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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES   VOLUME LXVI-4 (2025)

FIBRATIONS OF DOUBLE GROUPOIDS, II: CONNECTIONS TO SIMPLICIAL SETS AND TOPOLOGICAL SPACES.

Antonio M. CEGARRA

Résumé. Dans ce deuxième article d'une série en deux parties sur les fibrations de groupoïdes doubles, nous explorons la relation entre les fibrations de groupoïdes doubles et les fibrations de Moerdijk d'ensembles bisimpliciaux ainsi que les fibrations de Hurewicz d'espaces topologiques.

Abstract. In this second paper of a two-part series on double groupoid fibrations, we explore the relationship between double groupoid fibrations and both (Moerdijk) fibrations of bisimplicial sets and (Hurewicz) fibrations of topological spaces.

Keywords. Double groupoid, Fibration, Homotopy groups, Crossed module, Geometric realization.

Mathematics Subject Classification (2020). 18N10, 20L05, 55P15, 55R15.

Introduction and summary.

In this companion paper to [4], we demonstrate the relationship between fibrations of double groupoids and simplicial as well as topological fibrations. We will adhere to the notation introduced in the aforementioned paper.

The construction of a topological space $|\mathcal{A}|$ from a double groupoid $\mathcal{A}$ is well-known and proceeds as follows: the *geometric realization* $|\mathcal{A}| = |\mathrm{diagNN}\mathcal{A}|$ is the space defined by first taking the double nerve, which is a bisimplicial set, and then geometrically realizing the diagonal simplicial set. A main result in [5] and [3] states that the functor $\mathcal{A} \mapsto |\mathcal{A}|$ induces an equivalence between the homotopy category of double groupoids with the filling condition and the category of homotopy 2-types. The *homotopy double groupoid* construction $X \mapsto \Pi^{(2}X$, for topological spaces X, provides a quasi-inverse functor.

Moving from double groupoids to spaces, the main result here asserts that a double functor between double groupoids $F : \mathcal{A} \to \mathcal{B}$ is a fibration if and only if the induced simplicial map $\mathrm{diagNN}F : \mathrm{diagNN}\mathcal{A} \to \mathrm{diagNN}\mathcal{B}$ is a Kan fibration. Hence, each fibration of double groupoids $F : \mathcal{A} \to \mathcal{B}$ induces a Hurewicz fibration on geometric realizations $|F| : |\mathcal{A}| \to |\mathcal{B}|$. Both the Mayer-Vietoris and homotopy sequences defined by fibrations of double groupoids in [4] provide purely algebraic descriptions of the corresponding Mayer-Vietoris and homotopy sequences defined in the topological context by taking geometric realizations.

Conversely, we prove that if $f : X \to Y$ is a Hurewicz fibration of topological spaces, then the induced double functor on the homotopy double groupoids $\Pi^{(2}f : \Pi^{(2}X \to \Pi^{(2}Y$ is a fibration. For applications, we show that both the lower end up to dimension two of the homotopy sequence and the homotopy 2-groupoid of a Hurewicz fibration of spaces can be recovered using a fibration of double groupoids.

1. Relationship between fibrations of double groupoids and Moerdijk bisimplicial fibrations.

Except for explicit references, we refer to [8] for all related to (bi)simplicial sets that we use in the paper.

For a double groupoid $\mathcal{A}$, let $\mathrm{NN}\mathcal{A}$ denote its double nerve; that is, the bisimplicial set where a (p, q)-simplex is a matrix (α_{ij}) of $p \times q$ horizontally and vertically composable boxes of the form

$$\begin{tikzcd}
a_{pq} \arrow[r, phantom, ""] & a_{p\text{-}1q} \arrow[l, "f_{pq}"'] & \cdots & a_{1q} & a_{0q} \arrow[l, "f_{1q}"'] \\
a_{pq\text{-}1} \arrow[u, "x_{pq}"] \arrow[r, phantom, "\alpha_{p,q}"] & a_{p\text{-}1q\text{-}1} \arrow[l, "f_{pq\text{-}1}"] \arrow[u, "x_{p\text{-}1q}"'] & \cdots & a_{1q} \arrow[u, "x_{1q}"] \arrow[r, phantom, "\alpha_{1,q}"] & a_{0q\text{-}1} \arrow[l, "f_{1q\text{-}1}"] \arrow[u, "x_{0q}"'] \\
\vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] \\
a_{p1} \arrow[u] & a_{p\text{-}11} \arrow[l, "f_{p1}"'] \arrow[u] & \cdots & a_{11} \arrow[u] & a_{01} \arrow[l, "f_{11}"'] \arrow[u] \\
a_{p0} \arrow[u, "x_{p1}"] \arrow[r, phantom, "\alpha_{p,1}"] & a_{p\text{-}10} \arrow[l, "f_{p0}"] \arrow[u, "x_{p\text{-}11}"'] & \cdots & a_{10} \arrow[u, "x_{11}"] \arrow[r, phantom, "\alpha_{1,1}"] & a_{00} \arrow[l, "f_{10}"] \arrow[u, "x_{01}"']
\end{tikzcd}$$

The bisimplicial face maps are the natural ones, induced by horizontal and vertical composition of boxes in $\mathcal{A}$, and the degeneracy ones by appropriate identity boxes. We picture $\mathrm{NN}\mathcal{A}$ so that the set of (p,q)-simplices is the set in the p-th row and q-th column. Thus, its p-th column, $\mathrm{NN}\mathcal{A}_{p\bullet}$, is the nerve of the "vertical" groupoid whose objects are strings $\cdot \overset{f_p}{\leftarrow} \cdots \overset{f_1}{\leftarrow} \cdot$ of p composable horizontal arrows in $\mathcal{A}$ and whose arrows are length p sequences of horizontally composable boxes

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, "g_p"'] \arrow[r, dotted, no head] & \cdot & \cdot \arrow[l] & \cdot \arrow[l, "g_1"'] \\
\cdot \arrow[u] \arrow[r, phantom, "\alpha_p"] & \cdot \arrow[l, "f_p"] \arrow[u] \arrow[r, phantom, "\cdots"] \arrow[r, dotted, no head] & \cdot \arrow[u] \arrow[r, phantom, "\alpha_2"] & \cdot \arrow[l] \arrow[u] \arrow[r, phantom, "\alpha_1"] & \cdot \arrow[l, "f_1"] \arrow[u]
\end{tikzcd}$$

Analogously, the q-th column, $\mathrm{NN}\mathcal{A}_{\bullet q}$, is the nerve of the "horizontal" groupoid whose objects are length q sequences of composable vertical morphisms in $\mathcal{A}$ and whose arrows are sequences of q vertically composable boxes. In particular, $\mathrm{NN}\mathcal{A}_{0\bullet}$ and $\mathrm{NN}\mathcal{A}_{\bullet 0}$ are, respectively, the nerves of the groupoids of vertical and horizontal morphisms of $\mathcal{A}$.

Similar to the nerve functor with groupoids and simplicial sets, the double nerve functor embeds the category of double groupoids as a full subcategory into the category of bisimplicial sets.

There is a closed model structure in the category of bisimplicial sets, the so-called *Moerdijk structure* [12], [8, Chap. IV, §3.3], in which a fibration is a bisimplicial map $f : X \to Y$ which induces a Kan fibration

$\mathrm{diag}f : \mathrm{diag}X \to \mathrm{diag}Y$ between the associated diagonal simplicial sets. The following theorem generalizes to double groupoids the fact proven by Moerdijk and Svensson for 2-groupoids in [13, Proposition 2.1 (ii)].

Theorem 1.1. *A double functor between double groupoids $F : \mathcal{A} \to \mathcal{B}$ is a fibration if and only if the induced simplicial map*

$$\mathrm{diagNN}F : \mathrm{diagNN}\mathcal{A} \to \mathrm{diagNN}\mathcal{B}$$

is a fibration of simplicial sets.

Proof. (i)$\Rightarrow$ (ii): We must prove that every lifting-extension problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathrm{diagNN}\mathcal{A} \arrow[d, "\mathrm{diagNN}F"] \\ \Delta^n \arrow[r] \arrow[ur, dotted, "?"] & \mathrm{diagNN}\mathcal{B} \end{tikzcd}$$

has a solution. For $n \geq 3$, this is a direct consequence Lemma 8 of [5], namely: *If $\mathcal{G}$ is a double groupoid and $n \geq 3$, then every extension problem*

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathrm{diagNN}\mathcal{G} \\ \Delta^n \arrow[ur, dotted, "?"] & \end{tikzcd}$$

has a solution and it is unique.

For $n = 2$ and $k = 0$, let us consider the lifting-extension problem

$$\begin{tikzcd} \Lambda^2_0 \arrow[rr, "{(-,\alpha_1,\alpha_2)}"] \arrow[d, hook] & & \mathrm{diagNN}\mathcal{A} \arrow[d, "\mathrm{diagNN}F"] \\ \Delta^2 \arrow[rr, "\begin{pmatrix}\beta_{22} & \beta_{12}\\ \beta_{21} & \beta_{11}\end{pmatrix}"'] \arrow[urr, dotted, "\begin{pmatrix}\alpha_{22} & \alpha_{12}\\ \alpha_{21} & \alpha_{11}\end{pmatrix}?"] & & \mathrm{diagNN}\mathcal{B} \end{tikzcd} \tag{1}$$

Then, α_1 and α_2 are boxes of $\mathcal{A}$ of the form

$$\begin{tikzcd} \cdot & \cdot \arrow[l] \\ \cdot \arrow[u] \arrow[ur, phantom, "\alpha_1"] & \cdot \arrow[l, "f_1"] \arrow[u, "x_1"'] \end{tikzcd} \quad a \qquad \begin{tikzcd} \cdot & \cdot \arrow[l] \\ \cdot \arrow[u] \arrow[ur, phantom, "\alpha_2"] & \cdot \arrow[l, "f_2"] \arrow[u, "x_2"'] \end{tikzcd} \quad a$$

for some horizontal morphisms f_1 and f_2 and some vertical morphisms x_1 and x_2, all with the same domain, say a, such that

$$F\alpha_1 = \begin{array}{ccccc} \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot \\ \uparrow & \beta_{22} & \uparrow & \beta_{12} & \uparrow y_{02} \\ \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot \\ \uparrow & \beta_{21} & \uparrow & \beta_{11} & \uparrow y_{01} \\ \cdot & \underset{g_{20}}{\longleftarrow} & \cdot & \underset{g_{10}}{\longleftarrow} & \cdot \end{array} \quad \text{and} \quad F\alpha_2 = \beta_{11}.$$

In particular, $Fx_1 = y_{02}\, y_{01}$, $Ff_1 = g_{20}\, g_{10}$, $Fx_2 = y_{01}$, and $Ff_2 = g_{10}$.

Since F is a fibration, we can choose boxes σ and τ in $\mathcal{A}$ of the form

$$\begin{array}{ccc} \cdot & \longleftarrow & \cdot \\ x_1\uparrow & \sigma & \uparrow \\ a & \underset{f_2^{-1}}{\longleftarrow} & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \overset{f_1}{\longleftarrow} & a \\ \uparrow & \tau & \uparrow x_2^{-1} \\ \cdot & \longleftarrow & \cdot \end{array}$$

such that

$$F\sigma = \begin{array}{ccc} \cdot & \overset{g_{12}^{-1}}{\longleftarrow} & \cdot \\ y_{02}\uparrow & \beta_{12}^{\text{-h}} & \uparrow \\ \cdot & \longleftarrow & \cdot \\ y_{01}\uparrow & \beta_{11}^{\text{-h}} & \uparrow \\ \cdot & \underset{g_{10}^{-1}}{\longleftarrow} & \cdot \end{array} \quad \text{and} \quad F\tau = \begin{array}{ccccc} \cdot & \overset{g_{20}}{\longleftarrow} & \cdot & \overset{g_{10}}{\longleftarrow} & \cdot \\ \uparrow & \beta_{21}^{\text{-v}} & \uparrow & \beta_{11}^{\text{-v}} & \uparrow y_{01}^{-1} \\ \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot \end{array}$$

The boxes α_1, α_2, σ, and τ fit together defining a 2-simplex $\left(\begin{smallmatrix}\alpha_{22} & \alpha_{12}\\ \alpha_{21} & \alpha_{11}\end{smallmatrix}\right)$ of $\mathrm{diagNN}\mathcal{A}$, in which

$$\alpha_{11} = \begin{array}{ccc} \cdot & \longleftarrow & \cdot \\ \uparrow & \alpha_2 & \uparrow x_2 \\ \cdot & \underset{f_2}{\longleftarrow} & a \end{array} \qquad \alpha_{21} = \begin{array}{ccccc} \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot \\ \uparrow & \tau^{\text{-v}} & \uparrow & \alpha_2^{\text{-h}} & \uparrow \\ \cdot & \underset{f_1}{\longleftarrow} & a & \underset{f_2^{-1}}{\longleftarrow} & \cdot \end{array}$$

$$\alpha_{12} = \begin{array}{ccc} \cdot & \longleftarrow & \cdot \\ \uparrow & \sigma^{\text{-h}} & \uparrow x_1 \\ \cdot & \longleftarrow & a \\ \uparrow & \alpha_2^{\text{-v}} & \uparrow x_2^{-1} \\ \cdot & \longleftarrow & \cdot \end{array} \qquad \alpha_{22} = \begin{array}{ccccc} \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot \\ \uparrow & \alpha_1 & \uparrow & \sigma & \uparrow \\ \cdot & \longleftarrow & a & \longleftarrow & \cdot \\ \uparrow & \tau & \uparrow & \alpha_2^{\text{-hv}} & \uparrow \\ \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot \end{array}$$

This solves the lifting-extension problem (1) since, on the one hand, $\alpha_{11} = \alpha_2$ and

$$\begin{array}{|c|c|}\hline \alpha_{22} & \alpha_{12} \\ \hline \alpha_{21} & \alpha_{11} \\ \hline \end{array} \quad = \quad \begin{array}{|c|c|c|}\hline \alpha_1 & \sigma & \sigma^{\text{-h}} \\ \hline \tau & \alpha_2^{\text{-hv}} & \alpha_2^{\text{-v}} \\ \hline \tau^{\text{-v}} & \alpha_2^{\text{-h}} & \alpha_2 \\ \hline \end{array} = \alpha_1$$

and, on the other hand, $F\alpha_{11} = F\alpha_2 = \beta_{11}$,

$$F\alpha_{12} = \begin{array}{|c|}\hline \beta_{12} \\ \hline \beta_{11} \\ \hline \beta_{11}^{\text{-v}} \\ \hline \end{array} = \beta_{12}, \quad F\alpha_{21} = \begin{array}{|c|c|c|}\hline \beta_{21} & \beta_{11} & \beta_{11}^{\text{-h}} \\ \hline \end{array} = \beta_{21},$$

and

$$F\alpha_{22} = \begin{array}{|c|c|c|}\hline \beta_{22} & \beta_{12} & \beta_{12}^{\text{-h}} \\ \hline \beta_{21} & \beta_{11} & \beta_{11}^{\text{-h}} \\ \hline \beta_{21}^{\text{-v}} & \beta_{11}^{\text{-v}} & \beta_{11}^{\text{-hv}} \\ \hline \end{array} = \beta_{22}.$$

For $n = 2$, the case in which $k = 2$ is dual of the case $k = 0$ above, and the case in which $k = 1$ is easier: let us consider the lifting-extension problem

$$\begin{tikzcd}
\Lambda^2_1 \arrow[rr, "{(\alpha_0,-,\alpha_2)}"] \arrow[d, hook] & & \mathrm{diagNN}\mathcal{A} \arrow[d, "\mathrm{diagNN}F"] \\
\Delta^2 \arrow[rru, dotted, "{\begin{pmatrix}\alpha_{22} & \alpha_{12}\\ \alpha_{21} & \alpha_{11}\end{pmatrix}?}"] \arrow[rr, "{\begin{pmatrix}\beta_{22} & \beta_{12}\\ \beta_{21} & \beta_{11}\end{pmatrix}}"'] & & \mathrm{diagNN}\mathcal{B}
\end{tikzcd} \tag{2}$$

Then, α_0 and α_2 are boxes of $\mathcal{A}$ of the form

$$\begin{tikzcd} \cdot & \cdot \arrow[l] \\ \cdot \arrow[u] \arrow[ur, phantom, "\alpha_0"] & a \arrow[l, "f_0"] \arrow[u, "x_0"'] \end{tikzcd} \qquad \begin{tikzcd} a & \cdot \arrow[l, "f_2"'] \\ \cdot \arrow[u, "x_2"] \arrow[ur, phantom, "\alpha_2"] & \cdot \arrow[l] \arrow[u] \end{tikzcd}$$

for some object a, such that $F\alpha_0 = \beta_{22}$ and $F\alpha_2 = \beta_{11}$. Since F is a fibration, we can choose boxes α_{12} and α_{21} in $\mathcal{A}$ of the form

$$\begin{tikzcd} \cdot & \cdot \arrow[l] \\ a \arrow[u, "x_0"] \arrow[ur, phantom, "\alpha_{12}"] & \cdot \arrow[l, "f_2"] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \cdot & a \arrow[l, "f_0"'] \\ \cdot \arrow[u] \arrow[ur, phantom, "\alpha_{21}"] & \cdot \arrow[l] \arrow[u, "x_2"'] \end{tikzcd}$$

such that $F\alpha_{12} = \beta_{12}$ and $F\alpha_{21} = \beta_{21}$. Then, the 2-simplex $\left(\begin{smallmatrix} \alpha_0 & \alpha_{12} \\ \alpha_{21} & \alpha_2 \end{smallmatrix}\right)$ of $\mathrm{diagNN}\mathcal{A}$ solves the lifting-extension problem (2).

Let us discuses finally the case in which $n = 1$ and $k = 0$ (the case $k = 1$ is parallel). Consider the extension-lifting problem

$$\begin{tikzcd} \Lambda^1_0 \arrow[r, "(-{,}a_1)"] \arrow[d, hook] & \mathrm{diagNN}\mathcal{A} \arrow[d, "\mathrm{diagNN}F"] \\ \Delta^1 \arrow[r, "\beta"'] \arrow[ur, dotted, "\alpha?"] & \mathrm{diagNN}\mathcal{B} \end{tikzcd} \tag{3}$$

given by an object a_1 of $\mathcal{A}$ and a box β of $\mathcal{B}$ of the form

$$\begin{tikzcd} \cdot & \cdot \arrow[l] \\ \cdot \arrow[u] \arrow[ur, phantom, "\beta"] & Fa_1 \arrow[l, "g"] \arrow[u, "y"'] \end{tikzcd}$$

Since F is a fibration, we can first choose a vertical morphism x and a horizontal morphism f, both with source a_1, such that $Fx = y$ and $Ff = g$, and then to find a box α of $\mathcal{A}$ of the form

$$\begin{tikzcd} \cdot & \cdot \arrow[l] \\ \cdot \arrow[u] \arrow[ur, phantom, "\alpha"] & a_1 \arrow[l, "f"] \arrow[u, "x"'] \end{tikzcd}$$

such that $F\alpha = \beta$. This α is a solution to the lifting-extension problem (3).

(ii) $\Rightarrow$ (i): To prove that F restricts to a fibration between the groupoids of horizontal morphisms, let a be an object of $\mathcal{A}$ and let $g : Fa \to b$ a horizontal morphism of $\mathcal{B}$. Since $\mathrm{diagNN}F : \mathrm{diagNN}\mathcal{A} \to \mathrm{diagNN}\mathcal{B}$ is a fibration of simplicial sets, there is a solution, say α, to the extension-lifting problem (3) in which $a_1 = a$ and $\beta = \mathrm{I}^{\mathrm{v}}_g$. Then α is a box of $\mathcal{A}$ of the form

$$\begin{tikzcd} \cdot & \cdot \arrow[l] \\ \cdot \arrow[u] & a \arrow[l] \arrow[u] \arrow[ul, phantom, "\alpha"] \end{tikzcd}$$

and satisfies that $F\alpha = \mathrm{I}^{\mathrm{v}}_g$. In particular, if $f : a \to a'$ is the horizontal source of α, then $Ff = g$. Dually, we prove that prove that F restricts to a fibration between the groupoids of vertical morphisms.

Suppose now the lifting problem for F

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "f"'] & & \cdot & \cdot \arrow[l, "Ff"'] \\ \cdot \arrow[u, dotted] & \cdot \arrow[l, dotted] \arrow[u, "x"'] \arrow[ul, phantom, "\exists?"] & {} \arrow[r, mapsto, "F"] & {} & \cdot \arrow[u] & \cdot \arrow[l] \arrow[u, "Fx"'] \arrow[ul, phantom, "\beta"] \end{tikzcd}$$

Since $\mathrm{diagNN}F$ is a fibration, there is a solution, say $\begin{pmatrix} \alpha_{22} & \alpha_{12} \\ \alpha_{21} & \alpha_{11} \end{pmatrix}$, to the extension-lifting problem (1) in which $\alpha_0 = \mathrm{I}^{\mathrm{v}}_f$, $\alpha_2 = \mathrm{I}^{\mathrm{h}}_x$, $\beta_{22} = \mathrm{I}^{\mathrm{v}}_{Ff}$, $\beta_{12} = \mathrm{I}_{Fa}$, $\beta_{21} = \beta$, and $\beta_{11} = \mathrm{I}^{\mathrm{h}}_{Fx}$. Then, the box α_{21} satisfies that $F\alpha_{21} = \beta_{21} = \beta$ and, since $\alpha_{22} = \mathrm{I}^{\mathrm{v}}_f$. Moreover, since $\alpha_{11} = \mathrm{I}^{\mathrm{h}}_x$, α_{21} is of the required form

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "f"'] \\ \cdot \arrow[u, dotted] & \cdot \arrow[l, dotted] \arrow[u, "x"'] \arrow[ul, phantom, "\alpha_{21}"] \end{tikzcd}$$

$\square$

Theorem 1.1 above, for the double functor $\mathcal{A} \to *$, gives the following corollary. (Cf. [5, Theorem 8].)

Corollary 1.2. *A double groupoid $\mathcal{A}$ satisfies the filling condition if and only if* $\mathrm{diagNN}A$ *is a Kan complex.*

By [5, Theorem 9], for every object a of a double groupoid $\mathcal{A}$ with the filling property, there are natural isomorphisms

$$\pi_n(\mathrm{diagNN}A, a) \cong \begin{cases} \pi_n(\mathcal{A}, a) & 0 \leq n \leq 2, \\ 0 & n \geq 3. \end{cases} \tag{4}$$

Then, for every fibration of double groupoids $F : \mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ has the filling property and each object a of $\mathcal{A}$, the 9-term exact sequence in [4, Theorem 5.1] gives the exact sequence on homotopy groups of the induced fibre sequence of simplicial sets

$$(\mathrm{diagNN}\mathcal{F}_b, a) \hookrightarrow (\mathrm{diagNN}\mathcal{A}, a) \xrightarrow{\mathrm{diagNN}F} (\mathrm{diagNN}\mathcal{B}, b),$$

where $b = Fa$ and $\mathcal{F}_b$ is the fibre of F over b (see, e.g., [8, Lemma 7.3]).

2. Relationship between fibrations of double groupoids and topological Hurewicz fibrations.

Let $\mathcal{A}$ be a double groupoid. When the bisimplicial set $\mathrm{NN}\mathcal{A}$ is regarded as a simplicial object in the simplicial set category and one takes geometric realizations, then one obtains a simplicial space $\Delta^{\mathrm{op}} \to \mathbf{Top}$, $\mathbf{p} \mapsto |\mathrm{NN}\mathcal{A}_{p,\bullet}|$, whose Segal realization is taken to be $|\mathcal{A}|$, the geometric realization of $\mathcal{A}$. As there are natural homeomorphisms [15, Lemma in page 86]

$$|\mathbf{p} \mapsto |\mathrm{NN}\mathcal{A}_{p,*}|| \cong |\mathrm{diagNN}\mathcal{A}| \cong |\mathbf{p} \mapsto |\mathrm{NN}\mathcal{A}_{*,p}||,$$

one usually takes

$$|\mathcal{A}| = |\mathrm{diagNN}\mathcal{A}|.$$

By [14], [16], and [6], the functor geometric realization $|\ \ |$ carries fibrations of simplicial sets to Hurewicz fibrations of spaces. Hence Theorem 1.1 gives the following.

Theorem 2.1. *If $F : \mathcal{A} \to \mathcal{B}$ is a fibration of double groupoids, then the induced map $|F| : |\mathcal{A}| \to |\mathcal{B}|$ is a Hurewicz fibration.*

By [5, Theorem 9], for every object a of a double groupoid $\mathcal{A}$ with the filling property, there are natural isomorphisms

$$\pi_n(|\mathcal{A}|, |a|) \cong \begin{cases} \pi_n(\mathcal{A}, a) & 0 \le n \le 2, \\ 0 & n \ge 3. \end{cases} \tag{5}$$

By [11, Corollary 11.6], the functor geometric realization $\mathcal{A} \mapsto |\mathcal{A}|$, from the category of double groupoids to the category of compactly generated Hausdorff spaces, preserves pullbacks. Then, in the situation of double functors between double groupoids

$$\begin{tikzcd} & \mathcal{A} \arrow[d, "F"] \\ \mathcal{B}' \arrow[r, "G"] & \mathcal{B} \end{tikzcd}$$

where F is a fibration and both $\mathcal{B}'$ and $\mathcal{B}$ have the filling property, the Mayer-Vietoris exact sequences in [4, Theorem 4.1] give the Mayer-Vietoris exact sequences of the pullback of spaces

$$\begin{tikzcd} |\mathcal{B}'| \times_{|\mathcal{B}|} |\mathcal{A}| \arrow[r] \arrow[d] & |\mathcal{A}| \arrow[d, "|F|"] \\ |\mathcal{B}'| \arrow[r, "|G|"] & |\mathcal{B}| \end{tikzcd}$$

(Cf. [7, Theorem], [1, Corollary 4.2]).

In particular, for every fibration of double groupoids $F : \mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ has the filling property, and every object a of $\mathcal{A}$, the 9-term exact sequence in [4, Theorem 5.1] gives the exact sequence on homotopy groups of the induced fibre sequence

$$(\mathcal{F}_{|b|}, |a|) \hookrightarrow (|\mathcal{A}|, |a|) \xrightarrow{|F|} (|\mathcal{B}|, |b|),$$

where $b = Fa$ and $\mathcal{F}_{|b|} = |F|^{-1}|b|$ is the fibre of $|F|$ over $|b|$.

Transitioning now from topological spaces, every space X has an associated *homotopy double groupoid*, which captures the homotopy 2-type of the space. The data of this double groupoid, denoted by

$$\Pi^{(2}X,$$

are as follows. (For details, see [5, §4].)

The objects of $\Pi^{(2}X$ are the paths in X, that is, the continuous maps $u : I \to X$, from the interval $I = [0, 1]$ to X.

The groupoid of horizontal morphisms in $\Pi^{(2}X$ is the category with a unique morphism $(u',u):u\to u'$ for each pair (u',u) of paths in X such that $u'(1)=u(1)$. Composition is defined by $(u'',u')(u',u)=(u'',u)$. Similarly, the groupoid of vertical morphisms in $\Pi^{(2}X$ is the category having a unique morphism $(v,u):u\to v$ for each pair (v,u) of paths in X such that $v(0)=u(0)$.

A box $[\alpha]$ in $\Pi^{(2}X$ with boundary as in

$$\begin{tikzcd} v' & v \arrow[l] \\ u' \arrow[u] & u \arrow[l] \arrow[u] \arrow[ul, phantom, "{[\alpha]}"] \end{tikzcd}$$

is the equivalence class of a square in X of the form

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "v'"'] \arrow[d, "u'"] \\ \cdot \arrow[u, "v"] \arrow[r, "u"'] \arrow[ur, phantom, "\alpha"] & \cdot \end{tikzcd}$$

that is, a map $\alpha:I^2\to X$, whose effect on the boundary ∂I^2 is

$$\begin{cases} \alpha(t,0)=u(t),\\ \alpha(0,t)=v(t),\\ \alpha(1,t)=u'(1-t),\\ \alpha(t,1)=v'(1-t). \end{cases}$$

Two such mappings α,α' are equivalent, and then represent the same box in $\Pi^{(2}X$, whenever they are related by a homotopy relative to the sides of the square. Horizontal and vertical composition of boxes in $\Pi^{(2}X$

$$\begin{tikzcd} v'' & v' \arrow[l] & v \arrow[l] \\ u'' \arrow[u] & u' \arrow[l] \arrow[u] \arrow[ul, phantom, "{[\alpha']}"] & u \arrow[l] \arrow[u] \arrow[ul, phantom, "{[\alpha]}"] \end{tikzcd} \qquad \begin{tikzcd} w' & w \arrow[l] \\ v' \arrow[u] & v \arrow[l] \arrow[u] \arrow[ul, phantom, "{[\alpha']}"] \\ u' \arrow[u] & u \arrow[l] \arrow[u] \arrow[ul, phantom, "{[\alpha]}"] \end{tikzcd}$$

are defined in the traditional way by pasting squares in X along a common pair of sides

$$
\begin{tikzcd}
\cdot & & \cdot \arrow[ll, "v''"'] \arrow[dd, "u''"] \\
& \cdot \arrow[ul, "v'"] \arrow[dr, "u'"] \arrow[ur, phantom, "\alpha'"] \arrow[dl, phantom, "\alpha"] & \\
\cdot \arrow[uu, "v"] \arrow[rr, "u"'] & & \cdot
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\cdot & & \cdot \arrow[ll, "w'"'] \arrow[dd, "u'"] \arrow[dl, "v'"'] \\
& \cdot \arrow[ul, phantom, "\alpha'"] \arrow[dr, phantom, "\alpha"] & \\
\cdot \arrow[uu, "w"] \arrow[rr, "u"'] \arrow[ur, "v"] & & \cdot
\end{tikzcd}
$$

The horizontal and vertical identity boxes

$$
\mathrm{I}^{\mathrm{h}}_{(v,u)} = \begin{tikzcd} v \arrow[r, equal] & v \\ u \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "{[C^{\mathrm{h}}]}"] & u \arrow[u] \end{tikzcd}
\qquad
\mathrm{I}^{\mathrm{h}}_{(u',u)} = \begin{tikzcd} u' & u \arrow[l] \arrow[d, equal] \\ u' \arrow[u, equal] \arrow[ur, phantom, "{[C^{\mathrm{v}}]}"] & u \arrow[l] \end{tikzcd}
$$

are respectively defined by the squares $C^{\mathrm{h}}, C^{\mathrm{v}} : I^2 \to X$ given by

$$
C^{\mathrm{h}}(t,t') = \begin{cases} v(t'-t) & t \leq t', \\ u(t-t') & t \geq t'. \end{cases}
\quad
C^{\mathrm{v}}(t,t') = \begin{cases} u(t+t') & t+t' \leq 1, \\ u'(2-t-t') & t+t' \geq 1. \end{cases}
$$

For a point $x \in X$, let $c_x : I \to X$ denote the constant path $c_x(t) = x$. Then, the identity box of c_x in $\Pi^{(2}X$ is

$$
\mathrm{I}_{c_x} = \begin{tikzcd} c_x \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "{[C_x]}"] & c_x \arrow[d, equal] \\ c_x \arrow[r, equal] & c_x \end{tikzcd}
$$

where $C_x : I^2 \to X$ is the constant square $C_x(t,t') = x$.

Theorem 2.2. *If $f : X \to Y$ is a Hurewicz fibration of spaces, then the induced double functor $f_* : \Pi^{(2}X \to \Pi^{(2}Y$ is a fibration of double groupoids.*

Proof. The fact that the restrictions of f_* to the groupoids of morphisms of $\Pi^{(2}X$ and $\Pi^{(2}Y$ are fibrations follows directly from the path-lifting property of f.

Suppose given a lifting problem for f_*

$$
\begin{tikzcd} v' & v \arrow[l] \\ \cdot \arrow[u, dotted] \arrow[ur, phantom, "{[\alpha]?}"] & u \arrow[u] \arrow[l, dotted] \end{tikzcd}
\overset{f_*}{\longmapsto}
\begin{tikzcd} fv' & fv \arrow[l] \\ \bar{u} \arrow[u] \arrow[ur, phantom, "{[\beta]}"] & fu \arrow[u] \arrow[l] \end{tikzcd}
$$

defined by paths $u, v, v' : I \to X$, with $u(0) = v(0)$ and $v(1) = v'(1)$, and a square $\beta : I^2 \to Y$ of the form

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "fv'"'] \arrow[d, "\bar{u}"] \\ \cdot \arrow[u, "fv"] \arrow[r, "fu"'] \arrow[ur, phantom, "\beta"] & \cdot \end{tikzcd}$$

Let $\alpha' : (0 \times I) \cup (I \times \partial I) \to X$ be the map given by

$$\begin{cases} \alpha'(0,t) = v(t), \\ \alpha(t,0) = u(t), \\ \alpha(t,1) = v'(1-t). \end{cases}$$

Since f is a fibration and $f\alpha' = \beta|_{(0\times I)\cup(I\times\partial I)}$, the extension-lifting problem

$$\begin{tikzcd} (0 \times I) \cup (I \times \partial I) \arrow[r, "\alpha'"] \arrow[d, hook, "in"'] & X \arrow[d, "f"] \\ I \times I \arrow[r, "\beta"'] \arrow[ur, dotted, "\alpha"] & Y \end{tikzcd}$$

has a solution, say α. Then α is a square in X of the form

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "v'"'] \arrow[d, "u'"] \\ \cdot \arrow[u, "v"] \arrow[r, "u"'] \arrow[ur, phantom, "\alpha"] & \cdot \end{tikzcd}$$

which represents a box of $\Pi^{(2}X$

$$\begin{tikzcd} v' & v \arrow[l] \\ u' \arrow[u] \arrow[ur, phantom, "{[\alpha]}"] & u \arrow[l] \arrow[u] \end{tikzcd}$$

such that $f_*[\alpha] = [f\alpha] = [\beta]$. □

From here on, we consider a pointed Hurewicz fibration of topological spaces

$$f : (X, x) \to (Y, y).$$

Let $F_y = f^{-1}(y)$ be the fibre of f over y, so that we have the fibre sequence of pointed spaces

$$(F_y, x) \hookrightarrow (X, x) \xrightarrow{f} (Y, y), \tag{6}$$

and let $\mathcal{F}_{c_y} = f_*^{-1}(c_y)$ be the double groupoid fibre of $f_* : \Pi^{(2}X \to \Pi^{(2}Y$ over c_y, so that we have the fibre sequence of pointed double groupoids

$$(\mathcal{F}_{c_y}, c_x) \hookrightarrow (\Pi^{(2}X, c_x) \xrightarrow{f_*} (\Pi^{(2}Y, c_y). \tag{7}$$

Theorem 2.3. (i) *The double groupoid $\mathcal{F}_{c_y}$ has the filling property.*

(ii) *There is an injective on objects equivalence of groupoids*

$$c : \Pi F_y \simeq \Pi \mathcal{F}_{c_y},$$

between the fundamental groupoid of F_y and the fundamental groupoid of $\mathcal{F}_{c_y}$, which carries every point $x \in F_y$ to the constant path c_x and a morphism $[v] : x_0 \to x_1$ of ΠF_y to the morphism $[(c_{x_1}, v), (v, c_{x_0})] : c_{x_0} \to c_{x_1}$ of $\Pi \mathcal{F}_{c_y}$ defined by the path

$$((c_{x_1}, v), (v, c_{x_0})) : c_{x_0} \curvearrowright c_{x_1} \qquad \begin{tikzcd} c_{x_1} & v \arrow[l] \\ & c_{x_0} \arrow[u] \end{tikzcd} \tag{8}$$

(iii) *There are natural isomorphisms*

$$\begin{aligned} &\pi_0(\mathcal{F}_{c_y}, [c_x]) \cong \pi_0(F_y, [x]) \\ &\pi_1(\mathcal{F}_{c_y}, c_x) \cong \pi_1(F_y, x), \\ &\pi_2(\mathcal{F}_{c_y}, c_x) \cong \mathrm{Im}\big(i_* : \pi_2(F_y, x) \to \pi_2(X, x)\big). \end{aligned}$$

Proof. (i) This follows from Theorem 2.2 and [4, Proposition 2.6].

(ii) The objects of the double groupoid $\mathcal{F}_{c_y}$ are the paths $I \to F_y$. We define the functor $c : \Pi F_y \to \Pi \mathcal{F}_{c_y}$ on objects by $x \mapsto c_x$. For any two points $x_0, x_1 \in F_y$, a morphism in $\Pi \mathcal{F}_{c_y}$ from c_{x_0} to c_{x_1} is the homotopy class of a path in $\mathcal{F}_{c_y}$ as in (8), which is determined by a path $v : I \to F_y$ such that $v(0) = x_0$ and $v(1) = x_1$. If v' is any other path in F_y from x_0 to x_1, then $[(c_{x_1}, v), (v, c_{x_0})] = [(c_{x_1}, v'), (v', c_{x_0})]$ in $\Pi \mathcal{F}_{c_y}$ if and only if there is a box in $\mathcal{F}_{c_y}$ of the form

$$\begin{tikzcd} v \arrow[d, equal] & v' \arrow[l] \arrow[dl, phantom, "{[\alpha]}"] \\ v \arrow[r, equal] & v \arrow[u] \end{tikzcd}$$

That is, if and only if there is a square $\alpha : I^2 \to X$ of the form

$$\begin{tikzcd}
x_1 \arrow[dr, phantom, "\alpha"] & x_0 \arrow[l, "v"'] \arrow[d, "v"] \\
x_0 \arrow[u, "v'"] \arrow[r, "v"'] & x_1
\end{tikzcd}$$

such that there is a homotopy $f\alpha \simeq C_y$ rel ∂I^2. By Lemma 2.4 below, that happens if and only if there is a similar square $\alpha' : I^2 \to F_y$ with the same sides. Since this condition simply means that $[v'] = [v]$ in the fundamental groupoid ΠF_y, we conclude with a bijection

$$c : \mathrm{Hom}_{\Pi F_y}(x_0, x_1) \cong \mathrm{Hom}_{\Pi \mathcal{F}_{c_y}}(c_{x_0}, c_{x_1}), \qquad [v] \mapsto [(c_{x_1}, v), (v, c_{x_0})].$$

To see that we are in presence of a functor, let $v_1, v_2 : I \to F_y$ be paths with $v_1(0) = x_0$, $v_1(1) = x_1 = v_2(0)$, and $v_2(1) = x_2$. Then, in $\Pi\mathcal{F}_{c_y}$,

$$[(c_{x_2}, v_2), (v_2, c_{x_1})] \cdot [(c_{x_1}, v_1), (v_1, c_{x_0})] = [(c_{x_2}, v), (v, c_{x_0})],$$

where v occurs in a configuration such as in

$$\begin{tikzcd}
c_{x_2} & v_2 \arrow[l] \arrow[dr, phantom, "{[\alpha]}"] & v \arrow[l] \\
 & c_{x_1} \arrow[u] & v_1 \arrow[l] \arrow[u] \\
 & & c_{x_0} \arrow[u]
\end{tikzcd}$$

for some square $\alpha : I^2 \to F_y$ with boundary as in

$$\begin{tikzcd}
x_2 \arrow[dr, phantom, "\alpha"] & x_1 \arrow[l, "v_2"'] \arrow[d, equal, "c_{x_1}"] \\
x_0 \arrow[u, "v"] \arrow[r, "v_1"'] & x_1
\end{tikzcd}$$

Hence, in ΠF_y, $[v] = [v_2] \cdot [v_1]$.

