Action groupoid in protomodular categories II

Dominique Bourn

Abstract

We give here some examples of non pointed protomodular categories $\mathbb{C}$ satisfying a property similar to the property of representation of actions which holds for the pointed protomodular category Gp of groups : any slice category of Gp, any category of groupoids with a fixed set of objects, any essentially affine category. This property gives rise to an internal construction of the center of any object X, and consequently to a specific characterization of the abelian objects in $\mathbb{C}$.

The motivations of this work are detailed in the announcement of the Charles Ehresmann's birthday Meeting [7].

1 Action groupoid.

From now on we shall consider a protomodular category $\mathbb{C}$ [5], see also [1] where the fundamentals on this notion are collected. An internal groupoid $Z_\bullet$ in $\mathbb{C}$ will be presented (see [4]) as a reflexive graph endowed with an operation ζ_2:

$$\begin{tikzcd}
R^2[z_0] \arrow[r, shift left=2, "z_2"] \arrow[r, "z_1"] \arrow[r, shift right=2, "z_0"'] \arrow[r, bend left=60, "R(\zeta_2)"] & R[z_0] \arrow[r, shift left=2, "\zeta_2"] \arrow[r, "z_1"] \arrow[r, shift right=2, "z_0"'] & Z_1 \arrow[r, shift left=2, "z_1"] \arrow[r, shift right=2, "z_0"'] & Z_0 \arrow[l, "s_0"']
\end{tikzcd}$$

making the previous diagram satisfy all the simplicial identities (including the ones involving the degeneracies), where $R[z_0]$ is the kernel equivalence relation of the map z_0. In the set theoretical context, this operation ζ_2 associates the composite $g.f^{-1}$ with any pair (f, g) of arrows with same domain. Actually, when the category $\mathbb{C}$ is protomodular, and thus Mal'cev, we can even truncate this diagram at level 2 [9].

Definition 1.1. *An object X in $\mathbb{C}$ is said action representative, or to have an action groupoid, when there is an internal groupoid $D_\bullet(X)$:*

$$\begin{tikzcd}
R[d_0] \arrow[r, shift left=2, "\delta_2"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_0"'] & D_1(X) \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & D(X) \arrow[l, "s_0"']
\end{tikzcd}$$

endowed with a canonical discrete fibration $j : \nabla X \to D_\bullet(X)$:

$$\begin{tikzcd}
R[p_0] \arrow[r, shift left=2, "\delta_2"] \arrow[r, "p_1" description] \arrow[r, shift right=2, "p_0"'] \arrow[d, "R(\tilde{j})"'] & X \times X \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[d, "\tilde{j}"] & X \arrow[l, "s_0" description] \arrow[d, "j"] \\
R[d_0] \arrow[r, shift left=2, "\delta_2"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_0"'] & D_1(X) \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & D(X) \arrow[l, "s_0" description]
\end{tikzcd}$$

where ∇X is the indiscrete equivalence relation associated with X, satisfying the following property : given any other internal groupoid $Z_\bullet$ endowed with a discrete fibration $k_\bullet = (k_0, k_1) : \nabla X \to Z_\bullet$, there is a unique internal functor $\check{k}_\bullet = (\check{k}_0, \check{k}_1) : Z_\bullet \to D_\bullet$ such that $\check{k}_\bullet . k_\bullet = j_\bullet$. The category $\mathbb{C}$ is said to be action representative when any object X has an action groupoid.

This implies that $\check{k}_\bullet$ is itself a discrete fibration ($\mathbb{C}$ being protomodular). When $\mathbb{C} = Gp$, this action groupoid is just the internal groupoid associated with the canonical crossed module $X \to AutX$. In the pointed protomodular case many examples are given in [3] and [2]. We have moreover :

Proposition 1.1. *Let $\mathbb{C}$ be an action representative protomodular category. Then the category $\mathbb{C}_*$ of pointed objects in $\mathbb{C}$ is still action representative, and then any object in $\mathbb{C}_*$ has a split extension classifier [2].*

Proof. The category $\mathbb{C}_*$ is nothing but the coslice category $1\backslash\mathbb{C}$ of maps with domain the terminal object. It has the same underlying products and pullbacks as $\mathbb{C}$. So it is straitforward that if $(X, e), e : 1 \to X$ is an object in $\mathbb{C}_*$, we have $D(X, e) = (D(X), j.e)$ and $D_1(X, e) = (D_1(X), s_0.j.e)$. □

Remark. The same result holds for any coslice category $Y\backslash\mathbb{C}$.

