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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

VOLUME LXVI-1 (2025)

THE TOPOLOGY OF CRITICAL PROCESSES, III (COMPUTING HOMOTOPY)

Marco GRANDIS

Résumé. La topologie algébrique dirigée étudie des espaces équipés d'une forme de direction, avec l'objectif d'inclure les processus non réversibles. Dans l'extension présente nous voulons couvrir aussi les *processus critiques*, indivisibles et inarrêtables.

Les parties précédentes de cette série ont introduit les *espaces contrôlés* et leur catégorie fondamentale. Ici on étudie comment calculer cette dernière. La structure d'homotopie de ces espaces sera examinée dans la Partie IV.

Abstract. Directed Algebraic Topology studies spaces equipped with a form of direction, to include models of non-reversible processes. In the present extension we also want to cover *critical processes*, indecomposable and unstoppable.

The previous parts of this series introduced *controlled spaces* and their fundamental category. Here we study how to compute the latter. The homotopy structure of these spaces will be examined in Part IV.

Keywords. Directed algebraic topology, homotopy theory, fundamental category, concurrent process.

Mathematics Subject Classification (2010). 55M, 55P, 55Q, 68Q85.

Introduction

0.1 Directed and controlled spaces

Directed Algebraic Topology is an extension of Algebraic Topology, dealing with 'spaces' where the paths need not be reversible; the general aim is including the representation of *irreversible processes*. A typical setting for this study, the category $\mathrm{d}\mathsf{Top}$ of directed spaces, or d-spaces, was introduced and studied in [G1]–[G3]; it is often employed in the theory of concurrency, cf. [FGHMR].

The present series is devoted to a further extension, where the paths can also be non-decomposable in order to include *critical processes*, indivisible and unstoppable – either reversible or not. For instance: quantum effects, the onset of a nerve impulse, the combustion of fuel in a piston, the switch of a thermostat, the change of state in a memory cell, the action of a siphon, moving in a no-stop road, etc.

To this effect the category of d-spaces was extended in Part I [G4] to the category $\mathrm{c}\mathsf{Top}$ of *controlled spaces*, or *c-spaces*: an object is a topological space equipped with a set $X^\sharp$ of continuous mappings $a\colon [0,1] \to X$, called *controlled paths*, or *c-paths*, which are closed under concatenation and global reparametrisation (by surjective increasing endomaps of the interval) and include all the constant paths at the endpoints of c-paths.

A *map of c-spaces*, or *c-map*, is a continuos mapping which preserves the selected paths. Their category $\mathrm{c}\mathsf{Top}$ contains the category $\mathrm{d}\mathsf{Top}$ of d-spaces as a full subcategory, reflective and coreflective: a c-space is a d-space if and only if it is *flexible*, which means that each point is flexible (its trivial loop is controlled) and every controlled path is flexible (all its restrictions are controlled).

Every c-space X has two associated d-spaces, the generated d-space $\hat{X}$ and the flexible part $\mathsf{Fl}\,X$, by the reflector and coreflector of the embedding $\mathrm{d}\mathsf{Top} \to \mathrm{c}\mathsf{Top}$ (Section 1.2 of Part I).

0.2 The fundamental category

Part II [G5] defines and studies the fundamental category of controlled spaces, as a functor

$$\uparrow\!\Pi_1\colon \mathrm{c}\mathsf{Top} \to \mathsf{Cat}, \tag{1}$$

that extends the fundamental category of d-spaces [G1, G3] and the fundamental groupoid of topological spaces.

There are two natural transformations (see Section 5.2 of Part II)

$$\uparrow\!\Pi_1(\mathsf{Fl}\,X) \longrightarrow \uparrow\!\Pi_1(X) \longrightarrow \uparrow\!\Pi_1(\hat{X}) \qquad (2)$$

induced by the embeddings $\mathsf{Fl}\,X \to X \to \hat{X}$ (the counit of the coreflector and the unit of the reflector of d-spaces).

These functors need not be faithful, as we shall see in 1.3, but Theorem 5.3(b) of Part II says that $\uparrow\!\Pi_1(X) \to \uparrow\!\Pi_1(\hat{X})$ is a full embedding when the c-space X is *preflexible*, that is all the c-paths of $\hat{X}$ between flexible points of X are already controlled in the latter.

The present Part III is an immediate continuation of Part II, devoted to computing the fundamental category of c-spaces. The definitions and results of Part II are taken for granted and only referred to.

Part IV will study the homotopy structure of c-spaces, their homotopy equivalences and their links with cubical sets. In particular, we shall analyse the formal theory of homotopy in $\mathrm{c}\mathsf{Top}$, following the classification of directed settings in [G3].

