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dirigés par Andrée CHARLES EHRESMANN

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXVI-3 (2025)

# A (MATHEMATICAL) TRIBUTE TO RONNIE BROWN: 1935 – 2024

*Tim Porter*

**Résumé.** Une esquisse de la vie de Ronnie Brown ainsi que quelques aspects importants de son œuvre et une liste de ses publications.
**Abstract.** We provide a sketch of the life of Ronnie Brown, presenting some important aspects of his work and provide a list of his publications.
**Keywords.** Crossed complexes, $\omega$-groupoids, topological groupoids, holonomy, monodromy, popularisation, teaching innovation.
**Mathematics Subject Classification (2010).** 18B40, 18F15, 18G99, 18N99, 55P99, . . .

Ronald (Ronnie) Brown, who was a prolific researcher and a frequent contributor to the categorical and topological literature, died on 5 December 2024, aged 89. He was also well known as a populariser of mathematics and an innovator in teaching the subject.

Ronnie Brown was born on 4 January 1935 to parents who were first generation emigrés to the U. K. from Romania. Aged about six, he spent some time in the USA, staying with family, whilst his father was in the army. Coming back to the UK in 1944, his family moved around quite a bit, but, eventually, Ronnie went to Alleyn's School, Dulwich, London, where his interest in mathematics was encouraged. He gained a place in New College, Oxford, in 1953. In Oxford, he met Margaret, also a student of mathematics. They married in 1958 and later had eight children.

After completing his undergraduate degree at Oxford, Ronnie started as a postgraduate, again at Oxford, where his supervisor was Henry Whitehead.

Professor Whitehead died suddenly in 1960 and Ronnie continued under the supervision of Michael Barratt. His thesis title was '*Some problems in Algebraic Topology: A study of function spaces, function complexes and FD-complexes*'. The title shows several aspects of Ronnie's research interest, not only in algebraic topology, but also in what is now seen as part of the categorical side of topology. That latter area was exemplified, from the start, in the subject matter of his first two papers, [R1, R2], published in 1963 and 1964, which discussed properties of various topologies on products of topological spaces, and the corresponding relationship with function spaces and product topologies, especially with regard to the crucial 'exponential law',

$$X^{(Y\times Z)} \cong (X^Y)^Z,$$

for function spaces. Much later, when Ronnie was attending an international category theory meeting[1], he was surprised to discover that these, his first two papers, were considered of foundational importance for a large area of categorical topology, and also for the study of non-cartesian monoidal structures on categories.

At that time, (1959 - 1964), Ronnie was an Assistant Lecturer, and then Lecturer, at Liverpool University. Slightly later, he became, from 1964 to 1970, a Senior Lecturer, then Reader, at the University of Hull. In 1968, he published a text book, '*Elements of Modern Topology*', [B1], which was to become very influential, both in the teaching of the subject, and in the direction that Ronnie's own research took subsequently. That book was revised and updated twice over a period of years, as [B3] and [B4], yet its contents and style, suitably evolved, still prove important for more recent generations.

It was while writing that book, and, as he said later, 'to clarify certain points relating to the calculation of the fundamental group of a circle', that he started taking an interest in groupoids. Significantly one of his two publications at this time, [R8], discussed the groupoid version of van Kampen's theorem, which extended the classical form to unions of *non-connected* spaces, and which allowed the direct calculation of the fundamental group of the circle, and more. The proofs suggested to him there might be higher dimensional versions of that theorem and the search for those was one of the themes of Ronnie's research for the next nearly 20 years.

[1] I think it was in Amiens.

In 1970, Ronnie moved to North Wales to take up the post of Professor of Pure Mathematics at the then University College of North Wales, part of the University of Wales. The family moved to Anglesey, to Benllech and to a house just a short distance from the beach.

From 1974 onwards, Ronnie started exploring the topic of higher dimensional analogues of the groupoid van Kampen's theorem, working with Phil Higgins, initially from King's College London, later at Durham University. This work also involved input from several postgraduate students, often in crucial ways, and also from other collaborators such as Hans Baues, and later Jean-Louis Loday. The main thrust of this research lasted over 20 years, culminating in the book, [B5], with Phil Higgins and Rafael Sivera. This collected up the interlocking theories related to the proofs of a higher dimensional form of van Kampen's theorem, but also the interactions of those new theories with other more classical themes. It also involved some ideas, especially those around *crossed modules* and what became known as *crossed complexes*, which had been developed by Henry Whitehead in the 1940s and 50s from ideas of both Whitehead and Reidemeister in the 1930s, but which had mostly lain fallow in the intervening period. The strong connection with Whitehead's work only became clear as the collaboration with Higgins progressed. This linked the new Brown-Higgins theory into the vision of Whitehead for an *Algebraic Homotopy Theory*.

At this point we should mention the highly influential paper, [R56], with Marek Golasiński, in which a Quillen model category structure was specified on the category of crossed complexes, as this started the exploration of the link between a Whitehead-style algebraic homotopy theory and the theory of model categories.

For Ronnie, there was a very interesting spin off from the study of crossed modules and crossed complexes. With Johannes Huebschmann, [R35], he examined pre-war work by Reidemeister and his student Peiffer on identities among relations in presentations of groups, pushing that theory forward in several ways. These relied on a detailed analysis of the notion of free crossed modules and their interaction with various concepts from combinatorial group theory. It also used earlier work by Ronnie and Chris Spencer, [R20] and [R21], giving the first published detailed proof of a theorem mentioned by Verdier and Grothendieck, which showed the link between internal categories in the category of groups and crossed modules.

As mentioned above, further collaboration, (1983-87), again within this general area of '*higher dimensional group theory*', occurred in collaboration with Jean-Louis Loday, and centred on his models for homotopy $n$-types, called $cat^{n-1}$-*groups* and crossed $(n-1)$-cubes of groups, which generalise internal categories in groups, and crossed modules respectively. Via a van Kampen style theorem for these models, the collaboration revealed a new type of tensor product-like construction available when two (possibly non-abelian) groups act on each other in a compatible way. This further lead to work with Dave Johnson and Edmund Robertson, [R52], on more purely group theoretic aspects of this theory, whilst work by Ronnie's former research student, Graham Ellis, and many others, pushed the theory forward, linking it with several important areas of group theory.

One of the papers with Graham Ellis, [R55], explored higher dimensional analogues of the famous Hopf formula, which gives an expression for the second homology group of a group on being given a presentation of the group. As with many of Ronnie's papers, this has been very influential, leading to many developments, this time in categorical and 'higher dimensional' algebra. These continue today nearly 40 years later.

Although that area of 'higher dimensional algebra' was an important theme of Ronnie's research, from about 1975, he also continued to work[2] on the theory and application of topological groupoids, extending ideas of Charles Ehresmann and Jean Pradines and interacting with ideas on foliations and orbifolds. Much of this later work focussed on the ideas of Monodromy and Holonomy groupoids, interacting with Kirill MacKenzie in Sheffield; see [R71], [R118], and [R145].

In 1982, Ronnie wrote a letter to Alexander Grothendieck, and this started a very fruitful and amicable exchange of letters, [L], that lasted until 1991. His motivation had been to ask Grothendieck about his interest in notions of $\infty$-groupoids and categories, as certain types of $\infty$-groupoids corresponded to crossed complexes. This motivated Grothendieck to follow up on some ideas he had had some years earlier and to start on the famous set of typed notes known as *Pursuing Stacks*, published in book form only in 2022, but distributed from Bangor with Grothendieck's permission, as a photocopy of

[2]with others, including ex-research students, Hardy, Aof, Mucuk and İçen; see the references.

the original as Grothendieck wrote it.

The models for (some) homotopy types that Ronnie was putting forward were *strict* $\infty$-groupoids and categories, corresponding to crossed complexes in one of their manifestations. They thus did not fully answer to all the requirements of Grothendieck's wider programme, which needed some weaker form of $\infty$-groupoids to handle *all* homotopy types. None-the-less this provided additional evidence of the importance of the Brown-Higgins theory of what they now started referring to as *nonabelian algebraic topology*, *c.f.* [B5]. This was seen as an intermediate stage between a classical *homological*, and thus *abelian*, methodology and the vision of the theory proposed by Grothendieck that aimed at modelling *all* homotopy types by means of *weak* $\infty$-groupoids. The work of Ronnie with Jean-Louis Loday also fitted into this overall perspective.

At about this same period, Ronnie was reminded of one of his earliest papers. In [R5], he had examined some methods introduced by Shih in 1962, for calculating with E. H. Brown's notion of a twisted tensor product that dated from 1959, and also the related results of Barratt, Gugenheim and Moore from the same year. This was for examining the structure of the homology of the total space of a fibration in terms of the homology of the base and fibre of the fibration. Slightly later, in 1972, Gugenheim rediscovered these results and the two papers together came to form the basis for what is now called *Homological Perturbation Theory*. This has turned out to be an important theoretical and computational tool, and with computational input from Chris Wensley and Larry Lambe, Ronnie started on adapting various computer algebra packages with the aim of obtaining insight into the structure of algebraic homotopical invariants and algebraic homotopy types using methods from combinatorial group theory, and also from the theory of Gröbner bases; see [R101, R113].

Although this is not exhaustive as a description of Ronnie's research interests, rather than continuing with more such, we will pause and leave the reader to peruse the list of his research papers, noting how the various themes, topological and categorical, theoretical and computational, intertwine and interact throughout his research. We will return to this later on, but considering his impact on other aspects of the subject area, it would not be correct to omit either his work in popularisation or on teaching innovation.

Moreover these activities interacted with his research interests.

Ronnie's involvement in the popularisation of mathematics is well known. In about 1985, with a group of colleagues and with the assistance of local schools in North West Wales, he initiated a series of Mathematical Masterclasses for Young People in the area, under the aegis of the Royal Institution of Great Britain. This led to the development in Bangor of material that was designed to be useful in some of those Masterclass sessions, and also to the preparation of a set of exhibition boards on the theme of 'Maths and Knots' for use when giving talks in the masterclasses, and, increasingly, elsewhere; see [PP3].

Very quickly, this project was to acquire another component. As Ronnie says in [PP1]: '*One day in May 1985, I was walking down Albermarle Street*[3] *from a meeting on masterclasses at the Royal Institiution. As I passed the Freeland Gallery, ... and with some time to spare, I decided to wander inside, enticed by the sculptures of children and animals shown in the window. To my amazement I found also some strong and beautifully crafted knot sculptures*'. Thus started a collaboration and friendship between Ronnie and the sculptor, John Robinson, and a widening out of the scope of his popularising work on the theme of 'How Mathematics gets into Knots'. This led to Bangor's involvement in the *Pop Maths Roadshow*, and in a pan-European project on *Raising Public Awareness of Mathematics* for European National Science Week, 2000. This involved lots of fun interaction with lots of interesting people ... and a lot of thought and hard work!

Partly as a result of working with the younger students in the masterclasses, and with the general public through popular lectures and the exhibition, Ronnie, with myself and others, started thinking about the *context* of the mathematics that was being taught to our own students. We realised, for instance, that even our own students were not really aware that new mathematics was being discovered / created all the time, nor how that was done. We felt '*We should also popularise mathematics to (our own) students*[4]', as an antidote to the pressures for undergraduate courses to become 'a bare skeleton of technicalities', rather than an exciting endeavour involving human interaction. We thus developed a new type of course: 'Mathematics in Context'. The style was very informal and in [T3], we describe the reac-

[3] in central London

[4] from [T18]

tion of the students. We discussed historical, cultural and scientific issues, and got very good and interesting input and feedback from the students. The course did evolve in the years that followed with use of some external speakers, for instance from industry, and with colleagues describing and discussing some of their research projects in talks at the level of ones for a general educated audience. The sessions were very enjoyable and were, in general, very well received by both the students and, when the assessment was completed, by the external examiners for the degree.

The idea of adding 'context' explicitly into courses was addressed in [T18], and interacted with discussions on the *methodology of mathematics*, which coalesced in published versions, for instance, [T6, T7, T11], and [T20]. The multiple versions were due to requests from various sources to republish the original paper in their own journals for local consumption, as the originals had not always been that easy to find.

This emphasis on methodology and on making research more approachable started to feedback into Ronnie's research. It meant that he had a very clear idea on how to explain methodological ideas to non-mathematicians, partially bridging the well known gap between a mathematical view and the viewpoints of scientists from other branches of enquiry.

Running through Ronnie's research, there are some themes that repeat, and these also pervade other branches of science. One is the idea of '*local to global*', so how 'local' information about an object coalesces to give global information. Another, which is almost the converse of that, was the idea that '*subdivision is an inverse to composition*'. For Ronnie, the local to global paradigm was exemplified by the various higher algebraic structures that grew out of the quest for generalisations of van Kampen's theorem, but at the same time, how that worked involved a detailed analysis of cubical subdivisions. The local to global aspect provided insights into various philosophical and scientific questions, which were illustrative of the methodology of mathematics, for example, abstraction, modelling geometric situations via algebra, and the development of new concepts.

In discussions, he realised how these paradigmatic problems were important in other very interesting areas. In computer science, for instance, the local to global problem interacted with ideas on concurrent computing in a very geometric way, whilst ideas on 'composition versus subdivision' led to discussions on modelling neuro-systems, and more general hierarchical systems in biology and biocomputing. Philosophical / psychological contexts in which the problems of abstraction and the refinement of concepts occurred, also mirrored quite closely those encountered in mathematics. These discussions led to several papers involving Ray Paton, [R90, R111], of the Biocomputing and Computational Biology group at Liverpool University. This direction was cut short by Ray's untimely death in 2002, but was taken further with different emphasis in joint work with Jim Glazebrook and Ion Baianu, for instance, [R116, R123, R126] or [R138].

Throughout his life as a research mathematician, Ronnie served in editorial roles for international journals. From 1975 to 1994, he was on the editorial advisory board for the London Mathematical Society. He was a founding member of the editorial board of *Theory and Applications of Categories*, then on the editorial board of *Applied Categorical Structures*. In 1999, he helped found the electronic journal, *Homology, Homotopy and Applications* and then, from 2006, was an editor for the *Journal of Homotopy and Related Structures*. In 2016, a special volume of that journal was dedicated to him and his work on the occasion of his 80th birthday.

Throughout his career, he supervised many postgraduate students, who contributed greatly to the various research themes of interest at Bangor. In all there were 24 such, as listed by the Mathematics Genealogy Project. With many of these he developed long lasting collaborations as can be seen from the publication list.

Ronnie retired from full time teaching in 1999, although he continued as a half-time research professor until 2001. This, of course, did not reduce the amount of time and effort he put into his research and in particular into the preparation of the book, [B5], with Phil Higgins and Rafael Sivera, which regrouped the results from the long series of articles with Higgins, developing their theory in a coherent way in one source. He also developed new collaborations as Coordinator for an INTAS funded project, 'Algebraic K-theory, groups and categories', linking Bangor, with the University of Bielefeld, the Georgian Mathematical Institute, the State Universities of Moscow and of St. Petersburg, and the Steklov Institute, St. Petersburg. This involved research visits to Bielefeld and visits of their team to Bangor.

In 2016, Ronnie was elected to Fellowship of the Learned Society of

Wales and he was a lifelong member of the London Mathematical Society.

By about 1998, Ronnie had become hard of hearing on one side, and during one of the visits to Bielefeld for discussions with Tony Bak, he suffered from a severe loss of balance. On his return to North Wales, he was diagnosed as having an acoustic neuroma, which is a non-cancerous growth covering the acoustic nerve. He was treated for this in February 2000, and for a time was feeling a bit better. He then began to suffer from double vision. This was caused by some of the debris from the treatment of the neuroma. This itself was treated in 2001, and he gradually recovered and continued to work on his research projects.

Although he and Margaret had eight children, throughout his life he had found time to play firstly table tennis, then later on squash and to go swimming, both in the sea and in the lakes of North Wales. He also spent time gardening, making home made beer, searching for mushrooms in season, and exploring the region with the children when they were young.

On retirement, he and Margaret left Anglesey to move to Deganwy, near Llandudno, still in North Wales, and to a house with a beautiful view westward along the coast. They also had a small cottage for family reunions, and went on several Mediterranean cruises, including one during which he was roped in to explain some aspects of Greek mathematics! They both participated in the cultural life of the area.

Margaret died in 2020, and Ronnie's mobility had become reduced after a stroke. He was cared for by their eldest son, and enjoyed excursions using a mobility scooter, including to see seals in a cove a short drive away from their home, to visit a nature reserve on the Conwy estuary or the beautiful gardens at Bodnant, a short distance south along the valley of the river Conwy. It was after one of the visits to see the seals that he passed away peacefully, but suddenly, at home in Deganwy, aged 89 years. He will be missed by his seven surviving children, and his grandchildren.

Perhaps the last word should be left to his children who said that on journeys, their father would often deviate from the main roads to "take the scenic route" – sometimes getting lost in the process. But his attitude to life and, in particular, to mathematics was always “take the scenic route”. That love of exploring ideas or places for their own sake permeated his life both in his research career and in the teaching that he loved.

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXVI-3 (2025)

# WHEN A MATRIX CONDITION IMPLIES THE MAL'TSEV PROPERTY

*Michael Hoefnagel and Pierre-Alain Jacqmin*

**Résumé.** Les conditions matricielles étendent les conditions de Mal'tsev linéaires de l'algèbre universelle aux propriétés d'exactitude en théorie des catégories. Certaines peuvent être énoncées dans le contexte finiment complet alors que, en général, elles peuvent être énoncées seulement pour les catégories régulières. Nous étudions quand une telle condition matricielle implique la propriété de Mal'tsev. Nos résultats principaux affirment que, pour les deux types de matrices, cette implication est équivalente à l'implication correspondante restreinte au contexte des variétés d'algèbres universelles.
**Abstract.** Matrix conditions extend linear Mal'tsev conditions from Universal Algebra to exactness properties in Category Theory. Some can be stated in the finitely complete context while, in general, they can only be stated for regular categories. We study when such a matrix condition implies the Mal'tsev property. Our main results assert that, for both types of matrices, this implication is equivalent to the corresponding implication restricted to the context of varieties of universal algebras.
**Keywords.** Mal'tsev category, Mal'tsev condition, matrix property, cube term, finitely complete category, regular category, essentially algebraic category.
**Mathematics Subject Classification (2020).** 18E13, 08B05, 03C05, 18-08 (primary); 18A35, 18E08, 18C05, 08A55, 08C10 (secondary).

# Introduction

Given a *simple extended matrix of variables*

$$M = \left[\begin{array}{ccc|c} x_{11} & \cdots & x_{1m} & y_1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nm} & y_n \end{array}\right]$$

where the $x_{ij}$'s and the $y_i$'s are (not necessarily distinct) variables from $\{x_1, \dots, x_k\}$, one can associate the following linear Mal'tsev condition (in the sense of [34]) on a variety of universal algebras $\mathbb{V}$: the algebraic theory of $\mathbb{V}$ admits an $m$-ary term $p$ such that, for each $i \in \{1, \dots, n\}$, the equation

$$p(x_{i1}, \dots, x_{im}) = y_i$$

holds in $\mathbb{V}$. As shown in [25], this Mal'tsev condition is equivalent to the condition that, for each homomorphic $n$-ary relation $R \subseteq A^n$ on an algebra $A$ of $\mathbb{V}$, given any function $f\colon \{x_1, \dots, x_k\} \to A$ interpreting the variables in $A$, the implication

$$\left\{ \begin{bmatrix} f(x_{11}) \\ \vdots \\ f(x_{n1}) \end{bmatrix}, \dots, \begin{bmatrix} f(x_{1m}) \\ \vdots \\ f(x_{nm}) \end{bmatrix} \right\} \subseteq R \Longrightarrow \begin{bmatrix} f(y_1) \\ \vdots \\ f(y_n) \end{bmatrix} \in R$$

holds. While the above linear Mal'tsev condition does not make sense in an arbitrary category, the above condition on relations can be stated in any finitely complete category $\mathbb{C}$ using internal relations and generalized elements. If this condition is satisfied, we say that $\mathbb{C}$ *has $M$-closed relations*.

One of the most famous examples of such a condition is given by the matrix

$$\mathsf{Mal} = \left[\begin{array}{ccc|c} x_1 & x_2 & x_2 & x_1 \\ x_1 & x_1 & x_2 & x_2 \end{array}\right].$$

A finitely complete category has $\mathsf{Mal}$-closed relations if and only if it is a Mal'tsev category [10], i.e., if and only if every binary internal relation is difunctional in the sense of [33], which occurs if and only if every binary reflexive internal relation is an equivalence relation. A variety $\mathbb{V}$ has $\mathsf{Mal}$-closed relations if and only if its theory admits a ternary operation $p$ satisfying the axioms $p(x_1, x_2, x_2) = x_1$ and $p(x_1, x_1, x_2) = x_2$. Such varieties are characterized by the fact that the composition of congruences on any algebra in $\mathbb{V}$ is commutative [28] and are also called 2-permutable varieties.

Another example of matrix condition is given by

$$\mathsf{Ari} = \left[\begin{array}{ccc|c} x_1 & x_2 & x_2 & x_1 \\ x_1 & x_1 & x_2 & x_2 \\ x_1 & x_2 & x_1 & x_1 \end{array}\right]$$

where the notion of a finitely complete category with Ari-closed relations extends to the finitely complete context the notion of an arithmetical category in the sense of [30]. One can also mention

$$\mathsf{Maj} = \left[\begin{array}{ccc|c} x_1 & x_1 & x_2 & x_1 \\ x_1 & x_2 & x_1 & x_1 \\ x_2 & x_1 & x_1 & x_1 \end{array}\right]$$

for which a finitely complete category has Maj-closed relations if and only if it is a majority category in the sense of [13].

The paper [16] describes an algorithm to decide whether one matrix condition implies another one in the finitely complete context, i.e., given two simple extended matrices $M_1$ and $M_2$, whether each finitely complete category with $M_1$-closed relations has $M_2$-closed relations, which we denote by $M_1 \Rightarrow_{\mathsf{lex}} M_2$. We have also shown that this algorithm cannot be used in the varietal context. That is, the statement $M_1 \Rightarrow_{\mathsf{lex}} M_2$ is in general stronger than the statement that any variety with $M_1$-closed relations has $M_2$-closed relations, which we abbreviate as $M_1 \Rightarrow_{\mathsf{alg}} M_2$. Moreover, a general algorithm to decide $M_1 \Rightarrow_{\mathsf{alg}} M_2$ still does not exist. However, the results of [29] can be used to extract an algorithm for some matrices $M_2$, including the Mal'tsev matrix Mal. Surprisingly, in the case $M_2 = \mathsf{Mal}$, this algorithm reduces to the algorithm from [16] for $M_1 \Rightarrow_{\mathsf{lex}} \mathsf{Mal}$. This thus means that $M_1 \Rightarrow_{\mathsf{lex}} \mathsf{Mal}$ is equivalent to $M_1 \Rightarrow_{\mathsf{alg}} \mathsf{Mal}$, which is quite particular to the Mal'tsev matrix Mal. In that case, the algorithm to decide whether $M_1 \Rightarrow_{\mathsf{lex}} \mathsf{Mal}$ reduces to find two (not necessarily distinct) rows of $M_1$ such that, when reducing $M_1$ to those two rows, its right column cannot be found among its left columns. The number of operations required by this algorithm is bounded by a polynomial in the numbers of rows and of columns of the matrix $M_1$. In addition, using this algorithm and the results of [16], we can show that, given a finite number of simple extended matrices

$M_1, \ldots, M_d$, if each finitely complete category with $M_i$-closed relations for all $i \in \{1, \ldots, d\}$ is a Mal'tsev category, then there exists $i \in \{1, \ldots, d\}$ such that $M_i \Rightarrow_{\mathsf{lex}} \mathsf{Mal}$.

The linear Mal'tsev conditions arising from simple extended matrices only have equations of the form

$$p(x_1, \ldots, x_m) = y$$

but not of the form

$$p(x_1, \ldots, x_m) = p'(x'_1, \ldots, x'_{m'})$$

for (not necessarily distinct) variables $x_1, \ldots, x_m, x'_1, \ldots, x'_{m'}$ and $y$. In order to take this second kind of equation into account, one needs to consider (not necessarily simple) *extended matrices of variables*

$$M = \left[ \begin{array}{ccc|ccc} x_{11} & \cdots & x_{1m} & y_{11} & \cdots & y_{1m'} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n1} & \cdots & x_{nm} & y_{n1} & \cdots & y_{nm'} \end{array} \right]$$

as introduced in [27], where the $x_{ij}$'s are variables from $\{x_1, \ldots, x_\ell\}$ and the $y_{ij}$'s are variables from $\{x_1, \ldots, x_\ell, \ldots, x_k\}$ (where $k \geqslant \ell$). The linear Mal'tsev condition on a variety $\mathbb{V}$ associated to such an $M$ is: the algebraic theory of $\mathbb{V}$ contains $m$-ary terms $p_1, \ldots, p_{m'}$ and $\ell$-ary terms $q_1, \ldots, q_{k-\ell}$ such that, for each $i \in \{1, \ldots, n\}$ and each $j \in \{1, \ldots, m'\}$,

$$p_j(x_{i1}, \ldots, x_{im}) = \begin{cases} x_a & \text{if } y_{ij} = x_a \in \{x_1, \ldots, x_\ell\} \\ q_{a-\ell}(x_1, \ldots, x_\ell) & \text{if } y_{ij} = x_a \in \{x_{\ell+1}, \ldots, x_k\} \end{cases}$$

is an equation in the variables $x_1, \ldots, x_\ell$ that holds in the algebraic theory of $\mathbb{V}$. As shown in [27], this is equivalent to the condition that, for any homomorphic $n$-ary relation $R \subseteq A^n$ on an algebra $A$ of $\mathbb{V}$, given any function $f \colon \{x_1, \ldots, x_\ell\} \to A$, the implication

$$\left\{ \begin{bmatrix} f(x_{11}) \\ \vdots \\ f(x_{n1}) \end{bmatrix}, \ldots, \begin{bmatrix} f(x_{1m}) \\ \vdots \\ f(x_{nm}) \end{bmatrix} \right\} \subseteq R$$

$$\Longrightarrow \begin{array}{l} \exists\, g \colon \{x_1, \ldots, x_k\} \to A \\ \text{extension of } f \end{array} \mid \left\{ \begin{bmatrix} g(y_{11}) \\ \vdots \\ g(y_{n1}) \end{bmatrix}, \ldots, \begin{bmatrix} g(y_{1m'}) \\ \vdots \\ g(y_{nm'}) \end{bmatrix} \right\} \subseteq R$$

holds. In view of the existential quantifier in the above formula, a natural categorical context in which to extend this condition is the context of regular categories in the sense of [4]. In addition to the examples of matrix properties mentioned above, one has now also the example of $n$-permutable categories [8]. The exactness properties on a regular category being expressible by finite conjunctions of such matrix conditions have been semantically characterized in [23].

Given two such extended matrices $M_1$ and $M_2$, a general algorithm to decide whether each regular category with $M_1$-closed relations has $M_2$-closed relations, denoted as $M_1 \Rightarrow_{\mathsf{reg}} M_2$, is yet to be found. However, using the embedding theorems from [18, 21], the statement $M_1 \Rightarrow_{\mathsf{reg}} M_2$ is equivalent (assuming the *axiom of universes* [3]) to the statement, denoted by $M_1 \Rightarrow_{\mathsf{reg\,ess\,alg}} M_2$, that any regular essentially algebraic category (in the sense of [1, 2]) with $M_1$-closed relations has $M_2$-closed relations. Using this equivalence and the results from [29], we could prove that, when $M_2 = \mathsf{Mal}$, the statement $M_1 \Rightarrow_{\mathsf{reg}} \mathsf{Mal}$ is equivalent to the statement $M_1 \Rightarrow_{\mathsf{alg}} \mathsf{Mal}$.

Our two main theorems, the first one stating the equivalence of $M_1 \Rightarrow_{\mathsf{lex}} \mathsf{Mal}$ and $M_1 \Rightarrow_{\mathsf{alg}} \mathsf{Mal}$ for a simple extended matrix $M_1$ and the second one stating the equivalence of $M_1 \Rightarrow_{\mathsf{reg}} \mathsf{Mal}$ and $M_1 \Rightarrow_{\mathsf{alg}} \mathsf{Mal}$ for a (general) extended matrix $M_1$ are quite surprising and particular to the Mal'tsev case. Indeed, as it is the general philosophy of the papers [18, 19, 20, 21, 22, 23, 24], to prove the validity of many statements about exactness properties, one is often required to produce a proof in the essentially algebraic context (and not just in the varietal context as it is the case in the present situation). Actually, we prove these two theorems not only for the Mal'tsev matrix $\mathsf{Mal}$, but for the matrix $\mathsf{Cube}_n$ for each $n \geqslant 2$, describing the Mal'tsev condition of having an $n$-cube term [5]. The Mal'tsev case is then recovered in the case $n = 2$.

Let us stress here the fact that our results are proved in the context of 'non-pointed' matrices, i.e., each entry in our matrices is a variable. This is in contrast with, e.g., [15] where entries can also be the constant symbol $*$ representing the zero morphisms in a pointed category.

This paper is organized as follows. In Section 1, we recall the necessary material from other papers. In particular, we explain the theory of matrix conditions, in the finitely complete, regular and varietal contexts. We also recall the algorithm from [16] to decide for an implication $M_1 \Rightarrow_{\mathsf{lex}} M_2$ in the finitely complete context and conclude the section with a reminder on essentially algebraic theories. Section 2 contains the main new results of the paper and is divided in two parts. In the first one, we prove Theorem 2.4 which states that given a simple extended matrix $M$ and an integer $n \geqslant 2$, the statement $M \Rightarrow_{\mathsf{lex}} \mathsf{Cube}_n$ is equivalent to $M \Rightarrow_{\mathsf{alg}} \mathsf{Cube}_n$. We also obtain an easy algorithm to decide when these conditions hold. From this algorithm, we deduce (Theorem 2.7) that a finite conjunction of conditions induced by simple extended matrices implies the Mal'tsev property if and only if one of these matrix conditions alone already implies the Mal'tsev property. The second part of Section 2 deals with (general) extended matrices and we prove that for such a matrix $M$ and an integer $n \geqslant 2$, the statement $M \Rightarrow_{\mathsf{reg}} \mathsf{Cube}_n$ is equivalent to $M \Rightarrow_{\mathsf{alg}} \mathsf{Cube}_n$ (see Theorem 2.8).