2 Slice categories in pointed case.

Suppose now $\mathbb{C}$ is an action representative pointed protomodular category. The slice categories $\mathbb{C}/Y$ are no longer pointed.

Proposition 2.1. *When $\mathbb{C}$ is an action representative pointed protomodular category, the slice categories $\mathbb{C}/Y$ are still action representative. For any $h : Y' \to Y$, the change of base functor $h^* : \mathbb{C}/Y \to \mathbb{C}/Y'$ preserves the action groupoids.*

Proof. Let $f : X \to Y$ be an object of $\mathbb{C}/Y$. Consider the following diagram where K is the kernel of f :

$$\begin{tikzcd}
K \times K \arrow[r, tail, "R(k)"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & R[f] \arrow[r, dotted, "\check{k}_1"] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "f_1"] & D_1(K) \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
K \arrow[r, tail, "k"'] \arrow[d] & X \arrow[r, dotted, "\check{k}"'] \arrow[d, "f"] & D(K) \\
0 \arrow[r, tail, "\alpha_Y"'] & Y &
\end{tikzcd}$$

Then the upper left hand side part of this diagram determines a discrete fibration and thus produces the internal functor $\check{k}_\bullet$ (actually a discrete fibration). Whence the following diagram in $\mathbb{C}/Y$:

$$\begin{tikzcd}
R[f] \arrow[r, "(\check{k}_1, f.f_0)"] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "f_1"] & D_1(K) \times Y \arrow[d, shift right, "d_0 \times Y"'] \arrow[d, shift left, "d_1 \times Y"] \\
X \arrow[r, "(\check{k}, f)"'] \arrow[dr, "f"'] & D(K) \times Y \arrow[d, "p_Y"] \\
& Y
\end{tikzcd}$$

This is a discrete fibration since the composite with the projection towards $D_\bullet(K)$ is the discrete fibration $\check{k}_\bullet$. We claim that the internal groupoid in $\mathbb{C}/Y$ on the right is the action groupoid of the object f, the equivalence relation $R[f]$ being, in the category $\mathbb{C}/Y$, the indiscrete equivalence relation associated with this object f. The second point of the statement comes from the fact that the image $h^*(f)$ of the object f in $\mathbb{C}/Y$ by the change of base functor h^* has the same kernel K (in $\mathbb{C}$) as f. $\square$

We thus get the following :

Corollary 2.1. *Let $\mathbb{C}$ be an action representative pointed protomodular category; and $\pi : Pt\mathbb{C} \to \mathbb{C}$ [5] the associated fibration of pointed objects. Then its fibers are action representative and its change of base functors preserve the action groupoids and the split extension classifiers.*

Proof. The fiber $Pt_Y\mathbb{C}$ above Y is nothing but the category $(\mathbb{C}/Y)_*$ of points of $\mathbb{C}/Y$ (in other words : split epimorphisms with codomain Y). Then by Propositions 2.1 nad 1.1, it is action representative. The change of base functors are given by pullbacks, so, by Proposition 2.1, they preserve the action groupoids and consequently the split extension classifiers which are part of this structure, see [2]. $\square$

3 The fibers of the fibration $()_0 : Grd \to Set$.

Let Set and Grd be respectively the categories of sets and groupoids, and $()_0 : Grd \to Set$ the forgetful functor associating the object of objects Z_0 with any groupoid $Z_\bullet$. This functor is a fibration whose fiber above the singleton 1 is nothing but the category Gp of groups which is action representative. On the other hand, any fiber Grd_X above a set X is protomodular [5] and clearly non pointed. We are going to show that it is still action representative. This fiber has an initial object ΔX, namely the discrete equivalence relation on X, and a final object ∇X the indiscrete equivalence relation on X. For any groupoid $Z_\bullet$ in Grd_X, we shall need the subgroupoid defined by the following pullback in Grd_X (namely the subgroupoid of endomaps of $Z_\bullet$):

$$\begin{tikzcd} AutZ_\bullet \arrow[r, tail] \arrow[d] & Z_\bullet \arrow[d] \\ \Delta X \arrow[r, tail] & \nabla X \end{tikzcd}$$