0.3 Outline

In Section 1 we calculate the fundamental category of the c-spaces introduced so far, and others, applying Theorems 5.3 (on preflexible c-spaces) and 5.8 (on covering maps of c-spaces) of Part II, and developing peculiar techniques adequate to the present framework. The relationship between the fundamental category of c-spaces and d-spaces is discussed in 1.6, where we show that the theorem of Seifert-van Kampen fails for c-spaces.

In the same line, Section 2 briefly considers how the analysis of obstructions, a typical problem in concurrency, can be dealt with replacing the d-spaces used in [G3], Chapter 3 (and elsewhere) with rigid c-spaces. This leads to a far simpler analysis, but a less rich one.

Finally, in Section 3, we prove that the fundamental category of a border flexible c-space can be simply defined by general deformations of controlled paths, instead of using their flexible deformations – as in the general case.

Acknowledgments. The author is indepted to the Referee for many helpful suggestions.

0.4 Notation and conventions

A continuous mapping between topological spaces is called a *map*. $\mathbb{R}$ denotes the euclidean line as a topological space, and $\mathbb{I}$ the standard euclidean interval $[0, 1]$. The identity path $\mathrm{id}\,\mathbb{I}$ is written as $\underline{i}$. The open and semiopen intervals of the real line are denoted by square brackets, like $]0, 1[$, $[0, 1[$ etc.

A *preorder* relation is assumed to be reflexive and transitive; an *order* is also anti-symmetric. A mapping which preserves (resp. reverses) preorders is said to be *increasing* (resp. *decreasing*), always used in the weak sense.

As usual, a preordered set X is identified with the small category whose objects are the elements of X, with one arrow $x \to x'$ when x precedes x' and none otherwise.

The binary variable α takes values $0, 1$, which are generally written as $-, +$ in superscripts and subscripts. The symbol $\subset$ denotes weak inclusion.

The previous papers [G4, G5] of this series are cited as Part I and Part II, respectively; the reference I.2 or II.3.4, for instance, points to Section 2 of Part I or Subsection 3.4 of Part II.

1. Calculating the fundamental category

This section studies how to compute the fundamental category of c-spaces. Using Theorem II.5.3(b) on preflexible c-spaces, many of these results can be deduced from the fundamental category of the generated d-spaces, already computed in [G3]; but a direct calculation can often be simple and more significant.

The new aspects which appear here, with respect to the theory of d-spaces, are highlighted in 1.6.

The symbols $\mathbf{2}, \mathbf{3}, \mathbf{N}, \mathbf{Z}, \mathbf{R}$ denote ordered sets, and the associated categories; the ordered sets 2, 3 and $D|\mathbf{Z}|$ are discrete. $\mathbb{N}$ is the one-object category associated to the additive monoid of the natural numbers.

1.1 Elementary calculations

We begin by examining the basic c-spaces, showing that many of them are 1-simple, in the sense of II.5.1: their fundamental category is a preorder; of course, the controlled circles $\mathrm{c}\mathbb{S}^1$ and $\mathrm{c}_n\mathbb{S}^1$ are not. (Some of these results are already in II.5.9.)

(a) The fundamental categories of $\mathrm{c}\mathbb{I}$, $\mathrm{c}\mathbb{J}$, $\mathrm{c}\mathbb{R}$ are the following ordered sets:

$$\uparrow\!\Pi_1(\mathrm{c}\mathbb{I}) = \mathbf{2}, \qquad \uparrow\!\Pi_1(\mathrm{c}\mathbb{J}) = \mathbf{3}, \qquad \uparrow\!\Pi_1(\mathrm{c}\mathbb{R}) = \mathbf{Z}. \tag{3}$$

As to $\mathrm{c}\mathbb{I}$, the identity $\underline{i}\colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}$ is 2-equivalent to any other c-path $\rho\colon 0 \to 1$, by Lemma II.4.6(c): in fact, ρ is a global reparametrisation, and therefore $\rho = \underline{i}\rho \sim_2 \underline{i}$, so that there is precisely one arrow $[\underline{i}]$ from 0 to 1, in the fundamental category. At each flexible point, 0 or 1, there is only one loop $\mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}$, the trivial one.

As to $\mathrm{c}\mathbb{J}$ and $\mathrm{c}\mathbb{R}$, two c-paths $a, b\colon x \to y$ in any of them are always 2-equivalent, since they are in the one-jump c-structure of $[x, y]$, isomorphic to $\mathrm{c}\mathbb{I}$.

