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Synthetic Diœerential Geometry of Higher-Order Total Diœerentials

Abstract

Given microlinear spaces M, N with $x \in M$ and $y \in N$, we have investigated in our previous paper [Beitrge zur Algebra und Geometrie, 45 (2004), 677-696] a certain kind of mappings from the totality $\mathbf{T}_x^{D^n}(M)$ of D^n-microcubes on M at x to the totality $\mathbf{T}_y^{D^n}(N)$ of D^n-microcubes on N at y, called n˘th order preconnections there and D^n-tangentials here, as a germ-free generalization of n˘th order total diœerentials. In this paper, after studying the above germ-free generalization of n˘th order total diœerentials further, we propose a certain kind of mappings from the totality $\mathbf{T}_x^{D_n}(M)$ of D_n-microcubes on M at x to the totality $\mathbf{T}_y^{D_n}(N)$ of D_n-microcubes on N at y, called D_n-tangentials, as another germ-free generalization of n˘th order total diœerentials. Then we study the relationship between D^n-tangentials and D_n-tangentials, ørstly in case that coordinates are not available (i.e., M and N are general microlinear spaces without further conditions imposed) and secondly in case that coordinates are available (i.e., M and N are formal manifolds). In the former case we have a natural mapping from D^n-tangentials into D_n-tangentials, while in the latter case the natural mapping is shown to be injective. Our ideas will be presented within our favorite framework of synthetic diœerential geometry, but they are readily applicable to such generalizations of smooth manifolds as diœerentiable spaces and suitable inønite-dimensional manifolds with due modiøcations. This paper is to be looked upon as a microlinear generalization of Kock's [1978] perspicacious considerations on Taylor series calculus.

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In teaching diœerential calculus of several variables, mathematicians are expected to exhort freshmen or sophomores majoring in science, engineering etc. to understand that it is not partial derivatives but total diœerentials that are of intrinsic meaning, while partial derivatives are used for computational purposes. If we want to discuss not only ørst-order total diœerentials but higher-order ones, we have to resort to the theory of jets initiated by Ehresmann, though it is not easy to generalize it beyond the scope of ønite-dimensional smooth manifolds so as to encompass diœerentiable spaces and suitable inønie-dimensional manifolds, for which the reader is referred, e.g., to Navarro and Sancho de Salas [2003] and Libermann [1971].

The then moribund notion of nilpotent inønitesimals in diœerential geometry was retrieved by Lawvere in the middle of the preceding century, while Robinson revived invertible inønitesimals in analysis, and Grothendieck authenticated nilpotent inønitesimals in algebraic geometry. Kock [1977, 1978], following the new directions in diœerential geometry enunciated by Lawvere as synthetic diœerential geometry (usually abbreviated to SDG), has investigated diœerential calculus from this noble standpoint as the foundations of SDG. For readable textbooks on SDG, the reader is referred to Kock [1981], Lavendhomme [1996] and Moerdijk and Reyes [1991].

Kock [1978] has shown that the inønitesimal space D_n captures n-th order diœerential calculus. To show this, he had to exploit the fact that another inønitesimal space $D^n = D \times ... \times D$ (the product of n copies of D) has a good grasp of n-th order diœerential calculus. In our previous paper (Nishimura [2004]) we have demonstrated that, given microlinear spaces M, N with $x \in M$ and $y \in N$, n˘th order total diœerentials can be captured as a certain kind of mappings from the totality $\mathbf{T}_x^{D^n}(M)$ of D^n-microcubes on M at x to the totality $\mathbf{T}_y^{D^n}(N)$ of D^n-microcubes on N at y, which were called n˘th order preconnections there and are to be called D^n-tangentials here. In this paper we propose another kind of n˘th order total diœerentials as a certain kind of mappings from the totality $\mathbf{T}_x^{D_n}(M)$ of D_n-microcubes on M at x to the totality $\mathbf{T}_y^{D_n}(N)$ of D_n-microcubes on N at y, which are to be called D_n-tangentials. Then we study the relationship between D^n-tangentials and D_n-tangentials, ørstly in case that coordinates are not available (i.e., M and N are general microlinear spaces without further conditions imposed) and secondly in case that coordinates are available (i.e., M and N are formal manifolds). In the former case we have a natural mapping from D^n-tangentials into D_n-tangentials, while in the latter case the natural mapping is shown to be injective. Our ideas will be presented within our favorite framework of synthetic diœerential geometry, but they are readily applicable to such generalizations of smooth manifolds as diœerentiable spaces and suitable inønite-dimensional manifolds with due modiøcations. This paper is to be looked upon as a microlinear generalization of Kock's [1978] perspicacious considerations on Taylor series calculus.

1 Preliminaries

1.1 Microcubes

Let $\mathbb{R}$ be the extended set of real numbers with cornucopia of nilpotent inønitesimals, which is expected to acquiesce in the so-called general Kock axiom (cf. Lavendhomme [1996, 2.1]). We denote by D_1 or D the totality of elements of $\mathbb{R}$ whose squares vanish. More generally, given a natural number n, we denote by D_n the set

$$\{d \in \mathbb{R} | d^{n+1} = 0\}.$$

Given natural numbers m, n, we denote by $D(m)_n$ the set

$$\{(d_1, ..., d_m) \in D^m | d_{i_1} ... d_{i_{n+1}} = 0\},$$

where $i_1, ..., i_{n+1}$ shall range over natural numbers between 1 and m including both ends. We will often write $D(m)$ for $D(m)_1$. By convention $D^0 = D_0 = \{0\}$.

Simplicial inønitesimal spaces are spaces of the form

$$\begin{aligned} & D(m; \mathcal{S}) \\ = \quad & \{(d_1, ..., d_m) \in D^m | d_{i_1} ... d_{i_k} = 0 \text{ for any } (i_1, ..., i_k) \in \mathcal{S}\}, \end{aligned}$$

where $\mathcal{S}$ is a ønite set of sequences $(i_1, ..., i_k)$ of natural numbers with $1 \leq i_1 < ... < i_k \leq m$. To give an example, we have $D(2) = D(2; (1, 2))$ and $D(3) = D(3; (1, 2), (1, 3), (2, 3))$. The number m is called the degree of $D(m; \mathcal{S})$, in notation: $m = \deg D(m; \mathcal{S})$, while the maximum number n such that there exists a sequence $(i_1, ..., i_n)$ of natural numbers of length n with $1 \leq i_1 < ... < i_n \leq m$ containing no subsequence in $\mathcal{S}$ is called the dimension of $D(m; \mathcal{S})$, in notation: $n = \dim D(m; \mathcal{S})$. By way of example, $\deg D(3) = \deg D(3; (1, 2)) = \deg D(3; (1, 2), (1, 3)) = \deg D^3 = 3$, while $\dim D(3) = 1$, $\dim D(3; (1, 2)) = \dim D(3; (1, 2), (1, 3)) = 2$ and $\dim D^3 = 3$. Inønitesimal spaces of the form D^m are called basic inønitesimal spaces. Given two simplicial inønitesimal spaces $D(m; \mathcal{S})$ and $D(m'; \mathcal{S}')$, a mapping $\varphi = (\varphi_1, ..., \varphi_{m'}) : D(m; \mathcal{S}) \to D(m'; \mathcal{S}')$ is called a monomial mapping if every φ_j is a monomial in $d_1, ..., d_m$ with coeŒcient 1.

Given a microlinear space M and an inønitesimal space $\mathbb{E}$, a mapping γ from $\mathbb{E}$ to M is called an $\mathbb{E}$-microcube on M. D^n-microcubes are often called n-microcubes. In particular, 1-microcubes are usually called tangent vectors, and 2-microcubes are often referred to as microsquares. We denote by $\mathbf{T}^{\mathbb{E}}(M)$ the totality of $\mathbb{E}$-microcubes on M. Given $x \in M$, we denote by $\mathbf{T}^{\mathbb{E}}_x(M)$ the totality of $\mathbb{E}$-microcubes γ on M with $\gamma(0,...,0) = x$.

We denote by $\mathfrak{S}_n$ the symmetric group of the set $\{1,...,n\}$, which is well known to be generated by $n-1$ transpositions $< i, i+1 >$ exchanging i and $i+1 (1 \leq i \leq n-1)$ while keeping the other elements øxed. Given $\sigma \in \mathfrak{S}_n$ and $\gamma \in \mathbf{T}^{D^n}_x(M)$, we døne $\Sigma_\sigma(\gamma) \in \mathbf{T}^{D^n}_x(M)$ to be

$$\Sigma_\sigma(\gamma)(d_1,...,d_n) = \gamma(d_{\sigma(1)},...,d_{\sigma(n)})$$

for any $(d_1,...,d_n) \in D^n$. Given $\alpha \in \mathbb{R}$ and $\gamma \in \mathbf{T}^{D^n}_x(M)$, we døne $\alpha \underset{i}{\cdot} \gamma \in \mathbf{T}^{D^n}_x(M)$ $(1 \leq i \leq n)$ to be

$$(\alpha \underset{i}{\cdot} \gamma)(d_1,...,d_n) = \gamma(d_1,...,d_{i-1},\alpha d_i,d_{i+1},...,d_n)$$

for any $(d_1,...,d_n) \in D^n$. Given $\alpha \in \mathbb{R}$ and $\gamma \in \mathbf{T}^{D_n}_x(M)$, we døne $\alpha\gamma \in \mathbf{T}^{D_n}_x(M)$ $(1 \leq i \leq n)$ to be

$$(\alpha\gamma)(d) = \gamma(\alpha d)$$

for any $d \in D_n$. The restriction mapping $\gamma \in \mathbf{T}^{D_{n+1}}_x(M) \mapsto \gamma|_{D_n} \in \mathbf{T}^{D_n}_x(M)$ is often denoted by $\pi_{n+1,n}$.

Between $\mathbf{T}^{D^n}_x(M)$ and $\mathbf{T}^{D^{n+1}}_x(M)$ there are $2n+2$ canonical mappings:

$$\mathbf{T}^{D^{n+1}}_x(M) \quad \underset{\mathbf{s}_i}{\overset{\mathbf{d}_i}{\rightleftarrows}} \quad \mathbf{T}^{D^n}_x(M) \qquad (1 \leq i \leq n+1)$$

For any $\gamma \in \mathbf{T}^{D^n}_x(M)$, we døne $\mathbf{s}_i(\gamma) \in \mathbf{T}^{D^{n+1}}_x(M)$ to be

$$\mathbf{s}_i(\gamma)(d_1,...,d_{n+1}) = \gamma(d_1,...,d_{i-1},d_{i+1},...,d_{n+1})$$

for any $(d_1,...,d_{n+1}) \in D^{n+1}$. For any $\gamma \in \mathbf{T}^{D^{n+1}}_x(M)$, we døne $\mathbf{d}_i(\gamma) \in \mathbf{T}^{D^n}_x(M)$ to be

$$\mathbf{d}_i(\gamma)(d_1,...,d_n) = \gamma(d_1,...,d_{i-1},0,d_i,...,d_n)$$

for any $(d_1,...,d_n) \in D^n$. These operations satisfy the so-called simplicial identities (cf. Goerss and Jardine [1999, p.4]).

For any $\gamma \in \mathbf{T}_x^{D_n}(M)$ and any $d \in D_n$, we deøne $\mathbf{i}_d(\gamma) \in \mathbf{T}_x^{D_{n+1}}(M)$ to be

$$\mathbf{i}_d(\gamma)(d') = \gamma(dd')$$

for any $d' \in D_{n+1}$.

1.2 Quasi-Colimit Diagrams

Proposition 1 The diagram

$$\begin{array}{ccc} D_n \times D_n & \xrightarrow{\mathfrak{m}_n} & D_n \\ \mathfrak{m}_n \downarrow & & \downarrow \mathrm{id}_{D_n} \\ D_n & \xrightarrow[\mathrm{id}_{D_n}]{} & D_n \end{array}$$

is a quasi-colimit diagram, where $\mathfrak{m}_n(d_1, d_2) = d_1 d_2$ for any $(d_1, d_2) \in D_n \times D_n$ and id_{D_n} is the identity mapping of D_n. In other words, $\mathbb{R}$ believes that the multiplication $\mathfrak{m}_n : D_n \times D_n \to D_n$ is surjective.