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## 1. Preliminaries

By a *variety*, we mean a one-sorted finitary variety of universal algebras. By a *regular* category, we mean a regular category in the sense of [4], i.e., a finitely complete category with coequalizers of kernel pairs and pullback stable regular epimorphisms. Regular categories have been introduced as a context where finite limits and regular epimorphisms behave in a similar way as finite limits and surjections behave in the category of sets. In particular, every variety is a regular category. By a *pointed* category, we mean a category with a *zero object*, i.e., an object which is both terminal and initial. A variety is pointed if and only if its algebraic theory contains a unique constant term.

Let us fix throughout this paper an infinite sequence $x_1, x_2, x_3, \dots$ of pairwise distinct variables.

## Matrix conditions

Let us start by recalling the theory of matrix conditions on finitely complete and regular categories as introduced in [25, 26, 27]. We only treat 'non-pointed' matrices in this paper, in contrast with [15]. An *extended matrix* $M$ *of variables* (or simply an *extended matrix* for short) is given by integer parameters $n \geqslant 1$, $m \geqslant 0$, $m' \geqslant 0$ and $k \geqslant \ell \geqslant 0$ and by a $n \times (m + m')$ matrix

$$\left[\begin{array}{ccc|ccc} x_{11} & \cdots & x_{1m} & y_{11} & \cdots & y_{1m'} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n1} & \cdots & x_{nm} & y_{n1} & \cdots & y_{nm'} \end{array}\right] \tag{1}$$

where the $x_{ij}$'s are (not necessarily distinct) variables from $\{x_1, \dots, x_\ell\}$ and the $y_{ij}$'s are (not necessarily distinct) variables from $\{x_1, \dots, x_\ell, \dots, x_k\}$. When the parameters $n, m, m', \ell, k$ are clear from the context, we will omit them and we will represent an extended matrix $M$ just by its matrix part; this will be the case when the conditions $m + m' > 0$,

$$\{x_{ij} \,|\, i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\} = \{x_1, \dots, x_\ell\}$$

and

$$\{x_1, \dots, x_\ell\} \cup \{y_{ij} \,|\, i \in \{1, \dots, n\}, j \in \{1, \dots, m'\}\} = \{x_1, \dots, x_k\}$$

are all satisfied. The first $m$ columns of $M$ will be called its *left columns*, while its last $m'$ columns will be called its *right columns*. Given an object $A$ in a finitely complete category $\mathbb{C}$, each variable $x$ in $\{x_1, \dots, x_\ell\}$ gives rise to the corresponding projection $x^A \colon A^\ell \to A$ from the $\ell$-th power of $A$ (and similarly, each variable $x$ in $\{x_1, \dots, x_k\}$ gives rise to the corresponding projection $x^A \colon A^k \to A$). Given such an extended matrix $M$, an $n$-ary internal relation $r \colon R \rightarrowtail A^n$ in a regular category $\mathbb{C}$ is said to be $M$-*closed* if, when we consider the pullbacks

$$\begin{array}{ccc} P & \xrightarrow{\quad f' \quad} & R^m \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle r^m} \\ A^\ell & \xrightarrow[\left[\begin{smallmatrix} x^A_{11} & \cdots & x^A_{1m}\\ \vdots & & \vdots \\ x^A_{n1} & \cdots & x^A_{nm}\end{smallmatrix}\right]]{\qquad} & (A^n)^m \end{array} \qquad \begin{array}{ccc} Q & \xrightarrow{\quad g' \quad} & R^{m'} \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle r^{m'}} \\ A^k & \xrightarrow[\left[\begin{smallmatrix} y^A_{11} & \cdots & y^A_{1m'}\\ \vdots & & \vdots \\ y^A_{n1} & \cdots & y^A_{nm'}\end{smallmatrix}\right]]{\qquad} & (A^n)^{m'} \end{array}$$

and

$$\begin{array}{ccc} T & \xrightarrow{\quad h' \quad} & Q \\ & \lrcorner & \downarrow{\scriptstyle g} \\ {\scriptstyle h}\downarrow & & A^k \cong A^\ell \times A^{k-\ell} \\ & & \downarrow{\scriptstyle \pi_1 = (x_1^A, \ldots, x_\ell^A)} \\ P & \xrightarrow[\quad f \quad]{} & A^\ell \end{array}$$

then $h$ is a regular epimorphism (or, in other words, $f$ factors through the image of $\pi_1 g$). We say that the regular category $\mathbb{C}$ *has $M$-closed relations* if any internal $n$-ary relation $r\colon R \rightarrowtail A^n$ in $\mathbb{C}$ is $M$-closed. If $\mathbb{C} = \mathbb{V}$ is a variety, an internal relation is a homomorphic relation. An $n$-ary homomorphic relation $R \subseteq A^n$ on an algebra $A$ is $M$-closed when, for each function $f\colon \{x_1, \ldots, x_\ell\} \to A$ such that

$$\left\{ \begin{bmatrix} f(x_{11}) \\ \vdots \\ f(x_{n1}) \end{bmatrix}, \ldots, \begin{bmatrix} f(x_{1m}) \\ \vdots \\ f(x_{nm}) \end{bmatrix} \right\} \subseteq R,$$

there exists an extension $g\colon \{x_1, \ldots, x_k\} \to A$ of $f$ (i.e., $g(x_i) = f(x_i)$ for each $i \in \{1, \ldots, \ell\}$) such that

$$\left\{ \begin{bmatrix} g(y_{11}) \\ \vdots \\ g(y_{n1}) \end{bmatrix}, \ldots, \begin{bmatrix} g(y_{1m'}) \\ \vdots \\ g(y_{nm'}) \end{bmatrix} \right\} \subseteq R.$$

This description can be used to prove the following theorem characterizing varieties with $M$-closed relations via a linear Mal'tsev condition.

**Theorem 1.1** ([27]). *Let $M$ be an extended matrix as in (1). A variety $\mathbb{V}$ has $M$-closed relations if and only if the algebraic theory of $\mathbb{V}$ contains $m$-ary terms $p_1, \dots, p_{m'}$ and $\ell$-ary terms $q_1, \dots, q_{k-\ell}$ such that, for each $i \in \{1, \dots, n\}$ and each $j \in \{1, \dots, m'\}$,*

$$p_j(x_{i1}, \dots, x_{im}) = \begin{cases} x_a & \text{if } y_{ij} = x_a \in \{x_1, \dots, x_\ell\} \\ q_{a-\ell}(x_1, \dots, x_\ell) & \text{if } y_{ij} = x_a \in \{x_{\ell+1}, \dots, x_k\} \end{cases}$$

*is a theorem of the algebraic theory of $\mathbb{V}$ in the variables $x_1, \dots, x_\ell$.*

For such a matrix $M$, we will denote by $\mathbb{V}_M$ the variety whose basic operations are the $m$-ary terms $p_1, \dots, p_{m'}$ and the $\ell$-ary terms $q_1, \dots, q_{k-\ell}$ and whose axioms are the theorems described in Theorem 1.1. Obviously, $\mathbb{V}_M$ has $M$-closed relations.

## Simple matrix conditions

An extended matrix $M$ as above will be said to be *simple* when $k = \ell$ and $m' = 1$. We can display such a matrix $M$ as

$$\left[\begin{array}{ccc|c} x_{11} & \cdots & x_{1m} & y_1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nm} & y_n \end{array}\right] \tag{2}$$

where the $x_{ij}$'s and the $y_i$'s are variables from $\{x_1, \dots, x_k\}$. In that case, the notion of $n$-ary $M$-closed relations can be extended to the finitely complete context as follows. An $n$-ary internal relation $r\colon R \rightarrowtail A^n$ in a finitely complete category $\mathbb{C}$ is said to be *$M$-closed* when, given any object $B$ and any function $f\colon \{x_1, \dots, x_k\} \to \mathbb{C}(B, A)$ such that the induced morphism

$$\begin{bmatrix} f(x_{1j}) \\ \vdots \\ f(x_{nj}) \end{bmatrix} : B \to A^n$$

factor through $r$ for each $j \in \{1, \ldots, m\}$, then so does the morphism

$$\begin{bmatrix} f(y_1) \\ \vdots \\ f(y_n) \end{bmatrix} : B \to A^n.$$

For a simple extended matrix $M$, we say that the finitely complete category $\mathbb{C}$ has *$M$-closed relations* when each internal $n$-ary relation $r\colon R \rightarrowtail A^n$ is $M$-closed. If $\mathbb{C} = \mathbb{V}$ is a variety, a homomorphic relation $R \subseteq A^n$ is $M$-closed when, for each function $f\colon \{x_1, \ldots, x_k\} \to A$, the implication

$$\left\{ \begin{bmatrix} f(x_{11}) \\ \vdots \\ f(x_{n1}) \end{bmatrix}, \ldots, \begin{bmatrix} f(x_{1m}) \\ \vdots \\ f(x_{nm}) \end{bmatrix} \right\} \subseteq R \Longrightarrow \begin{bmatrix} f(y_1) \\ \vdots \\ f(y_n) \end{bmatrix} \in R$$

holds. Particularizing Theorem 1.1 to this simpler situation, one gets the following.

**Theorem 1.2** ([25]). *Let $M$ be a simple extended matrix as in (2). A variety $\mathbb{V}$ has $M$-closed relations if and only if the algebraic theory of $\mathbb{V}$ contains an $m$-ary term $p$ such that, for each $i \in \{1, \ldots, n\}$,*

$$p(x_{i1}, \ldots, x_{im}) = y_i$$

*is a theorem of the algebraic theory of $\mathbb{V}$ in the variables $x_1, \ldots, x_k$.*

Before describing some examples of matrix conditions, let us introduce some notation. We denote by $\mathsf{lex}$ (respectively by $\mathsf{lex}_*$, $\mathsf{reg}$, $\mathsf{reg}_*$, $\mathsf{alg}$ and $\mathsf{alg}_*$) the collection of finitely complete categories (respectively of finitely complete pointed categories, regular categories, regular pointed categories, varieties and pointed varieties). The notation $\mathsf{lex}$ abbreviates 'left exact categories' which is another name for finitely complete categories. Given two extended matrices $M_1$ and $M_2$ and a sub-collection $\mathcal{C}$ of $\mathsf{reg}$ (respectively, two simple extended matrices $M_1$ and $M_2$ and a sub-collection $\mathcal{C}$ of $\mathsf{lex}$), we write $M_1 \Rightarrow_{\mathcal{C}} M_2$ to mean that any category in $\mathcal{C}$ with $M_1$-closed relations has $M_2$-closed relations. We write $M_1 \Leftrightarrow_{\mathcal{C}} M_2$ for the conjunction of the statements $M_1 \Rightarrow_{\mathcal{C}} M_2$ and $M_2 \Rightarrow_{\mathcal{C}} M_1$. We also write $M_1 \not\Rightarrow_{\mathcal{C}} M_2$ for the negation of the statement $M_1 \Rightarrow_{\mathcal{C}} M_2$.

## Examples

**Example 1.3.** Let $\mathsf{Mal}$ be the simple extended matrix given by

$$\mathsf{Mal} = \left[\begin{array}{ccc|c} x_1 & x_2 & x_2 & x_1 \\ x_1 & x_1 & x_2 & x_2 \end{array}\right].$$

A finitely complete category has $\mathsf{Mal}$-closed relations if and only if it is a Mal'tsev category as introduced in [10] (and in [9] in the regular context). A variety has $\mathsf{Mal}$-closed relations if and only if its theory admits a Mal'tsev term, i.e., if and only if it is 2-permutable [28]. We refer the reader to [6, 7] for surveys on Mal'tsev categories.

**Example 1.4.** More generally, for any $r \geqslant 2$, let $\mathsf{Perm}_r$ be the extended matrix given by

$$\mathsf{Perm}_r = \left[\begin{array}{ccc|ccccc} x_1 & x_2 & x_2 & x_1 & x_3 & x_4 & \cdots & x_r \\ x_1 & x_1 & x_2 & x_3 & x_4 & \cdots & x_r & x_2 \end{array}\right].$$

A regular category has $\mathsf{Perm}_r$-closed relations if and only if it is an $r$-permutable category as introduced in [8], generalizing the notion of an $r$-permutable variety.

**Example 1.5.** Let $\mathsf{Ari}$ be the simple extended matrix given by

$$\mathsf{Ari} = \left[\begin{array}{ccc|c} x_1 & x_2 & x_2 & x_1 \\ x_1 & x_1 & x_2 & x_2 \\ x_1 & x_2 & x_1 & x_1 \end{array}\right].$$

The notion of a finitely complete category with $\mathsf{Ari}$-closed relations extends to the finitely complete context the notions of an arithmetical category in the sense of [30] and of an equivalence distributive Mal'tsev category in the sense of [12]. A variety $\mathbb{V}$ has $\mathsf{Ari}$-closed relations if and only if its theory admits a Pixley term [32].

**Example 1.6.** Let $\mathsf{Maj}$ be the simple extended matrix given by

$$\mathsf{Maj} = \left[\begin{array}{ccc|c} x_1 & x_1 & x_2 & x_1 \\ x_1 & x_2 & x_1 & x_1 \\ x_2 & x_1 & x_1 & x_1 \end{array}\right].$$

A finitely complete category has Maj-closed relations if and only if it is a majority category as introduced in [13, 14]. A variety $\mathbb{V}$ is a majority category if and only if its theory admits a majority term [31].

**Example 1.7.** For any $n, k \geqslant 2$, let $\mathsf{Cube}_{n,k}$ be the simple extended matrix with $n$ rows, $k^n - 1$ left columns, one right column and $k = \ell$ variables defined by taking as left columns (ordered lexicographically) all possible $n$-tuples of elements of $\{x_1, \ldots, x_k\}$ except $(x_1, \ldots, x_1)$, which is used as right column. As we shall see below (see Corollary 2.3), for any $n, k_1, k_2 \geqslant 2$, one has $\mathsf{Cube}_{n,k_1} \Leftrightarrow_{\mathsf{lex}} \mathsf{Cube}_{n,k_2}$. In view of this, we abbreviate $\mathsf{Cube}_{n,2}$ by $\mathsf{Cube}_n$. Up to permutation of left columns and change of variables, $\mathsf{Cube}_2$ is the matrix Mal from Example 1.3, and therefore, $\mathsf{Cube}_2 \Leftrightarrow_{\mathsf{lex}} \mathsf{Mal}$. The matrix $\mathsf{Cube}_3$ is the matrix

$$\mathsf{Cube}_3 = \left[\begin{array}{ccccccc|c} x_1 & x_1 & x_1 & x_2 & x_2 & x_2 & x_2 & x_1 \\ x_1 & x_2 & x_2 & x_1 & x_1 & x_2 & x_2 & x_1 \\ x_2 & x_1 & x_2 & x_1 & x_2 & x_1 & x_2 & x_1 \end{array}\right].$$

For $n \geqslant 2$, a variety has $\mathsf{Cube}_n$-closed relations if and only if its theory admits an $n$-cube term in the sense of [5].

**Example 1.8.** For any $n \geqslant 2$, let $\mathsf{Edge}_n$ be the simple extended matrix with $n$ rows, $n+1$ left columns, one right column and $k = \ell = 2$ variables defined by

$$\mathsf{Edge}_n = \left[\begin{array}{cccccc|c} x_2 & x_2 & x_1 & x_1 & \cdots & x_1 & x_1 \\ x_2 & x_1 & x_2 & x_1 & \cdots & x_1 & x_1 \\ x_1 & x_1 & x_1 & x_2 & \cdots & x_1 & x_1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1 & x_1 & x_1 & x_1 & \cdots & x_2 & x_1 \end{array}\right]$$

where the entries in positions $(1,1)$, $(2,1)$ and $(i, i+1)$ for all $i \in \{1, \ldots, n\}$ are $x_2$'s and the other ones are $x_1$'s. Up to permutation of left columns and change of variables, $\mathsf{Edge}_2$ is the matrix Mal, and thus $\mathsf{Edge}_2 \Leftrightarrow_{\mathsf{lex}} \mathsf{Mal}$. For a general $n \geqslant 2$, a variety has $\mathsf{Edge}_n$-closed relations if and only if its theory admits an $n$-edge term in the sense of [5]. Therefore, as it is shown in [5], one has $\mathsf{Cube}_n \Leftrightarrow_{\mathsf{alg}} \mathsf{Edge}_n$. Using Proposition 1.7 of [26], we know that $\mathsf{Edge}_n \Rightarrow_{\mathsf{lex}} \mathsf{Cube}_n$. However, for a general $n$, the converse is not true. For instance, from the computer-based results of [16], we can see that

$\mathsf{Cube}_3 \not\Rightarrow_{\mathsf{lex}} \mathsf{Edge}_3$. This is an additional example of the context dependency of the implications between matrix properties.

## The algorithm in the finitely complete context

Given two extended matrices $M_1$ and $M_2$, as far as we know, algorithms to decide whether $M_1 \Rightarrow_{\mathsf{reg}} M_2$ or whether $M_1 \Rightarrow_{\mathsf{alg}} M_2$ do not exist yet. On the contrary, in [16], an algorithm to decide whether $M_1 \Rightarrow_{\mathsf{lex}} M_2$ for two simple extended matrices $M_1$ and $M_2$ has been developed. Since we will need it further, let us recall it here.

A simple extended matrix $M$ is called *trivial* if any finitely complete category with $M$-closed relations is a *preorder* (i.e., a category with whose hom-sets contain at most one morphism). As examples, one can cite

$$\mathrm{T}_1 = \left[\begin{array}{c|c} x_2 & x_1 \end{array}\right] \quad \text{and} \quad \mathrm{T}_2 = \left[\begin{array}{cc|c} x_2 & x_1 & x_1 \\ x_1 & x_2 & x_1 \end{array}\right]$$

for which a finitely complete category has $\mathrm{T}_1$-closed relations if and only if it has $\mathrm{T}_2$-closed relations, which occurs if and only if it is a preorder. In addition, we have the example with zero left columns

$$\mathrm{T}_0 = \left[\begin{array}{c|c} & x_1 \end{array}\right]$$

for which a finitely complete category has $\mathrm{T}_0$-closed relations if and only if each hom-set contains exactly one morphism, i.e., if and only if it is equivalent to the terminal category.

In order to state the characterization of trivial matrices from [16], we need the following notation, using the presentation of $M$ as in (2). Given $i \in \{1, \dots, n\}$, $R_{M_i}$ denotes the equivalence relation on the set $\{1, \dots, m\}$ defined by $j_1 R_{M_i} j_2$ if and only if $x_{ij_1} = x_{ij_2}$. Given two equivalence relations $R$ and $S$ on the same set, $R \vee S$ denotes the smallest equivalence relation containing both $R$ and $S$. Finally, we denote by $\mathbf{Set}^{\mathsf{op}}$ the dual of the category $\mathbf{Set}$ of sets.

**Theorem 1.9** ([16]). *For a simple extended matrix $M$ as in (2), the following conditions are equivalent:*

*(a) $M$ is* not *a trivial matrix.*

*(b)* $\mathbf{Set}^{\mathsf{op}}$ *has $M$-closed relations.*

*(c) For all $i, i' \in \{1, \dots, n\}$, there exist $j, j' \in \{1, \dots, m\}$ such that $x_{ij} = y_i$, $x_{i'j'} = y_{i'}$ and $jR_{M_i} \vee R_{M_{i'}} j'$.*

For instance, one can see that $\mathrm{T}_2$ from above is indeed a trivial matrix by taking $i = 1$ and $i' = 2$. In that case, both $R_{M_i}$ and $R_{M_{i'}}$ are the discrete equivalence relation on $\{1, 2\}$ and there exist no $j, j' \in \{1, 2\}$ such that $x_{1j} = x_1$, $x_{2j'} = x_1$ and $jR_{M_1} \vee R_{M_2} j'$ (i.e. $j = j'$). On the other hand, one can see that the matrix $\mathsf{Mal}$ from Example 1.3 is not trivial by case analysis on all $i, i' \in \{1, 2\}$. If $i = 1$ and $i' = 2$, one can take $j = 1$ and $j' = 3$ since $x_{11} = x_1 = y_1$, $x_{23} = x_2 = y_2$ and $1R_{M_2}2R_{M_1}3$. Symmetrically, if $i = 2$ and $i' = 1$, one can take $j = 3$ and $j' = 1$; while if $i = i'$, the condition simply means that the right entry of the $i$-th row can be found in the left entries of that row.

Let $M_1$ and $M_2$ be two simple extended matrices with parameters $n_1$, $m_1, m'_1 = 1, k_1 = \ell_1$ and $n_2, m_2, m'_2 = 1, k_2 = \ell_2$. We know that if $m_1 = 0$, then $M_1$ is trivial, we always have $M_1 \Rightarrow_{\mathsf{lex}} M_2$ and we have $M_2 \Rightarrow_{\mathsf{lex}} M_1$ if and only if $m_2 = 0$. If $m_1 > 0$ and $M_1$ is trivial, then we have $M_1 \Rightarrow_{\mathsf{lex}} M_2$ if and only if $m_2 > 0$ and we have $M_2 \Rightarrow_{\mathsf{lex}} M_1$ if and only if $M_2$ is trivial. It thus remains to explain, in the case where neither $M_1$ nor $M_2$ is trivial, how to decide whether $M_1 \Rightarrow_{\mathsf{lex}} M_2$. In order to describe this algorithm, we need the following notion. Given $i \in \{1, \dots, n_1\}$ and a set $S$, an *interpretation of type $S$* of the $i$-th row of $M_1$ is an $(m_1 + 1)$-tuple

$$\left[\begin{array}{ccc|c} f(x^1_{i1}) & \dots & f(x^1_{im_1}) & f(y^1_i) \end{array}\right]$$

formed by applying a function $f \colon \{x_1, \dots, x_{k_1}\} \to S$ to the entries of the $i$-th row of $M_1$. Now, if neither $M_1$ nor $M_2$ is a trivial matrix, the algorithm from [16] to decide whether $M_1 \Rightarrow_{\mathsf{lex}} M_2$ is the following:

> Keep expanding the set of left columns of $M_2$, until it is no more possible, with right columns of $n_2 \times (m_1 + 1)$ matrices
>
> $$\left[\begin{array}{ccc|c} f_{i_1}(x^1_{i_1 1}) & \dots & f_{i_1}(x^1_{i_1 m_1}) & f_{i_1}(y^1_{i_1}) \\ \vdots & & \vdots & \vdots \\ f_{i_{n_2}}(x^1_{i_{n_2} 1}) & \dots & f_{i_{n_2}}(x^1_{i_{n_2} m_1}) & f_{i_{n_2}}(y^1_{i_{n_2}}) \end{array}\right]$$

for which each row is an interpretation of type $\{x_1, \ldots, x_{k_2}\}$ of a row of $M_1$, each of the first $m_1$ left columns is in the expansion of the set of left columns of $M_2$ but the right column is not. Then $M_1 \Rightarrow_{\mathsf{lex}} M_2$ holds if and only if the right column of $M_2$ is contained in the left columns of the expanded matrix $M_2$.

To illustrate this algorithm, let us consider the matrices Mal and Cube$_3$ of Examples 1.3 and 1.7. They can be shown to be non-trivial via Theorem 1.9. One can show that

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_2 & x_1 \\ x_1 & x_1 & x_2 & x_2 \end{array}\right] \Rightarrow_{\mathsf{lex}} \left[\begin{array}{ccccccc|c} x_1 & x_1 & x_1 & x_2 & x_2 & x_2 & x_2 & x_1 \\ x_1 & x_2 & x_2 & x_1 & x_1 & x_2 & x_2 & x_1 \\ x_2 & x_1 & x_2 & x_1 & x_2 & x_1 & x_2 & x_1 \end{array}\right]$$

i.e., that Mal $\Rightarrow_{\mathsf{lex}}$ Cube$_3$ in one step by considering the matrix

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_2 & x_1 \\ x_2 & x_2 & x_1 & x_1 \\ x_1 & x_1 & x_1 & x_1 \end{array}\right]$$

where the first row is the first row of Mal, the second row is an interpretation of the second row of Mal where $x_1$ is interpreted as $x_2$ and vice-versa and the third row is the first (or second) row of Mal where both variables are interpreted as $x_1$. Since the left columns of that matrix are left columns of Cube$_3$ and its right column is the right column of Cube$_3$, this shows the announced implication.

As a two-step example, one can cite the implication

$$M_1 = \left[\begin{array}{cccc|c} x_1 & x_1 & x_2 & x_2 & x_1 \\ x_1 & x_2 & x_1 & x_2 & x_1 \\ x_2 & x_3 & x_3 & x_1 & x_1 \end{array}\right] \Rightarrow_{\mathsf{lex}} M_2 = \left[\begin{array}{ccccc|c} x_2 & x_2 & x_1 & x_3 & x_3 & x_1 \\ x_1 & x_1 & x_2 & x_2 & x_1 & x_1 \\ x_3 & x_1 & x_2 & x_3 & x_2 & x_1 \end{array}\right]$$

appearing in [16]. These matrices are non-trivial by Theorem 1.9. To show the implication, one first considers the matrix

$$\left[\begin{array}{cccc|c} x_3 & x_3 & x_2 & x_2 & x_3 \\ x_2 & x_2 & x_1 & x_1 & x_2 \\ x_3 & x_3 & x_3 & x_1 & x_1 \end{array}\right]$$

where the first two rows are interpretations of the first row of $M_1$ ($x_1$ is interpreted respectively as $x_3$ in the first row and as $x_2$ in the second row, and $x_2$ is interpreted respectively as $x_2$ in the first row and as $x_1$ in the second row), and the third row is an interpretation of the third row of $M_1$ ($x_1$ is interpreted as $x_1$ and both $x_2$ and $x_3$ are interpreted as $x_3$). The left columns of that matrix are all left columns of $M_2$, so that now, we can expand $M_2$ as

$$\left[\begin{array}{cccccc|c} x_2 & x_2 & x_1 & x_3 & x_3 & x_3 & x_1 \\ x_1 & x_1 & x_2 & x_2 & x_1 & x_2 & x_1 \\ x_3 & x_1 & x_2 & x_3 & x_2 & x_1 & x_1 \end{array}\right].$$

One now concludes the proof of the implication by considering the matrix

$$\left[\begin{array}{cccc|c} x_2 & x_3 & x_3 & x_1 & x_1 \\ x_1 & x_1 & x_2 & x_2 & x_1 \\ x_1 & x_2 & x_1 & x_2 & x_1 \end{array}\right]$$

where the first row is the third row of $M_1$, the second row is the first row of $M_1$ and the third row is the second row of $M_1$. Notice that the left columns of that matrix are left columns of the expansion of $M_2$ above and the right column is the right column of $M_2$.

## Essentially algebraic categories

Let us conclude this preliminary section with a reminder on (many-sorted) essentially algebraic categories [1, 2] (or in other words locally presentable categories [11]) as we will need them to prove Theorem 2.8. They are described by *essentially algebraic theories*, i.e., quintuples

$$\Gamma = (S, \Sigma, E, \Sigma_t, \mathsf{Def})$$

where

- $S$ is a set of sorts;
- $\Sigma$ is an $S$-sorted signature of algebras, i.e., a set of operation symbols $\sigma$ with prescribed arity $\sigma\colon \prod_{u\in U} s_u \to s$ where $U$ is a set, $s_u \in S$ for each $u \in U$ and $s \in S$;

- $E$ is a set of $\Sigma$-equations;
- $\Sigma_t$ is a subset of $\Sigma$ called the set of *total operation symbols*;
- Def is a function assigning to each operation symbol $\sigma\colon \prod_{u\in U} s_u \to s$ in $\Sigma \setminus \Sigma_t$ a set $\mathsf{Def}(\sigma)$ of $\Sigma_t$-equations $t_1 = t_2$ where $t_1$ and $t_2$ are $\Sigma_t$-terms $\prod_{u\in U} s_u \to s'$.

A $\Gamma$-*model* is an $S$-sorted set $A = (A_s)_{s\in S}$ together with, for each operation symbol $\sigma\colon \prod_{u\in U} s_u \to s$ in $\Sigma$, a partial function $\sigma^A\colon \prod_{u\in U} A_{s_u} \to A_s$ such that:

1. for each $\sigma \in \Sigma_t$, $\sigma^A$ is totally defined;
2. given $\sigma\colon \prod_{u\in U} s_u \to s$ in $\Sigma \setminus \Sigma_t$ and a family $(a_u \in A_{s_u})_{u\in U}$ of elements, $\sigma^A((a_u)_{u\in U})$ is defined if and only if the identity
$$t_1^A((a_u)_{u\in U}) = t_2^A((a_u)_{u\in U})$$
holds for each $\Sigma_t$-equation of $\mathsf{Def}(\sigma)$;
3. $A$ satisfies the equations of $E$ wherever they are defined.

