# Cartesian Bicategories II

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The notion of *cartesian bicategory*, introduced in [C&W] for locally ordered bicategories, is extended to general bicategories. Bicategories of spans are characterized as cartesian bicategories in which every object is discrete, every comonad has an Eilenberg-Moore object, and for every object $X$, the left adjoint arrow $X \rightarrow I$, where $I$ is terminal with respect to left adjoints, is comonadic. Bicategories of relations are revisited from the present point of view. This report is an informal version, without proofs, of a forthcoming longer article by the authors.

## 1 Introduction

We recall that in [C&W] a locally ordered bicategory $\mathbf{B}$ was said to be *cartesian* if the subbicategory of left adjoints, $\mathrm{Map}\mathbf{B}$, had finite products; each hom-category $\mathbf{B}(B, C)$ had finite products; and a certain derived tensor product on $\mathbf{B}$, extending the product structure of $\mathrm{Map}\mathbf{B}$, was pseudofunctorial. It was shown that cartesian structure provides an elegant base for sets of axioms characterizing bicategories of

- i) relations in a regular category
- ii) ordered objects and order ideals in an exact category
- iii) additive relations in an abelian category
- iv) relations in a Grothendieck topos.

Notable was an axiom, here called *groupoidalness*, that captures the *discrete* objects in a cartesian locally ordered bicategory and gives rise to a very satisfactory approach to duals.

It was predicted in [C&W] that the notion of cartesian bicategory would be developable without the restriction of local orderedness, so as to capture

- v) spans in a category with finite limits
- vi) profunctors in an elementary topos.

It is this development of the unrestricted notion of cartesian bicategory that is our present concern. It turns out that the description of cartesianness given in the first sentence above, which was only an alternative characterization in [C&W], carries over more easily to the general case than the original definition in [C&W].

Merely assuming of a bicategory $\mathbf{B}$ that $\mathrm{Map}\mathbf{B}$ has finite products (in the sense appropriate for bicategories) and that each hom-category $\mathbf{B}(B, C)$ has finite products — in which case we say that $\mathbf{B}$ is *precartesian* — it is possible to define canonical lax functors $\otimes$ and $I$ and lax natural transformations $\mathbf{t}$ and $\mathbf{u}$ as below, where $\times$ and $\mathbf{1}$ are the pseudofunctors providing the finite products for $\mathrm{Map}\mathbf{B}$ and $i$ is the inclusion.

$$\begin{tikzcd}
\mathbf{B}\times\mathbf{B} \arrow[r, "\otimes"] & \mathbf{B} & \\
& & \mathbf{1} \arrow[ul, "I"'] \arrow[dl, "1"] \\
\mathrm{Map}\mathbf{B}\times\mathrm{Map}\mathbf{B} \arrow[uu, "i\times i"] \arrow[r, "\times"'] & \mathrm{Map}\mathbf{B} \arrow[uu, "i"] &
\arrow[from=3-1, to=1-2, phantom, "\mathbf{t}\Uparrow"]
\arrow[from=3-2, to=2-3, phantom, "\mathbf{u}\Uparrow"]
\end{tikzcd}$$

Now a bicategory $\mathbf{B}$ is *cartesian* if it is precartesian and moreover the constraints $\otimes^\circ : 1_{B\otimes C} \to 1_B \otimes 1_C$, $\widetilde{\otimes} : (T \otimes U)(R \otimes S) \to (TR) \otimes (US)$ of $\otimes$ and the constraint $I^\circ : 1_I \to \top = I(*)$ of $I$ are invertible (the last implying the invertibility of the constraint $\widetilde{I} : \top\top \to \top$). Invertibility of these 2-cells makes $\otimes$ and $I$ pseudofunctors and it also makes $\mathbf{t}$ and $\mathbf{u}$ invertible pseudonatural transformations. Note that when $\mathrm{Map}\mathbf{B}$ has finite products, the existence of finite products in the $\mathbf{B}(B,C)$ is equivalent to the existence of finite products in the Grothendieck bicategory arising from the pseudofunctor $\mathbf{B}(-,-){:}(\mathrm{Map}\mathbf{B})^{\mathrm{op}} \times \mathrm{Map}\mathbf{B} \to \mathbf{CAT}$ and their preservation by the associated pseudofibration; this explains the close connection between cartesian bicategories and the cartesian objects studied in [CKW] and [CKVW]. These considerations comprise the main thrust of Section 2.

The structure $(\mathbf{B}, \otimes, I)$ admits canonical constraints, of associativity and so on, that make it a monoidal bicategory. Thus $\otimes$ may be regarded as an 'outer level' composition and the *dual* $B^\circ$ of an object $B$ can be defined in terms of arrows $N{:}I \to B^\circ \otimes B$ and $E{:}B \otimes B^\circ \to I$ satisfying the *triangle equations* to within coherent canonical isomorphisms. The groupoidal condition of [C&W] can be expressed in the general context, and the groupoidal objects have duals for which $B^\circ = B$. An object $B$ is *posetal* if the unit $\eta_B{:}1_B \to d_B^* d_B$ of the adjunction for the diagonal map $d_B{:}B \to B \otimes B$ is invertible. The *discrete* objects in a general cartesian bicategory are those which are both groupoidal and posetal, these playing a major role in the characterizations of the bicategories of v) and vi) above. We defer our treatment of vi) to [CKW2]. Discreteness is the central matter of Section 3. It is to be noted that both its aspects are defined in terms of data that are available in virtue of cartesianness.

A cartesian bicategory becomes rather special when *every object is discrete* and this becomes a further axiom when we turn to the characterizations of bicategories of spans and of relations. Any characterization of bicategories in which the arrows are some sort of span must involve a procedure for associating to a general arrow a span of more specialized arrows. In Section 4 we show that comonads and their coalgebras admit a particularly simple description in a cartesian bicategory in which the left adjoints $X \to I$, where $I$ is terminal for left adjoints, are comonadic. In fact, an arrow carries at most one comonad structure so that being a comonad is then a *property* rather than additional structure. In this context we are able to use Eilenberg-Moore coalgebras for comonads to provide tabulations of general arrows. Our last axiom for characterizing bicategories of spans is *every comand admits an Eilenberg-Moore object and every left adjoint $X \to I$, as above, is comonadic*.

The considerations above are brought together in our final Section 5 in which we state the characterization theorems.

## 2 Precartesian and Cartesian Bicategories

Let $\mathbf{B}$ be a bicategory. We write $X, Y, Z, ...$ for the objects, $R, S, T, ...$ for the arrows, and $\alpha, \beta, \gamma, ...$ for the 2-cells of **B**. We sometimes omit parentheses in three-fold composites but in such cases $RST$ is to be understood as $(RS)T$. We use this choice in defining the hom pseudofunctor

$$\mathbf{B}(-,-){:}\mathbf{B}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{CAT} \tag{1}$$

where $\mathbf{CAT}$ denotes the 2-category of categories. By *equivalence of categories* or *equivalence in a bicategory*, we will mean, unless otherwise stated, an arrow $F$ that has an equivalence inverse $U$ with isomorphisms $\epsilon : FU \to 1$ and $\alpha : 1 \to UF$. It is a bicategorical formality that in such a situation one can find invertible $\eta{:}1 \to UF$ with $\eta, \epsilon{:}F \dashv U$.