Let $Z_\bullet$ be a groupoid such that $Z_0 = X$. We shall denote by $Z(x,x')$ the set of arrows from x to x'. Its action groupoid in Grd_X, if ever it exists, must be an internal groupoid in Grd_X, which is nothing but a 2-groupoid with X as object of objects. Let us denote by $D(Z_\bullet)$ the groupoid whose object of objects is X, and arrows $\phi : x \to x'$ are the group isomorphisms $\phi : Z(x,x) \to z(x',x')$. There is a canonical bijective on objects functor $j_\bullet : Z_\bullet \to D(Z_\bullet)$: given a map $f : x \to x'$ in $Z_\bullet$ its image $j(f) : x \to x'$ in $D(Z_\bullet)$ is the group isomorphism $j(f) : Z(x,x) \to z(x',x')$ given by $j(f)(\alpha) = f.\alpha.f^{-1}$, $\forall \alpha \in Z(x,x)$, i.e. such that the following diagram commutes in the groupoid $Z_\bullet$:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\alpha"'] & x' \arrow[d, "j(f)(\alpha)"] \\ x \arrow[r, "f"'] & x' \end{tikzcd}$$

This groupoid $D(Z_\bullet)$ is actually underlying a 2-groupoid. A 2-cell $\nu : \phi \Rightarrow \psi$ is given by a map $\nu \in Z(x',x')$ such that $\forall \alpha \in Z(x,x)$ we have $\psi(\alpha) = \nu.\phi(\alpha).\nu^{-1}$. The "vertical" composition is given by the composition in $Z_\bullet$, the "horizontal" one :

$$\begin{tikzcd} x \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\Downarrow \nu"] & x' \arrow[r, bend left, "\phi'"] \arrow[r, bend right, "\psi'"'] \arrow[r, phantom, "\Downarrow \nu'"] & x'' \arrow[r, mapsto] & x \arrow[r, bend left, "\phi'.\phi"] \arrow[r, bend right, "\psi'.\psi"'] \arrow[r, phantom, "\Downarrow \nu'.\nu"] & x'' \end{tikzcd}$$

is defined by $\nu'.\nu = \psi'(\nu).\nu'(= \nu'.\phi'(\nu))$. Accordingly, we define the groupoid $D_1(Z_\bullet)$ as the groupoid whose object of objects is X, and arrows $x \to x'$ are the pairs (ϕ, ν), with $\phi : x \to x'$ an arrow of $D(Z_\bullet)$ and $\nu \in z(x',x')$. The composition is defined by $(\phi',\nu').(\phi,\nu) = (\phi'.\phi, \nu'.\phi'(\nu))$. The bijective on objects functors $d_i : D_1(Z_\bullet) \to D(Z_\bullet)$ are defined by $d_0(\phi, \nu) = \phi$ and $d_1(\phi, \nu) = \psi$ with $\psi(\alpha) = \nu.\alpha.\nu^{-1}$. We have also a bijective on objects functor :

$$\tilde{j}_\bullet : Z_\bullet \times_X Z_\bullet \to D_1(Z_\bullet)$$

where $Z_\bullet \times_X Z_\bullet$ denotes the product in the fiber Grd_X, which is defined by $\tilde{j}(f, g) = (j(f), g.f^{-1})$, for any parallel pair of arrows $(f, g) : x \rightrightarrows x'$ in $Z_\bullet$. Whence the following commutative diagram which actually is underlying a discrete fibration in Grd_X:

$$\begin{tikzcd}
R[p_0] \arrow[r, shift left=2, "p_2"] \arrow[r, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[d, "R(\tilde{j}_\bullet)"'] & Z_\bullet \times_X Z_\bullet \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[d, "\tilde{j}_\bullet"] & Z_\bullet \arrow[l, "s_0"'] \arrow[d, "j_\bullet"] \\
R[d_0] \arrow[r, shift left=2, "\delta_2"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_0"'] & D_1(Z_\bullet) \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & D(Z_\bullet) \arrow[l, "s_0"']
\end{tikzcd}$$