Finally, since for any path $u : I \to F_y$, there is the horizontal morphism

$$(u, c_{x_1}) : c_{u(1)} \to u,$$

the functor $c : \Pi F_y \to \Pi\mathcal{F}_{c_y}$ is actually an equivalence of categories.

(iii) Both the bijection $\pi_0\mathcal{F}_{c_y} \cong \pi_0 F_y$ and the isomorphism $\pi_1(\mathcal{F}_{c_y}, c_x) \cong \pi_1(F_y, x)$ are derived from part (ii) above, which has already been proven. Let us consider the case $n = 2$. The abelian group $\pi_2(\mathcal{F}_{c_y}, c_x)$ consists of boxes $[\alpha]$ in $\Pi^{(2}X$ of the form

$$\begin{array}{ccc} c_x & = & c_x \\ \| & [\alpha] & \| \\ c_x & = & c_x \end{array}$$

such that $f_*[\alpha] = [C_y]$ or, equivalently, of relative to ∂I^2 homotopy classes of squares $\alpha : I^2 \to X$ which are constant x along the four sides of the square and such that there is a homotopy $f\alpha \simeq C_y$ relative to ∂I^2. Then, by Lemma 2.4 below, $\pi_2(\mathcal{F}_{c_y}, c_x)$ can be described as the set of of relative to ∂I^2 homotopy classes of squares $\alpha : I^2 \to X$ which are constant x along the four sides of the square and such that $\alpha(I^2) \subseteq F_y$, that is, the image of $i_* : \pi_2(F_y, x) \to \pi_2(X, x)$. □

Lemma 2.4. *If $\alpha : I^2 \to X$ be a square in X such that $\alpha(\partial I^2) \subseteq F_y$ and there is a homotopy $H : f\alpha \simeq C_y$ rel ∂I^2, then there is a square $\alpha' : I^2 \to F_y$ such that $\alpha'|_{\partial I^2} = \alpha|_{\partial I^2}$ and a homotopy $H' : \alpha \simeq \alpha'$ rel ∂I^2 such that $fH' = H$.*

Proof. Let $h : (0 \times I^2) \cup (I \times \partial I^2) \to X$ be the map given by

$$\begin{aligned} &h(0, s, t) = \alpha(s, t), \quad h(r, 0, t) = \alpha(0, t), \quad h(r, 1, t) = \alpha(1, t), \\ &h(r, s, 0) = \alpha(s, 0), \quad h(r, s, 1) = \alpha(s, 1). \end{aligned}$$

Since f is a fibration and $fh = H|_{(0\times I^2)\cup(I\times\partial I^2)}$, the extension-lifting problem

$$\begin{tikzcd} (0 \times I^2) \cup (I \times \partial I^2) \arrow[r, "h"] \arrow[d, hook, "in"'] & X \arrow[d, "f"] \\ I \times I \times I \arrow[r, "H"'] \arrow[ur, dotted, "H'"] & Y \end{tikzcd}$$

has a solution, say H'. Now define $\alpha' : I^2 \to X$ by $\alpha'(s, t) = H'(1, s, t)$. Since,

$$f\alpha'(s, t) = fH'(1, s, t) = H(1, s, t) = C_y(s, t) = y,$$

α' is actually a square in the fibre F_y. □

If we apply Theorem 2.3 to the fibration $X \to *$, then obtain the following.

Theorem 2.5. *Let X be a topological space.*

(i) *The double groupoid $\Pi^{(2}X$ has the filling property.*

(ii) *There is an injective on objects equivalence of groupoids*

$$c : \Pi X \simeq \Pi(\Pi^{(2}X),$$

between the fundamental groupoid of X and the fundamental groupoid of $\Pi^{(2}X$, which carries every point $x \in X$ to the constant path c_x and a morphism $[u] : x_0 \to x_1$ of ΠX to the morphism $[(c_{x_1}, u), (u, c_{x_0})] : c_{x_0} \to c_{x_1}$ of ΠX defined by the path

$$((c_{x_1}, u), (u, c_{x_0})) : c_{x_0} \curvearrowright c_{x_1} \qquad \begin{tikzcd} c_{x_1} & u \arrow[l] \\ & c_{x_0} \arrow[u] \end{tikzcd}$$

(iii) *For every $x \in X$, there are natural isomorphisms*

$$\begin{aligned} \pi_0(\Pi^{(2}X, [c_x]) &\cong \pi_0(X, [x]) \\ \pi_1(\Pi^{(2}X, c_x) &\cong \pi_1(X, x), \\ \pi_2(\Pi^{(2}X, c_x) &\cong \pi_2(X, x). \end{aligned}$$

Lemma 2.6. *Let $u : I \to X$ be a path in X with $u(0) = x_0$ and $u(1) = x_1$. Let $y_0 = f(x_0)$ and $y_1 = f(x_1)$, and let $v : I \to F_{y_0}$ be a loop in F_{y_0} based at x_0. Then, the action* [4, (19)] *of the morphism*

$$[(c_{x_1}, u), (u, c_{x_0})] : c_{x_0} \to c_{x_1}$$

of $\Pi(\Pi^{(2}X)$ on

$$[(c_{x_0}, v), (v, c_{x_0})] \in \pi_1(\mathcal{F}_{c_{y_0}}, c_{x_0})$$

is given by

$$^{[(c_{x_1},u),(u,c_{x_0})]}[(c_{x_0}, v), (v, c_{x_0})] = [(c_{x_1}, v'), (v', c_{x_1})] \in \pi_1(\mathcal{F}_{c_{y_1}}, c_{x_1})$$

where $v' : I \to F_{y_1}$ is a loop in F_{y_1} based at x_1 such that, in ΠX,

$$[v'] = [u] \cdot [v] \cdot [u]^{-1}.$$

Proof. As an object of $\Pi^{(2}X$, the path $v' : I \to X$ with $fv' = c_{y_1}$, i.e. with $Im(v') \subseteq F_{y_1}$, occurs in a diagram

$$\begin{tikzcd}
c_{x_1} & u \arrow[l] & w \arrow[l] & v' \arrow[l] \\
 & c_{x_0} \arrow[u] \arrow[l, phantom] \arrow[ur, phantom, "{[\alpha]}"] & v \arrow[l] \arrow[u] \arrow[ur, phantom, "{[\beta]}"] & \\
 & & u \arrow[u] & c_{x_1} \arrow[l] \arrow[uu]
\end{tikzcd}$$

for some squares $\alpha : I^2 \to X$ and $\beta : I^2 \to X$ of the form

$$\begin{tikzcd}
x_1 & x_0 \arrow[l, "u"'] \\
x_0 \arrow[u, "w"] \arrow[r, "v"'] \arrow[ur, phantom, "\alpha"] & x_0 \arrow[u, equal]
\end{tikzcd}
\qquad
\begin{tikzcd}
x_1 & x_0 \arrow[l, "w"'] \arrow[d, "u"] \\
x_1 \arrow[u, "v'"] \arrow[r, equal] \arrow[ur, phantom, "\beta"] & x_1
\end{tikzcd}$$

Then, v' is a path in F_{y_1} based at x_1 and, in ΠX,

$$[v'] = [w] \cdot [u]^{-1} = ([u] \cdot [v]) \cdot [u]^{-1}.$$

□

By Theorems 2.3 and 2.5, the 9-term exact sequence in [4, Theorem 5.1] defined by the fibre sequence of pointed double groupoids (7),

$$(\mathcal{F}_{c_y}, c_x) \hookrightarrow (\Pi^{(2}X, c_x) \xrightarrow{f_*} (\Pi^{(2}Y, c_y),$$

gives the 9-term exact sequence

$$\begin{tikzcd}
\pi_2(F_y, x) \arrow[r] & \pi_2(X, x) \arrow[r] & \pi_2(Y, y) \arrow[dll] \\
\pi_1(F_y, x) \arrow[r] & \pi_1(X, x) \arrow[r] & \pi_1(Y, y) \arrow[dll] \\
\pi_0(F_y, [x]) \arrow[r] & \pi_0(X, [x]) \arrow[r] & \pi_0(Y, [y]),
\end{tikzcd}$$

which is the bottom end up to dimension 2 of the long exact sequence on homotopy groups defined by the fibre sequence of pointed spaces (6)

$$(F_y, x) \hookrightarrow (X, x) \xrightarrow{f} (Y, y).$$

(See, for example, [17, Chap. IV, (8.6)]). Further, by the equivalence between crossed modules and cat^1-groups, the fundamental crossed module in [4, Proposition 5.2 (i)] gives Loday's fundamental crossed module [10, 2.7]

$$\pi_1(F_y, x) \to \pi_1(X, x).$$

(See [2, §2.6].) Even more, by the equivalence between crossed modules over groupoids and 2-groupoids, we can see that the fundamental crossed module in [4, Theorem 5.7] for the fibration $f_* : \Pi^{(2}X \to \Pi^{(2}Y$ correspond, up to equivalence, to the homotopy 2-groupoid of the fibration $f : X \to Y$ by Kamps and Porter [9]. In effect, the fundamental $\Pi(\Pi^{(2}X)$-crossed module of the fibration $f_* : \Pi^{(2}X \to \Pi^{(2}Y$,

$$\pi_1\mathcal{F} \to \pi_1(\Pi^{(2}X), \quad u \mapsto \big(\pi_1(\mathcal{F}_{fu}, u) \to \pi_1(\Pi^{(2}X, u)\big),$$

by the equivalence $c : \Pi X \hookrightarrow \Pi(\Pi^{(2}X)$, gives the ΠX-crossed module

$$c^*\pi_1\mathcal{F} \to c^*\pi_1(\Pi^{(2}X), \quad x \mapsto \big(\pi_1(\mathcal{F}_{c_{fx}}, c_x) \to \pi_1(\Pi^{(2}X, c_x)\big).$$

This, by Theorems 2.3 and 2.5 and Lemma 2.6 above, is isomorphic to the crossed module over the fundamental groupoid ΠX,

$$\pi_1 F \to \pi_1 X, \quad x \mapsto \big(\pi_1(F_{fx}, x) \to \pi_1(X, x)\big),$$

which justly defines the homotopy 2-groupoid of the fibration $f : X \to Y$ by Kamps and Porter.

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXVI-4 (2025)

CLASSIFICATION DIAGRAMS OF SIMPLICIAL CATEGORIES

Kensuke ARAKAWA

Résumé. Nous montrons que le diagramme de classification d'une ∞-catégorie relative issu d'une catégorie simpliciale relative est équivalent au nerf degré par degré. Les applications incluent la comparaison de la diagonale du nerf degré par degré et du nerf homotopique cohérent, ainsi qu'un résultat sur les localisations degré par degré des catégories simpliciales.
Abstract. We show that the classification diagram of a relative ∞-category arising from a relative simplicial category is equivalent to the levelwise nerve. Applications include the comparison of the diagonal of the levelwise nerve and the homotopy coherent nerve, and a result on the levelwise localizations of simplicial categories.
Keywords. classification diagram, homotopy coherent nerve, classifying space.
Mathematics Subject Classification (2020). 18N60, 55U40, 55U35.

1. Introduction

Given a category $\mathcal{C}$, the **nerve** of $\mathcal{C}$ is the simplicial set $N(\mathcal{C})$ whose n-simplices are the set of n composable arrows $X_0 \to \cdots \to X_n$ in $\mathcal{C}$. It is immediate from the definitions that the association $\mathcal{C} \mapsto N(\mathcal{C})$ defines a fully faithful functor

$$N\colon \mathsf{Cat} \to \mathsf{sSet}$$

from the category of small categories to that of simplicial sets. As elementary as this observation may be, it puts forward an amazing fact: A structured object (a 1-category) can be presented as a family of less structured objects (sets, or 0-categories). Experience suggests that the latter is generally easier to work with, and this explains the prevalence of the nerve construction in homotopy theory.

It is natural to look for a higher-categorical analog of the nerve construction. For example, we may ask whether it is possible to present an $(\infty, 1)$-category as a simplicial $(\infty, 0)$-category, or a simplicial space. The answer is yes. Recall that **∞-categories** (alias quasicateogries [12]) are a certain class of simplicial sets modeling $(\infty, 1)$-categories. Given an ∞-category $\mathcal{C}$, its **classifying diagram** $\mathrm{Cls}(\mathcal{C})$ is the simplicial space (or a bisimplicial set, to be more precise) whose nth space $\mathrm{Cls}(\mathcal{C})_n = \mathrm{Cls}(\mathcal{C})_{n,*}$ is the maximal sub Kan complex of $\mathrm{Fun}(\Delta^n, \mathcal{C})$. Joyal and Tierney showed in [14] that the association $\mathcal{C} \to \mathrm{Cls}(\mathcal{C})$ defines a fully faithful functor from the homotopy category of ∞-categories to that of simplicial spaces, thereby giving an analog of the nerve construction; the essential image of this functor consists of Rezk's **complete Segal spaces** [21].

A generalization of the classifying diagram construction has proved to be especially important in the theory of localizations of ∞-categories (in the sense of [15, Definition 2.4.2]). Given an ∞-category $\mathcal{C}$ and a subcategory $\mathcal{W} \subset \mathcal{C}$ containing all objects, their **classification diagram** $\mathrm{Cls}(\mathcal{C}, \mathcal{W})$ [21] is the simplicial ∞-category (hence a bisimplicial set) whose nth ∞-category is given by

$$\mathrm{Cls}(\mathcal{C}, \mathcal{W})_n = \mathrm{Cls}(\mathcal{C}, \mathcal{W})_{n,*} = \mathrm{Fun}(\Delta^n, \mathcal{C}) \times_{\mathcal{C}^{n+1}} \mathcal{W}^{n+1}.$$

For example, if $\mathcal{W}$ is the subcategory of equivalences of $\mathcal{C}$, then $\mathrm{Cls}(\mathcal{C}, \mathcal{W})$ is nothing but the classifying diagram of $\mathcal{C}$. Inspired by earlier works such as [21, 6, 4], Mazel-Gee showed[1] in [20] (see also [1, 3] for alternative arguments and generalizations) that a functor $f\colon \mathcal{C} \to \mathcal{D}$ of ∞-categories which carries $\mathcal{W}$ into $\mathcal{D}^{\simeq}$ is a localization with respect to $\mathcal{W}$ if and only if the map $\mathrm{Cls}(\mathcal{C}, \mathcal{W}) \to \mathrm{Cls}(\mathcal{D}, \mathcal{D}^{\simeq}) = \mathrm{Cls}(\mathcal{D})$ is a weak equivalence of the complete Segal space model structure [21, Theorem 7.2]. In other words, up to fibrant replacement, $\mathrm{Cls}(\mathcal{C}, \mathcal{W})$ computes the localization of $\mathcal{C}$. This is useful

[1]To be more precise, Mazel-Gee proved this in the case where $\mathcal{W}$ contains all equivalences (which suffices for most applications). The general case was proved by the author in [1].

because in some cases, the fibrant replacement of $\mathrm{Cls}(\mathcal{C}, \mathcal{W})$ have explicit descriptions; see, e.g., [16].

Now recall the **homotopy coherent nerve functor**

$$N_{\mathrm{hc}} \colon \mathsf{Cat}_{\Delta} \to \mathsf{sSet},$$

due to Cordier [7], is a right Quillen equivalence from Bergner's model structure on the category of simplicial categories [5] to Joyal's model structure on the category of simplicial sets [17, §2.2.5]. Many important ∞-categories arise as the homotopy coherent nerves of simplicial categories, so we frequently want to consider bisimplicial sets of the form $\mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}), N_{\mathrm{hc}}(\mathcal{W}))$, where $\mathcal{C}$ is a fibrant simplicial category and $\mathcal{W}$ is its simplicial subcategory such that $N_{\mathrm{hc}}(\mathcal{W})$ is a subcategory of $N_{\mathrm{hc}}(\mathcal{C})$. However, as is anything constructed from homotopy coherent nerves, this bisimplicial set is somewhat difficult to manipulate by hands. It will therefore be nice if there is an alternative, preferably simpler, presentation of this bisimplicial set. The goal of this paper is to show that this is possible, at least if we consider the *marked* version of classification diagrams.

A **marked bisimplicial set** is a pair (X, S) of a bisimplicial set X and a simplicial subset $S \subset X_1 = X_{1,*}$ which contains the image of the map $X_0 \to X_1$. Equivalently, it is a simplicial object $\{(X_{*,n}, S_n)\}_{n \geq 0}$ in the category of marked simplicial sets. One may argue that the natural place where classification diagrams live is not the category of bisimplicial sets, but the category of marked bisimplicial sets. Indeed, there is a very natural functor

$$\mathrm{Cls}^{+} \colon \mathsf{sSet}^{+} \to \left(\mathsf{sSet}^{+}\right)^{\mathbf{\Delta}^{\mathrm{op}}} = \mathsf{bsSet}^{+}$$

from the category of marked simplicial sets to the category of marked bisimplicial sets, defined by $(X, S) \mapsto (X, S)^{(\Delta^{\bullet})}$. (Here we wrote $(X, S)^{\Delta^n} = (X, S)^{(\Delta^n)^{\sharp}}$, where $(\Delta^n)^{\sharp}$ denotes the standard simplex Δ^n with all edges marked. It is the cotensor of (X, S) by the simplicial set Δ^n with respect to the simplicial enrichment $\mathrm{Map}^{\sharp}(-,-)$ of [17, §3.1.3].) Unwinding the definitions, we find that $\mathrm{Cls}(\mathcal{C}, \mathcal{W})$ is nothing but the underlying bisimplicial set of $\mathrm{Cls}^{+}(\mathcal{C}, \mathcal{W}_1)$.

In [1], the author showed that marked bisimplicial sets are to complete Segal spaces what marked simplicial sets are to ∞-categories. More precisely, [1, Theorems 2.9 and 3.4] state that bsSet^{+} admits a model structure, denoted by $\mathsf{bsSet}^{+}_{\mathrm{CSS}}$, such that:

- The functor $\mathrm{Cls}^+ \colon \mathsf{sSet}^+ \to \mathsf{bsSet}^+$ is a right Quillen equivalence , where sSet^+ carries the model structure for marked simplicial sets [18, Proposition 3.1.3.7, Remark 3.1.4.6].[2]
- The forgetful functor $\mathsf{bsSet}^+_{\mathrm{CSS}} \to \mathsf{bsSet}_{\mathrm{CSS}}$ is also a right Quillen equivalence, where $\mathsf{bsSet}_{\mathrm{CSS}}$ denotes the category of bisimplicial sets equipped with the model structure for complete Segal spaces.

What we will do is to construct a relatively simple marked bisimplicial set which is weakly equivalent to $\mathrm{Cls}^+(N_{\mathrm{hc}}(\mathcal{C}), N_{\mathrm{hc}}(\mathcal{W}))$ in $\mathsf{bsSet}^+_{\mathrm{CSS}}$. To give a precise statement of the main theorem, we must introduce some notation and terminology.

Definition 1.1. Let $\mathcal{C}$ be a simplicial category. A simplicial subcategory $\mathcal{W} \subset \mathcal{C}$ is said to be **wide** if it satisfies the following pair of conditions:

- $\mathcal{W}$ contains all objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathcal{C}$, the simplicial subset $\mathcal{W}(X,Y) \subset \mathcal{C}(X,Y)$ is a union of components of $\mathcal{C}(X,Y)$.

The pair $(\mathcal{C}, \mathcal{W})$ of a simplicial category and its wide simplicial subcategory will be called a **relative simplicial category**.

Definition 1.2. Let $\mathcal{C}$ be a simplicial category. For each $n \geq 0$, we let $\mathcal{C}_n$ denote the ordinary category constructed from the objects of $\mathcal{C}$ and the n-simplices of the hom-simplicial sets of $\mathcal{C}$. The **binerve** (or the **levelwise nerve**) of $\mathcal{C}$ is the bisimplicial set $N_{\mathrm{bi}}(\mathcal{C})$ whose nth row $N_{\mathrm{bi}}(\mathcal{C})_{*,n}$ is the nerve of $\mathcal{C}_n$. Note that for each $n \geq 1$, the nth column $N_{\mathrm{bi}}(\mathcal{C})_{n,*}$ of $N_{\mathrm{bi}}(\mathcal{C})$ is the disjoint union

$$\coprod_{X_0,\dots,X_n \in \mathcal{C}} \mathcal{C}(X_{n-1}, X_n) \times \cdots \times \mathcal{C}(X_0, X_1).$$

If $\mathcal{W} \subset \mathcal{C}$ is a wide simplicial subcategory, we let $N^+_{\mathrm{bi}}(\mathcal{C}, \mathcal{W})$ denote the marked bisimplicial set $\left(N_{\mathrm{bi}}(\mathcal{C}), \coprod_{X,Y \in \mathcal{C}} \mathcal{W}(X,Y)\right)$.

[2] In [1], the functor Cls is denoted by N and the functor Cls^+ is denoted by $(t^+)^!$. We changed the notation in the hope that the paper will be more readable, for we will consider many variations of nerves in this paper.

We can now state the main result of this paper.

Theorem 1.3 (Theorem 3.1). *Let $\mathcal{C}$ be a fibrant simplicial category and let $\mathcal{W} \subset \mathcal{C}$ be a wide simplicial subcategory. There is a weak equivalence*

$$N^+_{\mathrm{bi}}(\mathcal{C}, \mathcal{W}) \to \mathrm{Cls}^+(N_{\mathrm{hc}}(\mathcal{C}), \mathrm{mor}\,\mathcal{W}_0)$$

of $\mathsf{bsSet}^+_{\mathrm{CSS}}$ which is natural in $(\mathcal{C}, \mathcal{W})$, where $\mathrm{mor}\,\mathcal{W}_0$ denotes the set of morphisms of $\mathcal{W}_0$.

Notice how Theorem 1.3 simplifies the clunky marked bisimplicial set $\mathrm{Cls}^+(N_{\mathrm{hc}}(\mathcal{C}), \mathrm{mor}\,\mathcal{W}_0)$: If we want to directly work $\mathrm{Cls}^+(N_{\mathrm{hc}}(\mathcal{C}), \mathrm{mor}\,\mathcal{W}_0)$, we have to construct coherent higher homotopies governing the homotopy coherent nerve, which is often a hard labor. In contrast, the rows of $N_{\mathrm{bi}}(\mathcal{C})$ are nerves of *ordinary* categories, which has no higher structures.

Remark 1.4. The idea of relating the localization of a relative simplicial category $(\mathcal{C}, \mathcal{W})$ (which corresponds to $\mathrm{Cls}^+(N_{\mathrm{hc}}(\mathcal{C}), \mathrm{mor}\,\mathcal{W}_0)$) to those of $(\mathcal{C}_n, \mathcal{W}_n)$ (which correspond to the rows of $N^+_{\mathrm{bi}}(\mathcal{C}, \mathcal{W})$) is reminiscent of the work of Dwyer and Kan: In [8], Dwyer and Kan defined the *simplicial localizations* of relative simplicial categories by first defining them for relative categories, and then applying them levelwise to define them for all cases. We can therefore interpret Theorem 1.3 as another manifestation of Dwyer and Kan's principle that the localization of a relative simplicial category is the totality of the levelwise localization.

Remark 1.5. By the works of Joyal [13] and Joyal and Tierney [14], it has been known that $N_{\mathrm{bi}}(\mathcal{C})$ and $\mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}))$ are weakly equivalent in the complete Segal space model structure. Theorem 1.3 may be regarded as a refinement of this.

In addition to simplifying the marked classification diagrams, Theorem 1.3 also has some interesting applications. We list two of them below.

The first one exploits the connection between localizations and levelwise localizations we observed in Remark 1.4:

Corollary 1.6 (Corollary 4.1). *Let $(\mathcal{C}, \mathcal{W})$ and $(\mathcal{C}', \mathcal{W}')$ be relative simplicial categories, where $\mathcal{C}$ and $\mathcal{C}'$ are fibrant. Let $f \colon \mathcal{C} \to \mathcal{C}'$ be a simplicial functor which carries $\mathcal{W}$ into $\mathcal{W}'$. Suppose that for each $n \geq 0$, the functor*

$$N(\mathcal{C}_n)[N(\mathcal{W}_n)^{-1}] \to N(\mathcal{C}'_n)[N(\mathcal{W}'_n)^{-1}]$$

is a categorical equivalence (i.e., a weak equivalence in the Joyal model structure). Then so is the functor

$$N_{\mathrm{hc}}(\mathcal{C})[N_{\mathrm{hc}}(\mathcal{W})^{-1}] \to N_{\mathrm{hc}}(\mathcal{C}')[N_{\mathrm{hc}}(\mathcal{W}')^{-1}].$$

Corollary 1.6 is especially useful when $\mathcal{C}$ is equal to the ordinary category $\mathcal{C}'_0$, the 0th level of $\mathcal{C}'$. In this case, the corollary says that if the functors $N(\mathcal{C}'_0)[N(\mathcal{W}'_0)^{-1}] \to N(\mathcal{C}'_n)[N(\mathcal{W}'_n)^{-1}]$ are all categorical equivalences, then so is the functor

$$N(\mathcal{C}'_0)N[(\mathcal{W}'_0)^{-1}] \to N_{\mathrm{hc}}(\mathcal{C}')[N(\mathcal{W}')^{-1}].$$

In other words, the localization of a relative ∞-category (the right hand side) is equivalent to a localization of an *ordinary* relative category (the left hand side). This recovers an observation made by Lurie in the proof of [18, Proposition 1.3.4.7]. We also remark that in the same part of loc.cit., Lurie establishes a very useful criterion for the maps $N(\mathcal{C}'_0)[N(\mathcal{W}'_0)^{-1}] \to N(\mathcal{C}'_n)[N(\mathcal{W}'_n)^{-1}]$ to be a categorical equivalence: This happens if $\mathcal{C}'$ admits tensoring by Δ^1 and $\mathcal{W}'$ contains all homotpy equivalences of $\mathcal{C}'$.

The second application relates the homotopy coherent nerve functor with Segal's classical construction of classifying spaces of simplicial categories.

Definition 1.7. [23] Let $\mathcal{C}$ be a simplicial category. We let $B(\mathcal{C})$ denote the diagonal of the bisimplicial set $N_{\mathrm{bi}}(\mathcal{C})$; it is the **classifying space** of $\mathcal{C}$.

In [10, 2.6.1], Hinich constructed a natural transformation $B \to N_{\mathrm{hc}}$. (In fact, there is only one such natural transformation, as we will see in Proposition 2.1.) We then prove the following comparison result:

Corollary 1.8 (Corollary 4.3)**.** *Let $\mathcal{C}$ be a fibrant simplicial category. The map*

$$B(\mathcal{C}) \to N_{\mathrm{hc}}(\mathcal{C})$$

is a weak homotopy equivalence.

In the special case where $\mathcal{C}$ is a simplicial groupoid, Corollary 1.8 is a consequence of [2, Theorem 3.6] and [9, A.5.1]. (It was also announced in [10, Corollary 2.6.3], but its proof relies on 2.6.2 of loc. cit., which has a gap, as pointed out in [2].) Our proof of Corollary 1.8 uses different machinery from these earlier results, and this is why we were able to relax the hypothesis.

Organization of the paper

In Section 2, we will construct the comparison map $B \to N_{\mathrm{hc}}$. In Section 3, we will prove the main result. Section 4 concerns the applications of the main result.

Notation and convention

- If $\mathcal{C}$ is a category, its nerve will be denoted by $N(\mathcal{C})$.
- By a simplicial category, we mean a category enriched over the category of simplicial sets.
- If $\mathcal{C}$ is a simplicial category and $X, Y \in \mathcal{C}$ are its objects, we will write $\mathcal{C}(X, Y)$ or $\mathrm{Map}(X, Y)$ for the hom-simplicial set from X to Y.
- We understand that the category Cat_Δ is equipped with the Bergner model structure [5].
- We understand that the category sSet$^+$ is equipped with the model structure for marked simplicial sets [18, proposition 3.1.3.7, Remark 3.1.4.6].
- If X is a bisimplicial set and $n \geq 0$ is an integer, then the nth **column** (resp. nth **row**) of X is the simplicial set $X_{n,*}$ (resp. $X_{*,n}$). The nth column of X is denoted by X_n.
- If $\mathcal{C}$ is a simplicial category, its **homotopy coherent nerve** $N_{\mathrm{hc}}(\mathcal{C})$ is the simplicial set whose n-simplices are the simplicial functors $\mathfrak{C}[\Delta^n] \to \mathcal{C}$. Here $\mathfrak{C}[\Delta^n]$ denotes the simplicial category whose objects are the integers $0, \ldots, n$. If $0 \leq i \leq j \leq n$ are integers, the hom-simplicial set $\mathfrak{C}[\Delta^n](i, j)$ is the nerve of the poset $P_{i,j} = \{S \subset \{0, \ldots, n\} \mid \min S = i, \max S = j\}$, ordered by inclusion. The composition of $\mathfrak{C}[\Delta^n]$ is induced by the operation of union.[3]

[3]It is also common to define $\mathfrak{C}[\Delta^n](i, j)$ to be the nerve of the opposite of $P_{i,j}$. Our convention follows [17]. The only part that will be affected by the choice of conventions is the proof of Proposition 2.1 and the descriptions of maps appearing in Remarks 2.4 and 2.5.

2. The comparison map

In this section, we will construct a comparison map $B(\mathcal{C}) \to N_{\mathrm{hc}}(\mathcal{C})$ (where B is as in Definition 1.7), which will be the source of all the other comparison maps we consider in this note. The comparison map is obtained as the composite of two natural transformation constructed in [10, 2.6.1], but we give a direct approach. In fact, there is only one natural transformation $B \to N_{\mathrm{hc}}$:

Proposition 2.1. *There is a unique natural transformation $B \to N_{\mathrm{hc}}$ of functors $\mathsf{Cat}_\Delta \to \mathsf{sSet}$.*

For the proof of Proposition 2.1 and for later discussions, we introduce a bit of notation.

Notation 2.2. Let $n \geq 0$ and let K be a simplicial set. We define a simplicial category $[n]_K$ as follows: Its objects are the integers $0, \dots, n$. The hom-simplicial sets are given by

$$[n]_K(i,j) = \begin{cases} \prod_{i<k\leq j} K & \text{if } i \leq j, \\ \emptyset & \text{if } i > j. \end{cases}$$

The composition is defined by concatenation.

Remark 2.3. Let $\mathcal{C}$ be a simplicial category and let K be a simplicial set. We can define an ordinary category $\mathcal{C}_K$ as follows: The objects of $\mathcal{C}_K$ are the objects of $\mathcal{C}$. The hom-sets are given by $\mathcal{C}_K(X,Y) = \mathsf{sSet}(K, \mathcal{C}(X,Y))$. For each $n \geq 0$, a functor $[n] \to \mathcal{C}_K$ can be identified with a simplicial functor $[n]_K \to \mathcal{C}$.

Proof. For each $n \geq 0$, set $\mathfrak{B}[\Delta^n] = [n]_{\Delta^n}$. By the Yoneda lemma, the simplicial categories $\{\mathfrak{B}[\Delta^n]\}_{n\geq 0}$ can be organized into a cosimplicial object of Cat_Δ in such a way that the functor B is naturally isomorphic to $\mathsf{Cat}_\Delta(\mathfrak{B}[\Delta^\bullet], -)$. It suffices to show that there is a unique morphism $\mathfrak{C}[\Delta^\bullet] \to \mathfrak{B}[\Delta^\bullet]$ of cosimplicial simplicial categories.

We begin by showing the uniqueness. Suppose there is a map $f \colon \mathfrak{C}[\Delta^\bullet] \to \mathfrak{B}[\Delta^\bullet]$ of cosimplicial simplicial categories. For each $n \geq 0$, the map

$f_n\colon \mathfrak{C}[\Delta^n] \to \mathfrak{B}[\Delta^n]$ must act on the identity maps on objects because f_n is natural in n. If $0 \leq i \leq j \leq n$ are integers, then the map

$$f_n\colon \mathfrak{C}[\Delta^n](i,j) \to \mathfrak{B}[\Delta^n](i,j)$$

is completely determined by its values of the vertices, for both $\mathfrak{C}[\Delta^n](i,j)$ and $\mathfrak{B}[\Delta^n](i,j)$ are nerves of posets. Since every vertex of $\mathfrak{C}[\Delta^n](i,j)$ is a composition of morphisms in the image of maps $\mathfrak{C}[\Delta^1] \to \mathfrak{C}[\Delta^n]$, we deduce that f_n is completely determined by f_1. Now there are exactly two simplicial functors $\mathfrak{C}[\Delta^1] \to \mathfrak{B}[\Delta^1]$ which are bijective on objects (because $\mathfrak{C}[\Delta^1](0,1) = \Delta^0$ and $\mathfrak{B}[\Delta^1](0,1) = \Delta^1$.) If f_1 carried the unique vertex of $\mathfrak{C}[\Delta^1](0,1)$ to the vertex $1 \in \mathfrak{B}[\Delta^1](0,1)$, then the map

$$f_2\colon \mathfrak{C}[\Delta^2](0,2) \to \mathfrak{B}[\Delta^2](0,2)$$

would carry the vertices $\{0,2\}$ and $\{0,1,2\}$ to $(2,2)$ and $(2,1)$, respectively. But since there is no edge $(2,2) \to (2,1)$ in $\mathfrak{B}[\Delta^2](0,2)$, this is impossible. Hence there is only a unique choice for f_1, completing the proof of the uniqueness.

For existence, define $f_n\colon \mathfrak{C}[\Delta^n] \to \mathfrak{B}[\Delta^n]$ as follows: On objects, f_n acts by the identity map. For each $0 \leq i \leq j \leq n$, the map $\mathfrak{C}[\Delta^n](i,j) \to \mathfrak{B}[\Delta^n](i,j)$ is the nerve of the poset map $P_{i,j} \to \underbrace{[n] \times \cdots \times [n]}_{j-i \text{ times}}$ which assigns to each element $\{i = i_0 < \cdots < i_k = j\} \in P_{i,j}$ the element

$$(\underbrace{i_{k-1}, \dots, i_{k-1}}_{i_k - i_{k-1} \text{ times}}, \dots, \underbrace{i_0, \dots, i_0}_{i_1 - i_0 \text{ times}}) \in [n] \times \cdots \times [n].$$

It is easy to check that the simplicial functors $\{f_n\}_{n\geq 0}$ indeed define a map of cosimplicial objects of Cat_Δ. The claim follows. $\square$

Remark 2.4. Recall that the diagonal of a bisimplicial set X is equal to the coend $\int^{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_{*,n} \times \Delta^n$. Therefore, given a simplicial category $\mathcal{C}$, the map $B(\mathcal{C}) \to N_{\mathrm{hc}}(\mathcal{C})$ of Proposition 2.1 may equally well be specified by a compatible family of maps $\{\phi_n\colon N(\mathcal{C}_n) \times \Delta^n \to N_{\mathrm{hc}}(\mathcal{C})\}_{n\geq 0}$. Unwinding the definitions, the composite $N(\mathcal{C}_n) \times \{i\} \hookrightarrow N(\mathcal{C}_n) \times \Delta^n \to N_{\mathrm{hc}}(\mathcal{C})$ is equal to the composite

$$N(\mathcal{C}_n) \xrightarrow{i^*} N(\mathcal{C}_0) \to N_{\mathrm{hc}}(\mathcal{C}),$$

where the first map is the restriction along the inclusion $[0] \cong \{i\} \hookrightarrow [n]$ and the second map is induced by the simplicial functor $\mathcal{C}_0 \to \mathcal{C}$. In other words, the map ϕ_n is the canonical natural transformation between $n+1$ functors $N(\mathcal{C}_n) \to N_{\mathrm{hc}}(\mathcal{C})$ corresponding to the $n+1$ elements of $[n]$.

Remark 2.5. Recall that the diagonal of a bisimplicial set X is equal to the coend $\int^{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \Delta^n \times X_{n,*}$. Therefore, given a simplicial category $\mathcal{C}$, the map $B(\mathcal{C}) \to N_{\mathrm{hc}}(\mathcal{C})$ of Proposition 2.1 may equally well be specified by a compatible family of maps $\{\psi_n \colon \Delta^n \times N_{\mathrm{bi}}(\mathcal{C})_n \to N_{\mathrm{hc}}(\mathcal{C})\}_{n\geq 0}$. Unwinding the definitions, the map ψ_n admits the following description:

1. If $n = 0$, then ψ_n is the inclusion $N_{\mathrm{bi}}(\mathcal{C})_0 \cong \operatorname{ob}\mathcal{C} \hookrightarrow N_{\mathrm{hc}}(\mathcal{C})$.

2. Let $n \geq 1$ and let $\sigma\colon \Delta^m \to N_{\mathrm{bi}}(\mathcal{C})_n$ be an m-simplex of $N_{\mathrm{bi}}(\mathcal{C})_n$, corresponding to a simplicial functor $\sigma'\colon [n]_{\Delta^m} \to \mathcal{C}$. Then the composite

$$\Delta^n \times \Delta^m \xrightarrow{\mathrm{id}\times\sigma} \Delta^n \times N_{\mathrm{bi}}(\mathcal{C})_n \xrightarrow{\psi_n} N_{\mathrm{hc}}(\mathcal{C})$$

is adjoint to the composite

$$\mathfrak{C}[\Delta^n \times \Delta^m] \xrightarrow{\chi} [n]_{\Delta^m} \xrightarrow{\sigma'} \mathcal{C}.$$

Here χ is defined on objects by $\chi(i,j) = i$. Given integers $0 \leq i \leq i' \leq n$ and $0 \leq j \leq j' \leq m$, let $P_{(i,j),(i',j')}$ denote the poset of linearly ordered subsets $S \subset [n] \times [m]$ such that $\min S = (i,j)$ and $\max S = (i',j')$. Then the simplicial set $\mathfrak{C}[\Delta^n \times \Delta^m]((i,j),(i',j'))$ is the nerve of $P_{(i,j),(i',j')}$, and the map $\mathfrak{C}[\Delta^n \times \Delta^m]((i,j),(i',j')) \to [n]_{\Delta^m}(i,i')$ is induced by the poset map

$$\begin{aligned} P_{(i,j),(i',j')} &\to \prod_{i<p\leq i'} [m], \\ S &\mapsto \left(\max\{q \in [m] \mid (i'',q) \in S \text{ for some } i'' < p\}\right)_{i<p\leq i'}. \end{aligned}$$

3. Main result

In this section, we state and prove the main result of this paper.

