On centres and lax centres for promonoidal categories *

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In celebration of the hundreth anniversary of Charles Ehresmann's birth

1 Introduction

Braidings for monoidal categories were introduced in [7] and its forerunners. The centre $\mathcal{Z}\mathscr{X}$ of a monoidal category $\mathscr{X}$ was introduced in [6] in the process of proving that the free tortile monoidal category has another universal property. The centre of a monoidal category is a braided monoidal category. What we now call lax braidings were considered tangentially by Yetter [13]. What we now call the lax centre $\mathcal{Z}_l\mathscr{X}$ of $\mathscr{X}$ was considered under the name "weak centre" by P. Schauenburg [12]. The purpose of this work is to highlight the notions of lax braiding and lax centre for monoidal categories $\mathscr{X}$ and more generally for promonoidal categories $\mathscr{C}$. Indeed we further generalize to the $\mathscr{V}$-enriched context. Lax centres turn out to be lax braided monoidal categories. Generally the centre is a full subcategory of the lax centre, however it is sometimes the case that the two coincide. We have two such theorems under different hypotheses, one in the case sufficient dual objects exist in the additive context, and the other in the cartesian context. We examine when the centre of $[\mathscr{C}, \mathscr{V}]$ with a convolution monoidal structure (in the sense of [1]) is again a functor category $[\mathscr{D}, \mathscr{V}]$.

One reason for being interested in the lax centre of $\mathscr{X}$ is that, if an object X of $\mathscr{X}$ is equipped with the structure of monoid in $\mathcal{Z}_l\mathscr{X}$, then tensoring with X defines a monoidal endofunctor $-\otimes X$ of $\mathscr{X}$; this has applications in cases where the lax centre can be explicitly identified.

2 Review of definitions

The context in which we work is enriched category theory in the sense of [10]. The base monoidal category $\mathscr{V}$ is symmetric, closed, complete and cocomplete. The tensor product of $\mathscr{V}$ is denoted by $\otimes : \mathscr{V} \times \mathscr{V} \longrightarrow \mathscr{V}$, the unit by I, and the associativity and unital isomorphisms will be regarded as canonical (and so unnamed).

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A $\mathscr{V}$*-multicategory* is a $\mathscr{V}$-category $\mathscr{C}$ equipped with a sequence of $\mathscr{V}$-functors

$$P_n : \underbrace{\mathscr{C}^{\mathrm{op}} \otimes \ldots \otimes \mathscr{C}^{\mathrm{op}}}_{n} \otimes \mathscr{C} \longrightarrow \mathscr{V},$$

where we write J for $P_0 : \mathscr{C} \longrightarrow \mathscr{V}$, where P_1 is the hom $\mathscr{V}$-functor $\mathscr{C}(-, \sim) : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \longrightarrow \mathscr{V}$, and where we write P for P_2. Furthermore, there are *substitution operations*, which include $\mathscr{V}$-natural families

$$\int^X P(X, C; D) \otimes P(A, B; X) \xrightarrow{\mu_1} P_3(A, B, C; D) \xleftarrow{\mu_2} \int^Y P(A, Y; D) \otimes P(B, C; Y)$$

$$\int^X P(X, A; B) \otimes JX \xrightarrow{\eta_1} \mathscr{C}(A, B) \xleftarrow{\eta_2} \int^Y P(A, Y; B) \otimes JY,$$

satisfying associativity and unital conditions. For $\mathscr{V} = \mathbf{Set}$, this is a multicategory in the sense of [11].

A *promonoidal* $\mathscr{V}$*-category* [1] is a $\mathscr{V}$-multicategory $\mathscr{C}$ for which $\mu_1, \mu_2, \eta_1, \eta_2$ are invertible. In this case, P_n is determined up to isomorphism by P_0, P_1, P_2.

A *monoidal* $\mathscr{V}$*-category* is a promonoidal $\mathscr{V}$-category $\mathscr{C}$ for which P and J are representable. That is, there are $\mathscr{V}$-natural isomorphisms

$$P(A, B; C) \cong \mathscr{C}(A \boxtimes B, C), \qquad JC \cong \mathscr{C}(U, C)$$

for some $A \boxtimes B$ (depending on the choice of A and B) and some U. Monoidal structures on $\mathscr{C}$ are in bijection with monoidal structures on $\mathscr{C}^{\mathrm{op}}$.

