Weak Quantum Theory: Formal Framework and Selected Applications

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Abstract. Two key concepts of quantum theory, complementarity and entanglement, are considered with respect to their significance in and beyond physics. An axiomatically formalized, weak version of quantum theory, more general than the ordinary quantum theory of physical systems, is described. Its mathematical structure generalizes the algebraic approach to ordinary quantum theory. The crucial formal feature leading to complementarity and entanglement is the non-commutativity of observables.

The ordinary Hilbert space quantum mechanics can be recovered by stepwise adding the necessary features. This provides a hierarchy of formal frameworks of decreasing generality and increasing specificity. Two concrete applications, more specific than weak quantum theory and more general than ordinary quantum theory, are discussed: (i) complementarity and entanglement in classical dynamical systems, and (ii) complementarity and entanglement in the bistable perception of ambiguous stimuli.

INTRODUCTION

Key Concepts of Quantum Theory: Complementarity and Entanglement

Quantum theory has revolutionized our understanding of the physical world in both scientific and epistemological respects. It was developed in the third decade of the 20th century as a theory describing the behavior of atomic systems. Subsequently, its range of validity turned out to be much wider. Not only are nuclei and elementary particles, more than seven orders of magnitude smaller than atomic systems, governed by quantum theory, but also macroscopic phenomena like superconductivity or superfluidity are successfully described and understood.

The conceptual structure and axiomatic foundations of quantum theory have revealed that it is a logical consequence of some rather simple and plausible basic assumptions, most importantly the non-commutativity of observables. In this framework, classical physics represents the special case in which all observables commute with each other. The axiomatic foundations of quantum theory have shown how deeply rooted some of its apparently bizarre concepts really are. Two basic notions of quantum theory which are, in this sense, most different from our classical understanding of nature, are those of complementarity and entanglement. They were both introduced fairly early in the development of the theory, and much, if not most, of the conceptual insight received from quantum theory is due to their significance.

Complementarity in quantum physcis was proposed by Bohr in 1927 [15] to highlight crucial features of quantum theory (for more details see [27, 33, 34]). Bohr used the concept of complementarity to indicate a relationship between apparently opposing, contradictory notions which in fact should be considered in terms of a relationship of polarity. Complementary features typically exclude each other, but at the same time complement each other mutually to give a complete view of the phenomenon under study. A well-known physical example is the complementarity of non-commuting observables, such as position and momentum, implying both ontic indeterminacies and epistemic uncertainties. From another viewpoint, complementarity arises whenever a system requires a non-Boolean description, which is at variance with a classical Boolean picture.

Entanglement is formally related to complementarity and characterizes the fact that a system in a pure state in general cannot be simply decomposed into subsystems with pure states. In a certain sense, such subsystems do not exist *a priori* but must be generated by appropriate procedures. This has consequences first pointed out by Einstein *et al.* [19]; the term entanglement (German: Verschränkung) itself was coined by Schrödinger in 1935 [43]. Theoretical progress due to Bell [13] and experimental results due to Aspect *et al.* [1] confirmed, beyond any reasonable doubt, the entangled (holistic) character of quantum systems, exhibiting so-called nonlocal (holistic) correlations between subsystems. Popular misconceptions notwithstanding, it is illegitimate to interpret these correlations due to causal interactions between the subsystems.

Why Generalize Ordinary Quantum Theory?

Since the early days of quantum theory, Bohr (and others) entertained the idea that the notions of complementarity and entanglement might be meaningful and important even beyond the realm of the physical systems. For Bohr himself this was clearly motivated by his extraphysical interests, which in fact raised his awareness for complementarity before he transferred it into physics [23]. Among many other examples, he was familiar with the bistable perception of ambiguous stimuli through his psychologist friend Rubin [35]. Together with Bohr's studies of the philosophical writings of Kierkegaard, Høffding and James, this played an important role in the complicated genesis of complementarity in quantum physics.

Although the best formalized examples of complementary pairs of notions refer to pairs of non-commuting observables of a system, Bohr insisted that the significance of complementarity goes beyond this. He often emphasized the fundamental idea of complementarity due to a holistic entanglement of knowledge and action. In this respect, Bohr's complementarity refers to a key element of the pragmatist tradition, the reflective relation between the immediate experience of an object and the awareness of its objectification [28].

It is clear that covering a wide range of phenomena that ultimately even includes topics of psychology and philosophy, or of artistic and religious experience, by formal frameworks such as those of physics and mathematics is highly problematic. If one wants to maintain the rigor of a formal approach one must, therefore, restrict oneself to situations that are formally tractable in a minimal sense, which may be (much) weaker than in physics. This basic idea was first carried out in [10] and led to a well-defined, axiomatic mathematical structure preserving those elements necessary for the notions of complementarity and entanglement independent of their physical embedding and related applications.

The main ingredients of this structure, called weak quantum theory, will be reviewed in the first part of this contribution. Subsequently, two examples will be presented which were to some extent addressed before [10, 8]: (i) specific features of information processing dynamical systems, where weak quantum theory is applied within physics, but beyond ordinary quantum physics, and (ii) the bistable perception of ambiguous stimuli, where weak quantum theory is applied within cognitive science, that is outside physics.