Here is the statement of the main result.

Theorem 3.1. *Let $\mathcal{C}$ be a fibrant simplicial category and let $\mathcal{W} \subset \mathcal{C}$ be a wide simplicial subcategory. The maps $\{\psi_n \colon \Delta^n \times N_{\mathrm{bi}}(\mathcal{C})_n \to N_{\mathrm{hc}}(\mathcal{C})\}_{n \geq 0}$ of Remark 2.5 induces a weak equivalence*

$$\theta \colon N^+_{\mathrm{bi}}(\mathcal{C}, \mathcal{W}) \to \mathrm{Cls}^+(N_{\mathrm{hc}}(\mathcal{C}), \mathrm{mor}\, \mathcal{W}_0)$$

of $\mathsf{bsSet}^+_{\mathrm{CSS}}$.

The remainder of this section is devoted to the proof of Theorem 3.1. We begin by establishing the unmarked version of Theorem 3.1:

Proposition 3.2. *Let $\mathcal{C}$ be a fibrant simplicial category. The maps $\{\psi_n \colon \Delta^n \times N_{\mathrm{bi}}(\mathcal{C})_n \to N_{\mathrm{hc}}(\mathcal{C})\}_{n \geq 0}$ of Remark 2.5 induces a weak equivalence*

$$N_{\mathrm{bi}}(\mathcal{C}) \to \mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}))$$

of $\mathsf{bsSet}_{\mathrm{CSS}}$.

Proof. Observe that the Reedy fibrant replacement $N^f_{\mathrm{bi}}(\mathcal{C})$ of $N_{\mathrm{bi}}(\mathcal{C})$ is a Segal space; indeed, for each $n \geq 2$, the square

$$\begin{array}{ccc} N_{\mathrm{bi}}(\mathcal{C})_n & \longrightarrow & N_{\mathrm{bi}}(\mathcal{C})_{n-1} \\ \downarrow & & \downarrow \\ N_{\mathrm{bi}}(\mathcal{C})_1 & \longrightarrow & N_{\mathrm{bi}}(\mathcal{C})_0 \end{array}$$

induced by the inclusions $[1] \cong \{n-1 < n\} \hookrightarrow [n] \hookleftarrow [n-1]$ is homotopy cartesian. Therefore, by [21, Theorem 7.7], it suffices to show that the induced map $N^f_{\mathrm{bi}}(\mathcal{C}) \to \mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}))$ is a Dwyer–Kan equivalence. Since the map $N_{\mathrm{bi}}(\mathcal{C})_{0,0} \to \mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}))_{0,0}$ is bijective, it suffices to show that the square

$$\begin{array}{ccc} N_{\mathrm{bi}}(\mathcal{C})_1 & \longrightarrow & \mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}))_1 \\ \downarrow & & \downarrow \\ N_{\mathrm{bi}}(\mathcal{C})_0 \times N_{\mathrm{bi}}(\mathcal{C})_0 & \longrightarrow & \mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}))_0 \times \mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}))_0 \end{array}$$

is homotopy cartesian. For each pair of objects $X, Y \in \mathcal{C}$, the induced map between the fibers of the vertical arrows over (X, Y) can be identified with the homotopy equivalence

$$\mathcal{C}(X, Y) \to \mathrm{Hom}_{N_{\mathrm{hc}}(\mathcal{C})}(X, Y)$$

of [19, Tag 01LF] (see Remark 2.5). Hence the square is homotopy cartesian, as required. □

Before we proceed, we recall a few facts on marked bisimplicial sets.

Notation 3.3. [1, Definition 3.1] We let $\mathrm{diag}^+ \colon \mathsf{bsSet}^+ \to \mathsf{sSet}^+$ denote the functor

$$\mathrm{diag}^+(X, S) = (\mathrm{diag}(X), S_1),$$

where $\mathrm{diag}(X) = \{X_{n,n}\}_{n \geq 0}$ denotes the diagonal of X. The functor diag^+ is the left adjoint of the functor $\mathrm{Cls}^+ \colon \mathsf{sSet}^+ \to \mathsf{bsSet}^+$.

Theorem 3.4. *There is a simplicial model structure on* bsSet^+*, denoted by* $\mathsf{bsSet}^+_{\mathrm{CSS}}$*, which has the following properties:*

1. *The cofibrations are the monomorphisms.*
2. *The fibrant objects are the marked bisimplicial sets of the form* $\mathcal{X}^\natural = (\mathcal{X}, \mathcal{X}_{\mathrm{hoeq}})$*, where* $\mathcal{X}$ *is a complete Segal space and* $\mathcal{X}_{\mathrm{hoeq}} \subset \mathcal{X}_1$ *is the union of components spanned by homotopy equivalences of* $\mathcal{X}$*.*

Moreover, the model structure has the following additional properties:

3. *If* $\mathcal{X}$ *is a complete Segal space and* (A, S) *is a marked bisimplicial set, then* $\mathrm{Map}\big((A, S), \mathcal{X}^\natural\big)$ *is the component of* $\mathrm{Map}(A, \mathcal{X})$ *spanned by the maps* $A \to \mathcal{X}$ *which carries* S *into* $\mathcal{X}_{\mathrm{hoeq}}$*. Here* $\mathrm{Map}(A, \mathcal{X})$ *denotes the simplicial enrichment of* bsSet *adapted to the complete Segal space model structure [21, §2.3].*
4. *The adjunction* $\mathrm{diag}^+ \colon \mathsf{bsSet}^+_{\mathrm{CSS}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{sSet}^+ \colon \mathrm{Cls}^+$ *is a Quillen equivalence.*
5. *The model structure* $\mathsf{bsSet}^+_{\mathrm{CSS}}$ *is a Bousfield localization of the Reedy model structure on* $\mathsf{bsSet}^+ = \big(\mathsf{sSet}^+\big)^{\mathbf{\Delta}^{\mathrm{op}}}$*.*
6. *The functor* $\mathrm{Cls}^+ \colon \mathsf{sSet}^+ \to \mathsf{bsSet}^+_{\mathrm{CSS}}$ *preserves and reflects all weak equivalences.*

Proof. The first half is established in [1, Theorem 2.9]. Point (3) is proved in [1, Remark 2.7], points (4) and (5) are proved in [1, Theorem 3.4], and point (6) is proved in [1, Theorem 4.2]. □

Notation 3.5. Let $(\mathcal{C}, \mathcal{W})$ be a relative simplicial category. We define a marked bisimplicial set $B^+(\mathcal{C}, \mathcal{W})$ by

$$B^+(\mathcal{C}, \mathcal{W}) = \operatorname{diag}^+ N^+_{\mathrm{bi}}(\mathcal{C}, \mathcal{W}).$$

We now arrive at the proof of Theorem 3.1.

Proof of Theorem 3.1. We prove the theorem in four steps.

(Step 1) Assume first that $\mathcal{W} \subset \mathcal{C}$ is the smallest wide simplicial subcategory containing all homotopy equivalences of $\mathcal{C}$. If $\mathcal{X}$ is a complete Segal space, then every morphism $N_{\mathrm{bi}}(\mathcal{C}) \to \mathcal{X}$ of bisimplicial sets induces a map $N^+_{\mathrm{bi}}(\mathcal{C}, \mathcal{W}) \to \mathcal{X}^\natural$ of marked bisimplicial sets. Indeed, we only have to show that the induced map $N(\mathcal{C}_0) \to \mathcal{X}_{*,0}$ between the 0th row respects the markings (because $\mathcal{X}_{\mathrm{hoeq}} \subset \mathcal{X}$ is a union of components), which is obvious. Likewise, any map $\mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C})) \to \mathcal{X}$ lifts to a morphism $\mathrm{Cls}^+(N_{\mathrm{hc}}(\mathcal{C}), N_{\mathrm{hc}}(\mathcal{W})) \to \mathcal{X}^\natural$. Therefore, by properties (1) through (3) of Theorem 3.4, it suffices to show that the underlying map

$$N_{\mathrm{bi}}(\mathcal{C}) \to \mathrm{Cls}(N_{\mathrm{hc}}(\mathcal{C}))$$

of θ is a weak equivalence of $\mathsf{bsSet}_{\mathrm{CSS}}$. This is nothing but Proposition 3.2.

(Step 2) According to Theorem 3.4, the functor $\mathrm{Cls}^+ \colon \mathsf{sSet}^+ \to \mathsf{bsSet}^+_{\mathrm{CSS}}$ is a right Quillen equivalence and preserves all weak equivalences. Therefore, θ is a weak equivalence of $\mathsf{bsSet}^+_{\mathrm{CSS}}$ if and only if the map

$$B^+(\mathcal{C}, \mathcal{W}) \to (N_{\mathrm{hc}}(\mathcal{C}), \operatorname{mor} \mathcal{W}_0)$$

which is adjoint to θ is a weak equivalence of sSet^+.

(Step 3) Suppose that $\mathcal{W}$ contains all homotopy equivalences of $\mathcal{C}$. We observe that:

- The marked edges of $B^+(\mathcal{C}, \mathcal{W})$ are precisely the inverse image of $\operatorname{mor} \mathcal{W}_0$ under the map $B(\mathcal{C}) \to N_{\mathrm{hc}}(\mathcal{C})$; this is because $(\mathcal{C}, \mathcal{W})$ is a relative simplicial category.
- The map $B(\mathcal{C}) \to N_{\mathrm{hc}}(\mathcal{C})$ induces a surjection between the set of edges; this follows by inspection.

Since $\operatorname{mor} \mathcal{W}_0$ contains all equivalences of $N_{\mathrm{hc}}(\mathcal{C})$, the claim now follows by combining the above observations, Steps 1 and 2, and the definition of weak equivalences of marked simplicial sets.

(Step 4) We prove the theorem in the general case. Let $\mathcal{W}' \subset \mathcal{C}$ denote the smallest wide simplicial subcategory containing $\mathcal{W}$ and all homotopy equivalences of $\mathcal{C}$. The map $N_{\mathrm{bi}}^{+}(\mathcal{C}, \mathcal{W}) \to N_{\mathrm{bi}}^{+}(\mathcal{C}, \mathcal{W}')$ is a weak equivalence of $\mathsf{bsSet}_{\mathrm{CSS}}^{+}$, and the map $(N_{\mathrm{hc}}(\mathcal{C}), \operatorname{mor} \mathcal{W}_0) \to (N_{\mathrm{hc}}(\mathcal{C}), \operatorname{mor} \mathcal{W}'_0)$ is a weak equivalence of sSet^{+}. The claim now follows from part (6) of Theorem 3.4.

□

Remark 3.6. Let $\mathcal{C}$ be a fibrant simplicial category and let $\mathcal{W} \subset \mathcal{C}$ be a wide simplicial subcategory. As we saw in Step 2 of the proof of Theorem 3.1, Theorem 3.1 implies that the map

$$B^{+}(\mathcal{C}, \mathcal{W}) \to (N_{\mathrm{hc}}(\mathcal{C}), \operatorname{mor} \mathcal{W}_0)$$

is a weak equivalence of marked simplicial sets. Since the geometric realization functor models homotopy colimits in simplicial model categories ([22, Theorem 5.2.3], [11, Lemma 15.3.9]), and since $\operatorname{diag}^{+}\colon \mathsf{bsSet}^{+} \to \mathsf{sSet}^{+}$ is nothing but the geometric realization functor, we may interpret Theorem 3.1 as saying that the localization $N_{\mathrm{hc}}(\mathcal{C})[N_{\mathrm{hc}}(\mathcal{W})^{-1}]$ is a homotopy colimit of the simplicial ∞-category $[n] \mapsto N(\mathcal{C}_n)[N(\mathcal{W}_n)^{-1}]$. This point of view was articulated by Lurie in [18, Proposition 1.3.4.14]; in fact, Theorem 3.1 can also be proved using Lurie's result (and Theorem 3.4), though the proof will be a little longer.

4. Applications

We now list two applications of Theorem 3.1.

Corollary 4.1. *Let $(\mathcal{C}, \mathcal{W})$ and $(\mathcal{C}', \mathcal{W}')$ be relative simplicial categories, where $\mathcal{C}$ and $\mathcal{C}'$ are fibrant. Let $f\colon \mathcal{C} \to \mathcal{C}'$ be a simplicial functor which carries $\mathcal{W}$ into $\mathcal{W}'$. If for each $n \geq 0$, the map*

$$(N(\mathcal{C}_n), \operatorname{mor} \mathcal{W}_n) \to (N(\mathcal{C}'_n), \operatorname{mor} \mathcal{W}'_n)$$

is a weak equivalence of marked simplicial sets, then the map

$$(N_{\mathrm{hc}}(\mathcal{C}), \operatorname{mor} \mathcal{W}_0) \to (N_{\mathrm{hc}}(\mathcal{C}'), \operatorname{mor} \mathcal{W}_0')$$

is also a weak equivalence.

Proof. By hypothesis, the map $N^+_{\mathrm{bi}}(\mathcal{C}, \mathcal{W}) \to N^+_{\mathrm{bi}}(\mathcal{C}', \mathcal{W}')$ induces a weak equivalence of marked simplicial sets in each row. Thus, by part (5) of Theorem 3.4, it is a weak equivalence of $\mathsf{bsSet}^+_{\mathrm{CSS}}$. It follows from Theorem 3.1 that the map $\mathrm{Cls}^+(N_{\mathrm{hc}}(\mathcal{C}), \operatorname{mor} \mathcal{W}_0) \to \mathrm{Cls}^+(N_{\mathrm{hc}}(\mathcal{C}'), \operatorname{mor} \mathcal{W}_0')$ is a weak equivalence of $\mathsf{bsSet}^+_{\mathrm{CSS}}$. Using part (6) of Theorem 3.4, we deduce that the map $(N_{\mathrm{hc}}(\mathcal{C}), \operatorname{mor} \mathcal{W}_0) \to (N_{\mathrm{hc}}(\mathcal{C}'), \operatorname{mor} \mathcal{W}_0')$ is also a weak equivalence of sSet^+, and we are done. □

Remark 4.2. We do not know if the converse of Corollary 4.1 holds. We expect that this is false, given that the proof of the corollary relies on point (5) of Theorem 3.4, which only gives a *sufficient* condition for a map of $\mathsf{bsSet}^+_{\mathrm{CSS}}$ to be a weak equivalence. However, we are not aware of explicit counterexamples.

Corollary 4.3. *Let $\mathcal{C}$ be a fibrant simplicial category. The map*

$$B(\mathcal{C}) \to N_{\mathrm{hc}}(\mathcal{C})$$

of Proposition 2.1 is a weak homotopy equivalence.

Proof. By definition, every weak equivalence of marked simplicial sets induces a weak homotopy equivalence between the underlying simplicial sets [17, Proposition 3.1.3.3]. It will therefore suffice to show that the map $B^+(\mathcal{C}, \mathcal{C}^{\simeq}) \to (N_{\mathrm{hc}}(\mathcal{C}), (\mathcal{C}^{\simeq})_0) = N_{\mathrm{hc}}(\mathcal{C})^{\natural}$ is a weak equivalence of marked simplicial sets, where $\mathcal{C}^{\simeq} \subset \mathcal{C}$ denotes the smallest wide simplicial subcategory containing all equivalences of $\mathcal{C}$. This is immediate from Theorem 3.1 (and part (4) of Theorem 3.4). □

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXVI-4 (2025)

ALL SEGAL OBJECTS ARE GENERALISED MONADS IN SPANS

David KERN

Résumé. Nous étendons la construction par Barwick et Haugseng d'une ∞-catégorie double de correspondances dans une ∞-catégorie $\mathfrak{C}$ admettant les produits fibrés à des formes plus générales : pour une large classe de patrons algébriques $\mathfrak{P}$, nous définissons une ∞-catégorie $\mathfrak{P}$-monoïdale de correspondances $\mathfrak{P}$-modelées dans $\mathfrak{C}$, et identifions les $\mathfrak{P}$-monades dedans avec les $\mathfrak{P}$-objets de Segal dans $\mathfrak{C}$. Pour le patron cellulaire Θ^{op}, cela recouvre une reformulation homotopique de la définition originale de Batanin des ω-catégories faibles, et en général peut être vu comme une variante des multicatégories généralisées de Burroni, Hermida, Leinster et Cruttwell–Shulman.
Abstract. We extend Barwick's and Haugseng's construction of the double ∞-category of spans in a pullback-complete ∞-category $\mathfrak{C}$ to more general shapes: for a large class of algebraic patterns $\mathfrak{P}$, we define a $\mathfrak{P}$-monoidal ∞-category of $\mathfrak{P}$-shaped spans in $\mathfrak{C}$, and we identify $\mathfrak{P}$-monads in it with Segal $\mathfrak{P}$-objects in $\mathfrak{C}$. For the cell pattern Θ^{op}, this recovers a homotopical reformulation of Batanin's original definition of weak ω-categories, and in general can be seen as a variant of the generalised multicategories of Burroni, Hermida, Leinster and Cruttwell–Shulman.
Keywords. Segal objects, spans, double categories, weak ω-categories, multicategories.
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1. Introduction

1.1 Algebraic structures for higher categories

The various definitions of higher categories come in two families: algebraic definitions specify the minimal amount of shape data (for ℓ-categories, an ℓ-graph, comprised only of elementary cells) and add the structure of all the composition operations and their higher coherences, while geometric definitions start from a bigger shape containing all the possible pasting diagrams of cells and simply impose conditions to ensure that they come from decompositions into compatible elementary cells.

For example, the standard definition of an internal category, in a category $\mathfrak{C}$ admitting finite pullbacks, is as a Δ^{op}-shaped object $X_\bullet$ of $\mathfrak{C}$ — where Δ is the category of free categories on 1-dimensional pasting diagrams, that is sequences of composable arrows — satisfying the Segal decomposition condition which expresses each value X_n on a pasting of n consecutive arrows as $X_1 \times_{X_0} \cdots \times_{X_0} X_1$. This can be reinterpreted in a more algebraic way as giving a graph $X_\bullet|_{\{[0],[1]\}}$ in $\mathfrak{C}$ and a certain kind of algebra structure on it, subject to the simplicial identities. To make good sense of this algebra structure, it was noticed by [Bén67] that a graph in $\mathfrak{C}$ is nothing but an endomorphism in the bicategory (or better, the double category) of spans in $\mathfrak{C}$, and the required algebra structure is none other than a structure of monad on this endomorphism.

For *strict* higher categories, the situation generalises directly: on the one hand, [Joy97] introduced a category Θ of free ω-categories on ω-categorical pasting diagrams, so that strict ω-categories in any category with fibre products $\mathfrak{C}$ are exactly $\mathfrak{C}$-valued presheaves on Θ satisfying a Segal condition. On the other hand, [Bat98] constructed an internal (strict) ω-category in $\mathfrak{Cat}$ (a globular object in $\mathfrak{Cat}$ equipped with compositions) $\mathbb{Span}_\infty(\mathfrak{C})$ of infinitely iterated spans in $\mathfrak{C}$, so that globular monads in it are exactly strict ω-categories internal to $\mathfrak{C}$.

The key insight of [Bat98] is then that, using the higher structure naturally present in globular categories, one can refine the teminal globular operad to a suitable contractible globular operad $\mathscr{A}_\infty^{\mathbb{G}}$, which contains enough coherence data for $\mathscr{A}_\infty^{\mathbb{G}}$-algebras in $\mathbb{Span}_\infty(\mathfrak{C})$ to be a good definition of weak ω-categories in $\mathfrak{C}$.

While the presence of higher cells in globular sets allows one to make sense of $\mathscr{A}_\infty^{\mathbb{G}}$ as an algebraic resolution of the terminal globular operad, eschewing any homotopical machinery, formulating things in a setting of homotopy theory allows many constructions to become simpler, and more widely applicable. Indeed, the logic of using the higher cells to tame the infinite towers of coherences needed for a resolution only works for full ω-categories, but breaks down if trying to define weak ℓ-categories for some $\ell < \omega$. Nonetheless, [Hau21] showed that the situation for (weak) 1-categories can be dealt with using ∞-categories: category objects in an $(\infty, 1)$-category $\mathfrak{C}$ are identified with homotopy-coherent monads in the double $(\infty, 1)$-category of spans in $\mathfrak{C}$.

In this note, we extend this result (as a direct application of Theorem 5.7) to a characterisation of ℓ-category objects as ℓ-globular monads in ℓ-times iterated spans, which both extends Batanin's definition of weak ω-categories to one for weak ℓ-categories for any $\ell \leq \omega$, and also simplifies it by removing the need to resolve the terminal globular ∞-operad by a more complicated one.

1.2 Multicategories and algebraic patterns

In order to understand how to construct categories of generalised spans, let us switch gears to another categorical structure that can be defined in a similar way: multicategories, or coloured operads. It was noticed by [Bur71, Her04, Lei98] that multicategories can be defined as monads in a double category of Kleisli $\mathscr{M}$-spans, where $\mathscr{M}$ is the "free monoid" monad on $\mathfrak{Set}$, fitting in a more general framework of $\mathscr{T}$-multicategories, for a cartesian monad $\mathscr{T}$, as monads in a double category of Kleisli $\mathscr{T}$-spans, whose morphisms are the spans twisted by $\mathscr{T}$ on their source, and whose composition uses $\mathscr{T}$'s monad structure. In particular, Batanin's globular operads can also be obtained in this way.

Unfortunately, this double category of Keisli $\mathscr{T}$-spans is not characterised by a clear universal property (see [CS10, Remark 4.2]), which makes constructing it in the ∞-categorical world very difficult. Because of this, we will instead use a different kind of structure to organise the generalised spans.

To explain the idea, let us keep focusing of the example of multicategories. An $\mathscr{M}$-span from a set Y_0 to a set X_0 is given by a span $\mathscr{M}Y_0 \leftarrow X_1 \to X_0$, which we interpret as a multispan (as championed by [Baa19] for the study of hyperstructures) of some arbitrary arity $a+1$, one of whose legs (the root) goes to X_0 and the a others (the leaves) to Y_0. To compose it with an $\mathscr{M}$-span $\mathscr{M}Z_0 \leftarrow Y_1 \to Y_0$, one forms the $\mathscr{M}$-span

$$\begin{tikzcd}[column sep=small]
 & & & \mathscr{M}Y_1 \underset{\mathscr{M}Y_0}{\times} X_1 \arrow[dl] \arrow[dr] & & \\
 & & \mathscr{M}Y_1 \arrow[dl] \arrow[dr] & & X_1 \arrow[dl] \arrow[dr] & \\
\mathscr{M}Z_0 & \mathscr{M}^2 Z_0 \arrow[l, "\mu"'] & & \mathscr{M}Y_0 & & X_0
\end{tikzcd} \tag{1}$$

expressing that one takes a copies of the multispan corresponding to $\mathscr{M}Z_0 \leftarrow Y_1 \to Y_0$ and glues their distinguished roots to the various leaves of $\mathscr{M}Y_0 \leftarrow X_1 \to X_0$.

As is usual in operad theory, one also, instead of blowing up the situation globally, glue a single new span to one leaf of $\mathscr{M}Y_0 \leftarrow X_1 \to X_0$; the composition operation defined in this way, leaf by leaf, will no longer be a categorical composition, but indeed an operadic one. Thus, multispans can be organised, instead of in a double category, in a categorical operad (internal category in the category of operads).

While there are many different approaches to operadic structures in the 1-categorical setting, in the ∞-categorical one a very convenient and powerful framework is that of the algebraic patterns of [CH21], which extract the necessary data on a category of shapes to speak of Segal decompositions (inert morphisms from elementary objects) and keep additional algebraic operations (active morphisms): in other words, they give a geometric presentation of ∞-operadic structure, while remembering what is the algebraic part. In the approach that we will follow in this note, the choice of an algebraic pattern will play the role of the choice of the cartesian monad $\mathscr{T}$ in the story sketched above.

We will then construct in section 4, for any algebraic pattern $\mathfrak{P}$ (satisfying the very mild condition of soundness — that will be verified in all examples we know of, in particular in section 3 for ω-categories) and any complete enough $(\infty,1)$-category $\mathfrak{C}$, a Segal $\mathfrak{P}$-object $\mathbb{Span}_{\mathfrak{P}}(\mathfrak{C})$ in $(\infty,1)$-$\mathfrak{Cat}$

of $\mathfrak{P}$-shaped spans in $\mathfrak{C}$, by adapting the construction of [Hau18a] with the ideas raised in [Str00] and expanded upon in [Web07, Example 4.8]. We will continue in section 5 by showing the result promised above:

Theorem A. *(cf. Theorem 5.7) There is an equivalence between $\mathfrak{P}$-monads in the $\mathfrak{P}$-monoidal ∞-category $\mathbb{Span}_{\mathfrak{P}}(\mathfrak{C})$ and Segal $\mathfrak{P}$-objects in $\mathfrak{C}$.*

Then, in section 6, we will spell out the meaning of this result in examples of interest, in particular (∞,ℓ)-categories.

1.3 Aknowledgements

This note was closely inspired by the ideas of [Hau18a, Hau21] and [Str00], and would not exist without the insights developed in these works. Thanks are also due to Damien Calaque for discussions about algebras in iterated spans, to Hugo Pourcelot for conversations about differently-shaped spans, and to Reuben Stern for useful comments about the interpretation of fibrous Θ^{op}-patterns. Many thanks to Jan Steinebrunner and Thomas Blom for pointing out the necessity of soundness in lemma 4.7 and the automaticity of global saturation, and to Jan especially sharing material on sound patterns and a proof of global saturation. I thank the anonymous reviewer for several corrections and improvements.

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2. Algebraic and fibrous patterns

Definition 2.1 (Algebraic pattern)**.** An **algebraic pattern** is a diagram of inclusions of $(\infty,1)$-categories

$$\begin{tikzcd} & & \mathfrak{P} & \\ \mathfrak{P}^{\mathrm{el}} \arrow[r, hook] & \mathfrak{P}^{\mathrm{inrt}} \arrow[ur, hook] & & \mathfrak{P}^{\mathrm{act}} \arrow[ul, hook'] \end{tikzcd} \tag{2}$$

where the wide sub-$(\infty,1)$-categories $(\mathfrak{P}^{\mathrm{inrt}}, \mathfrak{P}^{\mathrm{act}})$ form an orthogonal factorisation system on $\mathfrak{P}$ and $\mathfrak{P}^{\mathrm{el}} \subset \mathfrak{P}^{\mathrm{inrt}}$ is a full sub-$(\infty,1)$-category.

The **inert** arrows (those in $\mathfrak{P}^{\mathrm{inrt}}$) are denoted as $\rightarrowtail$ and the **active** ones (those in $\mathfrak{P}^{\mathrm{act}}$) are denoted as $\rightsquigarrow$, while the objects in $\mathfrak{P}^{\mathrm{el}}$ are known as **elementary**.

Notation 2.2. For any $P \in \mathfrak{P}$, we write $\mathfrak{P}^{\mathrm{el}}_{P/} = \mathfrak{P}^{\mathrm{el}} \times_{\mathfrak{P}^{\mathrm{inrt}}} \mathfrak{P}^{\mathrm{inrt}}_{P/}$.

An $(\infty,1)$-category $\mathfrak{C}$ is said to be **$\mathfrak{P}$-complete** if it admits limits of diagrams of shape $\mathfrak{P}^{\mathrm{el}}_{P/}$ for any $P \in \mathfrak{P}$.

Definition 2.3 (Segal object)**.** Let $\mathfrak{P}$ be an algebraic pattern and $\mathfrak{C}$ a $\mathfrak{P}$-complete $(\infty,1)$-category. A **Segal $\mathfrak{P}$-object** in $\mathfrak{C}$ is a functor $\mathscr{X}\colon \mathfrak{P} \to \mathfrak{C}$ such that $\mathscr{X}|_{\mathfrak{P}^{\mathrm{inrt}}}$ is the right Kan extension of its restriction to $\mathfrak{P}^{\mathrm{el}}$, which means that for any $P \in \mathfrak{P}$, the canonical arrow

$$\mathscr{X}(P) \to \lim_{E \in \mathfrak{P}^{\mathrm{el}}_{P/}} \mathscr{X}(E) \tag{3}$$

is an equivalence.

The full sub-$(\infty,1)$-category of the functor $(\infty,1)$-category $\left\{\mathfrak{P},\mathfrak{C}\right\}$ on the Segal objects is denoted $\mathfrak{Seg}_{\mathfrak{P}}(\mathfrak{C})$.

Example 2.4 (Product patterns). The $(\infty,1)$-category of algebraic patterns admits all limits, which can be computed at the level of the underlying $(\infty,1)$-categories. In particular, it admits products, and these are compatible with currying, in that if $\mathfrak{P}$ and $\mathfrak{Q}$ are two algebraic patterns and $\mathfrak{C}$ is $\mathfrak{P}\times\mathfrak{Q}$-complete, then $\mathfrak{Seg}_{\mathfrak{Q}}(\mathfrak{C})$ is $\mathfrak{P}$-complete and there is an equivalence $\mathfrak{Seg}_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{C}) \simeq \mathfrak{Seg}_{\mathfrak{P}}(\mathfrak{Seg}_{\mathfrak{Q}}(\mathfrak{C}))$.

Example 2.5 ($\mathfrak{P}$-graphs). As observed in [CH21, beginning of §8], any algebraic pattern $\mathfrak{P}$ restricts to a pattern structure on $\mathfrak{P}^{\mathrm{inrt}}$, whose only active morphisms are the equivalences, and further restricts to $\mathfrak{P}^{\mathrm{el}}$. Evidently, the restriction–right Kan extension adjunctions along $\mathfrak{P}^{\mathrm{inrt,el}} = \mathfrak{P}^{\mathrm{el}} \hookrightarrow \mathfrak{P}^{\mathrm{inrt}}$ and $\mathfrak{P}^{\mathrm{el,el}} = \mathfrak{P}^{\mathrm{el}} \hookrightarrow \mathfrak{P}^{\mathrm{el}}$ induce equivalences $\mathfrak{Seg}_{\mathfrak{P}^{\mathrm{inrt}}}(\mathfrak{C}) \simeq \left\{\mathfrak{P}^{\mathrm{el}},\mathfrak{C}\right\}$ and $\mathfrak{Seg}_{\mathfrak{P}^{\mathrm{el}}}(\mathfrak{C}) \simeq \left\{\mathfrak{P}^{\mathrm{el}},\mathfrak{C}\right\}$ for any $\mathfrak{P}$-complete $(\infty,1)$-category $\mathfrak{C}$. We will refer to (necessarily Segal) $\mathfrak{P}^{\mathrm{el}}$-objects as **$\mathfrak{P}$-graph**, and to the restriction of a Segal $\mathfrak{P}$-object to $\mathfrak{P}^{\mathrm{el}}$ as its **underlying $\mathfrak{P}$-graph**.

When $\mathfrak{C}$ is $(\infty,1)$-$\mathfrak{Cat}$, Segal $\mathfrak{P}$-objects $\mathfrak{P} \to (\infty,1)$-$\mathfrak{Cat}$ can also be seen as categoy objects in the ∞-category $\mathfrak{Seg}_{\mathfrak{P}}(\infty\text{-}\mathfrak{Grpd})$ of Segal $\mathfrak{P}$-∞-groupoids, and as such will generally be written as $\mathbb{X},\mathbb{Y},\ldots$, in the font reserved for internal categories. Such an object $\mathbb{X}\colon \mathfrak{P} \to (\infty,1)$-$\mathfrak{Cat}$ can be recast as a cocartesian fibration $\mathfrak{X} = \int^P \mathbb{X} \to \mathfrak{P}$ satisfying the Segal condition for its fibres. We call such fibrations **Segal $\mathfrak{P}$-fibrations**. A certain weakening of this notion turns out to be extremely useful, in particular to define lax morphisms between Segal fibrations.

Definition 2.6 (Fibrous pattern)**.** Let $\mathfrak{P}$ be an algebraic pattern. A **fibrous $\mathfrak{P}$-pattern** is an $(\infty, 1)$-functor $\not{f}\colon \mathfrak{X} \to \mathfrak{P}$ such that:

1. for every object $X \in \mathfrak{X}$, every inert arrow $i\colon \not{f}X \to P$ in $\mathfrak{P}$ admits a $\not{f}$-cocartesian lift $i_!\colon X \to i_!X$;

2. for every $P \in \mathfrak{P}$, the commutative square

$$\begin{tikzcd} \mathfrak{X} \underset{\mathfrak{P}}{\times} \mathfrak{P}^{\mathrm{act}}_{/P} \arrow[r] \arrow[d] & \lim_{E \in \mathfrak{P}^{\mathrm{el}}_{P/}} \mathfrak{X} \underset{\mathfrak{P}}{\times} \mathfrak{P}^{\mathrm{act}}_{/E} \arrow[d] \\ \mathfrak{P}^{\mathrm{act}}_{/P} \arrow[r, "\lim_{E \in \mathfrak{P}^{\mathrm{el}}_{P/}} (P \rightarrowtail E)_!"'] & \lim_{E \in \mathfrak{P}^{\mathrm{el}}_{P/}} \mathfrak{P}^{\mathrm{act}}_{/E} \end{tikzcd} \tag{4}$$

   is cartesian.

A **morphism of fibrous $\mathfrak{P}$-patterns** from $\mathfrak{X} \to \mathfrak{P}$ to $\mathfrak{Y} \to \mathfrak{P}$ is an ∞-functor $\mathfrak{X} \to \mathfrak{Y}$ over $\mathfrak{P}$ preserving cocartesian arrows over inert arrows of $\mathfrak{P}$.

Morphisms from $\mathfrak{X} \to \mathfrak{P}$ to $\mathfrak{Y} \to \mathfrak{P}$ are also called $\mathfrak{X}$-algebras in $\mathfrak{Y}$, and their $(\infty, 1)$-category is denoted $\mathfrak{Alg}_{\mathfrak{X}}(\mathfrak{Y})$.

Lemma 2.7 ([CH21, Lemma 9.10])**.** *The domain of fibrous pattern $\not{f}\colon \mathfrak{X} \to \mathfrak{P}$ admits a structure of algebraic pattern, where an arrow is active if it is over an active arrow of $\mathfrak{P}$, inert if it is $\not{f}$-cocartesian and lies over an inert arrow, and an object is elementary if it lies over an elementary of $\mathfrak{P}$. In particular, Segal morphisms between (sources of) fibrous $\mathfrak{P}$-patterns are exactly their morphisms of fibrous $\mathfrak{P}$-patterns.*

If $\mathfrak{X} \to \mathfrak{P}$ and $\mathfrak{Y} \to \mathfrak{P}$ are Segal $\mathfrak{P}$-fibrations, with corresponding $\mathfrak{P}$-monoidal $(\infty, 1)$-categories $\mathbb{X}$ and $\mathbb{Y}$, morphisms of fibrous patterns $\mathfrak{X} \to \mathfrak{Y}$ can be seen as the lax morphisms $\mathbb{X} \to \mathbb{Y}$.

Definition 2.8 ($\mathfrak{P}$-Monads)**.** Let $\not{\ell}\colon \mathfrak{X} \to \mathfrak{P}$ be a fibrous $\mathfrak{P}$-pattern. A **$\mathfrak{P}$-monad** in $\mathfrak{X}$ is a morphism from the terminal (weak) Segal $\mathfrak{P}$-fibration $\mathfrak{P} \xrightarrow{\mathrm{id}} \mathfrak{P}$ to $\not{\ell}$.

In other words, a $\mathfrak{P}$-monad is a $\mathfrak{P}$-algebra in $\mathfrak{X}$.

Remark 2.9. When $\mathfrak{P}$ is the algebraic pattern $\Delta^{\mathrm{op}\natural}$ for internal categories (recalled in section 6.2), this recovers the usual definition of monads in double ∞-categories. More generally, for enrichable patterns, typically those denoted with a $(-)^\natural$ superscript in [CH21], $\mathfrak{P}$-monads can be thought of as a kind of $\mathfrak{P}$-shaped generalisation of monads, as will be explained in examples in section 6. On the other hand, for the associated cartesian patterns $\mathfrak{P}^\flat$, then $\mathfrak{P}^\flat$-monads correspond rather to a kind of monoids, recovering the etymologically motivating observation of [Bén67, § (5.4.1)].

An important technical condition on algebraic patterns will be that of soundness from [BHS22], which we will introduce with an alternative (equivalent) presentation due to [BS≥25] that is convenient to handle.

Construction 2.10. Let $\not{\ell}\colon \mathfrak{O} \to \mathfrak{P}$ be a morphism of algebraic patterns. The **inert factorisation** $(\infty,1)$**-category** of $\not{\ell}$ at $O \in \mathfrak{O}$ and $v\colon \not{\ell}O \rightarrowtail E \in \mathfrak{P}^{\mathrm{el}}_{\not{\ell}O/}$ is

$$\mathfrak{Fact}^{\mathrm{inrt}}_{\not{\ell}}(O,v) := \mathfrak{O}^{\mathrm{el}}_{O/} \underset{\mathfrak{P}^{\mathrm{inrt}}_{\not{\ell}O/}}{\times} \mathfrak{Fact}_{\mathfrak{P}^{\mathrm{inrt}}}(v) \tag{5}$$

where $\mathfrak{Fact}_{\mathfrak{P}^{\mathrm{inrt}}}(v) = \{v\} \times_{\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})} \left\{\mathbb{3}, \mathfrak{P}^{\mathrm{inrt}}\right\}$ is the $(\infty,1)$-category of factorisations of v in $\mathfrak{P}^{\mathrm{inrt}}$ (the pullback being defined relative to the functor $\mathbb{2} \to \mathbb{3}$ which is the unique endpoints-preserving one, encoding composition of a composable pair).

Definition 2.11 (Sound patterns)**.** An algebraic pattern $\mathfrak{P}$ is **sound** if for every $f\colon P \rightsquigarrow P'$ in $\mathfrak{Ar}_{\mathrm{act}}(\mathfrak{P})$ and any $h\colon \mathrm{ev}_{\mathrm{o}} f = P \rightarrowtail E$ in $\mathfrak{P}^{\mathrm{el}}_{P/}$ the ∞-category $\mathfrak{Fact}^{\mathrm{inrt}}_{\mathrm{ev}_{\mathrm{o}}\colon \mathfrak{Ar}_{\mathrm{act}}(\mathfrak{P}) \to \mathfrak{P}}(f\colon P \rightsquigarrow P', h\colon P \rightarrowtail E)$ is contractible.