A $\Sigma$-term $t\colon \prod_{u\in U} s_u \to s$ will be said to be *everywhere-defined* if, for each $\Gamma$-model $A$, the induced function $t^A\colon \prod_{u\in U} A_{s_u} \to A_s$ is totally defined (see [18, 19] for more details). A *homomorphism* $f\colon A \to B$ of $\Gamma$-models is an $S$-sorted function $(f_s\colon A_s \to B_s)_{s\in S}$ such that, given $\sigma\colon \prod_{u\in U} s_u \to s$ in $\Sigma$ and a family $(a_u \in A_{s_u})_{u\in U}$ such that $\sigma^A((a_u)_{u\in U})$ is defined in $A$, then $\sigma^B((f_{s_u}(a_u))_{u\in U})$ is defined in $B$ and the identity
$$f_s(\sigma^A((a_u)_{u\in U})) = \sigma^B((f_{s_u}(a_u))_{u\in U})$$
holds. The $\Gamma$-models and their homomorphisms form the category $\mathsf{Mod}(\Gamma)$. A category which is equivalent to a category $\mathsf{Mod}(\Gamma)$ for some essentially algebraic theory $\Gamma$ is called *essentially algebraic*. These are exactly the locally presentable categories. Note that essentially algebraic categories are in general not regular but have a (strong epimorphism, monomorphism)-factorization system. Each variety is a regular essentially algebraic category. The category $\mathbf{Cat}$ of small categories is a non-regular essentially algebraic category. We denote by $\mathsf{ess\,alg}$ (respectively $\mathsf{ess\,alg}_*$, $\mathsf{reg\,ess\,alg}$ and

$\mathsf{reg\,ess\,alg}_*$) the collection of essentially algebraic categories (respectively of essentially algebraic pointed categories, regular essentially algebraic categories and of regular essentially algebraic pointed categories). Therefore, for extended matrices $M_1$ and $M_2$, the notation $M_1 \Rightarrow_{\mathsf{reg\,ess\,alg}} M_2$ means that any regular essentially algebraic category with $M_1$-closed relations has $M_2$-closed relations.

# 2. Main results

## Results for simple matrices

We know from Theorem 4.2 in [15] that, given two simple extended matrices $M_1$ and $M_2$ where $M_2$ has at least one left column, then $M_1 \Rightarrow_{\mathsf{lex}} M_2$ holds if and only if $M_1 \Rightarrow_{\mathsf{lex}_*} M_2$ holds. Let us now prove the analogous result for varieties.

**Proposition 2.1.** *Let $M_1$ and $M_2$ be two simple extended matrices such that $M_2$ has at least one left column. Then, the following statements are equivalent:*

*(a)* $M_1 \Rightarrow_{\mathsf{alg}} M_2$

*(b)* $M_1 \Rightarrow_{\mathsf{alg}_*} M_2$

*Proof.* The implication (a)⇒(b) being trivial, let us prove (b)⇒(a). Let us denote the parameters of $M_i$ (for $i \in \{1,2\}$) by $n_i \geqslant 1$, $m_i \geqslant 0$, $m'_i = 1$ and $k_i = \ell_i \geqslant 1$. We recall that $\mathbb{V}_{M_1}$ is the variety with one $m_1$-ary basic operation $p$ and the axioms are the identities described in Theorem 1.2 for $M_1$. We also need the pointed variety $\mathbb{V}^*_{M_1}$ constructed from $\mathbb{V}_{M_1}$ by adding one constant symbol $0$ and the axiom $p(0,\dots,0) = 0$. Let us denote by $\mathrm{Fr}_{M_1} \colon \mathbf{Set} \to \mathbb{V}_{M_1}$ and $\mathrm{Fr}^*_{M_1} \colon \mathbf{Set} \to \mathbb{V}^*_{M_1}$ the left adjoints to the respective forgetful functors. For some distinct variables $z_0, z_1, \dots, z_{m_2}$, the $\mathbb{V}_{M_1}$-algebra $\mathrm{Fr}_{M_1}(z_0, z_1, \dots, z_{m_2})$ can be considered as a $\mathbb{V}^*_{M_1}$-algebra by considering $z_0$ as the constant $0$ (one has $p(z_0,\dots,z_0) = z_0$ since $n_1 \geqslant 1$ and there is thus at least one axiom as in Theorem 1.2). Moreover, the $\mathbb{V}^*_{M_1}$-algebra $\mathrm{Fr}^*_{M_1}(z_1,\dots,z_{m_2})$ can be considered as a $\mathbb{V}_{M_1}$-algebra in the obvious way. Let

$$f \colon \mathrm{Fr}^*_{M_1}(z_1,\dots,z_{m_2}) \to \mathrm{Fr}_{M_1}(z_0, z_1, \dots, z_{m_2})$$

be the unique homomorphism of $\mathbb{V}^*_{M_1}$-algebras such that $f(z_j) = z_j$ for each $j \in \{1, \dots, m_2\}$. Let also

$$g\colon \mathrm{Fr}_{M_1}(z_0, z_1, \dots, z_{m_2}) \to \mathrm{Fr}_{M_1}(z_1, \dots, z_{m_2})$$

be the unique homomorphism of $\mathbb{V}_{M_1}$-algebras such that $g(z_0) = z_1$ and $g(z_j) = z_j$ for each $j \in \{1, \dots, m_2\}$ (note that we need $m_2 > 0$ here).

Since $\mathbb{V}^*_{M_1}$ is a pointed variety with $M_1$-closed relations, it has $M_2$-closed relations assuming that $M_1 \Rightarrow_{\mathsf{alg}_*} M_2$. By Theorem 1.2, there exists an $m_2$-ary term $q \in \mathrm{Fr}^*_{M_1}(z_1, \dots, z_{m_2})$ satisfying the identities of Theorem 1.2 for $M_2$. Let $q' = g(f(q)) \in \mathrm{Fr}_{M_1}(z_1, \dots, z_{m_2})$. We would like to prove that this $m_2$-ary term of $\mathbb{V}_{M_1}$ also satisfies the identities of Theorem 1.2 for $M_2$. Fixing a row $i$ of $M_2$, we consider the function $\iota\colon \{z_1, \dots, z_{m_2}\} \to \{x_1, \dots, x_{k_2}\}$ given by $\iota(z_j) = x^2_{ij}$, where $x^2_{ij}$ is the corresponding entry of $M_2$. We consider also the homomorphisms of $\mathbb{V}^*_{M_1}$-algebras

$$\iota_1\colon \mathrm{Fr}^*_{M_1}(z_1, \dots, z_{m_2}) \to \mathrm{Fr}^*_{M_1}(x_1, \dots, x_{k_2})$$

and

$$f'\colon \mathrm{Fr}^*_{M_1}(x_1, \dots, x_{k_2}) \to \mathrm{Fr}_{M_1}(x_0, x_1, \dots, x_{k_2})$$

such that $\iota_1(z_j) = \iota(z_j)$ for each $j \in \{1, \dots, m_2\}$ and $f'(x_u) = x_u$ for each $u \in \{1, \dots, k_2\}$ (where $\mathrm{Fr}_{M_1}(x_0, x_1, \dots, x_{k_2})$ is considered as a $\mathbb{V}^*_{M_1}$-algebra with $x_0$ as constant $0$). Finally, we consider the homomorphisms of $\mathbb{V}_{M_1}$-algebras

$$\iota_2\colon \mathrm{Fr}_{M_1}(z_0, z_1, \dots, z_{m_2}) \to \mathrm{Fr}_{M_1}(x_0, x_1, \dots, x_{k_2}),$$
$$\iota_3\colon \mathrm{Fr}_{M_1}(z_1, \dots, z_{m_2}) \to \mathrm{Fr}_{M_1}(x_1, \dots, x_{k_2})$$

and

$$g'\colon \mathrm{Fr}_{M_1}(x_0, x_1, \dots, x_{k_2}) \to \mathrm{Fr}_{M_1}(x_1, \dots, x_{k_2})$$

such that $\iota_2(z_0) = x_0$, $\iota_2(z_j) = \iota(z_j)$ and $\iota_3(z_j) = \iota(z_j)$ for each $j \in \{1, \dots, m_2\}$, $g'(x_0) = \iota(z_1)$ and $g'(x_u) = x_u$ for each $u \in \{1, \dots, k_2\}$. Note that $\iota_2$ is also a homomorphism of $\mathbb{V}^*_{M_1}$-algebras. The left-hand square in

$$\begin{array}{ccccc}
\mathrm{Fr}^*_{M_1}(z_1, \dots, z_{m_2}) & \xrightarrow{f} & \mathrm{Fr}_{M_1}(z_0, z_1, \dots, z_{m_2}) & \xrightarrow{g} & \mathrm{Fr}_{M_1}(z_1, \dots, z_{m_2}) \\
\iota_1 \downarrow & & \downarrow \iota_2 & & \downarrow \iota_3 \\
\mathrm{Fr}^*_{M_1}(x_1, \dots, x_{k_2}) & \xrightarrow[f']{} & \mathrm{Fr}_{M_1}(x_0, x_1, \dots, x_{k_2}) & \xrightarrow[g']{} & \mathrm{Fr}_{M_1}(x_1, \dots, x_{k_2})
\end{array}$$

commutes in $\mathbb{V}^*_{M_1}$ (and thus in $\mathbb{V}_{M_1}$), while the right-hand square commutes in $\mathbb{V}_{M_1}$. Since $q$ is sent by $\iota_1$ to $y_i^2$ (i.e., the right entry of the $i$-th row of $M_2$), $q' = g(f(q))$ is sent by $\iota_3$ to $y_i^2$, proving that $q'$ satisfies the equations of Theorem 1.2 for $M_2$. Therefore, $\mathbb{V}_{M_1}$ has $M_2$-closed relations. Since any variety with $M_1$-closed relations admits a forgetful functor to $\mathbb{V}_{M_1}$ (Theorem 1.2), this shows $M_1 \Rightarrow_{\mathsf{alg}} M_2$. □

Let us make explicit that, as in the finitely complete context, the above theorem cannot be generalized to the situation where $M_2$ has no left columns. A counter-example is given by the implication $\mathrm{T}_1 \Rightarrow_{\mathsf{alg}_*} \mathrm{T}_0$ which holds whereas the implication $\mathrm{T}_1 \Rightarrow_{\mathsf{alg}} \mathrm{T}_0$ does not. A variety has $\mathrm{T}_1$-closed relations if and only if the identity $x = y$ holds (i.e., each algebra is either empty or a singleton); while a variety has $\mathrm{T}_0$-closed relations if and only if there is a constant symbol $0$ such that the identity $0 = y$ holds (so each algebra is a singleton). In the pointed varietal context, both properties are equivalent to the identity $0 = y$ for the unique constant $0$.

Let us now turn our attention to the case where $M_2 = \mathsf{Cube}_{n,k}$ is the matrix from Example 1.7. In that case, the algorithm for deciding $M_1 \Rightarrow_{\mathsf{lex}} M_2$ can be nicely simplified.

**Proposition 2.2.** *Let $M$ be a simple extended matrix (with parameters $n \geqslant 1$, $m \geqslant 0$, $m' = 1$ and $k = \ell \geqslant 1$)*

$$M = \left[\begin{array}{ccc|c} x_{11} & \cdots & x_{1m} & y_1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nm} & y_n \end{array}\right]$$

*and let $n', k' \geqslant 2$ be integers. The implication $M \Rightarrow_{\mathsf{lex}} \mathsf{Cube}_{n',k'}$ holds if and only if there exist $i_1, \ldots, i_{n'} \in \{1 \ldots, n\}$ such that there does not exist $j \in \{1, \ldots, m\}$ for which $x_{i_a j} = y_{i_a}$ for each $a \in \{1, \ldots, n'\}$.*

*Proof.* Firstly, let us notice that Theorem 1.9 implies that $\mathsf{Cube}_{n',k'}$ is not a trivial matrix. Let us now assume that $M$ is trivial. In that case, we have $M \Rightarrow_{\mathsf{lex}} \mathsf{Cube}_{n',k'}$. By Theorem 1.9, there must exist $i, i' \in \{1, \ldots, n\}$ such that for all $j, j' \in \{1, \ldots, m\}$, if $x_{ij} = y_i$ and $x_{i'j'} = y_{i'}$ then $j$ is not related to $j'$ by $R_{M_i} \vee R_{M_{i'}}$. If the left part of the $i$-th row of $M$ does not contain $y_i$ as an entry, the second condition is satisfied with $i_1 = \cdots = i_{n'} = i$. Similarly, if the left part of the $i'$-th row of $M$ does not contain $y_{i'}$ as an entry, the second condition is satisfied with $i_1 = \cdots = i_{n'} = i'$. Otherwise, there exist $j, j' \in \{1, \ldots, m\}$ such that $x_{ij} = y_i$ and $x_{i'j'} = y_{i'}$ and thus $j$ is not related to $j'$ by $R_{M_i} \vee R_{M_{i'}}$. Since $n' \geqslant 2$, the second condition in the statement is then satisfied with $i_1 = i$ and $i_2 = \cdots = i_{n'} = i'$. Indeed, if there exists $j'' \in \{1, \ldots, m\}$ such that $x_{ij''} = y_i$ and $x_{i'j''} = y_{i'}$, then $x_{ij} = x_{ij''}$ and $x_{i'j'} = x_{i'j''}$, which would imply $j R_{M_i} j'' R_{M_{i'}} j'$, a contradiction.

We can thus suppose without loss of generality that $M$ is not trivial. In the algorithm to decide whether $M \Rightarrow_{\mathsf{lex}} \mathsf{Cube}_{n',k'}$, there is only one column that could be added to $\mathsf{Cube}_{n',k'}$, i.e., the column of $x_1$'s. This column can indeed be added if and only if we can find $i_1, \ldots, i_{n'} \in \{1, \ldots, n\}$ and functions $f_{i_1}, \ldots, f_{i_{n'}} \colon \{x_1, \ldots, x_k\} \to \{x_1, \ldots, x_{k'}\}$ such that the left columns of the matrix

$$\left[\begin{array}{ccc|c} f_{i_1}(x_{i_1 1}) & \ldots & f_{i_1}(x_{i_1 m}) & f_{i_1}(y_{i_1}) \\ \vdots & & \vdots & \vdots \\ f_{i_{n'}}(x_{i_{n'} 1}) & \ldots & f_{i_{n'}}(x_{i_{n'} m}) & f_{i_{n'}}(y_{i_{n'}}) \end{array}\right]$$

are different from the $n'$-tuple $(x_1, \ldots, x_1)$, but the right column is equal to it. This condition clearly implies the one in the statement. Conversely, from the condition in the statement, one can construct such a matrix by considering, for each $a \in \{1, \ldots, n'\}$, the function $f_{i_a} \colon \{x_1, \ldots, x_k\} \to \{x_1, \ldots, x_{k'}\}$ which sends $y_{i_a}$ to $x_1$ and the other elements of the domain to $x_2$ (using the fact that $k' \geqslant 2$). □

Since the above condition characterizing $M \Rightarrow_{\mathsf{lex}} \mathsf{Cube}_{n',k'}$ does not depend on $k'$, one immediately has the following corollary.

**Corollary 2.3.** *For $n, k_1, k_2 \geqslant 2$, one has* $\mathsf{Cube}_{n,k_1} \Leftrightarrow_{\mathsf{lex}} \mathsf{Cube}_{n,k_2}$.

As mentioned in Example 1.7, this corollary is the reason why we abbreviate $\mathsf{Cube}_{n,2}$ as $\mathsf{Cube}_n$ since then, $\mathsf{Cube}_n \Leftrightarrow_{\mathsf{lex}} \mathsf{Cube}_{n,k}$ for any $k \geqslant 2$.

Putting together Propositions 2.1 and 2.2 and the results of [29], we can easily prove the following theorem. We recall that for an extended matrix $M$, we have defined after Theorem 1.1 the variety $\mathbb{V}_M$ as the 'generic' one with $M$-closed relations. If $M$ is simple, $\mathbb{V}_M$ is thus obtained with a single basic operation and one axiom for each row of $M$ as described in Theorem 1.2.

**Theorem 2.4.** *Let $M$ be a simple extended matrix (with parameters $n \geqslant 1$, $m \geqslant 0$, $m' = 1$ and $k = \ell \geqslant 1$)*

$$M = \left[\begin{array}{ccc|c} x_{11} & \cdots & x_{1m} & y_1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nm} & y_n \end{array}\right]$$

*and let $n' \geqslant 2$ be an integer. The following statements are equivalent:*

*(a)* $M \Rightarrow_{\mathsf{lex}} \mathsf{Cube}_{n'}$

*(b)* $M \Rightarrow_{\mathsf{lex}_*} \mathsf{Cube}_{n'}$

*(c)* $M \Rightarrow_{\mathsf{reg}} \mathsf{Cube}_{n'}$

*(d)* $M \Rightarrow_{\mathsf{reg}_*} \mathsf{Cube}_{n'}$

*(e)* $M \Rightarrow_{\mathsf{alg}} \mathsf{Cube}_{n'}$

*(f)* $M \Rightarrow_{\mathsf{alg}_*} \mathsf{Cube}_{n'}$

*(g)* $M \Rightarrow_{\mathsf{ess\,alg}} \mathsf{Cube}_{n'}$

*(h)* $M \Rightarrow_{\mathsf{ess\,alg}_*} \mathsf{Cube}_{n'}$

*(i)* $M \Rightarrow_{\mathsf{reg\,ess\,alg}} \mathsf{Cube}_{n'}$

*(j)* $M \Rightarrow_{\mathsf{reg\,ess\,alg}_*} \mathsf{Cube}_{n'}$

*(k) There exist $i_1, \dots, i_{n'} \in \{1, \dots, n\}$ such that there does not exist $j \in \{1, \dots, m\}$ for which $x_{i_a j} = y_{i_a}$ for each $a \in \{1, \dots, n'\}$.*

*(l) There does* not *exist a function $p\colon \{0,1\}^m \to \{0,1\}$ making $(\{0,1\}, p)$ an algebra of $\mathbb{V}_M$ such that its induced $n'$-power $(\{0,1\}^{n'}, p^{n'})$ is compatible with the $n'$-ary relation $R_{n'} = \{0,1\}^{n'} \setminus \{(0,\dots,0)\}$ (i.e., $p^{n'}(r_1, \dots, r_m) \in R_{n'}$ for each $r_1, \dots, r_m \in R_{n'}$).*

*Proof.* The equivalence (e)⇔(l) is an immediate application of Lemma 3.5 and Proposition 7.7 of [29] applied to the variety $\mathbb{V}_M$. The equivalence (a)⇔(k) is Proposition 2.2 with $k' = 2$. It is trivial that (a) implies all the statements (b)–(j) and any of these statements implies (f). The equivalence (e)$\Leftrightarrow$(f) is an immediate application of Proposition 2.1. It thus suffices to show the implication (l)$\Rightarrow$(k). By contradiction, let us suppose (k) does not hold and let us prove (l) does not hold neither. We define $p\colon \{0,1\}^m \to \{0,1\}$ on an $m$-tuple $(b_1,\dots,b_m)$ by $p(b_1,\dots,b_m)=0$ if and only if there exist $i \in \{1,\dots,n\}$ and a function $f\colon \{x_1,\dots,x_k\} \to \{0,1\}$ such that $f(y_i)=0$ and $(b_1,\dots,b_m)=(f(x_{i1}),\dots,f(x_{im}))$. To prove that $(\{0,1\},p)$ indeed forms an algebra of $\mathbb{V}_M$, we need to show that for any $i \in \{1,\dots,n\}$ and any function $f\colon \{x_1,\dots,x_k\} \to \{0,1\}$, one has

$$p(f(x_{i1}),\dots,f(x_{im})) = f(y_i).$$

If $f(y_i)=0$, this is immediate from the definition of $p$. If $f(y_i)=1$, we need to show there do not exist $i' \in \{1,\dots,n\}$ and $g\colon \{x_1,\dots,x_k\} \to \{0,1\}$ such that $g(y_{i'})=0$ and $(f(x_{i1}),\dots,f(x_{im}))=(g(x_{i'1}),\dots,g(x_{i'm}))$. But if this was the case, using that $n' \geqslant 2$ and that (k) is false with $i_1=i$ and $i_2=\dots=i_{n'}=i'$, we obtain a $j \in \{1,\dots,m\}$ such that $x_{ij}=y_i$ and $x_{i'j}=y_{i'}$, contradicting $f(x_{ij})=g(x_{i'j})$. It remains to prove that $p^{n'}$ is compatible with $R_{n'}$. The only way for it not to be so is that there exist $i_1,\dots,i_{n'} \in \{1,\dots,n\}$ and functions $f_1,\dots,f_{n'}\colon \{x_1,\dots,x_k\} \to \{0,1\}$ such that $f_a(y_{i_a})=0$ for each $a \in \{1,\dots,n'\}$ and such that the matrix

$$\begin{bmatrix} f_1(x_{i_1 1}) & \cdots & f_1(x_{i_1 m}) \\ \vdots & & \vdots \\ f_{n'}(x_{i_{n'}1}) & \cdots & f_{n'}(x_{i_{n'}m}) \end{bmatrix}$$

does not contain a column of 0's. But this is impossible by our assumption that (k) is false. $\square$

The situation described by Theorem 2.4 is very particular to the matrices $\mathsf{Cube}_{n'}$. Indeed, as we have already remarked in Example 1.8, one has $\mathsf{Cube}_3 \Rightarrow_{\mathsf{alg}} \mathsf{Edge}_3$ but $\mathsf{Cube}_3 \not\Rightarrow_{\mathsf{lex}} \mathsf{Edge}_3$. Other examples of this phenomenon are given in [16].

Since it is our most interesting case, let us specify some of the statements of Theorem 2.4 in the case where $n'=2$, i.e., when $\mathsf{Cube}_{n'}=\mathsf{Cube}_2$ describes the Mal'tsev property.

**Corollary 2.5.** *Let $M$ be a simple extended matrix (with parameters $n \geqslant 1$, $m \geqslant 0$, $m' = 1$ and $k = \ell \geqslant 1$)*

$$M = \left[\begin{array}{ccc|c} x_{11} & \cdots & x_{1m} & y_1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nm} & y_n \end{array}\right].$$

*The following statements are equivalent:*

*(a) Each finitely complete category with $M$-closed relations is a Mal'tsev category.*

*(b) Each variety with $M$-closed relations is a Mal'tsev variety.*

*(c) There exist $i, i' \in \{1\dots, n\}$ such that there is no $j \in \{1, \dots, m\}$ for which $x_{ij} = y_i$ and $x_{i'j} = y_{i'}$.*

**Remark 2.6.** It is shown in [17] that if $M$ is a simple extended matrix as in Corollary 2.5 with $k = 2$, then the equivalent conditions in that corollary are further equivalent to:

(d) There is a finitely complete majority category which does not have $M$-closed relations.

In other words, if $k = 2$, $M \Rightarrow_{\mathsf{lex}} \mathsf{Mal}$ if and only if $\mathsf{Maj} \not\Rightarrow_{\mathsf{lex}} M$. It is also shown that if $k > 2$, this equivalence is no longer true.

Statement (k) of Theorem 2.4 provides an algorithm to decide whether $M \Rightarrow_{\mathsf{alg}} \mathsf{Cube}_{n'}$ (or equivalently $M \Rightarrow_{\mathsf{lex}} \mathsf{Cube}_{n'}$). It is easy to see that this algorithm requires at most $m \times n^{n'}$ comparisons of columns, and thus at most $n' \times m \times n^{n'}$ comparisons of elements. For a fixed $n'$, this is thus a polynomial-time algorithm in the parameters $n$ and $m$ of the input matrix $M$. For $n' = 2$, statement (c) of Corollary 2.5 thus provides an algorithm to decide whether $M \Rightarrow_{\mathsf{alg}} \mathsf{Mal}$ (or equivalently $M \Rightarrow_{\mathsf{lex}} \mathsf{Mal}$) which requires at most $2mn^2$ comparisons of elements.

To illustrate Theorem 2.4 and Corollary 2.5, we can see that $\mathsf{Maj} \Rightarrow_{\mathsf{lex}} \mathsf{Cube}_3$ and $\mathsf{Maj} \Rightarrow_{\mathsf{alg}} \mathsf{Cube}_3$ by taking $i_1 = 1$, $i_2 = 2$ and $i_3 = 3$ in statement (k) of Theorem 2.4 since the right column

$$\begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix}$$

of Maj as presented in Example 1.6 does not appear as one of its left columns. Besides, one has Maj $\not\Rightarrow_{\mathsf{lex}}$ Mal and Maj $\not\Rightarrow_{\mathsf{alg}}$ Mal since, given any two rows of Maj, they contain

$$\left[\begin{array}{c} x_1 \\ x_1 \end{array}\right]$$

in their left part.

Theorem 3.6 in [16] states that the intersection of finitely many matrix properties induced by simple extended matrices is again a matrix property induced by a simple extended matrix. Given, for each $i \in \{1,2\}$, such a matrix $M_i$ with parameters $n_i$, $m_i$, $m'_i = 1$ and $k_i = \ell_i$, we can form a matrix $M$ with parameters $n = n_1 + n_2$, $m = m_1 \times m_2$, $m' = 1$ and $k = \ell = \max(k_1, k_2)$ as follows: the left columns of $M$ are indexed by the pairs consisting of a left column of $M_1$ and a left column of $M_2$ and are obtained by superposing this column of $M_1$ over this column of $M_2$. The right column of $M$ is obtained by superposing the right column of $M_1$ over the right column of $M_2$. Then, a finitely complete category has $M$-closed relations if and only if it has $M_1$-closed relations and $M_2$-closed relations. For instance, if $M_1 = \mathsf{Mal}$ as in Example 1.3 and if $M_2 = \mathsf{Maj}$ as in Example 1.6, one has

$$M = \left[\begin{array}{ccc:ccc:ccc|c} x_1 & x_1 & x_1 & x_2 & x_2 & x_2 & x_2 & x_2 & x_2 & x_1 \\ x_1 & x_1 & x_1 & x_1 & x_1 & x_1 & x_2 & x_2 & x_2 & x_2 \\ \hdashline x_1 & x_1 & x_2 & x_1 & x_1 & x_2 & x_1 & x_1 & x_2 & x_1 \\ x_1 & x_2 & x_1 & x_1 & x_2 & x_1 & x_1 & x_2 & x_1 & x_1 \\ x_2 & x_1 & x_1 & x_2 & x_1 & x_1 & x_2 & x_1 & x_1 & x_1 \end{array}\right]$$

which turns out to satisfy $M \Leftrightarrow_{\mathsf{lex}}$ Ari.

**Theorem 2.7.** *Let $d \geqslant 0$ be an integer and $(M_i)_{i\in\{1,\dots,d\}}$ be a finite family of simple extended matrices. For $n' \geqslant 2$, the following statements are equivalent:*

*(a) Each finitely complete category with $M_i$-closed relations for all $i \in \{1,\dots,d\}$ has $\mathsf{Cube}_{n'}$-closed relations.*

*(b) There exists $i \in \{1,\dots,d\}$ such that $M_i \Rightarrow_{\mathsf{lex}} \mathsf{Cube}_{n'}$.*

*Proof.* The statement being trivial for $d = 0$ and $d = 1$, let us assume without loss of generality that $d \geqslant 2$. Furthermore, since the intersection of finitely many matrix properties induced by simple extended matrices is again a matrix property induced by a simple extended matrix, using induction, we can assume without loss generality that $d = 2$. The implication (b)⇒(a) being trivial, we assume (a) and we shall prove (b). Let $M$ be the simple extended matrix as constructed above from $M_1$ and $M_2$. We shall use the same notation as above for the parameters of $M_1$, $M_2$ and $M$. Moreover, for $i \in \{1, 2\}$, we denote the entries of $M_i$ as

$$M_i = \left[\begin{array}{ccc|c} x^i_{11} & \cdots & x^i_{1m_i} & y^i_1 \\ \vdots & & \vdots & \vdots \\ x^i_{n_i 1} & \cdots & x^i_{n_i m_i} & y^i_{n_i} \end{array}\right]$$

and we denote the entries of $M$ as

$$M = \left[\begin{array}{ccc|c} x_{11} & \cdots & x_{1m} & y_1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nm} & y_n \end{array}\right].$$

We thus assume that $M \Rightarrow_{\mathsf{lex}} \mathsf{Cube}_{n'}$ and we shall prove that either $M_1 \Rightarrow_{\mathsf{lex}} \mathsf{Cube}_{n'}$ or $M_2 \Rightarrow_{\mathsf{lex}} \mathsf{Cube}_{n'}$. By Theorem 2.4, we know that there exist $i_1, \dots, i_{n'} \in \{1, \dots, n_1 + n_2\}$ such that there does not exist $j \in \{1, \dots, m_1 \times m_2\}$ for which $x_{i_a j} = y_{i_a}$ for each $a \in \{1, \dots, n'\}$. Let us denote by $S_1$ the set $S_1 = \{a \in \{1, \dots, n'\} \,|\, i_a \in \{1, \dots, n_1\}\}$ and by $S_2$ the set $S_2 = \{1, \dots, n'\} \setminus S_1$. If there exist $j_1 \in \{1, \dots, m_1\}$ and $j_2 \in \{1, \dots, m_2\}$ such that $x^1_{i_a j_1} = y^1_{i_a}$ for each $a \in S_1$ and $x^2_{(i_a - n_1) j_2} = y^2_{i_a - n_1}$ for each $a \in S_2$, by choosing $j \in \{1, \dots, m_1 \times m_2\}$ as the index of the left column of $M$ obtained by superposing the $j_1$-th left column of $M_1$ over the $j_2$-th left column of $M_2$, one obtains that $x_{i_a j} = y_{i_a}$ for each $a \in \{1, \dots, n'\}$, contradicting our hypothesis. By symmetry, we can therefore assume without loss of generality that there does not exist $j_1 \in \{1, \dots, m_1\}$ for which $x^1_{i_a j_1} = y^1_{i_a}$ for each $a \in S_1$. Let us define $i'_a \in \{1, \dots, n_1\}$ for each $a \in \{1, \dots, n'\}$ by

$$i'_a = \begin{cases} i_a & \text{if } a \in S_1 \\ 1 & \text{if } a \in S_2. \end{cases}$$

The indices $i'_1, \dots, i'_{n'} \in \{1, \dots, n_1\}$ satisfy condition (k) of Theorem 2.4 and therefore one has $M_1 \Rightarrow_{\mathsf{lex}} \mathsf{Cube}_{n'}$. □

## Results for general matrices

We now tackle the question to describe when $M \Rightarrow_{\mathsf{reg}} \mathsf{Cube}_n$ for a (not necessarily simple) extended matrix $M$. In general, we do not yet know an algorithm to decide whether a general matrix condition implies another one. In the following theorem, we thus have to use another technique than the one used to prove Theorem 2.4. We will use here, in addition to the results of [29], the embedding theorems of [21]. In order to do so, we will need the axiom of universes [3], which will only be used to prove the implication (b)⇒(a) (the equivalences (b)⇔(c)⇔(d) do not rely on the axiom of universes).