AXIOMATIC FRAMEWORK

The Mathematical Structure of Weak Quantum Theory

A first requirement for the scientific description of any system Σ, physical or otherwise, is its separation from the rest of the universe of discourse. A system Σ is considered as a part of reality in a very general sense, i.e. it can be the object of attention and investigation beyond the realm of quantum physics, possibly even beyond the limitations set by the notion of a material reality. Even though the isolation of parts of reality is often problematic, its possibility, at least in some approximate sense, is a prerequisite for any act of observation. Moreover, it is implicit in the epistemic split between subjects and objects of cognition.

In a second step, a set $\mathscr{A}$ of observables and a set Z of states is assigned to every system Σ. An observable is a property of the system Σ which can be investigated in a given context. Non-trivial observables must exist, whenever Σ has enough internal structure to be a possible object of a meaningful study. To every observable A there should belong a set $specA$ of possible results of an investigation of A.

It must at least be conceivable that the system Σ exists in different states. They should be reflected in different outcomes of observations associated with observables A. The possibility of different states is indispensible for discussing stability criteria for Σ, which has to maintain its identity under "unsubstantial" changes. In addition to such an ontic understanding, the notion of a state also has epistemic aspects, reflecting various degrees of knowledge about the ontic state of Σ [9]. As in ordinary quantum theory, we call a state pure if it encodes maximal information about Σ.

In the following we characterize the general structure of $\mathscr{A}$ and Z. The first property of $\mathscr{A}$ that we formulate is:

Axiom I. To every observable $A \in \mathscr{A}$ belongs a set *specA*, the set of possible outcomes of a "measurement" of A.

Quantum observables can be identified with functions $A : Z \to Z$ on the set of states. This fact, which underlines the active, operational character of observations, can be formulated as:

Axiom II. Observables are (identifyable with) *mappings* $A : Z \to Z$, which associate to every state z another state $A(z)$.

Axiom II implies that observables can be composed as maps on Z, where the map AB is defined by first applying B and then A. We shall assume:

Axiom III. With A and B, AB is an observable as well.

A direct consequence of Axiom III is the *associativity* of the composition of observables:

$$A(BC) = (AB)C \tag{1}$$

Moreover, we can postulate:

Axiom IV. There is a unit observable $\mathbb{1}$ such that $\mathbb{1}A = A\mathbb{1} = A \;\forall\, A \in \mathscr{A}$.

$\mathbb{1}$ is the operation on Z that does not change any state. It corresponds to a proposition which is always true, so $spec\mathbb{1} = \{\text{true}\}$. Axioms I–IV imply that the set of observables has the structure of a *monoid*, also called a *semigroup with unity* or an *associative magma with unity*.

For formal completeness we also need an "impossible" *zero state* $z = o$ and a *zero observable* 0 with $spec0 = \{\text{false}\}$, corresponding to an always false proposition:

Axiom V. There are a zero state o and a zero observable 0 such that

$$\begin{aligned} 0(z) &= o \;\forall\, z \in Z, \\ A(o) &= o \;\forall\, A \in \mathscr{A}, \\ A0 = 0A &= 0 \;\forall\, A \in \mathscr{A}. \end{aligned} \tag{2}$$

In the generalized framework of weak quantum theory there is no evident place for the *addition* of observables. As a consequence, the set of states cannot be presupposed to be convex as in ordinary quantum theory. This relates to the fact that a probability interpretation is not feasible (but can be implemented, see below).

There is no reason to assume commutativity, $AB = BA$, for all $A, B \in \mathscr{A}$. Rather there will be both commutative (compatible) and non-commutative (incompatible) pairs of observables, depending on whether $AB = BA$ or $AB \neq BA$. This implies that the monoid structure of $\mathscr{A}$, however general, contains the notions of complementarity and entanglement as essential features of quantum theory.

In addition to the structure defined by axioms I–V, we introduce *propositions* P in the set of observables $\mathscr{A}$, which play a distinguished role. A proposition $P \neq 0, \mathbb{1}$ is an observable whose outcome is either true or false:

$$specP = \{\text{true}, \text{false}\} \text{ for } P \neq 0, \mathbb{1} \tag{3}$$

Moreover, to every proposition P there must be a negation $\overline{P}$, providing "false" if and only if P provides "true". We give a few rather evident axioms assumed to hold for propositions:

Axiom VIa.

$$\begin{aligned} P^2 &= P, \\ \overline{\overline{P}} = P,\; \overline{\mathbb{1}} &= 0, \\ P\overline{P} = \overline{P}P &= 0. \end{aligned} \tag{4}$$

For *compatible* propositions $P_1, P_2, P_1P_2 = P_2P_1$ we can define a *conjunction*

$$P_1 \wedge P_2 = P_2 \wedge P_1 = P_1P_2 \tag{5}$$

and an *adjunction*

$$P_1 \vee P_2 = \overline{\overline{P}_1\overline{P}_2} = P_2 \vee P_1 \tag{6}$$

with the usual properties.

The meaning of P as verification is postulated by:

Axiom VIb. If $P(z) \neq o$, then $P(z)$ is a state in which P is true with certainty.