We tend to suppress the prefix *bi* whenever doing so seems unlikely to cause confusion. Thus, when we say that $\mathbf{B}$ has binary *products* we refer to what some authors call binary *biproducts*, meaning that, for each pair of objects $X, Y$ in $\mathbf{B}$, there is an object $X \times Y$ and arrows $P{:}X \times Y \to X$ and $R{:}X \times Y \to Y$ so that, for each $B$ in $\mathbf{B}$, the functor

$$\langle \mathbf{B}(B, P), \mathbf{B}(B, R)\rangle{:}\mathbf{B}(B, X \times Y) \to \mathbf{B}(B, X) \times \mathbf{B}(B, Y) \tag{2}$$

is an *equivalence of categories*, whose specified inverse sends the pair of arrows $(F, G)$ to an arrow $\langle F, G\rangle$. Of course a bicategory has finite products if and only if it has binary products and a terminal object.

An arrow $R$ is called a *map* if it has a right adjoint. We will usually denote maps by lowercase Roman letters; and if $r$ is a map, write $\eta_r, \epsilon_r : r \dashv r^*$ for a chosen adjunction that makes it so. We write $\mathrm{Map}\mathbf{B}$ for the locally full subbicategory of $\mathbf{B}$ determined by the maps. Of particular interest for us will be the $\mathbf{CAT}$-valued pseudofunctor of two variables

$$\mathbf{B}(-,-){:}(\mathrm{Map}\mathbf{B})^{\mathrm{op}} \times \mathrm{Map}\mathbf{B} \to \mathbf{CAT} \tag{3}$$

obtained by restricting the hom pseudofunctor (1) of $\mathbf{B}$. Applying a two-variable version of the Grothendieck construction to (3) gives a bicategory that we call $\mathbf{B}'$ and strict pseudofunctors $\partial_0$ and $\partial_1$ as shown in (4) below.

$$\begin{tikzcd}
 & \mathbf{B}' \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & \\
\mathrm{Map}\mathbf{B} & & \mathrm{Map}\mathbf{B}
\end{tikzcd} \tag{4}$$

Thus an object of $\mathbf{B}'$ consists of a pair of objects $X$ and $A$ of $\mathrm{Map}\mathbf{B}$ (whose objects are all those of $\mathbf{B}$) and a general arrow $R{:}X \to A$ of $\mathbf{B}$. An arrow from $R{:}X \to A$ to $S{:}Y \to B$ in $\mathbf{B}'$ is given by a map $f{:}X \to Y$, a map $u{:}A \to B$, and a 2-cell $\alpha{:}uR \to Sf$ of $\mathbf{B}$ which is illustrated by the square in (5) below.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "R"'] \arrow[dr, phantom, "\xrightarrow{\alpha}"] & Y \arrow[d, "S"] \\
A \arrow[r, "u"'] & B
\end{tikzcd} \tag{5}$$

A 2-cell from $(f, \alpha, u)$ to $(f', \alpha', u')$ is a pair of 2-cells $\phi : f \to f'$, $\psi : u \to u'$ satisfying $S\phi.\alpha = \alpha'.\psi R$. Vertical composition of 2-cells in $\mathbf{B}'$ is given by vertical composition in $\mathbf{B}$. Horizontal composition in $\mathbf{B}'$ is given by horizontal pasting of squares. The constraints of $\mathbf{B}'$ are the obvious ones inherited from $\mathrm{Map}\mathbf{B}$ and hence from $\mathbf{B}$. With these notations the pseudofunctor $\partial_0$ is 'domain'-like with

$$\partial_0[(\phi, \psi){:}(f, \alpha, u) \to (f', \alpha', u'){:}R \to S] = \phi{:}f \to f'{:}X \to Y$$

and similarly $\partial_1$ is 'codomain'-like with

$$\partial_1[(\phi,\psi){:}(f,\alpha,u) \to (f',\alpha',u'){:}R \to S] = \psi{:}u \to u'{:}A \to B$$

Later, we will find it convenient to write $\mathbf{B}'(R,S)_{f,u}$ for the subcategory of $\mathbf{B}'(R,S)$ given by the arrows of the form $(f,\alpha,u)$, for some $\alpha$ and the 2-cells of the form $(1_f,1_u)$. In other words, $\mathbf{B}'(R,S)_{f,u}$, clearly discrete, is the *set* of arrows that are $\partial_0$-over $f$ and $\partial_1$-over $u$.

**Proposition 1** *The typical arrow $(f,\alpha,u)$ of $\mathbf{B}'$, as in (5), is an equivalence if and only if $f$ and $u$ are equivalences in $\mathrm{Map}\mathbf{B}$ and $\alpha$ is invertible in $\mathbf{B}(X,B)$.*

In the terminology of [ST2], adapted for bicategories, the span $(\partial_0,\partial_1)$ is a pseudofibration from $\mathrm{Map}\mathbf{B}$ to $\mathrm{Map}\mathbf{B}$, directed from right to left in the display (4). In particular, $\partial_0$ is a pseudofibration and $\partial_1$ is a pseudo-opfibration, satisfying the compatibility condition described in [ST2]. Applying general theorems about fibrations we are able to show:

**Proposition 2** *If the bicategory $\mathrm{Map}\mathbf{B}$ has finite products, then the categories $\mathbf{B}(X,A)$ have finite products if and only if $\mathbf{B}'$ has finite products preserved by $\partial_0$ and $\partial_1$.*

**Definition 3** *A bicategory $\mathbf{B}$ is said to be* precartesian *when it has the following properties:*

*1)* $\mathrm{Map}\mathbf{B}$ *has finite products;*

*2) Each hom category $\mathbf{B}(X,Y)$ has finite products.*

We can also express *1)* of Definition 3 by saying that we have pseudofunctors

$$\mathrm{Map}\mathbf{B} \times \mathrm{Map}\mathbf{B} \xrightarrow{\times} \mathrm{Map}\mathbf{B} \xleftarrow{1} \mathbf{1}$$

which are right (pseudo)adjoint to $\Delta$ and !. However, it is convenient to write $X \otimes Y$ for the value of the binary product on objects and $I$ for the terminal object of $\mathrm{Map}\mathbf{B}$. The counit of the adjunction $\times \dashv \Delta$ has (pseudonatural) components which we write as $p{:}X \leftarrow X \otimes Y \to Y{:}r$ and call projection maps, while the unit has components written $d{:}X \to X \otimes X$ and called diagonal maps. The unit for the adjunction $! \dashv 1$ has components $t{:}X \to I$ called terminal maps. In the context of *2)* of Definition 3, we write $R \wedge S$ for the binary products and $\top$ for the terminal objects of the $\mathbf{B}(X,Y)$, with the projections now written $\pi{:}R \leftarrow R \wedge S \to S{:}\rho$, with the diagonals $\delta{:}R \to R \wedge R$, and with the terminal 2-cells $\tau{:}R \to \top$. We speak of $\wedge_{X,Y}$ and $\top$ as *local* products.