Proof. By the same token as in the proof of Proposition 1 of Lavendhomme [1996, 2.2]. ■

Proposition 2 The diagram

$$\begin{array}{ccc} D^n & \xrightarrow{\mathfrak{a}_n} & D_n \\ \mathfrak{a}_n \downarrow & & \downarrow \mathrm{id}_{D_n} \\ D_n & \xrightarrow[\mathrm{id}_{D_n}]{} & D_n \end{array}$$

is a quasi-colimit diagram, where $\mathfrak{a}_n(d_1, ..., d_n) = d_1 + ... + d_n$ for any $(d_1, ..., d_n) \in D^n$ and id_{D_n} is the identity mapping of D_n. In other words, $\mathbb{R}$ believes that the addition $\mathfrak{a}_n : D^n \to D_n$ is surjective.

Proof. By the same token as in the proof of Proposition 2 of Lavendhomme [1996, 2.2]. ■

Corollary 3 The diagram

$$\begin{array}{ccc} D_n \times D^n & \xrightarrow{\qquad\mathfrak{ma}_n\qquad} & D_n \\ \mathfrak{ma}_n \downarrow & & \downarrow \mathrm{id}_{D_n} \\ D_n & \xrightarrow[\qquad\mathrm{id}_{D_n}\qquad]{} & D_n \end{array}$$

is a quasi-colimit diagram, where $\mathfrak{ma}_n(d, d_1, ..., d_n) = d(d_1 + ... + d_n)$ for any $(d, d_1, ..., d_n) \in D_n \times D^n$ and id_{D_n} is the identity mapping of D_n. In other words, $\mathbb{R}$ believes that the mapping $\mathfrak{ma}_n : D_n \times D^n \to D_n$ is surjective.

Proof. This follows from Propositions 1 and 2. ■

Proposition 4 The diagram

$$\begin{array}{ccccc} & \xrightarrow{\mathrm{id}_{D^n}} & & & \\ D^n & \overset{\tau_1}{\rightarrow} & D^n & \underset{\rightarrow}{a_n} & D_n \\ & \vdots & & & \\ & \xrightarrow{\tau_{n-1}} & & & \end{array}$$

is a quasi-colimit diagram, where $\tau_i : D^n \to D^n$ is the mapping permuting the i-th and $(i+1)$-th components of D^n while øxing the other components.

Proof. By the same token as in the proof of Proposition 3 of Lavendhomme [1996, 2.2]. ■

Proposition 5 The diagram

$$D_n \times D_{n+1} \times D_{n+1} \begin{array}{c} \xrightarrow{\mathfrak{m}_{n,n+1} \times \mathrm{id}_{D_{n+1}}} \\ \xrightarrow[\mathrm{id}_{D_n} \times \mathfrak{m}_{n+1}]{} \end{array} D_n \times D_{n+1} \underset{\longrightarrow}{\mathfrak{m}_{n,n+1}} D_n$$

is a quasi-colimit diagram, where $\mathfrak{m}_{n,n+1}(d_1, d_2) = d_1 d_2$ for any $(d_1, d_2) \in D_n \times D_{n+1}$.

Proof. By the same token as in the proof of Proposition 5 of Lavendhomme [1996, 2.2]. ■

The following theorem will play a predominant role in this paper.

Figure 1:

Theorem 6 Any simplicial inønitesimal space $\mathcal{D}$ of dimension n is the quasi-colimit of a ønite diagram whose objects are of the form D^k's $(0 \leq k \leq n)$ and whose arrows are natural injections.

Proof. Here we deal only with $\mathcal{D} = D(3)_2$ as an illustration, leaving the general proof to the reader. The diagram

is the desired quasi-colimit representation of $D(3)_2$, where

1. the j-th and k-th components of $i_{jk}(d_1, d_2) \in D(3)_2$ are d_1 and d_2 respectively, while the remaining component is 0;

2. the j-th component of $i_j(d) \in D^2$ is d, while the other component is 0.

■

In the proof of the above theorem we have represented $D(3)_2$ as an overlapping family of three copies of D^2. Generally speaking, there are multiple ways of representation of a given simplicially inønitesimal space as an overlapping family of basic inønitesimal spaces. By way of example, two representations of $D(3;(2,3))$ $(= (D \times D) \vee D)$ were given in Lavendhomme [1996, pp.92-93]. The above choice of representation of $D(3)_2$ as an overlapping family of basic inønitesimal spaces might be called the standard respresentaion. An interesting nonstandard representation of $D(6;(1,2),(3,4),(5,6))$ as an overlapping family of basic inønitesimal spaces is obtained from the standard one (as the overlapping family of eight copies of D^3) by taking into account the automorphism of $D(6;(1,2),(3,4),(5,6))$

$$\begin{aligned}(d_1,d_2,d_3,d_4,d_5,d_6) &\in D(6;(1,2),(3,4),(5,6)) \longmapsto \\ (d_1,d_2-d_1,d_3,d_4-d_3,d_5,d_6-d_5) &\in D(6;(1,2),(3,4),(5,6)),\end{aligned}$$

whose inverse automorphism is

$$\begin{aligned}(d_1,d_2,d_3,d_4,d_5,d_6) &\in D(6;(1,2),(3,4),(5,6)) \longmapsto \\ (d_1,d_1+d_2,d_3,d_3+d_4,d_5,d_5+d_6) &\in D(6;(1,2),(3,4),(5,6)).\end{aligned}$$

The reader should note that the nonstandard representation of $D(6;(1,2),(3,4),(5,6))$ as an overlapping family of basic inønitesimal spaces obtained from the standard one via the latter automorphism contains

$$\begin{aligned}(d_1,d_2,d_3) &\in D^3 \longmapsto \\ (d_1,d_1,d_2,d_2,d_3,d_3) &\in D(6;(1,2),(3,4),(5,6)).\end{aligned}$$

Kock and Lavendhomme [1984] have provided a synthetic rendering of the notion of strong diœerence for microsquares, a good exposition of which can be seen in Lavendhomme [1996, 3.4]. Given two microsquares γ_+ and γ_- on M, their strong diœerence $\gamma_+\dot{-}\gamma_-$ is deøned exactly when $\gamma_+|_{D(2)} = \gamma_-|_{D(2)}$, and it is a tangent vector to M with $(\gamma_+\dot{-}\gamma_-)(0) = \gamma_+(0,0) = \gamma_-(0,0)$. Given $t \in \mathbf{T}_x^D(M)$ and $\gamma \in \mathbf{T}_x^{D^2}(M)$, the strong addition $t\dot{+}\gamma$ is deøned to be a microsquare on M with $(t\dot{+}\gamma)|_{D(2)} = \gamma|_{D(2)}$. With respect to these operations Kock and Lavendhomme [1984] have shown that

Theorem 7 The canonical projection $\mathbf{T}_x^{D^2}(M) \to \mathbf{T}_x^{D(2)}(M)$ is an aŒne bundle over the trivial vector bundle $\mathbf{T}_x^D(M) \times \mathbf{T}_x^{D(2)}(M) \to \mathbf{T}_x^{D(2)}(M)$.

These considerations can be generalized easily to n-microcubes for any natural number n. More speciøcally, given two n-microcubes γ_+ and γ_- on M, their strong diœerence $\gamma_+ \dot{-} \gamma_-$is deøned exactly when $\gamma_+|_{D(n)_{n-1}} = \gamma_-|_{D(n)_{n-1}}$, and it is a tangent vector to M with $(\gamma_+ \dot{-} \gamma_-)(0) = \gamma_+(0,...,0) = \gamma_-(0,...,0)$. Given $t \in \mathbf{T}_x^D(M)$ and $\gamma \in \mathbf{T}_x^{D^n}(\gamma)$, the strong addition $t \dot{+} \gamma$ is deøned to be an n-microcube on M with $(t \dot{+} \gamma)|_{D(n)_{n-1}} = \gamma|_{D(n)_{n-1}}$. So as to deøne $\dot{-}$ and $\dot{+}$, we need the following two lemmas. Their proofs are akin to their counterparts of microsquares (cf. Lavendhomme [1996, pp.92-93]).

Lemma 8 (cf. Nishimura [1997. Lemma 5.1] and Lavendhomme and Nishimura [1998, Proposition 3]). The diagram

$$\begin{array}{ccc} D(n)_{n-1} & \underrightarrow{i} & D^n \\ i \downarrow & & \downarrow \psi \\ D^n & \underrightarrow{\varphi} & D^n \vee D \end{array}$$

is a quasi-colimit diagram, where $i : D(n;n) \to D^n$ is the canonical injection, $D^n \vee D = D(n+1;(1,n+1),...,(n,n+1))$, $\varphi(d_1,...,d_n) = (d_1,...,d_n,0)$ and $\psi(d_1,...,d_n) = (d_1,...,d_n,d_1...d_n)$ for any $(d_1,...,d_n) \in D^n$.

Given two *n-microsquares* γ_+ and γ_- on M with $\gamma_+|_{D(n)_{n-1}} = \gamma_-|_{D(n)_{n-1}}$, there exists a unique function $\mathfrak{F} : D^n \vee D \to M$ with $\mathfrak{F} \circ \psi = \gamma_+$ and $\mathfrak{F} \circ \varphi = \gamma_-$. We deøne $(\gamma_+ \dot{-} \gamma_-)(d) = \mathfrak{F}(0,...,0,d)$ for any $d \in D$.

Lemma 9 The diagram

$$\begin{array}{ccc} 1 & \underrightarrow{i} & D \\ i \downarrow & & \downarrow \xi \\ D^n & \underrightarrow{\varphi} & D^n \vee D \end{array}$$

is a quasi-colimit diagram, where $i : 1 \to D^n$ and $i : 1 \to D$ are the canonical injections and $\xi(d) = (0,...,0,d)$ for any $d \in D$.

Given $t \in \mathbf{T}_x^D(M)$ and $\gamma \in \mathbf{T}_x^{D^n}(M)$, there exists a unique function $\mathfrak{G} : D^n \vee D \to M$ with $\mathfrak{G} \circ \varphi = \gamma$ and $\mathfrak{G} \circ \xi = t$. We deøne $(t \dot{+} \gamma)(d_1, ..., d_n) = \mathfrak{G}(d_1, ..., d_n, d_1 ... d_n)$ for any $(d_1, ..., d_n) \in D^n$.

We proceed as in the case of microsquares to get

Theorem 10 The canonical projection $\mathbf{T}_x^{D^n}(M) \to \mathbf{T}_x^{D(n)_{n-1}}(M)$ is an aŒne bundle over the trivial vector bundle $\mathbf{T}_x^D(M) \times \mathbf{T}_x^{D(n)_{n-1}}(M) \to \mathbf{T}_x^{D(n)_{n-1}}(M)$.

Now we are going to discuss D_n versions of Lemmas 8 and 9 and Theorem 10. The following two lemmas can be established by the same token as in Lavendhomme [1996, pp.92-93].

Lemma 11 The diagram

$$\begin{array}{ccc} D_{n-1} & \xrightarrow{i} & D_n \\ i \downarrow & & \downarrow \psi \\ D_n & \xrightarrow{\varphi} & D_n \vee D \end{array}$$

is a quasi-colimit diagram, where $i : D_{n-1} \to D^n$ is the canonical injection, $D_n \vee D = \{(d, d') \in D_n \times D | dd' = 0\}$, $\varphi(d) = (d, 0)$ and $\psi(d) = (d, d^n)$ for any $d \in D_n$.

Given two D_n-microcubes γ_+ and γ_- on M with $\gamma_+|_{D_{n-1}} = \gamma_-|_{D_{n-1}}$, there exists a unique function $\mathfrak{F} : D^n \vee D \to M$ with $\mathfrak{F} \circ \psi = \gamma_+$ and $\mathfrak{F} \circ \varphi = \gamma_-$. We deøne $(\gamma_+ \dot{-} \gamma_-)(d) = \mathfrak{F}(0, d)$ for any $d \in D$.

Lemma 12 The diagram

$$\begin{array}{ccc} 1 & \xrightarrow{i} & D \\ i \downarrow & & \downarrow \xi \\ D_n & \xrightarrow{\varphi} & D_n \vee D \end{array}$$

is a quasi-colimit diagram, where $i : 1 \to D_n$ and $i : 1 \to D$ are the canonical injections and $\xi(d) = (0, d)$ for any $d \in D$.

Given $t \in \mathbf{T}_x^D(M)$ and $\gamma \in \mathbf{T}_x^{D_n}(M)$, there exists a unique function $\mathfrak{G} : D^n \vee D \to M$ with $\mathfrak{G} \circ \varphi = \gamma$ and $\mathfrak{G} \circ \xi = t$. We deøne $(t \dot{+} \gamma)(d_1, ..., d_n) = \mathfrak{G}(d, d^n)$ for any $d \in D_n$.

We can proceed as in the proof of Proposition 4 of Lavendhomme [1996, 3.4] to get

Theorem 13 The canonical projection $\mathbf{T}_x^{D_n}(M) \to \mathbf{T}_x^{D_{n-1}}(M)$ is an aŒne bundle over the trivial vector bundle $\mathbf{T}_x^D(M) \times \mathbf{T}_x^{D_{n-1}}(M) \to \mathbf{T}_x^{D_{n-1}}(M)$.