For these preflexible spaces the components of the natural transformations $\uparrow\!\Pi_1(\mathsf{Fl}\,X) \to \uparrow\!\Pi_1(X) \to \uparrow\!\Pi_1(\hat{X})$ of (2) become inclusions of ordered sets:

$$2 \to \mathbf{2} \to [0, 1], \qquad 3 \to \mathbf{3} \to [0, 2], \qquad D|\mathbf{Z}| \to \mathbf{Z} \to \mathbf{R}. \tag{4}$$

(b) The argument used above for $\uparrow\!\Pi_1(\mathrm{c}\mathbb{I})$ also applies to the delayed intervals $\mathrm{c}_-\mathbb{I}$ and $\mathrm{c}_+\mathbb{I}$, in II.1.3(e)

$$\uparrow\!\Pi_1(\mathrm{c}_-\mathbb{I}) = \uparrow\!\Pi_1(\mathrm{c}_+\mathbb{I}) = \mathbf{2}, \tag{5}$$

whose c-structure is also generated by a single map $\mathbb{I} \to \mathbb{I}$. These c-spaces are not preflexible, but their fundamental category is still full in $\uparrow\!\Pi_1(\uparrow\!\mathbb{I})$.

(c) The fundamental category of the directed circle $\uparrow\!\mathbb{S}^1$, as described in [G3], 3.2.7(d), is the subcategory of the groupoid $\Pi_1\mathbb{S}^1$ formed of the classes of anticlockwise paths (in $\mathbb{R}^2$). Each monoid $\uparrow\!\pi_1(\uparrow\!\mathbb{S}^1, x)$ is isomorphic to the additive monoid $\mathbb{N}$ of natural numbers.

Applying Theorem II.5.3(b), the fundamental category of the one-stop circle $\mathrm{c}\mathbb{S}^1$ amounts to the fundamental monoid at the unique flexible point x_0 (the point 1 of the complex plane)

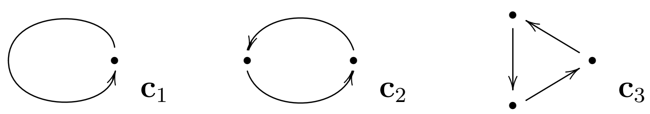
$$\uparrow\!\Pi_1(\mathrm{c}\mathbb{S}^1)(x_0, x_0) = \uparrow\!\pi_1(\uparrow\!\mathbb{S}^1, x_0) = \mathbb{N}. \tag{6}$$

Without using $\uparrow\!\Pi_1(\uparrow\!\mathbb{S}^1)$ this is also proved by Theorem II.5.8(b) applied to the exponential map $\mathrm{c}\mathbb{R} \to \mathrm{c}\mathbb{S}^1$.

Therefore two c-loops a, b in $\mathrm{c}\mathbb{S}^1$ are 2-equivalent if and only if they have the same length $2k\pi$ (in radians), if and only if they both turn k times ($k \geqslant 0$) around the circle, anticlockwise.

(d) More generally, the fundamental category of the preflexible n-stop circle $\mathrm{c}_n\mathbb{S}^1$ (see II.1.4(d) is the full subcategory of the fundamental category of $(\mathrm{c}_n\mathbb{S}^1)\hat{} = \uparrow\mathbb{S}^1 = \uparrow\mathbb{R}/\mathbb{Z}$ on n flexible points, the vertices $[i/n]$ (for $i = 0, ..., n-1$) of an inscribed n-gon.

$\uparrow\Pi_1(\mathrm{c}_n\mathbb{S}^1)$ is thus the category $\mathbf{c}_n$ freely generated by n arrows disposed as follows on the edges of an n-gon



(7)

Again, this result can also be obtained using the covering map of c-spaces $p_n \colon \mathrm{c}_n\mathbb{R} \to \mathrm{c}_n\mathbb{S}$.

(e) For the preflexible c-space X on the euclidean interval $[0, 3]$ described in I.2.3(e) we have a mixed situation; essentially, the paths in $[1, 2]$ behave as in $\mathrm{c}\mathbb{I}$, while those in $[0, 1]$ or $[2, 3]$ behave as in $\uparrow\mathbb{I}$.