It is also convenient to have notation for the finite products of $\mathbf{B}'$ as in Proposition 2 and to relate these to the local finite products of $\mathbf{B}$ by the formulae that are implicit in the proof of Proposition 2. In describing $\mathbf{B}' \times \mathbf{B}' \xrightarrow{\times} \mathbf{B}' \xleftarrow{1} \mathbf{1}$ we will extend the $\otimes$-notation so that $R{:}X \to A$ and $S{:}Y \to B$, seen as objects of $\mathbf{B}'$, have a product $R \otimes S{:}X \otimes Y \to A \otimes B$, with projections given by

$$\begin{tikzcd}
X \otimes Y \arrow[r, "p_{X,Y}"] \arrow[d, "R\otimes S"'] \arrow[dr, phantom, "\overset{p_{R,S}}{\rightarrow}"] & X \arrow[d, "R"] & X \otimes Y \arrow[r, "r_{X,Y}"] \arrow[d, "R\otimes S"'] \arrow[dr, phantom, "\overset{r_{R,S}}{\rightarrow}"] & Y \arrow[d, "S"] \\
A \otimes B \arrow[r, "p_{A,B}"'] & A & A \otimes B \arrow[r, "r_{A,B}"'] & B
\end{tikzcd} \tag{6}$$

Observe that the arrow components of these projections are as in $\mathrm{Map}\mathbf{B}$ since $\partial_0$ and $\partial_1$ preserve finite products. Similar remarks apply to the terminal object of $\mathbf{B}'$ and it is convenient to record the components of the units for $\Delta \dashv \times : \mathbf{B}' \times \mathbf{B}' \to \mathbf{B}'$ and $! \dashv 1 : \mathbf{1} \to \mathbf{B}'$ as

$$\begin{tikzcd}
X \arrow[r, "d_X"] \arrow[d, "R"'] \arrow[dr, phantom, "\xrightarrow{d_R}"] & X \otimes X \arrow[d, "R \otimes R"] & X \arrow[r, "t_X"] \arrow[d, "R"'] \arrow[dr, phantom, "\xrightarrow{t_R}"] & I \arrow[d, "\top = \top_{I,I}"] \\
A \arrow[r, "d_A"'] & A \otimes A & A \arrow[r, "t_A"'] & I
\end{tikzcd} \tag{7}$$

In terms of the local products of *2)* of Definition 3, (the proof of) Proposition 2 gives

$$R \otimes S = p^* Rp \wedge r^* Sr \tag{8}$$

(equality at this stage being a possible choice) with the terminal object of $\mathbf{B}'$ being $\top_{I,I} : I \to I$ as displayed in (7). On the other hand, Proposition 2 also allows us to recover the binary product of $R$ and $S$ in $\mathbf{B}(X, A)$ from their product in $\mathbf{B}'$ by

$$R \wedge S \cong d_A^*(R \otimes S)d_X \tag{9}$$

and the local terminal $\top_{X,A}$ as an isomorph of the composite

$$X \xrightarrow{t} I \xrightarrow{\top} I \xrightarrow{t^*} A \tag{10}$$

From the finite products given by Definition 3 we now construct lax functors (also called morphisms of bicategories)

$$\mathbf{B} \times \mathbf{B} \xrightarrow{\otimes} \mathbf{B} \xleftarrow{I} \mathbf{1} \tag{11}$$

for a precartesian bicategory $\mathbf{B}$. On objects $X \otimes Y$ is the product as in *1)* of Definition 3 while

$$\otimes_{(X,Y),(A,B)} : \mathbf{B}(X, A) \times \mathbf{B}(Y, B) \to \mathbf{B}(X \otimes Y, A \otimes B),$$

given by the composite of $\mathbf{B}(X, A) \times \mathbf{B}(Y, B) \xrightarrow{\mathbf{B}(p,p^*) \times \mathbf{B}(r,r^*)} \mathbf{B}(X \otimes Y, A \otimes B) \times \mathbf{B}(X \otimes Y, A \otimes B)$ and $\mathbf{B}(X \otimes Y, A \otimes B) \times \mathbf{B}(X \otimes Y, A \otimes B) \xrightarrow{\wedge} \mathbf{B}(X \otimes Y, A \otimes B)$, provides the effect on homs, with $\wedge$ as in *2)* of Definition 3. In particular, $\otimes$ is defined on arrows by (8), and thus agrees with the product of objects in $\mathbf{B}'$, with a similar formula for 2-cells.

For $R : X \to A$ and $S : Y \to B$, along with $T : A \to L$ and $U : B \to M$, we have the constraint

$$\widetilde{\otimes} : (T \otimes U)(R \otimes S) \to (TR) \otimes (US)$$

which, its codomain being a product in $\mathbf{B}'$, we describe in terms of its components, namely

$$\begin{tikzcd}
X \otimes Y \arrow[r, "p"] \arrow[d, "R \otimes S"'] \arrow[dr, phantom, "\xrightarrow{p_{R,S}}"] & X \arrow[d, "R"] & X \otimes Y \arrow[r, "r"] \arrow[d, "R \otimes S"'] \arrow[dr, phantom, "\xrightarrow{r_{R,S}}"] & Y \arrow[d, "S"] \\
A \otimes B \arrow[r, "p" description] \arrow[d, "T \otimes U"'] \arrow[dr, phantom, "\xrightarrow{p_{T,U}}"] & A \arrow[d, "T"] & A \otimes B \arrow[r, "r" description] \arrow[d, "T \otimes U"'] \arrow[dr, phantom, "\xrightarrow{r_{T,U}}"] & B \arrow[d, "U"] \\
L \otimes M \arrow[r, "p"'] & L & L \otimes M \arrow[r, "r"'] & M
\end{tikzcd} \tag{12}$$

For $(X, Y)$ in $\mathbf{B} \times \mathbf{B}$ we have the constraint

$$\otimes^\circ : 1_{X \otimes Y} \to 1_X \otimes 1_Y$$

which, its codomain again being a product in $\mathbf{B}'$ we describe in terms of its components:

$$\begin{tikzcd}
X \otimes Y \arrow[r, "p"] \arrow[d, "1_{X\otimes Y}"'] \arrow[dr, phantom, "\cong"] & X \arrow[d, "1_X"] & X \otimes Y \arrow[r, "r"] \arrow[d, "1_{X\otimes Y}"'] \arrow[dr, phantom, "\cong"] & Y \arrow[d, "1_Y"] \\
X \otimes Y \arrow[r, "p"'] & X & X \otimes Y \arrow[r, "r"'] & Y
\end{tikzcd} \tag{13}$$

**Proposition 4** *For a precartesian* $\mathbf{B}$*, the function* $(X, Y) \mapsto X \otimes Y$*, the functors*

$$\otimes_{(X,Y),(A,B)} : \mathbf{B}(X, A) \times \mathbf{B}(Y, B)) \to \mathbf{B}(X \otimes Y, A \otimes B)$$

*and the constraints* $\widetilde{\otimes}$ *and* $\otimes^\circ$ *constitute a lax functor* $\otimes : \mathbf{B} \times \mathbf{B} \to \mathbf{B}$.

Next, for $\mathbf{B}$ a precartesian bicategory, we define a lax functor $\mathbf{1} \to \mathbf{B}$, which amounts to giving an object of $\mathbf{B}$ and a monad on this.

**Proposition 5** *For a precartesian* $\mathbf{B}$*, the object* $I$ *of* $\mathbf{B}$*, the arrow* $\top{:}I \to I$*, and the unique 2-cells* $\widetilde{I}{:}\top\top \to \top$ *and* $I^\circ{:}1_I \to \top$ *constitute a lax functor* $I{:}\mathbf{1} \to \mathbf{B}$.

