From jets to synthetic differential geometry

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Abstract

La notion de *jet* au sens d'Ehresmann est rappelée et il est observé que l'axiome de Kock–Lawvere de la géométrie différentielle synthétique revient essentiellement à assurer la représentabilité du foncteur *jets*.

Let us make our life easy and let us consider first a surface $\mathcal{V}$ in the three dimensional real space $\mathbb{R}^3$: a sphere, a cylinder, a torus, whatever you like. Provided the equation (of whatever type) of that surface is sufficiently differentiable, all of us know how to define the tangent plane to $\mathcal{V}$ at some point $P \in \mathcal{V}$. This tangent plane is some plane in $\mathbb{R}^3$, which "touches soflty" $\mathcal{V}$ at the point P, a property that we can express precisely in terms of derivatives. Of course, the tangent plane at P does not live *in* the surface $\mathcal{V}$: it lives *outside of it*, in the surrounding space $\mathbb{R}^3$.

So, the tangent plane to the surface is something that we describe a priori using the embedding of the 2-dimensional surface $\mathcal{V}$ in the 3-dimensional surrounding space $\mathbb{R}^3$. But the development of modern differential geometry has rapidly led to consider surfaces as objects "existing on their own" and not necessarily as subsets of some space $\mathbb{R}^n$. For example, all of us have studied the projective plane $\mathbb{P}_2(\mathbb{R})$ and certainly, we consider it as a very decent and interesting two-dimensional space. But I am sure that very few of us have ever considered the possible presentation of the projective plane as – for example – a 2-dimensional surface embedded in the six dimensional space $\mathbb{R}^6$. The projective plane exists by itself, is worth to be studied for itself, independently of whatever more or less natural or artificial embedding in some space $\mathbb{R}^n$. All of us agree on this.

In modern differential geometry, a surface is thus simply a space which, locally, looks like a piece of the ordinary plane $\mathbb{R}^2$. More precisely, a 2-dimensional manifold – or a presentation of it – is a topological space $\mathcal{V}$ provided with an open covering $\mathcal{V} = \bigcup_{i \in I} V_i$, and for each piece V_i of that covering, a homeomorphism

$$\varphi_i \colon V_i \longrightarrow U_i \subseteq \mathbb{R}^2$$

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where U_i is now an open subset of the plane $\mathbb{R}^2$. For example, the sphere can be reconstructed by deforming two open discs of the real plane and glueing them together along some neighbourhood of the equator. The "quality of this glueing" is known as the class of differentiability of the manifold. More precisely, given two pieces V_i and V_j of the covering, we get a composite function

$$\mathbb{R}^2 \supseteq U_i \supseteq \varphi_i(V_i \cap V_j) \xrightarrow{\varphi_i^{-1}} V_i \cap V_j \xrightarrow{\varphi_j} \varphi_j(V_i \cap V_j) \subseteq U_j \subseteq \mathbb{R}^2;$$

when all these "transitions functions" are k times continuously differentiable, the manifold $\mathcal{V}$ is said to be of class $\mathcal{C}^k$. In this note, I shall only consider manifolds of class $\mathcal{C}^\infty$. In an analogous way, switching back to the open subsets $U_i \subseteq \mathbb{R}^2$, one defines the class of differentiability of a function $f\colon \mathcal{V} \longrightarrow \mathcal{W}$ between two manifolds – and again in this note, I shall only be interested in functions of class $\mathcal{C}^\infty$.

Of course, in what has just been said, there is no problem at all to replace the open subsets of $\mathbb{R}^2$ by open subsets of $\mathbb{R}^n$, for an arbitrary integer $n \neq 0$: one obtains in that way the notion of n-dimensional manifold. The 1-dimensional manifolds are simply called *curves* and as already indicated, the 2-dimensional manifolds are called *surfaces*.

All right! But now that surfaces are defined on their own, not via an embedding in some surrounding space $\mathbb{R}^n$, we have lost the notion of tangent plane, which should precisely be a plane of $\mathbb{R}^n$, somewhere "outside the surface". In fact, the problem is much richer than that. Let us come back to the case of an ordinary surface embedded in $\mathbb{R}^3$. The tangent plane varies when we pass from one point to another one and the way this tangent plane varies gives us very important information concerning the shape of the surface. In fact, all the tangent planes to the surface can be glued together to produce now a 4-dimensional manifold which gives us full information on the way the tangent plane varies from one point to another one, and thus gives us a lot of interesting information on the shape of the surface. This is the so-called *tangent bundle* of the surface.