One can see upon examination that the $(\infty,1)$-category $\mathfrak{Fact}^{\mathrm{inrt}}_{\mathrm{ev}_{\mathrm{o}}}(f,h)$, whose objects are diagram of the form

$$\begin{tikzcd} P \arrow[r, tail] \arrow[d, squiggly, "f"'] & M \arrow[r, tail] \arrow[d, squiggly] & E, \\ P' \arrow[r, tail] & E' & \end{tikzcd} \tag{6}$$

is equivalent to that denoted $\mathfrak{P}^{\mathrm{el}}_h(f) = \mathfrak{P}^{\mathrm{el}}_{P'/} \times_{\mathfrak{P}^{\mathrm{inrt}}_{P/}} (\mathfrak{P}^{\mathrm{inrt}}_{P/})_{/h}$ of [BHS22, Lemma 3.3.9. (2)], so that this definition recovers the notion of soundness from Definition 3.3.4 of *ibid*.

Remark 2.12. If $\mathfrak{P}$ is a sound algebraic pattern, fibrous $\mathfrak{P}$-patterns coincide with the more familiar (as a generalisation of the ∞-operads of [Lur17]) weak Segal $\mathfrak{P}$-fibrations (also called $\mathfrak{P}$-operads) of [CH21]. In particular, fibrous $\mathfrak{P}$-patterns which are also cocartesian fibrations are then the same thing as Segal $\mathfrak{P}$-fibrations.

Lemma 2.13. *A filtered colimit of sound algebraic patterns is sound.*

Proof. Let $\mathfrak{I}$ be a filtered $(\infty,1)$-category and $\mathscr{P}\colon \mathfrak{I} \to \mathfrak{AlgPatt}$ be a diagram of algebraic patterns all of which are sound. For any pattern $\mathfrak{P}$, the every ∞-category $\mathfrak{Fact}^{\mathrm{inrt}}_{\mathrm{ev}_0\colon \mathfrak{Ar}_{\mathrm{act}}(\mathfrak{P})\to\mathfrak{P}}(f,h)$ are pullbacks of powers of $\mathfrak{P}$ by finite categories, so finite weighted limits, and hence their construction commutes with filtered colimits (in $(\infty,1)$-$\mathfrak{Cat}$, and since limits and filtered colimits of algebraic patterns are computed in $(\infty,1)$-$\mathfrak{Cat}$). Since a filtered colimit of contractible $(\infty,1)$-categories is contractible, as all the terms $\mathscr{P}I$ are sound, we do obtain that the inert factorisation ∞-categories of $\operatorname{colim}_{I\in\mathfrak{I}} \mathscr{P}I$ are contractible. □

Finally, we describe a property of algebraic patterns which will be paramount for the construction and Segality of the categories of spans.

Notation 2.14 (Co-internalisation of a category). For any $(\infty,1)$-category $\mathfrak{E}$, we will let $\mathfrak{E}_{-/}$ denote the ∞-functor $\mathfrak{E}^{\mathrm{op}} \to (\infty,1)$-$\mathfrak{Cat}$ taking an object $E \in \mathfrak{E}$ to the slice $\mathfrak{E}_{E/}$ and an arrow $f\colon E \to E'$ to the **codependent coproduct** (or, plainly, precomposition by f) $\Sigma^f = (- \circ f)\colon \mathfrak{E}_{E'/} \to \mathfrak{E}_{E/}$. We refer to it as the **co-internalisation** of $\mathfrak{E}$, though it differs from the internalisation of $\mathfrak{E}^{\mathrm{op}}$ considered in [Str00] in that the latter, defined for $\mathfrak{E}$ admitting pushouts, has functoriality along f given by the right-adjoint (co-base change) of Σ^f — however, it is related, after passing to presheaves, to the co-internalisation of $\left\{\mathfrak{E}, \infty\text{-}\mathfrak{Grpd}\right\}$.

Recall that in [CH21, Proposition 14.16. (2)], an algebraic pattern $\mathfrak{P}$ is said to be **saturated** if the inclusion $\mathfrak{P}^{\mathrm{el}} \hookrightarrow \mathfrak{P}$ is codense[1], and under

[1]It is written there as $\mathfrak{P}^{\mathrm{inrt}} \hookrightarrow \mathfrak{P}$ being codense, but the proof of Proposition 14.20 immediately confirms this as a typo. In addition, extendability is required as part of the definition, but it is not necessary for this characterisation.

the mild assumptions of $\mathfrak{P}$ being slim and extendable, this is equivalent by [CH21, Proposition 14.20] to the more convenient condition of $\mathfrak{P}^{\mathrm{el}} \hookrightarrow \mathfrak{P}^{\mathrm{inrt}}$ being codense. If the relevant limits $P \simeq \lim_{E\in\mathfrak{P}^{\mathrm{el}}_{P/}} E$ are co-Van Kampen (*i.e.* preserved by the co-internalisation, which again, due to the functoriality used, is different from being Van Kampen colimits in the opposite category, and is in fact rare), taking "global sections" of the co-internalisation $(\mathfrak{P}^{\mathrm{el}}_{P/})_{-/}$ produces for every $P \in \mathfrak{P}$ an equivalence $\mathrm{colim}_{E\in(\mathfrak{P}^{\mathrm{el}}_{P/})^{\mathrm{op}}} \mathfrak{P}^{\mathrm{el}}_{E/} \simeq \mathfrak{P}^{\mathrm{el}}_{P/}$ which we may thus think of as **global saturation** for the pattern $\mathfrak{P}$. As has been observed independently by Thomas Blom and Jan Steinebrunner (who graciously provided the following proof), this weaker property turns out to be satisfied by every algebraic pattern, regardless of saturation and preservation.

Proposition 2.15. *For any $(\infty, 1)$-category $\mathfrak{E}$, the functor* $\mathrm{colim}_{\mathfrak{E}^{\mathrm{op}}} \mathfrak{E}_{-/} \to \mathfrak{E}$ *induced by the projections $\mathfrak{E}_{E/} \to \mathfrak{E}$ is an equivalence.*

Proof [Ste25]. The colimit of the co-internalisation functor $\mathrm{colim}_{E\in\mathfrak{E}^{\mathrm{op}}} \mathfrak{E}_{E/}$ can be computed in two steps: first take its lax colimit, which by [GHN17, Corollary 7.6] is its Grothendieck construction $\mathrm{ev}_0 \colon \mathcal{A}\mathrm{r}(\mathfrak{E}) \to \mathfrak{E}$ of [Lur09, Corollary 2.4.7.11], and then rectify by localising at the ev_0-cartesian arrows. By [Lur09, Lemma 2.4.7.5] or [RV22, Lemma 7.4.3. (iii)], these are precisely those whose image under ev_1 is an equivalence. Thus, we need only exhibit $\mathrm{ev}_1 \colon \mathcal{A}\mathrm{r}(\mathfrak{E}) \to \mathfrak{E}$ as such a localisation, which it is by [Cis19, Proposition 7.1.12] because it is a cocartesian fibration (whence smooth) whose fibres $\mathfrak{E}_{/E}$, admitting terminal objects, are contractible. □

Note that, as is seen by unfolding the formulas, this is also the content of [GHN17, Corollary 7.5].

Corollary 2.16. *Any algebraic pattern $\mathfrak{P}$ is globally saturated, that is: for any $P \in \mathfrak{P}$ the canonical map*

$$\operatorname*{colim}_{E\in(\mathfrak{P}^{\mathrm{el}}_{P/})^{\mathrm{op}}} \mathfrak{P}^{\mathrm{el}}_{E/} \to \mathfrak{P}^{\mathrm{el}}_{P/} \tag{7}$$

induced by the Σ^u for each $u \colon P \rightarrowtail E$ is an equivalence.

Proof [Ste25]. All the ∞-categories appearing in the colimit can be rewritten as slices $\mathfrak{P}^{\mathrm{el}}_{E/} \simeq (\mathfrak{P}^{\mathrm{el}}_{P/})_{E/}$ of $\mathfrak{P}^{\mathrm{el}}_{P/}$. Therefore, we can simply apply the preceding Proposition to $\mathfrak{E} = \mathfrak{P}^{\mathrm{el}}_{P/}$. □

This argument establishing global saturation for all algebraic patterns is somewhat inexplicit, relying on technical properties of localisation functors, so when applying it to specific algebraic patterns it can often be instructive to verify the reason for the property through direct examination of the pattern so as to get a more thorough understanding of its inner workings.

Example 2.17. [Hau18a, Proposition 5.13] shows explicitly how the algebraic pattern $\Delta^{\mathrm{op}\natural}$ for internal categories is globally saturated, in a way that we will now generalise to internal higher categories.

3. Saturation properties of the cell category Θ

The main example of interest for applying the result on general Segal objects we will establish in section 5 is that of (internal) ω-categories. In order to provide a good understanding of it for readers less familiar with Joyal's cell category, we will here study "by hand" the global saturation — and, along the way, saturation — of the relevant algebraic pattern.

Construction 3.1. Recall that the (non-reflexive) **globe category** $\mathbb{G}$ is generated by objects $\overline{n}$, for all $n \in \mathbb{N}$, and arrows $i_n^{\pm} \colon \overline{n} \to \overline{n+1}$, as presented in the graph

$$\overline{0} \underset{i_0^-}{\overset{i_0^+}{\rightrightarrows}} \overline{1} \underset{i_1^-}{\overset{i_1^+}{\rightrightarrows}} \cdots \underset{i_{n-1}^-}{\overset{i_{n-1}^+}{\rightrightarrows}} \overline{n} \underset{i_n^-}{\overset{i_n^+}{\rightrightarrows}} \cdots, \tag{8}$$

with the relations $i_{n+1}^+ i_n^\varepsilon = i_{n+1}^- i_n^\varepsilon$ for any $n \in \mathbb{N}$ and any $\varepsilon \in \{+,-\}$. A **globular object** in an $(\infty,1)$-category $\mathfrak{C}$ is a $\mathfrak{C}$-valued presheaf on $\mathbb{G}$. A **strict ω-category** is a globular set equipped with units and composition operations satisfying certain equations (spelled out for example in [Str00, p. 300]); such structure is monadic over $\left\{\mathbb{G}^{\mathrm{op}}, \mathfrak{Set}\right\}$, with monad $\mathscr{F}_\omega$.

The **cell category** Θ (first introduced in [Joy97]) has as objects the globular sets that are pastings of appropriately composable globes — a condition encoded precisely as the notion of globular sums in the sense of [Ara10, §2.1.1] or [Lou23, §1.1.2.2] — and as morphisms the morphisms of strict ω-categories between their associated free ω-categories. A morphism f is **inert** (also called an immersion) if it is the image by $\mathscr{F}_\omega$ of a morphism of globular sets, and **active** if in any factorisation $f = ia$ with i inert, i must be an identity (by [Ara10, Proposition 3.3.11], they correspond to the maps also known as algebraic, or covers). By [Ber02, Lemma 1.11] or [Ara10, Proposition 3.3.10], the classes of active and inert morphisms form a unique factorisation system, so in particular an orthogonal one, on Θ.

Notation 3.2 (Generic n-cells). For any $n \in \mathbb{N}$, the representable presheaf $\mathbb{G}(-,\overline{n})$ is canonically endowed with a structure of strict ω-category (which comes from viewing it as the restriction to $\mathbb{G}^{\mathrm{op}}$ of $\Theta(-,\overline{n})$). We denote this ω-category $\mathfrak{D}_n$; it is known as the ***n*-globe**, or as the generic (or "walking") n-cell.

Definition 3.3. The algebraic pattern $\Theta^{\mathrm{op}\natural}$ is the category Θ^{op}, endowed with the inert–active factorisation system described above, and with elementary objects the ℓ-globes (so that $\Theta^{\mathrm{op}\natural,\mathrm{el}} \simeq \mathbb{G}^{\mathrm{op}}$).

It is an immediate consequence of the definition (and of the fact that all inert maps into a globe in Θ also have to be from a globe) that Segal $\Theta^{\mathrm{op}\natural}$-objects are exactly what are called Θ-models in [Ber02].

Lemma 3.4 ([Ara10, Proposition 2.3.18])**.** *The pattern $\Theta^{\mathrm{op}\natural}$ is saturated.*

Proof. This is essentially a consequence of the definition of globular sums: any such globular set T can be written as an iterated pushout $T \simeq \mathfrak{D}_{i_1} \amalg_{\mathfrak{D}_{i'_1}} \cdots \amalg_{\mathfrak{D}_{i'_{p-1}}} \mathfrak{D}_{i_p}$, and by [Ara10, Lemme 2.3.22], the immersions $\mathfrak{D}_i \rightarrowtail T$ featuring in this pushout define a cofinal subcategory of $\Theta^{\natural,\mathrm{el}}_{/T}$. □

It follows from this that the definition of Segal $\Theta^{\mathrm{op}\natural}$-objects in a complete $(\infty,1)$-category $\mathfrak{C}$ coincides with that of (weak) ω-categories in $\mathfrak{C}$ in the sense of [Lou23], albeit without the Rezk-completeness (or univalent completeness) condition — so that, to be more precise, they correspond to flagged ω-categories as in [AF18].

Remark 3.5. For any $\ell \in \mathbb{N} \cup \{\omega\}$, we let

$$\Theta_\ell = \Theta \cap (\infty,\ell)\text{-}\mathfrak{Cat} \tag{9}$$

be the ℓ-dimensional cell category used in [Rez10]; the pattern structure of Definition 3.3 restricts to one on ${\Theta_\ell}^{\mathrm{op}}$ whose Segal objects are internal flagged ℓ-categories (and we obviously have $\Theta_\omega = \Theta$).

There is an obvious filtration ${\Theta_1}^{\mathrm{op}\natural} \simeq \Delta^{\mathrm{op}\natural} \hookrightarrow {\Theta_2}^{\mathrm{op}\natural} \hookrightarrow {\Theta_3}^{\mathrm{op}\natural} \hookrightarrow \cdots$ and we recover $\Theta^{\mathrm{op}\natural}$ as its colimit. In particular, since each ${\Theta_\ell}^{\mathrm{op}\natural}$ for ℓ finite is known to be sound from [BHS22, Example 3.3.18], it follows from Lemma 2.13 that $\Theta^{\mathrm{op}\natural}$ is sound as well.

Construction 3.6. Since the cells in a pasting diagram are unlabelled, the standard representation of objects of Θ contains redundant information. A more minimal presentation, suggested by [Bat98] and developed more thoroughly in [Ber02] and [Ara10], of these objects is as level trees, functors from some $[\ell]^{\mathrm{op}}$ (ℓ being the categorical dimension) to Δ whose value at the terminal object 0 is $[0]$: the cells in the corresponding pasting diagram can be all recovered as the sectors in the tree.

This description makes it easier to get a handle on the structure of these trees and their categories of inert morphisms: for a tree $T\colon [\ell]^{\mathrm{op}} \to \Delta$, for $k \leq \ell$, we set $|T|(k)$ to be the reunion, over $i \in T(k)$, of the $T(k+1)_i + 1$, where $T(k+1)_i$ is the fibre of $T(k+1) \to T(k)$ at i (and where we decreed $T(\ell+1)$ to be $[-1] = \varnothing$). Note that the assignment $[\ell]^{\mathrm{op}} \ni k \mapsto |T|(k)$ is *not* functorial; however ${\mathbb{G}_{\leq \ell}}^{\mathrm{op}} \ni \mathfrak{D}_k \mapsto |T|(k)$ can be made functorial.

Remark 3.7. The objects of $|T|(k)$ can be understood as the **sectors** at level k as defined by [Ber02] (and, likewise, their ordering is the natural left-to-right ordering of sectors in each fibre), so that $|T|$ coincides with the globular set denoted T^* in [Bat98].

In the dictionary between globular sums and trees, it is the sectors of a tree that correspond to the cells of the corresponding globular sum.

We will now use the decompositions provided by the proof of Lemma 3.4 to understand the categories $\Theta^{\natural,\mathrm{el}}_{/T}$.

Lemma 3.8. *Let $T \in \Theta$ be any globular sum. Then $\Theta^{\natural,\mathrm{el}}_{/T}$ is equivalent to the Grothendieck construction of the globular set $|T|$.*

Proof. We will exhibit an explicit isomorphism between $|T|$ and the underlying $\Theta^\natural$-graph of the presheaf represented by T, since then its Grothendieck construction is indeed the slice. Consider an object of $\Theta^{\natural,\mathrm{el}}_{/T}$, given by a map $\mathfrak{D}_i \rightarrowtail T$, *i.e.* an element of $T(\mathfrak{D}_i)$. Since $\mathfrak{D}_i$ is the free i-cell, this map is uniquely characterised by a choice of an i-cell in T. In terms of the associated trees, D_i is a linear tree and so such a map is characterised by a choice of a branch at level i and a sector around its top point. It follows from Remark 3.7 that these are exactly counted by the elements of $|T|(\mathfrak{D}_i)$. $\square$

Example 3.9. For any elementary $\mathfrak{D}_\ell$, the category $\Theta^{\natural,\mathrm{el}}_{/\mathfrak{D}_\ell}$ is freely generated by the graph

$$\begin{tikzcd}
 & \ulcorner C_\ell \urcorner & \\
\ulcorner C^-_{\ell-1} \urcorner \arrow[ur] & & \ulcorner C^+_{\ell-1} \urcorner \arrow[ul] \\
\vdots \arrow[u] \arrow[urr] & & \vdots \arrow[u] \arrow[ull] \\
\ulcorner C^-_1 \urcorner \arrow[u] \arrow[urr] & & \ulcorner C^+_1 \urcorner \arrow[u] \arrow[ull] \\
\ulcorner C^-_0 \urcorner \arrow[u] \arrow[urr] & & \ulcorner C^+_0 \urcorner \arrow[u] \arrow[ull]
\end{tikzcd} \tag{10}$$

where we recall that $\mathfrak{D}_\ell$ has a unique ℓ-cell C_ℓ and, for any $0 \le i < \ell$, two i-cells $C_i^\pm$ serving has source and target for the higher cells, and $\ulcorner C_i^\pm \urcorner \colon \mathfrak{D}_i \to \mathfrak{D}_\ell$ denotes the (inert) map selecting the corresponding cell. In other words, $\Theta^{\natural,\mathrm{el}}_{/\mathfrak{D}_\ell}$ is the free-living ℓ-iterated cospan, so that the category we are ultimately interested in, $\Theta^{\mathrm{op}\,\natural,\mathrm{el}}_{\mathfrak{D}_\ell/}$, which is its opposite, will be the free-living ℓ-iterated span.

Lemma 3.10. *Let T be a globular sum of the form $\mathfrak{D}_m \amalg_{\mathfrak{D}_\ell} \mathfrak{D}_n$. Then $\Theta^{\natural,\mathrm{inrt}}_{/T}$ is the strict pushout of 1-categories $\Theta^{\natural,\mathrm{inrt}}_{/\mathfrak{D}_m} \amalg_{\Theta^{\natural,\mathrm{inrt}}_{/\mathfrak{D}_\ell}} \Theta^{\natural,\mathrm{inrt}}_{/\mathfrak{D}_n}$*

Proof. Let us call E_i^ε the cells of T in $\mathfrak{D}_m$, F_i^ε those in $\mathfrak{D}_n$, and C_i^ε those in $\mathfrak{D}_\ell$, so that we have $E_i^\varepsilon = C_i^\varepsilon = F_i^\varepsilon$ for $i < \ell$ and $E_\ell^+ = C_\ell = F_\ell^-$. The matter is then that of enumerating the cells and their relations, for which no listing can be as clear as simply drawing a generating graph:

$$\begin{tikzcd}
& E_m & & & & F_n & \\
E^-_{m-1} \arrow[ur] & & E^+_{m-1} \arrow[ul] & & F^-_{n-1} \arrow[ur] & & F^+_{n-1} \arrow[ul] \\
\vdots \arrow[u] \arrow[urr] & & \vdots \arrow[u] \arrow[ull] & & \vdots \arrow[u] \arrow[urr] & & \vdots \arrow[u] \arrow[ull] \\
E^-_{\ell+1} \arrow[u] \arrow[urr] & & E^+_{\ell+1} \arrow[u] \arrow[ull] & & F^-_{\ell+1} \arrow[u] \arrow[urr] & & F^+_{\ell+1} \arrow[u] \arrow[ull] \\
E^-_{\ell} \arrow[u] \arrow[urr] & & & C_\ell \arrow[ul] \arrow[ur] \arrow[ulll] \arrow[urrr] & & & F^+_{\ell} \arrow[u] \arrow[ull] \\
& C^-_{\ell-1} \arrow[ul] \arrow[urr] \arrow[urrrrr] & & & & C^+_{\ell-1} \arrow[ur] \arrow[ull] \arrow[ulllll] & \\
& \vdots \arrow[u] \arrow[urrrr] & & & & \vdots \arrow[u] \arrow[ullll] & \\
& C^-_0 \arrow[u] \arrow[urrrr] & & & & C^+_0 . \arrow[u] \arrow[ullll] &
\end{tikzcd} \tag{11}$$

Since the Grothendieck construction takes colimits of presheaves (of sets) to strict colimits of 1-categories, and our slices are categories of elements as in Lemma 3.8, one can indeed recognise in eq. (11) a strict pushout of three versions of eq. (10). □

Proposition 3.11. *The algebraic pattern $\Theta^{\mathrm{op}\natural}$ is globally saturated.*

Proof. Again, we can use the decomposition $T \simeq \mathfrak{D}_{i_1} \amalg_{\mathfrak{D}_{i'_1}} \cdots \amalg_{\mathfrak{D}_{i'_{p-1}}} \mathfrak{D}_{i_p}$ since it is cofinal, so that all we have to prove is that

$$\Theta^{\natural,\mathrm{el}}_{/T} \simeq \Theta^{\natural,\mathrm{el}}_{/\mathfrak{D}_{i_1}} \coprod_{\Theta^{\natural,\mathrm{el}}_{/\mathfrak{D}_{i'_1}}} \cdots \coprod_{\Theta^{\natural,\mathrm{el}}_{/\mathfrak{D}_{i'_{p-1}}}} \Theta^{\natural,\mathrm{el}}_{/\mathfrak{D}_{i_p}} . \tag{12}$$

To compute this pushout of $(\infty, 1)$-categories, we will use the Joyal model structure for quasicategories. Letting $N_\bullet\mathfrak{C}$ denote the nerve of an $(\infty, 1)$-category $\mathfrak{C}$, it is clear — since $i'_{j\pm1} < i_j$ for all j in the decomposition —

that the maps of quasicategories $N_\bullet\Theta^{\natural,\mathrm{el}}_{/\mathbb{D}_{i'_{j\pm1}}} \to N_\bullet\Theta^{\natural,\mathrm{el}}_{/\mathbb{D}_{i_j}}$ are injective in every degree, *i.e.* cofibrations in the Joyal model structure, so that the pushout will coincide with the pushout of 1-categories. The result for this strict pushout is then established *via* Lemma 3.10. $\square$

4. Generalised spans

For this section, we fix an algebraic pattern $\mathfrak{P}$ and a $\mathfrak{P}$-complete $(\infty,1)$-category $\mathfrak{C}$. We will adapt to $\mathfrak{P}$ the constructions and arguments of [Hau18a, §5].

Recall that $\mathfrak{Ar}(\mathfrak{P}) := \left\{\mathbb{2},\mathfrak{P}\right\} \xrightarrow{\mathrm{ev}_0} \left\{\mathbb{1},\mathfrak{P}\right\} \simeq \mathfrak{P}$ is a cartesian fibration classifying the ∞-functor $\mathfrak{P}_{-/}\colon \mathfrak{P}^{\mathrm{op}} \to (\infty,1)\text{-}\mathfrak{Cat}$. We let $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ be the full sub-$(\infty,1)$-category of $\mathfrak{Ar}(\mathfrak{P})$ on the inert arrows — which, by the dual of [BHS22, Proposition 2.2.2], still defines a cartesian fibration.

Our first goal is to show that $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \xrightarrow{\mathrm{ev}_0} \mathfrak{P}$ classifies an ∞-functor $\mathfrak{P}^{\mathrm{op}} \to (\infty,1)\text{-}\mathfrak{Cat}$ whose action on objects is $P \mapsto \mathfrak{P}^{\mathrm{inrt}}_{P/}$.

Construction 4.1. Since the factorisation system of $\mathfrak{P}$ is functorial, projection onto the inert part of an arrow defines a functor $\mathit{inrt}\colon \left\{\mathbb{2},\mathfrak{P}\right\} \to \left\{\mathbb{3},\mathfrak{P}\right\} \to \left\{\mathbb{2},\mathfrak{P}\right\}$, which preserves the image of ev_0 so defines a morphism of categories over $\left\{\mathbb{1},\mathfrak{P}\right\}$ (but not of cartesian fibrations over $\mathfrak{P}$, as it does not preserve cartesian lifts of non-inert morphisms). We let $\mathit{inrt}(\mathfrak{Ar}(\mathfrak{P}))$ denote its essential image, whose objects are then the inert arrows of $\mathfrak{P}$ while morphisms are the squares all of whose edges are inert — so that, in particular, the fibre of $\mathrm{ev}_0|_{\mathit{inrt}(\mathfrak{Ar}(\mathfrak{P}))}$ at $P \in \mathfrak{P}$ is $(\mathfrak{P}^{\mathrm{inrt}})_{P/} = \mathfrak{P}^{\mathrm{inrt}}_{P/}$.

Lemma 4.2. *Consider a commuting triangle of inert arrows below-left*

$$\begin{tikzcd} & Q \arrow[d, tail, "h"] \\ P \arrow[ur, tail, "g"] \arrow[dr, tail, "g'"'] & Q' \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail, "\mathit{inrt}(gf)"'] \arrow[rr, bend left, "\Sigma^f g = gf"] \arrow[d, equal] & M \arrow[r, squiggly] \arrow[d, "\mathit{inrt}(\Sigma^f h)"] & Q \arrow[d, tail, "h"] \\ O \arrow[r, tail, "\mathit{inrt}(g'f)"] \arrow[rr, bend right, "\Sigma^f g' = hgf"'] & M' \arrow[r, squiggly] & Q' \end{tikzcd} \tag{13}$$

defining a morphism in $\mathfrak{P}^{\mathrm{inrt}}_{P/}$*, and let* $O \xrightarrow{f} P$ *be any arrow of* $\mathfrak{P}$*, with inert–active factorisation of* $\Sigma^f h$ *as above-right. Then* $\mathit{inrt}(\Sigma^f h)$ *is inert.*

Proof. This is a direct application of the left-cancellability property for the left class of an orthogonal factorisation system (see for example [Lur09, Proposition 5.2.8.6. (4)] or [Lou23, Proposition 4.1.2.12]). □

Corollary 4.3. *The projection* $\mathit{inrt}(\mathfrak{Ar}(\mathfrak{P})) \to \mathfrak{P}$ *is a cartesian fibration, and coincides with* $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \to \mathfrak{P}$. □

We thus obtain an ∞-functor $\mathfrak{P}^{\mathrm{inrt}}_{-/} \colon \mathfrak{P}^{\mathrm{op}} \to (\infty, 1)\text{-}\mathfrak{Cat}$ (whose restriction to $(\mathfrak{P}^{\mathrm{inrt}})^{\mathrm{op}}$ is the co-internalisation of $\mathfrak{P}^{\mathrm{inrt}}$).

Definition 4.4. We denote $\overline{p} \colon \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C}) \to \mathfrak{P}$ the cocartesian fibration classifying the ∞-functor $\left\{\mathfrak{P}^{\mathrm{inrt}}_{-/}, \mathfrak{C}\right\} \colon \mathfrak{P} \to (\infty, 1)\text{-}\mathfrak{Cat}$.

Recall that by [Bar22, Proposition 2.37], for any $P \in \mathfrak{P}$ there is an algebraic pattern structure on the slice $\mathfrak{P}_{P/}$, where an object (resp. an arrow) is elementary (resp. inert, resp. active) if and only if its image by ev_1 is so in $\mathfrak{P}$. Furthermore, by [Kos21, Proposition 2.14 and Proposition 2.4], it restricts to an algebraic pattern structure on $\mathfrak{P}^{\mathrm{inrt}}_{P/}$ (which has no non-trivial active morphisms).

Definition 4.5. We call $\mathfrak{Span}_{\mathfrak{P}}(\mathfrak{C})$ the full sub-$(\infty, 1)$-category of $\overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$ on the objects $(P, \mathscr{F} \colon \mathfrak{P}^{\mathrm{inrt}}_{P/} \to \mathfrak{C})$ such that $\mathscr{F}$ is a Segal $\mathfrak{P}^{\mathrm{inrt}}_{P/}$-object.

Remark 4.6. An alternate construction of $\mathfrak{Span}_{\mathfrak{P}}(\mathfrak{C})$ is provided by [Kos21, Corollary 2.16].

We let $\iota^{\mathrm{inrt}}_P \colon \mathfrak{P}^{\mathrm{inrt}}_{P/} \to \mathfrak{P}_{P/}$ denote the canonical inclusion (induced under slicing by $\mathfrak{P}^{\mathrm{inrt}} \hookrightarrow \mathfrak{P}$). By [Kos21, Proposition 2.15] (which is formulated in the case of $\mathfrak{P} = \Delta^{\mathrm{op}\natural}$ but only relies on the factorisation system), for any arrow $f \colon O \to P$ in $\mathfrak{P}$, the induced ∞-functor $\Sigma^{f,*} \colon \left\{\mathfrak{P}_{O/}, \mathfrak{C}\right\} \to \left\{\mathfrak{P}_{P/}, \mathfrak{C}\right\}$ sends the image of $\iota^{\mathrm{inrt}}_{O,!}$ into the image of $\iota^{\mathrm{inrt}}_{P,!}$.

We can then let $\overline{\mathfrak{PreSpan}}_{\mathfrak{P}}(\mathfrak{C}) \to \mathfrak{P}$ denote the Grothendieck construction of the ∞-functor $\left\{\mathfrak{P}_{-/}, \mathfrak{C}\right\} \colon \mathfrak{C} \to (\infty, 1)\text{-}\mathfrak{Cat}$, and $\overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$ is the full sub-$(\infty, 1)$-category of $\overline{\mathfrak{PreSpan}}_{\mathfrak{P}}(\mathfrak{C})$ on those objects $(P, \mathscr{F} \colon \mathfrak{P}_{P/} \to \mathfrak{C})$ such that $\mathscr{F}$ is in the image of $\iota^{\mathrm{inrt}}_{P,!}$ (so that it is determined by its restriction $\mathfrak{P}^{\mathrm{inrt}}_{P/} \to \mathfrak{C}$).

Lemma 4.7. *Assume the algebraic pattern $\mathfrak{P}$ is sound. The restricted projection $\not p\colon \mathfrak{Span}_{\mathfrak{P}}(\mathfrak{C}) \hookrightarrow \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C}) \xrightarrow{\overline{\not p}} \mathfrak{P}$ is a cocartesian fibration.*

Proof. As explained in the proof of [Hau18a, Corollary 5.12], since $\mathfrak{Span}_{\mathfrak{P}}(\mathfrak{C})$ is a full sub-$(\infty,1)$-category of $\overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$, all we need to do is check that if $(P,\mathscr{F}) \to (Q,\mathscr{G})$ is a $\overline{\not p}$-cocartesian morphism in $\overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$ such that $\mathscr{F}$ is Segal, then $\mathscr{G}$ is Segal as well. Note also that such a cocartesian morphism consists of an arrow $f\colon P \to Q$ in $\mathfrak{P}$ with $\mathscr{G} \simeq \Sigma^{f,*}\mathscr{F} = \mathscr{F} \circ (\Sigma^f)$: in other words, we must show that Segal objects are preserved by composition with codependent coproduct. That is, if $\mathscr{F}\colon \mathfrak{P}^{\mathrm{inrt}}_{P/} \to \mathfrak{C}$ satisfies the Segal condition, then for every $g\colon Q \rightarrowtail Q'$ we must have

$$\mathscr{F}\Big(\mathit{inrt}(P \xrightarrow{f} Q \overset{g}{\rightarrowtail} Q')\Big) \xrightarrow{\simeq} \lim_{(Q \overset{hg}{\rightarrowtail} E) \in (\mathfrak{P}^{\mathrm{inrt}}_{Q/})^{\mathrm{el}}_{g/}} \mathscr{F}\Big(\mathit{inrt}(P \xrightarrow{f} Q \overset{hg}{\rightarrowtail} E)\Big). \tag{14}$$

By the functoriality of the construction $\Sigma^{(-),*}$ and the fact that active and inert morphisms for a factorisation system, we only need to check the comparison in the cases where f is either purely inert or purely active. If f is inert, then inrt acts as the identity on the composition, and the functor $\Sigma^{f,*}$ is even iso-Segal (since both $(\mathfrak{P}^{\mathrm{inrt}}_{Q/})^{\mathrm{el}}_{g/}$ and $(\mathfrak{P}^{\mathrm{inrt}}_{P/})^{\mathrm{el}}_{gf/}$ are then equivalent to $\mathfrak{P}^{\mathrm{el}}_{Q'/}$ thanks to all maps being inert), which is strictly stronger than preserving Segal objects. The case of f being active is where the soundness assumption comes into play.

Write $P \overset{\mathit{inrt}(hgf)}{\rightarrowtail} M_{hg} \rightsquigarrow E$ the inert-active factorisation of the composition $hgf\colon P \rightsquigarrow Q \rightarrowtail E$, for any $Q \rightarrowtail E$ in $(\mathfrak{P}^{\mathrm{inrt}}_{Q/})^{\mathrm{el}}_{g/}$ given by $h\colon Q' \rightarrowtail E$. Then, since $\mathscr{F}$ is Segal, the right-hand side limit in eq. (14) becomes a double limit

$$\lim_{(Q \overset{hg}{\rightarrowtail} E) \in (\mathfrak{P}^{\mathrm{inrt}}_{Q/})^{\mathrm{el}}_{g/}} \; \lim_{(M_{hg} \rightarrowtail E') \in (\mathfrak{P}^{\mathrm{inrt}}_{P/})^{\mathrm{el}}_{M_{hg}/}} \mathscr{F}(P \rightarrowtail M_{hg} \rightarrowtail E'), \tag{15}$$

summarised by the dashed arrows in the diagram

$$\begin{tikzcd}
P \arrow[r, tail] \arrow[d, squiggly, "f"'] & M_g \arrow[r, tail, "\mathit{inrt}(h)"] \arrow[d, squiggly] & M_{hg} \arrow[r, tail, dashed] \arrow[d, squiggly] & E', \\
Q \arrow[r, tail, "g"'] & Q' \arrow[r, tail, "h"'] & E &
\end{tikzcd} \tag{16}$$

where the inert transport arrow $\mathscr{inrt}(h)$ comes from lemma 4.2. In particular, we can see that $(\mathfrak{P}^{\mathrm{inrt}}_{Q/})^{\mathrm{el}}_{g/} \simeq \mathfrak{P}^{\mathrm{el}}_{Q'/}$ and $(\mathfrak{P}^{\mathrm{inrt}}_{P/})^{\mathrm{el}}_{M_{hg}/} \simeq \mathfrak{P}^{\mathrm{el}}_{M_{hg}/}$.

Then, as explained in [BHS22, Observation 3.3.6], soundness of $\mathfrak{P}$ allows this double limit to be computed as a limit over $(M_g \rightarrowtail E') \in \mathfrak{P}^{\mathrm{el}}_{M_g/}$. But this is precisely what we obtain from the Segal decomposition for $\mathscr{F}(P \rightarrowtail M_g)$ in the left-hand side term of eq. (14). □

Proposition 4.8. *Let $\mathfrak{P}$ be a sound algebraic pattern. The cocartesian fibration $\not{p}\colon \mathfrak{Span}_{\mathfrak{P}}(\mathfrak{C}) \to \mathfrak{P}$ is a Segal fibration, that is the ∞-functor $\mathbb{Span}_{\mathfrak{P}}(\mathfrak{C})\colon \mathfrak{P} \to (\infty,1)\text{-}\mathfrak{Cat}$ it classifies defines a $\mathfrak{P}$-monoidal $(\infty,1)$-category.*

Proof. To make the definition explicit, we need to show that for any $P \in \mathfrak{P}$,

$$\mathfrak{Seg}_{\mathfrak{P}^{\mathrm{inrt}}_{P/}}(\mathfrak{C}) \to \lim_{E \in \mathfrak{P}^{\mathrm{el}}_{P/}} \mathfrak{Seg}_{\mathfrak{P}^{\mathrm{inrt}}_{E/}}(\mathfrak{C}) \tag{17}$$

is an equivalence. Since $\mathfrak{P}^{\mathrm{inrt}}_{P/}$ only has inert morphisms, the right Kan extension ∞-functor

$$\mathfrak{Seg}_{\mathfrak{P}^{\mathrm{el}}_{P/}}(\mathfrak{C}) \simeq \left\{\mathfrak{P}^{\mathrm{el}}_{P/}, \mathfrak{C}\right\} \to \mathfrak{Seg}_{\mathfrak{P}^{\mathrm{inrt}}_{P/}}(\mathfrak{C}) \tag{18}$$

is an equivalence. Similarly, every factor $\mathfrak{Seg}_{\mathfrak{P}^{\mathrm{inrt}}_{E/}}(\mathfrak{C})$ in eq. (17) is equivalent to $\left\{\mathfrak{P}^{\mathrm{el}}_{E/}, \mathfrak{C}\right\}$, and so the map of eq. (17) takes the form

$$\left\{\mathfrak{P}^{\mathrm{el}}_{P/}, \mathfrak{C}\right\} \to \lim_{E \in \mathfrak{P}^{\mathrm{el}}_{P/}} \left\{\mathfrak{P}^{\mathrm{el}}_{E/}, \mathfrak{C}\right\}. \tag{19}$$

By the property of global saturation from corollary 2.16, and since enriched homs (or cotensors) send colimits in the first variable to limits, this map is an equivalence. □

5. $\mathfrak{P}$-Monads in $\mathfrak{P}$-spans

This section will follow very closely the structure of [Hau21, §3].

Lemma 5.1. *The ∞-functor $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \xrightarrow{\mathrm{ev}_1} \mathfrak{P}$ admits a right adjoint right inverse.*

Proof. The functor $\ulcorner 1 \urcorner \colon \mathbb{1} \to \mathbb{2}$ has a retraction $\mathbb{2} \xrightarrow{!_{\mathbb{2}}} \mathbb{1}$, which upgrades in fact to a left adjoint left inverse: we clearly have $!_{\mathbb{2}} \circ \ulcorner 1 \urcorner = !_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$, while there is a (unique, since $\mathbb{2}$ is posetal) natural transformation $\mathrm{id}_{\mathbb{2}} \Rightarrow \ulcorner 1 \urcorner \circ !_{\mathbb{2}} = \mathcal{const}_1$, and it is easily checked (by unicity of !) that these two transformations satisfy the triangle identities.