Note that $\widetilde{I}$ is invertible when $I^\circ$ is so.

We now describe the lax natural transformations $\mathbf{t} : i.\times \to \otimes.i \times i$ and $\mathbf{u} : i.1 \to I$ mentioned in the Introduction. For each pair of maps $f{:}X \to A$, $g{:}Y \to B$ in a precartesian bicategory $\mathbf{B}$, we have the map $f \times g = \langle fp, gr\rangle{:}X \otimes Y \to A \otimes B$ and the arrow $f \otimes g = p^* fp \wedge r^* gr{:}X \otimes Y \to A \otimes B$. We will now define a 2-cell $\mathbf{t}_{f,g}{:}f \times g \to f \otimes g$. First notice that the product diagram

$$\begin{tikzcd}
A \otimes B \arrow[r, "p_{A,B}"] \arrow[d, "1_A\otimes 1_B"'] \arrow[dr, phantom, "\xrightarrow{p_{1_A,1_B}}"] & A \arrow[d, "1_A"] & A \otimes B \arrow[r, "r_{A,B}"] \arrow[d, "1_A\otimes 1_B"'] \arrow[dr, phantom, "\xrightarrow{r_{1_A,1_B}}"] & B \arrow[d, "1_B"] \\
A \otimes B \arrow[r, "p_{A,B}"'] & A & A \otimes B \arrow[r, "r_{A,B}"'] & B
\end{tikzcd}$$

constructed like any binary product in $\mathbf{B}'$ from a local product in $\mathbf{B}$, is preserved by precomposition with the map $f \times g{:}X \otimes Y \to A \otimes B$ so that the following is also a product diagram in $\mathbf{B}'$

$$\begin{tikzcd}
X \otimes Y \arrow[r, "p_{X,Y}"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\cong"] \arrow[dd, bend right=70, "f\otimes g"'] \arrow[dd, phantom, bend right=40, "\cong"] & X \arrow[d, "f"] \arrow[dd, bend left=70, "f"] \arrow[dd, phantom, bend left=40, "\cong"] & X \otimes Y \arrow[r, "r_{X,Y}"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\cong"] \arrow[dd, bend right=70, "f\otimes g"'] \arrow[dd, phantom, bend right=40, "\cong"] & Y \arrow[d, "g"] \arrow[dd, bend left=70, "g"] \arrow[dd, phantom, bend left=40, "\cong"] \\
A \otimes B \arrow[r, "p_{A,B}"] \arrow[d, "1_A\otimes 1_B"'] \arrow[dr, phantom, "\xrightarrow{p_{1_A,1_B}}"] & A \arrow[d, "1_A"] & A \otimes B \arrow[r, "r_{A,B}"] \arrow[d, "1_A\otimes 1_B"'] \arrow[dr, phantom, "\xrightarrow{r_{1_A,1_B}}"] & B \arrow[d, "1_B"] \\
A \otimes B \arrow[r, "p_{A,B}"'] & A & A \otimes B \arrow[r, "r_{A,B}"'] & B
\end{tikzcd}$$

We define $\mathbf{t}_{f,g}{:}f \times g \to f \otimes g$ (to within canonical isomorphism) by the pasting

$$\begin{tikzcd}
X\otimes Y \arrow[r, "1"] \arrow[d, "f\times g"'] \arrow[dd, bend right=80, "f\times g"'] \arrow[dd, bend right=40, phantom, "\cong"] \arrow[rd, phantom, "\cong"] & X\otimes Y \arrow[d, "f\times g"] \arrow[dd, bend left=80, "f\otimes g"] \arrow[dd, bend left=40, phantom, "\cong"] \\
A\otimes B \arrow[r, "1"] \arrow[d, "1_{A\otimes B}"'] \arrow[rd, phantom, "\overset{\otimes^\circ}{\Rightarrow}"] & A\otimes B \arrow[d, "1_A\otimes 1_B"] \\
A\otimes B \arrow[r, "1"'] & A\otimes B
\end{tikzcd} \tag{14}$$

For $X$, $Y$ a pair of objects of $\mathrm{Map}\mathbf{B}$ we define $\mathbf{t}_{X,Y} = 1_{X\otimes Y}{:}X \times Y \to X \otimes Y$ and, for any precartesian $\mathbf{B}$, these data constitute a lax natural transformation $\mathbf{t}{:}i.\times \to \otimes.i\times i$ as in the Introduction, which is pseudonatural if and only if the constraints $\otimes^\circ$ are invertible, and then $\mathbf{t}$ itself is invertible.

For a precartesian bicategory $\mathbf{B}$, we have the arrow $\mathbf{u}_* = 1_I : I \to I$ and the 2-cell $\mathbf{u} = \mathbf{u}_{1_*} = \tau{:}1_I \to \top$ and these data also constitute a lax natural transformation $\mathbf{u}{:}1.i \to I$ as in the Introduction, which is pseudonatural if and only if the constraint $I^\circ : 1_I \to \top$ is invertible, and then $\mathbf{u}$ itself is invertible.

**Definition 6** *A precartesian bicategory* $\mathbf{B}$ *is said to be* cartesian *when the*

$$\begin{tikzcd}
\mathbf{B}\times\mathbf{B} \arrow[r, "\otimes"] & \mathbf{B} & \mathbf{1} \arrow[l, "I"']
\end{tikzcd}$$

*are pseudofunctors, meaning that* $\widetilde{\otimes}$, $\otimes^\circ$, $I^\circ$ *(and hence* $\widetilde{I}$*) are invertible.*

The pseudofunctors $\otimes$ and $I$ do in fact endow $\mathbf{B}$ with a *canonical* monoidal bicategory structure wherein all the constraints arise from the universal properties.

**Proposition 7** *For* $X$ *and* $Y$ *in a cartesian bicategory there is a natural isomorphism*

$$\begin{tikzcd}
& \mathbf{B}(X,I)\times\mathbf{B}(I,Y) \arrow[ld, "\otimes"'] \arrow[rd, "\circ"] \arrow[d, phantom, "\cong"] & \\
\mathbf{B}(X\otimes I, I\otimes Y) \arrow[rr, "\mathbf{B}(p^*{,}r)"'] & {} & \mathbf{B}(X,Y)
\end{tikzcd}$$

*where we have used* $\circ$ *to denote composition in* $\mathbf{B}$. *Moreover, the 2-cells*

$$\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "R"'] \arrow[rd, phantom, "\cong"] & X \arrow[d, "R"] \arrow[dd, bend left=70, "R"] \arrow[dd, bend left=35, phantom, "\cong"] \\
I \arrow[r, "1"] \arrow[d, "S"'] \arrow[rd, phantom, "\xrightarrow{\tau}"] & I \arrow[d, "\top"] \\
Y \arrow[r, "t"'] & I
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "t"] \arrow[d, "R"'] \arrow[rd, phantom, "\xrightarrow{\tau}"] & I \arrow[d, "\top"] \arrow[dd, bend left=70, "S"] \arrow[dd, bend left=35, phantom, "\cong"] \\
I \arrow[r, "1"] \arrow[d, "S"'] \arrow[rd, phantom, "\cong"] & I \arrow[d, "S"] \\
Y \arrow[r, "1"'] & Y
\end{tikzcd}$$

*provide product projections for* $SR{:}X \to Y$ *seen as a product of* $R$ *and* $S$ *in* $\mathbf{B}'$.