For any small promonoidal $\mathscr{V}$-category $\mathscr{C}$, there is a *convolution* monoidal structure on the $\mathscr{V}$-functor $\mathscr{V}$-category $\mathscr{F} = [\mathscr{C}, \mathscr{V}]$ defined (following [1]) by

$$(F * G)C = \int^{A,B} P(A, B; C) \otimes FA \otimes GB.$$

The unit J and $\mathscr{F}$ is closed (by which we always mean "on both sides"):

$$\mathscr{F}(F, [G, H]_l) \cong \mathscr{F}(F * G, H) \cong \mathscr{F}(G, [F, H]_r)$$

where

$$[G, H]_l A = \int_{B,C} \mathscr{V}(P(A, B; C) \otimes GB, HC) \quad \text{and}$$

$$[F, H]_r B = \int_{A,C} \mathscr{V}(P(A, B; C) \otimes FA, HC).$$

Conversely, every closed monoidal structure $*$, J on $\mathscr{F} = [\mathscr{C}, \mathscr{V}]$ for a small $\mathscr{V}$-category $\mathscr{C}$ defines a promonoidal structure on $\mathscr{C}$ where

$$P_n(A_1, \ldots, A_n; B) = P_n(\mathscr{C}(A_1, -), \ldots, \mathscr{C}(A_n, -); \mathscr{C}(B, -)).$$

That is, we restrict the promonoidal structure along the Yoneda embedding $Y : \mathscr{C} \longrightarrow \mathscr{F}^{\mathrm{op}}$.

A *lax braiding* for a promonoidal $\mathscr{V}$-category $\mathscr{C}$ is a $\mathscr{V}$-natural family of morphisms

$$c_{A,B;C} : P(A,B;C) \longrightarrow P(B,A;C)$$

such that the following diagrams commute.

$$\begin{array}{ccc}
\int^{U} P(U,C;D) \otimes P(A,B;U) & \xrightarrow{\int^{U} c \otimes 1} & \int^{U} P(C,U;D) \otimes P(A,B;U) \\
\cong \downarrow & & \uparrow \cong \\
\int^{V} P(A,V;D) \otimes P(B,C;V) & & \int^{W} P(W,B;D) \otimes P(C,A;W) \\
\int^{V} 1 \otimes c \downarrow & & \uparrow \int^{W} 1 \otimes c \\
\int^{V} P(A,V;D) \otimes P(C,B;V) & \xrightarrow{\cong} & \int^{W} P(W,B;D) \otimes P(A,C;W)
\end{array}$$

$$\begin{array}{ccc}
\int^{V} P(A,V;D) \otimes P(B,C;V) & \xrightarrow{\int^{V} c \otimes 1} & \int^{V} P(V,A;D) \otimes P(B,C;V) \\
\cong \downarrow & & \uparrow \cong \\
\int^{U} P(U,C;D) \otimes P(A,B;U) & & \int^{W} P(B,W;D) \otimes P(C,A;W) \\
\int^{U} 1 \otimes c \downarrow & & \uparrow \int^{W} 1 \otimes c \\
\int^{U} P(U,C;D) \otimes P(B,A;U) & \xrightarrow{\cong} & \int^{W} P(B,W;D) \otimes P(A,C;W)
\end{array}$$

$$\begin{array}{ccc}
\int^{U} P(U,A;B) \otimes JU & \xrightarrow{\int^{U} c \otimes 1} & \int^{U} P(A,U;B) \otimes JU \\
 & \cong \searrow \quad \swarrow \cong & \\
 & \mathscr{C}(A,B) &
\end{array}$$

$$\begin{array}{ccc}
\int^{U} P(A,U;B) \otimes JU & \xrightarrow{\int^{U} c \otimes 1} & \int^{U} P(U,A;B) \otimes JU \\
 & \cong \searrow \quad \swarrow \cong & \\
 & \mathscr{C}(A,B) &
\end{array}$$

When $\mathscr{C}$ is monoidal, the lax braiding is induced by a $\mathscr{V}$-natural family of morphisms

$$c_{A,B} : A \boxtimes B \longrightarrow B \boxtimes A$$

which we also call the lax braiding in this case. For general promonoidal $\mathscr{C}$, lax braidings on the convolution monoidal $\mathscr{V}$-category $\mathscr{F} = [\mathscr{C}, \mathscr{V}]$ are in bijection with lax braidings on $\mathscr{C}$: the Yoneda embedding $Y : \mathscr{C} \longrightarrow \mathscr{F}^{\mathrm{op}}$ is a lax-braided promonoidal functor.