**Theorem 2.8.** *Let $M$ be an extended matrix (with parameters $n \geqslant 1$, $m \geqslant 0$, $m' \geqslant 0$ and $k \geqslant \ell \geqslant 0$)*

$$M = \left[\begin{array}{ccc|ccc} x_{11} & \cdots & x_{1m} & y_{11} & \cdots & y_{1m'} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n1} & \cdots & x_{nm} & y_{n1} & \cdots & y_{nm'} \end{array}\right]$$

*and let $n' \geqslant 2$ be an integer. The following statements are equivalent:*

*(a)* $M \Rightarrow_{\mathsf{reg}} \mathsf{Cube}_{n'}$

*(b)* $M \Rightarrow_{\mathsf{reg\,ess\,alg}} \mathsf{Cube}_{n'}$

*(c)* $M \Rightarrow_{\mathsf{alg}} \mathsf{Cube}_{n'}$

*(d) There do* not *exist functions*

$$p_1, \ldots, p_{m'} \colon \{0,1\}^m \to \{0,1\}$$

*and*

$$q_1, \ldots, q_{k-\ell} \colon \{0,1\}^\ell \to \{0,1\}$$

*making $A = (\{0,1\}, p_1, \ldots, p_{m'}, q_1, \ldots, q_{k-\ell})$ an algebra of $\mathbb{V}_M$ such that the $n'$-ary relation $R_{n'} = \{0,1\}^{n'} \backslash \{(0,\ldots,0)\}$ is a homomorphic relation on $A$.*

*Proof.* The equivalence (c)⇔(d) is an immediate application of Lemma 3.5 and Proposition 7.7 of [29] applied to the variety $\mathbb{V}_M$. Under the axiom of universes, the implication (b)⇒(a) follows immediately from Theorem 4.6 of [21]. The implications (a)⇒(b)⇒(c) being trivial, it remains to prove (d)⇒(b). Let us thus assume that (d) holds and let us consider an essentially algebraic theory $\Gamma = (S, \Sigma, E, \Sigma_t, \mathsf{Def})$ such that $\mathsf{Mod}(\Gamma)$ is a regular category with $M$-closed relations. We shall prove that it has $\mathsf{Cube}_{n'}$-closed relations. By Theorem 3.3 in [21], we know that, for each sort $s \in S$, there exist in $\Gamma$:

- for each $j \in \{1, \ldots, m'\}$, a term $p_j^s \colon s^m \to s$,
- for each $a \in \{1, \ldots, k-\ell\}$, a term $q_a^s \colon s^\ell \to s$

such that, for each $i \in \{1, \ldots, n\}$ and each $j \in \{1, \ldots, m'\}$,

- if $y_{ij} = x_a \in \{x_1, \ldots, x_\ell\}$, the term $p_j^s(x_{i1}, \ldots, x_{im}) \colon s^\ell \to s$ (in the variables $x_1, \ldots, x_\ell$) is everywhere-defined and equal to $x_a$,
- if $y_{ij} = x_a \in \{x_{\ell+1}, \ldots, x_k\}$, the term $p_j^s(x_{i1}, \ldots, x_{im}) \colon s^\ell \to s$ (in the variables $x_1, \ldots, x_\ell$) is defined for an $\ell$-tuple $(b_1, \ldots, b_\ell) \in B_s^\ell$ in a $\Gamma$-model $B$ if and only if $q_{a-\ell}^s(b_1, \ldots, b_\ell)$ is defined, and in that case they are equal.

Let now $T$ be a homomorphic $n'$-ary relation on the $\Gamma$-model $B$. Let $s \in S$ and let $b_0, b_1 \in B_s$ be such that any $n'$-tuple of elements of $\{b_0, b_1\}$, except maybe $(b_0, \ldots, b_0)$, is in $T_s$. We shall prove that $(b_0, \ldots, b_0) \in T_s$. For each $j \in \{1, \ldots, m'\}$, let us define the function $p_j \colon \{0,1\}^m \to \{0,1\}$ on $(c_1, \ldots, c_m) \in \{0,1\}^m$ by $p_j(c_1, \ldots, c_m) = 0$ if $p_j^s(b_{c_1}, \ldots, b_{c_m})$ is defined and equal to $b_0$, otherwise we set $p_j(c_1, \ldots, c_m) = 1$. Similarly, for each $a \in \{1, \ldots, k-\ell\}$, let us define the function $q_a \colon \{0,1\}^\ell \to \{0,1\}$ on $(c_1, \ldots, c_\ell) \in \{0,1\}^\ell$ by $q_a(c_1, \ldots, c_\ell) = 0$ if $q_a^s(b_{c_1}, \ldots, b_{c_\ell})$ is defined and equal to $b_0$, otherwise we set $q_a(c_1, \ldots, c_\ell) = 1$. It is easy to see that $A = (\{0,1\}, p_1, \ldots, p_{m'}, q_1, \ldots, q_{k-\ell})$ forms an algebra of $\mathbb{V}_M$. Let us consider the bijection $f \colon \{0,1\} \to \{b_0, b_1\}$ defined by $f(0) = b_0$ and $f(1) = b_1$ and the induced bijection $f^{n'} \colon \{0,1\}^{n'} \to \{b_0, b_1\}^{n'}$. Since (d) holds, either there exist $j \in \{1, \ldots, m'\}$ and elements $r_1, \ldots, r_m \in R_{n'}$ such that $p_j^{n'}(r_1, \ldots, r_m) = (0, \ldots, 0)$ or there exist $a \in \{1, \ldots, k-\ell\}$ and elements

$r_1, \dots, r_\ell \in R_{n'}$ such that $q_a^{n'}(r_1, \dots, r_\ell) = (0, \dots, 0)$ (where $p_j^{n'}$ and $q_a^{n'}$ are the operations induced on $\{0,1\}^{n'}$ by $p_j$ and $q_a$ respectively). We thus have either that $(p_j^s)^{n'}(f^{n'}(r_1), \dots, f^{n'}(r_m))$ is defined and equal to $(b_0, \dots, b_0)$, or that $(q_a^s)^{n'}(f^{n'}(r_1), \dots, f^{n'}(r_\ell))$ is defined and equal to $(b_0, \dots, b_0)$. Since $T$ is a homomorphic relation on $B$ and either $f^{n'}(r_1), \dots, f^{n'}(r_m) \in T_s$ or $f^{n'}(r_1), \dots, f^{n'}(r_\ell) \in T_s$, we can conclude in both cases that $(b_0, \dots, b_0) \in T_s$. □

Let us notice that condition (d) provides a (finite-time) algorithm to decide whether $M \Rightarrow_{\mathsf{reg}} \mathsf{Cube}_{n'}$ (or equivalently $M \Rightarrow_{\mathsf{alg}} \mathsf{Cube}_{n'}$), but it seems this is not a polynomial-time algorithm.

Again, since this is our most interesting case, let us emphasize the case $n' = 2$ of Theorem 2.8.

**Corollary 2.9.** *For an extended matrix $M$, the following statements are equivalent:*

*(a) Every regular category with $M$-closed relations is a Mal'tsev category.*

*(b) Every variety with $M$-closed relations is a Mal'tsev variety.*

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXVI-3 (2025)

# CONNECTEDNESS THROUGH DECIDABLE QUOTIENTS

*Enrique RUIZ-HERNÁNDEZ and Pedro SOLÓRZANO*

In memory of William Lawvere

**Résumé.** En considérant des quotients décidables, on fournit une condition suffisante (1) pour garantir que la sous-catégorie pleine des objets décidables d'un topos soit un idéal exponentiel et (2) pour que la notion classique de connexité pour un objet X coïncide avec $\Pi X = 1$, où $\Pi$ est le foncteur adjoint à gauche de l'inclusion des décidables.

L'ajout de cette condition-ci dans le contexte de l'axiomatique de McLarty pour la Géométrie Différentielle Synthétique rend tout topos qui la satisfait précohésif sur le topos de ses objets décidables. Une réciproque est également fournie.

**Abstract.** By looking at decidable quotients, a sufficient condition is provided to guarantee that (1) the full subcategory of decidable objects of a topos is an exponential ideal and that (2) the classical notion of connectedness for an object $X$ coincides with $\Pi X = 1$, where $\Pi$ is the left-adjoint functor of the inclusion of the decidable objects.

The addition of this condition to McLarty's axiomatic set up for Synthetic Differential Geometry makes any topos that satisfies it precohesive over the topos of its decidable objects. A converse is also provided.

**Keywords.** Topos, connectedness, precohesion.
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## Motivation

Colin McLarty [7] formalizes a development of the notion of set out of that of space through topos theory: he considers a topos of *spaces* from which it is possible to get a category of sets, a topos of sets. In that paper, he posits two axiom systems, one for the topos of sets and another for the topos of spaces, which he denotes $\mathrm{SDG}$—for several of the postulates pertain specifically to the topic of Synthetic Differential Geometry. There are two postulates in this latter system, $\mathrm{SDG}_6$ and $\mathrm{SDG}_7$ which he further studies in [6]. There they are presented as

(NS) Any object is either initial or has a global element,

and

(DSO) There exists a unique decidable subobject for any given object containing all of its points.

In McLarty's words, intuitively, the Nullstellensatz (NS) "says points are the smallest spaces, so two points in any space are either wholly coincident or wholly disjoint"; that is, "for every space $B$ and points $b_1 \in B$ and $b_2 \in B$, '$b_1 = b_2 \vee \neg(b_1 = b_2)$' is true even if $B$ is not discrete (decidable) and the corresponding sentence with variables over $B$ is not true" ([7, p. 81]).

Toposes satisfying NS abound: Any topos $\mathcal{E}$ precohesive over a topos $\mathcal{S}$ that satisfies NS must also satisfy NS—thus any topos that is precohesive over $\mathbf{Set}$ satisfies NS. Indeed, by Lawvere's Nullstellensatz, $f_!(X)$ is initial if $f_*(X)$ is initial. In such a case, by the strictness of $0$, $X$ would be initial too. Hence $f_*(X)$ is not initial whenever $X$ is not initial either.

These two axioms imply that the topos of spaces we begin with has a local geometric morphism to a category of—decidable—sets, as proved in [6]. There is just one functor missing to aspire to have precohesion in the sense of William Lawvere's program [4].

## Main Results

The purpose of this communication is to ultimately focus in such a functor, first in isolation, and then in the precohesive context. To this effect, proceeding à la McLarty, consider the following postulate:

(DQO) There exists a unique decidable quotient for any given object along which any arrow to 2 factors uniquely.

In the presence of DQO, let $p_X : X \to \Pi(X)$ be the corresponding quotient map. It follows that $\Pi 1 \cong 1$, $\Pi X = 0$ if and only if $X = 0$, and $\Pi\Pi X \cong \Pi X$ for every object $X \in \mathcal{E}$.

It is not immediately apparent that DQO implies that $\Pi$ is indeed the object part of a left adjoint of the inclusion of $\mathrm{Dec}(\mathcal{E})$ in $\mathcal{E}$. This is actually the case, as established by the following result.

**Theorem A.** *For a topos $\mathcal{E}$ that satisfies* NS *and* DQO*, $\Pi$ is the object part of a finite-product-preserving functor left adjoint to the inclusion of* $\mathrm{Dec}(\mathcal{E})$.

Conversely, by 2.3, DQO holds in the presence of NS as soon as the inclusion $\mathrm{Dec}(\mathcal{E})$ into $\mathcal{E}$ has a left adjoint. Therefore,

**Theorem B.** *For a topos $\mathcal{E}$ that satisfies* NS *,* $\mathrm{Dec}(\mathcal{E})$ *is an exponential ideal as soon as it is reflective.*

It is not immediately apparent that a topos $\mathcal{E}$ precohesive over $\mathrm{Dec}(\mathcal{E})$ satisfies McLarty's DSO. Section 3 is devoted to establishing this. It should be noted that the proof requires DQO. Hence,

**Theorem C.** *Let $\mathcal{E}$ be a topos satisfying NS. Then $\mathcal{E}$ satisfies* DQO *and McLarty's* DSO *if and only if $\mathcal{E}$ is precohesive over* $\mathrm{Dec}(\mathcal{E})$.

In an extensive category, an object is connected if it has exactly two complemented subobjects. On the other hand, Lawvere [4] calls an object connected in the context of precohesion if its image under the left-most adjoint is terminal. In this report, for an object $X$, the latter corresponds in a weak sense to $\Pi(X) = 1$. In fact, in the presence of NS and DQO both agree (see 1.1). Notice that to require DQO but not NS is not be enough: In $\mathbf{Set} \times \mathbf{Set}$, DQO holds and thus $\Pi(1) = 1$, yet $1$ has four complemented subobjects.

In view of these observations, the addition of DQO is a natural extension of McLarty's axioms for SDG that frames it in a context of Lawvere's precohesion in which these two notions of connectedness agree.

**Aknowledgements.**

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## 1. Connectedness

An object $X$ is *connected* if it has exactly two complemented subobjects. Let $\mathrm{Sub}_c(X)$ be the collection of complemented subobjects of $X$. These are evidently classified by 2. The DQO axiom requires the complemented subobjects of $X$ to be in bijective correspondence with those of its decidable quotient $\Pi X$. In fact,

**1.1 Proposition.** *Let $\mathcal{E}$ satisfy NS and DQO. For an object $X$, $\Pi X = 1$ if and only if $\mathrm{Sub}_c(X)$ has exactly two elements, i.e. if $X$ is connected.*

*Proof.* For the necessity, since $\Pi X = 1$ and NS guarantees that $\mathrm{Sub}_c(1) = 2$, there are exactly two arrows $X \to 2$. So, as $X, 0 \in \mathrm{Sub}_c(X)$, these are all of the complemented subobjects of $X$.

For the sufficiency, since $X \neq 0$, there is an arrow $1 \to X$. Now, by considering the composite $1 \to X \to 1$, it follows that $!_X : X \to 1$ is epic. Assuming that $\mathrm{Sub}_c(X)$ has exactly two elements, the two arrows $X \to 2$ are the constants, which factor through $!_X$; thus by DQO, the arrow $\Pi X \to 1$ is an isomorphism. □

**1.2 Proposition** (Schanuel)**.** *Let $\mathcal{E}$ satisfy NS and DQO. The finite product of connected objects is connected.*

*Proof.* The argument syntactically coincides with that of [2, Theorem 12.1.1], as expected since the Nullstellensatz therein coincides with NS.

Let $Z \rightarrowtail X \times Y$ be complemented in $X \times Y$ and different from $\varnothing$ and $X \times Y$. By NS, there are points $\langle a, b\rangle : 1 \to X \times Y$ and $\langle c, d\rangle : 1 \to X \times Y$ that factor through $Z$ and $Z^c$ respectively.

The decomposition given by $Z$ induces a decomposition of $X$ via the map $\langle 1, b!_X\rangle : X \to X \times Y$ and a decomposition of $Y$ via the map $\langle c!_Y, 1\rangle : Y \to X \times Y$. Now, since $X$ and $Y$ are connected, these decompositions are trivial. Therefore, $\langle 1, b!_X\rangle$ factors through $Z$ and $\langle c!_Y, 1\rangle$ factors through $Z^c$. This implies that $\langle b, c\rangle : 1 \to X \times Y$ factors through $Z \cap Z^c$. A contradiction.

Therefore there does not exist a nontrivial complemented subobject of $X \times Y$, and thus the conclusion follows. □

Before the next definition, let $P_c(X)$ be the subobject of the power object $P(X)$,

$$\{u \in P(X) : u \cup u^c = X\} \rightarrowtail P(X), \tag{1}$$

where $u^c$ stands for $\{x \in X : x \notin u\}$. Observe that $\mathrm{Sub}_c(X)$ is in one-to-one correspondence with $\mathcal{E}(1, P_c(X))$.

Also, recall that in a topos, morphisms can be described as in set theory via their graphs: appropriate subobjects of the product of their domain and codomain (See Exercise VI.11 in the textbook by Mac Lane and Moerdijk [5]): the subobject

$$G \rightarrowtail X \times Y$$

is the graph of an arrow $X \to Y$ if and only if

$$\exists! y(\langle x, y\rangle \in G \tag{2}$$

is universally valid for $x \in X$. In particular, for an arrow $f : X \to Y$ one writes $|f| \rightarrowtail X \times Y$ for its graph and $f^{-1}(y)$ instead of $\{x \in X : \langle x, y\rangle \in |f|\}$ for its standard fiber.

**1.3 Definition.** *A map $f : X \to Y$ has* pneumoconnected fibers *if the formula*

$$\neg\neg(f^{-1}(y) \cap w = \varnothing \vee f^{-1}(y) \cap w^c = \varnothing) \tag{3}$$

*is universally valid, with $y \in Y$ and $w \in P_c(X)$.*

Intuitively, it says that any generic fiber $f^{-1}(y)$ is very close to being connected: Except for the double negation, it reads that fibers cannot be separated through complemented objects. One cannot rid oneself from the double negation in the definition, since $\neg\neg\alpha \Rightarrow \alpha$ is not universally valid in the internal logic of a non-boolean topos. Yet to prove an assertion of the form $\neg\alpha$ (e.g. in the previous or in the following two propositions), a perfectly valid intuitionistic argument (so long as one refrains from invoking the axiom of choice and the excluded middle) is to assume $\alpha$ and arrive at a contradiction $\perp$, since $\neg\alpha$ is equivalent to $\alpha \Rightarrow \perp$.

When considering global elements (3) does capture the connectendess of fibers, as seen in the next few results.

**1.4 Proposition.** *Let $\mathcal{E}$ satisfy NS and DQO. And let $f : X \to Y$ have pneumoconnected fibers. For any point $b : 1 \to Y$, its fiber $f^{-1}(b)$, given by the pullback diagram*

$$\begin{array}{ccc} f^{-1}(b) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ 1 & \xrightarrow[b]{} & Y, \end{array}$$

*does not have nontrivial complemented subobjects, i.e.* $\Pi(f^{-1}(b)) = 1$.

*Proof.* Let $A \rightarrowtail f^{-1}(b)$ such that $f^{-1}(b) = A + A^c$. If this is a nontrivial decomposition, then both $A$ and $A^c$ are not initial. By NS, there are points $a : 1 \to A$ and $a' : 1 \to A^c$ such that $f \circ a = f \circ a' = b$. Therefore, $f^{-1}(b) \cap A \neq \varnothing$ and $f^{-1}(b) \cap A^c \neq \varnothing$, which proves that

$$\neg(f^{-1}(b) \cap A = \varnothing \vee f^{-1}(b) \cap A^c = \varnothing),$$

which contradicts (3). □

**1.5 Proposition.** *Let $\mathcal{E}$ satisfy NS and DQO. Let $f : X \to Y$ and $g : X' \to Y'$ be two arrows with pneumoconnected fibers. Then $f \times g$ has pneumoconnected fibers.*

*Proof.* Define

$$\vartheta := (f \times g)^{-1}(\langle z, w\rangle) \cap v \neq \varnothing \wedge (f \times g)^{-1}(\langle z, w\rangle) \cap v^c \neq \varnothing$$

and

$$R := \{\langle z, w\rangle \in Y \times Y' : \exists v \in P_c(X \times X').\vartheta\}.$$

Suppose for contradiction that $R$ is not initial. By the NS, there exist points $a : 1 \to Y$, $b : 1 \to Y'$ and a complemented $D \rightarrowtail X \times X'$ such that

$$(f \times g)^{-1}(\langle a, b\rangle) \cap D \neq \varnothing \wedge (f \times g)^{-1}(\langle a, b\rangle) \cap D^c \neq \varnothing. \tag{4}$$

On the other hand, $\Pi(f^{-1}(a)) = 1 = \Pi(g^{-1}(b))$ and thus

$$(f \times g)^{-1}(\langle a, b\rangle) = f^{-1}(a) \times g^{-1}(b)$$

is also connected. Therefore, $(f \times g)^{-1}(\langle a, b\rangle) \cap D = \varnothing$ or $(f \times g)^{-1}(\langle a, b\rangle) \cap D^c = \varnothing$, which is a contradiction to (4). Therefore, $R$ cannot have points and by NS it must be initial. That is that

$$\neg\exists v \in P_c(X \times X').\vartheta$$

is universally valid for $\langle z, w\rangle \in Y \times Y'$. Or, equivalently,

$$\forall v \in P_c(X \times X').\neg\vartheta$$

is universally valid for $\langle z, w\rangle \in Y \times Y'$.

Or, equivalently,

$$\neg\neg((f \times g)^{-1}(\langle z, w\rangle) \cap v = \varnothing \vee (f \times g)^{-1}(\langle z, w\rangle) \cap v^c = \varnothing)$$

is universally valid, with $z \in Y$, $w \in Y'$ and $v \in P_c(X \times X')$, as required by (3). □

Lastly, another result that can be proved along the same lines is the following.

**1.6 Proposition.** *Let $\mathcal{E}$ satisfy NS and DQO. Any pullback of an arrow that has pneumoconnected fibers has pneumoconnected fibers.*

*Proof.* Straightforward. □

## 2. Fiber Pneumoconnectedness Lemma

The purpose of this section is to state and prove the following result. It is central to several arguments in this report.

**2.1 Theorem** (Fiber Pneumoconnectedness Lemma)**.** *Let $\mathcal{E}$ be a topos that satisfies NS. Let $q : X \twoheadrightarrow Q$ be epic. Then the following statements are equivalent:*

*(i) Every arrow $X \to 2$ factors through $q$.*

*(ii) The map $q$ has pneumoconnected fibers.*

*(iii) Every arrow $X \to Y$ with $Y$ decidable factors through $q$.*

The rest of the section is devoted to some of its applications. Afterwards, a proof of 2.1 will be given towards the end of the section.

**2.2 Corollary.** *Le $\mathcal{E}$ satisfy NS and DQO and let $p_X : X \to \Pi(X)$ be the corresponding quotient map. Then $\Pi(X \times Y) \cong \Pi X \times \Pi Y$.*

*Proof.* By 2.1 both projections $p_X$ and $p_Y$ have pneumoconnected fibers, hence by 1.5 so does the epic arrow $p_X \times p_Y : X \times Y \to \Pi X \times \Pi Y$. By DQO, since $\Pi X \times \Pi Y$ is a decidable quotient that factors arrows to $2$, it coincides with $\Pi(X \times Y)$. □

*Proof of Theorem A.* For an arrow $f : X \to Y$ with $Y \in \mathrm{Dec}(\mathcal{E})$, by 2.1(iii) there exists a unique $f' : \Pi X \to Y$ such that $f' \circ p_X = f$. So $\Pi$ is functorial and $\Pi \dashv \mathcal{I}$. By 2.2 it also preserves products. □

**2.3 Corollary.** *Let $\mathcal{E}$ satisfy NS with $\mathrm{Dec}(\mathcal{E})$ reflective. Then DQO holds.*

*Proof.* Let $\Pi \dashv \mathcal{I}$ be the reflection. Let $p : 1 \to \mathcal{I}\Pi$ be its unit. Since every arrow $X \to 2$ in $\mathcal{E}$ factors through $p_X$ and thus also through its image. Since the image is also decidable, then it is universal and hence the unit of the adjunction. Whence $p_X$ is epic. Thus there exists a decidable quotient that factors arrows to $2$. By 2.1, $p_X$ has pneumoconnected fibers.

To verify uniqueness, let $q : X \twoheadrightarrow Q$ and $q' : X \twoheadrightarrow Q'$ with $Q$ and $Q'$ decidable be two quotients satisfying the factorization property of DQO. Then, by (iii), there are arrows $Q \to Q'$ and $Q' \to Q$ which are necessarily inverses of each other. Thus one verifies DQO. □

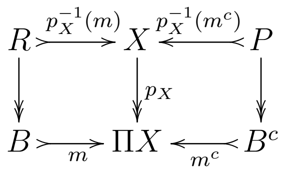
In his context, McLarty [6] proves that $\mathrm{Dec}(\mathcal{E})$ is actually a topos (see Menni [8] for some generalizations).

The following result also invokes 2.1 and provides necessary and sufficient conditions for $\mathrm{Dec}(\mathcal{E})$ to be a topos.

**2.4 Corollary.** *Let $\mathcal{E}$ be a nondegenerate topos satisfying NS and* DQO. *The category $\mathrm{Dec}(\mathcal{E})$ is a topos if and only if the arrow $\Pi(f)$ is epic for every $\neg\neg$-dense arrow $f$.*

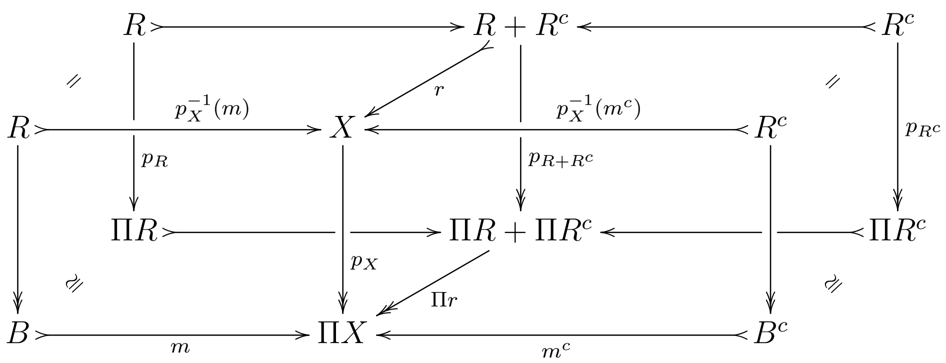
*Proof.* Proposition VI.1 in [5] establishes several equivalences for a topos to be boolean. Among which is that every subobject is complemented. Also, that the operator $\neg\neg$ is the identity, i.e. there are no nontrivial dense subobjects.

Suppose that $\Pi(f)$ is epic for every $\neg\neg$-dense arrow $f : A \to X$. Now, let $m : B \rightarrowtail \Pi X$ be a monic arrow. Consider the following pullback diagrams:




Since inverse images preserve pseudocomplements, $p_X^{-1}(m^c) \cong p_X^{-1}(m)^c$, without loss of generality $P = R^c$, and, by 1.6, $B \cong \Pi R$ and $B^c \cong \Pi R^c$.

Now, as $r : R + R^c \rightarrowtail X$ is $\neg\neg$-dense, $\Pi(r)$ is epic. Consider the following commutative diagram:




Therefore, as $\Pi R^c \cong B^c$ and accordingly $\Pi R + \Pi R^c \cong \Pi R \cup \Pi R^c$, then $\Pi r$ is monic. So $B$ is complemented, and thus $\mathrm{Dec}(\mathcal{E})$ is a topos with $2$ as its subobject classifier.

Conversely, suppose $\mathrm{Dec}(\mathcal{E})$ is a topos, then it must be boolean (see Acuña Ortega and Linton[1, Observation 2.6]). Let $f : X \to Y$ be $\neg\neg$-dense, since the composition of $\neg\neg$-dense arrows is $\neg\neg$-dense, it follows that $\Pi(f) \circ p_X = p_Y \circ f$ is dense. Hence it must be epic. $\square$

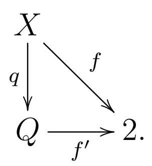
The following result shows that the property of having pneumoconnected fibers is also present in the canonical map from an object to its sheafification.

**2.5 Corollary.** *Let $\mathcal{E}$ be topos satisfying NS and let $m^X_{\neg\neg} : X \twoheadrightarrow M_{\neg\neg}X$ be the reflector of the inclusion of category of $\neg\neg$-separated objects of $\mathcal{E}$. Then $m_{\neg\neg}$ has pneumoconnected fibers. Consequently, the sheafification functor also has pneumoconnected fibers*[1].

*Proof.* Immediate from 2.1, since every decidable object is $\neg\neg$-separated. □

The proof of 2.1 is split into the next two results.

**2.6 Proposition.** *Let $\mathcal{E}$ be a topos satisfying NS. Let $q : X \to Q$ be such that for every arrow $f : X \to 2$ there is an arrow $f' : Q \to 2$ making the following diagram commutative:*




*Then $q$ has pneumoconnected fibers.*

*Proof.* Define

$$R := \{z \in Q : \exists v \in P_c(X)((q^{-1}(z) \cap v \neq \varnothing) \wedge (q^{-1}(z) \cap v^c \neq \varnothing))\}.$$

Suppose for contradiction that $R$ is not initial. Then, there is a point $a : 1 \to Q$ and a complemented $A \rightarrowtail X$ such that

$$q^{-1}(a) \cap A \neq \varnothing \wedge q^{-1}(a) \cap A^c \neq \varnothing. \tag{5}$$

For $A$ corresponds an arrow $\xi : X \to 2$. Let $\xi' : Q \to 2$ be such that $\xi = \xi' q$. That means that $\xi' a$ must be both $0!$ and $1!$. A contradiction. Therefore, $R$ is initial. This means that

$$\forall v \in P_c(X)\neg((q^{-1}(z) \cap v \neq \varnothing) \wedge (q^{-1}(z) \cap v^c \neq \varnothing)$$

[1]In the presence of NS, this suggests there ought to be a description that characterizes the behavior of the fibers of $m^X_j : X \to M_j X$ for an arbitrary local operator $j$, which might then provide a description for the required behavior of the fibers of the unit of $f^* f_!$ of an arbitrary precohesion $f$. Nothing thus far eases the work required to syntactically describe the image of $f^*$.

is universally valid for $z \in Q$, which in turn means that

$$\neg\neg((q^{-1}(z) \cap v = \varnothing) \vee (q^{-1}(z) \cap v^c = \varnothing))$$

is universally valid for $z \in Q$ and $v \in P_c(X)$, as promised. □

**2.7 Proposition.** *Let $\mathcal{E}$ be a topos satisfying NS and $q : X \twoheadrightarrow Q$ be epic with pneumoconnected fibers. Every arrow $X \to Y$ with $Y$ decidable factors through $q$.*

*Proof.* Given an arrow $f : X \to Y$ with $Y$ decidable, let $|f| \rightarrowtail X \times Y$ and $|q| \rightarrowtail X \times Q$ be the graphs of $f$ and $q$ respectively. The goal is to find $f'$ through it graph. Consider the subobject

$$G = \{\langle z, y\rangle : \exists x(\langle x, z\rangle \in |q| \wedge \langle x, y\rangle \in |f|)\} \rightarrowtail Q \times Y.$$

Since $q$ is epic,

$$\exists y(\langle z, y\rangle \in G). \tag{6}$$

is universally valid for $z \in Q$. To see that $G$ is indeed the graph of a function $f'$—as per (2)—, what remains to verify is uniqueness in $y$. Let

$$R := \{z \in Q : \exists\langle y, y'\rangle \in \Delta_Y^c.\langle z, y\rangle \in G \wedge \langle z, y'\rangle \in G\}$$

Assume for contradiction that $R$ is not initial. Then there are points $a : 1 \to Q$, $b, c : 1 \to Y$ such that $b$ is distinct from $c$ and such that $\langle a, b\rangle$ and $\langle a, c\rangle$ factor through $G$. That means that there exists points $d : 1 \to q^{-1}(a) \cap f^{-1}(b)$ and $e : 1 \to q^{-1}(a) \cap f^{-1}(c)$.