Finally, we formulate an axiom replacing the spectral theorem of ordinary quantum theory. Every observable A should be equivalent to a set of mutually exclusive propositions. More precisely, let A be an observable and let $\alpha \in specA$. A_α denote the proposition that the outcome of a measurement of A is $\alpha \in specA$. Then we have:

Axiom VIc.

$$A_\alpha A_\beta = A_\beta A_\alpha = 0 \text{ for } \alpha \neq \beta, \quad AA_\alpha = A_\alpha A, \quad \bigvee_{\alpha \in specA} A_\alpha = \mathbb{1}. \tag{7}$$

A and B are compatible if and only if A_α and B_β are compatible for all $\alpha \in specA$ and $\beta \in specB$. In general, incompatible observables do not have simultaneously definite values, and the associated states are not dispersion-free.

Comparison with Ordinary Quantum Theory

Although weak quantum theory as defined by axioms I–VI is considerably more general than ordinary quantum theory, they share the following two characteristic features.

- Incompatibility and complementarity arise due to the non-commutativity of the multiplication of observables, implying that the associated states are generally not dispersion-free.
- Holistic correlations and entanglement arise due to the incompatibility of observables. In particular, entanglement arises if for a composite system observables pertaining to the system as a whole are incompatible with observables of its subsystems.

In the latter context, it should be emphasized that weak quantum theory refers to the description of the system as a whole. Any identification of parts or subsystems implies a specific choice of representation in terms of partial monoids. This choice remains open in the general framework, where the absence of a vector space structure (in particular of a Hilbert space structure) implies that there is no tensor product construction for the set of observables of a composite system. In general, we can only expect:

$$\mathscr{A} \supset \mathscr{A}_1 \times \mathscr{A}_2, \; Z \supset Z_1 \times Z_2, \tag{8}$$

$$\mathscr{A}_1(Z_1) \subset Z_1, \; \mathscr{A}_2(Z_2) \subset Z_2. \tag{9}$$

A similar remark applies to the specific form of the dynamical evolution of (sub-)systems in weak quantum theory. The dynamics of a system is generally described by a one-parameter (semi-)group of endomorphisms. The process generating subsystems (e.g., by measurement) and the dynamics of interacting subsystems depends on details of the considered system and its decomposition. In particular, the Schrödinger equation of ordinary quantum theory is not included in weak quantum theory.

In addition, a number of other important characteristics of ordinary quantum theory are not shared by weak quantum theory.

- There is no quantity like Planck's action h which in ordinary quantum theory quantifies the degree of non-commutativity of two given observables. This indicates that, in the generalized theory, complementarity and entanglement are not restricted to a particular degree of non-commutativity.
- Weak quantum theory does not include a Hilbert space representation. Therefore, the compatibility or incompatibility of observables canot be characterized in terms of shared eigenfunctions. As a consequence, complementarity cannot be distinguished as maximal incompatibility, where observables have no eigenfunction at all in common. Hence, the notions of incompatibility and complementarity are synonymous in weak quantum theory.
- Since the addition of observables is not defined in weak quantum theory, we do neither have a von Neumann algebra of observables nor Heisenberg uncertainty relations. There is no convex set of states, there are no linear expectation value functionals, and there is no probability interpretation (no Born rule). Probability distributions on the sets *specA* do not occur and are not calculable in weak quantum theory. As a matter of fact, the concept of probability will be absent whenever a quantitative valuation of observables of a system is inappropriate or impossible.

- There is no way to generalize Bell's inequalities up to the general framework of weak quantum theory, and there is no way to argue that complementarity and indeterminacy in weak quantum theory are of ontic rather than epistemic nature. On the contrary, one would expect them to be of epistemic origin in many cases, for instance, due to incomplete knowledge of the system or uncontrollable perturbations by observation. An ontic interpretation is clearly appropriate if the state of the system is dispersion-free. If a state is dispersive, an ontic interpretation is still appropriate for pure (individual) states. Otherwise, if dispersive states are mixed (statistical), the proper interpretation is epistemic.

Axioms I–VI can be regarded as minimal requirements for a meaningful general theory of observables and states of systems showing complementarity and entanglement. Between the weak version of quantum theory and its ordinary version, there are intermediate theories which can be obtained by enriching the system of axioms stepwise. Let us first discuss enrichments of the propositional axiom VI.

One evident option is to postulate that the conjunction and adjunction of propositions is also defined in the less intuitive case of *incompatible* P_1 and P_2 such that propositions $P_1 \wedge P_2$ and $P_1 \vee P_2 = \overline{\overline{P}_1 \wedge \overline{P}_2}$ always fulfil the conditions of axiom VIa. In addition, it is natural to postulate

$$\begin{aligned} P_1 \wedge (P_1 \vee P_2) &= (P_1 \vee P_2) \wedge P_1 = P_1, \\ P_1 \vee (P_1 \wedge P_2) &= (P_1 \wedge P_2) \vee P_1 = P_1 \wedge P_2. \end{aligned} \tag{10}$$

The stronger distributivity condition

$$\begin{aligned} P_1 \wedge (P_2 \vee P_3) &= (P_1 \wedge P_2) \vee (P_1 \wedge P_3), \\ P_1 \vee (P_2 \wedge P_3) &= (P_1 \vee P_2) \wedge (P_1 \vee P_3), \end{aligned} \tag{11}$$

is not even satisfied in ordinary quantum theory. If every propositional subsystem generated by two compatible propositions with $P_1 \wedge P_2 = P_1$ and their negations are Boolean, then (modulo some technical complications) the propositional system is already isomorphic to a system of orthogonal projectors in a Hilbert space [38, 47]. This Boolean property does not follow from axioms I – VI.