Remark. Actually the definition of the lower groupoid depends uniquely on $AutZ_\bullet$. The only comparison j depends on $Z_\bullet$. This observation will be essential for the proof of the uniqueness in the following proposition:

Proposition 3.1. *The diagram above determines the lower groupoid as the action 2-groupoid $D_\bullet(Z_\bullet)$ associated with the groupoid $Z_\bullet$ in the protomodular fiber Grd_X. Accordingly the fibers Grd_X are action representative.*

Proof. Suppose we are given a discrete fibration in Grd_X:

$$\begin{tikzcd}
R[p_0] \arrow[r, shift left=2, "p_2"] \arrow[r, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[d, "R(k_{\bullet,1})"'] & Z_\bullet \times_X Z_\bullet \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[d, "k_{\bullet,1}"] & Z_\bullet \arrow[l, "s_0"'] \arrow[d, "k_{\bullet,0}"] \\
R[w_0] \arrow[r, shift left=2, "w_2"] \arrow[r, "w_1"] \arrow[r, shift right=2, "w_0"'] & W_{\bullet,1} \arrow[r, shift left=2, "w_1"] \arrow[r, shift right=2, "w_0"'] & W_{\bullet,0} \arrow[l, "s_0"']
\end{tikzcd}$$

The fact that this is a fibration means that any 2-cell in the 2-groupoid $W_\bullet$:

$$\begin{tikzcd}
x \arrow[r, bend left, "k(f)"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow \nu"] & x'
\end{tikzcd}$$

determines a unique arrow $g : x \to x'$ in the groupoid $Z_\bullet$ such that $k(g) = h$. In particular any 2-cell in $W_\bullet$:

$$\begin{tikzcd}
x \arrow[r, bend left, "1_x"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow \nu"] & x
\end{tikzcd}$$

produces a unique arrow $g : x \to x$ in the groupoid $Z_\bullet$ such that $k(g) = h$. We must now define a 2-functor:

$$\begin{tikzcd}
R[w_0] \arrow[r, shift left=2, "w_2"] \arrow[r, "w_1"] \arrow[r, shift right=2, "w_0"'] \arrow[d, "R(\check{k}_{\bullet,1})"'] & W_{\bullet,1} \arrow[r, shift left=2, "w_1"] \arrow[r, shift right=2, "w_0"'] \arrow[d, "\check{k}_{\bullet,1}"] & W_{\bullet,0} \arrow[l, "s_0"'] \arrow[d, "\check{k}_{\bullet,0}"] \\
R[d_0] \arrow[r, shift left=2, "\delta_2"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_0"'] & D_1(Z_\bullet) \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & D(Z_\bullet) \arrow[l, "s_0"']
\end{tikzcd}$$

Let $h : x \to x'$ be an arrow in $W_\bullet$; let us define $\check{k}(h) : Z(x,x) \to Z(x',x')$. So consider $\alpha : x \to x$ in $Z_\bullet$. We have a 2-cell $k(1_x, \alpha)$ in $W_\bullet$. Consider the following diagram:

$$\begin{tikzcd}
x \arrow[r, bend left, "1_x"] \arrow[r, phantom, "\Downarrow k(1_x,\alpha)"] \arrow[r, bend right, "k(\alpha)"'] \arrow[d, "h"'] & x \arrow[d, "h"] \\
x' \arrow[r, bend left, "1_{x'}"] \arrow[r, phantom, "\Downarrow h.k(1_x,\alpha).h^{-1}"] \arrow[r, bend right, "h.k(\alpha).h^{-1}"'] & x'
\end{tikzcd}$$

Then there is a unique arrow $\check{k}(h)(\alpha) : x' \to x'$ in $Z_\bullet$ such that $k(\check{k}(h)(\alpha)) = h.k(\alpha).h^{-1}$. The unicity of this map assures that $\check{k}(h)$ is a group homomorphism, and the last equation that $\check{k}_{\bullet,0}.k_{\bullet,0} = j_\bullet$. It is easy to check that the construction $\check{k}_\bullet : W_{\bullet,0} \to D(Z_\bullet)$ is functorial.