1.3 Forms

A D^n-form from (M, x) to (N, y) is a mapping η from $\mathbf{T}_x^{D^n}(M)$ to $\mathbf{T}_y^D(N)$ such that for any $\gamma \in \mathbf{T}_x^{D^n}(M)$, any $\gamma' \in \mathbf{T}_x^{D^{n-1}}(M)$, any $\alpha \in \mathbb{R}$ and any $\sigma \in \mathfrak{S}_n$, we have

$$\eta(\alpha \underset{i}{\cdot} \gamma) = \alpha\eta(\gamma) \qquad (1 \leq i \leq n) \tag{f1}$$

$$\eta(\Sigma_\sigma(\gamma)) = \eta(\gamma) \tag{f2}$$

$$\eta((d_1, ..., d_n) \in D^n \longmapsto \gamma'(d_1, ..., d_{n-2}, d_{n-1}d_n) \in M) = 0 \tag{f3}$$

We denote by $\mathbf{F}^n(M, x; N, y)$ the totality of D^n-forms from (M, x) to (N, y).

A D_n-form from (M, x) to (N, y) is a mapping η from $\mathbf{T}_x^{D_n}(M)$ to $\mathbf{T}_y^D(N)$ such that for any $\gamma \in \mathbf{T}_x^{D^n}(M)$, any $\alpha \in \mathbb{R}$, any natural number $m \leq n$ and any $\gamma' \in \mathbf{T}_x^{D_l}(M)$ with $l = [\frac{n}{m}]$ ([] stands for Gauss's symbol), we have

$$\eta(\alpha\gamma) = \alpha^n\eta(\gamma) \tag{f4}$$

$$\eta(d \in D_n \longmapsto \gamma'(d^m) \in M) = 0 \tag{f5}$$

We denote by $\mathbb{F}^n(M, x; N, y)$ the totality of D_n-forms from (M, x) to (N, y).

1.4 Convention

Unless stated to the contrary, M and N are microlinear spaces with $x \in M$ and $y \in N$.

2 The First Kind of Tangentials

Let n be a natural number. A D^n-pseudotangential from (M,x) to (N,y) is a mapping $\mathsf{f} : \mathbf{T}_x^{D^n}(M) \to \mathbf{T}_y^{D^n}(N)$ such that for any $\gamma \in \mathbf{T}_x^{D^n}(M)$, any $\alpha \in \mathbb{R}$ and any $\sigma \in \mathfrak{S}_n$, we have the following:

$$\mathsf{f}(\alpha \underset{i}{\cdot} \gamma) = \alpha \underset{i}{\cdot} \mathsf{f}(\gamma) \qquad (1 \leq i \leq n) \tag{a1}$$

$$\mathsf{f}(\Sigma_\sigma(\gamma)) = \Sigma_\sigma(\mathsf{f}(\gamma)) \tag{a2}$$

We denote by $\hat{\mathbf{J}}^n(M,x;N,y)$ the totality of D^n-pseudotangentials from (M,x) to (N,y).

The following lemma and two propositions have been established in Nishimura [2004].

Lemma 14 Let f be a D^{n+1}-pseudotangential from (M,x) to (N,y). Let $\gamma \in \mathbf{T}_x^{D^n}(M)$ and $(d_1,...,d_{n+1}) \in D^{n+1}$. Then $\mathsf{f}(\mathbf{s}_{n+1}(\gamma))(d_1,...,d_n,d_{n+1})$ is independent of d_{n+1}, so that we can put down $\mathsf{f}(\mathbf{s}_{n+1}(\gamma))$ at $\mathbf{T}_y^{D^n}(N)$.

Proposition 15 The assignment $\gamma \in \mathbf{T}_x^{D^n}(M) \longmapsto \mathsf{f}(\mathbf{s}_{n+1}(\gamma)) \in \mathbf{T}_y^{D^n}(N)$ in the above lemma is a D^n-pseudotangential from (M,x) to (N,y).

By the above proposition we have the canonical projection $\hat{\pi}_{n+1,n} : \hat{\mathbf{J}}^{n+1}(M,x;N,y) \to \hat{\mathbf{J}}^n(M,x;N,y)$, so that

$$\mathsf{f}(\mathbf{s}_{n+1}(\gamma)) = \mathbf{s}_{n+1}(\hat{\pi}_{n+1,n}(\mathsf{f})(\gamma))$$

for any $\mathsf{f} \in \hat{\mathbf{J}}^{n+1}(M,x;N,y)$ and any $\gamma \in \mathbf{T}_x^{D^n}(M)$. For any natural numbers n, m with $m \leq n$, we deøne $\hat{\pi}_{n,m} : \hat{\mathbf{J}}^n(M,x;N,y) \to \hat{\mathbf{J}}^m(M,x;N,y)$ to be $\hat{\pi}_{m+1,m} \circ ... \circ \hat{\pi}_{n,n-1}$.

Proposition 16 Let f be a D^{n+1}-pseudotangential from (M,x) to (N,y). Let i be a natural number with $1 \leq i \leq n+1$.Then the following diagrams are commutative:

$$
\begin{array}{ccc}
\mathbf{T}_x^{D^{n+1}}(M) & \xrightarrow{\mathsf{f}} & \mathbf{T}_y^{D^{n+1}}(N) \\
\mathbf{s}_i \uparrow & & \uparrow \mathbf{s}_i \\
\mathbf{T}_x^{D^n}(M) & \xrightarrow[\hat{\pi}_{n+1,n}(\mathsf{f})]{} & \mathbf{T}_y^{D^n}(N)
\end{array}
$$

$$
\begin{array}{ccc}
\mathbf{T}_x^{D^{n+1}}(M) & \xrightarrow{\mathsf{f}} & \mathbf{T}_y^{D^{n+1}}(N) \\
\mathbf{d}_i \downarrow & & \downarrow \mathbf{d}_i \\
\mathbf{T}_x^{D^n}(M) & \xrightarrow[\hat{\pi}_{n+1,n}(\mathsf{f})]{} & \mathbf{T}_y^{D^n}(N)
\end{array}
$$

Interestingly enough, any D^n-pseudotangential naturally gives rise to what might be called a $\mathfrak{D}$-pseudotangential for any simplicial inønitesimal space $\mathfrak{D}$ of dimension less than or equal to n.

Proposition 17 Let n be a natural number. Let $\mathfrak{D}$ be a simplicial inønitesimal space $\mathfrak{D}$ of dimension less than or equal to n. Any D^n-pseudotangential f from (M,x) to (N,y) naturally induces a mapping $\mathbf{T}_x^{\mathfrak{D}}(M) \to \mathbf{T}_y^{\mathfrak{D}}(N)$ abiding by the following condition:

$$\mathsf{f}(\alpha \underset{i}{\cdot} \gamma) = \alpha \underset{i}{\cdot} \mathsf{f}(\gamma) \text{ for any } \alpha \in \mathbb{R} \text{ and any } \gamma \in \mathbf{T}_x^{\mathfrak{D}}(M).$$

In particular, given a natural number m, f gives rise to a mapping $\mathbf{T}_x^{D(m)_n}(M) \to \mathbf{T}_y^{D(m)_n}(N)$ abiding by the following condition besides the above one:

$$\mathsf{f}(\Sigma_\sigma(\gamma)) = \Sigma_\sigma(\mathsf{f}(\gamma))$$

for any $\sigma \in \mathfrak{S}_n$ and any $\gamma \in \mathbf{T}_x^{D(m)_n}(M)$.

Proof. For the sake of simplicity, we deal only with the case that $\mathfrak{D} = D(3)_2$, for which the standard quasi-colimit representation was given in the proof of Theorem . Therefore, giving $\gamma \in \mathbf{T}_x^{D(3)_2}(M)$ is equivalent to giving $\gamma_{12}, \gamma_{13}, \gamma_{23} \in \mathbf{T}_x^{D^2}(M)$ with $\mathbf{d}_2(\gamma_{12}) = \mathbf{d}_2(\gamma_{13})$, $\mathbf{d}_1(\gamma_{12}) = \mathbf{d}_2(\gamma_{23})$ and $\mathbf{d}_1(\gamma_{13}) = \mathbf{d}_1(\gamma_{23})$. By Proposition, we have

$$\begin{aligned} \mathbf{d}_2(\mathsf{f}(\gamma_{12})) &= \mathsf{f}(\mathbf{d}_2(\gamma_{12})) = \mathsf{f}(\mathbf{d}_2(\gamma_{13})) = \mathbf{d}_2(\mathsf{f}(\gamma_{13})), \\ \mathbf{d}_1(\mathsf{f}(\gamma_{12})) &= \mathsf{f}(\mathbf{d}_1(\gamma_{12})) = \mathsf{f}(\mathbf{d}_2(\gamma_{23})) = \mathbf{d}_2(\mathsf{f}(\gamma_{23})), \text{ and} \\ \mathbf{d}_1(\mathsf{f}(\gamma_{13})) &= \mathsf{f}(\mathbf{d}_1(\gamma_{13})) = \mathsf{f}(\mathbf{d}_1(\gamma_{23})) = \mathbf{d}_1(\mathsf{f}(\gamma_{23})), \end{aligned}$$

which determines a unique $\mathsf{f}(\gamma) \in \mathbf{T}_y^{D(3)_2}(N)$ with $\mathbf{d}_1(\mathsf{f}(\gamma)) = \mathsf{f}(\gamma_{23})$, $\mathbf{d}_2(\mathsf{f}(\gamma)) = \mathsf{f}(\gamma_{13})$ and $\mathbf{d}_3(\mathsf{f}(\gamma)) = \mathsf{f}(\gamma_{12})$. The proof that $\mathsf{f}(\gamma)$ acquiesces in the desired two properties is safely left to the reader. ■

Remark 18 It is very important to note that in the above proposition, the D^n-pseudotangential f from (M,x) to (N,y) gives rise to the same mapping for all nonstandard representations of a given simplicial inønitesimal space as an overlapping family of basic inønitesimal spaces as well as the standard one.

The notion of a D^n-tangential from (M,x) to (N,y) is deøned inductively on n. The notion of a D^0-tangential from (M,x) to (N,y) and that of a D^1-tangential from (M,x) to (N,y) shall be identical with that of a D^0-pseudotangential from (M,x) to (N,y) and that of a D^1-pseudotangential from (M,x) to (N,y) respectively. Now we proceed by induction. A D^{n+1}-pseudotangential $\mathsf{f} : \mathbf{T}_x^{D^{n+1}}(M) \to \mathbf{T}_y^{D^{n+1}}(N)$ from (M,x) to (N,y) is called a D^{n+1}-tangential from (M,x) to (N,y) if it acquiesces in the following two conditions:

1. $\hat{\pi}_{n+1,n}(\mathsf{f})$ is a D^n-tangential from (M,x) to (N,y).

2. For any $\gamma \in \mathbf{T}_x^{D^n}(M)$, we have

$$\begin{aligned} \mathsf{f}((d_1,...,d_{n+1}) &\in D^{n+1} \longmapsto \gamma(d_1,...,d_{n-1},d_n d_{n+1}) \in M) \\ &= (d_1,...,d_{n+1}) \in D^{n+1} \longmapsto \hat{\pi}_{n+1,n}(\mathsf{f})(\gamma)(d_1,...,d_{n-1},d_n d_{n+1}) \in N \end{aligned} \quad (a4)$$

We denote by $\mathbf{J}^n(M,x;N,y)$ the totality of D^n-tangentials from (M,x) to (N,y). By the very deønition of a D^n-tangential, the projection $\hat{\pi}_{n+1,n} : \hat{\mathbf{J}}^{n+1}(M,x;N,y) \to \hat{\mathbf{J}}^n(M,x;N,y)$ is naturally restricted to a mapping $\pi_{n+1,n} : \mathbf{J}^{n+1}(M,x;N,y) \to \mathbf{J}^n(M,x;N,y)$. Similarly for $\pi_{n,m} : \mathbf{J}^n(M,x;N,y) \to \mathbf{J}^m(M,x;N,y)$ with $m \leq n$.

Proposition 19 Let L, M, N be microlinear spaces with $x \in L, y \in M$ and $z \in N$. If f is a D^n-tangential from (L,x) to (M,y) and g is a D^n-tangential from (M,y) to (N,z), then the composition $\mathsf{g} \circ \mathsf{f}$ is a D^n-tangential from (L,x) to (N,z), and $\pi_{n,n-1}(\mathsf{g}\circ\mathsf{f}) = \pi_{n,n-1}(\mathsf{g})\circ\pi_{n,n-1}(\mathsf{f})$ provided that $n \geq 1$.