Since $Y$ is decidable, $b$ complemented and thus so is $f^{-1}(b)$. As $f^{-1}(c)$ is a subobject of $f^{-1}(b)^c$, $(q^{-1}(a) \cap (f^{-1}(b))^c \neq \varnothing)$. This means that $q^{-1}(a)$ would not be connected. A contradiction.

Therefore, by NS, $R$ must be initial. Thus,

$$\neg\exists\langle y, y'\rangle \in \Delta_Y^c.\langle z, y\rangle \in G \wedge \langle z, y'\rangle \in G$$

is universally valid for $z \in Q$. Wherefrom,

$$\langle y, y'\rangle \in \Delta_Y^c \Rightarrow \neg(\langle z, y\rangle \in G \wedge \langle z, y'\rangle \in G)$$

is universally valid for $z \in Q$ and $\langle y, y' \rangle \in Y \times Y$. By contrapositive,

$$(\langle z, y \rangle \in G \wedge \langle z, y' \rangle \in G) \Rightarrow \neg\neg(y = y')$$

is universally valid for $z \in Q$ and $\langle y, y' \rangle \in Y \times Y$, since $\alpha$ always implies $\neg\neg\alpha$. But using the decidability once more,

$$(\langle z, y \rangle \in G \wedge \langle z, y' \rangle \in G) \Rightarrow y = y'$$

is universally valid for $z \in Q$ and $\langle y, y' \rangle \in Y \times Y$. Which yields uniqueness. Therefore, $G$ is the graph of an arrow $f' : Q \to Y$ that factors $f$. □

By delving deeper into the internal logic of the topos it is possible to do away with NS in the previous proof, yet this would go beyond the present purposes.

*Proof of the Fiber Pneumoconnectedness Lemma 2.1.* 2.6 yields (i)⇒(ii), 2.7 yields (ii)⇒(iii). Finally (iii)⇒(i) is trivial since 2 is decidable. □

## 3. Precohesiveness

Recall that a topos $\mathcal{E}$ is precohesive over a topos $\mathcal{S}$ if there is a string of adjunctions

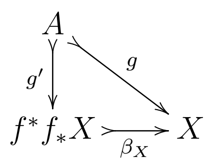
$$f_! \dashv f^* \dashv f_* \dashv f^! : \mathcal{E} \to \mathcal{S} \tag{7}$$

such that $f^*$ is fully faithful, $f_!$ preserves finite products, and that the counit $f^* f_* \to 1$ is monic (See Lawvere and Menni [3, Lemma 3.2]).

From Theorem A and from the results in [6] it is evident that NS + DSO + DQO yields that $\mathcal{E}$ is precohesive over $\mathrm{Dec}(\mathcal{E})$.

To provide a converse in the presence of NS, let $f$ be as in (7) over a boolean base. It is proved in 2.3 that DQO holds on $\mathcal{E}$.

Since this means in particular that the unit $\sigma : 1 \to f^* f_!$ is epic, by [3, Proposition 2.2]) this is equivalent to the counit $\beta : f^* f_* \to 1$ being monic. To verify the uniqueness in DSO, let $g : A \rightarrowtail X$ with $A$ decidable have the same factoring property for arrows from 1. Then there is a unique arrow $g' : A \rightarrowtail f^* f_* X$ such that the following diagram commutes:



It remains to see that $g'$ is epic. By virtue of 2.4, it suffices to verify that it is dense, since then

$$\begin{array}{ccc} A & \xrightarrow{\;g'\;} & f_*X \\ {\scriptstyle \cong}\downarrow{\scriptstyle \sigma_A} & & {\scriptstyle \sigma_{f_*X}}\downarrow{\scriptstyle \cong} \\ f_!A & \underset{f_!g'}{\twoheadrightarrow} & f_!f_*X \end{array}$$

$f_!g'$ is epic and thus so is $g'$.

To this effect, notice that by NS, exactly one of the following is universally valid: $f^*f_*X \cap (A^{\neg\neg})^c = \varnothing$ or $\neg(f^*f_*X \cap (A^{\neg\neg})^c = \varnothing)$, since neither has free variables and are thus interpreted as points in $\Omega$. Assuming for contradiction the latter holds, there exists a global element $a : 1 \to f^*f_*X \cap (A^{\neg\neg})^c$. But since $f^*f_*X \cap (A^{\neg\neg})^c \rightarrowtail X$, $a$ factors through $A$. That is,

$$a \in A \wedge \neg(a \in A)$$

would hold—a contradiction. Therefore, $f_*X \cap (A^{\neg\neg})^c = \varnothing$ holds. Whence $A$ is $\neg\neg$-dense in $f_*X$. This finishes the proof of Theorem C. $\square$

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# FIBRATIONS OF DOUBLE GROUPOIDS, I: ALGEBRAIC PROPERTIES AND HOMOTOPY SEQUENCES.

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**Résumé.** Nous introduisons la notion de fibration de double groupoïdes, que nous définissons comme un foncteur double possédant une propriété spécifique de remplissage-relèvement. Nous en étudions les propriétés fondamentales, notamment en établissant des suites exactes d'homotopie, parmi lesquelles figurent une suite de Mayer-Vietoris associée à un changement de base, ainsi que la suite d'homotopie propre à une fibration. Nous construisons également le module croisé fondamental d'une fibration.

**Abstract.** We introduce the notion of fibrations of double groupoids, defined as double functors with a specific filling-lifting property, and study their main properties. In particular, we establish exact homotopy sequences, including a Mayer-Vietoris sequence arising from a change of base, and the homotopy sequence associated to a fibration. We also construct the fundamental crossed module of a fibration.

**Keywords.** Double groupoid, Fibration, Homotopy groups, Crossed module, Geometric realization.

**Mathematics Subject Classification (2020).** 18N10, 20L05, 55P15, 55R15.

## Introduction and summary.

Double groupoids, that is, groupoid objects in the category of groupoids, were introduced by Ehresmann [22, 23, 24] in the late 20th century and have since been studied by several researchers due to their connections with various areas of mathematics. In particular, (small) double groupoids have garnered interest in algebraic topology, largely thanks to the work of Brown, Higgins, Porter, and others, where the connection of double groupoids with crossed modules and a higher Seifert-van Kampen Theory has been established (see, for example, the survey [7] and the references therein).

This is the third paper in a series exploring some purely algebraic properties of double groupoids using methods inspired by the topological context. In [17], we addressed the homotopy types obtained from double groupoids satisfying a quite natural filling condition. Like topological spaces, these double groupoids have associated homotopy groups, which are defined combinatorially using only their algebraic structure. Thus, the notion of weak equivalence between such double groupoids arises, and a corresponding homotopy category is defined. A main result states that the homotopy category of double groupoids with the filling property is equivalent to the homotopy category of all topological spaces with the property that the $n$th homotopy group at any base point vanishes for $n \geq 3$ (that is, the category of homotopy 2-types). Similar to the theory of Postnikov invariants with homotopy 2-types, the paper [19] provides a precise and purely algebraic classification for weak equivalence classes of double groupoids by three-cohomology classes.

This work and its companion paper [18] deal with *fibrations of double groupoids*, which we introduce as those double functors between double groupoids $F : \mathcal{A} \to \mathcal{B}$ that always solve certain filling-lifting problems on morphisms and boxes (see Definition 2.1 for precision). For instance, a double groupoid $\mathcal{A}$ has the filling property if and only if the double functor $\mathcal{A} \to *$, from $\mathcal{A}$ to the final double groupoid $*$, is a fibration. If $\mathcal{A}$ and $\mathcal{B}$ are 2-groupoids, regarded as double groupoids where one of the side groupoids of morphisms is discrete, then a fibration $F : \mathcal{A} \to \mathcal{B}$ in our sense is the same as a fibration of 2-groupoids in the sense of Moerdijk and Svensson[33, 34], Hermida [28], Buckley [15], or Hardie, Kamps, and Kieboom [27]. By the equivalence between crossed complexes over groupoids and 2-groupoids, our concept of fibration also generalizes the notion of fibration of crossed modules over groupoids by Brown [6]. In particular, if both $\mathcal{A}$ and $\mathcal{B}$ are groupoids, viewed as double groupoids whose vertical morphisms are all identities and whose boxes are all vertical identities, then a fibration $F : \mathcal{A} \to \mathcal{B}$ in our sense is the same as a fibration of groupoids in the sense of Grothendieck [25] and Brown [2]. However, our concept of fibration is more restrictive than the notion of *double fibration* proposed by Cruttwell, Lambert, Pronk, and Szyld in [20].

After Section 1, where we briefly establish some notational conventions on double groupoids, Sections 2 and 3 present the concept of fibration between double groupoids and study its basic properties, such as the change of base property, the filling property of fibres, and the path-lifting and homotopy-lifting properties. In Section 3, we also review several necessary definitions and results for the fundamental groupoid and the homotopy groups of a double groupoid. In Section 4, we show how a Mayer-Vietoris type exact sequence on homotopy groups arises from a change in the base of a fibration of double groupoids. This is used in Section 5 to derive a 9-term exact sequence on homotopy groups from a fibre sequence of double groupoids. This section also includes additional information about this homotopy sequence that relates to the actions of fundamental groupoids on the homotopy groups of fibres. In particular, we construct the fundamental crossed module over groupoids of a fibration of double groupoids. Our results in Sections 4 and 5 are deeply inspired by those we generalize, stated by Brown in [2] and Brown, Heath, and Kamps in [10] for groupoids, by Brown in [6] and Howie in [29] for crossed modules over groupoids, and by Hardie, Kamps, and Kieboom in [27] and Kamps and Porter in [30] for 2-groupoids.

Concerning the relationship between fibrations of double groupoids and simplicial and topological fibrations, we refer the reader to the companion paper [18].

## 1. Some conventions on double groupoids.

The notion of a double groupoid is well-known; in this preliminary section, we specify some basic terminology and notational conventions. We will work only with small double groupoids, so that in a double groupoid $\mathcal{A}$ we have a set of objects (usually denoted by $a, b, c, \ldots$), horizontal morphisms between them $(f, g, h, \ldots)$, vertical morphisms between them $(x, y, z, \ldots)$, both with composition written by juxtaposition, and boxes $(\alpha, \beta, \gamma, \ldots)$, usually depicted as

$$\begin{array}{ccc} d & \xleftarrow{\;f\;} & b \\ {\scriptstyle y}\uparrow & \alpha & \uparrow{\scriptstyle x} \\ c & \xleftarrow[\;g\;]{} & a \end{array} \tag{1}$$

where the horizontal morphisms $f$ and $g$ are, respectively, its vertical target and source, and the vertical morphisms $y$ and $x$ are its respective horizontal target and source. The horizontal composition of boxes is denoted by the symbol $\circ_{\mathrm{h}}$:

$$\begin{array}{ccccc} \cdot & \xleftarrow{\;f'\;} & \cdot & \xleftarrow{\;f\;} & \cdot \\ {\scriptstyle z}\uparrow & \alpha' & \uparrow & \alpha & \uparrow{\scriptstyle x} \\ \cdot & \xleftarrow[\;g'\;]{} & \cdot & \xleftarrow[\;g\;]{} & \cdot \end{array} \quad \mapsto \quad \begin{array}{ccc} \cdot & \xleftarrow{\;f'f\;} & \cdot \\ {\scriptstyle z}\uparrow & \alpha' \circ_{\mathrm{h}} \alpha & \uparrow{\scriptstyle x} \\ \cdot & \xleftarrow[\;g'g\;]{} & \cdot \end{array}$$

and the vertical composition of boxes is denoted by the symbol $\circ_{\mathrm{v}}$:

$$\begin{array}{ccc} \cdot & \xleftarrow{\;f\;} & \cdot \\ {\scriptstyle y}\uparrow & \alpha & \uparrow{\scriptstyle x} \\ \cdot & \longleftarrow & \cdot \\ {\scriptstyle y'}\uparrow & \alpha' & \uparrow{\scriptstyle x'} \\ \cdot & \xleftarrow[\;h\;]{} & \cdot \end{array} \quad \mapsto \quad \begin{array}{ccc} \cdot & \xleftarrow{\;f\;} & \cdot \\ {\scriptstyle yy'}\uparrow & \alpha \circ_{\mathrm{v}} \alpha' & \uparrow{\scriptstyle xx'} \\ \cdot & \xleftarrow[\;h\;]{} & \cdot \end{array}$$

Horizontal and vertical identities on objects and morphisms are respectively denoted by $\mathrm{I}^{\mathrm{h}}_a$, $\mathrm{I}^{\mathrm{v}}_a$, $\mathrm{I}^{\mathrm{v}}_f$, $\mathrm{I}^{\mathrm{h}}_x$, and $\mathrm{I}_a = \mathrm{I}^{\mathrm{v}}_{\mathrm{I}^{\mathrm{h}}_a} = \mathrm{I}^{\mathrm{h}}_{\mathrm{I}^{\mathrm{v}}_a}$, depicted as

$$a = a \qquad \begin{array}{c} a \\ \| \\ a \end{array} \qquad \begin{array}{ccc} \cdot & \xleftarrow{\;f\;} & \cdot \\ \| & \mathrm{I}^{\mathrm{v}}_f & \| \\ \cdot & \xleftarrow[\;f\;]{} & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & = & \cdot \\ {\scriptstyle x}\uparrow & \mathrm{I}^{\mathrm{h}}_x & \uparrow{\scriptstyle x} \\ \cdot & = & \cdot \end{array} \qquad \begin{array}{ccc} a & = & a \\ \| & \mathrm{I}_a & \| \\ a & = & a \end{array}$$

and horizontal and vertical inverses of boxes are respectively denoted by $\alpha^{-\mathrm{h}}$, $\alpha^{-\mathrm{v}}$, and $\alpha^{-\mathrm{hv}} = (\alpha^{-\mathrm{h}})^{-\mathrm{v}} = (\alpha^{-\mathrm{v}})^{-\mathrm{h}}$; that is, for $\alpha$ as in (1),

$$\begin{array}{ccc} \cdot & \xleftarrow{\;f^{-1}\;} & \cdot \\ {\scriptstyle x}\uparrow & \alpha^{-\mathrm{h}} & \uparrow{\scriptstyle y} \\ \cdot & \xleftarrow[\;g^{-1}\;]{} & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \xleftarrow{\;g\;} & \cdot \\ {\scriptstyle y^{-1}}\uparrow & \alpha^{-\mathrm{v}} & \uparrow{\scriptstyle x^{-1}} \\ \cdot & \xleftarrow[\;f\;]{} & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \xleftarrow{\;g^{-1}\;} & \cdot \\ {\scriptstyle x^{-1}}\uparrow & \alpha^{-\mathrm{hv}} & \uparrow{\scriptstyle y^{-1}} \\ \cdot & \xleftarrow[\;f^{-1}\;]{} & \cdot \end{array}$$

We will frequently use the coherence theorem by Dawson and Paré [21, Theorem 1.2], which ensures that *if a compatible arrangement of boxes in a double groupoid is composable in two different ways, the resulting pasted boxes are equal*. Throughout the paper, when we refer to an equality between pasting diagrams of boxes in a double groupoid, we means that the resulting pasted boxes are the same.

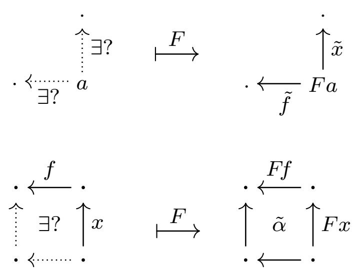
## 2. Fibrations between double groupoids.

A double functor $F : \mathcal{A} \to \mathcal{B}$ between double groupoids maps objects, horizontal and vertical morphisms, and boxes in $\mathcal{A}$ to objects, horizontal and vertical morphisms, and boxes in $\mathcal{B}$, respectively, in such a way that it preserves compositions and identities.

**Definition 2.1.** *A double functor $F : \mathcal{A} \to \mathcal{B}$ between double groupoids is a* fibration *if all lifting problems*

*(i)*

*(ii)*



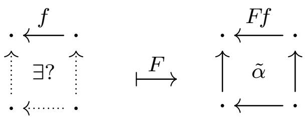

*have solution. That is,*

*(i) If $a$ is an object of $\mathcal{A}$, for any horizontal (resp. vertical) morphism $\tilde{f}$ (resp. $\tilde{x}$) in $\mathcal{B}$ with source $Fa$, there is a horizontal (resp. vertical) morphism $f$ (resp. $x$) in $\mathcal{A}$ with source $a$ such that $Ff = \tilde{f}$ (resp. $Fx = \tilde{x}$).*

*(ii) If $f$ is a horizontal morphism of $\mathcal{A}$ and $x$ is a vertical morphism of $\mathcal{A}$ whose target is the source of $f$, for any box $\tilde{\alpha}$ of $\mathcal{B}$ with vertical target $Ff$ and horizontal source $Fx$, there is a box $\alpha$ in $\mathcal{A}$ with vertical target $f$ and horizontal source $x$ such that $F\alpha = \tilde{\alpha}$.*

The above fibration condition $(i)$ means that the restrictions of $F$ to the respective groupoids of horizontal and vertical morphisms of $\mathcal{A}$ and $\mathcal{B}$ are both fibrations of groupoids in the sense of Brown [2] or Grothendieck [25]. In fact, if both $\mathcal{A}$ and $\mathcal{B}$ are groupoids, considered as double groupoids with all vertical morphisms as identities and all boxes as vertical identities, then a fibration $F : \mathcal{A} \to \mathcal{B}$ in the sense of Definition 2.1 is the same as a fibration of groupoids. Furthermore, if $\mathcal{A}$ and $\mathcal{B}$ are 2-groupoids, regarded as double groupoids whose vertical arrows are all identities, then a fibration $F : \mathcal{A} \to \mathcal{B}$ is the same as a fibration of 2-groupoids in the sense of Moerdijk and Svensson [33, 34] (see [33, Lemma 1.7.3]) or, equivalently, a 2-fibration as defined by Hermida [28] or Buckley [15].

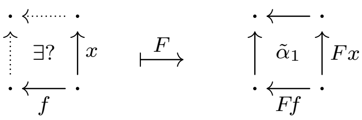
However, our concept of fibration between double groupoids is more restrictive than the notion of *double fibration* proposed by Cruttwell, *et al.* [20]. A double functor between double groupoids $F : \mathcal{A} \to \mathcal{B}$ is a double fibration in the sense of [20, Definition 2.25] whenever its restriction to the groupoids of vertical morphisms is a fibration, and every lifting problem



has a solution. If $F$ is a fibration as in Definition 2.1, we can first select a vertical morphism $x$ with target the source of $f$, which is carried by $F$ to the vertical target of $\tilde{\alpha}$ and then to find a box $\alpha$ in $\mathcal{A}$ with vertical target $f$ and horizontal source $x$ such that $F\alpha = \tilde{\alpha}$. Thus, every fibration between double groupoids is a double fibration. However, the converse is false because, for example, double fibration does not necessary restrict fibration between the groupoids of horizontal morphisms.

The fibration conditions are more symmetric than they appears:

**Lemma 2.2.** *If $F : \mathcal{A} \to \mathcal{B}$ is a fibration of double groupoids, then any of the three lifting problems below has a solution.*



$$\begin{array}{ccc} \cdot & \xleftarrow{f} & \cdot \\ {\scriptstyle x}\uparrow & \exists ? & \vdots \\ \cdot & \leftarrow\cdots & \cdot \end{array} \quad \overset{F}{\longmapsto} \quad \begin{array}{ccc} \cdot & \xleftarrow{Ff} & \cdot \\ {\scriptstyle Fx}\uparrow & \tilde{\alpha}_2 & \uparrow \\ \cdot & \longleftarrow & \cdot \end{array}$$

$$\begin{array}{ccc} \cdot & \leftarrow\cdots & \cdot \\ {\scriptstyle x}\uparrow & \exists ? & \vdots \\ \cdot & \xleftarrow[f]{} & \cdot \end{array} \quad \overset{F}{\longmapsto} \quad \begin{array}{ccc} \cdot & \longleftarrow & \cdot \\ {\scriptstyle Fx}\uparrow & \tilde{\alpha}_3 & \uparrow \\ \cdot & \xleftarrow[Ff]{} & \cdot \end{array}$$

*Proof.* Since $F$ is a fibration, there are boxes in $\mathcal{A}$

$$\begin{array}{ccc} \cdot & \xleftarrow{f} & \cdot \\ \uparrow & \alpha_1 & \uparrow{\scriptstyle x^{-1}} \\ \cdot & \longleftarrow & \cdot \end{array}, \quad \begin{array}{ccc} \cdot & \xleftarrow{f^{-1}} & \cdot \\ \uparrow & \alpha_2 & \uparrow{\scriptstyle x} \\ \cdot & \longleftarrow & \cdot \end{array}, \quad \begin{array}{ccc} \cdot & \xleftarrow{f^{-1}} & \cdot \\ \uparrow & \alpha_3 & \uparrow{\scriptstyle x^{-1}} \\ \cdot & \longleftarrow & \cdot \end{array}$$

such that $F\alpha_1 = \tilde{\alpha}_1^{-\mathrm{v}}$, $F\alpha_2 = \tilde{\alpha}_2^{-\mathrm{h}}$, and $F\alpha_3 = \tilde{\alpha}_3^{-\mathrm{hv}}$. Then, $\alpha_1^{-\mathrm{v}}$, $\alpha_2^{-\mathrm{h}}$, and $\alpha_3^{-\mathrm{hv}}$ are solutions to the respective lifting problems. □

Let $*$ denote the final double groupoid; that is, the double groupoid with only one object, $*$, one vertical morphism, $\mathrm{I}^{\mathrm{v}}_*$, one horizontal morphism, $\mathrm{I}^{\mathrm{h}}_*$, and one box

$$\begin{array}{ccc} * & = & * \\ \| & \mathrm{I}_* & \| \\ * & = & * \end{array}$$

If $\mathcal{A}$ is a double groupoid, then the double functor $\mathcal{A} \to *$ is a fibration if and only if $\mathcal{A}$ has the so-called *filling property*: *Any filling problem*

$$\begin{array}{ccc} \cdot & \xleftarrow{f} & \cdot \\ \vdots & \exists ? & \uparrow{\scriptstyle x} \\ \cdot & \leftarrow\cdots & \cdot \end{array}$$

*has a solution*. This filling condition on double groupoids is often assumed in the case of double groupoids arising in different areas of mathematics, such as in weak Hopf algebra theory or in differential geometry (see, for instance, Andruskiewitsch and Natale [1] and Mackenzie [32]). It is also satisfied for those double groupoids that have emerged with an interest in algebraic topology, mainly thanks to the work of Brown, Higgins, Spencer, *et al.* (see the papers by Brown [3, 4, 7, 8] and the references given there). Thus, the filling condition is easily proven for edge symmetric double groupoids (also called special double groupoids) with connections (see Brown and Higgins [12], Brown and Spencer [14], Brown, Hardie, Kamps and Porter [9], and Brown, Kamps and Porter [13]), for double groupoid objects in the category of groups (also termed $\mathrm{cat}^2$-groups by Loday [31], see also Porter [35] and Bullejos, Cegarra and Duskin [16]), or, for example, for 2-groupoids (regarded as double groupoids where one of the side groupoids of morphisms is discrete (see for instance Moerdijk and Svensson [34] and Hardie, Kamps and Kieboom [26]).

Lemma 2.2 implies the following useful result by Andruskiewitsch and Natale [1, Lemma 1.12].

**Corollary 2.3.** *In a double groupoid satisfying the filling condition, any of the filling problems below has a solution:*

$$\begin{array}{ccc} \cdot & \overset{}{\dashleftarrow} & \cdot \\ \vdots\!\uparrow & \exists? & \uparrow x \\ \cdot & \underset{f}{\leftarrow} & \cdot \end{array}\;,\qquad \begin{array}{ccc} \cdot & \dashleftarrow & \cdot \\ x\uparrow & \exists? & \vdots\!\uparrow \\ \cdot & \underset{f}{\leftarrow} & \cdot \end{array}\;,\qquad \begin{array}{ccc} \cdot & \overset{f}{\leftarrow} & \cdot \\ x\uparrow & \exists? & \vdots\!\uparrow \\ \cdot & \dashleftarrow & \cdot \end{array}\;,$$

**Proposition 2.4.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a fibration of double groupoids.*

*(i) If $\mathcal{B}$ has the filling property, then $\mathcal{A}$ also has the filling property.*

*(ii) If $\mathcal{A}$ has the filling property and $F$ is onto on objects, then $\mathcal{B}$ has the filling property.*

*Proof.* $(i)$ Suppose $\mathcal{B}$ has the filling property, and let

$$\begin{array}{ccc} \cdot & \overset{f}{\leftarrow} & \cdot \\ \vdots\!\uparrow & \exists? & \uparrow x \\ \cdot & \dashleftarrow & \cdot \end{array}$$

be a filling problem in $\mathcal{A}$. Choose a box $\tilde{\alpha}$ in $\mathcal{B}$ of the form

$$\begin{array}{ccc} \cdot & \overset{Ff}{\leftarrow} & \cdot \\ \uparrow & \tilde{\alpha} & \uparrow Fx \\ \cdot & \leftarrow & \cdot \end{array}$$

Then, as $F$ is a fibration, we may choose a box $\alpha$ in $\mathcal{A}$ with vertical target $f$ and horizontal source $x$ such that $F\alpha = \tilde{\alpha}$. In particular, $\alpha$ solves the given filling problem in $\mathcal{A}$.

$(ii)$ Suppose $\mathcal{A}$ has the filling property, and let

$$\begin{array}{ccc} \cdot & \xleftarrow{\tilde{f}} & \cdot \\ \uparrow & \exists? & \uparrow{\scriptstyle \tilde{x}} \\ \cdot & \longleftarrow & \cdot \end{array}$$

be a filling problem in $\mathcal{B}$. Since $F$ is onto on objects, we can choose an object $a$ of $\mathcal{A}$ such that $Fa$ is the source of $\tilde{x}$. Using that $F$ is a fibration, we can choose a vertical morphism $x$ of $\mathcal{A}$ with source $a$ such that $fx = \tilde{x}$, as well as a horizontal morphism with source the target of $x$ such that $Ff = \tilde{f}$. Since $\mathcal{A}$ has the filling property, we can choose a box $\alpha$ of $\mathcal{A}$

$$\begin{array}{ccc} \cdot & \xleftarrow{f} & \cdot \\ \uparrow & \alpha & \uparrow{\scriptstyle x} \\ \cdot & \longleftarrow & \cdot \end{array}$$

whose respective horizontal and vertical sources are $f$ and $x$. Obviously, $F\alpha$ solves the given filling problem in $\mathcal{B}$. □

**Proposition 2.5.** *In a pullback square of double groupoids*

$$\begin{array}{ccc} \mathcal{B}' \times_{\mathcal{B}} \mathcal{A} & \xrightarrow{G'} & \mathcal{A} \\ {\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{B}' & \xrightarrow[G]{} & \mathcal{B} \end{array}$$

*if $F$ is a fibration, then so also is $F'$.*

*Proof.* (i) By [2, Prop. 2.8], the projection $F'$ restricts giving fibrations both between the groupoids of horizontal and vertical morphisms.

(ii) Suppose given a box $\alpha'$ of $\mathcal{B}'$ and morphisms $f$ and $x$ in $\mathcal{A}$ as in

$$\begin{array}{ccc} \cdot & \xleftarrow{f'} & \cdot \\ \uparrow & \alpha' & \uparrow{\scriptstyle x'} \\ \cdot & \longleftarrow & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \xleftarrow{f} & \cdot \\ & & \uparrow{\scriptstyle x} \\ & & \cdot \end{array}$$

such that $Gf' = Ff$ and $Gx' = Fx$. Since $F$ is a fibration, there is a solution in $\mathcal{A}$, say $\alpha$, to the lifting problem

$$\begin{array}{ccc} \begin{array}{ccc} \cdot & \xleftarrow{f} & \cdot \\ \uparrow & \exists? & \uparrow x \\ \cdot & \longleftarrow & \cdot \end{array} & \xmapsto{F} & \begin{array}{ccc} \cdot & \longleftarrow & \cdot \\ \uparrow & G\alpha' & \uparrow \\ \cdot & \longleftarrow & \cdot \end{array} \end{array}$$

Then $(\alpha', \alpha)$ is a box in $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$ satisfying that $F'(\alpha', \alpha) = \alpha'$. □

Let $F : \mathcal{A} \to \mathcal{B}$ be a fibration of double groupoids. If $b$ is an object of $\mathcal{B}$, let $\mathcal{F}_b = F^{-1}(b)$ denote the *double groupoid fibre* of $F$ over $b$. That is, $\mathcal{F}_b$ is the double subgroupoid of $\mathcal{A}$ with objects those $a$ of $\mathcal{A}$ such that $Fa = b$, vertical morphisms those vertical morphisms $x$ of $\mathcal{A}$ such that $Fx = \mathrm{I}_b^{\mathrm{v}}$, horizontal morphisms those horizontal morphisms $f$ of $\mathcal{A}$ such that $Ff = \mathrm{I}_b^{\mathrm{h}}$, and boxes those $\alpha$ of $\mathcal{A}$ such that $F\alpha = \mathrm{I}_b$. For every object $a$ of $\mathcal{F}_b$, we call the sequence

$$(\mathcal{F}_b, a) \hookrightarrow (\mathcal{A}, a) \xrightarrow{F} (\mathcal{B}, b)$$

a *fibre sequence* of double groupoids.