We must now extend $\check{k}$ to the 2-cells. So let $\nu : h \Rightarrow h'$ be a 2-cell in $W_\bullet$. Then $\nu.h^{-1} : 1_{x'} \Rightarrow h'.h^{-1}$ is a 2-cell in $W_\bullet$ which determines a unique map $\check{\nu} : x' \to x'$ in $Z_\bullet$ such that $k(\check{\nu}) = h'.h^{-1}$. We define $\check{k}(\nu)$ as the 2-cell $(\check{k}(h), \check{\nu})$ in $D(Z_\bullet)$. This completes the 2-functor $\check{k}_{\bullet,\bullet}$ we were looking for.

To prove the unicity of this factorization, let us look at the following diagram:

$$\begin{tikzcd}
R[p_0] \arrow[r, shift left=2, "p_2"] \arrow[r, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[d] & AutZ_\bullet \times_X AutZ_\bullet \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[d] & AutZ_\bullet \arrow[l, "s_0"'] \arrow[d] \\
R[p_0] \arrow[r, shift left=2, "p_2"] \arrow[r, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[d, "R(k_{\bullet,1})"'] & Z_\bullet \times_X Z_\bullet \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[d, "k_{\bullet,1}"] & Z_\bullet \arrow[l, "s_0"'] \arrow[d, "k_{\bullet,0}"] \\
R[w_0] \arrow[r, shift left=2, "w_2"] \arrow[r, "w_1"] \arrow[r, shift right=2, "w_0"'] \arrow[d, "R(\check{k}_{\bullet,1})"'] & W_{\bullet,1} \arrow[r, shift left=2, "w_1"] \arrow[r, shift right=2, "w_0"'] \arrow[d, "\check{k}_{\bullet,1}"] & W_{\bullet,0} \arrow[l, "s_0"'] \arrow[d, "\check{k}_{\bullet,0}"] \\
R[d_0] \arrow[r, shift left=2, "\delta_2"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_0"'] & D_1(Z_\bullet) \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & D(Z_\bullet) \arrow[l, "s_0"']
\end{tikzcd}$$

The upper part of this diagram produced from the inclusion $AutZ_\bullet \rightarrowtail Z_\bullet$ is actually a discrete fibration. Moreover, as we noticed in the previous remark the lower 2-groupoid is also the action 2-groupoid associated with $AutZ_\bullet$. So the unicity of the factoriation $\check{k}_\bullet$ can be equally checked from $AutZ_\bullet$. This is relatively easy, since $AutZ_\bullet$ is nothing but a family of ordinary groups. $\square$

Remark. As in any protomodular category, there is, in the fibers Grd_X, an intrinsic notion of normal subobject which is explicited in [6] (Theorem 3). The normal subobjects are closely related to the action groupoids, see [2]. Let us quickly mention here, that a subgroupoid $V_\bullet \rightarrowtail Z_\bullet$ in Grd_X is normal if and only if, given any map $f : x \to x'$ in $Z_\bullet$ the restriction of the isomorphism $j(f) : Z(x,x) \to Z(x',x')$ to $V(x,x)$ takes its values in $V(x',x')$.

4 Action groupoid and centrality.

In any protomodular category $\mathbb{C}$, there is an intrinsic notion of abelian objects, see for instance [8] or [1]. The existence of action groupoids allows us to measure the obstruction to abelianity:

Proposition 4.1. *Suppose the object X in $\mathbb{C}$ admits an action groupoid. The kernel relation $R[j]$ of the map $j : X \to D(X)$ is the centre of X, i.e. the greatest central equivalence relation on X.*