Proof. In case of $n = 0$, there is nothing to prove. It is easy to see that if f is a D^n-tangential from (L,x) to (M,y) and g is a D^n-tangential from (M,y) to (N,z), then the composition $\mathsf{g} \circ \mathsf{f}$ satisøes conditions ?? and ??. For any $\gamma \in \mathbf{T}_x^{D^n}(M)$, if f is a D^{n+1}-tangential from (L,x) to (M,y) and g is a D^{n+1}-tangential from (M,y) to (N,z), we have

$$\begin{aligned} & \mathsf{g} \circ \mathsf{f}(\mathbf{s}_{n+1}(\gamma)) \\ = \; & \mathsf{g}(\mathsf{f}(\mathbf{s}_{n+1}(\gamma))) \\ = \; & \mathsf{g}(\mathbf{s}_{n+1}(\pi_{n+1,n}(\mathsf{f})(\gamma))) \\ = \; & \mathbf{s}_{n+1}(\pi_{n+1,n}(\mathsf{g}) \circ \pi_{n+1,n}(\mathsf{f})(\gamma)), \end{aligned}$$

which implies that $\pi_{n+1,n}(\mathsf{g} \circ \mathsf{f}) = \pi_{n+1,n}(\mathsf{g}) \circ \pi_{n+1,n}(\mathsf{f})$. Therefore we have

$$
\begin{aligned}
\mathsf{g}\circ\mathsf{f}((d_1,...,d_{n+1}) &\in D^{n+1}\mapsto\gamma(d_1,...,d_{n-1},d_nd_{n+1})\in L) \\
&= \mathsf{g}(\mathsf{f}((d_1,...,d_{n+1})\in D^{n+1}\longmapsto\gamma(d_1,...,d_{n-1},d_nd_{n+1})\in L)) \\
&= \mathsf{g}((d_1,...,d_{n+1})\in D^{n+1}\longmapsto\pi_{n+1,n}(\mathsf{f})(\gamma)(d_1,...,d_{n-1},d_nd_{n+1})\in M) \\
&= (d_1,...,d_{n+1})\in D^{n+1}\longmapsto\pi_{n+1,n}(\mathsf{g})\circ\pi_{n+1,n}(\mathsf{f})(\gamma)(d_1,...,d_{n-1},d_nd_{n+1})\in N \\
&= (d_1,...,d_{n+1})\in D^{n+1}\longmapsto\pi_{n+1,n}(\mathsf{g}\circ\mathsf{f})(\gamma)(d_1,...,d_{n-1},d_nd_{n+1})\in N,
\end{aligned}
$$

which implies that the composition $\mathsf{g}\circ\mathsf{f}$ satisøes condition ?? . Now we can prove by induction on

n that $\hat{\pi}_{n+1,n}(\mathsf{g}\circ\mathsf{f})$ is a D^n-tangential from (L,x) to (N,z), so that it is a D^{n+1}-tangential from (L,x) to (N,z). ■

The following simple proposition may help the reader understand where our locution of D^n-tangential has originated.

Proposition 20 Let M,N be microlinear spaces with $x\in M$ and $y\in N$. If f is a mapping from (M,x) to (N,y), then the assignment of $f\circ\gamma\in\mathbf{T}_y^n(N)$ to each $\gamma\in\mathbf{T}_x^n(M)$, denoted by $\mathbf{D}^nf$ and called the D^n-prolongation of f, is a D^n-tangential from (M,x) to (N,y). We have $\mathbf{D}^nf=\pi_{n+1,n}(\mathbf{D}^{n+1}f)$. If L is another microlinear space with $z\in L$ and g is a mapping from (N,y) to (L,z), then we have $\mathbf{D}^n(g\circ f)=(\mathbf{D}^ng)\circ(\mathbf{D}^nf)$.

Proof. It is easy to see that $\mathbf{D}^nf$ abides by conditions ?? and ??. Trivially $\mathbf{D}^nf=\pi_{n+1,n}(\mathbf{D}^{n+1}f)$ and $\mathbf{D}^n(g\circ f)=(\mathbf{D}^ng)\circ(\mathbf{D}^nf)$. For any $\gamma\in\mathbf{T}_x^n(M)$, we have

$$\begin{aligned}\mathbf{D}^{n+1}f((d_1,...,d_{n+1}) &\in D^{n+1}\mapsto\gamma(d_1,...,d_{n-1},d_nd_{n+1}))\\ &= (d_1,...,d_{n+1})\in D^{n+1}\mapsto f(\gamma(d_1,...,d_{n-1},d_nd_{n+1}))\\ &= (d_1,...,d_{n+1})\in D^{n+1}\mapsto \mathbf{D}^n f(\gamma)(d_1,...,d_{n-1},d_nd_{n+1})\\ &= (d_1,...,d_{n+1})\in D^{n+1}\mapsto \pi_{n+1,n}(\mathbf{D}^{n+1}f)(\gamma)(d_1,...,d_{n-1},d_nd_{n+1}),\end{aligned}$$

which implies that $\mathbf{D}^{n+1}f$ abides by condition ?? for any natural number n. By dint of $\mathbf{D}^n f = \pi_{n+1,n}(\mathbf{D}^{n+1}f)$ again, we can prove by induction on n that $\hat{\pi}_{n+1,n}(\mathbf{D}^{n+1}f)$ is a D^n-tangential from (M,x) to (N,y), so that $\mathbf{D}^{n+1}f$ is a D^{n+1}-tangential from (M,x) to (N,y). ■

We have established the following proposition in Nishimura [2004]:

Proposition 21 Let m,n be natural numbers with $m\leq n$. Let $k_1,...,k_m$ be positive integers with $k_1+\ ...\ +k_m=n$. For any $\mathsf{f}\in\mathbf{J}^n(M,x;N,y)$, any $\gamma\in\mathbf{T}_x^{D^n}(M)$ and any $\sigma\in\mathfrak{S}_n$, we have

$$\begin{aligned}\mathsf{f}((d_1,...,d_n) &\in D^n\longmapsto\gamma(d_{\sigma(1)}...d_{\sigma(k_1)},d_{\sigma(k_1+1)}...d_{\sigma(k_1+k_2)},...,\\ &\quad d_{\sigma(k_1+...+k_{m-1}+1)}...d_{\sigma(n)}))\\ &= (d_1,...,d_n)\in D^n\longmapsto\pi_{n,m}(\mathsf{f})(\gamma)(d_{\sigma(1)}...d_{\sigma(k_1)},\\ &\quad d_{\sigma(k_1+1)}...d_{\sigma(k_1+k_2)},...,d_{\sigma(k_1+...+k_{m-1}+1)}...d_{\sigma(n)})\end{aligned}$$

Remark 22 This proposition not only derives from ?? and ?? but also subsumes them.

With due regard to Proposition 17, the above proposition has the following far-Æung generalization:

Proposition 23 Let f be a D^n-tangential from (M, x) to (N, y). Let $\mathfrak{D}$ and $\mathfrak{D}'$ be simplicially inønitesimal spaces of dimension less than or equal to n with $\deg \mathfrak{D} = k$ and $\deg \mathfrak{D}' = m$. Let χ be a monomial mapping from $\mathfrak{D}$ to $\mathfrak{D}'$. Let $\gamma \in \mathbf{T}_x^{\mathfrak{D}'}(M)$. For simplicity we denote the induced mappings $\mathbf{T}_x^{\mathfrak{D}}(M) \to \mathbf{T}_y^{\mathfrak{D}}(N)$ and $\mathbf{T}_x^{\mathfrak{D}'}(M) \to \mathbf{T}_y^{\mathfrak{D}'}(N)$ in Proposition 17 by the same symbol f. Then we have

$$\mathsf{f}(\gamma \circ \chi) = \mathsf{f}(\gamma) \circ \chi$$

Remark 24 The reader should note that the above far-Æung generalization of Proposition 22 subsumes not only Proposition 22 (subsuming ?? and ??) but also Proposition 16.

Proof. In place of giving a general proof with formidable notation, we satisfy ourselves with an illustration. Here we deal only with the case that $\mathfrak{D} = D^3$, $\mathfrak{D}' = D(3)$ and $\chi(d_1, d_2, d_3) = (d_1 d_2, d_1 d_3, d_2 d_3)$ for any $(d_1, d_2, d_3) \in D^3$, assuming that $n \geq 3$. We note ørst that the monomial mapping $\chi : D^3 \to D(3)$ is the composition of two monomial mappings $\chi_1 : D^3 \to D(6; (1,2), (3,4), (5,6))$ and $\chi_2 : D(6; (1,2), (3,4), (5,6)) \to D(3)$ with $\chi_1(d_1, d_2, d_3) = (d_1, d_1, d_2, d_2, d_3, d_3)$ for any $(d_1, d_2, d_3) \in D^3$ and $\chi_2(d_1, d_2, d_3, d_4, d_5, d_6) = (d_1 d_3, d_2 d_5, d_4 d_6)$ for any $(d_1, d_2, d_3, d_4, d_5, d_6) \in D(6; (1,2), (3,4), (5,6))$. Therefore it suŒces to prove that

$$\mathsf{f}(\gamma' \circ \chi_1) = \mathsf{f}(\gamma') \circ \chi_1$$

for any $\gamma' \in \mathbf{T}_x^{D(6;(1,2),(3,4),(5,6))}(M)$ and

$$\mathsf{f}(\gamma'' \circ \chi_2) = \mathsf{f}(\gamma'') \circ \chi_2$$

for any $\gamma'' \in \mathbf{T}_x^{D(3)}(M)$. The ørst identity is merely the deønition of $\mathsf{f}(\gamma')$ (cf. Remark 24). In order to prove the second identity, it suŒces to note that

$$
\begin{aligned}
\mathsf{f}(\gamma'' \circ \chi_2) \circ i_{135} &= \mathsf{f}(\gamma'') \circ \chi_2 \circ i_{135} \\
\mathsf{f}(\gamma'' \circ \chi_2) \circ i_{136} &= \mathsf{f}(\gamma'') \circ \chi_2 \circ i_{136} \\
\mathsf{f}(\gamma'' \circ \chi_2) \circ i_{145} &= \mathsf{f}(\gamma'') \circ \chi_2 \circ i_{145} \\
\mathsf{f}(\gamma'' \circ \chi_2) \circ i_{146} &= \mathsf{f}(\gamma'') \circ \chi_2 \circ i_{146} \\
\mathsf{f}(\gamma'' \circ \chi_2) \circ i_{235} &= \mathsf{f}(\gamma'') \circ \chi_2 \circ i_{235} \\
\mathsf{f}(\gamma'' \circ \chi_2) \circ i_{236} &= \mathsf{f}(\gamma'') \circ \chi_2 \circ i_{236} \\
\mathsf{f}(\gamma'' \circ \chi_2) \circ i_{245} &= \mathsf{f}(\gamma'') \circ \chi_2 \circ i_{245} \\
\mathsf{f}(\gamma'' \circ \chi_2) \circ i_{246} &= \mathsf{f}(\gamma'') \circ \chi_2 \circ i_{246},
\end{aligned}
$$

where $i_{jkl} : D^3 \to D(6; (1,2), (3,4), (5,6))$ $(1 \leq j < k < l \leq 6)$ is a mapping with $i_{jkl}(d_1, d_2, d_3) = (..., \underset{j}{d_1}, ..., \underset{k}{d_2}, ..., \underset{l}{d_3}, ...)$ (d_1, d_2 and d_3 are inserted at the j-th, k-th and l-th positions respectively, while the other components are øxed at 0). Here we deal only with the ørst identity among the eight similar ones. Since

$$
\mathsf{f}(\gamma'' \circ \chi_2) \circ i_{135} = \mathsf{f}(\gamma'' \circ \chi_2 \circ i_{135}),
$$

it suŒces to show that

$$
\mathsf{f}(\gamma'' \circ \chi_2 \circ i_{135}) = \mathsf{f}(\gamma'') \circ \chi_2 \circ i_{135}.
$$

However the last identity follows at once by simply observing that the mapping $\chi_2 \circ i_{135} : D^3 \to D(3)$ is the mapping

$$
(d_1, d_2, d_3) \in D^3 \longmapsto (d_1 d_2, 0, 0) \in D(3),
$$

which is the successive composition of the following three mappings:

$$
\begin{aligned}
(d_1, d_2, d_3) &\in D^3 \longmapsto (d_1, d_2) \in D^2 \\
(d_1, d_2) &\in D^2 \longmapsto d_1 d_2 \in D \\
d &\in D \longmapsto (d, 0, 0) \in D(3).
\end{aligned}
$$

■

We have established the following proposition and theorem in our previous paper [Nishimura, 2004].