## 3 Groupoidal, Posetal, and Discrete Objects

For each object $X$ in a cartesian bicategory, we have an identity 2-cell

$$\begin{tikzcd}
X \otimes X \arrow[r, "d\otimes 1"] & (X \otimes X) \otimes X \arrow[dr, "a"] & \\
& | \Uparrow & X \otimes (X \otimes X) \\
X \arrow[uu, "d"] \arrow[rr, "d"'] & & X \otimes X \arrow[u, "1\otimes d"']
\end{tikzcd}$$

which has a mate $\delta_X : dd^* \to (1 \otimes d^*)a(d \otimes 1)$.

**Definition 8** *$X$ is said to be* groupoidal *if $\delta_X$ is invertible.*

In any precartesian bicategory we have, for each object $X$, the following arrows:

$$N_X \quad = \quad I \xrightarrow{t_X^*} X \xrightarrow{d_X} X \otimes X \qquad \text{and} \qquad E_X \quad = \quad X \otimes X \xrightarrow{d_X^*} X \xrightarrow{t_X} I$$

**Proposition 9** *For a groupoidal object $X$ in a cartesian bicategory, $N_X$ and $E_X$ satisfy the* triangle equations *(the $s$ being instances of symmetry) associated with adjunctions:*

$$(E_X \otimes X)(X \otimes N_X) \cong s_{X,I} \quad \textit{and} \quad (X \otimes E_X)(N_X \otimes X) \cong s_{I,X}$$

*to within coherent canonical isomorphisms.*

**Proposition 10** *In a cartesian bicategory $\mathbf{B}$, the groupoidal objects are closed under finite products.*

Write $\mathrm{Grp}\mathbf{B}$ for the full subbicategory of $\mathbf{B}$ determined by the groupoidal objects. It follows immediately from Proposition 10 that

**Proposition 11** *For a cartesian bicategory $\mathbf{B}$, the full subbicategory $\mathrm{Grp}\mathbf{B}$ is a cartesian bicategory in which every object is groupoidal.*

**Proposition 12** *For a cartesian bicategory $\mathbf{B}$ in which every object is groupoidal there is an involutory pseudofunctor*

$$(-)^\circ : \mathbf{B}^{\mathrm{op}} \to \mathbf{B}$$

*which is the identity on objects.*

**Proposition 13** *For a cartesian bicategory $\mathbf{B}$ in which every object is groupoidal there are equivalences of hom-categories, pseudonatural in each of the three variables,*

$$\mathbf{B}(X \otimes Y, Z) \underset{(\check{-})}{\overset{(\hat{-})}{\rightleftarrows}} \mathbf{B}(X, Z \otimes Y) \tag{15}$$

*where, for $R{:}X \otimes Y \to Z$, $\hat{R}$ is the composite*

$$X \xrightarrow{p^*} X \otimes I \xrightarrow{1_X \otimes N_Y} X \otimes Y \otimes Y \xrightarrow{R \otimes 1_Y} Z \otimes Y$$

*and $(\check{-})$ is defined similarly in terms of $E_Y$.*

**Proposition 14** *For an arrow $R : X \to A$ in a cartesian bicategory, with both $X$ and $A$ groupoidal, if the $d_R$ and $t_R$ of (7) are invertible then we can construct squares $N_R$ and $E_R$*

$$N_R = \begin{tikzcd} I \arrow[r, "1_I"] \arrow[d, "t^*_X"'] \arrow[dd, bend right=60, "N_X"'] \arrow[dr, phantom, "\xrightarrow{t^*_R}"] & I \arrow[d, "t^*_A"] \arrow[dd, bend left=60, "N_A"] \\ X \arrow[r, "R"] \arrow[d, "d_X"'] \arrow[dr, phantom, "\xrightarrow{d_R^{-1}}"] & A \arrow[d, "d_A"] \\ X \otimes X \arrow[r, "R \otimes R"'] & A \otimes A \end{tikzcd} \qquad E_R = \begin{tikzcd} X \otimes X \arrow[r, "R \otimes R"] \arrow[d, "d^*_X"'] \arrow[dd, bend right=60, "E_X"'] \arrow[dr, phantom, "\xrightarrow{d^*_R}"] & A \otimes A \arrow[d, "d^*_A"] \arrow[dd, bend left=60, "E_A"] \\ X \arrow[r, "R"] \arrow[d, "t_X"'] \arrow[dr, phantom, "\xrightarrow{t_R^{-1}}"] & A \arrow[d, "t_A"] \\ I \arrow[r, "1_I"'] & I \end{tikzcd}$$

*where $t^*_R$ is the mate of $t_R$ and $d^*_R$ is the mate of $d_R$, which when tensored with the identity square $R{:}R1_X \to 1_A R$, satisfy the following equations (in which $\otimes$ is suppressed):*

$$R = \begin{tikzcd} X \arrow[r, "R"] \arrow[d, "N_X X"'] \arrow[dd, bend right=70, "X"', "\cong"] \arrow[dr, phantom, "\xrightarrow{N_R R}"] & A \arrow[d, "N_A A"] \arrow[dd, bend left=70, "A", "\cong"'] \\ XXX \arrow[r, "RRR"] \arrow[d, "XE_X"'] \arrow[dr, phantom, "\xrightarrow{RE_R}"] & AAA \arrow[d, "AE_A"] \\ X \arrow[r, "R"'] & A \end{tikzcd} \qquad R = \begin{tikzcd} X \arrow[r, "R"] \arrow[d, "XN_X"'] \arrow[dd, bend right=70, "X"', "\cong"] \arrow[dr, phantom, "\xrightarrow{RN_R}"] & A \arrow[d, "AN_A"] \arrow[dd, bend left=70, "A", "\cong"'] \\ XXX \arrow[r, "RRR"] \arrow[d, "E_X X"'] \arrow[dr, phantom, "\xrightarrow{E_R R}"] & AAA \arrow[d, "E_A A"] \\ X \arrow[r, "R"'] & A \end{tikzcd} \tag{16}$$

**Proposition 15** *For an arrow $R : X \to A$ in a cartesian bicategory, with both $X$ and $A$ groupoidal, $R$ is a map if and only if $d_R$ and $t_R$ are invertible, in which case $R \dashv R^\circ$.*

From Proposition 15 it follows that for a map $f{:}X \to A$, with $X$ and $A$ groupoidal in a cartesian bicategory, we have $f^\circ \cong f^*$.