Proof. Recall that an equivalence relation $(r_0, r_1) : R \rightrightarrows X$ on X is central when there is a "connector" between the equivalence realtion R and ∇_X, which is a map $p : R \times X \to X$, satisfying internally the Mal'cev equations $p(x,y,y) = x$ and $p(x,x,y) = y$, see [8]. Now let us consider the following diagram:

$$\begin{tikzcd}
R[\tilde{j}] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[d, shift right, "R(p_0)"'] \arrow[d, shift left, dotted, "R(p_1)"] & X \times X \arrow[r, "\tilde{j}"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, dotted, "p_1"] & D_1(X) \arrow[d, shift right, "d_0"'] \arrow[d, shift left, dotted, "d_1"] \\
R[j] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] & X \arrow[r, "j"'] & D(X)
\end{tikzcd}$$

Since the downward right hand side square is a pullback (as a part of a discrete fibration), the downward left hand side squares are pullbacks, $R[\tilde{j}]$ is isomorphic then to $R \times X$ and consequently the map $p_0.R(p_1) : R[\tilde{j}] \to X$ is a connector. This make $R[j]$ central.

If R is another central relation, the connector p makes the following central squares commute and determine internal functors:

$$\begin{tikzcd}
X \times X \arrow[r, dotted, "s_0 \times X"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & R \times X \arrow[r, shift left, "r_1 \times X"] \arrow[r, shift right, "(r_0,p)"'] \arrow[d, shift right, "p_R"'] \arrow[d, shift left, "(p,p_X)"] & X \times X \arrow[r, dotted, "\tilde{j}"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & D_1(X) \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
X \arrow[r, dotted, "s_0"'] & R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_0"'] & X \arrow[r, dotted, "j"'] & D(X)
\end{tikzcd}$$

On the other hand, the dotted arrows on the left determine a discrete fibration, while the composition by s_0 (resp. by $s_0 \times X$) equalizes the horizontal arrows. Accordingly both maps $j.r_0$ and $j.r_1$ are the classifier of this dotted discrete fibration, and are consequently equal. Accordingly $R \leq R[j]$. $\square$

In this way we obtain a characterization of abelian objects in $\mathbb{C}$, where, classically an object is called abelian when the indiscrete equivalence relation ∇_X is central:

Corollary 4.1. *Let the object X have an action groupoid. Then the following conditions are equivalent:*
1) the object X is abelian in $\mathbb{C}$
2) $d_0 = d_1$ (in other words the action groupoid $D_\bullet(X)$ is absolutely disconnected)

Proof. Suppose X abelian. Then its centre is the coarse relation ∇_X, and according to the previous proposition $R[j] = \nabla_X$. It is the case if and only if $j.p_0 = j.p_1 : X \times X \rightrightarrows X \to D(X)$. Now, considering the following diagram:

$$\begin{tikzcd} X \times X \arrow[r, "\tilde{j}"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & D_1(X) \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ X \arrow[r, "j"'] & D(X) \end{tikzcd}$$

we have always $d_i.\tilde{j} = p_i.j$. So that X is abelian if and only if $d_0.\tilde{j} = d_1.\tilde{j}$. But the pair $(s_0, \tilde{j})$ is jointly strongly epic. Since d_0 and d_1 are clearly equalized by s_0, this last equality holds if and only if $d_0 = d_1$. $\square$

5 Essentially affine categories.

A category $\mathbb{C}$ is essentially affine [5] when it admits pullbacks of split epimorphisms, pushouts of split monomorphisms and is such that, given any commutative square of split epimorphisms:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\ Y \arrow[r, "v"'] \arrow[u, shift right, "s"'] & Y' \arrow[u, shift right, "s'"'] \end{tikzcd}$$

the downward square is a pullback if and only if the upward square is a pushout. A pointed finitely complete category $\mathbb{C}$ is essentially affine if and only if it is additive. The slice categories of any finitely complete category are essentially affine. On the other hand, any essentially affine category is necessarily protomodular and Naturally Mal'cev in the sense of [10]. In the protomodular context, this last condition exactly means that any object is abelian.