Proposition 25 Let $\mathsf{f} \in \mathbf{J}^n(M,x;N,y)$, $t \in \mathbf{T}_x^D(M)$ and γ, γ_+, $\gamma_- \in \mathbf{T}_x^{D^n}(M)$ with $\gamma_+|_{D(n;(1,\ldots,n))} = \gamma_-|_{D(n;(1,\ldots,n))}$. Then we have

$$\begin{aligned} \mathsf{f}(\gamma_+)\dot{-}\mathsf{f}(\gamma_-) &= \pi_{n,1}(\mathsf{f})(\gamma_+\dot{-}\gamma_-) \\ \pi_{n,1}(\mathsf{f})(t)\dot{+}\mathsf{f}(\gamma) &= \mathsf{f}(t\dot{+}\gamma) \end{aligned}$$

Theorem 26

1. Let $\mathsf{f}^+, \mathsf{f}^- \in \mathbf{J}^{n+1}(M,x;N,y)$ with $\hat{\pi}_{n+1,n}(\mathsf{f}^+) = \hat{\pi}_{n+1,n}(\mathsf{f}^-)$. Then the assignment $\gamma \in \mathbf{T}_x^{D^{n+1}}(M) \longmapsto \mathsf{f}^+(\gamma)\dot{-}\mathsf{f}^-(\gamma)$, to be denoted by $\mathsf{f}^+\dot{-}\mathsf{f}^-$, belongs to $\mathbf{F}^{n+1}(M,x;N,y)$.
2. Let $\mathsf{f} \in \mathbf{J}^{n+1}(M,x;N,y)$ and $\eta \in \mathbf{F}^{n+1}(M,x;N,y)$. Then the assignment $\mathsf{f} \in \mathbf{T}_x^{D^{n+1}}(M) \longmapsto \eta(\gamma)\dot{+}\mathsf{f}(\gamma)$, to be denoted by $\eta\dot{+}\mathsf{f}$, belongs to $\mathbf{J}^{n+1}(M,x;N,y)$.
3. The bundle $\pi_{n+1,n} : \mathbf{J}^{n+1}(M,x;N,y) \to \mathbf{J}^n(M,x;N,y)$ is an aŒne bundle over the trivial vector bundle $\mathbf{J}^n(M,x;N,y) \times \mathbf{F}^{n+1}(M,x;N,y) \to \mathbf{J}^n(M,x;N,y)$ with respect to the above two operations.

3 The Second Kind of Tangentials

Let n be a natural number. A D_n-pseudotangential from (M,x) to (N,y) is a mapping $\mathfrak{f} : \mathbf{T}_x^{D_n}(M) \to \mathbf{T}_y^{D_n}(N)$ such that for any $\gamma \in \mathbf{T}_x^{D_n}(M)$ and any $\alpha \in \mathbb{R}$, we have the following:

$$\mathfrak{f}(\alpha\gamma) = \alpha\mathfrak{f}(\gamma) \tag{b1}$$

We denote by $\hat{\mathbb{J}}^n(M,x;N,y)$ the totality of D_n-pseudotangentials from (M,x) to (N,y).

Lemma 27 Let $\mathfrak{f}$ be a D_{n+1}-pseudotangential from (M,x) to (N,y) and $\gamma \in \mathbf{T}_x^{D_n}(M)$. Then there exists a unique $\gamma' \in \mathbf{T}_y^{D_n}(N)$ such that for any $d \in D_n$, we have

$$\mathfrak{f}(\mathbf{i}_d(\gamma)) = \mathbf{i}_d(\gamma')$$

Proof. For any $d' \in D_{n+1}$, we have

$$\begin{aligned} & \mathfrak{f}(\mathbf{i}_{d'd}(\gamma)) \\ = & \ \mathfrak{f}(d'(\mathbf{i}_d(\gamma))) \\ = & \ d'(\mathfrak{f}(\mathbf{i}_d(\gamma))) \end{aligned}$$

so that the lemma follows from Proposition 5. ■

Proposition 28 The assignment $\gamma \in \mathbf{T}_x^{D_n}(M) \longmapsto \gamma' \in \mathbf{T}_y^{D_n}(N)$ in the above lemma is a D_n-pseudotangential from (M,x) to (N,y).

Proof. For any $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} & \mathbf{i}_d(\alpha(\hat{\pi}_{n+1,n}(\mathfrak{f})(\gamma))) \\ = & \ \alpha(\mathbf{i}_d(\hat{\pi}_{n+1,n}(\mathfrak{f})(\gamma))) \\ = & \ \alpha(\mathfrak{f}(\mathbf{i}_d(\gamma))) \\ = & \ \mathfrak{f}(\alpha(\mathbf{i}_d(\gamma))) \\ = & \ \mathfrak{f}(\mathbf{i}_d(\alpha\gamma)), \end{aligned}$$

which establishes the proposition. ■

By the above proposition we have the canonical projection $\hat{\pi}_{n+1,n} : \hat{\mathbb{J}}^{n+1}(M,x;N,y) \rightarrow \hat{\mathbb{J}}^{n}(M,x;N,y)$, so that

$$\mathfrak{f}(\mathbf{i}_d(\gamma)) = \mathbf{i}_d(\hat{\pi}_{n+1,n}(\mathfrak{f})(\gamma))$$

for any $\mathfrak{f} \in \hat{\mathbb{J}}^{n+1}(M,x;N,y)$, any $d \in D_n$ and any $\gamma \in \mathbf{T}_x^{D_n}(M)$. For any natural numbers n, m with $m \leq n$, we deøne $\hat{\pi}_{n,m} : \hat{\mathbb{J}}^{n}(M,x;N,y) \rightarrow \hat{\mathbb{J}}^{m}(M,x;N,y)$ to be $\hat{\pi}_{m+1,m} \circ ... \circ \hat{\pi}_{n,n-1}$.

Proposition 29 Let $\mathfrak{f}$ be a D_{n+1}-pseudotangential from (M,x) to (N,y) and $d \in D_n$. Then the following diagrams are commutative:

$$\begin{array}{ccc} \mathbf{T}_x^{D_{n+1}}(M) & \xrightarrow{\mathfrak{f}} & \mathbf{T}_y^{D_{n+1}}(N) \\ \mathbf{i}_d \uparrow & & \uparrow \mathbf{i}_d \\ \mathbf{T}_x^{D_n}(M) & \xrightarrow[\hat{\pi}_{n+1,n}(\mathfrak{f})]{} & \mathbf{T}_y^{D_n}(N) \end{array}$$

$$\begin{array}{ccc} \mathbf{T}_x^{D_{n+1}}(M) & \xrightarrow{\mathfrak{f}} & \mathbf{T}_y^{D_{n+1}}(N) \\ \boldsymbol{\pi}_{n+1,n} \downarrow & & \downarrow \boldsymbol{\pi}_{n+1,n} \\ \mathbf{T}_x^{D_n}(M) & \xrightarrow[\hat{\pi}_{n+1,n}(\mathfrak{f})]{} & \mathbf{T}_y^{D_n}(N) \end{array}$$

Proof. The commutativity of the ørst diagram is exactly the deønition of $\hat{\pi}_{n+1,n}(\mathfrak{f})$. For the sake of commutativity of the second diagram, it suŒces to note by dint of Proposition 1 that for any $d \in D_n$, we have

$$\begin{aligned} & \mathbf{i}_d(\hat{\pi}_{n+1,n}(\mathfrak{f})(\boldsymbol{\pi}_{n+1,n}(\gamma))) \\ = \; & \mathfrak{f}(\mathbf{i}_d(\boldsymbol{\pi}_{n+1,n}(\gamma))) \\ = \; & \mathfrak{f}(d\gamma) \\ = \; & d(\mathfrak{f}(\gamma)) \\ = \; & \mathbf{i}_d(\boldsymbol{\pi}_{n+1,n}(\mathfrak{f}(\gamma))). \end{aligned}$$

■

Corollary 30 Let $\mathfrak{f}$ be a D_{n+1}-pseudotangential from (M, x) to (N, y). For any $\gamma, \gamma' \in \mathbf{T}_x^{D_{n+1}}(M)$, if $\gamma|_{D_n} = \gamma'|_{D_n}$, then $\mathfrak{f}(\gamma)|_{D_n} = \mathfrak{f}(\gamma')|_{D_n}$.

Proof. By the above proposition, we have

$$\begin{aligned} & \boldsymbol{\pi}_{n+1,n}(\mathfrak{f}(\gamma)) \\ = & \ \hat{\pi}_{n+1,n}(\mathfrak{f})(\boldsymbol{\pi}_{n+1,n}(\gamma)) \\ = & \ \hat{\pi}_{n+1,n}(\mathfrak{f})(\boldsymbol{\pi}_{n+1,n}(\gamma')) \\ = & \ \boldsymbol{\pi}_{n+1,n}(\mathfrak{f}(\gamma')), \end{aligned}$$

which establishes the desired proposition. ■

The notion of a D_n-tangential from (M, x) to (N, y) is deøned inductively on n. The notion of a D_0-tangential from (M, x) to (N, y) and that of a D_1-tangential from (M, x) to (N, y) shall be identical with that of a D_0-pseudotangential from (M, x) to (N, y) and that of a D_1-pseudotangential from (M, x) to (N, y) respectively. Now we proceed by induction. A D_{n+1}-pseudotangential $\mathfrak{f} : \mathbf{T}_x^{D_{n+1}}(M) \to \mathbf{T}_y^{D_{n+1}}(N)$ from (M, x) to (N, y) is called a D_{n+1}-tangential from (M, x) to (N, y) if it acquiesces in the following two conditions:

1. $\hat{\pi}_{n+1,n}(\mathfrak{f})$ is a D_n-tangential from (M, x) to (N, y).
2. For any natural number $m \leq n + 1$ and any $\gamma \in \mathbf{T}_x^l(M)$ with $l = [\frac{n+1}{m}]$, where [] stands for Gauss's symbol, we have

$$\begin{aligned} \mathfrak{f}(d \ \in \ & D_{n+1} \longmapsto \gamma(d^m) \in M) \\ = \ & d \in D_{n+1} \longmapsto \pi_{n+1,l}(\mathfrak{f})(\gamma)(d^m) \in N \end{aligned} \qquad \text{(b3)}$$

We denote by $\mathbb{J}^n(M,x;N,y)$ the totality of D_n-tangentials from (M,x) to (N,y). By the very deønition of a D_n-tangential, the projection $\hat{\pi}_{n+1,n} : \hat{\mathbb{J}}^{n+1}(M,x;N,y) \rightarrow \hat{\mathbb{J}}^n(M,x;N,y)$ is naturally restricted to a mapping $\pi_{n+1,n} : \mathbb{J}^{n+1}(M,x;N,y) \rightarrow \mathbb{J}^n(M,x;N,y)$. Similarly for $\pi_{n,m} : \mathbb{J}^n(M,x;N,y) \rightarrow \mathbb{J}^m(M,x;N,y)$ with $m \leq n$.

The proofs of the following two propositions are similar to those of the corresponding Propositions but simpler, so they are safely left to the reader.

Proposition 31 Let L, M, N be microlinear spaces with $x \in L, y \in M$ and $z \in N$. If $\mathfrak{f}$ is a D_n-tangential from (L,x) to (M,y) and $\mathfrak{g}$ is a D_n-tangential from (M,y) to (N,z), then the composition $\mathfrak{g} \circ \mathfrak{f}$ is an D_n-tangential from (L,x) to (N,z), and $\pi_{n,n-1}(\mathfrak{g} \circ \mathfrak{f}) = \pi_{n,n-1}(\mathfrak{g}) \circ \pi_{n,n-1}(\mathfrak{f})$, provided that $n \geq 1$.

Proposition 32 Let M, N be microlinear spaces with $x \in M$ and $y \in N$. If f is a mapping from (M,x) to (N,y), then the assignment of $f \circ \gamma \in \mathbf{T}_y^{D_n}(N)$ to each $\gamma \in \mathbf{T}_x^{D_n}(M)$, denoted by $\mathbf{D}^{D_n} f$ and called the D_n-prolongation of f, is a D_n-tangential from (M,x) to (N,y). We have $\mathbf{D}^{D_n} f = \pi_{n+1,n}(\mathbf{D}^{D_{n+1}} f)$. If L is another microlinear space with $z \in L$ and g is a mapping from (N,y) to (L,z), then we have $\mathbf{D}^{D_n}(g \circ f) = (\mathbf{D}^{D_n} g) \circ (\mathbf{D}^{D_n} f)$.