**Proposition 16** *For groupoidal objects $X$ and $A$ in a cartesian bicategory $\mathbf{B}$, the hom-category $\mathrm{Map}(\mathbf{B})(X, A)$ is a groupoid.*

We now extend the transfer of variables provided by Proposition 13 to the bicategory $\mathbf{B}'$

**Proposition 17** *For a cartesian bicategory $\mathbf{B}$ in which every object is groupoidal, for all maps $f{:}X \to A$, $g{:}Y \to B$, and $h{:}Z \to C$, there are bijections*

$$\mathbf{B}'(R, S)_{f \times g, h} \xrightarrow{(\bar{-})} \mathbf{B}'(\hat{R}, \hat{S})_{f, h \times g} \qquad \mathbf{B}'(T, U)_{f, h \times g} \xrightarrow{(\tilde{-})} \mathbf{B}'(\check{T}, \check{U})_{f \times g, h} \tag{17}$$

*satisfying*

$$\begin{tikzcd}
\mathbf{B}'(R,S)_{f\times g,h} \arrow[r, "(\hat{-})"] \arrow[dr, "\mathbf{B}'(i_R,i_S^{-1})_{f\times g,h}"'] & \mathbf{B}'(\hat{R},\hat{S})_{f,h\times g} \arrow[d, "(\check{-})"] \\
& \mathbf{B}'(\check{\hat{R}},\check{\hat{S}})_{f\times g,h}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{B}'(T,U)_{f,h\times g} \arrow[r, "(\check{-})"] \arrow[dr, "\mathbf{B}'(j_T,j_U^{-1})_{f,h\times g}"'] & \mathbf{B}'(\check{T},\check{U})_{f\times g,h} \arrow[d, "(\hat{-})"] \\
& \mathbf{B}'(\hat{\check{T}},\hat{\check{U}})_{f,h\times g}
\end{tikzcd}$$

*where $i$ is the unit and $j$ is the counit for the equivalence $(\check{-}) \dashv (\hat{-})$. Moreover, the mate $hR(f\times g)^* \to S$ of $\alpha{:}hR \to S(f\times g)$, with respect to the adjunction $f\times g \dashv (f\times g)^*$, is invertible if and only if the mate $(h\times g)\hat{R}f^* \to \hat{S}$ of $\bar{\alpha}{:}(h\times g)\hat{R} \to \hat{S}f$, with respect to the adjunction $f \dashv f^*$, is invertible.*

Our interest in Proposition 17 will be in the following:

**Corollary 18** *For a cartesian bicategory in which every object is groupoidal, for every object $T$, there is a bijective correspondence between 2-cells of the form $\alpha$ and 2-cells of the form $\beta$ as below.*

$$\begin{tikzcd}
T \arrow[r, "x"] \arrow[d, "1_T"'] \arrow[dr, phantom, "\xrightarrow{\alpha}"] & X \arrow[d, "R"] \\
T \arrow[r, "a"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "t_T"] \arrow[d, "1_T"'] \arrow[dr, phantom, "\xrightarrow{\beta}"] & I \arrow[d, "\hat{R}"] \\
T \arrow[r, "{\langle a,x\rangle}"'] & A\otimes X
\end{tikzcd}$$

*Moreover, the mate $ax^* \to R$ of $\alpha$ is invertible if and only if the mate $\langle a,x\rangle t_T^* \to \hat{R}$ of $\beta$ is invertible.*

**Definition 19** *An object $X$ in a cartesian bicategory is said to be* posetal *if the unit $\eta_{d_X}$ : $1_X \to d_X^* d_X$ for the adjunction $d_X \dashv d_X^*$ is invertible.*

**Proposition 20** *In a cartesian bicategory $\mathbf{B}$, the posetal objects are closed under finite products.*

Write $\mathrm{Ord}\mathbf{B}$ for the full subbicategory of $\mathbf{B}$ determined by the posetal objects. It follows immediately from Proposition 20 that

**Proposition 21** *For a cartesian bicategory $\mathbf{B}$, the full subbicategory $\mathrm{Ord}\mathbf{B}$ is a cartesian bicategory in which every object is posetal.*

**Proposition 22** *For a posetal object $A$ and any other object $X$ in a cartesian bicategory $\mathbf{B}$, the hom-category $\mathrm{Map}\mathbf{B}(X,A)$ is an ordered set, meaning that the category structure forms a reflexive, transitive relation.*

**Definition 23** *An object $X$ in a cartesian bicategory is said to be* discrete *if it is both groupoidal and posetal. We write $\mathrm{Dis}\mathbf{B}$ for the full subbicategory of $\mathbf{B}$ determined by the discrete objects.*

From Propositions 20 and 10 it is immediate that:

**Proposition 24** *For discrete objects $X$ and $A$ in a cartesian bicategory $\mathbf{B}$, the hom category* $\mathrm{Map}\mathbf{B}(X, A)$ *is an equivalence relation (on the set of maps from $X$ to $A$).*

and

**Proposition 25** *For a cartesian bicategory $\mathbf{B}$, the full subbibcategory* $\mathrm{Dis}\mathbf{B}$ *of discrete objects is a cartesian bicategory in which every object is discrete.*

**Proposition 26** *For any cartesian bicategory $\mathbf{B}$, there is a biequivalence*

$$\mathrm{MapDis}\mathbf{B} \xrightarrow{\sim} \mathcal{E}$$

*where $\mathcal{E}$ is a mere category regarded as a locally discrete bicategory.*

# 4 Comonads and Tabulation

In the terminology of [ST1] we have:

**Definition 27** *An* Eilenberg-Moore object *for a comonad $(X, G)$ in $\mathbf{B}$ is an object $X_G$ of $\mathbf{B}$ together with a $G$-coalgebra $(g, \gamma)$ with domain $X_G$*

$$\begin{tikzcd}
 & X_G \arrow[dl, "g"'] \arrow[dr, "g"] \arrow[d, phantom, "\xrightarrow{\gamma}"] & \\
X & & X \arrow[ll, "G"]
\end{tikzcd} \tag{18}$$

*such that, for all $Z$ in $\mathbf{B}$,*

$$\mathbf{B}(Z, (g, \gamma)){:}\mathbf{B}(Z, X_G) \to \mathbf{B}_{G\text{-}\mathbf{coalg}}(Z, X) \tag{19}$$

*is an equivalence of categories.*

It is a bicategorical generality that the structure arrow $g$ for an Eilenberg-Moore object has a right adjoint. Thus $g{:}X_G \to X$ is a map (which is why we used a lower case Roman letter). Furthermore, we have $G\epsilon_g.\gamma g^*{:}gg^* \to G$ an isomorphism.

We now consider the simplifications arising in the theory of comonads in a cartesian bicategory when:

$$\text{The 2-cells } \tau_{1_X}{:}1_X \to \top_{X,X} \text{ are monomorphisms.} \tag{20}$$

In the first instance, existence of a copoint for an arrow $G{:}X \to X$ is then a property (rather than a structure). The property in question is whether $\tau_G : G \to \top_{X,X}$ factors through $\tau_{1_X} : 1_X \to \top_{X,X}$. Since, assuming (20), any copoint $\epsilon{:}G \to 1_X$ is uniquely determined, we will write it as $\epsilon = \epsilon_G : G \to 1_X$. Note that if we have $\epsilon : G \to 1_X$ then we have $\epsilon^\circ{:}G^\circ \to 1_X^\circ \cong 1_X$. Moreover:

**Proposition 28** *If $G, H{:}X \to X$ are copointed in a cartesian $\mathbf{B}$ satisfying (20) then*

$$\begin{tikzcd}
 & GH \arrow[dl, "G\epsilon_H"'] \arrow[dr, "\epsilon_G H"] & \\
G \arrow[dr, "\epsilon_G"'] & & H \arrow[dl, "\epsilon_H"] \\
 & 1_X &
\end{tikzcd} \tag{21}$$

*provides a pullback and $G\epsilon_H{:}G \leftarrow GH \to H{:}\epsilon_G H$ a product in $\mathbf{B}(X,X)$.*

**Proposition 29** *For a copointed arrow $G{:}X \to X$ on a discrete object in a cartesian bicategory, there is an isomorphism $G \cong G^\circ$ given by the following composite*

$$G \cong G \wedge 1_X \cong d_X^*(G \otimes 1_X)d_X \xrightarrow{d_X^* \tilde{d_G}} d_X^*(1_X \otimes G^\circ)d_X G$$

$$\cong (1_X \wedge G^\circ)G \xrightarrow{(1_X \wedge G^\circ)\epsilon_G} 1_X \wedge G^\circ \cong G^\circ$$

*(where $\tilde{d_G}$ is constructed from $d_G$ using Proposition 13).*

It follows from Proposition 28 that an arrow $G{:}X \to X$ in a cartesian bicategory satisfying (20) is a comonad as soon it enjoys the property of being copointed, the comultiplication then being given by the diagonal $\delta : G \to G \wedge G$. Moreover, if $G, H : X \to X$ are both copointed then it follows that any $\alpha{:}G \to H$ is a comonad homomorphism.

