, T)$ is a topological category (in the sense: category internal to Top, i.e. the domain, codomain and composition maps are continuous [8]); let T_0 be the topology induced by T on A_0.

2° (A^+, T) is a topological groupoid (hence the addition and the opposite map are continuous); let T_0^+ be the topology induced by T on the set A_0^+ of objects of A^+, which is the set of 0-morphisms of A.

3° The continuous map $0_{ee'} \mapsto (e, e')$ from T_0^+ to $T_0 \times T_0$ is a homeomorphism.

These conditions imply that $A(e, e')$ becomes a topological group for the topology $T(e, e')$ induced by T.

EXAMPLES. 1° A topological (unitary) ring is a topological ringoid, with only one object.

2° If M is a differentiable manifold, the topological category $J^1(M)$ of 1-jets from M to M underlies a topological ringoid [9].

3° To a vector bundle is associated the topological ringoid of homomorphisms from fibre to fibre.

4° If E is a set, we have the ringoid A of couples of elements of E whose set of objects is E, the group $A(e, e')$ being reduced to its zero (e, e') for any pair of objects. If T is a topology on E, then $(A, T \times T)$ is a topological ringoid, called the *topological ringoid of pairs of* T.

General results on sketched structures (see also [18]) assert that the category of topological ringoids, denoted by RdT, admits both projective and inductive limits. The faithful functors from RdT to Rd and to Top preserve projective limits, and the first one is an initial-structure functor [23] (topological functor in the terminology of Herrlich [14], which is contradictory with ours). RdT is the category of 1-morphisms of a 2-category.

Let $\mathsf{A} = (A, T)$ be a topological ringoid. If we equip the ringoids of squares of A with the topology $\square T$ induced by the product topology T^4, we get two topological ringoids $\boxminus\!\!\!\!\square \mathsf{A}$ and $\boxminus \mathsf{A}$, whose couple is the *topological double ringoid of squares of* A.

Let $A' = (A', T')$ be a topological ringoid ; we denote by $Hom(A, A')$ the subringoid of $Hom(A, A')$ of continuous additive functors from A to A'. Let σ be a c-stable set of subsets of A. Identifying a morphism $\bar{F}$ of $Hom(A,A')$, i.e. a continuous additive natural transformation, with the corresponding continuous additive functor $\bar{F}: A \to \boxminus A'$, we equip $Hom(A,A')$ with the topology induced by $C_\sigma(T, \square T')$ and get the topological ringoid [17] $Hom_\sigma(A,A')$. We have the endofunctor $Hom_\sigma(A,-)$ of RdT such that

$$Hom_\sigma(A, F'): Hom_\sigma(A, A') \to Hom_\sigma(A, A''): \bar{F} \mapsto F' \circ \bar{F},$$

if $F': A' \to A''$, where $\circ$ is the total law of the 2-category on RdT.

C. Tensor products of topological rings.

Let $A = (A, T)$ and $A' = (A', T')$ be topological ringoids and σ be a set of subsets of A whose union is A.

If $A'' = (A'', T'')$ is a topological ringoid, we say that

$$F: (A', A)_\sigma \to A''$$

is a σ*-continuous bi-additive functor* if it is a bi-additive functor from (A', A) to A'' which is continuous from $T' \times_\sigma T$ to T''.

THEOREM 1. *1° There exists a finest topology* $\hat{T}$ *on the ringoid* $A' \otimes A$*, such that* $(A' \otimes A, \hat{T})$ *be a topological ringoid, denoted by* $A' \otimes_\sigma A$*, and*

$$J: (A', A)_\sigma \to A' \otimes_\sigma A: (a', a) \mapsto a' \otimes a$$

a σ*-continuous bi-additive functor.*

2° The σ*-continuous bi-additive functors from* $(A', A)_\sigma$ *to* A'' *are in 1-1 correspondence with the continuous additive functors from* $A' \otimes_\sigma A$ *to* A''*, for each topological ringoid* A''.

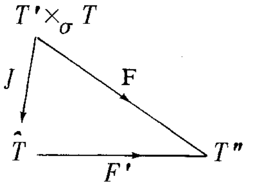
PROOF. Let L be the class of all σ-continuous bi-additive functors

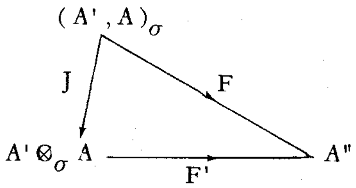
$$F: (A', A)_\sigma \to A'' = (A'', T'').$$

Each F in L determines the additive functor

$$F': A' \otimes A \to A'': a' \otimes a \mapsto F(a', a).$$

Let $\hat{T}$ be the initial topology associated to the family $(F', T'')_{F \in L}$ (i.e. the coarser topology on $A' \otimes A$ such that $F': \hat{T} \to T''$ be continuous for any in L). The forgetful functor $RdT \to Rd$ being an initial-structure functor, nd the functor $RdT \to Top$ preserving initial-structures, $(A' \otimes A, \hat{T})$ is a ›pological ringoid, $A' \otimes_\sigma A$, which is the initial topological ringoid assoiated to the family $(F', A'')_{F \in L}$. So by construction, each F in L de- :rmines the continuous additive functor $F': A' \otimes_\sigma A \to A''$.





- Let $J: (A', A) \to A' \otimes A$ be the canonical bi-additive functor. Each F ı L being continuous from $T' \times_\sigma T$ to T'' and factorizing through J, the niversal property of the initial topology implies that $J: T' \times_\sigma T \to \hat{T}$ is coninuous; it follows that $\hat{T}$ is the finest ringoid topology such that

$$J: (A', A)_\sigma \to A' \otimes_\sigma A$$

e a continuous bi-additive functor J. ∎

:OROLLARY 1. *With the notations of Theorem 1, the topology* $\hat{T}_0$ *induced* ›n $\hat{A}_0 = A'_0 \times A_0$ *by* $A' \otimes_\sigma A$ *is finer than the topology* $\hat{T}'_0$ *induced by* $T' \times T$ *nd coarser than that* $\hat{T}''_0$ *induced by* $T' \times_\sigma T$. *Hence if* T_0 *and* T'_0 *are Hausorff (resp. discrete) topologies, so is* $\hat{T}_0$.

›ROOF. $J: T' \times_\sigma T \to \hat{T}$ being continuous, its restriction to $\hat{A}_0$ which is the dentity on $\hat{A}_0$ is continuous from $\hat{T}''_0$ to $\hat{T}_0$. On the other hand, let B be he topological ringoid of pairs of $\hat{T}'_0$ (Example 4 above). There exists a ·i-additive functor $G: (A', A) \to B$ which maps

$$(a', a) \text{ onto } ((u', u), (e', e)),$$

f $a: e \to u$ in A and $a': e' \to u'$ in A'. It is continuous from $T' \times T$ to $'_0 \times \hat{T}'_0$ (since the maps domain and codomain are continuous in A' and n A), and a fortiori σ-continuous. Hence G factors through a continuous dditive functor $G': A' \otimes_\sigma A \to B$; the identity of $\hat{A}_0$ being the restriction ›f G' to $\hat{A}_0$, it is continuous from $\hat{T}_0$ to $\hat{T}'_0$. Finally, $\hat{T}''_0 \to \hat{T}_0 \to \hat{T}'_0$. ∎

:XAMPLE. If A and A' are topological rings, so is $A' \otimes_\sigma A$.

THEOREM 2 *(Unitarity). Let* Z *be the ring of integers, with the discrete topology. Then*

$$Z \otimes_\sigma A \sim A \sim A \otimes_s Z .$$

PROOF. We shall construct a σ-continuous bi-additive functor

$$H : (Z, A)_\sigma \to A$$

and prove that each σ-continuous bi-additive functor from $(Z, A)_\sigma$ factors through it. From the universal property of $Z \otimes_\sigma A$, it will follow that A is isomorphic to this tensor product. Indeed, there exists a bi-additive functor

$$H : (Z, A) \to A : (z, a) \mapsto z a .$$

Since Z is discrete, the topology $Z \times_\sigma T$ is the coproduct of the topologies $(\{z\} \times T)_{z \in Z}$. The addition on A being continuous, each map

$$H(z, -) : T \to T : a \mapsto z a$$

is continuous, so that $H : Z \times_\sigma T \to T$ is continuous.

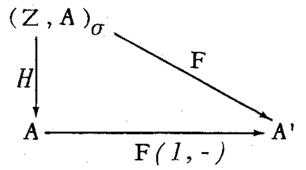
- Let $F : (Z, A)_\sigma \to A'$ be a σ-continuous bi-additive functor. In particular, $F(1, -) : A \to A'$ is a continuous additive functor. The composite

$$(Z, A)_\sigma \xrightarrow{H} A \xrightarrow{F(1,-)} A'$$

maps (z, a) onto

$$F(1, z a) = z F(1, a) = F(z, a)$$

(we use the bi-additivity of F), hence it is identical with F, and F factors through A .



- A similar method proves that A is isomorphic with $A \otimes_s Z$. ■

If $F' : A' \to A''$ is a continuous additive functor, the map sending (a', a) onto $F'(a') \otimes a$ defines a σ-continuous bi-additive functor

$$(A', A)_\sigma \xrightarrow{F' \times Id} (A'', A)_\sigma \xrightarrow{J'} A'' \otimes_\sigma A ,$$

so that it factors through an additive functor

$$F' \otimes_\sigma A : A' \otimes_\sigma A \to A'' \otimes_\sigma A .$$

This determines an endofunctor $- \otimes_\sigma A$ of $Rd\,T$.

D. Some canonical isomorphisms.

THEOREM 3. *If* $A = (A, T)$ *is a topological ringoid and* σ *a c-stable set of subsets of* A, *then the functor* $- \otimes_\sigma A$ *is a left adjoint of the functor*

$$\mathrm{Hom}_\sigma (A, -) : Rd\,T \to Rd\,T .$$

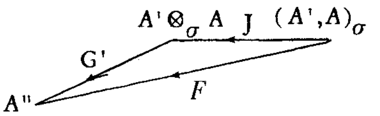
PROOF. We denote by $J : (A', A)_\sigma \to A' \otimes_\sigma A$ the canonical projection. Let

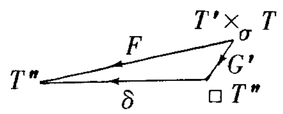
$$G : A' \to \mathrm{Hom}_\sigma (A, A''),$$

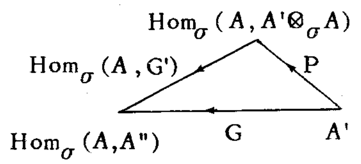
where $A' = (A', T')$ and $A'' = (A'', T'')$, be a continuous additive functor. Then G determines an additive functor from A' to $Hom(A, A'')$, hence a unique additive functor $G' : A' \otimes A \to A''$ (universal property of the tensor product). The composite

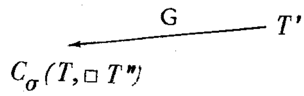
$$F : (A', A) \xrightarrow{J} A' \otimes A \xrightarrow{G'} A'' : (a', a) \mapsto G'(a' \otimes a)$$

defines a bi-additive functor. If we show that F is σ-continuous, it follows from Theorem 1 that G' defines a continuous additive functor from $A' \otimes_\sigma A$, to A'' , denoted by G'.









- Indeed, by construction of $\mathrm{Hom}_\sigma (A, A'')$, we have the continuous map

$$G : T' \to C_\sigma (T, \square T'').$$

As σ is c-stable, this implies that the map $(a', a) \mapsto G(a')(a)$ is continuous from $T' \times_\sigma T$ to $\square T''$. The diagonal map $\delta : \square T'' \to T''$ is continuous so the map

$$(a', a) \mapsto \delta\, G(a')(a) = G'(a' \otimes a)$$

is also continuous from $T' \times_\sigma T$ to T'' ; this map is F . Hence F is σ-continuous. We have constructed a canonical bijection

$$\mathrm{Hom}(A', \mathrm{Hom}_\sigma(A, A''))_0 \to \mathrm{Hom}(A' \otimes_\sigma A, A'')_0 : G \mapsto G',$$

whose inverse maps $H: A' \otimes_\sigma A \to A''$ onto

$$A' \xrightarrow{P} \mathrm{Hom}_\sigma(A, A' \otimes_\sigma A) \xrightarrow{\mathrm{Hom}_\sigma(A,H)} \mathrm{Hom}_\sigma(A, A''),$$

where P is the «liberty morphism» defined by

$$P(a'): A \to A' \otimes_\sigma A : a \mapsto a' \otimes a. \quad \blacksquare$$

1

Now we lift the canonical isomorphisms into topological ones. Suppose σ' is a c-stable set of subsets of A'. For each topological ringoid A'' the σ-continuous bi-additive functors $F : (A', A)_\sigma \to A''$ are objects of the ringoid $\mathrm{Hom}((A', A)_\sigma, A'')$, whose morphisms from F to G are identified with the σ-continuous bi-additive functors $\bar{F} : (A', A)_\sigma \to \boxminus A''$ such that

$$\bar{F}(a', a) = \begin{array}{ccc} & \bar{F}(\beta a', \beta a) & \\ G(a', a)\uparrow & \longleftarrow & \uparrow F(a', a) \\ & \longleftarrow & \\ & \bar{F}(\alpha a', \alpha a) & \end{array}$$

(α and β being the domain and codomain maps). By this identification we equip $\mathrm{Hom}((A', A)_\sigma, A'')$ with the topology induced by $C_{\sigma' \times \sigma}(T' \times_\sigma T, \square T'')$. As $\sigma' \times \sigma$ is c-stable (Section 1), so is constructed a topological ringoid denoted by $\mathrm{Hom}_{\sigma'}((A', A)_\sigma, A'')$.

We consider the set $\sigma' \otimes \sigma$ of subsets of $A' \otimes A$ formed by the sets

$$\Sigma' \otimes \Sigma = J(\Sigma' \times \Sigma), \text{ where } \Sigma' \in \sigma', \Sigma \in \sigma,$$

and by the one-point sets $\{y\}$, where y is not in the image of the canonical projection $J : (A', A)_\sigma \to A' \otimes_\sigma A$.

THEOREM 4. *If σ and σ' are c-stable, the 1-1 correspondence η_0 between the σ-continuous bi-additive functors from $(A', A)_\sigma$ to A'' and the continuous additive functors from $A' \otimes_\sigma A$ to A'' extends into an isomorphism*

$$\eta : \mathrm{Hom}_{\sigma'}((A',A)_\sigma , A'') \to \mathrm{Hom}_{\sigma'\otimes\sigma}(A'\otimes_\sigma A, A'').$$

PROOF. 1° There is clearly a ringoid isomorphism η. We have to show that it is an homeomorphism from the topology

$$S \text{ induced by } C_{\sigma'\times\sigma}(T'\times_\sigma T, \square T'')$$

to the topology

$$S' \text{ induced by } C_{\sigma'\otimes\sigma}(\hat{T}, \square T'').$$

This will imply that $\mathrm{Hom}(A'\otimes_\sigma A, A'')$ equipped with S' is a topological ringoid, yet denoted by $\mathrm{Hom}_{\sigma'\otimes\sigma}(A'\otimes_\sigma A, A'')$, and that η is a topological isomorphism. (Remark that the existence of this topological ringoid is not obvious a priori, since $\sigma'\otimes\sigma$ is not always c-stable, and the construction of $\mathrm{Hom}_\sigma(A,-)$ uses the preservation of pullbacks by $C_\sigma(T,-)$.)

2° $$\eta^{-1} : S' \to S : \bar{F}' \mapsto \bar{F}' \circ J$$

is continuous. Indeed, it is sufficient to see that the image by η of each elementary open set of S,

$$<\Sigma'\times\Sigma, U> = \{ \bar{F} \mid \bar{F}(\Sigma'\times\Sigma) \subset U \},$$

where U open in $\square T''$ and $\Sigma' \epsilon \sigma'$, $\Sigma \epsilon \sigma$, is open in S'. This is true, since:

$$\eta(<\Sigma'\times\Sigma, U>) = \{ \bar{F}' \mid \bar{F}' J(\Sigma'\times\Sigma) \subset U \} = <\Sigma'\otimes\Sigma, U>.$$

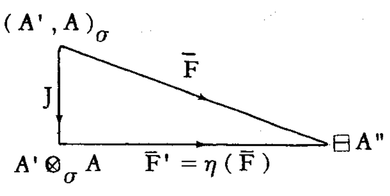
3° $\eta : S \to S'$ is continuous. Indeed, the elementary open sets of S' are of the form

$$<\Sigma'\otimes\Sigma, U> \text{ or } <\{y\}, U> \text{ with } y \notin J(A'\times A).$$

It suffices to show that the image by η^{-1} of these sets are open sets in S. From Part 2:

$$\eta^{-1}(<\Sigma'\otimes\Sigma, U>) = <\Sigma'\times\Sigma, U>$$

is open in S. We are going to show that $\eta^{-1}(<\{y\}, U>)$ is a neighborhood of each of its elements $\bar{F}$. As $J(A'\times A)$ additively generates $A'\otimes A$, there



exist $x_1, \ldots, x_n \in A' \times A$ such that

$$y = \mathrm{J}(x_1) + \ldots + \mathrm{J}(x_n).$$

$\bar{\mathrm{F}} \in \eta^{-1}(<\{y\}, U>)$ implies $\eta(\bar{\mathrm{F}})(y) \in U$. We have $\eta(\bar{\mathrm{F}}) \circ \mathrm{J} = \bar{\mathrm{F}}$, so that:

$$\eta(\bar{\mathrm{F}})(y) = \eta(\bar{\mathrm{F}})(\mathrm{J}(x_1) + \ldots + \mathrm{J}(x_n)) = \bar{\mathrm{F}}(x_1) + \ldots + \bar{\mathrm{F}}(x_n) \in U.$$

Since the addition of $\boxminus \mathrm{A}''$ is continuous, there exist open neighborhoods U_i of $\bar{\mathrm{F}}(x_i)$ in $\square T''$, $i = 1, \ldots, n$, such that $U_1 + \ldots + U_n \subset U$. Each x_i is contained in a $\tilde{\Sigma}_i \in \sigma' \times \sigma$. Since $\sigma' \times \sigma$ is c-stable and $\bar{\mathrm{F}}^{-1}(U_i)$ is an open neighborhood of x_i, there exist

$$\tilde{\Sigma}'_i \in \sigma' \times \sigma \quad \text{such that} \quad x_i \in \tilde{\Sigma}'_i \subset \bar{\mathrm{F}}^{-1}(U_i) \cap \tilde{\Sigma}_i.$$

Therefore the set $\bigcap_{i=1}^{n} <\tilde{\Sigma}'_i, U_i>$ is an open neighborhood V of $\bar{\mathrm{F}}$ in S. It is included in $\eta^{-1}<\{y\}, U>$, because $\bar{\mathrm{G}} \in V$ implies

$$\bar{\mathrm{G}}(x_i) \in \bar{\mathrm{G}}(\tilde{\Sigma}'_i) \subset U_i,$$

and so

$$\eta(\bar{\mathrm{G}})(y) = \bar{\mathrm{G}}(x_1) + \ldots + \bar{\mathrm{G}}(x_n) \in U_1 + \ldots + U_n \subset U. \quad \blacksquare$$

A set σ of subsets of A is called *rc-stable for* A if it is c-stable and if the images of each $\Sigma \in \sigma$ by the maps domain α and codomain β of A are in σ. For example such is the case if $\sigma = s$, or if $\sigma = c$ and T is a Hausdorff space.

If $\mathrm{A}'' = (A'', T'')$ and $\mathrm{B} = (B, S)$ are topological ringoids, we say that $\mathrm{F}: ((\mathrm{A}'', \mathrm{A}')_{\sigma'}, \mathrm{A})_{\sigma} \to \mathrm{B}$ is a *(σ', σ)-continuous tri-additive functor*, if F is a tri-additive functor, continuous from $(T'' \times_{\sigma'} T') \times_{\sigma} T$ to S.

THEOREM 5. *Let σ be rc-stable for* A *and σ' be rc-stable for* A'; *then:*

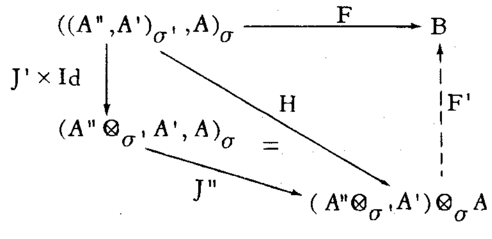
1° Each (σ', σ)-continuous tri-additive functor factors through the tensor product $(\mathrm{A}'' \otimes_{\sigma'} \mathrm{A}') \otimes_{\sigma} \mathrm{A}$.

2° There exists a continuous additive «associativity» functor:

$$\gamma : (\mathrm{A}'' \otimes_{\sigma'} \mathrm{A}') \otimes_{\sigma} \mathrm{A} \to \mathrm{A}'' \otimes_{\sigma' \otimes \sigma} (\mathrm{A}' \otimes_{\sigma} \mathrm{A}) :$$
$$(a'' \otimes a') \otimes a \mapsto a'' \otimes (a' \otimes a),$$

which is an isomorphism if $\sigma' \otimes \sigma$ is c-stable.

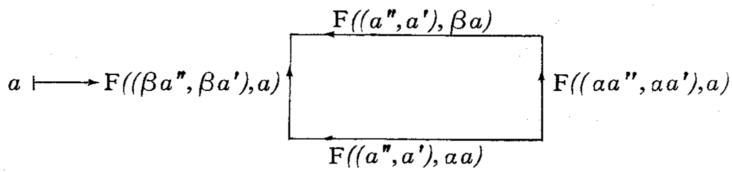
PROOF. 1° Let $F: ((A'', A')_{\sigma'}, A)_\sigma \to B = (B, S)$ be a (σ', σ)-continuous tri-additive functor. We want to show the existence of the broken line in the diagram (*):



in which J' and J'' are the canonical projections; the composite H:

$$((a'', a'), a) \mapsto (a'' \otimes a') \otimes a$$

is a (σ', σ)-continuous tri-additive functor. Since F is tri-additive, it determines the bi-additive functor $G: (A'', A') \to Hom(A, B)$, which maps (a'', a') onto the additive functor $G(a'', a'): A \to \boxminus B$:



Suppose proven that $G: (A'', A')_{\sigma'} \to Hom_\sigma(A, B)$ is σ'-continuous. Then it factors through a continuous additive functor

$$G': A'' \otimes_{\sigma'} A' \to Hom_\sigma(A, B),$$

to which is associated by Theorem 3 the continuous additive functor

$$F': (A'' \otimes_{\sigma'} A') \otimes_\sigma A \to B : (a'' \otimes a') \otimes a \mapsto F((a'', a'), a).$$

- Hence it suffices to prove that $G: T'' \times_{\sigma'} T' \to C_\sigma(T, \square S)$ is continuous. Indeed, σ' being stable by α, the map

$$Id \times \alpha: T'' \times_{\sigma'} T' \to T'' \times_{\sigma'} T'$$

is continuous. As $- \times_{\sigma'} T$ and $- \times_\sigma T$ are endofunctors of Top, we have the continuous map

$$f_\alpha: (T'' \times_{\sigma'} T') \times_\sigma T \xrightarrow{(\alpha \times \alpha) \times Id} (T'' \times_{\sigma'} T') \times_\sigma T \xrightarrow{F} S$$

$$((a'', a'), a) \mapsto F((\alpha a'', \alpha a'), a).$$

Using the stability of σ by α, we find that

$$g_\alpha : (T'' \times_{\sigma'} T') \times_\sigma T \xrightarrow{Id \times \alpha} (T'' \times_{\sigma'} T') \times_\sigma T \xrightarrow{F} S :$$

$$((a'', a'), a) \mapsto F((a'', a'), \alpha a)$$

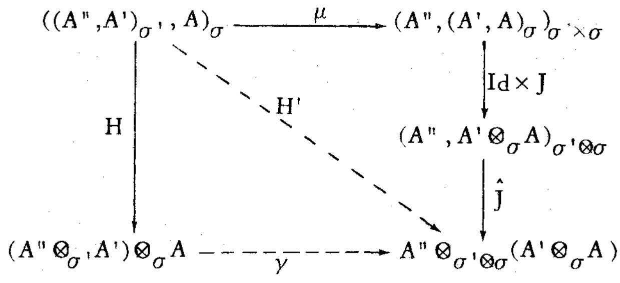
is continuous. Let f_β and g_β be the similar maps with respect to β. These maps determine the continuous map

$$[f_\beta, g_\beta, g_\alpha, f_\alpha] : (T'' \times_{\sigma'} T') \times_\sigma T \to \square S \hookrightarrow S^4 :$$

$$((a'', a'), a) \mapsto G(a'', a')(a),$$

from which follows the continuity of $G : T'' \times_{\sigma'} T' \to C_\sigma(T, \square S)$.

2° We have the following diagram:



in which μ is the homeomorphism (cf. Section 1)

$$\mu : (T'' \times_{\sigma'} T') \times_\sigma T \to T'' \times_{\sigma' \times \sigma} (T' \times_\sigma T)$$

and J and $\hat{J}$ are the canonical projections; by definition, J maps $\sigma' \times \sigma$ into $\sigma' \otimes \sigma$, so that

$$Id \times J : T'' \times_{\sigma' \times \sigma} (T' \times_\sigma T) \to T'' \times_{\sigma' \otimes \sigma} \hat{T}$$

is continuous, where $\hat{T}$ is the topology of $A' \otimes_\sigma A$. Therefore H':

$$((a'', a'), a) \mapsto a'' \otimes (a' \otimes a)$$

is a (σ', σ)-continuous tri-additive functor, and Part 1 implies that it factors through H to give the continuous additive functor γ.

3° Suppose that $\sigma' \otimes \sigma$ is c-stable. To prove that γ is an isomorphism, it suffices to prove that each (σ', σ)-continuous tri-additive functor F as above also factors through H'. Indeed, by a method similar to that used in Part 1 we associate to F the continuous additive functor

$$K : A'' \to \mathrm{Hom}_{\sigma'}((A', A)_\sigma, B)$$

such that $K(a''): T' \times_\sigma T \to \square S$ maps (a', a) onto the square $G(a'', a')(a)$ drawn in Part 1. As $\sigma' \otimes \sigma$ is supposed to be c-stable, Theorem 3 associates to the continuous additive functor

$$A'' \xrightarrow{K} \mathrm{Hom}_{\sigma'}((A', A)_\sigma, B) \xrightarrow{\eta} \mathrm{Hom}_{\sigma' \otimes \sigma}(A' \otimes_\sigma A, B)$$

(where η is defined in Theorem 4) a continuous additive functor

$$F'' : A'' \otimes_{\sigma' \otimes \sigma} (A' \otimes_\sigma A) \to B : a'' \otimes (a' \otimes a) \mapsto F((a'', a'), a)$$

whose composite with H' is F. ■

COROLLARY 1. *There exists an associativity isomorphism*

$$\gamma : (A'' \otimes_s A') \otimes_s A \to A'' \otimes_s (A' \otimes_s A).$$

PROOF. This follows from Theorem 5 applied in the case $\sigma = s$ and $\sigma' = s$, in which $\sigma' \otimes \sigma = s$ is c-stable. In this case there is a simple proof of Part 1 (and similarly of Part 3). Indeed, given the diagram (*) above, F defines a bi-additive functor

$$L : (A'' \otimes A', A) \to B : (a'' \otimes a', a) \mapsto F((a'', a'), a).$$

L is s-continuous, since the (s, s)-continuity of F implies the continuity of the maps:

for each $a \in A$, $L(-, a) = F(-, a) : T'' \times_s T' \to S$,
for each $x \in A'' \times A'$, $L(J'(x), -) = F(x, -) : T \to S$,
for each $y \in A'' \otimes A'$, $L(y, -) : T \to S$, since there exist $x_i \in A'' \times A'$ with $y = J'(x_1) + \ldots + J'(x_n)$, and $L(y, -) = F(x_1, -) + \ldots + F(x_n, -)$.

Hence L factors through H. ■

COROLLARY 2. *If σ and σ' are rc-stable, and if $\sigma' \otimes \sigma$ is c-stable, there exist isomorphisms*

$$\omega : \mathrm{Hom}_{\sigma'}(A', \mathrm{Hom}_\sigma(A, A'')) \to \mathrm{Hom}_{\sigma' \otimes \sigma}(A' \otimes_\sigma A, A''),$$

$$\omega' : \mathrm{Hom}_{\sigma'}(A', \mathrm{Hom}_\sigma(A, A'')) \to \mathrm{Hom}_{\sigma'}((A', A)_\sigma, A''),$$

E.g. they exist if $\sigma = s$ and $\sigma' = s$.

PROOF. ω is constructed from the identity of $\mathrm{Hom}_{\sigma'}(A', \mathrm{Hom}_\sigma(A, A''))$, by repeated use of the adjunction and «associativity» maps. Then ω' is the composite $\eta^{-1} \circ \omega$ (cf. Theorem 4). As $\sigma' \otimes \sigma$ is c-stable, ω^{-1} is deduced in a similar way from the identity of $\mathrm{Hom}_{\sigma' \otimes \sigma}(A' \otimes_\sigma A, A'')$. ■

From Theorem 3 and Corollary 1 of Theorem 5, we obtain:

THEOREM 6. *RdT admits a symmetric monoidal closed structure whose tensor product $\otimes_S$ extends the functors $- \otimes_S A : RdT \to RdT$ and whose internal Hom extends the functors* $\mathrm{Hom}_S(-, A)$.

1+

3. HAUSDORFF RINGOIDS AND *Top*-RINGOIDS.

We study here two subcategories of RdT, a reflective one and a coreflective one.

A. Hausdorff ringoids.

A *Hausdorff ringoid* is defined as a topological ringoid A whose topology T is a Hausdorff topology.

We denote by RdH the full subcategory of RdT whose objects are the Hausdorff ringoids. It is complete and cocomplete, and the forgetful functors toward Rd and Top preserve projective limits.

General existence theorems prove that RdH is a reflective subcategory of RdT. Let $A = (A, T)$ be a topological ringoid and $P : A \to \tilde{A}$ the reflection morphism; its restriction $P_0 : A_0 \to \tilde{A}_0$ is onto: otherwise the restriction $P' : A \to \tilde{A}'$ of P to the full subringoid of $\tilde{A}$ such that $\tilde{A}'_0 = P(A_0)$ could not factor through P though $\tilde{A}'$ be a Hausdorff ringoid.

THEOREM 1. *If* $A = (A, T)$ *is a topological ringoid such that T_0 be a Hausdorff topology, then:*

1° $P : A \to \tilde{A} = (\tilde{A}, \tilde{T})$ *is onto and* $P_0 : T_0 \to \tilde{T}_0$ *is a homeomorphism.*

2° If σ is a c-stable set of subsets of A, *for each Hausdorff ringoid* A' *there is an isomorphism*

$$\zeta : \mathrm{Hom}_{\tilde{\sigma}}(\tilde{A}, A') \to \mathrm{Hom}_\sigma(A, A') : \bar{F} \mapsto \bar{F} \circ P,$$

where $\tilde{\sigma} = \{ P(\Sigma) \mid \Sigma \in \sigma \}$.

PROOF. 1° Let B be the topological ringoid of pairs of T_0 (Example 4-2). Its topological space of objects is T_0. The continuous additive functor:

$$G : A \to B : a \mapsto (\beta a, \alpha a)$$

admits a factorization

$$G : A \xrightarrow{P} \tilde{A} \xrightarrow{G'} B$$

(since B is Hausdorff), and its restriction to the objects

$$G_0 : T_0 \xrightarrow{P_0} \tilde{T}_0 \xrightarrow{G'_0} T_0$$

is an identity; hence the onto map $P_0 : T_0 \to \tilde{T}_0$ is an homeomorphism (and P will be chosen so that P_0 be an identity). It follows that P(A) is a Hausdorff subringoid of $\tilde{A}$, hence $P(A) = \tilde{A}$.

2° The canonical 1-1 correspondence ζ deduced from the universal property of the reflection is an isomorphism, since it maps the set of elementary open sets

$$<P(\Sigma), U>, \text{ where } \Sigma \epsilon \sigma \text{ and } U \text{ open in } \boxminus A',$$

of $\mathrm{Hom}_{\tilde{\sigma}}(\tilde{A}, A')$ onto the set of elementary open sets of $\mathrm{Hom}_\sigma(A, A')$:

$$<\Sigma, U> = \zeta(<P(\Sigma), U>). \blacksquare$$

Let $A = (A, T)$ be a Hausdorff ringoid. Then $\boxminus A$ is also a Hausdorff ringoid. If σ is c-stable on A, the σ-open topology $C_\sigma(T, S)$ is a Hausdorff topology if S is a Hausdorff topology. It follows that, for each Hausdorff ringoid A', $\mathrm{Hom}_\sigma(A, A')$ is a Hausdorff ringoid; hence the functor $\mathrm{Hom}_\sigma(A, -)$ admits as a restriction an endofunctor of *RdH*.

On the other hand let σ be a set of subsets of A whose union is A, and let A' be a Hausdorff ringoid. The tensor product $A' \otimes_\sigma A$ is not necessarily a Hausdorff ringoid, but the set of its objects has a Hausdorff topology (Corollary 1 Theorem 1-2). We denote by $A' \tilde{\otimes}_\sigma A$ the Hausdorff ringoid associated with $A' \otimes_\sigma A$, and call it the *Hausdorff σ-tensor product of* A' and A. Theorem 1 asserts that the reflection morphism

$$P : A' \otimes_\sigma A \to A' \tilde{\otimes}_\sigma A$$

is onto and that its restriction to the objects is a homeomorphism.

$A' \tilde{\otimes}_\sigma A$ solves the universal problem to render continuous additive the σ-continuous bi-additive functors from $(A', A)_\sigma$ to Hausdorff ringoids. We denote by $- \tilde{\otimes}_\sigma A$ the composite functor (where ρ is the reflector):

$$RdH \hookrightarrow RdT \xrightarrow{-\otimes_\sigma A} RdT \xrightarrow{\rho} RdH.$$

From Theorem 3-2 and transitivity of adjunctions, we get:

THEOREM 2. *If σ is c-stable, the functor $-\tilde{\otimes}_\sigma A$ is a left adjoint of the functor* $\mathrm{Hom}_\sigma(A, -): RdH \to RdH$.

Let σ' be a c-stable set of subsets of A'. We denote by $\sigma' \tilde{\otimes} \sigma$ the set formed by the $P(\Sigma' \otimes \Sigma)$, where $\Sigma \in \sigma$ and $\Sigma' \in \sigma'$.

THEOREM 3. *Theorems 2, 4 and 5 of Section 2 are yet valid if we replace in them $\otimes$ by $\tilde{\otimes}$ and topological ringoid by Hausdorff ringoid.*

PROOF. From Theorems 4-2 and 1, we deduce the isomorphism

$$\mathrm{Hom}_{\sigma'}((A', A)_\sigma, A'') \xrightarrow{\eta} \mathrm{Hom}_{\sigma' \otimes \sigma}(A' \otimes_\sigma A, A'')$$

$$\downarrow \zeta^{-1}$$

$$\mathrm{Hom}_{\sigma' \tilde{\otimes} \sigma}(A' \tilde{\otimes}_\sigma A, A'').$$

The other results are proved as in Section 2. ∎

COROLLARY. *1° RdH admits a symmetric monoidal closed structure whose tensor product $\tilde{\otimes}_s$ extends the functors $-\tilde{\otimes}_s A$ and whose internal Hom is a restriction of* Hom_s.

2° RdH admits a semi-associative monoidal closed structure whose tensor product $\tilde{\otimes}_c$ extends the functors $-\tilde{\otimes}_c A$ and whose internal Hom extends the functors $\mathrm{Hom}_c(A, -): RdH \to RdH$.

B. ***Top*-ringoids.**

A *Top-ringoid* is the data consisting of a ringoid A and of a topological group $\mathrm{A}(e, e')$ on $A(e, e')$ for each couple (e, e') of objects of A, such that, for each triple (e, e', e'') of objects, the composition map:

$$\mathrm{A}(e, e') \times \mathrm{A}(e', e'') \to \mathrm{A}(e, e''): (a, b) \mapsto b.a$$

be continuous.

To each topological ringoid $\mathrm{A} = (A, T)$ is associated the *Top*-ringoid obtained by taking A and on each group $A(e, e')$ the topology induced by T; this *Top*-ringoid entirely determines A if the topology induced by T on the set A_0 of objects is discrete.

Conversely, if $(A, \mathrm{A}(e, e'))$ is a *Top*-ringoid and if we equip A with the topology S coproduct of the topologies $\mathrm{A}(e, e')$, we obtain a topological ringoid in which the topological space S_0 of objects is discrete. Hence we identify the *Top*-ringoids with the topological ringoids whose topological space of objects is discrete. 1

We denote by *T-Rd* the full subcategory of *Rd T* whose objects are the *Top*-ringoids. It is a coreflective subcategory, the coreflection of A being the *Top*-ringoid associated above to A and the coreflection morphism being defined by the identity of A.

Let A be a *Top*-ringoid and σ a set of subsets of A whose union is A.

THEOREM 4. *1°* $\mathrm{A}' \otimes_\sigma \mathrm{A}$ *is a Top-ringoid, for each Top-ringoid* A'.

2° If σ is c-stable, the functor $- \otimes_\sigma \mathrm{A} : T\text{-}Rd \to T\text{-}Rd$ *admits as a right adjoint the functor*

$$\mathrm{H}_\sigma(\mathrm{A}, -) : T\text{-}Rd \hookrightarrow Rd\,T \xrightarrow{\mathrm{Hom}_\sigma(\mathrm{A},-)} Rd\,T \xrightarrow{\nu} T\text{-}Rd,$$

where ν is the coreflector.

PROOF. Corollary 1, Theorem 1-2 asserts that the topological space of objects of $\mathrm{A}' \otimes_\sigma \mathrm{A}$ is discrete, so that $\mathrm{A}' \otimes_\sigma \mathrm{A}$ is a *Top*-ringoid. The second assertion comes from the transitivity of adjunctions. ■

COROLLARY. *T-Rd is a symmetric monoidal closed category for the tensor product restriction of* $\otimes_s$ *and for an internal Hom extending the functors* $\mathrm{H}_s(\mathrm{A}, -)$.

REMARK. The topological ringoids $\mathrm{Hom}_\sigma(\mathrm{A}, \mathrm{A}')$ are not *Top*-ringoids (in general) since even the simplest of them □ A is a *Top*-ringoid iff the topology of A is discrete.

Similar results for *H Top*-ringoids are deduced from A.

C. Examples.

1° The category of topological rings TR is a full subcategory of the category $T\text{-}Rd$ of *Top*-ringoids. If A is a topological ring and σ a set of subsets of A whose union is A, the functor $-\otimes_\sigma A$ admits as a restriction an endofunctor of TR. In particular, TR admits a symmetric monoidal (not closed) structure whose tensor product is a restriction of $\otimes_s$, and also a semi-associative monoidal structure for the tensor product $-\otimes_\pi-$ obtained by taking on each A the set π of all its subsets.

2° A topological abelian group B may be identified with the *Top*-ringoid $\hat{B}$ admitting only two objects u and u' and such that $\hat{B}(u,u') = B$ and $\hat{B}(u,u)$ and $\hat{B}(u',u')$ are discrete groups with two elements.

Let σ be a set of subsets of B whose union is B. If B' is a topological abelian group, by a method similar to that of Theorem 1-2 it is constructed a topological abelian group, denoted by $B'\otimes_\sigma B$, such that each σ-continuous bi-homomorphism from (B',B) to a topological abelian group B" factors through $B'\otimes_\sigma B$ into a continuous homomorphism toward B".

So is defined an endofunctor $-\otimes_\sigma B$ on the category TAb of topological abelian groups.

If σ is c-stable, $-\otimes_\sigma B$ *admits a right adjoint* $\mathrm{Hom}_\sigma(B,-)$ *such that* $\mathrm{Hom}_\sigma(B,B")$ *be the group of continuous homomorphisms from* B *to* B", *equipped with the topology induced by the σ-open topology* $C_\sigma(B,B")$, *for each topological abelian group* B".

It follows that TAb admits a symmetric monoidal closed structure with tensor product $-\otimes_s-$ and the internal Hom functor $\mathrm{Hom}_s(-,-)$.

It also admits *a symmetric semi-associative monoidal (not closed)* structure $(TAb)_\pi$ for the tensor product $-\otimes_\pi-$, where π associates to B the set of all its subsets. A bi-homomorphism from (B',B) is π-continuous iff it is continuous for the product topology $B'\times B$ and it then factors through $B'\otimes_\pi B$. Hence, the *Top*-ringoids may be identified with the $(TAb)_\pi$-categories (categories enriched in $(TAb)_\pi$).

4. RINGOIDS IN A CATEGORY.

A realization A of the sketch of ringoids in a category X is called a *ringoid in(ternal to)* X. Let *Rd*X be the category of ringoids in X and suppose X equipped with an intial-structure functor $\chi : \mathrm{X} \to Set$.

Then the methods and results of Section 2 may be generalized. More precisely, let A be a ringoid in X ; it is entirely determined by the couple (A, X), where A is the ringoid defined by the realization $\chi \circ \mathrm{A}$ and where $X \in \mathrm{X}_0$ is the «object of morphisms» (see [18]).

1° If $-\& X$ is an endofunctor of X such that

$$\mathrm{X} \xrightarrow{-\& X} \mathrm{X} \xrightarrow{\chi} Set = \mathrm{X} \xrightarrow{\chi} Set \xrightarrow{-\times\chi(X)} Set$$

we construct as in Theorem 1-2 an endofunctor $-\&\mathrm{A}$ of *Rd*X such that the ringoid underlying $\mathrm{A}'\&\mathrm{A}$ be $A'\otimes A$.

2° To A is associated the double ringoid $\square\mathrm{A}$ in X , over $\square A$.

3° Let $M(X,-)$ be an endofunctor of X preserving pullbacks. If A' is a ringoid in X , the realization $M(X,-)\circ\mathrm{A}'$ is a ringoid $M(X,\mathrm{A}')$ in X . Its object of morphisms is $M(X,X')$. We'll suppose moreover that

$$\mathrm{X} \xrightarrow{M(X,-)} \mathrm{X} \xrightarrow{\chi} Set = Hom_{\mathrm{X}}(X,-).$$

In this case, $M(X,\boxplus\mathrm{A}')$ admits a subringoid $M(\mathrm{A},\mathrm{A}')$ in X over the ringoid of morphisms from A to A' (whose morphisms are the $F : \mathrm{A} \to \boxminus\mathrm{A}'$).

4° If $M(X,-)$ is a right adjoint of $-\& X$, then $-\&\mathrm{A}$ admits a right adjoint $M(\mathrm{A},-)$. If $(\mathrm{X}, \&, M(-,-))$ is a monoidal closed category, the functors $-\&\mathrm{A}$ and $M(\mathrm{A},-)$ extend to give a monoidal closed structure on *Rd*X . 1

For instance, the ringoids in the cartesian closed category *Ke* (see Section 1) of Kelley spaces form a monoidal closed category. (Remark that a Kelley ringoid is not necessarily a topological ringoid, pullbacks in *Ke* differing from pullbacks in *Top* .) The ringoids in the categories of limit-spaces, or of pseudo-topologies, or of Spanier quasi-topologies,... form also monoidal closed categories.

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MULTIPLE FUNCTORS

I. LIMITS RELATIVE TO DOUBLE CATEGORIES

by Andrée BASTIANI and Charles EHRESMANN

INTRODUCTION.

Double categories are sets equipped with two laws of categories satisfying the «axiom of permutability». This axiom was first exhibited in [E7] for the two laws on the set of natural transformations from a category C to itself and in [E8] for the two laws on the set of commutative squares of C. The general definition of a double category (and by induction of a multiple category) was given in [E2], as a category internal to the category $\mathcal{F}$ of categories or, more precisely, as a structured category relative to the faithful functor from $\mathcal{F}$ to the category of sets. 2-categories are those double categories whose identities for the second law are also identities for the first law (but they are most often defined as categories enriched in the cartesian closed category $\mathcal{F}$); they have been considered by many authors [GZ, G1, G2, G3, Bo, S] as well as the double categories of squares of a 2-category [GZ, G1, Pa]. Benabou's bicategories [B2] are a «laxification» of 2-categories (and double categories may be laxified in a similar way, as done in [Ch, M]).

While a substantial and extensive theory of 2-categories has been given by Gray [G1, 2, 3], no such theory exists for double categories. We 1
are going to generalize here some of the numerous fine results of Gray in the frame of double categories, using a method outlined in [E2] and whose main idea is to associate to a category A and to a double category D a category $T(D, A)$ which plays the same role as the category of natural transformations (to which it reduces if D is the double category of commutative squares of a category).

In chapter 0 are gathered some complements about sketched structures (used in particular later on to construct internal multiple categories). In chapter I we study the functor $T(-, A)$ from the category of double functors to $\mathcal{F}$; it associates to D the category formed by the functors from A to the first category underlying D, and whose law is deduced from the second law of D; it admits an adjoint $- \blacksquare A$. Free objects relative to the canonical functor from the first category of 1-morphisms of D toward $T(D, A)$ are called D-wise limits. The main theorem, proved in chapter II, asserts that, if D is representable (i.e. there exist D-wise limits indexed by $\mathbf{2}$) and if the second category of 1-morphisms of D admits small limits, then all small D-wise limits exist. If D is the double category of up-squares of a representable 2-category, D is representable and the theorem reduces to a theorem of Gray, D-wise limits being cartesian quasi-limits of [G1].

This paper is the first part of a work whose other parts will appear in the following issues of the «Cahiers».

- In the second part, the present results are generalized to n-fold categories: the category of all multiple categories is equipped with a monoidal closed structure, whose internal Hom associates to the $n+m$-fold category M and to the m-fold category B the n-fold category $T(M, B)$ of generalized transformations; the tensor product $\blacksquare$ is only symmetrical «up to an interchange of the laws». As before, M-wise limits are defined and there is a similar theorem of existence of M-wise limits when there exist M-wise limits indexed by $\mathbf{2}^{\blacksquare n} = \mathbf{2} \blacksquare \dots \blacksquare \mathbf{2}$ (this theorem is proved using a result of Appelgate-Tierney [AT] and the fact that each n-fold category is generated from $\mathbf{2}^{\blacksquare n}$ by inductive limits).

- In the third part, we will describe different monoidal closed structures on the category of double functors: its cartesian closed structure (whose existence is proved in [BE]), whose internal Hom maps (D', D) on the double category of double functors from D to the 4-fold category of squares of squares of D'; two monoidal closed structures non symmetrical which occur when double natural transformations are laxified (and which generalize the monoidal closed structure on the category of 2-functors considered by Gray [G1]). These results will then be applied to the study of structures
1 defined as realizations or lax realizations of «double sketches».

0. COMPLEMENTS ABOUT SKETCHED STRUCTURES

A. Notations.

1. We denote by $\mathcal{U}$ a universe and a set is said *small* if it is an element of this universe. The category of maps between small sets is denoted by $\mathcal{M}$.

A *small category* is a category whose set of morphisms is small.

2. Since we will have to consider several categories with the same set of morphisms, we will often denote a category by a symbol $A^{\cdot}$, where A is the set of its morphisms and «.» the symbol of its law of composition (i. e. the composite of (y, x) is written $y.x$). Then:

$\alpha^{\cdot}$, $\beta^{\cdot}$ and $\kappa^{\cdot}$ are its maps source, target and law of composition,
$A^{\cdot}_0$ is the set of its objects, $A^{\cdot} * A^{\cdot}$ the set of its composable pairs,
$A^{\cdot *}$ its dual category.

But often we also denote a category by a unique letter (an italic or a greek letter or, for «big» categories, a script letter). In that case, if C is a category, its set of morphisms is denoted by $\underline{C}$, its symbol of composition by «.», its set of objects by C_0, the dual category by C^*, and the set of morphisms from e to e' by $C(e', e)$ or by $e'.C.e$, and $x: e \to e'$ 1
is read $x \in e'.C.e$. If the sets $C(e', e)$ are small, the *Hom* functor from $C \times C^*$ to $\mathcal{M}$ is denoted by $C(-,-): C \times C^* \to \mathcal{M}$.

3. A functor f from A to C is also denoted by (C, ϕ, A), where ϕ is the map from $\underline{A}$ to $\underline{C}$ defining it (sometimes we put $\underline{f} = \phi$). If f is constant on an object e of C, we write $f = e^{\hat{}}$.

The category of functors between small categories (i. e. of small functors) is denoted by $\mathcal{F}$, the composite functor:

$$A \xrightarrow{\;f\;} C \xrightarrow{\;f'\;} D$$

being written $f'.f$ or, more often, $f'f$.

There are two «canonical» functors from $\mathcal{F}$ to $\mathcal{M}$:

the faithful functor $p\mathcal{F}$ which associates to $f: A \to C$ the map $\underline{f}: \underline{A} \to \underline{C}$;
the functor $\dot{p}\mathcal{F}$ associating to $f: A \to C$ the map $f_0: A_0 \to C_0$ restriction of f to the sets of objects.

The functor $p\dot{\mathcal{F}}$ admits an adjoint functor, mapping the small set M on the *discrete category on* M (each element of M is an identity) which will be denoted by M^0. It also admits a coadjoint which associates to M the *groupoid of pairs* $(M \times M)^{\circ}$.

The functor $p\mathcal{F}$ has no coadjoint (since it does not preserve coequalizers). But it admits an adjoint functor, which associates to M the category $\mathbf{2} \times M^0$, coproduct of M copies of the category $\mathbf{2}$, where

$$\mathbf{2} \quad \text{is} \quad 1 \xleftarrow{\;z\;} 0 \;.$$

1 4. If A and C are categories, we denote by C^A the category of natural transformations between functors from A to C. If $t = (f', \underline{t}, f)$ is the natural transformation from the functor f to f' defined by the map $\underline{t}$ from A_0 to C, we write $t(u) = \underline{t}(u)$ for each object u of A, and

$$t : f \to f' : A \rightrightarrows C \;, \quad \text{or} \quad t : A \rightrightarrows C .$$

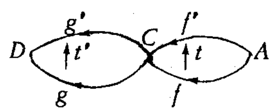
If $t' : f' \to f''$ is another natural transformation, then

$$t' \square\square\, t : f \to f''$$

denotes their composite in C^A. Identical natural transformations are identified with functors.

On the set of all small natural transformations we have two laws:
$\mathcal{N}^{\square\square}$ is the category coproduct of the categories C^A for all small categories A and C;
$\mathcal{N}^{\cdot}$ is the category, admitting $\mathcal{F}$ as a sub-category, in which the composite of $t : f \to f' : A \rightrightarrows C$ and $t' : g \to g' : C \rightrightarrows D$ is

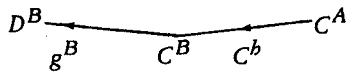
$t' . t : gf \to g'f' : A \rightrightarrows D$, where
$(t' . t)(u) = t'(f'(u)) . g(t(u))$,
for each object u of A.

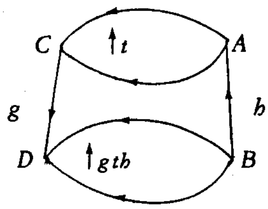


This composite is sometimes written $t't$, especially when t or t' is an identical transformation. We have:

$$t' . t = (g't) \square\square (t'f) = (t'f') \square\square (gt).$$

If $h : B \to A$ is a functor, the functor $t \to th$ from C^A to C^B is denoted by C^h. In the same way, $g^A : C^A \to D^A$ is the functor associating gt to $t: A \rightrightarrows C$. Finally, $g^h : C^A \to D^B$ is the composite functor $g^B C^h$:

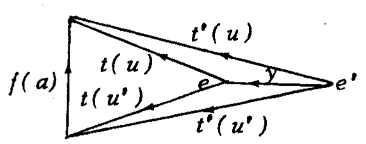


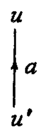


5. Let $f: A \to C$ be a functor. A natural transformation $t: e^{\wedge} \to f$, where $e^{\wedge}$ is a constant functor, is called *a projective cone indexed by A*, with vertex e and basis f. If $y: e' \to e$ is a morphism of C, we denote by ty the cone with basis f and vertex e' such that

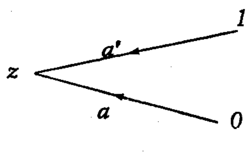
$$(ty)(u) = t(u).y \quad \text{for each object } u \text{ of } A.$$

If t is a limit-cone and t' a projective cone with basis f, the unique y such that $ty = t'$ is called *the factor of t' relative to t.*

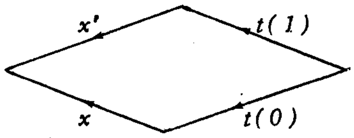




In particular, let us take for A the category



and for f the functor mapping a and a' onto x and x'. If t is a projective limit-cone with basis f, we also say that



is a pullback P of (x, x'). If t' is a projective cone with basis f (i.e. if $x.t'(0) = x'.t'(1)$), the factor y of t' relative to t is denoted by

$[t'(0), t'(1)]$ and called the factor of $(t'(0), t'(1))$ relative to P.

Similar notations are used for inductive cones $\bar{t}: f \to \hat{e}$.

B. Sketched structures.

1. We recall [BE] that a (projective) limit-bearing category σ is a category Σ equipped with a set Γ of distinguished (projective) limit-cones; the set of the indexing categories of the cones $\gamma \in \Gamma$ is called the set of indexing categories of σ.

If Σ' is a category, a *σ-structure in* Σ' is a functor $\phi: \Sigma \to \Sigma'$ such that $\phi\gamma$ is a limit-cone for each $\gamma \in \Gamma$. We denote by Σ'^{σ} the category of σ-morphisms in Σ', which is the full sub-category of Σ'^{Σ} whose objects are the σ-structures in Σ'.

If $\psi: \Sigma \to \Sigma'^{*}$ is a σ-structure in the dual Σ'^{*} of Σ', then the dual functor $\psi^{*}: \Sigma^{*} \to \Sigma'$ is called a *σ-costructure in* Σ'.

σ-structures are called *sketched structures* (this terminology is justified by Proposition 8-I [BE]).

PROPOSITION 1. *If σ is a projective limit-bearing category (Σ, Γ) and Σ' a category, there exists a functor $\theta: \Sigma'^{\sigma} \times \Sigma'^{*} \to \mathfrak{M}^{\sigma}$ associating to an object (ϕ, ω) the σ-structure $\Sigma'(-,\omega)\phi: \Sigma \to \mathfrak{M}$.*

Δ. We consider the following functors:

the insertion $\iota: \Sigma'^{\sigma} \to \Sigma'^{\Sigma}$,

the Yoneda embedding $Y': \Sigma'^{*} \to \mathfrak{M}^{\Sigma'}$,

the «composition functor» $\lambda: \Sigma'^{\Sigma} \times \mathfrak{M}^{\Sigma'} \to \mathfrak{M}^{\Sigma}$ which associates to the pair (τ, τ') of natural transformations their composite $\tau' . \tau$.

The composite functor θ':

$$\Sigma'^{\sigma} \times \Sigma'^{*} \xrightarrow{\iota \times Y'} \Sigma'^{\Sigma} \times \mathfrak{M}^{\Sigma'} \xrightarrow{\lambda} \mathfrak{M}^{\Sigma}$$

maps the pair (ϕ, ω), where ϕ is a σ-structure in Σ' and ω an object of Σ', on the functor

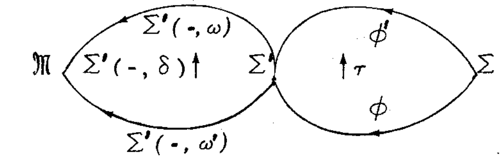
$$\lambda(\phi, Y'(\omega)) = \Sigma'(-,\omega)\phi: \Sigma \to \mathfrak{M}$$

which is a σ-structure in $\mathfrak{M}$, since $\Sigma'(-,\omega)$ preserves projective limits.

Hence θ' takes its values in $\mathcal{M}^{\sigma}$ and it admits as a restriction

$$\theta : \Sigma'^{\sigma} \times \Sigma'^{*} \longrightarrow \mathcal{M}^{\sigma}.$$

If $\tau : \phi \to \phi'$ is a σ-morphism in Σ' and $\delta : \omega \to \omega'$ a morphism in Σ', then $\theta(\tau, \delta) = \Sigma'(-, \delta) . \tau$:



∇

2. Let σ be a projective limit-bearing category (Σ, Γ) and Σ' a category admitting projective limits indexed by the indexing categories of σ. For each object ω of Σ, let $\nu_{\omega} : \mathcal{M}^{\sigma} \to \mathcal{M}$ be the functor «value in ω», which maps $\tau : \Sigma \rightrightarrows \mathcal{M}$ onto $\tau(\omega)$.

PROPOSITION 2. *1° $(\mathcal{M}^{\sigma})^{\Sigma'^{*}}$ and $(\mathcal{M}^{\Sigma'^{*}})^{\sigma}$ are isomorphic.*

2° Σ'^{σ} is equivalent to the full sub-category $\mathcal{R}$ of $(\mathcal{M}^{\sigma})^{\Sigma'^{*}}$ whose objects are the functors $\psi : \Sigma'^{*} \to \mathcal{M}^{\sigma}$ such that $\nu_{\omega}\psi : \Sigma'^{*} \to \mathcal{M}$ is representable, for each $\omega \in \Sigma_0$.

Δ. 1° We denote by μ the canonical isomorphism

$$\mu : (\mathcal{M}^{\Sigma'^{*}})^{\Sigma} \to (\mathcal{M}^{\Sigma})^{\Sigma'^{*}}$$

and by $\nu'_{\omega'} : \mathcal{M}^{\Sigma'^{*}} \to \mathcal{M}$ the functor value in $\omega' \in \Sigma'_0$. Let $\phi : \Sigma \to \mathcal{M}^{\Sigma'^{*}}$ be a functor. We have $\nu'_{\omega'} \phi = \mu(\phi)(\omega')$. If $\gamma : I \rightrightarrows \Sigma$ is a limit-cone, limits in $\mathcal{M}^{\Sigma'^{*}}$ being computed termwise, $\phi\gamma$ is a limit-cone iff

$$\nu'_{\omega'} \phi\gamma = \mu(\phi)(\omega')\gamma : I \rightrightarrows \mathcal{M}$$

is a limit-cone for each $\omega' \in \Sigma'_0$. Hence ϕ is a σ-structure iff

$$\mu(\phi)(\omega') \text{ is a } \sigma\text{-structure in } \mathcal{M}, \text{ for each } \omega' \in \Sigma'_0,$$

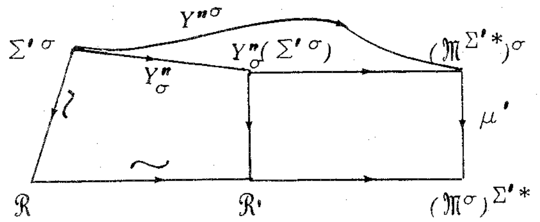
i.e. iff $\mu(\phi)$ takes its values in $\mathcal{M}^{\sigma}$. So μ admits as a restriction an isomorphism $\mu' : (\mathcal{M}^{\Sigma'^{*}})^{\sigma} \to (\mathcal{M}^{\sigma})^{\Sigma'^{*}}$.

2° Let $Y'' : \Sigma' \to \mathcal{M}^{\Sigma'^{*}}$ be the Yoneda embedding. It gives an isomorphism $Y''_{\sigma} : \Sigma'^{\sigma} \to Y''(\Sigma')^{\sigma} \cong Y''_{\sigma}(\Sigma'^{\sigma})$ = sub-category of $(\mathcal{M}^{\Sigma'^{*}})^{\sigma}$, the insertion $Y''(\Sigma') \hookrightarrow \mathcal{M}^{\Sigma'^{*}}$ preserving projective limits. The isomorphism

μ' maps $Y''_\sigma(\Sigma'^\sigma)$ onto the full sub-category $\mathfrak{R}'$ of $(\mathfrak{M}^\sigma)^{\Sigma'^*}$ whose objects are the functors $\psi: \Sigma'^* \to \mathfrak{M}^\sigma$ such that $\mu'^{-1}(\psi): \Sigma \to \mathfrak{M}^{\Sigma'^*}$ takes its values in $Y''(\Sigma')$, i.e. such that

$$\mu'^{-1}(\psi)(\omega) = \nu_\omega \psi: \Sigma'^* \to \mathfrak{M}$$

is an object of $Y''(\Sigma')$ for each $\omega \in \Sigma_0$. Hence Σ'^σ is isomorphic with $\mathfrak{R}'$. As $Y''(\Sigma')$ is equivalent to the full sub-category of $\mathfrak{M}^{\Sigma'^*}$ whose objects are the representable functors, $\mathfrak{R}'$ is equivalent to the category $\mathfrak{R}$ defined in the Proposition. So Σ'^σ is equivalent to $\mathfrak{R}$.



∇

3. *Projective closure of a set.*

Let σ be a limit-bearing category (Σ, Γ) and Ω a sub-set of Σ_0. We define by induction a transfinite increasing sequence of full sub-categories Σ_ξ of Σ as follows:

Σ_0 is the full sub-category of Σ admitting Ω as its set of objects;

$\Sigma_\xi = \bigcup_{\zeta < \xi} \Sigma_\zeta$, if ξ is an ordinal without a predecessor;

if Σ_ξ is defined, then $\Sigma_{\xi+1}$ is the full sub-category of Σ whose objects are the vertices of the distinguished cones $\gamma \in \Gamma$ whose bases take their values in Σ_ξ, and the objects of Σ_ξ.

DEFINITION. We say that Σ is the Γ*-closure of* Ω if there exists an ordinal δ such that $\Sigma = \Sigma_\delta$; then $(\Sigma_\xi)_{\xi \leq \delta}$ is said to Γ-generate Σ.

If Σ is the Γ-closure of Ω, it is also the Γ'-closure of Ω, for each set Γ' of limit-cones including Γ.

PROPOSITION 3. *Let σ be a projective limit-bearing category (Σ, Γ) and Σ' a category admitting projective limits indexed by the indexing categories of σ. If Σ is the Γ-closure of a sub-set Ω of Σ_0, then Σ'^σ*

is equivalent to the full sub-category of $(\mathfrak{M}^{\sigma})^{\Sigma'^*}$ *whose objects are the functors* ψ *such that* $\nu_\omega \psi$ *is representable fo each* $\omega \in \Omega$, *where* ν_ω *is the functor value in* ω *from* $\mathfrak{M}^{\sigma}$ *to* $\mathfrak{M}$. 1

Δ. Let Ω_ξ be the set of objects of the sub-category Σ_ξ of Σ defined above and δ the smallest ordinal such that $\Sigma = \Sigma_\delta$. Then the union of the transfinite sequence of sets $(\Omega_\xi)_{\xi \leqslant \delta}$ is Σ_0. In view of Proposition 2, it suffices to prove that, if $\psi : \Sigma'^* \to \mathfrak{M}^{\sigma}$ is a functor such that $\nu_\omega \psi$ be representable for each $\omega \in \Omega = \Omega_0$, the set Π of objects ω' of Σ such that $\nu_{\omega'} \psi$ be representable is equal to Σ_0. This will be proved by induction:

Ω_0 is included in Π.

If ξ has no predecessor and if Ω_ζ is included in Π for each ordinal $\zeta < \xi$, then the union Ω_ξ of $(\Omega_\zeta)_{\zeta < \xi}$ is included in Π.

Now let us suppose that Ω_ξ is included in Π for some ordinal $\xi < \delta$ and that $\omega' \in \Omega_{\xi+1} \setminus \Omega_\xi$. So ω' is the vertex of a cone $\gamma \in \Gamma$ whose basis ρ takes its values in Σ_ξ. Let ϕ be the σ-structure in $\mathfrak{M}^{\Sigma'^*}$ associated to ψ by the isomorphism μ'^{-1} of Proposition 2. The cone $\phi\gamma$ is a limit-cone in $\mathfrak{M}^{\Sigma'^*}$ with vertex $\phi(\omega') = \nu_{\omega'} \psi$ and the induction hypothesis implies that its basis $\phi\rho$ takes its values in the sub-category of $\mathfrak{M}^{\Sigma'^*}$, whose objects are the representable functors. A projective limit of representable functors being a representable functor, this sub-category is closed for projective limits, so that the vertex $\nu_{\omega'} \psi$ of $\phi\gamma$ belongs to it. Therefore $\omega' \in \Pi$. It follows that $\Omega_{\xi+1}$ is included in Π.

By induction this proves that $\Pi = \Sigma_0$. ∇

DEFINITION. Let Σ be a category and Ω a sub-set of Σ_0. We say that Σ is the *projective* (resp. *inductive*, resp. *connected projective*) *closure* of Ω if Σ is the L-closure of Ω, where L is the set of all small limit-cones in Σ which are projective (resp. inductive, resp. projective and indexed by a connected category).

PROPOSITION 4. *Let* σ *be a projective limit-bearing category* (Σ, Γ) *and* $Y : \Sigma^* \to \mathfrak{M}^{\Sigma}$ *the Yoneda embedding.*

1° Y admits as a restriction an injective σ*-costructure* $\overline{Y}$ *in* $\mathfrak{M}^{\sigma}$ *and*

1 *each* σ*-structure* ϕ *in* $\mathfrak{M}$ *is equivalent to* $\mathfrak{M}^\sigma(\phi, \cdot)\overline{Y}^*$.

2° $\mathfrak{M}^\sigma$ *is the inductive closure of* $Y(\Sigma_0)$.

3° If Σ *is the* Γ*-closure of a sub-set* Ω *of* Σ_0, *then* $\mathfrak{M}^\sigma$ *is the inductive closure of* $Y(\Omega)$.

Δ . 1° For each $\omega \in \Sigma_0$, the functor

$$Y(\omega) = \Sigma(\cdot, \omega): \Sigma \to \mathfrak{M},$$

which preserves projective limits, is a σ-structure in $\mathfrak{M}$, so that $Y(\Sigma)$ is included in the full sub-category $\mathfrak{M}^\sigma$ of $\mathfrak{M}^\Sigma$. The restriction

$$\overline{Y}: \Sigma^* \longrightarrow \mathfrak{M}^\sigma \quad \text{of } Y$$

is a σ-costructure, since Y sends projective limit-cones belonging to Γ on inductive limit-cones in $\mathfrak{M}^\sigma$, according to a result of [Lm]. Hence $\overline{Y}$ is a σ-costructure in $\mathfrak{M}^\sigma$.

2° Let ϕ be a σ-structure in $\mathfrak{M}$. The Yoneda lemma asserts that ϕ is equivalent to

$$\mathfrak{M}^\Sigma(\phi, \cdot)\, Y^*: \Sigma \xrightarrow{\ Y^*\ } (\mathfrak{M}^\Sigma)^* \xrightarrow{\ \mathfrak{M}^\Sigma(\phi, \cdot)\ } \mathfrak{M},$$

which is equal to $\mathfrak{M}^\sigma(\phi, \cdot)\overline{Y}^*$ since $\mathfrak{M}^\sigma$ is a full sub-category of $\mathfrak{M}^\Sigma$. On the other hand, in $\mathfrak{M}^\Sigma$ the object ϕ is the inductive limit of the functor $Y h^*$:

$$H^* \xrightarrow{\ \ h^*\ \ } \Sigma^* \xrightarrow{\ \ Y\ \ } \mathfrak{M}^\Sigma,$$

where $h: H \to \Sigma$ is the discrete fibration (or «hypermorphism functor» [E1]) associated to $\phi: \Sigma \to \mathfrak{M}$. The functor $Y h^*$ admits as a restriction a functor $k: H^* \to \mathfrak{M}^\sigma$ which takes its values in $Y(\Sigma)$. The sub-category $\mathfrak{M}^\sigma$ being full, its object ϕ is also the inductive limit of k. Hence $\mathfrak{M}^\sigma$ is the inductive closure of $Y(\Sigma_0)$. (In fact, the closure operation takes only one step in this case.)

3° Let Σ be the Γ-closure of Ω. The restriction $\overline{Y}$ of Y maps injectively a full sub-category of Σ onto a full sub-category of $\mathfrak{M}^\sigma$ and sends each cone of Γ onto an inductive limit-cone of $\mathfrak{M}^\sigma$. Hence $\overline{Y}$ maps the Γ-closure Σ of Ω into the inductive closure of $Y(\Omega)$ in $\mathfrak{M}^\sigma$, so that the inductive closure $\mathfrak{M}^\sigma$ of $Y(\Sigma_0)$ is also the inductive closure of $Y(\Omega)$. ∇

PROPOSITION 5. *If* Σ *is the projective connected closure of a sub-set* Ω *of* Σ_0 *and if* Σ' *is a category which is the projective connected closure of a sub-set* Ω' *of* Σ'_0, *then* $\Sigma' \times \Sigma$ *is the projective connected closure of* $\Omega' \times \Omega$.

Δ. Let $(\Sigma_\xi)_{\xi \leqslant \delta}$ and $(\Sigma'_\xi)_{\xi \leqslant \delta'}$ be the canonical increasing transfinite sequences of full sub-categories of Σ and Σ', where

$$\Sigma = \Sigma_\delta \quad \text{and} \quad \Sigma' = \Sigma'_{\delta'} \,;$$

we may suppose that $\delta = \delta'$. Then we have an increasing transfinite sequence $(\Sigma'_\xi \times \Sigma_\xi)_{\xi \leqslant \delta}$ of full sub-categories of $\Sigma' \times \Sigma$ satisfying:

$$\Sigma' \times \Sigma = \Sigma'_\delta \times \Sigma_\delta \,.$$

If (ω', ω) is an object of $\Sigma'_{\xi+1} \times \Sigma_{\xi+1}$, there exist projective limit-cones γ in Σ and γ' in Σ', with vertices ω and ω', whose bases

$$\rho : I \to \Sigma \quad \text{and} \quad \rho' : I' \to \Sigma'$$

take their values in Σ_ξ and Σ'_ξ respectively, and whose indexing categories I and I' are connected. The product functor

$$\rho' \times \rho : I' \times I \to \Sigma' \times \Sigma$$

takes its values in $\Sigma'_\xi \times \Sigma_\xi$ and it admits (ω', ω) as its projective limit; its indexing category $I' \times I$ is connected, I and I' being connected. This proves that the connected projective closure Π of $\Omega' \times \Omega$ in $\Sigma' \times \Sigma$ contains $\Sigma'_{\xi+1} \times \Sigma_{\xi+1}$ as soon as it contains $\Sigma'_\xi \times \Sigma_\xi$. By induction it follows that Π contains $\Sigma'_\delta \times \Sigma_\delta = \Sigma' \times \Sigma$; whence $\Pi = \Sigma' \times \Sigma$. ∇

4. *Tensor product of cone-bearing categories.*

Let $\sigma = (\Sigma, \Gamma)$ and $\sigma' = (\Sigma', \Gamma')$ be two projective cone-bearing categories. Conduché [C] and Lair [L] have proved that there exists a cone-bearing category $\sigma' \otimes \sigma$ on $\Sigma' \times \Sigma$ satisfying the universal property: 1+

Let H be a category admitting projective limits indexed by the indexing categories of σ and of σ'. Then the canonical isomorphism

$$(H^{\Sigma'})^{\Sigma} \xrightarrow{\sim} H^{\Sigma' \times \Sigma}$$

admits as a restriction an isomorphism from $(H^{\sigma'})^{\sigma}$ onto $H^{\sigma' \otimes \sigma}$.

They have given the following explicit construction of $\sigma' \otimes \sigma$:

- The underlying category is $\Sigma' \times \Sigma$.

If $\omega' \in \Sigma'_0$ and if $\gamma \in \Gamma$ is a cone with basis $\phi : I \to \Sigma$ and vertex ω, let γ'' be the cone $[\omega'^{\wedge}, \gamma] : I \to \Sigma' \times \Sigma$, with basis $[\omega'^{\wedge}, \phi]$, vertex (ω', ω), and such that

$$\gamma''(i) = (\omega', \gamma(i)) \quad \text{for each} \quad i \in I_0 .$$

If I is connected, this cone is a limit-cone, when γ is a limit-cone.

We define in a similar way the cone $[\gamma', \omega^{\wedge}]$, where

$$\gamma' \in \Gamma' \quad \text{and} \quad \omega \in \Sigma_0 .$$

- The set $\Gamma' \otimes \Gamma$ of cones is formed by all the cones $[\omega'^{\wedge}, \gamma]$ and $[\gamma', \omega^{\wedge}]$, for $\gamma \in \Gamma$, $\gamma' \in \Gamma'$, $\omega' \in \Sigma'_0$ and $\omega \in \Sigma_0$.

If all the indexing categories of σ and of σ' are connected, then $\sigma' \otimes \sigma$ is a limit-bearing category, when so are σ and σ'.

DEFINITION. $\sigma' \otimes \sigma$ is called *the tensor product of* (σ', σ).

If $(\sigma_i)_{i<n}$ is a finite sequence of cone-bearing categories, their tensor product, denoted by

$$\bigotimes_{i<n} \sigma_i \quad \text{or} \quad \sigma_0 \otimes \ldots \otimes \sigma_{n-1} ,$$

is defined by induction from the formula:

$$\bigotimes_{i<m+1} \sigma_i = (\bigotimes_{i<m} \sigma_i) \otimes \sigma_m \quad \text{for each} \quad m < n-1 .$$

If $\sigma_i = \sigma$ for each $i < n$, then $\bigotimes_{i<n} \sigma_i$ is also written $\overset{n}{\otimes} \sigma$.

The underlying category of $\bigotimes_{i<n} \sigma_i$ is the category $\mathop{\mathrm{X}}_{i<n} \Sigma_i$, defined by induction from the formula:

$$\mathop{\mathrm{X}}_{i<m+1} \Sigma_i = (\mathop{\mathrm{X}}_{i<m} \Sigma_i) \times \Sigma_m \quad \text{for each} \quad m < n-1 .$$

The word «tensor product» is well justified. Indeed, Lair proves in [L] that the category of morphisms between cone-bearing categories is equipped with a symmetrical monoidal closed structure, whose tensor product maps (σ', σ) onto $\sigma' \otimes \sigma$. From the general properties of symmetrical monoidal closed categories, we get:

PROPOSITION 6. *Let* $(\sigma_i)_{i<n}$ *be a sequence of projective limit-bearing categories and let* H *be a category admitting projective limits indexed by the indexing categories of* σ_i, *for each integer* $i<n$. *For each permutation* f *of* $\{0, \dots, n\text{-}1\}$ *and each sequence*

$$0 = n_0 < n_1 < \dots < n_m < n_{m+1} = n$$

of integers, there exists a canonical isomorphism

$$H^{\sigma_0 \otimes \dots \otimes \sigma_{n-1}} \xrightarrow{\sim} (\dots((H^{\sigma'_{n_0}})^{\sigma'_{n_1}})\dots)^{\sigma'_{n_m}},$$

where $\sigma'_{n_j} = \sigma_{f(n_j)} \otimes \dots \otimes \sigma_{f(n_{j+1}-1)}$.

PROPOSITION 7. *Let* n *be an integer,* $\sigma = (\Sigma, \Gamma)$ *a projective limit-bearing category whose indexing categories are connected,* $\sigma' = (\Sigma', \Gamma') = \overset{n}{\otimes}\sigma$ *and* Ω *a sub-set of* Σ_0.

1° If Σ *is the connected projective closure of* Ω, *then* Σ' *is the connected projective closure of* $\Omega' = \overset{n}{\times}\Omega$.

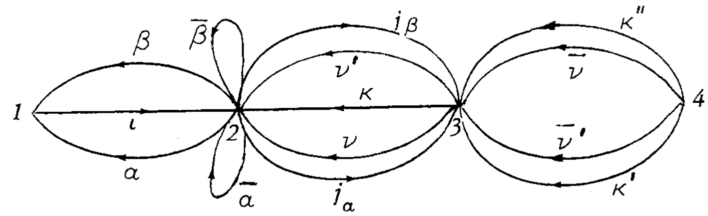
2° If Σ *is the* Γ*-closure of* Ω, *then* Σ' *is the* Γ'*-closure of* Ω' *and* $\mathfrak{M}^{\sigma'}$ *is the inductive closure of* $Y'(\Omega')$ *where* Y' *is the Yoneda embedding.*

Δ. By induction, part 1 follows from Proposition 5, part 2 from Proposition 4, since $(\Sigma_0 \times \Sigma_0, \dots, \Sigma_\delta \times \Sigma_0, \Sigma \times \Sigma_1, \dots, \Sigma \times \Sigma_\delta)$ is Γ'-generating Σ' for $n = 2$, if $(\Sigma_\xi)_{\xi \leqslant \delta}$ is Γ-generating Σ. ∇

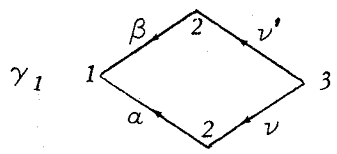
C. Internal categories. 1

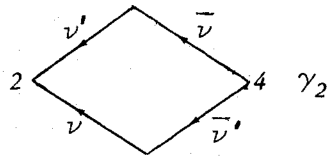
1. We denote by $\sigma\mathcal{F} = (\Sigma\mathcal{F}, \Gamma\mathcal{F})$ the sketch of categories [BE] which is the following limit-bearing category:

$\Sigma\mathcal{F}$ is the dual of the full sub-category of the simplicial category Δ whose objects are the natural integers $1, 2, 3$ and 4; its main morphisms are denoted according to the following diagram, where $\bar{\alpha} = \iota.\alpha$, $\bar{\beta} = \iota.\beta$:

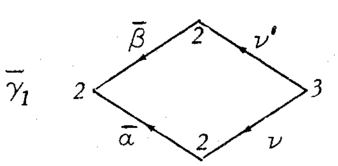


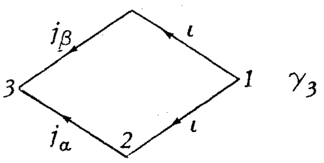
The only distinguished cones are the two pullbacks:





Since ι is a right inverse of α, it is an absolute equalizer of the pairs $(2, \bar{\alpha})$ and $(\bar{\beta}, 2)$, and we have the pullbacks





in $\Sigma_{\mathcal{G}}$. We write $\bar{\Gamma}_{\mathcal{G}} = \{\bar{\gamma}_1, \gamma_2, \gamma_3\}$ and $\bar{\sigma}_{\mathcal{G}} = (\Sigma_{\mathcal{G}}, \bar{\Gamma}_{\mathcal{G}})$. Then $\Sigma_{\mathcal{G}}$ is the $\bar{\Gamma}_{\mathcal{G}}$-closure of $\{2\}$. So Propositions 3, 4, 7 may be applied to $\bar{\sigma}_{\mathcal{G}}$.

2. Let $\mathcal{H}$ be a category with pullbacks. A $\sigma_{\mathcal{G}}$-structure in $\mathcal{H}$ is called a *category internal to* (or *in*) $\mathcal{H}$; other names: category object in $\mathcal{H}$ for [Gr], «catégorie structurée généralisée dans $\mathcal{H}$» for [E3].

A $\sigma_{\mathcal{G}}$-morphism in $\mathcal{H}$ is called a *functor internal to* (or *in*) $\mathcal{H}$. We denote by $\mathcal{F}(\mathcal{H})$ the category $\mathcal{H}^{\sigma\mathcal{F}}$ of the functors in $\mathcal{H}$. It is equal to the category $\mathcal{H}^{\bar{\sigma}\mathcal{F}}$; indeed, if $\phi\colon \Sigma_{\mathcal{G}} \to \mathcal{H}$ is a functor, $\phi\gamma_3$ is a pullback, γ_3 being an absolute pullback, and, $\phi(\iota)$ being a monomorphism, $\phi\bar{\gamma}_1$ is a pullback iff $\phi\gamma_1$ is a pullback in $\mathcal{H}$.

If ψ is a category in the dual of $\mathcal{H}$, the dual functor $\psi^*\colon \Sigma^*_{\mathcal{G}} \to \mathcal{H}$ of ψ is called a *cocategory in* $\mathcal{H}$.

There exists a unique category δ in $\Sigma_{\mathcal{G}}$ mapping ι and κ on themselves and interchanging α and β, ν and ν'. If ϕ is a category in $\mathcal{H}$, then $\phi\delta$ is also a category in $\mathcal{H}$; we denote it by ϕ_* and call it the *dual* of ϕ. We get the «duality isomorphism» from $\mathcal{F}(\mathcal{H})$ onto $\mathcal{F}(\mathcal{H})$ by sending ϕ onto ϕ_* and the functor (ϕ', τ, ϕ) in $\mathcal{H}$ onto (ϕ'_*, τ, ϕ_*).

3. The categories $\mathcal{F}$ and $\mathcal{F}(\mathcal{M})$ are equivalent [E3, BE]. We will use the following canonical equivalences:

a) If C is a small category, there exists a unique category in $\mathfrak{M}$, denoted by $\eta_1(C)$ and called *the category in* $\mathfrak{M}$ *associated to* C, which transforms the pullbacks γ_1 and γ_2 into canonical pullbacks in $\mathfrak{M}$ and which maps α, β, κ and ι respectively onto the maps source, target, law of composition of C, and insertion from C_0 into C.

If $f: A \to C$ is a functor, $\eta_1(f)$ will denote the unique natural transformation (or functor internal to $\mathfrak{M}$)

$$\eta_1(f): \eta_1(A) \to \eta_1(C) \quad \text{such that} \quad \eta_1(f)(2) = \underline{f}.$$

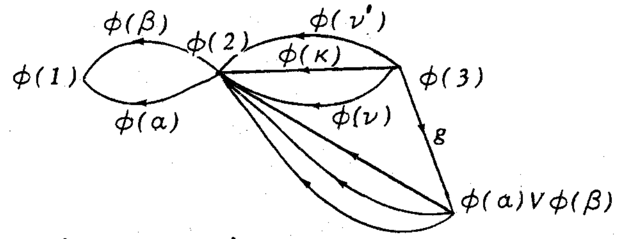
In this way, we get an equivalence $\eta_1: \mathcal{F} \to \mathcal{F}(\mathfrak{M})$. This equivalence admits as a restriction an isomorphism from $\mathcal{F}$ onto the full sub-category of $\mathcal{F}(\mathfrak{M})$ whose objects are the categories in $\mathfrak{M}$ mapping γ_1 and γ_2 on canonical pullbacks in $\mathfrak{M}$ and ι on an insertion.

b) On the other hand, we have an equivalence ζ_1 from $\mathcal{F}(\mathfrak{M})$ onto $\mathcal{F}$, which maps:

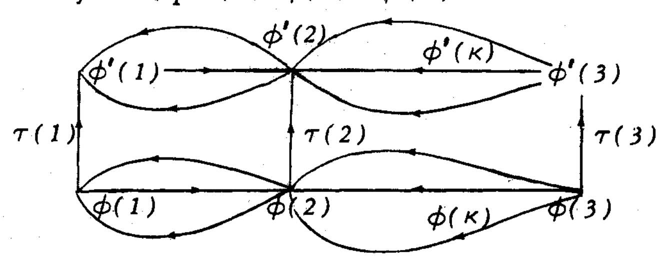
the category ϕ in $\mathfrak{M}$ on the category $\zeta_1(\phi)$, called the *category associated to* ϕ, whose underlying set is $\phi(2)$ and whose law of composition if $\phi(\kappa).g^{-1}$, where g is the bijection:

$$x \mapsto (\phi(\nu)(x), \phi(\nu')(x))$$

from $\phi(3)$ onto the canonical pullback of $(\phi(\alpha), \phi(\beta))$,



the functor $\tau: \phi \to \phi'$ internal to $\mathfrak{M}$ on the functor from $\zeta_1(\phi)$ to $\zeta_1(\phi')$ defined by the map $\tau(2): \phi(2) \to \phi'(2)$.



In particular, if M is a small set, the constant functor $M^{\wedge}: \Sigma_{\mathcal{F}} \to \mathfrak{M}$ is a category in $\mathfrak{M}$ whose associated category is the discrete category M^0.