Now note that $\mathrm{ev}_1 \colon \mathfrak{Ar}(\mathfrak{P}) = \left\{\mathbb{2}, \mathfrak{P}\right\} \to \left\{\mathbb{1}, \mathfrak{P}\right\}$ is exactly given by $\left\{\ulcorner 1 \urcorner, \mathfrak{P}\right\}$, and so, as powering with $\mathfrak{P}$ is $(\infty, 2)$-functorial (that is, as an ∞-functor $\left\{(-), \mathfrak{P}\right\} \colon (\infty, 1)\text{-}\mathfrak{Cat}^{\mathrm{op}} \to (\infty, 1)\text{-}\mathfrak{Cat}$, it is $(\infty, 1)$-$\mathfrak{Cat}$-linear, and so upgrades to an $(\infty, 2)$-functor), it has a right adjoint right inverse given by $\left\{!_{\mathbb{2}}, \mathfrak{P}\right\}$. The latter ∞-functor can be described very explicitly: it maps an object $P \in \mathfrak{P}$ to its identity arrow $\mathrm{id}_P \in \mathfrak{Ar}(\mathfrak{P})$.

In particular, it factors through $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ — as identity arrows are inert — and since this sub-$(\infty, 1)$-category of $\mathfrak{Ar}(\mathfrak{P})$ is full, the astriction of $\left\{!_{\mathbb{2}}, \mathfrak{P}\right\}$ to it furnishes the desired right adjoint right inverse to $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \xrightarrow{\mathrm{ev}_1} \mathfrak{P}$. $\square$

Given its description, we will denote $\ulcorner \mathrm{id} \urcorner \colon \mathfrak{P} \to \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ the right adjoint right inverse to ev_1. The unit will simply be known as $\eta \colon \mathrm{id}_{\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})} \Rightarrow \ulcorner \mathrm{id} \urcorner \circ \mathrm{ev}_1$; its component at $(P \rightarrowtail Q) \in \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ is the square

$$\eta_{(P \rightarrowtail Q)} \colon \begin{tikzcd} P \arrow[r, tail] \arrow[d, tail] & Q \arrow[d, equal] \\ Q \arrow[r, equal] & Q. \end{tikzcd} \tag{20}$$

Proposition 5.2. *The ∞-functor $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \xrightarrow{\mathrm{ev}_1} \mathfrak{P}$ exhibits $\mathfrak{P}$ as the localisation of $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ at the set $\mathcal{I}$ of* ev_0*-cartesian morphisms lying over inert arrows of $\mathfrak{P}$.*

Proof. Let $\mathcal{W}$ be the set of morphisms in $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ inverted by ev_1; by [Lur09, Corollary 2.4.7.11 and Lemma 2.4.7.] (*cf.* also [BHS22, Proposition 2.2.2.(2)]), W consists exactly of the ev_0-cartesian morphisms, so that we do have $\mathcal{I} \subset \mathcal{W}$. If $(f, g) \colon (P \rightarrowtail Q) \to (P' \rightarrowtail Q')$ is a morphism in $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ lifting $f \colon P \to P'$ in $\mathfrak{P}$, it is in $\mathcal{W}$ if and only if $g \colon Q \to Q'$ is an equivalence so that we have a commutative square

$$\begin{tikzcd}
(P \rightarrowtail Q) \arrow[r, "{(f,g)}"] \arrow[d] & (P' \rightarrowtail Q') \arrow[d] \\
(Q = Q) \arrow[r, "\simeq"] & (Q' = Q')
\end{tikzcd} \tag{21}$$

in which the two vertical morphisms are in $\mathcal{I}$. Any ∞-functor from $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ to some $(\infty,1)$-category $\mathfrak{C}$ inverting the morphisms in $\mathcal{I}$ will then send the square of eq. (21) to a square whose veritcal arrows (in addition to the lower horizontal one) are equivalences, whence its upper horizontal is one as well since equivalences always satisfy the 2-of-3 property. This means that such an ∞-functor automatically inverts all the morphisms in $\mathcal{W}$, and we only need to show that ev_1 is a localisation, along $\mathcal{W}$. This follows readily from the fact that it has a right adjoint right inverse (in fact it is equivalent to it), but in our specific situation it can be seen in a more explicit way.

Let $\mathfrak{C}$ be again any $(\infty,1)$-category and let us consider the comparison ∞-functor $\left\{\mathrm{ev}_1,\mathfrak{C}\right\}\colon \left\{\mathfrak{P},\mathfrak{C}\right\} \to \left\{\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}),\mathfrak{C}\right\}_{(\mathcal{W})}$, where the target denotes the full sub-$(\infty,1)$-category of $\left\{\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}),\mathfrak{C}\right\}$ on the ∞-functors inverting the morphisms in $\mathcal{W}$ (through which $\left\{\mathrm{ev}_1,\mathfrak{C}\right\}$ does factor by definition of $\mathcal{W}$). The crux of the matter is that the components of the unit transformation η all belong to $\mathcal{I}$ — as can be seen in eq. (20) — and so *a fortiori* to $\mathcal{W}$. Hence, the adjunction $\left\{\mathrm{ev}_1,\mathfrak{C}\right\} \dashv \left\{\ulcorner\mathrm{id}\urcorner,\mathfrak{C}\right\}$ restricts on $\left\{\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}),\mathfrak{C}\right\}_{(\mathcal{W})}$ to an equivalence (as its counit was already an identity, and its unit becomes one after this restriction), which means that ev_1 is a localisation along $\mathcal{W}$. $\square$

Construction 5.3. Let $\not{f}\colon \mathfrak{X} \to \mathfrak{P}$ be an ∞-functor such that $\mathfrak{X}$ admits $\not{f}$-

cocartesian lifts of inert morphisms. Consider the solid pullback

$$
\begin{tikzcd}
 & \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \arrow[ddl, bend left, "\mathrm{ev}_0"] & & \mathfrak{X}\times_{\mathfrak{P}}\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \arrow[ll, "\mathrm{ev}_0^* \mathscr{f}"'] \arrow[ddl, bend left, "\mathscr{f}^* \mathrm{ev}_0"] \\
\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \arrow[ur, bend right, "\mathrm{id}"'] \arrow[ur, dashed, bend left=60, "\ulcorner \mathrm{id} \urcorner \circ \mathrm{ev}_1"] \arrow[ur, phantom, "\Leftarrow \eta"] \arrow[d, "\mathrm{ev}_0"'] \arrow[dr, phantom, "\Rightarrow \mathrm{ev}_0 \eta"] & & \mathfrak{X}\times_{\mathfrak{P}}\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \arrow[ll, "\mathrm{ev}_0^* \mathscr{f}"] \arrow[dll, phantom, very near start, "\lrcorner"] \arrow[d, "\mathscr{f}^* \mathrm{ev}_0"'] \arrow[ur, bend right, "\mathscr{f}^* \mathrm{id} = \mathrm{id}"'] \arrow[ur, dotted, bend left] \arrow[ur, phantom, "\Leftarrow \eta_!"] & \\
\mathfrak{P} & & \mathfrak{X} \arrow[ll, "\mathscr{f}"] &
\end{tikzcd}
\tag{22}
$$

which is a (strongly) commutative diagram in the $(\infty,2)$-category $(\infty,1)$-$\mathfrak{Cat}$. Adding $\ulcorner \mathrm{id} \urcorner \circ \mathrm{ev}_1$, represented as a dashed arrow, the induced back-left triangle does not commute; however, adding as well the unit cell η and its whiskering $\mathrm{ev}_0 \eta \colon\ \mathrm{ev}_0 = \mathrm{ev}_0 \circ \mathrm{id}_{\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})} \Rightarrow \mathrm{ev}_0 \circ \ulcorner \mathrm{id} \urcorner \circ \mathrm{ev}_1$ we obtain a "2-commutative" pasting diagram.

Now as $\mathscr{f}$ admits cocartesian lifts of inert arrows so does its base-change $\mathrm{ev}_0^* \mathscr{f}$ (since cocartesian lifts are stable by pullback, by the co-dual of [RV22, Proposition 5.2.4]), and so, using the formulation of cocartesian lifts from [RV22, Definition 5.4.2], the transformation $\eta(\mathrm{ev}_0 \mathscr{f}^*)$, whose components are inert, admits an $\mathrm{ev}_0 \mathscr{f}^*$-cocartesian dotted lift $\mathrm{id}_{\mathfrak{X}\times_{\mathfrak{P}}\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})} \overset{\eta_!}{\Longrightarrow} (\mathrm{id}_{\mathfrak{X}\times_{\mathfrak{P}}\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})})_\eta =: \mathscr{f}^\eta(\ulcorner \mathrm{id} \urcorner \circ \mathrm{ev}_1)$.

We can finally define

$$\mathscr{f}^* \mathrm{ev}_1 \coloneqq (\mathscr{f}^* \mathrm{ev}_0) \circ \mathscr{f}^\eta(\ulcorner \mathrm{id} \urcorner \circ \mathrm{ev}_1). \tag{23}$$

Explicitly, $\mathscr{f}^* \mathrm{ev}_1$ sends an object $(X, \jmath \colon \mathscr{f}X \rightarrowtail Q) \in \mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ to $(X_Q, Q = Q)$ where $X \xrightarrow{\jmath_!} X_Q$ is a cocartesian lift of $\jmath$. By construction it comes equipped with a natural transformation that we will call $\mathscr{f}^* \alpha \coloneqq$

$(\ell^* \mathrm{ev}_0)\eta_! \colon \ell^* \mathrm{ev}_0 \Rightarrow \ell^* \mathrm{ev}_1$, sitting in the diagram

$$\begin{tikzcd}
 & \mathfrak{P} \arrow[ddl, bend left] & & \mathfrak{X} \times_{\mathfrak{P}} \mathfrak{P} \simeq \mathfrak{X} \arrow[ll] \arrow[ddl, bend left, "\ell^* \mathrm{id}_{\mathfrak{P}} = \mathrm{id}_{\mathfrak{X}}"] \\
\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \arrow[ur, "\mathrm{ev}_1"] \arrow[d, "\mathrm{ev}_0"'] \arrow[ur, phantom, shift right=3, "\Uparrow \alpha"'] & & \mathfrak{X} \underset{\mathfrak{P}}{\times} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \arrow[ll] \arrow[ur, "\ell^* \mathrm{ev}_1"] \arrow[d, "\ell^* \mathrm{ev}_0"'] \arrow[ur, phantom, shift right=3, "\Uparrow \ell^* \alpha"'] \arrow[dll, phantom, "\llcorner" very near start] & \\
\mathfrak{P} & & \mathfrak{X} \arrow[ll, "\ell"] &
\end{tikzcd} \tag{24}$$

whose front and back squares are cartesian, but whose top square is not — and where the natural transformation α comes from cotensoring with $\mathfrak{P}$ the canonical 2-cell $\ulcorner 0 < 1 \urcorner \colon \ulcorner 0 \urcorner \Rightarrow \ulcorner 1 \urcorner \colon \mathbb{1} \to \mathbb{2}$ (in particular, it is easily checked that the adjunction $!_{\mathbb{2}} \dashv \ulcorner 1 \urcorner$ lives under $\mathbb{1}$ so that $\mathrm{ev}_1 \dashv \ulcorner \mathrm{id} \urcorner$ lives over $\mathfrak{P}$). Conjecturally, the right face of eq. (24) could be seen in terms of the $(\infty, 3)$-topos of $(\infty, 2)$-categories as the strong base change, along ℓ admitting enough cocartesian lifts, between fibrational lax slice $(\infty, 2)$-categories, justifying our notation, though since the conditions for its construction are rather specific we will not pursue this point of view in further generality.

Lemma 5.4. *The ∞-functor $\ell^* \mathrm{ev}_1$ admits a right adjoint right inverse.*

Proof. Note that in addition to being a right adjoint right inverse to ev_1, the map $\ulcorner \mathrm{id} \urcorner$ is also a left adjoint right inverse to ev_0. We will denote the counit of this adjunction κ. Since the identity unit exhibits $\mathrm{ev}_0 \circ \ulcorner \mathrm{id} \urcorner = \mathrm{id}_{\mathfrak{P}}$, the ∞-functor $\ulcorner \mathrm{id} \urcorner$ lifts strongly to $\ell^* \ulcorner \mathrm{id} \urcorner \colon \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$: the equivalent of eq. (24) with ev_1 replaced by $\ulcorner \mathrm{id} \urcorner$ (and α replaced by the identity unit, *mutatis mutandis*) is a strongly commutative diagram, and fully cartesian. We claim that $\ell^* \ulcorner \mathrm{id} \urcorner$ is the sought-after right adjoint right inverse to $\ell^* \mathrm{ev}_1$.

To see this, we will show that the transformation $\eta_!$ constructed in eq. (22) works as a unit with identity counit; it requires first identifying its target $\ell^{\eta}(\ulcorner \mathrm{id} \urcorner \circ \mathrm{ev}_1)$ as $\ell^* \ulcorner \mathrm{id} \urcorner \circ \ell^* \mathrm{ev}_1$. This is in fact trivial, because the transformations $\eta_! \colon \mathrm{id} \Rightarrow \ell^{\eta}(\ulcorner \mathrm{id} \urcorner \circ \mathrm{ev}_1)$ and $(\ell^* \ulcorner \mathrm{id} \urcorner)(\ell^* \mathrm{ev}_0)\eta_! \colon \mathrm{id} \Rightarrow \ell^* \ulcorner \mathrm{id} \urcorner \circ \ell^* \mathrm{ev}_1$ are both ℓ-cocartesian lifts of $\eta \colon \mathrm{id} \Rightarrow \ulcorner \mathrm{id} \urcorner \circ \mathrm{ev}_1$, but there is another interesting way of seeing it, that we detail in the next paragraph.

Since the unit of the adjunction $\ulcorner \mathrm{id} \urcorner \dashv \mathrm{ev}_0$ is an equivalence, the triangle identities imply that the whiskering $\kappa \ulcorner \mathrm{id} \urcorner$ is the identity transformation of $\ulcorner \mathrm{id} \urcorner$, and also $\kappa \ulcorner \mathrm{id} \urcorner \mathrm{ev}_1 \simeq \mathrm{id}_{\ulcorner \mathrm{id} \urcorner \mathrm{ev}_1}$. There are now two things we can do: since $\mathrm{id}_{\ell^\eta(\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1)}$ is a cocartesian lift of $\mathrm{id}_{\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1}$, the transformation $\mathrm{id}_{\ell^\eta(\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1)}$ factors through a unique lift $\ell^\eta(\kappa\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1)$ of $\kappa\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1$, which because of the factorisation has to be an identity. At the same time, one can take a cocartesian lift of $\kappa\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1$, which is easily seen to coincide with $\ell^\eta(\kappa\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1)$; as a cocartesian lift of an identity, it is, again, an identity. We thus have an equivalence

$$\begin{aligned}(\ell^*\ulcorner\mathrm{id}\urcorner)\circ(\ell^*\mathrm{ev}_1) = (\ell^*\ulcorner\mathrm{id}\urcorner)\circ(\ell^*\mathrm{ev}_0)\circ\ell^\eta(\ulcorner\mathrm{id}\urcorner\circ\mathrm{ev}_1)&\\ \xrightarrow[\ell^\eta(\kappa\ulcorner\mathrm{id}\urcorner\mathrm{ev}_1)]{\simeq} \ell^\eta(\ulcorner\mathrm{id}\urcorner\circ\mathrm{ev}_1),&\end{aligned} \tag{25}$$

expressing the decomposition we needed.

Furthermore, constructing the equivalent of eq. (22) but with $\mathrm{ev}_1\circ\ulcorner\mathrm{id}\urcorner$ in place of $\mathrm{id}_{\mathfrak{P}}$ (so with structure map to $\mathfrak{P}$ given by $\mathrm{id}_{\mathfrak{P}}$ instead of ev_0), and with the identity counit $\varepsilon\colon\ \mathrm{ev}_1\circ\ulcorner\mathrm{id}\urcorner \overset{=}{\Longrightarrow} \mathrm{id}_{\mathfrak{P}}$ instead of η, we obtain, after strongly pulling back $\mathrm{ev}_1\circ\ulcorner\mathrm{id}\urcorner$, an ℓ-cocartesian transformation

$$\varepsilon_!\colon\ \ell^*(\mathrm{ev}_1\circ\ulcorner\mathrm{id}\urcorner)\Rightarrow(\ell^*(\mathrm{ev}_1\circ\ulcorner\mathrm{id}\urcorner))_\varepsilon = \mathrm{id}_{\mathfrak{X}}, \tag{26}$$

which as a cocartesian lift of ε which is an identity, is itself an equivalence.

Finally, the fact that $\ell^*\eta := \eta_!$ and $\varepsilon_!$ satisfy the triangle identities is a consequence of the triangle identities for η and ε, to which is applied the same reasoning we used to obtain eq. (25). □

It is worthwhile to note that the component of $\ell^*\eta\colon\ \mathrm{id}\Rightarrow\ell^*\ulcorner\mathrm{id}\urcorner\circ\ell^*\mathrm{ev}_1$ at an object $(X,\ell X\overset{\jmath}{\rightarrowtail}Q)\in\mathfrak{X}\times_{\mathfrak{P}}\mathcal{A}\mathrm{r}_{\mathrm{inrt}}(\mathfrak{P})$ is

$$\ell^*\eta_{(X,\ell X\rightarrowtail Q)}\colon \left(X,\ \begin{array}{c}\ell X\\ \downarrow\\ Q\end{array}\right) \begin{array}{c}\xrightarrow{(X,\jmath)}\\ \\ \xrightarrow[(X,\mathrm{id}_Q)]{}\end{array} \left(X,\ \begin{array}{c}Q\\ \|\\ Q\end{array}\right). \tag{27}$$

Proposition 5.5. *Let $\ell\colon\mathfrak{X}\to\mathfrak{P}$ be an ∞-functor such that $\mathfrak{X}$ admits ℓ-cocartesian lifts of inert morphisms. The ∞-functor $\ell^*\mathrm{ev}_1\colon \mathfrak{X}\times_{\mathfrak{P}}\mathcal{A}\mathrm{r}_{\mathrm{inrt}}(\mathfrak{P})\to\mathfrak{X}$ exhibits $\mathfrak{X}$ as the localisation of $\mathfrak{X}\times_{\mathfrak{P}}\mathcal{A}\mathrm{r}_{\mathrm{inrt}}(\mathfrak{P})$ at the set $\mathfrak{I}_{\mathfrak{X}}$ of morphisms $(X;(\ell(X)\rightarrowtail Q))\to(X';(\ell(X')\rightarrowtail Q'))$ such that*

- $X \to X'$ *is* $\not{p}$*-cocartesian and*
- $(\not{p}(X) \rightarrowtail Q) \to (\not{p}(X') \rightarrowtail Q')$ *is* ev_0*-cartesian and* ev_0*-over an inert arrow.*

Proof. The proof follows the lines of that of Proposition 5.2. Let $\mathcal{W}_{\mathfrak{X}}$ be the class of morphisms inverted by $\not{p}^* \mathrm{ev}_1$. A morphism of $\mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$, of the form (ξ, θ) where $\xi \colon X \to Y$ in $\mathfrak{X}$ and θ sits is a commutative square

$$\begin{tikzcd} \not{p}X \arrow[r, "\not{p}\xi"] \arrow[d, tail, "\jmath"'] & \not{p}X' \arrow[d, tail, "\jmath'"] \\ Q \arrow[r, "\theta"'] & Q' \end{tikzcd} \tag{28}$$

in $\mathfrak{P}$, is in $\mathcal{W}_{\mathfrak{X}}$ if and only if θ is an equivalence $Q \simeq Q'$, so that it induces a commutative square

$$\begin{tikzcd} (X, \not{p}X \rightarrowtail Q) \arrow[r, "{(\xi,\theta)}"] \arrow[d, "\jmath_!"'] & (X', \not{p}X' \rightarrowtail Q') \arrow[d, "\jmath'_!"] \\ (X_Q, \not{p}(X_Q) = Q) \arrow[r, "{(\jmath_!\xi,\theta)}"'] & (X'_{Q'}, \not{p}(X'_{Q'} = Q')) \end{tikzcd} \tag{29}$$

where $X \xrightarrow{\jmath_!} X_Q$ and $X' \xrightarrow{\jmath'_!} X'_{Q'}$ are cocartesian lifts of $\jmath$ and $\jmath'$, and $\jmath_!\xi$ is the arrow $X_Q \to X'_{Q'}$ uniquely induced by cartesianity, which is invertible since it lifts the isomorphism $Q \simeq Q'$. The vertical morphisms are in $\mathcal{I}_{\mathfrak{X}}$ by construction, so it follows from the 2-of-3 property of equivalences that any ∞-functor that inverts the morphisms in $\mathcal{I}_{\mathfrak{X}}$ will invert the morphisms in $\mathcal{W}_{\mathfrak{X}}$, and that the localisations along $\mathcal{I}_{\mathfrak{X}}$ and $\mathcal{W}_{\mathfrak{X}}$ coincide.

But again, it can be seen in eq. (27) that the components of $\not{p}^*\eta$ are in $\mathcal{I}_{\mathfrak{X}}$ whence in $\mathcal{W}_{\mathfrak{X}}$, so $\not{p}^* \mathrm{ev}_1$ is indeed a localisation along $\mathcal{W}_{\mathfrak{X}}$. □

From now on, we assume that $\mathfrak{P}$ is a sound pattern, and we let $\mathfrak{C}$ be a $\mathfrak{P}$-complete $(\infty, 1)$-category.

Corollary 5.6. *Let* $\not{p} \colon \mathfrak{X} \to \mathfrak{P}$ *be an* ∞*-functor such that* $\mathfrak{X}$ *admits* $\not{p}$*-cocartesian lifts of inert morphisms. There is a fully faithful* ∞*-functor* $\left\{\mathfrak{X}, \mathfrak{C}\right\} \hookrightarrow \left\{\mathfrak{X}, \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})\right\}_{/\mathfrak{P}}$ *whose essential image is spanned by the* ∞*-functors preserving cocartesian morphisms over inert morphisms of* $\mathfrak{P}$.

Proof. Direct application of [GHN17, Proposition 7.3] shows that for any $(\infty, 1)$-category $\mathfrak{X}$ over $\mathfrak{P}$ there is an equivalence

$$\left\{\mathfrak{X}, \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})\right\}_{/\mathfrak{P}} \simeq \left\{\mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}), \mathfrak{C}\right\}, \tag{30}$$

in which an ∞-functor $\mathcal{S} \colon \mathfrak{X} \to \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$ over $\mathfrak{P}$ (so mapping $X \in \mathfrak{X}$ to $\mathcal{S}(X) \colon \mathfrak{P}^{\mathrm{inrt}}_{\not{p}X/} \to \mathfrak{C}$) corresponds to $\widetilde{\mathcal{S}} \colon \mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}) \to \mathfrak{C}$ mapping

$$(X, \not{p}X \rightarrowtail Q) \mapsto \mathcal{S}(X)(\not{p}X \rightarrowtail Q). \tag{31}$$

In addition, by the description of $\overline{\not{p}}$-cocartesian morphisms in $\overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$ provided by [Lur09, Corollary 3.2.2.13], one sees that an ∞-functor $\mathcal{S} \colon \mathfrak{X} \to \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$ takes an arrow $\xi \colon X \to X'$ to a cocartesian arrow in $\overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$ if and only if the corresponding $\widetilde{\mathcal{S}}$ takes all morphisms (ξ, θ) where θ is ev_0-cartesian in $\mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P})$ to equivalences in $\mathfrak{C}$.

By Proposition 5.5, $\left\{\mathfrak{X}, \mathfrak{C}\right\}$ identifies as the full sub-$(\infty, 1)$-category of $\left\{\mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}), \mathfrak{C}\right\}$ on those ∞-functors inverting all morphisms in $\mathfrak{I}_{\mathfrak{X}}$. More precisely, the equivalence of eq. (30) sits in the sequence

$$\left\{\mathfrak{X}, \mathfrak{C}\right\} \simeq \left\{\mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}), \mathfrak{C}\right\}_{(\mathfrak{I}_{\mathfrak{X}})} \hookrightarrow \left\{\mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}), \mathfrak{C}\right\} \simeq \left\{\mathfrak{X}, \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})\right\}_{/\mathfrak{P}}. \tag{32}$$

One then only needs to observe that an ∞-functor $\widetilde{\mathcal{S}} \in \left\{\mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}), \mathfrak{C}\right\}$, corresponding to $\mathcal{S} \in \left\{\mathfrak{X}, \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})\right\}_{/\mathfrak{P}}$, is in $\left\{\mathfrak{X} \times_{\mathfrak{P}} \mathfrak{Ar}_{\mathrm{inrt}}(\mathfrak{P}), \mathfrak{C}\right\}_{(\mathfrak{I}_{\mathfrak{X}})}$ if and only if for any $\xi \colon X \to X'$ in $\mathfrak{X}$ that is $\not{p}$-cocartesian and any θ as in eq. (28) that is ev_0-cartesian and ev_0-over an inert arrow, $\widetilde{\mathcal{S}}(\xi, \theta)$ is an equivalence, which is exactly the description given above of $\mathcal{S}$ taking $\not{p}$-cocartesian morphisms $\not{p}$-over (since $\mathrm{ev}_0(\theta) = \not{p}(\xi)$) an inert arrow to $\overline{\not{p}}$-cocartesian arrows. □

We can now arrive at our main result.

Theorem 5.7. *Let $\mathfrak{X} \to \mathfrak{P}$ be a fibrous $\mathfrak{P}$-pattern, with $\mathfrak{P}$ sound, and $\mathfrak{C}$ a $\mathfrak{P}$-complete $(\infty, 1)$-category. There is an equivalence of $(\infty, 1)$-categories*

$$\mathfrak{Seg}_{\mathfrak{X}}(\mathfrak{C}) \simeq \mathfrak{Alg}_{\mathfrak{X}}(\mathfrak{Span}_{\mathfrak{P}}(\mathfrak{C})). \tag{33}$$

In particular, taking $\mathfrak{X} = \mathfrak{P}$ to be the terminal fibrous $\mathfrak{P}$-pattern, we obtain theorem A.

Proof. Let $\mathscr{S}\colon \mathfrak{X} \to \overline{\mathfrak{Span}}_{\mathfrak{P}}(\mathfrak{C})$ be an ∞-functor over $\mathfrak{P}$ that preserves cocartesian morphisms over inert arrows (so corresponds to $\widetilde{\mathscr{S}}\colon \mathfrak{X} \to \mathfrak{C}$). It factors through $\mathfrak{Span}_{\mathfrak{P}}(\mathfrak{C})$ if and only if for every $X \in \mathfrak{X}$, the $\mathfrak{P}^{\mathrm{inrt}}_{\ell X/}$-object $\mathscr{S}(X)$ in $\mathfrak{C}$ is Segal.

By [Bar22, Lemma 2.39], for any map of algebraic patterns $\mathfrak{O} \to \mathfrak{P}$ and any $P \in \mathfrak{P}$, the projection $\mathfrak{O} \times_{\mathfrak{P}} \mathfrak{P}_{P/} \to \mathfrak{P}$ is an iso-Segal morphism. Applying this to $\mathfrak{O} = \mathfrak{P}^{\mathrm{inrt}}$ and $P = \ell X$ (for any X), we find that the above condition is equivalent to $\widetilde{\mathscr{S}}$ being a Segal $\mathfrak{X}$-object. □

6. Some examples: flavours of generalised multicategories

Remark 6.1 (Graphs and endomorphisms). Since the Segal condition for a pattern $\mathfrak{P}^{\mathrm{el}}$ with only elementary objects and inert morphisms is trivial, the underlying $\mathfrak{P}$-graph of the $\mathfrak{P}$-monoidal $(\infty,1)$-category $\mathbb{Span}_{\mathfrak{P}}(\mathfrak{C})$ is $\mathbb{Span}_{\mathfrak{P}^{\mathrm{el}}}(\mathfrak{C})$, which is directly given by the ∞-functor $\left\{\mathfrak{P}^{\mathrm{el}}_{-/}, \mathfrak{C}\right\}$. Since $\mathfrak{P}^{\mathrm{el}}_{E/}$, for any elementary E, generally has a simple form, this will make the underlying $\mathfrak{P}$-graph of $\mathfrak{P}$-spans easy to describe.

Furthermore, since the "algebraic operations" in Segal $\mathfrak{P}$-objects come from active morphisms, a $\mathfrak{P}^{\mathrm{el}}$-monad carries no algebraic structure and can simply be seen as a $\mathfrak{P}$-endomorphism. The statement of Theorem 5.7 thus restricts to saying that $\mathfrak{P}$-endomorphisms in $\mathbb{Span}_{\mathfrak{P}}(\mathfrak{C})$ are exactly $\mathfrak{P}$-graphs in $\mathfrak{C}$.

6.2 Categories and multiple categories

Take $\mathfrak{P}$ to be the pattern $\Delta^{\mathrm{op}\natural}$, consisting of the simplicial indexing category Δ^{op} with its usual inert-active factorisation system (where a map $[n] \to [m]$ in Δ is inert if it is a subinterval inclusion and active if it is endpoints-preserving), and $[0]$ and $[1]$ as elementary objects. Its Segal objects are internal categories.

Remark 6.2.1. Direct comparison shows that for any $[n] \in \Delta^{\mathrm{op}}$, the category $(\Delta^{\mathrm{op}\natural})^{\mathrm{inrt}}_{[n]/}$ is equivalent (in fact isomorphic) to the twisted arrow category of $\mathbb{n} + \mathbb{1} = [n]$, as has been previously noticed in [Hau18a, Remark 5.4] and implicitly used in [Kos21, Remark 2.18]: more precisely, a morphism in

$\mathfrak{Tw}(\mathbb{m}+\mathbb{1})$ represented by a factorising square in $\mathbb{m}+\mathbb{1}$ below-left

$$\begin{tikzcd} i & i' \arrow[l, "i'\leq i"'] \arrow[d, "i'\leq j'"] \\ j \arrow[u, <-, "i\leq j"'] \arrow[r, "j\leq j'"'] & j' \end{tikzcd} \qquad \begin{tikzcd} {[n]} \arrow[rr, equal] & & {[n]} \\ {[j-i]\simeq\{i,\dots,j\}} \arrow[u, hook] \arrow[rr, hook] & & {[j'-i']\simeq\{i',\dots,j'\}} \arrow[u, hook] \end{tikzcd} \tag{34}$$

corresponds to the morphism in $(\Delta^{\mathrm{op}\natural})^{\mathrm{inrt}}_{[n]/}$ represented as the commutative square (in Δ) above-right.

Thus for any $(\infty,1)$-category $\mathfrak{C}$ admitting finite fibre products, $\mathbb{Span}_{\Delta^{\mathrm{op}\natural}}(\mathfrak{C})$ is the double $(\infty,1)$-category of spans in $\mathfrak{C}$ constructed in [Bar13] and [Hau18a] (and denoted $\mathfrak{SPAN}_1^+(\mathfrak{C})$ there).

Now, we also note that fibrous $\Delta^{\mathrm{op}\natural}$-patterns are virtual double ∞-categories (also referred to as generalised non-symmetric ∞-operads in [GH15]) so that morphisms of fibrous $\Delta^{\mathrm{op}\natural}$-patterns correspond to "lax double functors", and in particular $\Delta^{\mathrm{op}\natural}$-monads recover the usual notion of monad in a virtual double $(\infty,1)$-category. In conclusion, Theorem 5.7 applied to the pattern $\Delta^{\mathrm{op}\natural}$ recovers the main theorem of [Hau21], that monads (or algebras) in spans are internal categories.

Example 6.2.2. More generally, using products of algebraic patterns (*cf.* Example 2.4), one sees that for any $d\in\mathbb{N}$, the Segal $\Delta^{\mathrm{op}\natural,d}$-$(\infty,1)$-category $\mathbb{Span}_{\Delta^{\mathrm{op}\natural,d}}(\mathfrak{C})$ is the $(d+1)$-uple $(\infty,1)$-category $\mathfrak{SPAN}_d^+(\mathfrak{C})$ of iterated spans also constructed in [Hau18a].

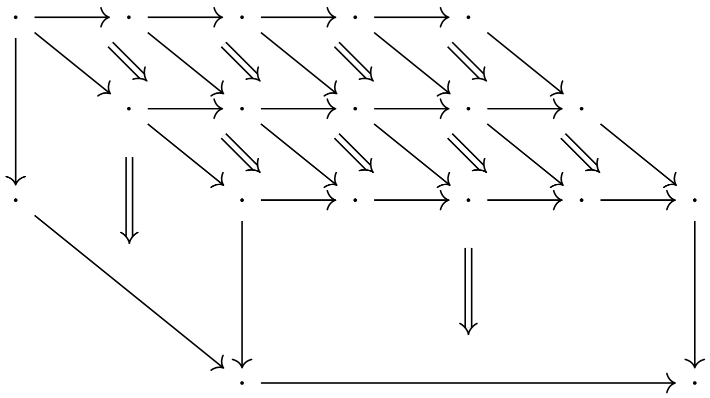
We now explain how lax Segal $\Delta^{\mathrm{op}\natural,d}$-fibrations should be seen as virtual $(d+1)$-uple ∞-categories. When viewing (strong) Segal $\Delta^{\mathrm{op}\natural,d}$-fibrations as $(d+1)$-uple categories, one should separate the d directions coming from Δ^d, which we dub the **algebraic** directions, from the last one coming from straightening the cocartesian fibration, which we will know as the categorical, or **transversal**, direction. A lax Segal $\Delta^{\mathrm{op}\natural,d}$-fibration $\mathfrak{X}$ is then virtual in all the algebraic directions: it has, for all $n\leq d$, algebraic n-cells in the usual directions for d-uple categories, and it has transversal cells from any n-dimensional grid of n-cells to a single n-cell. We stress that, for the domains of the transversal n-cells, we only require grids rather than the more general n-uple pasting diagrams of [Rui22], as the grids are the objects of Δ^d.

Let us represent the low dimensions; for ease of viewing we shall draw the transversal direction vertically, from top to bottom (since drawing it transversally would hide the face with the most information in the back).

For $d = 1$, the description — of virtual double ∞-categories — is well-known: there are objects and algebraic arrows, and in addition there are transversal arrows between objects and transversal cells from any pasting diagram (*i.e.* composable sequence) of algebraic arrows to one algebraic arrow, drawn as 2-cells in

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d, phantom, "\Downarrow"] & \cdot \arrow[r] & \cdot \arrow[d] \\ \cdot \arrow[rrr] & & & \cdot \end{tikzcd}. \tag{35}$$

For $d = 2$, we similarly have objects, two kinds of algebraic 1-arrows, and algebraic squares or 2-arrows, and in addition transversal 1-arrows between objects, two kinds of transversal 2-cells, corresponding to the two directions of algebraic arrows, and finally transversal cubes or 3-cells for any grid of composable squares, as represented in



(36)

where the 3-cell is not visible but fills the cube.

A $\Delta^{\mathrm{op}\natural,d}$-monad then consists of monads (whose structure cells are transversal) in all possible algebraic directions and throughout the different dimensions, resembling (a less lax version of) the intermonads of [GP17, §7.1].

Example 6.2.3. If one takes instead the pattern $\Delta^{\mathrm{op}^\flat}$, which has the same underlying category and factorisation system but only $[1]$ as elementary object — whose Segal objects are internal categories $X_\bullet$ with trivial object

X_0 of objects, so internal associative monoids — then the monoidal $(\infty,1)$-category $\mathbb{Span}_{\Delta^{\mathrm{op}\flat}}(\mathfrak{C})$, for $\mathfrak{C}$ admitting finite products (for this is what $\Delta^{\mathrm{op}\flat}$-completeness means) is $\mathfrak{C}$ itself seen with its cartesian monoidal structure.

Generalising to $\Delta^{\mathrm{op}\flat,n}$ (whose Segal objects are n-iterated associative monoids, so $\mathscr{E}_n$-monoids), we have that $\mathbb{Span}_{\Delta^{\mathrm{op}\flat,n}}(\mathfrak{C})$ is $\mathfrak{C}$ seen with its cartesian structure as an $\mathscr{E}_n$-monoidal structure. In this case, Theorem 5.7 simply recovers the fact that Segal $\Delta^{\mathrm{op}\flat,n}$-objects in a cartesian $(\infty,1)$-category $\mathfrak{C}$ are n-uply commutative (meaning $\mathscr{E}_n$-) algebras in the cartesian monoidal $(\infty,1)$-category $\mathfrak{C}^\times$ (*i.e.* $\mathscr{E}_n$-monoids in $\mathfrak{C}$).

6.3 Commutative monoids

Take $\mathfrak{P}$ to be the pattern $\Gamma^{\mathrm{op}\flat}$ where $\Gamma^{\mathrm{op}} \simeq \mathfrak{Fin}_*$ is the opposite of Segal's category, which is equivalent to the category of pointed finite sets, with its usual inert-active factorisation system, and $\langle 1\rangle$ as the only elementary object. Its Segal objects are commutative (or $\mathscr{E}_\infty$) monoids. As explained in [CH21, Example 14.22], this algebraic pattern is not saturated; however its global saturation is easily seen from the fact that $\Gamma^{\mathrm{op}\flat,\mathrm{el}}_{\langle n\rangle/}$ is a set of n elements.

It also follows that for any $\Gamma^{\mathrm{op}\flat}$-complete (*i.e.* admitting finite products) $(\infty,1)$-category $\mathfrak{C}$, $\mathbb{Span}_{\Gamma^{\mathrm{op}\flat}}(\mathfrak{C})$ is again $\mathfrak{C}$ itself equipped with its cartesian symmetric monoidal structure. Since fibrous $\Gamma^{\mathrm{op}\flat}$-patterns are ∞-operads in the sense of [Lur17] and $\Gamma^{\mathrm{op}\flat}$-monads are commutative algebras, we recover that Segal $\Gamma^{\mathrm{op}\flat}$-objects in $\mathfrak{C}$ are commutative monoids in $\mathfrak{C}$ (where it is again understood that the term "monoid" refers to an algebra in a cartesian monoidal ∞-category).

Remark 6.3.1. For the product pattern $\mathfrak{P} = \Gamma^{\mathrm{op}\flat} \times \Delta^{\mathrm{op}\natural}$, whose Segal objects are internal symmetric monoidal categories, we recover as $\mathbb{Span}_{\mathfrak{P}}(\mathfrak{C})$ the double $(\infty,1)$-category of spans in $\mathfrak{C}$, endowed with its symmetric monoidal structure coming from the cartesian product in $\mathfrak{C}$.