A *braiding* is a lax braiding for which each $c_{A,B;C}$ (and hence each $c_{A,B}$ in the monoidal case) is invertible. The third and fourth conditions on a lax braiding are automatic in this case.

In the presence of duals in a monoidal $\mathscr{C}$ (more precisely, $\mathscr{C}$ should be right autonomous in the sense of [7]), *every lax braiding is automatically a braiding* (see [8, Section 10, Proposition 8], [13, Proposition 7.1], [7, Propositions 7.1 and 7.4]).

3 Lax centres

The *lax centre* $\mathcal{Z}_l\mathscr{C}$ of a monoidal $\mathscr{V}$-category $\mathscr{C}$ is the lax-braided monoidal $\mathscr{V}$-category defined as follows. The objects are pairs (A, u) where A is an object of $\mathscr{C}$ and u is a $\mathscr{V}$-natural family of morphisms

$$u_B : A \boxtimes B \longrightarrow B \boxtimes A$$

such that the following two diagrams commute:

$$\begin{array}{ccccc} & (A \boxtimes B) \boxtimes C & \xrightarrow{u_B \boxtimes 1} & (B \boxtimes A) \boxtimes C & \\ \cong \swarrow & & & & \searrow \cong \\ A \boxtimes (B \boxtimes C) & & & & B \boxtimes (A \boxtimes C) \\ \downarrow u_{B \boxtimes C} & & & & \downarrow 1 \boxtimes u_C \\ (B \boxtimes C) \boxtimes A) & & \xrightarrow[\cong]{\qquad\qquad} & & B \boxtimes (C \boxtimes A) \end{array}$$

$$\begin{array}{ccc} A \boxtimes U & \xrightarrow{u_U} & U \boxtimes A \\ \cong \searrow & & \swarrow \cong \\ & A & \end{array}$$

(where the marked isomorphisms are induced by the substitution operations μ and η and their inverses). The hom object $\mathcal{Z}_l\mathscr{C}((A, u), (A', u'))$ is defined to be the equalizer in $\mathscr{V}$ of the two composed paths around the following square.

$$\begin{array}{ccc} \mathscr{C}(A, A') & \xrightarrow{-\boxtimes B} & \mathscr{C}(A \boxtimes B, A' \boxtimes B) \\ {\scriptstyle B\boxtimes -}\downarrow & & \downarrow{\scriptstyle \mathscr{C}(1, u'_B)} \\ \mathscr{C}(B \boxtimes A, B \boxtimes A') & \xrightarrow[\mathscr{C}(u_B, 1)]{} & \mathscr{C}(A \boxtimes B, B \boxtimes A') \end{array}$$

Composition in $\mathcal{Z}_l\mathscr{C}$ is defined so that we have the obvious faithful $\mathscr{V}$-functor $\mathcal{Z}_l\mathscr{C} \longrightarrow \mathscr{C}$ taking (A, u) to A.

The monoidal structure on $\mathcal{Z}_l\mathscr{C}$ is defined on objects by

$$(A, u) \boxtimes (B, v) = (A \boxtimes B, w)$$

where $w_C : (A \boxtimes B) \boxtimes C \longrightarrow C \boxtimes (A \boxtimes B)$ is the composite

$$A \boxtimes (B \boxtimes C) \xrightarrow{1 \boxtimes v_C} A \boxtimes (C \boxtimes B) \xrightarrow{\cong} (A \boxtimes C) \boxtimes B \xrightarrow{u_C \boxtimes 1} (C \boxtimes A) \boxtimes B$$

conjugated by canonical isomorphisms. The unit object is U equipped with the family of canonical isomorphisms $U \boxtimes C \cong C \boxtimes U$. The faithful $\mathscr{V}$-functor $Z_l\mathscr{C} \longrightarrow \mathscr{C}$ is strong monoidal.