All that is nice, but we are still faced with the question: what is the tangent plane at a point P of a surface $\mathcal{V}$, if the surface is defined as a 2-dimensional manifold, independently of any embedding in some surrounding space $\mathbb{R}^n$? Well, in the classical case of a surface embedded in $\mathbb{R}^3$, the tangent plane at P is the set of all tangent vectors to $\mathcal{V}$ at P. Clearly, every tangent vector to $\mathcal{V}$ is also the tangent vector to some curve on $\mathcal{V}$, passing through the point P. But of course, different curves on $\mathcal{V}$ through P can have the same tangent vector at P. A tangent vector to $\mathcal{V}$ at P can thus be defined as an equivalence class of curves on $\mathcal{V}$ passing through P: it suffices to declare two curves equivalent when they have the same tangent vector at P. Good idea ... except that this definition does not make any sense since it uses the notion of tangent vector at P to define the notion of tangent vector at P!

But this difficulty is very easy to overcome. Two curves defined on the surface have the same tangent vector at P precisely when their Taylor developments at P coincide at the order one. So declare equivalent two curves through P on $\mathcal{V}$ which admit the same Taylor development limited at the order 1: the equivalence classes for that equivalence relation are in bijection with the tangent vectors to $\mathcal{V}$ at P. Thus formally, a tangent vector can be defined as being such an equivalence class. Elementary isn't it? Yes, like the egg of Christopher Columbus! But this is the egg – or more precisely the *jet* – of Charles Ehresmann. In the early '50s, Charles Ehresmann introduced the notion of *r-jet* between two manifolds; the case in which I am mainly interested here is that of a *1-jet*. Let me thus explain this general notion of *jet* in the case of a manifold, independently of any embedding in some space $\mathbb{R}^n$.

First, a special case. Consider two functions of class $\mathcal{C}^\infty$

$$f, g\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$$

defined on a neighbourhood of a point $P \in \mathbb{R}^n$. The functions f and g are called 1-*equivalent* at P when their Taylor developments at P, limited at the order 1, are equal. That is

$$f(P) = g(P), \quad \frac{\partial f_i}{\partial x_j}(P) = \frac{\partial g_i}{\partial x_j}(P) \ \ i = 1, \ldots, m, \ j = 1, \ldots, n.$$

This is clearly an equivalence relation and a corresponding equivalence class of $\mathcal{C}^\infty$-functions is called a *1-jet* at P. Why this prefix 1? Simply because we have used a Taylor development limited at the order 1: limiting the Taylor development at the order r yields the notion of *r-jet*.

Let us now switch to the general case and consider two $\mathcal{C}^\infty$-functions

$$f, g\colon \mathcal{V} \rightrightarrows \mathcal{W}$$

between two $\mathcal{C}^\infty$-manifolds of respective dimensions n and m. Suppose furhter that these two functions f, g coincide at some point $P \in \mathcal{V}$:

$$f(P) = Q = g(P) \in \mathcal{W}.$$

In the presentations of $\mathcal{V}$ and $\mathcal{W}$ as manifolds, let us choose open neighbourhoods V of P and W of Q, homeomorphic to open subsets $U \subseteq \mathbb{R}^n$ and $U' \subseteq \mathbb{R}^m$. The following composites are correctly defined on some sufficiently small neighbourhood of $\varphi(P)$

$$\mathbb{R}^n \supseteq U \xrightarrow{\ \varphi^{-1}\ } V \underset{g}{\overset{f}{\rightrightarrows}} W \xrightarrow{\ \varphi'\ } U' \subseteq \mathbb{R}^m$$

and we say that f and g are 1-*equivalent* at P when this is the case of these two composites, which are now $\mathcal{C}^\infty$-functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. It is easily seen that this definition is independent of the various choices which have been made.