For a probability interpretation of states, one does not loose much by assuming $specA \in \mathbb{C}$, because it is plausible that the set of outcomes of A can be mapped onto the complex numbers in a one-to-one way. Assuming this, introducing a probability interpretation amounts to postulating for every $z \neq 0$ the existence of an expectation value functional

$$\begin{aligned} E_z : \mathscr{A} \to \mathbb{C}, \\ A \mapsto E_z(A) \in \mathbb{C}, \end{aligned} \tag{12}$$

with

$$E_z(\mathbb{1}) = 1. \tag{13}$$

The existence of an expectation value functional has far reaching consequences

- Addition of observables and multiplication of observables with complex numbers can now be defined by postulating
$$E_z(\alpha A + \beta B) = \alpha E_z(A) + \beta E_z(B) \tag{14}$$
for all E_z.
- As the mean value of a probability distribution, $E_z(A)$ has to obey reality and positivity conditions. The only evident way to achieve this is the introduction of a star-involution $A \to A^*$ (A^*A has to be self-adjoint also if A and A^* do not commute) implying a C^*-algebra of observables.
- The set of all expectation value functionals will be convex such that pure states can be defined as in ordinary quantum theory.

For the discussion of concrete applications, it has to expected that the full generality of weak quantum theory has to be restricted in order to formulate a proper level between weak and ordinary QT

SELECTED APPLICATIONS

The examples that we are going to discuss in the following refer to (i) the temporal evolution of (nonlinear) dynamical systems, briefly addressed under the notion of information dynamics, and (ii) to a model for the bistable perception of ambiguous stimuli inspired by the quantum Zeno effect. Their proper treatment requires specific details not included in the minimal framework of weak quantum theory as specified in the axioms discussed above. Most importantly, the addition of observables is defined and a probability interpretation is adopted in both examples.

Their difference from ordinary quantum theory is best characterized by their particular way to implement non-commuting observables. While in ordinary quantum theory Planck's action represents a *universal* commutator of *canonically conjugate* observables, complementarity in dynamical systems is expressed by a *system-specific* commutator, the dynamical entropy, of different types of *generators* of the dynamics. Our cognitive example is more general in the sense that no particular commutator needs to be specified at all.

Information Dynamics

Complementarity of Liouville Dynamics and Information Dynamics

Generalizing earlier work by Misra and colleagues [29, 30], an information theoretical description of chaotic systems (including K-systems) was found to provide a commutation relation between the Liouville operator L for such systems and a suitably defined information operator M [11]. The definition of L is, as usually, given by

$$L\,\rho = i\frac{\partial}{\partial t}\,\rho \tag{15}$$

where L acts on distributions ρ representing the states of a system in a probability space (not in a Hilbert space). The dynamics of the system is thus characterized in terms of an automorphism A.

The continuous spectrum of M derives from the time-dependent information $I(t)$ which can be gained by measuring particular properties of a system at time t in comparison with its predicted properties:

$$M\,\rho = I(t)\,\rho = (I(0) + Kt)\,\rho\ . \tag{16}$$

K is the Kolmogorov-Sinai entropy, a statistical dynamical invariant of the system. It is experimentally available by Grassberger-Procaccia type algorithms [22]. $K > 0$ only for chaotic systems with intrinsically unstable dynamics. In an information theoretical interpretation [44], K characterizes the rate at which the system generates information along its unstable manifolds. Kt is the information generated by the system after a time interval $[0,t]$. This means that the accuracy of a prediction decreases with increasing prediction time.

In simple cases, the commutator of L and M is straightforwardly given by the rate of information generation, namely the Kolmogorov-Sinai entropy (cf. subsequent subsection and [3] for more details):

$$i[L,M] = K\mathbb{1}\ . \tag{17}$$

The two operators L and M commute if the considered system does not generate information, i.e., if it is intrinsically stable and $K = 0$. If $K > 0$, the dynamical descriptions due to L and M are different with respect to the prediction of a future state of the system. This is a consequence of the increasing uncertainty of a predicted state of the system as time proceeds. Whenever $K > 0$, the state $\rho(t)$ of a system cannot be predicted as accurate as initial conditions have been measured or otherwise fixed at $t = 0$.

The commutation relation of L and M resembles corresponding commutation relations in ordinary quantum theory, but there are differences. First of all, since K is explicitly system- and parameter-dependent (i.e. highly contextual), the "degree" of non-commutativity of L and M is not universally determined. This situation is at variance with conventional quantum mechanics with h as a universal commutator. Moreover, K is a statistical quantity specifying the average flow of information in chaotic systems, while h is a non-statistical constant of nature.

As a consequence of (17), L and M provide complementary modes of description. There are two basic features of this complementarity. (i) While L refers to a global description of the system as a whole, M refers to a description with respect to its unstable manifold(s) only. (ii) While a description in terms of L is time-reversal symmetric (reversible), this symmetry is broken by a description in terms of M, thus implying irreversibility.