**Proposition 30** *If the map $t_X{:}X \to I$ is comonadic then property (20) holds for $X$.*

The Eilenberg-Moore construction in display (18) associates to the comonad $G$ a span of maps. We next consider a generalization of this for an arbitrary arrow $R : X \to A$ in a bicategory $\mathbf{B}$. We define an *$R$-tabulation with domain $M$*, for $M$ an object of $\mathbf{B}$, to be a pair of maps $a : A \leftarrow M \to X : x$ and a 2-cell $\mu : a \to Rx$. If $(a; x, \mu)$ and $(a'; x', \mu')$ are $R$-tabulations with domain $M$, an $R$-homomorphism from $(a; x, \mu)$ to $(a'; x', \mu')$ is a pair of 2-cells $\alpha{:}a \to a'$ and $\xi{:}x \to x'$ satisfying

$$\begin{tikzcd}
 & M \arrow[dl, bend right=60, "a"'] \arrow[dl, "a'"] \arrow[dl, phantom, bend right=30, "\xrightarrow{\alpha}"] \arrow[dr, "x'"] \arrow[d, phantom, "\xrightarrow{\mu'}"] & & & & M \arrow[dl, "a"'] \arrow[dr, "x"] \arrow[dr, bend left=60, "x'"] \arrow[dr, phantom, bend left=30, "\xrightarrow{\xi}"] \arrow[d, phantom, "\xrightarrow{\mu}"] & \\
A & {} & X \arrow[ll, "R"] & = & A & {} & X \arrow[ll, "R"]
\end{tikzcd} \tag{22}$$

Observe that if

$$(\alpha, \xi){:}(a; x, \mu) \to (a'; x', \mu')$$

is an $R$-homomorphism between $R$-tabulations with domain $M$, and $S : N \to M$ is any arrow, then

$$(\alpha S, \xi S){:}(aS; xS, \mu S) \to (a'S; x'S, \mu' S)$$

is an $R$-homomorphism between $R$-tabulations with domain $N$. For the evident category of $R$-tabulations with domain $M$ and $R$-homomorphisms, write $\mathbf{B}_{R\text{-}\mathbf{tab}}(M, A; X)$. For $(a; x, \mu)$ an $R$-tabulation with domain $M$ write

$$\mathbf{B}(N, (a; x, \mu)){:}\mathbf{B}(N, M) \to \mathbf{B}_{R\text{-}\mathbf{tab}}(N, A; X)$$

for the functor given by composing with $(a; x, \mu)$.

**Definition 31** *A* tabulating object *for an arrow $R : X \to A$ in $\mathbf{B}$ is an object $\tau R$ of $\mathbf{B}$, together with an $R$-tabulation $(p; r, \sigma)$ with domain $\tau R$ as in*

$$\begin{tikzcd} & \tau R \arrow[dl, "p"'] \arrow[dr, "r"] \arrow[d, phantom, "\xrightarrow{\sigma}"] & \\ A & {} & X \arrow[ll, "R"] \end{tikzcd} \tag{23}$$

*for which the composite*

$$pr^* \xrightarrow{\sigma r^*} Rrr^* \xrightarrow{R\epsilon_r} R$$

*(the mate of $\sigma{:}p \to Rr$ under $r \dashv r^*$) is invertible, and such that, for all $N$ in $\mathbf{B}$,*

$$\mathbf{B}(N, (p; r, \sigma)){:}\mathbf{B}(N, \tau R) \to \mathbf{B}_{R\text{-}\mathbf{tab}}(N, A; X) \tag{24}$$

*is an equivalence of categories.*

**Proposition 32** *For an arrow $G{:}X \to X$ which carries a comonad structure, in a cartesian $\mathbf{B}$ with every object discrete, the span in (18) provides a tabulation of $G$.*

In fact we have an isomorphism of categories $\mathbf{B}_{G\text{-}\mathbf{tab}}(Z, X; X) \simeq \mathbf{B}_{G\text{-}\mathbf{coalg}}(Z, X)$ and commutativity of the following

$$\begin{tikzcd} & \mathbf{B}(Z, X_G) \arrow[dl, "{\mathbf{B}(Z,(g;g,\gamma))}"'] \arrow[dr, "{\mathbf{B}(Z,(g,\gamma))}"] & \\ \mathbf{B}_{G\text{-}\mathbf{tab}}(Z, X; X) \arrow[rr, "\sim"'] & & \mathbf{B}_{G\text{-}\mathbf{coalg}}(Z, X) \end{tikzcd}$$

**Proposition 33** *For a cartesian bicategory in which every object is discrete, $\alpha$ exhibits $T$ as a tabulating object for $R{:}X \to A$ if and only if $\beta$ exhibits $T$ as a tabulating object for $\hat{R}{:}I \to A \otimes X$.*

$$\begin{tikzcd} & T \arrow[dl, "p"'] \arrow[dr, "r"] \arrow[d, phantom, "\xrightarrow{\alpha}"] & \\ A & {} & X \arrow[ll, "R"] \end{tikzcd} \qquad \begin{tikzcd} & T \arrow[dl, "{\langle p,r \rangle}"'] \arrow[dr, "t_T"] \arrow[d, phantom, "\xrightarrow{\beta}"] & \\ A \otimes X & {} & I \arrow[ll, "\hat{R}"] \end{tikzcd}$$

*where $\alpha$ and $\beta$ are related as in Proposition 18*

**Proposition 34** *For a cartesian bicategory* $\mathbf{B}$ *and an arrow* $R{:}I \to X$*, the arrow* $d_X^*(X \otimes R)p_{X,I}^*{:}X \to X$ *is copointed and hence by Proposition 28 carries a unique comonad structure.*

**Proposition 35** *For a cartesian bicategory* $\mathbf{B}$ *in which every object is discrete and an arrow* $R{:}I \to X$*, if an Eilenberg-Moore object for the comonad* $G = d_X^*(X \otimes R)p_{X,I}^*$ *is provided by*

$$\begin{tikzcd}
& X_G \arrow[dl, "g"'] \arrow[dr, "g"] & \\
X & \xrightarrow{\gamma} & X \arrow[ll, "G"]
\end{tikzcd}$$