Example 6.3.2. As a further variant, one may consider the algebraic pattern $\Gamma^{\mathrm{op}\natural}$, which is like $\Gamma^{\mathrm{op}\flat}$ but also has $\langle 0\rangle$ as an additional elementary object. Its Segal objects in a $\Gamma^{\mathrm{op}\natural}$-complete $(\infty,1)$-category $\mathfrak{C}$ are commutative monoids in a slice of $\mathfrak{C}$, which it is convenient to interpret as families of commutative monoids in $\mathfrak{C}$ indexed by an object of $\mathfrak{C}$. In the same spirit, fibrous $\Gamma^{\mathrm{op}\natural}$-patterns are generalised ∞-operads of [Lur17], which are the same thing as families of ∞-operads.

For any object $\langle n\rangle$, the category $\Gamma^{\mathrm{op}\natural,\mathrm{el}}_{\langle n\rangle/}$ is

$$\begin{tikzcd}
\rho_1 \arrow[drr] & \rho_2 \arrow[dr] & \cdots & \rho_{n-1} \arrow[dl] & \rho_n \arrow[dll] \\
 & & \left(\langle n\rangle \xrightarrow{!} \langle 0\rangle\right) & &
\end{tikzcd} \tag{37}$$

where $\rho_i \colon \langle n\rangle \to \langle 1\rangle$ sends i to 1 and all the other elements of $\langle n\rangle$ to 0, from which it is seen that the pattern $\Gamma^{\mathrm{op}\natural}$ is globally saturated. More generally, any inert map $\langle n\rangle \to \langle k\rangle$ (with necessarily $k \leq n$) determines and is uniquely determined by a k-element subset of $n = \langle n\rangle \setminus \{0\}$, so that writing $\wp(n)$ for the powerset of n (equipped with its natural order), we have $\Gamma^{\mathrm{op}\natural,\mathrm{el}}_{\langle n\rangle/} \simeq \wp(n)^{\mathrm{op}}$. For example, for $n = 3$ the poset $\Gamma^{\mathrm{op}\natural,\mathrm{el}}_{\langle 3\rangle/}$ is

$$\begin{tikzcd}
 & & \{1,2,3\} \arrow[dl] \arrow[d] \arrow[dr] & & \\
 & \{1,2\} \arrow[dl] \arrow[dr] & \{1,3\} \arrow[dll] \arrow[drr] & \{2,3\} \arrow[dl] \arrow[dr] & \\
\{1\} \arrow[drr] & & \{2\} \arrow[d] & & \{3\} \arrow[dll] \\
 & & \{0\} & &
\end{tikzcd} \tag{38}$$

(containing copies of $\Gamma^{\mathrm{op}\natural,\mathrm{el}}_{\langle 2\rangle/}$, $\Gamma^{\mathrm{op}\natural,\mathrm{el}}_{\langle 1\rangle/}$, and $\Gamma^{\mathrm{op}\natural,\mathrm{el}}_{\langle 0\rangle/}$ on the left). We can thus see that $\mathbb{Span}_{\Gamma^{\mathrm{op}\natural}}(\mathfrak{C})$ is the family of slices of $\mathfrak{C}$, each equipped with its monoidal structure given by the pullbacks in $\mathfrak{C}$, and Theorem 5.7 recovers the description of Segal $\Gamma^{\mathrm{op}\natural}$-objects given above.

6.4 Higher categories and iterated spans

We now take $\mathfrak{P}$ to be the pattern ${\Theta_\ell}^{\mathrm{op}\natural}$ of Remark 3.5, for some $\ell \in \mathbb{N} \cup \{\omega\}$.

It follows from the description given in eq. (10) that $\mathbb{Span}_{\Theta_\ell^{\mathrm{op}\natural}}(\mathfrak{C})$ is a cellular $(\infty,1)$-category of ℓ-times iterated spans: for any $k \leq \ell$, the $(\infty,1)$-category of k-cells has as objects the spans between the apices of two $(k-1)$-iterated spans, and as morphisms the morphisms between spans. In other words, it is a categorical enhancement of the $(\infty,\ell+1)$-category $\mathfrak{Span}_\ell^+(\mathfrak{C})$ of ℓ-iterated spans from [Hau18a, Definition 5.16, Remark 5.17], obtained by discarding all the extraneous "algebraic" directions of the $(\ell+1)$-uple one as in [*ibid.*] but still retaining the transversal one.

Remark 6.4.1. At the level of the underlying $\Theta^{\mathrm{op}\natural}$-graph, the fact that our construction of the globular category of iterated spans through slices of $\Theta^{\mathrm{op}\natural,\mathrm{el}} = \mathbb{G}^{\mathrm{op}}$ recovers the combinatorial one given in [Bat98, Definition 3.2] was already observed in [Str00].

Example 6.4.2. For any $k \leq \ell$, we can also define the pattern $({\Theta_\ell}^{\mathrm{op}})^{\Sigma^k\natural}$ to consist of the same structure as $\Theta^{\mathrm{op}\natural}$ but only the globes $\mathfrak{D}_n$ with $n \geq k$ as elementaries. For example, if ℓ is finite, taking $k = \ell$ recovers the pattern denoted $\Theta_\ell^{\mathrm{op}\flat}$ in [CH21]. Segal objects for $({\Theta_\ell}^{\mathrm{op}})^{\Sigma^k\natural}$ are $\mathscr{E}_k$-monoidal internal $(\ell-k)$-categories.

As noted in [CH21, Example 9.8. (iv)], fibrous ${\Theta_\ell}^{\mathrm{op}\natural}$-patterns are an ∞-categorical version of the ℓ-globular multicategories or many-sorted ℓ-globular operads of [Lei04, p. 273] and [CS10, Example 4.11], themselves a many-sorted, or coloured, version of the ℓ-globular operads of [Bat98]. They are similar to the fibrous $\Delta^{\mathrm{op}\natural,\ell}$-patterns described in EXAMPLE 6.2.2, but where the domain of a transversal n-cell is an n-categorical pasting diagram instead of an n-dimensional grid (and its codomain is a single n-globe rather than an n-cube).

Warning 6.4.3. Despite their name of "ℓ-operads" in [Bat98], fibrous $\Delta^{\mathrm{op}\natural,\ell}$-patterns should not be thought of as a kind of (∞,ℓ)-operads, meaning ∞-operads enriched in $(\infty,\ell-1)$-$\mathfrak{Cat}$. Indeed, as seen from the description above, they contain more data and structure than (∞,ℓ)-operads.

Likewise, the strong Segal ${\Theta_\ell}^{\mathrm{op}\natural}$-fibrations, known as "monoidal ℓ-globular categories" in [*ibid.*], are really categorical (∞,ℓ)-categories. In particular, ${\Theta_\ell}^{\mathrm{op}\natural}$-monads are very different from any kind of usual ℓ-categorical monads that could be made sense of (for example following the philosophy of [Hau18b] identifying Segal ${\Theta_{\ell+1}}^{\mathrm{op}\natural}$-objects with reduced categorical ${\Theta_\ell}^{\mathrm{op}\natural}$-objects) in categorical (∞,ℓ)-categories: the ${\Theta_\ell}^{\mathrm{op}\natural}$-monad structure associates to any configuration of (algebraic) n-cells a *transversal* cell, so is really independent of the (∞,ℓ)-categorical structure.

As such, we will only refer to ${\Theta_\ell}^{\mathrm{op}\natural}$-monads as **$\ell$-globular monads**.

We then obtain by applying Theorem 5.7 that ${\Theta_\ell}^{\mathrm{op}\natural}$-monads in the categorical (∞,ℓ)-category $\mathbb{Span}_{{\Theta_\ell}^{\mathrm{op}\natural}}(\mathfrak{C})$ are Segal ${\Theta_\ell}^{\mathrm{op}\natural}$-objects in $\mathfrak{C}$, or in more evocative language:

Corollary 6.4.4. *There is an equivalence of $(\infty,1)$-categories between ℓ-globular monads in the categorical (∞,ℓ)-category $\mathbb{Span}^+_\ell(\mathfrak{C})$ of ℓ-iterated spans in $\mathfrak{C}$, and internal (∞,ℓ)-categories in $\mathfrak{C}$.*

For $\ell = \omega$, this recovers a homotopical formulation of the definition of weak ω-categories given by [Bat98] as well as that of [Lei04] (*cf.* [CL04] for an explanation of the different definitions of ω-categories).

6.5 Multicategories and multispans

We finish by considering the algebraic pattern $\Omega^{\mathrm{op}\natural}$ (resp. $\Omega^{\mathrm{op}\natural}_{\mathrm{pl.}}$) whose Segal objects are internal coloured operads (resp. internal coloured planar operads). Here, Ω is the dendroidal category, whose objects are rooted trees (resp. with planar structure), henceforth referred to as dendrices to avoid confusion with the objects of Θ, and whose morphisms express the grafting of dendrices — in contrast with the morphisms of Θ which express the pasting of trees. The algebraic pattern structure is given by having the inert morphisms be the sub-dendrex inclusions, the active morphisms the boundary-preserving maps, and the elementary objects be the corollas $\star_a$ (determined by their arities $a \in \mathbb{N}$) and the nodeless edge η. As noted in [CH21, Examples 14.21], the pattern $\Omega^{\mathrm{op}\natural}$ is saturated because any dendrex can be decomposed as a gluing of corollas along edges, and the same argument shows that it is also globally saturated.

To understand the dendroidal $(\infty,1)$-category $\mathbb{Span}_{\Omega^{\mathrm{op}\natural}}(\mathfrak{C})$, let us first describe its underlying categorical $\Omega^{\mathrm{op}\natural}$-graph $\mathbb{Span}_{\Omega^{\mathrm{op}\natural,\mathrm{el}}}(\mathfrak{C})$. At the level of colours, we just have $\Omega^{\natural,\mathrm{el}}_{/\eta} = \{\mathrm{id}_\eta\}$. At the level of operations, writing

$e_1, \ldots, e_a$ the leaves of the corolla $\star_a$ and r its root, we find that $\Omega^{\natural,\mathrm{el}}_{/\star_a}$ is the category

$$\begin{tikzcd}
(\eta \xrightarrow{\ulcorner e_1 \urcorner} \star_a) \arrow[dr] & \cdots & (\eta \xrightarrow{\ulcorner e_a \urcorner} \star_a) \arrow[dl] \\
& \mathrm{id}_{\star_a} & \\
& (\eta \xrightarrow{\ulcorner r \urcorner} \star_a) \arrow[u] &
\end{tikzcd} \tag{39}$$

of $(a+1)$-ary multicospans.

The structure of category objects in multicategories (coloured non-symmetric operads) was studied in [CGR14, Definition 3.9]. In our case, we get for $\mathbb{Span}_{\Omega^{\mathrm{op}\natural}}(\mathfrak{C})$ an operadic composition of multispans by fibre products along the relevant legs, where each multispan has a distinguished root as seen in eq. (39).

Example 6.5.1. It is also possible to replace $\Omega^{\mathrm{op}\natural}$ by the pattern $\Xi^{\mathrm{op}\natural}$ of [HRY19], whose Segal objects are cyclic operads. We obtain for $\mathbb{Span}_{\Xi^{\mathrm{op}\natural}}(\mathfrak{C})$ the same structure as above, except that the $(a+1)$-ary spans come without a choice of root. Note also that in Ξ, the nodeless edge η is equipped with an involution, which for Segal objects becomes a "duality" operation on colours. In our case, it acts as the identity.

Going further, we may also use the pattern $\Upsilon^{\mathrm{op}\natural}$ of [HRY20] (denoted $\mathbb{U}$ there), whose Segal objects are modular operads. The categorical modular ∞-operad $\mathbb{Span}_{\Upsilon^{\mathrm{op}\natural}}(\mathfrak{C})$ works much as $\mathbb{Span}_{\Xi^{\mathrm{op}\natural}}(\mathfrak{C})$, but with additional contraction operations that turn the abstract self-duality of objects into an actual self-duality (in the usual monoidal, or rather properadic, sense).

The fibrous $\Omega^{\mathrm{op}\natural}$-patterns were identified in [Ber22] as "tree-hyperoperads", which are cumbersome to describe in detail (*cf.* [GK98, §4.1] or [MSS02, Definition 5.45] for the modular generalisation, simply called hyperoperads). Nevertheless, we still obtain from Theorem 5.7 that dendroidal monads in the categorical ∞-operad of multispans in $\mathfrak{C}$ are internal operads in $\mathfrak{C}$.

Remark 6.5.2. The definition of operads, and more general multicategorical structures, as monads in multispans is well-known: it dates to [Bur71], and was independently rediscovered by both [Her04] and [Lei98], and then further systematised by [Lei04] and [CS10]. From a cartesian monad $\mathscr{T}$ on a category $\mathfrak{C}$, one constructs a double category of Kleisli $\mathscr{T}$-spans, whose objects are those of $\mathfrak{C}$ and morphisms from C to D are spans from $\mathscr{T}C$ to D, composition of spans using the monad structure.

For example, taking $\mathscr{T}$ to be the monad $\mathscr{F}_{\mathbb{F}^{\mathrm{op}\flat}}$ for free monoids, a Kleisli $\mathscr{T}$-span is a multispan of arbitrary arity, and monads (in the double-categorical sense) are coloured operads. Generally speaking, if $\mathscr{T}$ is the monad $\mathscr{F}_{\mathfrak{P}}$ for free Segal $\mathfrak{P}$-objects on $\mathfrak{P}$-graphs for some appropriate algebraic pattern $\mathfrak{P}$, we expect that monads in Kleisli $\mathscr{F}_{\mathfrak{P}}$-spans should be fibrous $\mathfrak{P}$-patterns, obtained as the Segal objects for a plus construction $\mathfrak{P}^+$ of $\mathfrak{P}$ (as in [Ker23, Proposition 3.2.10]), so as $\mathfrak{P}^+$-monads in $\mathfrak{P}^+$-spans.

However, Kleisli $\mathscr{F}_{\mathbb{F}^{\mathrm{op}\flat}}$-spans and $\Omega^{\mathrm{op}\natural}$-spans, while both admitting a natural interpretation as multispans, form markedly different structures. On the one hand, $\mathbb{S}\mathrm{pan}_{\Omega^{\mathrm{op}\natural}}(\mathfrak{C})$ is a categorical ∞-operad, whose operadic composition is given (leg by leg) by simple pullbacks. On the other hand, Keisli $\mathscr{F}_{\mathbb{F}^{\mathrm{op}\flat}}$-spans only form a double category, but its composition is more complex and makes full use of the monad structure on $\mathscr{F}_{\mathbb{F}^{\mathrm{op}\flat}}$. For a general algebraic pattern $\mathfrak{P}$, the difference will be similar: we think of it as moving the structure from the microcosm (on the Keisli $\mathscr{F}_{\mathfrak{P}}$-spans side) to the macrocosm (on the $\mathfrak{P}^+$-spans side).

It is nonetheless unclear what the precise relation between the two constructions is, if there even is one: the Keisli-type construction can be abstracted away from a span setting by using general monads acting on virtual double categories, but it is unlikely to be able to handle non-directed structures such as the cyclic and modular ∞-operads of EXAMPLE 6.5.1.

7. Conclusion: A fibrational perspective

Notation 7.1. In this section, we will identify $(\infty,1)$-categories with internal categories in ∞-$\mathfrak{Grpd}$, where internal categories are by definition Segal $\mathbb{\Delta}^{\mathrm{op}\natural}$-objects satisfying Rezk's univalence-completeness condition. It will also be convenient to see Segal $\mathfrak{P}$-objects in $(\infty,1)$-$\mathfrak{Cat}$ — such as, in particular, the $\mathfrak{P}$-$(\infty,1)$-categories of $\mathfrak{P}$-spans — as internal categories in $\mathfrak{Seg}_{\mathfrak{P}}(\infty\text{-}\mathfrak{Grpd})$.

Recall that, for any regular cardinal κ, the $(\infty,1)$-category $\infty\text{-}\mathfrak{Grpd}^{(\kappa)} \subset$

$\infty\text{-}\mathfrak{Grpd}$ is the base of the universal discrete cocartesian fibration with κ-small fibres $\infty\text{-}\mathfrak{Grpd}_{\bullet}^{(\kappa)} \to \infty\text{-}\mathfrak{Grpd}^{(\kappa)}$ in the $(\infty,2)$-topos $(\infty,1)\text{-}\mathfrak{Cat}$, just as $\mathfrak{Set}^{(\kappa)} \subset \mathfrak{Set}$ is the universal κ-small discrete cocartesian fibration in the $(2,2)$-topos $\mathfrak{Cat}$. In [Web07, Examples 4.7 and 4.8], it is explained that, for an algebraic pattern $\mathfrak{P}^{\mathrm{el}}$ in which all objects are elementary and all morphisms inert, the construction $\mathbb{Span}_{\mathfrak{P}^{\mathrm{el}}}(-)$ preserves classifying discrete fibrations, so that the 2-topos $\mathfrak{Cat}\left(\left\{\mathfrak{P}^{\mathrm{el}}, \mathfrak{Set}\right\}\right)$ has a sufficient family of classifying discrete cocartesian fibrations given by $\mathbb{Span}_{\mathfrak{P}^{\mathrm{el}}}(\mathfrak{Set}_{\bullet}^{(\kappa)}) \to \mathbb{Span}_{\mathfrak{P}^{\mathrm{el}}}(\mathfrak{Set}^{(\kappa)})$ (where "sufficient" means that every discrete cocartesian fibration is classified by one in the family).

In the ∞-categorical setting, the properties of universal (or "classifying") fibrations are captured by the notion of univalence, which we restate from [GK16] (see also [Ras21b, Theorem 4.4]) in the internal setting.

Construction 7.2. Let $\mathfrak{C}$ be a finitely complete $(\infty,1)$-category. Recall that a **discrete cocartesian fibration** in $\mathfrak{C}$ is an internal functor $\mathscr{f}\colon \mathbb{E} \to \mathbb{B}$ such that $(d_1, \mathscr{f}_1)\colon \mathbb{E}_1 \to \mathbb{E}_0 \times_{\mathbb{B}_0} \mathbb{B}_1$ is an equivalence.

Lifting the construction of [GK16, Theorem 2.10] to the cartesian closed $(\infty,2)$-category $\mathfrak{Cat}(\mathfrak{C})$, one can construct for any discrete cocartesian fibration $\mathscr{f}\colon \mathbb{E} \to \mathbb{B}$ in $\mathfrak{C}$ an internal category $\mathbb{Eq}_{/\mathbb{B}\times\mathbb{B}}(\varpi_1^*\mathbb{E}, \varpi_2^*\mathbb{E})$ over $\mathbb{B}\times\mathbb{B}$ (where $\varpi_1, \varpi_2\colon \mathbb{B}\times\mathbb{B} \to \mathbb{B}$ are the two projections) characterising equivalences between fibres of $\mathscr{f}$.

Definition 7.3 (Univalent fibration)**.** A discrete cocartesian fibration $\mathbb{E} \to \mathbb{B}$ internal to $\mathfrak{C}$ is **univalent** if $\mathbb{B} \to \mathbb{Eq}_{/\mathbb{B}\times\mathbb{B}}(\varpi_1^*\mathbb{E}, \varpi_2^*\mathbb{E})$ is an equivalence.

Example 7.4. It is shown in [Cis19, Proposition 5.3.13] that the universal discrete cocartesian fibration $\infty\text{-}\mathfrak{Grpd}_{\bullet} \to \infty\text{-}\mathfrak{Grpd}$ (in the $(\infty,2)$-category $(\infty,1)\text{-}\mathfrak{Cat} = \mathfrak{Cat}(\infty\text{-}\mathfrak{Grpd})$) is univalent.

Using this characterisation, one can show (though we omit the proof here as this result is only used for motivation) that for any sound algebraic pattern $\mathfrak{P}^{\mathrm{inrt}}$ all of whose morphisms are inert, the construction

$$\mathbb{Span}_{\mathfrak{P}^{\mathrm{inrt}}}(-)\colon (\infty,1)\text{-}\mathfrak{Cat}^{(\mathfrak{P}\text{-cplt})} \to \mathfrak{Cat}(\mathfrak{Seg}_{\mathfrak{P}^{\mathrm{inrt}}}(\infty\text{-}\mathfrak{Grpd})) \qquad (40)$$

preserves classifying discrete cocartesian fibrations:

Proposition 7.5. *Let $\mathfrak{G}_\bullet \to \mathfrak{G}$ be a univalent discrete cocartesian fibration. Then $\mathbb{S}\mathrm{pan}_{\mathfrak{P}^{\mathrm{inrt}}}(\mathfrak{G}_\bullet) \to \mathbb{S}\mathrm{pan}_{\mathfrak{P}^{\mathrm{inrt}}}(\mathfrak{G})$ is a univalent discrete cocartesian fibration internally to $\mathfrak{Seg}_{\mathfrak{P}^{\mathrm{inrt}}}(\mathfrak{G})$.* □

For an algebraic pattern with non-trivial active morphisms, the situation becomes richer and goes beyond the $(\infty, 2)$-topos theory of internal categories in presheaf $(\infty, 1)$-topoi. Indeed, Theorem 5.7 shows that, even when restricting our attention as we are doing here from fibrous patters to Segal fibrations, the morphisms of interest will be the lax morphisms, the maps of underlying fibrous patterns. We will thus use a notion of lax univalence, obtained by replacing strong morphisms by general lax morphisms of categorical Segal $\mathfrak{P}$-∞-groupoids in the definition of univalence for fibrations in $\mathfrak{Seg}_{\mathfrak{P}}(\infty\text{-}\mathfrak{Grpd})$.

Conjecture 7.6. Let $\mathfrak{G}_\bullet \to \mathfrak{G}$ be a univalent discrete cocartesian fibration. Then $\mathbb{S}\mathrm{pan}_{\mathfrak{P}}(\mathfrak{G}_\bullet) \to \mathbb{S}\mathrm{pan}_{\mathfrak{P}}(\mathfrak{G})$ is a lax-univalent discrete cocartesian fibration internally to $\mathfrak{Seg}_{\mathfrak{P}}(\mathfrak{G})$.

This Conjecture states that $\mathbb{S}\mathrm{pan}_{\mathfrak{P}}(\mathfrak{G}_\bullet) \to \mathbb{S}\mathrm{pan}_{\mathfrak{P}}(\mathfrak{G})$ classifies a class of discrete cocartesian fibrations. It remains to see that *every* such class is classified by a universal fibration of this form.

Conjecture 7.7. Suppose $\left(\mathfrak{G}_\bullet^{(\kappa)} \to \mathfrak{G}^{(\kappa)}\right)_{\kappa \in K}$ is a sufficient family of univalent fibrations for $(\infty, 1)$-$\mathfrak{Cat}$. Then $\left(\mathbb{S}\mathrm{pan}_{\mathfrak{P}}(\mathfrak{G}_\bullet^{(\kappa)}) \to \mathbb{S}\mathrm{pan}_{\mathfrak{P}}(\mathfrak{G}^{(\kappa)})\right)_{\kappa \in K}$ provides enough lax-univalent fibrations for $\mathfrak{Cat}(\mathfrak{Seg}_{\mathfrak{P}}(\infty\text{-}\mathfrak{Grpd}))$.

Corollary 7.8. *Let $\mathfrak{P}$ be a sound algebraic pattern and $\mathfrak{X} \to \mathfrak{P}$ be a Segal $\mathfrak{P}$-fibration, and assume that Conjecture 7.6 and Conjecture 7.7 hold. Then Segal $\mathfrak{X}$-objects in ∞-$\mathfrak{Grpd}$ are internal discrete cocartesian fibrations over the straightening $\mathbb{X}$ of $\mathfrak{X}$.*

Proof. The key point is that, by [GK16, Proposition 3.8], if $\mathfrak{G}_\bullet^{(\kappa)} \to \mathfrak{G}^{(\kappa)}$ is univalent then $\mathfrak{G}^{(\kappa)}$ is a full sub-$(\infty, 1)$-category of ∞-$\mathfrak{Grpd}$, from which it follows that lax morphisms $\mathfrak{X} \to \mathfrak{Span}_{\mathfrak{P}}(\mathfrak{G}^{(\kappa)})$ can be seen as lax morphisms $\mathfrak{X} \to \mathfrak{Span}_{\mathfrak{P}}(\infty\text{-}\mathfrak{Grpd})$. By the two conjectures, discrete cocartesian fibrations over $\mathbb{X}$ are the same thing as lax morphisms $\mathfrak{X} \to \mathfrak{Span}_{\mathfrak{P}}(\infty\text{-}\mathfrak{Grpd})$ (factoring through some $\mathfrak{Span}_{\mathfrak{P}}(\mathfrak{G}^{(\kappa)})$). At the same time, by Theorem 5.7, the latter are the same thing as Segal $\mathfrak{X}$-objects in ∞-$\mathfrak{Grpd}$ (or, to be precise, in some $\mathfrak{G}^{(\kappa)}$), which proves the result. □

Example 7.9 (Double fibrations and Segal fibrations). For the algebraic pattern, Corollary 7.8 says explicitly that discrete cocartesian fibrations of double ∞-categories correspond biunivocally to lax double ∞-functors to the double ∞-category of spans (of ∞-groupoids). This is precisely an ∞-categorical version of the main construction of [Lam21]. This use of internal discrete fibrations is also very similar to how [Ras21a] deals with fibrations of Segal spaces (see also [Lou23, §6.1.1] for the version for fibrations of ω-categories).

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CAHIERS DE TOPOLOGIE ET
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MODULE COTANGENT RELATIF

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Résumé. Pour tout objet x dans une catégorie $\mathbf{C}$ il est possible de définir la catégorie des *modules de Beck* au dessus de x, comme la catégorie $(\mathbf{C}_{/x})_{ab}$ des objets en groupe abélien de la catégorie $\mathbf{C}_{/x}$. Nous pouvons en déduire, au moins pour toute catégorie localement présentable, la notion de module *cotangent* Ω_x de x dans $(\mathbf{C}_{/x})_{ab}$.

Dans le cas de la catégorie $\mathbf{Alg}_k$ des k-algèbres commutatives au-dessus d'un anneau k, la catégorie des modules de Beck $(\mathbf{Alg}_{k/A})_{ab}$ au-dessus d'une k-algèbre A est équivalente à la catégorie $\mathbf{Mod}_A$ des A-modules et le module cotangent est égal au module des *différentielles de Kähler* de A.

Le but de cet article est de montrer pour toute catégorie localement présen -table des résultats qui généralisent les propriétés classiques des modules des différentielles de Kähler.

Abstract. For any object x in a category $\mathbf{C}$ it is possible to define the category of *Beck modules* over x as the category $(\mathbf{C}_{/x})_{ab}$ of abelian group objects in the category $\mathbf{C}_{/x}$. We can deduce from this construction, at least for any locally presentable category, the notion of *cotangent* module Ω_x of x in $(\mathbf{C}_{/x})_{ab}$.

In the case of the category $\mathbf{Alg}_k$ of commutative k-algebras over a ring k, the category of Beck modules $(\mathbf{Alg}_{k/A})_{ab}$ over a k-algebra A is equivalent to the category $\mathbf{Mod}_A$ of A-modules and the cotangent module is equal to the module of *Kähler differentials* of A.

The aim of this article is to prove for any locally presentable category some results which generalize the classical properties of modules of Kähler differentials.

Keywords. Module de Beck, module cotangent.

Mathematics Subject Classification (2010). 13A18 (12J10 14E15).

1. Introduction

Soit X une variété différentiable, analytique ou algébrique, pour tout point x de X nous pouvons définir un espace tangent T_xX et pour tout morphisme $g : X \longrightarrow Y$ nous pouvons définir une application différentielle $dg_x : T_xX \longrightarrow T_{g(x)}Y$ qui est une application linéaire qui approxime g au voisinage de $x \in X$. Nous pouvons ainsi considérer les notions d'espace tangent et de différentielle comme une manière de linéariser une catégorie. Pour pouvoir généraliser cette construction nous allons d'abord rappeler la construction de l'espace tangent et de l'application différentielle dans le cadre algébrique.

Soient k un anneau commutatif et A une k-algèbre, alors il existe un A-module $\Omega_{A/k}$, le *module des différentielles de Kähler* ou *module cotangent*, qui représente le foncteur

$$\begin{array}{rccc} Der_k(A,-): & \mathbf{Mod}_A & \longrightarrow & \mathbf{Ens} \\ & M & \mapsto & Der_k(A,M), \end{array}$$

où pour tout A-module M nous notons $Der_k(A,M)$ l'ensemble des *k-dérivations* de A à valeurs dans M.

Alors si X est une variété algébrique sur k, nous pouvons définir le faisceau $\Omega_{X/k}$ des différentielles de Kähler de X comme le faisceau quasi-cohérent sur X défini par le A-module $\Omega_{A/k}$ sur tout ouvert affine $Spec(A)$ de X, et le fibré tangent $TX \longrightarrow X$ est égal par définition au fibré $\mathbb{V}(\Omega_{X/k}) := \mathrm{Spec}(\mathrm{Sym}(\Omega_{X/k}))$.

Pour tout morphisme $f : A \rightarrow B$ dans la catégorie $\mathbf{Alg}_k$ des k-algèbres commutatives il existe une application de B-modules

$$\delta_f : \Omega_A \otimes_A B \longrightarrow \Omega_B \ ,$$

nous en déduisons que pour tout morphisme $g : X \longrightarrow Y$ nous avons un diagramme commutatif

$$\begin{array}{ccc} TX & \xrightarrow{dg} & TY \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & Y \end{array}$$

où le morphisme $TX \longrightarrow TY \times_Y X$ correspond à l'application $g^*(Sym(\Omega_Y)) \longrightarrow Sym(\Omega_X)$ induite par l'application naturelle $\delta_g : g^*(\Omega_Y) \longrightarrow \Omega_X$.

Le module cotangent, ou plus généralement le complexe cotangent qui est la version *dérivée* du module cotangent dans le cas d'une variété singulière, permet d'étudier les déformations de la variété, et plus précisément d'étudier les déformations infinitésimales. Pour cela il est utile de considérer les *extensions de carré nul* d'un anneau A. Nous rappelons que pour tout A-module M nous pouvons munir le A-module $A \oplus M$ d'une structure de k-algèbre commutative en posant $(a, m).(a', m') = (aa', am' + a'm)$.

Nous considérons la k-algèbre $B = A \oplus M$ comme un objet de la catégorie $\mathbf{Alg}_{k/A}$, c'est-à-dire comme une k-algèbre munie d'un morphisme $u_B : B \longrightarrow A$, et dont les morphismes sont les triangles commutatifs

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "u_B"'] & & C \arrow[dl, "u_C"] \\ & A & \end{tikzcd}$$

Nous remarquons alors que $B = A \oplus M$ n'est pas un objet arbitraire de la catégorie $\mathbf{Alg}_{k/A}$, mais est muni d'une structure *d'objet en groupe abélien*, et que le foncteur qui envoie M sur $A \oplus M$ définit une équivalence de catégories entre la catégorie $\mathbf{Mod}_A$ des A-modules et la catégorie $(\mathbf{Alg}_{k/A})_{ab}$ des objets en groupe abélien dans $\mathbf{Alg}_{k/A}$.

Le foncteur oubli de la structure d'objet en groupe abélien

$$U_A : (\mathbf{Alg}_{k/A})_{ab} \longrightarrow \mathbf{Alg}_{k/A}$$

admet un foncteur adjoint à gauche

$$L_A : \mathbf{Alg}_{k/A} \longrightarrow (\mathbf{Alg}_{k/A})_{ab}$$

et nous avons l'égalité $\Omega_{A/k} = L_A(\star_{\mathbf{Alg}_{k/A}})$, où $\star_{\mathbf{Alg}_{k/A}}$ est l'objet final $id_A : A \longrightarrow A$ de la catégorie $\mathbf{Alg}_{k/A}$.

Il est possible de généraliser cette construction, nous pouvons définir les objets en groupe abélien dans toute catégorie $\mathbf{C}$ comme les objets a de $\mathbf{C}$

tel que le préfaisceau représentable associé par le morphisme de Yoneda $h_a : \mathbf{C}^{op} \longrightarrow \mathbf{Ens}$ est un préfaisceau en groupes abéliens. Alors pour tout objet x de $\mathbf{C}$ nous définissons la catégorie des *modules de Beck* au-dessus de x comme la catégorie $(\mathbf{C}_{/x})_{ab}$ des objets en groupe abélien dans la catégorie $\mathbf{C}_{/x}$. De plus la catégorie $(\mathbf{C}_{/x})_{ab}$ est une catégorie additive, qui est abélienne si la catégorie $\mathbf{C}$ est une catégorie exacte.

Alors si le foncteur *oubli* $U_x : (\mathbf{C}_{/x})_{ab} \longrightarrow \mathbf{C}_{/x}$ admet un adjoint à gauche $L_x : \mathbf{C}_{/x} \longrightarrow (\mathbf{C}_{/x})_{ab}$, nous appelons *module cotangent* de x l'image par L_x de l'objet final $\star_{\mathbf{C}_{/x}} = id_x : x \longrightarrow x$ de $\mathbf{C}_{/x}$, et nous notons

$$\Omega_x = L_x(\star_{\mathbf{C}_{/x}}) .$$

Nous renvoyons aux articles de J.M. Beck [Be] et de D.G. Quillen [Qu] pour le lien entre la catégorie des objets en groupe abélien et la catégorie des modules et à l'article de M. Barr [Ba 1] pour les propriétés de la catégorie des modules de Beck.

Comme dans le cas des k-algèbres commutatives, nous voulons alors définir pour tout morphisme $f : x \longrightarrow y$ dans $\mathbf{C}$ une application entre les modules cotangents Ω_x et Ω_y. Pour f dans $\mathbf{C}$ il existe un foncteur additif $f^*_{ab} : (\mathbf{C}_{/y})_{ab} \longrightarrow (\mathbf{C}_{/x})_{ab}$ entre les catégories des modules de Beck au dessus respectivement de y et de x et nous construisons une application naturelle $[\delta_f] : \Omega_x \longrightarrow f^*_{ab}(\Omega_y)$ dans $(\mathbf{C}_{/x})_{ab}$. Alors si le foncteur f^*_{ab} admet un adjoint à gauche $f^{ab}_! : (\mathbf{C}_{/x})_{ab} \longrightarrow (\mathbf{C}_{/y})_{ab}$ nous en déduisons une application $[\tilde{\delta_f}] : f^{ab}_!(\Omega_x) \longrightarrow \Omega_y$ dans $(\mathbf{C}_{/y})_{ab}$ qui correspond à l'application $\delta_f : \Omega_A \otimes_A B \longrightarrow \Omega_B$ dans le cas de la catégorie $\mathbf{Alg}_k$ des k-algèbres commutatives.

Dans le cadre de la catégorie des k-algèbres l'application δ_f donne des informations sur le morphisme $f : A \longrightarrow B$. En particulier nous avons deux suites exactes fondamentales entre les modules des différentielles de Kähler, et un des objectif de cet article est de généraliser ces suites exactes au cadre d'une catégorie $\mathbf{C}$ localement présentable quelconque.

Pour énoncer ces résultats nous devons d'abord définir pour tout morphisme $f : x \longrightarrow y$ dans la catégorie $\mathbf{C}$ l'analogue du B-module des différentielles relatives $\Omega_{B/A}$ défini pour tout morphisme $f : A \longrightarrow B$ de k-algèbres. Pour cela nous construisons le module Ω_f comme objet dans la catégorie des modules de Beck au dessus de f considéré comme objet de la catégorie $\mathbf{C}_{x/}$ (cf. définition 3.14).

Alors le théorème 3.21 généralise la *première suite exacte fondamentale*, ou *suite exacte de Jacobi-Zariski*, définie entre les modules des différentielles dans le cadre de $\mathbf{Alg}_k$, le théorème 3.22 est inspiré par la *deuxième suite exacte fondamentale* et la compatibilité du module des différentielles avec la localisation, enfin le théorème 3.23 est la généralisation du fait que les modules de différentielles commutent avec *l'extension des scalaires*.

Dans la dernière partie nous étudions comment ces résultats s'énoncent dans les cas des catégories des ensembles, des monoïdes commutatifs et nous revenons sur l'exemple de la catégorie des k-algèbres commutatives.

Je remercie le rapporteur pour ses remarques et ses suggestions qui m'ont permis d'apporter des précisions au texte et j'espère de le rendre plus lisible.

2. Rappels sur les foncteurs adjoints

Soient $\mathbf{C}$ et $\mathbf{D}$ deux catégories, si nous avons une paire de foncteurs adjoints

$$F : \mathbf{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{D} : G$$

nous notons respectivement $\eta_x : x \longrightarrow GFx$ et $\epsilon_y : FGy \longrightarrow y$ l'unité et la counité de l'adjonction. Nous rappelons que les morphismes composés $\epsilon_{Fx} \circ F\eta_x : Fx \longrightarrow Fx$ et $G\epsilon_y \circ \eta_{Gy} : Gy \longrightarrow Gy$ sont les morphismes identités.

Les bijections naturelles entre $Hom_{\mathbf{C}}(x, Gy)$ et $Hom_{\mathbf{D}}(Fx, y)$ sont définies par :

$$\begin{array}{rcl} Hom_{\mathbf{C}}(x, Gy) & \rightleftarrows & Hom_{\mathbf{D}}(Fx, y) \\ \alpha & \longmapsto & \epsilon_y \circ F\alpha \\ G\beta \circ \eta_x & \longleftarrow\!\shortmid & \beta \end{array}$$

Soit $\mathbf{C}$ une catégorie, pour tout objet x de $\mathbf{C}$ nous notons respectivement $\mathbf{C}_{/x}$ et $\mathbf{C}_{x/}$ les catégories des objets au-dessus de x, c'est-à-dire la catégorie formée des flèches $u : z \longrightarrow x$, et la catégorie des objets au-dessous de x, c'est-à-dire la catégorie formée des flèches $v : x \longrightarrow z$. Pour tout couple d'objets (x, y) de $\mathbf{C}$ nous définissons aussi la catégorie $\mathbf{C}_{x/./y}$ comme la catégorie formée des diagrammes $x \xrightarrow{v} z \xrightarrow{u} y$.

Alors la catégorie $\mathbf{Pt_C}(x)$ des points au dessus de x est la sous-catégorie pleine de $\mathbf{C}_{x/./x}$ formée des diagrammes $x \xrightarrow{v} z \xrightarrow{u} x$, vérifiant $u \circ v = id_x$ (cf. [Bo-Bo], Chapitre 2).