The lax braiding on $Z_l\mathscr{C}$ is defined to be the family of morphisms

$$c_{(A,u),(B,v)} : (A \boxtimes B, w) \longrightarrow (B \boxtimes A, \tilde{w})$$

lifting $u_B : A \boxtimes B \longrightarrow B \boxtimes A$ to $Z_l\mathscr{C}$.

The *centre* $Z\mathscr{C}$ of $\mathscr{C}$ is the full monoidal sub-$\mathscr{V}$-category of $Z_l\mathscr{C}$ consisting of the objects (A, u) with each u_B invertible. Clearly $Z\mathscr{C}$ is a braided monoidal $\mathscr{V}$-category.

There are interesting cases where the centre $Z\mathscr{C}$ is actually equal to the lax centre. Much as a lax braiding in the presence of duals is a braiding, we have that, if $\mathscr{C}$ is right autonomous, then $Z\mathscr{C} = Z_l\mathscr{C}$ (see [2, Proposition 3.1]).

It is worth noting that for a closed monoidal $\mathscr{C}$ with a dense full sub-$\mathscr{V}$-category, the objects (A, u) of $Z_l\mathscr{C}$ are determined by the restriction of u_B to those B in the dense sub-$\mathscr{V}$-category (see [2, Proposition 3.1 and 3.3]).

We can generalize the lax centre construction to promonoidal $\mathscr{V}$-categories $\mathscr{C}$. It is defined as a $\mathscr{V}$-multicategory to be the pullback $Z_l\mathscr{C}$

$$\begin{array}{ccc} (\mathcal{Z}_l\mathscr{C}) & \xrightarrow{\Psi} & (\mathcal{Z}_l\mathscr{F})^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathscr{C} & \xrightarrow[\mathrm{Y}]{} & \mathscr{F}^{\mathrm{op}} \end{array}$$

of $\mathscr{V}$-categories and $\mathscr{V}$-functors. The multicategory structure is defined by restriction along the fully faithful Ψ (where $\mathscr{F} = [\mathscr{C}, \mathscr{V}]$ with convolution).

Similarly $Z\mathscr{C}$ is defined by replacing $Z_l\mathscr{F}$ by $Z\mathscr{F}$ in the pullback.

It is frequently the case that $Z_l\mathscr{C}$ is promonoidal, not merely a multicategory; moreover, the forgetful $\mathscr{V}$-functor $Z_l\mathscr{C} \longrightarrow \mathscr{C}$ is strong promonoidal. If $\mathscr{C}$ is monoidal, this $Z_l\mathscr{C}$ agrees with the definition in Section 2.

The objects of $Z_l\mathscr{C}$ are pairs (A, α) where A is an object of $\mathscr{C}$ and α is a $\mathscr{V}$-natural family of morphisms $\alpha_{X;Y} : P(A, X; Y) \longrightarrow P(X, A; Y)$ such that the pair $(\mathscr{C}(A, -), u)$ is an object of $(Z_l\mathscr{F})^{\text{op}}$, where u is determined by

$$u_{\mathscr{C}(X,-)} = \alpha_{X;-} : \mathscr{C}(A, -) * \mathscr{C}(X, -) \longrightarrow \mathscr{C}(X, -) * \mathscr{C}(A, -).$$

If $Z_l\mathscr{C}$ is promonoidal, it has a lax braiding

$$c_{(A,\alpha),(B,\beta);(C,\gamma)} : P((A, \alpha), (B, \beta); (C, \gamma)) \longrightarrow P((B, \beta), (A, \alpha); (C, \gamma))$$

obtained by restriction of $\alpha_{B;C} : P(A, B; C) \longrightarrow P(B, A; C)$ to the equalizers.

The $\mathscr{V}$-functor Ψ induces an adjunction $\hat{\Psi} \dashv \tilde{\Psi}$:

$$Z_l[\mathscr{C}, \mathscr{V}] \underset{\tilde{\Psi}}{\overset{\hat{\Psi}}{\rightleftarrows}} [Z_l\mathscr{C}, \mathscr{V}]$$

where

$$\hat{\Psi}(G) = \int^{(A,\alpha)} G(A, \alpha) \otimes \Psi(A, \alpha) \quad \text{and} \quad \tilde{\Psi}(F, \theta)(A, \alpha) = Z_l[\mathscr{C}, \mathscr{V}](\Psi(A, \alpha), (F, \theta)).$$

The last object can be obtained as the equalizer of two morphisms out of FA. If $Z_l\mathscr{C}$ is promonoidal, this is a lax-braided monoidal adjunction. We shall see that the adjunction can be an equivalence of lax-braided monoidal $\mathscr{V}$-categories.