**Proposition 2.6.** *In any fibre sequence as above, the double groupoid fibre $\mathcal{F}_b$ has the filling property.*

*Proof.* This follows from Proposition 2.5, since $\mathcal{F}_b$ occurs in the pullback square of double groupoids

$$\begin{array}{ccc} \mathcal{F}_b & \hookrightarrow & \mathcal{A} \\ \downarrow & & \downarrow F \\ * & \xrightarrow{b} & \mathcal{B} \end{array}$$

□

## 3. Paths, loops, homotopies, homotopy groups.

In this section, we work under the assumption that the double groupoids satisfy the filling condition.

Let $\mathcal{A}$ be a double groupoid. A *path* in $\mathcal{A}$ from an objet $a$ to an object $a'$ [17, §2], denoted by

$$(f, x) : a \curvearrowright a',$$

is a pair of morphisms $(f, x)$ where $x$ is a vertical morphism from $a$ and $f$ is a horizontal morphism from the target of $x$ to $a'$; that is, a pair of morphisms of the form

$$\begin{array}{ccc} a' & \xleftarrow{f} & \cdot \\ & & \uparrow x \\ & & a \end{array} \tag{2}$$

When $a' = a$, we say that $(f, x) : a \curvearrowright a$ is a *loop* with base point $a$. The *identity loop* at an object $a$ is the loop $(\mathrm{I}^{\mathrm{h}}_a, \mathrm{I}^{\mathrm{v}}_a) : a \curvearrowright a$, which is depicted as

$$\begin{array}{ccc} a & = & a \\ & & \| \\ & & a \end{array}$$

**Proposition 3.1** (Path-lifting property)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a fibration of double groupoids. For every object $a$ of $\mathcal{A}$ and every path $(\tilde{f}, \tilde{x}) : Fa \curvearrowright b$ in $\mathcal{B}$, there exists a path $(f, x) : a \curvearrowright a'$ such that $(Ff, Fx) = (\tilde{f}, \tilde{x})$.*

*Proof.* Since $F$ is a fibration, we can choose a vertical morphism in $\mathcal{A}$ with source $a$, say $x : a \to a''$, such that $Fx = \tilde{x}$. Since $Fa''$ is the source of $\tilde{f}$, we can also choose a horizontal morphism $f : a'' \to a'$ in $\mathcal{A}$ such that $Ff = \tilde{f}$. Thus $(f, x) : a \curvearrowright a'$ is a path in $\mathcal{A}$ such that $Ff = \tilde{f}$ and $Fx = \tilde{x}$. □

If $(f, x), (g, y) : a \curvearrowright a'$ are two paths in $\mathcal{A}$, then $(f, x)$ is *homotopic* to $(g, y)$, denoted by $(f, x) \simeq (g, y)$, whenever there is a box $\alpha$ in $\mathcal{A}$ of the form

$$\begin{array}{ccc} \cdot & \xleftarrow{f^{-1}g} & \cdot \\ \| & \alpha & \uparrow yx^{-1} \\ \cdot & = & \cdot \end{array} \tag{3}$$

that is, whose horizontal target and vertical sources are identities, its horizontal source is $yx^{-1}$, and its vertical target is $f^{-1}g$ (see [17, §2]). We call such a box a *homotopy*, and we often write $\alpha : (f, x) \simeq (g, y)$ whenever we wish to display the homotopy.

**Proposition 3.2** (Path homotopy-lifting property)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a fibration of double groupoids. Suppose $(g, y) : a \curvearrowright a'$ is a path in $\mathcal{A}$, $(\tilde{f}, \tilde{x}) : Fa \curvearrowright Fa'$ is a path in $\mathcal{B}$, and $\tilde{\alpha} : (\tilde{f}, \tilde{x}) \simeq (Fg, Fy)$ is a homotopy in $\mathcal{B}$. Then, there is a path $(f, x) : a \curvearrowright a'$ in $\mathcal{A}$ such that $(Ff, Fx) = (\tilde{f}, \tilde{x})$ and there is a homotopy $\alpha : (f, x) \simeq (g, y)$ such that $F\alpha = \tilde{\alpha}$.*

*Proof.* Using the filling property, we can select a box $\tilde{\beta}$ in $\mathcal{B}$ of the form

$$\begin{array}{ccc} Fa' & \overset{\tilde{f}}{\longleftarrow} & \cdot \\ {\scriptstyle \tilde{z}}\uparrow & \tilde{\beta} & \uparrow{\scriptstyle \tilde{x}} \\ \cdot & \underset{\tilde{h}}{\longleftarrow} & Fa \end{array}$$

and construct the box $\tilde{\gamma}$ of $\mathcal{B}$ by

$$\begin{array}{ccc} Fa' & \overset{Fg}{\longleftarrow} & \cdot \\ {\scriptstyle \tilde{z}}\uparrow & \tilde{\gamma} & \uparrow{\scriptstyle Fy} \\ \cdot & \underset{\tilde{h}}{\longleftarrow} & Fa \end{array} \quad = \quad \begin{array}{ccccc} Fa' & \overset{\tilde{f}}{\longleftarrow} & \cdot & \overset{\tilde{f}^{-1}\,Fg}{\longleftarrow} & \cdot \\ \uparrow & & \| & \tilde{\alpha} & \uparrow{\scriptstyle Fy\,\tilde{x}^{-1}} \\ {\scriptstyle \tilde{z}} & \tilde{\beta} & \cdot & = & \cdot \\ | & & {\scriptstyle \tilde{x}}\uparrow & \mathrm{I^h} & \uparrow{\scriptstyle \tilde{x}} \\ \cdot & \underset{\tilde{h}}{\longleftarrow} & Fa & = & Fa \end{array}$$

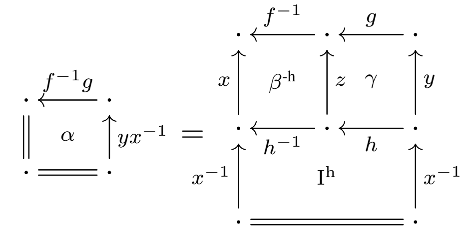
Since $F$ is a fibration, we can choose a box $\gamma$ in $\mathcal{A}$ of the form

$$\begin{array}{ccc} a' & \overset{g}{\longleftarrow} & \cdot \\ {\scriptstyle z}\uparrow & \gamma & \uparrow{\scriptstyle y} \\ \cdot & \underset{h}{\longleftarrow} & a \end{array}$$

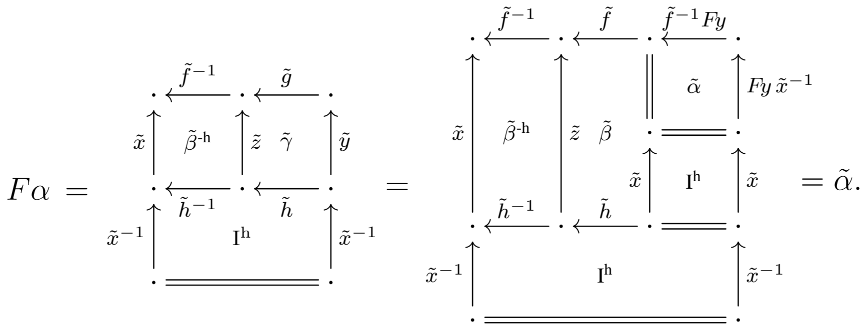
such that $F\gamma = \tilde{\gamma}$, and then (since $Fz = \tilde{z}$ and $Fh = \tilde{h}$) we can also choose a box $\beta$ of the form

$$\begin{array}{ccc} a' & \overset{f}{\longleftarrow} & \cdot \\ {\scriptstyle z}\uparrow & \beta & \uparrow{\scriptstyle x} \\ \cdot & \underset{h}{\longleftarrow} & a \end{array}$$

such that $F\beta = \tilde{\beta}$. Then $(f, x) : a \curvearrowright a'$ is a path in $\mathcal{A}$ satisfying that $(Ff, Fx) = (\tilde{f}, \tilde{x})$. Furthermore, if $\alpha : (f, x) \simeq (g, y)$ is the homotopy defined by



then,



□

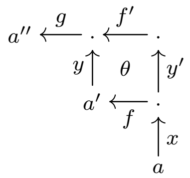
For every pair of objects $a$ and $a'$ of a double groupoid $\mathcal{A}$, by [17, Lemma 2.1], homotopy is an equivalence relation on the set of paths in $\mathcal{A}$ from $a$ to $a'$, and we write $[f, x]$ to denote the homotopy class of a path $(f, x) : a \curvearrowright a'$ in $\mathcal{A}$. These homotopy classes of paths are the morphisms

$$[f, x] : a \to a'$$

of the *fundamental groupoid* of the double groupoid, which is denoted by

$$\Pi\mathcal{A}.$$

The composition of two morphisms $[f, x] : a \to a'$ and $[g, y] : a' \to a''$ in $\Pi\mathcal{A}$ is defined as follows: By the filling condition, we can select a box $\theta$ in $\mathcal{A}$ whose horizontal target is $y$ and whose vertical source is $f$. Thus, we have a diagram in $\mathcal{A}$ of the form



(4)

and we define the composite

$$[g, y] \cdot [f, x] = [gf', y'x] : a \curvearrowright a''. \tag{5}$$

By [17, Lemma 2.2], this composition is well-defined, that is, it *does not depend on the representative paths or on the selection made of the box $\theta$ in* (4). By [17, Theorem 2.3], with this composition, $\Pi\mathcal{A}$ is actually a groupoid. The *identity* of an object $a$ is the morphism represented by the identity loop at $a$,

$$[\mathrm{I}^{\mathrm{h}}_a, \mathrm{I}^{\mathrm{v}}_a] : a \to a.$$

The *inverse* in $\Pi\mathcal{A}$ of a morphism $[f, x] : a \to a'$ represented by a path $(f, x) : a \curvearrowright a'$ is the morphism

$$[f, x]^{-1} = [f', x'] : a' \to a$$

represented by the path $(f', x') : a' \curvearrowright a$ defined by the vertical target and the horizontal source of a (any) box $\gamma$ in $\mathcal{A}$ whose horizontal target is $x^{-1}$ and whose vertical source is $f^{-1}$, that is, of the form

$$\begin{array}{ccc} a & \xleftarrow{f'} & \cdot \\ {\scriptstyle x^{-1}}\uparrow & \gamma & \uparrow{\scriptstyle x'} \\ \cdot & \xleftarrow[f^{-1}]{} & a' \end{array}$$

The set $\pi_0\mathcal{A}$ [19, §3,1], *of path-connected classes of objects* of a double groupoid $\mathcal{A}$, is

$$\pi_0\mathcal{A} = \pi_0(\Pi\mathcal{A}),$$

the set of iso-classes of objects of its fundamental groupoid.

The group $\pi_1(\mathcal{A}, a)$ [19, §3,2], *of homotopy classes of loops in $\mathcal{A}$ based at $a$*, is

$$\pi_1(\mathcal{A}, a) = \mathrm{Aut}_{\Pi\mathcal{A}}(a),$$

the group of automorphisms of $a$ in the fundamental groupoid $\Pi\mathcal{A}$.

The abelian group

$$\pi_2(\mathcal{A}, a)$$

[19, §3,3] consists of all boxes of $\mathcal{A}$ whose horizontal source and target are both $\mathrm{I}^{\mathrm{v}}_a$, the vertical identity of $a$, and whose vertical source and target are both $\mathrm{I}^{\mathrm{h}}_a$, the horizontal identity of $a$; that is, of the form

$$\begin{array}{ccc} a & = & a \\ \| & \sigma & \| \\ a & = & a \end{array}$$

By the Eckman-Hilton argument, the interchange law on $\mathcal{A}$ implies that the operations $\circ_{\mathrm{h}}$ and $\circ_{\mathrm{v}}$ on $\pi_2(\mathcal{A}, a)$ coincide and are commutative. Thus, $\pi_2(\mathcal{A}, a)$ is an abelian group with addition

$$\sigma + \tau = \sigma \circ_{\mathrm{h}} \tau = \sigma \circ_{\mathrm{v}} \tau,$$

zero $0 = \mathrm{I}_a$, and opposites $-\sigma = \sigma^{-\mathrm{v}} = \sigma^{-\mathrm{h}}$.

Every double functor $F : \mathcal{A} \to \mathcal{B}$ induces a functor between the fundamental groupoids

$$F_* : \Pi\mathcal{A} \to \Pi\mathcal{B},$$

which carries a morphism $[f, x] : a \to a'$ to the morphism

$$F_*[f, x] = [Ff, Fx] : Fa \to Fa'.$$

Hence, for every object $a$ of $\mathcal{A}$, $F$ induces a pointed map

$$F_* : \pi_0(\mathcal{A}, [a]) \to \pi_0(\mathcal{B}, [Fa])$$

and a homomorphism of groups

$$F_* : \pi_1(\mathcal{A}, a) \to \pi_1(\mathcal{B}, Fa).$$

Clearly, there is also an induced homomorphism

$$F_* : \pi_2(\mathcal{A}, a) \to \pi_2(\mathcal{B}, Fa)$$

given by

$$\begin{array}{ccc} a & = & a \\ \| & \sigma & \| \\ a & = & a \end{array} \quad \mapsto \quad \begin{array}{ccc} Fa & = & Fa \\ \| & F\sigma & \| \\ Fa & = & Fa \end{array}$$

## 4. The Mayer-Vietoris sequence.

Throughout this section, we consider a pullback square of double groupoids

$$\begin{array}{ccc} \mathcal{B}'\times_{\mathcal{B}}\mathcal{A} & \xrightarrow{G'} & \mathcal{A} \\ {\scriptstyle F'}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ \mathcal{B}' & \xrightarrow{G} & \mathcal{B} \end{array} \tag{6}$$

where $F$ is a fibration and both $\mathcal{B}$ and $\mathcal{B}'$ have the filling property. By Proposition 2.5 $F'$ is a fibration and, by Proposition 2.4, both $\mathcal{A}$ and $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$ have the filling property.

Moreover, we fix an object $(b', a)$ of $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$, so that $b'$ is an object of $\mathcal{B}'$ $a$ is an object of $\mathcal{A}$ such that $Gb' = Fa$, and let $b = Fa$.

**Theorem 4.1** (Mayer-Vietoris sequence)**.** *There is an exact sequence of homomorphisms of groups and pointed maps*

$$0 \to \pi_2\big(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}, (b', a)\big) \xrightarrow{(F'_*, G'_*)} \pi_2(\mathcal{B}', b')\times\pi_2(\mathcal{A}, a) \xrightarrow{-G_*+F_*} \pi_2(\mathcal{B}, b)$$
$$\xrightarrow{\partial_2} \pi_1\big(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}, (b', a)\big) \xrightarrow{(F'_*, G'_*)} \pi_1(\mathcal{B}', b')\times\pi_1(\mathcal{A}, a) \xrightarrow{G_*^{-1}\cdot F_*} \pi_1(\mathcal{B}, b)$$
$$\xrightarrow{\partial_1} \pi_0\big(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}, [b', a]\big) \xrightarrow{(F'_*, G'_*)} (\pi_0\mathcal{B}'\times_{\pi_0\mathcal{B}}\pi_0\mathcal{A}, ([b'], [a])) \to 1$$

*Furthermore,* $[\tilde{f}_1, \tilde{x}_1], [\tilde{f}_2, \tilde{x}_2] \in \pi_1(\mathcal{B}, b)$ *satisfy* $\partial_1[\tilde{f}_1, \tilde{x}_2] = \partial_1[\tilde{f}_2, \tilde{f}_2]$ *if and only if there are* $[f', x'] \in \pi_1(\mathcal{B}', b')$ *and* $[f, x] \in \pi_1(\mathcal{A}, a)$ *such that*

$$[\tilde{f}_2, \tilde{x}_2] = G_*[f', x']^{-1}\cdot[\tilde{f}_1, \tilde{x}_1]\cdot F_*[f, x].$$

The meaning of the maps in the sequence is clarified in the proof provided in the following subsections 4.1, 4.2, and 4.3.

If the pullback square (6) is a pullback of groupoids, regarded as double groupoids where the vertical morphisms are all identities and the boxes are all vertical identities, then the Mayer-Vietoris sequence in Theorem 4.1 specializes to the Mayer-Vietoris sequence of Brown, Heath, and Kamps [10, Theorem 2.2] (see also [5, 10.7.6]).

### 4.1 The connecting homomorphism $\partial_2 : \pi_2(\mathcal{B}, b) \to \pi_1\big(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}, (b', a)\big)$.

Let $\beta \in \pi_2(\mathcal{B}, b)$. Since $F$ is a fibration, the lifting problem

$$\begin{array}{ccc} a & \xleftarrow{\cdots} & \cdot \\ \| & \exists ? & \uparrow \\ a & = & a \end{array} \quad \overset{F}{\longmapsto} \quad \begin{array}{ccc} b & = & b \\ \| & \beta & \| \\ b & = & b \end{array}$$

has solution. Thus, we can choose a box $\alpha_\beta$ in $\mathcal{A}$ of the form

$$\begin{array}{ccc} a & \xleftarrow{f_\beta} & \cdot \\ \| & \alpha_\beta & \uparrow x_\beta \\ a & = & a \end{array} \tag{7}$$

such that $F\alpha_\beta = \beta$. Since $Ff_\beta = \mathrm{I}^{\mathrm{h}}_b = G\mathrm{I}^{\mathrm{h}}_{b'}$ and $Fx_\beta = \mathrm{I}^{\mathrm{v}}_b = G\mathrm{I}^{\mathrm{v}}_{b'}$, we have that

$$\big((\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta), (\mathrm{I}^{\mathrm{v}}_{b'}, x_\beta)\big) : (b', a) \curvearrowright (b', a) \tag{8}$$

is a loop in the double groupoid $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$.

**Lemma 4.2.** *If the choice of $\alpha_\beta$ in* (7) *is changed, then the loop* (8) *is changed to a homotopic loop in $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$.*

*Proof.* Suppose any other box in $\mathcal{A}$

$$\begin{array}{ccc} a & \xleftarrow{f'} & \cdot \\ \| & \alpha' & \uparrow x' \\ a & = & a \end{array}$$

such that $F\alpha' = \beta$. We define a homotopy $\alpha : (f_\beta, x_\beta) \simeq (f', x')$ in $\mathcal{A}$ by

$$\begin{array}{ccc} \cdot & \xleftarrow{f_\beta^{-1} f'} & \cdot \\ \| & \alpha & \uparrow x' x_\beta^{-1} \\ \cdot & = & \cdot \end{array} \quad = \quad \begin{array}{ccccc} \cdot & \xleftarrow{f_\beta^{-1}} & a & \xleftarrow{f'} & \cdot \\ x_\beta \uparrow & \alpha_\beta^{-\mathrm{h}} & \| & \alpha' & \uparrow x' \\ a & = & a & = & a \\ x_\beta^{-1} \uparrow & & \mathrm{I}^{\mathrm{h}} & & \uparrow x_\beta^{-1} \\ \cdot & & = & & \cdot \end{array}$$

Since

$$F\alpha = \begin{array}{ccccc} b & = & b & = & b \\ \| & \beta^{-h} & \| & \beta & \| \\ b & = & b & = & b \\ \| & & \mathrm{I}_b & & \| \\ b & & = & & b \end{array} = \mathrm{I}_b = G\mathrm{I}_{b'}$$

$(\mathrm{I}_{b'},\alpha):\big((\mathrm{I}^{\mathrm{h}}_{b'},f_\beta),(\mathrm{I}^{\mathrm{v}}_{b'},x_\beta)\big)\simeq\big((\mathrm{I}^{\mathrm{h}}_{b'},f'),(I^{\mathrm{v}}_{b'},x')\big)$ is a homotopy in the pull-back double groupoid $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$. □

**Definition 4.3.** *The map $\partial_2:\pi_2(\mathcal{B},b)\to\pi_1(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},a')$ is given, on every $\beta\in\pi_2(\mathcal{B},b)$, by*

$$\partial_2\beta=[(\mathrm{I}^{\mathrm{h}}_{b'},f_\beta),(\mathrm{I}^{\mathrm{v}}_{b'},x_\beta)]:(b',a)\to(b',a),$$

*where $(f_\beta,x_\beta):a\curvearrowright a$ is the loop of $\mathcal{A}$ given by the vertical target $f_\beta$ and the horizontal source $x_\beta$ of a (any) box $\alpha_\beta$, as in* (7)*, such that $F\alpha_\beta=\beta$.*

Let us stress that, by Lemma 4.2, $\partial_2$ is a well-defined map.

**Proposition 4.4.** $\partial_2:\pi_2(\mathcal{B},b)\to\pi_1(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},(b',a))$ *is a homomorphism of groups.*

*Proof.* Suppose $\beta,\gamma\in\pi_2(\mathcal{B},b)$. Let

$$\begin{array}{ccc} a & \xleftarrow{f_\beta} & \cdot \\ \| & \alpha_\beta & \uparrow{\scriptstyle x_\beta} \\ a & = & a \end{array}\qquad\qquad \begin{array}{ccc} a & \xleftarrow{f_\gamma} & \cdot \\ \| & \alpha_\gamma & \uparrow{\scriptstyle x_\gamma} \\ a & = & a \end{array}$$

be boxes of $\mathcal{A}$ such that $F\alpha_\beta=\beta$ and $F\alpha_\gamma=\gamma$. Since $F$ is a fibration, $Fx_\beta=\mathrm{I}^{\mathrm{v}}_b$, and $Ff_\gamma=\mathrm{I}^{\mathrm{h}}_b$, we can choose a box $\theta$ in $\mathcal{A}$ of the form

$$\begin{array}{ccc} \cdot & \xleftarrow{f'} & \cdot \\ {\scriptstyle x_\beta}\uparrow & \theta & \uparrow{\scriptstyle x'} \\ a & \xleftarrow[f_\gamma]{} & \cdot \end{array}$$

such that $F\theta=\mathrm{I}_b$. Hence $F\theta=G\mathrm{I}_{b'}$ and the diagram in $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$

$$\begin{array}{ccccc} (b',a) & \xleftarrow{(\mathrm{I}^{\mathrm{h}}_{b'},f_\beta)} & (b',\cdot) & \xleftarrow{(\mathrm{I}^{\mathrm{h}}_{b'},f')} & (b',\cdot) \\ & & {\scriptstyle (\mathrm{I}^{\mathrm{v}}_{b'},x_\beta)}\uparrow & (\mathrm{I}_{b'},\theta) & \uparrow{\scriptstyle (\mathrm{I}^{\mathrm{v}}_{b'},x')} \\ & & (b',a) & \xleftarrow[(\mathrm{I}^{\mathrm{h}}_{b'},f_\gamma)]{} & (b',\cdot) \\ & & & & \uparrow{\scriptstyle (\mathrm{I}^{\mathrm{v}}_{b'},x_\gamma)} \\ & & & & (b',a) \end{array}$$

tell us that, in the group $\pi_1(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},(b',a))$,

$$\begin{aligned}\partial_2\beta\cdot\partial_2\gamma &= [(\mathrm{I}^{\mathrm{h}}_{b'},f_\beta),(\mathrm{I}^{\mathrm{v}}_{b'},x_\beta)]\cdot[(\mathrm{I}^{\mathrm{h}}_{b'},f_\gamma),(\mathrm{I}^{\mathrm{v}}_{b'},x_\gamma)]\\ &= [(\mathrm{I}^{\mathrm{h}}_{b'},f_\beta f'),(\mathrm{I}^{\mathrm{v}}_{b'},x'x_\gamma)].\end{aligned}$$

Now, we have the box $\alpha_{\beta+\gamma}$ of $\mathcal{A}$ defined by

$$\begin{array}{ccc} a & \xleftarrow{f_\beta f'} & \cdot \\ \| & \alpha_{\beta+\gamma} & \uparrow{\scriptstyle x'x_\gamma} \\ a & = & a \end{array} \;=\; \begin{array}{ccccc} a & \xleftarrow{f_\beta} & \cdot & \xleftarrow{f'} & \cdot \\ \| & \alpha_\beta & \uparrow & \theta & \uparrow{\scriptstyle x'} \\ a & = & a & \xleftarrow{f_\gamma} & \cdot \\ \| & & \alpha_\gamma & & \uparrow{\scriptstyle x_\gamma} \\ a & & = & & a \end{array}$$

which satisfies that

$$F(\alpha_{\beta+\gamma}) = \begin{array}{ccccc} b & = & b & = & b \\ \| & \beta & \| & \mathrm{I}_b & \| \\ b & = & b & = & b \\ \| & & \gamma & & \| \\ b & & = & & b \end{array} = \beta+\gamma.$$

Hence, by Lemma 4.2, $\partial_2(\beta+\gamma) = [(\mathrm{I}^{\mathrm{h}}_{b'},f_\beta f'),(\mathrm{I}^{\mathrm{v}}_{b'},x'x_\gamma)] = \partial_2\beta\cdot\partial_2\gamma$. $\square$

### 4.2 The connecting map $\partial_1:\pi_1(\mathcal{B},b)\to\pi_0(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},[b',a])$.

Let $(\tilde f,\tilde x): b \curvearrowright b$ be a loop in $\mathcal{B}$ based at $b$

$$\begin{array}{ccc} b & \xleftarrow{\tilde f} & \cdot \\ & & \uparrow{\scriptstyle \tilde x} \\ & & b \end{array}$$

By Proposition 3.1, we can choose a path $(f,x): a \curvearrowright a_{\tilde f,\tilde x}$ in $\mathcal{A}$

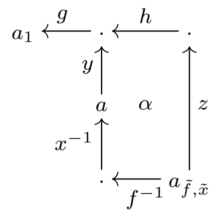
$$\begin{array}{ccc} a_{\tilde f,\tilde x} & \xleftarrow{f} & \cdot \\ & & \uparrow{\scriptstyle x} \\ & & a \end{array} \tag{9}$$

such that $(Ff,Fx)=(\tilde f,\tilde x)$. Since $Fa_{\tilde f,\tilde x}=b$, the pair $(b',a_{\tilde f,\tilde x})$ is an object of the pullback double groupoid $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$.

**Lemma 4.5.** *$(i)$ If the choice of $(f, x)$ in* (9) *is changed, then $(b', a_{\tilde{f},\tilde{x}})$ is changed to a path-connected object in $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$.*

*$(ii)$ If $(\tilde{g}, \tilde{y}) : b \curvearrowright b$ is a loop in $\mathcal{B}$ homotopic to $(\tilde{f}, \tilde{x})$, then a suitable selection of the lifting path of $(\tilde{g}, \tilde{y})$ leaves the object $a_{\tilde{f},\tilde{x}}$ unaltered.*

*Proof.* $(i)$ Suppose $(g, y) : a \curvearrowright a_1$ other path in $\mathcal{A}$ such that $(Fg, Fy) = (\tilde{f}, \tilde{x})$. Since $Fy = \tilde{x} = Fx$ and $F$ is a fibration, we can choose a box $\alpha$ in $\mathcal{A}$ as in the diagram



such that $F\alpha = \mathrm{I}^{\mathrm{h}}_{\tilde{f}^{-1}}$. Since $F(gh) = Fg\,Fh = \tilde{f}\,\tilde{f}^{-1} = \mathrm{I}^{\mathrm{h}}_{b} = G(\mathrm{I}^{\mathrm{h}}_{b'})$ and $Fz = \mathrm{I}^{\mathrm{v}}_{b} = G(\mathrm{I}^{\mathrm{v}}_{b'})$, the path $\big((\mathrm{I}^{\mathrm{h}}_{b'}, gh), (\mathrm{I}^{\mathrm{v}}_{b'}, z)\big) : (b', a_{\tilde{f},\tilde{x}}) \curvearrowright (b', a_1)$ belongs to the pullback double groupoid $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$, so that $[b', a_{f,x}] = [b', a_1]$ in $\pi_0(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A})$.

$(ii)$ If $(\tilde{g}, \tilde{y}) : b \curvearrowright b$ is a loop homotopic to $(\tilde{f}, \tilde{x})$ in $\mathcal{B}$, by Proposition 3.2, there is a path $(g, y) : a \curvearrowright a_{\tilde{f},\tilde{x}}$, homotopic to $(f, x)$, such that $(Fg, Fy) = (\tilde{g}, \tilde{y})$. Choosing this lifting path, we have $a_{\tilde{g},\tilde{y}} = a_{\tilde{f},\tilde{x}}$. □

**Definition 4.6.** *The map $\partial_1 : \pi_1(\mathcal{B}, b) \to \pi_0\big(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}, [b', a]\big)$ is given, for every loop $(\tilde{f}, \tilde{x}) : b \curvearrowright b$ of $\mathcal{B}$, by*

$$\partial_1[\tilde{f}, \tilde{x}] = [b', a_{\tilde{f},\tilde{x}}]$$

*where $a_{\tilde{f},\tilde{x}}$ is the end of a (any) path $(f, x) : a \curvearrowright a_{\tilde{f},\tilde{x}}$ in $\mathcal{A}$, as in* (9)*, such that $(Ff, Fx) = (\tilde{f}, \tilde{x})$.*

Remark that, by Lemma 4.2, $\partial_1$ is a well-defined map. Moreover, since $(F\mathrm{I}^{\mathrm{h}}_{a}, F\mathrm{I}^{\mathrm{v}}_{a}) = (\mathrm{I}^{\mathrm{h}}_{b}, \mathrm{I}^{\mathrm{v}}_{b})$, we have $\partial_1[\mathrm{I}^{\mathrm{h}}_{b}, \mathrm{I}^{\mathrm{v}}_{b}] = [b', a]$, that is, $\partial_1$ is a pointed map.