Proposition 33 Let $\mathfrak{f} \in \mathbb{J}^{n+1}(M,x;N,y)$, $t \in \mathbf{T}_x^D(M)$ and γ, γ_+, $\gamma_- \in \mathbf{T}_x^{D_{n+1}}(M)$ with $\gamma_+|_{D_n} = \gamma_-|_{D_n}$. Then we have

$$\begin{aligned} \mathfrak{f}(\gamma_+) \dot{-} \mathfrak{f}(\gamma_-) &= \pi_{n+1,1}(\mathfrak{f})(\gamma_+ \dot{-} \gamma_-) \\ \pi_{n+1,1}(\mathfrak{f})(t) \dot{+} \mathfrak{f}(\gamma) &= \mathfrak{f}(t \dot{+} \gamma) \end{aligned}$$

Proof. It is an easy exercise of aŒne geometry to show that these two statements are equivalent. Therefore it suŒces to deal only with the former statement. Apply the discussion in the proof of Proposition to the quasi-colimit diagram so as to guarantee that there exists what is to be denoted by $\mathfrak{f}(\widetilde{\gamma}) \in \mathbf{T}_y^{D_{n+1} \vee D}(N)$ such that

$$\begin{aligned} \mathfrak{f}(\gamma_+)(d) &= \mathfrak{f}(\widetilde{\gamma})(d, d^{n+1}) \text{ and} \\ \mathfrak{f}(\gamma_-)(d) &= \mathfrak{f}(\widetilde{\gamma})(d, 0) \end{aligned}$$

for any $d \in D_{n+1}$. Then we apply the discussion in the proof of Proposition to the mapping

$$d \in D \longmapsto (0, d) \in D_{n+1} \vee D$$

so as to get the desired conclusion. ■

The proof of the following theorem is similar to that of Theorem, which has been given in our previous paper [Nishimura, 2004], so that it is completely omitted.

Theorem 34

1. Let $\mathfrak{f}^+, \mathfrak{f}^- \in \mathbb{J}^{n+1}(M, x; N, y)$ with $\hat{\pi}_{n+1,n}(\mathfrak{f}^+) = \hat{\pi}_{n+1,n}(\mathfrak{f}^-)$. Then the assignment $\gamma \in \mathbf{T}_x^{D_{n+1}}(M) \longmapsto \mathfrak{f}^+(\gamma) \dot{-} \mathfrak{f}^-(\gamma)$, to be denoted by $\mathfrak{f}^+ \dot{-} \mathfrak{f}^-$, belongs to $\mathbb{F}^{n+1}(M, x; N, y)$.

2. Let $\mathfrak{f} \in \mathbb{J}^{n+1}(M, x; N, y)$ and $\eta \in \mathbb{F}^{n+1}(M, x; N, y)$. Then the assignment $\mathfrak{f} \in \mathbf{T}_x^{D_{n+1}}(M) \longmapsto \eta(\gamma) \dot{+} \mathfrak{f}(\gamma)$, to be denoted by $\eta(\gamma) \dot{+} \mathfrak{f}$, belongs to $\mathbb{J}^{n+1}(M, x; N, y)$.

3. The bundle $\pi_{n+1,n} : \mathbb{J}^{n+1}(M, x; N, y) \to \mathbb{J}^n(M, x; N, y)$ is an aŒne bundle over the trivial vector bundle $\mathbb{J}^n(M, x; N, y) \times \mathbb{F}^{n+1}(M, x; N, y) \to \mathbb{J}^n(M, x; N, y)$.

4 The Relationship between the Two Kinds of Tangentials without Coordinates

Lemma 35 Let $\mathfrak{f}$ be a D^n-pseudotangential from (M, x) to (N, y) and $\gamma \in \mathbf{T}_x^{D_n}(M)$. Then there exists a unique $\gamma' \in \mathbf{T}_y^{D_n}(N)$ such that

$$
\begin{aligned}
\mathsf{f}((d_1,...,d_n) &\in D^n \longmapsto \gamma(d_1+...+d_n) \in M) \\
&= (d_1,...,d_n) \in D^n \longmapsto \gamma'(d_1+...+d_n) \in N
\end{aligned}
$$

Proof. By Proposition 4. ■

We will denote by $\hat{\varphi}_n(\mathsf{f})(\gamma)$ the unique γ' in the above lemma, thereby getting a function $\hat{\varphi}_n(\mathsf{f}) : \mathbf{T}_x^{D_n}(M) \to \mathbf{T}_y^{D_n}(N)$.

Corollary 36 Let $\mathfrak{D}$ be a simplicial inønitesimal space of dimension n and degree m. Let f be a D^n-pseudotangential from (M,x) to (N,y) and $\gamma \in \mathbf{T}_x^{\mathfrak{D}}(M)$. Then we have

$$
\begin{aligned}
\mathsf{f}((d_1,...,d_m) &\in \mathfrak{D} \longmapsto \gamma(d_1+...+d_m) \in M) \\
&= (d_1,...,d_m) \in \mathfrak{D} \longmapsto \hat{\varphi}_n(\mathsf{f})(d_1+...+d_m) \in N
\end{aligned}
$$

Proposition 37 For any $\mathsf{f} \in \hat{\mathbf{J}}^n(M,x;N,y)$, we have $\hat{\varphi}_n(\mathsf{f}) \in \hat{\mathbb{J}}^n(M,x;N,y)$.

Proof. It suŒces to note that for any $\alpha \in \mathbb{R}$ and any $\gamma \in \mathbf{T}_x^{D_n}(M)$, we have

$$\begin{aligned}
(d_1,...,d_n) &\in D^n \longmapsto \hat{\varphi}_n(\mathsf{f})(\alpha\gamma)(d_1+...+d_n) \in N \\
&= \mathsf{f}((d_1,...,d_n) \in D^n \longmapsto (\alpha\gamma)(d_1+...+d_n) \in M) \\
&= \mathsf{f}((d_1,...,d_n) \in D^n \longmapsto \gamma(\alpha d_1+...+\alpha d_n) \in M) \\
&= \mathsf{f}(\alpha \underset{1}{\cdot} ...\alpha \underset{n}{\cdot} (d_1,...,d_n) \in D^n \longmapsto \gamma(d_1+...+d_n) \in M)) \\
&= \alpha \underset{1}{\cdot} ...\alpha \underset{n}{\cdot} (\mathsf{f}((d_1,...,d_n) \in D^n \longmapsto \gamma(d_1+...+d_n) \in M)) \\
&= \alpha \underset{1}{\cdot} ...\alpha \underset{n}{\cdot} ((d_1,...,d_n) \in D^n \longmapsto \hat{\varphi}_n(\mathsf{f})(\gamma)(d_1+...+d_n) \in N) \\
&= (d_1,...,d_n) \in D^n \longmapsto \hat{\varphi}_n(\mathsf{f})(\gamma)(\alpha d_1+...+\alpha d_n) \in N \\
&= (d_1,...,d_n) \in D^n \longmapsto \alpha(\hat{\varphi}_n(\mathsf{f})(\gamma))(d_1+...+d_n) \in N,
\end{aligned}$$

which implies that

$$\hat{\varphi}_n(\mathsf{f})(\alpha\gamma) = \alpha(\hat{\varphi}_n(\mathsf{f})(\gamma))$$

■

Proposition 38 The diagram

$$\begin{array}{ccc}
\hat{\mathbf{J}}^{n+1}(M,x;N,y) & \xrightarrow{\hat{\varphi}_{n+1}} & \hat{\mathbb{J}}^{n+1}(M,x;N,y) \\
\hat{\pi}_{n+1,n} \downarrow & & \downarrow \hat{\pi}_{n+1,n} \\
\hat{\mathbf{J}}^{n}(M,x;N,y) & \xrightarrow[\hat{\varphi}_{n}]{} & \hat{\mathbb{J}}^{n}(M,x;N,y)
\end{array}$$

is commutative.

Proof. For any $d \in D_n$, we have

$$
\begin{aligned}
&(d_1,...,d_n)\in D^n \longmapsto \mathbf{i}_d(\hat{\pi}_{n+1,n}(\hat{\varphi}_{n+1}(\mathsf{f}))(\gamma))(d_1+...+d_n)\in N\\
&\quad= \mathbf{d}_{n+1}((d_1,...,d_{n+1})\in D^{n+1}\longmapsto \mathbf{i}_d(\hat{\pi}_{n+1,n}(\hat{\varphi}_{n+1}(\mathsf{f}))(\gamma))(d_1+...+d_{n+1})\in N)\\
&\quad= \mathbf{d}_{n+1}((d_1,...,d_{n+1})\in D^{n+1}\longmapsto \hat{\varphi}_{n+1}(\mathsf{f})(\mathbf{i}_d(\gamma))(d_1+...+d_{n+1})\in N)\\
&\quad= \mathbf{d}_{n+1}(\mathsf{f}((d_1,...,d_{n+1})\in D^{n+1}\longmapsto \mathbf{i}_d(\gamma)(d_1+...+d_{n+1})\in M))\\
&\quad= \hat{\pi}_{n+1,n}(\mathsf{f})((d_1,...,d_n)\in D^n\longmapsto \mathbf{i}_d(\gamma)(d_1+...+d_n)\in M)\\
&\quad= \hat{\pi}_{n+1,n}(\mathsf{f})((d_1,...,d_n)\in D^n\longmapsto (d\gamma)(d_1+...+d_n)\in M)\\
&\quad= (d_1,...,d_n)\in D^n\longmapsto \hat{\varphi}_n(\hat{\pi}_{n+1,n}(\mathsf{f}))(d\gamma)(d_1+...+d_n)\in N\\
&\quad= (d_1,...,d_n)\in D^n\longmapsto d(\hat{\varphi}_n(\hat{\pi}_{n+1,n}(\mathsf{f}))(\gamma))(d_1+...+d_n)\in N\\
&\quad= (d_1,...,d_n)\in D^n\longmapsto \mathbf{i}_d(\hat{\varphi}_n(\hat{\pi}_{n+1,n}(\mathsf{f}))(\gamma))(d_1+...+d_n)\in N,
\end{aligned}
$$

which implies by Proposition 2 that

$$\hat{\pi}_{n+1,n}(\varphi_{n+1}(\mathsf{f}))=\varphi_n(\hat{\pi}_{n+1,n}(\mathsf{f}))$$

■

Theorem 39 For any $\mathsf{f}\in\mathbf{J}^n(M,x;N,y)$, we have $\hat{\varphi}_n(\mathsf{f})\in\mathbb{J}^n(M,x;N,y)$.

Proof. In view of Proposition 37, it suŒces to show that

$$
\begin{aligned}
\hat{\varphi}_n(\mathsf{f})(d &\in D_n\longmapsto\gamma(d^m)\in M)\\
&= d\in D_n\longmapsto \pi_{n,l}(\hat{\varphi}_n(\mathsf{f}))(\gamma)(d^m)\in N
\end{aligned}
$$

for any natural number $m\leq n$ and any $\gamma\in\mathbf{T}^l_x(M)$ with $l=[\frac{n}{m}]$. Here we deal only with the case that $n=3$ and $m=2$, leaving the general treatment to the reader. Since

$$(d_1 + d_2 + d_3)^2 = 2(d_1d_2 + d_1d_3 + d_2d_3)$$

for any $(d_1, d_2, d_3) \in D^3$, we have

$$(d_1, d_2, d_3) \in D^3 \longmapsto \hat{\varphi}_3(\mathsf{f})(d \in D_3 \longmapsto \gamma(d^2) \in M)(d_1 + d_2 + d_3) \in N$$

$$= \mathsf{f}((d_1, d_2, d_3) \in D^3 \longmapsto \gamma((d_1 + d_2 + d_3)^2) \in M)$$

$$= \mathsf{f}(((d_1, d_2, d_3, d_4, d_5, d_6) \in D(6) \longmapsto \gamma(d_1 + d_2 + d_3 + d_4 + d_5 +$$

$$((d_1, d_2, d_3) \in D^3 \longmapsto (d_1d_2, d_1d_2, d_1d_3, d_1d_3, d_2d_3, d_2d_3) \in D(6))$$

$$= \pi_{3,1}(\mathsf{f})((d_1, d_2, d_3, d_4, d_5, d_6) \in D(6) \longmapsto \gamma(d_1 + d_2 + d_3 + d_4 +$$

$$((d_1, d_2, d_3) \in D^3 \longmapsto (d_1d_2, d_1d_2, d_1d_3, d_1d_3, d_2d_3, d_2d_3) \in D(6))$$

$$= ((d_1, d_2, d_3, d_4, d_5, d_6) \in D(6) \longmapsto \varphi_1(\pi_{3,1}(\mathsf{f}))(\gamma)(d_1 + d_2 + d_3 +$$

$$N) \circ ((d_1, d_2, d_3) \in D^3 \longmapsto (d_1d_2, d_1d_2, d_1d_3, d_1d_3, d_2d_3, d_2d_3) \in$$

$$= ((d_1, d_2, d_3, d_4, d_5, d_6) \in D(6) \longmapsto \pi_{3,1}(\hat{\varphi}_3(\mathsf{f}))(\gamma)(d_1 + d_2 + d_3 +$$

$$N) \circ ((d_1, d_2, d_3) \in D^3 \longmapsto (d_1d_2, d_1d_2, d_1d_3, d_1d_3, d_2d_3, d_2d_3) \in$$

$$= (d_1, d_2, d_3) \in D^3 \longmapsto \pi_{3,1}(\hat{\varphi}_3(\mathsf{f}))(\gamma)((d_1 + d_2 + d_3)^2) \in N,$$

which implies the desired identity. ■

Thus the mapping $\hat{\varphi}_n : \mathbf{J}^n(M, x; N, y) \to \mathbb{J}^n(M, x; N, y)$ is naturally restricted to a mapping $\varphi_n : \mathbf{J}^n(M, x; N, y) \to \mathbb{J}^n(M, x; N, y)$.