1.2 Higher dimensional c-spaces

(a) Applying Theorem II.5.6 on cartesian products, we get the following fundamental categories

$$\begin{gathered}\uparrow\Pi_1(\mathrm{c}\mathbb{I}^n) = \mathbf{2}^n, \qquad \uparrow\Pi_1(\mathrm{c}\mathbb{J}^n) = \mathbf{3}^n,\\ \uparrow\Pi_1(\mathrm{c}\mathbb{I}\times\mathrm{c}\mathbb{J}) = \mathbf{2}\times\mathbf{3},\\ \uparrow\Pi_1(\mathrm{c}\mathbb{R}^n) = \mathbf{Z}^n, \qquad \uparrow\Pi_1(\mathrm{c}\mathbb{T}^n) = \mathbb{N}^n,\end{gathered} \tag{8}$$

which are (partially) ordered sets, except the last. The controlled n-torus $\mathrm{c}\mathbb{T}^n$ was defined in I.2.6(d) as the cartesian power $(\mathrm{c}\mathbb{S}^1)^n$, or equivalently as the orbit c-space $(\mathrm{c}\mathbb{R}^n)/\mathbb{Z}^n$; its fundamental category amounts to the monoid $\mathbb{N}^n$ at the only flexible point.

(b) The fundamental category of all the higher c-spheres $\mathrm{c}\mathbb{S}^n$, for $n \geqslant 2$, is trivial: the discrete category $\mathbf{1}$.

In fact, there is one flexible point, $*$. Every c-path of $\mathrm{c}\mathbb{S}^n$ is a general concatenation of a finite family of c-loops of the form pa, where $a\colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}^n$ is a c-path of the controlled n-cube, and it is sufficient to prove that each of them is 2-equivalent to the trivial loop (at $*$).

If the path a lies in a face of the cube, pa is already the trivial loop. Otherwise, it is a path $(0, ..., 0) \to (1, ..., 1)$, and it is 2-equivalent to the concatenation $b = b_1 * b_2$ of two c-paths living in some faces, and collapsed to the trivial loop in the quotient c-space. For instance one can take $b_1(t) = (t, 0, ..., 0)$ (on an edge) and $b_2(t) = (1, t, ..., t)$ (in the face $t_1 = 1$).

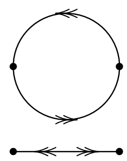
1.3 Other calculations

The following computations of the fundamental category give a better understanding of the natural transformations $\uparrow\!\Pi_1(\mathsf{Fl}\,X) \to \uparrow\!\Pi_1(X) \to \uparrow\!\Pi_1(\hat{X})$ of (2). Moreover, they are based on topological arguments which will also be useful in other cases.

(a) The reversible c-interval $\mathrm{c}\mathbb{I}^{\sim}$ of II.1.3(d) has a c-structure generated by the identity path $\underline{i}$ and the reversion $r\colon \mathbb{I} \to \mathbb{I}$; the flexible points are 0 and 1.

Each c-path $x \to y$ (between flexible points) has an integral length, which is even if $x = y$ and odd if $x \neq y$. We prove below, in Theorem 1.7, that this length is constant up to 2-equivalence, and determines the class of a path in $\uparrow\!\Pi_1(\mathrm{c}\mathbb{I}^{\sim})(x, y)$.

In other words, we shall prove that the obvious c-map $p\colon \mathrm{c}_2\mathbb{S}^1 \to \mathrm{c}\mathbb{I}^{\sim}$



$$p(x, y) = (x + 1)/2 \tag{9}$$

induces an isomorphism $p_*\colon \uparrow\!\Pi_1(\mathrm{c}_2\mathbb{S}^1) \to \uparrow\!\Pi_1(\mathrm{c}\mathbb{I}^{\sim})$ defined on the category $\mathbf{c}_2$ described in (7). Let us note that p is not a covering map: the flexible points of the basis are not evenly covered; loosely speaking, the selection of c-paths in the domain and codomain 'mends' this failure.

Thus the category $\uparrow\!\Pi_1(\mathrm{c}\mathbb{I}^{\sim})$ is freely generated by two arrows, the classes $[\underline{i}]\colon 0 \to 1$ and $[r]\colon 1 \to 0$; at each vertex it has a fundamental monoid isomorphic to the additive monoid $\mathbb{N}$.

The generated d-space $(\mathrm{c}_S\mathbb{I})\hat{} = \mathbb{I}^\sim$ is the reversible d-interval of I.2.4(c), whose fundamental category is the indiscrete groupoid on two objects (with one arrow between any pair of objects).

In this case the functor $\uparrow\!\Pi_1(X) \to \uparrow\!\Pi_1(\hat{X})$ is not faithful; moreover $\underline{i}$ and r are reversible c-paths of X whose classes in $\uparrow\!\Pi_1(X)$ are not invertible.