4. *The functor toward $\mathcal{F}$ associated to a category in $\mathcal{H}$.*

PROPOSITION 8. *Let $\mathcal{H}$ be a category admitting pullbacks. The category $\mathcal{F}(\mathcal{H})$ is equivalent to the full sub-category $\mathcal{K}$ of $\mathcal{F}^{\mathcal{H}^*}$ whose objects are the functors $\phi: \mathcal{H}^* \to \mathcal{F}$ whose composite $p_{\mathcal{F}}\phi$ with the forgetful functor $p_{\mathcal{F}}: \mathcal{F} \to \mathfrak{M}$ is representable.*

Δ. Since $\Sigma_{\mathcal{F}}$ is the $\overline{\Gamma}_{\mathcal{F}}$-closure of $\{2\}$ and $\mathcal{H}^{\sigma\mathcal{F}} = \mathcal{H}^{\overline{\sigma}\mathcal{F}}$, Proposition 3 asserts that $\mathcal{F}(\mathcal{H}) = \mathcal{H}^{\sigma\mathcal{F}}$ is isomorphic with the full sub-category $\mathcal{R}$ of $\mathcal{F}(\mathfrak{M})^{\mathcal{H}^*}$ whose objects are the functors $\psi: \mathcal{H}^* \to \mathcal{F}(\mathfrak{M})$ such that $\nu\psi$ is representable, $\nu: \mathcal{F}(\mathfrak{M}) \to \mathfrak{M}$ denoting the functor value in 2 which sends τ onto $\tau(2)$. If $\zeta_1: \mathcal{F}(\mathfrak{M}) \to \mathcal{F}$ is the equivalence constructed in 3 above, the composite functor

$$\mathcal{F}(\mathfrak{M}) \xrightarrow{\zeta_1} \mathcal{F} \xrightarrow{p_{\mathcal{F}}} \mathfrak{M}$$

is equal to ν, so that $\nu\psi$ is representable iff $p_{\mathcal{F}}\zeta_1\psi$ is representable. The equivalence

$$\zeta_1^{\mathcal{H}^*}: \mathcal{F}(\mathfrak{M})^{\mathcal{H}^*} \to \mathcal{F}^{\mathcal{H}^*}$$

associating $\zeta_1\psi$ to $\psi: \mathcal{H}^* \to \mathcal{F}(\mathfrak{M})$, it admits as a restriction an equivalence from $\mathcal{R}$ onto the full sub-category $\mathcal{K}$ of $\mathcal{F}^{\mathcal{H}^*}$. Hence $\mathcal{F}(\mathcal{H})$ and $\mathcal{K}$ are equivalent. ∇

5. *The canonical cocategory in $\mathcal{F}$.*

If n is a natural integer, the composite functor

$$\Sigma_{\mathcal{F}} \hookrightarrow \Delta^* \xrightarrow{\Delta(n,\cdot)} \mathfrak{M}$$

is a category in $\mathfrak{M}$, since the pullbacks γ_1 and γ_2 in $\Sigma_{\mathcal{F}}$ are also pullbacks in Δ^*. Its associated category is the category **n** defining the canonical order of the ordinal $n = \{0, \dots, n-1\}$; the morphisms of **n** are the pairs (m', m) of integers such that $m \leqslant m' < n$.

If $f: n \to m$ is a morphism of $\Sigma_{\mathcal{F}}$, i.e. if f defines an increasing map from $(n, \leqslant)$ to $(m, \leqslant)$, the composite natural transformation

$$\Sigma_{\mathcal{F}} \hookrightarrow \Delta^* \underset{}{\overset{\Delta(f,\cdot)}{\rightrightarrows}} \mathfrak{M}$$

is a functor internal to $\mathfrak{M}$, to which is associated the functor $\mathbf{f}$:

$$(j,i) \to (f(j), f(i)) \quad \text{from } \mathbf{n} \text{ to } \mathbf{m}$$

(defined by the map $\Delta(f, 2)$).

PROPOSITION 9. *There exists a cocategory in* $\mathcal{F}$ *admitting as a restriction an isomorphism from* $\Sigma^*_{\mathcal{F}}$ *onto the full sub-category* $\overline{\Sigma}^*_{\mathcal{F}}$ *of* $\mathcal{F}$ *whose objects are* **1, 2, 3** *and* **4**. $\mathcal{F}$ *is the inductive closure of* $\{\mathbf{2}\}$.

Δ. From Proposition 4, it follows that the Yoneda embedding Y_1 from $\Sigma^*_{\mathcal{F}}$ to $\mathfrak{M}^{\Sigma_{\mathcal{F}}}$ admits as a restriction a cocategory $\overline{Y}_1$ in $\mathcal{F}(\mathfrak{M})$ and that $\mathcal{F}(\mathfrak{M}) = \mathfrak{M}^{\overline{\sigma}\mathcal{F}}$ is the inductive closure of $\{Y_1(2)\}$. As ζ_1 is an equivalence, $\mathcal{F}$ is the inductive closure of $\{\zeta_1 \overline{Y}_1(2)\}$ and the composite $\zeta_1 \overline{Y}_1$:

$$\Sigma^*_{\mathcal{F}} \xrightarrow{\overline{Y}_1} \mathcal{F}(\mathfrak{M}) \xrightarrow{\zeta_1} \mathcal{F}$$

is a cocategory in $\mathcal{F}$. It admits as a restriction an isomorphism from $\Sigma^*_{\mathcal{F}}$ onto the full sub-category of $\mathcal{F}$ whose objects are the categories

$$\zeta_1 \overline{Y}_1(n), \quad \text{where} \quad n \in \{1, 2, 3, 4\}.$$

So, it remains only to prove that the category $\zeta_1 \overline{Y}_1(n)$ is identical with $\mathbf{n}$. Indeed, this category is the category associated to the category in $\mathfrak{M}$:

$$Y_1(n) = \Sigma_{\mathcal{F}}(\cdot, n) : \Sigma_{\mathcal{F}} \to \mathfrak{M}.$$

Since $\Sigma_{\mathcal{F}}$ is a full sub-category of Δ^*, we have $Y_1(n)$ equal to the composite functor:

$$\Sigma_{\mathcal{F}} \hookrightarrow \Delta^* \xrightarrow{\Delta(n,\cdot)} \mathfrak{M},$$

to which is associated, by definition, the category $\mathbf{n}$. ∇

REMARK. The above constructed cocategory in $\mathcal{F}$ is a restriction of the canonical embedding of the simplicial category Δ into $\mathcal{F}$, which defines $\mathcal{F}$ as a category admitting as models the categories $\mathbf{n}$, for all the integers n. The corresponding «singular functor» from $\mathcal{F}$ to the category $\mathcal{S}$ of sim-

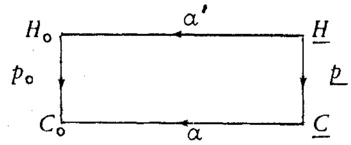
plicial maps sends a category C onto the corresponding simplicial object; the homology of this simplicial object is called the homology of C [Gr]. The singular functor admits an adjoint, the realization functor, which associates to a simplicial object F the category canonically associated to F; the groupoid projection of this category is the fundamental groupoid of F
1+ (see [GZ]).

D. Internal discrete fibrations.

1. It is known [E1] that the three following notions are equivalent, where C is a category:

a) A functor from C to the category $\mathfrak{M}$ of maps.

b) A discrete fibration (or hypermorphism functor [E1]) over C, which is a functor $p: H \to C$ such that

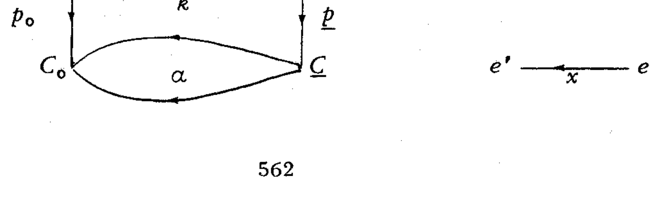


is a pullback, where α and α' are the source maps of C and H and p_0 the restriction of p to the objects; this means that, if s is an object of H and $x: p(s) \to e'$ a morphism in C, there exists one and only one morphism y in H admitting s as its source and satisfying $p(y) = x$.

c) A left action k' of C on a set A, also called a category action
2 (or an operator category on A, or a species of structures in [E1]): then k' is a map $(x, s) \mapsto xs$ from a sub-set K' of $\underline{C} \times \underline{A}$ to A satisfying the following axioms: there exists a map $p_0: A \to C_0$ such that K' is the canonical pullback of (α, p_0) and that:

$es = s$ if $s \in A$ and $e = p_0(s)$,

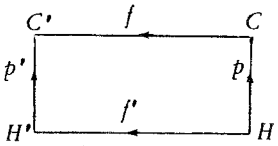
$x'(xs) = (x'.x)s$ if $x'.x$ exists in C and if $(x,s) \in K'$



(the map p_0 is uniquely determined by these conditions, which imply that $p_0(xs)$ is the target of x). The associated discrete fibration is the functor $p: C*A \to C$, where $C*A$ is the category on K' such that:

$$(x', s').(x, s) = (x'.x, s) \quad \text{iff } x'.x \text{ exists in } C \text{ and } s' = xs.$$

2. We denote by $\square\mathfrak{F}$ the horizontal category of commutative squares (or quartets [E1]) of the category $\mathfrak{F}$ of small functors whose objects are the small functors, the morphisms from p to p' being the commutative squares (p', f, f', p).



We denote by $\mathfrak{A}$ the full sub-category of $\square\mathfrak{F}$ whose objects are the discrete fibrations; its morphisms are called morphisms between discrete fibrations.

The category $\mathfrak{A}$ is equivalent to the category of covariant maps between category actions (see [E1]).

We denote by $p_{\mathfrak{A}}$ and $p_{\mathring{\mathfrak{A}}}$ the functors from $\mathfrak{A}$ to $\mathfrak{M}$ sending the morphism (p', f, f', p) respectively onto the map $\underline{f}$ defining f and onto the map $f'_0 : H_0 \to H'_0$ restriction of f' to the objects.

$\mathfrak{F}$ will be identified with the full sub-category of $\mathfrak{A}$ whose objects are the identical fibrations.

Let C be a small category. $\mathfrak{A}$ admits as a «non-full» sub-category the category $\mathfrak{A}_C$ of morphisms over C, whose elements are the morphisms (p', f, f', p) such that f is the identity of C (such a morphism identifies with the triangle (p', f', p)). There exists an equivalence from $\mathfrak{M}^C$ toward $\mathfrak{A}_C$ which sends a functor $\phi : C \to \mathfrak{M}$ onto the discrete fibration h_ϕ, from H_ϕ to C, associated to it (the morphisms of H_ϕ are the pairs

$$(x, s), \text{ where } x \in C \text{ and } s \in \phi(\alpha(x)),$$

and $h_\phi(x, s) = x$).

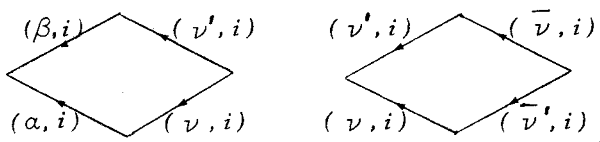
$\mathfrak{A}_C$ is also equivalent to the category of covariant maps over C.

3. *The sketch of discrete fibrations.*

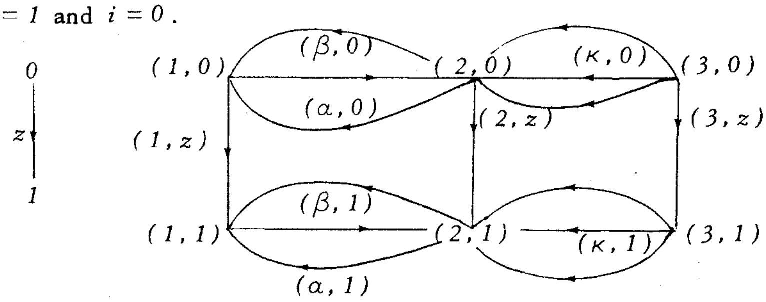
We denote by **2** the category

$$1 \xleftarrow{\;z\;} 0$$

as well as the limit-bearing category on **2** without any cone. The tensor product $\sigma\mathcal{F} \otimes \mathbf{2}$ is the category $\Sigma\mathcal{F} \times \mathbf{2}$ equipped with the pullbacks



for $i = 1$ and $i = 0$.



PROPOSITION 10. *There is a canonical equivalence which is surjective*

$$\zeta' : \mathfrak{M}^{\sigma\mathcal{F}\otimes\mathbf{2}} \xrightarrow{\sim} \square\mathcal{F}.$$

Δ. Let $\psi : \Sigma\mathcal{F} \times \mathbf{2} \to \mathfrak{M}$ be a $\sigma\mathcal{F}\otimes\mathbf{2}$-structure in $\mathfrak{M}$. Then

$$\psi(-,1) : \Sigma\mathcal{F} \to \mathfrak{M} \quad \text{and} \quad \psi(-,0) : \Sigma\mathcal{F} \to \mathfrak{M}$$

are categories in $\mathfrak{M}$; let C and H be the associated categories. The map $\psi(2,z)$ defines a functor $\zeta'(\psi)$ from H to C.

If $\tau : \psi \to \psi'$ is a $\sigma\mathcal{F}\otimes\mathbf{2}$-morphism, then

$$\zeta'(\psi) : H \to C \quad \text{and} \quad \zeta'(\psi') : H' \to C'$$

are functors and the maps $\tau(2,1)$ and $\tau(2,0)$ define, respectively, functors $f : C \to C'$ and $f' : H \to H'$. Then $\zeta'(\tau)$ is the commutative square

$$(\zeta'(\psi'), f, f', \zeta'(\psi)). \quad \nabla$$

DEFINITION. We define the *sketch of discrete fibrations* as the limit-bearing category $\sigma_\phi = (\Sigma_\phi, \Gamma_\phi)$ got by equipping the category $\Sigma_\phi = \Sigma\mathcal{J} \times \mathbf{2}$ with the set $\Gamma\mathcal{J} \otimes \emptyset$ of the distinguished cones of $\sigma\mathcal{J} \otimes \mathbf{2}$ and the pullback

$$\gamma_4 : \quad \begin{array}{ccc} (1,0) & \xleftarrow{(a,0)} & (2,0) \\ {\scriptstyle (1,z)} \downarrow & & \downarrow {\scriptstyle (2,z)} \\ (1,1) & \xleftarrow[(a,1)]{} & (2,1) \end{array}$$

Let $\overline{\Gamma}_\phi$ be the set $(\overline{\Gamma}\mathcal{J} \otimes \emptyset) \cup \{\gamma_4\}$ of 7 cones, among them the absolute pullbacks $[\gamma_3, 0^\cdot]$ and $[\gamma_3, 1^\cdot]$, and $\overline{\sigma}_\phi = (\Sigma_\phi, \overline{\Gamma}_\phi)$. Then Σ_ϕ is the $\overline{\Gamma}_\phi$-closure of $\{(2,1),(1,0)\}$, since $\Sigma\mathcal{J}$ is the $\overline{\Gamma}\mathcal{J}$-closure of $\{2\}$ and γ_4 is a pullback. Moreover $\mathfrak{M}^{\sigma_\phi} = \mathfrak{M}^{\overline{\sigma}_\phi}$.

PROPOSITION 11. *The category $\mathfrak{A}$ is equivalent to $\mathfrak{M}^{\sigma_\phi}$ and it is the inductive closure of $\{\mathbf{1}, \mathfrak{2}\}$, where $\mathfrak{2}$ is the void fibration from $\emptyset$ to $\mathbf{2}$.*

Δ. 1° If $\psi : \Sigma\mathcal{J} \times \mathbf{2} \to \mathfrak{M}$ is a $\sigma\mathcal{J} \otimes \mathbf{2}$-structure in $\mathfrak{M}$, it is a σ_ϕ-structure iff it maps γ_4 on a pullback in $\mathfrak{M}$, i.e. iff the functor $\zeta'(\psi)$ is a discrete fibration, where ζ' is the equivalence defined in Proposition 10. Hence ζ' admits as a restriction an equivalence ζ'' from the full sub-category $\mathfrak{M}^{\sigma_\phi}$ of $\mathfrak{M}^{\sigma\mathcal{J} \otimes \mathbf{2}}$ onto the full sub-category $\mathfrak{A}$ of $\square\square\mathcal{F}$.

2° Since $\mathfrak{M}^{\sigma_\phi} = \mathfrak{M}^{\overline{\sigma}_\phi}$ and Σ_ϕ is the $\overline{\Gamma}_\phi$-closure of $\{(1,0),(2,1)\}$, by Proposition 4, the category $\mathfrak{M}^{\sigma_\phi}$ is the inductive closure of

$$\{ Y(1,0),\ Y(2,1) \}, \quad \text{where} \quad Y : \Sigma^*_\phi \to \mathfrak{M}^{\sigma_\phi}$$

is the Yoneda embedding. Using the equivalence ζ'', we deduce that $\mathfrak{A}$ is the inductive closure of $\{ \zeta'' Y(1,0),\ \zeta'' Y(2,1) \}$.

As $(1,0)$ is an initial object of Σ_ϕ, it is mapped by Y on a final object of $\mathfrak{M}^{\sigma_\phi}$, and by $\zeta'' Y$ on a final object of $\mathfrak{A}$. Hence $\zeta'' Y(1,0)$ is isomorphic with the identical fibration $\mathbf{1}$.

The category associated to $Y(2,1)(\cdot, 1) : \Sigma\mathcal{J} \to \mathfrak{M}$ is $\mathbf{2} \times \{1\}$, for

$$Y(2,1)(m,1) = \Sigma_\phi((m,1),(2,1)) = \Sigma\mathcal{J}(m,2) \times \{1\}$$

for each $m \in \{1,2,3,4\}$. In the same way, the category associated to $Y(2,1)(\cdot, 0) : \Sigma\mathcal{J} \to \mathfrak{M}$ is void. Therefore, $\zeta'' Y(2,1)$ is the discrete fibration from the void category to $\mathbf{2} \times \{1\}$ (isomorphic to $\mathbf{2}$). This fibration is isomorphic in $\mathcal{A}$ with the fibration $\mathbf{2}$.

If follows that $\mathcal{A}$ is the inductive closure of $\{1, \mathbf{2}\}$. ∇

4. Discrete fibrations in a category $\mathcal{H}$.

We suppose that $\mathcal{H}$ is a category admitting pullbacks. A σ_ϕ-structure in $\mathcal{H}$ is called *a discrete fibration in* $\mathcal{H}$. We denote by $\mathcal{A}(\mathcal{H})$ the category of σ_ϕ-morphisms in $\mathcal{H}$, which is equal to $\mathcal{H}^{\overline{\sigma}\phi}$.

PROPOSITION 12. *1° $\mathcal{A}(\mathcal{H})$ is equivalent to the full sub-category $\mathcal{R}$ of $\mathcal{A}^{\mathcal{H}^*}$ whose objects are the functors $\rho : \mathcal{H}^* \to \mathcal{A}$ such that $p_{\mathcal{A}}\rho$ and $\mathring{p}_{\mathcal{A}}\rho$ are representable (where $p_{\mathcal{A}}$ and $\mathring{p}_{\mathcal{A}}$ are the forgetful functors from $\mathcal{A}$ to $\mathcal{M}$ defined in 2).*

2° If ψ and ψ' are two discrete fibrations in $\mathcal{H}$ such that

$$\psi(\cdot, 1) = \psi'(\cdot, 1) : \Sigma_{\mathcal{F}} \to \mathcal{H}, \quad \psi'\gamma_4 = \psi\gamma_4, \quad \psi(\beta, 0) = \psi'(\beta, 0),$$

then ψ and ψ' are isomorphic in $\mathcal{A}(\mathcal{H})$.

Δ. 1° As $\mathcal{A}(\mathcal{H}) = \mathcal{H}^{\overline{\sigma}\phi}$ and Σ_ϕ is the $\overline{\Gamma}_\phi$-closure of $\{(1,0), (2,1)\}$, Proposition 3 asserts that $\mathcal{A}(\mathcal{H})$ is equivalent to the full sub-category $\mathcal{R}'$ of $\mathcal{A}(\mathcal{M})^{\mathcal{H}^*}$ whose objects are the functors $\rho' : \mathcal{H}^* \to \mathcal{A}(\mathcal{M})$ such that $q\rho'$ and $q^\circ\rho'$ are representable, where q° and q are the value functors from $\mathcal{A}(\mathcal{M})$ to $\mathcal{M}$ associating to τ respectively $\tau(1,0)$ and $\tau(2,1)$. If ζ'' is the canonical equivalence (Proposition 11), then

q is the composite functor $\mathcal{A}(\mathcal{M}) \xrightarrow{\zeta''} \mathcal{A} \xrightarrow{p_{\mathcal{A}}} \mathcal{M}$,

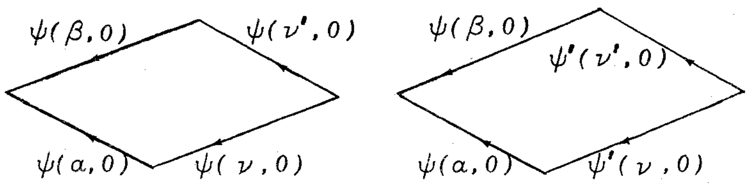
q° is the composite functor $\mathcal{A}(\mathcal{M}) \xrightarrow{\zeta''} \mathcal{A} \xrightarrow{\mathring{p}_{\mathcal{A}}} \mathcal{M}$.

It follows that a functor $\rho' : \mathcal{H}^* \to \mathcal{A}(\mathcal{M})$ is an object of $\mathcal{R}'$ iff the functor $\zeta''\rho'$ is such that $p_{\mathcal{A}}\zeta''\rho'$ and $\mathring{p}_{\mathcal{A}}\zeta''\rho'$ are representable, i.e. iff $\zeta''\rho'$ is an object of the category $\mathcal{R}$ defined in the Proposition. Hence the equivalence $\zeta''^{\mathcal{H}^*} : \mathcal{A}(\mathcal{M})^{\mathcal{H}^*} \to \mathcal{A}^{\mathcal{H}^*}$ admits as a restriction an equivalence from $\mathcal{R}'$ to $\mathcal{R}$. Finally, $\mathcal{A}(\mathcal{H})$ is equivalent to $\mathcal{R}$.

2° Let ψ and ψ' be discrete fibrations in $\mathcal{H}$ satisfying

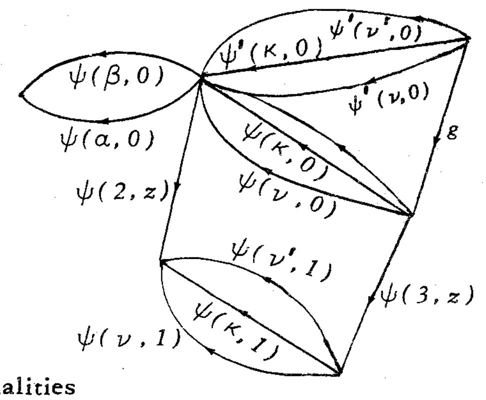
$$\psi(\cdot, 1) = \psi'(\cdot, 1), \qquad \psi\gamma_4 = \psi'\gamma_4, \qquad \psi(\beta, 0) = \psi'(\beta, 0).$$

Since



are two pullbacks, there exists an unique isomorphism g of $\mathcal{H}$ such that:

$$\psi(\nu_i, 0).\, g = \psi'(\nu_i, 0) \quad \text{for } \nu_0 = \nu \text{ and } \nu_1 = \nu'.$$



From the equalities

$$(\nu_i, 1).(3, z) = (2, z).(\nu_i, 0)$$

for $i = 0$ and $i = 1$, we deduce

$$\psi'(\nu_i, 1).\psi'(3, z) = \psi'(2, z).\psi'(\nu_i, 0) = \psi'(2, z).\psi(\nu_i, 0).\, g =$$
$$= \psi(2, z).\psi(\nu_i, 0).\, g = \psi'(\nu_i, 1).\psi(3, z).\, g$$

for i equal to 0 and 1. This implies (unicity of the factor relative to a pullback):

$$\psi'(3, z) = \psi(3, z).\, g.$$

In the same way, from

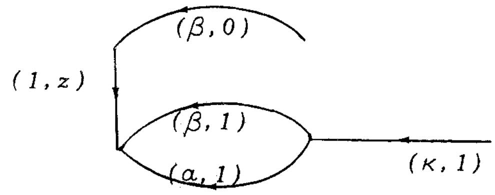
$$(\alpha, 0).(\kappa, 0) = (\alpha, 0).(\nu, 0) \quad \text{and} \quad (2, z).(\kappa, 0) = (\kappa, 1).(3, z),$$

we get

$$\psi'(\kappa, 0) = \psi(\kappa, 0).\, g,$$

the functors ψ and ψ' taking the same values on $(\alpha, 0)$, on $(2, z)$ and on $(\kappa, 1)$. It follows that the categories $\psi(\cdot, 0)$ and $\psi'(\cdot, 0)$ in $\mathcal{H}$ are equivalent, whence ψ and ψ' are equivalent, i.e. isomorphic in $\mathcal{Q}(\mathcal{H})$. ∇

The preceding proof shows that σ_ϕ admits as its «idea»



Hence a discrete fibration ψ in $\mathcal{H}$ is determined up to an isomorphism by $(\psi(-,1), \psi(1,z), \psi(\beta,0))$. This leads to the following definition:

DEFINITION. We say that *(ϕ, h, k') is a category action in $\mathcal{H}$* if:

1° ϕ is a category in $\mathcal{H}$,

2° h and k' are morphisms of $\mathcal{H}$,

3° there exists a discrete fibration ψ in $\mathcal{H}$ such that

$$\psi(-,1) = \phi, \quad \psi(1,z) = h, \quad \psi(\beta,0) = k'.$$

If ϕ is a category in $\mathcal{H}$, let ϕ^* be its dual (section C-2). If we have a category action (ϕ_*, h, k') in $\mathcal{H}$, we also say that (k', h, ϕ) is *a right category action in* $\mathcal{H}$.

EXEMPLE. Category actions were introduced in [E4] as an axiomatization of the notion of a fiber-bundle. Indeed, topological (resp. r-differentiable) fiber-bundles correspond exactly to the category actions in the category $\mathcal{T}$ of continuous maps (resp. $\mathcal{D}^r$ of r-differentiable maps between manifolds) such that the operating topological (resp. differentiable) category be a locally trivial groupoid [E4, 5]. 1

5. *Distributors in* $\mathcal{H}$.

If B and C are categories, the following notions are equivalent:

a) A distributor from B to C, which is defined [B1] as a functor from $C^* \times B$ to $\mathcal{M}$.

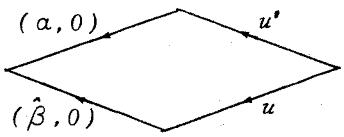
b) A pair of category actions on a set (introduced in [E1] under the name of «couple de catégories d'opérateurs»), i. e. a pair

$$((B, A, \kappa'), (C^*, A, \kappa''))$$

of category actions such that

$(xs)x' = x(sx')$ whenever the composites
$xs = \kappa'(x, s)$ and $sx' = \kappa''(x', s)$ are defined.

This last notion is easily «internalized» in a category $\mathcal{H}$ admitting pullbacks by defining a limit-bearing category σ_δ whose realizations in $\mathfrak{M}$ are pairs of category actions on a set. We will not formally construct σ_δ here; its description is given in [V]. Intuitively, it is got by gluing together along $(1, 0)$ the sketch of a discrete fibration and the sketch of a discrete fibration over the dual of a category (in which (μ, y) is replaced by $(\hat{\mu}, y)$), and by adding the pullback

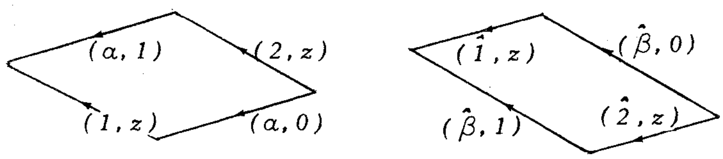


,

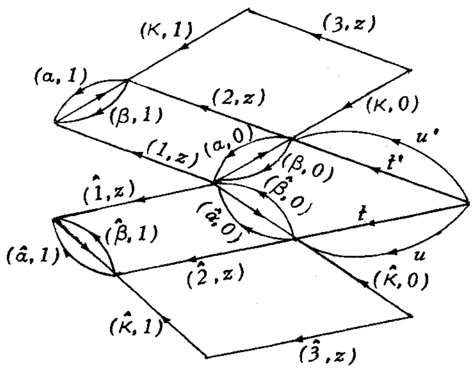
and the factors

$$t' = [(2, z).u', (\hat{\alpha}, 0).u] \quad \text{and} \quad t = [(\beta, 0).u', (\hat{2}, z).u]$$

relative to the pullbacks



of $((\alpha, 1), (1, z))$ and $((\hat{1}, z), (\hat{\beta}, 1))$,
the axiom $(\hat{\alpha}, 0).t = (\beta, 0).t'$.



The realizations of σ_δ in $\mathcal{H}$ are called [B1] distributors in $\mathcal{H}$:

DEFINITION. Let $\mathcal{H}$ be a category admitting pullbacks. A *distributor in* $\mathcal{H}$ is defined as a sextuple $(\phi', h', k'', k', h, \phi)$, where:

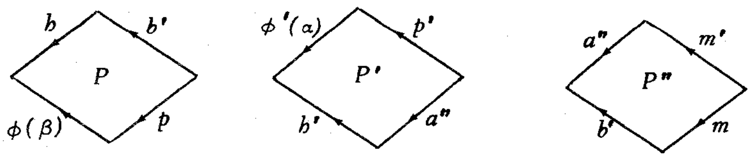
1° (ϕ', h', k'') is a category action in $\mathcal{H}$,

2° (k', h, ϕ) is a right category action in $\mathcal{H}$,

3° $k''.\,l' = k'.\,l$, where

$$l = [\,k''.\,m', p.\,m\,] \quad \text{and} \quad l' = [\,p'.\,m', k'.\,m\,]$$

are the factors relative to the pullbacks respectively P of $(h, \phi(\beta))$ and P' of $(\phi'(\alpha), h')$, where we have the pullbacks:



$\mathfrak{D}$ denoting the category $\mathfrak{M}^{\sigma_\delta}$ of morphisms between distributors, it follows from Propositions 2 and 12 that $\mathcal{H}^{\sigma_\delta}$ is equivalent to the full sub-category of $\mathfrak{D}^{\mathcal{H}^*}$ whose objects are the $\psi: \mathcal{H}^* \to \mathfrak{D}$ such that $\psi(-)(\omega)$ is representable, for $\omega \in \{(\hat{2},1), (2,1), (1,0)\}$.

REMARKS. 1° To a distributor $\delta: C^* \times B \to \mathfrak{M}$ is associated a functor Δ, from B to $\mathfrak{M}^{C^*}$ and, since $\mathfrak{M}^{C^*}$ and $\mathfrak{A}_{C^*}$ are equivalent, a functor from B to $\mathfrak{A}_{C^*}$. More generally, problems in Differential Geometry and in Analysis led to consider functors from a category B to $\mathfrak{A}$. Such a functor associates to each $e \in B_0$ a category action (C_e, A_e, k'_e); then B operates on the category sum of the C_e and on the set A sum of the A_e. This situation is easily internalized in $\mathcal{H}$ and enriched by giving supplementary structures on the A_e. In fact, it was this more general notion (suggested by that of a sheaf of operators on a sheaf) which was first introduced (in [AB] to define distributions on infinite dimensional vector spaces) under the name of «catégorie de catégories d'opérateurs» and which is studied in [E1,5]
1 (and called espèce de structures dominée par des applications covariantes).

2° Distributors are the 1-morphisms of a bicategory (see [B1]), for a
2 law which can be suggested by that of the category of atlases of a category defined in [E6].

1. THE CATEGORY OF DOUBLE FUNCTORS

A. Double categories.

1. In this section, we recall the initial «naive» definition of double categories, as it is given in [E2].

DEFINITION. *A double category* is defined as a pair $(\Sigma^o, \Sigma^\cdot)$ of categories with the same set of morphisms, satisfying the following conditions:

1° The maps source and target of $\Sigma^\cdot$ define functors from Σ^o onto a sub-category of Σ^o.

2° The law of composition of $\Sigma^\cdot$ defines a functor toward Σ^o from the sub-category of $\Sigma^o \times \Sigma^o$ formed by the pairs of morphisms composable in the category $\Sigma^\cdot$.

$(\Sigma^o, \Sigma^\cdot)$ is then called a double category *on* Σ, and the categories Σ^o and $\Sigma^\cdot$ are respectively its *first category* and its *second category*. A double category on Σ is said *small* if Σ is a small set.

In [E2] it is shown that the axioms 1 and 2 are equivalent to the following ones, where α, β and α^o, β^o denote the maps source and target in $\Sigma^\cdot$ and in Σ^o respectively:

1° For each $d \in \Sigma$, we have

$$\alpha(\alpha^o(d)) = \alpha^o(\alpha(d)), \quad \alpha(\beta^o(d)) = \beta^o(\alpha(d)),$$
$$\beta(\alpha^o(d)) = \alpha^o(\beta(d)), \quad \beta(\beta^o(d)) = \beta^o(\beta(d)).$$

2° If the composite $d' \circ d$ exists in Σ^o, then

$$\alpha(d' \circ d) = \alpha(d') \circ \alpha(d) \quad \text{and} \quad \beta(d' \circ d) = \beta(d') \circ \beta(d);$$

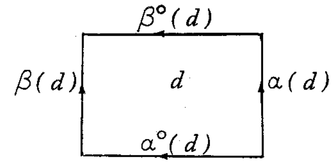
if the composite $\hat{d}.d$ exists in $\Sigma^\cdot$, then

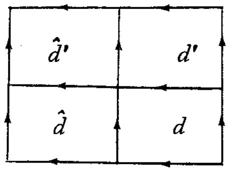
$$\alpha^o(\hat{d}.d) = \alpha^o(\hat{d}).\alpha^o(d) \quad \text{and} \quad \beta^o(\hat{d}.d) = \beta^o(\hat{d}).\beta^o(d).$$

3° Permutability axiom: If the composites $d' \circ d$, $\hat{d}' \circ \hat{d}$, $\hat{d}.d$, $\hat{d}'.d'$ are defined, then the composites

$$(\hat{d}' \circ \hat{d}).(d' \circ d) \quad \text{and} \quad (\hat{d}'.d') \circ (\hat{d}.d)$$

are defined and both are equal.





This set of axioms being symmetrical relative to $\Sigma^{\cdot}$ and to Σ°,, it follows that $(\Sigma^{\circ}, \Sigma^{\cdot})$ is a double category iff $(\Sigma^{\cdot}, \Sigma^{\circ})$ is a double category; these two double categories are said *symmetrical.*

2. *Notations.*

A double category $(\Sigma^{\circ}, \Sigma^{\cdot})$ is generally denoted by a unique italic letter, for example D. In that case:

The underlying set Σ is denoted by $\underline{D}$.

The first category Σ° is also denoted by D^1, its symbol of composition by $\circ_1$ (instead of $\circ$), its mappings source, target and law of composition by α^1, β^1 and κ^1.

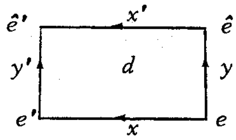
The second category $\Sigma^{\cdot}$ is denoted by D^2, its symbol of composition by $\circ_2$ (instead of $.$), its mappings source, target and law of composition by α^2, β^2 and κ^2.

The set of objects of D^1 defines a sub-category of D^2, which is denoted by $D^1_{\circ}$ and called the *second category of 1-morphisms of* D.

The set of objects of D^2 defines a sub-category of D^1, which is denoted by $D^2_{\circ}$ and called *the first category of 1-morphisms of* D.

The categories $D^1_{\circ}$ and $D^2_{\circ}$ have the same set of objects, which is written $D_{\circ\circ}$ and called *the set of vertices of* D. The elements of $\underline{D}$ which are not objects for D^1 nor D^2 are called *2-blocks of* D.

Let d be a 2-block of D. As a morphism of D^1, it admits a source $x = \alpha^1(d)$ and a target $x' = \beta^1(d)$ and we write $d: x \to x'$. As a morphism of D^2, it admits a source $y = \alpha^2(d)$ and a target $y' = \beta^2(d)$ and we write $d: y \rightrightarrows y'$.



3. *Examples.*

a) 2-categories are defined as the double categories D such that the objects of D^2 are also objects of D^1, so that $D_{oo} = D_o^2 \subset D_o^1$; a 2-block of D is then called a 2-cell. For example, we have the 2-category of natural transformations (between small categories), denoted by $(\mathfrak{N}^{\square\square}, \mathfrak{N}^{\cdot})$, or $\mathfrak{N}$, whose second category of 1-morphisms is $\mathcal{F}$.

b) If Σ^o is a category and Σ^0 the discrete category on Σ, then the pair (Σ^o, Σ^0) is a double category, called the *discrete double category on* Σ^o. Similarly, (Σ^0, Σ^o) is a 2-category, called the *discrete 2-category on* Σ^o.

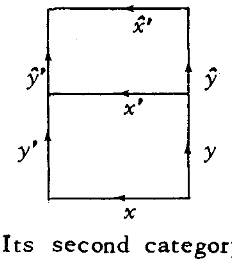
c) If A is a category, (A, A) is a double category iff A is a commutative category, i. e. a category coproduct of commutative monoids.

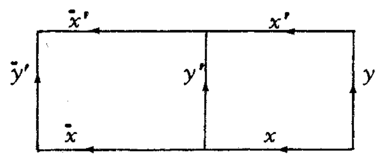
If D is a double category such that D_o^1 and D_o^2 are discrete categories, then $D^1 = D^2$.

d) Let A be a category. We denote by $\square A$ *the double category of commutative squares of* A. Its underlying set is the set of commutative squares (or quartets) of A, which are the 4-tuples (y', x', x, y) such that the composites $y'.x$ and $x'.y$ are defined and equal.

Its first category, denoted by $\boxminus A$, is called the *vertical category of squares*; its law of composition is:

$$(\hat{y}', \hat{x}', \hat{x}, \hat{y}) \boxminus (y', x', x, y) = (\hat{y}'.y', \hat{x}', x, \hat{y}.y) \text{ iff } x' = \hat{x}.$$





Its second category, denoted by $\square\square A$, is called the *horizontal category of squares*; its law of composition is:

$$(\bar{y}', \bar{x}', \bar{x}, \bar{y}) \square\square (y', x', x, y) = (\bar{y}', \bar{x}'.x', \bar{x}.x, y) \text{ iff } \bar{y} = y'.$$

There is an isomorphism $(y', x', x, y) \mapsto (x', y', y, x)$ from $\square\square A$ onto $\boxminus A$.

With similar laws, the set of all (non commutative) squares of A also becomes a double category.

e) Let D be a double category. If $\underline{C}$ is a sub-set of $\underline{D}$ which defines a sub-category C^1 of D^1 and a sub-category C^2 of D^2, then (C^1, C^2) is a double category C, called the *double sub-category of* D defined by $\underline{C}$ (or by C^1, or by C^2).

In particular, among all the double sub-categories of D which are 2-categories, there is a greatest one, namely that defined by the full sub-category of D^2 whose objects are the vertices of D.

The full sub-category of D^1 whose objects are all the vertices of D also defines a double sub-category of D, whose symmetrical double category is the greatest sub-2-category of the symmetrical of D.

f) Let D be a double category. Then,

$$(D^{1*}, D^2), \quad (D^1, D^{2*}) \quad \text{and} \quad (D^{1*}, D^{2*})$$

are double categories, called respectively *the first dual*, the *second dual* and the *dual of* D.

4. *Double functors.*

DEFINITION. We say that (D, ϕ, C) is a *double functor* if C and D are double categories and if ϕ is a map from $\underline{C}$ to $\underline{D}$ defining a functor from C^1 to D^1 and a functor from C^2 to D^2.

A double functor (D, ϕ, C) will often be denoted by an italic letter f. In that case:

the map ϕ is also denoted by $\underline{f}$,
the functor (D^1, ϕ, C^1) by f^1,
the functor (D^2, ϕ, C^2) by f^2.

Moreover, we say that

$$f: C \to D \quad \text{is a double functor,}$$

or that ϕ defines a double functor from C to D.

EXAMPLES. a) The double functors between 2-categories are called 2-functors.

b) Let Σ^{o} and Σ'^{o} be categories. A map $\phi : \Sigma \to \Sigma'$ defines a functor $f: \Sigma^{o} \to \Sigma'^{o}$ iff it defines a double functor from the discrete double category (Σ^{o}, Σ^{0}) on Σ^{o} toward the discrete double category on Σ'^{o}. In that case, there exists a double functor from the double category of commutative squares $\square\Sigma^{o}$ to $\square\Sigma'^{o}$ defined by the restriction « to the commutative squares» of the product map $\phi\times\phi\times\phi\times\phi$. This double functor is denoted by $\square f$.

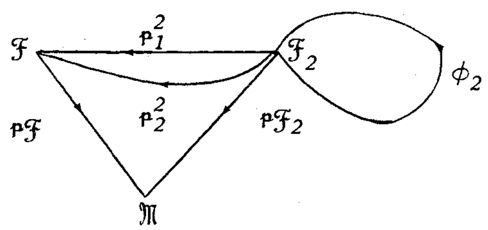
The double functors between small double categories are the morphisms of the *category* $\mathfrak{F}_2$ *of (small) double functors*, whose objects are the small double categories.

This category is equipped with the following forgetful functors:

$p_1^2: \mathfrak{F}_2 \to \mathfrak{F}$, which associates to the double functor f the functor f^1,

$p_2^2: \mathfrak{F}_2 \to \mathfrak{F}$, which associates f^2 to f,

$p_{\mathfrak{F}_2}: \mathfrak{F}_2 \to \mathfrak{M}$, which associates to f the underlying map $\underline{f}$.



Moreover, there is an isomorphism $\phi_2: \mathfrak{F}_2 \to \mathfrak{F}_2$, which is its own inverse, mapping the double category D on its symmetrical one (denoted D^{21}) and associating

$$(D^{21}, \phi, C^{21}) \text{ to } (D, \phi, C) \in \mathfrak{F}_2 .$$

We have the equality $p_1^2 \phi_2 = p_2^2$.

B. Double categories as sketched structures.

Double categories may be considered both as categories in $\mathfrak{F}$ or as $\sigma_{\mathfrak{F}}\otimes\sigma_{\mathfrak{F}}$-structures in $\mathfrak{M}$ (called double categories in $\mathfrak{M}$).

1. *Categories in* $\mathfrak{F}$.

Let $\mathfrak{F}(\mathfrak{F})$ be the category of functors in(ternal to) $\mathfrak{F}$ (this category has been defined in 0-C).

PROPOSITION 1. *The category* $\mathcal{F}_2$ *of double functors is equivalent to the category* $\mathcal{F}(\mathcal{F})$ *and isomorphic to a full sub-category of* $\mathcal{F}(\mathcal{F})$.

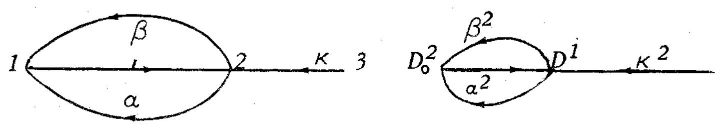
Δ. We are going to construct two canonical equivalences, which will be used later on.

1° a) Let D be a small double category. There exists a unique functor $\eta_{11}(D): \Sigma_{\mathcal{F}} \to \mathcal{F}$ mapping the two distinguished cones γ_1 and γ_2 of $\sigma_{\mathcal{F}}$ on canonical pullbacks in $\mathcal{F}$ and associating to the morphisms ι, α, β and κ of $\sigma_{\mathcal{F}}$ respectively:

the insertion from the sub-category D_0^2 of D^1 into D^1,

the functors from D^1 to D_0^2 defined by the mappings source and target α^2 and β^2 of D^2.

the functor defined by the law of composition κ^2 of D^2 from the sub-category $(D^2 * D^2)^1$ of $D^1 \times D^1$ on the set of composable pairs of D^2 to D^1.



Hence, $\eta_{11}(D)$ is the unique category in $\mathcal{F}$ such that $\eta_{11}(D)(2)$ is the first category D^1 and that $p_{\mathcal{F}}\,\eta_{11}(D)$ is the category $\eta_1(D^2)$ in $\mathfrak{M}$ associated to the second category D^2 (cf. 0-C-3).

b) If $f: C \to D$ is a double functor, we have a unique functor

$$\eta_{11}(f): \eta_{11}(C) \to \eta_{11}(D)$$

internal to $\mathcal{F}$ such that $\eta_{11}(f)(2) = f^1$.

c) We have so defined a functor η_{11}:

$$f \mapsto \eta_{11}(f) \text{ from } \mathcal{F}_2 \text{ to } \mathcal{F}(\mathcal{F}).$$

It satisfies $p_{\mathcal{F}}^{\sigma_{\mathcal{F}}}\,\eta_{11} = \eta_1$, where $p_{\mathcal{F}}^{\sigma_{\mathcal{F}}}$ is the functor:

$$\tau \mapsto p_{\mathcal{F}}\,\tau \text{ from } \mathcal{F}(\mathcal{F}) \text{ to } \mathcal{F}(\mathfrak{M}).$$

Since η_1 admits as a restriction an isomorphism from $\mathcal{F}$ onto a full sub-category of $\mathcal{F}(\mathfrak{M})$, the functor η_{11} admits as a restriction an isomorphism from $\mathcal{F}_2$ onto the full sub-category $\mathcal{F}(p_{\mathcal{F}})$ of $\mathcal{F}(\mathcal{F})$ whose objects are the categories ϕ in $\mathcal{F}$ such that $p_{\mathcal{F}}\,\phi$ is the category $\eta_1(\Sigma)$ in $\mathfrak{M}$ associated to a category Σ (such a category in $\mathfrak{F}$ is called a $p\mathfrak{F}$-structured category in [E2]).

2° We define now a functor ζ_{11} from $\mathfrak{F}(\mathfrak{F})$ onto $\mathfrak{F}_2$.

a) Let $\phi: \Sigma_{\mathfrak{F}} \to \mathfrak{F}$ be a category in $\mathfrak{F}$. Then $\phi(2)$ is a category Σ° and $p\mathfrak{F}\phi$ is a category in $\mathfrak{M}$; the associated category $\zeta_1(p\mathfrak{F}\phi)$ (defined in 0-C-3) is denoted by $\Sigma^{\cdot}$. The pair $(\Sigma^\circ, \Sigma^{\cdot})$ is a double category, whose image by η_{11} is a category in $\mathfrak{F}$ equivalent to ϕ (by its construction). In particular, noting $(\Sigma^\circ, \Sigma^{\cdot})$ by $\zeta_{11}(\phi)$, we have

$$\zeta_{11}(\eta_{11}(C)) = C \quad \text{for each double category } C.$$

b) If $\tau: \phi \to \phi'$ is a functor in $\mathfrak{F}$, then

$$\zeta_{11}(\tau) = (\zeta_{11}(\phi'), \tau(2), \zeta_{11}(\phi))$$

is a double functor; in this way we have defined a surjective functor ζ_{11} from $\mathfrak{F}(\mathfrak{F})$ to $\mathfrak{F}$. The composite functor

$$\zeta_{11}\eta_{11}: \mathfrak{F}_2 \xrightarrow[\eta_{11}]{} \mathfrak{F}(\mathfrak{F}) \xrightarrow[\zeta_{11}]{} \mathfrak{F}_2$$

is an identity functor, while

$$\eta_{11}\zeta_{11}: \mathfrak{F}(\mathfrak{F}) \longrightarrow \mathfrak{F}(\mathfrak{F})$$

is equivalent to an identity. ∇

COROLLARY. *$\mathfrak{F}_2$ is equivalent to the category* $\mathfrak{M}^{\sigma\mathfrak{F}\otimes\sigma\mathfrak{F}}$.

Δ. From Proposition 6-0, we know that $\mathfrak{M}^{\sigma\mathfrak{F}\otimes\sigma\mathfrak{F}}$ is isomorphic with $(\mathfrak{M}^{\sigma\mathfrak{F}})^{\sigma\mathfrak{F}}$; this last category is equivalent to $\mathfrak{F}^{\sigma\mathfrak{F}} = \mathfrak{F}(\mathfrak{F})$, and therefore to $\mathfrak{F}_2$, according to the Proposition. ∇

DEFINITION. If D is a double category, $\eta_{11}(D)$ is called *the category in $\mathfrak{F}$ associated to D*. If ϕ is a category in $\mathfrak{F}$, then $\zeta_{11}(\phi)$ is called *the double category associated to ϕ.*

2. *The sketch of double categories.*

Since $\mathfrak{F}_2$ is equivalent to $\mathfrak{M}^{\sigma\mathfrak{F}\otimes\sigma\mathfrak{F}}$, it is natural to give the

DEFINITION. The tensor product $\sigma\mathfrak{F}\otimes\sigma\mathfrak{F}$ is called *the sketch of double* *categories*; it is denoted by $\sigma\mathcal{F}_2$ and its underlying category by $\Sigma\mathcal{F}_2$.- A $\sigma\mathcal{F}_2$-structure (resp. -morphism) in a category $\mathcal{H}$ is called a *double category* (resp. a double *functor) in* $\mathcal{H}$.

The category $\mathcal{H}^{\sigma\mathcal{F}_2}$ of double functors in $\mathcal{H}$ is denoted by $\mathcal{F}_2(\mathcal{H})$. It is equal to $\mathcal{H}^{\bar{\sigma}\mathcal{F}\otimes\bar{\sigma}\mathcal{F}}$, since $(\mathcal{H}^{\bar{\sigma}\mathcal{F}})^{\bar{\sigma}\mathcal{F}} = (\mathcal{H}^{\sigma\mathcal{F}})^{\sigma\mathcal{F}}$. Proposition 7-0 asserts that $\Sigma\mathcal{F}_2$ is the $\bar{\Gamma}\mathcal{F}\otimes\bar{\Gamma}\mathcal{F}$-closure of $\{(2,2)\}$.

PROPOSITION 2. *There exist a surjective equivalence* $\zeta_2 : \mathcal{F}_2(\mathcal{M}) \to \mathcal{F}_2$ *and an equivalence* $\eta_2 : \mathcal{F}_2 \to \mathcal{F}_2(\mathcal{M})$ *such that* $\zeta_2\eta_2$ *be an identity.*

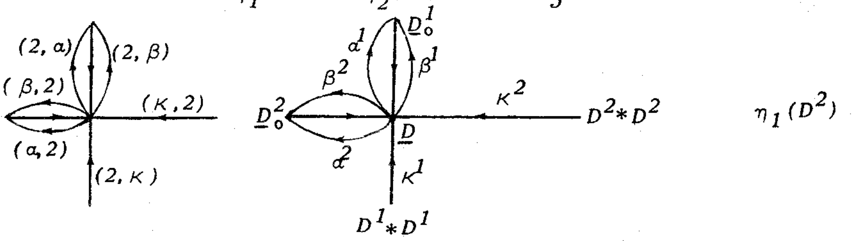
Δ. From Proposition 1 and from 0-C-3, we get the equivalence η_2:

$$\mathcal{F}_2 \xrightarrow{\eta_{11}} \mathcal{F}(\mathcal{F}) \xrightarrow{\eta_1^{\sigma\mathcal{F}}} (\mathcal{M}^{\sigma\mathcal{F}})^{\sigma\mathcal{F}} \xrightarrow{\sim} \mathcal{M}^{\sigma\mathcal{F}\otimes\sigma\mathcal{F}} = \mathcal{F}_2(\mathcal{M}).$$

which is constructed as follows:

If D is a small double category, $\eta_2(D)$ is the unique double category in $\mathcal{M}$ mapping the distinguished cones of $\sigma\mathcal{F}_2$ on canonical pullbacks in $\mathcal{M}$, mapping the morphisms (ι, n) and (n, ι), for $n \in \{1, 2, 3, 4\}$, on insertions and such that

$$\eta_1(D^1) = \eta_2(D)(2, \cdot) : \Sigma\mathcal{F} \to \mathcal{M},$$
$$\eta_1(D^2) = \eta_2(D)(\cdot, 2) : \Sigma\mathcal{F} \to \mathcal{M}.$$



If $f : C \to D$ is a double functor, $\eta_2(f)$ is the unique double functor

$$\tau : \eta_2(C) \to \eta_2(D) \text{ in } \mathcal{M}$$

such that $\tau(2,2)$ is the map $\underline{f}$ defining f.

We construct now a surjective equivalence $\zeta_2 : \mathcal{F}_2(\mathcal{M}) \to \mathcal{F}_2$:

If $\phi : \Sigma\mathcal{F}_2 \to \mathcal{M}$ is a double category in $\mathcal{M}$, then

$$\phi_1 = \phi(2, \cdot) : \Sigma\mathcal{F} \to \mathcal{M} \text{ and } \phi_2 = \phi(\cdot, 2) : \Sigma\mathcal{F} \to \mathcal{M}$$

are categories in $\mathcal{M}$, and the pair of their associated categories

$$\zeta_2(\Phi) = (\zeta_1(\Phi_1), \zeta_1(\Phi_2))$$

is a double category on $\Phi(2,2)$.

If $\tau : \Phi \to \Phi'$ is a double functor in $\mathfrak{M}$, the map $\tau(2,2)$ defines a double functor $\zeta_2(\tau) : \zeta_2(\Phi) \to \zeta_2(\Phi')$.

We have so defined the functor $\zeta_2 : \mathcal{F}_2(\mathfrak{M}) \to \mathcal{F}_2$.

Since $\zeta_1 \eta_1 : \mathcal{F} \to \mathcal{F}$ is an identity, the functor $\zeta_2 \eta_2$:

$$\mathcal{F}_2 \xrightarrow{\eta_2} \mathcal{F}_2(\mathfrak{M}) \xrightarrow{\zeta_2} \mathcal{F}_2$$

is an identity, and $\eta_2(\mathcal{F}_2)$ defines a full sub-category of $\mathcal{F}_2(\mathfrak{M})$, isomorphic with $\mathcal{F}_2$. ∇

3. General results about σ-structures in $\mathfrak{M}$ may be applied to the category $\mathcal{F}_2$, according to Proposition 2. In particular:

PROPOSITION 3. *1° $\mathcal{F}_2$ is a category admitting small projective limits and small inductive limits.*

2° The forgetful functor toward $\mathfrak{M}$ as well as the two forgetful functors p_1^2 and p_2^2 toward $\mathcal{F}$ preserve projective limits and filtered inductive limits.

3° The forgetful functor toward $\mathfrak{M}$ admits quasi-quotient structures, i. e. [E1] if D is a small double category on $\underline{D}$ and r an equivalence on the set $\underline{D}$, there exists a small double category quasi-quotient of D by r.

These results are deduced in [BE1] from general theorems about internal categories (which would also apply to $\mathcal{F}_2(\mathcal{H})$).

C. Categories of generalized natural transformations.

If D is a double category and A a category, the functors from A to the first category of 1-morphisms of D are the objects of a category, denoted by $T(D,A)$, whose law is deduced from that of the second category D^2 underlying D. Functors from a category B to $T(D,A)$ may be identified with double functors toward D from the «square product» $B \blacksquare A$.

1. *The functor T_{11}.*

PROPOSITION 4. *There exists a functor $T_{11} : \mathcal{F}_2 \times \mathcal{F}^* \to \mathcal{F}$ mapping the ob-*

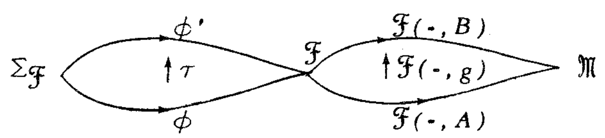
ject (D, A) *onto the category* $T(D, A)$ *got by equipping the set of functors from* A *to* D^1 *with the law:*

$$t' \circ_2 t \text{ is defined iff } \alpha^2 \underline{t}' = \beta^2 \underline{t} \text{ and is then equal to the functor}$$
$$a \mapsto t'(a) \circ_2 t(a) \text{ from } A \text{ to } D^1.$$

Δ. 1° From Proposition 1-0 there exists a functor

$$\theta : \mathcal{F}(\mathcal{F}) \times \mathcal{F}^* \to \mathcal{F}(\mathfrak{M})$$

mapping $(\tau, g) \in \mathcal{F}(\mathcal{F}) \times \mathcal{F}$ onto the natural transformation $\mathcal{F}(\cdot, g) . \tau$:



(where $g : B \to A$). We denote by T_{11} the composite functor:

$$\mathcal{F}_2 \times \mathcal{F}^* \xrightarrow{\eta_{11} \times \mathcal{F}^*} \mathcal{F}(\mathcal{F}) \times \mathcal{F}^* \xrightarrow{\theta} \mathcal{F}(\mathfrak{M}) \xrightarrow{\zeta_1} \mathcal{F} .$$

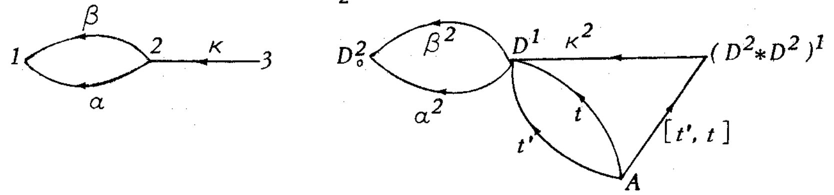
2° Let D be a small double category and A a small category. Then $T_{11}(D, A)$ is the category $T(D, A)$ associated to $\mathcal{F}(-, A)\ \phi : \Sigma_{\mathcal{F}} \to \mathfrak{M}$, where ϕ is the category in $\mathcal{F}$ associated to D:

- Its set of morphisms is $\mathcal{F}(\phi(2), A) = \mathcal{F}(D^1, A) = L$.
- Its law $\mathcal{F}(\phi(\kappa), A)$ is defined on the pullback

$$\mathcal{F}(\phi(\alpha), A) \vee \mathcal{F}(\phi(\beta), A) = \{ (t', t) \in L \times L \mid \alpha^2 \underline{t}' = \beta^2 \underline{t} \},$$

and it maps (t', t) onto the functor $\phi(\kappa) . [t', t]$:

$$a \mapsto t'(a) \circ_2 t(a) \text{ from } A \text{ to } D^1.$$

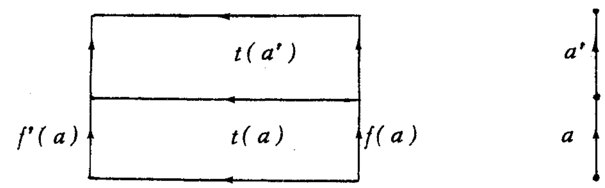


3° Let $h : D \to E$ be a double functor and $g : B \to A$ a functor. $T_{11}(h, g)$ is the functor from $T(D, A)$ to $T(E, B)$ defined by the map

$$\mathcal{F}(-, g)\eta_{11}(h)(2) = \mathcal{F}(-, g)(h^1) = \mathcal{F}(h^1, g),$$

which associates $h^1 t g$ to $t \in \mathcal{F}(D^1, A)$. ∇

DEFINITION. The category $T(D, A)$ defined above is called *the category of D-wise transformations from A to D*. A functor $t: A \to D^1$ is called a *D-wise transformation from f to f'*, if f is its source and f' its target in $T(D, A)$.



$\mathcal{F}(D_o^2, A)$ is the set of objects of $T(D, A)$. This definition, given in [E2] (where $T(D, A)$ was constructed directly), has been inspired by the following example:

EXAMPLES. 1° Let B be a category, $\square B$ the double category of its commutative squares. If A is a category, $T(\square B, A)$ is identified with the category B^A of natural transformations, by identifying a functor from A to $\boxminus B$ (i.e. a $\square B$-wise transformation) with a natural transformation between functors from A to B.

2° For any double category D, the category $T(D, \mathbf{2})$ is isomorphic with D^2.

2. *The square product of categories.*

We are going to construct an adjoint to the «partial» functor

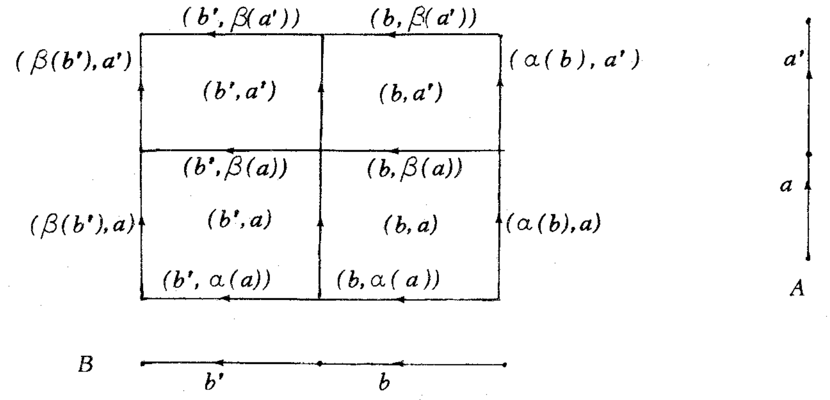
$$T_{11}(-, A): \mathcal{F}_2 \to \mathcal{F}, \text{ for each small category } A.$$

DEFINITION. Let A and B be categories. We call the *square product of* (B, A), denoted by $B \blacksquare A$, the double category $(\underline{B}^0 \times A, B \times \underline{A}^0)$ (where $\underline{A}^0$ and $\underline{B}^0$ are the discrete categories on the sets of morphisms of A and B respectively).

$B \blacksquare A$ is a double category, since it is the product in $\mathcal{F}_2$ of the double categories $(\underline{B}^0, B)$ and $(A, \underline{A}^0)$. Its laws are:

$(b', a') \circ_1 (b, a) = (b, a'.a)$ iff $b' = b$ and $a'.a$ exists in A,

$(b', a') \circ_2 (b, a) = (b'.b, a)$ iff $a' = a$ and $b'.b$ exists in B. 1



REMARK. If we identify the block (b, a) (sometimes written $b \blacksquare a$) with its frame

$$(\beta(b) \blacksquare a,\ b \blacksquare \beta(a),\ b \blacksquare \alpha(a),\ \alpha(b) \blacksquare a),$$

we get an isomorphism from $B \blacksquare A$ onto a double sub-category of the double category $\square(B \times A)$.

DEFINITION. We say that $(D, \phi, (B, A))$ is an *alternative double functor*, or that ϕ defines an alternative double functor *from* (B, A) *to* D if:

1° A and B are categories on $\underline{A}$ and $\underline{B}$;

2° D is a double category on $\underline{D}$ and $\phi: \underline{B} \times \underline{A} \to \underline{D}$ a map;

3° the partial map $\phi(b, -): \underline{A} \to \underline{D}$ defines a functor from A to D^1 for every b in B;

4° the partial map $\phi(-, a): \underline{B} \to \underline{D}$ defines a functor from B to D^2 for every a in A.

PROPOSITION 5. *Let A and B be categories on $\underline{A}$ and $\underline{B}$. The double category $B \blacksquare A$ is characterized by each of the following conditions:*

1° If D is a double category, a map $\phi: \underline{B} \times \underline{A} \to \underline{D}$ defines an alternative double functor from (B, A) to D iff ϕ defines a double functor from $B \blacksquare A$ to D.

2° $B \blacksquare A$ is a free object associated to B relative to the partial functor $T_{11}(-, A): \mathfrak{F}_2 \to \mathfrak{F}$.

Δ. 1° Let D be a double category and $\phi: \underline{B} \times \underline{A} \to \underline{D}$ a map.

a) The category $\underline{B}^0 \times A$ being the coproduct category $\coprod_{b \in B} \{b\} \times A$, the map ϕ defines a functor from $\underline{B}^0 \times A$ to D^1 iff the map

$$\phi(b,-): a \mapsto \phi(b,a) \text{ from } \underline{A} \text{ to } \underline{D}$$

defines a functor from A to D^1, for each b in B. In the same way, since $B \times \underline{A}^0 = \coprod_{a \in A} B \times \{a\}$, the map ϕ defines a functor from $B \times \underline{A}^0$ to D^2 iff $\phi(-,a)$ defines a functor from B to D^2 for each a in A. Hence $B \blacksquare A$ satisfies the first property.

b) Suppose that $\phi(b,-)$ defines a functor $f'(b): A \to D^1$ for each b in B. The map

$$f': b \mapsto f'(b) \text{ from } B \text{ to } \mathcal{F}(D^1, A)$$

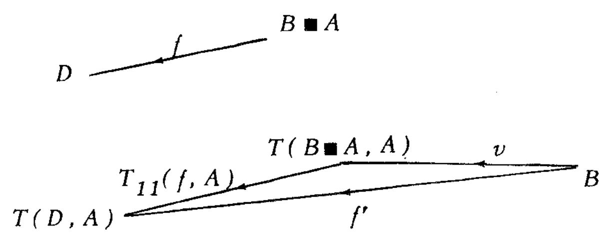
defines a functor from B to $T(D,A)$ iff:

- For each object e of B, $f'(e)$ is an object of $T(D,A)$, which means that $f'(e)(a) = \phi(e,a)$ is an object of D^2, for any a in A.
- For each composite $b'.b$ in B, we have $f'(b'.b) = f'(b') \circ_2 f'(b)$, i.e. $\phi(b'.b, a) = \phi(b', a) \circ_2 \phi(b, a)$ for each a in A.

These conditions are equivalent to say that $\phi(-,a)$ defines a functor from B to D^2 for each a in A. In view of Part a, they are verified iff ϕ defines a double functor from $B \blacksquare A$ to D.

2° By the preceding method, we associate to the identity of $B \blacksquare A$ a functor $v: B \to T(B \blacksquare A, A)$ such that $v(b)$ be the functor

$$a \mapsto (b,a) \text{ from } A \text{ to } (B \blacksquare A)^1, \text{ for each } b \in B.$$



If $f': B \to T(D,A)$ is a functor, it follows from Part 1-b that the map ϕ:

$$(b,a) \mapsto f'(b)(a) \text{ from } \underline{B} \times \underline{A} \text{ to } \underline{D}$$

defines a double functor $f: B \blacksquare A \to D$. Then, the functor $T_{11}(f,A).v$, from B to $T(D,A)$, maps b onto the functor

$$T_{11}(f, A)(v(b)) = f^1 v(b) : A \to D^1 .$$

which associates $f(b, a) = f'(b)(a)$ to $a \in A$, and hence is equal to $f'(b)$. So v defines $B \blacksquare A$ as a free object associated to B relative to the functor $T_{11}(\cdot, A)$. ∇

COROLLARY 1. *Let A, B and C be categories. There are bijections*

$$\mathfrak{F}_2(\square C, B \blacksquare A) \to \mathfrak{F}(C^A, B) \to \mathfrak{F}(C, B \times A).$$

Δ. This results from Proposition 5, since C^A is isomorphic with the category $T(\square C, A)$. The canonical composite bijection maps $f: B \blacksquare A \to \square C$ onto the functor g:

$$(b, a) \mapsto f(b, \beta(a)).f(\alpha(b), a) \text{ from } B \times A \text{ to } C. \quad \nabla$$

COROLLARY 2. *Let A and B be categories. If D is a double category, there are canonical bijections:*

$$\mathfrak{F}(T(D^{21}, B), A) \xrightarrow{\sim} \mathfrak{F}_2(D^{21}, A \blacksquare B) \xrightarrow{\sim} \mathfrak{F}_2(D, B \blacksquare A) \xrightarrow{\sim} \mathfrak{F}(T(D, A), B).$$

Δ. Since $(B \blacksquare A)^{21} = (B \times \underline{A}^0, \underline{B}^0 \times A)$, there exists an isomorphism

$$h: (b, a) \mapsto (a, b) \text{ from } (B \blacksquare A)^{21} \text{ onto } A \blacksquare B,$$

and

$$\mathfrak{F}_2(D^{21}, h): \mathfrak{F}_2(D^{21}, A \blacksquare B) \xrightarrow{\sim} \mathfrak{F}_2(D^{21}, (B \blacksquare A)^{21})$$

is a bijection l. Now, by sending a double functor from $(B \blacksquare A)^{21}$ to D^{21} onto the functor from $B \blacksquare A$ to D defined by the same map we get a bijection

$$l': \mathfrak{F}_2(D^{21}, (B \blacksquare A)^{21}) \xrightarrow{\sim} \mathfrak{F}_2(D, B \blacksquare A).$$

From Proposition 5, there are canonical bijections

$$l'': \mathfrak{F}(T(D^{21}, B), A) \xrightarrow{\sim} \mathfrak{F}_2(D^{21}, A \blacksquare B),$$

$$l''': \mathfrak{F}_2(D, B \blacksquare A) \xrightarrow{\sim} \mathfrak{F}(T(D, A), B).$$

Composing all these bijections, we get the bijection

$$\gamma_{B,A} = l''' l' l l'': \mathfrak{F}(T(D^{21}, B), A) \xrightarrow{\sim} \mathfrak{F}(T(D, A), B),$$

which sends the functor $f': A \to T(D^{21}, B)$ onto the functor f'', from B to $T(D, A)$, such that $f''(b)$ be the functor

$$a \mapsto f'(a)(b) \text{ from } A \text{ to } D^1. \quad \nabla$$

COROLLARY 3. *$T_{11}(D, \cdot): \mathfrak{F}^* \to \mathfrak{F}$ is coadjoint to the dual of $T_{11}(D^{21}, \cdot)$*

for each double category D.

Δ. The canonical bijections $\gamma_{B,A}$ defined above determine an equivalence $\gamma: \mathfrak{F}(T_{11}(D^{21},=),-) \to \mathfrak{F}^*(=, T_{11}(D,-)): \mathfrak{F}^* \times \mathfrak{F}^* \to \mathfrak{M}$. ∇

3. *The functor* $\blacksquare : \mathfrak{F} \times \mathfrak{F} \to \mathfrak{F}_2$.

PROPOSITION 6. *There exists a functor* $\blacksquare$ *from* $\mathfrak{F}\times\mathfrak{F}$ *to* $\mathfrak{F}_2$ *such that the partial functor* $-\blacksquare A$ *be an adjoint of* $T_{11}(-,A)$ *for each small category* A. *If* $\mathfrak{F}_{\mathcal{C}}$ *denotes the full sub-category of* $\mathfrak{F}$ *whose objects are the small connected categories, then* $\blacksquare$ *maps* $\mathfrak{F}_{\mathcal{C}} \times \mathfrak{F}_{\mathcal{C}}$ *onto a full sub-category of* $\mathfrak{F}_2$. 1

Δ. 1° If $g: A \to A'$ and $h: B \to B'$ are functors, the product map $\underline{h}\times\underline{g}$ defines a double functor $h \blacksquare g: B \blacksquare A \to B' \blacksquare A'$. We so define the functor

$$\blacksquare : (h, g) \mapsto h \blacksquare g \text{ from } \mathfrak{F}\times\mathfrak{F} \text{ to } \mathfrak{F}_2 .$$

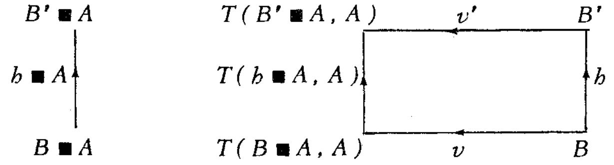
2° The «canonical» adjoint of $T_{11}(-,A): \mathfrak{F}_2 \to \mathfrak{F}$ maps $h: B \to B'$ onto the double functor $h': B \blacksquare A \to B' \blacksquare A$ associated to the functor $v'h$, where

$$v: B \to T(B \blacksquare A, A) \text{ and } v': B' \to T(B' \blacksquare A, A),$$

are the functors defining $B \blacksquare A$ and $B' \blacksquare A$ as free objects. As $v'h$ maps $b \in B$ onto the functor

$$a \mapsto (h(b), a) \text{ from } A \text{ to } \underline{B}'^{0} \times A,$$

the functor h' maps (b, a) onto $(h(b), a)$, and $h' = h \blacksquare A$. Hence the partial functor $-\blacksquare A: \mathfrak{F} \to \mathfrak{F}_2$ is the canonical adjoint of $T_{11}(-,A)$.



3° Let A, B, A' and B' be small connected categories and suppose that $f: B \blacksquare A \to B' \blacksquare A'$ is a double functor. Since A and A' are connected, the components

- of $\underline{B}^0 \times A$ are the sets $\{b\}\times\underline{A}$, where $b \in B$,
- of $\underline{B}'^0 \times A'$ are the sets $\{b'\}\times\underline{A}'$, where $b' \in B'$.

The functor $f^1: \underline{B}^0 \times A \to \underline{B}'^0 \times A'$ mapping a component into a component, for each $b \in B$, there exists a unique $b'_b \in B'$ such that

$$f(\{b\} \times \underline{A}) \subset \{b'_b\} \times \underline{A}'.$$

In the same way $\underline{f}$ defining a functor $f^2 : B \times \underline{A}^0 \to B' \times \underline{A}'^0$, for each $a \in A$ there exists a unique $a'_a \in A'$ such that

$$f(\underline{B} \times \{a\}) \subset \underline{B}' \times \{a'_a\}.$$

Hence

$$f(b, a) = (b'_b, a'_a) \quad \text{for each} \quad (b, a) \in B \times A,$$

which implies $\underline{f} = \underline{h} \times \underline{g}$, where the map

$\underline{h} : \underline{B} \to \underline{B}'$ associates b'_b to $b \in B$,

$\underline{g} : \underline{A} \to \underline{A}'$ associates a'_a to $a \in A$.

These maps define functors $h : B \to B'$ and $g : A \to A'$, and $f = h \blacksquare g$. ∇

D. Some applications.

1. *The canonical double cocategory in* $\mathcal{F}_2$.

PROPOSITION 7. *1° There exists a double cocategory* ι_2 *in* $\mathcal{F}_2$ *which defines an isomorphism from* $\Sigma\mathcal{F}_2^*$ *onto the full sub-category* $\overline{\Sigma}\mathcal{F}_2^*$ *of* $\mathcal{F}_2$ *whose objects are the double categories* $\mathbf{m} \blacksquare \mathbf{n}$, *for* m *and* n *in* $\{1, 2, 3, 4\}$.

2° $\mathcal{F}_2$ *is the inductive closure of* $\{\mathbf{2} \blacksquare \mathbf{2}\}$.

Δ. 1° From Proposition 4-0, the restriction $\overline{Y}_2 : \Sigma\mathcal{F}_2^* \to \mathcal{F}_2(\mathcal{M})$ of the Yoneda embedding is a double cocategory in $\mathcal{F}_2(\mathcal{M})$. The composite

$$\iota_2 : \Sigma\mathcal{F}_2^* \xrightarrow{\overline{Y}_2} \mathcal{F}_2(\mathcal{M}) \xrightarrow{\zeta_2} \mathcal{F}_2,$$

where ζ_2 is the canonical equivalence (Proposition 2), is a double cocategory in $\mathcal{F}_2$, and $\overline{Y}_2$ defines an isomorphism from $\Sigma\mathcal{F}_2^*$ onto a full sub-category of $\mathcal{F}_2(\mathcal{M})$ which is mapped by the surjective equivalence ζ_2 onto a full sub-category of $\mathcal{F}_2$. The equivalence ζ_2 being faithful, so is ι_2.

2° We are going to prove that

$$\mathbf{m} \blacksquare \mathbf{n} = \iota_2(m, n), \quad \text{for } m \text{ and } n \text{ in } \{1, 2, 3, 4\}.$$

Indeed, $\Sigma\mathcal{F}_2 = \Sigma\mathcal{F} \times \Sigma\mathcal{F}$, so that $\overline{Y}_2(m, n) : \Sigma\mathcal{F}_2 \to \mathcal{M}$ maps the pair

$$(\mu, \nu) \in \Sigma\mathcal{F}_2 \quad \text{onto}$$

$$\Sigma_{\mathcal{F}_2}((\mu,\nu),(m,n)) = \Sigma_{\mathcal{F}}(\mu,m)\times\Sigma_{\mathcal{F}}(\nu,n) =$$
$$= Y_1(m)(\mu)\times Y_1(n)(\nu),$$

where $Y_1 : \Sigma_{\mathcal{F}}^* \to \mathfrak{M}^{\Sigma_{\mathcal{F}}}$ is the Yoneda embedding. From the construction of ζ_2 it follows that $\iota_2(m,n)$ is the double category

$$(\zeta_1(Y_1(m)(2)\times Y_1(n)(\cdot)),\ \zeta_1(Y_1(m)(\cdot)\times Y_1(n)(2))),$$

where $\zeta_1 : \mathcal{F}(\mathfrak{M}) \to \mathcal{F}$ is the canonical equivalence (Proposition 8-0). The set $Y_1(m)(2)$ is the set $\underline{\mathbf{m}}$ underlying $\mathbf{m}$ and $\zeta_1(Y_1(n))$ is the category $\mathbf{n}$ (see 0-C), so that

$$\zeta_1(Y_1(m)(2)\times Y_1(n)(\cdot)) = \underline{\mathbf{m}}^0\times\mathbf{n}.$$

In the same way

$$\zeta_1(Y_1(m)(\cdot)\times Y_1(n)(2)) = \mathbf{m}\times\underline{\mathbf{n}}^0.$$

Hence

$$\iota_2(m,n) = (\underline{\mathbf{m}}^0\times\mathbf{n},\ \mathbf{m}\times\underline{\mathbf{n}}^0) = \mathbf{m}\,\blacksquare\,\mathbf{n}.$$

3° The preceding results imply that ι_2 maps $\Sigma_{\mathcal{F}_2}$ onto the full subcategory of $\mathcal{F}_2$ whose objects are the double categories $\mathbf{m}\,\blacksquare\,\mathbf{n}$. As

$$\mathbf{m}\,\blacksquare\,\mathbf{n} = \mathbf{m}'\,\blacksquare\,\mathbf{n}' \text{ iff } m = m' \text{ and } n = n',$$

the faithful functor ι_2 is injective on the objects, whence injective.

4° $\Sigma_{\mathcal{F}}$ being the $\Gamma_{\mathcal{F}}$-closure of $\{2\}$ (see 0-C), Proposition 7-0 asserts that $\mathcal{F}_2(\mathfrak{M}) = \mathfrak{M}^{\sigma_{\mathcal{F}}\otimes\sigma_{\mathcal{F}}} = \mathfrak{M}^{\overline{\sigma}_{\mathcal{F}}\otimes\overline{\sigma}_{\mathcal{F}}}$ is the inductive closure of the set $\{\overline{Y}_2(2,2)\}$. The image $\mathcal{F}_2$ of $\mathcal{F}_2(\mathfrak{M})$ by the equivalence ζ_2 is then the inductive closure of the set whose unique element is

$$\zeta_2(\overline{Y}_2(2,2)) = \iota_2(2,2) = \mathbf{2}\,\blacksquare\,\mathbf{2}. \quad \nabla$$

2. *Generalized limits.*

By analogy with the usual definition of a limit for a functor we define limits relative to D for functors toward the first category of 1-morphisms of the double category D.

Let D be a double category and A a category. The category $T(D,A)$ of D-wise transformations (see C-1) admits for objects the functors from

A to the first category D^2_o of 1-morphisms of D.

The alternative double functor

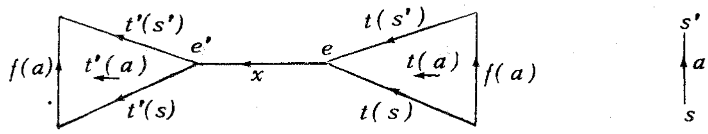
$$(x, a) \mapsto x \quad \text{from} \quad (D^1_o, A) \quad \text{to} \quad D$$

determines a functor $d_{DA}: D^1_o \to T(D, A)$ (Proposition 5-C). This functor maps

- the vertex e of D onto the constant functor $e\hat{}: A \to D^2_o$,
- the morphism $x: e \rightrightarrows e'$ of D^1_o onto the constant functor $x\hat{}: A \to D^1$, which is a D-wise transformation form $e\hat{}$ to $e'\hat{}$.

As for natural transformations, we will use a more «geometrical» language: Let $f: A \to D^2_o$ be a functor.

- If $t: f \to e\hat{}$ is a D-wise transformation toward a constant functor, we say that t is an *inductive D-wise cone, indexed by A, with vertex e and basis f.*
- A D-wise transformation $t': e'\hat{} \to f$ is called *a projective D-wise cone with vertex e' and basis f.*



- Let $x: e \rightrightarrows e'$ be a morphism of D^1_o. If $t: f \to e\hat{}$ is an inductive D-wise cone, we denote by xt the inductive D-wise cone

$$x\hat{} \circ_2 t: f \to e'\hat{} \quad \text{such that} \quad xt(a) = x \circ_2 t(a) \quad \text{for each } a \in A.$$

Dually, if $t': e'\hat{} \to f$ is a projective D-wise cone, then $t'x: e\hat{} \to f$ is the projective D-wise cone $t' \circ_2 x\hat{}$ such that

$$(t'x)(a) = t'(a) \circ_2 x \quad \text{for each} \quad a \in A.$$

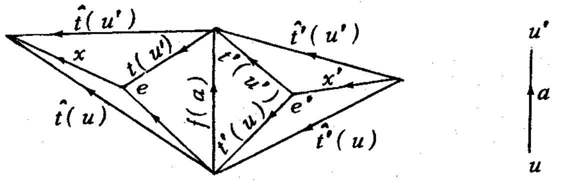
DEFINITION. Let $f: A \to D^2_o$ be a functor. If $t: f \to e\hat{}$ (resp. $t: e\hat{} \to f$) is a D-wise cone defining e as a free (resp. a cofree) object generated by f relative to the functor d_{DA}, then e is called an *inductive* (resp. a *projective*) *D-wise limit of f* and t is called an *inductive* (resp. a *projective*) *D-wise limit-cone.*

REMARKS. Limits relative to a double category were introduced by Ehresmann in [E2] and some general properties of these limits are given in [Le].

Quasi-limits of Gray [G1], analimits and catalimits of Bourn [Bo] are examples of such limits which will be studied later on.

Let $f: A \to D_o^2$ be a functor. The inductive D-wise cone t with vertex e and basis f is a D-wise limit-cone iff, for each inductive D-wise cone $\hat{t}$ with basis f, there exists a unique morphism x in D_o^1, called the *factor of* $\hat{t}$ *relative to* t, such that $\hat{t} = xt$.

The projective D-wise cone t' with basis f is a D-wise limit-cone iff, for each projective D-wise cone $\hat{t}'$ with basis f, there exists a unique morphism x' in D_o^1, called the *factor of* $\hat{t}'$ *relative to* t', such that $\hat{t}' = t'x'$.



The terminology is justified by the following examples.

EXAMPLES. 1° If B is a category and $\square B$ the double category of its commutative squares, a functor $f: A \to B$ admits a projective (resp. an inductive) $\square B$-wise limit e iff e is a (usual) projective (resp. inductive) limit of f. Indeed, if we identify $T(\square B, A)$ with B^A and B with the second category of 1-morphisms of $\square B$, the functor $d_{\square BA}$ is identified with the «diagonal» functor from B to B^A.

2° If I^0 is the discrete category on I, a projective D-wise cone t indexed by I^0 and with vertex e is identified with the family $(t(i))_{i \in I}$ of 1-morphisms $t(i): e \rightrightarrows e_i$ of D; hence t is a D-wise limit-cone iff e is a product of $(e_i)_{i \in I}$ in D_o^1, the $t(i)$'s being the projections.

3° If D^* is the double category (D^1, D^{2*}) which is the second dual of D, then $T(D^*, A)$ is the dual of $T(D, A)$, so that a D-wise cone t is an inductive D-wise limit-cone iff t is a projective D^*-wise limit-cone.

DEFINITION. We say that D_o^2 *admits inductive* (resp. *projective*) D*-wise* A*-limits* if d_{DA} admits an adjoint (resp. a coadjoint), which is then called a D*-wise* A*-limit functor.* If D_o^2 admits inductive (resp. projective) D-wise limits for each small (or finite,...) category A, we say that D_o^2 *admits inductive* (resp. *projective*) D*-wise small* (or *finite*,...) *limits.*

2. REPRESENTABLE DOUBLE CATEGORIES

We are going to study the double categories D whose first category of 1-morphisms D_o^2 admits D-wise **2**-limits. For them, the existence of D-wise limits reduces to the existence of «enough» usual limits in D_o^1. Fundamental examples of such double categories are the double categories of squares of a representable (in the sense of Gray) 2-category.

In all this chapter, we denote by D a double category, by « $\circ$» and «.» respectively its first and its second law.

A. Representation of a 1-morphism.

DEFINITION. The double category D is said *representable* (resp. *corepresentable*)
1+ if D_o^2 admits projective (resp. inductive) D-wise **2**-limits.

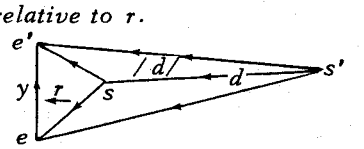
D is corepresentable iff its second dual is representable.