Complementary Partitions and Non-Hyperbolic Manifolds

The definition of M depends decisively on the choice of a state space partition. Since the Kolmogorov-Sinai entropy K is defined as the supremum of the dynamical entropy $H(P,A)$ with respect to all possible partitions P,

$$K = \sup_P H(P,A), \text{ where } H(P,T) = \lim_{n\to\infty} \frac{1}{n} H(P \vee AP \vee ... \vee A^{n-1}P), \tag{18}$$

the proper choice for M is the partition which maximizes the dynamical entropy. This partition is called a generating partition. It is defined such that boundaries between its cells are mapped onto each other under the dynamics of the system. This entails that the cells are constructed such that correlations between points within cells are maximized and correlations between points in different cells are minimized. Put differently, epistemic states defined by the cells of a generating partition are stable under the dynamics. Well-known examples of generating partitions are Markov partitions.

As has been shown recently [12], different partitions (and associated descriptions) are complementary, or incompatible, with respect to each other if they are not generating. This formal result demonstrates how complementary epistemic state space descriptions can arise although the underlying ontic description is purely classical, i.e. rigorously commutative.

Due to a theorem by Bowen [16], all hyperbolic systems have generating partitions; there are even possibilities to construct them explicitly (e.g., by shadowing [21] or by template analysis [39]). Nevertheless, generating partitions are known only for a few specific systems such as the logistic map or the Henon map. The basic fact distinguishing hyperbolic systems is that they can be represented in terms of a direct sum $E = E^s \oplus E^u$ of stable and unstable manifolds with respect to a fixed point x:

$$E^s(x) = \{y \in \mathbb{R}^m : f^n(y) \to x \text{ for } n \to \infty\} \tag{19}$$

$$E^u(x) = \{y \in \mathbb{R}^m : f^{-n}(y) \to x \text{ for } n \to \infty\}. \tag{20}$$

In hyperbolic systems, trajectories belonging to E^s and E^u can only intersect transversely. If a system is not hyperbolic (e.g. in case of homoclinic or heteroclinic tangencies), stable and unstable manifolds are not strictly separable but entangled, and a direct sum decomposition is not possible.

It is highly nontrivial to find or approximate generating partitions for non-hyperbolic systems (see [48]). Since most systems pf physical or biological interest are (or must be assumed to be) non-hyperbolic, this is a serious obstacle to a suitable definition of robust epistemic states for such systems. There is considerable current interest in these and related problems, and there is much to be explored. It may be conjectured that the complementarity of partitions is basically related to another, more fundamental, complementarity of stable and unstable manifolds in non-hyperbolic systems.

Practical applications of this theoretical topic can be found in the discussion of emergent levels of descriptions as outlined in [6]. For instance, cognitive states are most often defined in terms of equivalence classes of (fine-grained) neuronal states [20], thus utilizing the strategy of partitioning the underlying (neuronal) state space. The partitioning is usually chosen by empirical plausibility; the robustness of the resulting cells and, therefore, of cognitive states associated with them is usually not addressed. If it is true that biological systems are generically non-hyperbolic, so that generating partitions are difficult to obtain (or even unobtainable), this could explain the observation that psychological descriptions are generically complementary, i.e. incompatible, with each other.

Non-Boolean Logic of Dynamical Systems

A general interpretation of the commutation relation between L and M in terms of propositions is possible in terms of a lattice theoretical analysis. Analogous to the work of Birkhoff and von Neumann [14], which pioneered the non-Boolean logic of quantum theory, such an analysis provides basic logical features of information processing systems. Following an idea by Krueger [26], it was shown that the temporal evolution of information processing systems is governed by a non-Boolean logic [4]. More precisely, the propositional lattice characterizing such a logic is complemented but not distributive. This non-distributivity shows a subtle but important difference as compared with the non-distributivity due to ordinary quantum theory.

A fundamental feature of lattices as mathematical structures is the duality of their properties. Formally this means that each true proposition is transformed into another true propositon by exchanging the dual operations defined in lattice theory. It turns out that the difference between the ordinary quantum theoretical non-distributivity and the non-distributivity due to information processing systems precisely accounts for this duality. While ordinary quantum theory provides non-distributivity relations of the form

$$\begin{aligned} & a > (a \wedge b) \vee (a \wedge b') \\ \wedge \quad & b > (b \wedge a) \vee (b \wedge a') \end{aligned} \tag{21}$$

(a' ist the complement of proposition a, b' is the complement of proposition b), information processing systems satisfy non-distributivity relations of the form:

$$\begin{aligned} & a < (a \vee b) \wedge (a \vee b') \\ \vee \quad & b < (b \vee a) \wedge (b \vee a')\ . \end{aligned} \tag{22}$$

In contrast to (21), its dual version (22) requires only one of the two inequalities to be satisfied. A detailed analysis [4] shows that this is indeed crucial for the non-distributivity of information processing systems. It is therefore possible to consider the logics of ordinary quantum systems and of information processing systems as dual aspects of one underlying non-distributive lattice.

Temporal Order Threshold in Perception

There is an interesting relation between (17) and another commutation relation between L and a time operator T introduced by Misra and colleagues [29, 30]:

$$i[L,T] = \mathbb{1}\ . \tag{23}$$

T is well-defined if $K > 0$. Since L, in addition to its role as an evolution operator as in (15), can also be interpreted as an energy difference due to $L\rho = [H,\rho]$ for a Hamiltonian H, (23) indicates a complementarity between energy and time for chaotic systems. This suggests the idea of a temporal entanglement for such systems. This entanglement can be interpreted as a temporal nonlocality [31] due to a coarse grained phase space; for a more detailed discussion see [3]. It should be emphasized that this nonlocality is epistemic and must not be confused with the ontic nonlocality of ordinary quantum theory.