Soit $f : x \longrightarrow y$ un morphisme dans $\mathbf{C}$, alors si la catégorie $\mathbf{C}$ admet des limites finies nous avons une paire de foncteurs adjoints

$$f_! : \mathbf{C}_{/x} \rightleftarrows \mathbf{C}_{/y} : f^*$$

définie par

$$f_! : (z \xrightarrow{u} x) \longmapsto (z \xrightarrow{f \circ u} y)$$
$$f^* : (t \xrightarrow{v} y) \longmapsto (x \times_y t \xrightarrow{p_1} x)$$

De même si la catégorie $\mathbf{C}$ admet des colimites finies nous avons une paire de foncteurs adjoints

$$f_* : \mathbf{C}_{x/} \rightleftarrows \mathbf{C}_{y/} : f^!$$

définie par

$$f_* : (x \xrightarrow{v} t) \longmapsto (y \xrightarrow{q_1} y \textstyle\coprod_x t)$$
$$f^! : (y \xrightarrow{u} z) \longmapsto (x \xrightarrow{u \circ f} z)$$

En particulier dans les cas où y est l'objet final ou dans le cas où x est l'objet initial nous trouvons les paires de foncteurs adjoints suivantes :

$$[x]_! : \mathbf{C}_{/x} \rightleftarrows \mathbf{C} : [x]^* \qquad \text{et} \qquad [y]_* : \mathbf{C} \rightleftarrows \mathbf{C}_{y/} : [y]^!$$

Soit $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$ une paire de foncteurs adjoints, alors pour tout y dans $\mathbf{D}$ nous avons une nouvelle paire de foncteurs adjoints

$$F_{G(y)} : \mathbf{C}_{/G(y)} \rightleftarrows \mathbf{D}_{/y} : G_y$$

définie par

$$\begin{aligned} F_{G(y)} : (x \xrightarrow{v} G(y)) &\longmapsto (F(x) \xrightarrow{\epsilon_y \circ F(v)} y) \\ G_y : (z \xrightarrow{u} y) &\longmapsto (G(z) \xrightarrow{G(u)} G(y)) \ . \end{aligned}$$

De même pour tout x dans $\mathbf{C}$ nous avons une paire de foncteurs adjoints

$$F_x : \mathbf{C}_{x/} \rightleftarrows \mathbf{D}_{F(x)/} : G_{F(x)}$$

définie par

$$\begin{aligned} F_x : (x \xrightarrow{v} z) &\longmapsto (F(x) \xrightarrow{F(v)} F(z)) \\ G_{F(x)} : (F(x) \xrightarrow{u} y) &\longmapsto (x \xrightarrow{G(u)\circ\eta_x} G(y)) \ . \end{aligned}$$

Remarque 2.1. Considérons un morphisme $g : z \longrightarrow y$ dans une catégorie $\mathbf{C}$ admettant des colimites finies et la paire de foncteurs adjoints

$$F = [z]_* : \mathbf{C} \rightleftarrows \mathbf{E} = \mathbf{C}_{z/} : [z]^! = G \ .$$

L'image de $(z \xrightarrow{g} y)$ par le foncteur G est y et nous en déduisons la nouvelle paire de foncteurs adjoints $F_y : \mathbf{C}_{/y} \rightleftarrows \mathbf{E}_{/g} : G_g$ définie de la manière suivante :

$$F_y : \quad (b \xrightarrow{l} y) \quad \longmapsto \quad \begin{tikzcd} z \arrow[r, equal] \arrow[d, "q_1"] & z \arrow[d, "g"] \\ z \coprod b \arrow[r, "(g,l)"'] & y \end{tikzcd} \qquad \text{et}$$

$$G_g : \quad \begin{tikzcd} z \arrow[r, equal] \arrow[d, "u"] & z \arrow[d, "g"] \\ a \arrow[r, "h"] & y \end{tikzcd} \quad \longmapsto \quad (a \xrightarrow{h} y)$$

La catégorie $\mathbf{E}_{/g}$ définie par $\mathbf{E}_{/g} = (\mathbf{C}_{z/})_{/g}$ est égale à la catégorie $(\mathbf{C}_{/y})_{g/}$, c'est la catégorie des diagrammes $z \xrightarrow{v_a} a \xrightarrow{u_a} y$ avec l'égalité $u_a \circ v_a = g$.

La paire de foncteurs adjoints $F_y : \mathbf{C}_{/y} \rightleftarrows \mathbf{E}_{/g} : G_g$ définie précédemment peut aussi être définie comme la paire

$$F_y = [g]_* : \mathbf{C}_{/y} \rightleftarrows \mathbf{E}_{/g} := [g]^! = G_g \quad .$$

3. Modules de Beck

3.1 Objets en groupe abélien et modules de Beck

Pour toute catégorie $\mathbf{Y}$ la catégorie $(\mathbf{Y})_{ab}$ des objets en groupe abélien dans $\mathbf{Y}$ est la sous-catégorie formée des objets y de $\mathbf{Y}$ tels que le préfaisceau $h_y : \mathbf{Y}^{op} \longrightarrow \mathbf{Ens}$ représenté par y est un préfaisceau en groupes abéliens. Cela signifie que pour tout z dans $\mathbf{Y}$ l'ensemble $h_y(z) = Hom_{\mathbf{Y}}(z, y)$ est un groupe abélien et que pour tout morphisme $u : z \longrightarrow z'$ dans $\mathbf{Y}$ l'application associée $u^* : h_y(z') \longrightarrow h_y(z)$ est un morphisme de groupes. Un morphisme dans $(\mathbf{Y})_{ab}$ est un morphisme de préfaisceaux en groupes.

Si la catégorie $\mathbf{Y}$ admet des limites finies, il suffit de supposer que $\mathbf{Y}$ admet des produits finis et a un objet final $\star_{\mathbf{Y}}$, un objet en groupe abélien est un objet y de $\mathbf{Y}$ et la donnée de morphismes $m_y : y \times y \longrightarrow y$, $i_y : y \longrightarrow y$ et $e_y : \star_{\mathbf{Y}} \longrightarrow y$ tels que m_y définit une loi de groupe abélien avec i_y pour l'inverse et e_y pour l'élément neutre. Un morphisme dans $(\mathbf{Y})_{ab}$ de y dans y' est un morphisme $f : y \longrightarrow y'$ dans $\mathbf{Y}$ tel que nous ayons $f \circ m_y = m_{y'} \circ (f \times f)$, $f \circ i_y = i_{y'} \circ f$ et $f \circ e_y = e_{y'}$.

Nous rappelons le résultat suivant (cf. [Ba 1], Théorèmes 1.5 et 2.4 du chapitre 2).

Théorème 3.1. *Pour toute catégorie $\mathbf{Y}$ ayant des limites finies la sous-catégorie $(\mathbf{Y})_{ab}$ des objets en groupe abélien est une catégorie additive, qui est localement présentable si $\mathbf{Y}$ l'est.*

De plus si la catégorie $\mathbf{Y}$ est exacte, la catégorie $(\mathbf{Y})_{ab}$ est une catégorie abélienne.

Soit $\mathbf{Y}$ une catégorie admettant des limites finies, alors l'*élément nul* $0_{\mathbf{Y}}$ de la catégorie additive $(\mathbf{Y})_{ab}$ est l'objet final $\star_{\mathbf{Y}}$ muni de la structure de groupe triviale.

Pour tout x dans $(\mathbf{Y})_{ab}$, le morphisme $x \longrightarrow 0_{\mathbf{Y}}$ dans $(\mathbf{Y})_{ab}$ est le morphisme induit par le morphisme $x \longrightarrow \star_{\mathbf{Y}}$ dans $\mathbf{Y}$, et pour tout y dans $(\mathbf{Y})_{ab}$ le morphisme $0_{\mathbf{Y}} \longrightarrow y$ dans $(\mathbf{Y})_{ab}$ est le morphisme induit par l'élément neutre $e_y : \star_{\mathbf{Y}} \longrightarrow y$ de la structure de groupe de y.

En particulier l'élément neutre 0 du groupe abélien $Hom_{(\mathbf{Y})_{ab}}(x, y)$ est le morphisme g dans $\mathbf{Y}$ qui se factorise par l'objet final

$$\star_{\mathbf{Y}} : \begin{tikzcd} x \arrow[r] \arrow[rr, bend left, "g"] & \star_{\mathbf{Y}} \arrow[r] & y \end{tikzcd} .$$

Il existe un foncteur *oubli* $U : (\mathbf{Y})_{ab} \longrightarrow \mathbf{Y}$, et si la catégorie $\mathbf{Y}$ est localement présentable le foncteur U admet un adjoint à gauche L, le foncteur *objet en groupe abélien libre* (cf. [Ba 1], Exercice 1.5-3 du chapitre 6).

Un foncteur exact à gauche $G : \mathbf{Y} \longrightarrow \mathbf{X}$ préserve les limites finies et l'image d'un objet en groupe abélien est un objet en groupe abélien, il induit alors un foncteur additif $G_{ab} : (\mathbf{Y})_{ab} \longrightarrow (\mathbf{X})_{ab}$ tel que le diagramme suivant est commutatif

$$\begin{tikzcd} (\mathbf{Y})_{ab} \arrow[r, "G_{ab}"] \arrow[d, "U"] & (\mathbf{X})_{ab} \arrow[d, "U"] \\ \mathbf{Y} \arrow[r, "G"] & \mathbf{X} \end{tikzcd}$$

Soit $\mathbf{C}$ une catégorie admettant des limites finies.

Définition 3.2. *La catégorie des modules de Beck au-dessus d'un objet x de $\mathbf{C}$ est la catégorie des objets en groupe abélien dans la catégorie $\mathbf{C}_{/x}$ des objets au-dessus de x : $B_{\mathbf{C}}(x) = (\mathbf{C}_{/x})_{ab}$.*

Nous pouvons définir un foncteur *oubli* U_x de la catégorie des modules de Beck $(\mathbf{C}_{/x})_{ab}$ au-dessus de x dans la catégorie $\mathbf{C}_{/x}$. Si la catégorie $\mathbf{C}$ est localement présentable, il en est de même de la catégorie $\mathbf{C}_{/x}$ et le foncteur U_x admet un adjoint à gauche L_x, qui est appelé *foncteur d'abélianisation* et est noté Ab_x dans [Fr] :

$$L_x : \mathbf{C}_{/x} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathbf{C}_{/x})_{ab} : U_x$$

Soit $G : \mathbf{D} \longrightarrow \mathbf{C}$ un foncteur exact à gauche, pour tout y dans $\mathbf{D}$ le foncteur $G_y : \mathbf{D}_{/y} \longrightarrow \mathbf{C}_{/G(y)}$ préserve l'objet final et les produits finis, l'image d'un objet en groupe abélien est un objet en groupe abélien. Nous avons donc un foncteur $(G_y)_{ab}$ de la catégorie des modules de Beck au-dessus de y, $B_{\mathbf{D}}(y) = (\mathbf{D}_{/y})_{ab}$, dans la catégorie des modules de Beck au-dessus de $G(y)$, $B_{\mathbf{C}}(G(y)) = (\mathbf{C}_{/G(y)})_{ab}$, qui est additif, tel que le diagramme suivant soit commutatif :

$$\begin{tikzcd}
(\mathbf{D}_{/y})_{ab} \arrow[r, "(G_y)_{ab}"] \arrow[d, "U_y"] & (\mathbf{C}_{/G(y)})_{ab} \arrow[d, "U_{G(y)}"] \\
\mathbf{D}_{/y} \arrow[r, "G_y"] & \mathbf{C}_{/G(y)}
\end{tikzcd}$$

Nous noterons $[a] = (a \underset{u_a}{\overset{e_a}{\rightleftarrows}} x)$ un élément de la catégorie $(\mathbf{C}_{/x})_{ab}$, cela correspond à un objet a de $\mathbf{C}$ avec un morphisme $u_a : a \longrightarrow x$, à la donnée des morphismes $m_a : a \times_x a \longrightarrow a$, $i_a : a \longrightarrow a$ et $e_a : x \longrightarrow a$ donnés par la structure de groupe, et nous avons l'égalité $u_a \circ e_a = id_x$,

Remarque 3.3. L'existence du morphisme *élément neutre* $e_a : x \longrightarrow a$ définit un foncteur naturel $O_x : (\mathbf{C}_{/x})_{ab} \longrightarrow \mathbf{Pt}_{\mathbf{C}}(x)$, qui est fidèle, et le foncteur oubli $U_x : (\mathbf{C}_{/x})_{ab} \longrightarrow \mathbf{C}_{/x}$ se factorise par le foncteur canonique $G_x : \mathbf{Pt}_{\mathbf{C}}(x) \longrightarrow \mathbf{C}_{/x}$.

De plus le foncteur induit $(G_x)_{ab} : (\mathbf{Pt}_{\mathbf{C}}(x))_{ab} \longrightarrow (\mathbf{C}_{/x})_{ab}$ est une équivalence de catégories.

De même nous noterons $[\alpha]$ un morphisme de $Hom_{(\mathbf{C}_{/x})_{ab}}([a], [b])$, c'est la donnée d'un morphisme α dans $Hom_{\mathbf{C}}(a, b)$ tel que nous ayons le diagramme commutatif

$$\begin{tikzcd}
 & a \arrow[dr, "u_a"] \arrow[dd, "\alpha"] & \\
x \arrow[ur, "e_a"] \arrow[dr, "e_b"'] & & x \\
 & b \arrow[ur, "u_b"'] &
\end{tikzcd}$$

et compatible aux morphismes m_a et m_b, i_a et i_b.

Lemme 3.4. *Soit α un morphisme dans $Hom_{(\mathbf{C}_{/x})_{ab}}([a],[b])$, alors avec les notations précédentes nous avons*

$$[\alpha] = [0] \iff \alpha = e_b \circ u_a \ .$$

Démonstration. Le morphisme $[\alpha]$ est le morphisme nul dans le groupe abélien $Hom_{(\mathbf{C}_{/x})_{ab}}([a],[b])$ s'il se factorise par $0_{\mathbf{C}_{/x}}$, c'est-à-dire si nous pouvons trouver un diagramme commutatif

$$\begin{tikzcd}
 & a \arrow[d, "\alpha_1"] \arrow[dr, "u_a"] & \\
x \arrow[ur, "e_a"] \arrow[r, "id_x"] \arrow[dr, "e_b"'] & x \arrow[r, "id_x"] \arrow[d, "\alpha_2"] & x \\
 & b \arrow[ur, "u_b"'] &
\end{tikzcd}$$

$\square$

Définition 3.5. *(1) Le module cotangent, Ω_x de x est le module de Beck au-dessus de x obtenu comme image par L_x de l'objet final $\star_{\mathbf{C}_{/x}} = (\ x \xrightarrow{id_x} x\ )$ de $\mathbf{C}_{/x}$, $\Omega_x = L_x(\star_{\mathbf{C}_{/x}})$.*

(2) Pour tout $[b]$ appartenant à $(\mathbf{C}_{/x})_{ab}$, nous appelons ensemble des dérivations de Beck *de x à valeurs dans $[b]$ l'ensemble*

$$Der_{\mathbf{C}}(x,[b]) := Hom_{(\mathbf{C}_{/x})_{ab}}(\Omega_x,[b]) \simeq Hom_{\mathbf{C}_{/x}}(\star_{\mathbf{C}_{/x}}, b)$$

avec $b = U_x([b])$.

Le module des différentiel Ω_x correspond à un élément $[\omega_x]$, avec ω_x dans $\mathbf{C}$, et par définition nous avons $U_x(\Omega_x) = (\ \omega_x \xrightarrow{u_{\omega_x}} x\ )$. En particulier l'unité $\eta_{\star_{\mathbf{C}_{/x}}}$ de l'adjonction $L_x \dashv U_x$ correspond à un morphisme $\eta_x : x \longrightarrow \omega_x$ vérifiant $u_{\omega_x} \circ \eta_x = id_x$.

Une dérivation θ dans $Der_{\mathbf{C}}(x,[b])$ correspond à un morphisme $[\delta]$ dans $Hom_{(\mathbf{C}_{/x})_{ab}}([\omega_x],[b])$, c'est-à-dire correspond à un morphisme $\delta : \omega_x \longrightarrow b$ tel que le diagramme suivant soit commutatif

$$\begin{tikzcd}
 & \omega_x \arrow[d, "\delta"] \arrow[dr, "u_{\omega_x}"] & \\
x \arrow[ur, "e_{\omega_x}"] \arrow[dr, "e_b"'] & & x \\
 & b \arrow[ur, "u_b"'] &
\end{tikzcd}$$

Par adjonction le morphisme $[\delta]$ correspond à un morphisme β dans $Hom_{\mathbf{C}_{/x}}(\star_{\mathbf{C}_{/x}}, b)$, qui est défini par $\beta = U([\delta]) \circ \eta_{\star_{\mathbf{C}_{/x}}}$, c'est-à-dire à un morphisme $\beta : x \longrightarrow b$ vérifiant $u_b \circ \beta = id_x$ tel que nous ayons l'égalité $\beta = \delta \circ \eta_x$. Et réciproquement par adjonction nous associons à un morphisme β dans $Hom_{\mathbf{C}_{/x}}(\star_{\mathbf{C}_{/x}}, b)$ le morphisme $[\delta]$ dans $Hom_{(\mathbf{C}_{/x})_{ab}}([\omega_x], [b])$ défini par $[\delta] = \epsilon_{[b]} \circ L_x(\beta)$.

Remarque 3.6. Pour tout $[b]$ appartenant à $(\mathbf{C}_{/x})_{ab}$, l'ensemble des dérivations de Beck $Der_{\mathbf{C}}(x, [b])$ est canoniquement isomorphe à l'ensemble des sections du morphisme $u_b : b \longrightarrow x$, pour $b = U_x([b])$. Mais la structure de groupe abélien sur cet ensemble dépend de la structure d'objet en groupe abélien dont est muni l'objet b.

Proposition 3.7. *Un élément θ du groupe abélien $Der_{\mathbf{C}}(x, [b])$ est l'élément nul si et seulement si le morphisme β de $Hom_{\mathbf{C}_{/x}}(\star_{\mathbf{C}_{/x}}, b)$ correspondant à θ est égal au morphisme e_b élément neutre de la structure de groupe abélien de $[b]$.*

Démonstration. Par définition l'élément θ est l'élément nul si nous avons $[\delta] = [0]$.

D'après le lemme 3.4, si $[\delta] = [0]$ nous avons $\delta = e_b \circ u_{\omega_x}$ et nous en déduisons $\beta = e_b \circ u_{\omega_x} \circ \eta_x = e_b$.

Réciproquement si $\beta = e_b$ alors β est l'image par U_x du morphisme nul $[0] : 0_{\mathbf{C}_{/x}} \longrightarrow [b]$, nous avons alors le diagramme commutatif suivant

$$\begin{tikzcd}
\omega_x = L_x U_x 0_{\mathbf{C}_{/x}} \arrow[r, "L_x(\beta)"] \arrow[d, "\epsilon_{0_{\mathbf{C}_{/x}}}"'] \arrow[dr, "{[\delta]}"] & L_x U_x [b] \arrow[d, "\epsilon_{[b]}"] \\
0_{\mathbf{C}_{/x}} \arrow[r, "{[0]}"'] & {[b]}
\end{tikzcd}$$

et nous en déduisons que $[\delta] = [0]$.

□

Remarque 3.8. D'après la proposition 3.7 l'élément θ du groupe abélien $Der_{\mathbf{C}}(x, [b])$ est l'élément nul si et seulement si la section β du morphisme $u_b : b \longrightarrow x$ est égale à l'élément neutre e_b.

En particulier si x est l'objet initial de la catégorie $\mathbf{C}$ pour tout objet $u_b : b \longrightarrow x$ dans $\mathbf{C}_{/x}$ il existe un seul morphisme $\beta : x \longrightarrow b$ tel que $u_b \circ \beta = id_x$, par conséquent le groupe abélien $Der_{\mathbf{C}}(x, [b])$ est réduit à l'élément nul. En particulier nous avons $\Omega_x = (0)$.

3.2 Catégorie en groupe

Nous considérons maintenant une catégorie $\mathbf{C}$ telle que tout objet a admet une structure de groupe, non nécessairement commutatif, et telle que les morphismes respectent cette structure. Plus précisément nous supposons que pour tout objet a dans $\mathbf{C}$ il existe des morphismes $\mu_a : a \times a \longrightarrow a$, $\iota_a : a \longrightarrow a$ et $\varepsilon_a : \star_{\mathbf{C}} \longrightarrow a$ définissant une structure d'objet en groupe et que pour tout morphisme $f : b \longrightarrow a$ dans $\mathbf{C}$ nous avons $f \circ \mu_b = \mu_a \circ (f \times f)$, $f \circ \iota_b = \iota_a \circ f$ et $f \circ \varepsilon_b = \varepsilon_a$. Et nous notons $c_a : a \times a \longrightarrow a \times a$ le morphisme qui échange les facteurs, en particulier la loi de groupe est commutative si nous avons $\mu_a \circ c_a = \mu_a$.

Soit x un objet dans $\mathbf{C}$ et nous notons comme précédemment m_a, i_a et e_a les morphismes définissant la structure d'objet en groupe abélien d'un objet $[a]$ dans $(\mathbf{C}_{/x})_{ab}$, avec $u_a : a \longrightarrow x$.

Nous avons le diagramme commutatif

$$\begin{tikzcd} a \times_x a \arrow[r, "p_1"] \arrow[d, "p_2"'] & a \arrow[d, "u_a"] \\ a \arrow[r, "u_a"] & x \end{tikzcd}$$

et nous notons $g_a : a \longrightarrow a$ le morphisme $g_a = e_a \circ u_a$, $q_a : a \times_x a \longrightarrow a$ le morphisme $g_a \circ p_1 = g_a \circ p_2$, et $\mu_a^{(3)} : a \times a \times a \longrightarrow a$ le morphisme $\mu_a(\mu_a, id_a) = \mu_a(id_a, \mu_a)$

Proposition 3.9. *Nous avons l'égalité* $m_a = \mu_a^{(3)} \circ \phi$ *où le morphisme* $\phi : a \times_x a \longrightarrow a \times a \times a$ *est défini par* $\phi = (p_2, \iota_a \circ q_a, p_1)$.

Démonstration. C'est un résultat classique qui est une conséquence des remarques suivantes.

Comme les morphismes dans $\mathbf{C}$ respectent la loi de composition μ_a nous avons la commutativité du diagramme suivant

$$\begin{tikzcd}
(a \times_x a) \times (a \times_x a) \arrow[rr, "\simeq"] \arrow[d, "m_a \times m_a"'] & & (a \times a) \times_{(x \times x)} (a \times a) \arrow[d, "\mu_a \times \mu_a"] \\
a \times a \arrow[dr, "\mu_a"] & & a \times_x a \arrow[dl, "m_a"'] \\
& a &
\end{tikzcd}$$

et nous en déduisons que le morphsime composé

$$a \times a \xrightarrow{\psi} (a \times a) \times_{(x \times x)} (a \times a) \xrightarrow{m_a \circ (\mu_a \times \mu_a)} a \ ,$$

avec $\psi = ((g_a \circ p_2, p_1), (p_2, g_a \circ p_1))$, est égal à $\mu_a \circ c_a$.

□

Si nous supposons que les objets de la catégorie $\mathbf{C}$ sont des ensembles munis d'une loi de groupe notée multiplicativement les relations précédentes peuvent s'écrire :

- $m_a(r_1.s_1, r_2.s_2) = m_a(r_1, s_1).m_a(r_2, s_2)$ pour r_1, r_2, s_1, s_2 dans l'ensemble a avec $u_a(r_1) = u_a(s_1)$ et $u_a(r_2) = u_a(s_2)$, et
- $m_a(g_a(s).r, s.g_a(r)) = s.r$ pour r et s quelconques dans a,

et l'égalité de la proposition peut s'écrire

$$m_a(r, s) = s.e_a(t)^{-1}.r \ ,$$

pour r et s appartenant à l'ensemble a avec $t = u_a(r) = u_a(s)$.

Nous avons vu qu'il existe un foncteur oubli de la catégorie $(\mathbf{C}_{/x})_{ab}$ dans la catégorie $\mathbf{Pt_C}(x)$ des points de $\mathbf{C}$ au-dessus de x, et dans le cas d'une catégorie en groupe nous avons le résultat plus précis suivant.

Corollaire 3.10. *La foncteur* $O_x : (\mathbf{C}_{/x})_{ab} \longrightarrow \mathbf{Pt_C}(x)$ *est pleinement fidèle.*

Démonstration. En effet nous déduisons de la proposition précédente que la structure d'objet en groupe abélien de $[a]$ est entièrement déterminée par les morphisme $e_a : x \longrightarrow a$ et $u_a : a \longrightarrow x$.

□

3.3 Suite exacte des modules cotangents

Soit $\mathbf{C}$ une catégorie localement présentable, et soit $f : x \longrightarrow y$ dans $\mathbf{C}$, nous voulons comparer les modules cotangents Ω_x et Ω_y et définir un module cotangent relatif $\Omega_{x/y}$.

Le foncteur $f^* : \mathbf{C}_{/y} \longrightarrow \mathbf{C}_{/x}$ est un adjoint à droite, il est donc exact à gauche et induit un foncteur additif $f^*_{ab} : (\mathbf{C}_{/y})_{ab} \longrightarrow (\mathbf{C}_{/x})_{ab}$ tel que le diagramme suivant soit commutatif

$$\begin{tikzcd}
(\mathbf{C}_{/y})_{ab} \arrow[r, "f^*_{ab}"] \arrow[d, "U_y"] & (\mathbf{C}_{/x})_{ab} \arrow[d, "U_x"] \\
\mathbf{C}_{/y} \arrow[r, "f^*"] & \mathbf{C}_{/x}
\end{tikzcd}$$

Grâce aux adjonctions $L_y \dashv U_y$ et $L_x \dashv U_x$ nous définissons la transformation de Beck-Chevalley à gauche $L_x f^* \Longrightarrow f^*_{ab} L_y$ ([Ma] **2.16**). Nous déduisons de l'isomorphisme canonique $\star_{\mathbf{C}_{/x}} \xrightarrow{\simeq} f^*(\star_{\mathbf{C}_{/y}})$ un morphisme naturel dans $(\mathbf{C}_{/x})_{ab}$:

$$[\delta_f] : \Omega_x = L_x(\star_{\mathbf{C}_{/x}}) \xrightarrow{\simeq} L_x f^*(\star_{\mathbf{C}_{/y}}) \longrightarrow f^*_{ab} L_y(\star_{\mathbf{C}_{/y}}) = f^*_{ab}(\Omega_y) \ .$$

Remarque 3.11. Si la catégorie $\mathbf{C}$ est une catégorie localement présentable, pour tout objet x dans $\mathbf{C}$ l'adjonction $L_x : \mathbf{C}_{/x} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathbf{C}_{/x})_{ab} : U_x$ est monadique et la catégorie $(\mathbf{C}_{/x})_{ab}$ est encore localement présentable (**Theorem 1** de [Po] [1]).

Nous déduisons alors du théorème **4.5.6.** de [Bo] que le foncteur f^*_{ab} admet toujours un adjoint à gauche $f^{ab}_! : (\mathbf{C}_{/x})_{ab} \longrightarrow (\mathbf{C}_{/y})_{ab}$.

Nous déduisons de l'existence du foncteur adjoint à gauche

$$f^{ab}_! : (\mathbf{C}_{/x})_{ab} \longrightarrow (\mathbf{C}_{/y})_{ab}$$

1. Je remercie Marino Gran pour m'avoir indiqué cette référence

un morphisme $[\tilde{\delta_f}] : f_!^{ab}(\Omega_x) \longrightarrow \Omega_y$ dans $(\mathbf{C}_{/y})_{ab}$, qui peut être construit directement à partir du diagramme commutatif

$$\begin{tikzcd} (\mathbf{C}_{/x})_{ab} \arrow[r, "f_!^{ab}"] & (\mathbf{C}_{/y})_{ab} \\ \mathbf{C}_{/x} \arrow[u, "L_x"] \arrow[r, "f_!"] & \mathbf{C}_{/y} \arrow[u, "L_y"] \end{tikzcd}$$

Remarque 3.12. Le morphisme $[\tilde{\delta_f}]$ est égal au morphisme $L_y(\vartheta_f)$ où ϑ_f est le morphisme canonique $\vartheta_f : f_!(\star_{\mathbf{C}_{/x}}) \longrightarrow \star_{\mathbf{C}_{/y}}$ dans $\mathbf{C}_{/y}$.

Le morphisme $[\delta_f]$ induit un morphisme

$$U_x(\Omega_x) \longrightarrow U_x f_{ab}^*(\Omega_y) = f^* U_y(\Omega_y)$$

dans $\mathbf{C}_{/x}$, c'est-à-dire un morphisme $\delta_f : \omega_x \longrightarrow x \times_y \omega_y$ au dessus de x compatible avec les structures d'objets en groupe abélien de ω_x et de $x \times_y \omega_y$. Alors le morphisme $\psi_f : x \longrightarrow x \times_y \omega_y$ dans $Hom_{\mathbf{C}_{/x}}(\star_{\mathbf{C}_{/x}}, U_x f_{ab}^*(\Omega_y))$ correspondant par l'adjonction $L_x \dashv U_x$ au morphisme $[\delta_f]$ est défini par $\psi_f = \delta_f \circ \eta_x$.

Par construction ce morphisme ψ_f est défini par le diagramme commutatif suivant

$$\begin{tikzcd} x \arrow[dr, "\psi_f"] \arrow[drr, bend left, "\eta_y \circ f"] \arrow[ddr, bend right, "id_x"'] & & \\ & x \times_y \omega_y \arrow[r, "p_2"] \arrow[d, "p_1"] & \omega_y \arrow[d, "u_{\omega_y}"] \\ & x \arrow[r, "f"] & y \end{tikzcd}$$

et comme nous avons l'égalité $id_y = u_{\omega_y} \circ \eta_y$ nous en déduisons le diagramme suivant formé de carrés cartésiens :

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\psi_f"] & y \arrow[d, "\eta_y"] \\ x \times_y \omega_y \arrow[r, "p_2"] \arrow[d, "p_1"] & \omega_y \arrow[d, "u_{\omega_y}"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

et nous pouvons ainsi écrire $\delta_f \circ \eta_x = \psi_f = f^*(\eta_y)$.

Pour tout $[b]$ dans $(\mathbf{C}_{/y})_{ab}$ le morphisme $[\delta_f] : \Omega_x \longrightarrow f^*_{ab}(\Omega_y)$ définit un morphisme de groupes abéliens $[\Delta_f]$ de l'ensemble $Der_{\mathbf{C}}(y, [b])$ dans l'ensemble $Der_{\mathbf{C}}(x, f^*_{ab}([b])$:

$$\begin{array}{rcl} [\Delta_f] : Hom_{(\mathbf{C}_{/y})_{ab}}(\Omega_y, [b]) & \longrightarrow & Hom_{(\mathbf{C}_{/x})_{ab}}(\Omega_x, f^*_{ab}([b])) \\ [\delta] & \longmapsto & f^*_{ab}([\delta]) \circ [\delta_f] \end{array}$$

Le morphisme dans $Hom_{\mathbf{C}_{/y}}(\star_{\mathbf{C}_{/y}}, b)$ associé à $[\delta]$ par l'adjonction $L_y \dashv U_y$ est égal à $\beta = \delta \circ \eta_y$, où $\delta = U_y([\delta])$, et de même le morphisme dans $Hom_{\mathbf{C}_{/x}}(\star_{\mathbf{C}_{/x}}, f^*(b))$ associé à $f^*_{ab}([\delta]) \circ [\delta_f]$ par l'adjonction $L_x \dashv U_x$ est égal à $f^*(\delta) \circ \delta_f \circ \eta_x$. Nous déduisons alors de ce qui précède que l'application associée au morphisme de groupes $[\Delta_f]$ par les adjonctions $L_y \dashv U_y$ et $L_x \dashv U_x$ est l'application induite par le foncteur f^* :

$$\begin{array}{rcl} \Delta_f : Hom_{\mathbf{C}_{/y}}(\star_{\mathbf{C}_{/y}}, b) & \longrightarrow & Hom_{\mathbf{C}_{/x}}(\star_{\mathbf{C}_{/x}}, f^*(b)) \\ \beta & \longmapsto & f^*(\beta) \end{array}$$

qui correspond au morphisme canonique entre l'ensemble des sections du morphisme $u_b : b \longrightarrow y$ et l'ensemble des sections du morphisme $u_{x \times_y b} : x \times_y b \longrightarrow x$.

Proposition 3.13. *L'élément $[\delta]$ de $Der_{\mathbf{C}}(y, [b])$ appartient au noyau de $[\Delta_f]$ si et seulement si nous avons l'égalité $e_b \circ f = \beta \circ f$, où β est le morphisme de $Hom_{\mathbf{C}_{/y}}(\star_{\mathbf{C}_{/y}}, b)$ associé à $[\delta]$ et où e_b est l'élément neutre de la structure de groupe abélien de $[b]$.*

Démonstration. Soit $[b]$ un objet de $(\mathbf{C}_{/y})_{ab}$ correspondant à un objet $u_b = b \longrightarrow y$ de $\mathbf{C}_{/y}$, alors l'élément neutre $e_a : x \longrightarrow a = x \times_y b$ de l'image $[a] = f^*_{ab}([b])$ est défini par le diagramme commutatif suivant

$$\begin{tikzcd}
x \arrow[dr, "e_a"] \arrow[drr, bend left, "e_b \circ f"] \arrow[ddr, bend right, "id_x"'] & & \\
& x \times_y b \arrow[r, "p_2"] \arrow[d, "p_1"] & b \arrow[d, "u_b"] \\
& x \arrow[r, "f"] & y
\end{tikzcd}$$

où $e_b : y \longrightarrow b$ est l'elément neutre de $[b]$.

La proposition est alors une conséquence de la proposition 3.7 et de la description précédente de l'application Δ_f associée au morphisme $[\Delta_f]$.

□

Pour tout morphisme $f : x \longrightarrow y$ dans $\mathbf{C}$, nous pouvons considérer f comme un objet de la catégorie $\mathbf{D} = \mathbf{C}_{x/}$, comme précédemment nous avons le couple de foncteurs adjoints

$$L_f : \mathbf{D}_{/f} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathbf{D}_{/f})_{ab} : U_f$$

et nous avons la définition suivante.

Définition 3.14. *Le module cotangent Ω_f est le module de Beck au-dessus de f dans $(\mathbf{D}_{/f})_{ab}$ obtenu comme image par L_f de l'objet final $\star_{\mathbf{D}_{/f}} = (\ f \xrightarrow{id_f} f\ )$ de $\mathbf{D}_{/f}$, $\Omega_f = L_f(\star_{\mathbf{D}_{/F}})$.*

Nous considérons alors la paire de foncteurs adjoints

$$F = [x]_* : \mathbf{C} \rightleftarrows \mathbf{C}_{x/} = \mathbf{D} : [x]^! = G$$

où l'image de $(f : x \longrightarrow y)$ par le foncteur G est y, et nous en déduisons la paire de foncteurs adjoints

$$F_y : \mathbf{C}_{/y} \rightleftarrows \mathbf{D}_{/f} : G_f\ .$$

Les objets de la catégorie $\mathbf{D}_{/f} = (\mathbf{C}_{x/})_{/f}$ sont les diagrammes commutatifs dans $\mathbf{C}$:

$$\begin{tikzcd} x \arrow[r, "v_z"] \arrow[d, equal] & z \arrow[d, "u_z"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

que nous notons (v_z, z, u_z).

Un morphisme de (v_{z_1}, z_1, u_{z_1}) dans (v_{z_2}, z_2, u_{z_2}) est un morphisme $g : z_1 \longrightarrow z_2$ dans $\mathbf{C}$ vérifiant $v_{z_2} = g \circ v_{z_1}$ et $u_{z_1} = u_{z_2} \circ g$, et l'objet final $\star_{\mathbf{D}_{/f}}$ de $\mathbf{D}_{/f}$ est le diagramme (f, y, id_y). La donnée d'un morphisme $[\alpha]$ de $\star_{\mathbf{D}_{/f}}$ dans l'objet (v_z, z, u_z) est la donnée d'un morphisme $s : y \longrightarrow z$ dans $\mathbf{C}$ vérifiant $u_z \circ s = id_y$ et $s \circ f = v_z$.

De même les objets de la catégorie $\mathbf{C}_{/y}$ sont les morphismes $u_t : t \longrightarrow y$ dans $\mathbf{C}$ que nous notons (t, u_t). Un morphisme de (t_1, u_{t_1}) dans (t_2, u_{t_2}) est un morphisme $h : t_1 \longrightarrow t_2$ vérifiant $u_{t_1} = u_{t_2} \circ h$, et l'objet final $\star_{\mathbf{C}_{/y}}$ de $\mathbf{C}_{/y}$ est le morphisme (id_y, y). La donnée d'un morphisme $[\beta]$ de $\star_{\mathbf{C}_{/y}}$ dans l'objet (t, u_t) est la donnée d'un morphisme $s : y \longrightarrow t$ dans $\mathbf{C}$ vérifiant $u_t \circ s = id_y$.

Comme précédement nous notons $[(v_z, z, u_z)]$ (resp. $[(t, u_t)]$), un élément de la catégorie $(\mathbf{D}_{/f})_{ab}$ (resp. $(\mathbf{C}_{/y})_{ab}$), d'image (v_z, z, u_z) (resp. (t, u_t)).

Le foncteur $G_f : \mathbf{D}_{/f} \longrightarrow \mathbf{C}_{/y}$ induit un foncteur $(G_f)_{ab}$ entre les sous-catégories des objets en groupe abéliens, c'est-à-dire nous avons le diagramme commutatif suivant :

$$\begin{tikzcd} (\mathbf{D}_{/f})_{ab} \arrow[r, "(G_f)_{ab}"] \arrow[d, "U_f"] & (\mathbf{C}_{/y})_{ab} \arrow[d, "U_y"] \\ \mathbf{D}_{/f} \arrow[r, "G_f"] & \mathbf{C}_{/y} \end{tikzcd}$$

Proposition 3.15. *(1) Le foncteur $G_f : \mathbf{D}_{/f} \longrightarrow \mathbf{C}_{/y}$ est fidèle.*

(2) Le foncteur $(G_f)_{ab} : (\mathbf{D}_{/f})_{ab} \longrightarrow (\mathbf{C}_{/y})_{ab}$ est une équivalence de catégories.

Démonstration. (1) Il suffit de remarquer que le foncteur $G_f : \mathbf{D}_{/f} \longrightarrow \mathbf{C}_{/y}$ envoie (v_z, z, u_z) sur (z, u_z), et un morphisme $g : z_1 \longrightarrow z_2$ sur lui-même.