Let $v: T(D, \mathbf{2}) \to D^2$ be the canonical isomorphism mapping t onto $t(z)$, where z always denotes the morphism from 0 to 1 in **2**. The composite functor

$$D_o^1 \xrightarrow{d_{D\mathbf{2}}} T(D, \mathbf{2}) \xrightarrow{v} D^2$$

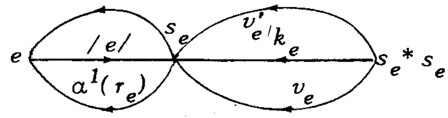
is the insertion into D^2 of its sub-category D_o^1. Hence D is representable (resp. corepresentable) iff the insertion $D_o^1 \hookrightarrow D^2$ admits a coadjoint (resp. an adjoint). In particular, for 2-categories, these definitions are equivalent to that given by Gray [G2].

Let D be a representable double category. If $y: e \to e'$ is a morphism of D_o^2 and if $r: s \rightrightarrows y$ is a 2-block of D defining s as a cofree object generated by y relative to the insertion $D_o^1 \hookrightarrow D^2$, we call r *a representation of y in D*. If $d: s' \rightrightarrows y$ is a 2-block, there exists a unique 1-morphism $x: s' \rightrightarrows s$ such that $r.x = d$; this x is denoted by $/d/$ and called the *factor of d relative to r*.



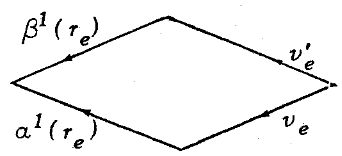
If $r_e : s_e \rightrightarrows e$ is a representation of a vertex e in D, it is also the representation of e in the greatest sub-2-category C of D; it follows that C is also representable. From Gray's results [G2], we know that, if D_0^1 (which is the category of 1-morphisms of C) admits pullbacks, there exists a category ϕ_e in D_0^1 mapping the morphisms $\alpha, \beta, \iota, \nu, \nu', \kappa$ of $\Sigma\mathcal{F}$ respectively on:

$a_e = \alpha^1(r_e)$, $b_e = \beta^1(r_e)$,



the factor $/e/$ relative to r_e,

the canonical projections v_e and v'_e of the pullback in D_0^1:



the factor $k_e = /(r_e . v_e) \circ (r_e . v'_e)/$ relative to r_e.

Indeed, let $D(e, \cdot)$ be the functor from the dual of D_0^1 to $\mathcal{F}$ which maps:

the vertex s of D on the sub-category of D^1 defined by $D^2(e, s)$,

the morphism $x : s \rightrightarrows s'$ of D_0^1 on the functor $c \mapsto c.x$ from $D(e, s')$ to $D(e, s)$.

The functor $p_{\mathcal{F}} D(e, \cdot)$ is equal to $D^2(e, \cdot)$ which, as r_e is a representation of e in D, is equivalent to $D_0^1(s_e, \cdot)$, whence representable. Then Proposition 8-0 associates to $D(e, \cdot)$ a category in D_0^1, which is ϕ_e.

PROPOSITION 1. *Let D be a representable double category such that D_0^1 admits pullbacks. If $y : e \to e'$ is a morphism of D^2 and if ϕ_e and $\phi_{e'}$ are the categories in D_0^1 associated above to e and e', there exists a distributor in D_0^1:*

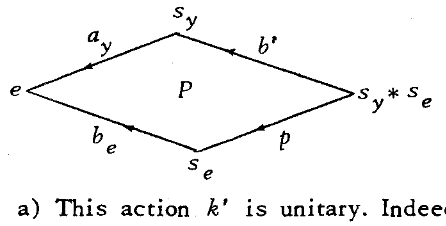
$$\delta_y = (\phi_{e'}, b_y, k'', k', a_y, \phi_e),$$

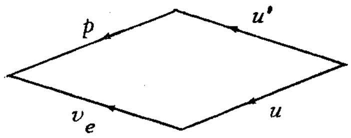
where $r : a_y \to b_y : s_y \rightrightarrows y$ is a representation of y.

Δ. 1° We get a right category action (k', a_e, ϕ_e) in D_0^1 by considering the factor relative to r:

$$k' = /(r.b') \circ (r_e . p)/$$

and the pullbacks in D_0^1





a) This action k' is unitary. Indeed, if i' is the factor $[s_y, /e/ . a_y]$ relative to the pullback P, then $k'. i' = s_y$ follows from the equalities

$$r. k'. [s_y, /e/ . a_y] = ((r. b') \circ (r_e . p)). [s_y, /e/ . a_y] =$$
$$= r \circ (r_e . /e/ . a_y) = r \circ a_y = r$$

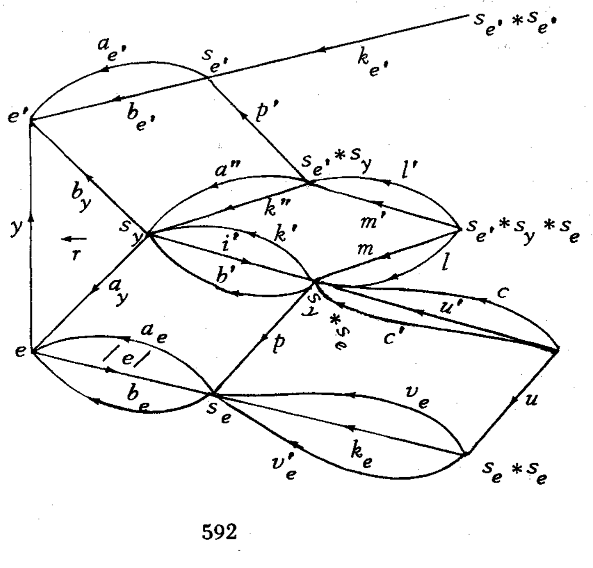
and from the unicity of the factor relative to r.

b) To show the associativity of the action, we consider the factors

$$c = [\, b'. u', k_e . u\,] \quad \text{and} \quad c' = [\, k'. u', v_e' . u\,]$$

relative to the pullback P, and we have to prove that $k'. c = k'. c'$. This is deduced from the equalities:

$$\begin{aligned} r. k'. c &= ((r. b') \circ (r_e . p)). [\, b'. u', k_e . u\,] \\ &= (r. b'. u') \circ (r_e . k_e . u) \\ &= (r. b'. u') \circ ((r_e . v_e) \circ (r_e . v_e')). u \\ &= (r. b'. u') \circ (r_e . v_e . u) \circ (r_e . v_e' . u)), \end{aligned}$$



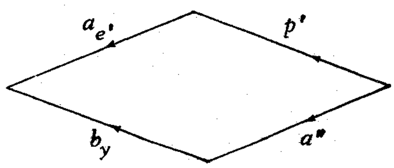
$$r.k'.c' = ((r.b')\circ(r_e.p)).[k'.u', v'_e.u]$$
$$= (r.k'.u')\circ(r_e.v'_e.u)$$
$$= (((r.b')\circ(r_e.p)).u')\circ(r_e.v'_e.u)$$
$$= (r.b'.u')\circ(r_e.v_e.u)\circ(r_e.v'_e.u) = r.k'.c,$$

since $p.u' = v_e.u$.

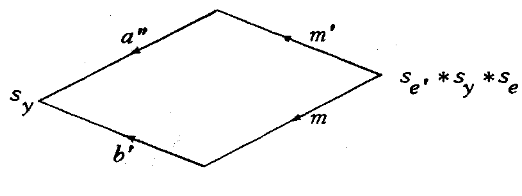
2° A similar proof shows that $(\phi_{e'}, b_y, k'')$ is a category action in D^1_0, where k'' is the factor

$$k'' = /(r_{e'}.p')\circ(r.a'')/$$

relative to r and where we have the pullback P' in D^1_0:



3° For $(\phi_{e'}, b_y, k'', k', a_y, \phi_e)$ to be a distributor, it remains to prove the «compatibility» of the two actions, i.e. $k''.l' = k'.l$, where



is a pullback and where

$$l' = [p'.m', k'.m] \quad \text{and} \quad l = [k''.m', p.m]$$

are the factors relative to the pullbacks P' and P. Indeed, we get the equalities:

$$r.k''.l' = ((r_{e'}.p')\circ(r.a'')).[p'.m', k'.m]$$
$$= (r_{e'}.p'.m')\circ(r.k'.m)$$
$$= (r_{e'}.p'.m')\circ(((r.b')\circ(r_e.p)).m)$$
$$= (r_{e'}.p'.m')\circ(r.b'.m)\circ(r_e.p.m),$$

$$\begin{aligned} r.k'.l &= ((r.b')\circ(r_e.p)).[k''.m', p.m] \\ &= (r.k''.m')\circ(r_e.p.m) \\ &= (((r_{e'}.p')\circ(r.a'')).m')\circ(r_e.p.m) \\ &= (r_{e'}.p'.m')\circ(r.a''.m')\circ(r_e.p.m) = r.k''.l', \end{aligned}$$

since $b'.m = a''.m'$. ∇

REMARK. δ_y may be defined as the distributor in D_o^1 associated (p. 24) to the canonical functor ψ from the dual of D_o^1 to the category $\mathfrak{M}^{\sigma}\delta$ which maps the vertex s on the distributor $(D(e',s), \beta_{y,s}, \kappa''_s, \kappa'_s, \alpha_{y,s}, D(e,s))$, where κ'_s and κ''_s are restrictions of the law of D^1 and where $\alpha_{y,s}$ and $\beta_{y,s}$ are the maps from $D^2(y,s)$ to $D_o^1(e,s)$ and $D_o^1(e',s)$ restrictions of α^1 and β^1. Indeed, $\psi(\cdot)(1,0)$ is represented by s_y, $\psi(\cdot)(\hat{2},1)$ and $\psi(\cdot)(2,1)$ by s_e and $s_{e'}$ respectively.

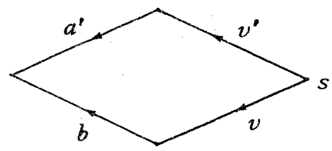
B. Existence of limits relative to a representable double category.

PROPOSITION 2. *If D is a representable double category and if D_o^1 admits pullbacks, then D_o^2 admits projective D-wise* **3***-limits.*

Δ. We denote by z, z' and z'' the morphisms of the category **3**:

Functors from **3** to a category A are in bijection with pairs of composable morphisms of A. Let $f: \mathbf{3} \to D_o^2$ be a functor.

1° Let $r: a \to b$ and $r': a' \to b'$ be representations of $f(z)$ and $f(z')$ in D. By hypothesis there exists a pullback

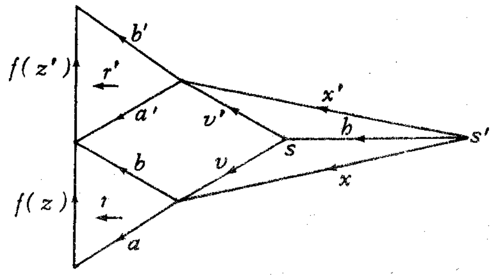


P of (a', b) in D_o^1. We put

$$t(z) = r.v : s \Rightarrow f(z) \quad \text{and} \quad t(z') = r'.v' : s \Rightarrow f(z').$$

Since

$$\alpha^1(t(z')) = \alpha^1(r'.v') = \alpha^1(r').v' = a'.v' = b.v = \beta^1(r.v) = \beta^1(t(z)),$$



there exists a composite $t(z'') = t(z') \circ t(z)$ in D^1. We have so defined a D-wise cone t with vertex s and basis f.

2° t is a limit-cone. Indeed, let t' be a projective D-wise cone with basis f and vertex s'. The 2-block $t'(z) : s' \Rightarrow f(z)$ admits a factor x relative to r and $t'(z') : s' \Rightarrow f(z')$ admits a factor x' relative to r'. Since

$$a'.x' = \alpha^1(r').x' = \alpha^1(r'.x') = \alpha^1(t'(z')) = \beta^1(t'(z)) = b.x,$$

there exists a factor $h = [x', x]$ relative to P. From the equalities

$$t(z).h = r.v.h = r.x = t'(z),$$

$$t(z').h = r'.v'.h = r'.x' = t'(z'),$$

$$t(z'').h = (t(z') \circ t(z)).h = (t(z').h) \circ (t(z).h) =$$
$$= t'(z') \circ t'(z) = t'(z''),$$

we deduce that h is the unique morphism of D_o^1 satisfying $th = t'$. ∇

COROLLARY. *If D is a corepresentable double category and if D_o^1 admits pushouts, then D_o^2 admits D-wise inductive **3**-limits.*

Δ. This results from Proposition 2 applied to the second dual of D, which is representable. ∇

PROPOSITION 3. *Let D be a representable double category and A a small (resp. a finite) category. If D_o^1 admits small (resp. finite) projective limits, then D_o^2 admits projective D-wise A-limits.*

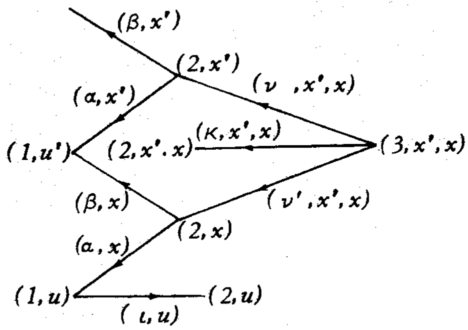
Δ. 1° *Some notations.*

a) The category H: Let $\psi_A : \Sigma\mathcal{F} \to \mathfrak{M}$ be the category in $\mathfrak{M}$ associated to A and ψ its restriction to the sub-category Σ of $\Sigma\mathcal{F}$ generated by $\{\alpha, \beta, \iota, \kappa, \nu, \nu'\}$. We denote by H the source of the discrete fibration $\eta : H \to \Sigma$ associated to ψ. Then H is generated by the morphisms:

(ν, x', x), (ν', x', x), (κ, x', x) from $(3, x', x)$ to: $(2, x')$, $(2, x)$ and $(2, x'.x)$ respectively, where (x', x) is any pair of composable morphisms of A,

(α, x) and (β, x) from $(2, x)$ to $(1, u)$ and $(1, u')$, where x is any morphism in A, from u to u',

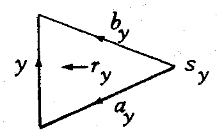
$(\iota, u) : (1, u) \to (2, u)$, for any object u of A.



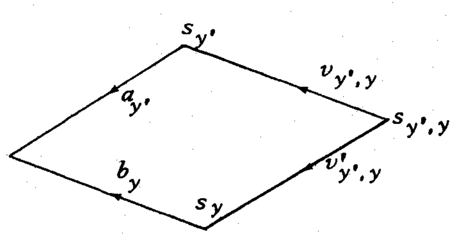
Since $\Sigma\mathcal{F}$ is a finite category, H is small or finite when so is A.

b) For each morphism y in D_o^2, we choose a representation of y in D:

$r_y : a_y \to b_y : s_y \rightrightarrows y$



and for each pair (y', y) of composable morphisms of D_o^1, we choose a pullback $P_{y',y}$ of $(a_{y'}, b_y)$ in D_o^1:



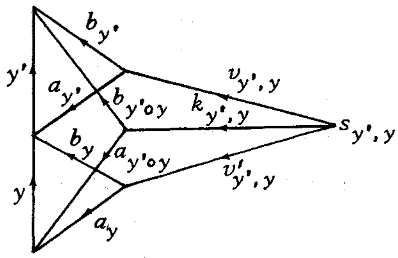
(This pullback exists, pullbacks being finite projective limits.) Since

$$\beta^1(r_y . v'_{y',y}) = \beta^1(r_y).v'_{y',y} = b_y . v'_{y',y} = a_{y'}.v_{y',y} = \alpha^1(r_{y'}.v_{y',y}),$$

there exists a composite

$$(r_{y'}.v_{y',y}) \circ (r_y . v'_{y',y}): s_{y',y} \rightrightarrows y' \circ y$$

in D^1, and it admits a factor relative to $r_{y' \circ y}$, which will be denoted by $k_{y',y}$.



2° Let $f: A \to D^2_0$ be a functor. We are going to construct a projective D-wise cone t with basis f.

a) There exists a functor $p: H \to D^1_0$ defined as follows: it maps

(α, x) and (β, x) on $a_{f(x)}$ and on $b_{f(x)}$, for each x in A,

(ι, u) on the factor $/f(u)/$ relative to $r_{f(u)}$, for each $u \in A_0$,

(κ, x', x) on $k_{y',y}$, (ν, x', x) on $v_{y',y}$ and (ν', x', x) on $v'_{y',y}$,

for each pair (x', x) of composable morphisms of A, where $y = f(x)$ and $y' = f(x')$.

Since H is small (resp. finite), there exists a projective limit-cone l with basis p and vertex s (in the usual meaning).

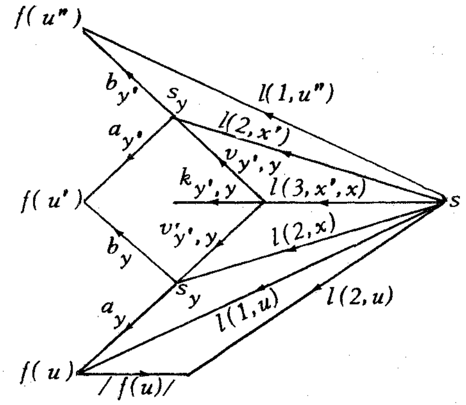
b) For each morphism $x: u \to u'$ in A, we define $t(x) = r_{f(x)}. l(2,x)$. The map associating $t(x)$ to x in A defines a functor $t: A \to D^1$. Indeed, if u is an object of A, we get

$$t(u) = r_{f(u)}. l(2,u) = r_{f(u)}./f(u)/ . l(1,u) = l(1,u) \in D^1_0,$$

since, l being a cone with basis p, we have

$$l(2,u) = p(\iota, u). l(1,u) = /f(u)/. l(1,u).$$

On the other hand, if $x: u \to u'$ and $x': u' \to u''$ are morphisms of A and if



$y = f(x)$ and $y' = f(x')$, the equalities

$$\alpha^1(t(x')) = \alpha^1(r_{y'}).\, l(2,x') = a_{y'}.\, l(2,x') = p(\alpha, x').\, l(2,x') = l(1,u')$$

and

$$\beta^1(t(x)) = b_y.\, l(2,x) = p(\beta, x).\, l(2,x) = l(1,u') = \alpha^1(t(x'))$$

imply that the composite $t(x') \circ t(x)$ is defined in D^1. From

$$l(2,x') = p(\nu, x', x).\, l(3,x',x) = v_{y',y}.\, l(3,x',x),$$

$$l(2,x) = p(\nu', x', x).\, l(3,x',x) = v'_{y',y}.\, l(3,x',x),$$

$$l(2,x'.x) = p(\kappa, x', x).\, l(3,x',x) = k_{y',y}.\, l(3,x',x)$$

and from the definition of $k_{y',y}$ as a factor relative to $r_{y',y}$, we deduce

$$t(x') \circ t(x) = (r_{y'}.\, l(2,x')) \circ (r_y.\, l(2,x)) =$$

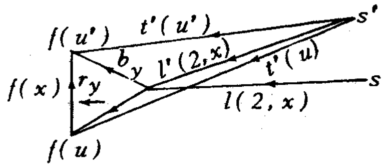
$$= ((r_{y'}.\, v_{y',y}) \circ (r_y.\, v'_{y',y})).\, l(3,x',x) = r_{y'\circ y}.\, k_{y',y}.\, l(3,x',x) =$$

$$= r_{y'\circ y}.\, l(2,x'.x) = t(x'.x).$$

Hence $t: A \to D^1$ is a D-wise cone. As

$$t(x) = r_{f(x)}.\, l(2,x) : s \rightrightarrows f(x),$$

this D-wise cone admits f as its basis.



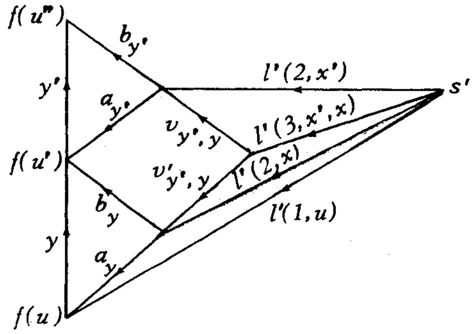
3° We are going to prove that t is a D-wise limit-cone. For this, we suppose that t' is a projective D-wise cone with vertex s' and basis f.

a) We first construct a (usual) cone l' with vertex s' and basis p as follows:

for each object u of A, we define $l'(1,u)$ as the 1-morphism

$$l'(1,u) = t'(u): s' \rightrightarrows f(u);$$

$l'(2,x)$, for each morphism x of A, is the factor of the 2-block $t'(x): s' \rightrightarrows f(x)$ relative to $r_{f(x)}$.



If $x: u \to u'$ and $x': u' \to u''$ are composable morphisms of A and if $y = f(x)$ and $y' = f(x')$, we have

$$a_{y'}.\, l'(2,x') = \alpha^1(r_{y'}.\, l'(2,x')) = \alpha^1(t'(x')) =$$
$$= \beta^1(t(x)) = b_y.\, l'(2,x),$$

so that there exists a factor $l'(3,x',x) = [\,l'(2,x'),\, l(2,x)\,]$ relative to the pullback $P_{y',y}$.

b) We prove now that in this way we get a cone l' with vertex s' and basis p. Indeed:

If $x: u \to u'$ in A, then

$$p(\alpha,x).\, l'(2,x) = a_{f(x)}.\, l'(2,x) = \alpha^1(t'(x)) = t'(u) = l'(1,u),$$
$$p(\beta,x).\, l'(2,x) = b_{f(x)}.\, l'(2,x) = \beta^1(t'(x)) = t'(u') = l'(1,u').$$

If u is an object of A, we get

$$p(\iota,u).\, l'(1,u) = /f(u)/\,.\, t'(u) = /f(u).\, t'(u)/ = \surd t'(u)/ = l'(2,u)$$

the factors being relative to $r_{f(u)}$.

Let $x: u \to u'$ and $x': u' \to u''$ be composable morphisms of A and write $y = f(x)$ and $y' = f(x')$. By definition of $l'(3, x', x)$, we have

$$p(\nu', x', x).\, l'(3, x', x) = \nu'_{y', y}.\, l'(3, x', x) = l(2, x),$$

$$p(\nu, x', x).\, l'(3, x', x) = \nu_{y', y}.\, l'(3, x', x) = l(2, x').$$

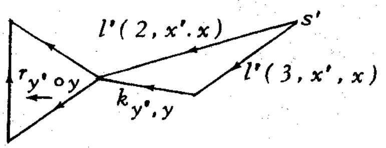
Finally, $p(\kappa, x', x).\, l'(3, x', x)$ and $l'(2, x'.x)$ are equal, since both are factors relative to $r_{y' \circ y}$ of

$$r_{y' \circ y}.\, p(\kappa, x', x).\, l'(3, x', x) = r_{y' \circ y}.\, k_{y', y}.\, l'(3, x', x) =$$

$$= ((r_{y'}.\, \nu_{y', y}) \circ (r_y.\, \nu'_{y', y})).\, l'(3, x', x) =$$

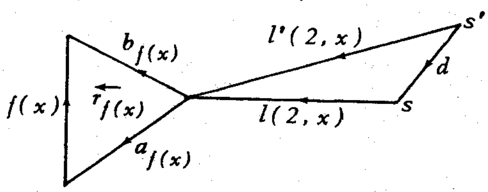
$$= (r_{y'}.\, l'(2, x')) \circ (r_y.\, l'(2, x)) = t'(x') \circ t'(x) =$$

$$= t'(x'.x) = r_{y' \circ y}.\, l'(2, x'.x).$$



c) The projective usual cone l' with basis p admits a factor $d: s' \rightrightarrows s$ relative to the limit-cone l, so that $l' = l d$. For x in A, we have

$$t(x).\, d = r_{f(x)}.\, l(2, x).\, d = r_{f(x)}.\, l'(2, x) = t'(x).$$



Hence d is the unique morphism of D^1_0 satisfying the equality $t' = t d$. This ends the proof. ∇

More precisely, we have proved:

COROLLARY 1. *If D is a representable double category, A a category, and if D^1_0 admits pullbacks and projective H-limits (where H is the category defined in the preceding proof), then D^2_0 admits projective D-wise A-limits.*

COROLLARY 2. *If D is a representable double category and if D^1_0 admits* *small connected projective limits (resp. pullbacks and equalizers), then* D_0^2 *admits small (resp. finite) connected projective* D*-wise limits.*

Δ. If A is connected, $\Sigma_{\mathfrak{F}}$ being connected it is easily seen that H is also connected. Now a category admitting pullbacks and equalizers has connected finite projective limits. So Corollary 1 implies Corollary 2. ∇

By duality, it follows from Proposition 3:

PROPOSITION 4. *If* D *is a corepresentable double category and if* D_0^1 *admits small (resp. finite) inductive limits, then* D_0^2 *admits small (resp. finite) inductive* D*-wise limits.*

COROLLARY. *If* D *is a corepresentable double category and if* D_0^1 *admits pushouts and cokernels,* D_0^2 *admits finite connected inductive* D*-wise limits.*

REMARK. 1° In the proof of Proposition 3, instead of H we could have used the source $\hat{H}$ of the discrete fibration associated to the category in $\mathfrak{M}$ associated to A. Indeed, the functor p constructed in this proof extends in a functor $\hat{p}: \hat{H} \to D_0^1$. As H is a cofinal sub-category of $\hat{H}$, the functor p has the same limit as $\hat{p}$, and this limit is the D-wise limit of f.

2° The preceding remark leads to a more abstract proof of Proposition
3 (which will be explicited later on for multiple categories). This proof pro- 1
ceeds as follows: Let Ω be the set of categories A such that D_0^2 admits D-wise A-limits. As $\mathfrak{F}$ is the inductive closure of $\{\mathbf{2}\}$ and as $\mathbf{2}$ belongs to Ω (by definition of a representable double category), we will have $\Omega = \mathfrak{F}_0$ if B belongs to Ω when B is the vertex of an inductive limit-cone $c: I \rightrightarrows \mathfrak{F}$ whose basis w satisfies:

$$w(i) \in \Omega \quad \text{for each object } i \text{ of } I.$$

Indeed, the functor $T_{11}(D,-): \mathfrak{F}^* \to \mathfrak{F}$, coadjoint to $T_{11}(D^{21},-)^*$, transforms c into a projective limit-cone $\bar{c}$ with basis $\bar{w} = T_{11}(D, w-): I^* \to \mathfrak{F}$ and vertex $T(D,B)$. The canonical functor d_{DB} is the factor relative to $\bar{c}$ of the projective cone c' with basis $\bar{w}$ defined by:

$$c'(i) = d_{D\,w(i)} \quad \text{for each object } i \text{ of } I.$$

Since $c'(i)$ admits a coadjoint for each i, a theorem of Appelgate-Tierney [AT] asserts that the factor of c' also has a coadjoint; hence $B \in \Omega$.

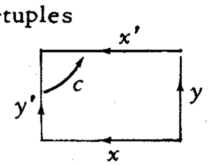
C. The double category of squares of a 2-category.

In this section we give a fundamental example of a representable double category.

We denote by C a 2-category, by «∘» and by «.» the symbols of the laws of the categories C^1 and C^2.

1. *The double category of up-squares of* C is [GZ] the following double category, which is denoted by $Q(C)$:

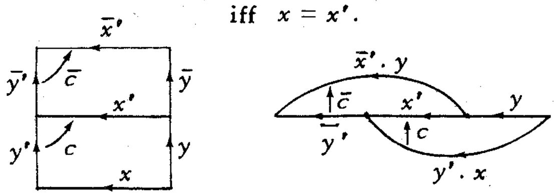
- Its 2-blocks, called *up-squares of* C, are the 5-tuples $q = (y', x', c, x, y)$, where (y', x', x, y) is a (non-commutative) square of C^1_0 and $c: y'.x \to x'.y$ a 2-cell of C.



- The first law, said *vertical composition* and denoted by $\boxminus$, is

$$(\bar{y}', \bar{x}', \bar{c}, \bar{x}, \bar{y}) \boxminus (y', x', c, x, y) = (\bar{y}'.y', \bar{x}', (\bar{c}.y) \circ (\bar{y}'.c), x, \bar{y}.y)$$

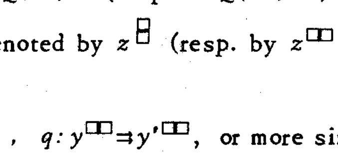
iff $\bar{x} = x'$.



- The second law, the *horizontal composition*, denoted by $\square\!\square$, is:

$$(\hat{y}', \hat{x}', \hat{c}, \hat{x}, \hat{y}) \,\square\!\square\, (y', x', c, x, y) = (\hat{y}', \hat{x}'.x', (\hat{x}'.c) \circ (\hat{c}.x), \hat{x}.x, y)$$

iff $\hat{y} = y'$.

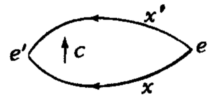


The first and the second categories underlying $Q(C)$ will be denoted by $Q(C)^{\boxminus}$ and $Q(C)^{\square\!\square}$. Both admit as objects the 1-morphisms of C. The identity of $Q(C)^{\boxminus}$ (resp. of $Q(C)^{\square\!\square}$) corresponding to the 1-morphism z will be denoted by $z^{\boxminus}$ (resp. by $z^{\square\!\square}$), or sometimes even by z. Hence we write

$$q: x^{\boxminus} \to x'^{\boxminus}, \quad q: y^{\square\!\square} \rightrightarrows y'^{\square\!\square}, \text{ or more simply } q: x \to x', \; q: y \rightrightarrows y'.$$

We may identify C with the greatest sub-2-category of $Q(C)$ by identifying

$c: x \to x': e \rightrightarrows e'$ with (e', x', c, x, e).



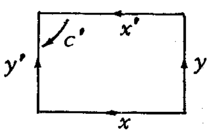
The double category $\square C_0^1$ of commutative squares of C_0^1 is identified with the double sub-category of $Q(C)$ formed by the up-squares of the form $(y', x', y'.x, x', y)$ (i.e. the up-squares q such that c be a 1-morphism of C). In particular, this double sub-category is equal to $Q(C)$ iff $C_0^1 = C^1$, i.e. iff C is the discrete 2-category on C^2. 1+

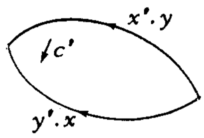
2. *The double category of down-squares of* C.

This double category, denoted by $Q_{\downarrow}(C)$, is defined as the double category of up-squares of the first dual (C^{1*}, C^2) of C. Hence its 2-blocks, called *down-squares of* C, are the 5-tuples (y', x', c', x, y), where

(y', x', x, y) is a (non-commutative) square of C_0^1,

$c': x'.y \to y'.x$ is a 2-cell of C.

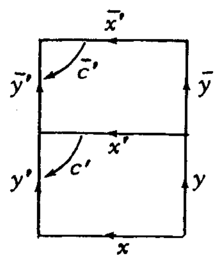


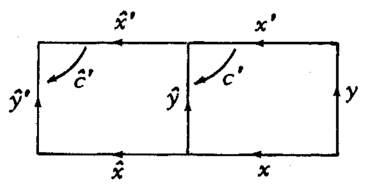


The two laws, expressed with the laws of C only, are:

$$(\bar y', \bar x', \bar c', \bar x, \bar y) \boxminus (y', x', c', x, y) = (\bar y'.y', \bar x', (\bar y'.c') \circ (\bar c'.y), x, \bar y.y)$$
iff $x' = \bar x$,

$$(\hat y', \hat x', \hat c', \hat x, \hat y) \square\!\square (y', x', c, x, y) = (\hat y', \hat x'.x', (\hat c'.x) \circ (\hat x'.c'), \hat x.x, y)$$
iff $\hat y = y'$.





The double category $\square C_0^1$ is also identified with a double sub-category of $Q_{\downarrow}(C)$.

The bijection:

$$(y', x', c, x, y) \to (x', y', c, y, x)$$

from the set of up-squares of C onto the set of down-squares of C defines a canonical isomorphism from $Q(C)$, to the double category symmetrical of the double category $Q_{\downarrow}(C)$.

3. Let $f: C \to K$ be a 2-functor. We have double functors

$$Q(f): Q(C) \to Q(K) \quad \text{and} \quad Q_{\downarrow}(f): Q_{\downarrow}(C) \to Q_{\downarrow}(K)$$

associating $(f(y'), f(x'), f(c), f(x), f(y))$ to (y', x', c, x, y).

In this way are defined two functors $Q(\cdot)$ and $Q_{\downarrow}(\cdot)$ from the category of small 2-functors into the category $\mathcal{F}_2$ of small double functors.

4. *Limits in* $Q(C)^{\square\square}$.

PROPOSITION 5. *If* C_0^1 *admits projective* A*-limits preserved by the insertion* $i: C_0^1 \hookrightarrow C^2$, *then* $\square\square C_0^1$ *admits projective* A*-limits which are preserved by the insertions* j *and* j' *into* $Q(C)^{\square\square}$ *and* $Q_{\downarrow}(C)^{\square\square}$.

Δ. We denote by $\bar{\alpha}$ and $\bar{\beta}$ the functors from the category $K = \square\square C_0^1$ to C_0^1 defined by the maps source and target of the vertical category $\boxminus C_0^1$ (whose objects are identified with 1-morphisms of C). Let F be a functor from A to K. We write:

$$f = \bar{\alpha} F \quad \text{and} \quad \hat{f} = \bar{\beta} F.$$

Since K is isomorphic with the category $(C_0)^2$, there exists a projective limit-cone T with basis F and the cones

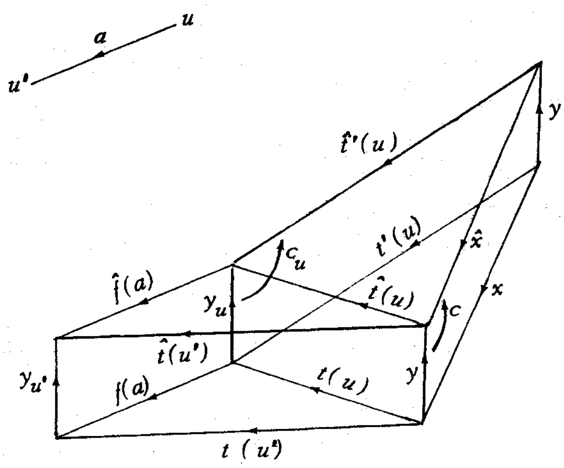
$$t = \bar{\alpha} T \quad \text{and} \quad \hat{t} = \bar{\beta} T$$

are limit-cones with bases f and $\hat{f}$.

1° jT is a limit-cone. Indeed, let $T': A \to Q(C)^{\square\square}$ be a projective cone with basis jF. As $t' = \bar{\alpha} T'$ is a projective cone with basis f, there exists a factor x of t' relative to t. There also exists a factor $\hat{x}$ of $\hat{t}'$ relative to $\hat{t}$, where $\hat{t}' = \bar{\beta} T'$. We have $T'(u) = (y_u, \hat{t}'(u), c_u, t'(u), y')$, for each $u \in A_0$, where $c_u: y_u . t'(u) \to \hat{t}'(u) . y'$ is a 2-cell. The equality

$$F(a) \square\square T'(u) = T'(u') \quad \text{implies} \quad \hat{f}(a) . c_u = c_{u'}$$

for each morphism $a: u \to u'$ in A. Hence, there exists a projective cone



$t'': A \rightrightarrows C^2$ with basis $i\hat{f}$ such that

$$t''(u) = c_u \quad \text{for each object } u \text{ of } A.$$

Since $\hat{t}$ is a limit-cone, the hypothesis asserts that $i\hat{t}$ is a projective limit-cone and there exists a factor c of t'' relative to $i\hat{t}$. From

$$\hat{t}(u).\,\alpha^1(c) = \alpha^1(\hat{t}(u).\,c) = \alpha^1(c_u) = y_u\,.\,t'(u) = \\ = y_u\,.\,t(u).\,x = \hat{t}(u).\,y.\,x,$$

we deduce, $\hat{t}$ being a limit-cone, $\alpha^1(c) = y.\,x$. Similarly, $\beta^1(c) = \hat{x}.\,y'$. It follows that $(y, \hat{x}, c, x, y')$ is an up-square q and, by its construction, it is the unique up-square satisfying

$$T(u) \,\square\!\square\, q = T'(u) \quad \text{for each object } u \text{ of } A.$$

2° The category $\underset{\downarrow}{Q}(C)^{\square\!\square}$ being identical with $Q(C^*)^{\square\!\square}$, where C^* is the first dual of C, the preceding proof applied to this dual shows that $j'T$ is also a limit-cone. ∇

COROLLARY 1. *If C^1_0 admits projective A-limits preserved by the insertion into C^2, then $\boxminus\, C^1_0$ admits projective A-limits which are preserved by the insertions into $Q(C)^{\boxminus}$ and $\underset{\downarrow}{Q}(C)^{\boxminus}$.*

Δ. This corollary is deduced from the proposition, via the canonical isomorphism from $Q(C)^{\square\!\square}$ onto $Q(C)^{\boxminus}$ (resp. from $\underset{\downarrow}{Q}(C)^{\square\!\square}$ onto $\underset{\downarrow}{Q}(C)^{\boxminus}$) which maps $\square\!\square\, C^1_0$ onto $\boxminus\, C^1_0$. ∇

COROLLARY 2. *If* C_o^1 *admits inductive A-limits preserved by the insertion into* C^2, *then* $\square\square C_o^1$ *admits inductive A-limits preserved by the insertions into* $Q(C)^{\square\square}$ *and* $Q_{\downarrow}(C)^{\square\square}$.

Δ. This results from Proposition 5 applied to the second dual of C. ∇

COROLLARY 2. *If* C_o^1 *admits I-products (resp. I-sums) preserved by the insertion into* C^2, *then* $Q(C)^{\square\square}$, $Q_{\downarrow}(C)^{\square\square}$, $Q(C)^{\boxminus}$ *and* $Q_{\downarrow}(C)^{\boxminus}$ *admit I-products (resp. I-sums).*

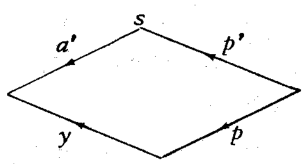
Δ. This comes from Proposition 5 and its corollaries, applied to the discrete category I^0 on I. ∇

REMARK. $Q(C)^{\square\square}$ does not always admit pullbacks, for up-squares which are not commutative squares.

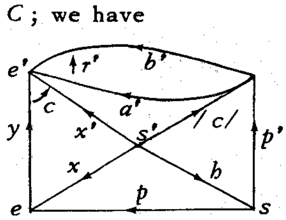
5. *Representability of* $Q(C)$.

PROPOSITION 6. *If* C *is a representable 2-category and if* C_o^1 *admits pullbacks, then* $Q(C)$ *is a representable double category.*

Δ. We consider an object of $Q(C)^{\square\square}$, identified with a 1-morphism y of C, where $y: e \rightrightarrows e'$. In the 2-category C, there exists a representation $r': a' \to b'$ of e'; there exists also in C_o^1 a pullback P



of (a', y). Then $q = (y, b'.p', r'.p', p, s)$ is an up-square of C. We are going to prove that q is a representation of y in $Q(C)$. Indeed, let $q' = (y, x', c, x, s')$ be an up-square of C, where s' is a vertex of C. Since $c: s' \rightrightarrows e'$ is a 2-cell of C, it admits a factor $/c/$ relative to the representation r' of e' in C; we have



$$a'./c/= \alpha^1(r')./c/ = \alpha^1(r'./c/) = \alpha^1(c) = y.x,$$

so that there exists a factor h of $(/c/, x)$ relative to P. As

$$b'.p'.h = b'./c/ = \beta^1(r'./c/) = \beta^1(c) = x',$$

$$r'.p'.h = r'./c/ = c \text{ and } p.h = x,$$

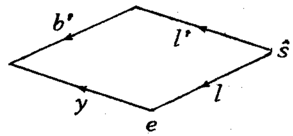
we get

$$q \,\square\!\square\, h^{\boxminus} = (y, b'.p'.h, r'.p'.h, p.h, s') = q'.$$

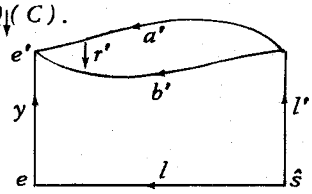
The unicity of the factors asserts the unicity of the 1-morphism h satisfying $q \,\square\!\square\, h^{\boxminus} = q'$. ∇

COROLLARY. *If C is a representable 2-category and if C_0^1 admits pullbacks, then $Q_{\downarrow}(C)$ is a representable double category.*

Δ. Since C is representable, so is its first dual. This dual admitting also C_0^1 as its category of 1-morphisms, the double category of its up-squares, which is $Q_{\downarrow}(C)$ by definition, is representable. More precisely, a representation of $y: e \rightrightarrows e'$ is constructed as follows: Let $r': a' \to b'$ be a representation of e' in C; then r' is also a representation of e' in the first dual of C, but its source in C^{1*} is b'. Let



be a pullback in C_0^1. The down-square $(y, a'.l', r'.l', l, \hat{s})$ of C is a representation of y in $Q_{\downarrow}(C)$.

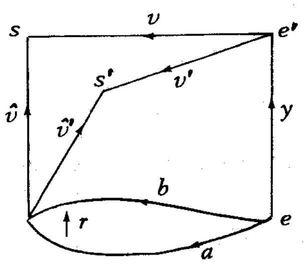


∇

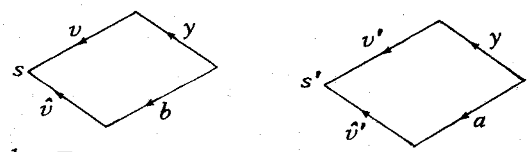
PROPOSITION 7. *If C is a corepresentable 2-category and if C_0^1 admits pushouts, then $Q(C)$ and $Q_{\downarrow}(C)$ are corepresentable double categories.*

Δ. A proof similar to that of Proposition 6 and of its Corollary shows that the 1-morphism $y: e \rightrightarrows e'$ of C admits:

- as a corepresentation in $Q(C)$ the up-square $(s, v, \hat{v}.r, \hat{v}.a, y)$,



- as a corepresentation in $Q_{\downarrow}(C)$ the down-square $(s', v', \hat{v}' . r, \hat{v}' . b, y)$ where $r: a \to b$ is a corepresentation of e in C and



are pushouts in C_0^1 . ∇

6. *Limits relative to the double category $Q(C)$.*

PROPOSITION 8. *If C is a representable 2-category such that C_0^1 admits connected (resp. small, resp. finite) projective limits, then C_0^1 admits connected (resp. small, resp. finite) projective $Q(C)$-wise and $Q_{\downarrow}(C)$-wise limits.*

Δ. This follows from Proposition 3, since $Q(C)$ and $Q_{\downarrow}(C)$ are both representable double categories (Proposition 7) whose second categories of 1-morphisms are isomorphic to C_0^1 . ∇

COROLLARY. *If C is a corepresentable 2-category such that C_0^1 admits connected (resp. small, resp. finite) inductive limits, then C_0^1 admits connected (resp. small, resp. finite) inductive $Q(C)$-wise and $Q_{\downarrow}(C)$-wise limits.* ∇

We are going to look more closely to $Q(C)$-wise limits and to compare them with generalized limits introduced by Gray.

Let A be a category. The category $T(Q(C), A)$ of $Q(C)$-wise transformations indexed by A admits as objects the functors from A to C_0^1, since C_0^1 is canonically isomorphic with the first category of 1-morphisms of the double category $Q(C)$. So a $Q(C)$-wise transformation $t: f \to f'$

is equivalent to the following data:

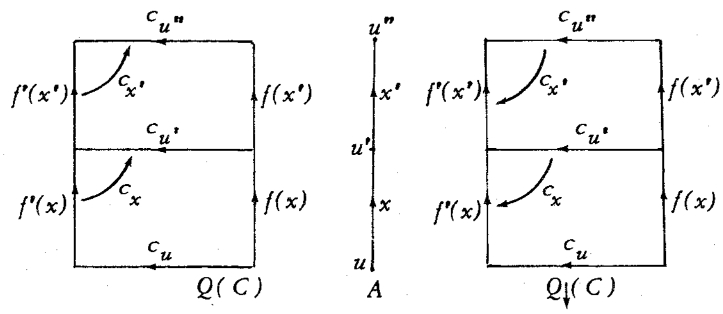
1° Functors f and f' from A to C_0^1.

2° For each object u of A a 1-morphism $c_u : f(u) \to f'(u)$ of C.

3° For each $x: u \to u'$ in A, a 2-cell $c_x : f'(x).c_u \to c_{u'}.f(x)$ such that, if $x': u' \to u''$ in A, then $c_{x'.x} = (c_{x'}.f(x)) \circ (f'(x').c_x)$.

Indeed, these conditions mean that there exists a functor t from A to $Q(C)^{\boxminus}$ such that

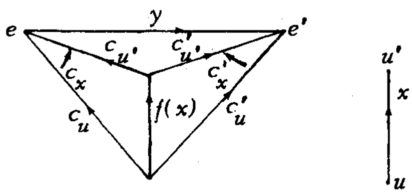
$$t(x) = (f'(x), c_{u'}, c_x, c_u, f(x)) \quad \text{for each } x: u \to u' \text{ in } A.$$



We have a similar description for $Q_{\downarrow}(C)$-wise transformations, except that c_x goes «down» instead of «up».

In other words, if $\hat{f}$ and $\hat{f}'$ are the 2-functors from the discrete 2-category on A toward C defined by f and f', the $Q_{\downarrow}(C)$-transformations from f to f' correspond to the quasi-natural transformations from $\hat{f}$ to $\hat{f}'$ defined by Gray [G1], called anadeses by Bourn [Bo], while the $Q(C)$-wise transformations from f to f' correspond to the quasi-natural transformations from $\hat{f}$ to $\hat{f}'$ of Gray or to the catadeses of Bourn. (The way the diagrams are drawn explains why we call «up» what these authors consider as being «down».)

Let $f: A \to C_0^1$ be a functor considered as an object of $T(Q(C), A)$. An inductive $Q(C)$-wise cone t with basis f and vertex e corresponds to



a family $(c_x)_{x \in A}$ of 2-cells of C such that:

$c_u : f(u) \Rightarrow e$ is a 1-morphism of C, for each object u of A,

$c_x : c_u \to c_{u'} . f(x)$ is a 2-cell, for $x : u \to u'$ in A,

$c_{x'.x} = (c_{x'} . f(x)) \circ c_x$, if $x' : u' \to u''$ in A

(the corresponding cone t associates to $x : u \to u'$ the up-square:

$$t(x) = (e, c_{u'}, c_x, c_u, f(x))).$$

This family corresponds to an inductive $Q(C)$-wise limit-cone if, for each family $(c'_x)_{x \in A}$ satisfying the same conditions, there exists one and only one 1-morphism y of C such that $y . c_x = c'_x$ for each x in A.

With this formulation, we see that the inductive (resp. projective) $Q(C)$-wise limits «are» the cartesian quasi-colimits (resp. quasi-limits) of Gray [G1] and also the inductive (resp. projective) catalimits of Bourn [Bo], for 2-functors from a discrete 2-category. Hence Proposition 8 has been announced by Gray [G2] and proved by Bourn [Bo] (in a more gene-
1+ ral case which will be considered later on).

D. Examples and Applications to sketched structures.

1. *Limits relative to the double category of quintets.*

The 2-category $\mathfrak{N}$ of small natural transformations admits the category $\mathfrak{F}$ of functors as its category of 1-morphisms. It is representable and corepresentable, a small category A admitting:

as a representation the natural transformation $r_A : \boxminus A \Rightarrow A$ associated to the identity functor of $\boxminus A$,

as a corepresentation the natural transformation $r'_A : A \Rightarrow A \times \mathbf{2}$, from $v = [\cdot, 0^{\frown}]$ to $[\cdot, 1^{\frown}]$ such that $r'_A(u) = (u, z)$ for each $u \in A_0$.

An up-square of $\mathfrak{N}$ is called a *quintet* and we denote by $\mathfrak{Q}$ the double category $\mathfrak{Q}(\mathfrak{N})$ of quintets (following [E2], where this double category was introduced, as well as its sub-2-category $\mathfrak{N}$). Let $\mathfrak{Q}_{\downarrow}$ be the double category of down-squares of $\mathfrak{N}$.

As $\mathfrak{F}$ admits small projective and inductive limits, preserved by the insertion into $\mathfrak{N}^{\cdot}$ (which admits an adjoint and a coadjoint), Proposition 5 asserts that $\square\square\mathfrak{F}$ admits small projective and inductive limits preserved by the insertions into $\mathfrak{Q}^{\square\square}$ and into $\mathfrak{Q}_{\downarrow}^{\square\square}$.

REMARK. The category $\mathfrak{N}^{\cdot}$ is cartesian closed; it may be shown that $\mathfrak{Q}^{\square\square}$ is «partially» cartesian closed. More precisely:

Let $f: A \to B$ be a small functor and K a small category; if there exist left Kan extensions along f for functors from A to K, each small functor $g: H \to K$ admits a cofree object G relative to the partial product functor $\cdot \times f: \mathfrak{Q}^{\square\square} \to \mathfrak{Q}^{\square\square}$.

Indeed, G is the composite functor:

$$H^A \xrightarrow{g^A} K^A \xrightarrow{L} K^B ,$$

where L is the left Kan extension functor (adjoint to K^f). There is a similar result replacing $\mathfrak{Q}$ by $\mathfrak{Q}_{\downarrow}$ and left Kan extensions by right Kan extensions.

From Propositions 6 and 8, it follows:

PROPOSITION 9. *The double category $\mathfrak{Q}$ is representable and corepresentable and $\mathfrak{F}$ admits small projective and inductive $\mathfrak{Q}$-wise limits.*

In fact, Gray has given in [G1] an explicit construction of $\mathfrak{Q}$-wise limits: Let $F: A \to \mathfrak{F}$ be a functor, where A is a small category.

1° F admits as an inductive $\mathfrak{Q}$-wise limit the source $K(F)$ (denoted by $[1, F]$ in Gray) of the fibration $k_F: K(F) \to A$ associated to F. (The category $K(F)$ is called in [E1] the «catégorie produit croisé associée à l'espèce de morphismes» F.) The category $F(u)$, for each object u of A, is identified to a sub-category of $K(F)$. From a general result of Gray (the Yoneda-like lemma [G1]), it follows that, if $F': A \to \mathfrak{F}$ is a functor, the $\mathfrak{Q}$-wise transformations from F to F' are in a one-to-one correspondence (a restriction of the adjoint K of $d\mathfrak{Q}_A: \mathfrak{F} \to T(\mathfrak{Q}, A)$) with the functors $h: K(F) \to K(F')$ such that $k_{F'} . h = k_F$.

2° F admits as a projective $\mathfrak{Q}$-wise limit the sub-category $L(F)$ of $K(F)^A$ formed by the natural transformations $t: A \Rightarrow K(F)$ such that $k_F\, t$ is an identity. $L(F)$ is isomorphic with the category of crossed transformations, whose objects are the crossed homomorphisms (defined in [E1]);

the set of components of its greatest sub-groupoid is called in [E1] the *first non-abelian cohomology class of* F, by analogy with the case where A is a group and F a A-module. This remark might be helpful to define the higher order non-abelian cohomology classes of F (see also the Appen-
1 dix of Bourn [Bo]).

2. *Limits relative to a sub-2-category.*

The following criterium is often useful in applications, for example we will use it in the next section.

PROPOSITION 10. *Let* C *be a 2-category and* H *a full sub-2-category (i. e.* H^1 *and* H^2 *are full sub-categories of* C^1 *and* C^2 *). If the insertion* $j: H^1_0 \hookrightarrow C^1_0$ *admits an adjoint (resp. a coadjoint) and if* C^1_0 *admits* $Q(C)$*-wise inductive (resp. projective)* A*-limits, then* H^1_0 *admits* $Q(H)$*-wise inductive (resp. projective)* A*-limits.*

Δ. Since H is a full sub-2-category of C, the double category $Q(H)$ of the up-squares of H is a full double sub-category of $Q(C)$, and the category $T(Q(H), A)$ is identified with a full sub-category of $T(Q(C), A)$. The hypotheses imply that the composite functor:

$$H^1_0 \longrightarrow C^1_0 \xrightarrow{\sim} Q(C)^{\square\square}_0 \xrightarrow{d_{Q(C)A}} T(Q(C), A)$$

admits an adjoint (resp. a coadjoint). This functor taking its values into the full sub-category $T(Q(H), A)$, we deduce that its restriction from H^1_0 to $T(Q(H), A)$ also admits an adjoint (resp. a coadjoint). Hence H^1_0 admits inductive (resp. projective) $Q(H)$-wise A-limits. ∇

3. *Limits relative to 2-categories of bimorphisms between sketches.*

In [BE] we have defined the category $\mathfrak{F}m''$ of morphisms between small mixed cone-bearing categories, and its full sub-categories:

$\mathfrak{F}m'$, whose objects are the presketches σ (i. e. two distinguished cones of σ have different bases),

$\mathfrak{P}m'$, whose objects are the limit-bearing categories,

$\mathfrak{P}m = \mathfrak{F}m' \cap \mathfrak{P}m'$, whose objects are the prototypes,

$\mathfrak{F}m''^{\mathfrak{I}\mathfrak{J}}$ (resp. $\mathfrak{F}^{\mathfrak{I}\mathfrak{J}}$), whose objects are the $(\mathfrak{I}, \mathfrak{J})$-cone-bearing categories (resp. the $(\mathcal{I},\mathcal{J})$-types), where $\mathcal{I}$ and $\mathcal{J}$ are small sets of small categories.

These different categories $\mathcal{X}$ admit small projective and inductive limits, and the following insertion functors admit adjoints:

$$\begin{array}{ccccc} & & \mathcal{P}m' & & \\ & \nearrow & & \searrow & \\ \mathcal{P}m & & & & \mathcal{F}m'' \\ & \searrow & & \nearrow & \\ & & \mathcal{F}m' & & \end{array} \qquad \mathcal{F}^{\mathcal{I}\mathcal{J}} \hookrightarrow \mathcal{F}m''^{\mathcal{I}\mathcal{J}}$$

$\mathcal{X}$ is the category of 1-morphisms of a 2-category $\mathcal{N}\mathcal{X}$, whose double category of up-squares will be denoted by $\mathcal{Q}\mathcal{X}$.

Proposition 18-2 [BE] asserts that $\mathcal{N}\mathcal{X}$ is a representable (except for $\mathcal{X} = \mathcal{F}m'$) and corepresentable 2-category, so that we deduce from Proposition 8:

PROPOSITION 11. *$\mathcal{X}$ admits small projective (resp. inductive) $\mathcal{Q}\mathcal{X}$-wise limits, for $\mathcal{X} = \mathcal{F}m''$, $\mathcal{P}m'$, $\mathcal{P}m$, $\mathcal{F}m''^{\mathcal{I}\mathcal{J}}$, $\mathcal{F}^{\mathcal{I}\mathcal{J}}$ (resp. $\mathcal{F}^{\mathcal{I}\mathcal{J}}$ and $\mathcal{F}m'$).*

Δ. Using the preceding results, we may give an explicit construction of some of these limits. Let $S: A \to \mathcal{X}$ be a functor, where A is a small category. We denote by F the functor from A to $\mathcal{F}$ got by composing S with the forgetful functor from $\mathcal{X}$ to $\mathcal{F}$. If $\mathcal{X}$ is a proper sub-category of $\mathcal{F}m''$, we consider the composite functor $\hat{S}$:

$$A \xrightarrow{S} \mathcal{X} \hookrightarrow \mathcal{F}m''.$$

1° If $\mathcal{X} = \mathcal{F}m''$ or $\mathcal{F}m''^{\mathcal{I}\mathcal{J}}$, then S admits:

as an inductive $\mathcal{Q}\mathcal{X}$-wise limit the cone-bearing category $K(S)$ got by equipping $K(F)$ (the inductive $\mathcal{Q}$-wise limit of F) with all the cones $i_u \gamma_u$, where $i_u: F(u) \to K(F)$ is the insertion and where γ_u is a distinguished cone of $S(u)$, for each object u of A;

as a projective $\mathcal{Q}\mathcal{X}$-wise limit the cone-bearing category $L(S)$ got by equipping $L(F)$ (the $\mathcal{Q}$-wise projective limit of F) with the cones T such that $v_u T$ be a distinguished cone of $S(u)$ for each object u of A, where $v_u: L(F) \to F(u)$ is the valuation functor, which maps the natural transformation $t: A \rightrightarrows K(F)$ onto $t(u)$.

2° If $\mathcal{X} = \mathcal{F}_{m}'$, then $K(\hat{S})$ is a presketch, which is the $\mathcal{Q}\mathcal{X}$-wise inductive limit of S. If $\mathcal{X} = \mathcal{P}_m$ or $\mathcal{P}_m'$ (resp. $= \mathcal{F}^{\mathcal{I}\mathcal{J}}$), it follows from part 1 and Proposition 10 that S admits as an inductive $\mathcal{Q}\mathcal{X}$-wise limit the limit-bearing category (resp. the $(\mathcal{I},\mathcal{J})$-type) freely associated to $K(\hat{S})$.

3° The insertion functors i_u, for $u \in A_0$, preserve connected limits; if A defines a preorder on the set of its objects, they preserve all limits. Using these facts we deduce:

a) Let us suppose that $\mathcal{X} = \mathcal{P}_m'$ (resp. $\mathcal{P}_m$) and that the indexing categories of $S(u)$ are connected for each object u of A, or that A defines a preorder. Then $K(\hat{S})$ is a limit-bearing category (resp. a prototype), so that it is the inductive $\mathcal{Q}\mathcal{X}$-wise limit of S. Moreover the insertion from the category $L(F)$ into $K(F)^A$ reflecting limits, $L(\hat{S})$ is also a limit-bearing category (resp. a prototype), projective $\mathcal{Q}\mathcal{X}$-wise limit of S.

b) Finally, if $\mathcal{X} = \mathcal{F}^{\mathcal{I}\mathcal{J}}$ and if $\mathcal{I}$ and $\mathcal{J}$ are sets of connected categories, or if A defines a preorder, $L(\hat{S})$ is a $(\mathcal{I},\mathcal{J})$-type, which is the $\mathcal{Q}\mathcal{X}$-wise projective limit of S. ∇

4. *Lax morphisms between sketched structures.*

In this section σ will be a projective limit-bearing category (Σ, Γ) and $\mathcal{I}$ is the set of its indexing categories.

DEFINITION. If D is a double category and if ϕ and ϕ' are σ-structures in the first category D_0^2 of 1-morphisms of D, a *D-wise σ-morphism from ϕ to ϕ'* is defined as a σ-structure τ in D^1 such that

$$\phi = \alpha^2 \tau \quad \text{and} \quad \phi' = \beta^2 \tau .$$

EXAMPLE. If B is a category and $\square B$ the double category of its commutative squares, the $\square B$-wise σ-morphisms are identified with σ-morphisms in B (by identifying a functor to $\boxminus B$ with a natural transformation to B).

PROPOSITION 12. *If D is a double category and if the functors*

$$\alpha^2 : D^1 \to D_0^2, \quad \beta^2 : D^1 \to D_0^2 \quad \textit{and} \quad \kappa^2 : (D^2 * D^2)^1 \to D^1$$

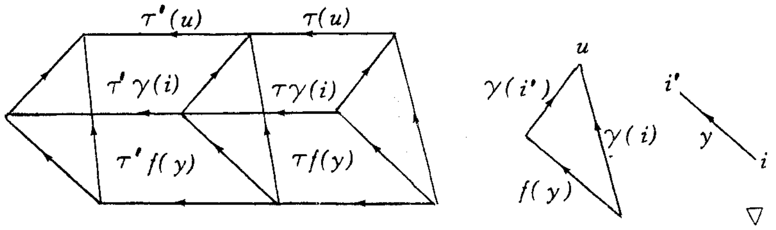
preserve projective limits indexed by elements of $\mathcal{I}$, then the D-wise σ-morphisms define a sub-category of $T(D, \Sigma)$.

Δ. Let τ be a D-wise σ-morphism from ϕ to ϕ' and τ' a D-wise

σ-morphism from ϕ' to ϕ''. They have a composite $\tau'' = \kappa^2\,[\tau', \tau]$ in $T(D, \Sigma)$, which is a D-wise transformation from ϕ to ϕ''. Let $\gamma \in \Gamma$ a cone with basis $f: I \to \Sigma$; the cone $\tau''\gamma = \kappa^2\,[\tau', \tau]\,\gamma$ is the image by κ^2 of the cone γ', with basis $[\tau' f, \tau f] : I \to (D^2 * D^2)^1$ such that

$$\gamma'(i) = (\tau'\gamma(i), \tau\gamma(i)) \text{ for each } i \in I_0 .$$

Since $\tau\gamma$ and $\tau'\gamma$ are limit-cones in D^1, the cone γ' is a limit-cone in the category $(D^2 * D^2)^1$, which is the pullback of (α^2, β^2); its image by κ^2, which is $\tau''\gamma$, is a limit-cone in D^1. Hence τ'' is a σ-structure in D^1, i. e. a D-wise σ-morphism from ϕ to ϕ''.



∇

We consider now the case where D is the double category of up-squares of a 2-category C.

DEFINITION. Let C be a 2-category, ϕ and ϕ' two σ-structures in C_0^1. *A C-lax σ-morphism from ϕ to ϕ'* is defined as a $Q(C)$-wise σ-morphism τ from ϕ to ϕ' such that $\tau f(y)$ be a commutative square for any morphism y of I, if $f: I \to \Sigma$ is the basis of a cone $\gamma \in \Gamma$. 1

PROPOSITION 13. *If C is 2-category and if C_0^1 admits projective limits indexed by elements of $\mathfrak{I}$ and preserved by the insertion into C^2, then the C-lax σ-morphisms define a sub-category of $T(Q(C), \Sigma)$.*

Δ. Let τ and τ' be C-lax σ-morphisms from ϕ to ϕ' and from ϕ' to ϕ'', and τ'' their composite in $T(Q(C), \Sigma)$. If $f: I \to \Sigma$ is the basis of a cone $\gamma \in \Gamma$, since $\tau f(y)$ and $\tau' f(y)$ are commutative squares, so is

$$\tau'' f(y) = \tau' f(y) \boxminus \tau f(y) \text{ for each } y \text{ in } I.$$

C_0^1 admitting projective I-limits preserved by the insertion into C^2, the functor from I to $\boxminus C_0^1$ restriction of τf admits a projective limit which is also a projective limit of τf (Prop. 5); hence the limit-cone $\tau\gamma$ takes its values in $\boxminus C_0^1$, as well as $\tau'\gamma$. The composite

$$\tau''\gamma(i) = \tau'\gamma(i) \boxbox \tau\gamma(i), \quad \text{for each } i \in I_0,$$

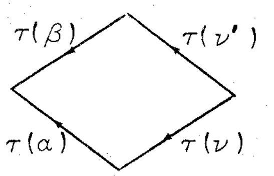
is a commutative square, so that $\tau''\gamma$ takes its values in $\boxminus C_0^1$ and, considered as a cone in $\boxminus C_0^1$, it is a limit-cone (limits in $\boxminus C_0 = (C_0^1)^2$ being computed pointwise). Hence (Proposition 5) $\tau''\gamma$ is a limit-cone in the category $Q(C)^{\boxminus}$. This proves that τ'' is a $Q(C)$-wise σ-morphism, i. e. a C-lax σ-morphism from ϕ to ϕ''. ∇

5. *Lax double functors.*

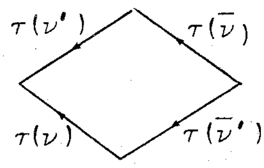
We apply here the preceding results to the sketch $\sigma\mathfrak{F} = (\Sigma\mathfrak{F}, \Gamma\mathfrak{F})$ of categories.

DEFINITION. Let A and B be double categories, ϕ_A and ϕ_B the corresponding categories in $\mathcal{F}$. A $\mathfrak{N}$-lax $\sigma\mathfrak{F}$-morphism from ϕ_A to ϕ_B is called *a lax double functor from A to B*.

The lax double functors from A to B are exactly the $\mathcal{Q}$-wise transformations (where $\mathcal{Q}$ is always the double category of quintets) τ such that $\tau(\mu)$ be a commutative square for $\mu \in \{\alpha, \beta, \nu, \nu', \bar{\nu}, \bar{\nu}'\}$. Indeed, these conditions imply that



and



are pullbacks in $\boxminus\mathcal{F}$ (and therefore in $\mathcal{Q}^{\boxminus}$), since pullbacks in $\boxminus\mathcal{F} = \mathcal{F}^2$ are computed pointwise and ϕ_A and ϕ_B are $\sigma\mathfrak{F}$-structures.

It follows from Proposition 13 that the lax double functors between small double categories define a sub-category of $T(\mathcal{Q}, \Sigma\mathfrak{F})$.

Let A and B be double categories, ϕ_A and ϕ_B the associated categories in $\mathcal{F}$; we denote by «$\circ$» the laws of A^1 and B^1, by «$\cdot$» those of A^2 and B^2, by a, b, i, k and a', b', i', k' respectively the images of $\alpha, \beta, \iota, \kappa$ by ϕ_A and ϕ_B.

PROPOSITION 14. *The lax double functors from A to B are in one-to-*

one correspondence with the 4-tuples (g_0, g, t, t'), *where*

1° $g : A^1 \to B^1$ *and* $g_0 : A^2_0 \to B^2_0$ *are functors such that*

$$a' g = g_0 a \quad \text{and} \quad b' g = g_0 b .$$

This implies the existence of a functor g':

$$(x', x) \mapsto (g(x'), g(x)) \quad \text{from } (A^2 * A^2)^1 \text{ to } (B^2 * B^2)^1 .$$

2° $t : i' g_0 \to g i$ *and* $t' : k' g' \to g k$ *are natural transformations.*

3° *The following coherence axioms are satisfied:*

$$(u) \quad t'(x, e).(g(x) \circ t(e)) = g(x) = t'(e', x).(t(e') \circ g(x))$$

$$\text{for each } x : e \to e' \text{ in } A^2_0 .$$

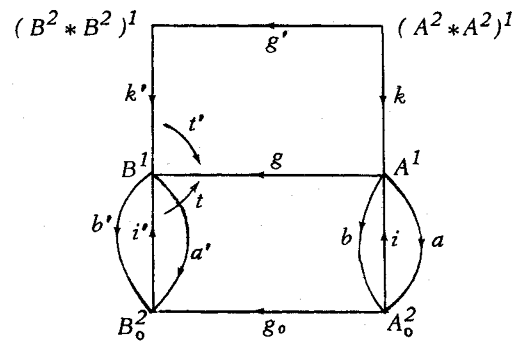
$$(a) \quad t'(x'', x' \circ x).(g(x'') \circ t'(x', x)) = t'(x'' \circ x', x).(t'(x'', x') \circ g(x))$$

$$\text{for each path } (x'', x', x) \text{ in } A^2_0 .$$

Δ. Let τ be a lax double functor from A to B. We take for g and for g_0 the functors $\tau(2)$ and $\tau(1)$, for t and t' the natural transformations arising in the quintets $\tau(\iota)$ and $\tau(\kappa)$. Condition 1 is satisfied, $\tau(\alpha)$ and $\tau(\beta)$ being commutative squares. The two coherence axioms are respectively deduced by pointwise computation from the axioms

$$\tau(\kappa) \boxminus \tau(j_\alpha) = \tau(2) = \tau(\kappa) \boxminus \tau(j_\beta),$$

$$\tau(\kappa) \boxminus \tau(\kappa') = \tau(\kappa) \boxminus \tau(\kappa'').$$



2° If (g_0, g, t, t') is given, we construct as follows a lax double functor τ:

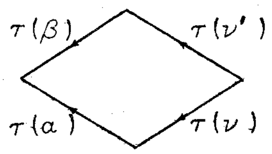
$\tau(\alpha)$ and $\tau(\beta)$ «are» the commutative squares

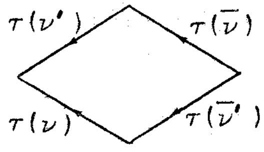
$$(a', g_0, g, a) \quad \text{and} \quad (b', g_0, g, b),$$

and g' is their canonical pullback in $\boxminus\mathcal{F}$ (and also in $\mathcal{Q}^{\boxminus}$),

$$\tau(\iota) = (i', g, t, g_0, i) \quad \text{and} \quad \tau(\kappa) = (k', g, t, g', k).$$

As $\Sigma\mathcal{G}$ is «generated» by $\alpha, \beta, \iota, \kappa$, the other quintets $\tau(\lambda)$, for λ in $\Sigma\mathcal{G}$, are then deduced as composites or factors relative to the pullbacks:



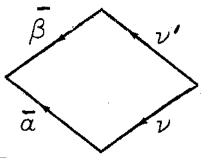


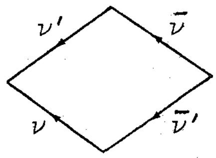
(in $\boxminus\mathcal{F}$). The axioms *(u)* and *(a)* imply that we have so defined a functor $\tau : \Sigma\mathcal{G} \to \mathcal{Q}^{\boxminus}$. ∇

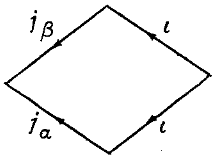
REMARKS. 1° Let A and B be 2-categories. The 4-tuples considered in Proposition 14 are then the morphisms of bicategories from the bicategory A to B defined by Bénabou [B2] (called pseudo-functors in [G1]); as a natural transformation toward the discrete category A_0^2 is an identity, any $\mathcal{Q}$-wise $\sigma\mathcal{G}$-morphism from ϕ_A to ϕ_B is a lax double functor.

2° By a process of «laxification» similar to that leading from 2-categories to bicategories and from 2-functors to morphisms of bicategories,
1 Moreau [M] defines lax double functors between dicategories, i. e. categories equipped with a second law which is unitary and associative «up to isomorphisms» which reduce for double categories to those considered here; he generalizes Proposition 14 to the case where A and B are dicategories.

3° The $\sigma\mathcal{G}$-morphisms are identical to the $\bar{\sigma}\mathcal{G}$-morphisms (see Part C-0), where $\bar{\sigma}\mathcal{G}$ is the sketch $(\Sigma\mathcal{G}, \bar{\Gamma}\mathcal{G})$ in which the pullbacks are







But $\mathcal{N}$-lax $\bar{\sigma}\mathcal{G}$-morphisms are only those lax double functors τ corresponding to 4-tuples (g_0, g, t, t') such that t is an identity (since the factors $\tau(j_\alpha)$ and $\tau(j_\beta)$ must be commutative squares); they are said *unitary*.

Let A and B be double categories. We denote by

$$k_A : K(A) \to \Sigma\mathcal{F} \quad \text{and} \quad k_B : K(B) \to \Sigma\mathcal{F}$$

the fibrations corresponding to ϕ_A and ϕ_B. With the notations of [E1] , a morphism of $K(A)$ is a triple $m = (z, \mu, s)$, where $\mu : \omega \to \omega'$ is a morphism of $\Sigma\mathcal{F}$, where s is an object of $\phi_A(\omega)$ and

$$z : s' \to s'' \text{ in } \phi_A(\omega'), \text{ if } s' = \phi_A(\mu)(s).$$

Identifying $\phi_A(\omega)$ to a sub-category of $K(A)$ and the «cartesian» morphism (s', μ, s) to (μ, s), we get $m = z.(\mu, s)$ in $K(A)$.

PROPOSITION 15. *There is a bijection from the set of lax double functors from A to B onto the set of functors $h : K(A) \to K(B)$ such that:*

(1) $$k_B h = k_A \text{ and } h(\mu, s) = (\mu, h(s)),$$

for each cartesian morphism (μ, s), where $\mu \in \{\alpha, \beta, \nu, \nu', \bar{\nu}, \bar{\nu}'\}$.

Δ. This bijection is a restriction of the bijection K' (considered after Proposition 9) from the set of $\mathcal{Q}$-wise transformations from ϕ_A to ϕ_B onto the set of functors from $K(A)$ to $K(B)$ commuting with the fibrations to $\Sigma\mathcal{F}$. Indeed K' maps τ onto the functor h whose restriction to $\phi_A(\omega)$ is $\tau(\omega)$ for each object ω of $\Sigma\mathcal{F}$ and such that

$$h(\mu, s) = (t_\mu(s), \mu, h(s)),$$

if t_μ is the natural transformation arising in the quintet $\tau(\mu)$. Hence, h satisfies the condition *(1)* iff t_μ is an identity (i.e. iff $\tau(\mu)$ is a commutative square) for $\mu \in \{\alpha, \beta, \nu, \nu', \bar{\nu}, \bar{\nu}'\}$. ∇

This proposition reduces the study of lax double functors to that of ordinary functors. 1+

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MULTIPLE FUNCTORS
II. THE MONOIDAL CLOSED CATEGORY OF MULTIPLE CATEGORIES

by Andrée and Charles EHRESMANN

This paper is the Part II of our work on multiple functors, which was announced in Part I [5].

In this Part II we define directly (i.e., without reference to sketched structures) and study the category *MCat* of multiple categories. *MCat* is partially monoidal closed, for the «square product» which associates to an m-fold category A and an n-fold category B an $(n+m)$-fold category B ■ A, and for a closure functor *Hom* such that *Hom*(A, B), the $(n-m)$-fold category of «generalized natural transformations», is the set of multiple functors from A to B with compositions deduced «pointwise» from the $(n-m)$ last compositions of B.

One application is a criterium for the existence of colimits in *MCat*, which suggests the introduction of «infinite-fold» categories to embed *MCat* into a complete and cocomplete category. Another one is an existence theorem for generalized limits in n-fold categories, which admits as a particular case a result of Gray [13] and Bourn [3] on representable 2-categories (generalized in Part I to double categories); however the proof given here is more «structural» (and much shorter!).

Other applications are the descriptions of the cartesian closed structure of the category of n-fold categories, and of a monoidal closed structure which «laxifies» it. Part III (to appear in Vol. XIX-4) is devoted to them.

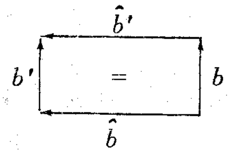
In an Appendix, the constructions of B ■ A and of *Hom*(A, B) are translated in terms of sketched structures. This leads to similar results on internal multiple sketched structures (in particular internal multiple categories), which will be given in a subsequent paper. 1

Notations for Hom have been «inversed» relatively to Part I, in order to conform to more usual conventions.

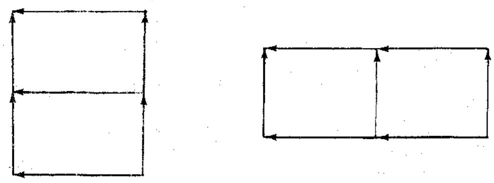
0. Motivating examples.

n-fold categories were introduced in [7] by induction, as categories internal to the category of $(n-1)$-fold categories. They are also defined as realizations in the category of sets of the sketch of n-fold categories, which is the n-th tensor power of the sketch of categories (see [5]). In this Part, we define and study them directly (i. e., without using the theory of sketched structures).

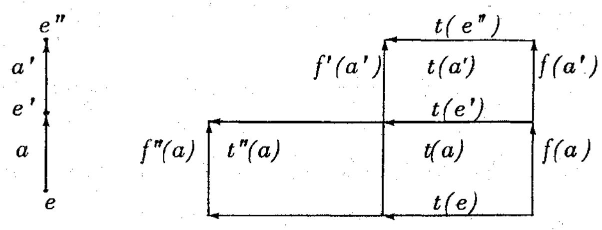
Double categories introduce themselves very naturally as soon as natural transformations are considered. Indeed, if B is a category, its commutative squares



form a double category $\square B$ for the «vertical and horizontal» compositions :

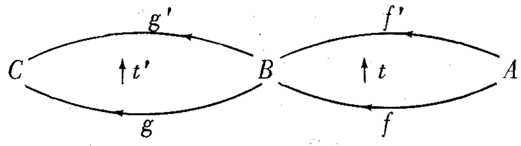


A natural transformation $t: f \to f': A \to B$ may be seen as a functor from A to the vertical category of squares of B, while the composition of natural transformations is deduced from the horizontal composition :



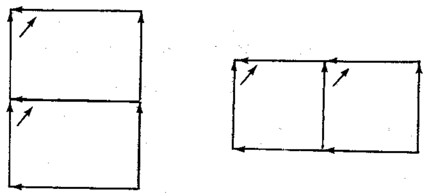
By induction, one defines (see [7], page 398) the multiple category of squares of squares..., which intervene to define transformations between natural transformations and so on... . We will generalize this construction in Part 2.

Other «usual» double categories are the 2-categories (considered by many authors), which are those double categories in which the objects for the second composition are also objects for the first one. For example, natural transformations between small categories form a 2-category, Nat.



There is also the 2-category of homotopy classes of continuous mappings, very useful in Algebraic Topology. 1

To a 2-category M is canonically associated the double category $Q(\mathrm{M})$ of its (lax-)squares, with the vertical and horizontal compositions:



(see [8], where $Q(\mathrm{Nat})$ is introduced in 1963 under the name of double category of «quintets», and [11, 2]); such double categories are characterized in [15].

More generally, n-categories are special n-fold categories, in which objects for some of the compositions are also objects for the other ones, and the lax-squares will be generalized in Part IV.

A. The category of n-fold categories.

Let n be a positive integer.

An n-fold category A (on the set $\underline{A}$) is a sequence of n categories $(A^0, \dots, A^{n-1})$ with the same set $\underline{A}$ of morphisms, satisfying the *permutability axiom:*

(P) (A^i, A^j) is a double category for each pair (i, j) of integers, such that $i \neq j$, $0 \leqslant i < n$, $0 \leqslant j < n$ (see [5]).

An element of $\underline{A}$ is called a *block of* A, and A^i is the *i-th category of* A. We also say that A is *a multiple category, of multiplicity n*.

The axiom (P) means that, for each i, $0 \leqslant i \leqslant n-1$, the maps source (or domain), target (or codomain) and composition of A^i define functors with respect to the $(n-1)$ other categories A^j. In particular, it follows that the set of objects of A^i defines a subcategory of A^j, for each $j \neq i$. Moreover two of the categories A^i and A^j for $j \neq i$ are identical iff $A^i = A^j$ is a commutative category (i.e., a coproduct of commutative monoids). For example, if C is a commutative monoid, then $(\underbrace{C, \dots, C}_{n \text{ times}})$ is an n-fold category.

In the definition of the n-fold category A, the sequence of categories $(A^0, \dots, A^{n-1})$ is well given. If γ is a permutation of the set

$$n = \{0, 1, \dots, n-1\},$$

then $(A^{\gamma(0)}, \dots, A^{\gamma(n-1)})$ is also an n-fold category on A, but it is different from A as an n-fold category and we denote it A^γ. If $(i_1, \dots, i_m)$ is a sequence of m distinct elements of n, then $(A^{i_1}, \dots, A^{i_m})$ is an m-fold category, denoted more simply by $A^{i_1, \dots, i_m}$. If $\underline{A}^{dis}$ denotes the discrete category on the set $\underline{A}$ (there are only objects), then

$$(A^0, \dots, A^{n-1}, \underbrace{\underline{A}^{dis}, \dots, \underline{A}^{dis}}_{m \text{ times}})$$

is an $(n+m)$-fold category, whatever be the integer m.

If A and B are n-fold categories, an *n-fold functor* $f: A \to B$ from A to B is defined by a map $f: \underline{A} \to \underline{B}$ defining a functor

$$f: A^i \to B^i \quad \text{for each } i < n.$$

Let Cat_n be the category whose objects are the small n-fold categories (i.e., the n-fold categories on small sets, small meaning that they belong to a given universe), and whose morphisms are the n-fold functors between them. By convention, a 0-fold category is a set, a 1-fold category is a category. So Cat_0 is the category Set of (small) sets and Cat_1, the category of (small) categories.

For a permutation γ of the set n, we denote by $\tilde{\gamma}: Cat_n \to Cat_n$ the isomorphism «permutation of the compositions»:

$$(f: \mathrm{A} \to \mathrm{B}) \longmapsto (f: \mathrm{A}^\gamma \to \mathrm{B}^\gamma).$$

These isomorphisms will be useful, since they permit to change the order of compositions when necessary.

PROPOSITION 1. *Cat_n is complete and, for each $i < n$, limits are preserved by the functor $U^i: Cat_n \to Cat$:*

$$(f: \mathrm{A} \to \mathrm{B}) \longmapsto (f: \mathrm{A}^i \to \mathrm{B}^i)$$

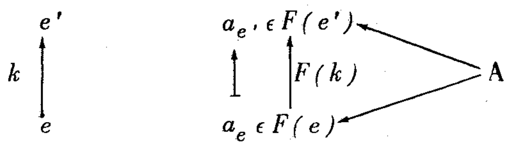
forgetting the compositions other than the i-th one.

PROOF. Let $F: K \to Cat_n$ be a functor indexed by a small category K. For each i the composite functor

$$K \xrightarrow{F} Cat_n \xrightarrow{U^i} Cat$$

admits a (projective) limit A^i on the set $\underline{\mathrm{A}}$ of families $(a_e)_e$ indexed by the objects e of K, such that:

$$a_e \in F(e) \text{ and } F(k)(a_e) = a_{e'} \text{ for each } k: e \to e' \text{ in } K.$$



It is easily seen that $(\mathrm{A}^0, \ldots, \mathrm{A}^{n-1})$ is an n-fold category A, which is the limit of F. ∇

The following proposition will be used to prove Proposition 3.

PROPOSITION 2. *Let A be an n-fold category and M an infinite subset of $\underline{\mathrm{A}}$. Then the n-fold subcategory M of A generated by M is such that $\underline{\mathrm{M}}$ is equipotent with M.*

PROOF. $\underline{\mathrm{M}}$ is constructed as the union of the increasing sequence of sets M_l, $l \in \mathrm{N}$, defined by induction as follows: $M_0 = M$; if M_l is defined, then M_{l+1} is obtained by adding to M_l, for each $i < n$:

- the source and target in A^i of the blocks m in M_l,

- the composites in A^i of all the couples (m', m) of blocks in M_l admitting a composite in A^i.

Since M is infinite, it is seen by induction that M_{l+1} is equipotent to M_l, hence to M. It follows that $\underline{M} = \bigcup_{l \in N} M_l$ is also equipotent to M. ∇

Let m be an integer, $m < n$. There is a faithful functor

$$U_{n,m}: Cat_n \to Cat_m ,$$

which «forgets the $(n-m)$ first compositions»: it maps A onto $A^{n-m, \dots, n-1}$ and $f: A \to B$ onto $f: U_{n,m}(A) \to U_{n,m}(B)$.

From Proposition 1, it follows that the functors $U_{n,m}$ preserve limits. We shall prove in Section D that they admit left adjoints.

By composing $U_{n,m}$ with the isomorphism $\tilde{\gamma}: Cat_n \to Cat_n$ corresponding to a permutation γ of the set n (see before Proposition 1), we obtain faithful functors $Cat_n \to Cat_m$ mapping A onto the m-fold category $A^{i_1, \dots, i_m}$ for every sequence $(i_1, \dots, i_m)$ of m distinct elements of n.

In particular, the functor $U_{n,0}: Cat_n \to Set$ is defined by:

$$(f: A \to B) \longmapsto (f: \underline{A} \to \underline{B}).$$

PROPOSITION 3. *This faithful functor $U_{n,0}: Cat_n \to Set$ admits quasi-quotient objects.*

PROOF. This assertion is deduced from the general existence theorem of quasi-quotient objects of [9], whose hypotheses are satisfied due to Propositions 1 and 2. In fact, we deduce from it the more precise result (used later on):

Let r be a relation on a set $\underline{H}$ and suppose given a sequence H of n structures of neocategories (i.e., we do not impose unitarity nor associativity) H^i on $\underline{H}$. Then there exists a universal solution to the problem of finding an n-fold category A and a map $f: \underline{H} \to \underline{A}$ compatible with r and defining a neofunctor $f: H^i \to A^i$ for each $i < n$. If $\hat{r}: H \to B$ is such a universal solution (i.e., every other solution factors through it uniquely), B is an *n-fold category quasi-quotient of* H *by* r. ∇

PROPOSITION 4. *Cat_n is cocomplete. The functor $U_{n,m}: Cat_n \to Cat_m$ preserves coproducts (but not every colimit).*

PROOF. 1° A family $(\mathrm{A}_\lambda)_{\lambda \in \Lambda}$ of n-fold categories admits as a coproduct the n-fold category A on the set

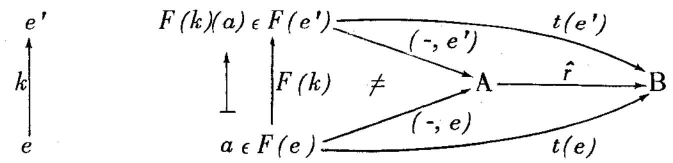
$$\{ (a, \lambda) \mid a \in \underline{\mathrm{A}}_\lambda,\ \lambda \in \Lambda \}$$

such that A^i is the category coproduct of the categories A^i_λ, $\lambda \in \Lambda$.