### 4.3 The exactness of the Mayer-Vietoris sequence.

**Proposition 4.7.** *The sequence of group homomorphisms below is exact.*

$$0 \to \pi_2\big(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}, (b', a)\big) \xrightarrow{(F'_*, G'_*)} \pi_2(\mathcal{B}', b') \times \pi_2(\mathcal{A}, a) \xrightarrow{F_* - G_*} \pi_2(\mathcal{B}, b)$$

*Proof.* Exactness of the sequence above means that the homomorphisms $F'_*$ and $G'_*$ induce an isomorphism

$$\pi_2\big(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}, (b', a)\big) \cong \pi_2(\mathcal{A}, a) \times_{\pi_2(\mathcal{B},b)} \pi_2(\mathcal{B}', b'),$$

which follows directly from the definition of $\pi_2$. □

**Proposition 4.8.** *The following sequence of group homomorphisms is exact.*

$$\pi_2(\mathcal{B}', b') \times \pi_2(\mathcal{A}, a) \xrightarrow{F_* - G_*} \pi_2(\mathcal{B}, b) \xrightarrow{\partial_2} \pi_1\big(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}, (b', a)\big)$$

*Proof.* If $\sigma$ is an element of the group $\pi_2(\mathcal{A}, a)$, then we can choose the box $\alpha_{F\sigma} = \sigma$ in (7). Hence, $\partial_2(F\sigma) = [(\mathrm{I}^{\mathrm{h}}_{b'}, \mathrm{I}^{\mathrm{h}}_{a}), (\mathrm{I}^{\mathrm{h}}_{b'}, \mathrm{I}^{\mathrm{h}}_{a})] = [\mathrm{I}^{\mathrm{h}}_{(b',a)}, \mathrm{I}^{\mathrm{v}}_{(b',a)}]$. So $\mathrm{Im}F_* \subseteq \mathrm{Ker}\partial_2$. Let $\beta'$ be an element of the group $\pi_2(\mathcal{B}', b')$. For any chosen box in $\mathcal{A}$ as in (7)

$$\begin{array}{ccc} a & \xleftarrow{f} & \cdot \\ \| & \alpha & \uparrow x \\ a & = & a \end{array}$$

such that $F\alpha = G\beta'$, the box of $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$

$$\begin{array}{ccc} (b',a) & \xleftarrow{(\mathrm{I}^{\mathrm{h}}_{b'},f)} & (b',\cdot) \\ \| & (\beta',\alpha) & \uparrow (\mathrm{I}^{\mathrm{v}}_{b'},x) \\ (b',a) & = & (b',a) \end{array}$$

is a homotopy $(\beta', \alpha) : (\mathrm{I}^{\mathrm{h}}_{(b',a)}, \mathrm{I}^{\mathrm{v}}_{(b',a)}) \simeq \big((\mathrm{I}^{\mathrm{h}}_{b'}, f), (\mathrm{I}^{\mathrm{v}}_{b'}, x)\big)$. Hence,

$$\partial_2(G\beta') = [(\mathrm{I}^{\mathrm{h}}_{b'}, f), (\mathrm{I}^{\mathrm{v}}_{b'}, x)] = [\mathrm{I}^{\mathrm{h}}_{(b',a)}, \mathrm{I}^{\mathrm{v}}_{(b',a)}].$$

So $\mathrm{Im}G_* \subseteq \mathrm{Ker}\partial_2$.

We now prove $\mathrm{Ker}\partial_2 \subseteq \mathrm{Im}F_* + \mathrm{Im}G_*$: Suppose $\beta \in \pi_2(\mathcal{B}, b)$ such that $\partial_2\beta = [\mathrm{I}^{\mathrm{h}}_{(b',a)}, \mathrm{I}^{\mathrm{v}}_{(b',a)}]$. As in (7), let

$$\begin{array}{ccc} a & \xleftarrow{f_\beta} & \cdot \\ \| & \alpha_\beta & \uparrow x_\beta \\ a & = & a \end{array}$$

be a box of $\mathcal{A}$ such that $F\alpha_\beta = \beta$; so that $\partial_2\beta = [(\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta), (\mathrm{I}^{\mathrm{v}}_{b'}, x_\beta)]$. There is then a homotopy $(\mathrm{I}^{\mathrm{h}}_{(b',a)}, \mathrm{I}^{\mathrm{v}}_{(b',a)}) \simeq \big((\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta), (\mathrm{I}^{\mathrm{v}}_{b'}, x_\beta)\big)$; that is, a box in $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$ of the form

$$\begin{array}{ccc} (b',a) & \xleftarrow{(\mathrm{I}^{\mathrm{h}}_{b'},f_\beta)} & (b',\cdot) \\ \| & (\beta',\alpha) & \uparrow{\scriptstyle (\mathrm{I}^{\mathrm{v}}_{b'},x_\beta)} \\ (b',a) & = & (b',a) \end{array}$$

for some boxes $\beta'$ of $\mathcal{B}'$ and $\alpha$ of $\mathcal{A}$ of the form

$$\begin{array}{ccc} b' & = & b' \\ \| & \beta' & \| \\ b' & = & b' \end{array} \qquad \begin{array}{ccc} a & \xleftarrow{f_\beta} & \cdot \\ \| & \alpha & \uparrow{\scriptstyle x_\beta} \\ a & = & a \end{array}$$

satisfying that $G\beta' = F\alpha$. Define $\sigma = \alpha_\beta \circ_{\mathrm{h}} \alpha^{-\mathrm{h}} \in \pi_2(\mathcal{A}, a)$

$$\begin{array}{ccc} a & = & a \\ \| & \sigma & \| \\ a & = & a \end{array} \quad = \quad \begin{array}{ccccc} a & \xleftarrow{f_\beta} & \cdot & \xleftarrow{f_\beta^{-1}} & a \\ \| & \alpha_\beta & \uparrow & \alpha^{-\mathrm{h}} & \| \\ a & = & a & = & a \end{array}$$

Then $F\sigma = F\alpha_\beta - F\alpha = \beta - G\beta'$; so that $\beta = F_*(\sigma) + G_*(\beta') \in \mathrm{Im}F_* + \mathrm{Im}G_*$. $\square$

**Proposition 4.9.** *The sequence of group homomorphisms below is exact.*

$$\pi_2(\mathcal{B}, b) \xrightarrow{\partial_2} \pi_1\big(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}, (b',a)\big) \xrightarrow{(F'_*,G'_*)} \pi_1(\mathcal{B}', b')\times\pi_1(\mathcal{A}, a)$$

*Proof.* For every $\beta \in \pi_2(\mathcal{B}, b)$, the box $\alpha_\beta$ in (7) is actually a homotopy in $\mathcal{A}$, $\alpha_\beta : (\mathrm{I}^{\mathrm{h}}_a, \mathrm{I}^{\mathrm{v}}_a) \simeq (f_\beta, x_\beta)$. Hence,

$$\begin{aligned} G'_*(\partial_2\beta) &= G'_*[(\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta), (\mathrm{I}^{\mathrm{v}}_{b'}, x_\beta)] = [f_\beta, x_\beta] = [\mathrm{I}^{\mathrm{h}}_a, \mathrm{I}^{\mathrm{v}}_a], \\ F'_*(\partial_2\beta) &= F'_*[(\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta), (\mathrm{I}^{\mathrm{v}}_{b'}, x_\beta)] = [\mathrm{I}^{\mathrm{h}}_{b'}, \mathrm{I}^{\mathrm{v}}_{b'}]. \end{aligned}$$

So $\mathrm{Im}\partial_2 \subseteq \mathrm{Ker}\, F'_* \cap \mathrm{Ker}\, G'_*$. For the opposite inclusion, let

$$\big((f', f), (x', x)\big) : (b', a) \curvearrowright (b', a)$$

be a loop in $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$ such that $[f',x'] = [\mathrm{I}^{\mathrm{h}}_{b'},\mathrm{I}^{\mathrm{v}}_{b'}]$ in $\pi_1(\mathcal{B}',b')$ and $[f,x] = [\mathrm{I}^{\mathrm{h}}_{a},\mathrm{I}^{\mathrm{v}}_{a}]$ in $\pi_1(\mathcal{A},a)$. Choose homotopies $\beta' : (\mathrm{I}^{\mathrm{h}}_{b'},\mathrm{I}^{\mathrm{v}}_{b'}) \simeq (f',x')$ in $\mathcal{B}'$ and $\alpha : (\mathrm{I}^{\mathrm{h}}_{a},\mathrm{I}^{\mathrm{v}}_{a}) \simeq (f,x)$ in $\mathcal{A}$; that is, boxes of the form

$$\begin{array}{ccc} b' & \xleftarrow{f'} & \cdot \\ \| & \beta' & \uparrow x' \\ b' & = & b' \end{array} \qquad \begin{array}{ccc} a & \xleftarrow{f} & \cdot \\ \| & \alpha & \uparrow x \\ a & = & a \end{array}$$

Since $Gf' = Ff$ and $Gx' = Fx$, the box $\beta = F\alpha \circ_{\mathrm{h}} G\beta'^{-\mathrm{h}}$

$$\begin{array}{ccc} b & = & b \\ \| & \beta & \| \\ b & = & b \end{array} \;=\; \begin{array}{ccccc} b & \xleftarrow{Ff} & \cdot & \xleftarrow{Gf'^{-1}} & b \\ \| & F\alpha & \uparrow & G\beta'^{-\mathrm{h}} & \| \\ b & = & b & = & b \end{array}$$

belongs to $\pi_2(\mathcal{B},b)$. Let

$$\begin{array}{ccc} a & \xleftarrow{f_\beta} & \cdot \\ \| & \alpha_\beta & \uparrow x_\beta \\ a & = & a \end{array}$$

be a box of $\mathcal{A}$ such that $F\alpha_\beta = \beta$, so that $\partial_2\beta = [(\mathrm{I}^{\mathrm{h}}_{b'}, f_\beta),(\mathrm{I}^{\mathrm{v}}_{b'},x_\beta)]$. We can construct a homotopy $\alpha_1 : (f_\beta,x_\beta) \simeq (f,x)$ in $\mathcal{A}$ by

$$\begin{array}{ccc} \cdot & \xleftarrow{f_\beta^{-1}f} & \cdot \\ \| & \alpha_1 & \uparrow xx_\beta^{-1} \\ \cdot & = & \cdot \end{array} \;=\; \begin{array}{ccccc} \cdot & \xleftarrow{f_\beta^{-1}} & \cdot & \xleftarrow{f} & \cdot \\ x_\beta\uparrow & \alpha_\beta^{-\mathrm{h}} & \| & \alpha & \uparrow x \\ \cdot & = & \cdot & = & \cdot \\ x_\beta^{-1}\uparrow & & \mathrm{I}^{\mathrm{h}} & & \uparrow x_\beta^{-1} \\ \cdot & & = & & \cdot \end{array}$$

Since

$$F\alpha_1 = \begin{array}{ccccc} b & = & b & = & b \\ \| & \beta^{-\mathrm{h}} & \| & F\alpha & \| \\ b & = & b & = & b \\ \| & & \mathrm{I}_b & & \| \\ b & & = & & b \end{array} = G\beta' \circ_{\mathrm{h}} F\alpha^{-\mathrm{h}} \circ_{\mathrm{h}} F\alpha = G\beta',$$

the pair $(\beta',\alpha_1) : \big((\mathrm{I}^{\mathrm{h}}_{b'},f_\beta),(\mathrm{I}^{\mathrm{v}}_{b'},x_\beta)\big) \simeq \big((f',f),(x',x)\big)$ is a loop homotopy in $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$. Thus, $[(f',f),(x',x)] = [(\mathrm{I}^{\mathrm{h}}_{b'},f_\beta),(\mathrm{I}^{\mathrm{v}}_{b'},x_\beta)] = \partial_2\beta =\in \mathrm{Im}\partial_2$. $\square$

**Proposition 4.10.** *The sequence*

$$\pi_1\big(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},(b',a)\big)\overset{(F'_*,G'_*)}{\longrightarrow}\pi_1(\mathcal{B}',b')\times\pi_1(\mathcal{A},a)\overset{G_*^{-1}\cdot F_*}{\longrightarrow}\pi_1(\mathcal{B},b),$$

*where* $(F'_*,G'_*)$ *is a homomorphism and* $G_*^{-1}\cdot F_*$ *is a pointed map, is exact.*

*Proof.* Exactness of the sequence above means that the homomorphisms $F'_*$ and $G'_*$ induce an epimorphism

$$\pi_1\big(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},(b',a)\big)\twoheadrightarrow\pi_1(\mathcal{A},a)\times_{\pi_1(\mathcal{B},b)}\pi_1(\mathcal{B}',b').$$

To prove this, let $(g,y):a\curvearrowright a$ and $(f',x'):b'\curvearrowright b'$ be loops, in $\mathcal{A}$ and $\mathcal{B}'$ respectively, such that $[Fg,Fy]=[Gf',Gx']$ in $\pi_1(\mathcal{B},b)$. By Proposition 3.2, there is a loop $(f,x):a\curvearrowright a$ in $\mathcal{A}$ such that $[f,x]=[g,y]$ and $(Ff,Fx)=(Gf',Gx')$. Then, $\big((f',f),(x',x)\big):(b',a)\curvearrowright(b',a)$ is a loop in $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$ and $F'_*[(f',f),(x',x)]=[f',x']$ and $G'_*[(f',f),(x',x)]=[f,x]=[g,y]$. □

**Proposition 4.11.** *The following sequence of pointed maps is exact.*

$$\pi_1(\mathcal{B}',b')\times\pi_1(\mathcal{A},a)\overset{G_*^{-1}\cdot F_*}{\longrightarrow}\pi_1(\mathcal{B},b)\overset{\partial_1}{\longrightarrow}\pi_0\big(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A},[b',a]\big)$$

*Further,* $[\tilde{f}_1,\tilde{x}_1],[\tilde{f}_2,\tilde{x}_2]\in\pi_1(\mathcal{B},b)$ *satisfy* $\partial_1[\tilde{f}_1,\tilde{x}_2]=\partial_1[\tilde{f}_2,\tilde{f}_2]$ *if and only if there are* $[f',x']\in\pi_1(\mathcal{B}',b')$ *and* $[f,x]\in\pi_1(\mathcal{A},a)$ *such that*

$$[\tilde{f}_2,\tilde{x}_2]=G_*[f',x']^{-1}\cdot[\tilde{f}_1,\tilde{x}_1]\cdot F_*[f,x]. \tag{10}$$

*Proof.* Given $(\tilde{f}_1,\tilde{x}_1),(\tilde{f}_2,\tilde{x}_2):b\curvearrowright b$ loops in $\mathcal{B}$, let us choose paths in $\mathcal{A}$ $(f_1,x_1):a\curvearrowright a_1$ and $(f_2,x_2):a\curvearrowright a_2$ such that $(Ff_1,Fx_1)=(\tilde{f}_1,\tilde{x}_1)$ and $(Ff_2,Fx_2)=(\tilde{f}_2,\tilde{x}_2)$; so that $\partial_1[\tilde{f}_1,\tilde{x}_1]=[b',a_1]$ and $\partial_1[\tilde{f}_2,\tilde{x}_2]=[b',a_2]$.

Suppose there are loops $(f',x'):b'\curvearrowright b'$ in $\mathcal{B}'$ and $(f,x):a\curvearrowright a$ in $\mathcal{A}$ such that (10) holds. Choose $(g,y):a_2\curvearrowright a_1$ a path in $\mathcal{A}$ representative of the composite morphism $[f_1,x_1]\cdot[f,x]\cdot[f_2,x_2]^{-1}:a_2\to a_1$ of $\Pi\mathcal{A}$. Since

$$\begin{aligned}[Gf',Gx']&=F_*[f_1,x_1]\cdot F_*[f,x]\cdot F_*[f_2,x_2]^{-1}\\&=F_*\big([f_1,x_1]\cdot[f,x]\cdot[f_2,x_2]^{-1}\big)=[Fg,Fy],\end{aligned}$$

by Proposition 3.2, there is a path $(g',y'):a_2\curvearrowright a_1$ which is homotopic to $(g,y)$ and satisfies $(Fg',Fy')=(Gf',Gx')$. Then,

$$\big((f',g'),(x',y')\big):(b',a_2)\curvearrowright(b',a_1) \tag{11}$$

is a path in $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$ and therefore $[b', a_1] = [b', a_2]$.

Conversely, assume that $[b', a_1] = [b', a_2]$, so that there is a path in the pullback $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$ as (11), for some loop $(f', x') : b' \curvearrowright b'$ in $\mathcal{B}'$ and some path $(g', y') : a_2 \curvearrowright a_1$ in $\mathcal{A}$ such that $(Gf', Gx') = (Fg', Fy')$. The composite morphism in $\Pi\mathcal{A}$

$$[f, x] = [f_1, x_1]^{-1} \cdot [g', y'] \cdot [f_2, x_2] : a \to a$$

is then an element of $\pi_1(\mathcal{A}, a) = \mathrm{Aut}_{\Pi\mathcal{A}}(a)$ and

$$\begin{aligned}[\tilde{f}_2, \tilde{x}_2] &= F_*[f_2, x_2] = F_*[g'.y']^{-1} \cdot F_*[f_1, x_1] \cdot F_*[f, x] \\ &= G_*[f', x']^{-1} \cdot [\tilde{f}_1, \tilde{x}_1] \cdot F_*[f, x].\end{aligned}$$

□

**Proposition 4.12.** *The following sequence of pointed maps is exact.*

$$\pi_1(\mathcal{B}, b) \xrightarrow{\partial_1} \pi_0\big(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}, [b', a]\big) \xrightarrow{(F'_*, G'_*)} \big(\pi_0\mathcal{B}' \times_{\pi_0\mathcal{B}} \pi_0\mathcal{A}, ([b'], [a])\big)$$

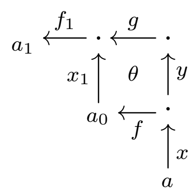
*Proof.* For every loop $(\tilde{f}, \tilde{x}) : b \curvearrowright b$ in $\mathcal{B}$, the path $(f, x) : a \to a_{\tilde{f},\tilde{x}}$ in (9) tell us that the objects $a$ and $a_{\tilde{f},\tilde{x}}$ are path connected in $\mathcal{A}$. Hence

$$(F'_*, G'_*)\partial_1[\tilde{f}, \tilde{x}] = (F'_*, G'_*)[b', a_{\tilde{f},\tilde{x}}] = ([b'], [a_{\tilde{f},\tilde{x}}]) = ([b'], [a]).$$

So $\mathrm{Im}\partial_1 \subseteq (F'_*, G'_*)^{-1}([b'], [a])$. For the opposite inclusion, let $(b'_0, a_0)$ be an object of $\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}$ such that $[b'_0] = [b']$ in $\pi_0\mathcal{B}'$ and $[a_0] = [a]$ in $\pi_0\mathcal{A}$. Choose paths $(f', x') : b'_0 \curvearrowright b'$ in $\mathcal{B}'$ and $(f, x) : a \curvearrowright a_0$ in $\mathcal{A}$. Since $Gb'_0 = Fa_0$ and $F$ is a fibration, we can select a path $(f_1, x_1) : a_0 \curvearrowright a_1$ such that

$$(Ff_1, Fx_1) = (Gf', Gx') : Fa_0 \curvearrowright b.$$

Further, because of the filling property, we can choose now a box $\theta$ in $\mathcal{A}$ as in the diagram



This way, we find the path $(f_1g, yx) : a \curvearrowright a_1$ in $\mathcal{A}$ which is a lifting of the loop $(\tilde{f}, \tilde{x}) = (F(f_1g), F(yx)) : b \curvearrowright b$ in $\mathcal{B}$. Hence $\partial_1[\tilde{f}, \tilde{x}] = [b', a_1]$. As

$$\big((f', f_1), (x', x_1)\big) : (b'_0, a_0) \curvearrowright (b', a_1)$$

is a path in $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$, $[b'_0, a_0] = [b', a_1]$ in $\pi_0(\mathcal{B}'\times_{\mathcal{B}}\mathcal{A})$, and we finally conclude that $\partial_1[\tilde{f}, \tilde{x}] = [b'_0, a_0]$. Thus, $[b'_0, a_0] \in \mathrm{Im}\partial_1$, as claimed. □

**Proposition 4.13.** *The map* $(F'_*, G'_*) : \pi_0(\mathcal{B}' \times_{\mathcal{B}} \mathcal{A}) \to \pi_0\mathcal{B}' \times_{\pi_0\mathcal{B}} \pi_0\mathcal{A}$ *is surjective.*

*Proof.* Suppose objects $a_0$ of $\mathcal{A}$ and $b'_0$ of $\mathcal{B}'$ such that $[Fa_0] = [Gb'_0]$ in $\pi_0\mathcal{B}$. Then, we can choose is a loop $(\tilde{f}, \tilde{x}) : Fa_0 \curvearrowright Gb'_0$ in $\mathcal{B}$ and, by Proposition 3.1, we can also choose a loop $(f, x) : a_0 \curvearrowright a_1$ in $\mathcal{A}$ such that $(Ff, Fx) = (\tilde{f}, \tilde{x})$. Then, since $Fa_1 = Gb'_0$, the pair $(b'_0, a_1)$ is an object of $\mathcal{B}'\times_{\mathcal{B}}\mathcal{A}$ and $F'_*[b'_0, a_1] = [b'_0]$, $G'_*[b'_0, a_1] = [a_1] = [a_0]$. □

## 5. The homotopy sequence.

Throughout this section, we consider a given fibration of double groupoids $F : \mathcal{A} \to \mathcal{B}$, assuming that $\mathcal{B}$ has the filling property. For an object $a$ in $\mathcal{A}$, let $b = Fa$ and $\mathcal{F}_b = F^{-1}(b)$ be the corresponding double groupoid fibre over $b$. This setup leads to the following fibre sequence of pointed double groupoids, where Propositions 2.4 and 2.6 ensure that both $\mathcal{A}$ and $\mathcal{F}_b$ have the filling property:

$$(\mathcal{F}_b, a) \overset{i}{\hookrightarrow} (\mathcal{A}, a) \overset{F}{\longrightarrow} (\mathcal{B}, b) \tag{12}$$

**Theorem 5.1.** *The fibre sequence* (12) *gives rise to an exact sequence (of groups and pointed sets)*

$$\begin{array}{l} 0 \longrightarrow \pi_2(\mathcal{F}_b, a) \xrightarrow{i_*} \pi_2(\mathcal{A}, a) \xrightarrow{F_*} \pi_2(\mathcal{B}, b) \\ \xrightarrow{\partial_2} \pi_1(\mathcal{F}_b, a) \xrightarrow{i_*} \pi_1(\mathcal{A}, a) \xrightarrow{F_*} \pi_1(\mathcal{B}, b) \\ \xrightarrow{\partial_1} \pi_0(\mathcal{F}_b, [a]) \xrightarrow{i_*} \pi_0(\mathcal{A}, [a]) \xrightarrow{F_*} \pi_0(\mathcal{B}, [b]). \end{array} \tag{13}$$

*Furthermore,* $[\tilde{f}_1, \tilde{x}_1], [\tilde{f}_2, \tilde{x}_2] \in \pi_1(\mathcal{B}, b)$ *satisfy* $\partial_1[\tilde{f}_1, \tilde{x}_2] = \partial_1[\tilde{f}_2, \tilde{f}_2]$ *if and only if there is an* $[f, x] \in \pi_1(\mathcal{A}, a)$ *such that*

$$[\tilde{f}_2, \tilde{x}_2] = [\tilde{f}_1, \tilde{x}_1] \cdot F_*[f, x].$$

*Proof.* This result follows from the Mayer-Vietoris sequence stated in Theorem 4.1 above, because $\mathcal{F}_b$ appears in the pullback square of double groupoids depicted below. □

$$\begin{array}{ccc} \mathcal{F}_b \cong * \times_{\mathcal{B}} \mathcal{A} & \xrightarrow{i} & \mathcal{A} \\ \downarrow & & \downarrow{\scriptstyle F} \\ * & \xrightarrow{b} & \mathcal{B} \end{array}$$

If $(\mathcal{F}_b, a) \hookrightarrow (\mathcal{A}, a) \xrightarrow{F} (\mathcal{B}, b)$ is a pointed Moerdijk fibration of 2-groupoids, regarded as double groupoids whose vertical arrows are all identities, then the associated 9-term exact sequence (13) yields the exact sequence constructed by Hardie, Kamps, and Kieboom in [27]. In particular, if $F : \mathcal{A} \to \mathcal{B}$ is a Grothendieck fibration of groupoids, viewed as double groupoids whose vertical morphisms are all identities and whose boxes are all vertical identities, then (13) specializes to the 6-term exact sequence due to Brown [2, Theorem 4.3], [5, 7.2.9].

The following proposition provides further relevant information about the 9-term sequence.

**Proposition 5.2.** *(i) There is a group action of the group* $\pi_1(\mathcal{A}, a)$ *on the group* $\pi_1(\mathcal{F}_b, a)$ *making the homomorphism* $i_* : \pi_1(\mathcal{F}_b, a) \to \pi_1(\mathcal{A}, a)$ *into a crossed module.*

*(ii) There is a canonical action of the group* $\pi_1(B, b)$ *on the set* $\pi_0\mathcal{F}_b$ *such that the boundary* $\partial_1$ *is given by* $\partial_1[\tilde{f}, \tilde{x}] = {}^{[\tilde{f},\tilde{x}]}[a]$.

*(iii)* $[a], [a'] \in \pi_0\mathcal{F}_b$ *satisfy* $i_*[a] = i_*[a']$ *if and only if* $[a'] = {}^{[\tilde{f},\tilde{x}]}[a]$*, for some* $[\tilde{f}, \tilde{x}] \in \pi_1(\mathcal{B}, b)$.

*Proof.* In the following subsections, these issues are addressed in a more general setting, as detailed in Theorems 5.7 and 5.9, and Proposition 5.10 below. □

If $F : \mathcal{A} \to \mathcal{B}$ is a fibration of 2-groupoids, viewed as double groupoids whose vertical morphisms are all identities, and we consider the equivalence between 2-groupoids and crossed modules over groupoids as established by Brown and Higgins [11], then Proposition 5.2 leads to the analogous statements for fibrations of crossed modules over groupoids, as demostrated by Howie [29, Theorem 3.1] and Brown [6, Theorem 2.11].

### 5.1 The fundamental crossed module $\pi_1\mathcal{F} \to \pi_1\mathcal{A}$.

We begin by fixing some notations concerning crossed modules over groupoids. If $\mathsf{P}$ is a groupoid, a (left) $\mathsf{P}$-*group* is a functor from $\mathsf{P}$ to the category $\mathsf{Gr}$ of groups. For every $\mathsf{P}$-group $H : \mathsf{P} \to \mathsf{Gr}$, each morphism $\phi : a \to b$ in $\mathsf{P}$, and each $h \in H(a)$, we denote by ${}^{\phi}h$ the value of $H(\phi)$ at $h$ and call it *the action of* $\phi$ *on* $h$. Thus, a $\mathsf{P}$-group $H$ provides groups $H(a)$, one for each $a \in \mathrm{Ob}\mathsf{P}$, and action homomorphisms

$$H(a) \to H(b), \quad h \mapsto {}^{\phi}h,$$

one for each morphism $\phi : a \to b$ in $\mathsf{P}$, satisfying ${}^{\psi}({}^{\phi}h) = {}^{\psi\phi}h$ and ${}^{1}h = h$. For instance, let

$$\pi_1\mathsf{P} : \mathsf{P} \to \mathsf{Gr}, \quad a \mapsto \pi_1(\mathsf{P}, a) = \mathrm{Aut}_{\mathsf{P}}(a), \tag{14}$$

denote the $\mathsf{P}$-group that attaches to each object $a$ the group of its automorphisms in $\mathsf{P}$. The action of a morphism $\phi : a \to b$ on an automorphism $\psi : a \to a$ is given by conjugation in $\mathsf{P}$, that is, ${}^{\phi}\psi = \phi\,\psi\phi^{-1}$. If $\mathsf{P} = G$ is a group regarded as a groupoid with only one object, then $\pi_1 G = G$ with the action on itself by conjugation.

A *morphism of* $\mathsf{P}$-*groups* $\mu : H \to H'$ is a natural transformation, so it consists of homomorphisms $\mu = \mu_a : H(a) \to H'(a)$, one for each object $a$ of $\mathsf{P}$, such that, for every $\phi : a \to b$ in $\mathsf{P}$, $\mu({}^{\phi}h) = {}^{\phi}\mu(h)$.

A $\mathsf{P}$-*crossed module* (or *crossed module of groupoids over* $\mathsf{P}$) is a morphism of $\mathsf{P}$-groups

$$H \xrightarrow{\mu} \pi_1\mathsf{P}$$

such that, for every $h, h' \in H(a)$, $a \in \mathrm{Ob}\mathsf{P}$, the equation below holds:

$${}^{\mu(h)}h'\,h = h\,h'. \tag{15}$$

Thus, for every object $a$ of $\mathsf{P}$, $H(a) \xrightarrow{\mu} \pi_1(\mathsf{P}, a)$ is a crossed module over the group of automorphism of $a$ in $\mathsf{P}$.