We showed in [2] that when the category $\mathbb{C}$ is additive, then, for each object X, the action groupoid structure is nothing but the canonical (internal) abelian group structure on X, namely:

$$\begin{tikzcd} X \times X \arrow[r, shift left=2, "d"] \arrow[r, "p_1"] \arrow[r, shift right=2, "p_0"'] & X \arrow[r, shift right, "\tau_X"'] & 0 \arrow[l, shift right, "\alpha_X"'] \end{tikzcd}$$

where $d = p_1 - p_0$. In the same order of ideas, we have:

Proposition 5.1. *Any essentially affine category* $\mathbb{C}$ *with a terminal object* 1 *is action representative.*

Proof. Given any object X let us consider the following pushout of the split monomorphism s_0 along the terminal map:

$$\begin{tikzcd} X \times X \arrow[r, "d"] \arrow[d, dotted, shift right=2, "p_0"'] \arrow[d, dotted, shift left=2, "p_1"] & \bar{X} \arrow[d, dotted, shift right] \\ X \arrow[u, "s_0"] \arrow[r] & 1 \arrow[u, shift right, "e"'] \end{tikzcd}$$

The object $\bar{X}$ being pointed by e (and abelian), it is canonically endowed with an internal (abelian) group structure, and the map d determines an internal functor $d_\bullet : \nabla_X \to \bar{X}$. Moreover the downward squares are pullbacks and this functor is a dicrete fibration. Let us show that this dicrete fibration makes the group strucure on $\bar{X}$ be the action groupoid of X. So consider any other discrete fibration $k_\bullet$:

$$\begin{tikzcd} X \times X \arrow[r, "k_1"'] \arrow[rr, bend left, "d"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & W_1 \arrow[r, dotted, "\check{k}_1"'] \arrow[d, shift right=2, "w_0"'] \arrow[d, shift left=2, "w_1"] & \bar{X} \arrow[d, shift right] \\ X \arrow[u] \arrow[r, "k_0"'] & W_0 \arrow[u] \arrow[r, dotted, "\tau"'] & 1 \arrow[u, shift right, "e"'] \end{tikzcd}$$

We must explicit a unique dotted factorization. Clearly $\check{k}_0$ is the terminal map τ. Since $k_\bullet$ is a discrete fibration, the downward left hand side squares are pullbacks, and consequently the upward left hand side square is a pushout. Accordingly there is a unique map $\check{k}_1$ such that $\check{k}_1.k_1 = d$ and $\check{k}_1.s_0 = e.\tau$. It is easy to check that this diagram is actually underlying an internal functor $\check{k}_\bullet : W_\bullet \to \bar{X}$. $\square$

References

[1] F. Borceux and D. Bourn, Mal'cev, protomodular, homological and semi-abelian categories, Kluwer, *Mathematics and its applications*, vol. **566**, 2004.

[2] F. Borceux and D. Bourn *Split extension classifier and centrality*, Preprint 2005.

[3] F. Borceux, G. Janelidze and G.M. Kelly *On the representability of actions in a semi-abelian category*, to appear in Theory and Applications of Categories, 2005.

[4] D. Bourn, *The shift functor and the comprehensive factorization for internal groupoids*, Cahiers de Top. et Géom. Diff. Catégoriques, **28**, 1987, 197-226.

[5] D. Bourn, *Normalization equivalence, kernel equivalence and affine categories*, Springer Lecture Notes Math. 1488, 1991, 43-62.

[6] D. Bourn, *Aspherical abelian groupoids and their directions*, J. Pure Appl. Algebra, **168**, 2002, 133-146.

[7] D. Bourn, *Action groupoid in protomodular categories*, Cahiers de Top. Géom. Diff. Catégoriques, **x**, 2005,

[8] D. Bourn and M. Gran, *Centrality and normality in protomodular categories*, Th. and Applications of Categories, **9**, 2002, 151-165.

[9] A. Carboni, M.C. Pedicchio and N. Pirovano, *Internal graphs and internal groupoids in Mal'cev categories*, CMS Conference Proceedings, **13**, 1992, 97-109.

[10] P.T. Johnstone, *Affine categories and naturally Mal'cev categories*, J. Pure Appl. Algebra, **61**, 1989, 251-256.

Université du Littoral, Laboratoire de Mathématiques Pures et Appliquées
Bat. H. Poincaré, 50 Rue F. Buisson BP 699, 62228 Calais - France.
bourn@lmpa.univ-littoral.fr

Key words : Protomodular categories; representation of actions; internal groupoids; abelian objects; central relations and center.
[2000]primary: ; secondary: .