2° Let $F: K \to Cat_n$ be a functor indexed by a small category K, and let A be the n-fold category coproduct of the n-fold categories $F(e)$, for all objects e of K. Let r be the relation on $\underline{\mathrm{A}}$ defined by:

$$(a, e) \sim (F(k)(a), e') \quad \text{for each } k: e \to e' \text{ in } K \text{ and } a \in F(e).$$

According to Proposition 3, there exists an n-fold category B quasi-quotient of A by r. From the general construction of colimits from coproducts



and quasi-quotients [9] it follows that B is a colimit of F, the colimit cone being $t: F \Longrightarrow \mathrm{B}$, where

$$t(e) = (F(e) \xrightarrow{(-,e)} \mathrm{A} \xrightarrow{\hat{r}} \mathrm{B}). \qquad \nabla$$

REMARK. Since the functor $U_{n,m}$ does not preserve all colimits, it does not admit a right adjoint.

B. The monoidal category of multiple categories.

In this section, we consider the category *MCat* of multiple categories, defined as follows:

- Its objects are all the small n-fold categories, for every integer n (hence sets, categories, double categories, ... are objects);

- Let A be an m-fold category and B an n-fold category. If $m \leqslant n$, the morphisms $f: \mathrm{A} \to \mathrm{B}$, called *multiple functors*, are the m-fold functors f, from A to the m-fold category $\mathrm{B}^{0,\dots,m-1}$ (in which the $(n-m)$ last compositions of B are forgotten). If $m > n$, there is no morphism from A to B.

- The composition is trivially deduced from the composition of maps.

For each integer n, the category Cat_n is a full subcategory of the category $MCat$.

PROPOSITION 5. *1° MCat is complete and the faithful functor*

$$U: MCat \to Set: (f: \mathrm{A} \to \mathrm{B}) \longmapsto (f: \underline{\mathrm{A}} \to \underline{\mathrm{B}})$$

admits quasi-quotient objects.

2° For each integer n, the insertion $Cat_n \hookrightarrow MCat$ preserves limits, colimits and quasi-quotient objects.

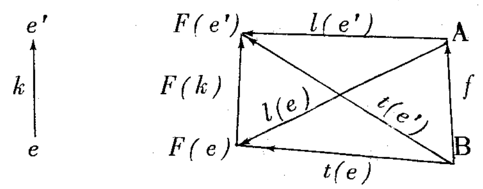
PROOF. 1° Let $F: K \to MCat$ be a functor indexed by a small category K. For each object e of K, let n_e be the multiplicity of the multiple category $F(e)$. Let n be the least of the integers n_e, for all objects e of K. By the definition of the multiple functors, we have, for each $m \leqslant n$, a functor $F_m: K \to Cat_m$ such that

$$(k: e \to e') \longmapsto F(k): F(e)^{0,\dots,m-1} \to F(e')^{0,\dots,m-1}.$$

It follows from Proposition 1 that F_n is the basis of a limit cone in Cat_n, say $l: \mathrm{A} \Longrightarrow F_n$ and that $\mathrm{A}^{0,\dots,m-1}$ is the limit of F_m for each $m < n$.

a) We prove that A is also the limit of F in $MCat$. Indeed, for each object e of K, $l(e): \mathrm{A} \to F(e)$ is a multiple functor, the multiplicity n of A being lesser than n_e, so that $l: \mathrm{A} \Longrightarrow F$ is also a cone in $MCat$. Let $t: \mathrm{B} \Longrightarrow F$ be a cone in $MCat$. Since $t(e): \mathrm{B} \to F(e)$ is a multiple functor, the multiplicity m of B is lesser than each n_e; hence $m \leqslant n$ and $t: \mathrm{B} \Longrightarrow F_m$ is a cone in Cat_m. There is a unique $f: \mathrm{B} \to \mathrm{A}^{0,\dots,m-1}$ such that

$$(t: \mathrm{B} \Longrightarrow F_m) = (\mathrm{B} \xrightarrow{f} \mathrm{A}^{0,\dots,m-1} \overset{l}{\Longrightarrow} F_m),$$

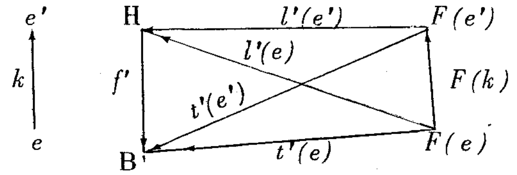


and $f: A \to B$ is the unique morphism such that

$$(t: B \Longrightarrow F) = (B \xrightarrow{f} A \overset{l}{\Longrightarrow} F).$$

b) Consider now the case where $n_e = n$ for each object e of K, so that F takes its values in Cat_n. According to Proposition 4, there exists a colimit cone $l': F_n \Longrightarrow H$ in Cat_n. Then $l': F \Longrightarrow H$ is a colimit cone in $MCat$. Indeed, let $t': F \Longrightarrow B'$ be an inductive cone, with vertex the p-fold category B'. Then $n \leqslant p$, and $t': F_n \Longrightarrow B'^{0,\dots,n-1}$ is an inductive cone which factorizes through H:

$$(t': F_n \Longrightarrow B'^{0,\dots,n-1}) = (F_n \overset{l'}{\Longrightarrow} H \xrightarrow{f'} B'^{0,\dots,n-1}).$$

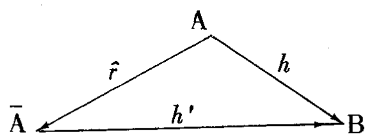


So $f': H \to B'$ is the unique morphism such that

$$(t': F \Longrightarrow B') = (F \overset{l'}{\Longrightarrow} H \xrightarrow{f'} B').$$

3° Let A be an n-fold category, r a relation on $\underline{A}$ and $\bar{A}$ the n-fold category quasi-quotient of A by r (which exists, Proposition 3). Then $\bar{A}$ is also an object quasi-quotient of A by r with respect to the functor U. Indeed, let $h: A \to B$ be a multiple functor compatible with r; the multiplicity of B must be greater than n, so that there exists in Cat_n a factorization:

$$(h: A \to B^{0,\dots,n-1}) = (A \xrightarrow{\hat{r}} \bar{A} \xrightarrow{h'} B^{0,\dots,n-1})$$



where $\hat{r}: A \to \bar{A}$ is the canonical multiple functor. Then $h': \bar{A} \to B$ is the unique morphism factorizing h through $\bar{A}$ in $MCat$. ∇

REMARK. $MCat$ is not cocomplete. In Proposition 10 we shall prove that a functor $F: K \to MCat$ admits a colimit iff the multiplicities of all the $F(e)$ for e object of K are bounded.

There is a partial monoidal structure on $MCat$, whose tensor product extends the square product $B \blacksquare A$ of two categories defined in [5] as being the double category $(\underline{B}^{dis} \times A,\ B \times \underline{A}^{dis})$, where $\underline{B}^{dis}$ denotes the discrete category on $\underline{B}$.

DEFINITION. Let A be an m-fold category and B an n-fold category. We call *square product of* (B, A), denoted by $\mathrm{B} \blacksquare \mathrm{A}$, the $(n+m)$-fold category on the product of sets $\underline{\mathrm{B}} \times \underline{\mathrm{A}}$, defined as follows:

- if $0 \leqslant i < m$, its i-th category is the product $\underline{\mathrm{B}}^{dis} \times \mathrm{A}^i$,
- if $0 \leqslant j < n$, its $(m+j)$-th category is the product $\mathrm{B}^j \times \underline{\mathrm{A}}^{dis}$.

This defines an $(n+m)$-fold category, which is the product of the $(n+m)$-fold categories:

$$(\underbrace{\underline{\mathrm{B}}^{dis}, \dots, \underline{\mathrm{B}}^{dis}}_{m \text{ times}}, \mathrm{B}^0, \dots, \mathrm{B}^{n-1}) \text{ and } (\mathrm{A}^0, \dots, \mathrm{A}^{m-1}, \underbrace{\underline{\mathrm{A}}^{dis}, \dots, \underline{\mathrm{A}}^{dis}}_{n \text{ times}}).$$

EXAMPLE. If E is a set, $\mathrm{B} \blacksquare E$ is the n-fold category whose j-th category is $\mathrm{B}^j \times E^{dis}$, for $0 \leqslant j < n$.

If H is a p-fold category, a map $g: \underline{\mathrm{B}} \times \underline{\mathrm{A}} \to \underline{\mathrm{H}}$ defines a multiple functor $g: \mathrm{B} \blacksquare \mathrm{A} \to \mathrm{H}$ iff the following conditions are satisfied:

(A1) $m + n \leqslant p$.

(A2) For each block b of B,

$$g(b, -): \mathrm{A} \to \mathrm{H}: a \mapsto g(b, a)$$

is a multiple functor.

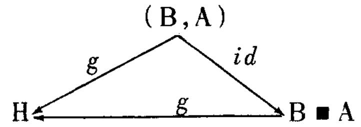
(A3) For each block a of A,

$$g(-, a): \mathrm{B} \to \mathrm{H}^{m, \dots, p-1}: b \mapsto g(b, a)$$

is a multiple functor.

In this case we say that $g: (\mathrm{B}, \mathrm{A}) \to \mathrm{H}$ *is an alternative functor.*

In particular, the identity of $\underline{\mathrm{B}} \times \underline{\mathrm{A}}$ defines an alternative functor $id: (\mathrm{B}, \mathrm{A}) \to \mathrm{B} \blacksquare \mathrm{A}$, and any alternative functor $g: (\mathrm{B}, \mathrm{A}) \to \mathrm{H}$ factors through it.



In other words, B ■ A is the solution of the universal problem «to transform an alternative functor into a multiple functor».

PROPOSITION 6. *There is a functor* ■: $MCat \times (\coprod_n Cat_n) \to MCat$ *extending the square product, with a restriction giving to* $\coprod_n Cat_n$ *a monoidal structure symmetric «up to an interchange of the compositions». (We say that MCat is partially monoidal.)* 1

PROOF. 1° We define a functor ■ : $MCat \times (\coprod_n Cat_n) \to MCat$ as follows: If $f: A \to A'$ and $g: B \to B'$ are multiple functors with A and A' of the same multiplicity (this last condition is essential), then

$$g \times f : B \blacksquare A \to B' \blacksquare A' : (b, a) \mapsto (g(b), f(a))$$

is a multiple functor $g \blacksquare f$. The map $(g, f) \mapsto g \blacksquare f$ defines the required functor ■ .

2° The square product admits as a unit the set $1 = \{0\}$, the «unitarity isomorphisms» being:

$$A \to A \blacksquare 1 : a \mapsto (a, 0) \quad \text{and} \quad A \to 1 \blacksquare A : a \mapsto (0, a),$$

for each multiple category A . It is associative up to the «associativity isomorphisms»:

$$(B' \blacksquare B) \blacksquare A \to B' \blacksquare (B \blacksquare A) : ((b', b), a) \mapsto (b', (b, a))$$

for any multiple categories A, B, B'.

3° The square product is not symmetric in the usual sense, but there is, if A is an m-fold category and B an n-fold category, the isomorphism:

$$B \blacksquare A \to (A \blacksquare B)^{\gamma} : (b, a) \mapsto (a, b),$$

where $(A \blacksquare B)^{\gamma}$ is deduced from $A \blacksquare B$ by the interchange of compositions corresponding to the permutation

$$\gamma : (0, \ldots, m+n-1) \longmapsto (n, \ldots, n+m-1, 0, \ldots, n-1). \quad \nabla$$

The square product being associative «up to an isomorphism», a sequence $(A_q, \dots, A_1)$ of multiple categories admits several composites, depending on the position of the parentheses. Any two of these composites are related by a canonical isomorphism, since $(\coprod_n Cat_n, \blacksquare)$ is monoidal. In particular, all these composites are canonically isomorphic with

$$(\dots((A_q \blacksquare A_{q-1}) \blacksquare \dots) \blacksquare A_2) \blacksquare A_1 .$$

This composite will be denoted by $A^{\blacksquare q}$, if $A_q = \dots = A_1 = A$; it is then also defined by induction:

$$A^{\blacksquare 1} = A, \quad A^{\blacksquare q} = A^{\blacksquare q-1} \blacksquare A .$$

C. The internal Hom on *MCat.*

Now we define an «internal Hom functor» on the category of multiple categories, so that *MCat* becomes partially monoidal closed. In particular this Hom associates to a category A and to a double category B the category of B-wise transformations from A (denoted by $T(B, A)$ in [5]), i.e. the set of functors $f: A \to B^0$ equipped with the composition deduced «pointwise from B^1»:

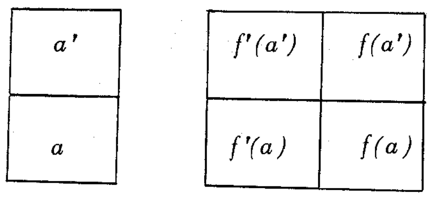
$$f' \circ_1 f: A \to B^0 : a \mapsto f'(a) \circ_1 f(a).$$

DEFINITION. Let A be an m-fold category and B an n-fold category. We call *multiple category of multiple functors from* A *to* B, and we denote by $Hom(A, B)$:

- if $m > n$, the void set;
- if $m \leqslant n$, the $(n-m)$-fold category, on the set of the multiple functors $f: A \to B$, whose j-th composition, for $0 \leqslant j < n-m$, is

$$(f', f) \mapsto (f' \circ_j f : A \to B : a \mapsto f'(a) \circ_{j+m} f(a)),$$

iff the composite $f'(a) \circ_{j+m} f(a)$ exists in B^{j+m} for each block a of A.



So, for each pair

$$(i, j), \quad 0 \leqslant i < m, \quad 0 \leqslant j < n-m,$$

the category $Hom(\mathrm{A},\mathrm{B})^j$ is a subcategory of the category of $(\mathrm{B}^i, \mathrm{B}^{m+j})$-wise transformations from A^i to $(\mathrm{B}^i, \mathrm{B}^{m+j})$. The permutability axiom is satisfied by $Hom(\mathrm{A},\mathrm{B})$ since it is satisfied by B and the compositions are defined «pointwise» from that of B.

EXAMPLES. 1° If E is a set, $Hom(E, \mathrm{B})$ is the n-fold category B^E, product of E copies of B (i.e., product in Cat_n of the family $(\mathrm{B}_e)_{e \in E}$, with $\mathrm{B}_e = \mathrm{B}$ for each e in E).

2° If A is a category and B is the double category of squares of a category C, then $Hom(A, \mathrm{B})$ is the category C^A of natural transformations between functors from A to C.

REMARK. In fact, Example 2 motivated the introduction of $Hom(\mathrm{A},\mathrm{B})$, which was generally defined in 1963 [7], under the name «multiple category of generalized transformations», represented by $\mathfrak{F}(\mathrm{B},\mathrm{A})$. We interchange here A and B in the notation to adopt a more usual convention.

If $g\colon \mathrm{A}' \to \mathrm{A}$ is an m-fold functor and $h\colon \mathrm{B} \to \mathrm{B}'$ a multiple functor,

$$(f\colon \mathrm{A} \to \mathrm{B}) \longmapsto (\mathrm{A}' \xrightarrow{g} \mathrm{A} \xrightarrow{f} \mathrm{B} \xrightarrow{h} \mathrm{B}')$$

defines a multiple functor

$$Hom(g, h)\colon Hom(\mathrm{A},\mathrm{B}) \to Hom(\mathrm{A}',\mathrm{B}').$$

This determines the functor

$$Hom\colon (\coprod_n Cat_n)^{op} \times MCat \to MCat\colon (g, h) \longmapsto Hom(g, h).$$ 1

PROPOSITION 7. *The partial functor* $- \blacksquare \mathrm{A}\colon MCat \to MCat$, *for each multiple category* A, *admits* $Hom(\mathrm{A}, -)\colon MCat \to MCat$ *as a right adjoint. (We say that* $(MCat, \blacksquare, Hom)$ *is a partial monoidal closed category.) In particular* $\coprod_n Cat_n$, *equipped with restrictions of* $\blacksquare$ *and Hom, is a monoidal closed category.*

PROOF. Let H be a p-fold category.

1° The evaluation $ev\colon (f, a) \longmapsto f(a)$ defines an alternative functor

$ev: (Hom(A,H), A) \to H$ since:

- for each block a of A,

$$ev(-,a): Hom(A,H) \to H^{m,\dots,p-1} : f \mapsto f(a)$$

is a multiple functor, by the «pointwise» definition of the compositions of $Hom(A,H)$,

- for each f in $Hom(A,H)$,

$$ev(f,-) = f: A \to H$$

is a multiple functor.

From the universal property of the square product, it follows that

$$ev: Hom(A,H) \blacksquare A \to H$$

is a multiple functor, which will be the coliberty morphism which defines $Hom(A,H)$ as the cofree object generated by H.

2° Let B be an n-fold category. Then $g: B \blacksquare A \to H$ is a multiple functor iff $g: (B,A) \to H$ is an alternative functor, i.e., iff:

- $m+n \leqslant p$ (condition $A1$),
- there is a map

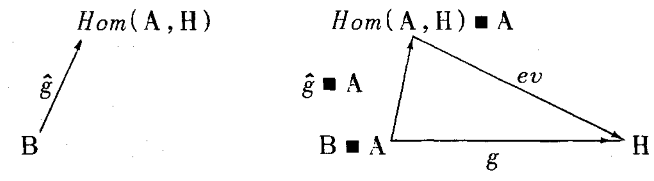
$$\hat{g}: b \mapsto g(b,-): A \to H$$

from $\underline{B}$ to the set of multiple functors from A to H (condition (A2)),

- for each block a of A, the composite

$$g(-,a) = (B \xrightarrow{\hat{g}} Hom(A,H) \xrightarrow{ev(-,a)} H^{m,\dots p-1})$$

is a multiple functor (condition (A3));



$$MCat \xrightarrow{\ -\blacksquare A\ } MCat$$

this is equivalent to say that $\hat{g}: B \to Hom(A,H)$ is a multiple functor, due to the pointwise definition of the compositions of $Hom(A,H)$. ∇

COROLLARY 1. *Let* A *be an m-fold category; then the «partial» functor*

$-\blacksquare A: Cat_n \to Cat_{m+n}$ *admits the functor* $Hom(A,-): Cat_{m+n} \to Cat_n$ *as a right adjoint.* ∇

REMARK. For $m = 1$ and $n = 1$, this Corollary has been proved in [5].

COROLLARY 2. *Let* A, B, H *be multiple categories of multiplicities* m, n *and* p. *There exists a canonical isomorphism*

$$\lambda: Hom(B \blacksquare A, H) \overset{\sim}{\to} Hom(B, Hom(A,H)).$$

If $p \geq m+n$, *there is also a canonical isomorphism*

$$Hom(B, Hom(A,H)) \overset{\sim}{\to} Hom(A, Hom(B, H^{\pi})),$$

where H^{π} *is deduced from* H *by the interchange of compositions corresponding to the permutation*

$$\pi: (0, \dots, p\text{-}1) \mapsto (m, \dots, m+n\text{-}1, 0, \dots, m\text{-}1, m+n, \dots, p\text{-}1).$$

PROOF. 1° It is well-known for monoidal closed categories [10] that the one-one correspondence

$$(g: B \blacksquare A \to H) \mapsto (\hat{g}: B \to Hom(A,H): b \mapsto g(b,-): A \to H)$$

resulting from the adjunction (see Proof Proposition 7) defines an isomorphism

$$\lambda: Hom(B \blacksquare A, H) \overset{\sim}{\to} Hom(B, Hom(A,H)).$$

(This is also expressed by saying that $Hom(A,-)$ is a right $MCat$-adjoint of $-\blacksquare A$.) This result extends here (with the same proof).

2° Assume $p \geq m+n$. We have the «semi-symmetry» isomorphism

$$\sigma: B \blacksquare A \to (A \blacksquare B)^{\gamma}: (b,a) \mapsto (a,b)$$

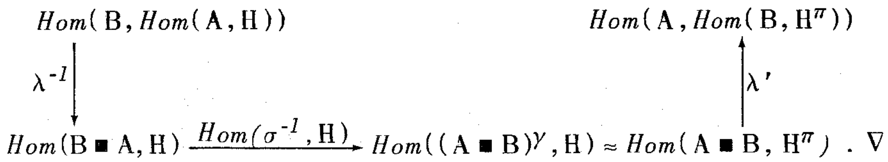
(Proposition 6), where γ is the permutation

$$\gamma: (0, \dots, m+n\text{-}1) \mapsto (n, \dots, m+n\text{-}1, 0, \dots, n\text{-}1).$$

For each $(m+n)$-fold category K we have the identification

$$Hom(K^{\gamma}, H) \approx Hom(K, H^{\pi}),$$

which comes from the definition of Hom and from the fact that the inverse of γ is a restriction of π and that π is the identity on $(m+n, \dots, p\text{-}1)$. So, we get the following string of isomorphisms:

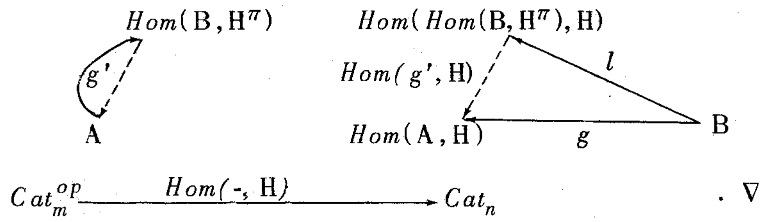


The existence of this composite canonical isomorphism can yet be expressed in the following form, if $p = m+n$.

COROLLARY 3. *Let* H *be a p-fold category, with* $p = m+n$, *and* H^{π} *the p-fold category deduced from* H *as in Corollary 2. Then the partial functor* $Hom(-,\mathrm{H}): Cat_m^{op} \to Cat_n$ *admits as a left adjoint the opposite of the functor* $Hom(-,\mathrm{H}^{\pi}): Cat_n^{op} \to Cat_m$.

PROOF. The liberty morphism corresponding to the n-fold category B is

$$l: \mathrm{B} \to Hom(Hom(\mathrm{B},\mathrm{H}^{\pi}),\mathrm{H}): b \mapsto [\, f \mapsto f(b)\,].$$

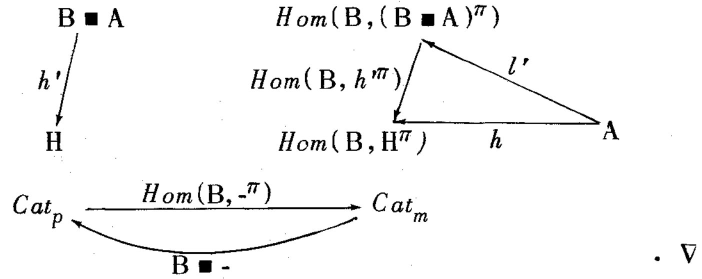


COROLLARY 4. *Let* B *be an n-fold category,* $p = m+n$ *and* π *the permutation* $(0,\ldots,p-1) \mapsto (m,\ldots,p-1,0,\ldots,m-1)$. *Then the partial functor* $\mathrm{B} \blacksquare -: Cat_m \to Cat_p$ *is a left adjoint of the functor*

$$Hom(\mathrm{B},-^{\pi}) = (\, Cat_p \xrightarrow{\tilde{\pi}} Cat_p \xrightarrow{Hom(\mathrm{B},-)} Cat_m\,).$$

PROOF. The liberty morphism corresponding to the m-fold category A is

$$l': \mathrm{A} \to Hom(\mathrm{B},(\mathrm{B} \blacksquare \mathrm{A})^{\pi}): a \mapsto [\, b \mapsto (b,a)\,].$$



EXAMPLES.

a) Let E be a set and E_n the n-fold category on E whose categories are all discrete. The partial square product functor $- \blacksquare E : Cat_n \to Cat_n$ is identical with the partial product functor $- \times E_n : Cat_n \to Cat_n$. So Corollary 1 implies that the functor $- \times E_n$ admits as a right adjoint the «power functor» $-^E : Cat_n \to Cat_n$: mapping $f: \mathrm{B} \to \mathrm{B}'$ onto

$$f^E : \mathrm{B}^E \to \mathrm{B}'^E : (b_e)_{e \in E} \mapsto (f(b_e))_{e \in E} .$$

More generally, we shall prove in Part III that the partial product functor $- \times \mathrm{B} : Cat_n \to Cat_n$ admits a right adjoint for each n-fold category B, i.e., that Cat_n is cartesian closed.

b) *Functors «forgetting some compositions»:*

We denote by 2 the category

$$1 \xleftarrow{\quad (1,0) \quad} 0 ,$$

by $2^{\blacksquare m}$ the m-fold category defined by induction (see end of Section B):

$$2^{\blacksquare 1} = 2, \quad 2^{\blacksquare q} = 2^{\blacksquare q-1} \blacksquare 2 \quad \text{for each integer } q > 1 .$$

If B is an n-fold category, a multiple functor $f: 2 \to \mathrm{B}$ is identified with a block $f(1,0)$ of B, and $Hom(2, \mathrm{B})$ is identified with $\mathrm{B}^{1,\dots,n-1}$. So $Hom(2,-): MCat \to MCat$ «is» the functor U_0 «forgetting the 0-th composition» (and mapping a set on the void set). By Proposition 7, this functor U_0 admits as a left adjoint the functor $- \blacksquare 2 : MCat \to MCat$.

Let $U_m : MCat \to MCat$ be the composite of U_0 by itself m times: it maps the p-fold category H on:

- the void set if $p < m$,
- $\mathrm{H}^{m,\dots,p-1}$ if $p \geq m$.

It admits as a left adjoint the composite of $- \blacksquare 2 : MCat \to MCat$ by itself m times, and this functor maps the n-fold category B onto the $(n+m)$-fold category $(\dots(\mathrm{B} \blacksquare 2) \blacksquare \dots 2) \blacksquare 2$, which is canonically isomorphic (end of Section B) with $\mathrm{B} \blacksquare 2^{\blacksquare m}$. Hence U_m also admits as a left adjoint the functor $- \blacksquare 2^{\blacksquare m} : MCat \to MCat$, and U_m may be identified with the functor

$$Hom(2^{\blacksquare m}, -): MCat \to MCat.$$

Taking restrictions of these functors, we get the first assertion of:

PROPOSITION 8. *The functor* $U_{m+n,n}: Cat_{m+n} \to Cat_n$ *forgetting the* m *first compositions admits as a left adjoint the partial functor*

$$-\blacksquare 2^{\blacksquare m}: Cat_n \to Cat_{m+n}.$$

The functor $U'_{m+n,n}: Cat_{m+n} \to Cat_n$ *forgetting the* m *last compositions admits as a left adjoint the partial functor*

$$2^{\blacksquare m} \blacksquare -: Cat_n \to Cat_{m+n}.$$

PROOF. We prove the second assertion. From Corollary 4, Proposition 7, it follows that the functor $2^{\blacksquare m} \blacksquare -: Cat_n \to Cat_{m+n}$ is a left adjoint of

$$Hom(2^{\blacksquare m}, -^{\pi}) = (Cat_{m+n} \xrightarrow{\tilde{\pi}} Cat_{m+n} \xrightarrow{Hom(2^{\blacksquare m},-)} Cat_n)$$

where $\tilde{\pi}$ is the isomorphism associated to the permutation

$$\pi: (0, \dots, m+n-1) \mapsto (n, \dots, m+n-1, 0, \dots, n-1),$$

and this composite functor identifies with

$$U'_{m+n,n} = (Cat_{m+n} \xrightarrow{\tilde{\pi}} Cat_{m+n} \xrightarrow{U_{m+n,n}} Cat_n). \qquad \nabla$$

c) «*Objects - functors*»:

Let 1_m be the «unique» m-fold category on the set $1 = \{0\}$. A multiple functor $f: 1_m \to B$, where B is an n-fold category, is identified with a block $f(0)$ of B which is moreover an object for the m first categories B^i. Hence the functor $Hom(1_m, -): MCat \to MCat$ maps B onto:

- the void set if $n < m$,

- if $n \geqslant m$, the $(n-m)$-fold subcategory of $B^{m,\dots,n-1}$ formed by the blocks of B which are objects for each category B^i, for $0 \leqslant i < m$; we will denote it by $|B|^{m,\dots,n-1}$.

The functor $Hom(1_m, -)$ admits as a left adjoint $-\blacksquare 1_m: MCat \to MCat$ which maps the n-fold category B onto the $(n+m)$-fold category $B \blacksquare 1_m$, wich is identified with the $(n+m)$-fold category on $\underline{B}$ whose m first categories are discrete and whose $(m+j)$-th category is B^j, for $0 \leqslant j < n$.

PROPOSITION 9. *The functor* $|U_{n+m,n}|: Cat_{m+n} \to Cat_n$ *restriction of the functor* $Hom(1_m, -)$ *admits both a left and a right adjoint.*

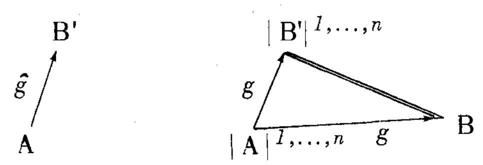
PROOF. The left adjoint is the restriction of the functor $-\blacksquare\, 1_m$, described above. Since $|U_{n+m,n}|$ is equal to the composite

$$Cat_{n+m} \xrightarrow{|U_{n+m,n+m-1}|} Cat_{n+m-1} \to \dots \xrightarrow{|U_{n+1,n}|} Cat_n ,$$

it suffices to prove the existence of a right adjoint for

$$|U_{n+1,n}|: Cat_{n+1} \to Cat_n .$$

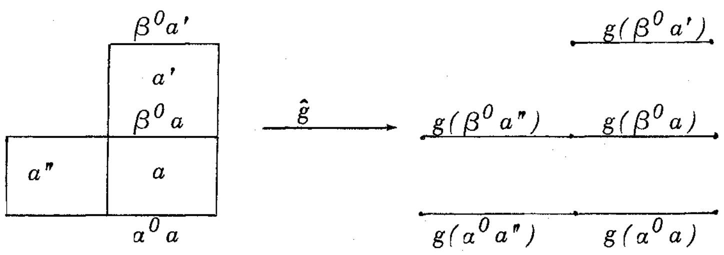
For this, let B be an n-fold category. There is an $(n+1)$-fold category B' on the product $\underline{B}\times\underline{B}$ whose 0-th category is the groupoid of couples of B, and whose $(i+1)$-th category is the product category $B^i \times B^i$, for $0 \leqslant i < n$. The image $|B'|^{1,\dots,n}$ of B' by $|U_{n+1,n}|$ is identified with B by identifying B with the set of objects for the groupoid of its couples. We say that B' is the cofree object generated by B . Indeed, if A is an $(n+1)$-fold category, α^0 and β^0 the source and target of A^0 , then a map g defines an n-



$$Cat_{n+1} \xrightarrow{|U_{n+1,n}|} Cat_n$$

fold functor $g: |A|^{1,\dots,n} \to B$ iff the map

$$\hat{g}: a \mapsto (g(\beta^0 a), g(\alpha^0 a))$$



defines an $(n+1)$-fold functor $\hat{g}: A \to B'$. ∇

In particular, the «object-functor» $Cat \to Set$ which maps a category on the set of its objects has a left adjoint mapping the set E onto the discrete category E^{dis} , and a right adjoint mapping E onto the groupoid of its couples.

D. Some applications to the existence of colimits.

1. Construction of colimits in *MCat*.

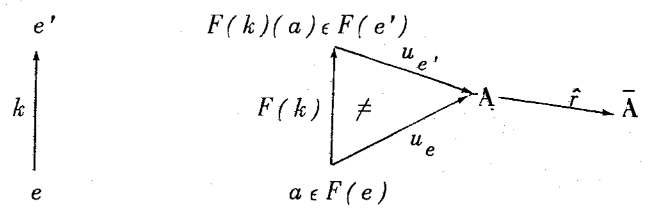
We have seen (Proposition 5) that *MCat* is complete. It is not co-complete; however, using Proposition 8, we are going to prove:

PROPOSITION 10. *Let $F: K \to MCat$ be a functor, where K is a small category. Then F admits a colimit iff the multiplicities of the multiple categories $F(e)$, for all objects e of K, are bounded.*

PROOF. The condition is clearly necessary. On the other hand, if there exists a coproduct A of the multiple categories $F(e)$ for all objects e of K, then F will admit as a colimit the multiple category $\bar{A}$ quasi-quotient of A by the relation r:

$$u_e(a) \sim u_{e'}(F(k)(a)) \text{ if } k: e \to e' \text{ in } K \text{ and } a \in F(e),$$

where $u_e: F(e) \to A$ is the canonical injection into the coproduct:



So it suffices to prove the existence of a coproduct for a family $(A_\lambda)_{\lambda \in \Lambda}$ such that A_λ is an m_λ-fold category and that there exists $n = \sup_{\lambda \in \Lambda} m_\lambda$.

For this, let $B_\lambda = 2^{\blacksquare^{n-m_\lambda}} \blacksquare A_\lambda$ be the free object generated by A_λ with respect to the functor $U'_{n,m_\lambda}: Cat_n \to Cat_{m_\lambda}$ forgetting the $(n-m_\lambda)$ last compositions (see Proposition 8); let $l_\lambda: A_\lambda \to B_\lambda^{0,\dots,m_\lambda-1}$ be the liberty morphism. The family $(B_\lambda)_{\lambda \in \Lambda}$ admits as a coproduct in *MCat* its coproduct B in Cat_n (by Proposition 5), the canonical injection being

$$v_\lambda: B_\lambda \to B: b \mapsto (b, \lambda).$$

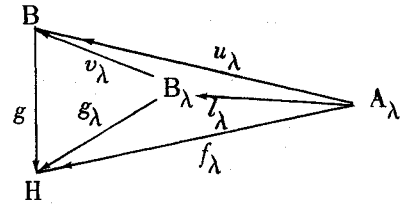
We say that B is also the coproduct of $(B_\lambda)_{\lambda \in \Lambda}$ in *MCat*, the canonical injection being

$$(u_\lambda: A_\lambda \to B) = (A_\lambda \xrightarrow{l_\lambda} B_\lambda \xrightarrow{v_\lambda} B).$$

Indeed, let H be a p-fold category and $f_\lambda: A_\lambda \to H$ a multiple functor for each $\lambda \in \Lambda$. Then $m_\lambda \leq p$ for each λ implies $n \leq p$, and by definition of a free object generated by A_λ, there exists a unique $g_\lambda: B_\lambda \to H^{0,\dots,n-1}$ with

$$(f_\lambda: A_\lambda \to H) = (A_\lambda \xrightarrow{l_\lambda} B_\lambda \xrightarrow{g_\lambda} H).$$

The factor $g: B \to H$ of the family $(g_\lambda)_{\lambda \in \Lambda}$ through the coproduct B is the unique morphism rendering commutative the diagram



i.e., factorizing $(f_\lambda)_{\lambda \in \Lambda}$ through B. ∇

2. Generalized limits.

Motivated by the example of the category of natural transformations from a category A to a category C, which is identified with the category $Hom(A, \square C)$, the following terminology was generally introduced in [7], and precised in [5] for double categories.

In this section, B denotes an m-fold category and H an $(m+1)$-fold category such that B is the m-fold subcategory $|B|^{0,\dots,m-1}$ of $H^{0,\dots,m-1}$ formed by those blocks of H which are objects for the last category H^m. Let $|H|^m$ be the subcategory of H^m formed by those blocks of H which are objects for the m first categories H^i. The objects of $|H|^m$ (hence the blocks of H which are objects for all the categories H^j) are called *vertices of* H.

Let A be an m-fold category. The objects of the category $Hom(A, H)$ are the multiple functors $f: A \to H$ taking their values in $|H|^{0,\dots,m-1} = B$; they are identified with the m-fold functors $f: A \to B$. Then, if $g: A \to H$ is a multiple functor, its source in $Hom(A, H)$ is

$$\alpha^m g = (A \xrightarrow{g} H^{0,\dots,m-1} \xrightarrow{\alpha^m} B),$$

and its target is

$$\beta^m g = (\, \mathrm{A} \xrightarrow{g} \mathrm{H}^{0,\dots,m-1} \xrightarrow{\beta^m} \mathrm{B}\,),$$

where α^m and β^m are the maps source and target of H^m. We say that g *is a* H-*wise transformation from* $\alpha^m g$ *to* $\beta^m g$, denoted by $g: \alpha^m g \to \beta^m g$.

There is a canonical functor, called the *diagonal functor,*

$$d_{\mathrm{AH}} : |\mathrm{H}|^m \to Hom(\mathrm{A}, \mathrm{H})$$

(which is the functor associated to the alternative functor

$$(\,|\mathrm{H}|^m, \mathrm{A}) \to \mathrm{H} : (u, a) \vdash u\,).$$

This functor maps an object u of $|\mathrm{H}|^m$, i.e., a vertex of H, onto the constant functor

$$u^\wedge : \mathrm{A} \to \mathrm{B} : a \mapsto u,$$

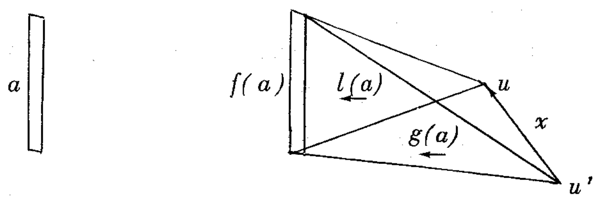
and it maps the morphism $x: u \to u'$ of $|\mathrm{H}|^m$ onto the H-wise transformation «constant on x», denoted by $x^\wedge : u^\wedge \to u'^\wedge$.

DEFINITION. Let $f: \mathrm{A} \to \mathrm{B} = |\mathrm{H}|^{0,\dots,m-1}$ be an m-fold functor. If u is a free (resp. cofree) object generated by f with respect to the diagonal functor $d_{\mathrm{AH}} : |\mathrm{H}|^m \to Hom(\mathrm{A}, \mathrm{H})$, then u is called an H-*wise colimit* (resp. *limit*) *of* f.

If u is a vertex of H and $g: u^\wedge \to f$ an H-wise transformation, we also say (by reference with the case of natural transformations) that $g: u \Rightarrow f$ is a projective cone. Then u is a limit of $f: \mathrm{A} \to \mathrm{B}$ iff there exists a projective cone $l: u \Rightarrow f$, called a *limit-cone*, such that each projective cone $g: u' \Rightarrow f$ factors in a unique way through l, i.e., there exists a unique morphism $x: u' \to u$ of $|\mathrm{H}|^m$ satisfying:

$$g = (\, u'^\wedge \xrightarrow{d_{\mathrm{AH}}(x)} u^\wedge \xrightarrow{l} f\,)$$

(this means $g(a) = l(a) \circ_m x$ for each block a of A).



If the diagonal functor d_{AH} admits a right (resp. left) adjoint, so that each m-fold functor $f: \mathrm{A} \to \mathrm{B}$ admits a limit (resp. a colimit), we say that B *admits* H-*wise* A-*limits* (resp. A-*colimits*). If B admits H-wise A-limits for each small (resp. finite) m-fold category A, we say that B is H-*wise complete* (resp. *finitely complete*). Similarly is defined the notion: H-wise (finitely) cocomplete.

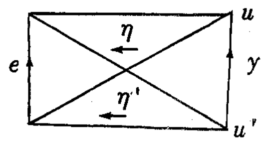
EXAMPLES. 1° If H is a double category $(\mathrm{H}^0, \mathrm{H}^1)$ and B is the category of 1-morphisms obtained by equipping the set of objects of H^1 with the composition induced by H^0 (denoted by H^1_o in [5]), these definitions coincide with those given in [5].

2° If $\mathrm{B} = |\mathrm{H}|^{0,\ldots,m-1}$ admits H-wise $2^{\blacksquare m}$-limits, we also say that H is a *representable (m+1)-fold category*, by extension of the notion of a representable 2-category introduced by Gray [13] and generalized in [5] to double categories. This means that the insertion functor $|\mathrm{H}|^m \hookrightarrow \mathrm{H}^m$ admits a right adjoint (since $Hom(2^{\blacksquare m}, \mathrm{H})$ is identified with H^m). In other words, for each object e of H^m, there exists a vertex u of H, called the *representant of* e, and a block η of H with $\alpha^m \eta = u$, $\beta^m \eta = e$, such that, for each block η' of H with

$$\beta^m \eta' = e \quad \text{and} \quad \alpha^m \eta' = u' = \text{vertex of H},$$

there exists a unique

$$y: u' \to u \ \text{ in } |\mathrm{H}|^m \ \text{ with } \eta' = \eta \circ_m y.$$



Dually, H *is corepresentable* if the insertion $|\mathrm{H}|^m \hookrightarrow \mathrm{H}^m$ admits a left adjoint.

The next proposition gives an existence theorem for H-wise limits. It utilizes the following Lemma, whose proof is given in the Appendix (since it considers multiple categories as sketched structures).

LEMMA. *Cat_m is the inductive closure of* $\{2^{\blacksquare^m}\}$ *(i. e., Cat_m is the smallest subcategory of Cat_m containing $2^{\blacksquare^m}$ and closed by colimits).*

PROPOSITION 11. *Let* H *be a representable $(m+1)$-fold category and let* $B = |H|^{0,\dots,m-1}$. *If* $|H|^m$ *is complete (resp. finitely complete), then* B *is* H*-wise complete (resp. finitely complete).*

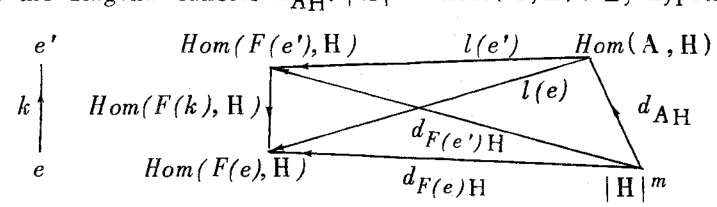
PROOF. Let Ω be the full subcategory of Cat_m whose objects are the m-fold categories P such that B is H-wise P-complete. To say that H is representable means that $2^{\blacksquare^m}$ is an object of Ω. Let A be an m-fold category which is the colimit of a functor $F: K \to \Omega$, where K is small (resp. finite); if we prove that such an A is an object of Ω, it will follow that B is H-wise complete (resp. finitely complete), since Cat_m is the inductive closure of $\{2^{\blacksquare^m}\}$ by the preceding Lemma. For this, let $l': F \Longrightarrow A$ be the colimit cone. Since the functor $Hom(-,H): (Cat_m)^{op} \to Cat$ admits a left adjoint (by Corollary 3, Proposition 7), it transforms the colimit cone l' in Cat_m into a limit cone

$$l: Hom(A,H) \Longrightarrow Hom(F-,H).$$

We have a cone $d: |H|^m \Longrightarrow Hom(F-,H)$ such that

$$d(e) = d_{F(e)H}: |H|^m \to Hom(F(e),H),$$

for each object e of K. The factor of this cone with respect to the limit cone l is the diagonal functor $d_{AH}: |H|^m \to Hom(A,H)$. By hypothesis,



$F(e)$ belonging to Ω, each diagonal functor $d(e)$ admits a right adjoint, and $|H|^m$ admits K-limits. Hence a theorem of Appelgate-Tierney [1] asserts that the factor d_{AH} also admits a right adjoint, i.e., B admits H-wise A-limits. Therefore A is also an object of Ω, and a fortiori B is H-wise complete (resp. finitely complete). In fact, if $f: A \to B$ is an m-fold functor, its H-wise limit u is constructed as follows [1]: let u_e be a H-wise limit of the m-fold functor

$$l(e)(f) = (F(e) \xrightarrow{l'(e)} \mathrm{A} \xrightarrow{f} \mathrm{B}).$$

By the universal property of the limit, there exists a unique functor

$$G: K \to |\mathrm{H}|^m \quad \text{such that} \quad G(e) = u_e$$

for each object e of K. This functor G admits a limit u, which is a H-wise limit of $f: \mathrm{A} \to \mathrm{B}$. ∇

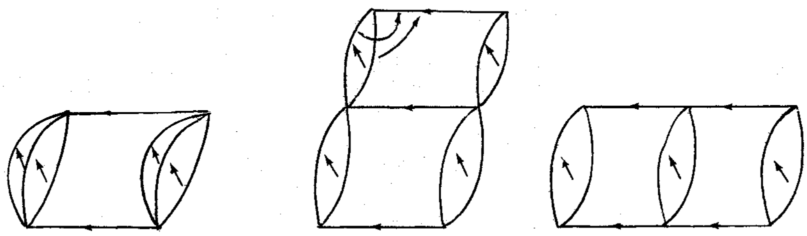
Dually, we prove by a similar method:

PROPOSITION 12. *If* H *is a corepresentable* $(m+1)$*-fold category and if* $|\mathrm{H}|^m$ *is (finitely) cocomplete, then the* m*-fold category* $\mathrm{B} = |\mathrm{H}|^{0,\dots,m-1}$ *is* H*-wise (finitely) cocomplete.* ∇

EXAMPLES.

a) If H is a double category, we find anew Proposition 3-2 [5] (with a much simpler proof). So if H is the double category $Q(\mathrm{K})$ of up-squares of a 2-category K, it reduces to Gray's Theorem of existence of cartesian quasi-limits [13], as explained in [5], page 64.

b) Let K be a 2-category. There is a triple category H, called the *tri-* 1
ple category of squares of $Q(\mathrm{K})$, such that $\mathrm{H}^{0,2}$ is the double category of squares of the vertical category $Q(\mathrm{K})^{\boxminus}$ and that the composition of H^{1} is deduced pointwise from that of the horizontal category $Q(\mathrm{K})^{\boxplus}$; its greatest 3-category is the 3*-category of cylinders of* K, defined in [2]:



If K is representable, so is $Q(\mathrm{K})$ (by Proposition 6-2 [5]), and also H
(this will be proved in Part III, where we construct more generally the mul- 2
tiple category of squares of an n-fold category). If A is a 2-category, an

object of $Hom(\mathrm{A}, \mathrm{H})$ is identified with a 2-functor $f: \mathrm{A} \to \mathrm{K}$; a H-wise limit of f is then a catalimit of f in the sense of Bourn [3] . Analimits are obtained by taking down-squares instead of up-squares. So Proposition 11 then reduces to Proposition 7 of Bourn [3], whose proof, of the same type than
1+ that of Proposition 3-2 [5], is less «structural» and therefore longer.

E. Infinite-fold categories.

MCat does not admit coproducts for families $(\mathrm{A}_\lambda)_{\lambda \in \Lambda}$ such that the multiplicities of the multiple categories A_λ are not bounded; indeed, such a coproduct should have «an infinity» of compositions. This leads to extend as follows *MCat* into a complete and cocomplete category *VMCat* which is partially monoidal closed.

DEFINITION. An N-*fold category* X on the set $\underline{X}$ is an infinite sequence $(\mathrm{X}^i)_{i \in \mathrm{N}}$ of categories with the same set of morphisms $\underline{X}$, such that, for each pair (i, j) of distinct integers, $(\mathrm{X}^i, \mathrm{X}^j)$ is a double category. If X' is also an N-fold category, $h: \mathrm{X} \to \mathrm{X}'$ is an N-*fold functor* if $h: \mathrm{X}^i \to \mathrm{X}'^i$ is a functor for each integer i .

EXAMPLES.

a) If A is an m-fold category, there is an N-fold category X on $\underline{A}$ with

$$\mathrm{X}^i = \mathrm{A}^i \quad \text{for} \quad 0 \leqslant i < m,$$
$$\mathrm{X}^j = \underline{A}^{dis} \text{ (discrete category on } \underline{A} \text{) for } m \leqslant i \in \mathrm{N}.$$

b) Let $(C_n)_{n \in \mathrm{N}}$ be a sequence of categories; we define an N-fold category on the set product of the sets $\underline{C}_n$ of morphisms of C_n by taking as i-th category the product category

$$\prod_{n \in \mathrm{N}} K_n, \quad \text{where } K_i = C_i \text{ and } K_n = \underline{C}_n^{dis} \text{ if } n \neq i.$$

In particular, if $C_n = 2$ for each integer n, we so obtain the N-fold category, denoted by 2_N , whose i-th category is

$$\underline{2}^{dis} \times \ldots \times \underline{2}^{dis} \times \underset{i\text{-th position}}{2} \times \underline{2}^{dis} \times \ldots \qquad (\text{with } \underline{2} = \{0, 1, (1, 0)\});$$

its unique non-degenerate «block» is $(u_n)_{n \in \mathrm{N}}$, where $u_n = (1, 0)$ for each integer n. Hence, an N-fold functor $h: 2_{\mathrm{N}} \to \mathrm{X}$, where X is an N-fold category, may be identified with the block $h((u_n)_{n \in \mathrm{N}})$ of X, image by h of the unique non-degenerate block $(u_n)_{n \in \mathrm{N}}$. 1+

The N-fold functors between small N-fold categories form a category Cat_{N}. For each integer m, there is the faithful functor

$$U'_{\mathrm{N},m}: Cat_{\mathrm{N}} \to Cat_m ,$$

which maps the N-fold category X onto the m-fold category $\mathrm{X}^{0,\dots,m-1}$ obtained by «keeping only the m first compositions».

REMARK. Cat_{N} may also be defined as the limit of the functor:

$$(n, m) \mapsto U'_{n,m}: Cat_n \to Cat_m$$

(where $U'_{n,m}$ is the functor «forgetting the $(n-m)$ last compositions» defined in Proposition 8), from the category of couples defining the order of N toward the category of categories associated to a universe containing the universe of small sets (if the existence of such a universe is assumed!).

PROPOSITION 13. *Cat_{N} is complete, cocomplete, and the faithful functor $U'_{\mathrm{N},0}: Cat_{\mathrm{N}} \to Set$ «forgetting all the compositions» admits quasi-quotient objects.*

PROOF. 1° From Proposition 1, it follows that, if $F: K \to Cat_{\mathrm{N}}$ is a functor, where K is small, it admits as a limit the N-fold category X such that $\mathrm{X}^{0,\dots,m-1}$ is the limit of the functor

$$K \xrightarrow{F} Cat_{\mathrm{N}} \xrightarrow{U'_{\mathrm{N},m}} Cat_m ,$$

for each integer m.

2° If $(\mathrm{X}_\lambda)_{\lambda \in \Lambda}$ is a family of N-fold categories, it admits as a coproduct in Cat_{N} the N-fold category X such that X^i is the coproduct of the family of categories $(\mathrm{X}^i_\lambda)_{\lambda \in \Lambda}$.

3° The existence of quasi-quotient objects, and then of colimits, is proved by a method analogous to that used in Propositions 2, 3, 4 to prove similar results with respect to Cat_n, showing first by the same construction the following assertion:

The N-fold subcategory of an N-fold category X generated by an infinite subset M of $\underline{X}$ is equipotent with M. ∇

Let *VMCat* be the category whose objects are the small multiple categories and the small N-fold categories, and of which *MCat* and Cat_N are full subcategories, the only other morphisms being the $g: A \to X$, where A is an m-fold category and $g: A \to X^{0,\dots,m-1}$ an m-fold functor. We shall extend «partially» to *VMCat* the square product and the internal Hom functor of *MCat*.

DEFINITION. If X is an N-fold category and A an m-fold category, the *square product* $X \blacksquare A$ of (X, A) will be the N-fold category on the product set $\underline{X} \times \underline{A}$ whose i-th category is

$$\underline{X}^{dis} \times A^i \text{ if } 0 \leqslant i < m, \quad X^{i-m} \times \underline{A}^{dis} \text{ if } m \leqslant i \in N.$$

So $X \blacksquare A$ is the N-fold category such that, for each integer $i > m$:

$$(X \blacksquare A)^{0,\dots,i} = X^{0,\dots,i} \blacksquare A.$$

It follows that a map $g: \underline{X} \times \underline{A} \to P$ defines a morphism $g: X \blacksquare A \to P$ iff: P is an N-fold category,

$$g(x,-): A \to P: a \mapsto g(x,a)$$

is a morphism for each block x of X, and for each block a of A,

$$g(-,a): X^i \to P^{m+i}: x \mapsto g(x,a)$$

is a functor for each integer i. Then we say that $g: (X, A) \to P$ *is an alternative morphism.*

In particular, the alternative morphism $id: (X, A) \to X \blacksquare A$ gives the universal solution of the problem of transforming alternative morphisms into N-fold functors.

$$\begin{array}{ccc} & (X,A) & \\ {}^{g}\swarrow & & \searrow^{id} \\ P & \xleftarrow[g]{} & X \blacksquare A \end{array}$$

DEFINITION. If X is an N-fold category and A an m-fold category, we denote by $Hom(A, X)$ the N-fold category on the set of morphisms from A to

X , whose i-th composition is deduced «pointwise» from that of X^{m+i} , so that, for each integer i :

$$Hom(A, X)^{0,\dots,i-1} = Hom(A, X^{0,\dots,m+i-1}) .$$

PROPOSITION 14. *The square product functor and the internal Hom functor of MCat extend into functors, still denoted:*

$$\blacksquare : VMCat \times \coprod_n Cat_n \to VMCat \text{ and } Hom: (\coprod_n Cat_n)^{op} \times VMCat \to VMCat.$$

For each multiple category A *the partial functor*

$$Hom(A, -): VMCat \to VMCat$$

is right adjoint to $- \blacksquare A : VMCat \to VMCat$.

PROOF. The proof is similar to that of Proposition 7. The extended functor ■ maps $(h: X \to X', f: A \to A')$ onto the N-fold functor

$$h \times f: X \blacksquare A \to X' \blacksquare A' : (x, a) \mapsto (h(x), f(a)).$$

The extended functor Hom maps $(f': B \to A, h: X \to X')$ onto the morphism $Hom(f', h): Hom(A, X) \to Hom(B, X')$ defined by

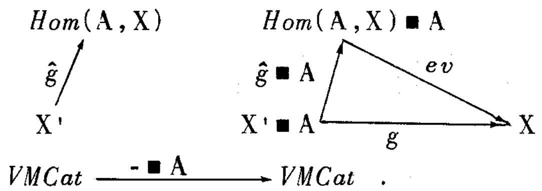
$$(g: A \to X) \mapsto (B \xrightarrow{f'} A \xrightarrow{g} X \xrightarrow{h} X').$$

If A is an m-fold category and X an N-fold category, $Hom(A, X)$ is the cofree object generated by X with respect to the partial functor

$$- \blacksquare A : VMCat \to VMCat ,$$

the coliberty morphism being

$$ev: Hom(A, X) \blacksquare A \to X : (f, a) \mapsto f(a).$$



∇

COROLLARY. *The functor* $U'_{N,m}: Cat_N \to Cat_m$ *«keeping only the m first compositions» admits as a left adjoint the functor* $2_N \blacksquare - : Cat_m \to Cat_N$.

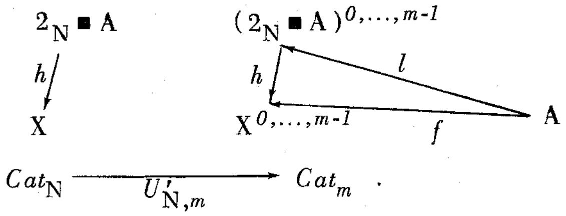
PROOF. Let A be an m-fold category; then $2_N \blacksquare A$ (where 2_N is defined in Example b) is a free object generated by A with respect to $U'_{\mathrm{N},m}$, the liberty morphism being:

$$l: \mathrm{A} \to (2_{\mathrm{N}} \blacksquare \mathrm{A})^{0,\dots,m-1} : a \mapsto ((u_n)_{n\in\mathrm{N}}, a),$$

where $u_n = (1,0): 0 \to 1$ for each integer n. Indeed, let X be an N-fold category. By the proposition, there is a canonical 1-1 correspondence between N-fold functors $h: 2_{\mathrm{N}} \blacksquare \mathrm{A} \to \mathrm{X}$ and N-fold functors $2_{\mathrm{N}} \to Hom(\mathrm{A},\mathrm{X})$, which are identified with blocks of $Hom(\mathrm{A},\mathrm{X})$, i.e., with m-fold functors $f: \mathrm{A} \to \mathrm{X}^{0,\dots,m-1}$. The morphism associated to $h: 2_{\mathrm{N}} \blacksquare \mathrm{A} \to \mathrm{X}$ is

$$f: \mathrm{A} \to \mathrm{X}^{0,\dots,m-1} : a \mapsto h((u_n)_{n\in\mathrm{N}}, a),$$

and h if the unique factor of f through l.



∇

This Corollary, similar to Proposition 8, is used to prove:

PROPOSITION 15. *VMCat is complete, cocomplete, and the functor «forgetting all the compositions» U: VMCat → Set admits quasi-quotient objects.*

PROOF. The proof is analogous to that of Propositions 5 and 10, using the fact that Cat_{N} and Cat_m, for each integer m, are complete and cocomplete. More precisely:

1° The functor $F: K \to VMCat$, where K is small, admits as a limit in $VMCat$:

- if F takes its values in Cat_{N}, the limit in Cat_{N} of the restriction $F: K \to Cat_{\mathrm{N}}$,

- otherwise, let n be the least of the multiplicities (finite or infinite) of the objects $F(e)$, for all objects e of K; then the limit of F in $VMCat$ is the limit of the composite functor:

$$K \xrightarrow{F} VMCat \xrightarrow{U'_{\mathrm{N},n}} Cat_n .$$

2° F admits as a colimit the quasi-quotient of the coproduct P of the objects $F(e)$, e object of K, in $VMCat$, this quasi-quotient being computed in Cat_{N} if P is an N-fold category, in $MCat$ otherwise.

3° A family $(P_\lambda)_{\lambda\in\Lambda}$ of objects of $VMCat$ admits as a coproduct:

- its coproduct in $MCat$ if the multiplicities of the objects P_λ are all finite and bounded;

- and otherwise the coproduct of $(X_\lambda)_{\lambda\in\Lambda}$ in Cat_{N}, where $X_\lambda = P_\lambda$ if P_λ is an N-fold category, and $X_\lambda = 2_{\mathrm{N}} \blacksquare P_\lambda$ if P_λ is an n_λ-fold category for some integer n_λ. ∇

REMARK.

The functor $X \blacksquare - : \coprod_n Cat_n \to VMCat$, where X is an N-fold category, cannot be extended into a functor from $VMCat$, since to define $X \blacksquare A$ we have first considered «all the compositions of A». In the same way, the functor $Hom(-, X): (\coprod_n Cat_n)^{op} \to VMCat$ cannot be extended trivially into a functor from $(VMCat)^{op}$. However, we may define as follows an internal Hom functor

$$Hom_{\mathrm{N}}: (Cat_{\mathrm{N}})^{op} \times Cat_{\mathrm{N}} \to Cat_{\mathrm{N}}$$

and a functor $\blacklozenge: Cat_{\mathrm{N}} \times Cat_{\mathrm{N}} \to Cat_{\mathrm{N}}$ such that the partial functors

$$-\blacklozenge X,\ Hom_{\mathrm{N}}(X, -): Cat_{\mathrm{N}} \to Cat_{\mathrm{N}}$$

are adjoint, for each N-fold category X. If X' is also an N-fold category:

$X' \blacklozenge X$ is the N-fold category whose $2i$-th category is $\underline{X}'^{dis} \times X^i$ and whose $(2i+1)$-th category is $X'^i \times \underline{X}^{dis}$;

denoting by X'^{even} and X'^{odd} respectively the N-fold categories

$$X'^{even} = (X'^{2i})_{i\in\mathrm{N}} \quad\text{and}\quad X'^{odd} = (X'^{2i+1})_{i\in\mathrm{N}},$$

we take for $Hom_{\mathrm{N}}(X, X')$ the N-fold category on the set of N-fold functors $h: X \to X'^{even}$ whose compositions are deduced «pointwise» from that of X'^{odd}, so that

$$h' \circ_i h: X \to X'^{even}: x \mapsto h'(x) \circ_{2i+1} h(x) \text{ iff this is defined.}$$

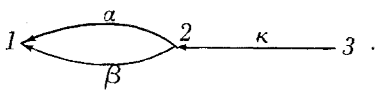
But this does not give a monoidal closed structure on Cat_{N}. It is not associative nor unitary (up to isomorphisms or interchange of compositions).

APPENDIX

In this paper, we have defined multiple categories directly, but they can also be considered (in several ways) as sketched structures. Here we interpret the constructions of the square product and of *Hom* in terms of «multiple internal categories».

A. Multiple categories as sketched structures.

For the notations and results on sketched structures and internal categories, we refer to Section 0 [5]. We only recall that the category underlying the sketch $\sigma\mathcal{G}$ of categories (denoted more simply $\sigma = (\Sigma, \Gamma)$) is the full subcategory Σ of the opposite of the simplicial category whose obtects are the integers $1, 2, 3, 4$. The «idea» of this sketch is



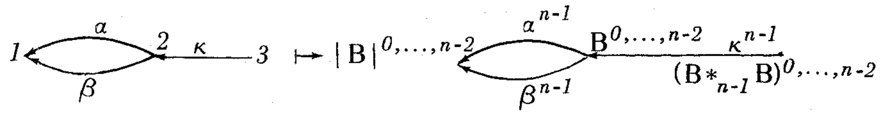
This means that a realization $F: \sigma \to K$, or «category in(ternal to) K» is uniquely determined by $F(\alpha)$, $F(\beta)$, $F(\kappa)$, whatever be the category K.

If C is a category, the realization $\sigma \to Set$ canonically associated to C maps α, β, κ respectively on the maps source, target and composition of C.

Multiple categories appear as sketched structures in three different ways:

1° *The category* Cat_n *of n-fold categories is equivalent to the category* Cat^{σ}_{n-1} *of categories in* Cat_{n-1}.

Indeed, if B is an n-fold category, the realization $\sigma \to Cat_{n-1}$ canonically associated to B maps α, β, κ on the maps source α^{n-1}, target β^{n-1} and composition κ^{n-1} of B^{n-1}, considered as $(n-1)$-fold functors with respect to the $(n-1)$ first compositions of B, so that:



where $(B *_{n-1} B)^{0,\dots,n-2}$ is the *(n-1)*-fold subcategory of the product *(n-1)*-fold category $B^{0,\dots,n-2} \times B^{0,\dots,n-2}$ formed by the couples (b', b) having a composite $b' \circ_{n-1} b$ in B^{n-1}.

2° *Cat_n is equivalent to the category Set^{σ_n} of realizations in Set of the «sketch of n-fold categories» σ_n.*

Indeed, σ_n is the n-th tensor power $\overset{n}{\otimes}\sigma$ of σ (see [5]) defined inductively by:

$$\sigma_1 = \sigma \quad \text{and} \quad \sigma_n = \sigma_{n-1} \otimes \sigma .$$

Its underlying category Σ_n is:

$$\Sigma_n = \Sigma_{n-1} \times \Sigma = (\dots(\Sigma \times \Sigma) \times \dots \times \Sigma) \times \Sigma \ ;$$

a morphism of Σ_n will be more simply written as a sequence $(x_0, \dots, x_{n-1})$ of morphisms of Σ (i.e., we omit the parentheses).

For $0 \leqslant i < n$, there is a one-one functor $\delta_n^i : \Sigma \to \Sigma_n$, which maps x onto the sequence $(2, \dots, 2, x, 2, \dots, 2)$ in which all the factors are 2 except the i-th one, which is x. This functor defines a morphism of sketches $\delta_n^i : \sigma \to \sigma_n$. If $F : \sigma_n \to K$ is a realization in a category K, also called an *n-fold category in* K, then F is uniquely determined by the n categories F^i in K such that

$$F^i = (\sigma \xrightarrow{\delta_n^i} \sigma_n \xrightarrow{F} K), \ \text{ for } 0 \leqslant i < n.$$

If B is an n-fold category, the realization $B : \sigma_n \to Set$ (canonically) associated to B is such that

$$B^i = (\sigma \xrightarrow{\delta_n^i} \sigma_n \xrightarrow{B} Set)$$

is the realization in Set associated to the category B^i, for each $i < n$. This determines an equivalence $\eta_n : Cat_n \to Set^{\sigma_n}$.

3° *For each integer $m < n$, the category Cat_n is equivalent to the category $(Set^{\sigma_m})^{\sigma_{n-m}}$,* and *to the category $(Cat_m)^{\sigma_{n-m}}$ of (n-m)-fold categories in Cat_m.*

Indeed, from the universal property of the tensor product of sketches (which equips the category of sketches of a monoidal closed structure, see

[4,14]), we deduce the canonical isomorphisms $\sigma_{n+m} \xrightarrow[\sim]{associativity} \sigma_n \otimes \sigma_m$,

$$Set^{\sigma_n} \tilde{\to} Set^{\sigma_m \otimes \sigma_{n-m}} \tilde{\to} (Set^{\sigma_m})^{\sigma_{n-m}} .$$

More precisely, let B be an n-fold category; then the realization $\hat{B}: \sigma_{n-m} \to Cat_m$ (canonically) associated to B maps $(2, \ldots, 2)$ onto the m-fold category $B^{0,\ldots,m-1}$ and it is determined by the fact that for $j < n-m$, the composite

$$\sigma \xrightarrow{\delta^j_{n-m}} \sigma_{n-m} \xrightarrow{\hat{B}} Cat_m$$

is the category in Cat_m associated (as in 1 above) to the $(m+1)$-fold category $B^{0,\ldots,m-1,m+j}$, so that it is defined by:

$$\alpha, \beta \xleftarrow{\kappa} \quad \mapsto \quad |B|^{0,\ldots,m-1} \; \alpha^{m+j}, \beta^{m+j} \; B^{0,\ldots,m-1} \xleftarrow{\kappa^{m+j}} (B *_{m+j} B)^{0,\ldots,m-1}$$

The realization $\bar{B}: \sigma_{n-m} \to Set^{\sigma_m}$ associated to B is the composite of $\hat{B}$ with the equivalence $\eta_m : Cat_m \to Set^{\sigma_m}$ (defined in 2), so that

$$\sigma \xrightarrow{\delta^j_{n-m}} \sigma_{n-m} \xrightarrow{\bar{B}} Set^{\sigma_m},$$

for $0 \leqslant j < n-m$, is the category in Set^{σ_m} associated to $B^{0,\ldots,m-1,m+j}$.

B. Realizations associated to B ■ A and to $Hom(A, B)$.

In this section, we denote by A an m-fold category, by B an n-fold category, by

$$A: \sigma_m \to Set \quad \text{and} \quad B: \sigma_n \to Set$$

the associated realizations in Set.

PROPOSITION 1. *The realization in Set associated to* B ■ A *is*

$$P = (\sigma_{n+m} \xrightarrow{\lambda} \sigma_n \otimes \sigma_m \xrightarrow{B \times A} Set \times Set \xrightarrow{- \times -} Set),$$

where λ *is the isomorphism*

$$(x_0, \ldots, x_{n+m-1}) \mapsto ((x_m, \ldots, x_{n+m-1}), (x_0, \ldots, x_{m-1}))$$

and where the last functor is the (cartesian) product functor.

PROOF. We will use the following facts:

- If K and K' are categories with associated realizations K, K' from σ in Set, then the realization associated to the product category K × K' is

$$(K, K') : \sigma \to Set : x \mapsto K(x) \times K'(x).$$

- If E is a set, the discrete category E^{dis} admits as its associated realization $E\hat{}: \sigma \to Set$, where $E\hat{}$ is the functor «constant on E».

Now, we have the functor $P : \Sigma_{n+m} \to Set$ defined by:

$$(x_0, \dots, x_{n+m-1}) \mapsto B(x_m, \dots, x_{n+m-1}) \times A(x_0, \dots, x_{m-1}).$$

The composite P^i:

$$\Sigma \xrightarrow{\delta^i_{n+m}} \Sigma_{n+m} \xrightarrow{P} Set$$

is defined by:

1° $x \mapsto B(2, \dots, 2) \times A(2, \dots, x, \dots, 2) = \underline{B} \times A(\delta^i_m(x))$, ($x$ in i-th position)

if $0 \leqslant i < m$, so that P^i is then the realization from $\sigma^{\cdot}$ associated to the product category $\underline{B}^{dis} \times A^i$;

2° $x \mapsto B(2, \dots, x, \dots, 2) \times A(2, \dots, 2) = B(\delta^j_n(x)) \times \underline{A}$, ($x$ in j-th position)

if $m \leqslant i = j+m < n+m$, so that P^{j+m} is the realization associated to the product category $B^j \times \underline{A}^{dis}$.

Hence, the realization associated to B ■ A is $P : \sigma_{n+m} \to Set$. ∇

COROLLARY. *The $(n+m)$-fold category* K *whose associated realization is*

$$\sigma_{n+m} \xrightarrow[\sim]{ass.} \sigma_n \otimes \sigma_m \xrightarrow{B \times A} Set \times Set \xrightarrow{-\times-} Set$$

is deduced from A ■ B *by the «symmetry isomorphism»* $(a, b) \mapsto (b, a)$.

If K is a category, for each object e of K we denote the partial Hom functor by $K(e, -) : K \to Set$.

PROPOSITION 2. *If $m < n$, the realization in Set associated to the $(n\text{-}m)$-fold category Hom(A, B) is*

$$H = (\sigma_{n-m} \xrightarrow{\hat{B}} Cat_m \xrightarrow{Cat_m(A, -)} Set),$$

where $\hat{B}$ is the $(n\text{-}m)$-fold category in Cat_m associated to B *(by A-3); it is equivalent to the realization*

$$H' = (\sigma_{n-m} \xrightarrow{\hat{B}} Cat_m \xrightarrow{\eta_m} Set^{\sigma_m} \xrightarrow{Set^{\sigma_m}(A, -)} Set)$$

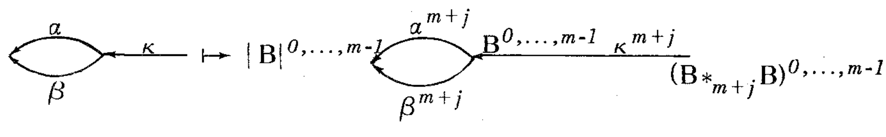
(where η_m is the equivalence defined in A-2*).*

PROOF. $\hat{B}$ is a realization and a partial Hom functor preserves limits, so that H and H' are realizations.

1° Since $\hat{B}(2,\dots,2) = \mathrm{B}^{0,\dots,m-1}$, the functor H maps $(2,\dots,2)$ onto $Cat_m(\mathrm{A},\mathrm{B}^{0,\dots,m-1})$, which is the set of multiple functors from A to B. For $0 \leq j < n-m$, let us consider the category H^j whose associated realization is:

$$\sigma \xrightarrow{\delta^j_{n-m}} \sigma_{n-m} \xrightarrow{\hat{B}} Cat_m \xrightarrow{Cat_m(\mathrm{A},-)} Set.$$

The composite of the two first functors is defined by:



It follows that the composition map of H^j is

$$Cat_m(\mathrm{A},\kappa^{m+j}) : Cat_m(\mathrm{A},(\mathrm{B}*_{m+j}\mathrm{B})^{0,\dots,m-1}) \to Cat_m(\mathrm{A},\mathrm{B}^{0,\dots,m-1}) \quad ;$$

an element of $Cat_m(\mathrm{A},(\mathrm{B}*_{m+j}\mathrm{B})^{0,\dots,m-1})$ is identified with a couple (f',f) of multiple functors from A to B such that $\alpha^{m+j}f' = \beta^{m+j}f$; by $Cat_m(\mathrm{A},\kappa^{m+j})$, it is mapped onto

$$\kappa^{m+j}\circ(f',f) : \mathrm{A} \to \mathrm{B} : a \mapsto f'(a) \circ_{m+j} f(a),$$

which is equal to the composite $f' \circ_j f$ in $Hom(\mathrm{A},\mathrm{B})^j$. Therefore, we have $\mathrm{H}^j = Hom(\mathrm{A},\mathrm{B})^j$ for each j, and H is the realization associated to the $(n\text{-}m)$-fold category $Hom(\mathrm{A},\mathrm{B})$.

2° H' is equivalent to H. Indeed, let A' be an m-fold category and $A' : \sigma_m \to Set$ the associated realization. The composite

$$Cat_m \xrightarrow{\eta_m} Set^{\sigma_m} \xrightarrow{Set^{\sigma_m}(A,-)} Set$$

maps A' onto the set $Set^{\sigma_m}(A,A')$ of natural transformations from A to A', which is in 1-1 correspondence with the set $Cat_m(\mathrm{A},\mathrm{A}')$ of m-fold functors from A to A'. Hence the above composite is equivalent to

$$Cat_m(\mathrm{A},-) : Cat_m \to Set.$$

It follows that H' is equivalent to H. ∇

REMARK. The reason for introducing H' in the above proposition is that it is constructed by using only realizations associated to A and B (while A itself remains in H). Propositions 1 and 2 suggest definitions of the square product and of the functor Hom for general internal multiple sketched structures; in this way all the results of the present paper may be extended, as will be shown in a subsequent paper. 1

C. An application.

This Section is devoted to prove that Cat_n is «generated from $2^{\blacksquare n}$ by colimits».

We denote by $Y_n : \Sigma_n^{op} \to Set^{\Sigma_n}$ (= category of natural transformations from Σ_n into Set) the Yoneda embedding, which maps an object u of Σ_n onto the partial Hom functor $\Sigma_n(u,-): \Sigma_n \to Set$. It is known [6, 5] that Y_n defines a σ_n-costructure in Set^{σ_n} (i.e., a realization

$$Y_n : \sigma_n \to (Set^{\sigma_n})^{op}),$$

called the *Yoneda* σ_n-costructure, denoted by $Y_n : \sigma_n^{op} \to Set^{\sigma_n}$. Since Cat_n is equivalent to Set^{σ_n}, there is also a canonical σ_n-costructure in Cat_n, defined by:

$$Y'_n = (\sigma_n^{op} \xrightarrow{Y_n} Set^{\sigma_n} \xrightarrow{\zeta_n} Cat_n),$$

where ζ_n is the canonical equivalence (see A-2).

In particular, if $n = 1$, the σ-costructure Y'_1 in Cat maps the integer q, for $q = 1, 2, 3, 4$, onto the category q defining the usual order on $q = \{0, \ldots, q\text{-}1\}$ (see Proposition 9-0 [5]).

More generally, we have the following result, used in Proposition 4.

PROPOSITION 3. *The canonical σ_n-costructure Y'_n in Cat_n maps an object $(q_0, \ldots, q_{n-1})$ of Σ_n onto an n-fold category isomorphic with*

$$\mathrm{q}_{n-1} \blacksquare (\ldots \blacksquare \mathrm{q}_0).$$

PROOF. The proof is by induction. As said above, the assertion is true for $n = 1$. Let us assume it is true for $(n\text{-}1)$-fold categories. Let u be an object $(q_0, \ldots, q_{n-1})$ of Σ_n; by Y'_n, it is mapped onto the n-fold category whose associated realization is $\Sigma_n(u,-): \sigma_n \to Set$. As $\Sigma_n = \Sigma_{n-1} \times \Sigma$, the partial Hom functor $\Sigma_n(u,-)$ is equal to the composite

$$\Sigma_n = \Sigma_{n-1} \times \Sigma \xrightarrow{\Sigma_{n-1}((q_0, \dots, q_{n-2}), -) \times \Sigma(q_{n-1}, -)} Set \times Set \xrightarrow{- \times -} Set.$$

The induction hypothesis indicates that

$$\Sigma_{n-1}((q_0, \dots, q_{n-2}), -): \sigma_{n-1} \to Set$$

is the realization associated to an $(n-1)$-fold category isomorphic with

$$q_{n-2} \blacksquare (\dots \blacksquare q_0),$$

and $\Sigma(q_{n-1}, -): \sigma \to Set$ is associated to q_{n-1}. Then Corollary, Proposition 1 asserts that the n-fold category whose associated realization is the above composite (equal to $\Sigma_n(u,-)$) is isomorphic with

$$q_{n-1} \blacksquare (q_{n-2} \blacksquare (\dots \blacksquare q_0)).$$

This achieves the proof by induction. ∇

PROPOSITION 4. *Cat_n is the inductive closure of* $\{ 2^{\blacksquare^n} \}$.

PROOF. In C-0 [5], it is proved that Σ is the Γ-closure of $\{2\}$ (where Γ is the set of distinguished cones of σ), so that by Proposition 7-0 [5] it follows that Set^{σ_n} is the inductive closure of $\{ Y_n(2, \dots, 2) \}$. Since

$$Y'_n = (\sigma_n^{op} \xrightarrow{Y_n} Set^{\sigma_n} \xrightarrow{\zeta_n} Cat_n),$$

where ζ_n is an equivalence, Cat_n is the inductive closure of

$$\{ \zeta_n(Y_n(2, \dots, 2)) = Y'_n(2, \dots, 2) \}.$$

By Proposition 3, $Y'_n(2, \dots, 2)$ is isomorphic with $\underbrace{2 \blacksquare (\dots \blacksquare 2)}_{n \text{ times}}$, which is isomorphic with

$$2^{\blacksquare^n} = \underbrace{(2 \blacksquare \dots) \blacksquare 2}_{n \text{ times}}.$$

More precisely, it is shown that the subcategory image of Y'_n is the pushout closure of $\{ 2^{\blacksquare^n} \}$, because q_j, for $q_j = 1, 2, 3, 4$, is deduced from 2 by pushouts [5], $q_j \blacksquare -$ preserve pushouts and $Y'_n(q_0, \dots, q_{n-1})$ is isomorphic to $q_{n-1} \blacksquare (\dots \blacksquare q_0)$. Then an n-fold category B is the colimit of the composite of Y'_n with the opposite of the discrete fibration $K_B \to \Sigma_n$ corresponding to the realization $B: \sigma_n \to Set$ associated to B. ∇

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MULTIPLE FUNCTORS
III. THE CARTESIAN CLOSED CATEGORY Cat_n

by Andrée and Charles EHRESMANN

INTRODUCTION.

This paper is Part III of our work on multiple functors [4, 5] and it is a direct continuation of Part II. It is devoted to an explicit description of the cartesian closed structure on Cat_n (= category of n-fold categories) which will be «laxified» in the Part IV [6] (this is a much more general result than that announced in Part I). The existence of such structures might be deduced from general theorems on sketched structures [7,14], but 1
this does not lead to concrete definitions. Here the construction uses the monoidal closed category $(\coprod_n Cat_n, \blacksquare, Hom)$ of multiple categories defined in Part II.

In the cartesian closed category Cat, the internal Hom functor maps (A, C) onto the category of natural transformations from A to C, which is identified with the category $Hom(A, \square C)$, where $\square C$ is the double category of squares of C.

To generalize this situation, the idea is to construct a functor $\square_n$ from Cat_n to Cat_{2n} (which reduces for $n = 1$ to the functor $\square : Cat \to Cat_2$), whose composite with the functor $Hom(A, -): Cat_{2n} \to Cat_n$ gives, for each n-fold category A, the partial internal Hom functor of the cartesian closed structure of Cat_n. In fact, we first define a pair of adjoint functors *Square* and *Link* between Cat_n and Cat_{n+1}, which has also some interest of its own; iteration of this process leads to a functor $\square_n : Cat_n \to Cat_{2n}$ whose left adjoint maps $B \blacksquare A$ onto the product $B \times A$, for each n-fold category B. Hence the functor

$$Hom(A, \square_n -): Cat_n \to Cat_n$$

is a right adjoint of the product functor $- \times A$, as desired.

The delicate point is the explicit construction of *Link*, which «is» a left inverse of *Square*. The category of components of a 2-category, as well as the crossed product category associated to the action [8] of a category on a category, appear as examples of *Link* A.

Finally Cat_n is «embedded» as the category of 1-morphisms in the $(n+1)$-category Nat_n of hypertransformations (or «natural transformation between natural transformations, between...»), whose n first categories form the n-fold category coproduct of $Hom_n(\mathrm{A}, \mathrm{B})$, for any n-fold categories A, B. The construction of Nat_n uses the equivalence (see Appendix) between categories enriched in a category V with commuting coproducts (in the sense of [21]) and categories internal to V whose object of objects is a coproduct of copies of the final object.

NOTATIONS.

The notations are those introduced in Part II. In particular, if B is an n-fold category, B^i denotes its i-th category for each integer $i < n$, and $\mathrm{B}^{i_0, \dots, i_{p-1}}$, for each sequence $(i_0, \dots, i_{p-1})$ of distinct integers $i_j < n$, is the p-fold category whose j-th category is B^{i_j}.

Let A be an m-fold category. The square product $\mathrm{B} \blacksquare \mathrm{A}$ is the $(n+m)$-fold category on the product set $\underline{\mathrm{B}} \times \underline{\mathrm{A}}$ (where $\underline{\mathrm{B}}$ always denotes the set of blocks of B) whose i-th category is:

$$\underline{\mathrm{B}}^{dis} \times \mathrm{A}^i \text{ for } i < m, \quad \mathrm{B}^{i-m} \times \underline{\mathrm{A}}^{dis} \text{ for } m \leqslant i < n+m$$

($\underline{\mathrm{B}}^{dis}$ is the discrete category on $\underline{\mathrm{B}}$).

If $m < n$, then $Hom(\mathrm{A}, \mathrm{B})$ is the $(n\text{-}m)$-fold category on the set of multiple functors $f: \mathrm{A} \to \mathrm{B}$ (i.e., on the set of m-fold functors f from A to $\mathrm{B}^{0, \dots, m-1}$) whose j-th composition is deduced «pointwise» from that of B^{m+j}, for each integer $j < n-m$.

The category $\coprod_n Cat_n$ of (all small) multiple categories, equipped with $\blacksquare$ and *Hom* is monoidal closed (Proposition 7 [5]), i.e., the partial functor $Hom(\mathrm{A}, -): Cat_{n+m} \to Cat_n$ is right adjoint to $- \blacksquare \mathrm{A}: Cat_n \to Cat_{n+m}$.

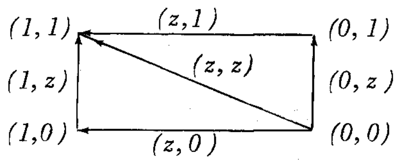
A. The adjoint functors Square and Link.

This Section is devoted to the construction of the functor *Square* from Cat_n to Cat_{n+q}, and of its left adjoint, the functor *Link*. For $n = 1$, the functor *Square* reduces to the functor $\square : Cat \to Cat_2$, whose definition is first recalled to fix the notations.

2 is always the category

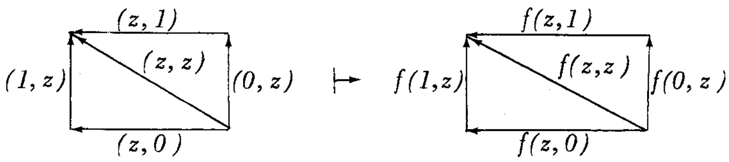
$$1 \xleftarrow{\;(1,0)\;} 0$$

so that 2×2 is represented by the commutative diagram :

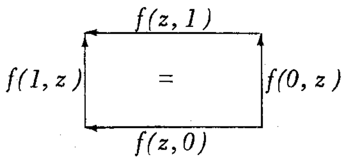


(where $z = (0, 1)$).

Let C be a category. A functor $f : 2 \times 2 \to C$:



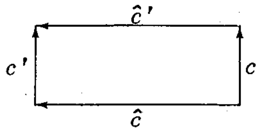
is entirely determined by the (commutative) square of C :



(since $f(z, z)$ is the «diagonal» of this square :

$$f(z, 1) f(0, z) = f(1, z) f(z, 0) \;);$$

and every square $(c', \hat{c}', \hat{c}, c)$

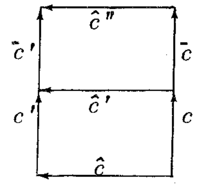


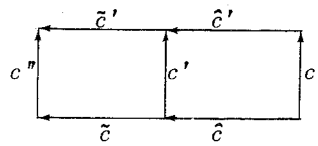
of C is obtained in this way. So we shall identify the set $Hom(2 \times 2, C)$ of functors from 2×2 to C with the set of squares of C.

On this set, the «vertical» and the «horizontal» compositions:

$$(\bar{c}', \hat{c}'', \hat{c}', \bar{c}) \boxminus (c', \hat{c}', \hat{c}, c) = (\bar{c}'c', \hat{c}'', \hat{c}, \bar{c}c),$$

$$(c'', \tilde{c}', \tilde{c}, c') \boxbar (c', \hat{c}', \hat{c}, c) = (c'', \tilde{c}'\hat{c}', \tilde{c}\hat{c}, c),$$





define categories $\boxminus$C and $\boxbar$C (which are both isomorphic, and also called by some authors category of arrows of C). The couple ($\boxminus$C, $\boxbar$C) is the *double category* $\square$C *of squares of* C.