Returning to the fibration $F : \mathcal{A} \to \mathcal{B}$, for $\mathsf{P} = \Pi\mathcal{A}$, the fundamental groupoid of $\mathcal{A}$, we denote the $\Pi\mathcal{A}$-group (14), i.e. $\pi_1(\Pi\mathcal{A})$, simply by

$$\pi_1\mathcal{A} : \Pi\mathcal{A} \to \mathsf{Gr}, \quad a \mapsto \pi_1(\mathcal{A}, a) = \mathrm{Aut}_{\Pi\mathcal{A}}(a). \tag{16}$$

The assignment $a \mapsto \pi_1(\mathcal{F}_{Fa}, a)$ defines the function on objects of a functor

$$\pi_1\mathcal{F} : \Pi\mathcal{A} \to \mathsf{Gr}$$

whose effect on morphisms is as follows: Suppose $[f, x] : a \to a'$ a morphism in $\Pi\mathcal{A}$, which is represented by a path $(f, x) : a \curvearrowright a'$ in $\mathcal{A}$, and let $[g, y] \in \pi_1(\mathcal{F}_{Fa}, a)$, represented by a loop $(g, y) : a \curvearrowright a$ in $\mathcal{F}_{Fa}$. Since $F$ is a fibration, we can choose boxes $\alpha$ and $\beta$ in $\mathcal{A}$ as in the diagram

$$\begin{array}{ccccccc}
a' & \xleftarrow{f} & \cdot & \xleftarrow{g'} & \cdot & \xleftarrow{f'} & \cdot \\
 & & {\scriptstyle x}\uparrow & \alpha & \uparrow{\scriptstyle x'} & & \\
 & & a & \xleftarrow[g]{} & \cdot & \beta & \uparrow{\scriptstyle y'} \\
 & & & & {\scriptstyle yx^{-1}}\uparrow & & \\
 & & & & \cdot & \xleftarrow[f^{-1}]{} & a'
\end{array} \tag{17}$$

such that $F\alpha = \mathrm{I}^{\mathrm{h}}_{Fx}$ and $F\beta = \mathrm{I}^{\mathrm{v}}_{Ff^{-1}}$. Since $Fy' = \mathrm{I}^{\mathrm{v}}_{Fa'}$ and $F(f\,g'f') = \mathrm{I}^{\mathrm{h}}_{Fa'}$, the loop

$$(f\,g'f', y') : a' \curvearrowright a' \tag{18}$$

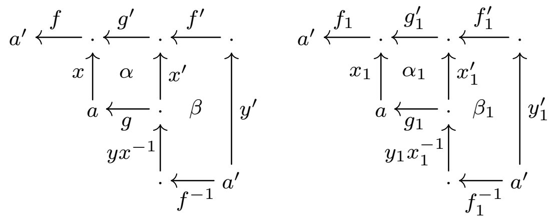
belongs to the double groupoid fibre $\mathcal{F}_{Fa'}$. We define the action of the morphism $[f, x] : a \to a'$ of $\Pi\mathcal{A}$ on $[g, y] \in \pi_1(\mathcal{F}_{Fa}, a)$ by

$$^{[f,x]}[g, y] = [f\,g'f', y'] \in \pi_1(\mathcal{F}_{Fa'}, a'). \tag{19}$$

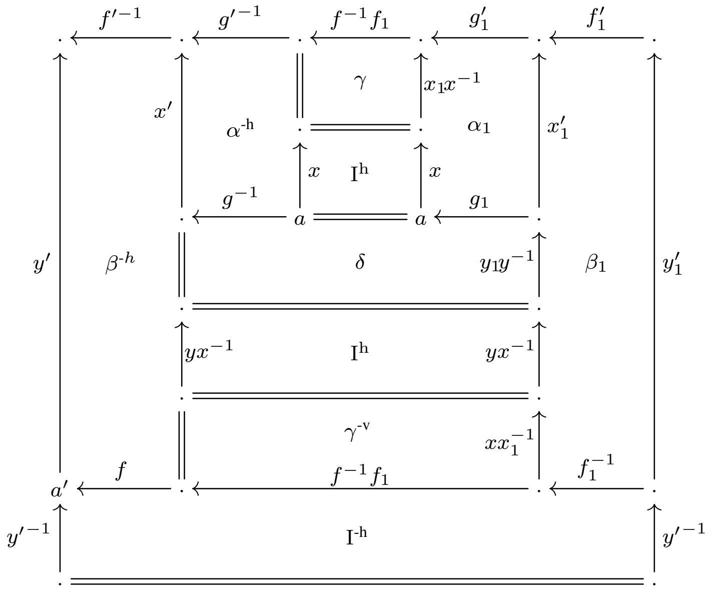
It follows from Lemmas 5.3, 5.4 and 5.5 below that this action is well defined and that $\pi_1\mathcal{F}$ is really a $\Pi\mathcal{A}$-group.

**Lemma 5.3.** $^{[f,x]}[g, y]$ *is independent of the choices of the representative path of* $[f, x]$ *in* $\mathcal{A}$*, of the representative loop of* $[g, y]$ *in* $\mathcal{F}_{Fa}$*, and of the boxes* $\alpha$ *and* $\beta$ *in* (17).

*Proof.* Let $\gamma : (f,x) \simeq (f_1,x_1)$ be a homotopy of paths from $a$ to $a'$ in $\mathcal{A}$ and let $\delta : (g,x) \simeq (g_1,x_1)$ be a homotopy of loops at $a$ in the double groupoid fibre $\mathcal{F}_{Fa}$. Suppose we have selected boxes $\alpha$, $\beta$, $\alpha_1$, and $\beta_1$, as in the diagrams



such that $F\alpha = \mathrm{I}^{\mathrm{h}}_{Fx}$, $F\beta = \mathrm{I}^{\mathrm{v}}_{Ff^{-1}}$, $F\alpha_1 = \mathrm{I}^{\mathrm{h}}_{Fx_1}$, and $F\beta_1 = \mathrm{I}^{\mathrm{v}}_{Ff_1^{-1}}$. Then, we get a homotopy of loops at $a'$ in the double groupoid fibre $\mathcal{F}_{Fa'}$ from $(f\,g'f', y')$ to $(f_1\,g'_1f'_1, y'_1)$ by pasting the diagram



□

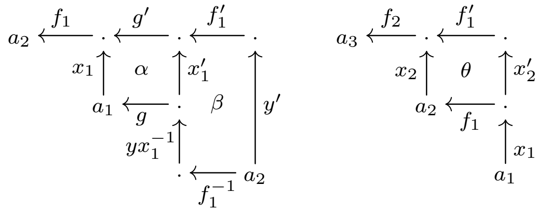
**Lemma 5.4.** *For every pair of paths in* $\mathcal{A}$

$$(f_1,x_1) : a_1 \curvearrowright a_2,\ (f_2,x_2) : a_2 \curvearrowright a_3$$

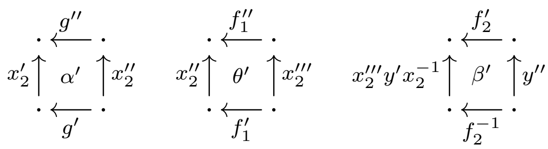
*and every loop* $(g, y) : a_1 \curvearrowright a_1$ *in the fibre* $\mathcal{F}_{Fa_1}$,

$$^{[f_2,x_2]}\big(^{[f_1,x_1]}[g, y]\big) = {}^{[f_2,x_2]\cdot[f_1,x_1]}[g, y].$$

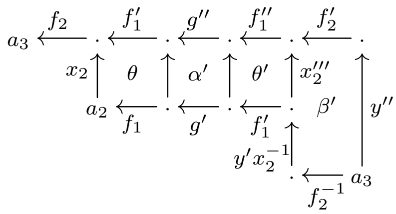
*Proof.* Let $\alpha$, $\beta$, and $\theta$ be boxes of $\mathcal{A}$ as in the diagrams



such that $F\alpha = \mathrm{I}^{\mathrm{h}}_{Fx_1}$ and $F\beta = \mathrm{I}^{\mathrm{v}}_{Ff_1^{-1}}$. Hence, $^{[f_1,x_1]}[g, y] = [f_1\, g' f_1', y']$ and $[f_2, x_2] \cdot [f_1, x_1] = [f_2 f_1', x_2' x_1]$. Since $F$ is fibration, we can successively choose boxes $\alpha'$, $\theta'$, and $\beta'$ in $\mathcal{A}$ of the form

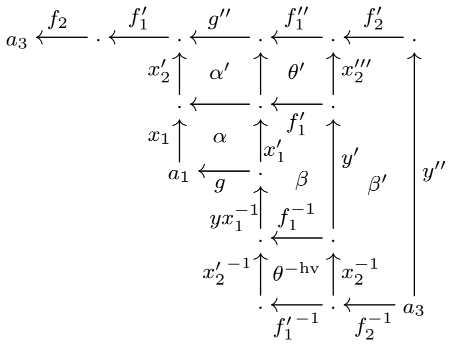


such that $F\alpha' = \mathrm{I}^{\mathrm{h}}_{Fx_2'}$, $F\theta' = \theta^{-\mathrm{h}}$, and $F\beta' = \mathrm{I}^{\mathrm{v}}_{Ff_2^{-1}}$. Then, the pasting diagram



tell us that $^{[f_2,x_2]}\big(^{[f_1,x_1]}[g, y]\big) = [f_2 f_1' g'' f_1'' f_2', y'']$, since $F(\theta \circ_{\mathrm{h}} \alpha' \circ_{\mathrm{h}} \theta') = \mathrm{I}^{\mathrm{h}}_{Fx_2}$

and $F\beta' = \mathrm{I}^{\mathrm{v}}_{Ff_2^{-1}}$, while the pasting diagram

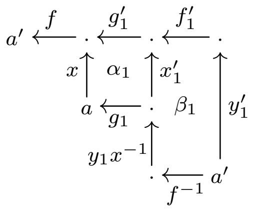


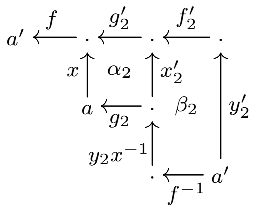
tell us that also ${}^{[f_2,x_2]\cdot[f_1,x_1]}[g,y] = [f_2 f_1' g'' f_1'' f_2', y'']$, since $F(\alpha' \circ_{\mathrm{v}} \alpha) = \mathrm{I}^{\mathrm{v}}_{F(x_2' x_1)}$ and $F\big((\theta' \circ_{\mathrm{v}} \beta \circ_{\mathrm{v}} \theta^{-\mathrm{hv}}) \circ_{\mathrm{h}} \beta'\big) = \mathrm{I}^{\mathrm{v}}_{F(f_2 f_1')^{-1}}$. □

**Lemma 5.5.** *For every pair of loops $(g_1, y_1), (g_2, y_2) : a \curvearrowright a$ in $\mathcal{F}_{Fa}$ and every path $(f, x) : a \curvearrowright a'$ in $\mathcal{A}$,*

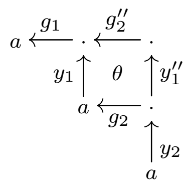
$$ {}^{[f,x]}\big([g_1, y_1] \cdot [g_2, y_2]\big) = {}^{[f,x]}[g_1, y_1] \cdot {}^{[f,x]}[g_2, y_2]. $$

*Proof.* Let $\alpha_1$, $\beta_1$, $\alpha_2$, and $\beta_2$ be boxes of $\mathcal{A}$ as in the diagrams

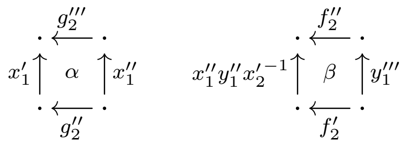




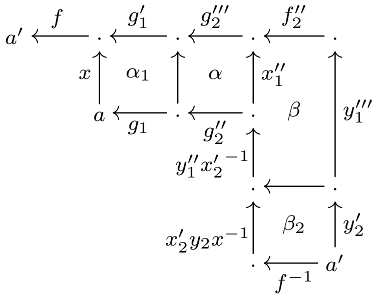
such that $F\alpha_1 = \mathrm{I}^{\mathrm{h}}_{Fx} = F\alpha_2$ and $F\beta_1 = \mathrm{I}^{\mathrm{v}}_{Ff^{-1}} = F\beta_2$, so that ${}^{[f,x]}[g_1, y_1] = [f g_1' f_1', y_1']$ and ${}^{[f,x]}[g_2, y_2] = [f g_2' f_2', y_2']$, and let $\theta$ be a box in $\mathcal{F}_{Fa}$ as in the diagram



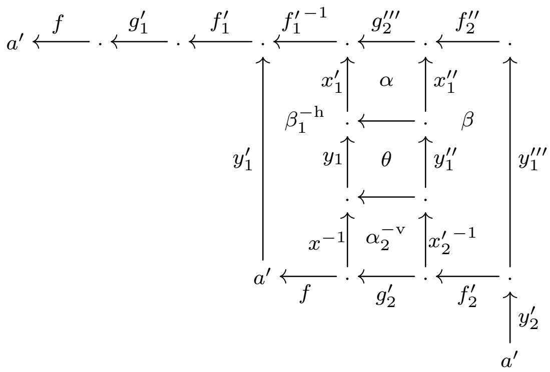
so that, $[g_1, y_1] \cdot [g_2, y_2] = [g_1 g_2'', y_1'' y_2]$ in the group $\pi_1(\mathcal{F}_{Fa}, a)$. Since $F$ is fibration, we can successively choose boxes $\alpha$ and $\beta$ in $\mathcal{A}$ of the form




such that $F\alpha = \mathrm{I}^{\mathrm{h}}_{Fx}$ and $F\beta = \mathrm{I}^{\mathrm{v}}_{Ff^{-1}}$. Then, on the one hand, the pasting diagram




tell us that ${}^{[f,x]}\big([g_1, y_1] \cdot [g_2, y_2]\big) = [f g_1' g_2''' f_2'', y_1''' y_2']$, since $F(\alpha_1 \circ_{\mathrm{h}} \alpha) = \mathrm{I}^{\mathrm{h}}_{Fx}$ and $F(\beta \circ_{\mathrm{v}} \beta_2) = \mathrm{I}^{\mathrm{v}}_{Ff^{-1}}$. On the other hand, the pasting diagram



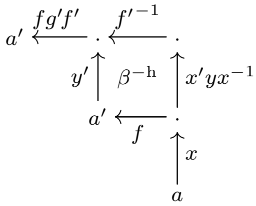

also tell us that ${}^{[f,x]}[g_1, y_1] \cdot {}^{[f,x]}[g_2, y_2] = [f g_1' g_2''' f_2'', y_1''' y_2']$, since the pasted box of the inner boxes belongs to $\mathcal{F}_{Fa'}$. $\square$

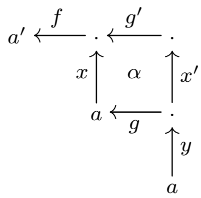
**Proposition 5.6.** *There is a morphism of* $\Pi\mathcal{A}$*-groups* $i_* : \pi_1\mathcal{F} \to \pi_1\mathcal{A}$ *which, at each object* $a \in \mathrm{Ob}\mathcal{A}$*, consists of the homomorphism induced by the inclusion* $i_* : \pi_1(\mathcal{F}_{Fa}, a) \to \pi_1(\mathcal{A}, a)$*.*

*Proof.* We must prove that, for every path $(f,x) : a \curvearrowright a'$ in $\mathcal{A}$ and every loop $(g,y) : a \curvearrowright a$ in $\mathcal{F}_{Fa}$, the equality

$$i_*\big({}^{[f,x]}[g,y]\big)\cdot[f,x] = [f,x]\cdot i_*[g,y]$$

holds in the fundamental groupoid $\Pi\mathcal{A}$. For, let us choose boxes $\alpha$ and $\beta$ as in diagram (17) such that $F\alpha = \mathrm{I}^{\mathrm{h}}_{Fx}$ and $F\beta = \mathrm{I}^{\mathrm{v}}_{Ff^{-1}}$; so that ${}^{[f,x]}[g,y] = [f\,g'f', y']$. We have the diagrams in $\mathcal{A}$




The first of them tell us that

$$i_*\big({}^{[f,x]}[g,y]\big)\cdot[f,x] = [fg'f'f'^{-1}, x'yx^{-1}x] = [fg', y'x],$$

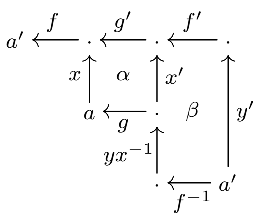
and the second one that also $[f,x]\cdot i_*[g,y] = [fg', x'y]$. $\square$

**Theorem 5.7.** *The morphism of* $\Pi\mathcal{A}$*-groups* $i_* : \pi_1\mathcal{F} \to \pi_1\mathcal{A}$ *is a crossed module over* $\Pi\mathcal{A}$*.*

*Proof.* We must prove that, for every pair of loops $(f,x),(g,y) : a \curvearrowright a$ in $\mathcal{F}_{Fa}$, the equality

$${}^{i_*[f,x]}[g,y]\cdot[f,x] = [f,x]\cdot[g,y]$$

holds in the fundamental group $\pi_1(\mathcal{F}_{Fa}, a)$. To do that, since the double groupoid fibre $\mathcal{F}_{Fa}$ has the filling property, we can choose boxes $\alpha$ and $\beta$ of $\mathcal{F}_{Fa}$ as in the diagram



Then ${}^{i_*[f,x]}[g,y] = [f\,g'f',y']$, and the diagrams in $\mathcal{F}_{Fa}$

$$
\begin{array}{ccccc}
a' & \xleftarrow{fg'f'} & \cdot & \xleftarrow{f'^{-1}} & \cdot \\
 & & {\scriptstyle y'}\big\uparrow & \beta^{-\mathrm{h}} & \big\uparrow{\scriptstyle x'yx^{-1}} \\
 & & a' & \xleftarrow[f]{} & \cdot \\
 & & & & \big\uparrow{\scriptstyle x} \\
 & & & & a
\end{array}
\qquad\qquad
\begin{array}{ccccc}
a' & \xleftarrow{f} & \cdot & \xleftarrow{g'} & \cdot \\
 & & {\scriptstyle x}\big\uparrow & \alpha & \big\uparrow{\scriptstyle x'} \\
 & & a & \xleftarrow[g]{} & \cdot \\
 & & & & \big\uparrow{\scriptstyle y} \\
 & & & & a
\end{array}
$$

tell us that ${}^{i_*[f,x]}[g,y]\cdot[f,x] = [fg',x'y] = [f,x]\cdot[g,y]$. □

## 5.2 The $\Pi\mathcal{B}$-set $\pi_0\mathcal{F}$.

If $\mathsf{P}$ is a groupoid, a (left) $\mathsf{P}$-*set* is a functor from $\mathsf{P}$ to the category $\mathsf{Set}$ of sets. For every $\mathsf{P}$-set $H:\mathsf{P}\to\mathsf{Set}$, each morphism $\phi: a\to b$ in $\mathsf{P}$, and each $h\in H(a)$, we denote by ${}^{\phi}h$ the value of $H(\phi)$ at $h$ and call it *the action of* $\phi$ *on* $h$. Thus, a $\mathsf{P}$-set $H$ provides of sets $H(a)$, one for each $a\in \mathrm{Ob}\mathsf{P}$, and action homomorphisms

$$H(a)\to H(b),\quad h\mapsto {}^{\phi}h,$$

one for each morphism $\phi: a\to b$ in $\mathsf{P}$, satisfying ${}^{\psi}({}^{\phi}h) = {}^{\psi\phi}h$ and ${}^{1}h = h$.

Returning to the fibration $F:\mathcal{A}\to\mathcal{B}$, the assignment $b\mapsto \pi_0\mathcal{F}_b$ defines the function on objects of a functor from the fundamental groupoid of $\mathcal{B}$

$$\pi_0\mathcal{F}:\Pi\mathcal{B}\to\mathsf{Set}$$

whose effect on morphisms is described as follows: Suppose $[\tilde f,\tilde x]: b\to b'$ a morphism in the fundamental groupoid $\Pi\mathcal{B}$ of $\mathcal{B}$, represented by a path $(\tilde f,\tilde x): b\curvearrowright b'$ in $\mathcal{B}$, and let $[a]\in\pi_0\mathcal{F}_b$, represented by an object $a$ of $\mathcal{F}_b$. Since $F$ is a fibration, we can choose a path in $\mathcal{A}$

$$(f,x): a\curvearrowright a_{\tilde f,\tilde x}$$

such that $(Ff,Fx) = (\tilde f,\tilde x)$. Then $Fa_{\tilde f,\tilde x} = b'$, so that the object $a_{\tilde f,\tilde x}$ belongs to fibre double groupoid $\mathcal{F}_{b'}$. We define the action of the morphism $[\tilde f,\tilde x]: b\to b'$ of $\Pi\mathcal{B}$ on $[a]\in\pi_0\mathcal{F}_b$ by

$$ {}^{[\tilde f,\tilde x]}[a] = [a_{\tilde f,\tilde x}]\in\pi_0\mathcal{F}_{b'}. \tag{20}$$

It follows from the lemma below that this action is well-defined.

**Lemma 5.8.** $[a_{\tilde{f},\tilde{x}}]$ *is independent of the choices of the representative object* $a$ *of* $[a]$ *in* $\mathcal{F}_b$*, of the representative path* $(\tilde{f},\tilde{x}) : b \curvearrowright b'$ *of* $[\tilde{f},\tilde{x}]$ *in* $\mathcal{B}$*, and of its lifted path* $(f,x) : a \curvearrowright a_{\tilde{f},\tilde{x}}$ *in* $\mathcal{A}$*.*

*Proof.* Let $(h,z) : a \curvearrowright a'$ be a path in $\mathcal{F}_b$ and let $\tilde{\alpha} : (\tilde{f},\tilde{x}) \simeq (\tilde{g},\tilde{y})$ be a homotopy of paths in $\mathcal{B}$ from $b$ to $b'$. Suppose we have chosen paths $(f,x) : a \curvearrowright a_{\tilde{f},\tilde{x}}$ and $(g,y) : a' \curvearrowright a'_{\tilde{g},\tilde{y}}$ in $\mathcal{A}$ such that $(Ff,Fx) = (\tilde{f},\tilde{x})$ and $(Fg,Fy) = (\tilde{g},\tilde{y})$. We must prove that there is a path $a_{\tilde{f},\tilde{x}} \curvearrowright a'_{\tilde{g},\tilde{y}}$ in $\mathcal{F}_{b'}$. For, we can proceed as follows: By the filling property, let us choose boxes $\alpha_1$ of $\mathcal{F}_b$ and $\alpha_2 \in \mathcal{A}$ of the form

$$
\begin{array}{ccc}
a' & \xleftarrow{\;h\;} & \cdot \\
{\scriptstyle z'}\uparrow & \alpha_1 & \uparrow{\scriptstyle z} \\
\cdot & \xleftarrow[\;h'\;]{} & a
\end{array}
\qquad\qquad
\begin{array}{ccc}
\cdot & \xleftarrow{\;f^{-1}\;} & a_{\tilde{f},\tilde{x}} \\
{\scriptstyle x}\uparrow & \alpha_2 & \uparrow{\scriptstyle x'} \\
a & \xleftarrow[\;f'\;]{} & \cdot
\end{array}
$$

Then, since $F$ is a fibration, we can select a box $\alpha_3$ in $\mathcal{A}$ of the form

$$
\begin{array}{ccc}
\cdot & \xleftarrow{\;f''\;} & \cdot \\
{\scriptstyle yz'}\uparrow & \alpha_3 & \uparrow{\scriptstyle x''} \\
\cdot & \xleftarrow[\;h'f'\;]{} & \cdot
\end{array}
$$

such that

$$
F\alpha_3 = \begin{array}{ccccc}
\cdot & \xleftarrow{\;\tilde{g}^{-1}\tilde{f}\;} & \cdot & \xleftarrow{\;\tilde{f}^{-1}\;} & b' \\
{\scriptstyle \tilde{y}\tilde{x}^{-1}}\uparrow & \tilde{\alpha}^{-\mathrm{h}} & \| & & \\
\cdot & = & \cdot & F\alpha_2 & \uparrow{\scriptstyle Fx'} \\
{\scriptstyle \tilde{x}}\uparrow & \mathrm{I}^{\mathrm{h}}_{\tilde{x}} & \uparrow{\scriptstyle \tilde{x}} & & \\
b & = & b & \xleftarrow[\;Ff'\;]{} & \cdot
\end{array}
$$

This way, we have the path $(gf'', x''x'^{-1}) : a_{\tilde{f},\tilde{x}} \curvearrowright a'_{\tilde{g},\tilde{y}}$

$$
\begin{array}{ccccc}
a'_{\tilde{g},\tilde{y}} & \xleftarrow{\;g\;} & \cdot & \xleftarrow{\;f''\;} & \cdot \\
 & & & & \uparrow{\scriptstyle x''} \\
 & & & & \cdot \\
 & & & & \uparrow{\scriptstyle x'^{-1}} \\
 & & & & a_{\tilde{f},\tilde{x}}
\end{array}
$$

which actually belongs to $\mathcal{F}_{b'}$, since $F(gf'') = \tilde{g}\,\tilde{g}^{-1} = \mathrm{I}^{\mathrm{h}}_{b'}$ and $F(x''x'^{-1}) = Fx'\,Fx'^{-1} = \mathrm{I}^{\mathrm{v}}_{b'}$. $\square$

**Theorem 5.9.** *$\pi_0\mathcal{F}$ is a $\Pi\mathcal{B}$-set.*

*Proof.* We must prove that, for every pair of paths $(\tilde{f},\tilde{x}) : b \curvearrowright b'$ and $(\tilde{g},\tilde{y}) : b' \curvearrowright b''$ in $\mathcal{B}$ and every object $a$ in $\mathcal{F}_b$,

$$^{[\tilde{g},\tilde{y}]}\big(^{[\tilde{f},\tilde{x}]}[a]\big) = \, ^{[\tilde{g},\tilde{y}]\cdot[\tilde{f},\tilde{x}]}[a].$$

For, using that $F$ is a fibration and $\mathcal{A}$ has the filling property, let us construct a diagram in $\mathcal{A}$ of the form

$$\begin{array}{ccccc} a'' & \xleftarrow{g} & \cdot & \xleftarrow{f'} & \cdot \\ & & {\scriptstyle y}\uparrow & \theta & \uparrow{\scriptstyle y'} \\ & & a' & \xleftarrow[f]{} & \cdot \\ & & & & \uparrow{\scriptstyle x} \\ & & & & a \end{array}$$

such that $(Ff, Fx) = (\tilde{f},\tilde{x})$ and $(Fg, Fy) = (\tilde{g},\tilde{y})$. Thus, on the one hand, $^{[\tilde{f},\tilde{x}]}[a] = [a']$ and $^{[\tilde{g},\tilde{y}]}\big(^{[\tilde{f},\tilde{x}]}[a]\big) = [a'']$. On the other hand, the induced diagram in $\mathcal{B}$

$$\begin{array}{ccccc} b'' & \xleftarrow{\tilde{g}} & \cdot & \xleftarrow{Ff'} & \cdot \\ & & {\scriptstyle \tilde{y}}\uparrow & F\theta & \uparrow{\scriptstyle Fy'} \\ & & b' & \xleftarrow[\tilde{f}]{} & \cdot \\ & & & & \uparrow{\scriptstyle \tilde{x}} \\ & & & & b \end{array}$$

tell us that, in the fundamental groupoid of $\mathcal{B}$, $[\tilde{g},\tilde{y}]\cdot[\tilde{f},\tilde{x}] = [\tilde{g}\,Ff', Fy'\,\tilde{x}] = [F(gf'), F(y'x)]$. So that $(gf', y'x) : a \curvearrowright a''$ is a lifting in $\mathcal{A}$ of a representative path in $\mathcal{B}$ of the composite morphism $[\tilde{g},\tilde{y}]\cdot[\tilde{f},\tilde{x}] : b \to b''$ of $\Pi\mathcal{B}$. Hence, $^{[\tilde{g},\tilde{y}]\cdot[\tilde{f},\tilde{x}]}[a] = [a'']$. $\square$

**Proposition 5.10.** *Let $b, b'$ be objects of $\mathcal{B}$ and let*

$$\pi_0\mathcal{F}_b \xrightarrow{i_*} \pi_0\mathcal{A} \xleftarrow{i'_*} \pi_0\mathcal{F}_{b'}$$

*be the induced maps by the inclusions $i : \mathcal{F}_b \hookrightarrow \mathcal{A}$ and $i' : \mathcal{F}_{b'} \hookrightarrow \mathcal{A}$. Then, $[a] \in \pi_0\mathcal{F}_b$ and $[a'] \in \pi_0\mathcal{F}_{b'}$ satisfy $i_*[a] = i'_*[a']$ if and only if*

$$[a'] = \, ^{[\tilde{f},\tilde{x}]}[a]$$

*for some morphism $[\tilde{f},\tilde{x}] : b \to b'$ of $\Pi\mathcal{B}$.*

*Proof.* Suppose $i_*[a] = i'_*[a']$, so that there is a path $(f,x) : a \curvearrowright a'$ in $\mathcal{A}$. Then, if $(\tilde{f},\tilde{x}) = (Ff, Fx) : b \curvearrowright b'$, we have ${}^{[\tilde{f},\tilde{x}]}[a] = [a']$.

Conversely, suppose $(\tilde{f},\tilde{x}) : b \curvearrowright b'$ is a path in $\mathcal{B}$ such that ${}^{[\tilde{f},\tilde{x}]}[a] = [a']$. If $(f,x) : a \curvearrowright a'_0$ is a path in $A$ such that $(Ff, Fx) = (\tilde{f},\tilde{x})$, then we have $i_*[a] = i'_*[a'_0]$ in $\pi_0\mathcal{A}$ and also $[a'_0] = [a']$ in $\pi_0\mathcal{F}_{b'}$. Hence, $i_*[a] = i'_*[a']$. □

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