Similar remarks apply to $Z\mathscr{C}$.

4 The cartesian case

For this section, suppose $\mathscr{V} = \mathbf{Set}$ with cartesian monoidal structure. Our concern is with the lax centre of cartesian monoidal categories $\mathscr{C}$: that is, $\mathscr{C}$ is a category with finite products regarded as a monoidal category whose tensor is product.

It is an easy exercise to see that an object (A, u) of $Z_l\mathscr{C}$ is such that $u_X : A \times X \longrightarrow X \times A$ is determined by its first projection $A \times X \longrightarrow X$. In fact, every family of morphisms $\theta_X : A \times X \longrightarrow X$ determines an object (A, u) of $Z_l\mathscr{C}$ via

$$u_X = (\theta_X, \text{pr}_1).$$

So we identify objects of $Z_l\mathscr{C}$ with pairs (A, θ).

We therefore see that the *core* $C_{\mathscr{C}}$ of the category $\mathscr{C}$ in the sense of [5] is precisely a terminal object of $Z_l\mathscr{C}$. If this core exists, we have the identification of the lax centre with a slice category

$$Z_l\mathscr{C} \cong \mathscr{C}/C_{\mathscr{C}}.$$

The monoidal structure on $\mathscr{C}/C_{\mathscr{C}}$ arises from a monoidal structure on $C_{\mathscr{C}}$ in $\mathscr{C}$: the multiplication $C_{\mathscr{C}} \times C_{\mathscr{C}} \longrightarrow C_{\mathscr{C}}$ in $\mathscr{C}$ is the unique morphism into the terminal object in $Z_l\mathscr{C}$.

If $\mathscr{C}$ is cartesian closed (with internal hom written as [X,Y]), we have the formula

$$C_{\mathscr{C}} \cong \int_X [X, X]$$

provided the end exists; it does when $\mathscr{C}$ has a small dense subcategory and is complete.

Now suppose $\mathscr{C}$ is any small category and we shall apply the considerations of this section to the cartesian monoidal category $\mathscr{F} = [\mathscr{C}, \mathbf{Set}]$.

The promonoidal structure on $\mathscr{C}$ that leads to the cartesian structure on $\mathscr{F}$ via convolution is defined by

$$P(A, B; C) = \mathscr{C}(A, C) \times \mathscr{C}(B, C).$$

(This is monoidal if and only if $\mathscr{C}$ has finite coproducts.) We can obtain the following explict descriptions of $Z_l\mathscr{C}$ and $Z\mathscr{C}$ in this case. The objects of $Z_l\mathscr{C}$ are pairs (A, ϕ) where A is an object $\mathscr{C}$ and ϕ is a family of morphisms

$$\phi_X : \mathscr{C}(A, X) \longrightarrow \mathscr{C}(X, X)$$

dinatural in X in the sense of [4]; that is,

$$f \circ \phi_X(u) = \phi_Y(f \circ u) \circ f$$

for $f : X \longrightarrow Y$. A morphism $g : (A, \phi) \longrightarrow (A', \phi')$ in $Z_l\mathscr{C}$ is a morphism $g : A \longrightarrow A'$ in $\mathscr{C}$ such that $\phi_X(v \circ g) = \phi'_X(v)$. The promonoidal structure on $Z_l\mathscr{C}$ is defined by

$$P((A, \phi), (B, \psi); (C, \chi)) =$$
$$\left\{ A \xrightarrow{u} C \xleftarrow{v} B \;\middle|\; \chi_X(f) = \phi_X(f \circ u) \circ \psi_X(f \circ v) \quad \text{for all} \quad C \xrightarrow{f} X \right\}.$$

The lax braiding on $Z_l\mathscr{C}$ is defined by

$$P((A, \phi), (B, \psi); (C, \chi)) \xrightarrow{c_{(A,\phi),(B,\psi);(C,\chi)}} P((B, \psi), (A, \phi); (C, \chi))$$
$$(A \xrightarrow{u} C \xleftarrow{v} B) \longmapsto (B \xrightarrow{\alpha_C(u) \circ v} C \xleftarrow{u} A).$$

An object (A, ϕ) of $Z_l\mathscr{C}$ is in $Z\mathscr{C}$ if and only if the function

$$\mathscr{C}(A, C) \times \mathscr{C}(B, C) \longrightarrow \mathscr{C}(B, C) \times \mathscr{C}(A, C) \; ; \; (u, v) \longmapsto (\alpha_C(u) \circ v, u)$$

is bijection for all B, C.