The functor $\square: Cat \to Cat_2$ maps $g: C \to C'$ onto

$$\square g: \square C \to \square C': (c', \hat{c}', \hat{c}, c) \mapsto (g(c'), g(\hat{c}'), g(\hat{c}), g(c)).$$

Now let n be an integer, $n > 1$. Let B be an n-fold category. Taking for C above the 0-th category B^0 of B, we have, on the set of squares of B^0 (to which are identified the functors $2 \times 2 \to B^0$), not only the double category $\square B^0$, but also the $(n-1)$-fold category $Hom(2 \times 2, B)$, whose i-th composition (deduced pointwise from that of B^{i+1}) is written with squares:

$$(b_1', \hat{b}_1', \hat{b}_1, b_1) \circ_i (b', \hat{b}', \hat{b}, b) = (b_1' \circ_{i+1} b', \hat{b}_1' \circ_{i+1} \hat{b}', \hat{b}_1 \circ_{i+1} \hat{b}, b_1 \circ_{i+1} b),$$

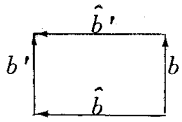
iff the four composites are defined in B^{i+1}.

DEFINITION. The *multiple category of squares of* B, denoted by SqB, is the $(n+1)$-fold category on the set of squares of B^0 such that:

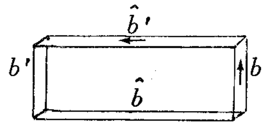
$$(Sq\mathrm{B})^{0,\dots,n-2} = Hom(2 \times 2, \mathrm{B}),\ (Sq\mathrm{B})^{n-1} = \boxminus \mathrm{B}^0,\ (Sq\mathrm{B})^n = \square \mathrm{B}^0$$

(the $(n-1)$ first compositions are those of $Hom(2 \times 2, B)$, the two last ones being the vertical and the horizontal compositions of squares).

To «visualize» this multiple category SqB, we shall also represent a square

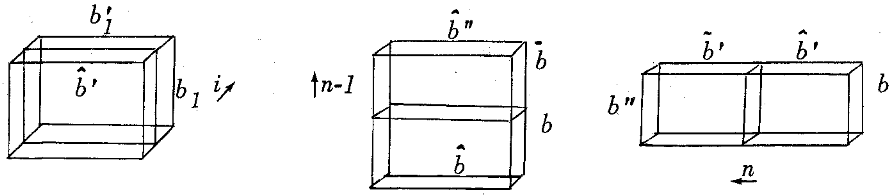


of B^0 by



;

then the compositions of $Sq\mathrm{B}$ are represented by:



REMARK (not used afterwards). The construction of $Sq\mathrm{B}$ may be interpreted in terms of sketched structures. To each category $\phi : \sigma \to \mathrm{V}$ internal to a category V with pullbacks, it is associated a category $\partial\phi : \sigma \to \mathrm{V}^{\sigma}$ internal to V^{σ} (Proposition 28 [7]). If $\phi : \sigma \to Cat_{n-1}$ is the category in Cat_{n-1} canonically associated to $\mathrm{B}^{1,\dots,n-1,0}$ (Appendix, Part II [5]), then

$$\sigma \xrightarrow{\partial\phi} Cat^{\sigma}_{n-1} \xrightarrow{\sim} Cat_n$$

is the category in Cat_n associated to $Sq\mathrm{B}$.

There is a functor from Cat_n to Cat_{n+1}, called the *functor Square*, and denoted by

$$Sq_{n,\,n+1} : Cat_n \to Cat_{n+1},$$

which maps an n-fold functor $g : \mathrm{B} \to \mathrm{B}'$ onto the $(n+1)$-fold functor

$$Sqg : Sq\mathrm{B} \to Sq\mathrm{B}' : (b', \hat{b}', \hat{b}, b) \mapsto (g(b'), g(\hat{b}'), g(\hat{b}), g(b))$$

(defined by $\square g : \square\mathrm{B}^0 \to \square\mathrm{B}'^0$).

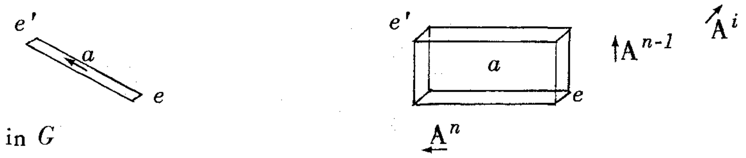
PROPOSITION 1. *The functor* $Sq_{n,n+1} : Cat_n \to Cat_{n+1}$ *admits a left adjoint* $Lk_{n+1,n} : Cat_{n+1} \to Cat_n$.

PROOF. The proof, quite long, will be decomposed in several steps. Let A be an $(n+1)$-fold category, α^i and β^i the maps source and target of A^i for each integer $i \leqslant n$.

1° *We define an n-fold category, called the multiple category of* $(n\text{-}1, n)$*-links of* A, denoted by $Lk\mathrm{A}$ (later on, it will be proved that $Lk\mathrm{A}$ is the free object generated by A with respect to the functor *Square*).

a) Consider the graph G whose vertices are those blocks e of A which are objects for the two last categories A^{n-1} and A^n, and whose edges $a : e \to e'$ from e to e' are the blocks a of A such that:

$$\alpha^n \alpha^{n-1}(a) = e \quad \text{and} \quad \beta^n \beta^{n-1}(a) = e'.$$



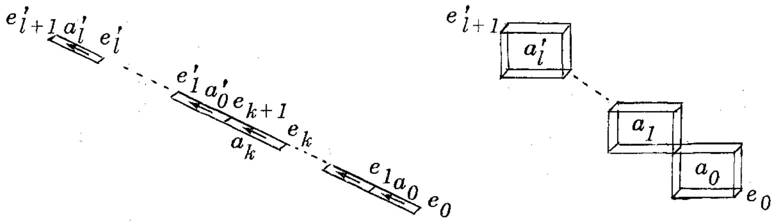
b) Let P^0 be the set $\underline{P}$ of all paths of the graph G (i. e. sequences

$$(a_k, \dots, a_0), \quad \text{where} \quad a_i : e_i \to e_{i+1} \quad \text{in } G),$$

equipped with the concatenation:

$$(a'_l, \dots, a'_0) \circ_0 (a_k, \dots, a_0) = (a'_l, \dots, a'_0, a_k, \dots, a_0)$$

$$\text{iff } \alpha^n \alpha^{n-1}(a'_0) = \beta^n \beta^{n-1}(a_k).$$

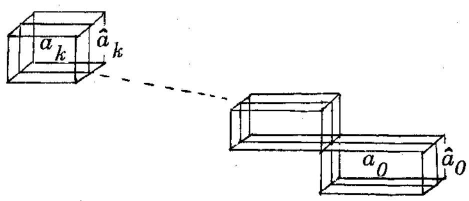


P^0 is an associative but non-unitary category (called a quasi-category in [10], where P^0 is shown to be the free quasi-category generated by G).

c) For each integer i with $0 \leqslant i < n-1$, there is a category P^{i+1} on $\underline{P}$ whose composition is deduced «pointwise» from that of A^i, which means:

$$(\hat{a}_l, \dots, \hat{a}_0) \circ_{i+1} (a_k, \dots, a_0) = (\hat{a}_k \circ_i a_k, \dots, \hat{a}_0 \circ_i a_0)$$

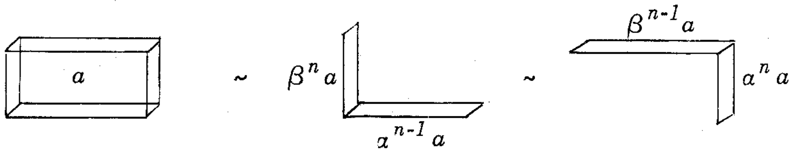
iff $l = k$ and the composites $\hat{a}_j \circ_i a_j$ are defined in A^i, for $j \leqslant k$.



REMARK. It is to be able to define P^i that we had to take all the paths of G, and not only the reduced ones (i.e., those without objects) which form the free category generated by G.

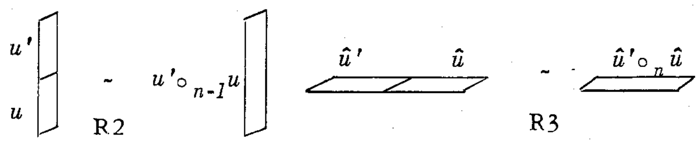
d) Consider on the set $\underline{P}$ of all the paths of G the relation r defined as follows:

(R 1) $(a) \sim (\beta^n a, \alpha^{n-1} a) \sim (\beta^{n-1} a, \alpha^n a)$ for each block a of $\mathbb{A}$.



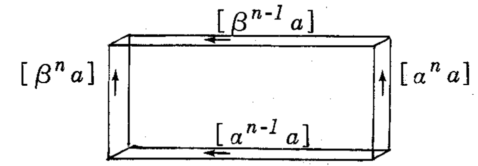
(R 2) $(u', u) \sim (u' \circ_{n-1} u)$ iff (u', u) is a couple of objects of $\mathbb{A}^n$ whose composite exists in the category $\mathbb{A}^{n-1}$.

(R 3) $(\hat{u}', \hat{u}) \sim (\hat{u}' \circ_n \hat{u})$ iff $(\hat{u}', \hat{u})$ is a couple of objects of $\mathbb{A}^{n-1}$ whose composite exists in the category $\mathbb{A}^n$.



e) According to the proof of Proposition 3 [5], there exists an n-fold category (called the *multiple category of $(n$-$1, n)$-links of* $\mathbb{A}$, denoted by $Lk\mathbb{A}$) quasi-quotient of $\mathrm{P} = (\mathrm{P}^0, \mathrm{P}^1, \dots, \mathrm{P}^{n-1})$ by r and such that the canonical morphism $\hat{r} : \mathrm{P} \to Lk\mathbb{A}$ defines a quasi-functor $\hat{r} : \mathrm{P}^0 \to Lk\mathbb{A}^0$ and a functor $\hat{r} : \mathrm{P}^i \to Lk\mathbb{A}^i$ for $1 \leqslant i < n$. The image $\hat{r}(a_k, \dots, a_0)$ is denoted by $[a_k, \dots, a_0]$; those blocks generate $Lk\mathbb{A}$ ($\hat{r}$ may not be onto).

2° *There is an $(n+1)$-fold functor $l : \mathbb{A} \to Sq(Lk\mathbb{A})$ which maps a block a of $\mathbb{A}$ onto the square $l(a)$ of $(Lk\mathbb{A})^0$ such that*

$l(a) =$ 

(intuitively, $l(a)$ «is» the frame of a in the double category $(\mathbb{A}^{n-1}, \mathbb{A}^n)$).

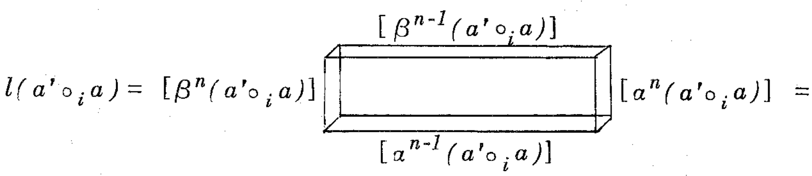
a) The map l is well-defined: The relation (R1) has been introduced so that $l(a)$ be a commutative square of $(Lk\mathbb{A})^0$, since

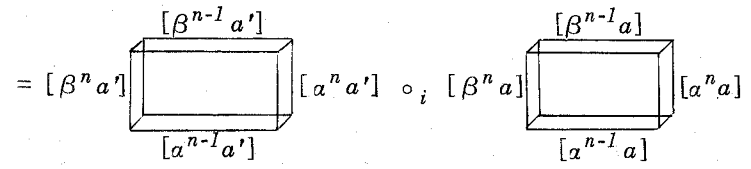
$$[\beta^{n-1} a] \circ_0 [\alpha^n a] = [(\beta^{n-1} a, \alpha^n a)] \underset{R1}{=} [(\beta^n a, \alpha^{n-1} a)] = [\beta^n a] \circ_0 [\alpha^{n-1} a].$$

b) For $0 \leqslant i < n\text{-}1$ the map l defines a functor $l: \mathrm{A}^i \to Sq(Lk\mathrm{A})^i$: The i-th composition of $Sq(Lk\mathrm{A})$ is deduced «pointwise» from the $(i+1)$-th composition of $Lk\mathrm{A}$, which is itself deduced «pointwise» from the composition of A^i. Suppose the composite $a' \circ_i a$ defined in A^i; as $\alpha^n : \mathrm{A}^i \to \mathrm{A}^i$ is a functor, we have

$$[\alpha^n(a' \circ_i a)] = [(\alpha^n a') \circ_i (\alpha^n a)] = [\alpha^n a'] \circ_{i+1} [\alpha^n a] ;$$

similar equalities are valid if we replace α^n by β^n, by α^{n-1} or by β^{n-1}. Hence:





$$= \quad l(a') \circ_i l(a).$$

c) The relation (R2) implies that $l: \mathrm{A}^{n-1} \to (Sq(Lk\mathrm{A}))^{n-1}$ is a functor: By definition,

$$(Sq(Lk\mathrm{A}))^{n-1} = \boxminus(Lk\mathrm{A})^0 .$$

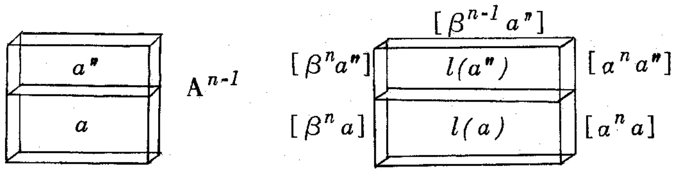
Suppose $a'' \circ_{n-1} a$ defined in A^{n-1}. As $\alpha^n : \mathrm{A}^{n-1} \to \mathrm{A}^{n-1}$ is a functor,

$$[\alpha^n(a'' \circ_{n-1} a)] = [\alpha^n(a'') \circ_{n-1} \alpha^n(a)] \underset{R2}{=} [(\alpha^n a'', \alpha^n a)] =$$

$$= [\alpha^n a''] \circ_0 [\alpha^n a] ;$$

and similarly with α^n replaced by β^n. Moreover:

$$[\alpha^{n-1}(a'' \circ_{n-1} a)] = [\alpha^{n-1} a], \quad [\beta^{n-1}(a'' \circ_{n-1} a)] = [\beta^{n-1} a''] .$$



It follows that

$$l(a''\circ_{n-1}a) = l(a'')\boxminus l(a).$$

d) Using the relation (R 3) instead of (R 2) it is proved analogously that $l: \mathrm{A}^n \to (Sq(Lk\mathrm{A}))^n = \boxplus Lk\mathrm{A}$ is a functor.

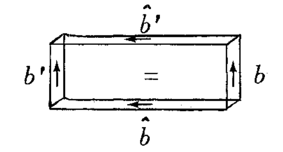
3° *$l: \mathrm{A} \to Sq(Lk\mathrm{A})$ is the liberty morphism defining $Lk\mathrm{A}$ as the free object generated by A with respect to $Sq_{n,n+1}: Cat_n \to Cat_{n+1}$.*

Indeed, let B be an n-fold category and $g: \mathrm{A} \to Sq\mathrm{B}$ an $(n+1)$-fold functor.

a) The «diagonal map» d sending a square s of B^0 onto its diagonal defines an $(n-1)$-fold functor

$$d: (Sq\mathrm{B})^{0,\dots,n-2} \to \mathrm{B}^{1,\dots,n-1} :$$

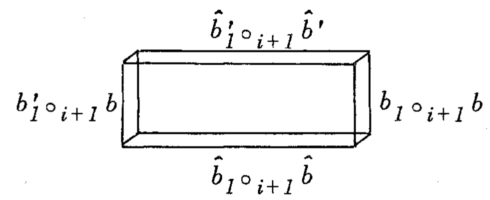
This map d sends the square

$s =$ 

of B^0 onto

$$d(s) = \hat{b}'\circ_0 b = b'\circ_0 \hat{b}.$$

For each integer $i < n-1$, the composition of $(Sq\mathrm{B})^i$ is deduced pointwise from that of B^{i+1}. As B is an n-fold category, the 0-th and $(i+1)$-th compositions of B satisfy the permutability axiom (P). Hence, if $s_1\circ_i s$ is defined in $(Sq\mathrm{B})^i$, then

 $= s_1\circ_i s$,

$$d(s_1\circ_i s) = (\hat{b}'_1\circ_{i+1}\hat{b}')\circ_0(b_1\circ_{i+1}b) \underset{\mathrm{P}}{=} (\hat{b}'_1\circ_0 b_1)\circ_{i+1}(\hat{b}'\circ_0 b)$$
$$= d(s_1)\circ_{i+1}d(s).$$

b) There is a unique morphism $h: \mathrm{P} \to \mathrm{B}$ extending the composite

$(n\text{-}1)$-fold functor

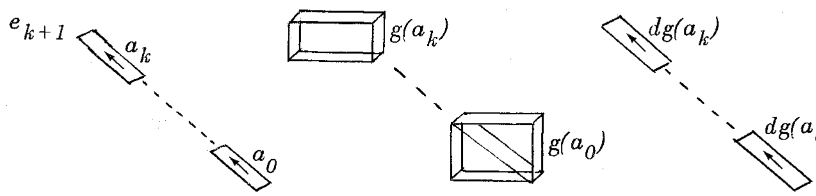
$$A^{0,\ldots,n-2} \xrightarrow{\ g\ } (Sq\mathrm{B})^{0,\ldots,n-2} \xrightarrow{\ d\ } \mathrm{B}^{1,\ldots,n-1}:$$

The edge $a: e \to e'$ of the graph G is mapped by dg onto the morphism

$$dg(a): dg(e) \to dg(e') \text{ of } \mathrm{B}^0.$$

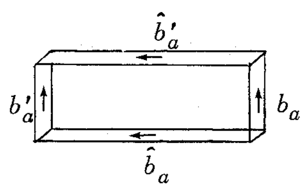
There is a unique quasi-functor $h: \mathrm{P}^0 \to \mathrm{B}^0$ extending dg (by the universal property of P^0) and h sends the path $p = (a_k, \ldots, a_0)$ onto the composite:

$$h(p) = dg(a_k) \circ_0 \cdots \circ_0 dg(a_0).$$



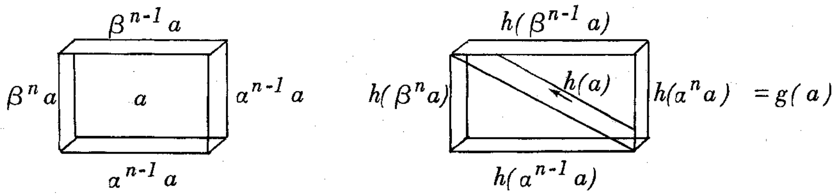
For $0 \leqslant i < n-1$, the composition of P^{i+1} is deduced pointwise from that of A^i and $dg: \mathrm{A}^i \to \mathrm{B}^{i+1}$ is a functor; it follows that $h: \mathrm{P}^{i+1} \to \mathrm{B}^{i+1}$ is a functor. Hence, $h: \mathrm{P} \to \mathrm{B}$ is a morphism.

c) $h: \mathrm{P} \to \mathrm{B}$ is compatible with the relation r used to define $Lk\mathrm{A}$: If a is a block of A, the square $g(a)$ of B^0 will be denoted by:

$g(a) =$ 

- As $g: \mathrm{A}^{n-1} \to \boxminus\mathrm{B}^0$ is a functor, $g(\alpha^{n-1}a)$ is the vertical source of the square $g(a)$, and its diagonal $h(\alpha^{n-1}a)$ is equal to $\hat{b}_a$. Similarly, $h(\beta^n a) = b'_a$, since $g: \mathrm{A}^n \to \boxplus\mathrm{B}^0$ is a functor. Therefore,

$$h(a) = b'_a \circ_0 \hat{b}_a = h(\beta^n a) \circ_0 h(\alpha^{n-1}a) = h(\beta^n a, \alpha^{n-1}a).$$



In an analogous way, we get

$$h(a) = \hat{b}'_a \circ_0 b_a = h(\beta^{n-1} a, \alpha^n a).$$

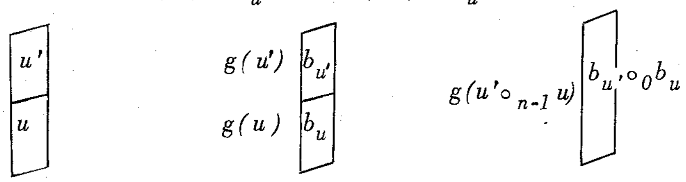
This proves that h is compatible with (R 1).

- Let the composite $u' \circ_{n-1} u$ be defined in A^{n-1}, with u and u' objects of A^n. Applying the functor $g: \mathrm{A}^{n-1} \to \boxminus \mathrm{B}^0$, we have

$$g(u' \circ_{n-1} u) = g(u') \boxminus g(u).$$

As $g: \mathrm{A}^n \to \boxplus \mathrm{B}^0$ is a functor, it maps the objects u and u' of A^n onto objects of $\boxplus \mathrm{B}^0$ whose diagonals are

$$h(u) = b_u \quad \text{and} \quad h(u') = b_{u'}.$$

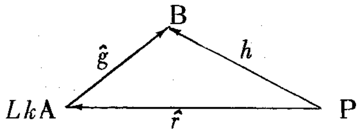


The composite $g(u' \circ_{n-1} u) = g(u') \boxminus g(u)$ is also an object of $\boxplus \mathrm{B}^0$ whose diagonal is $b_{u'} \circ_0 b_u$. It follows that

$$h(u' \circ_{n-1} u) = d(g(u') \boxminus g(u)) = b_{u'} \circ_0 b_u = h(u') \circ_0 h(u) = h(u', u).$$

Hence h is compatible with (R 2). The compatibility with (R 3) is proved by a similar method.

d) By the universal property of the quasi-quotient $Lk\mathrm{A}$ of P by r, there exists a unique n-fold functor $\hat{g}: Lk\mathrm{A} \to \mathrm{B}$ factorizing the morphism $h: \mathrm{P} \to \mathrm{B}$ compatible with r through the canonical morphism $\hat{r}: \mathrm{P} \to Lk\mathrm{A}$:

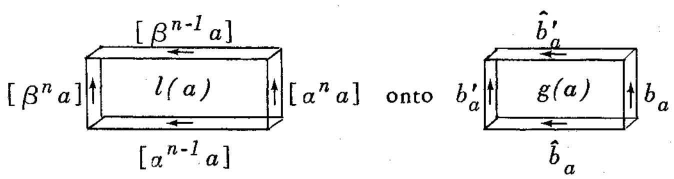


It maps the block $[a_k, \ldots, a_0]$ of $Lk\mathrm{A}$ onto $h(a_k) \circ_0 \cdots \circ_0 h(a_0)$. In particular, for each block a of A, we have

$$\hat{g}([\alpha^n a]) = h(\alpha^n a) = b_a, \quad \hat{g}([\beta^n a]) = b'_a,$$

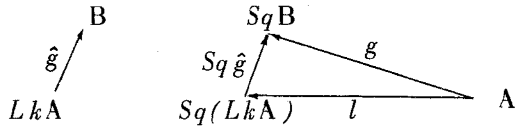
$$\hat{g}([\alpha^{n-1} a]) = \hat{b}_a, \quad \hat{g}([\beta^{n-1} a]) = \hat{b}'_a.$$

These equalities imply that $Sq\hat{g}: Sq(Lk\mathrm{A}) \to Sq\mathrm{B}$ maps



Therefore

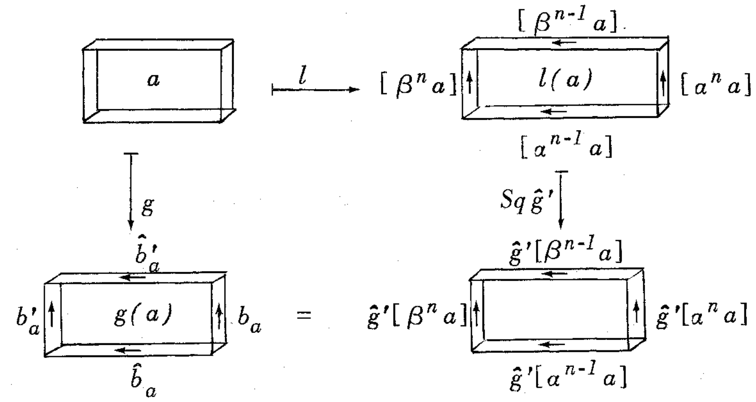
$$(\, g: A \to SqB\,) = (\, A \xrightarrow{\;l\;} Sq(LkA) \xrightarrow{\;Sq\hat{g}\;} SqB\,).$$



$$Cat_n \xrightarrow{\qquad Sq_{n,n+1} \qquad} Cat_{n+1}$$

e) Suppose that $\hat{g}': LkA \to B$ is an n-fold functor such that

$$(Sq\hat{g}')(l(a)) = g(a) \quad \text{for each block } a \text{ of } A\,.$$



In particular, this implies that $\hat{g}'(u) = b_u$ for each object u of A^n, and $\hat{g}'(\hat{u}) = \hat{b}_{\hat{u}}$ for each object $\hat{u}$ of A^{n-1}. Then:

$$\hat{g}'\hat{r}(a) \underset{R1}{=} \hat{g}'[\beta^n a, \alpha^{n-1}a] = \hat{g}'[\beta^n a] \circ_0 \hat{g}'[\alpha^{n-1} a] = \hat{b}'_a \circ_0 b_a = h(a),$$

i.e., the two morphisms

$$h: P \to B \quad \text{and} \quad (\, P \xrightarrow{\;\hat{r}\;} LkA \xrightarrow{\;\hat{g}'\;} B\,)$$

have the same «restriction» to the graph G. By the unicity of h (see b), it follows that they are equal, and $\hat{g}: LkA \to B$ is their unique factor through

$\hat{r}$. Hence, $\hat{g}' = \hat{g}$.

f) This proves that $Lk\mathrm{A}$ is the free object generated by A. The corresponding left adjoint of $Sq_{n,n+1}: Cat_n \to Cat_{n+1}$, denoted by

$$Lk_{n+1,n}: Cat_{n+1} \to Cat_n ,$$

maps the $(n+1)$-fold functor $f: \mathrm{A} \to \mathrm{A}'$ onto $Lkf: Lk\mathrm{A} \to Lk\mathrm{A}'$ such that

$$(Lkf)[a_k, \dots, a_0] = [f(a_k), \dots, f(a_0)]. \qquad \nabla$$

DEFINITION. The functor $Lk_{n+1,n}: Cat_{n+1} \to Cat_n$ defined above is called the *Link functor from* Cat_{n+1} *to* Cat_n.

COROLLARY 1. *The functor* $\square: Cat \to Cat_2$ *admits as a left adjoint the Link functor from* Cat_2 *to* Cat. ∇

By iteration, for each integer m, we define the functor $Sq_{n,n+m} =$

$$(Cat_n \xrightarrow{Sq_{n,n+1}} Cat_{n+1} \;\cdots\; Cat_{n+m-1} \xrightarrow{Sq_{n+m-1,\,n+m}} Cat_{n+m}).$$

COROLLARY 2. *The functor* $Sq_{n,n+m}$ *admits as a left adjoint the functor* $Lk_{n+m,n} =$

$$(Cat_{n+m} \xrightarrow{Lk_{n+m,\,n+m-1}} Cat_{n+m-1} \;\cdots\; Cat_{n+1} \xrightarrow{Lk_{n+1,n}} Cat_n). \quad \nabla$$

DEFINITION. $Sq_{n,n+m}$ will be called the *Square functor, from* Cat_n *to* Cat_{n+m}, and $Lk_{n+m,n}$ the *Link functor from* Cat_{n+m} *to* Cat_n.

These functors (for $n = m$) will be used as essential tools in Section C to describe the cartesian closed structure on Cat_n.

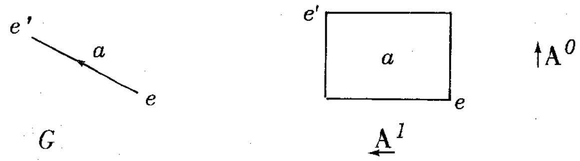
B. Some examples concerning double categories.

1° The category of links of a double category.

By Corollary 1, Proposition 1, the functor $\square: Cat \to Cat_2$ admits as a left adjoint the functor *Link* from Cat_2 to Cat. If A is a double category $(\mathrm{A}^0, \mathrm{A}^1)$, the category of its links $Lk\mathrm{A}$ may also be described as follows:

Let G be the graph associated to A in Proposition 1, whose vertices are the vertices e of A and whose edges $a: e \to e'$ are the blocks of A such that

$$\alpha^1 \alpha^0 a = e \quad \text{and} \quad \beta^1 \beta^0 a = e'.$$



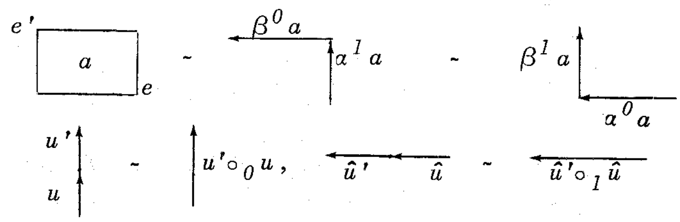
Let L be the free category generated by this graph; its objects are the vertices of A and its other morphisms are the «reduced» (i.e., with no factor a vertex) paths $(a_k, \dots, a_0)$ of G. Let R be the equivalence relation compatible with the composition of L generated by the relation r (introduced in Proposition 1):

$$(a) \sim (\beta^0 a,\ \alpha^1 a) \sim (\beta^1 a,\ \alpha^0 a),$$

for each block a of A which is not a vertex,

$$(u', u) \sim u' \circ_0 u, \quad \text{for } u' \text{ and } u \text{ objects of } A^1,$$

$$(\hat{u}', \hat{u}) \sim \hat{u}' \circ_1 \hat{u}, \quad \text{for } \hat{u}' \text{ and } \hat{u} \text{ objects of } A^0.$$

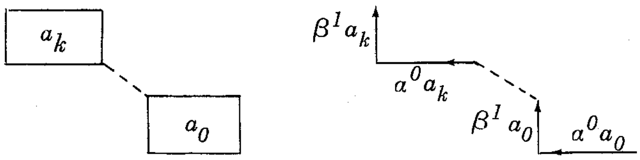


As distinct objects of L are not identified by r, and a fortiori by R, there exists a category L/R, quotient of L by R, whose morphisms are the equivalence classes modulo R, denoted by $[a_k, \dots, a_0]$. *The category* L *may be identified with* LkA.

Indeed, as we have remarked in the proof of Proposition 1, the quasi-category P^0 of all paths of G was introduced to insure that the compositions of A other than the last two ones give rise to categories P^i; here, there are only two compositions on A, so that it is equivalent to consider the «smallest» category L instead of P^0.

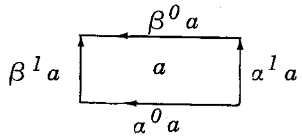
A morphism of L will be called a *simple path* if it is of the form $(v_l, \dots, v_0)$, where the factors v_i are objects of one and only one category

A^0 or A^1 and two successive factors are not objects of the same category. Any morphism $(a_k, \dots, a_0)$ of L is equivalent modulo R to at least one simple path. Indeed,



$$(a_k, \dots, a_0) \sim (\beta^1 a_k, \alpha^0 a_k, \dots, \beta^1 a_0, \alpha^0 a_0)$$

if this path is reduced; otherwise, there exist successive factors of this path, $(v_{j+m}, \dots, v_j)$, which are objects of the same category A^i; in this case, we replace $(v_{j+m}, \dots, v_j)$ by its composite $v_{j+m} \circ_i \dots \circ_i v_j$. The sequence thus obtained is a simple path, equivalent to $(a_k, \dots, a_0)$ modulo R. Hence the morphisms of $L \approx Lk A$ are of the form $[v_l, \dots, v_0]$, where $(v_l, \dots, v_0)$ is a simple path. Remark that two different simple paths may be equivalent modulo R, as shows the example of the double category $2 \blacksquare 2$ which has only one non-degenerate block a:



and in which

$$(\beta^1 a, \alpha^0 a) \text{ and } (\beta^0 a, \alpha^1 a)$$

are two simple paths which are equivalent modulo R.

REMARK. With the general hypotheses of Proposition 1, to each path p of G is also associated a «simple path» defined as above (with A^0 and A^1 replaced by A^{n-1} and A^n), and which is mapped by $\hat{r}: P \to Lk A$ onto the same block than p. But the compositions of $Lk A$ other than the 0-th one are not expressed easily on these simple paths.

2° Fibrations as categories of links.

Let $F: C \to Cat$ be a functor, where C is a small category (F is also called «une espèce de morphismes» [8]).

a) *F determines an action* κ' *of the category* C *on the category* S *coproduct of the categories* $F(u)$, *for all objects* u *of* C, defined by:

$$\kappa'(c,s) = F(c)(s) \quad (\text{written } cs)$$

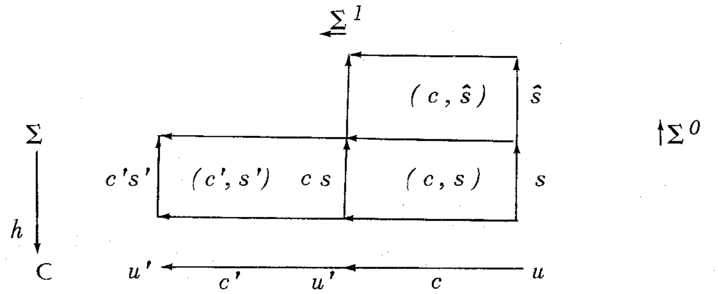
iff $c: u \to u'$ in C and s in $F(u)$.

Conversely, each action of a (small) category on a (small) category corresponds in this way to a functor toward *Cat* (see Chapter II [8]).

b) *To* F (or to the action κ' of C on S) *is also associated a double functor* $h: \Sigma \to (\underline{C}^{dis}, C)$ defined as follows:

- Let $h: \Sigma^1 \to C$ be the discrete fibration (or «foncteur d'hypermorphisme» in the terminology of [8]) associated to the action κ' of C on the set $\underline{S}$ of morphisms of S: the morphisms of Σ^1 are the couples (c, s) such that the composite $\kappa'(c,s) = cs$ is defined; the composition of Σ^1 is:

$$(c', s') \circ_1 (c, s) = (c'c, s) \quad \text{iff } s' = cs.$$



The object (u, s) of Σ^1 is identified with the morphism s of S. The functor $h: \Sigma^1 \to C$ maps (c, s) onto c.

- There is another category Σ^0 with the same set $\underline{\Sigma}$ of morphisms than Σ^1, whose composition is:

$$(\hat{c}, \hat{s}) \circ_0 (c, s) = (c, \hat{s}s) \quad \text{iff } c = \hat{c} \text{ and } \hat{s}s \text{ defined in S.}$$

The couple (Σ^0, Σ^1) is a double category Σ, and $h: \Sigma \to (\underline{C}^{dis}, C)$ is a double functor.

c) By the construction of b, we obtain every double functor $f: T \to K$ satisfying the two conditions:

(F 1) The 0-th category of K is discrete;

(F 2) The functor $f: T^1 \to K^1$ is a discrete fibration.

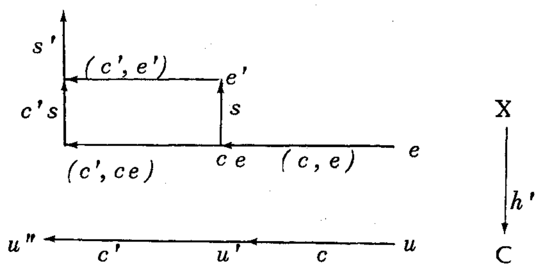
A double functor $f: T \to K$ satisfies (F 2) iff it is a *discrete fibration internal to Cat* (i.e., a realization in *Cat* of the sketch of discrete fibrations given in 0-D [4]), and then it is in 1-1 correspondence with a *category action in Cat* (in the sense of [4], page 22).

The category actions in *Cat* have been introduced in 1963 [9] under the name «catégories $\mathfrak{F}$-structurées d'opérateurs» or «$\mathfrak{F}$-espèces de morphismes»; in this Note, it was also indicated that the actions of a category on a category (or the functors toward *Cat*) are in 1-1 correspondence with the discrete fibrations internal to *Cat* over a double category whose 0-th category is discrete.

d) *To* F (or to the action κ' of C on S) *is also associated the (non-discrete) fibration* $h': X \to C$, where X is the *crossed product category* defined as follows (see Chapter II [8]);

- The morphisms of X are the triples (s, c, e) such that e is an object of S, the composite $ce = \kappa'(c, e)$ is defined and $s: ce \to e'$ is a morphism of S. The composition of X is:

$$(s', c', e').(s, c, e) = (s'(c's), c'c, e) \quad \text{iff} \quad s: ce \to e'.$$



- The category X is generated by the morphisms of one of the forms:

$(e', s, \hat{e})$, where $s: \hat{e} \to e'$ in S, identified with s,

(ce, c, e), denoted by (c, e).

The functor $h': X \to C$ maps (s, c, e) onto c.

Different characterizations of X have been indicated [15,16,17], and fibrations are of a great actuality [20,2]. Another characterization of X is given now:

PROPOSITION 2. *Let* $h: \Sigma \to (\underline{C}^{dis}, C)$ *be the discrete fibration internal to Cat associated (in b) to the action* κ' *of* C *on the category* S. *Then* $Lk\Sigma$ *is isomorphic with the crossed product category* X.

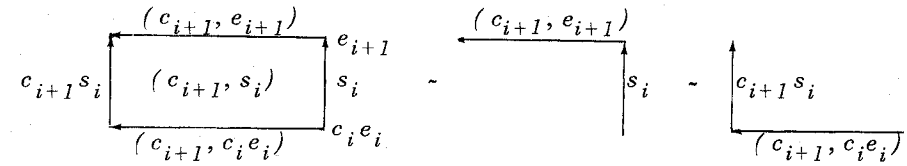
PROOF. 1° *Each morphism of the category* $Lk\Sigma$ *is of the form* $[s, (c, e)]$, *where* (s, c, e) *is a morphism of* X :

Indeed, the objects of Σ^1 are the morphisms of S, those of Σ^0 are the couples (c, e), where e is an object of S. So a simple path p is of the form

$$p = (s_k, (c_k, e_k), \dots, s_0, (c_0, e_0)),$$

where $s_i: c_i e_i \to e_{i+1}$ in S, for each $i \leqslant k$. We have

$$((c_{i+1}, e_{i+1}), s_i) \sim (c_{i+1}, s_i) \sim (c_{i+1} s_i, (c_{i+1}, c_i e_i))$$

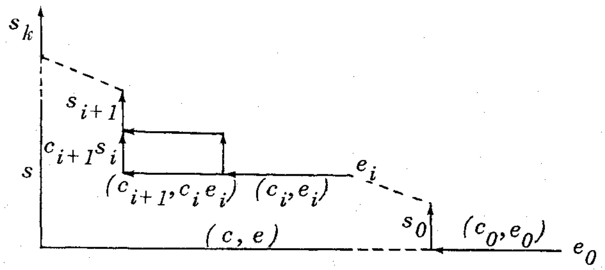


in the equivalence relation R defining $Lk\Sigma$ as a quotient of the category of paths (we use the «simplified» construction of $Lk\Sigma$ given in 1-B above). Moreover, in R, we have also :

$$(s_{i+1}, c_{i+1} s_i, (c_{i+1}, c_i e_i), (c_i, e_i)) \sim (s_{i+1}(c_{i+1} s_i), (c_{i+1} c_i, e_i)).$$

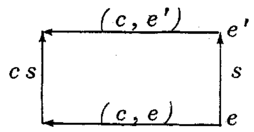
By iteration it follows that $p \sim (s, (c, e_0))$ where

$$s = s_k(c_k s_{k-1}) \dots (c_k \dots c_1) s_0, \quad c = c_k \dots c_0 .$$



Since each morphism of $Lk\Sigma$ is of the form $[p]$ for some simple path p, it is also of the form $[s, (c, e)]$, as announced.

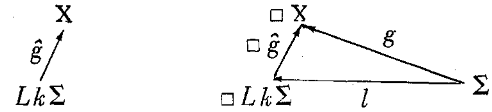
2° There is a double functor $g: \Sigma \to \square X$ mapping (c, s) onto the square



of X,

whose diagonal $d(g(c,s))$ is (cs, c, e). Since $Lk\Sigma$ is a free object generated by Σ with respect to $\square: Cat \to Cat_2$, there corresponds to g a unique functor $\hat{g}: Lk\Sigma \to X$ which maps $[s,(c,e)]$ onto

$$d(g(s)).d(g(c,e)) = s.(c,e) = (s,c,e):$$



This functor is 1-1 and onto, hence it is an isomorphism, whose inverse $\hat{g}^{-1}: X \to Lk\Sigma$ maps (s,c,e) onto $[s,(c,e)]$. ∇

COROLLARY. *With the hypotheses of Proposition 2,* X *is a free object generated by* Σ *with respect to* $\square: Cat \to Cat_2$. ∇

REMARK. The category of links of $(\underline{C}^{dis}, C)$ is identified with C, so that

$$Lkh: Lk\Sigma \to Lk(\underline{C}^{dis}, C) \approx C$$

is a fibration isomorphic with $h': X \to C$. This suggests the following generalization of Chapter II [8] : Let $f: T \to K$ be any discrete fibration internal to Cat. The functor $Lkf: Lk\,T \to Lk K$ «plays the role» of the fibration associated to the action of a category on a category. In particular, the equivalence classes of the sections of the functor Lkf could be called «classes of cohomology of f of order 1».

3° The multiple category of links of an $(n+1)$-category.

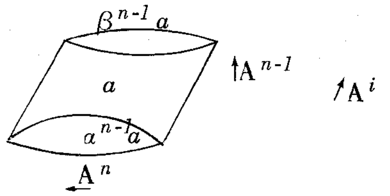
An $(n+1)$-fold category is called an *$(n+1)$-category* A if the objects of A^n are also objects of A^{n-1}. For $n = 1$, this reduces to the usual 1
notion of a 2-category. For $n = 2$, an example of a 3-category is provided by the 3-category of cylinders of a 2-category [1].

Let A be an $(n+1)$-category. Those blocks of A which are objects for A^{n-1} define an n-fold subcategory of $A^{0,\dots,n-2,n}$, denoted by

$$| A^{n-1} |^{0,\dots,n-2,n}, \text{ or more simply } | A^{n-1} | .$$

There exists (Proposition 3 [5]) an n-fold category quasi-quotient of $|A^{n-1}|$ by the relation:

$$\alpha^{n-1} a \sim \beta^{n-1} a \quad \text{for each block } a \text{ of } A ;$$



it will be called the *n-fold category of components of* A, denoted by ΓA. The canonical n-fold functor $\tilde{\rho} : | A^{n-1} | \to \Gamma A$ may not be onto, but its image generates the n-fold category ΓA. Remark that two objects of A^{n-1} which are in the same component of A^{n-1} have the same image by $\tilde{\rho}$.

EXAMPLE. Let A be a 2-category; then $| A^0 |^1 = | A^0 |$ is the category of 1-morphisms of A; the equivalence relation ρ generated on it by the relation (considered above):

$$\alpha^0 a \sim \beta^0 a \quad \text{for each block (or 2-cell) } a \text{ of } A$$

is defined by:

$$v \sim v' \text{ iff } v \text{ and } v' \text{ are in the same component of } A^0 .$$

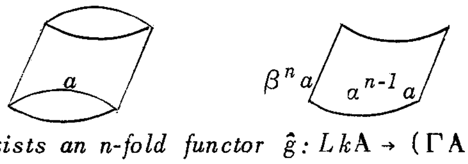
Since ρ is compatible with the composition of $| A^0 |^1$, the category ΓA of components of A is then the category quotient of $| A^0 |$ by ρ. So its morphisms are the components of A^0, and $\tilde{\rho} : | A^0 | \to \Gamma A$ is onto. It is this example which explains the name given to ΓA.

PROPOSITION 3. *Let* A *be an* $(n+1)$*-category,* ΓA *the n-fold category of its components. Then* LkA *is isomorphic to* $(\Gamma A)^{n-1,0,\dots,n-2}$*, which is deduced from* ΓA *by a permutation of the compositions.*

PROOF. 1° *The n-fold category* LkA *is generated by those blocks* $[v]$*, where* v *is an object of* A^{n-1}. With the notations of Proposition 1, Proof, 1°, LkA is generated by the blocks $[a]$, where a is a block of A, and

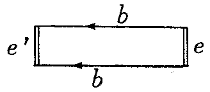
$$(a) \underset{R1}{\sim} (\beta^n a, \alpha^{n-1} a) \underset{R3}{\sim} (\beta^n a \circ_n \alpha^{n-1} a) = (\alpha^{n-1} a),$$

since $\beta^n a$ is also an object of A^{n-1} ; so $[a] = [a^{n-1}a]$.

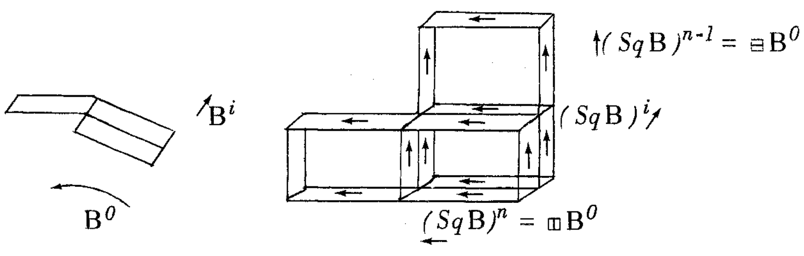


2° *There exists an n-fold functor* $\hat{g} : LkA \to (\Gamma A)^{n-1,0,\dots,n-2}$ *such that* $\hat{g}[v] = \tilde{\rho}(v)$ *for each object* v *of* A^{n-1}, where $\tilde{\rho} : |A^{n-1}| \to \Gamma A$ is the canonical n-fold functor.

a) For each n-fold category B , the n-fold subcategory of the n-fold category $(SqB)^{0,\dots,n-2,n}$ formed by the objects of $(SqB)^{n-1} = ⊟B^0$ (which are degenerate squares) is isomorphic with $B^{1,\dots,n-1,0}$, by the isomorphism mapping $b : e \to e'$ in B^0 onto the degenerate square

$b^{⊟} =$ 

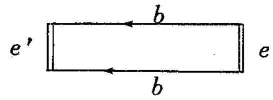
(since the composition of $(SqB)^i$, for $i < n-1$,is deduced pointwise from that of B^{i+1} and $(SqB)^n = ◫B^0$).



In particular, let B be the n-fold category $(\Gamma A)^{n-1,0,\dots,n-2}$; then

$$B^{1,\dots,n-1,0} = \Gamma A \quad \text{and} \quad B^0 = (\Gamma A)^{n-1} ,$$

so that the map

$b \mapsto b^{⊟} =$ 

(where $b : e \to e'$ in $(\Gamma A)^{n-1}$) defines an n-fold functor

$$-^{⊟} : \Gamma A \to (SqB)^{0,\dots,n-2,n} .$$

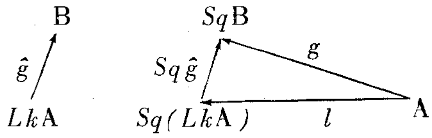
b) There is an $(n+1)$-fold functor $g : A \to SqB : a \mapsto \tilde{\rho}(a^{n-1}a)^{⊟}$: Indeed, the composite n-fold functor

$$A^{0,\dots,n-2,n} \xrightarrow{\alpha^{n-1}} |A^{n-1}|^{0,\dots,n-2,n} \xrightarrow{\tilde{\rho}} \Gamma A \xrightarrow{-^{\boxminus}} (SqB)^{0,\dots,n-2,n}$$

is defined by the map $g: a \mapsto \tilde{\rho}(\alpha^{n-1} a)^{\boxminus}$. The map $\tilde{\rho}\alpha^{n-1}$ is constant on each component of A^{n-1} (by definition of $\tilde{\rho}$) and $-^{\boxminus}$ takes its values in the set of objects of $(SqB)^{n-1}$ $(= \boxminus(\Gamma A)^{n-1})$; whence the functor

$$g: A^{n-1} \to (SqB)^{n-1}.$$

c) To $g: A \to SqB$ is canonically associated (by the adjunction between the *Square* and *Link* functors) the n-fold functor $\hat{g}: LkA \to B$ which maps $[v]$ onto the diagonal $\tilde{\rho}(v)$ of $g(v) = \tilde{\rho}(v)^{\boxminus}$ for each object v of A^{n-1} (Proof, Proposition 1).

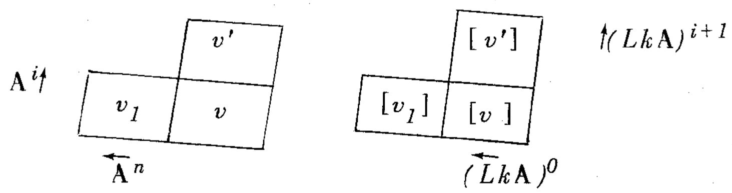


3° $\hat{g}: LkA \to B$ *is an isomorphism* and its inverse is constructed as follows, using the universal property of ΓA:

a) There is an n-fold functor

$$g': |A^{n-1}|^{0,\dots,n-2,n} \to (LkA)^{1,\dots,n-1,0}: v \mapsto [v]:$$

the $(i+1)$-th composition of LkA being deduced pointwise from that of A^i, the map $g': v \mapsto [v]$ defines a functor from the i-th category $|A^{n-1}|^i$ of $|A^{n-1}|^{0,\dots,n-2,n}$ to $(LkA)^{i+1}$ for $i \leqslant n-2$; it defines also a functor from $|A^{n-1}|^n$ to $(LkA)^0$, since



$$(v_1, v) \underset{R_3}{\sim} (v_1 \circ_n v) \quad \text{for } v \text{ and } v_1 \text{ objects of } A^{n-1}$$

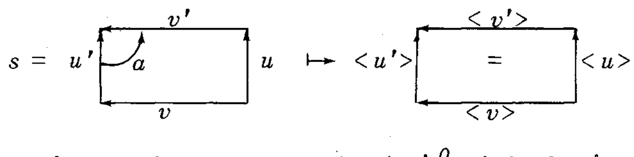
implies

$$g'(v_1 \circ_n v) = [v_1 \circ_n v] = [v_1] \circ_0 [v] = g'(v_1) \circ_0 g'(v).$$

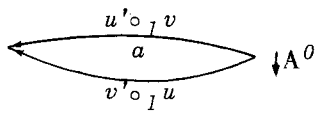
b) There is an n-fold functor $\hat{g}': B \to LkA$ such that $\hat{g}'\tilde{\rho}(v) = [v]$

where $w = u'_k \circ_1 v_k \circ_1 \dots \circ_1 u'_0 \circ_1 v_0$; hence $[s_k, \dots, s_0] = [w^{\boxminus}]$.

2° a) There is a functor $g: Q(\mathrm{A}) \to \square\Gamma\mathrm{A}$:



where $<u>$ denotes the component of u in A^0 ; indeed, $u' \circ_1 v$ and $v' \circ_1 u$, being the source and target of a in A^0, are in the same component of A^0,

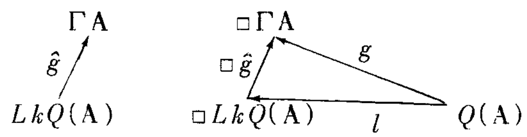


so that, in $\Gamma\mathrm{A}$,

$$<u'><v> = <u' \circ_1 v> = <v' \circ_1 u> = <v'><u>.$$

b) To $g: Q(\mathrm{A}) \to \square\Gamma\mathrm{A}$ corresponds (by the adjunction between the functors *Link* and $\square$) the functor

$$\hat{g}: LkQ(\mathrm{A}) \to \Gamma\mathrm{A} : [v^{\boxminus}] \mapsto <v>.$$



This functor is onto, each morphism of $\Gamma\mathrm{A}$ being of the form $<v>$ for some 1-morphism v of A (by Example 3-B). It is also 1-1, since $\hat{g}[v^{\boxminus}] = \hat{g}[v'^{\boxminus}]$ means $<v> = <v'>$, which implies $(v^{\boxminus}) \sim (v'^{\boxminus})$ modulo R, hence $[v^{\boxminus}] = [v'^{\boxminus}]$. This proves that $\hat{g}: LkQ(\mathrm{A}) \to \Gamma\mathrm{A}$ is an isomorphism. ∇

COROLLARY. *If* C *is a category, Lk(*$\square$*C) is isomorphic to* C.

PROOF. $\square$C is the double category of up-squares of the (trivial) 2-category $(\underline{C}^{dis}, \mathrm{C})$, whose category of components is (identified with) C. So, the corollary is a particular case of the Proposition 4. ∇

This Corollary means that each double functor $g: \square\mathrm{C} \to \square\mathrm{C}'$, where C and C' are categories, is of the form $\square f$, for a unique functor $f: \mathrm{C} \to \mathrm{C}'$. We use this result to generalize the Corollary as follows:

PROPOSITION 5. *Let* B *be an n-fold category; then* $Lk(Sq\mathrm{B})$ *is isomorphic to* B.

PROOF. It suffices to prove that B is also a free object generated by $Sq\mathrm{B}$ with respect to the functor

$$Sq_{n,n+1}: Cat_n \to Cat_{n+1},$$

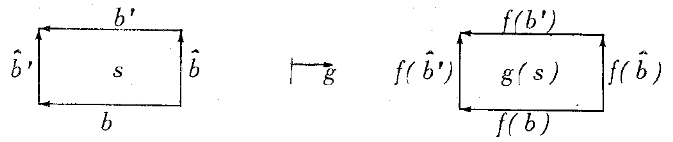
the liberty morphism being $id: Sq\mathrm{B} \to Sq\mathrm{B}$. For this, let H be an n-fold category and $g: Sq\mathrm{B} \to Sq\mathrm{H}$ an $(n+1)$-fold functor.

a) As g defines a double functor

$$g: (Sq\mathrm{B})^{n-1,n} = \square\mathrm{B}^0 \to (Sq\mathrm{H})^{n-1,n} = \square\mathrm{H}^0,$$

by the Corollary there exists a unique functor $f: \mathrm{B}^0 \to \mathrm{H}^0$ such that

$$g = \square f: \square\mathrm{B}^0 \to \square\mathrm{H}^0.$$



In particular, $g(b^{\boxminus}) = f(b)^{\boxminus}$ for each block b of B.

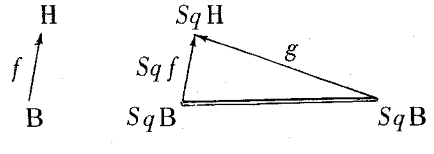
b) Let us prove that $f: \mathrm{B} \to \mathrm{H}$ is an n-fold functor. Indeed, denote by $|(Sq\mathrm{B})^{n-1}|$ the n-fold subcategory of $(Sq\mathrm{B})^{0,\dots,n-2,n}$ formed by the objects of $(Sq\mathrm{B})^{n-1} = \boxminus\mathrm{B}^0$ (i.e., formed by the degenerate squares $b^{\boxminus}$). There is an isomorphism

$$(-^{\boxminus})_{\mathrm{B}}: \mathrm{B}^{1,\dots,n-1,0} \xrightarrow{\sim} |(Sq\mathrm{B})^{n-1}|: b \mapsto b^{\boxminus}$$

(see Proof, Proposition 3). The composite functor

$$\mathrm{B}^{1,\dots,n-1,0} \xrightarrow{(-^{\boxminus})_{\mathrm{B}}} |(Sq\mathrm{B})^{n-1}| \xrightarrow{|g|} |(Sq\mathrm{H})^{n-1}| \xrightarrow{(-^{\boxminus})_{\mathrm{H}}^{-1}} \mathrm{H}^{1,\dots,n-1,0}$$

where $|g|$ is a restriction of g, maps b onto $f(b)$, since $g(b)^{\boxminus} = f(b)^{\boxminus}$. Hence it is defined by f, and this implies (after a permutation of compositions) that $f: \mathrm{B} \to \mathrm{H}$ is an n-fold functor. It is the unique n-fold functor such



where $w = u'_k \circ_1 v_k \circ_1 \dots \circ_1 u'_0 \circ_1 v_0$; hence $[s_k, \dots, s_0] = [w^{\boxminus}]$.

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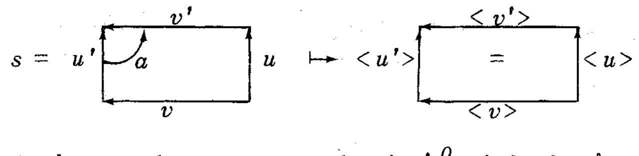

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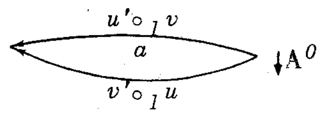

so that, in $\Gamma\mathrm{A}$,

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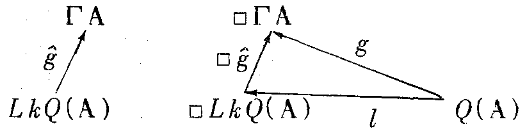

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PROOF. $\Box$C is the double category of up-squares of the (trivial) 2-category $(\underline{C}^{dis}, \mathrm{C})$, whose category of components is (identified with) C. So, the corollary is a particular case of the Proposition 4. ∇

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PROPOSITION 5. *Let* B *be an n-fold category; then* $Lk(Sq\mathrm{B})$ *is isomorphic to* B.

PROOF. It suffices to prove that B is also a free object generated by $Sq\mathrm{B}$ with respect to the functor

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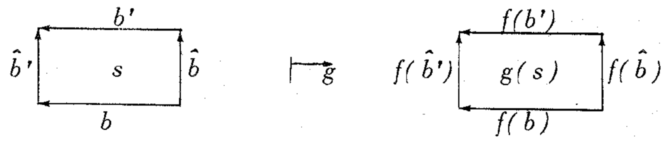
the liberty morphism being $id: Sq\mathrm{B} \to Sq\mathrm{B}$. For this, let H be an n-fold category and $g: Sq\mathrm{B} \to Sq\mathrm{H}$ an $(n+1)$-fold functor.

a) As g defines a double functor

$$g: (Sq\mathrm{B})^{n-1,n} = \square \mathrm{B}^0 \to (Sq\mathrm{H})^{n-1,n} = \square \mathrm{H}^0,$$

by the Corollary there exists a unique functor $f: \mathrm{B}^0 \to \mathrm{H}^0$ such that

$$g = \square f: \square \mathrm{B}^0 \to \square \mathrm{H}^0.$$



In particular, $g(b^{\boxminus}) = f(b)^{\boxminus}$ for each block b of B.

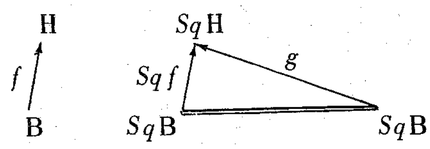
b) Let us prove that $f: \mathrm{B} \to \mathrm{H}$ is an n-fold functor. Indeed, denote by $|(Sq\mathrm{B})^{n-1}|$ the n-fold subcategory of $(Sq\mathrm{B})^{0,\dots,n-2,n}$ formed by the objects of $(Sq\mathrm{B})^{n-1} = \boxminus \mathrm{B}^0$ (i.e., formed by the degenerate squares $b^{\boxminus}$). There is an isomorphism

$$(-^{\boxminus})_{\mathrm{B}}: \mathrm{B}^{1,\dots,n-1,0} \xrightarrow{\sim} |(Sq\mathrm{B})^{n-1}|: b \mapsto b^{\boxminus}$$

(see Proof, Proposition 3). The composite functor

$$\mathrm{B}^{1,\dots,n-1,0} \xrightarrow{(-^{\boxminus})_{\mathrm{B}}} |(Sq\mathrm{B})^{n-1}| \xrightarrow{|g|} |(Sq\mathrm{H})^{n-1}| \xrightarrow{(-^{\boxminus})^{-1}_{\mathrm{H}}} \mathrm{H}^{1,\dots,n-1,0}$$

where $|g|$ is a restriction of g, maps b onto $f(b)$, since $g(b)^{\boxminus} = f(b)^{\boxminus}$. Hence it is defined by f, and this implies (after a permutation of compositions) that $f: \mathrm{B} \to \mathrm{H}$ is an n-fold functor. It is the unique n-fold functor such



that $Sqf = g: Sq\mathrm{B} \to Sq\mathrm{H}$. ∇

COROLLARY. *Whatever be the integers n and m, the Link functor from Cat_{n+m} to Cat_n is equivalent to a left inverse of $Sq_{n,n+m}: Cat_n \to Cat_{n+m}$.*

PROOF. Proposition 4 implies that the composite functor

$$Cat_n \xrightarrow{Sq_{n,n+1}} Cat_{n+1} \xrightarrow{Lk_{n+1,n}} Cat_n$$

is equivalent to the identity. By iteration, the same result is valid for the functors $Lk_{n+m,n}$ and $Sq_{n,n+m}$, due to their definition (end of Section A) as composites of functors $Lk_{p+1,p}$ and $Sq_{p,p+1}$ respectively. ∇

C. The cartesian closed structure of Cat_n.

Let n be an integer, $n > 1$. In this section we are going to show that the category Cat_n of n-fold categories is cartesian closed, by constructing the partial internal Hom functor $Hom_n(\mathrm{A}, -)$, for an n-fold category A, as the composite

$$Cat_n \xrightarrow{Sq_{n,2n}} Cat_{2n} \xrightarrow{\tilde{\gamma}} Cat_{2n} \xrightarrow{Hom(\mathrm{A},-)} Cat_n ,$$

where $Hom(\mathrm{A},-)$ is the Hom functor associated to the partial monoidal closed structure of $M\,Cat$ (defined in [5] and recalled on page 2) and where $\tilde{\gamma}$ is the isomorphism «permutation of compositions» associated to the permutation γ:

$$(0, \dots, 2n-1) \mapsto (0, 2, \dots, 2n-2, 1, 3, \dots, 2n-1)$$

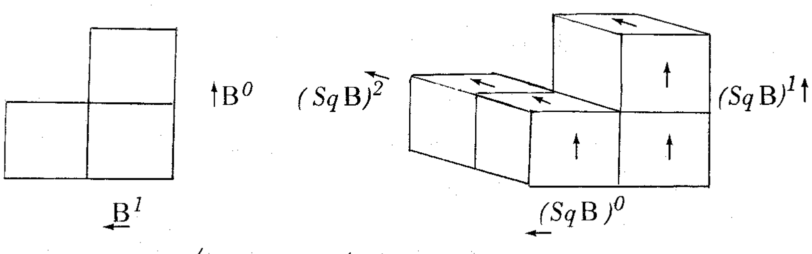
(which maps $g: \mathrm{H} \to \mathrm{K}$ onto $g: \mathrm{H}^\gamma \to \mathrm{K}^\gamma$, where

$$\mathrm{H}^\gamma = \mathrm{H}^{0,2,\dots,2n-2,1,3,\dots,2n-1}\,).$$

The necessity of introducing this isomorphism $\tilde{\gamma}$ is best understood on the Example here after and on the following Proposition.

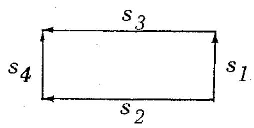
EXAMPLE: *The 4-fold category $SqSq\mathrm{B}$, where B is a double category.*

By definition, $Sq\mathrm{B}$ is the 3-fold category whose 1-st and 2-nd categories are the vertical and horizontal categories $\boxminus \mathrm{B}^0$ and $\boxdot \mathrm{B}^0$ of squares of the 0-th category B^0 of B, and whose 0-th composition is «deduced pointwise» from that of B^1.



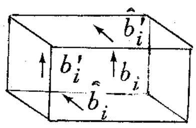
1 The 4-fold category $Sq_{2,4}(B)$ is constructed as follows:

- The set of its blocks is $\square(SqB)^0$, i.e., the blocks are the squares

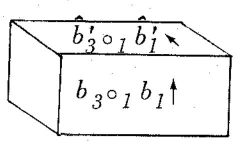


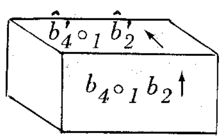
of $(SqB)^0$,

so that

$s_i =$ 

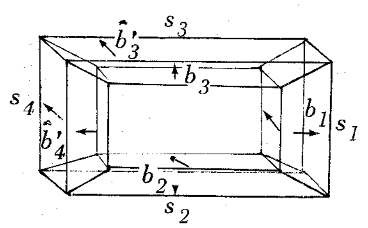
is a square of B^0 for $i = 1, 2, 3, 4$, and

$s_3 \circ_0 s_1 =$ 

$=$ 

$= s_4 \circ_0 s_2$.

Such a block will be represented by the «frame»

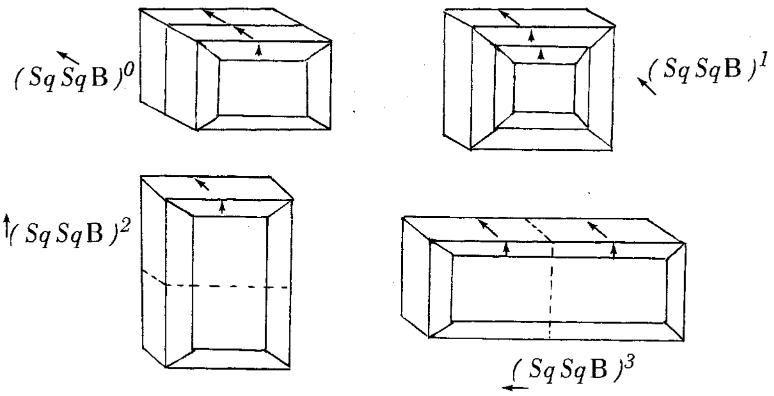


- The 0-th and 1-st compositions are deduced «pointwise» from that of

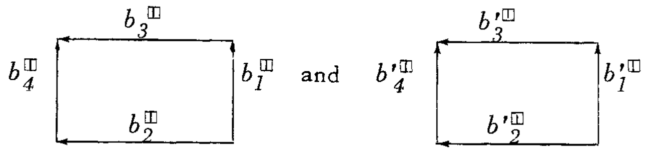
$$(SqB)^1 = \boxminus B^0 \quad \text{and} \quad (SqB)^2 = \boxplus B^0,$$

so that they consist in putting «one frame behind the other» and «one frame inside the other».

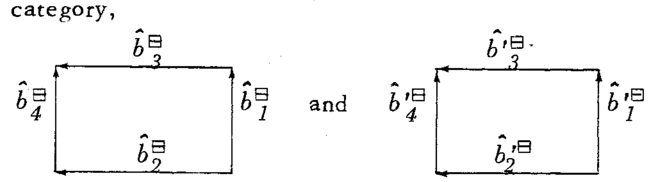
- The 2-nd and 3-rd compositions are the vertical and horizontal compositions of squares of $(Sq\mathrm{B})^0$ (whose composition is deduced from that of B^1) so that they consist in putting «one frame above the other» and «one frame beside the other» (the common border being «erased»).



- The sources and targets of (s_4, s_3, s_2, s_1) are respectively the degenerate frames:



for the 0-th category,



for the 1-st category,

$s_2^{\boxminus}$ and $s_3^{\boxminus}$ for the 2-nd category,

$s_1^{\boxbar}$ and $s_4^{\boxbar}$ for the 3-rd category.

Hence, the two first compositions are deduced from that of B^0, the two last ones being deduced from that of B^1.

More generally, if we consider $Sq_{n,2n}(\mathrm{B})$ for an n-fold category B, its $(2i)$-th and $(2i+1)$-th compositions are deduced from that of B^i, for each $i < n$. Therefore, $Sq_{n,2n}(\mathrm{B})^{\gamma}$ has its compositions deduced respectively of that of $\mathrm{B}^0, \ldots, \mathrm{B}^{n-1}, \mathrm{B}^0, \ldots, \mathrm{B}^{n-1}$.

The following proposition will be an essential tool to describe the cartesian closed structure of Cat_n.

PROPOSITION 6. *Let* A *and* B *be two n-fold categories; then the n-fold category product* B×A *is isomorphic to* $Lk((\mathrm{B} \blacksquare \mathrm{A})^{\gamma^{-1}})$, *where* B ■ A *is the square product and* γ^{-1} *the permutation*

$$(0, \dots, 2n-1) \mapsto (0, n, \dots, i, n+i, \dots, n-1, 2n-1).$$

PROOF. Remark firstly that γ^{-1} is the permutation inverse of the permutation (considered above) γ:

$$(0, \dots, 2n-1) \mapsto (0, 2, \dots, 2n-2, 1, 3, \dots, 2n-1).$$

We denote by H the $(2n)$-fold category $(\mathrm{B} \blacksquare \mathrm{A})^{\gamma^{-1}}$, so that:

$$\mathrm{H}^{2i} = \underline{\mathrm{B}}^{dis} \times \mathrm{A}^{i} \text{ and } \mathrm{H}^{2i+1} = \underline{\mathrm{B}}^{i} \times \underline{\mathrm{A}}^{dis}, \text{ for each } i < n.$$

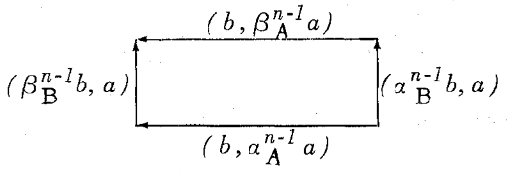
1° *LkH is isomorphic to the* $(2n-1)$*-fold category* K *on* $\underline{\mathrm{B}} \times \underline{\mathrm{A}}$ *such that*

$$\mathrm{K}^{0} = \mathrm{B}^{n-1} \times \mathrm{A}^{n-1} \text{ and } \mathrm{K}^{j+1} = \mathrm{H}^{j} \text{ for } 0 \leqslant j < 2n-2.$$

(hence $\mathrm{K} = (\mathrm{B}^{n-1} \times \mathrm{A}^{n-1}, \underline{\mathrm{B}}^{dis} \times \mathrm{A}^{0}, \mathrm{B}^{0} \times \underline{\mathrm{A}}^{dis}, \dots, \underline{\mathrm{B}}^{dis} \times \mathrm{A}^{n-2}, \mathrm{B}^{n-2} \times \underline{\mathrm{A}}^{dis})$).

a) There exists a $(2n)$-fold functor $g : \mathrm{H} \to Sq\mathrm{K}$:

$(b, a) \mapsto$



where $\alpha_{\mathrm{A}}^{n-1}$ and β_{A}^{n-1} denote the source and target maps of A^{n-1} :

(i) $g(b, a)$ is a square of $\mathrm{K}^{0} = \mathrm{B}^{n-1} \times \mathrm{A}^{n-1}$, for any blocks a of A and b of B.

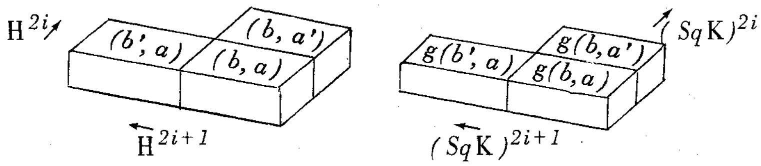
(ii) For $0 \leqslant j < 2n-2$, the j-th composition of $Sq\mathrm{K}$ is deduced pointwise from that of $\mathrm{K}^{j+1} = \mathrm{H}^{j}$; to prove that $g : \mathrm{H}^{j} \to (Sq\mathrm{K})^{j}$ is a functor, it suffices to show that the four maps:

$$\beta_{\mathrm{B}}^{n-1} \times id_{\mathrm{A}} : (b, a) \mapsto (\alpha_{\mathrm{B}}^{n-1} b, a), \quad \beta_{\mathrm{B}}^{n-1} \times id_{\mathrm{A}} : (b, a) \mapsto (\beta_{\mathrm{B}}^{n-1} b, a),$$

$$id_{\mathrm{B}} \times \alpha_{\mathrm{A}}^{n-1} : (b, a) \mapsto (b, \alpha_{\mathrm{A}}^{n-1} a), \quad id_{\mathrm{B}} \times \beta_{\mathrm{A}}^{n-1} : (b, a) \mapsto (b, \beta_{\mathrm{A}}^{n-1} a)$$

define functors from H^{j} to H^{j}. This comes from the following facts:

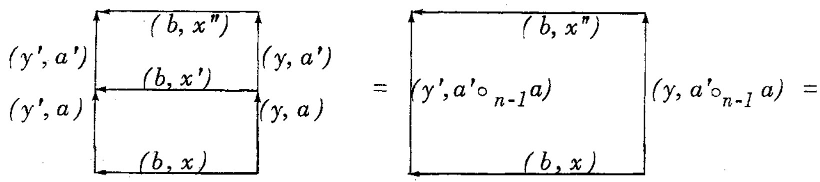
- $H^j = \underline{B}^{dis} \times A^i$ if $j = 2i$ and $H^j = B^i \times \underline{A}^{dis}$ if $j = 2i+1$,
- α^{n-1}_A and β^{n-1}_A define functors $A^j \to A^j$ and $\underline{A}^{dis} \to \underline{A}^{dis}$,
- α^{n-1}_B and β^{n-1}_B define functors $B^j \to B^j$ and $\underline{B}^{dis} \times \underline{B}^{dis}$.



(iii) $g: H^{2n-2} = \underline{B}^{dis} \times A^{n-1} \to (SqK)^{2n-2} = \boxminus K^0$ is a functor. Indeed, if $a: x \to x'$ and $a': x' \to x''$ in A^{n-1} and $b: y \to y'$ in B^{n-1}, then

$$(b, a') \circ_{2n-2} (b, a) = (b, a' \circ_{n-1} a)$$

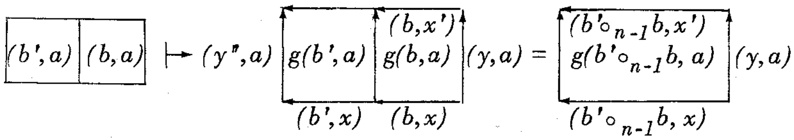
and $g(b, a') \boxminus g(b, a) =$



$$= g(b, a' \circ_{n-1} a).$$

(iv) A similar method gives the functor

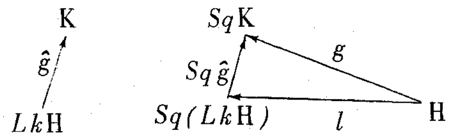
$$g: H^{2n-1} = B^{n-2} \times \underline{A}^{dis} \to (SqK)^{2n-1} = \boxplus K^0$$



($b': y' \to y''$ in B^{n-1}).

b) To $g: H \to SqK$ is canonically associated (by the adjunction between the *Link* and *Square* functors) a $(2n-1)$-fold functor $\hat{g}: LkH \to K$ such that $\hat{g}[b, a] = (b, a)$ for any (b, a) in $\underline{B} \times \underline{A}$, since (b, a) is the diagonal of the square $g(b, a)$ of $K^0 = B^{n-1} \times A^{n-1}$ (Proof, Proposition 1).

c) There exists a $(2n-1)$-fold functor



$$\hat{g}': \mathrm{K} \to Lk\mathrm{H} : (b, a) \mapsto [b, a].$$

(i) For $0 \leqslant j < 2n-2$, since $\mathrm{K}^{j+1} = \mathrm{H}^j$ and the composition of $(Lk\mathrm{H})^{j+1}$ is deduced pointwise from that of H^j, it follows that $\hat{g}': \mathrm{K}^{j+1} \to (Lk\mathrm{H})^{j+1}$ is a functor.

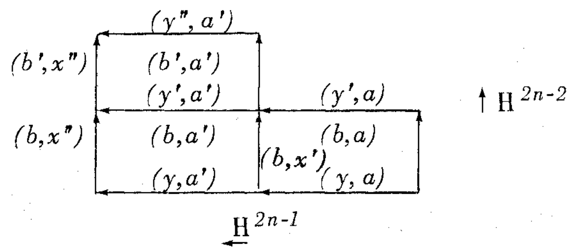
(ii) It remains to prove that $\hat{g}': \mathrm{K}^0 \to (Lk\mathrm{H})^0$ is a functor. For this, let the composite

$$(b', a') \circ_0 (b, a) = (b' \circ_{n-1} b,\ a' \circ_{n-1} a)$$

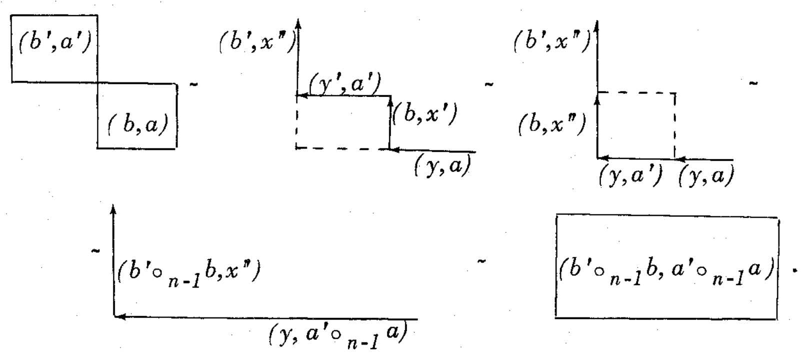
be defined in $\mathrm{K}^0 = \mathrm{B}^{n-1} \times \mathrm{A}^{n-1}$, so that

$a: x \to x'$ and $a': x' \to x''$ in A^{n-1}, $b: y \to y'$ and $b': y' \to y''$ in B^{n-1}.

Since $\mathrm{H}^{2n-2} = \underline{\mathrm{B}}^{dis} \times \mathrm{A}^{n-1}$ and $\mathrm{H}^{2n-1} = \mathrm{B}^{n-1} \times \underline{\mathrm{A}}^{dis}$, in the relation on paths



used to define $Lk\mathrm{H}$ (Proof, Proposition 1), we have successively



This implies

$$\hat{g}'(b', a') \circ_0 \hat{g}'(b, a) = [b', a'] \circ_0 [b, a] = [(b', a'), (b, a)] =$$

$$= [\, b' \circ_{n-1} b,\ a' \circ_{n-1} a\,] = \hat{g}'((b', a') \circ_0 (b, a)).$$

Hence $\hat{g}' : \mathrm{K}^0 \to (Lk\mathrm{H})^0$ is also a functor.

d) $\hat{g}'$ is the inverse of $\hat{g} : Lk\mathrm{H} \to \mathrm{K}$. Indeed,

$$\hat{g}\hat{g}'(b, a) = \hat{g}\ [\, b, a\,] = (b, a)$$

for each block (b, a) of K, so that $\hat{g}\hat{g}'$ is an identity. On the other hand, the equalities

$$\hat{g}'\hat{g}\,[\, b, a\,] = \hat{g}'(b, a) = [\, b, a\,]$$

imply that $\hat{g}'\hat{g}$ is also an identity, the blocks $[\, b, a\,]$ generating (by definition) the $(2n\text{-}1)$-fold category $Lk\mathrm{H}$. So $\hat{g} : Lk\mathrm{H} \to \mathrm{K}$ is an isomorphism.

2° Let us suppose proven that $Lk_{2n, 2n-m}\mathrm{H}$, for $1 \leqslant m \leqslant n-1$, is isomorphic to the $(2n\text{-}m)$-fold category K_m such that

$$(\mathrm{B}^{n-m} \times \mathrm{A}^{n-m}, \ldots, \mathrm{B}^{n-1} \times \mathrm{A}^{n-1}, \underline{\mathrm{B}}^{dis} \times \mathrm{A}^0, \ldots, \underline{\mathrm{B}}^{dis} \times \mathrm{A}^{n-m-1}, \mathrm{B}^{n-m-1} \times \underline{\mathrm{A}}^{dis}).$$

Then a proof similar to the preceding one proves that $Lk\mathrm{K}_m$, and a fortiori

$$Lk(Lk_{2n,\, 2n-m}\mathrm{H}) = Lk_{2n,\, 2n-m-1}\mathrm{H}$$

is isomorphic to the $(2n\text{-}m\text{-}1)$-fold category K_{m+1}. By induction, it follows that $Lk_{2n,n}\mathrm{H}$ is isomorphic to

$$\mathrm{B} \times \mathrm{A} = (\mathrm{B}^0 \times \mathrm{A}^0, \ldots, \mathrm{B}^{n-1} \times \mathrm{A}^{n-1}). \qquad \nabla$$

COROLLARY. *For each n-fold category* A, *the «partial» product functor* $-\times\mathrm{A} : Cat_n \to Cat_n$ *is equivalent to the composite functor*

$$Cat_n \xrightarrow{-\blacksquare \mathrm{A}} Cat_{2n} \xrightarrow{\tilde{\gamma}^{-1}} Cat_{2n} \xrightarrow{Lk_{2n,n}} Cat_n. \qquad \nabla$$

DEFINITION. The composite functor

$$Cat_n \xrightarrow{Sq_{n,2n}} Cat_{2n} \xrightarrow{\tilde{\gamma}} Cat_{2n}$$

will be called the *n-square functor*, denoted by $\square_n : Cat_n \to Cat_{2n}$.

PROPOSITION 7. Cat_n *is a cartesian closed category whose internal Hom functor* $Hom_n : (Cat_n)^{op} \times Cat_n \to Cat_n$ *is such that, for any n-fold category* A, *the partial functor* $Hom_n(\mathrm{A}, -)$ *is equal to the composite:*

$$Cat_n \xrightarrow{\square_n} Cat_{2n} \xrightarrow{Hom(\mathrm{A}, -)} Cat_n.$$ 1

PROOF. 1° Since Cat_n admits (finite) products, to prove that it is cartesian closed it suffices to show that the partial product functor $-\times A : Cat_n \to Cat_n$ admits a right adjoint [13]. By the Corollary of Proposition 6, this functor is equivalent to the composite of three functors:

$-\blacksquare A : Cat_n \to Cat_{2n}$ who has a right adjoint $Hom(A,-)$ (due to the partial monoidal closed structure of $M\,Cat$, Proposition 7 [5]),

$\tilde{\gamma}^{-1} : Cat_{2n} \to Cat_{2n}$ whose inverse $\tilde{\gamma}$ is a right adjoint,

$Lk_{2n,n} : Cat_{2n} \to Cat_n$ who admits $Sq_{n,2n}$ as a right adjoint.