> *A corresponding equivalence class of $\mathcal{C}^\infty$-functions is called a* 1-jet *from $\mathcal{V}$ to $\mathcal{W}$, with* domain P *and* target Q.

The definition of a tangent vector is now easy. The real line is trivially a 1-dimensional manifold and we define – or more precisely, Ehresmann defines –

> *A tangent vector at a point P of a manifold $\mathcal{V}$ is a 1-jet from $\mathbb{R}$ to $\mathcal{V}$, with source 0 and target P.*

Notice that somehow "dually", one can consider the jets from $\mathcal{V}$ to $\mathcal{R}$ with source P and target 0: they are called the *covectors* of the manifold and play a very important role in the study of differential forms.

The notion of *r-jet* between manifolds, obtained thus when working with Taylor developments limited at the order r, allows in particular a very elegant and simple way to define the successive derivatives of a $\mathcal{C}^\infty$-function between manifolds and has been largely used by many authors, in particular by René Thom and Andr Weil – for example to study the singularities of manifolds.

But let me switch now to the more recent idea of synthetic differential geometry, which essentially assumes that the notion of jet is representable by some adequate manifold. What does this "representability condition" intend to say?

> *There exists some manifold $\mathcal{D}$ and a point $Q \in \mathcal{D}$, such that, for every manifold $\mathcal{V}$ and every point $P \in \mathcal{V}$, there is a natural bijection between*
>
> - *the 1-jets from $\mathcal{D}$ to $\mathcal{V}$, with source Q and target P;*
> - *the $\mathcal{C}^\infty$-functions $f\colon \mathcal{D} \longrightarrow \mathcal{V}$ such that $f(Q) = P$.*

Categorically, this expresses indeed the representabilty of the functor which associates to a pointed manifold the set of its tangent vectors at the base point.

What is the idea hidden behind such a property? What could such a "representing manifold $\mathcal{D}$" possibly be? Well, when we study a jet

$$\mathbb{R} \longrightarrow \mathcal{V}$$

with source 0 and target P, the only thing which counts is what happens at the neighbourhood of 0. And thus two functions giving rise to the same jet become closer and closer when we consider smaller and smaller neighbourhoods of 0; as one often says "they become infinitely close to each other when we restrict our attention to infinitely small neighbourhoods of 0". Intuitively – if these were existing – the two functions would eventually become equal on the infinitesimal real numbers, that is, on those numbers which are infinitely close to 0.

But is it possible to express in a rigorous way this idea of being "infinitesimal"? Well, a 1-jet is defined via a Taylor development limited at the order 1. Thus in particular, a 1-jet

$$[f]\colon \mathbb{R} \longrightarrow \mathbb{R}$$

with source 0 and target 0 is also the jet corresponding to the function

$$\mathbb{R} \longrightarrow \mathbb{R}, \quad t \mapsto f(0) + tf'(0) = tf'(0).$$

Now look at the square of this jet, that is, the jet corresponding to the square of that function

$$\mathbb{R} \longrightarrow \mathbb{R}, \quad t \mapsto \big(tf'(0)\big)^2.$$

This squared jet is simply 0, because the first two terms of its Taylor development are 0. Thus the notion of jet provides us with a very natural and important situation where definitely non-zero elements – the 1-jets from $\mathbb{R}$ to $\mathbb{R}$ with source 0 and target 0 – turn out to have zero squares. These 1-jets are *nilpotents*.

So if we take seriously the intuitive idea that a tangent vector at $P \in \mathcal{V}$ should simply be a function

$$f\colon D \longrightarrow \mathcal{V}$$

where D is the set of infinitesimal real numbers, we are forced to conclude that in the case of 1-jets from $\mathbb{R}$ to $\mathbb{R}$, with source 0 and target 0 as studied above, we must necessarily have

$$\big(tf'(0)\big)^2 = 0$$

for every infinitesimal number $t \in D$. Since this must be the case for every $\mathcal{C}^\infty$-function f, let us simply choose $f(t) = t$, the identity function, so that $f'(t) = 1$. This leaves us with the condition that necessarily, $t^2 = 0$ for every infinitesimal real number $t \in D$.