In a recent paper [7] it has been suggested to apply this type of entanglement to experimental observations concerning the perception of the temporal sequence of successively presented stimuli. A number of corresponding studies (for reviews see [41, 40]) reveal the existence of a temporal window with a duration of approximately 20-40 msec in which individual stimuli can be discriminated but their sequence cannot be assigned properly. The size of this window, the so-called order threshold, is modality-independent and has been suggested to represent the duration of an extended "now" or "presence" [41]. It may be regarded as a cognitive example of temporal nonlocality or temporal holism.

Since there is a lot of evidence that many brain processes are chaotic [46], their Kolmogorov-Sinai entropy K is positive, and it is tempting to interpret the inverse of K in terms of the duration of the order threshold. In this spirit, a "temporal double slit" scenario has been proposed [7] on the basis of the complementarity of L and T as in (23), where L corresponds to an energy difference or, equivalently, frequency difference $\Delta\nu$. If one considers the numerical coincidence of $\Delta t \approx 30$ msec for the order threshold and the $\Delta\nu \approx 40$ Hz for the (γ-band) neuronal oscillations in terms of a generalized time-energy uncertainty relation, this leads to an interesting empirical prediction. According to such an uncertainty relation, the distribution of experimentally determined values of Δt should change in a well-defined manner if the distribution of $\Delta\nu$ γ-band frequencies is varied.

Further experimental work will be required to check this prediction. Another application of weak quantum theory to cognitive processes, which has already received empirical confirmation, will be described in the following section.

Necker-Zeno Model

Bistable Perception of Ambiguous Stimuli

Bistable perception arises whenever a stimulus can be interpreted in two different ways with approximately equal plausibility. A very simple and often investigated example of bistable perception is the so-called Necker cube. A grid of a cube in two-dimensional representation can be perceived as a three-dimensional object in two different perspectives, either as a cube seen from above or from below. The perception of the Necker cube switches back and forth between these two possible representations spontaneously and inevitably.

Recently [8] it was proposed to describe bistable perception with the formalism of a two-state system, where the two basis states correspond to the two different ways to represent the stimulus. Measurement is considered as the mental process determining in which way the figure is perceived. The switching between the different perceptions corresponds to transitions between the two states, which are eigenstates of the operator representing a particular perception and unstable under the time evolution of the system.

Such a description of bistable perception employs a non-minimal version of weak quantum theory with a linear structure and a two-dimensional linear state space. This version is relatively close to the structure of the full quantum theory used in physics. This does not imply, however, to understand bistable perception as a quantum phenomenon in the sense that the related brain processes are ordinary quantum processes (for an overview of corresponding approaches see [5]). Rather, we will discuss the quantum-like behavior of bistable perception as a result of the truncation of an extremely complicated system to a two-state system, into which the effect of many uncontrolled variables and influences is lumped in a global way.

For this purpose, we consider a system with a linear state space spanned by two states ψ_1 and ψ_2, neither of which is an eigenstate of the Hamiltonian H generating the evolution matrix $U(t) = \mathrm{e}^{\,\mathrm{i}Ht}$. If the system is initially in state ψ_1 and allowed to evolve freely according to $U(t)$, then its state will oscillate between ψ_1 and ψ_2. This oscillation can be slowed down by increasing the frequency at which the system is measured, asking whether it still resides in its initial state. In the limit of continuous measurement, the evolution of the system can be completely suppressed. In ordinary quantum mechanics, this phenomenon is known as the quantum Zeno effect. Its possible cognitive significance was indicated previously by Ruhnau [42] as well as by Stapp [45]. Concrete and quantitative predictions for cognitive systems were for the first time proposed in [8].

Complementarity of Observation and Dynamics

The quantum Zeno effect was originally introduced as the quantum Zeno "paradox" by Misra and Sudarshan [32] for the decay of unstable quantum systems. Its key meaning is that repeated observations of such systems decelerate the time evolution which they would undergo without observations, e.g. their decay. The metaphor "a watched pot never boils" paraphrases this behavior in the limit of continuous observation.

The situation addressed in the following refers to a quantum system oscillating between two non-stationary states. For this purpose, consider a two-state system with the following properties (the results apply to more general systems as well):

- An observation is represented by the operator

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{24}$$

  Immediately after an observation, the system will be in one of the corresponding eigenstates

$$\psi_1 = |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{or} \qquad \psi_2 = |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{25}$$

- Both σ_3-eigenstates may also be represented by their projection operators

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \tag{26}$$

- Without loss of generality, the Hamilton operator giving rise to transitions of the system can be written as

$$H = g\sigma_1 = g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{27}$$

  where g is a coupling constant. Hence, the unitary operator of time evolution is represented by

$$U(t) = \mathrm{e}^{\mathrm{i}Ht} = \begin{pmatrix} \cos gt & \mathrm{i}\sin gt \\ \mathrm{i}\sin gt & \cos gt \end{pmatrix} . \tag{28}$$

Note that σ_1 and σ_3 do not commute, thus motivating the framework of weak quantum theory, since observation and dynamics are complementary.

- In this model, ΔT defines the time interval between two successive observations, and T defines the time scale after which the state has changed with 50% probability. It is assumed that $T/\Delta T = N \gg 1$. (The cognitive interpretation of ΔT and T will be discussed in the next subsection.)