By transitivity of adjunctions, this implies that $-\times A$ admits as a right adjoint the composite $Hom_n(A,-) =$

$$Cat_n \xrightarrow{Sq_{n,2n}} Cat_{2n} \xrightarrow{\tilde{\gamma}} Cat_{2n} \xrightarrow{Hom(A,-)} Cat_n . \qquad (\square_n = \tilde{\gamma}\circ Sq_{n,2n} : Cat_n \to Cat_{2n})$$

2° The corresponding internal Hom functor (or closure functor)

$$Hom_n : (Cat_n)^{op} \times Cat_n \to Cat_n$$

maps the couple of n-fold functors $(f: A' \to A,\ g: B \to B')$ onto the n-fold functor

$$Hom_n(f,g) : Hom_n(A,B) = Hom(A, \square_n B) \to Hom_n(A',B')$$

mapping $h: A \to \square_n B$ onto

$$A' \xrightarrow{f} A \xrightarrow{h} \square_n B \xrightarrow{\square_n g} \square_n B' .$$

3° Let us describe more explicitly the adjunction between $-\times A$ and $Hom_n(A,-) : Cat_n \to Cat_n$. Let B be an n-fold category.

a) There is a map $\partial : \underline{\square_n B} \to \underline{B}$ (it is not a multiple functor, but a map between the sets of blocks) which maps an n-square of B onto «its diagonal» defined as follows: For each $i < n$, there is the diagonal map

$$d_i : \underline{Sq_{n,n+i+1} B} = \underline{Sq(Sq_{n,n+i} B)} \to \underline{Sq_{n,n+i} B}$$

which maps the square

$$\begin{array}{ccc} & \hat{s}' & \\ s' & \square & s \\ & \hat{s} & \end{array} \quad \text{of} \quad (Sq_{n,n+i} B)^0$$

onto its diagonal $\hat{s}' \circ_0 s = s' \circ_0 \hat{s}$. Then ∂ is the composite map $d_0 \ldots d_{n-1}$:

$$\underline{\square_n \mathrm{B}} = \underline{Sq_{n,2n}\mathrm{B}} \xrightarrow[d_{n-1}]{} \underline{Sq_{n,2n-1}\mathrm{B}} \to \dots \to \underline{Sq\mathrm{B}} \xrightarrow[d_0]{} \underline{\mathrm{B}}\,.$$

b) The 1-1 correspondence due to the adjunction between $-\times \mathrm{A}$ and $Hom_n(\mathrm{A},-)$ maps the n-fold functor $h: \mathrm{A}' \to Hom_n(\mathrm{A},\mathrm{B})$ onto the n-fold functor

$$\hat{h}: \mathrm{A}'\times\mathrm{A} \to \mathrm{B}: (a', a) \mapsto \partial(h(a')(a)).$$

Indeed, the adjunction between $Hom(\mathrm{A},-)$ and $-\blacksquare\,\mathrm{A}$ associates to h the n-fold functor

$$h_0^\gamma: \mathrm{A}'\blacksquare\,\mathrm{A} \to \square_n\mathrm{B}: (a',a) \mapsto h(a')(a),$$

and therefore the n-fold functor

$$h_0: (\mathrm{A}'\blacksquare\,\mathrm{A})^{\gamma^{-1}} \to (\square_n\mathrm{B})^{\gamma^{-1}} = Sq_{n,2n}\mathrm{B}\ ;$$

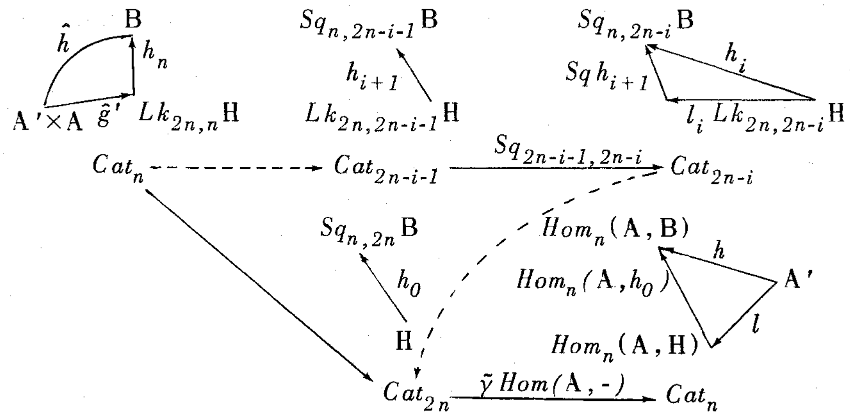
we write H instead of $(\mathrm{A}'\blacksquare\,\mathrm{A})^{\gamma^{-1}}$. By induction, we define

$$h_{i+1}: Lk_{2n,2n-i-1}\mathrm{H} = Lk(Lk_{2n,2n-i}\mathrm{H}) \to Sq_{n,2n-i-1}\mathrm{B},$$

for each $i < n$, as the $(2n-i-1)$-fold functor associated (by the adjunction between $Lk_{2n-i,2n-i-1}$ and $Sq_{2n-i-1,2n-i}: Cat_{2n-i-1} \to Cat_{2n-i}$) to

$$h_i: Lk_{2n,2n-i}\mathrm{H} \to Sq_{n,2n-i}\mathrm{B} = Sq(Sq_{n,2n-i-1}\mathrm{B})\ ;$$

by construction, h_{i+1} maps a block of $Lk_{2n,2n-i-1}\mathrm{H}$ of the form $[a', a]$ (see Proof, Proposition 6) onto the diagonal $d_{n-i-1}h_i[a', a]$ of the square



$h_i[a', a]$ of $(Sq_{n,2n-i-1}\mathrm{B})^0$. It follows that to h is associated

$$\hat{h} = (\mathrm{A}'\times\mathrm{A} \xrightarrow[\tilde{g}']{} Lk_{2n,n}\mathrm{H} \xrightarrow[h_n]{} \mathrm{B}),$$

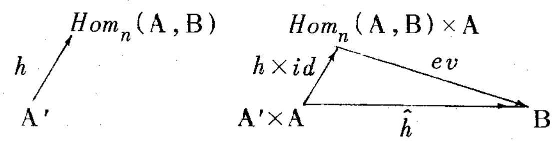
where $\hat{g}'$ is the canonical isomorphism $(a', a) \mapsto [a', a]$ (see Proof, Proposition 6); $\hat{h}$ maps (a', a) onto

$$d_0 \dots d_{n-1} h(a')(a) = \partial h(a')(a).$$

c) The coliberty morphism defining $Hom_n(\mathrm{A}, \mathrm{B})$ as a cofree object generated by B is the «evaluation»:

$$ev: Hom_n(\mathrm{A}, \mathrm{B}) \times \mathrm{A} \to \mathrm{B} : (f, a) \mapsto \partial f(a),$$

since it corresponds to the identity of $Hom_n(\mathrm{A}, \mathrm{B})$. In particular, if A is



the n-fold category $2^{\blacksquare^n}$ (see [5]), with only one non-degenerate block z, then

$$Hom_n(2^{\blacksquare^n}, \mathrm{B}) = Hom(2^{\blacksquare^n}, \square_n \mathrm{B})$$

is identified with $(\square_n \mathrm{B})^{n, \dots, 2n-1}$, and the evaluation becomes the n-fold functor $ev: (\square_n \mathrm{B})^{n, \dots, 2n-1} \times 2^{\blacksquare^n} \to \mathrm{B}$ such that the map

$$ev(-, z): \underline{\square_n \mathrm{B}} \to \underline{\mathrm{B}} : s \mapsto \partial s$$

is the diagonal map ∂ defined in a. ∇

COROLLARY 1. *The vertices of* $Hom_n(\mathrm{A}, \mathrm{B})$ *are identified with the n-fold functors from* A *to* B.

PROOF. The final object 1_n of Cat_n is the unique n-fold category on the set $1 = \{0\}$. The vertices of $Hom_n(\mathrm{A}, \mathrm{B})$ are identified [5] with the n-fold functors $1_n \to Hom_n(\mathrm{A}, \mathrm{B})$, which are in 1-1 correspondence (by adjunction) with the n-fold functors from $1_n \times \mathrm{A} \approx \mathrm{A}$ to B. To $f: \mathrm{A} \to \mathrm{B}$ corresponds the vertex of $Hom_n(\mathrm{A}, \mathrm{B})$ mapping a onto the degenerate n-square (vertex of $\square_n \mathrm{B}$) determined by $f(a)$. ∇

COROLLARY 2. *There is a canonical isomorphism*

$$\lambda: Hom_n(\mathrm{A}', Hom_n(\mathrm{A}, \mathrm{B})) \xrightarrow{\sim} Hom_n(\mathrm{A}' \times \mathrm{A}, \mathrm{B})$$

extending the 1-1 correspondence (Proof above):

$$(h: \mathrm{A}' \to Hom_n(\mathrm{A},\mathrm{B})) \mapsto (\hat{h}: \mathrm{A}'\times\mathrm{A} \to \mathrm{B}: (a', a) \mapsto \partial h(a')(a)).$$

PROOF. It is a general result on cartesian (as well as monoidal) closed categories [13] ; it means that $Hom_n(\mathrm{A}, -): Cat_n \to Cat_n$ is a Cat_n-right adjoint of $-\times\mathrm{A}$. ∇

COROLLARY 3. *There is a canonical n-fold « composition » functor*

$$\kappa_{\mathrm{A,B,B'}}: Hom_n(\mathrm{A},\mathrm{B})\times Hom_n(\mathrm{B},\mathrm{B}') \to Hom_n(\mathrm{A},\mathrm{B}'):$$
$$(f, f') \mapsto (f'': \mathrm{A} \to \square_n \mathrm{B}') \text{ with } \partial_{\mathrm{B}'} f'' = \partial_{\mathrm{B}'} f' \partial_{\mathrm{B}} f: \underline{\mathrm{A}} \to \underline{\mathrm{B}}.$$

PROOF. This is also a general result on cartesian closed categories ; in fact, $\kappa_{\mathrm{A,B,B'}}$ corresponds to the composite n-fold functor:

$$\begin{array}{c} (Hom_n(\mathrm{A},\mathrm{B})\times Hom_n(\mathrm{B},\mathrm{B}'))\times\mathrm{A} \cong Hom_n(\mathrm{B},\mathrm{B}')\times(Hom_n(\mathrm{A},\mathrm{B})\times\mathrm{A}) \\ \downarrow id\times ev_{\mathrm{A,B}} \\ Hom_n(\mathrm{B},\mathrm{B}')\times\mathrm{B} \xrightarrow[ev_{\mathrm{B,B'}}]{} \mathrm{B}' \end{array}$$

mapping (f, f', a) onto $\partial_{\mathrm{B}'} f'(\partial f(a))$. ∇

This Corollary 3 means that Cat_n is a Cat_n-category (i.e., a category enriched in the cartesian closed category Cat_n) and it will be used in Proposition 8.

REMARK. The *existence* of a cartesian closed structure on Cat_n may also be deduced, by induction, from Corollary 3, Proposition 23 [7] , as follows: since Cat is cartesian closed, the sketch σ of categories is cartesian [7] ; so, if Cat_i is cartesian closed, the category Cat_i^σ of categories in Cat_i is cartesian closed by this Corollary, as well as the equivalent category Cat_{i+1} (see Appendix [5]). However the *explicit construction of* Hom_n cannot be deduced from this (or from another) existence result.

EXAMPLE. *The cartesian closed category* Cat_2 :

Let A and B be double categories. Then $\square_2\mathrm{B}$ is the 4-fold category deduced from $SqSq\mathrm{B}$ (described in the Example above) by permutation of the 1-st and 2-nd compositions. Hence, $Hom_2(\mathrm{A},\mathrm{B})$ is constructed as follows:

- Its blocks are the double functors from A to the double category

$(SqSq\mathrm{B})^{0,2}$ of «frames» whose compositions are «one frame behind the the other» and «one frame above the other».

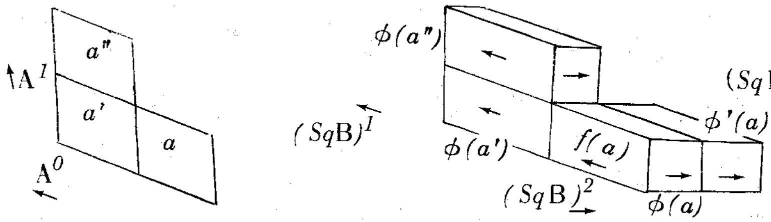
- Its compositions are deduced pointwise from the compositions «one frame inside the other» and «one frame beside the other».

- Its vertices «are» the double functors $f: \mathrm{A} \to \mathrm{B}$.

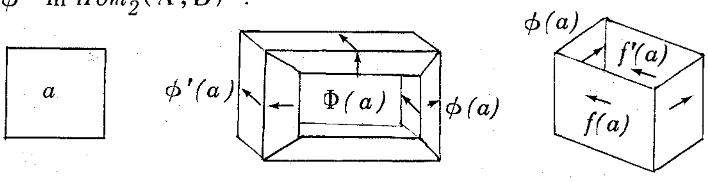
- The 3-fold subcategory $|(SqSq\mathrm{B})^3|^{0,2,1}$ of $(SqSq\mathrm{B})^{0,2,1}$ formed by the objects of $(SqSq\mathrm{B})^3$ is identified with $(Sq\mathrm{B})^{1,0,2}$ by the isomorphism

$$-^{\square}: (Sq\mathrm{B})^{1,0,2} \xrightarrow{\sim} |(SqSq\mathrm{B})^3| : s \mapsto s^{\square}.$$

Then an object of $Hom_2(\mathrm{A},\mathrm{B})^1$ (which is a double functor $\mathrm{A} \to (SqSq\mathrm{B})^{0,2}$ taking its values in $|(SqSq\mathrm{B})^3|$) will be identified with a double functor $\phi: \mathrm{A} \to (Sq\mathrm{B})^{1,0}$, and the subcategory of $Hom_2(\mathrm{A},\mathrm{B})^0$ formed by these objects «is» $Hom(\mathrm{A},(Sq\mathrm{B})^{1,0,2})$. The objects of this last category are



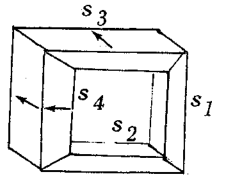
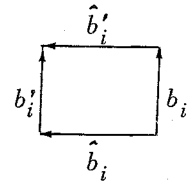
themselves identified with the double functors $f: \mathrm{A} \to \mathrm{B}$. With the terminology of [7], a double functor $\phi: \mathrm{A} \to (Sq\mathrm{B})^{1,0}$ is called a *double natural transformation* (i.e., a natural transformation internal to *Cat*) from f to f', if $\phi: f \to f'$ in $Hom(\mathrm{A},(Sq\mathrm{B})^{1,0,2})$. This may suggest to call the block $\Phi: \mathrm{A} \to \square_2\mathrm{B}$ of $Hom_2(\mathrm{A},\mathrm{B})$ *a hypertransformation from* ϕ *to* ϕ' where $\Phi: \phi \to \phi'$ in $Hom_2(\mathrm{A},\mathrm{B})^1$.



- If $h: \mathrm{A}' \to Hom_2(\mathrm{A},\mathrm{B})$ is a double functor, the double functor canonically associated (by adjunction) $\hat{h}: \mathrm{A}' \times \mathrm{A} \to \mathrm{B}$ maps (a', a) onto the diagonal of the frame $h(a')(a)$, which is equal to

$$(b_4' \circ_0 \hat{b}_4) \circ_1 (b_2' \circ_0 \hat{b}_2)$$

if

$h(a')(a) =$  where $s_i =$ 

APPLICATION. *The $(n+1)$-category* Nat_n *of hypertransformations.*

The following Proposition 8 shows that Cat_n is the category of 1-morphisms of an $(n+1)$-category Nat_n which, for $n = 1$, is the 2-category of natural transformations. It is based on the Lemma, whose proof is given in the Appendix:

LEMMA. *Let* V *denote a cartesian category with commuting coproducts (in the sense of Penon* [21]*) and* A *be a* V*-category. If* V *admits coproducts indexed by the class of objects of* A, *then there is a category in* V *whose object of morphisms is the coproduct of* $A(e, e')$, *for any objects* e *and* e' *of* A, *and whose composition «glues together» the composition morphisms*

$$\kappa_{e,e',e''} : A(e, e') \times A(e', e'') \to A(e, e'').$$

PROPOSITION 8. *There is an $(n+1)$-fold category* Nat_n *satisfying the following conditions:*

1° $(\mathrm{Nat}_n)^{0,\dots,n-1}$ *is the n-fold category coproduct of the n-fold categories* $Hom_n(\mathrm{A},\mathrm{B})$, *for any (small) n-fold categories* A, B.

2° Its n-th composition κ_n *is (notations Corollary 3, Proposition 7):*

$(f, f') \mapsto \kappa_{\mathrm{A},\mathrm{B},\mathrm{B}'}(f, f')$ *iff* f *in* $Hom_n(\mathrm{A},\mathrm{B})$ *and* f' *in* $Hom_n(\mathrm{B},\mathrm{B}')$.

3° Cat_n *is the category of 1-morphisms of* Nat_n.

PROOF. 1° Let $\hat{Cat}_n$ be the category of n-fold categories associated to a universe $\hat{U}$ to which belongs the universe U of small sets, and a fortiori the class of objects of Cat_n. Then $\hat{Cat}_n$ is also cartesian closed. The faithful functor «forgetting all the compositions» from $\hat{Cat}_n$ toward the category $\hat{Set}$ (of sets associated to the universe $\hat{U}$) preserves coproducts and limits, and it reflects isomorphisms (an n-fold functor defined by a 1-1 and onto map is an isomorphism); hence Corollary 1, Proposition 1-6 of Penon

[21] asserts that $\hat{Cat}_n$ has commuting coproducts (in [21] «small» is now to be replaced by: belonging to $\hat{U}$).

2° As Cat_n is cartesian closed, it «is» a Cat_n-category [3], and it determines also a $\hat{Cat}_n$-category, the insertion functor $Cat_n \hookrightarrow \hat{Cat}_n$ preserving the cartesian closed structure. More precisely, we have the $\hat{Cat}_n$-category H_n defined as follows:

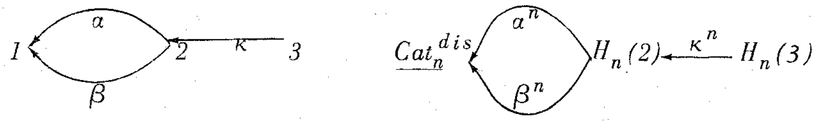
- its objects are the small n-fold categories A, B, ..., and

$$\mathrm{H}_n(\mathrm{A},\mathrm{B}) = Hom_n(\mathrm{A},\mathrm{B}) \;;$$

- the «unitarity» morphisms are of the form $j_{\mathrm{A}} : 1_n \to \mathrm{H}_n(\mathrm{A},\mathrm{A})$, where $j_{\mathrm{A}}(0)$ is the vertex of $Hom_n(\mathrm{A},\mathrm{A})$ identified with $id: \mathrm{A} \to \mathrm{A}$;

- the «composition» morphisms $\kappa_{\mathrm{A},\mathrm{B},\mathrm{B}'}$ are those defined in Corollary 3, Proposition 7.

3° The Lemma associates to H_n a category H_n in $\hat{Cat}_n$ defined as



follows:

- its object of morphisms $H_n(2)$ is the n-fold category $\coprod_{\mathrm{A},\mathrm{B}} Hom_n(\mathrm{A},\mathrm{B})$ coproduct of the n-fold categories $Hom_n(\mathrm{A},\mathrm{B})$, for any (small) n-fold categories A, B (as the sets $\underline{Hom_n(\mathrm{A},\mathrm{B})}$ are disjoint, this coproduct is on their union);

- its object of objects $H_n(1)$ is the «discrete» n-fold category on Cat_n (since it is the coproduct of $\underline{Cat_n}$ copies of the final object 1_n);

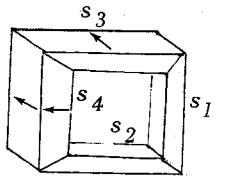
- the morphisms source α^n and target β^n send a block $f: \mathrm{A} \to \square_n \mathrm{B}$ of $Hom_n(\mathrm{A},\mathrm{B})$ onto A and B respectively;

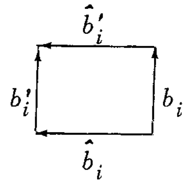
- the composition morphism κ^n is the union of the n-fold «composition» functors $\kappa_{\mathrm{A},\mathrm{B},\mathrm{B}'}$ (Corollary 3, Proposition 7).

4° By the equivalence between categories in $\hat{Cat}_n$ and $(n+1)$-fold categories (see Appendix [5]), $H_n : \sigma \to \hat{Cat}_n$ is the realization associated to the $(n+1)$-fold category Nat_n such that:

$$(\mathrm{Nat}_n)^{0,\ldots,n} = H_n(2) = \coprod_{\mathrm{A},\mathrm{B}} Hom_n(\mathrm{A},\mathrm{B}) \,,$$

if

$h(a')(a) =$ 

where $s_i =$ 

APPLICATION. *The $(n+1)$-category* Nat_n *of hypertransformations.*

The following Proposition 8 shows that Cat_n is the category of 1-morphisms of an $(n+1)$-category Nat_n which, for $n = 1$, is the 2-category of natural transformations. It is based on the Lemma, whose proof is given in the Appendix:

LEMMA. *Let* V *denote a cartesian category with commuting coproducts (in the sense of Penon* [21]*) and* A *be a* V*-category. If* V *admits coproducts indexed by the class of objects of* A*, then there is a category in* V *whose object of morphisms is the coproduct of* $A(e, e')$*, for any objects* e *and* e' *of* A*, and whose composition «glues together» the composition morphisms*

$$\kappa_{e,e',e''} : A(e, e') \times A(e', e'') \to A(e, e'').$$

PROPOSITION 8. *There is an $(n+1)$-fold category* Nat_n *satisfying the following conditions:*

1° $(\mathrm{Nat}_n)^{0,\dots,n-1}$ *is the n-fold category coproduct of the n-fold categories* $Hom_n(\mathrm{A}, \mathrm{B})$*, for any (small) n-fold categories* A, B.

2° Its n-th composition κ_n *is (notations Corollary 3, Proposition 7):*

$(f, f') \mapsto \kappa_{\mathrm{A},\mathrm{B},\mathrm{B}'}(f, f')$ *iff* f *in* $Hom_n(\mathrm{A}, \mathrm{B})$ *and* f' *in* $Hom_n(\mathrm{B}, \mathrm{B}')$.

3° Cat_n *is the category of 1-morphisms of* Nat_n.

PROOF. 1° Let $\hat{Cat}_n$ be the category of n-fold categories associated to a universe $\hat{\mathrm{U}}$ to which belongs the universe U of small sets, and a fortiori the class of objects of Cat_n. Then $\hat{Cat}_n$ is also cartesian closed. The faithful functor «forgetting all the compositions» from $\hat{Cat}_n$ toward the category $\hat{Set}$ (of sets associated to the universe $\hat{\mathrm{U}}$) preserves coproducts and limits, and it reflects isomorphisms (an n-fold functor defined by a 1-1 and onto map is an isomorphism); hence Corollary 1, Proposition 1-6 of Penon

[21] asserts that $\hat{Cat}_n$ has commuting coproducts (in [21] «small» is now to be replaced by: belonging to $\hat{U}$).

2° As Cat_n is cartesian closed, it «is» a Cat_n-category [3], and it determines also a $\hat{Cat}_n$-category, the insertion functor $Cat_n \hookrightarrow \hat{Cat}_n$ preserving the cartesian closed structure. More precisely, we have the $\hat{Cat}_n$-category H_n defined as follows:

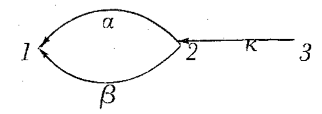
- its objects are the small n-fold categories A, B, ..., and

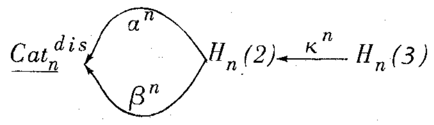
$$H_n(A, B) = Hom_n(A, B) ;$$

- the «unitarity» morphisms are of the form $j_A : 1_n \to H_n(A, A)$, where $j_A(0)$ is the vertex of $Hom_n(A, A)$ identified with $id : A \to A$;

- the «composition» morphisms $\kappa_{A,B,B'}$ are those defined in Corollary 3, Proposition 7.

3° The Lemma associates to H_n a category H_n in $\hat{Cat}_n$ defined as





follows:

- its object of morphisms $H_n(2)$ is the n-fold category $\coprod_{A,B} Hom_n(A, B)$ coproduct of the n-fold categories $Hom_n(A, B)$, for any (small) n-fold categories A, B (as the sets $\underline{Hom_n(A, B)}$ are disjoint, this coproduct is on their union);

- its object of objects $H_n(1)$ is the «discrete» n-fold category on Cat_n (since it is the coproduct of $\underline{Cat_n}$ copies of the final object 1_n);

- the morphisms source α^n and target β^n send a block $f: A \to \square_n B$ of $Hom_n(A, B)$ onto A and B respectively;

- the composition morphism κ^n is the union of the n-fold «composition» functors $\kappa_{A,B,B'}$ (Corollary 3, Proposition 7).

4° By the equivalence between categories in $\hat{Cat}_n$ and $(n+1)$-fold categories (see Appendix [5]), $H_n : \sigma \to \hat{Cat}_n$ is the realization associated to the $(n+1)$-fold category Nat_n such that:

$$(Nat_n)^{0,\dots,n} = H_n(2) = \coprod_{A,B} Hom_n(A, B) ,$$

between coproducts such that, for each $\lambda \in \Lambda$, the diagram

$$\begin{array}{ccc} \coprod_{\mu \in M} V'_\mu & \xleftarrow{\quad v \quad} & \coprod_{\lambda \in \Lambda} V_\lambda \\ \uparrow{\scriptstyle j_{\phi\lambda}} & & \uparrow{\scriptstyle j_\lambda} \\ V'_{\phi\lambda} & \xleftarrow{\quad v_\lambda \quad} & V_\lambda \end{array}$$

commutes, where j_λ and $j_{\phi\lambda}$ always denote the injections into the coproducts. Indeed, v, called the *factor of* $(v_\lambda)_\lambda$ *with respect to* ϕ, is defined as follows:

$$v = (\coprod_\lambda V_\lambda \xrightarrow{\sim} \coprod_\mu (\coprod_{\lambda \in \phi^{-1}(\mu)} V_\lambda) \xrightarrow{\coprod_\mu v^\mu} \coprod_\mu V'_\mu),$$

where $v^\mu : \coprod_{\lambda \in \phi^{-1}(\mu)} V_\lambda \to V'_\mu$ is the factor of $(v_\lambda)_\lambda$ through the coproduct $\coprod_{\lambda \in \phi^{-1}(\mu)} V_\lambda$.

2° *Construction of the category* ΓA in V, for a V-category A such that there exist in V coproducts indexed by the class A_0 of objects of A.

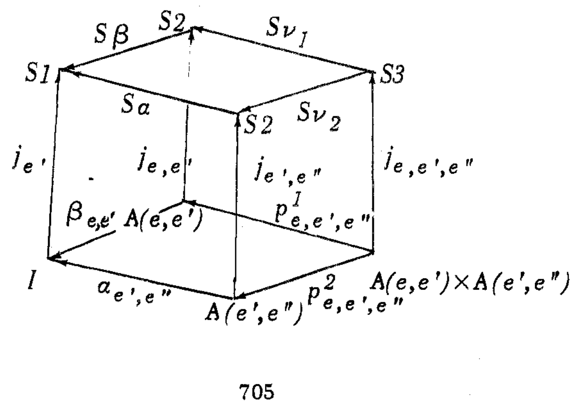
a) Since A_0 is finite or equipotent with $A_0 \times A_0$ and $A_0 \times A_0 \times A_0$, there exist in V coproducts:

S1 of the family $(I_e)_e$ indexed by A_0, where I_e is equal to the final object I of V for each object e of A,

S2 of the family $(A(e,e'))_{e,e'}$ indexed by $A_0 \times A_0$, where $A(e,e')$ is the «object of morphisms from e to e' in A»,

S3 of the family $(A(e,e') \times A(e',e''))_{e,e',e''}$ indexed by $A_0 \times A_0 \times A_0$.

b) (i) There exist unique morphisms $S\alpha$, $S\beta$, $S\nu_i$ rendering commutative the «cube»:



where $p^i_{e,e',e''}$ are the projections of the product, $\alpha_{e',e''}$ and $\beta_{e,e'}$ are the unique morphisms toward the final object I (the name of such a morphism will often be omitted). Indeed, $S\alpha$, $S\beta$, $S\nu_i$ are respectively the factors of:

$(\alpha_{e',e''})$ with respect to the projection $A_0 \times A_0 \to A_0 : (e', e'') \mapsto e'$,

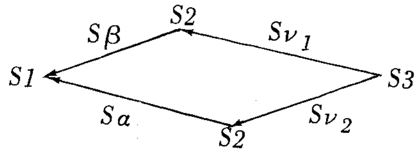
$(\beta_{e,e'})$ with respect to the map $A_0^2 \to A_0 : (e, e') \mapsto e'$,

$(p^i_{e,e',e''})$ with respect to the maps $q_i : A_0 \times A_0 \times A_0 \to A_0 \times A_0$ with

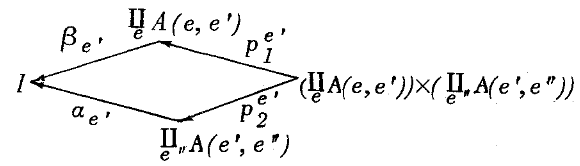
$$q_1(e, e', e'') = (e, e') \text{ and } q_2(e, e', e'') = (e', e'').$$

Since the down face of the cube commutes (there is only one morphism $I_{e,e',e''}: A(e,e') \times A(e', e'') \to I$), by unicity of the factor of $(I_{e,e',e''})$ with respect to the projection $A_0 \times A_0 \times A_0 \to A_0 : (e, e', e'') \mapsto e'$, the up face of the cube also commutes.

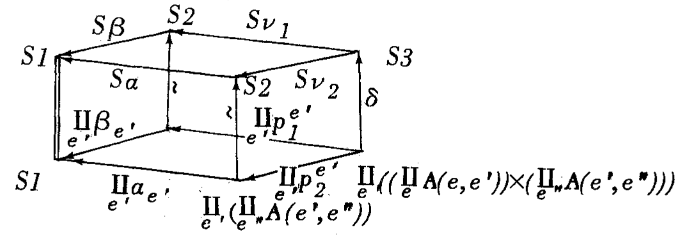
(ii) The square

(D) 

is a pullback. Indeed, for each object e' of A we have the pullback

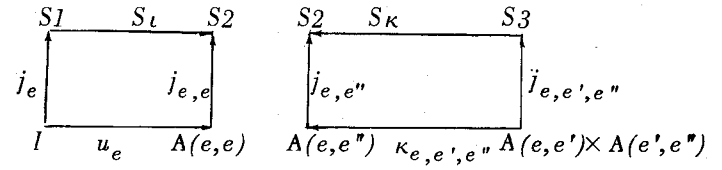
($D_{e'}$) 

where $p_i^{e'}$ are projections of the product, since I is a final object. V having commuting coproducts, the theorem of commutation of Penon (Corollary 3, Proposition 1-8 [21]) asserts that the square (D') coproduct of the squares ($D_{e'}$) is also a pullback. Now (D') is the down face of the cube



The vertical edges of this cube are canonical isomorphisms between coproducts (the existence of δ follows from the preservation of coproducts by the partial product functors in V). By construction of the factors $S\alpha$, $S\beta$, $S\nu_i$, this cube commutes, so that its up face (D) is also a pullback.

(iii) There exist unique morphisms $S\iota$ and $S\kappa$ rendering commutative the squares

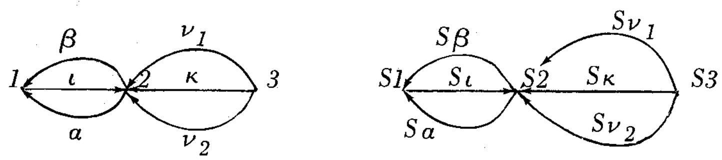


where u_e and $\kappa_{e,e',e''}$ are the «identity» morphisms and the «composition» morphisms of the V-category A . Indeed, $S\iota$ and $S\kappa$ are respectively the factors of

$(u_e)_e$ with respect to the map $A_0 \to A_0 \times A_0 : e \mapsto (e, e)$,

$(\kappa_{e,e',e''})$ with respect to $A_0 \times A_0 \times A_0 \to A_0 \times A_0 : (e, e', e'') \mapsto (e, e'')$.

c) This defines a category S in V , i.e., a realization $S: \sigma \to$ V of the sketch σ of categories (see [4] and [5] Appendix):



(i) For a couple (e, e') of objects of A , let $u_{e,e'}$ be equal to

$$A(e, e') \xrightarrow{\ \sim\ } I \times A(e, e') \xrightarrow{\ u_e \times id\ } A(e, e) \times A(e, e')$$

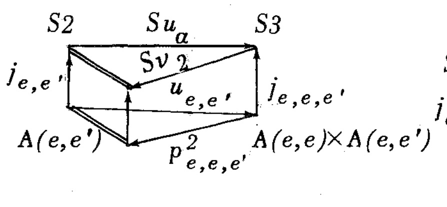
and Su_α be the factor of $(u_{e,e'})_{e,e'}$ with respect to the map

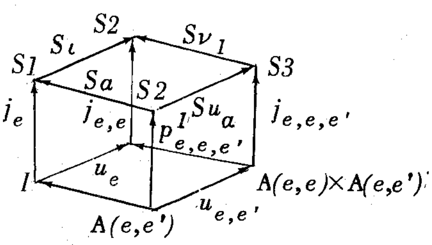
$$A_0 \times A_0 \to A_0 \times A_0 \times A_0 : (e, e') \mapsto (e, e, e').$$

Then

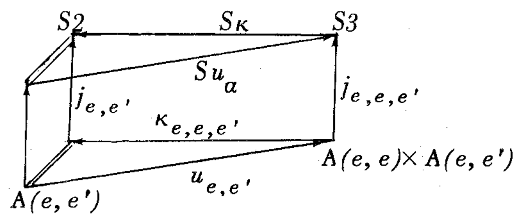
$$S\nu_1 . Su_\alpha = S\iota . S\alpha, \quad S\nu_2 . Su_\alpha = id_{S2} = S\kappa . Su_\alpha$$

(«source» unitarity axiom of an internal category). Indeed, by unicity of the factors and by definition of $u_{e,e'}$, for every objects e and e' of A the two following diagrams commute, so that the two first equalities are valid.





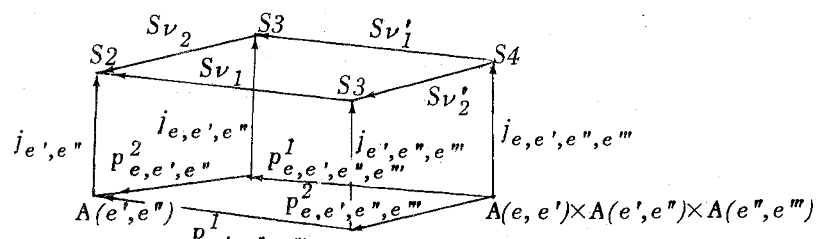
The validity of the third equation is deduced from the commutativity of the diagram



(whose down triangle commutes due to the unitarity axiom satisfied by A).

(ii) A similar proof shows that S satisfies the «target» unitarity axiom of an internal category.

(iii) S also satisfies the associativity axiom of an internal category. Indeed, for objects e, e', e'', e''' of A, there exists a commutative cube



where $j_{e,e',e'',e'''}$ is the injection toward the coproduct $S4$ of the family $(A(e,e')\times A(e',e'')\times A(e'',e'''))_{e,e',e'',e'''}$ indexed by A_0^4, and where $S\nu'_i$ is the factor of the family $(p^i_{e,e',e'',e'''})$ of projections with respect to the map $q'_i : A_0^4 \to A_0^3$ defined by

$$q'_1(e,e',e'',e''') = (e,e',e''), \quad q'_2(e,e',e'',e''') = (e',e'',e''').$$

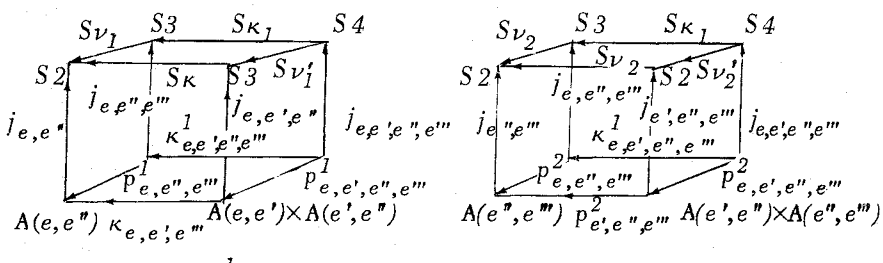
As the down face of this cube is a pullback, a proof analogous to that of Part b proves that the up face of this cube is a pullback. Now, let us denote by $\kappa^1_{e,e',e'',e'''}$ the composite

$$A(e,e')\times A(e',e'')\times A(e'',e''') \xrightarrow{\sim} (A(e,e')\times A(e',e''))\times A(e'',e''') \xrightarrow{\kappa_{e,e',e''}\times id} A(e,e'')\times A(e'',e''')$$

$S\kappa_1$ factor of the family $(\kappa^1_{e,e',e'',e'''})$ with respect to the projection

$$A^4_o \to A^3_o : (e,e',e'',e''') \mapsto (e,e'',e''')$$

renders commutative the cubes



(by definition of $\kappa^1_{e,e',e'',e'''}$ and of the different factors), so that

$$S\nu_1 . S\kappa_1 = S\kappa . S\nu_1' \quad \text{and} \quad S\nu_2 . S\kappa_1 = S\nu_2 . S\nu_2' .$$

In the same way, there is a factor $S\kappa_2 : S4 \to S3$ of the family of composites $\kappa^2_{e,e',e'',e'''} =$

$$A(e,e')\times A(e',e'')\times A(e'',e''') \xrightarrow{\sim} A(e,e')\times (A(e',e'')\times A(e'',e''')) \xrightarrow{id\times\kappa_{e',e'',e'''}} A(e,e')\times A(e',e''')$$

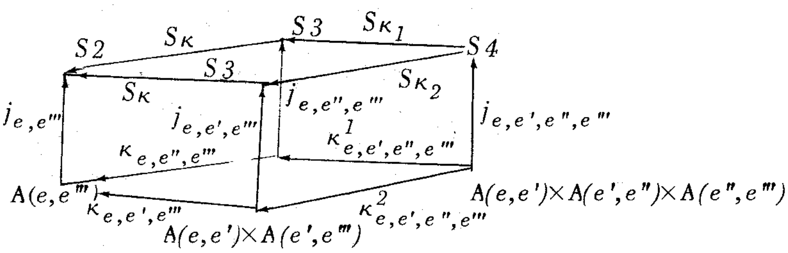
with respect to the projection

$$A^4_o \to A^3_o : (e,e',e'',e''') \mapsto (e,e',e'''),$$

and $S\kappa_2$ satisfies the equalities

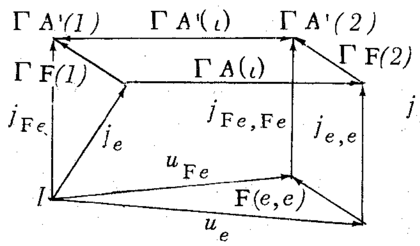
$$S\nu_1 . S\kappa_2 = S\nu_1 . S\nu_1' \quad \text{and} \quad S\nu_2 . S\kappa_2 = S\kappa . S\nu_2' .$$

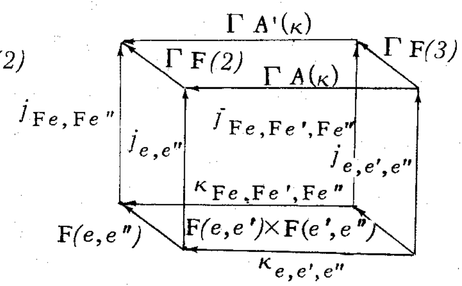
The associativity axiom $S\kappa . S\kappa_1 = S\kappa . S\kappa_2$ then follows from the unicity of factors and from the following cube, whose down face commutes due to the associativity axiom satisfied by the V-category A and whose lateral faces are commutative, by definition of the different factors. Hence, S defines a realization $S : \sigma \to V$ of σ in V, i.e., a category internal to V, which will be denoted by ΓA.



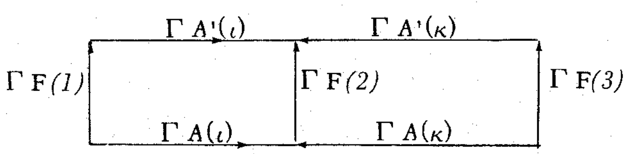
2° a) Let $F : A \to A'$ be a V-functor, $F_0 : A_0 \to A'_0 : e \mapsto Fe$ the map between objects and $F(e,e') : A(e,e') \to A(Fe, Fe')$ the canonical morphism, for every couple (e,e') of objects of A. There exist factors
$\Gamma F(2) : \Gamma A(2) \to \Gamma A'(2)$ of $(F(e,e'))_{e,e'}$ with respect to $F_0 \times F_0$,
$\Gamma F(1) : \Gamma A(1) \to \Gamma A'(1)$ of $(1_e = 1)_e$ with respect to F_0,
$\Gamma F(3) : \Gamma A(3) \to \Gamma A'(3)$ of $(F(e,e') \times F(e',e''))_{e,e',e''}$ with respect to $F_0 \times F_0 \times F_0$.

These factors render commutative the diagrams





whose down faces commute by definition of a V-functor. This proves that $\Gamma F : \Gamma A \to \Gamma A'$ is a functor in V.



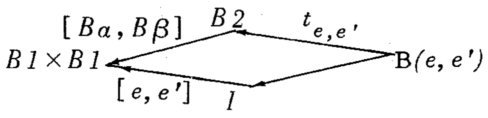
b) This defines a functor $\Gamma : V\text{-}Cat \to Cat\,V : F \mapsto \Gamma F$, due to the unicity of the factors defining $\Gamma F(i)$, $i = 1, 2, 3$. ∇

PROPOSITION B. *The functor* $\Gamma : V\text{-}Cat \to Cat\,V$ *constructed above admits*
1 *a right adjoint.*

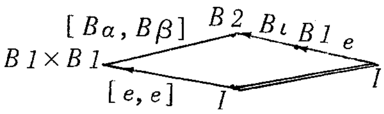
PROOF. Let B be a category in V.

1° We define a V-category $\mathrm{B} = \Gamma' B$. The class B_0 of its objects is the set of morphisms $e: I \to B1$. If $e: I \to B1$ and $e': I \to B1$ are such objects, $\mathrm{B}(e,e')$ is defined by the pullback

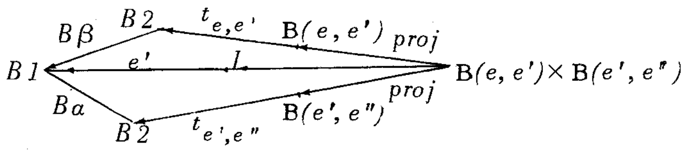
$(\mathrm{D}_{e,e'})$



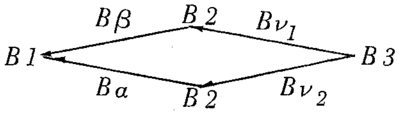
where $[e,e']$ and $[B\alpha, B\beta]$ are the factors of (e,e') and $(B\alpha, B\beta)$ through the product $B1\times B1$. There exists a factor $u_e: I \to \mathrm{B}(e,e)$, through the pullback $(\mathrm{D}_{e,e})$, of the diagram



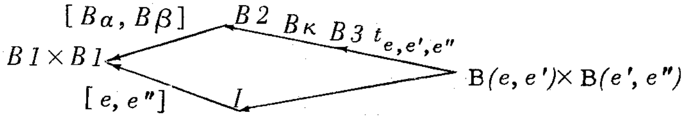
(which commutes, since $B\alpha . B\iota$ and $B\beta . B\iota$ are identities). Let e'' be another «object» $e'': I \to B1$. The commutative diagram



factors uniquely through the pullback



into $t_{e,e',e''}: \mathrm{B}(e,e')\times \mathrm{B}(e',e'') \to B3$, and the diagram



commutes (this uses the equalities

$$B\alpha . B\kappa = B\alpha . B\nu_1 \quad \text{and} \quad B\beta . B\kappa = B\beta . B\nu_2$$

of an internal category, and the commutativity of $(\mathrm{D}_{e,e'})$ and $(\mathrm{D}_{e',e''})$). Hence this diagram factors uniquely through the pullback $(\mathrm{D}_{e,e''})$ into

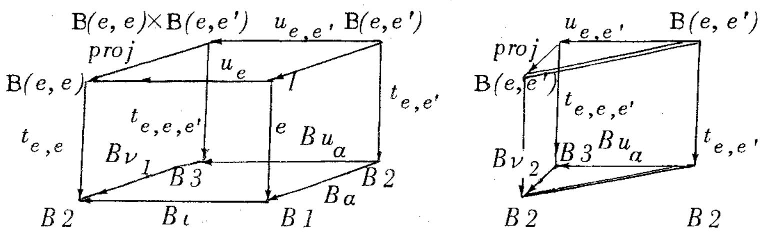
$$\kappa_{e,e',e''} : \mathrm{B}(e,e') \times \mathrm{B}(e',e'') \to \mathrm{B}(e,e'').$$

b) This defines a V-category B.

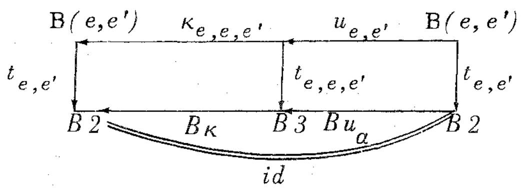
(i) Let us denote by $u_{e,e'}$ the composite

$$\mathrm{B}(e,e') \xrightarrow{\sim} I \times \mathrm{B}(e,e') \xrightarrow{u_e \times id} \mathrm{B}(e,e) \times \mathrm{B}(e,e').$$

In the diagrams

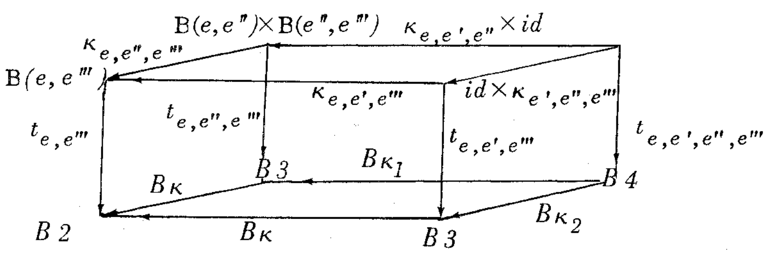


all the faces commute, except perhaps the back one; as $B\nu_i$ are projections of a pullback, it follows that this last face also commutes. So, we have the commutative diagram



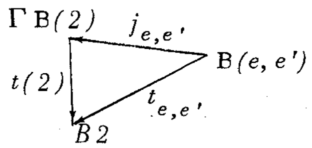
and the unicity of the factor through the pullback $\mathrm{B}(e,e')$ implies that $\kappa_{e,e,e'} . u_{e,e'}$ is an identity. Therefore, B satisfies the unitarity axiom.

(ii) A similar method proves that B satisfies the associativity axiom. It uses the fact that there is a cube

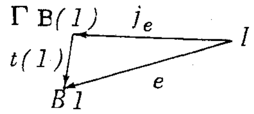


in which all the vertical edges are projections of pullbacks and all faces, except perhaps the up face commute; so this up face also commutes.

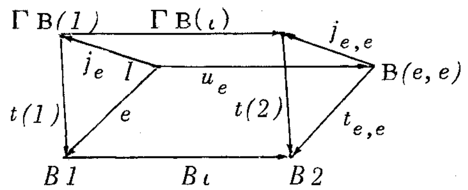
2° There is an internal functor $t: \Gamma B \to B$. Indeed, let $t(2)$ be the factor through the coproduct $\Gamma B(2)$ (constructed in Proposition A) of the family $(t_{e,e'}: B(e,e') \to B2)_{e,e'}$ indexed by $B_0 \times B_0$, so that:



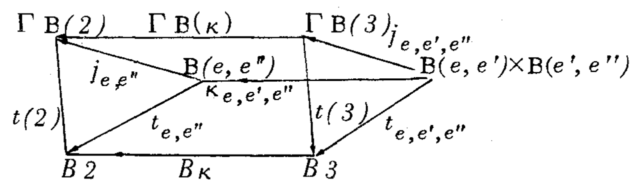
commutes. Let $t(1): \Gamma B(1) \to B1$ be the factor through the coproduct $\Gamma B(1)$ of the family $(e)_e$ indexed by B_0, so that



commutes. Then the back face of the diagram



commutes, because all the other faces commute and $\Gamma B(1)$ is a coproduct. Similarly, the back face of the diagram



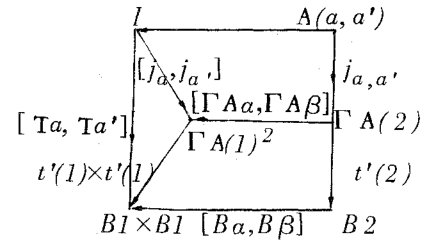
commutes, where $t(3)$ is the factor of $(t_{e,e',e''})$ through the coproduct $\Gamma B(3)$. We have so defined an internal to V functor $t: \Gamma B \to B$.

3° $t: \Gamma B \to B$ is the coliberty morphism defining B as a cofree object generated by B. Indeed, let A be a V-category and $t': \Gamma A \to B$ be a functor in V. We are going to construct a V-functor $T: A \to B$.

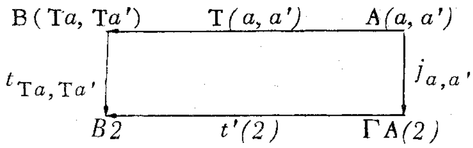
b) For each object a of A, let Ta be the object of B:

$$\mathrm{T}\,a = (I \xrightarrow{j_a} \Gamma A(1) \xrightarrow{t'(1)} B1)$$

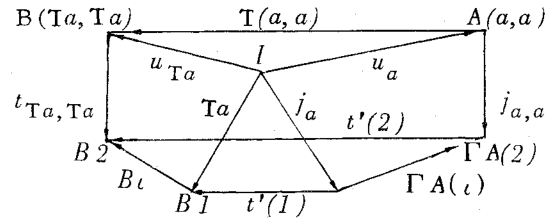
where j_a is always the injection into the coproduct; this defines a map $\mathrm{T}_0 : A_0 \to B_0$. If a and a' are objects of A, the two small squares of the diagram



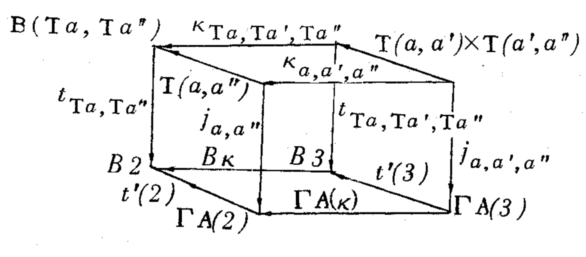
are commutative (by definition of ΓA and t' being an internal functor). Hence the exterior square is commutative, and it factors through the pull-back ($D_{\mathrm{T}a,\mathrm{T}a'}$) into a unique $\mathrm{T}(a,a') : A(a,a') \to B(\mathrm{T}a, \mathrm{T}a')$.



b) This defines a V-functor $\mathrm{T} : A \to B$. Indeed, for each object a of A, the up face of the diagram

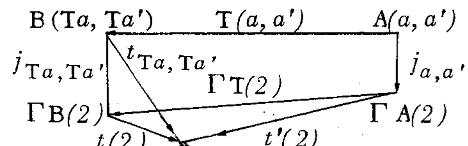


commutes, since all the other faces commute and $B(\mathrm{T}a, \mathrm{T}a')$ is a pull-back. Similarly, the up face of the following cube



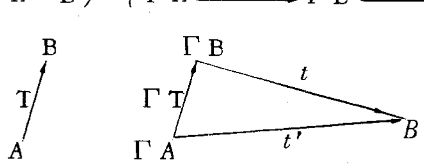
commutes, all the other faces commuting and $B(Ta, Ta'')$ being a pull-back. Hence, $T : A \to B$ is a V-functor.

c) The down face of the diagram



commutes, whatever be the objects a', a of A since the other faces commute and $\Gamma A(2)$ is the coproduct of $(A(a, a'))_{a,a'}$. It follows that

$$(t' : \Gamma A \to B) = (\Gamma A \xrightarrow{\Gamma T} \Gamma B \xrightarrow{t} B).$$



Finally, the unicity of the V-functor T satisfying this equality results from the unicity of the morphisms $T(a, a')$. So B is a cofree object generated by B with respect to $\Gamma : V\text{-}Cat \to Cat\,V$. ∇

DEFINITION. *A* category in V is called *pseudo-discrete* if its object of objects is a coproduct of copies of the final object I.

By the construction of the functor Γ (Proposition A), it takes its values into the full subcategory $PsCat\,V$ of $Cat\,V$ whose objects are the pseudo-discrete categories in V. Hence it admits as a restriction a functor, also denoted by $\Gamma : V\text{-}Cat \to PsCat\,V$. Remark that the existence of this functor is conjectured (without precise hypotheses) in the Appendix III of the book [8].

PROPOSITION C. *Let* V *be a category with commuting coproducts, I its final object. If the functor $Hom(I, -) : V \to V$ preserves coproducts, then the functor $\Gamma : V\text{-}Cat \to PsCat\,V$ is an equivalence.*

PROOF. Let $\Gamma' : PsCat\,V \to V\text{-}Cat$ be the right adjoint of Γ constructed in Proposition B.

1° *The composite*

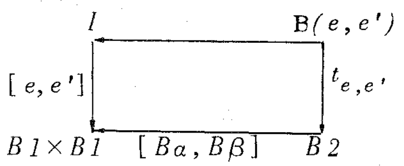
$$PsCat\,\mathrm{V} \xrightarrow{\Gamma'} \mathrm{V}\text{-}Cat \xrightarrow{\Gamma} PsCat\,\mathrm{V}$$

is equivalent to the identity:

Indeed, it suffices to prove that, for each pseudo-discrete category B in V, the coliberty morphism $t': \Gamma\Gamma' B \to B$ is an isomorphism. By hypothesis, $B1$ is the coproduct of a family $(I_\lambda = I)_{\lambda \in \Lambda}$ and

$$Hom(I, B1) \approx \coprod_\lambda Hom(I, I_\lambda) \approx \Lambda$$

since $Hom(I, I)$ is reduced to the identity of I; hence $B1$ is also a coproduct of the family $(I_e = I)_e$ indexed by the set $\mathrm{B}_0 = Hom(I, B1)$ of morphisms $e: I \to B1$, the e-th injection being e itself. As the partial product functors preserve coproducts, $B1 \times B1$ is the coproduct of the family $(I_{e,e'} = I)_{e,e'}$, indexed by $\mathrm{B}_0 \times \mathrm{B}_0$, the injections being the factor $[e, e']: I \to B1 \times B1$ into the product. We take the pullback



used to define $\mathrm{B} = \Gamma B$. The category V admitting commuting coproducts, by pulling back along $[B\alpha, B\beta]$ the coproduct $B1 \times B1$, we get $B2$ as a coproduct of $(\mathrm{B}(e, e'))_{e,e'}$, the injections being the morphisms $t_{e,e'}: \mathrm{B}(e, e') \to B2$. So the factor $t(2): \Gamma\,\mathrm{B}(2) \to B2$ of $(t_{e,e'})_{e,e'}$ is an isomorphism. This implies that $t: \Gamma\,\mathrm{B} \to B$ is an isomorphism.

2° *The composite*

$$\mathrm{V}\text{-}Cat \xrightarrow{\Gamma} PsCat\,\mathrm{V} \xrightarrow{\Gamma'} \mathrm{V}\text{-}Cat$$

is also equivalent to the identity, so that $PsCat\,\mathrm{V}$ and V-Cat are equivalent. Indeed, let A be a V-category; by adjunction, there is a V-functor $\mathrm{T}: A \to \Gamma'\Gamma A$ such that $\mathrm{T}a$ is the injection $j_a: I \to \Gamma A(1)$ for each object a of A and that the following diagram commutes, for each couple (a, a') of objects of A (we take up the notations of Proposition B, in which we choose $B = \Gamma A$).

$$\begin{array}{ccc} \Gamma'\Gamma A(\mathrm{T}a, \mathrm{T}a') & \xleftarrow{\mathrm{T}(a,a')} & A(a,a') \\ & {\scriptstyle t_{\mathrm{T}a,\mathrm{T}a'}} \searrow \quad \swarrow {\scriptstyle j_{a,a'}} & \\ & \Gamma A(2) & \end{array}$$

We are going to prove that T is an isomorphism.

a) $\mathrm{T}_0 : A_0 \to (\Gamma'\Gamma A)_0$ is 1-1 and onto: $\Gamma A(1)$ is the coproduct of the family $(I = I_a)_a$ indexed by the set A_0 of objects of A; since $Hom(I, -): \mathrm{V} \to \mathrm{V}$ preserves coproducts, we have

$$Hom(I, \Gamma A(1)) \approx \coprod_a Hom(I, I_a) \approx A_0 ,$$

so that T_0 is an isomorphism.

b) For every objects a, a' of A, there is a pullback

$$\begin{array}{ccc} I & \longleftarrow & \Gamma'\Gamma A(\mathrm{T}a, \mathrm{T}a') \\ {\scriptstyle [j_a, j_{a'}]} \downarrow & & \downarrow {\scriptstyle t_{\mathrm{T}a,\mathrm{T}a'}} \\ \Gamma A(1) \times \Gamma A(1) & \xleftarrow{[\Gamma A\alpha, \Gamma A\beta]} & \Gamma A(2) \end{array}$$

defining $\Gamma'\Gamma A(\mathrm{T}a, \mathrm{T}a')$. We deduce as in Part 1 that $\Gamma A(2)$ is the coproduct of $(\Gamma'\Gamma A(\mathrm{T}a, \mathrm{T}a'))_{a,a'}$ with injections $t_{\mathrm{T}a,\mathrm{T}a'}$. But (by definition) $\Gamma A(2)$ is also the coproduct of $(A(a, a'))_{a,a'}$, and the commutativity of the diagrams defining $\mathrm{T}(a, a')$ implies that the identity of $\Gamma A(2)$ is the coproduct of $(\mathrm{T}(a, a'))_{a,a'}$. So, by definition of a category with commuting coproducts, each $\mathrm{T}(a, a')$ is an isomorphism. Hence $\mathrm{T}: A \to \Gamma'\Gamma A$ is an isomorphism. ∇

COROLLARY. *If* V *is a category with commuting coproducts, the functor* Γ*:* V-*Cat* $\to$ *PsCat*V *is an equivalence iff the endofunctor* $Hom(I, -)$ *preserves coproducts of copies of the final object* I.

PROOF. The preceding proof shows that the condition is sufficient. On the other hand, let us suppose that Γ : V-*Cat* $\to$ *PsCat*V is an equivalence and let S be the coproduct of a family $(I_\lambda = I)_{\lambda \in \Lambda}$. There exists a V-category A (the «V-groupoid of pairs of Λ») such that Λ is the set of its objects and $A(\lambda, \lambda') = I$ for each couple (λ, λ') of objects. The canonical V-functor

$$\mathrm{T}: A(\lambda, \lambda') \to \Gamma'\Gamma A(\mathrm{T}\lambda, \mathrm{T}\lambda')$$

being an isomorphism by hypothesis, its «restriction to the objects»:

$$T_0 : (A_0 = \Lambda) \to (\Gamma'\Gamma A)_0 = Hom(I, S)$$

is an isomorphism, and $Hom(I,S) \approx \Lambda \approx \coprod_{\lambda \in \Lambda} Hom(I, I_\lambda)$. ∇

EXAMPLES.

1° There are many examples of categories V with commuting coproducts (see Penon [21]):

- the elementary topoi admitting coproducts,

- the categories admitting finite limits and coproducts and equipped with a faithful functor toward *Set* preserving pullbacks and coproducts and reflecting isomorphisms; in particular, the initialstructure categories (Wischnewsky [22], or topological categories in the sense of Herrlich [18]), the categories Cat_n for any integer n.

The condition that $Hom(I, -): V \to V$ preserves coproducts means that I is connected (in the sense of Hoffmann [19], see also Proposition 3-12 of Penon [21]). It is satisfied in the categories of a «topological nature», as well as in Cat_n. Remark that an $(n+1)$-fold category H (considered as a category in Cat_n, see Appendix [5]) is pseudo-discrete, and therefore «is» a Cat_n-category, by Proposition C, iff the objects of the last category H^n are also objects for the n first categories H^i (in an $(n+1)$-category, the objects of H^n are only supposed to be objects for H^{n-1}). The $(n+1)$-category Nat_n constructed in Proposition 8 «is» pseudo-discrete.

2° Proposition C is also valid if V is the category of r-differentiable manifolds (modelled on Banach spaces), though only some pullbacks exist in it (the pullbacks used in the proof will exist). Hence categories whose Hom are equipped with «compatible» r-differentiable structures «are» those r-differentiable categories (in the sense of [12]) in which the topology induced on the class of objects is discrete.

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MULTIPLE FUNCTORS
IV. MONOIDAL CLOSED STRUCTURES ON Cat_n

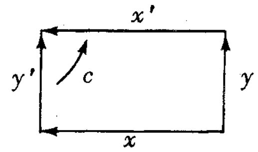
by Andrée and Charles EHRESMANN

INTRODUCTION.

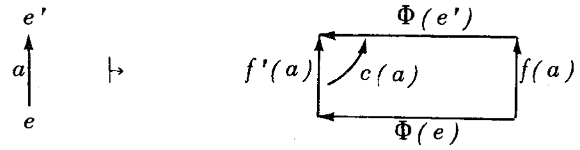
This paper is Part IV of our work on multiple categories whose Parts I, II and III are published in [3, 4, 5]. Here we «laxify» the constructions of Part III (replacing equalities by cells) in order to describe monoidal closed structures on the category Cat_n of n-fold categories, for which the internal Hom functors associate to (A, B) an n-fold category of «lax hypertransformations» between n-fold functors from A to B.

As an application, we prove that all double categories are (canonically embedded as) double sub-categories of the double category of squares of a 2-category; by iteration this gives a complete characterization of multiple categories in terms of 2-categories. Hence the study of multiple categories reduces «for most purposes» to that of 2-categories and of their squares, and generalized limits of multiple functors [4, 5] are just lax limits (in the sense of Gray [7], Bourn [2], Street [10],...), taking somewhat restricted values.

More precisely, if C is a category, the double category $Q(C)$ of its (up-)squares



is a laxification of the double category of commutative squares of the category of 1-morphisms of C ; a lax transformation Φ between two functors from a category A to $|C|^1$ «is» a double functor $\Phi : A \to Q(C)$:



(Φ «is» a natural transformation iff $c(a)$ is an identity for each a in A). Similarly, to an n-fold category A, we associate in Section A the $(n+1)$-fold category *Cub* B (of cubes of B), which is a laxification of the $(n+1)$-fold category *Sq* B (of squares of B) used in Part III to explicit the cartesian closure functor of Cat_n.

In Section B, the construction (given in Part III) of the left adjoint *Link* of the functor *Square* from Cat_n to Cat_{n+1} is laxified in order to get the left adjoint *LaxLink* of the functor *Cube*: $Cat_n \to Cat_{n+1}$. While *Link* A, for an $(n+1)$-fold category A, is generated by classes of strings of objects of the two last categories A^{n-1} and A^n, the n-fold category *LaxLink* A is generated by classes of strings of strings of objects of «alternately» A^{n-2} and A^{n-1} or A^n (so we introduce objects of A^{n-2} instead of equalities).

LaxLink is a left inverse (Section C) of the functor *Cylinder* from n-*Cat* to $(n+1)$-*Cat* associating to an n-category B the greatest $(n+1)$-category included in *Cub* B.

The functor $Cub_{n,m}$ from Cat_n to Cat_m is defined by iteration as well as its left adjoint. They give rise to a closure functor $LaxHom_n$ on Cat_n mapping the couple (A, B) of n-fold categories onto the n-fold category $Hom(A, (Cub_{n,2n}B)^{\gamma})$, where:

- *Hom* is the internal Hom of the monoidal closed category (considered in Part II) $(\coprod_n Cat_n, \blacksquare, Hom)$,

- $(Cub_{n,2n}B)^{\gamma}$ is the $2n$-fold category deduced from $Cub_{n,2n}B$ by the permutation of the compositions γ:

$$(0, \dots, 2n-1) \mapsto (0, 2, \dots, 2n-2, 1, 3, \dots, 2n-1).$$

The corresponding tensor product on Cat_n admits as a unit the n-fold category on 1.

«Less laxified» monoidal closed structures on Cat_n are defined by replacing at some steps the functor *Cube* by the functor *Square* ; the «most rigid» one is the cartesian closed structure (where only functors *Square* are considered [5]). For 2-categories, Gray's monoidal structure is also obtained.

Existence theorems for the «lax limits» corresponding to these closure functors are given in Section D. In fact, we prove that, if B is an n-fold category whose category $|B|^{n-1}$ of objects for the $(n-1)$-th first compositions admits (finite) usual limits, then the representability of B implies that of the $(n+1)$-fold categories $K = Sq\,B$, $Cub\,B$, $Cyl\,B$; therefore, according to the theorem of existence of generalized limits given in Part II, Proposition 11, all (finite) n-fold functors toward B admit K-wise limits. In particular, the existence theorem for lax limits of 2-functors given by Gray [7], Bourn [2], Street [10] is found anew, with a more structural (and shorter) proof (already sketched in Part I, Remark page 271, and exposed in our talk at the Amiens Colloquium in 1975) *. 1

The notations are those of Parts II and III. If B is an n-fold category, $\underline{B}$ is the set of its blocks and, for each sequence $(i_0, \dots, i_{p-1})$ of distinct integers lesser than n, the p-fold category whose j-th category is B^{i_j} is denoted by $B^{i_0, \dots, i_{p-1}}$.

* NOTE ADDED IN PROOFS. We have just received a mimeographed text of J.W. Gray, *The existence and construction of lax limits*, in which a very similar proof is given for this particular theorem. The only difference is that *Cat* is considered as the inductive closure of {1,2,3} (instead of {2}) and that the proof is not split in two parts:

1° existence of generalized limits (those limits are not used by Gray),

2° representability of $Q(C)$ and $Cyl\,C$ for a 2-category C (though this result is implicitly proved).

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A. The cubes of a multiple category.

The aim is to give to Cat_n a monoidal closed structure whose tensor product «laxifies» the (cartesian) product (by introducing non-degenerate blocks in place of some identities). The method is the same as that used in Part III to construct the cartesian closed structure of Cat_n.

The first step is the description of a functor *Cube* from Cat_n to Cat_{n+1}, admitting a left adjoint which maps an $(n+1)$-fold category $\mathbb{A}$ onto an n-fold category $LaxLk\mathbb{A}$, obtained by «laxification» of the construction of $Lk\mathbb{A}$.

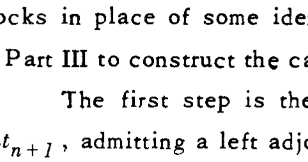
1° The «model» double category $\mathbb{M}$.

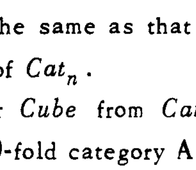
To define the *Square* functor, we used as a basic tool the double category of squares of a category $\mathbb{C}$, whose blocks «are» the functors from 2×2 to $\mathbb{C}$. The analogous tool will be here the triple category of cubes of a double category, obtained by replacing the category 2×2 by the «model» double category $\mathbb{M}$ described as follows:

Consider the 2-category Q with four vertices, six 1-morphisms

$$y, \; y', \; \hat{y}, \; \hat{y}', \; \hat{y}' \circ_1 y, \; y' \circ_1 \hat{y},$$

and only one non-degenerate 2-cell $x: y' \circ_1 \hat{y} \to \hat{y}' \circ_1 y$ in Q^0:

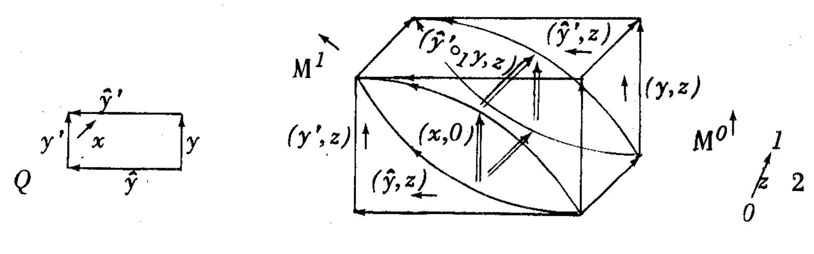


(also represented by 

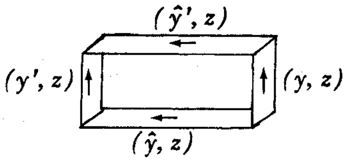
).

(Intuitively, Q consists of a square «only commutative up to a 2-cell».)

The model double category $\mathbb{M}$ is the double category $Q \times (2, \underline{2}^{dis})$, product of Q with the double category $(2, \underline{2}^{dis})$:

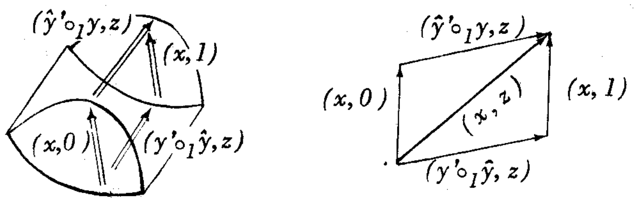


It is generated by the blocks forming the non-commutative square



of $M^1 = Q^1 \times \underline{2}^{dis}$

and those forming the commutative square («cylinder») of $M^0 = Q^0 \times 2$:

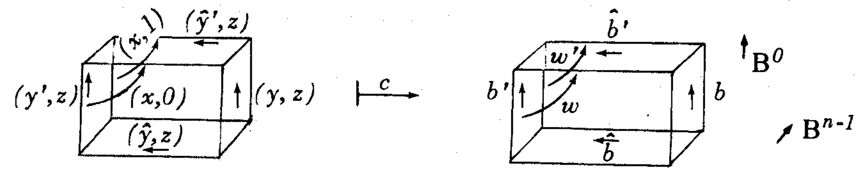


whose diagonal is (x, z).

2° The multiple category of cubes of an n-fold category.

Let n be an integer such that $n \geqslant 2$. We denote by B an n-fold category, by α^i and β^i the source and target maps of B^i, for $i < n$.

DEFINITION. A double functor $c: M \to B^{n-1,0}$ from the model double category M to the double category $B^{n-1,0}$ (whose compositions are the $(n-1)$-th and 0-th compositions of B) is called *a cube of* B .



The cube c will be identified with the 6-uple $(b', \hat{b}', w', w, \hat{b}, b)$ where

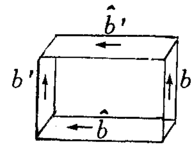
$$b = c(y, z), \quad b' = c(y', z), \quad \hat{b} = c(\hat{y}, z), \quad \hat{b}' = c(\hat{y}', z),$$
$$w = c(x, 0), \quad w' = c(x, 1)$$

(which determines the cube c uniquely).

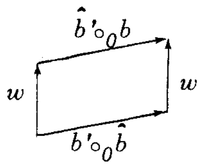
In other words, a cube c of B may also be defined as a 6-uple

$$c = (b', \hat{b}', w', w, \hat{b}, b)$$

of blocks of B such that



is a non-commutative square of B^0,



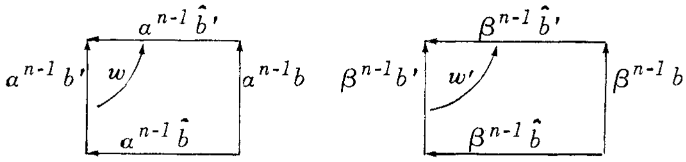
is a commutative square of B^{n-1}.

The diagonal of this last square:

$$(\hat{b}' \circ_0 b) \circ_{n-1} w = w' \circ_{n-1} (b' \circ_0 \hat{b})$$

is called the *diagonal of the cube* c, and denoted by ∂c.

Remark that w and w' are 2-cells of the greatest 2-category contained in $\mathrm{B}^{n-1,0}$, and that in the cube c (represented by a «geometric» cube), the «front» and «back» faces are up-squares of this 2-category:



On the set $Cub\mathrm{B}$ of cubes of B, we have the $(n\text{-}2)$-fold category $Hom(\mathrm{M}, \mathrm{B}^{n-1,0,1,\dots,n-2})$, whose i-th composition is deduced pointwise from the $(i+1)$-th composition of B, for $i < n-2$. With the notations above (we add everywhere indices if necessary), the i-th composition is written:

$$c_1 \circ_i c = (b'_1 \circ_{i+1} b', \hat{b}'_1 \circ_{i+1} \hat{b}', w'_1 \circ_{i+1} w', w_1 \circ_{i+1} w, \hat{b}_1 \circ_{i+1} \hat{b}, b_1 \circ_{i+1} b),$$

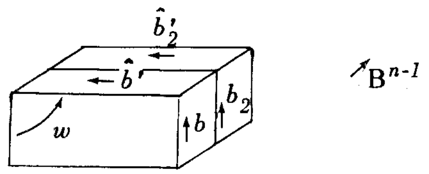
iff the six composites are defined.

Now, we define three other compositions on $Cub\mathrm{B}$ so that, by adding these new compositions, we obtain an $(n+1)$-fold category $Cub\mathrm{B}$:

- We denote by $(Cub\mathrm{B})^{n-2}$ the category whose composition is deduced «laterally pointwise» from that of B^{n-1}:

$$c_2 \circ_{n-2} c = (b'_2 \circ_{n-1} b', \hat{b}'_2 \circ_{n-1} \hat{b}', w'_2, w, \hat{b}_2 \circ_{n-1} \hat{b}, b_2 \circ_{n-1} b)$$

iff $w_2 = w'$ and the four composites are defined.

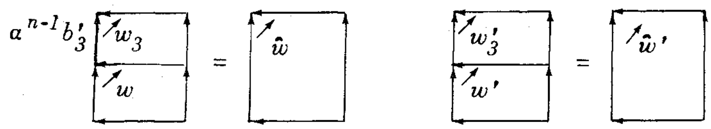


The source and target of c are the degenerate cubes determined by the front and back faces of c.

- Let $(Cub\mathrm{B})^{n-1}$ be the category whose composition is the «vertical» composition of cubes (also denoted by ⊟):

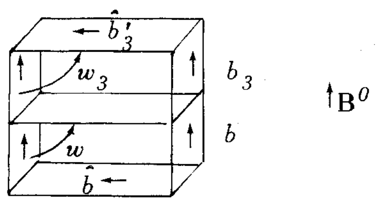
$$c_3 \circ_{n-1} c = (b'_3 \circ_0 b',\ \hat{b}'_3,\ \hat{w}',\ \hat{w},\ \hat{b},\ b_3 \circ_0 b) \quad \text{iff} \quad \hat{b}' = b_3,$$

where $\hat{w}$ and $\hat{w}'$ are the 2-cells of the vertical composites of the front and back up-squares:



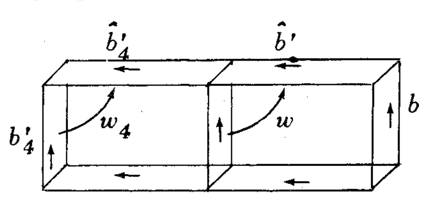
(hence:

$$\hat{w} = (w_3 \circ_0 \alpha^{n-1} b) \circ_{n-1} (\alpha^{n-1} b'_3 \circ_0 w),\quad \hat{w}' = (w'_3 \circ_0 \beta^{n-1} b) \circ_{n-1} (\beta^{n-1} b'_3 \circ_0 w')\ ).$$

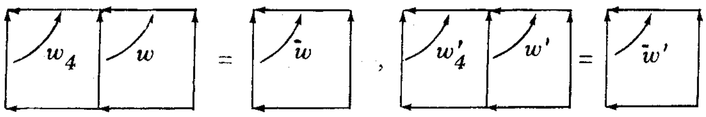


- Finally, $(Cub\mathrm{B})^n$ is the category whose composition is the «horizontal» composition of cubes (also denoted by ◫):

$$c_4 \circ_n c = (b'_4,\ \hat{b}'_4 \circ_0 \hat{b}',\ \bar{w}',\ \bar{w},\ \hat{b}_4 \circ_0 \hat{b},\ b) \quad \text{iff} \quad b' = b_4,$$



where $\bar{w}$ and $\bar{w}'$ are the 2-cells of the horizontal composites of the front and back up-squares:



REMARK. $(Cub\mathrm{B})^{n-1,n}$ is the double category of up-squares of the 2-category of cylinders $(Cyl\mathrm{B})^{n,n-1}$, which is the greatest 2-category contained in the double category $(Sq(\mathrm{B}^{n-1,0}))^{2,0}$ (with the notations of Section C).

From the permutability axiom satisfied by B it follows that we have an $(n+1)$-fold category on the set of cubes of B, denoted by $Cub\,\mathrm{B}$, such that:

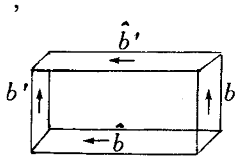
- $(Cub\mathrm{B})^{0,\dots,n-2} = Hom(\mathrm{M}, \mathrm{B}^{n-1,0,1,\dots,n-2})$,
- the $(n-2)$-th, $(n-1)$-th and n-th compositions are those defined above.

DEFINITION. This $(n+1)$-fold category $Cub\mathrm{B}$ is called the *$(n+1)$-fold category of cubes of* B.

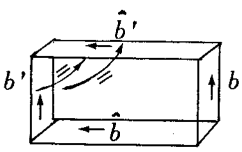
Summing up, the i-th category $(Cub\mathrm{B})^i$ is deduced pointwise from B^{i+1} for $i < n-2$ and «laterally pointwise» from B^{n-1} for $i = n-2$, while $(Cub\mathrm{B})^{n-1}$ and $(Cub\mathrm{B})^n$ are the «vertical» and «horizontal» categories of cubes.

EXAMPLE. If B is a double category, $Cub\mathrm{B}$ is a triple category whose 0-th composition is deduced laterally pointwise from B^1. 1+

If a square s of B^0,

$s =$ 

is identified with the cube

$c =$ 

(with the same «lateral» faces) in which w and w' are the degenerate 2-cells $\alpha^{n-1}(b' \circ_0 \hat{b})$ and $\beta^{n-1}(\hat{b}' \circ_0 b)$, then the $(n+1)$-fold category $Sq\mathrm{B}$

of squares of $\mathbf{B}$ (see Part III [5]) becomes an $(n+1)$-fold subcategory of $Cub\mathbf{B}$, which has the same objects than $Cub\mathbf{B}$ for the $(n-1)$-th and n-th categories. It follows that we may still consider the isomorphisms

$$-^{\boxminus} : \mathbf{B}^{1,\dots,n-1,0} \stackrel{\sim}{\to} |(Cub\mathbf{B})^{n-1}|^{0,\dots,n-2,n} : b \mapsto b^{\boxminus},$$

$$-^{\boxplus} : \mathbf{B}^{1,\dots,n-1,0} \stackrel{\sim}{\to} |(Cub\mathbf{B})^{n}|^{0,\dots,n-1} : b \mapsto b^{\boxplus}$$

from $\mathbf{B}^{1,\dots,n-1,0}$ onto the n-fold categories defined from $Cub\mathbf{B}$ by taking the objects of $(Cub\mathbf{B})^{n-1}$ and $(Cub\mathbf{B})^{n}$ respectively.

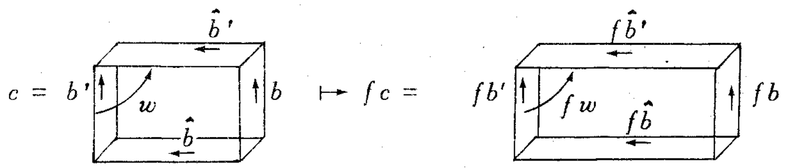
B. The adjoint functors *Cube* **and** *LaxLink* **.**

If $f: \mathbf{B} \to \mathbf{B}'$ is an n-fold functor, the $(n-2)$-fold functor

$$Hom(\mathbf{M}, f): Hom(\mathbf{M}, \mathbf{B}) \to Hom(\mathbf{M}, \mathbf{B}'): c \mapsto fc$$

underlies an $(n+1)$-fold functor $Cubf: Cub\mathbf{B} \to Cub\mathbf{B}'$ defined by:

$$c = (b', \hat{b}', w', w, \hat{b}, b) \mapsto fc = (fb', f\hat{b}', fw', fw, f\hat{b}, fb).$$



This determines the functor $Cub_{n,n+1}: Cat_n \to Cat_{n+1}$:

$$(f: \mathbf{B} \to \mathbf{B}') \mapsto (Cubf: Cub\mathbf{B} \to Cub\mathbf{B}'),$$

called the *functor Cube from* Cat_n *to* Cat_{n+1}.

PROPOSITION 1. *The functor* $Cub_{n,n+1}: Cat_n \to Cat_{n+1}$ *admits a left adjoint* $LaxLk_{n+1,n}: Cat_{n+1} \to Cat_n$.

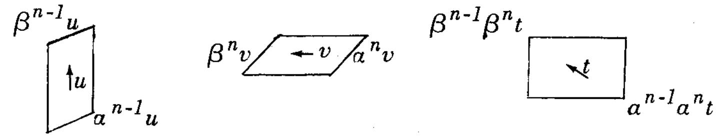
PROOF. Let $\mathbf{A}$ be an $(n+1)$-fold category, α^i and β^i the maps source and target of the i-th category $\mathbf{A}^i$.

1° *We define an n-fold category* $\bar{\mathbf{A}}$, which will be the free object generated by $\mathbf{A}$, as follows:

a) Let G be the graph whose vertices are those blocks e of $\mathbf{A}$ which are objects for both $\mathbf{A}^{n-1}$ and $\mathbf{A}^{n}$, the arrows ν from e to e' being the objects of either $\mathbf{A}^{n}$, $\mathbf{A}^{n-1}$ or $\mathbf{A}^{n-2}$ such that

$$\alpha^n \alpha^{n-1} \nu = e \quad \text{and} \quad \beta^n \beta^{n-1} \nu = e'.$$

Hence the arrows of G are of one of the three forms :



where u, v, t will always denote objects of A^n, A^{n-1}, A^{n-2}, respectively.

b) If K is an n-fold category, we say that *$f: G \to$ K is an admissible morphism* if $f: G \to \underline{K}$ is a map satisfying the 8 following conditions :

(i) If $\nu : e \to e'$ in G, then $f(\nu) : f(e) \to f(e')$ in K^0.

(ii) $|A^n|^{n-1} \hookrightarrow G \xrightarrow{f} K^0$ and $|A^{n-1}|^n \hookrightarrow G \xrightarrow{f} K^0$ are functors (where $|A^i|^j$ is the subcategory of A^j formed by the objects of A^i).

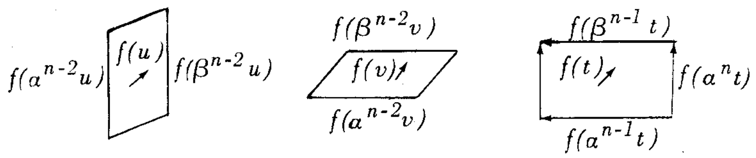
(iii) $|A^j|^i \hookrightarrow G \xrightarrow{f} K^{i+1}$ is a functor, for

$$i < n-2 \quad \text{and} \quad j = n,\ n\text{-}1 \text{ or } n\text{-}2.$$

(iv) For each arrow ν of G,

$$f(\nu) : f(\beta^n \alpha^{n-2} \nu) \circ_0 f(\alpha^{n-1} \alpha^{n-2} \nu) \to f(\beta^{n-1} \beta^{n-2} \nu) \circ_0 f(\alpha^n \beta^{n-2} \nu)$$

in the category K^{n-1}.



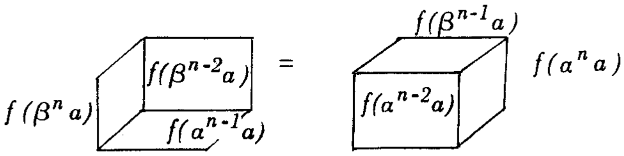
(v) $|A^n|^{n-2} \hookrightarrow G \xrightarrow{f} K^{n-1}$ and $|A^{n-1}|^{n-2} \hookrightarrow G \xrightarrow{f} K^{n-1}$ are functors.

(vi) For each block a of A ,

$$f(\beta^{n-2} a) \circ_{n-1} (f(\beta^n a) \circ_0 f(\alpha^{n-1} a)) = (f(\beta^{n-1} a) \circ_0 f(\alpha^n a)) \circ_{n-1} f(\alpha^{n-2} a)$$

(these composites are well-defined, due to conditions (i-iv-v) and to the fact that K is an n-fold category). This condition (vi) is equivalent to:

(vi') $c_f a = (f\beta^n a,\ f\beta^{n-1} a,\ f\beta^{n-2} a,\ f\alpha^{n-2} a,\ f\alpha^{n-1} a,\ f\alpha^n a)$

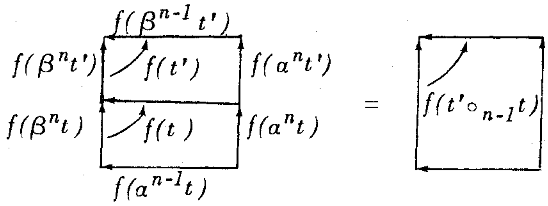


is a cube of K for each block a of A .

(vii) If $t' \circ_{n-1} t$ is defined in $|A^{n-2}|^{n-1}$, then

$$f(t' \circ_{n-1} t) = (f(t') \circ_0 f(a^n t)) \circ_{n-1} (f(\beta^n t') \circ_0 f(t)).$$

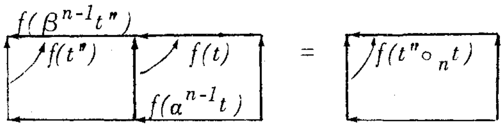
With (iv) this means that $f(t' \circ_{n-1} t)$ is the 2-cell of the vertical composite up-square of $K^{n-1,0}$:



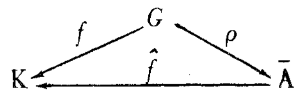
(viii) If $t'' \circ_n t$ is defined, then

$$f(t'' \circ_n t) = (f(\beta^{n-1} t'') \circ_0 f(t)) \circ_{n-1} (f(t'') \circ_0 f(a^{n-1} t)).$$

Hence, with (iv), in the horizontal category of up-squares of $K^{n-1,0}$:



c) By the general existence theorem of « universal solutions » [6], there exist: an n-fold category $\bar{A}$ and an admissible morphism $\rho : G \to \bar{A}$ such that any admissible morphism $f : G \to K$ factors uniquely through ρ into an n-fold functor $\hat{f} : \bar{A} \to K$.