Theorem 4.1. *Let $\mathscr{C}$ denote a small category with promonoidal structure such that the convolution structure on $[\mathscr{C}, \mathbf{Set}]$ is cartesian product.*

(a). *The adjunction* $\hat{\Psi} \dashv \tilde{\Psi}$ *defines an equivalence of lax-braided monoidal categories*

$$Z_l[\mathscr{C}, \mathbf{Set}] \simeq [Z_l\mathscr{C}, \mathbf{Set}]$$

which restricts to a braided monoidal equivalence

$$Z[\mathscr{C}, \mathbf{Set}] \simeq [Z\mathscr{C}, \mathbf{Set}].$$

(b). *If every endomorphism in the category* $\mathscr{C}$ *is invertible then* $Z_l\mathscr{C} = Z\mathscr{C}$.

(c). *If* $\mathscr{C}$ *is a groupoid then*

$$Z\mathscr{C} = Z_l\mathscr{C} = [\Sigma\mathbb{Z}, \mathscr{C}]$$

(where $\Sigma\mathbb{Z}$ *is the additive group of the integers as a one-object groupoid).*

5 The autonomous case

Suppose $\mathscr{C}$ is a closed monoidal $\mathscr{V}$-category with tensor product $\boxtimes$ and unit U. We write $[Y, Z]_l$ for the left internal hom. Put $X^l = [X, U]_l$. We have a canonical isomorphism $U^l \cong U$ and a canonical morphism $Y^l \boxtimes X^l \longrightarrow (X \boxtimes Y)^l$.

Define a $\mathscr{V}$-functor $M : \mathscr{C} \longrightarrow \mathscr{C}$ by

$$M(A) = \int^X X^l \boxtimes A \boxtimes X$$

when the coend exists (which it does when $\mathscr{C}$ is cocomplete and has a small dense sub-$\mathscr{V}$-category). Using the canonical isomorphism and morphism just mentioned, we obtain a monad structure on M. Notice that M preserves colimits.

Proposition 5.1. *If* $\mathscr{C}$ *has a small dense sub-*$\mathscr{V}$*-category of objects with left duals then* $Z_l\mathscr{C}$ *is isomorphic to the* $\mathscr{V}$*-category* $\mathscr{C}^M$ *of Eilenberg-Moore algebras for the monad* M.

We can apply this in the case where $\mathscr{C}$ is replaced by $\mathscr{F}$.

Theorem 5.2. $(\mathscr{V} = \mathbf{Vect}_k)$ *Suppose* $\mathscr{C}$ *is a promonoidal* k*-linear category with finite-dimensional homs. Let* $\mathscr{F} = [\mathscr{C}, \mathscr{V}]$ *have the convolution monoidal structure. Then*

$$Z\mathscr{F} = Z_l\mathscr{F} \cong \mathscr{F}^M \simeq [\mathscr{C}_M, \mathscr{V}]$$

where $\mathscr{C}_M$ *is the Kleisli category for the promonad* M *on* $\mathscr{C}$.

6 Monoids in the lax centre

Let $\mathscr{C}$ be a monoidal $\mathscr{V}$-category. Each monoid (A, u) in $Z_l\mathscr{C}$ determines a canonical enrichment of the $\mathscr{V}$-functor

$$- \boxtimes A : \mathscr{C} \longrightarrow \mathscr{C}$$

to a monoidal functor:

$$X \boxtimes A \boxtimes Y \boxtimes A \xrightarrow{1 \boxtimes u_Y \boxtimes 1} X \boxtimes Y \boxtimes A \boxtimes A \xrightarrow{1 \boxtimes 1 \boxtimes \mu} X \boxtimes Y \boxtimes A$$

$$U \xrightarrow{\eta} A \cong U \boxtimes A.$$

This becomes useful when $Z_l \mathscr{C}$ can be explicitly identified as in the last two sections.

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