The experimentally accessible quantities ΔT and T can be related to the evolution of the system under the hypothetical condition that no observations are performed at all [8]. In this empirically inaccessible case, the time evolution is solely given by $U(t)$ and the state of the system oscillates between the two eigenstates $|+\rangle$ and $|-\rangle$ with period t_0. As shown in [8], the Necker-Zeno model predicts that these three time scales are related by

$$t_0 = \frac{\pi}{4\sqrt{\ln 2}}\sqrt{T\,\Delta T} \approx \sqrt{T\,\Delta T}. \tag{29}$$

The derivation of this relation depends on two arbitrary choices: T is an expectation value determined from the condition that the probability of state flipping is $1/2$, and t_0 is determined from the condition that the oscillating state is a superposition of eigenstates of σ_3 with equal coefficients. Even if these conditions are varied, the general result remains unchanged. It entails the following two predictions:

1. As long as the time interval ΔT between two observations is non-zero, the states will spontaneously switch into each other after an average time T which is large compared to ΔT and t_0.
2. The relation between the time scales T, ΔT and t_0 is given by (29).

Cognitive Time Scales

In order to assign significance to the time scales T, ΔT and t_0 in terms of the process associated with bistable perception, corresponding cognitive time scales have to be identified. As discussed in detail in [8], there are natural choices.

One of the fairly invariant patterns in the perception of ambiguous stimuli is a remarkably stable rate of reversals for individual subjects, corresponding to a "mean first passage time" between 1 and 15 seconds. The duration after which the stimulus orientation spontaneously reverses was found to be gamma-distributed around a maximum of about 3 seconds [18] (recent results indicate alternative distributions [17]). This time scale can straightforwardly be attributed as the extended oscillation period T due to observations. Its ubiquity and basic significance (for more examples see [8]) suggests that $T \approx 3$ sec is also significant for cognitive processes beyond the bistable perception of ambiguous stimuli.

A reasonable estimate for the time between observations in the sense of the quantum Zeno effect, ΔT, is difficult to obtain from the phenomenology of bistable perception. It has to satisfy at least one condition: the perceptual system must be able to assign a temporal sequence to successive events, i.e. observations. In this respect, the order threshold in the perception of sequential stimuli [41, 40] is significant.

As indicated above, the sequential order of successive stimuli with a temporal interval smaller than approximately 30 msec cannot be properly recognized. This suggests to use the order threshold as a generic lower bound for the time ΔT between successive observations in the Necker-Zeno model. Observations with smaller temporal distance cannot be temporally ordered. The fact that the order threshold is modality-independent and its fundamental significance for the binding problem add to the plausibility of this suggestion.

Finally, the meaning of t_0 is that of an oscillation period of the transition process under the assumption of no observation, i.e. the evolution of the system is solely governed by $U(t)$. According to (29), observation leads to an increase of the effective oscillation time from t_0 to $T \approx t_0^2/\Delta T$. With $T \approx 3$ sec and $\Delta T \gtrapprox 30$ msec, this provides $t_0 \gtrapprox 300$ msec. Under the influence of observations with a temporal interval of 30 msec, the observation-free oscillation period of 300 msec due to $U(t)$ is increased to an oscillation period of 3 sec.

Several hundred milliseconds are the order of magnitude which is most often discussed as the time required for a stimulus to become consciously perceived (cf. the P300 component in event-related potentials). In contrast to T, which represents the "lifetime" (or mean first passage time) of each of the perceptual representations, t_0 can be regarded as the relaxation time into each of the representations, which is much shorter than the lifetime in each representation due to the Necker-Zeno model. Without the Zeno effect, the lifetime T would be identical with the transition time t_0.

It has been observed that the lifetime T for bistable Necker cube perception changes considerably if the stimulus is presented in a non-continuous way [36]. Particular combinations of on- and off-time intervals lead to significant changes of T. Recent results [24] show that T depends essentially on off-times rather than on on-times: T is maximal for long off-times (on the order of a second).

Long off-times obviously increase the interval after which a reversal of the Necker cube perception becomes possible at all. From the theoretical point of view outlined above, non-continuous presentation of the Necker cube with considerable off-times effectively modifies the Hamiltonian of the system, leading to an increased effective oscillation time t_0.

More precisely, this argument applies if off-times are greater than the value of t_0 under continuous presentation (with vanishing off-time). For non-continuous presentation, off-times that are long enough can therefore be used to "simulate" t_0 in an experimentally well-controlled fashion. For such a situation, Fig. 1 shows experimental results for $T = f(t_0)$ from [24, 36] together with a theoretically obtained curve according to (29) and with $\Delta T = 70$ msec. The theoretical curve fits the empirical results perfectly well. Using $\Delta T = 70$ msec to estimate t_0 for continuous presentation provides $t_0 \approx 460$ msec.