Indeed, if we take the set of all admissible morphisms $\phi : G \to K_\phi$ with K_ϕ a small n-fold category, there exists an n-fold category $\prod_\phi K_\phi$ product in the category of n-fold categories associated to a universe to which belongs the universe of small sets. The factor

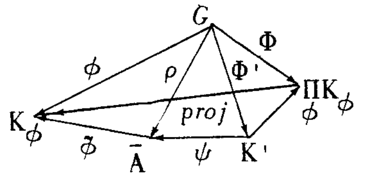
$$\Phi : G \to \prod_\phi K_\phi : \nu \mapsto (\phi(\nu))_\phi$$

of the family of maps ϕ is an admissible morphism, as well as its restriction $\Phi' : G \to K'$ to the n-fold subcategory K' of $\prod_\phi K_\phi$ generated by the image

$\Phi(G)$. As $\Phi(G)$ and K' are equipotent (by Proposition 2 [4]) and $\Phi(G)$ is of lesser cardinality than the small set G, it follows that there exists an isomorphism $\psi: \mathrm{K}' \to \bar{\mathrm{A}}$ onto a small n-fold category $\bar{\mathrm{A}}$. Then

$$\rho = (G \xrightarrow{\Phi'} \mathrm{K}' \xrightarrow{\psi} \bar{\mathrm{A}})$$

is a «universal» admissible morphism, since each admissible morphism



$\phi: G \to \mathrm{K}_\phi$ factors uniquely into

$$\phi = (G \xrightarrow{\rho} \bar{\mathrm{A}} \xrightarrow{\bar{\phi}} \mathrm{K}_\phi),$$

where

$$\bar{\phi} = (\bar{\mathrm{A}} \xrightarrow{\psi^{-1}} \mathrm{K}' \hookrightarrow \prod_\phi \mathrm{K}_\phi \xrightarrow{\text{projection}} \mathrm{K}_\phi).$$

Remark that the blocks $\rho(\nu)$, for any arrow ν of G, generate $\bar{\mathrm{A}}$.

d) An explicit construction of the universal admissible morphism $\rho: G \to \bar{\mathrm{A}}$ is sketched now (it will not be used later on).

(i) Let $\mathrm{P}(G)^0$ be the free quasi-category of paths $(\nu_k, \dots, \nu_0)$ of 1
the graph G; an arrow ν is identified to the path (ν). On the same set $\underline{\mathrm{P}}(G)$ of paths, there is a category $\mathrm{P}(G)^{i+1}$, whose composition is deduced pointwise from that of A^i, for each $i < n-2$. If r is the relation on $\underline{\mathrm{P}}(G)$ defined by:

$$(u', u) \sim u' \circ_{n-1} u \quad \text{if } u \text{ and } u' \text{ are objects of } \mathrm{A}^n,$$
$$(v', v) \sim v' \circ_n v \quad \text{if } v \text{ and } v' \text{ are objects of } \mathrm{A}^{n-1},$$

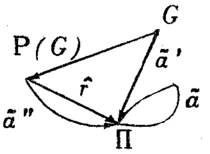
there exists an $(n-1)$-fold category Π quasi-quotient (Proposition 3 [4]) of $\mathrm{P}(G) = (\mathrm{P}(G)^0, \dots, \mathrm{P}(G)^{n-2})$ by r; the canonical morphism is denoted by $\hat{r}: \mathrm{P}(G) \to \Pi$.

(ii) We define a graph on $\underline{\Pi}$: Consider the morphism

$$\bar{a}': \nu \mapsto \hat{r}(\beta^n a^{n-2}\nu, a^{n-1}a^{n-2}\nu)$$

from the graph G to the graph (Π, a^0, β^0) underlying the category Π^0.

By the universal property of $P(G)$, $\tilde{a}'$ extends into a quasi-functor $\tilde{a}''$ from $P(G)^0$ to Π^0, and $\tilde{a}'': P(G) \to \Pi$ is also a morphism, due to the pointwise definition of $P(G)^{i+1}$. Moreover, $\tilde{a}''$ is seen to be compatible with r. Hence it factors uniquely into an $(n\text{-}1)$-fold functor $\tilde{a}: \Pi \to \Pi$. The



equality $\tilde{a}\tilde{a}\hat{r} = \tilde{a}\hat{r}$ implies $\tilde{a}\tilde{a} = \tilde{a}$. Similarly, there is an $(n\text{-}1)$-fold functor $\tilde{\beta}: \Pi \to \Pi$ such that

$$\tilde{\beta}\hat{r}(\nu) = \hat{r}(\beta^{n-1}\beta^{n-2}\nu, a^n\beta^{n-2}\nu)$$

for each arrow ν of G, and we have

$$\tilde{\beta}\tilde{\beta} = \tilde{\beta}, \quad \tilde{\beta}\tilde{a} = \tilde{a}, \quad \tilde{a}\tilde{\beta} = \tilde{\beta}.$$

These equalities mean that $(\underline{\Pi}, \tilde{a}, \tilde{\beta})$ is a graph, in which a block π of Π is an arrow $\pi: \tilde{a}(\pi) \to \tilde{\beta}(\pi)$.

(iii) Let $P(\Pi)^{n-1}$ be the free quasi-category of all paths $<\pi_k, \dots, \pi_0>$ of the graph $(\underline{\Pi}, \tilde{a}, \tilde{\beta})$ (equipped with the concatenation). A block π of Π is identified to the path $<\pi>$. On the set $\underline{P}(\Pi)$ of these paths, we consider the relation r' defined by:

$<\hat{r}(u'), \hat{r}(u)> \sim \hat{r}(u' \circ_{n-2} u)$, if u and u' are objects of A^n,

$<\hat{r}(v'), \hat{r}(v)> \sim \hat{r}(v' \circ_{n-2} v>$, if v and v' are objects of A^{n-1},

$<\hat{r}(\beta^{n-2}a), \hat{r}(\beta^n a)\circ_0 \hat{r}(a^{n-1}a)> \sim <\hat{r}(\beta^{n-1}a)\circ_0 \hat{r}(a^n a), \hat{r}(a^{n-2}a)>$

for each block a of A,

$\hat{r}(t' \circ_{n-1} t) \sim <\hat{r}(t')\circ_0 \hat{r}(a^n t), \hat{r}(\beta^n t')\circ_0 \hat{r}(t)>$,

if $t' \circ_{n-1} t$ is defined in $|A^{n-2}|^{n-1}$,

$\hat{r}(t'' \circ_n t) \sim <\hat{r}(\beta^{n-1} t'')\circ_0 \hat{r}(t), \hat{r}(t'')\circ_0 \hat{r}(a^{n-1} t)>$,

if $t'' \circ_n t$ is defined in $|A^{n-2}|^n$.

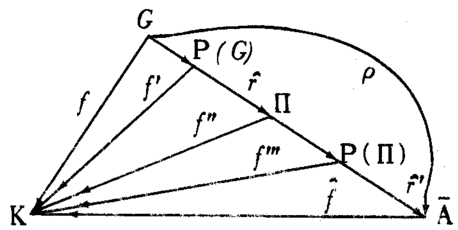
(iv) For $i < n-2$, there is also a category $P(\Pi)^i$ on $P(\Pi)$ whose composition is deduced pointwise from that of Π^i. There exists an n-fold category $\bar{A}$ quasi-quotient of $P(\Pi) = (P(\Pi)^0, \dots, P(\Pi)^{n-2}, P(\Pi)^{n-1})$

by r', the canonical morphism being $\hat{r}': \mathrm{P}(\Pi) \to \bar{\mathrm{A}}$. The composite map ρ:

$$G \hookrightarrow \mathrm{P}(G) \xrightarrow{\hat{r}} \Pi \hookrightarrow \mathrm{P}(\Pi) \xrightarrow{\hat{r}'} \bar{\mathrm{A}}$$

gives an admissible morphism $\rho: G \to \bar{\mathrm{A}}$ due to the construction of $\hat{r}$ and $\hat{r}'$.

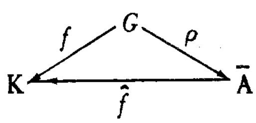
(v) $\rho: G \to \bar{\mathrm{A}}$ is a universal admissible morphism. Indeed, let $f: G \to \mathrm{K}$ be an admissible morphism. As f satisfies (i), it extends into a (quasi-) functor $f': \mathrm{P}(G)^0 \to \mathrm{K}^0$; by (iii), $f': \mathrm{P}(G) \to \mathrm{K}^{0,\dots,n-2}$ is a morphism which is compatible with r (according to (ii)). By the universal property of Π, there exists a factor $f'': \Pi \to \mathrm{K}^{0,\dots,n-2}$ of f' through $\hat{r}$. The con-



dition (iv) implies that f'' is a morphism of graphs

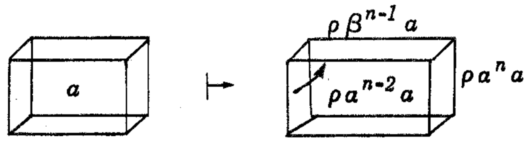
$$f'': (\Pi, \tilde{\alpha}, \tilde{\beta}) \to (\mathrm{K}^{n-1}, \alpha^{n-1}, \beta^{n-1})$$

so that it extends into a (quasi-)functor $f''': \mathrm{P}(\Pi)^{n-1} \to \mathrm{K}^{n-1}$, defining a morphism $f''': \mathrm{P}(\Pi) \to \mathrm{K}$ (the composition of $\mathrm{P}(\Pi)^i$ being deduced pointwise from that of Π^i). The conditions (v, vi, vii, viii) mean that f''' is compatible with r'. Hence f''' factors through $\hat{r}'$ into an n-fold functor $\hat{f}: \bar{\mathrm{A}} \to \mathrm{K}$; and $\hat{f}$ is the unique n-fold functor rendering commutative the diagram



2° *There exists an $(n+1)$-fold functor $l: \mathrm{A} \to Cub\bar{\mathrm{A}}$:*

$$a \mapsto l(a) = (\rho\beta^n a, \rho\beta^{n-1}a, \rho\beta^{n-2}a, \rho\alpha^{n-2}a, \rho\alpha^{n-1}a, \rho\alpha^n a)$$



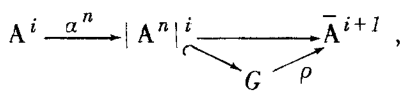
where $\rho: G \to \bar{\mathrm{A}}$ is a fixed universal admissible morphism.

a) As ρ satisfies (vi') and as $l(a)$ is the cube $c_\rho(a)$ considered in this condition, the map l is well-defined.

b) Suppose $i < n-2$. The composition of $(Cub\bar{A})^i$ being deduced pointwise from that of $\bar{A}^{i+1}$, for $l: A^i \to (Cub\bar{A})^i$ to be a functor, it suffices that the maps

$$\rho\alpha^n,\ \rho\beta^n,\ \rho\alpha^{n-1},\ \rho\beta^{n-1},\ \rho\alpha^{n-2},\ \rho\beta^{n-2}$$

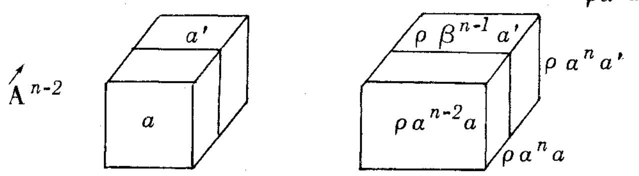
sending a onto each of the six factors of the cube $l(a)$ define functors $A^i \to \bar{A}^{i+1}$. Since $\alpha^n : A^i \to |A^n|^i$ is a functor and axiom (iii) is satisfied, $\rho\alpha^n$ defines the composite functor:



and similarly for the five other maps.

c) $l: A^{n-2} \to (Cub\bar{A})^{n-2}$ is a functor. Indeed, suppose $a' \circ_{n-2} a$ defined in A^{n-2}. The composition of $(Cub\bar{A})^{n-2}$ being deduced «laterally pointwise» from that of $\bar{A}^{n-1}$, there exists $l(a')\circ_{n-2} l(a) =$

$$(\rho\beta^n a' \circ_{n-1} \rho\beta^n a,\ \rho\beta^{n-1} a' \circ_{n-1} \rho\beta^{n-1} a,\ \rho\beta^{n-2} a',\ \rho\alpha^{n-2} a,\ \rho\alpha^{n-1} a' \circ_{n-1} \rho\alpha^{n-1} a,\ \rho\alpha^n a' \circ_{n-1} \rho\alpha^n a).$$



Now, by (v),

$$\rho\alpha^n a' \circ_{n-1} \rho\alpha^n a = \rho(\alpha^n a' \circ_{n-2} \alpha^n a) = \rho\alpha^n(a' \circ_{n-2} a),$$

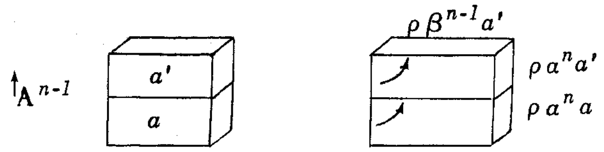
which is also the right lateral face of the cube $l(a' \circ_{n-2} a)$. Same proof for the other lateral faces. Finally,

$$\rho\alpha^{n-2} a = \rho\alpha^{n-2}(a' \circ_{n-2} a)$$

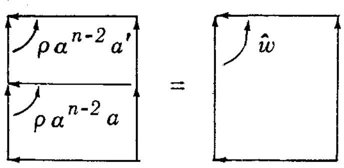
is the front face of both $l(a')\circ_{n-2} l(a)$ and $l(a' \circ_{n-2} a)$, whose back face is $\rho\beta^{n-2} a'$. Hence, $l(a' \circ_{n-2} a) = l(a')\circ_{n-2} l(a)$.

d) $l: A^{n-1} \to (Cub\bar{A})^{n-1}$ is a functor. Indeed, suppose $a' \circ_{n-1} a$ defined. The composition of $(Cub\bar{A})^{n-1}$ being the «vertical» composition, the composite

$$l(a') \boxminus l(a) = (\rho\beta^n a' \circ_0 \rho\beta^n a,\ \rho\beta^{n-1}a',\ \hat{w}',\ \hat{w},\ \rho\alpha^{n-1}a\ ,\ \rho\alpha^n a' \circ_0 \rho\alpha^n a)$$



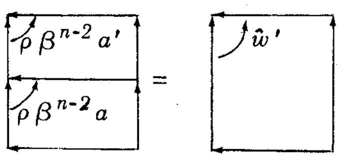
is defined; $\hat{w}$ is the 2-cell of the vertical composite up-square



which, by (vii), is equal to

$$\rho(\alpha^{n-2}a' \circ_{n-1} \alpha^{n-2}a) = \rho\alpha^{n-2}(a' \circ_{n-1} a),$$

and this is the 2-cell of the front face of $l(a' \circ_{n-1} a)$. Similarly, $\hat{w}'$ is the 2-cell of the back face of $l(a' \circ_{n-1} a)$.

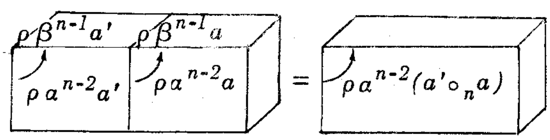


Using (ii), we get

$$\rho\alpha^n a' \circ_0 \rho\alpha^n a = \rho(\alpha^n a' \circ_{n-1} \alpha^n a) = \rho\alpha^n(a' \circ_{n-1} a)$$

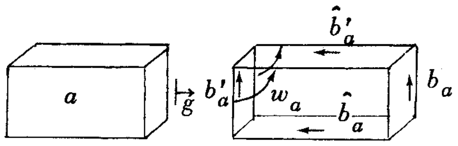
and idem with β instead of α. Hence $l(a') \boxminus l(a) = l(a' \circ_{n-1} a)$.

e) The same proof (using (viii) instead of (vii)) shows that l defines the functor $l: \mathrm{A}^n \to (Cub\overline{\mathrm{A}})^n$: if $a' \circ_n a$ is defined,

$l(a') \boxdot l(a) =$  $= l(a' \circ_n a)$.

3° *$l: \mathrm{A} \to Cub\overline{\mathrm{A}}$ is the liberty morphism defining $\overline{\mathrm{A}}$ as a free object generated by* A : Let B be an n-fold category and $g: \mathrm{A} \to Cub\mathrm{B}$ an $(n+1)$-fold functor. The cube $g(a)$ of B , for any block a of A , is written

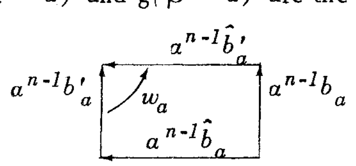
$$g(a) = (b'_a, \hat{b}'_a, w'_a, w_a, \hat{b}_a, b_a).$$

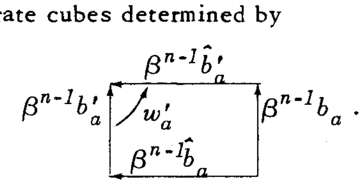


In particular,

$$g(\alpha^n a) = b_a^{\boxbar},\quad g(\alpha^{n-1}a) = \hat{b}_a^{\boxminus},\quad g(\beta^n a) = b_a'^{\boxbar},\quad g(\beta^{n-1}a) = \hat{b}_a'^{\boxminus},$$

$g(\alpha^{n-2}a)$ and $g(\beta^{n-2}a)$ are the degenerate cubes determined by



and


.

a) There is an admissible morphism $f: G \to B$ mapping ν onto the diagonal $\partial g(\nu)$ of the cube $g(\nu)$.

(i) As $\partial g(a) = (\hat{b}'_a \circ_0 b_a) \circ_{n-1} w_a$, we have

$$f(\alpha^n a) = \partial g(\alpha^n a) = b_a,\quad f(\alpha^{n-1}a) = \hat{b}_a,\quad f(\alpha^{n-2}a) = w_a,$$

$$f(\beta^n a) = b'_a,\quad f(\beta^{n-1}a) = \hat{b}'_a,\quad f(\beta^{n-2}a) = w'_a,$$

so that

$$c_f(a) = (f\beta^n a, f\beta^{n-1}a, f\beta^{n-2}a, f\alpha^{n-2}a, f\alpha^{n-1}a, f\alpha^n a) =$$
$$= (b'_a, \hat{b}'_a, w'_a, w_a, \hat{b}_a, b_a) = g(a)$$

is a cube, and f satisfies (vi). It also satisfies (i) and (iv), because it is more precisely defined by

$$f(u) = b_u,\quad f(v) = \hat{b}_v,\quad f(t) = w_t,$$

where u, v, t always denote objects of A^n, A^{n-1}, A^{n-2} respectively.

(ii) $|A^n|^{n-1} \hookrightarrow G \xrightarrow{f} B^0$ is a functor. Indeed, if $u' \circ_{n-1} u$ is defined,

$$g(u' \circ_{n-1} u) = g(u') \boxminus g(u) = b_{u'}^{\boxbar} \boxminus b_u^{\boxbar} = (b_{u'} \circ_0 b_u)^{\boxbar}$$

so that

$$f(u' \circ_{n-1} u) = \partial g(u' \circ_{n-1} u) = b_{u'} \circ_0 b_u = f(u') \circ_0 f(u).$$

Similarly, $|A^{n-1}|^n \hookrightarrow G \xrightarrow{f} B^0$ is a functor, since

$$g(v' \circ_n v) = g(v') \boxbar g(v) = (\hat{b}_{v'} \circ_0 \hat{b}_v)^{\boxminus},$$

so
$$f(v' \circ_n v) = \hat{b}_{v'} \circ_0 \hat{b}_v = f(v') \circ_0 f(v).$$

That $|A^n|^{n-2} \hookrightarrow G \xrightarrow{f} B^{n-1}$ and $|A^{n-1}|^{n-2} \hookrightarrow G \xrightarrow{f} B^{n-1}$ are functors is deduced from the equalities

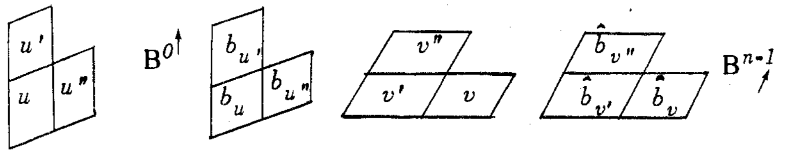
$$g(u'' \circ_{n-2} u) = g(u'') \circ_{n-2} g(u) = b^{\boxdot}_{u''} \circ_{n-2} b^{\boxdot}_{u} = (b_{u''} \circ_{n-1} b_u)^{\boxdot},$$

$$g(v'' \circ_{n-2} v) = g(v'') \circ_{n-2} g(v) = \hat{b}^{\boxminus}_{v''} \circ_{n-2} \hat{b}^{\boxminus}_{v} = (\hat{b}_{v''} \circ_{n-1} \hat{b}_v)^{\boxminus},$$

giving

$$f(u'' \circ_{n-2} u) = b_{u''} \circ_{n-1} b_u = f(u'') \circ_{n-1} f(u)$$

and $f(v'' \circ_{n-2} v) = f(v'') \circ_{n-1} f(v)$. Hence, f verifies (ii) and (v).



(iii) For $i < n-2$, there is a functor $\partial : (Cub\mathrm{B})^i \to \mathrm{B}^{i+1}$, since the pointwise deduction of the composition of $(Cub\mathrm{B})^i$ from that of B^{i+1}, and the permutability axiom in B imply:

$$\partial(c_1 \circ_i c) = ((\hat{b}'_1 \circ_{i+1} \hat{b}') \circ_0 (b_1 \circ_{i+1} b)) \circ_{n-1} (w_1 \circ_{i+1} w) =$$

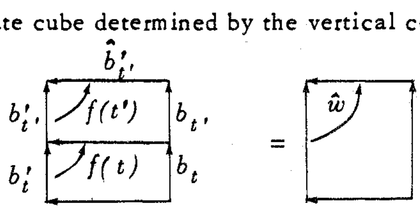
$$= ((\hat{b}'_1 \circ_0 b_1) \circ_{n-1} w_1) \circ_{i+1} ((\hat{b}' \circ_0 b) \circ_{n-1} w) = \partial c_1 \circ_{i+1} \partial c,$$

if $c_1 \circ_i c$ is defined in $(Cub\mathrm{B})^i$, with $c = (b', \hat{b}', w', w, \hat{b}, b)$ and idem for c_1 with indices. The composite functor

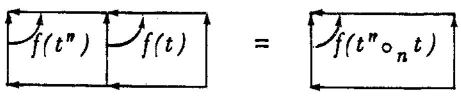
$$|A^j|^i \xrightarrow{g} (Cub\mathrm{B})^i \xrightarrow{\partial} \mathrm{B}^{i+1}$$

is defined by a restriction of f, for $j = n$, $n-1$ or $n-2$. So f satisfies (iii).

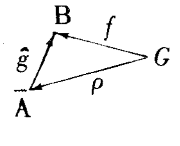
(iv) If $t' \circ_{n-1} t$ is defined in $|A^{n-2}|^{n-1}$, then $g(t' \circ_{n-1} t) = g(t') \boxminus g(t)$ is the degenerate cube determined by the vertical composite up-square

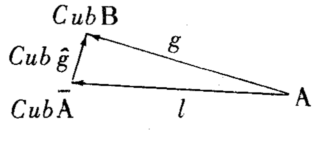


so that its diagonal $f(t' \circ_{n-1} t)$ is the 2-cell $\hat{w}$ of this composite. Therefore f satisfies (vii) and (by a similar proof) (viii).



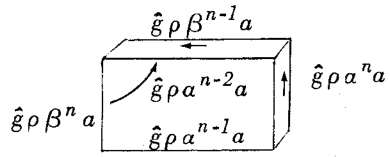
b) This proves that $f: G \to \mathrm{B}$ is an admissible morphism; so it factors uniquely through the universal admissible morphism $\rho: G \to \bar{\mathrm{A}}$ into an n-fold functor $\hat{g}: \bar{\mathrm{A}} \to \mathrm{B}$.





(i) For each block a of A, the cube

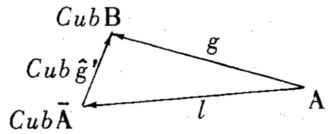
$\hat{g}l(a) =$



is identical to $c_f(a) = g(a)$ (see a), since $f = \hat{g}\rho$; so

$$(g: \mathrm{A} \to Cub\mathrm{B}) = (\mathrm{A} \xrightarrow{l} Cub\bar{\mathrm{A}} \xrightarrow{Cub\hat{g}} Cub\mathrm{B}).$$

(ii) Let $\hat{g}': \bar{\mathrm{A}} \to \mathrm{B}$ be an n-fold functor, rendering commutative the diagram



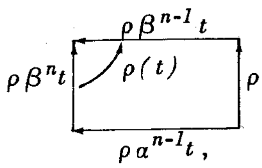
We are going to prove that $\hat{g}'\rho = f$; the unicity of the factor of f through ρ then implies $\hat{g}' = \hat{g}$. Indeed, for an object u of A^n, from the equalities

$$l(u) = \rho(u)^{\boxplus} \quad \text{and} \quad g(u) = \hat{g}'l(u) = (\hat{g}'\rho(u))^{\boxplus}$$

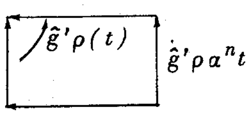
we deduce $f(u) = \partial g(u) = \hat{g}'\rho(u)$. If v is an object of A^{n-1}, then

$$l(v) = \rho(v)^{\boxminus}, \quad g(v) = (\hat{g}'\rho(v))^{\boxminus} \quad \text{and} \quad f(v) = \hat{g}'\rho(v).$$

If t is an object of A^{n-2}, the degenerate cubes $l(t)$ and $g(t) = \hat{g}'l(t)$ are determined by the up-squares



and



so that $f(t) = \partial g(t) = \hat{g}'\rho(t)$. Hence, $\hat{g}'\rho = f$, and $\hat{g}' = \hat{g}$.

REMARK. To prove that $\hat{g}'\rho = f$, we could have used the relations

$$\partial\, Cub\, \hat{g}' = \hat{g}'\bar{\partial} \quad \text{and} \quad \bar{\partial}\, l(\nu) = \rho(\nu) \quad \text{for each } \nu \text{ in } G,$$

where $\bar{\partial}$ is the diagonal map from $Cub\bar{\mathrm{A}}$ to $\bar{\mathrm{A}}$.

4° For each $(n+1)$-fold small category A, we choose a universal admissible morphism $\rho_{\mathrm{A}} : G_{\mathrm{A}} \to \bar{\mathrm{A}}$ (where G_{A} is the graph G above), for example the canonical one constructed in 1-c; by the preceding proof, $\bar{\mathrm{A}}$ is a free object generated by A with respect to the *Cube* functor. $\bar{\mathrm{A}}$ will be called the *multiple category of lax links of* A, denoted by $LaxLk\mathrm{A}$. The corresponding left adjoint

$$LaxLk_{n+1,n} : Cat_{n+1} \to Cat_n \quad \text{of} \quad Cub_{n,n+1} : Cat_n \to Cat_{n+1}$$

maps $h: \mathrm{A} \to \mathrm{A}'$ onto the unique n-fold functor

$$LaxLkh : LaxLk\mathrm{A} \to LaxLk\mathrm{A}'$$

satisfying

$$LaxLkh(\rho_{\mathrm{A}}\nu) = \rho_{\mathrm{A}'}h(\nu)$$

for each object ν of A^n, A^{n-1} or A^{n-2}. ∇

By iteration, for each integer $m > n$, we define the functors

$$Cub_{n,m} = (\, Cat_n \xrightarrow{Cub_{n,n+1}} Cat_{n+1} \to \dots \to Cat_{m-1} \xrightarrow{Cub_{m-1,m}} Cat_m),$$

$$LaxLk_{m,n} = (\, Cat_m \xrightarrow{LaxLk_{m,m-1}} Cat_{m-1} \to \dots \to Cat_{n+1} \xrightarrow{LaxLk_{n+1,n}} Cat_n \,).$$

DEFINITION. $Cub_{n,m}$ is called the *Cube* functor from Cat_n to Cat_m and $LaxLk_{m,n}$ the *LaxLink functor from* Cat_m *to* Cat_n.

COROLLARY. *The Cube functor from Cat_n to Cat_m admits as a left adjoint the LaxLink functor from Cat_m to Cat_n for any integer $m > n > 1$.*

This results from Proposition 1, since a composite of left adjoint functors is a left adjoint functor of the composite. ∇

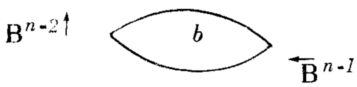
REMARK. If B is an n-fold category, in the $2n$-fold category $Cub_{n,2n}\mathrm{B}$ the $2i$-th and $(2i+1)$-th compositions are deduced respectively «verticaly» and «horizontaly» from the composition of B^i.

C. Cylinders of a multiple category.

We recall that an *n-category* is an n-fold category K whose objects for the last category K^{n-1} are also objects for K^{n-2}.

The full subcategory of Cat_n whose objects are the (small) n-categories is denoted by n-Cat. It is reflective and coreflective in Cat_n. More precisely, the insertion functor $n\text{-}Cat \hookrightarrow Cat_n$ admits:

- A right adjoint $\mu_n : Cat_n \to n\text{-}Cat$ mapping the n-fold category B onto *the greatest n-category included in* B, which is the n-fold subcategory of B formed by those blocks b of B such that $\alpha^{n-1} b$ and $\beta^{n-1} b$ are also objects of B^{n-2} (those blocks are called *n-cells of* B).



- A left adjoint $\lambda_n : Cat_n \to n\text{-}Cat$, whose existence follows from the general existence Theorem of free objects [6] (its hypotheses are satisfied, n-Cat being complete and each infinite subcategory of an n-category K generating an equipotent sub-n-category of K). In fact, $\lambda_n(B)$ is the n-category quasi-quotient of B by the relation:

$u \sim \alpha^{n-2} u$ for each object u of B^{n-1}.

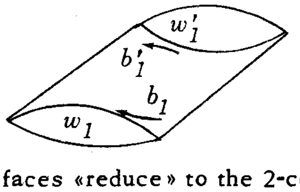
1° The multiple category CylB.

Let n be an integer, $n \geqslant 2$, and B be an n-fold category.

DEFINITION. The greatest $(n+1)$-category included in the $(n+1)$-fold category CubB of cubes of B is called the *$(n+1)$-category of cylinders of* B, denoted by Cyl B.

So a cylinder of B is a cube of the form

$$q_1 = (\beta^0 b'_1, b'_1, w'_1, w_1, b_1, \alpha^0 b_1)$$



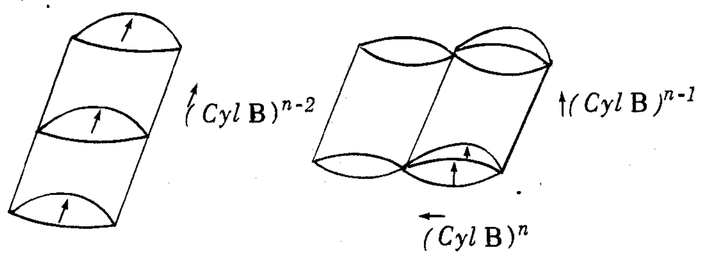
its front and back faces «reduce» to the 2-cells w_1 and w'_1 of the double category $\mathrm{B}^{n-1,0}$. We will write more briefly

$$q_1 = [\, b'_1, w'_1, w_1, b_1 \,].$$

The composition of $(Cyl\,\mathrm{B})^i$, for $i < n-2$, is deduced pointwise from that of B^{i+1}. The $(n-2)$-th composition of $Cyl\,\mathrm{B}$ is:

$$q_2 \circ_{n-2} q_1 = [\, b'_2 \circ_{n-1} b_1, w'_2, w_1, b_2 \circ_{n-1} b_1 \,] \quad \text{iff} \quad w'_1 = w_2,$$

so that the objects of $(Cyl\,\mathrm{B})^{n-2}$ are the degenerate cylinders «reduced to their front face» $[\beta^{n-1} w, w, w, \alpha^{n-1} w]$, denoted by w°, for any 2-cell w of $\mathrm{B}^{n-1,0}$.



The $(n-1)$-th composition of $Cyl\,\mathrm{B}$ is the vertical one:

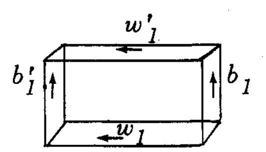
$$q_3 \boxminus q_1 = [\, b'_3, w'_3 \circ_{n-1} w'_1, w_3 \circ_{n-1} w_1, b_1 \,] \quad \text{iff} \quad b'_1 = b_3,$$

and its objects are the degenerate squares $b^{\boxminus}$, for any block b of B. The n-th composition of $Cyl\,\mathrm{B}$ is the horizontal one:

$$q_4 \boxplus q_1 = [\, b'_4 \circ_0 b'_1, w'_4 \circ_0 w'_1, w_4 \circ_0 w_1, b_4 \circ_0 b_1 \,] \quad \text{iff} \quad \beta^0 b'_1 = \alpha^0 b_4$$

(which is deduced pointwise from the composition of B^0); its objects are the degenerate squares $e^{\boxminus}$, for any object e of B^0.

REMARKS. 1° The cylinder q_1 of B may be identified with the square



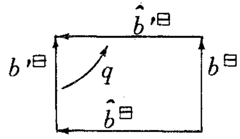
of B^{n-1}, in which w_1 and w'_1 are 2-cells of $\mathrm{B}^{n-1,0}$; in this way, $Cyl\,\mathrm{B}$ is identified with the greatest $(n+1)$-category included in

$$Sq(\mathrm{B}^{n-1,1,\dots,n-2,0})^{0,\dots,n-3,n-1,n,n-2}.$$

2° $(Cub\mathrm{B})^{n-1,n}$ is identified with the double category of up-squares of the 2-category $(Cyl\,\mathrm{B})^{n-1,n}$ by identifying the cube

$$c = (b', \hat{b}', w', w, \hat{b}, b) \quad \text{of } \mathrm{B}$$

with the up-square



where $q = [\,b' \circ_0 \hat{b}, w', w, \hat{b}' \circ_0 b\,]$.

2° The functor Cylinder.

If $f: \mathrm{B} \to \mathrm{B}'$ is an n-fold functor, there is an $(n+1)$-fold functor

$$Cyl f: Cyl\mathrm{B} \to Cyl\mathrm{B}': [\,b'_1, w'_1, w_1, b_1\,] \mapsto [\,f b'_1, f w'_1, f w_1, f b_1\,]$$

restriction of $Cub\, f$. This determines a functor

$$Cyl_{n,n+1}: Cat_n \to Cat_{n+1}: f \mapsto Cyl f,$$

called the *Cylinder functor from* Cat_n *to* Cat_{n+1}. Remark that this functor is equal to the composite

$$Cat_n \xrightarrow{Cub_{n,n+1}} Cat_{n+1} \xrightarrow{\mu_{n+1}} (n+1)\text{-}Cat \hookrightarrow Cat_{n+1}$$

where μ_{n+1} is the right adjoint of the insertion.

PROPOSITION 2. *The functor* $LaxLk_{n+1,n}: Cat_{n+1} \to Cat_n$ *is equivalent to a left inverse of* $Cyl_{n,n+1}: Cat_n \to Cat_{n+1}$.

PROOF. We are going to prove that, *for each n-fold category* B, *the n-fold category* $LaxLk(Cyl\,\mathrm{B})$ *is canonically isomorphic with* B. It follows that, in the construction of the *LaxLink* functor (Proof, Proposition 1), we may choose B as the free object generated by $Cyl\mathrm{B}$, for each n-fold category B (remark that $Cyl\,\mathrm{B}$ determines uniquely B); in this way, we obtain the identity as the composite

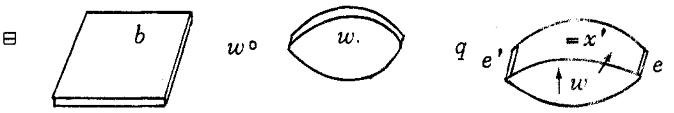
$$Cat_n \xrightarrow{Cyl_{n,n+1}} Cat_{n+1} \xrightarrow{LaxLk_{n+1,n}} Cat_n.$$

To prove the assertion, we take up the notations of Proposition 1, Proof, with $\mathrm{A} = Cyl\,\mathrm{B}$, $\bar{\mathrm{A}} = LaxLk\,\mathrm{A}$ and $\rho: G \to \bar{\mathrm{A}}$ the universal admissible morphism.

1° $\bar{\mathrm{A}}$ *is generated by the blocks* $\rho(b^{\boxminus})$, *for any block* b *of* B. Indeed, the arrows of the graph G are the objects $b^{\boxminus}$ of the vertical category of cylinders A^{n-1} and the objects w° of the category A^{n-2} (each object of A^n being also an object of A^{n-1}); the n-fold category $\bar{A}$ is generated by the blocks

$\rho(b^{\boxminus})$ for any block b of B and

$\rho(w^\circ)$ for any 2-cell w of the double category $B^{n-1,0}$.



Now, given the 2-cell w, there is a cylinder $q = [x', x', w, w]$ of B, where $x' = \beta^{n-1} w : e \to e'$ in B^0. Applying to q (considered as a cube) the axiom (vi) satisfied by the admissible morphism ρ, we get

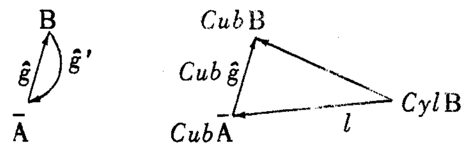
$$\rho(x'^{\boxminus}) \circ_{n-1} (\rho(e'^{\boxminus}) \circ_0 \rho(w^\circ)) = (\rho(e'^{\boxminus}) \circ_0 \rho(w^{\boxminus})) \circ_{n-1} \rho(x'^\circ);$$

as $\rho(x'^{\boxminus})$ is an object of $\bar{A}^{n-1}$ and $\rho(e^{\boxminus})$ an object of $\bar{A}^0$ (axioms (i) and (iv)), this equality gives $\rho(w^\circ) = \rho(w^{\boxminus})$. Hence $\bar{A}$ is generated by the sole blocks $\rho(b^{\boxminus})$.

2° a) To the insertion $Cyl\,B \hookrightarrow Cub\,B$ is associated (by the adjunction between the *Cube* and *Lax Link* functors, Proposition 1) the n-fold functor $\hat{g} : \bar{A} \to B$ such that

$$\hat{g}\rho(b^{\boxminus}) = \partial b^{\boxminus} = b \quad \text{for each block } b \text{ of } B$$

(this determines uniquely $\hat{g}$ by 1).



b) There is also an n-fold functor

$$\hat{g}' : B \to \bar{A} : b \mapsto \rho(b^{\boxminus}).$$

Indeed, $\hat{g}'$ is the composite functor

$$B \xrightarrow{-^{\boxminus}} |(Cyl\,B)^{n-1}|^{n,0,\dots,n-2} \xrightarrow{\rho'} \bar{A}, \qquad \rho' = \rho \circ (\,\cdot \to G)$$

where $-^{\boxminus}$ is the canonical isomorphism $b \mapsto b^{\boxminus}$ onto the n-fold category of objects of $(Cub\mathrm{B})^{n-1}$ (Section A-3) and where ρ' is a functor according to the axioms (ii, iii, v) satisfied by ρ.

c) $\hat{g}'$ is the inverse of $\hat{g}$. Indeed, for each block b of B we have

$$\hat{g}\hat{g}'(b) = \hat{g}\,\rho(b^{\boxminus}) = b \quad \text{and} \quad \hat{g}'\hat{g}(\rho(b^{\boxminus})) = \hat{g}'(b) = \rho(b^{\boxminus}).$$

These equalities mean that $\hat{g}\hat{g}'$ is an identity, as well as $\hat{g}'\hat{g}$, since the blocks $\rho(b^{\boxminus})$ generate $\bar{\mathrm{A}}$ by 1. So $\hat{g}' = \hat{g}^{-1}$.

3° Let $f: \mathrm{B} \to \mathrm{B}'$ be an n-fold functor, and

$$\hat{g}'_{\mathrm{B}'}: \mathrm{B}' \to LaxLk(Cyl\mathrm{B}'): b' \mapsto \rho_{\mathrm{B}'}(b'^{\boxminus})$$

the isomorphism similar to $\hat{g}'$. The square

$$\begin{array}{ccc} LaxLk(Cyl\mathrm{B}') & \xleftarrow{\ \hat{g}'_{\mathrm{B}'}\ } & \mathrm{B}' \\ \uparrow{\scriptstyle LaxLkf} & & \uparrow{\scriptstyle f} \\ \bar{\mathrm{A}} & \xleftarrow{\ \hat{g}'\ } & \mathrm{B} \end{array}$$

is commutative, since, for each block b of B,

$$LaxLkf(\hat{g}'(b)) = LaxLkf(\rho(b^{\boxminus})) = \rho_{\mathrm{B}'}(f(b)^{\boxminus}) = \hat{g}'_{\mathrm{B}'}f(b)$$

(by the construction of *LaxLink*, Proposition 1). This proves that the functor

$$Cat_n \xrightarrow{Cyl_{n,n+1}} Cat_{n+1} \xrightarrow{LaxLk_{n+1,n}} Cat_n$$

is equivalent to an identity. ∇

COROLLARY 1. *If $h: Cyl\mathrm{B} \to Cyl\mathrm{B}'$ is an $(n+1)$-fold functor, there exists a unique n-fold functor $f: \mathrm{B} \to \mathrm{B}'$ such that $h = Cylf$.*

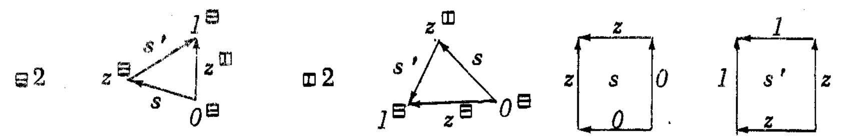
Indeed, this expresses the fact that B is a free object generated by $Cyl\mathrm{B}$ (Proof above) with respect to the *LaxLink* functor. ∇

COROLLARY 2. *For each integer $m > n > 1$, the LaxLink functor from Cat_n to Cat_m is equivalent to a left inverse of the functor $Cyl_{n,m}$ =*

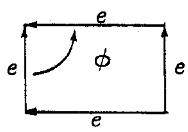
$$(Cat_n \xrightarrow{Cyl_{n,n+1}} Cat_{n+1} \to \dots \to Cat_{m-1} \xrightarrow{Cyl_{m-1,m}} Cat_m).\ \nabla$$

REMARK. Proposition 1 may be compared with the fact that the *Link* functor is equivalent to a left inverse of the *Square* functor (Proposition 5

[5]). However the *LaxLink* functor is not equivalent to a left inverse of the *Cube* functor. Indeed, B and $LaxLk(Cub\mathrm{B})$ are isomorphic iff each $(n+1)$-fold functor $h: Cub\mathrm{B} \to Cub\mathrm{B}'$ is of the form $Cub f$. A counter example is obtained as follows. Let B be the double category $(2, \underline{2}^{dis})$ so that $Cub\mathrm{B} = (\underline{\square 2}^{dis}, \boxminus 2, \boxplus 2)$, where $2 = 1 \xleftarrow{z} 0$,



Let B' be the 2-category (Z_2, Z_2), where Z_2 is the group $\{e, \phi\}$ of unit e. The unique triple functor $h: Cub\mathrm{B} \to Cub\mathrm{B}'$ mapping s and s' onto the degenerate cube



is not of the form $Cub f: Cub\mathrm{B} \to Cub\mathrm{B}'$ for any double functor $f: \mathrm{B} \to \mathrm{B}'$.

3° The functor *n-Cyl*.

The *Cylinder* functor from Cat_n to Cat_{n+1} taking its values in $(n+1)$-Cat, it admits as a restriction a functor

$$n\text{-}Cyl: n\text{-}Cat \to (n+1)\text{-}Cat.$$

PROPOSITION 3. *The functor n-Cyl: n-Cat → (n+1)-Cat admits a left adjoint which is equivalent to a left inverse of n-Cyl.*

PROOF. By definition of the *Cylinder* functor, *n-Cyl* is equal to the composite functor

$$n\text{-}Cat \hookrightarrow Cat_n \xrightarrow{Cub_{n,n+1}} Cat_{n+1} \xrightarrow{\mu_{n+1}} (n+1)\text{-}Cat,$$

where μ_{n+1} is the right adjoint of the insertion. So this functor admits as a left adjoint the composite functor

$$(n+1)\text{-}Cat \hookrightarrow Cat_{n+1} \xrightarrow{LaxLk_{n+1,n}} Cat_n \xrightarrow{\lambda_n} n\text{-}Cat,$$

where λ_n is a left adjoint of the insertion (which exists, as seen above). The free object $\overline{\mathrm{K}}$ generated by an $(n+1)$-category K with respect to

n-Cyl is the n-category reflection of the n-fold category $Lax\,Lk\mathrm{K}$. In particular, if $\mathrm{K} = Cyl\mathrm{B}$ for some n-category B, then $Lax\,Lk\mathrm{K}$ is isomorphic with B (by Proposition 2), hence is an n-category, and $\bar{\mathrm{K}}$ is also isomorphic with B. ∇

COROLLARY. *The composite functor* (n,m)-Cyl =

$$(\, n\text{-}Cat \xrightarrow{n\text{-}Cyl} (n+1)\text{-}Cat \to \dots \to (m-1)\text{-}Cat \xrightarrow{(m-1)\text{-}Cyl} m\text{-}Cat\,)$$

admits a left adjoint equivalent to a left inverse of (n,m)-Cyl. ∇

D. Some applications.

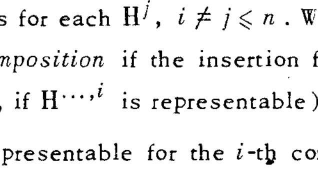
1° Existence of generalized limits.

An $(n+1)$-fold category H is *representable* (Section C-2 [4]) if the insertion functor $|\mathrm{H}|^n \hookrightarrow \mathrm{H}^n$ admits a right adjoint, where $|\mathrm{H}|^n$ is the subcategory of H^n formed by those blocks of H which are objects for the n first categories H^i; in this case, the greatest $(n+1)$-category included in H is also representable.

Remark that the order of the n first compositions of H does not intervene: H is representable iff so is $\mathrm{H}^{\gamma(0),\dots,\gamma(n-1),n}$ for any permutation γ of $\{0,\dots,n-1\}$. More generally:

DEFINITION. For each $i < n$, we denote by $\mathrm{H}^{\dots,i}$ the $(n+1)$-fold category $\mathrm{H}^{0,\dots,i-1,i+1,\dots,n,i}$ obtained by «putting the i-th composition at the last place», by $|\mathrm{H}|^i$ the subcategory of H^i formed by the blocks of H which are objects for each H^j, $i \neq j \leqslant n$. We say that H *is representable for the i-th composition* if the insertion functor $|\mathrm{H}|^i \hookrightarrow \mathrm{H}^i$ admits a right adjoint (i.e., if $\mathrm{H}^{\dots,i}$ is representable).

So, H is representable for the i-th composition iff, for each object e of H^i, there exists a morphism $\eta e : u \to e$ in H^i with u a vertex



of H, through which factors uniquely any morphism $\eta : u' \to e$ of H^i with

u' a vertex of H , so that

$$\eta = \eta e \circ_i /\eta/ \ , \quad \text{where } /\eta/ : u' \to u \text{ in } |H|^i .$$

ηe is called *an i-representing block for e*.

From Proposition 11 [4], we deduce that, if H is representable for the i-th composition and if $|H|^i$ is (finitely) complete, then the n-fold category $|H^i|^{0,\dots,i-1,i+1,\dots,n}$ formed by the objects of H^i is $H^{\dots,i}$-wise (finitely) complete.

Let B be an n-fold category, for an integer $n > 1$. The three following propositions are concerned with the representability of SqB, CylB and CubB for the three last compositions. From the isomorphism

$$B^{\dots,0} \xrightarrow{\boxminus} |(CubB)^{n-1}|^{0,\dots,n-2,n} = |(SqB)^{n-1}|^{0,\dots,n-2,n} : b \mapsto b^{\boxminus}$$

it follows that:

- $|CubB|^n = |SqB|^n$ is isomorphic with $|B|^0$,
- $|CubB|^{n-2}$ and $|SqB|^{n-2}$ are isomorphic with $|B|^{n-1}$.
- the vertices of CubB, SqB and CylB are the degenerate cubes $u^{\boxminus}$, where u is a vertex of B .

PROPOSITION 4. *1° If* B *is representable for the 0-th composition, then* SqB *is representable for the n-th and (n-1)-th compositions.*

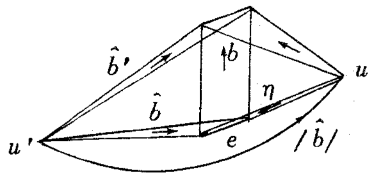
2° If B *is representable, then* CylB *is representable for the (n-2)-th composition.*

PROOF. 1° As the categories

$$(SqB)^n = \boxplus B^0 \quad \text{and} \quad (SqB)^{n-1} = \boxminus B^0$$

are isomorphic as well as $|SqB|^n$ and $|SqB|^{n-1}$ (isomorphic with $|B|^0$), the $(n+1)$-fold categories SqB and $(SqB)^{\dots,n-1}$ are simultaneously representable. Suppose that $B^{\dots,0}$ is representable and that $b^{\boxplus}$ is an object of $(SqB)^n$; let $\eta : u \to e$ be the 0-representing block for $e = \alpha^0 b$. Then $sb = (b, b\circ_0\eta, \eta, u)$ is a square and $\alpha^{\boxplus}(sb) = u^{\boxplus} = u^{\boxminus}$ is a vertex of SqB . If $s = (b, \hat{b}', \hat{b}, u')$ is a square of B with u' a vertex of B , and if $/\hat{b}/$ is the unique factor of $\hat{b}$ through η, then $/\hat{b}/^{\boxminus} : u'^{\boxminus} \to u^{\boxminus}$ is the unique morphism of $|SqB|^n$ such that $sb \boxplus /\hat{b}/^{\boxminus} = s$, since

$$\hat{b}' = b \circ_0 \hat{b} = b \circ_0 \eta \circ_0 /\hat{b}/ .$$

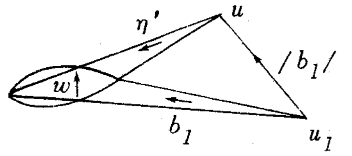


Hence sb is an n-representing square for $b^{\square}$.

2° Suppose that B is representable and that w° is an object of the category $(Cyl\mathrm{B})^{n-2}$ (so that w is a 2-cell of $\mathrm{B}^{n-1,0}$). The same method proves that there exists an $(n-2)$-representing cylinder for w°, which is

$$qw = [\, w \circ_0 \eta', \; w, u, \; \eta' \,] ,$$

where $\eta' : u \to e$ is the n-representing block for $e = \alpha^{n-1} w$. (This can



also be deduced from 1 using Remark 1-C, by a proof similar to that which will be used in Proposition 6.) ∇

COROLLARY. *1° If* $\mathrm{B}^{\dots,0}$ *is representable and if* $|\mathrm{B}|^0$ *admits (finite) limits, then* $\mathrm{B}^{\dots,0}$ *admits* $Sq\mathrm{B}$*-wise (finite) limits.*

2° If B *is representable and if* $|\mathrm{B}|^{n-1}$ *admits (finite) limits, then the greatest n-category included in* $\mathrm{B}^{\dots,0}$ *is* $(Cyl\mathrm{B})^{\dots,n-2}$*-wise (finitely) complete.*

PROOF. The first assertion comes from Proposition 4, and the remarks preceding it. The second one uses the fact that $|Cyl\mathrm{B}|^{n-2}$ is isomorphic with $|\mathrm{B}|^{n-1}$ and that $|(Cyl\mathrm{B})^{n-2}|^{0,\dots,n-3,n-1,n}$ is isomorphic with the greatest n-category included in $\mathrm{B}^{\dots,0}$. ∇

REMARKS. 1° $Cyl\mathrm{B}$ is not representable for the $(n-1)$-th composition.

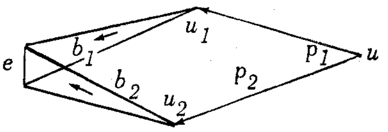
2° If C is a representable 2-category, the double category $Q(\mathrm{C})$ of its up-squares is also representable [3] and Part 2 of the preceding co-

rollary applied to $\mathrm{B} = Q(\mathrm{C})$ gives Bourn's Proposition 7 [2], since a 1
$(Cyl\mathrm{B})^{\dots, n-2}$-wise limit is an analimit in the sense of Bourn, $|\mathrm{B}|^{1}$ «is» the category of 1-morphisms of C and C «is» the greatest 2-category included in $Q(\mathrm{C})^{\boxplus, \boxminus}$.

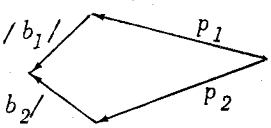
PROPOSITION 5. *If* B *is representable and if* $|\mathrm{B}|^{n-1}$ *admits pullbacks, then* $Cub\mathrm{B}$ *and* $Sq\mathrm{B}$ *are representable for the* $(n-2)$*-th composition.*

PROOF. For each object e of B^{n-1}, we denote by $\eta e : re \to e$ an $(n-1)$-representing block for e.

1° If $b_1 : u_1 \to e$ and $b_2 : u_2 \to e$ are morphisms of B^{n-1} with u_1, u_2 vertices of B, there exists a «universal» square



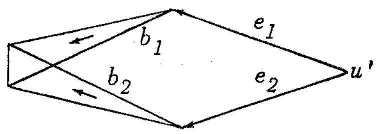
of B^{n-1} with p_1 and p_2 in $|\mathrm{B}|^{n-1}$ (called a $|\mathrm{B}|^{n-1}$*-pullback*). Indeed, by hypothesis, there exists in $|\mathrm{B}|^{n-1}$ a pullback



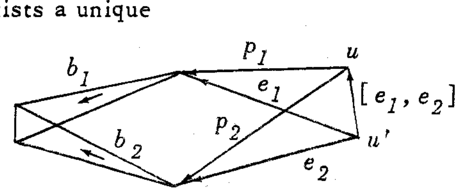
of the factors $/b_i/$ of b_i through ηe, and

$$b_2 \circ_{n-1} p_2 = \eta e \circ_{n-1} /b_2/ \circ_{n-1} p_2 = b_1 \circ_{n-1} p_1 .$$

If



is a square of B^{n-1} with e_1 and e_2 in $|\mathrm{B}|^{n-1}$, then $/b_1/ \circ_{n-1} e_1$ and $/b_2/ \circ_{n-1} e_2$ are both equal to the factor of $b_1 \circ_{n-1} e_1 = b_2 \circ_{n-1} e_2$ through ηe, so that there exists a unique

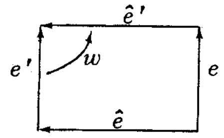


$$[e_1, e_2] : u' \to u \quad \text{in } |\mathrm{B}|^{n-1}$$

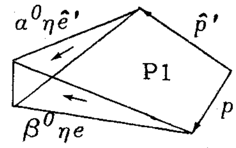
factorizing (e_1, e_2) through the pullback, i.e. satisfying

$$p_1 \circ_{n-1} [e_1, e_2] = e_1 \quad \text{and} \quad p_2 \circ_{n-1} [e_1, e_2] = e_2 .$$

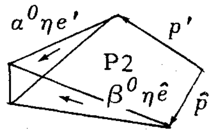
2° Let κ be an object of $(Cub\mathrm{B})^{n-2}$, which is a degenerate cube «reduced to its front face»



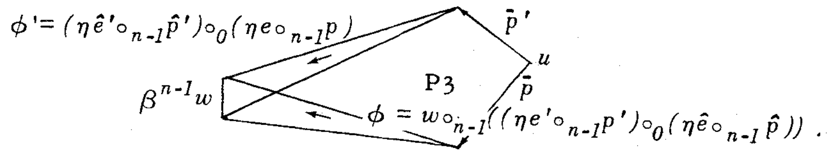
a) Construction of the $(n-2)$-representing cube for κ. By 1, there exist $|\mathrm{B}|^{n-1}$-pullbacks



and



As $p, p', \hat{p}, \hat{p}'$ are in particular objects of B^0, the composites ϕ and ϕ' are defined and admit a $|\mathrm{B}|^{n-1}$-pullback



The construction has been done so that $c\kappa =$

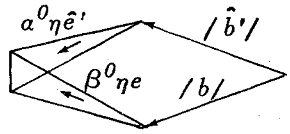
$$(\eta e' \circ_{n-1} p' \circ_{n-1} \bar{p}, \eta \hat{e}' \circ_{n-1} \hat{p}' \circ_{n-1} \bar{p}', w, u, \eta \hat{e} \circ_{n-1} \hat{p} \circ_{n-1} \bar{p}, \eta e \circ_{n-1} p \circ_{n-1} \bar{p}')$$

be a cube of B.

b) Universal property of $c\kappa$. Let $c = (b', \hat{b}', w, u', \hat{b}, b)$ be a cube with u' a vertex of B and $\beta^{n-2} c = \kappa$ (this means:

$$e = \beta^{n-1} b, \quad e' = \beta^{n-1} b', \quad \hat{e} = \beta^{n-1} \hat{b}, \quad \hat{e}' = \beta^{n-1} \hat{b}').$$

If $/b/$ and $/\hat{b}'/$ are the factors of b and $\hat{b}'$ through ηe and $\eta \hat{e}'$ there is a square



whose diagonal is

$$\alpha^0\eta\hat{e}'\circ_{n-1}/\hat{b}'/ = \alpha^0(\eta\hat{e}'\circ_{n-1}/\hat{b}'/) = \alpha^0\hat{b}' = \beta^0 b = \beta^0\eta e\circ_{n-1}/b/ .$$

By the universal property of the $|\mathrm{B}|^{n-1}$-pullback P1, there is a unique

$$[\hat{b}', b] : u' \to \alpha^{n-1}\mathrm{p} \quad \text{in} \quad |\mathrm{B}|^{n-1}$$

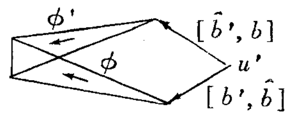
such that

$$\hat{p}'\circ_{n-1}[\hat{b}', b] = /\hat{b}'/ \quad \text{and} \quad \mathrm{p}\circ_{n-1}[\hat{b}', b] = /b/ .$$

In the same way, using the equality $\alpha^0 b' = \beta^0\hat{b}$, the factors $/b'/$ of b' through $\eta e'$ and $/\hat{b}/$ of $\hat{b}$ through $\eta\hat{e}$ factorize through the $|\mathrm{B}|^{n-1}$-pullback P2 into a unique

$$[b', \hat{b}] : u' \to \alpha^{n-1}\hat{p} \quad \text{in} \quad |\mathrm{B}|^{n-1} .$$

Using the permutability axiom in B and the fact that p' and $\hat{p}$ are objects of B^0, we find the square



since

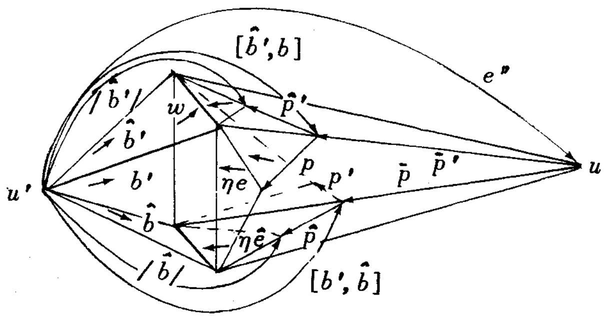
$$\phi'\circ_{n-1}[\hat{b}', b] = ((\eta\hat{e}'\circ_{n-1}\hat{p}')\circ_0(\eta e\circ_{n-1}\mathrm{p}))\circ_{n-1}[\hat{b}', b] =$$
$$= (\eta\hat{e}'\circ_{n-1}\hat{p}'\circ_{n-1}[\hat{b}', b])\circ_0((\eta e\circ_{n-1}\mathrm{p})\circ_{n-1}[\hat{b}', b]) =$$
$$= (\eta\hat{e}'\circ_{n-1}/\hat{b}'/)\circ_0(\eta e\circ_{n-1}/b/) = \hat{b}'\circ_0 b ,$$

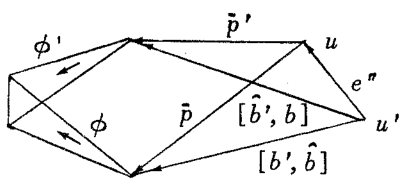
and similarly

$$\phi\circ_{n-1}[b', \hat{b}] = w\circ_{n-1}(b'\circ_0\hat{b}) = \partial c = \hat{b}'\circ_0 b .$$

This square factorizes through the $|\mathrm{B}|^{n-1}$-pullback P3 into a unique

$$e'' : u' \to u \quad \text{in} \quad |\mathrm{B}|^{n-1} .$$





We have $c\kappa \circ_{n-2} e''^{\boxminus} = c$, since

$$\eta e \circ_{n-1} p \circ_{n-1} \bar{p}' \circ_{n-1} e'' = \eta e \circ_{n-1} p \circ_{n-1} [\hat{b}', b] = \eta e \circ_{n-1} /b/ = b,$$

and idem for the other lateral faces. The unicity of the different factors implies that $e''^{\boxminus}$ is the unique cube $/c/ : u'^{\boxminus} \to u^{\boxminus}$ in $|\,Cub\mathrm{B}\,|^{n-2}$ (isomorphic with $|\,\mathrm{B}\,|^{n-1}$) such that $c\kappa \circ_{n-2} /c/ = c$. This proves that $c\kappa$ is an *(n-2)*-representing cube for κ .

3° Let κ be an object of $(Sq\mathrm{B})^{n-2}$. Then κ is of the form considered in 2 except that now

$$w = \hat{e}' \circ_0 e = e' \circ_0 \hat{e} .$$

The *(n-2)*-representing cube $c\kappa$ «reduces» to a square (w being an object of B^{n-1}), and it is also the *(n-2)*-representing square for κ . ∇

COROLLARY. *If* B *is representable and if* $|\,\mathrm{B}\,|^{n-1}$ *admits (finite) limits, then* $|\,(Cub\mathrm{B})^{n-2}\,|^{0,\dots,n-3,n-1,n}$ *and* $Sq(|\,\mathrm{B}^{n-1}\,|^{0,\dots,n-2})$ *admit respectively* $(Cub\mathrm{B})^{\dots,n-2}$*-wise and* $(Sq\mathrm{B})^{\dots,n-2}$*-wise (finite) limits.*

PROOF. This results from Proposition 5 and the remarks preceding Proposition 4. In fact, $|\,(Cub\mathrm{B})^{n-2}\,|^{0,\dots,n-3,n-1,n}$ «is formed» by the up-squares of the greatest 2-category included in $\mathrm{B}^{n-1,0}$, its two last compositions are the vertical and horizontal compositions of up-squares, and its i-th composition, for $i < n-2$, is deduced pointwise from that of B^{i+1} .

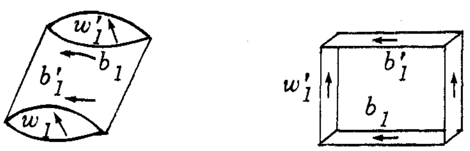
PROPOSITION 6. *If* B *is representable for the 0-th composition and if* $|\,\mathrm{B}\,|^{0}$ *admits pullbacks, then* $Cyl\,\mathrm{B}$, $\dot{Cub}\mathrm{B}$ *and* $(Cub\mathrm{B})^{\dots,n-1}$ *are representable.*

PROOF. 1° Let B' denote the n-fold category $\mathrm{B}^{n-1,1,\dots,n-2,0}$ deduced from $\mathrm{B}^{\dots,0}$ by the permutation

$$(1,\dots,n-1,0) \mapsto (n-1,1,\dots,n-2,0)$$

on the order of compositions. There is a canonical isomorphism

$$f: Cyl\,\mathrm{B} \xrightarrow{\sim} \mathrm{K}: [\,b'_1, w'_1, w_1, b_1\,] \to (w'_1, b'_1, b_1, w_1)$$



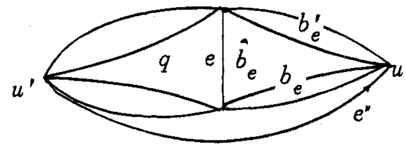
(Remark 1-C) onto the greatest $(n+1)$-category K included in the $(n+1)$-fold category $(Sq\mathrm{B}')^{\cdots, n-2}$. As $\mathrm{B}^{\cdots,0}$ is representable, so is B', and $|\mathrm{B}'|^{n-1} = |\mathrm{B}|^0$ admits pullbacks. By Proposition 5, $Sq\mathrm{B}'$ is representable for the $(n-2)$-th composition, as well as its greatest $(n+1)$-category K, and also the isomorphic $(n+1)$-category $Cyl\,\mathrm{B}$. More precisely, let $e^{\boxbar}$ be an object of $(Cyl\,\mathrm{B})^n$ (so that e is an object of B^0); then $e^{\boxminus} = f(e^{\boxbar})$ is an object of $(Sq\mathrm{B}')^{n-2}$ which admits an $(n-2)$-representing square

$$c\,e^{\boxminus} = (b'_e, \hat{b}'_e, \hat{b}_e, b_e): u^{\boxminus} \to e^{\boxminus} \quad \text{in} \quad \mathrm{K}^{n-2};$$

the cylinder of B :

$$f^{-1}(ce^{\boxminus}) = [\,\hat{b}'_e, b'_e, b_e, \hat{b}_e\,]$$

is the n-representing cylinder qe for $e^{\boxbar}$. If $q: u'^{\boxbar} \to e^{\boxbar}$ in $(Cyl\,\mathrm{B})^n$ with u' a vertex of B, its unique factor $e''^{\boxminus}$ through qe is such that $e''^{\boxbar}$ be the factor of $f(q)$ through $ce^{\boxminus}$.

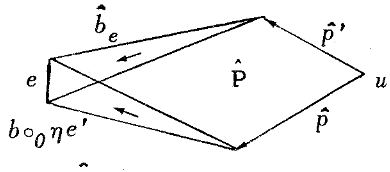


2° Let $b^{\boxbar}$ be an object of $(Cub\mathrm{B})^n$, $b \in \mathrm{B}$. We are going to construct an n-representing cube for $b^{\boxbar}$. Suppose $b: e' \to e$ in B^0.

a) By 1, there exists an n-representing cylinder

$$qe = [\,\hat{b}'_e, w'_e, w_e, \hat{b}_e\,] \quad \text{for} \quad e^{\boxbar}.$$

Applying Part 1 of the proof of Proposition 5 to $\mathrm{B}^{\cdots,0}$ instead of B (we interchange the 0-th and $(n-1)$-th compositions), there exists a $|\mathrm{B}|^0$-pullback $\hat{\mathrm{P}}$ of the following form, where $\eta e'$ denotes the 0-representing block for e':

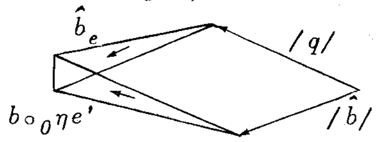


$$cb = (b, \hat{b}'_e \circ_0 \hat{p}', w'_e \circ_0 \hat{p}', w_e \circ_0 \hat{p}', \eta e' \circ_0 \hat{p}, u)$$

is a cube, since its diagonal ∂cb is:

$$(\hat{b}'_e \circ_0 \hat{p}') \circ_{n-1} (w_e \circ_0 \hat{p}') = (\hat{b}'_e \circ_{n-1} w_e) \circ_0 \hat{p}' = \partial qe \circ \hat{p}' =$$
$$= (w'_e \circ_{n-1} \hat{b}_e) \circ_0 \hat{p}' = (w'_e \circ_0 \hat{p}') \circ_{n-1} (b \circ_0 \eta e' \circ_0 \hat{p}).$$

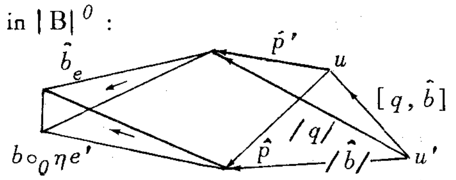
b) Let $c = (b, \hat{b}', w', w, \hat{b}, u')$ be a cube with u' a vertex of B. Then the factor $/q/$ of the cylinder $q = [\hat{b}', w', w, b \circ_0 \hat{b}]$ through qe and the factor $/\hat{b}/$ of $\hat{b}$ through $\eta e'$ determine the square



because

$$b \circ_0 \eta e' \circ_0 /\hat{b}/ = b \circ_0 \hat{b} = \hat{b}_e \circ_0 /q/ .$$

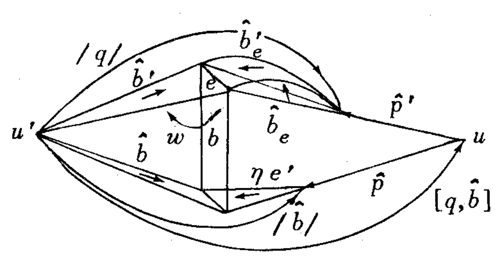
This square factors uniquely through the $|B|^0$-pullback $\hat{P}$ into a morphism $[q, \hat{b}]: u' \to u$ in $|B|^0$:



It follows from the construction that $cb \boxdot [q, \hat{b}]^{\boxminus} = c$, since

$$\eta e' \circ_0 \hat{p} \circ_0 [q, \hat{b}] = \eta e' \circ_0 /\hat{b}/ = \hat{b}, \quad w_e \circ_0 \hat{p}' \circ_0 [q, \hat{b}] = w_e \circ_0 /q/ = w,$$

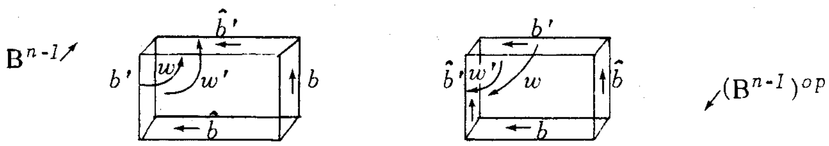
and idem for the other terms of c. Moreover, the unicity of the successive factors implies that $[q, \hat{b}]^{\boxminus}$ is the unique morphism $/c/$ of $|Cub B|^n$ sa-



tisfying $cb \,\square / c / = c$. Hence cb is a representing cube for $b^{\square}$.

c) $(Cub\mathrm{B})^{\dots, n-1}$ is representable. Indeed, let B^{op}_{n-1} be the n-fold category obtained from B by replacing the $(n-1)$-th category B^{n-1} by its opposite. B^{op}_{n-1} and B being simultaneously representable for the 0-th composition (B^{n-1} and $(\mathrm{B}^{n-1})^{op}$ have the same objects), $Cub(\mathrm{B}^{op}_{n-1})$ is representable by Part 2. There is a canonical isomorphism «reversing the cubes» $F : (Cub\,\mathrm{B})^{n-1} \to (Cub\,\mathrm{B}^{op}_{n-1})^{n}$:

$$(b', \hat{b}', w', w, \hat{b}, b) \mapsto (\hat{b}', b', w, w', b, \hat{b}),$$



which maps $|Cub\mathrm{B}|^{n-1}$ onto $|Cub\mathrm{B}^{op}_{n-1}|^{n}$. Hence $(Cub\mathrm{B})^{\dots, n-1}$ is also representable. ∇

REMARK. F defines an isomorphism $(Cub\mathrm{B})^{\dots, n-1} \to (Cub\,\mathrm{B}^{op}_{n-1})^{op}_{n-2}$. The $(n+1)$-fold category $Cub(\mathrm{B}^{op}_{n-1})$ might be called *the multiple category of down-cubes* of B (by analogy with the notion of a down-square of a 2-category), denoted by $Cub^{\downarrow}\mathrm{B}$.

COROLLARY. *If* B *is representable for the 0-th composition and if* $|\mathrm{B}|^{0}$ *admits (finite) limits, then* $\mathrm{B}^{\dots, 0}$ *admits* $Cub\mathrm{B}$*-wise (finite) limits.*

This results from Proposition 6, since $|(Cub\mathrm{B})^{n}|^{0, \dots, n-1}$ is isomorphic with $\mathrm{B}^{\dots, 0}$. ∇

2° A laxified internal Hom on Cat_n.

Imitating the construction of the cartesian closed structure on Cat_n given in Section C [5], we define a «closure» functor on Cat_n by replacing the *Square* functor and the *Link* functor respectively by the *Cube* functor and by the *LaxLink* functor.

Let $LaxHom_n : Cat_n^{op} \times Cat_n \to Cat_n$ be the composite functor

$$Cat_n^{op} \times Cat_n \xrightarrow{id \times Cub_{n,2n}} Cat_n^{op} \times Cat_{2n} \xrightarrow{id \times \bar{\gamma}} Cat_n^{op} \times Cat_{2n} \xrightarrow{Hom(-,-)} Cat_n$$

where :

- $\tilde{\gamma}: Cat_{2n} \to Cat_{2n}$ is the isomorphism «permutation of the compositions» associated to the permutation

$$\gamma : (0, \dots, 2n-1) \mapsto (0, 2, \dots, 2n-2, 1, 3, \dots, 2n-1),$$

which associates to the $2n$-fold category H the $2n$-fold category H^{γ} in which the i-th category is H^{2i} and the $(i+n)$-th category is H^{2i+1}, for each $i < n$.

- $Hom(-,-)$ is the restriction of the internal Hom functor of the monoidal closed category $(\coprod_n Cat_n, \blacksquare, Hom)$ (defined in [4]); it maps the couple (A, H) of an n-fold category A and a $2n$-fold category H onto the n-fold category $Hom(A, H)$ formed by the n-fold functors $f: A \to H^{0,\dots,n-1}$, the i-th composition being deduced pointwise from that of H^{n+i}, for $i < n$.

DEFINITION. The functor $LaxHom_n : Cat_n^{op} \times Cat_n \to Cat_n$ is called the *laxified internal Hom on* Cat_n.

If A and B are n-fold categories, then

$$LaxHom_n(A, B) = Hom(A, (Cub B)^{\gamma})$$

is formed by the n-fold functors

$$h: A \to (Cub_{n,2n} B)^{0,2,\dots,2n-2},$$

the i-th composition being deduced pointwise from the $(2i+1)$-th composition of $Cub_{n,2n} B$ (itself deduced «horizontaly» from the composition of B^i, as remarked at the end of Section B).

PROPOSITION 7. *For each n-fold category* A, *the partial functor*

$$LaxHom_n(A, -): Cat_n \to Cat_n$$

admits a left adjoint $-\otimes A : Cat_n \to Cat_n$. *The corresponding tensor product functor* $\otimes : Cat_n \times Cat_n \to Cat_n$ *admits as a unit the n-fold category* 1_n *on the set* $1 = \{0\}$.

PROOF. 1° a) Since $LaxHom_n(A, -)$ is equal to the composite

$$Cat_n \xrightarrow{Cub_{n,2n}} Cat_{2n} \xrightarrow{\tilde{\gamma}} Cat_{2n} \xrightarrow{Hom(A,-)} Cat_n,$$

it admits as a left adjoint, denoted by $-\otimes A : Cat_n \to Cat_n$, the composite functor

$$Cat_n \xrightarrow{-\blacksquare A} Cat_{2n} \xrightarrow{\tilde{\gamma}^{-1}} Cat_{2n} \xrightarrow{LaxLk_{2n,n}} Cat_n ,$$

where $-\blacksquare A$ is the partial square product functor, left adjoint of $Hom(A,-)$ (see [4]) and $LaxLk_{2n,n}$ is the left adjoint of $Cub_{n,2n}$ (Proposition 1, Corollary 1). So, if B is an n-fold category, we have

$$B\otimes A = LaxLk_{2n,n}(B \blacksquare A)^{\gamma^{-1}} ,$$

where $(B\blacksquare A)^{\gamma^{-1}}$ is the $2n$-fold category in which

- the $2i$-th category is $\underline{B}^{dis} \times A^i$,
- the $(2i+1)$-th category is $B^i \times \underline{A}^{dis}$, for $i < n$.

b) There exists a functor

$$\otimes : Cat_n \times Cat_n \to Cat_n$$

extending the functors $-\otimes A$, for any n-fold category A. This comes from the fact that the right adjoints $LaxHom_n(A,-)$ of $-\otimes A$ are all restrictions of the functor $LaxHom_n$. The functor $\otimes$ maps the couple

$$(f: A \to A',\ g: B \to B')$$

of n-fold functors onto the n-fold functor $g\otimes f: B\otimes A \to B'\otimes A'$ corresponding by adjunction to the composite n-fold functor:

$$B \xrightarrow{g} B' \xrightarrow{l} Hom(A', B'\otimes A') \xrightarrow{Hom(f, B'\otimes A')} Hom(A, B'\otimes A')$$

where l is the liberty morphism defining $B'\otimes A'$ as a free object generated by B' with respect to $Hom(A',-)$.

2° *$\otimes$ admits 1_n as a unit* (up to isomorphisms): We have to construct, for each n-fold category A, canonical isomorphisms

$$1_n \otimes A \cong A \cong A \otimes 1_n ,$$

where

$$1_n \otimes A = LaxLk_{2n,n}(1_n \blacksquare A)^{\gamma^{-1}} \text{ and } A\otimes 1_n = LaxLk_{2n,n}(A \blacksquare 1_n)^{\gamma^{-1}} .$$

Now, there are isomorphisms:

- $(0,a) \mapsto a$ from $(1_n \blacksquare A)^{\gamma^{-1}}$ onto the $2n$-fold category

$$\tilde{A} = (A^0, \underline{A}^{dis}, \dots, A^{n-1}, \underline{A}^{dis})$$

such that $\tilde{A}^{2i} = A^i$ and $\tilde{A}^{2i+1} = \underline{A}^{dis}$, for $i < n$,

- $(a,0) \mapsto a$ from $(A \blacksquare 1_n)^{\gamma^{-1}}$ onto the $2n$-fold category

$$\tilde{\tilde{A}} = (\underline{A}^{dis}, A^0, \ldots, \underline{A}^{dis}, A^{n-1})$$

such that $\tilde{\tilde{A}}^{2i} = \underline{A}^{dis}$ and $\tilde{\tilde{A}}^{2i+1} = A^i$, for $i < n$.

Hence, it suffices to construct isomorphisms

$$A \cong LaxLk_{2n,n}\tilde{A} \quad \text{and} \quad A \cong LaxLk_{2n,n}\tilde{\tilde{A}}.$$

For this, we first prove the assertions a and b:

a) *If* H *is an* $(m+1)$*-fold category such that* H^m *is the discrete category on* $\underline{H}$, *then* $LaxLkH \approx H^{m-1,0,\ldots,m-2}$.

Indeed, an $(m+1)$-fold functor $g: H \to CubK$, where K is an m-fold category, takes its values into the objects of $(CubK)^m$ (we use that H^m is discrete), so that it admits a restriction

$$g': H^{0,\ldots,m-1} \to |(CubK)^m|^{0,\ldots,m-1}.$$

Then,

$$\hat{g} = (H^{0,\ldots,m-1} \xrightarrow{g'} |(CubK)^m|^{0,\ldots,m-1} \xrightarrow{(-^{\boxplus})^{-1}} K^{1,\ldots,m-1,0})$$

is an m-fold functor, as well as

$$\hat{g}: H^{m-1,0,\ldots,m-2} \to K: \eta \mapsto k \quad \text{if } g(\eta) = k^{\boxplus}.$$

This determines a 1-1 correspondence $g \mapsto \hat{g}$ from the set of $(m+1)$-fold functors $g: H \to CubK$ onto the set of m-fold functors $H^{m-1,0,\ldots,m-2} \to K$. It follows that $H^{m-1,0,\ldots,m-2}$ is a free object generated by H with respect to the functor $Cub_{m,m+1}: Cat_m \to Cat_{m+1}$, and we can choose it as $LaxLkH$ (Proposition 1).

b) *If* H *is an* $(m+1)$*-fold category such that* H^{m-1} *is discrete, then* $LaxLkH \approx H^{m,0,\ldots,m-2}$. The proof is similar, using the isomorphism

$$|(CubK)^{m-1}|^{0,\ldots,m-2,m} \xrightarrow{(-^{\boxminus})} K^{1,\ldots,m-1,0}.$$

c) Applying a to the $2n$-fold category $\tilde{A}$ whose last composition is the discrete one, we find an isomorphism

$$LaxLk\tilde{A} \approx (A^{n-1}, A^0, \underline{A}^{dis}, \ldots, A^{n-2}, \underline{A}^{dis}),$$

and by iteration, $1_n \otimes A \approx LaxLk_{2n,n}\tilde{A}$ may be identified with A. Similarly, we deduce from b that

$$LaxLk\tilde{\tilde{A}} \approx (A^{n-1}, \underline{A}^{dis}, A^0, \dots, \underline{A}^{dis}, A^{n-2}),$$

and by iteration $A \otimes 1_n \approx LaxLk_{2n,n}\tilde{\tilde{A}}$ may be identified with A. ∇

COROLLARY. *The vertices of* $LaxHom_n(A, B)$ *are identified with the n-fold functors from* A *to* B.

PROOF. These vertices are identified [4] with the n-fold functors

$$f: 1_n \to LaxHom_n(A, B),$$

which by adjunction (Proposition 7) are in 1-1 correspondence with the n-fold functors $A \tilde{\to} 1_n \otimes A \to B$. ∇

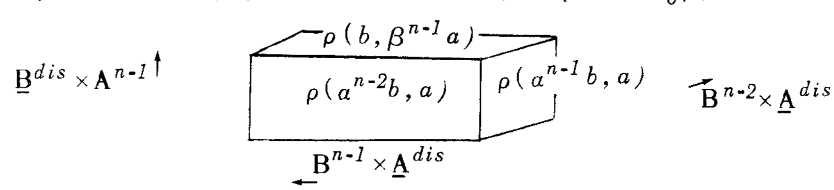
EXAMPLES.

1° Let A and B be n-fold categories. Then $L = LaxLk((B \blacksquare A)^{\gamma^{-1}})$ is generated by the blocks

$$\rho(u, a), \quad \rho(b, v), \quad \rho(t, a),$$

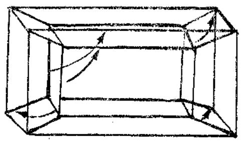
where a and b are blocks of A and B, where u, v, t are objects of B^{n-1}, A^{n-1} and B^{n-2} respectively, and where ρ is the universal admissible morphism used in the construction of *LaxLink* (Proof, Proposition 1). In particular, for any couple (b, a), there exist blocks of L

$$\rho(\alpha^{n-2}b, a), \quad \rho(\beta^{n-2}b, a), \quad \langle b, a\rangle = \rho(b, \beta^{n-1}a) \circ_0 \rho(\alpha^{n-1}b, a).$$



So L may be seen as an «enrichment» of $B \times A$ by the blocks $\rho(t, a)$, for each object t of B^{n-2}. By iteration, $B \otimes A$ is an «enrichment», or a « laxification» of $B \times A$.

2° For $n = 2$, the 4-fold category $(Cub_{2,4}A)^{\gamma}$ is defined in a similar way as the 4-fold category of frames $(Sq_{2,4}A)^{\gamma}$ (Example, Section C [5]), by replacing the frames, which are «squares of squares» by «full frames», which are «cubes of cubes». Then $LaxHom_2(A, B)$ has a description analogous to that given for $Hom_2(\mathbb{A},\mathbb{B})$, except that frames



are replaced by full frames; the vertices remain the double functors $\mathbb{A} \to \mathbb{B}$ (Corollary, Proposition 7). In particular, if $\mathbb{A}$ and $\mathbb{B}$ are 2-categories, the greatest 2-category included in $LaxHom_2(\mathbb{A},\mathbb{B})$ is the 2-category $Fun(\mathbb{A},\mathbb{B})$ introduced by Gray [7], and the tensor product $\mathbb{B} \otimes \mathbb{A}$ admits

1+ as a reflection the 2-category tensor product constructed by Gray [8].