5 The Relationship between the Two Kinds of Tangentials with Coordinates

The principal objective in this section is to show that the mapping $\varphi_n : \mathbf{J}^n(M, x; N, y) \to \mathbb{J}^n(M, x; N, y)$ is bijective for any natural number n in case that coordinates are available. We will assume that M and N are formal manifolds of dimensions p and q respectively. Since our considerations to follow are always inønitesimal, this means that we can assume without any loss of generality that $M = \mathbb{R}^p$ and $N = \mathbb{R}^q$. We will let i with or without subscripts range over natural numbers between 1 and p (including endpoints), while we will let j with or without subscripts range over natural numbers between 1 and q (including endpoints). Let $x = (x^i)$ and $y = (y^j)$. For any natural number n, we denote by $\mathcal{J}^n(p,q)$ the totality of $(\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 ... i_n}^j)$'s of $pq + p^2 q + ... + p^n q$ elements of $\mathbb{R}$ such that $\alpha_{i_1 ... i_k}^j$'s are symmetric with respect to subscripts, i.e., $\alpha_{i_{\sigma(1)} ... i_{\sigma(k)}}^j = \alpha_{i_1 ... i_k}^j$ for any $\sigma \in \mathfrak{S}_k$ $(2 \leq k \leq n)$. Therefore the number of independent components in $(\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 ... i_n}^j) \in \mathcal{J}^n(p,q)$ is $q \Sigma_{k=0}^n \binom{p+k-1}{p-1} - q = q\binom{p+n}{n} - q$. The canonical projection

$$
\begin{aligned}
(\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 ... i_n}^j, \alpha_{i_1 ... i_{n+1}}^j) &\in \mathcal{J}^{n+1}(p,q) \longmapsto \\
(\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 ... i_n}^j) &\in \mathcal{J}^n(p,q)
\end{aligned}
$$

is denoted by $\pi_{n+1,n}$. We will use Einstein's summation convention to suppress Σ.

Now we deøne mappings $\overline{\theta}_n : \mathcal{J}^n(p,q) \to \mathbf{J}^n(M,x;N,y)$ and $\overline{\varpi}_n : \mathcal{J}^n(p,q) \to \mathbb{J}^n(M,x;N,y)$. For the former, we ørstly deøne $\overline{\theta}_1 : \mathcal{J}^1(p,q) \to \mathbf{J}^1(M,x;N,y)$ to be

$$
\begin{aligned}
\overline{\theta}_1((\alpha_i^j))(d &\in D \longmapsto (x^i) + d(a^i) \in \mathbb{R}^p) \\
&= d \in D \longmapsto (y^j) + d(a^i \alpha_i^j) \in \mathbb{R}^q
\end{aligned}
$$

We deøne $\overline{\theta}_2 : \mathcal{J}^2(p,q) \to \mathbf{J}^2(M,x;N,y)$ to be

$$
\begin{aligned}
&\overline{\theta}_2((\alpha_i^j, \alpha_{i_1 i_2}^j))((d_1,d_2) \in D^2 \longmapsto (x^i) + d_1(a_1^i) + d_2(a_2^i) + d_1 d_2(a_{12}^i) \in \mathbb{R}^p) \\
&= (d_1,d_2) \in D^2 \longmapsto (y^j) + d_1(a_1^i \alpha_i^j) + d_2(a_2^i \alpha_i^j) + d_1 d_2(a_1^{i_1} a_2^{i_2} \alpha_{i_1 i_2}^j + a_{12}^i \alpha_i^j) \in \mathbb{R}^q
\end{aligned}
$$

Generally we deøne $\overline{\theta}_n : \mathcal{J}^n(p,q) \to \mathbf{J}^n(M,x;N,y)$ to be

$$
\begin{aligned}
&\overline{\theta}_n((\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 i_2 ... i_n}^j))((d_1, ..., d_n) \in D^n \longmapsto (x^i) + \Sigma_{r=1}^n \Sigma_{1 \leq k_1 < ... < k_r \leq n} d_{k_1} ... d_{k_r}(a_{k_1 ... k_r}^i) \in \mathbb{R}^p) \\
&= (d_1, ..., d_n) \in D^n \longmapsto (y^j) + \Sigma_{r=1}^n \Sigma_{1 \leq k_1 < ... < k_r \leq n} d_{k_1} ... d_{k_r}(\Sigma \alpha_{i_{\mathbf{J}_1} ... i_{\mathbf{J}_s}}^j a_{\mathbf{J}_1}^{i_{\mathbf{J}_1}} ... a_{\mathbf{J}_s}^{i_{\mathbf{J}_s}}) \in \mathbb{R}^q,
\end{aligned}
$$

where the last Σ is taken over all partitions of the set $\{k_1, ..., k_r\}$ into nonempty subsets $\{\mathbf{J}_1, ..., \mathbf{J}_s\}$, and if $\mathbf{J} = \{k_1, ..., k_t\}$ is a set of natural numbers with $k_1 < ... < k_t$, then $a_{\mathbf{J}}^{i_{\mathbf{J}}}$ denotes $a_{k_1 \cdots k_t}^{i_{\mathbf{J}}}$.

We have already established the following theorem in Nishimura [2004]:

Theorem 40 For any natural number n, the mapping $\overline{\theta}_n$ is a bijective correspondence from $\mathcal{J}^n(p,q)$ onto $\mathbf{J}^n(M,x;N,y)$.

Now we would like to deøne mappings $\overline{\omega}_n : \mathcal{J}^n(p,q) \to \mathbb{J}^n(M,x;N,y)$. We ørstly deøne $\overline{\omega}_1 : \mathcal{J}^1(p,q) \to \mathbb{J}^1(M,x;N,y)$ to be

$$\begin{aligned}\overline{\omega}_1((\alpha_i^j))(d \in D \longmapsto (x^i) + d(a_{\mathbf{1}}^i) \in \mathbb{R}^p)\\ = d \in D \longmapsto (y^j) + d(\alpha_i^j a_{\mathbf{1}}^i) \in \mathbb{R}^q.\end{aligned}$$

We deøne $\overline{\omega}_2 : \mathcal{J}^2(p,q) \to \mathbb{J}^2(M,x;N,y)$ to be

$$\begin{aligned}\overline{\omega}_2((\alpha_i^j, \alpha_{i_1 i_2}^j))(d \in D_2 \longmapsto (x^i) + d(a_{\mathbf{1}}^i) + \frac{d^2}{2!}(a_{\mathbf{2}}^i) \in \mathbb{R}^p)\\ = d \in D_2 \longmapsto (y^j) + d(\alpha_i^j a_{\mathbf{1}}^i) + \frac{d^2}{2!}(\alpha_{i_1 i_2}^j a_{\mathbf{1}}^{i_1} a_{\mathbf{1}}^{i_2} + \alpha_i^j a_{\mathbf{2}}^i) \in \mathbb{R}^q\end{aligned}$$

Generally we deøne $\overline{\omega}_n : \mathcal{J}^n(p,q) \to \mathbb{J}^n(M,x;N,y)$ to be

$$\begin{aligned}\overline{\omega}_n((\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 i_2 \ldots i_n}^j))(d \in D_n \longmapsto (x^i) + \Sigma_{r=1}^n \frac{d^r}{r!}(a_{\mathbf{r}}^i) \in \mathbb{R}^p)\\ = d \in D_n \longmapsto \Sigma_{r=1}^n \frac{d^r}{r!} \Sigma_{1 \le r_1 \le \ldots \le r_k \le n}(\frac{r!}{r_1! \ldots r_k!} \alpha_{i_1 \ldots i_k}^j a_{\mathbf{r}_1}^{i_1} \ldots a_{\mathbf{r}_k}^{i_k}) \in \mathbb{R}^q,\end{aligned}$$

where the last Σ is taken over all partitions of the positive integer r into positive integers $r_1, ..., r_k$ (so that $r = r_1 + ... + r_k$) with $r_1 \le ... \le r_k$.

Certainly it remains to show that we have $\overline{\omega}_n((\alpha_i^j, ..., \alpha_{i_1 \ldots i_n}^j)) \in \mathbb{J}^n(M,x;N,y)$ for any $(\alpha_i^j, ..., \alpha_{i_1 \ldots i_n}^j) \in \mathcal{J}^n(p,q)$. First of all it is easy to see that

Lemma 41 For any $(\alpha_i^j, ..., \alpha_{i_1 \ldots i_n}^j) \in \mathcal{J}^n(p,q)$, we have $\overline{\omega}_n((\alpha_i^j, ..., \alpha_{i_1 \ldots i_n}^j)) \in \hat{\mathbb{J}}^n(M,x;N,y)$.

Proof. It suŒces to note that for any $\alpha \in \mathbb{R}$, we have

$$\overline{\omega}_n((\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 i_2 \ldots i_n}^j))(\alpha(d \in D_n \longmapsto (x^i) + \Sigma_{r=1}^n \frac{d^r}{r!}(a_{\mathbf{r}}^i) \in$$

$$= \overline{\omega}_n((\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 i_2 \ldots i_n}^j))(d \in D_n \longmapsto (x^i)$$

$$= \overline{\omega}_n((\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 i_2 \ldots i_n}^j))(d \in D_n \longmapsto (x^i)$$

$$= d \in D_n \longmapsto \Sigma_{r=1}^n \frac{d^r}{r!} \Sigma_{1 \leq r_1 \leq \ldots \leq r_k \leq n}(\frac{r!}{r_1! \ldots r_k!} \alpha_i^j$$

$$= d \in D_n \longmapsto \Sigma_{r=1}^n \frac{(\alpha d)^r}{r!} \Sigma_{1 \leq r_1 \leq \ldots \leq r_k \leq n}(\frac{r!}{r_1! \ldots r_k}$$

$$= \alpha(d \in D_n \longmapsto \Sigma_{r=1}^n \frac{d^r}{r!} \Sigma_{1 \leq r_1 \leq \ldots \leq r_k \leq n}(\frac{r!}{r_1! \ldots r_k!}$$

■

It is easy to see that

Proposition 42 The following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{J}^{n+1}(p,q) & \xrightarrow{\overline{\omega}_{n+1}} & \hat{\mathbb{J}}^{n+1}(M,x;N,y) \\
\pi_{n+1,n} \downarrow & & \downarrow \hat{\pi}_{n+1,n} \\
\mathcal{J}^{n}(p,q) & \xrightarrow[\overline{\omega}_{n}]{} & \hat{\mathbb{J}}^{n}(M,x;N,y)
\end{array}$$

Proof. It suŒces to note that for any $d' \in D_n$, we have ■

$$\overline{\omega}_{n+1}((\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 i_2 \ldots i_{n+1}}^j))((d \in D_n \longmapsto (x^i) + \Sigma_{r=1}^n \frac{d^r}{r!}(a_{\mathbf{r}}^i) \in \mathbb{R}^p) \circ (d \in D_{n+1} \longmapsto d'd \in D_n))$$

$$= \overline{\omega}_{n+1}((\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 i_2 \ldots i_{n+1}}^j))(d \in D_{n+1} \longmapsto (x^i) + \Sigma_{r=1}^n \frac{(d'd)^r}{r!}(a_{\mathbf{r}}^i) \in \mathbb{R}^p)$$

$$= \overline{\omega}_{n+1}((\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 i_2 \ldots i_{n+1}}^j))(d \in D_{n+1} \longmapsto (x^i) + \Sigma_{r=1}^n \frac{d^r}{r!}((d')^r a_{\mathbf{r}}^i) \in \mathbb{R}^p)$$

$$= d \in D_{n+1} \longmapsto \Sigma_{r=1}^{n+1} \frac{d^r}{r!} \Sigma_{1 \leq r_1 \leq \ldots \leq r_k \leq n}(\frac{r!}{r_1! \ldots r_k!} \alpha_{i_1 \ldots i_k}^j (d')^{r_1} a_{\mathbf{r}_1}^{i_1} \ldots (d')^{r_k} a_{\mathbf{r}_k}^{i_k})$$

$$= d \in D_{n+1} \longmapsto \Sigma_{r=1}^n \frac{(d'd)^r}{r!} \Sigma_{1 \leq r_1 \leq \ldots \leq r_k \leq n}(\frac{r!}{r_1! \ldots r_k!} \alpha_{i_1 \ldots i_k}^j a_{\mathbf{r}_1}^{i_1} \ldots a_{\mathbf{r}_k}^{i_k}) \in \mathbb{R}^q$$

$$= \overline{\omega}_n(\pi_{n+1,n}((\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 i_2 \ldots i_{n+1}}^j)))(d \in D_n \longmapsto (x^i) + \Sigma_{r=1}^n \frac{d^r}{r!}(a_{\mathbf{r}}^i) \in \mathbb{R}^p) \circ$$

$$(d \in D_{n+1} \longmapsto d'd \in D_n)$$

Proposition 43 For any $(\alpha_i^j, ..., \alpha_{i_1 \ldots i_n}^j) \in \mathcal{J}^n(p,q)$, we have $\overline{\omega}_n((\alpha_i^j, ..., \alpha_{i_1 \ldots i_n}^j)) \in \mathbb{J}^n(M,x;N,y)$.