This is the idea of an *infinitesimal* number in synthetic differential geometry: when a number is small, its square is even smaller; when a number is very very small, its square becomes almost neglectable; and a number is infinitesimally small when its square becomes *equal* to zero. Moreover, as we have seen, the definition of a 1-jet forces two functions defining the same 1-jet to be equal on these infinitesimal numbers, that is, to have the same Taylor development limited at the order 1. In other words, every function *must* be linear on the set of infinitesimals numbers. This is the so-called *Kock-Lawvere axiom* of synthetic differential geometry.

> *There exists a ring* $\mathbf{R}$*, containing the rational numbers, and such that putting*
>
> $$D = \{r \in \mathbf{R} | r^2 = 0\},$$
>
> *every function* $f\colon D \longrightarrow \mathbf{R}$ *is linear, in a unique way:*
>
> $$\exists! b \in \mathbf{R} \;\; \forall t \in D \;\; f(t) = f(0) + tb.$$

Of course, the element b is called the *derivative* $b = f'(0)$ of f at 0. And the elements of D are called the *infinitesimals*.

Does such an axiom really make sense? At a first look, not really. Simply choose the function

$$f(t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t \neq 0 \end{cases}$$

and then trivially, the Kock–Lawvere axiom cannot hold, except if $D = \{0\}$. So there are no infinitesimal numbers at all, except of course 0 itself. Moreover, if the only infinitesimal number is 0, the uniqueness of b in the Kock–Lawvere axiom forces the existence of a unique element in $\mathbf{R}$: thus $\mathbf{R}$ itself is $\{0\}$. So the whole idea of working with infinitesimals vanishes at once.

But not quite! Because observe that in the trivial argument above, the construction of the counter-example function f uses in an essential way the law of excluded middle:

$$\models (t = 0) \ \text{ or } \ (t \neq 0).$$

And it is well-known that there are very natural universes, particularly well adapted to the study of geometry, whose internal logic does not satisfy the law of excluded middle. Think of the Grothendieck toposes, of the categories of sheaves, of the theory of schemes in algebraic geometry. In all these situations, one no longer handles fixed objects, but – for example – objects which vary continuously along the points of a topological space, like a sheaf of rings instead of a single ring. And two varying elements of such varying objects can very well be equal at the neighbourhood of some point of the space, but different at the neighbourhood of another point. Thus, in the internal logic of such continuously varying situations, the law of excluded middle no longer holds: two elements can be equal from some local point of view, but different from another local point of view. In such contexts, our counter-example to the Kock–Lawvere axiom no longer holds ... and certainly cannot be adapted, because there are very important and natural situations where the Kock–Lawvere axiom holds!

Let me sketch such an example. A ring is a set A provided with some operations, like

$$+ \colon A^2 \longrightarrow A, \quad - \colon A^1 \longrightarrow A, \quad \times \colon A^2 \longrightarrow A.$$

Analogously, a $\mathcal{C}^\infty$-algebra is a set A provided with an operation

$$\tau_f \colon A^n \longrightarrow A$$

for every $n \in \mathbb{N}$ and every $\mathcal{C}^\infty$-function $f \colon \mathbb{R}^n \longrightarrow \mathbb{R}$; the axioms for the operations τ_f are all the equalities valid between $\mathcal{C}^\infty$-funtions on $\mathbb{R}$. One defines easily a Grothendieck topology on the category $\mathcal{A}$ of finitely presentable $\mathcal{C}^\infty$-algebras – or more precisely, on its dual – and one works now in the topos of sheaves on this site. Among all the sheaves, we have a very trivial one: the sheaf whose value at a point $A \in \mathcal{A}$ is simply the algebra A itself.

$$\mathbf{R}(A) = A.$$

This is easily seen to be a sheaf of rings, that is, a ring which varies "continuously" along the objects of the site $\mathcal{A}$. This sheaf admits the more sophisticated description

$$\mathbf{R}(A) = \mathsf{Hom}\big(\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}), A\big) \cong A.$$

Via this description, it is now easy to compute that the corresponding subsheaf of infinitesimals

$$D = \{r \in \mathbf{R} | r^2 = 0\},$$

at some point $A \in \mathcal{A}$, is simply given by

$$D(A) = \mathsf{Hom}\big(\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})/x^2, A\big) \cong \{a \in A | a^2 = 0\}.$$

The quotient by x^2 is thus the classical external operation which, interpreted in the internal logic of our sheaves, yields the nilpotent elements of $\mathbf{R}$. And that ring $\mathbf{R}$ satisfies the Kock–Lawvere axiom, in an even stronger form that what I have explained above, a form which is reminiscent of the notion of Weil algebra.