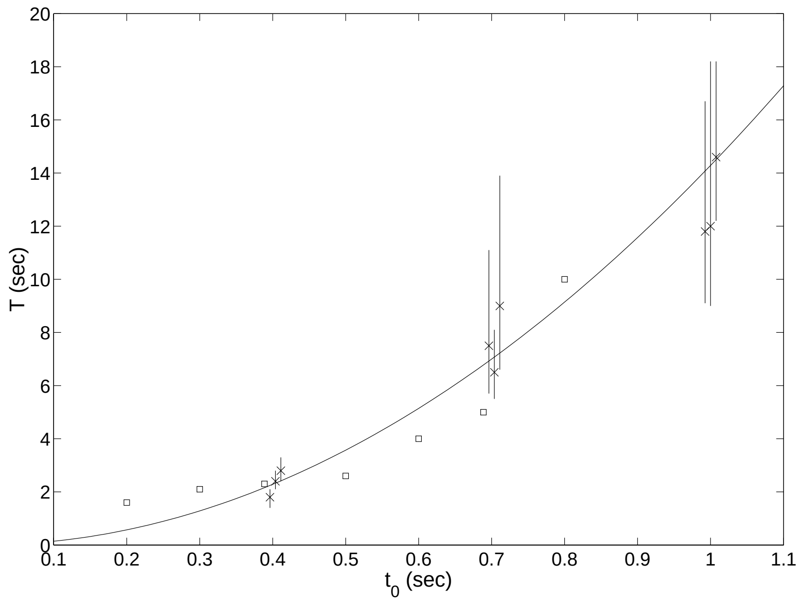

FIGURE 1. Experimentally obtained lifetimes T for the bistable perception of a non-continuously presented Necker cube. Crosses mark results from [24]: for each off-time t_0, T (including standard errors) is plotted for three on-times of 0.05 sec, 0.1 sec, and 0.4 sec. Squares mark results from [36] for an on-time of 0.3 sec (no errors indicated in [36]). The plotted curve shows T as a function of off-times t_0 according to (29) with $\Delta T = 70$ msec.

For off-times less than 300 msec, T starts to increase again with decreasing off-times. This behavior can be understood as a consequence of the unperturbed dynamics $U(t)$ in the model, where no Zeno effect plays a role. Recent empirical results for off-times smaller than 300 msec agree with corresponding theoretical predictions [25]. In addition, it is possible to derive the distribution of T (e.g., a gamma distribution) by a suitable choice for the transient initial behavior of $U(t)$ after each observation [25].

Since the T-distribution can strongly vary between different individuals, it would be interesting to study whether or not this corresponds to a covariation of ΔT. Dramatic increases of T by a factor of about 1000 for Tibetan monchs as subjects were recently reported by Pettigrew [37]. However, it is obvious that associated changes of ΔT are much more difficult to measure, and no such data are available so far.

Let us eventually remark that weak quantum theory indicates the possibility of superpositions of the eigenstates $|+\rangle$ and $|-\rangle$. One may speculate that such a type of entanglement may be an attractive candidate to model cognitive states which are in neither of the two categorial representations of the Necker cube. From the perspective of dynamical systems, such states would be intrinsically unstable and have been denoted as acategorial [2]. Further work will be necessary to develop this idea in more detail.

SUMMARY

The first part of this paper reviews the formulation of a weak version of quantum theory first proposed in [10]. It is motivated by the attempt to find a formal framework for the concepts of complementarity and entanglement not only within ordinary quantum physics, but also in more general contexts. The weak version of quantum theory is based on a minimal set of axioms forming the mathematical structure of a monoid. The key requirement for complementarity and entanglement in this framework is the non-commutativity of observables.

Ordinary quantum theory can be recovered from weak quantum theory by additional axioms, restrictions, and specifications. For example, the minimal version of weak quantum theory does not provide a von Neumann algebra and it does not include a Hilbert space representation. There is no Schrödinger equation for the dynamics and no Born rule for a probabilistic interpretation. In general, the non-commutativity of observables is not quantified by Planck's constant, the variance of observables is not given by Heisenberg uncertainty relations, and Bell-type inequalities cannot be formulated. Weak quantum theory is applicable at both ontic and epistemic levels of discussion.

In the second part, two applications are presented to demonstrate the viability of weak quantum theory. They refer to (i) complementary types of dynamical descriptions of classical dynamical systems, and (ii) the bistable preception of ambiguous stimuli. These examples are based on different levels of generalization between weak and ordinary quantum theory, depending on which restrictions are added to the minimal framework. The main formal difference between them can be expressed by the commutator of the non-commuting operators introduced. In example (i) the commutator is system-specific rather than universal, in example (ii) no commutator is specified at all. Both examples depend heavily on probabilistic concepts, and they refer to epistemic rather than ontic interpretations.

Both applications have been developed far enough that they can be related to empirical results. In example (i), the key quantity in this respect is the empirically accessible Kolmogorov-Sinai entropy of classical dynamical systems. It specifies the degree of non-commutativity of different dynamical descriptions if the underlying state space partition is generating. If this is not the case, the partitions themselves are in general complementary. The relation between complementary descriptions and a non-Boolean logic for dynamical systems is pointed out. Finally, it is indicated how non-commuting operators in classical dynamical systems can imply a temporal version of entanglement.

In example (ii) a Necker-Zeno model for bistable peception is presented as a generalization of the quantum Zeno effect. This model is based on the complementarity of an elementary cognitive observation process and the switching process between the different representations of an ambiguous stimulus. It provides a quantitative relation between three fundamentally different cognitive time scales of some ten milliseconds, some hundred milliseconds, and some seconds. Experimental results agree with this prediction. Superpositions of states in different representations are tentatively suggested as candidates for entangled cognitive states.

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