Proof. In view of Lemma 41, it suŒces to note that for any natural number m with $l = [\frac{n}{m}]$, we have

$$\overline{\omega}_n((\alpha^j_i,\alpha^j_{i_1i_2},...,\alpha^j_{i_1i_2...i_n}))(d\in D_n\longmapsto(d\in D_l\longmapsto(x^i)+\Sigma^l_{r=1}\tfrac{d^r}{r!}(a^i_{\mathbf{r}})\in\mathbb{R}^p)(d^m)\in\mathbb{R}^p)$$
$$=\overline{\omega}_n((\alpha^j_i,\alpha^j_{i_1i_2},...,\alpha^j_{i_1i_2...i_n}))(d\in D_n\longmapsto(x^i)+\Sigma^l_{r=1}\tfrac{d^{mr}}{r!}(a^i_{\mathbf{r}})\in\mathbb{R}^p)$$
$$=\overline{\omega}_n((\alpha^j_i,\alpha^j_{i_1i_2},...,\alpha^j_{i_1i_2...i_n}))(d\in D_n\longmapsto(x^i)+\Sigma^l_{r=1}\tfrac{d^{mr}}{(mr)!}(\tfrac{(mr)!}{r!}a^i_{\mathbf{r}})\in\mathbb{R}^p)$$
$$=d\in D_n\longmapsto\Sigma^l_{r=1}\tfrac{d^{mr}}{(mr)!}\Sigma_{1\leq r_1\leq...\leq r_k\leq l}(\tfrac{(mr)!}{(mr_1)!...(mr_k)!}\alpha^j_{i_1...i_k}\tfrac{(mr_1)!}{r_1!}a^{i_1}_{\mathbf{r}_1}...\tfrac{(mr_k)!}{r_k!}a^{i_k}_{\mathbf{r}_k})\in\mathbb{R}^q$$
$$=d\in D_n\longmapsto\Sigma^l_{r=1}\tfrac{d^{mr}}{r!}\Sigma_{1\leq r_1\leq...\leq r_k\leq l}(\tfrac{r!}{r_1!...r_k!}\alpha^j_{i_1...i_k}a^{i_1}_{\mathbf{r}_1}...a^{i_k}_{\mathbf{r}_k})\in\mathbb{R}^q$$
$$=d\in D_n\longmapsto\overline{\omega}_l((\alpha^j_i,\alpha^j_{i_1i_2},...,\alpha^j_{i_1i_2...i_l}))(d\in D_n\longmapsto(x^i)+\Sigma^l_{r=1}\tfrac{d^r}{r!}(a^i_{\mathbf{r}})\in\mathbb{R}^p)(d^m)$$
$$=d\in D_n\longmapsto\overline{\omega}_l(\pi_{n,l}((\alpha^j_i,\alpha^j_{i_1i_2},...,\alpha^j_{i_1i_2...i_n})))(d\in D_n\longmapsto(x^i)+\Sigma^l_{r=1}\tfrac{d^r}{r!}(a^i_{\mathbf{r}})\in\mathbb{R}^p)(d^m)$$
$$=d\in D_n\longmapsto\pi_{n,l}(\overline{\omega}_n((\alpha^j_i,\alpha^j_{i_1i_2},...,\alpha^j_{i_1i_2...i_n})))(d\in D_n\longmapsto(x^i)+\Sigma^l_{r=1}\tfrac{d^r}{r!}(a^i_{\mathbf{r}})\in\mathbb{R}^p)(d^m).$$

■

Now we have the following fundamental theorem, though its proof is easy:

Theorem 44 The following diagram is commutative:

$$\begin{array}{ccc} & \mathcal{J}^n(p,q) & \\ \overline{\theta}_n\swarrow & & \searrow\overline{\omega}_n \\ \mathbf{J}^n(M,x;N,y) & \xrightarrow[\varphi_n]{} & \mathbb{J}^n(M,x;N,y)\end{array}$$

Proof. We note the following well-known identity of inønitesimals:

$$\frac{(d_1+...+d_n)^k}{k!}=\sum_{1\leq l_1<...<l_k\leq n}d_{l_1}...d_{l_k}$$

for any natural number k and any $d_1,...,d_n\in D$, for which the reader is referred to Lavendhomme

[1996;p.10]. In view of this, we have

$$\overline{\theta}_n((\alpha^j_i, \alpha^j_{i_1 i_2}, ..., \alpha^j_{i_1 i_2 ... i_n}))((d \in D_n \longmapsto (x^i) + \Sigma^n_{r=1} \frac{d^r}{r!}(a^i_{\mathbf{r}}) \in \mathbb{R}^p) \circ$$

$$((d_1, ..., d_n) \in D^n \longmapsto d_1 + ... + d_n \in D_n))$$

$$= \overline{\theta}_n((\alpha^j_i, \alpha^j_{i_1 i_2}, ..., \alpha^j_{i_1 i_2 ... i_n}))((d_1, ..., d_n) \in D^n \longmapsto (x^i) + \Sigma^n_{r=1} \frac{(d_1+...+d_n)^r}{r!}(a^i_{\mathbf{r}}) \in$$

$$= \overline{\theta}_n((\alpha^j_i, \alpha^j_{i_1 i_2}, ..., \alpha^j_{i_1 i_2 ... i_n}))((d_1, ..., d_n) \in D^n \longmapsto (x^i) + \Sigma^n_{r=1} \Sigma_{1 \leq k_1 < ... < k_r \leq n} d_k$$

$$= (d_1, ..., d_n) \in D^n \longmapsto (y^j) + \Sigma^n_{r=1} \Sigma_{1 \leq k_1 < ... < k_r \leq n} d_{k_1} ... d_{k_r} \Sigma_{\substack{r_1 + ... + r_s = r \\ 1 \leq r_1 \leq ... \leq r_s \leq n}} (\frac{r!}{r_1! ... r$$

$$= (d_1, ..., d_n) \in D^n \longmapsto (y^j) + \Sigma^n_{r=1} \frac{(d_1+...+d_n)^r}{r!} \Sigma_{\substack{r_1 + ... + r_s = r \\ 1 \leq r_1 \leq ... \leq r_s \leq n}} (\frac{r!}{r_1! ... r_s!} \alpha^j_{i_1 ... i_s} a^{i_1}_{\mathbf{r}_1} ...$$

$$= \overline{\omega}_n((\alpha^j_i, \alpha^j_{i_1 i_2}, ..., \alpha^j_{i_1 i_2 ... i_n}))(d \in D_n \longmapsto (x^i) + \Sigma^n_{r=1} \frac{d^r}{r!}(a^i_{\mathbf{r}}) \in \mathbb{R}^p) \circ ((d_1, ..., d_n)$$

$$D^n \longmapsto d_1 + ... + d_n \in D_n)$$

■

In view of Theorems 40 and 44, in order to establish that the mapping $\varphi_n : \mathbf{J}^n(M, x; N, y) \to \mathbb{J}^n(M, x; N, y)$ is injective for any natural number n, it suffices to demonstrate the following theorem:

Theorem 45 For any natural number n, the mapping $\overline{\omega}_n : \mathcal{J}^n(p, q) \to \mathbb{J}^n(M, x; N, y)$ is injective.

The rest of this section is devoted to a proof of the above theorem. Now we are going to define mappings $\underline{\omega}_n : \mathbb{J}^n(M, x; N, y) \to \mathcal{J}^n(p, q)$ by induction on n such that the diagram

$$\begin{array}{ccc} \mathbb{J}^{n+1}(M, x; N, y) & \xrightarrow{\underline{\omega}_{n+1}} & \mathcal{J}^{n+1}(p, q) \\ \pi_{n+1,n} \downarrow & & \downarrow \pi_{n+1,n} \\ \mathbb{J}^n(M, x; N, y) & \xrightarrow[\underline{\omega}_n]{} & \mathcal{J}^n(p, q) \end{array}$$

is commutative. The mapping $\underline{\omega}_0 : \mathbb{J}^0(M, x; N, y) \to \mathcal{J}^0(p, q)$ shall be the trivial one. Assuming that $\underline{\omega}_n : \mathbb{J}^n(M, x; N, y) \to \mathcal{J}^n(p, q)$ is defined, we are going to define $\underline{\omega}_{n+1} : \mathbb{J}^{n+1}(M, x; N, y) \to \mathcal{J}^{n+1}(p, q)$, for which it suffices, by the required commutativity of the above diagram, only to give $\alpha^j_{i_1 ... i_{n+1}}$'s to each $\mathfrak{f} \in \mathbb{J}^{n+1}(M, x; N, y)$. Let $\mathbf{e}_i$ denote $(0, ..., 0, 1, 0, ..., 0) \in \mathbb{R}^p$, where 1 is inserted at the i-th position, while the other $p - 1$ elements are fixed zero. By the general Kock axiom (cf. Lavendhomme [1996, 2.1.3]), $\mathfrak{f}(d \in D_{n+1} \longmapsto (x^i) + d(d_1 \mathbf{e}_{i_1} + ... + d_{n+1} \mathbf{e}_{i_{n+1}}) \in M)$ should be a polynomial of $d, d_1, ..., d_{n+1}$, in which the coeŒcient of $d^{n+1}d_1...d_{n+1}$ should be of the form $m_1!...m_p!(\alpha^1_{i_1...i_{n+1}}, ..., \alpha^q_{i_1...i_{n+1}}) \in \mathbb{R}^q$, where m_i is the number of i_k's with $i = i_k$. We choose these $\alpha^j_{i_1...i_{n+1}}$'s as our desired $\alpha^j_{i_1...i_{n+1}}$'s.

Now we have

Proposition 46 For any $\mathfrak{f} \in \mathbb{J}^n(M, x; N, y)$, we have $\underline{\omega}_n(\mathfrak{f}) \in \mathcal{J}^n(p, q)$.

Proof. Trivial. ■

It is easy to see that

Proposition 47 The composition $\underline{\omega}_n \circ \overline{\omega}_n$ is the identity mapping of $\mathcal{J}^n(p, q)$.

Proof. Using the commutative diagram

$$\begin{array}{ccccc} \mathcal{J}^{n+1}(p,q) & \xrightarrow{\overline{\omega}_{n+1}} & \mathbb{J}^{n+1}(M,x;N,y) & \xrightarrow{\underline{\omega}_{n+1}} & \mathcal{J}^{n+1}(p,q) \\ \pi_{n+1,n}\downarrow & & \downarrow \pi_{n+1,n} & & \downarrow \pi_{n+1,n} \\ \mathcal{J}^{n}(p,q) & \xrightarrow[\overline{\omega}_{n}]{} & \mathbb{J}^{n}(M,x;N,y) & \xrightarrow[\underline{\omega}_{n}]{} & \mathcal{J}^{n}(p,q) \end{array}$$

we can easily establish the desired result by induction on n. ■

Now Theorem 45 follows readily from Proposition 47.

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