But there is much more than that! In the presence of the Kock–Lawvere axiom, one can develop *synthetic differential geometry*, that is, a differential geometry based on the consideration of these elements t such that $t^2 = 0$, the so-called *infinitesimals*.

Given an arbitrary object M in our universe, we can consider the object M^D of all functions from D to M, which we think thus as the "tangent vectors to M". This set M^D, together with the natural projection

$$\pi\colon M^D \longrightarrow M, \quad \pi(v) = v(0)$$

is called the *tangent bundle* of M. The "fibre" of this bundle at a point $P \in M$

$$\pi_P(M) = \{v \in M^D | \pi(v) = P\} = \{v\colon D \to M | v(0) = P\}$$

is called the *tangent space* to M at the point P.

Being a *synthetic manifold* – also called an *infinitesimally linear object* – is now defined in terms of the properties of the tangent bundle. Of course, the product of two infinitesimals is an infinitesimal

$$t^2 = 0 \text{ and } s^2 = 0 \;\Rightarrow\; (ts)^2 = 0$$

but in general, there is no reason to have also $ts = 0$. Let us put

$$D_2 = \big\{(t,s) \in D \times D \big| ts = 0\big\}.$$

In D_2, we find certainly the two axis and the two diagonals:

$$(t,0) \in D_2, \quad (0,t) \in D_2, \quad (t,t) \in D_2, \quad (t,-t) \in D_2.$$

More generally, one defines, for every $n \in \mathbb{N}$

$$D_n = \big\{(t_1, \ldots, t_n) \in D^n \big| t_1 \cdot t_2 \cdots t_n = 0\big\}.$$

We are now ready to define the *synthetic manifolds*:

> *An object M is a* synthetic manifold *when given tangent vectors at a point $P \in M$*
>
> $$v_1, \ldots, v_n \in \pi_P(M)$$
>
> *there exists a unique function*
>
> $$f\colon D_n \longrightarrow M$$
>
> *which for every index i, coincides with v_i on the i-th axis.*

It follows then quite easily that the tangent space $\pi_P(M)$ is naturally provided with the structure of an $\mathbf{R}$-module: given two tangent vectors $v, w \in \pi_P(M)$ and the corresponding function $f\colon D_2 \longrightarrow M$ given by the definition of a synthetic manifold, the tangent vector $v + w$ is defined as the composite

$$D \xrightarrow{\ \Delta\ } D_2 \xrightarrow{\ f\ } M$$

where Δ indicates the first diagonal. This generalizes the classical fact that the tangent space to a manifold is naturally provided with the structure of a real vector space.

Most notions and results valid in classical differential geometry can now be translated in this "synthetic" context and generally, the proofs become now very intuitive and simple. For example, a vector field on M is, as you can expect, a section of the tangent bundle: in synthetic differential geometry, this becomes simply a function

$$\sigma\colon M \longrightarrow M^D$$

such that $\pi \circ \sigma = \mathsf{id}_M$. And so on.

But a very striking result is the fact that most intuitive proofs, in terms of infinitesimals, constitute also actual proofs of the corresponding classical results in classical differential geometry. Why? Simply because there exists a very good embedding of the category of paracompact $\mathcal{C}^\infty$-manifolds in the category of manifolds *à la Kock–Lawvere*, considered in the topos of sheaves that I have indicated. This embedding preserves and reflects numerous notions and properties concerning manifolds and thus, when a theorem using these notions can be proved in synthetic differential geometry, it is automatically also a theorem in classical differential geometry.

Synthetic differential geometry is thus somehow based on the fundamental idea of a *jet*, due to Ehresmann, transposed in categorical terms in the convenient context of topos theory. *From differential geometry to category theory*: a slogan which also characterizes the mathematical career of Charles Ehresmann.