(2) Nous allons construire le foncteur $(F_y)_{ab}$ inverse du foncteur $(G_f)_{ab}$.
Soit $[(t, u_t)]$ un élément de $(\mathbf{C}_{/y})_{ab}$, le morphisme $e_t : \star_{\mathbf{C}_{/y}} \longrightarrow (t, u_t)$ induit un morphisme $e_t : y \longrightarrow t$ dans $\mathbf{C}$ vérifiant $u_t \circ e_t = id_y$. Nous définissons alors $(F_y)_{ab}([(t, u_t)]) = [(v_t, t, u_t)]$ avec $v_t = e_t \circ f$. Il suffit de vérifier alors que (v_t, t, u_t) est un objet en groupe abélien dans $\mathbf{D}_{/f}$, et que nous avons $(G_f)_{ab} \circ (F_y)_{ab} = id_{(\mathbf{C}_{/y})_{ab}}$ et $(F_y)_{ab} \circ (G_f)_{ab} = id_{(\mathbf{D}_{/f})_{ab}}$.

□

Nous pouvons aussi déduire l'équivalence de catégories

$$(G_f)_{ab} : (\mathbf{D}_{/f})_{ab} \longrightarrow (\mathbf{C}_{/y})_{ab}$$

de l'équivalence de catégories $\mathbf{Pt}_{\mathbf{D}}(f) \longrightarrow \mathbf{Pt}_{\mathbf{C}}(y)$ et des équivalences de catégories $(\mathbf{Pt}_{\mathbf{D}}(f))_{ab} \longrightarrow (\mathbf{D}_{/f})_{ab}$ et $(\mathbf{Pt}_{\mathbf{C}}(y))_{ab} \longrightarrow (\mathbf{C}_{/y})_{ab}$ (cf. remarque 3.3).

Définition 3.16. *Le* module cotangent relatif,*noté $\Omega_{y/x}$, est l'image du module cotangent Ω_f par le foncteur $(G_f)_{ab} : (\mathbf{D}_{/f})_{ab} \longrightarrow (\mathbf{C}_{/y})_{ab}$:*

$$\Omega_{y/x} = (G_f)_{ab}(\Omega_f) .$$

Pour x égal à l'objet initial de la catégorie $\mathbf{C}$, le module cotangent relatif $\Omega_{y/x}$ est canoniquement isomorphe au module cotangent Ω_y.

Remarque 3.17. Nous pouvons décrire tout objet en groupe dans $\mathbf{D}_{/f}$ par un diagramme $x \xrightarrow{v_z} z \underset{e_z}{\overset{u_z}{\rightleftarrows}} y$, avec $u_z \circ e_z = id_y$, $u_z \circ v_z = f$ et $e_z \circ f = v_z$, où le morphisme e_z est défini par l'élément neutre. En particulier le module cotangent Ω_f correspond à un diagramme $x \xrightarrow{v_{\omega_f}} \omega_f \underset{e_{\omega_f}}{\overset{u_{\omega_f}}{\rightleftarrows}} y$.

Nous déduisons du diagramme commutatif précédent et des adjonctions $L_y \dashv U_y$ et $L_x \dashv U_x$ la transformation de Beck-Chevalley à droite $[\Gamma]$: $L_y G_f \Rightarrow (G_f)_{ab} L_f$ ([Ma] **2.16**).

Proposition 3.18. *Pour tout a dans $\mathbf{D}_{/f}$ le morphisme*

$$[\gamma_a] \in Hom_{(\mathbf{C}_{/x})_{ab}}(L_y G_f(a), (G_f)_{ab} L_f(a))$$

défini par la transformation de Beck-Chevalley $[\Gamma]$ est un épimorphisme.

Démonstration. Pour tout a dans $\mathbf{D}_{/f}$ et tout $[b]$ nous déduisons de l'adjonction $L_f \dashv U_f$ et de la pleine fidélité du foncteur $(G_f)_{ab}$ les isomorphismes

$$\begin{aligned} Hom_{\mathbf{D}_{/f}}(a, U_f([b])) &\simeq Hom_{(\mathbf{D}_{/f})_{ab}}(L_f(a), [b]) \\ &\simeq Hom_{(\mathbf{C}_{/y})_{ab}}((G_f)_{ab} L_f(a), (G_f)_{ab}([b]))\ , \end{aligned}$$

et de la commutativité du diagramme précédent et de l'adjonction $L_y \dashv U_y$ les isomorphismes

$$\begin{aligned} Hom_{\mathbf{C}_{/y}}(G_f(a), G_f U_f([b])) &\simeq Hom_{\mathbf{C}_{/y}}(G_f(a), U_y (G_f)_{ab}([b])) \\ &\simeq Hom_{(\mathbf{C}_{/y})_{ab}}(L_y G_f(a), (G_f)_{ab}([b]))\ . \end{aligned}$$

Le foncteur G_f induit une application

$$Hom_{\mathbf{D}_{/f}}(a, U_f([b])) \longrightarrow Hom_{\mathbf{C}_{/y}}(G_f(a), G_f U_f([b]))\ ,$$

d'où une application $\left[\Gamma_a^{(G_f)_{ab}([b])}\right]$ de $Hom_{(\mathbf{C}_{/y})_{ab}}((G_f)_{ab} L_f(a), (G_f)_{ab}([b]))$ dans $Hom_{(\mathbf{C}_{/y})_{ab}}(L_y G_f(a), G_{f\,ab}([b]))$, qui correspond à la composition par le morphisme $[\gamma_a]$. Cette application correspond aussi, par les isomorphismes précédents à l'application

$$Hom_{\mathbf{D}_{/f}}(a, U_f([b])) \longrightarrow Hom_{\mathbf{C}_{/y}}(G_f(a), G_f U_f([b]))$$

induite par le foncteur G_f, donc est injective d'après la proposition 3.15 (1).

Nous déduisons alors de l'essentielle surjectivité du foncteur $(G_f)_{ab}$ que pour tout $[c]$ dans $(\mathbf{C}_{/y})_{ab}$ l'application

$$\begin{aligned} \left[\Gamma_a^{[c]}\right] : Hom_{(\mathbf{C}_{/y})_{ab}}((G_f)_{ab} L_f(a), [c]) &\longrightarrow Hom_{(\mathbf{C}_{/y})_{ab}}(L_y G_f(a), [c]) \\ \varphi &\longmapsto \varphi \circ [\gamma_a] \end{aligned}$$

est injective, par conséquent $[\gamma_a]$ est un épimorphisme.

□

Pour a égal à l'objet final $\star_{\mathbf{D}_{/f}}$, nous trouvons un épimorphisme naturel :

$$[\gamma_f] : \ \Omega_y \longrightarrow \Omega_{y/x} \ \ ,$$

et nous déduisons que pour tout $[b]$ dans $(\mathbf{D}_{/f})_{ab}$ nous avons un morphisme injectif entre les ensembles des dérivations

$$\left[\Gamma_f^{[b]}\right] : Der_{\mathbf{D}}(f,[b]) \ \hookrightarrow \ Der_{\mathbf{C}}(y,(G_f)_{ab}([b])) \ .$$

Remarque 3.19. Le morphisme $[\gamma_f]$ se factorise par

$$\Omega_y \longrightarrow L_y U_y(\Omega_{y/x}) \longrightarrow \Omega_{y/x} \ .$$

Définition 3.20. *Soient* $f : X \longrightarrow Y$ *et* $g : Y \longrightarrow Z$ *deux morphismes dans une catégorie additive* $\mathbf{A}$*, alors on dit que la suite*

$$X \xrightarrow{R} Y \xrightarrow{S} Z \longrightarrow 0$$

est exacte *dans* $\mathbf{A}$ *si pour tout* U *dans* $\mathbf{A}$ *nous avons une suite excate de groupes abéliens*

$$0 \longrightarrow Hom_{\mathbf{A}}(Z,U) \xrightarrow{S^*} Hom_{\mathbf{A}}(Y,U) \xrightarrow{R^*} Hom_{\mathbf{A}}(X,U) \ .$$

Si la catégorie $\mathbf{A}$ est de plus une catégorie exacte, c'est-à dire si $\mathbf{A}$ est une catégorie abélienne, nous retrouvons la définition usuelle de suite exacte.

Théorème 3.21. *Soit* $\mathbf{C}$ *une catégorie localement présentable, pour tout morphisme* $f : x \longrightarrow y$ *dans* $\mathbf{C}$*, nous avons une suite exacte dans* $\left(\mathbf{C}_{/y}\right)_{ab}$ *:*

$$f_!^{ab}(\Omega_x) \xrightarrow{\ [\tilde{\delta_f}]\ } \Omega_y \xrightarrow{\ [\gamma_f]\ } \Omega_{y/x} \longrightarrow 0$$

Démonstration. Par définition il faut montrer que pour tout $[c]$ dans $(\mathbf{C}_{/y})_{ab}$ la suite de morphismes induite

$$0 \longrightarrow Hom_{\left(\mathbf{C}_{/y}\right)_{ab}}(\Omega_{y/x},[c]) \xrightarrow{\ [\gamma_f]^*\ } Hom_{\left(\mathbf{C}_{/y}\right)_{ab}}(\Omega_y,[c]) \xrightarrow{\ [\tilde{\delta}_f]^*\ } Hom_{\left(\mathbf{C}_{/y}\right)_{ab}}(f_!^{ab}(\Omega_x),[c])$$

est une suite exacte de groupes abéliens. De ce qui précède nous en déduisons que c'est équivalent à monter que pour tout $[b]$ dans $(\mathbf{D}_{/f})_{ab}$ la suite de morphismes

$$0 \longrightarrow Hom_{(\mathbf{D}_{/f})_{ab}}(\Omega_f, [b]) \xrightarrow{[\Gamma_f]} Hom_{(\mathbf{C}_{/y})_{ab}}(\Omega_y, (G_f)_{ab}([b])) \xrightarrow{[\Delta_f]} Hom_{(\mathbf{C}_{/x})_{ab}}(\Omega_x, f^*_{ab}(G_f)_{ab}([b]))$$

est exacte.

Comme nous avons déjà vu par la proposition 3.18 que le morphisme $[\Gamma_f]$ est injectif il suffit de monter que nous avons l'égalité :

$$Ker([\Delta_f]) = Im([\Gamma_f]) \ .$$

Nous pouvons écrire le diagramme commutatif suivant

$$\begin{tikzcd}
Hom_{(\mathbf{D}_{/f})_{ab}}(\Omega_f, [b]) \arrow[r, "{[\Gamma_f]}"] \arrow[d, "\simeq"] & Hom_{(\mathbf{C}_{/y})_{ab}}(\Omega_y, (G_f)_{ab}([b])) \arrow[r, "{[\Delta_f]}"] \arrow[d, "\simeq"] & Hom_{(\mathbf{C}_{/x})_{ab}}(\Omega_x, f^*_{ab}(G_f)_{ab}([b])) \arrow[d, "\simeq"] \\
Hom_{\mathbf{D}_{/f}}(\star_{\mathbf{D}_{/f}}, U_f([b])) \arrow[r, "\Gamma_f"] & Hom_{\mathbf{C}_{/y}}(\star_{\mathbf{C}_{/y}}, U_y(G_f)_{ab}([b])) \arrow[r, "\Delta_f"] & Hom_{\mathbf{C}_{/x}}(\star_{\mathbf{C}_{/x}}, U_x f^*_{ab}(G_f)_{ab}([b]))
\end{tikzcd}$$

Avec les notations précédentes nous notons (v_b, b, u_b) l'élément $U_f([b])$ dans la catégorie $\mathbf{D}_{/f}$, (b, u_b) l'élément $U_y(G_f)_{ab}([b]) = G_f U_f([b])$ dans la catégorie $\mathbf{C}_{/y}$, et $(x \times_y b, p_1)$ l'élément $U_x f^*_{ab}(G_f)_{ab}([b]) = f^* G_f U_f([b])$ dans la catégorie $\mathbf{C}_{/x}$.

L'application Γ_f envoie un élément $[\alpha]$ de $Hom_{\mathbf{D}_{/f}}(\star_{\mathbf{D}_{/f}}, U_f([b]))$, correspondant à un morphisme $s : y \longrightarrow b$, sur l'élément $[\beta]$ appartenant à $Hom_{\mathbf{C}_{/y}}(\star_{\mathbf{C}_{/y}}, U_y(G_f)_{ab}([b]))$, correspondant au même morphisme s, en particulier nous retrouvons que l'application Γ_f est injective. Nous en déduisons qu'un élément $[\beta]$ de $Hom_{\mathbf{C}_{/y}}(\star_{\mathbf{C}_{/y}}, U_y(G_f)_{ab}([b]))$ appartient à l'image de l'application Γ_f si et seulement le morphisme $s : y \longrightarrow b$ correspondant vérifie l'égalité $s \circ f = v_b$, c'est-à-dire si et seulement si nous avons l'égalité $s \circ f = e_b \circ f$. Le résultat est alors une conséquence de la proposition 3.13.

□

Sous certaines conditions le module cotangent relatif $\Omega_{y/x}$ est nul ; plus précisément nous avons le résultat suivant.

Théorème 3.22. *Soit* $\mathbf{C}$ *une catégorie localement présentable, et soit* $f : x \longrightarrow y$ *dans* $\mathbf{C}$*, alors si* f *est un épimorphisme dans* $\mathbf{C}$*, l'application* $[\tilde{\delta_f}] : f_!^{ab}(\Omega_x) \longrightarrow \Omega_y$ *est un épimorphisme dans* $(\mathbf{C}_{/y})_{ab}$.

Démonstration. Si $f : x \longrightarrow y$ est un épimorphisme dans $\mathbf{C}$, le morphisme induit $\vartheta_f : f_!(\star_{\mathbf{C}_{/x}}) \longrightarrow \star_{\mathbf{C}_{/y}}$, correspondant à

$$\vartheta_f : (x \xrightarrow{f} y) \longrightarrow (y \xrightarrow{id_y} y)$$

est un épimorphisme dans $\mathbf{C}_{/y}$. Comme le foncteur L_y est un adjoint à gauche, il en est de même du morphisme $L_y(\vartheta_f)$, qui est égal au morphisme $[\tilde{\delta_f}]$ d'après la remarque 3.12.

$\square$

Nous considérons deux objets x et z de $\mathbf{C}$, $y = z \coprod x$, et les morphismes $f = q_2 : x \longrightarrow y$ et $g = q_1 : z \longrightarrow y$. Comme précédemment nous avons la paire de foncteurs adjoints

$$F^{(1)} = [z]_* : \mathbf{C} \rightleftarrows \mathbf{C}_{z/} = \mathbf{E} : [z]^! = G^{(1)} \quad ,$$

l'image de $(g : z \longrightarrow y)$ par le foncteur $G^{(1)}$ est y, nous en déduisons la paire de foncteurs adjoints

$$F_y^{(1)} : \mathbf{C}_{/y} \rightleftarrows \mathbf{E}_{/g} : G_g^{(1)} \quad .$$

Théorème 3.23. *Il existe dans* $(\mathbf{C}_{/y})_{ab}$ *un isomorphisme naturel*

$$f_!^{ab}(\Omega_x) \simeq \Omega_{y/z} \ .$$

Démonstration. Le foncteur $(G_g^{(1)})_{ab} : (\mathbf{E}_{/g})_{ab} \longrightarrow (\mathbf{C}_{/y})_{ab}$ est une équivalence de catégories, le foncteur inverse $(F_y^{(1)})_{ab}$ est l'adjoint à gauche et nous avons le diagramme commutatif suivant

$$\begin{tikzcd}
(\mathbf{C}_{/x})_{ab} \arrow[r, "f_!^{ab}"] & (\mathbf{C}_{/y})_{ab} \arrow[r, "(F_y^{(1)})_{ab}"] & (\mathbf{E}_{/g})_{ab} \\
\mathbf{C}_{/x} \arrow[u, "L_x"'] \arrow[r, "f_!"] & \mathbf{C}_{/y} \arrow[u, "L_y"'] \arrow[r, "F_y^{(1)}"] & \mathbf{E}_{/g} \arrow[u, "L_g"']
\end{tikzcd}$$

Par définition du foncteur $f_!$ nous avons $f_!(\star_{\mathbf{C}_{/x}}) = (f : x \longrightarrow y)$ et d'après la remarque 2.1 nous avons $F_y^{(1)}\,(f : x \longrightarrow y) = \star_{\mathbf{E}_{/g}}$, nous déduisons alors de la commutativité du diagramme précédent l'égalité

$$\left(F_y^{(1)}\right)_{ab} f_{!ab}(\Omega_x) = \Omega_g\ ,$$

et le théorème est une conséquence de l'égalité $\Omega_{y/z} = \left(G_g^{(1)}\right)_{ab}(\Omega_g)$.

□

4. Exemples

4.1 Catégorie des ensembles

Nous prenons comme catégorie $\mathbf{C}$ la catégorie $\mathbf{Ens}$ des ensembles. La catégorie des objets en groupe abélien dans $\mathbf{Ens}$ est la catégorie $\mathbf{Ab}$ des groupes abéliens.

Pour tout X dans $\mathbf{Ens}$ la catégorie $\left(\mathbf{Ens}_{/X}\right)_{ab}$ est la catégorie des applications $u : F \longrightarrow X$ telles que chaque fibre $F_x = u^{-1}(x)$ pour $x \in X$ est un groupe abélien, et les morphismes sont définis par

$$Hom_{\left(\mathbf{Ens}_{/X}\right)_{ab}}\left(\left(\,F^{(1)} \xrightarrow{u^{(1)}} X\,\right), \left(\,F^{(2)} \xrightarrow{u^{(2)}} X\,\right)\right) = \prod_{x \in X} Hom_{\mathbf{Ab}}(F_x^{(1)}, F_x^{(2)})\ .$$

Le foncteur $L_X : \mathbf{Ens}_{/X} \longrightarrow \left(\mathbf{Ens}_{/X}\right)_{ab}$ adjoint à gauche du foncteur $U_X : \left(\mathbf{Ens}_{/X}\right)_{ab} \longrightarrow \mathbf{Ens}_{/X}$ est défini par

$$L_X(\,E \xrightarrow{l} X) = (\,F = \mathbb{Z}_X[E] \xrightarrow{u} X)$$

avec $F_x = \mathbb{Z}[E_x]$ le groupe abélien libre engendré par la fibre E_x pour tout $x \in X$. Nous en déduisons que le module cotangent de l'ensemble X est défini par $\Omega_X = \ (X \times \mathbb{Z} \xrightarrow{p_1} X)$.

Soit $f : X \longrightarrow Y$ une application dans $\mathbf{Ens}$, alors le foncteur

$$f^*_{ab} : \left(\mathbf{Ens}_{/Y}\right)_{ab} \longrightarrow \left(\mathbf{Ens}_{/X}\right)_{ab}$$

envoie l'objet en groupe abélien $(G \xrightarrow{v} Y)$ sur $(F = X \times_Y G \xrightarrow{p_1} X)$ où pour tout $x \in X$ nous avons $F_x = G_{f(x)}$.

Le foncteur $f_!^{ab} : (\mathbf{Ens}_{/X})_{ab} \longrightarrow (\mathbf{Ens}_{/Y})_{ab}$ envoie l'objet en groupe abélien $(F \xrightarrow{u} X)$ sur $(G \xrightarrow{v} Y)$ où pour tout $y \in Y$ nous avons $G_y = \coprod_{f(x)=y} F_x$.

En effet pour $(F^{(1)} \xrightarrow{u^{(1)}} X)$ dans $(\mathbf{Ens}_{/X})_{ab}$ et pour $(G^{(2)} \xrightarrow{v^{(2)}} Y)$ dans $(\mathbf{Ens}_{/Y})_{ab}$ nous avons

$$Hom_{(\mathbf{Ens}_{/X})_{ab}}((\, F^{(1)} \xrightarrow{u^{(1)}} X\,), f_{ab}^*(\, G^{(2)} \xrightarrow{v^{(2)}} Y\,)) = \prod_{x \in X} Hom_{\mathbf{Ab}}(F_x^{(1)}, G_{f(x)}^{(2)})$$

et

$$Hom_{(\mathbf{Ens}_{/Y})_{ab}}(f_!^{ab}(\, F^{(1)} \xrightarrow{u^{(1)}} X\,), (\, G^{(2)} \xrightarrow{v^{(2)}} Y\,)) = \prod_{y \in Y} Hom_{\mathbf{Ab}}(\coprod_{f(x)=y} F_x^{(1)}, G_y^{(2)})\ .$$

Pour une application $f : X \longrightarrow Y$ nous avons alors

$$f_!^{ab}(\Omega_X) = (G \xrightarrow{v} Y)$$

avec pour tout $y \in Y$ la fibre $G_y = \coprod_{f(x)=y} \mathbb{Z}$ et le morphisme

$$[\tilde{\delta_f}] : f_!^{ab}(\Omega_X) \longrightarrow \Omega_Y$$

est le morphisme qui au dessus de y est défini par

$$\delta_y : \coprod_{f(x)=y} \mathbb{Z} \longrightarrow \mathbb{Z}\ .$$

Si l'ensemble $f^{-1}(y) = \{x \in X \mid f(x) = y\}$ est non vide nous avons $\delta_y : \coprod_{f(x)=y} \mathbb{Z} \longrightarrow \mathbb{Z}$ qui est défini par $id_{\mathbb{Z}}$ sur chaque facteur et δ_y est surjectif, et si l'ensemble $f^{-1}(y)$ est vide nous avons $\delta_y : (0) \longrightarrow \mathbb{Z}$. Nous en déduisons que le module cotangent relatif est défini par $\Omega_{Y/X} = (H \longrightarrow Y)$ avec $H_y = (0)$ si y appartient à l'image de l'application f et $H_y = \mathbb{Z}$ sinon.

Nous déduisons de ce qui précède le résultat suivant.

Proposition 4.1. *Le morphisme* $[\tilde{\delta_f}] : f_!^{ab}(\Omega_X) \longrightarrow \Omega_Y$ *est un épimorphisme (resp. un monomorphisme) si et seulement si l'application* $f : X \longrightarrow Y$ *est surjective (resp. injective).*

En particulier $[\tilde{\delta_f}] : f_!^{ab}(\Omega_X) \longrightarrow \Omega_Y$ *est un isomorphisme dans la catégorie* $(\mathbf{Ens}_{/Y})_{ab}$ *si et seulement si* f *est une bijection dans* $\mathbf{Ens}$.

4.2 Catégorie des monoïdes commutatifs

Nous considérons maintenant la catégorie $\mathbf{Com}$ des monoïdes associatifs, commutatifs unitaires. Un objet X de $\mathbf{Com}$ est un ensemble muni d'une loi de composition interne $\star : X \times X \longrightarrow X$ associative, commutative et muni d'un élément neutre 1_X pour la loi $\star$. Suivant Michael Barr nous avons la description suivante.

Soit X dans $\mathbf{Com}$, un objet de la catégorie $(\mathbf{Com}_{/X})_{ab}$ est la donnée d'un morphisme de monoïdes $u_A : A \longrightarrow X$ et de morphismes de monoïdes au-dessus de X $m_A : A \times_X A \longrightarrow A$, $i_A : A \longrightarrow A$ et $e_A : X \longrightarrow A$ qui définissent une structure d'objet en groupe abélien.

Nous en déduisons que pour tout $x \in X$ ces morphismes induisent une structure de groupe abélien sur la fibre $A_x = u_A^{-1}(x)$ notée $+_x$ dont l'élément neutre est $0_{A_x} = e_A(x)$. La composition par un élément x de X définit une application de A_y dans $A_{x\star y}$, et comme d'après la démonstration de la proposition 3.9 cette application est un morphisme de groupes abéliens. Cela détermine complètement les objets en groupe abélien dans $\mathbf{Com}_{/X}$, plus précisément nous avons le résultat suivant.

Proposition 4.2. *[Ba 2] Soit* X *un monoïde commutatif, un objet* A *de la catégorie* $(\mathbf{Com}_{/X})_{ab}$ *est la donnée pour tout* x *dans* X *d'un groupe abélien* A_x*, pour tout* x *et* y *dans* X *d'un morphisme de groupes* $h_x^{(A)} : A_y \longrightarrow A_{x\star y}$ *vérifiant* $h_x^{(A)} \circ h_y^{(A)} = h_{x\star y}^{(A)}$.

Nous notons un élément de A *sous la forme d'une paire* (x, a) *avec* $x = u_A(x, a)$ *dans* X*,* a *dans* A_x *et la structure de monoïde sur* A *est définie par* $(x, a) \star (y, b) = (x \star y, h_x^{(A)}(b) +_{x\star y} h_y^{(A)}(a))$.

Pour tout A dans $(\mathbf{Com}_{/X})_{ab}$ l'espace des dérivations $Der_{\mathbf{Com}}(X, A)$ est égal par définition à l'ensemble $Hom_{\mathbf{Com}_{/X}}(X, A)$. D'après ce qui précède un morphisme s dans $Hom_{\mathbf{Com}_{/X}}(X, A)$ correspond à la donnée pour tout x dans X d'un élément s_x dans A_x vérifiant la propriété suivante

$$s_{x\star y} = (h_x^{(A)}(s_y) +_{x\star y} h_y^{(A)}(s_x)) \ .$$

En particulier pour $x = y$ égal à l'élément neutre 1_X du monoïde X, nous avons $h_x^{(A)} = id$ et nous en déduisons s_{1_X} est égal à l'élément neutre $0_{A_{1_X}}$ du groupe abélien A_{1_X}.

Un morphisme $f : A \longrightarrow B$ dans $(\mathbf{Com}_{/X})_{ab}$ induit pour chaque x dans X un morphisme de groupes abéliens $f_x : A_x \longrightarrow B_x$. Comme f respecte la structure de monoïdes nous avons de plus pour tout a dans A_x et b dans A_y l'égalité $f_{x\star y}(h_x^{(A)}(b) +_{x\star y} h_y^{(A)}(a)) = h_x^{(B)}(f_y(b)) +_{x\star y} h_y^{(B)}(f_x(a))$, ce qui induit pour $b = 0_{A_y}$ l'égalité $f_{x\star y}(h_x^{(A)}(a)) = h_x^{(B)}(f_y(a))$. Nous avons alors le résultat suivant.

Proposition 4.3. *[Ba 2] Un morphisme $f : A \longrightarrow B$ dans $(\mathbf{Com}_{/X})_{ab}$ est défini par la donnée d'une famille de morphismes de groupes abéliens $f_x : A_x \longrightarrow B_x$ vérifiant la relation $f_{x\star y} \circ h_x^{(A)} = h_x^{(B)} \circ f_y$.*

Soit $f : X \longrightarrow Y$ un morphisme dans $\mathbf{Com}$, nous avons alors la paire de foncteurs adjoints

$$f_! : \mathbf{Com}_{/X} \rightleftarrows \mathbf{Com}_{/Y} : f^*$$

et le diagramme commutatif

$$\begin{array}{ccc} (\mathbf{Com}_{/Y})_{ab} & \xrightarrow{f^*_{ab}} & (\mathbf{Com}_{/X})_{ab} \\ \downarrow{\scriptstyle U_Y} & & \downarrow{\scriptstyle U_X} \\ \mathbf{Com}_{/Y} & \xrightarrow{f^*} & \mathbf{Com}_{/X} \end{array}$$

où le foncteur f^*_{ab} envoie l'objet B de $(\mathbf{Com}_{/Y})_{ab}$ défini par la famille de groupes abéliens $(B_y)_{y\in Y}$ et les morphismes $h_y^{(B)}$ sur l'objet $A = f^*_{ab}(B)$ de $(\mathbf{Com}_{/X})_{ab}$ défini par la famille de groupes abéliens $(A_x)_{x\in X}$ avec $A_x = B_{f(x)}$ et les morphismes $h_x^{(A)} = h_{f(x)}^{(B)}$.

Soit Ω_X le module cotangent de X et soit $\eta_X : X \longrightarrow \Omega_X$ l'unité de l'adjonction, alors Ω_X correspond à une famille $((\Omega_X)_x)_{x \in X}$ de groupes abéliens et à des morphismes $h_x^{(\Omega_X)} : (\Omega_X)_y \longrightarrow (\Omega_X)_{x\star y}$, et le morphisme η_X correspond à la donnée d'éléments η_x dans $(\Omega_X)_x$ vérifiant

$$\eta_{x\star y} = (h_x^{(\Omega_X)}(\eta_y) +_{x\star y} h_y^{(\Omega_X)}(\eta_x)) \ .$$

Nous considérons le monoïde $X = (\mathbb{N}, +)$, un objet A dans $(\mathbf{Com}_{/\mathbb{N}})_{ab}$ est défini par la donnée de groupes abéliens A_n et de morphismes de groupes $h^{(A)} : A_n \longrightarrow A_{n+1}$, pour tout n appartenant à $\mathbb{N}$, avec les notations précédentes nous avons $h_n^{(A)} = (h^{(A)})^n$. Et de la même manière un morphisme $f : A \longrightarrow B$ dans $(\mathbf{Com}_{/\mathbb{N}})_{ab}$ est défini par une famille de morphismes de groupes abéliens $f_n : A_n \longrightarrow B_n$ vérifiant $f_{n+1} \circ h^{(A)} = h^{(B)} \circ f_n$.

Proposition 4.4. *Le module cotangent $\Omega_{\mathbb{N}}$ est l'objet de $(\mathbf{Com}_{/\mathbb{N}})_{ab}$ défini par*

1. $(\Omega_{\mathbb{N}})_0 = (0)$
2. $(\Omega_{\mathbb{N}})_n = \mathbb{Z}$ *et* $h^{(\Omega_{\mathbb{N}})} : (\Omega_{\mathbb{N}})_n \xrightarrow{id_{\mathbb{Z}}} (\Omega_{\mathbb{N}})_{n+1}$ *pour* $n \geq 1$.

Démonstration. Pour tout A dans $(\mathbf{Com}_{/\mathbb{N}})_{ab}$ l'ensemble $Hom_{\mathbf{Com}_{/\mathbb{N}}}(\mathbb{N}, A)$ est isomorphe au groupe abélien A_1. En effet un élément s_1 dans A_1 détermine un morphisme $s : \mathbb{N} \longrightarrow A$ avec $s_0 = 0_{A_0}$ et pour tout $n \geq 1$ $s_n = n(h^{(A)})^{n-1}(s_1)$.

Un morphisme $f : \Omega_{\mathbb{N}} \longrightarrow A$ dans $(\mathbf{Com}_{/\mathbb{N}})_{ab}$ est entièrement déterminé par le morphisme $f_1 : (\Omega_{\mathbb{N}})_1 \longrightarrow A_1$, en effet nous avons $f_0 = 0$ et $f_n = (h^{(A)})^n \circ f_1$ pour tout $n \geq 1$. Nous déduisons de l'égalité $(\Omega_{\mathbb{N}})_1 = \mathbb{Z}$ que le morphisme f_1 est défini par $f_1(1)$ et nous trouvons alors que l'ensemble $Hom_{(\mathbf{Com}_{/\mathbb{N}})_{ab}}(\Omega_{\mathbb{N}}, A)$ est aussi isomorphe au groupe abélien A_1. □

4.3 Catégorie des anneaux commutatifs

Soient k un anneau commutatif et $\mathbf{Alg}_k$ la catégorie des k-algèbres commutatives, nous rappelons les définitions et les résulats suivants.

Soient A une k-algèbre et M un A-module, alors une *k-dérivation de A à valeurs dans M* est un morphisme de k-modules $d : A \longrightarrow M$ vérifiant la règle de Leibniz $d(aa') = ad(a') + a'd(a)$. Nous notons $Der_k(A, M)$ l'ensemble des k-dérivations.

Nous appelons *module des différentielles de Kähler* de A sur k le A-module $\Omega_{A/k}$ défini comme le A-module I/I^2, où nous notons I le noyau de la *multiplication* $m : A \otimes_k A \longrightarrow A$, et nous appelons d_A la k-dérivation à valeurs dans $\Omega_{A/k}$ définie par le morphisme qui envoie un élément a de A sur l'image de $1 \otimes a - a \otimes 1$ dans I/I^2.

Alors le foncteur

$$\begin{array}{rccc} Der_k(A, -) : & \mathbf{Mod}_A & \longrightarrow & \mathbf{Ens} \\ & M & \mapsto & Der_k(A, M), \end{array}$$

est représenté par le A-module $\Omega_{A/k}$, où l'isomorphisme naturel en M entre les ensembles $Hom_A(\Omega_{A/k}, M)$ et $Der_k(A, M)$ est défini par la composition par d_A.

Nous voulons comparer les notions introduites à la définition 3.5 avec les notions données ci-dessus.

Proposition 4.5. *Pour toute k-algèbre A la catégorie $(\mathbf{Alg}_{k/A})_{ab}$ des objets en groupe abélien de $\mathbf{Alg}_{k/A}$ est équivalente à la catégorie $\mathbf{Mod}_A$ des A-modules.*

Démonstration. Soit B un objet de la catégorie $(\mathbf{Alg}_{k/A})_{ab}$ des objets en groupe abélien de $\mathbf{Alg}_{k/A}$, nous avons alors le morphisme élément neutre $e_B : A \longrightarrow B$ et le morphisme composition pour la loi de groupe abélien $m_B : B \times_A B \longrightarrow B$. Comme le morphisme e_B est une section du morphisme $u_B : B \longrightarrow A$, nous en déduisons que B est isomorphe, en tant que A-module, à la somme directe $A \oplus M$ avec $M = Ker(\ u_B : B \longrightarrow A\ )$, et grâce à la proposition 3.9 appliquée à la loi de groupe commutatif sur un anneau nous trouvons que m_B est défini par $m_B(b_1, b_2) = b_1 + b_2 - e_B(a)$ avec $a = u_B(b_1) = u_B(b_2)$.

En appliquant l'égalité "$\mu_B \circ c_B = \mu_B \circ (\mu_B \times \mu_B) \circ \psi$" de la démonstration de la proposition 3.9 pour μ_B la multiplication dans B nous en déduisons que B est isomorphe à la k-algèbre $A \oplus M$ avec la structure de k-algèbre *extension de carré nul*, définie par $(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + a_2 m_1)$.

Réciproquement nous pouvons associer ainsi à tout A-module M un objet en groupe abélien $B = A \oplus M$, et l'équivalence de catégories est alors une conséquence du corollaire 3.10.

□

Corollaire 4.6. *1) Pour tout A-module M l'ensemble des dérivations de Beck $Der_{\mathbf{Alg}_k}(A, B)$ de A à valeurs dans l'objet en groupe abélien $B = A \oplus M$ est égal à l'ensemble $Der_k(A, M)$ des dérivations de A à valeurs dans M.*

2) Le module cotangent rΩ_A est naturellement isomorphe à la k-algèbre $A \oplus \Omega_{A/k}$, extension de carré nul de A par le module $\Omega_{A/k}$ des différentielles de Kähler.

Démonstration. 1) D'après la remarque 3.6 il suffit de montrer que l'ensemble des sections $s : A \longrightarrow B$ du morphisme $u_B : B \longrightarrow A$ est en bijection canonique avec l'ensemble des dérivations $d : A \longrightarrow M$ de A à valeurs dans M, bijection donnée par l'égalité $s(a) = (a, d(m))$.

2) Par définition la k-algèbre Ω_A et le A-module $\Omega_{A/k}$ représentent respectivement les foncteurs $Der_{\mathbf{Alg}_k}(A, -)$ et $Der_k(A, -)$, par conséquent l'isomorphisme entre Ω_A et $A \oplus \Omega_{A/k}$ est une conséquence de la proposition 4.5 et du résultat précédent.

□

Nous retrouvons la définition classique du module des différentielles de Kähler Ω_A au dessus d'un anneau A, et les théorèmes 3.21, 3.22 et 3.23 sont des généralisations des résultats classiques de le théorie des modules des différentielles de Kähler.

Le théorème 3.21 est une généralisation de la première suite exacte fondamentale, résultat qui énonce que pour tout morphisme $f : A \longrightarrow B$ dans $\mathbf{Alg}_k$ nous avons une suite exacte de B-modules

$$\Omega_{A/k} \otimes_A B \xrightarrow{\delta_f} \Omega_{B/k} \longrightarrow \Omega_{B/A} \longrightarrow 0 \,.$$

Le théorème 3.22 est une généralisation du résultat suivant concernant les épimorphismes $f : A \longrightarrow B$ dans $\mathbf{Alg}_k$. La classe des épimorphismes est engendrée par la classe des surjections $f : A \longrightarrow B$ avec $B = A/I$ où l'idéal I de A est le noyau de f, et la classe des localisations $f : A \longrightarrow B$ avec $B = S^{-1}A$ où S est une partie multiplicative de A.

Alors la *deuxième suite exacte fondamentale* énonce que pour toute surjection $f : A \longrightarrow B = A/I$ nous avons une suite exacte de B-modules

$$I/I^2 \longrightarrow \Omega_{A/k} \otimes_A B \xrightarrow{\delta_f} \Omega_{B/k} \longrightarrow 0 ,$$

et la compatibilité des modules des différentielles avec la localisation énonce que pour toute localisation $f : A \longrightarrow B = S^{-1}A$ nous avons un isomorphisme de B-modules

$$\delta_f : \Omega_{A/k} \otimes_A B \xrightarrow{\simeq} \Omega_{B/k} .$$

Nous en déduisons que pour tout épimorphisme $f : A \longrightarrow B$ l'application $\delta_f : \Omega_{A/k} \otimes_A B \longrightarrow \Omega_{B/k}$ est un épimorphisme dans $\mathbf{Mod}_B$.

Enfin le théorème 3.23 est la généralisation du fait que les modules de différentielles de Kähler commutent avec *l'extension des scalaires*, soient A une k-algèbre, $f : k \longrightarrow k'$ un morphisme d'anneaux et $A' = A \otimes_k k'$, alors nous avons l'isomorphisme de A'-modules

$$\Omega_{A/k} \otimes_A A' \xrightarrow{\simeq} \Omega_{A'/k'} .$$

Pour finir nous pouvons faire la remarque suivante.

Remarque 4.7. Comme la catégorie $(\mathbf{Alg}_{k/A})_{ab}$ est équivalente à la catégorie $\mathbf{Mod}_A$, elle est indépendante de l'anneau de base k. C'est une traduction du (2) de la proposition 3.15.

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXVI-4 (2025)

ERRATA FOR "LAX KAN EXTENSIONS FOR DOUBLE CATEGORIES (ON WEAK DOUBLE CATEGORIES, PART IV)"

Marco Grandis and Robert Paré

Résumé. Une erreur technique a causé la substitution de plusieurs symboles dans les formules de la version publiée de l'article [1]. Nous donnons des liens à la version originale.

Abstract. A technical glitch resulted in a substitution of symbols in many formulas of the published version of the article [1]. We provide links to the original version.

Keywords. Errata, lax Kan extension, double category

Mathematics Subject Classification (2010). 18N10, 18A40

1. Errata

It would serve no purpose to list all occurrences of these substitutions, appearing throughout the paper. Instead, we refer the interested reader to either of the following links where the original version may be found (and, incidentally, the first author's name is spelled correctly):

https://www.mscs.dal.ca/~pare/LaxKanExt.pdf
https://www.researchgate.net/publication/242233217

We thank Bryce Clarke for pointing out the problem to us.

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