*then a tabulating object for* $R{:}I \to X$ *is provided by*

$$\begin{tikzcd}
& & & X_G \arrow[dl, "g"'] \arrow[dr, "g"] \arrow[dddlll, bend right, "g"'] \arrow[dddrrr, bend left, "t_{X_G}"] & & & \\
& \cong & X \arrow[dl, "d"'] & \xrightarrow{\overline{\gamma}} & X \arrow[dr, "p^*"] & \cong & \\
& X \otimes X \arrow[dl, "r"'] & & & & X \otimes I \arrow[llll, "1_X \otimes R"'] \arrow[dr, "r"] & \\
X & & & & & & I \arrow[llllll, "R"]
\arrow[from=3-4, to=4-4, phantom, "\xrightarrow{r_{1_X,R}}"]
\end{tikzcd}$$

*where* $\overline{\gamma}$ *is the mate of* $\gamma{:}g \to d^*(1_X \otimes R)p^*g$ *with respect to* $d \dashv d^*$*, and conversely.*

**Proposition 36** *For a cartesian bicategory* $\mathbf{B}$ *in which every object is discrete and every arrow has a tabulation,* $\mathrm{Map}\mathbf{B}$ *has pullbacks satisfying the Beck-Chevalley condition, meaning that for a pullback square*

$$\begin{tikzcd}
& P \arrow[dl, "p"'] \arrow[dr, "q"] & \\
N \arrow[dr, "a"'] & & M \arrow[dl, "b"] \\
& A &
\end{tikzcd} \tag{25}$$

*the mate* $pr^* \to a^*b$ *of the identity* $ap \xrightarrow{=} br$ *is invertible.*

**Proposition 37** *In the context of Proposition 35, if the arrow* $R{:}I \to X$ *is a map* $x{:}I \to X$ *then* $I$ *together with* $(x; 1_I, 1_x)$ *is a tabulating object for* $x$ *and* $x$ *is comonadic.*

**Proposition 38** *For a cartesian bicategory in which every object is discrete, every comonad has an Eilenberg-Moore, and the* $t_X{:}X \to I$ *are comonadic; every map is comonadic and every span of maps* $a{:}A \leftarrow M \to X{:}x$ *tabulates the arrow* $ax^*{:}X \to A$.

# 5 Characterization of Bicategories of Spans and Relations

If $\mathbf{B}$ is a cartesian bicategory with $\mathrm{Map}\mathbf{B}$ essentially locally discrete then $\mathrm{Map}\mathbf{B}/A \times X$, for each pair $A, X$, is also essentially locally discrete and we write $\mathrm{SpanMap}\mathbf{B}(X, A)$ for the categories obtained by taking the quotients of the equivalence relations comprising the hom categories of the $\mathrm{Map}\mathbf{B}/A \times X$. Then we can construct functors $E_{X,A} : \mathrm{SpanMap}\mathbf{B}(X, A) \to \mathbf{B}(X, A)$, where for an arrow in $\mathrm{SpanMap}\mathbf{B}(X, A)$ as shown,

$$\begin{tikzcd}
& M \arrow[ld, "a"'] \arrow[rd, "x"] \arrow[dd, "h"] & \\
A & & X \\
& N \arrow[lu, "b"] \arrow[ru, "y"'] &
\end{tikzcd}$$

we define $E(y, N, b) = by^*$ and $E(h){:}ax^* = (bh)(yh)^* \cong bhh^*y^* \xrightarrow{b\epsilon_h y^*} by^*$. If $\mathrm{Map}\mathbf{B}$ is known to have pullbacks then the $\mathrm{SpanMap}\mathbf{B}(X, A)$ become the hom-categories for a bicategory $\mathrm{SpanMap}\mathbf{B}$ and we may consider whether the $E_{X,A}$ provide the effects on homs for an identity-on-objects pseudofunctor $E{:}\mathrm{SpanMap}\mathbf{B} \to \mathbf{B}$. Consider

$$\begin{tikzcd}
& & P \arrow[ld, "p"'] \arrow[rd, "r"] & & \\
& N \arrow[ld, "y"'] \arrow[rd, "b"] & & M \arrow[ld, "a"'] \arrow[rd, "x"] & \\
Y & & A & & X
\end{tikzcd} \tag{26}$$

where the square is a pullback. In somewhat abbreviated notation, what is needed further are coherent, invertible 2-cells $\widetilde{E}{:}EN.EM \to E(NM) = EP$, for each composable pair of spans $M$, $N$, and coherent, invertible 2-cells $E^\circ{:}1_A \to E(1_A)$, for each object $A$. Since the identity span on $A$ is $(1_A, A, 1_A)$, and $E(1_A) = 1_A.1_A^* \cong 1_A.1_A \cong 1_A$ we take the inverse of this composite for $E^\circ$. To give the $\widetilde{E}$ though is to give 2-cells $yb^*ax^* \to ypr^*x^*$ and since spans of the form $(1_N, N, b)$ and $(a, M, 1_M)$ arise as special cases, it is easy to verify that to give the $\widetilde{E}$ it is necessary and sufficient to give coherent, invertible 2-cells $b^*a \to pr^*$ for each pullback square in $\mathrm{Map}\mathbf{B}$. The inverse of such a 2-cell $pr^* \to b^*a$ is the mate of a 2-cell $bp \to aq$. But by discreteness a 2-cell $bp \to aq$ must be essentially an identity. Thus, definability of $\widetilde{E}$ is equivalent to the invertibility in $\mathbf{B}$ of the mate $pr^* \to b^*a$ of the identity $bp \to ar$, for each pullback square as displayed in (26). In short, if $\mathrm{Map}\mathbf{B}$ has pullbacks and these satisfy what we have called in Proposition 36 the Beck-Chevalley condition then we have a canonical pseudofunctor $E{:}\mathrm{SpanMap}\mathbf{B} \to \mathbf{B}$.

**Theorem 39** *For a bicategory* $\mathbf{B}$ *the following are equivalent:*

(1) *There is a biequivalence* $\mathbf{B} \simeq \mathrm{Span}\mathcal{E}$*, for* $\mathcal{E}$ *a category with finite limits;*

(2) *The bicategory* $\mathbf{B}$ *is cartesian with every object discrete, every comonad having an Eilenberg-Moore object, and the maps* $t_X{:}X \to I$ *being comonadic.*

*(3) The bicategory* $\mathrm{Map}\mathbf{B}$ *is an essentially locally discrete bicategory with finite limits, whose pullbacks satisfy the Beck-Chevalley condition in* $\mathbf{B}$*, and* $E{:}\mathrm{SpanMap}\mathbf{B} \twoheadrightarrow \mathbf{B}$ *is a biequivalence of bicategories.*

**Theorem 40** *For a bicategory* $\mathbf{B}$ *the following are equivalent:*

*(1) There is a biequivalence* $\mathbf{B} \simeq \mathrm{Rel}\mathcal{E}$*, for* $\mathcal{E}$ *a regular category;*

*(2) The bicategory* $\mathbf{B}$ *is cartesian and locally ordered with every object discrete and every comonad having an Eilenberg-Moore object.*

*(3) The bicategory* $\mathrm{Map}\mathbf{B}$ *is a regular essentially locally discrete bicategory, whose pullbacks satisfy the Beck-Chevalley condition in* $\mathbf{B}$*, and* $E{:}\mathrm{RelMap}\mathbf{B} \twoheadrightarrow \mathbf{B}$ *is a biequivalence of bicategories.*

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