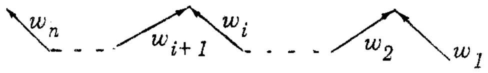
COMPLEMENTS. *Other closure functors.*

1° *A closure functor on the category n-Cat of n-categories* is defined by the same method as above, replacing the *Cube* functor $Cub_{n,2n}$ by the *Cylinder* functor *(n, 2n)-Cyl* (Section C), and there is also associated a tensor product on *n-Cat*.

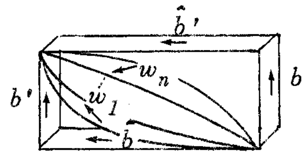
2° In the last Remark of 1-D, we have defined the *(n+1)*-fold category of down-cubes of $\mathbb{B}$; it gives rise to a functor «*Down-cube*» $Cub^{\downarrow}_{n,2n}$ from Cat_n to Cat_{2n}, and as above to a «*laxified*» *internal Hom functor on* Cat_n, denoted by $Lax\,Hom^{\downarrow}_n$, *for which Proposition 7 is also valid*, with a tensor product functor $\otimes^{\downarrow}$ having I_n as unit.

3° The tensor product functors $\otimes$ and $\otimes^{\downarrow}$ on Cat_n are not symmetric, one being in some sense the symmetric of the other. More generally,

2 we may replace the cubes by «laxified cubes» in which the 2-cells w' and w of $\mathbb{B}^{n-1,0}$ would be replaced by «strings of 2-cells of $\mathbb{B}^{n-1,0}$»



(with respect to the category $\mathbb{B}^{n-1}$).



This gives rise to an *(n+1)*-fold category $Lax\,Cub\,\mathbb{B}$, containing both

CubB and $Cub^{\downarrow}$B as $(n+1)$-fold subcategories. *The constructions of this paper may be generalized in this setting.*

4° *«Less-laxified» internal Hom functors on* Cat_n are defined by replacing in Proposition 7 the composite $Cub_{n,2n}$ of *Cube* functors by a composite in which at some steps $Cub_{m,m+1}$ is replaced by $Sq_{m,m+1}$. Then *Proposition 7 remains valid*, so that we obtain different tensor products of the couple (B, A) of n-fold categories, the «smallest» one being the cartesian product B × A (corresponding to the internal Hom functor constructed in [5], where only *Square* functors are taken), the «greatest» one being B ⊗ A (where only *Cube* functors are used); all admit 1_n as a unit up to isomorphisms. In Part III, we have constructed an $(n+1)$-category Nat_n «gluing together» the n-fold categories Hom_n(A, B), for any n-fold categories A and B. If $\hat{H}$ is an internal Hom functor other than the «cartesian closure functor» Hom_n, there is no $(n+1)$-fold category on the n-fold category coproduct of the multiple categories $\hat{H}$(A, B), the canonical composition functor

$$\hat{\kappa}: \hat{H}(A,B) \otimes \hat{H}(B,K) \to \hat{H}(A,K)$$

admitting as its domain a tensor product and not a cartesian product.

5° *The constructions of Square, Link, Cube, LaxLink, and so the results given in Parts III and IV may be «internalized» (without essential changes) for multiple categories in(ternal to) a category* V *with commuting coproducts* (see Penon [8] and Part III, Appendix) *and cokernels.* Indeed there exist then free categories in V generated by a graph in V and quasi-quotient categories in V.

3° Characterization of multiple categories in terms of 2-categories.

The construction of *LaxLink* will be used now to prove that each double category «is» a double sub-category of a double category of squares of a 2-category.

PROPOSITION 8. *Let* Q: 2-$Cat \to Cat_2$ *be the functor mapping a 2-category* C *onto the double category* Q(C) *of its (up-)squares. Then* Q *admits a left adjoint String:* $Cat_2 \to$ 2-Cat.

PROOF. Q may be seen as the composite of the four functors

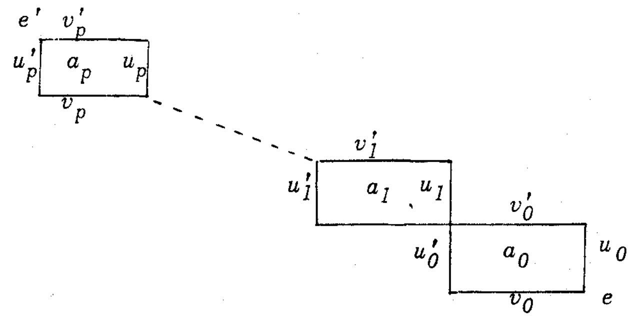
$$2\text{-}Cat \hookrightarrow Cat_2 \xrightarrow{\tilde{y}^{1,0}} Cat_2 \xrightarrow{Cub} Cat_3 \xrightarrow{|-|^{1,2}} Cat_2 ,$$

where $\tilde{y}^{1,0}$ is the isomorphism «interchanging the two compositions» and where $|-|^{1,2}$ is the functor mapping a triple category T onto the double category formed by the objects of the 0-th category T^0. These four functors admitting left adjoints, their composite Q admits a left adjoint, constructed as follows:

Let A be a double category and $\bar{A}$ be the triple category with the same blocks $(\underline{A}^{dis}, A^0, A^1)$ whose 0-th category is the discrete category on $\underline{A}$ (it is the free object generated by A with respect to $|-|^{1,2}$, by Proposition 9, Part II). The free object $(Lax\,Lk\bar{A})^{1,0}$ generated by $\bar{A}$ with respect to

$$Cat_2 \xrightarrow{\tilde{y}^{1,0}} Cat_2 \xrightarrow{Cub} Cat_3$$

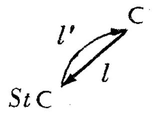
is a 2-category whose 1-morphisms are equivalence classes of strings of objects of alternately A^0 and A^1, and whose 2-cells from e to e' are classes of strings of blocks of A :

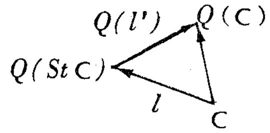


This 2-category is the free object generated by A with respect to Q. It
1+ will be called the *2-category of strings of* A, denoted by *St* A. ∇

COROLLARY. *The functor String:* $Cat_2 \to 2\text{-}Cat$ *is equivalent to a left inverse of the inclusion:* $2\text{-}Cat \hookrightarrow Cat_2$.

PROOF. It suffices to prove that, if C is a 2-category, *St* C is isomorphic to C. Indeed, let $l: C \to Q(St\,C)$ be the liberty double functor. As C is a 2-category, l takes its values into the greatest sub-2-category $St\,\mathsf{C}$ of $Q(St\,\mathsf{C})$, and its restriction $l: \mathsf{C} \to St\,\mathsf{C}$ admits as an inverse the 2-func-





tor $l': St\,\mathsf{C} \to \mathsf{C}$ associated by adjunction to the inclusion $\mathsf{C} \hookrightarrow Q(\mathsf{C})$. ∇

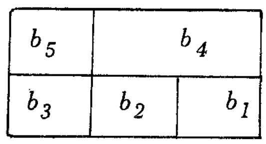
REMARK. If A is the double category $Q(\mathsf{C})$ of squares of a 2-category C then C is not isomorphic to $St\,\mathsf{A}$; counter example: C is the 2-category $(\underline{2}^{dis}, 2)$.

PROPOSITION 9. *If* A *is a double category, then it is canonically isomorphic to a double sub-category of the double category* $Q(St\mathsf{A})$ *of squares of the 2-category* $St\mathsf{A}$.

PROOF. The liberty double functor $l: \mathsf{A} \to Q(St\mathsf{A})$ is injective. Indeed, let a and a' be blocks of A such that $l(a) = l(a')$. By definition of the equivalence relation used to define $Lax\,Lk\bar{\mathsf{A}}$ (and therefore $St\mathsf{A}$), there exists a family (b_i) of «smaller» blocks of A admitting both a and a' as double composites. More precisely, let Λ be the free double non-associative category generated by the double graph underlying A, and $\lambda: \Lambda \to \mathsf{A}$ be the canonical non-associative double functor (for its existence, see [6]); then there exist blocks η and η' of Λ constructed on the family (b_i) and such that

$$a = \lambda(\eta) = \lambda(\eta') = a'.$$

(Example:



$$a = (b_5 \circ_0 b_4) \circ_1 (b_3 \circ_0 b_2 \circ_0 b_1) = (b_5 \circ_1 b_3) \circ_0 (b_4 \circ_1 (b_2 \circ_0 b_1)) = a'.)$$

So l is injective, and its image $l(\mathsf{A})$ is isomorphic to A. ∇

Hence all double categories «are» double sub-categories of double

1 categories of squares of a 2-category. This explains why it was difficult to find natural examples of double categories other than 2-categories and their squares! (Spencer [9] has characterized double categories of squares of a 2-category as those double categories admitting a special connection in the sense of Brown.)

It follows that, if $\mathbf{A}$ is a double category and $f: \mathrm{K} \to |\mathbf{A}^0|^1$ a functor, an $\mathbf{A}$-wise limit of f is simply a lax-limit (in the sense of Gray-Bourn-Street) of f considered as a 2-functor from $(\underline{\mathrm{K}}^{dis}, \mathrm{K})$ into the greatest 2-category included in $\mathbf{A}$, such that the 2-cells projections of the lax-limit take their values in $\mathbf{A}$; this is a restrictive condition, since $\mathbf{A}$ is only a double sub-category of $Q(St\mathbf{A})$. Hence generalized limits (defined in Part II) are just lax-limits «relativized to a double sub-category».

From Proposition 9, we deduce :

PROPOSITION 10. *Let* $\mathbf{A}$ *be an n-fold category, with* $n > 2$. *Then there exists a canonical embedding from* $\mathbf{A}$ *into an n-fold category of the form* $Cub_{2,n}Q(\mathrm{C})$, *where* C *is a 2-category.*

PROOF. The functor

$$2\text{-}Cat \xrightarrow{Q} Cat_2 \xrightarrow{Cub_{2,n}} Cat_n$$

admits a left adjoint which associates to $\mathbf{A}$ the 2-category

$$\mathrm{C} = St(LaxLk_{n,2}\mathbf{A}).$$

Remark that the corresponding liberty morphism $L: \mathbf{A} \to Cub_{2,n}Q(\mathrm{C})$ is generally not injective, since it factors through the liberty morphism l from $\mathbf{A}$ to $Cub(LaxLk\mathbf{A})$ which identifies (Proof, Proposition 1) two blocks of $\mathbf{A}$ having the same sources and targets for the last three compositions. ∇

COMPLEMENT. Proposition 10 does not give a complete characterization of n-fold categories, for $n > 2$, in terms of 2-categories, since the embedding L is generally not injective. However there is such a characteriza-
2 tion (which will be given elsewhere), obtained by laxifying at each step the construction of the functor *Cube*, in a way similar to that used to proceed from the functor *Square* to the functor *Cube*.

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Vol. XIV-2

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COLLOQUE SUR L'ALGÈBRE DES CATÉGORIES

AMIENS - 1973

RESUMÉS DES CONFÉRENCES

INTRODUCTION

Ce Colloque, organisé par l'équipe de recherche «Théorie et Applications des Catégories» (T. A. C.), s'est déroulé à la Faculté des Sciences d'Amiens, du 9 au 13 Juillet 1973. Il a réuni une cinquantaine de participants, venus des pays les plus divers: Australie, Canada, Etats-Unis, Europe.

En 5 jours, 28 conférences ont eu lieu; comme elles ont souvent duré plus que le temps prévu (une demi-heure à une heure), 5 chercheurs de l'équipe T. A. C., dont les conférences étaient inscrites au programme initial, ont renoncé à parler. Enfin, deux conférenciers attendus, M. Kock et M^{me} Preller, n'ont pu venir à Amiens par suite d'empêchements de dernière heure.

On trouvera dans les pages suivantes les résumés de ces conférences, et deux fascicules ultérieurs des «Cahiers» seront consacrés à la publication de textes développés (y compris les articles promis par MM. Joyal, Kock et Ulmer, dont les résumés ne sont pas parvenus à temps).

Le dynamisme actuel de la Théorie des Catégories s'est manifesté par la variété des sujets abordés. Bien qu'une telle classification soit très contestable, nous essayons de regrouper ci-dessous les conférences d'après leurs thèmes principaux.

Topoï:

- Topoï élémentaires: Joyal, Wraith, Mikkelsen, Tierney, Osius.
- Généralisations: Barthélémy, Penon.

Catégories enrichies.

- Catégories internes: Joyal, Wraith, Bastiani - Ehresmann, Tierney.
- Catégories monoïdales: Voréadou, Kelly, Chartrelle, Variot.
- V-catégories: Borceux, Lindner, Lavendhomme.
- 2-catégories: Kelly, Bourn, Gray, A. Burroni, Kock, Variot.
- Catégories additives: Baumgartner, Dartois.

Structures algébriques.

- Esquisses: Bastiani - Ehresmann, Lair.
- Structures 2-algébriques: Kelly, Gray, A. Burroni.
- Autres notions: E. Burroni, Guitart, Coppey, Wischnewsky.

Catégories et notions topologiques.

- Objets connexes: Hoffmann, Tanré.
- Faisceaux: Ulmer.
- Catégories topologiques: Lengagne, Ehresmann.

Applications des Catégories

- à la Logique: Joyal, Barthélémy, Osius.
- aux Automates: E. Burroni, Guitart.
- à l'Algèbre homologique: Kleisli, Hilton.
- à la Théorie de la mesure: Riguet, Lengagne.
- à la Géométrie différentielle: Ehresmann.

Avant de terminer cette introduction, nous tenons à remercier:

- Les participants, qui ont accepté de venir, bien que nous n'ayions pu envoyer les invitations que très tardivement.

- Tous ceux qui nous ont aidés à recevoir les invités: M^{mes} Bednarz et Leblond, MM. Cordier et Largillier, et plus particulièrement, MM. Chartrelle, Boidin et Tanré (qui ont été tous les trois sans cesse sur la brèche avant, pendant et après le Colloque, pour toutes sortes de tâches), M. Lair (qui a organisé un pique-nique à Faucoucourt le 14 Juillet) et aussi M^{elle} Normand et M. Schimel (deux jeunes étudiants qui, avec beaucoup de gentillesse, nous ont rendu d'innombrables services pendant toute la durée du Colloque).

- L'Université de Picardie, qui a entièrement et généreusement financé ce Colloque, en particulier M. le Président D. Taddéi ainsi que M. A. Chevalier, Directeur du Département de Mathématiques, qui a offert de compléter les crédits de recherche de l'équipe T.A.C. pour faciliter l'organisation de ce Colloque.

- Enfin, M. Bonvalet, Recteur de l'Académie d'Amiens, qui nous a encouragés et qui a accueilli lui-même les participants à la réception qu'il a offerte au Rectorat.

A. et C. Ehresmann

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DEUXIÈME COLLOQUE SUR L'ALGÈBRE DES CATÉGORIES
AMIENS - 1975

RÉSUMÉS DES CONFÉRENCES

INTRODUCTION

par Andrée et Charles EHRESMANN

C'est du 7 au 12 Juillet 1975 que nous avons organisé à Amiens un deuxième Colloque sur l'Algèbre des Catégories (annoncé à la fin du premier Colloque sur l'Algèbre des Catégories qui s'était déroulé à Amiens en Juillet 1973). Ce Colloque a réuni une cinquantaine de mathématiciens, parmi lesquels la plupart des participants du premier Colloque. En particulier, une trentaine de catégoristes étrangers sont venus d'Australie, des Etats-Unis, du Canada, d'Afrique, d'Europe. Il y aurait eu sans doute plus de participants si des difficultés matérielles ne nous avaient empêchés d'envoyer assez tôt les invitations.

Ci-après, on trouvera les résumés de la plupart des conférences données à ce Colloque (ainsi que trois résumés d'articles qui n'ont pu être exposés faute de temps). En comparant avec les exposés du Colloque de 1973 (voir *Cahiers de Topologie et Géométrie différentielle* XIV-2, 1973), on constate qu'en ces deux années la Théorie des Catégories s'est développée dans diverses directions, surtout en vue d'applications dans des domaines variés. Grosso modo, les sujets traités peuvent se répartir comme suit:

Structures algébriques et généralisations: Diers, Gray, Kelly, Kock, Lair, Meisen, Ulmer.

Catégories enrichies (catégories doubles, 2-catégories, V-catégories): Bastiani-Ehresmann, Borceux, Bourn, Lindner, Linton.

Topos élémentaires et généralisations: Bourn, Diaconescu, Fourman, Guitart, Sols, Stout.

Applications des catégories en

Logique : Blanc, Joyal, Mijoule, Reyes;

Théorie des groupes : Hilton, Kleisli, Wischnewsky;

Algèbre homologique : Baumgartner, Duskin, Lavendhomme, Mac Donald;

Topologie et Géométrie différentielle : Brümmer, Guitart, Hoff, Porter, Pradines, Tanré;

Combinatoire et Théorie des Jeux : Leroux, Riguet;

Théorie des Automates : Meseguer-Pfender, Sols.

Huit conférences ne sont pas résumées ici :

celles de Brümmer, Diaconescu, Duskin, Lavendhomme, Linton, Meisen, qui ne nous ont malheureusement pas envoyé de texte,

celle de Riguet, faite à Chantilly le Samedi, qui paraîtra dans le fascicule 4 (avec les résumés des exposés faits à Chantilly en Septembre),

la nôtre, qui est résumée dans l'introduction de notre article: Multiple functors, Part 1, *Cahiers de Topologie et Géométrie différentielle* XV-3.

Enfin, nous désirons remercier vivement tous ceux qui nous ont aidés dans l'organisation de ce Colloque: M. BONVALET, alors Recteur de l'Académie d'Amiens, qui a bien voulu présider le dîner officiel du Lundi, l'Université de Picardie, qui a financé ce Colloque, et avant tout son Président M. Roland PEREZ, qui a eu la gentillesse d'ouvrir ce Colloque et de venir à plusieurs reprises se joindre aux participants; les chercheurs de notre équipe de recherche T.A.C., et plus particulièrement Bernard FERRIF et Anne-Marie KEMPF, qui se sont chargés de nombreuses tâches matérielles pendant tout le Colloque; le Père BOUTRY qui a accueilli les participants au Centre «Les Fontaines» à Chantilly le Samedi.

CAHIERS DE TOPOLOGIE
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JOURNÉES T.A.C. DE CHANTILLY

RESUMES DES CONFÉRENCES

INTRODUCTION

C'est au Centre «Les Fontaines» de Chantilly, au milieu d'un magnifique parc, que s'est déroulée en Septembre 1975 la Semaine de l'Université de Picardie (sous les auspices du C.E.R.I.C., association créée par le Centre «Les Fontaines» et l'Université de Picardie).

Dans le cadre de cette Semaine culturelle ont eu lieu, le Lundi 15 et le Mardi 16 Septembre, deux journées «Théorie et Applications des Catégories» (T.A.C.). Les conférences données à cette occasion sont résumées ci-après, à l'exclusion de celle de C. Auderset dont nous n'avons pas reçu le résumé; les textes développés paraîtront ultérieurement.

Déjà la dernière journée du deuxième Colloque sur l'Algèbre des Catégories (Amiens 1975) s'était passée à Chantilly, le Samedi 12 Juillet; on trouvera également ici le résumé de l'exposé de J. Riguet, fait ce jour-là au Centre «Les Fontaines» (celui de F. Linton n'a pas été résumé).

Nous tenons à remercier les Pères Jésuites du Centre «Les Fontaines», et en particulier le Père Boutry, pour leur chaleureuse hospitalité. Nous espérons que d'autres mathématiciens auront l'idée d'organiser des Colloques dans ce lieu si propice à la discussion et à la méditation mathématique, non seulement dans les nombreuses salles de réunion, mais aussi dans les sentiers sous les arbres et au bord de l'étang.

A. et C. Ehresmann

*CAHIERS DE TOPOLOGIE
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DÉJÀ VINGT ANS...

par Charles et Andrée EHRESMANN

A l'occasion du vingtième anniversaire de notre publication, nous voudrions remercier tous ceux qui rendent possible la réalisation des «Cahiers», à savoir les auteurs qui envoient des articles intéressants et les abonnés sans lesquels ce périodique ne pourrait subsister.

C'est en effet en 1957-58 que le Volume I a été publié *; il s'insérait alors dans la série des «Séminaires de l'Institut Henri Poincaré»; le principe strict de cette intéressante série est que les textes sont des rédactions d'exposés faits dans des Séminaires. C'est pour avoir plus de liberté dans le choix des articles, par exemple pour pouvoir publier des thèses (la publication dans le Volume I de la thèse de troisième cycle de A. Bastiani - actuellement A. Ehresmann - avait soulevé des difficultés), que nous avons désiré créer une publication indépendante. La rencontre fortuite d'une machine Vari-Typer inutilisée a permis la réalisation de ce projet.

Jusqu'en 1966, les volumes ont paru sans périodicité stricte (environ un volume par an) et le titre initial de «Séminaire de Topologie et Géométrie Différentielle» s'est modifié peu à peu pour aboutir en 1966 au titre de : «Cahiers de Topologie et Géométrie Différentielle».

C'est depuis 1967 que les «Cahiers» paraissent sous leur présentation actuelle et qu'ils sont déclarés légalement comme périodique trimestriel. De 1967 à 1972, ils ont été édités par la Maison Dunod ; en 1972, nous avons décidé de les éditer nous-mêmes, à Paris jusqu'en 1975, ensuite à Amiens.

Jusqu'en 1975, une grande partie de la composition des textes a été effectuée par une Varitypiste mise à la disposition des «Cahiers» par le Centre National de la Recherche Scientifique. Lorsque nous nous sommes

* Il a été précédé de trois recueils d'articles publiés par Charles Ehresmann (seul) sous le titre «Colloque de Topologie de Strasbourg».

installés à Amiens fin 1975, nous avons dû renoncer à cette aide matérielle, dont le maintien aurait été subordonné à des conditions inacceptables pour nous.

Au cours des années, le contenu des «Cahiers» s'est un peu modifié, par suite du développement de la Théorie des Catégories et d'une évolution dans notre conception des Mathématiques. Aujourd'hui un titre mieux adapté serait sans doute «Théorie et Applications des Catégories», mais nous tenons à garder le titre actuel, marque de la continuité dans le changement.

Encore une fois merci à tous ceux qui s'intéressent aux «Cahiers» et nous permettent ainsi de poursuivre cette publication qui tient une place si importante dans notre vie depuis vingt ans.

Amiens, 1er Janvier 1978

COMMENTS ON PART IV-2

by Andrée CHARLES EHRESMANN

INTRODUCTION

The motivations and conventions of these comments are the same as in Part IV-1. Their characteristics, fewer but often longer, commentaries, stem from the specificity of the articles reproduced here :

- Technical remarks, which were numerous in the volumes containing more sketchy papers, are very few for three reasons :

a) the style and notations are almost standard ;

b) the proofs are explicitly written ;

c) since the papers have been published in our «*Cahiers*», I have corrected the small remaining typographical errors before reproduction.

- All the articles consist of a few main theorems with extensive proofs using notions already studied in older papers, and so commented upon in Parts III or IV-1.

- As the works are relatively recent (72-79), further developments only begin to appear and it is difficult to get a synthetical view of actual research on the subject. Hence the comments are centered on what I think to be the most promising domains for applications (mixed sketches and figurative algebras ; completions and lax limits ; homotopy theory and coherence via multiple categories).

The following Synopsis summarizes the main results.

CONVENTIONS. The Y^{th} comment of the X^{th} page is denoted by X.Y and + indicates more substantial addenda. Numbers between / / refer to Charles' list of publications (in Part IV-1) and numbers between [ ] to the Bibliography of Part IV-1 (up to 130) and of Part IV-2 (from 131 to 200).

GENERAL COMMENTS

ON / 115 / : CATEGORIES OF SKETCHED STRUCTURES.

The first part of this paper may be seen as a sequel and a generalization of / 102 /. In the late sixties we realized that the notations used in the preceding papers made the reading uneasy for most of the categoricians; it explained Charles's results were often ignored and found anew later on. In the present article, we tried to adopt a more standard terminology, as in Mac Lane [174]. The only exception is the set of morphisms $e \to e'$ in a category C still denoted by $C(e', e)$ (and not yet by $C(e, e')$, as it is usual).

Comments on sketched structures and on completions have already been given in O, III-2 and IV-1; we refer to them and here we just add some more specific or more recent results.

418.1. The formulation of the proposition stresses the fact that the construction does not depend upon a universe $\mathfrak{U}$ and that it is universal with respect to all universes. If we restrict ourselves to $\mathfrak{U}$, then the result means existence of a suitable adjoint; it is indicated in the Remarks 1 and 3, pages 426-427.

428.1. Same comment as above.

436.1 + *Other constructions of a loose type.*

1° In [165], Kelly gives the following construction of the loose $\mathfrak{I}$-type of a sketch $\sigma = (\Sigma, \Gamma)$, which he calls the *$\mathfrak{I}$-theory of σ* : Take the category Set^{σ} of models of σ in the category of sets, and identify Σ to its image via the Yoneda embedding $\Sigma \to Set^{\sigma}$. Let T be the closure of Σ into Set^{σ} under $\mathfrak{I}$-colimits; then the opposite T^{op} of T is a loose $\mathfrak{I}$-type of σ. Notice that this construction still requires a transfinite induction, to give an explicit description of T ; and it does not generalize to the case of mixed sketches (as does the construction of this paper; cf. Section III). However it is interesting to obtain a similar result for enriched sketches, as it is done by Kelly in [87].

436.1 ... A similar construction is given by Adamek [131] in a slightly more general situation : he does not necessarily add «all» the limits indexed by $\mathcal{J}$, but only some kinds of such limits for each category.

If $\mathcal{J}$ is the class Cat_0 of all small categories, the $\mathcal{J}$-loose type of σ (which is now in a larger universe) becomes $(Set^{\sigma})^{op}$ itself. This type had been considered by Gabriel & Ulmer [50] and by Lair in his specifiability Theorem [64, 170] ; but these authors did not mention the universal property of this type.

2. *Free completions*: If $\mathcal{J}$ is void, the $\mathcal{J}$-loose type of $\sigma = (\Sigma, \emptyset)$ reduces to a free $\mathcal{J}$-completion of Σ. In O, IV-1, Comment 199.1, several other constructions of such a completion $\hat{\Sigma}$ have been recalled: in particular, it may be obtained as the category $Pro_{\mathcal{J}}\Sigma$ of $\mathcal{J}$-pro-objects of Σ (Grothendieck [35], Vincent [129], Deleanu & Hilton [33]). In a just published paper [193], Tholen generalizes this construction to get a theorem which unifies both the free completion of a category and the universal completion of a concrete functor. More precisely, he proves the following result:

Let $\mathcal{J}$ be an admissible class of categories in the sense: 1 is in $\mathcal{J}$ and the class of all final functors with source in $\mathcal{J}$ is closed under $\mathcal{J}$-colimits; let B be a $\mathcal{J}$-complete category and $U: K \to B$ any given functor (eventually not faithful). Then U may be universally extended (up to an equivalence) into a functor $\hat{U}: \hat{K} \to B$ such that $\hat{K}$ is $\mathcal{J}$-complete and $\hat{U}$ preserves $\mathcal{J}$-limits.
If $B = 1$, then $\hat{K}$ is the free $\mathcal{J}$-completion (or loose $\mathcal{J}$-type) of K. If K is a concrete category via U, then $\hat{K}$ is equivalent to the universal $\mathcal{J}$-completion of U (constructed in [40]); for instance, if $\mathcal{J} = Cat_0$, then $\hat{K}$ is the universal initial completion of K, in the sense of Herrlich [68].

$\hat{K}$ may be described as follows (we slightly modify Tholen's construction by using the atlases defined in Comment 199.1). Let $V: C \to B$ be the projection functor of the comma category $Id_B \downarrow U$ and $Pro_{\mathcal{J}}C$ the category of $\mathcal{J}$-pro-objects of C; its objects are the functors F indexed by elements of $\mathcal{J}$, and its morphisms $F \to F'$ are the atlases

436.1 ... from F to F' (cf. Comment 199.1 in O, IV-1).

If $\Phi : \mathrm{I} \to \mathrm{K}$ is a functor and $\xi : B_\xi \to U.\Phi$ a cone in B, there is a functor $\Phi_\xi : \mathrm{I} \to \mathrm{C}$ defined by

$$I \mapsto (\xi_I : B_\xi \to U\Phi(I), \Phi(I)),$$

$$(k : I \to I') \mapsto \left(B_\xi \begin{array}{c} \xrightarrow{\xi_{I'}} \\ \xrightarrow[\xi_I]{} \end{array} \Big\uparrow U\Phi(k), \Phi(k) \right).$$

Then $\hat{\mathrm{K}}$ is the subcategory of $Pro_\mathfrak{J}\mathrm{C}$ whose objects are the functors Φ_ξ such that ξ be a limit-cone, and whose morphisms $\Phi_\xi \to \Psi_\eta$ are the atlases A such that $V(\mathrm{A})$ is reduced to a unique morphism a of B. The functor $\hat{U}$ sends Φ_ξ to B_ξ and A to a.

3. In [191] Street extends the free completion Theorem to the case of «variable categories».

449.1 + *Mixed sketches and figurative algebras:*

While projective sketches have been widely used, mixed sketches seemed too general a notion and were only used by a few authors (cf. O, IV-1, Comment 31.2). Even «mixed completion Theorems» (adding both limits and colimits) are scarce in the literature, because explicit constructions are not easy.

In the last years, Guitart & Lair [63, 65] have succeeded in developing a powerful theory of mixed sketched structures; it generalizes most of the results known for algebraic structures and also encompasses such structures as fields, topologies (Burroni [19]), local rings and Banach spaces, ... (Diers [34]), first-order logic theories [65] (cf. Comments 31.2, 31.3, 31.4). Here are some new results on them.

1. *The locally free diagram Theorem*: Let σ be a mixed sketch on Σ and $F : \Sigma \to Set$ a functor; then there exists a small diagram, say $D_F : I_F \to Set^\sigma$, in the category of Set-models of σ and a cone from F to D_F in Set^Σ which is sent to a colimit-cone by the functor $Set^\Sigma(\cdot, M)$, for each model M of σ; hence

$$Set^\Sigma(F, M) \approx \underset{\to}{Lim}\ Set^\sigma(D_F \cdot, M).$$

449.1 ... *Set* may be replaced by a suitable category H.

If σ is a projective sketch, $I_F = 1$ and $D_F(0)$ is the associated to F sheaf (cf. Comment 413.1). If the cocones of σ are discrete, the theorem has been obtained by Diers [34]. If the cocones of σ are monomorphic, the existence of this locally free diagram D_F is proved in [63] and, by different methods, in Guitart & Lair [163,I] and in Kelly [166]. The general case is given by Guitart-Lair [163, II] who work with concrete sketches (cf. Comment 31.3), and by Lair [170], who characterizes «mixed-sketchable» categories.

2. *Applications to ultraproducts* (Guitart [160]). The locally free diagram Theorem may be looked at as the existence of a multiadjoint to the functor *lim* from the category $Pro_{\{connected\}}\mathrm{X}$ to Y, where $\mathrm{X} = Set^{\sigma}$ and $\mathrm{Y} = Set^{\Sigma}$. It has suggested to Guitart the following definition: If $(X_\alpha)_{\alpha \in \mathrm{A}}$ is a family of objects of a category X and if U is an ultrafilter on A, the *ultraproduct of* $(X_\alpha)_{\alpha \in \mathrm{A}}$ with respect to U is the object W obtained through a reflection into the representables:

$$\operatorname*{colim}_{U \in \mathrm{U}}(\hat{cone}_{\mathrm{X}}(\cdot, (X_\alpha)_{\alpha \in U})) \to \mathrm{X}(-, W).$$

The generalized ultraproduct of Monk [168] is a reflection into coproducts of representables:

$$\operatorname*{colim}_{U \in \mathrm{U}}(\hat{cone}_{\mathrm{X}}(\cdot, (X_\alpha)_{\alpha \in U})) \to \coprod_{c \in C} \mathrm{X}(-, W_c).$$

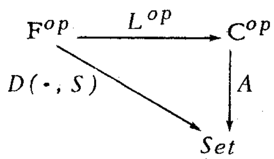
Hence the construction of ultraproducts is englobed in that of locally free diagrams.

In [160], there is defined a notion of «local concept» such that an ultraproduct derives from a «local concept of cone».

3. *Figurative algebras* (Guitart [159, 162]). The theory of mixed sketches is an analytic approach to a geometrical model theory. There is a corresponding synthetic approach, namely the (equivalent) theory of figurative algebras, based on calculus of contacts, incidence relations and motions. It leads to easy descriptions of concrete situa-

449.1 ... tions, such as computer programs [161], puzzles [162], problems in Biology (as I'll prove elsewhere).

A *figuration* [159] T consists of the following data: a category F of figures, a category S of supports, a functor $D: F^{op} \times S \to Set$ (the «drawing» functor) and a functor $L: F \to C$ which is 1-1 onto the objects (the morphisms of $C \backslash L(F)$ are the composition laws). A *figurative algebra* (S, A) of type T (or T-algebra) is a support S and a functor $A: C^{op} \to Set$ such that



commutes. A figure F such that $D(F, \cdot): S \to Set$ is not representable is called a *paradox*.

If there is no paradox for T, the T-algebras are the algebras of a monad on S; in this case, the arities of laws and the sets on which these laws act are symbolized by resp. the figures and the supports; for each support S, the possible domain of a law $c: F' \to F$ is given a priori as the set $D(F, S)$ of drawings, and the action of c becomes the map:

$$A(c)_S: D(F, S) \to D(F', S): d \mapsto dc.$$

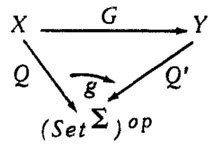
This is well explained in the paper [162], where Guitart vividly analyzes his motivations for introducing figurations (I regret mathematicians scarcely dare to write such direct papers).

The theory of sketched structures and the theory of figurative algebras are equivalent. Given the figuration T, a mixed sketch σ is constructed, whose *Set*-models correspond to T-algebras: its underlying category contains F^{op}, C and the drawings $D(F, S)$ as morphisms from F to S; its distinguished cones describe the «contacts» between figures and its distinguished cocones describe the potential «motions of figures»; generally σ is a large sketch with large indexed cocones.

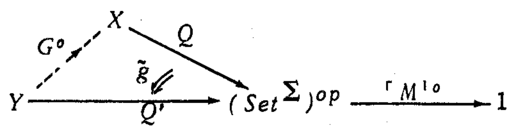
Conversely, if σ is a mixed sketch, its *Set*-models «are» the T-

449.1... algebras for suitable figurations T ; the proof of this result relies on the following remarks :

- There is a concrete sketch C_σ associated to σ (cf. [63] and O, IV-1 Comment 32.3) such that $M: \Sigma \to Set$ is a model of σ iff, for each formula



of C_σ, the composite

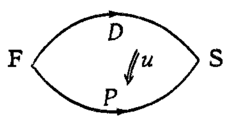


is an isomorphism, where $G \dashv G^o$ in the bicategory of distributors.

- To any figuration T, we associate the distributors

$$L' = \mathrm{C}(L\cdot, L.): \mathrm{F}^{op} \times \mathrm{F} \to Set \quad \text{and} \quad P = D \otimes L'$$

(tensor product of distributors). Then T is also determined by the 2-cell



and a T-algebra on S is identified with a natural transformation

$$a: P(\cdot, S) \to D(\cdot, S) \quad \text{satisfying} \quad a.u(\cdot, S) = Id.$$

Such an algebra may be interpreted as a cosection of

$$\ulcorner S \urcorner^o \otimes u: \ulcorner S \urcorner^o \otimes D \to \ulcorner S \urcorner^o \otimes P$$

(where $\ulcorner S \urcorner$ is the distributor $1 \to \mathrm{S}$ «naming» S), which extends the above notion of satisfaction of a formula.

455.1+ *Completions of categories :*

The corollaries may be translated in terms of completions and cocompletions of categories :

(a) The loose type is the universal, up to an equivalence, solution

455.1... of the problem: to embed a category Σ into a category admitting all $\mathfrak{J}$(-indexed) limits and all $\mathfrak{J}$(-indexed) colimits, with preservation of some given distinguished limits and colimits.

(b) The type is the universal, up to an *isomorphism*, solution of the problem: to embed a category Σ with a partial choice of $\mathfrak{J}$-limits and $\mathfrak{J}$-colimits into a category equipped with a total choice of $\mathfrak{J}$-limits and $\mathfrak{J}$-colimits, so that this total choice extends the given partial choice.

(c) The solution of Problem (b) is also a solution of Problem (a).

456.1. A proof is given in /114/.

457.1+ Models of a sketch which send the distinguished (co) cones to distinguished (co)limit-cones and not to *any* (co)limit-cones were introduced in /93/; the motivation was to get *Cat* as «the» category of models of the sketch σ_{Cat} of categories into *Set* equipped with canonical pullbacks; the category $Set^{\sigma_{Cat}}$ of models of σ_{Cat} into *Set* is only equivalent, not isomorphic, to *Cat*. In fact, the «canonical» models give a better description of concrete structures. But general theorems are more easily proved for the categories H^{σ} of all models of σ in a category H. Regular sketches cumulate both advantages, since the category of canonical models is equivalent to H^{σ}. Usual sketches are regular.

In [98], Lair proves that each mixed cone-bearing neocategory is universally embedded into a regular presketch over an associative neocategory with monomorphic cones and epimorphic cocones.

471.1. It is proved in /114/, Theorem 3 (O, IV-1).

474.1. P' commutes with $\mathfrak{J}$-inductive limits because $\mathfrak{M}^{\sigma}$ is cocomplete (cf. e.g. /114/, Theorem 3) and $\mathfrak{M}^{\sigma} \hookrightarrow \mathfrak{M}^{\Sigma}$ preserves inductive limits by hypothesis.

479.1+ At about the same time, Day [151] obtained a slightly stronger result: Let V' be a reflective subcategory of the symmetric monoidal closed category V and $J: V' \to V$ the reflection functor; let A be a dense subcategory of V and A' a strongly cogenerating class in V'; then V' admits a monoidal closed structure for which $V' \hookrightarrow V$ is

479.1... enriched iff the following condition is satisfied: for all objects s of A and s' of A', there exists an object $D(s', s)$ of V' and a natural isomorphism

$$Hom(J(e \otimes s), s') \xrightarrow{\simeq} Hom(J(e), D(s', s))$$

for each object e of V. If V is cartesian closed, this condition is also equivalent to: the reflection J preserves finite products. Day applies his results to construct cartesian closed categories of topological spaces.

487.1. To ask for $V^{\sigma} \hookrightarrow V^{\Sigma}$ to admit a right adjoint is a very strong condition. It will be relaxed in Section 11.

491.1+ This corollary and Proposition 21 have been refined by Street. In [123], Theorem 3.11, he proves the equivalence of the conditions:

1. Set^{σ} is cartesian closed (or equivalently, by Proposition 21, σ is cartesian);
2. V^{σ} is cartesian closed for a complete cartesian closed, locally small category V ;
3. The left adjoint to the inclusion $V^{\sigma} \hookrightarrow V^{\Sigma}$ preserves finite products for any cartesian closed and locally presentable category V.
4. For any category V as in 2 and any model F of σ in V, the functor $D(s, F\cdot)$ is a model of σ in V, where D is the internal Hom-functor in V and s an object of V.
5. Condition 3 (resp. 4) is just satisfied for V = *Set*.

492.1+ *On monoidal closed structures:*

The conditions given in Foltz & Lair's paper [46] do not insure that V^{σ} be monoidal closed; they have corrected this paper in [99]: From a double costructure C in V^{σ} (that is, a model in V^{σ} of the opposite of the tensor product $\sigma \otimes \sigma$), there is deduced an internal Hom functor

$$V^{\sigma}(-,-) : V^{\sigma} \times V^{\sigma\, op} \to V^{\sigma}$$

whose partial functors $V^{\sigma}(-, s)$ and $V^{\sigma}(s, -)$ admit adjoints for each object s of V. This biclosed structure is a monoidal closed

492.1... one iff C satisfies some symmetry, associativity, and unitarity conditions.

These conditions are analyzed with some detail in [101], thanks to the notion of a *virtual morphism* $\sigma \to \sigma'$ between sketches (it is a morphism $g: V^{\sigma'} \to V^{\sigma}$). They express the fact that σ is a virtual commutative monoid, for the composition $k: \sigma \otimes \sigma \to \sigma$ deduced from C; hence k is defined by

$$k(F)(x,y) = \int_z \mathrm{V}(\mathrm{C}(x,y)(z), F(z))$$

for each model $F: \sigma \to V$ and each $(x,y) \in \Sigma \times \Sigma$.

Applications are given in [46, 47]; in particular: the categories of groups, of monoids, of rings admit no monoidal biclosed structures; the categories of abelian groups and abelian monoids admit none but the classical one. (The absence of a symmetric monoidal closed structure on groups and monoids is also shown by Rosičky [182].) *Cat* admits two monoidal biclosed structures. The categories of non-commutative Lawvere algebraic structures do not admit a symmetric monoidal closed structure with the free object on 1 as a unit. This last result strengthens Linton's [173] who has proved that a category of Lawvere algebraic structures has a symmetric monoidal closed structure in which the tensor product represents the bihomomorphisms iff the theory is commutative.

For other generalizations of tensor products, cf. O, IV-1, Comment 160.1.

497.1. Read: B' and $e \in B'_0$ i.o. of B'.

498.1. The internal Hom functor on the cartesian closed category of multiple categories is described in /120/.

503.1. This equivalence, constructed in /93/, means that an internal category in V is an object of category in the sense of Grothendieck.

506.1. The internal category of quartets is also constructed in /113/.

507.1. The construction of E was suggested by the «concrete» construction of the internal Hom between structured (i.e., concrete internal) categories given in /109/.

ON / 118 / : TENSOR PRODUCTS OF TOPOLOGICAL RINGOIDS.

In [172], Lellahi asked for a construction of tensor products in the category of topological ringoids. The present paper gives an answer; it was published in the same issue of the *Cahiers* as [172].

519.1. These examples are developed in Lellahi's paper [172]. In his Thesis [106] he defined the sketch of *ringoids* whose models in *Set* are the additive categories (= *Ab*-categories); its models in a concrete category H are the *internal ringoids in* H; *topological ringoids* are the concrete internal ringoids in *Top*.

519.2. Cf. O, IV-1, Comment 160.1.

520.1 + *Monoidal closed structures on Top:*

We first refer to: O, II-2, Comment 672.2, to O, III-2, Comment 703.1 and to Comment 521.1 hereafter.

In [128], Greve proves that each partial monoidal closed structure on *Top* may be extended into a global one, and that there are as many monoidal closed structures on *Top* as there are functors $Top \to Set$.

In [184], Schwarz precises the fact that *Top* is not cartesian closed thanks to the following characterization of *exponentiable topological spaces: If T is a topological space, the product functor $T\times\cdot : Top \to Top$ has a right adjoint iff, for each topological space T', the quasi-topology of local convergence $\lambda(T, T')$ on the space of continuous maps $T \to T'$* (cf. /81, 92/) *is a topology.* In particular, if T is completely regular or is T_2, then T is exponentiable iff T is locally compact. The theorem is deduced from a characterization of exponentiable objects in initialstructured categories which generalizes Herrlich [167] and Nel's [179] result on cartesian closedness.

521.1. $T'\times_c T$ has been defined by Brown [142].

521.2 + *Other topologies on $E'\times E$:*

The topology $T'_{\sigma'}\times_\sigma T$ defined by Tanré [190] is the final topology with respect to all the insertions

$T'\times\Sigma \hookrightarrow E'\times E, \quad \Sigma'\times T \hookrightarrow E'\times E,$ where $\Sigma\in\sigma,\ \Sigma'\in\sigma'$.

Booth and Tillotson [135] consider another topology: they replace the

521.2... data of a class σ of subspaces of T by the data of a class K of topological spaces. Then the topology $T' \times_K T$ on $E' \times E$ is defined as the final topology with respect to all maps of the form:

$$Id_{E'} \times f : T' \times A \to E' \times E, \quad \text{where } f: A \to T, \ A \in K,$$
$$\text{and } \{x'\} \times T \hookrightarrow E' \times E, \quad \text{where } x' \in E'.$$

They prove the analogues of Theorem 3 and 4 for $T' \times_K T$ (the assertions are simplified thanks to the independence of K on T):
- If K is a class of locally compact spaces closed under finite cartesian products, the K-product $\times_K$ on *Top* is associative up to homeomorphism.
- K is regular (which corresponds to c-stable) if, for each $A \in K$, every neighborhood of $a \in A$ contains a closed neighborhood C with a surjective map $B \to C$ for a $B \in K$. Let K be a regular class of compact spaces; then the functor $- \times_K T: Top \to Top$ has a right adjoint $C_K(T, -)$; a basis of open neighborhoods of $g: T \to T'$ in $C_K(T, T')$ is generated by the

$$W(f, U) = \{ h: T \to T' \mid hf(A) \subset U \}$$

such that $gf(A) \subset U$, where $f: A \to T$, $A \in K$, and U is open in T'. Moreover, if K is closed under finite cartesian products, then *Top* has a monoidal closed structure with $\times_K$ as its tensor and $C_K(-,-)$ as its internal Hom.

If K is reduced to the one-point compactification of N with the discrete topology, the reflective hull of K in *Top* is the category *Seq* of sequential spaces; it is the category of metrizable spaces and their quotients. It is deduced from above that *Seq* is the smallest cartesian closed category which contains all CW-complexes, all differentiable manifolds and metric spaces; hence *Seq is the smallest convenient category* of topological spaces, in Steenrod's sense [189].

532.1. The definition of $A' \otimes_\sigma A$ has been suggested by the construction of the tensor product of topological vector spaces in [154] and the definition of $\mathrm{Hom}_\sigma(A, A')$ mimicks the construction of the internal Hom in the categories of concrete internal categories described in /109/.

538.1 + *Another monoidal closed structure on RdT:*

With the notations of Comment 521.2, let K be a regular class of com-

538.1... pact spaces closed under cartesian products. The results of Section 2 remain valid (with similar proofs) if everywhere we replace the topology $T' \times_\sigma T$ by $T' \times_K T$. We associate in this way, to a pair of topological ringoids $A = (A, T)$ and $A' = (A', T')$:

- a topological ringoid $A' \otimes_K A$ on $A' \otimes A$,
- a topological ringoid $\mathrm{Hom}_K(A, A')$ on $Hom(A, A')$, whose topology is induced by the topology $C_K(T, T'^4)$.

The same proof gives:

THEOREM. *The category* RdT *of topological ringoids admits a monoidal closed structure whose tensor is* $\otimes_K$ *and whose internal Hom is* Hom_K.

541.1. The equivalence between the notions of *Top*-ringoids and of topological ringoids with discrete objects is deduced from the equivalence between categories enriched in a cartesian category H with commuting coproducts (in Penon's sense [180]), and internal categories in H with a discrete (i.e., coproduct of 1) object of objects; cf. / 120 / Appendix and for a generalization, Proposition B-6, Synopsis of O, III-2.

543.1. Theorem of Comment 538.1 is also obtained as an application of 4 to the monoidal closed category $(Top, \times_K, C_K)$.

ON / 117 / : MULTIPLE FUNCTORS I.

This paper is the first paper of a 4 parts work. It was written in 74, as well as most of Part II, and we also obtained some intricate enough results related to monoidal closed structures on the category of double categories. However personal and professional problems, linked to Charles' retirement of the University Paris 7 and his subsequent teaching and administrative function in Amiens, prevented us to do much research work in 75-77. When we came back to the writing of the following parts in 1978, new ideas completely modified our initial project, and led us to a general study of multiple categories published in / 119, 120, 121/.

545.1. At the time this paper was written, there was some scepticism about the usefulness of double categories, except for 2-categories. Later on double categories and even n-fold categories were studied in connection with Homotopy Theory (cf. Comments 562.1, 603.1, 619.1, 648.1, 729.1).

546.1. The informations given here on the third part do not correspond to the published Parts III and IV. Indeed, a better understanding of the situation led us to an explicit description of the internal Hom of the cartesian closed category Cat_n of n-fold categories for any n (not only $n = 2$) via the Link functor and the category of multiple categories / 120 /. It suggested how to laxify the construction for getting monoidal closed structures on Cat_n. As a by-product, we characterized all double categories as being the double subcategories of the double category of squares of a 2-category. But we did not publish the applications to double-sketched structures.

547.1. It is the last paper where the notation $C(e', e)$ is used, instead of the standard one $C(e, e')$.

548.1. We often use this symbol, because it is both standard and adapted to the right-to-left notation we favored for morphisms.

553.1. Propositions 2 and 3 were suggested by Propositions 1.9 and 4.9 of / 93 / (summarized in Proposition 9 hereafter), which take the category case. Charles exposed them in his late sixties lectures.

554.1. Condition 1 «is» the *Yoneda Lemma for sketched structures.*

555.1+ *Tensor products of sketches:*

The tensor product sketch $\sigma' \otimes \sigma$, introduced by Conduché [26] and Lair [95], is used by Lair in [98] to equip categories of sketches with a monoidal closed structure. More precisely, he proves:

- The category $\mathcal{O}^{\mathfrak{J}\mathfrak{J}'}$ of $(\mathfrak{J}, \mathfrak{J}')$-cone bearing neocategories admits a monoidal closed structure whose tensor is the above $\otimes$ and whose internal Hom D is such that the underlying neocategory of $D(\sigma, \sigma')$ is the neocategory of morphisms $\sigma \to \sigma'$.

- An object σ of $\mathcal{O}^{\mathfrak{J}\mathfrak{J}'}$ is *strict* if its underlying neocategory Σ is associative, if its distinguished cones are monomorphic, its cocones epimorphic, and if there is at most one distinguished (co)cone with a given basis (whence σ is 1-1 embedded in its type); it is *complete* if Σ is an associative neocategory with a good definition of invertible morphisms and if every (co)cone isomorphic to a distinguished one is distinguished. Let

555.1... $\mathcal{Y}^{\mathfrak{gg}'}$ and $\mathcal{Z}^{\mathfrak{gg}'}$ be the full subcategories of $\mathcal{O}^{\mathfrak{gg}'}$ whose objects are resp., the complete and the strict cone-bearing neocategories. Both are reflective subcategories, and they admit a monoidal closed structure in which the internal Hom is a restriction of D and the tensor product of (σ, σ') is the reflection of $\sigma \otimes \sigma'$.

557.1. Internal categories are studied in / 113 /, which contains the results of this Section C.

562.1 + *Homotopy type of a category:*

The singular functor N from *Cat* to the category *Simp* of simplicial sets, generally called the *nerve functor*, is useful in Algebraic Topology. (Cf. e.g. Gabriel & Zisman [51].) Its adjoint $R_c : Simp \to Cat$ sends a simplicial set X to the category obtained as follows: GX is the graph with vertices the 0-simplices and arrows the 1-simplices x, looked at as $x : d_1 x \to d_0 x$; then R_cX is the quotient category of the free category on GX by the equivalence generated by:

$$d_1 \xi \sim (d_0 \xi, d_2 \xi) \quad \text{for each 2-simplex } \xi \text{ of X}.$$

Simp is a *closed model category* in the sense of Quillen [182]; its weak equivalences are those maps whose image by the geometric realization functor $R : Simp \to Top$ is a homotopy equivalence. The composite:

$$B = (Cat \xrightarrow{N} Simp \xrightarrow{R} Top)$$

is called the *classifying space functor*; the homotopy type of BC, for a category C, is the *homotopy type of* C (cp. with the homotopy of C defined by Evrard [155] and Hoff [169]).

A functor f is called a *weak homotopy equivalence* if Bf is a homotopy equivalence. These functors are part of a closed model structure on *Cat*, which is constructed as follows by Thomason [195]: He considers the functor

$$Ex^2 N = (Cat \xrightarrow{N} Simp \xrightarrow{Ex^2} Simp)$$

and its adjoint $R_c Sd^2$, where Sd^2 is the iterated «subdivision functor», and Ex^2 its right adjoint; the unit and counit of this adjunction are weak homotopy equivalences. The closed model structure of *Simp* is lifted by

562.1... this adjoint pair to the required model closed structure on *Cat*. In particular, the corresponding cofibrant categories are posets.

562.2. The terminology is different according to the papers. In earlier articles / 57, 122 / , C is called an operator category on A only if the projection $A \to C_0$ is onto ; and it is a species of structures if it exists any subcategory C' of C which acts on A (while here C' must contain the *f* with their source in the image of A). Cf. Comment 206.1 in O, II-1.

568.1. Comments on internal category actions are given in O, III-1, Comment 25.1 and in O, III-2, Comments 475.1 and 478.3.

570.1. Cf. O, III-2, Comments 490.1, 490.2 and 470.3.

570.2. Cf. O, III-2, Comment 491.1 and O, II-2, Comment 449.2.

581.1. The definition by Charles of the square product of two categories was an essential step toward our understanding of lax transformations. It led to the definition of the $(n+m)$-fold category square product of an n-fold category and an m-fold category, which is the tensor of a monoidal closed structure on the category of all multiple categories / 119/; the corresponding internal Hom is used to describe the internal Hom of the cartesian closed structure, and of its laxified monoidal closed structures, on the category of n-fold categories in / 120, 121 /.

Cf. Appendix of / 119 / for a more abstract definition of A ■ B.

585.1. ■ and T_{11} are restrictions of the tensor product and of the internal Hom of a monoidal closed structure on the category of all multiple categories (cf. / 119 /, Proposition 7).

590.1+ *Representable 2-categories* were introduced by Gray, and the definition given here is a generalization to the case of double categories.

The category of 1-morphisms of a representable 2-category is often monoidal closed. The converse assertion is studied by Variot in his thesis [197]. In particular, he proves the following theorem : Let V be a cartesian closed category equipped with a cocategory in V whose object of objects is the final object. Then V is the category of 1-morphisms of a representable and corepresentable 2-category.

601.1. This « structural » proof is generalized to obtain a lax-completeness theorem for multiple categories in Proposition 11 of / 119 /. Cf also 648.1.

603.1 + *Connections in double categories:*

Further results on the double categories of up-squares (or of « quintets » in the primitive terminology /64/) are recalled in O, III-1, Comment 105.1, in particular Spencer's characterization [187] of those double categories as the double categories with a « special » connection; hence the category DC of double categories with a connection is equivalent to the category of 2-categories. Cf. also Comment 766.1.

Connections on a double category are introduced by Brown [147] for generalizing homotopy theory; they are an abstraction of the *path-connections* defined by Virsik [198] in Differential Geometry.

It is proved in [147] that DC admits a full subcategory DG' equivalent to the category of crossed modules; its objects are the double groupoids with a connection and only one vertex. Later papers of Brown and Higgins give generalizations to higher order and applications in Algebraic Topology (cf. Comments 649.1 and 727.1).

610.1 + *Lax completion of a category:*

Proposition 8 implies the existence of quasi-limits (also called *lax limits*) for 2-functors from a discrete 2-category. This theorem had been exposed by Gray in 1971 (e.g., in his Paris lectures) and it has triggered much research afterwards, all the more since the sketchy proof given by Gray at that time was not easily understandable. It has been extended to 2-functors with domain *any* 2-category by Bourn [136] and, independently, by Street [190] and Gray [156]; the general case will also follow from Proposition 11 of / 119 / (cf. Example b, page 647), and more comments will be indicated there (Comment 648.1).

The problem of embedding a category A into a 2-category which admits lax colimits indexed by some types $\mathcal{J}$ of categories is universally solved by Guitart & Van den Bril [66]; the *$\mathcal{J}$-lax cocompletion of* A is the (full) sub-2-category $D_{\mathcal{J}}A$ of the 2-category of diagrams of A whose objects are the diagrams with domain in $\mathcal{J}$ (cf. O, IV-1, Comment 199.1).

612.1. The problem of defining a general cohomology has been tackled by many authors (cf. O, III-2, Comments 450.6, 579.3, and II-2, Comment 499.3).

I have recently heard of a series of letters from Grothendieck to Brynn in which he stresses the interest to develop a theory of multiple categories to study this problem. In [164] Guitart & Van den Bril suggest a solution via their satellisation process.

615.1. Lax morphisms between sketched structures are studied by Guitart & Lair in [64]; their idea is summarized in O, IV-1, Comment 32.1.

618.1. Dicategories realize the idea of a *homotopy associative category.* In [175], Malraison stressed the interest of such a structure for studying higher order homotopy.

619.1+ *Laxification and coherence problems:*

Proposition 15 replaces a lax double functor $A \to B$ by a cartesian functor $K(A) \to K(B)$; in this way, both the domain and codomain are modified. If A and B are 2-categories, a lax functor $A \to B$ may be replaced by a 2-functor with the same codomain B. More precisely, Gray [53] (and Bénabou, unpublished) associate to a 2-category A a universal 2-category ΔA such that, for any 2-category B, there is an isomorphism between the categories of lax functors $A \to B$ and of 2-functors $\Delta A \to B$. Explicit constructions of ΔA are given by Vaugelade [128], Penon and Bourn [136] (via the 2-sketch of adjoint functors), Street [190] (using the notion of a *computad*), Van den Bril in his Thesis [196].

Street's construction pasts together the 2-cells of A. If A is a category (looked at as a discrete 2-category), the 1-morphisms of ΔA are the paths in A, and the 2-morphisms

$$(g_k, \dots, g_0) \to (f_n, \dots, f_0)$$

correspond to the sequences $i_0 = 0 < i_1 < \dots < i_k < n+1 = i_{k+1}$ such that

$$g_j = f_{i_{j+1}-1} \cdots f_{i_j} \quad \text{for each } j \leq k.$$

ΔA is called the *laxification of* A.

Let N_* be the functor from the category *Cat-Cat* of 2-categories to the category *Simp-Cat* of categories enriched in the monoidal closed category of simplicial sets, deduced from the nerve functor $N: Cat \to Simp$ (cf. Comment 562.1). In [148] Cordier compares $N_*(\Delta A)$ with the simplicial category $S(A)$ associated to a category A by Dwyer & Kan [153]. He

619.1... uses this comparison to prove that the simplicial functors from $S(A)$ to the simplicial category Top_S on *Top* are in 1-1 correspondence with the *homotopically coherent diagrams* $A \to Top$ considered by Vogt [199]. Applications to the homotopy limits are given in [149] (cf. Comment 648.1).

ON / 119 / : MULTIPLE FUNCTORS II.

This paper is the sequel of / 117 /, though it may be read independently of it. Its final version has been written four years later. To help the reader we decided to adopt standard notations, in particular for the Hom sets.

623.1. A primitive version directly dealt with multiple internal sketched structures, but Charles thought it was too abstract (cf. Comment 659.1).

625.1. Other «topological» examples of double categories are given by Brown and Spencer [147].

633.1. ■ does not extend into a tensor product on *M Cat* itself: Indeed, if $f: A \to A'$ is a multiple functor in which A has a smaller multiplicity than A', then $g \times f$ does not define a multiple functor $B \blacksquare A \to B' \blacksquare A'$.

635.1. This functor does not extend into an internal Hom in *M Cat*, because $Hom(g, f)$ does not define a multiple functor $Hom(A, B) \to Hom(A', B')$ if A' has a smaller multiplicity than A.

647.1. Here, K^0 is the total category of the 2-category K and K^1 the coproduct of the Hom categories. The two compositions are inverted with respect to the convention adopted in / 117, 120, 121/.

647.2. Cf. / 121/, Proposition 4 and Remark 2, page 750.

648.1 + *Lax limits and homotopy limits:*

1° *Construction of lax limits.* Bourn [136] proves that a representable 2-category is (finitely) lax complete iff its category of 1-morphisms is (finitely) complete; his explicit construction of the lax limit has suggested the proof of Proposition 8 in / 117 /. At the same time, Gray did a similar construction to generalize his 1971 result (the ancestor of all those theorems) on the existence of lax limits for 2-functors with domain a (discrete 2-) category. (Cf. Comment 610.1.)

The idea of the more structural proof given here (in Proposition 11 for

648.1... multiple categories) is sketched in / 117 /, Remark 2, Page 601. Independently, Street [190] gave an abstract existence proof, and more generally he showed that a finitely lax complete 2-category admits all J-indexed limits for any finitary 2-functor $J : A \to Cat$; the indexed limit is explicitly constructed thanks to a presentation via computads of the 2-category cone of J.

Street's paper led Gray to about the same existence proof as ours; when we received the first version of Gray's paper [156], we were finishing / 121 /, so we added a Note in it (page 723) to compare both methods. In [156] Gray also gives the following algorithm for constructing lax limits: Let K be a representable 2-category; the lax-limit of a 2-functor $I: A \to K$ is the (ordinary) limit of the functor $PR(I): Prol\, A \to K_1$ constructed in 3 steps: a) the representability of K is used to define a lax functor l from K to the bicategory $Span\, K_1$ of spans of K_1 (= category of 1-morphisms of K); b) To a 2-category A there is (functorially) associated a category *Prol* A and a functor *R* from the category of lax functors $A \to Span\, K_1$ to the category of left exact functors $Prol\, A \to K_1$; if A «is» a category, *Prol* A is the discrete fibration associated to the corresponding *Set*-model of the sketch of categories; c) the functor $PR(I)$ is the image by *R* of the lax functor

$$A \xrightarrow{I} K \xrightarrow{l} Span\, K_1 .$$

Gray applies this algorithm to describe indexed limits, lax ends (which have also been studied in the Theses of Bozapalides [141] and Sirot [187] under the name of cartesian ends). In [140], Bozapalides constructs lax (= quasi-) limits in the bicategory of profunctors (or distributors) *Dist*, and in the category of V-profunctors, when V is a monoidal category. For limits in *Dist*, cf. also Vaugelade's Thesis [127].

2. *Homotopy limits.* There is a clear analogy between the notions of lax limits and of homotopy limits, which were introduced: by Puppe, in special cases [181], by Bousfield & Kan [139] and by Vogt [199] for homotopically coherent diagrams. In [194], Thomason proved that the Nerve functor carries the lax limits in *Cat* to homotopy limits in the category *Simp* of simplicial sets, up to homotopy. His paper led Gray [159] to

648.1... define homotopy (co)limits for simplicial functors $A \to B$, where A is a small simplicial category and B a complete and cocomplete simplicial category by means of indexed limits; they are calculated via the replacement scheme. The Bousfield & Kan notion is found anew if A is the free simplicial category on a small category, and the relations between homotopy limits and lax limits are exhaustively treated.

Gray's definition cuts off the coherence at level 2 and does not lead to an analogue of the Bousfield-Kan spectral sequences. Bourn and Cordier [138] propose a definition which remedies these defects (a special case of it appears in Segal [185]); they still define the homotopy limit of a simplicial functor as an indexed limit, but in the context of simplicial profunctors (instead of simplicial functors). They show how coherence in homotopy limits corresponds to the intuitive geometric idea of a homotopy cone, and how their definition «is» the replacement scheme. Gray's homotopy limits, and lax limits (in the case of 2-categories) are examples of this situation.

Vogt's definition is more topologically flavored, since it is given for homotopically coherent diagrams $F: A \to Top$. Cordier, who was motivated by strong shape theory, has proved in [148] that F may be replaced by a simplicial functor $S(A) \to Top_S$ (cf. Comment 619.1). In [149] he constructs a simplicial functor $\Phi: S(A) \to Simp$ (thanks to Artin & Mazur's total object of a bisimplicial space [133]) such that Vogt's homotopy limit of F appear as a Φ-indexed limit; thus the different definitions are unified.

In [137], Bourn shows that, if V is a monoidal category, and if $\Phi: A \to V$ is a V-functor, there is a V-monoid which acts on each Φ-indexed limits. In the case V is *Simp* and Φ is Gray's indexation of homotopy limits, it follows each homotopy limit is acted upon by the canonical simplicial monoid with one object $*$, generated by a 1-morphism t and a 2-cell $t \to *$. A consequence is that *coherent homotopy idempotents split in the category Ho-Top* (although homotopy idempotents may not split).

Homotopy pullbacks and pushouts have been studied by several authors, in particular Mather and Walkers [176, 177], Spencer and Wong [188, 200] (cf. Comment 729.1).

649.1 + *∞-categories and Van Kampen Theorems :*

∞-categories are those N-fold categories X such that the objects of the n-th category be also objects of the $(n+1)$-th category, for each integer n. The *∞-groupoids* considered by Brown and Higgins in several papers are the ∞-categories X such that all the categories X^n be groupoids and X be the union of their objects. They are used as an essential tool for getting an algebraisation of geometric constructions in higher order homotopy theory, and an n-dimensional Van Kampen Theorem.

A version of this theorem for $n = 2$ was proved by Brown & Spencer [147] by means of the equivalence :

crossed modules $\longleftrightarrow$ double groupoids with 1 vertex and a connection.

The generalization to any n lies on the equivalence of the 6 following categories :

$$\begin{array}{ccccc} \infty\text{-groupoids} & \overset{a}{\longleftrightarrow} & \omega\text{-groupoids} & \overset{b}{\longleftrightarrow} & \text{cubical T-complexes} \\ & & \updownarrow c & & \\ & & \text{crossed complexes} & \overset{d}{\longleftrightarrow} & \text{simplicial T-complexes.} \end{array}$$

ω-groupoids are cubical complexes whose n-cubes form an n-fold groupoid with $(n-1)$ commuting connections, and *T-complexes* are special Kan complexes. The equivalence a is proved in [146], b in [17], c in [144] and d in Ashley's Thesis [134]. (Cf. Brown [143] for an introduction to the simplicial T-complexes, defined by Dakin [150].)

The main results of [145] are the following ones : Let X be a filtered space such that each loop in X_0 is contractible in X_1. A new homotopy invariant, the *homotopy ω-groupoid* ρX, is defined; the *homotopy crossed complex* πX, image of ρX by the equivalence c, plays the rôle of the fundamental group for a topological space. The *n-dimensional Seifert-Van Kampen Theorem* describes πX as, under certain circumstances, the colimit of the homotopy crossed complexes of some filtered subspaces of X. As special cases, this theorem includes the Brouwer degree Theorem and the relative Hurewicz Theorem.

659.1. A first version of the intended paper was written in 78, but it was not completed because of Charles' illness. Essentially, it contained a generalization of the square product and of the internal «object of natural transformations» to σ-structures in a monoidal closed categroy V, for any sketch σ ; whence the monoidal closed category of multiple σ-structures in V, if V is complete enough.

ON / 120/ : MULTIPLE FUNCTORS III.

This paper is entirely devoted to the construction of the internal Hom of the cartesian closed category of n-fold categories. There are few Comments, because all the notions have already been introduced in the precedent papers, and commented upon there.

663.1. Cf. / 115 / , Part III, Propositions 26 and 29.

681.1. An n-category A is often defined more restrictively, requiring that the objects of A^{i+1} be objects of A^i for each $i < n$ (or with the inverse order on the compositions, so that the definition extends to the case of ∞-categories ; cf. Comment 649.1).

690.1. This multiple category of frames has recently been used by Cordier to study coherence problems.

695.1. Another construction of the internal Hom on Cat_n (lookedas the category of internal categories in Cat_{n-1}) may be deduced by iteration from / 115/ , Part III, Proposition 29.

704.1. The results of this Appendix are both generalized and simplified in the Synopsis of O, III-2, Section 6 ; cf. also the two following comments.

704.2. The following shorter and less computational construction is deduced from Corollary of Proposition B.6, Synopsis of O. III-2 :

Let $\pi : l\,A_0 \to \Sigma$ be the discrete fibration corresponding to the *Set* model of the sketch of categories which defines the groupoid of pairs of A_0 ; the V-category A looked at as an A_0-polyspan (in Bénabou's sense) determines a functor $a : l\,A_0 \to V$ (cf. O, III-2, page 857), then the left extension of a along π is the model of the internal category ΓA in V.

710.1. A «structural» proof is obtained from the more general Proposition A.6 and Corollary, Proposition B.6 of the Synopsis, O, III-2 : the V-category $\Gamma' B$ is a «relative» right Kan extension. This proof is valid as soon as V is cartesian (commuting coproducts are necessary to define the functor Γ, but not for Γ').

ON / 121 / : MULTIPLE FUNCTORS IV.

The last part of this paper was written at the hospital, during the final illness of Charles.

723.1. Cf. Comment 648.1.

729.1 + *Homotopy commutative cubes :*

Cubes in a double category with connection (hence in the double category of squares of a 2-category C [147]) are used by Spencer-Wong [188] to develop the abstract theory of homotopy pullbacks and pushouts introduced by Spencer in [187].

The usual notion of a homotopy commutative cube in the topological setting is found back if C is the 2-category of topological spaces (the 2-cells being homotopy classes); Leitch's generalization [171] also enters this frame.

Thanks to these (homotopy commutative) cubes, Mather's cube Theeorems [176] linking topological homotopy pullbacks and pushouts are transposed by Wong to chain homotopy pullbacks and pushouts [200], and by Müller in the frame of abelian categories with a homotopy system [178].

733.1. A quasi-category, or category without identities /93, 100/, is a graph equipped with an associative composition defined for adjacent arrows. The *free quasi-category of paths* of a graph G consists of *all* the paths of G, and not only paths without identities in them as the free category of paths of G.

751.1. This particular case (Bourn's Proposition) states that a representable 2-category whose category of 1-morphisms is (finitely) complete is lax (finitely) complete. Commentaries on this theorem and on lax limits are given in Comments 610.1 and 648.1.

762.1 + *The tensor product of 2-categories:*

The category *2-Cat* of 2-categories admits a non-symmetrical monoidal closed structure: the tensor product $B \hat{\otimes} A$ of the 2-categories B and A is the reflection in *2-Cat* of the double category $B \otimes A$, and the internal Hom is $Fun(A, B) \subset LaxHom_2(A, B)$. This monoidal closed category on *2-Cat* has been introduced by Gray; in [53], he gives two constructions of $B \hat{\otimes} A$: an explicit one in terms of cells and relations, and another one, using fibred categories (which he says was inspired by a more general unpublished assertion of Bénabou); he also examines the relations between the two internal Hom.

Note that the method of Proposition 7 leads to yet another explicit construction of $B \hat{\otimes} A$ as the 2-category reflection of $B \otimes A$. In fact, our study of monoidal closed structures on categories of *n*-fold categories stemmed from a desire to generalize Gray's results on 2-categories to double categories.

762.2. Laxified cubes were introduced in the hope of getting a notion of lax functors such that a composition be defined on them.

764.1 + *Lax bimodules:*

The String functor is used by Guitart and Van den Bril in a recent paper [164]. They prove that a *lax bimodule* $W: X \dashrightarrow Y$ between 2-categories is in 1-1 correspondence with a 2-functor

$$\hat{W}: C = String((\Delta X)^{2op} \blacksquare \Delta(Y^{op})) \to Cat,$$

where ΔX is the laxification of X (cf. Comment 619.1); hence C solves the coherence problems for lax bimodules.

If $X = Y$, they construct the *glueing* GW of W (or: *lax crossed product*) as the lax colimit of $\hat{W}$. The fibration (crossed product) associated to a *Cat*-valued functor and the Kleisli constructions on a monad or a comonad are examples of glueings. This construction leads to a presentation of a bimodule $Cat \dashrightarrow Cat$ which is used to define a non-abelian cohomology of any order in *Cat*, via satellisation [164].

766.1. Another characterization of double categories may also be deduced from Proposition 9: *Any double category is isomorphic to a double sub*

766.1... *category of the double category Q(Cat) of quintets* (= squares of the 2-category *Cat*); in other terms, *Q(Cat)* is a «universal double category». For the proof, cf. O, III-1, Comment 105.1.

766.2. The alluded to characterization is intricate enough; we have not published it at that time because of Charles' illness. Now I think it could be simplified if one used Brown & Higgins homotopy ω-groupoids (cf. [145] and Comment 649.1). These authors are working on explicit constructions of the tensor product of ∞-groupoids (private communication).

ON /142, 143, 144/.

/142/ and /143/ are Introductions to the Proceedings of the two «Colloques sur l'Algèbre des Catégories» which we organized in Amiens in 1973 and 1975 (a third one, dedicated to Charles, was held in 1980).

/144/ is an Introduction to the «Journées Théorie et Applications des Catégories» at Chantilly, which completed the 75 Amiens Conference.

ON /139/ : DÉJÀ 20 ANS.

This Note was written to commemorate the 20^{th} anniversary of the creation of the «Cahiers de Topologie et Géométrie Différentielle».

SYNOPSIS

The 6 papers reproduced in this volume, which represent Charles's last works (72-79), are devoted to the theory of sketched structures and its applications; they are long articles, with explicit proofs and standard terminology.

Parts I and II of / 115 / refine the completion theorems of /102, 106/ ; Part III gives existence theorems for monoidal closed structures on categories of sketched structures, with applications to categories of internal categories.

The 5 other papers deal with concrete constructions of such monoidal closed structures: on categories of topological ringoids in / 118 /, and specially on categories of multiple categories in the series / 117, 119, 120, 121/ which contains the more important results.

In the following summary, numbers between brackets refer to the main comments.

1. SKETCHES, TYPES AND COMPLETIONS / 115/.

Sketches are intended to offer one presentation of a general algebraic structure, while types correspond to its theory. A *sketch* was defined in / 106/ as a neocategory equipped with a partial choice of cones and cocones (= inductive cones), and it was proved that a sketch is universally embedded into a *prototype* and into a *type* (cf. Synopsis of Part IV - 1). The existence proof of / 106 / required the unicity of the distinguished (co)cone with a given basis. In / 115 / we succeed in deleting this restrictive, and often cumbersome, condition, thanks to a more constructive approach.

More precisely, a *mixed cone-bearing neocategory* σ is a neocategory Σ equipped with a set Γ of (projective) cones and a set Δ of cocones. It is a *limit-bearing category* if Σ is a category and if the (co)cones are (co)limit-cones, a *loose* (I, J)*-type* (or theory) if moreover each functor with its domain in the given class I (resp. J) is the basis of at least one distinguished cone (resp. cocone). Morphisms between cone-bearing neocategories are neofunctors which preserve the distinguished (co)cones.

In Parts I and II of / 115 /, we construct by transfinite induction an embedding of a cone-bearing neocategory σ into:

- a limit-bearing category $\bar{\sigma}$, a sketch $\bar{\bar{\sigma}}$, a prototype π, an (I,J)-type τ, which are universally defined up to an *isomorphism*,

- a loose (I,J)-type τ' which is universally defined up to an *equivalence.*

If the embedding $\sigma \to \pi$ is 1-1, it is proved that π and $\bar{\sigma}$ are isomorphic, while the type τ and the loose type τ' are equivalent.

If the distinguished (co)cones on the category Σ are (co)limitcones the loose type τ' affords a *universal, up to equivalence, (I,J)-completion* of Σ with preservation of the given (co)limits; the type τ is a universal, up to isomorphism, (I,J)-completion of Σ with preservation of a given partial choice of (co)limits.

Recently, more direct constructions of the loose projective I-type of a limit-bearing category σ (there are no distinguished cocones) have been given by several authors (436.1); they don't extend to the mixed case.

From these mixed completion theorems, we deduce a 2-adjunction between the representable and corepresentable 2-category of cone-bearing categories and some of its full sub-2-categories / 115/.

2. CATEGORIES OF SKETCHED STRUCTURES (/ 115 / , Part III).

Let σ be a cone-bearing category on Σ ; a *model of* σ in a category V (or σ-structure in V) is a functor $\Sigma \to V$ which sends the distinguished (co)cones on (co)limit cones; the category of models of σ in V is denoted by V^{σ}.

The theory of mixed sketched structures, which has recently been developed by Guitart-Lair (449.1) is equivalent to Guitart's theory of figurative algebras, more adapted to concrete situations (449.1).

In / 115 /, Part III, we only consider the projective case, that is σ is a category Σ equipped with a set of (projective) cones. Then, if V is monoidal closed category with «enough» limits, V^σ *admits a monoidal closed structure as soon as* Set^σ *is cartesian closed.* This structure is deduced from the monoidal closed structure on V^Σ defined by Day-Kelly, thanks to a theorem on reflective subcategories of a monoidal closed category (479.1). Other conditions for V^σ to be monoidal closed have been given by Foltz-Lair and by Street (491.1, 492.1).

If σ is the sketch of categories, there is deduced *a monoidal closed structure on the category* $Cat(V)$ of internal categories in V. An explicit construction of its internal Hom is given at the end of / 115 /; the method directly generalizes that used in / 109 / in the case V is a concrete cartesian closed category.

3. MULTIPLE CATEGORIES / 117, 119/.

The category of sketches is a monoidal closed category (555.1). If σ is a sketch, the models of its n-th tensor product $\overset{n}{\otimes}\sigma$ in a category V are called *n-fold σ-structures in* V. In particular, n-fold categories are obtained if σ is the sketch σ_{Cat} of categories.

n-fold categories were introduced by Charles in 1963 as the internal categories in the category of $(n-1)$-fold categories, or equivalently, as sequences $A = (A^i)_{i<n}$ of n pairwise commuting categories on the same set $\underline{A}$. Let Cat_n be the category of n-fold categories.

The categories Cat_n for all n are subcategories of the category *M Cat of multiple categories*, whose morphisms $A \to B$ from an n-fold category A to an m-fold category B are the n-fold functors $A \to (B^i)_{i<n}$, if $n \leq m$; otherwise, there are no morphisms.

M Cat is studied in / 119 / ; its subcategory coproduct of the Cat_n is equipped with a monoidal closed structure as follows:

- the tensor product of (B, A) is the $(n+m)$-fold category $B \blacksquare A$, in which the n first categories are $\underline{B}^{dis} \times A^i$, and the $(n+j)$-th category is $B^j \times \underline{A}^{dis}$, for $j < m$;

- if C is an $(n+p)$-fold category, the internal Hom Hom(A,C) is the p-fold category on the set of multiple functors $A \to C$ whose compotions are pointwise deduced from the p last compositions of C.

These constructions are given in /117/ in the case A and B are categories and C is a double category; then the category Hom(A, C) had been defined in /63/ as a generalization of the category of functors A → D to which it reduces if D is the double category of commutative squares of the category D.

M Cat is a complete category, but it does not admit coproducts of families of multiple categories whose multiplicities are not bounded. So, it is embedded in the complete and cocomplete category *VM Cat*, which also admits as objects the *infinite-fold categories*. *VM Cat* is equipped with a partial monoidal closed non-symmetrical structure defined as above.

Examples of infinite-fold categories are the ∞-groupoids which have been studied by Brown and Higgins to get an n-dimensional Seifert-van Kampen Theorem (649.1).

4. GENERALIZED LIMITS /117, 119, 121/.

Let A be an n-fold category and C an $(n+1)$-fold category. We denote by |C| the n-fold category of objects of the last category C^n on C.

Motivated by the $n = 1$ case (cf. above), we call a multiple functor T: A → C a *C-wise transformation* between the n-fold functors F and F' from A to |C| which are its domain and codomain in Hom(A, C). If F is constant, T is called a *C-wise cone with basis* F'; a *C-wise limit* of F' is a universal C-wise cone with basis F'.

An important theorem (Proposition 8 of /119/) states that |C| *admits all (finite) C-wise limits if* |C| *admits C-wise limits indexed by* $\overset{n}{\blacksquare}\, 2$ *and if the subcategory of* C^n *consisting of the objects for the n first compositions is (finitely) complete.*

The short existence proof uses the fact that *Cat* is the inductive closure if {2}, and an Appelgate-Tierney theorem [132]. For $n = 1$, a more constructive proof is given in /117/.

In the case A is a 2-category and C is the 3-category of cylinders of a 2-category D, a C-wise limit of F: A → D is usually called a *lax* *limit of* F. The above theorem reduces /119, 121/ to the Gray-Bourn-Street result which says that a representable 2-category D whose category of 1-morphisms is (finitely) complete is also (finitely) lax-complete (648.1). It leads to applications in Algebraic Topology, since lax limits may be related to homotopy limits and coherence problems (648.1, 619.1).

Existence theorems for several kinds of generalized limits are considered in /121/.

5. MONOIDAL CLOSED STRUCTURES ON Cat_n /120, 121/.

It follows from the general results of /115/ that Cat_n is a cartesian closed category. In /120/ we give an explicit construction of its internal Hom functor Hom_n : If A is an n-fold category, then

$$\mathrm{Hom}_n(A,-) = (Cat_n \xrightarrow{Square_{n,2n}} Cat_{2n} \xrightarrow{\tilde{\gamma}} Cat_{2n} \xrightarrow{\mathrm{Hom}(A,-)} Cat_n,$$

where :

- Hom(A,-) is a restriction of the internal Hom of *M Cat* (cf. n° 3),
- $\tilde{\gamma}$ is deduced from the permutation of the compositions :

  $(0, \dots, 2n-1) \mapsto (0, 2, \dots, 2n-2, 1, 3, \dots, 2n-1)$,

- $Square_{n,2n}$ is obtained by iteration of the Square functor

$$Square : Cat_n \longrightarrow Cat_{n+1}.$$

The $(n+1)$-category $Square(A)$ consists of the commutative squares of A^0 ; its $n-1$ first compositions are pointwise deduced from the compositions of A^i, $i > 0$, its 2 last compositions are the horizontal and vertical compositions of squares.

The difficult points of the proof are :

- the construction of the adjoint $Link : Cat_{n+1} \to Cat_n$ of the Square functor; in particular, it sends an $(n+1)$-category to the n-category of its components ;
- the proof that the functor $\square_n = \tilde{\gamma} . Square_{n,2n}$ maps the $2n$-fold category B ■ A on the n-fold category B × A, for any n-fold category B.

As an application, Cat_n is embedded in the $(n+1)$-category Nat_n

of «hypertransformations», using the fact that a V-category in a cartesian category V with commuting coproducts «is» an internal category in V (Appendix /120/ and O, III-2, Synopsis (704.2, 710.1)).

The construction of Hom_n is laxified in /121/ to get non-symmetrical monoidal closed structures on Cat_n. The same method is used, but the Square functor is replaced by the *Cube functor*; a cube consists of 6 faces which «commute up to an $(n-1)$-cell». For $n = 2$, these cubes generalize the notion of a homotopy commutative cube used in Algebraic Topology (729.1).

Here again, the main difficulty is the construction of the adjoint, *LaxLink*, to $Cube: Cat_n \to Cat_{n+1}$. While *Link* A is generated by classes of strings of objects of the two last categories A^n and A^{n-1}, the n-fold category *LaxLink* A is generated by classes of strings of strings of objects taken alternatively in A^{n-2} and in A^{n-1} or A^n (i.e., objects of A^{n-2} replace identities).

An important application is the following characterization of double categories: *any double category* D *is isomorphic to a double subcategory of the double category* $Q(String\,\mathrm{D})$, where $Q: 2\text{-}Cat \to Cat_2$ sends a 2-category on the double category of its squares, and

$$String: Cat_2 \longrightarrow 2\text{-}Cat$$

is the adjoint of Q (deduced from *LaxLink*). It follows that D is also isomorphic to a double subcategory of the double category $Q(Cat)$ of quintets (766.1). The double categories $Q(\mathrm{C})$ are the double categories with connections (C a 2-category), which provide a convenient setting for homotopy theory (603.1). The String functor is used by Guitart-van den Bril to study lax bimodules and non-abelian cohomology (764.1).

6. TOPOLOGICAL RINGOIDS /118/.

The paper /118/ was written to complete Lellahi's paper [172].

A *ringoid* is an *Ab*-category, looked at as a ring with several objects. A *topological ringoid* is an internal ringoid in the category *Top* of topological spaces.

In /118/, monoidal closed structures on the category of topological ringoids and some of its subcategories are deduced from partial monoidal closed structures on *Top* as follows.

Let A and A' be topological ringoids; let S be a covering of A by compact subspaces S such that each $x \in S$ admits a basis of neighborhood in S consisting of elements of S. We construct a topological ringoid $A' \otimes_S A$ which is a universal solution of the problem: for each topological ringoid A", the morphisms $A' \otimes_S A \to A''$ are in 1-1 correspondence with the biadditive continuous functors $A' \times_S A \to A''$, where $A' \times_S A$ is the topological ringoid on the product ringoid, equipped with the S-product topology [154]; or equivalently, they are in 1-1 correspondence with the morphisms $A' \to \mathrm{Hom}_S(A, A'')$, to the ringoid of morphisms $A \to A''$, equipped with the S-open topology on the set of morphisms $A \to A''^{2}$.

The S-product topology may be replaced by the topology on products of topological spaces associated by Booth - Tillotson to the data of a class K of topological spaces (i.o. subspaces of A; cf. 538.1).

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