(b) The fundamental category $\uparrow\!\Pi_1(\mathrm{c}_S\mathbb{I})$ of the growing-siphon interval (in I.3.3(a)) is generated by the following arrows (where r is the reversion path $r(t) = 1 - t$)

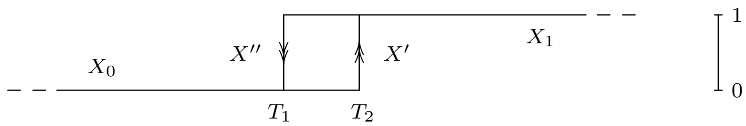
$$(x, x')\colon x \to x', \qquad [r]\colon 1 \to 0 \qquad\qquad (0 \leqslant x < x' \leqslant 1), \qquad (10)$$

under the relation $(x, x')(x', x'') = (x, x'')$, for $0 \leqslant x < x' < x'' \leqslant 1$.

The identity path $\underline{i}$ is flexible and reversible in $\mathrm{c}_S\mathbb{I}$, but is not flexibly reversible: the reversed path r is not flexible, and the associated arrow $[\underline{i}] = (0, 1)\colon 0 \to 1$ is not invertible. But it becomes invertible in the fundamental category of $(\mathrm{c}_S\mathbb{I})\hat{} = \mathbb{I}^\sim$: also here the functor $\uparrow\!\Pi_1(X) \to \uparrow\!\Pi_1(\hat{X})$ is not faithful.

1.4 On-off controller

We now examine the c-space X built in I.3.1(a) to model an on-off controller (e.g. a thermostat) that oversees a variable T (e.g. the temperature), counteracting its rising



(11)

On the left branch X_0 the system is in state 0: the cooling device is off; if the temperature grows to T_2 the device jumps to state 1; then, if the temperature cools to T_1, it goes back to state 0.

The support $|X|$ of our model is a one-dimensional subspace of $\mathbb{R}^2$. The c-structure of X is generated by the c-structures of:

- X_0, X_1, natural intervals where T can vary,

- X', X'', one-jump c-intervals, where T is constant and the state of the system varies.

The flexible part $X_0 + X_1$ of the c-space X is the sum of two natural intervals; its fundamental groupoid $\Pi_1(\mathsf{Fl}\, X)$ is the sum of the indiscrete groupoids on the same sets, categorically equivalent to the discrete groupoid $2 = \{0, 1\}$.

The fundamental category $\uparrow\!\Pi_1(X)$ is equivalent to its skeleton, the full subcategory on two points $x_0 \in X_0$ and $x_1 \in X_1$; the latter is isomorphic to the category $\mathbf{c}_2$ (see (7)).

1.5 Transport networks and labelled graphs

Transport networks are usually modelled in graph theory, in an effective way as far as they do not interact with continuous variation. They can also be modelled by c-spaces, which allows us to combine them with planar or three-dimensional regions, as we have discussed in I.3.4.

The fundamental category can be readily used to study such models. Controlled spaces can thus unify aspects of continuous and discrete mathematics, interacting with hybrid control systems and others sectors of Control Theory [Br, He].

1.6 Comments

(a) The main method of calculation of the fundamental category for complex spaces, the theorem of Seifert-van Kampen, holds true in $\mathrm{d}\mathsf{Top}$, in the fundamental-category version of [G3], 3.2.6, but fails here.

For instance, we have seen that the category $\uparrow\!\Pi_1(\mathrm{c}\mathbb{I}) = \mathbf{2}$ has one arrow $0 \to 1$. Now we can cover $\mathrm{c}\mathbb{I}$ with the open subspaces $U = [0, 1[$ and $V =]0, 1]$, which only inherit the trivial loops at 0 and 1, respectively. Their fundamental category has only these trivial arrows, and the pushout over $\uparrow\!\Pi_1(U \cap V)$ (the empty category) gives the discrete category 2.

(b) Nevertheless, we have seen that the fundamental category $\uparrow\!\Pi_1(X)$ of a rigid or 'partially rigid' c-space can be rather easy to compute without this tool – or using it on $\uparrow\!\Pi_1(\hat{X})$ when the original c-space is preflexible.

(c) In many cases $\uparrow\!\Pi_1(X)$ is very small and easy to analyse, while $\uparrow\!\Pi_1(\hat{X})$ gives a finer description, at the price of a complex analysis where the equivalence of categories is totally ineffective. This will show even more clearly in the next section.

1.7 Theorem

The projection $p\colon \mathrm{c}_2\mathbb{S}^1 \to \mathrm{c}\mathbb{I}^\sim$ *defined in* (9) *induces an isomorphism of categories* $p_*\colon {\uparrow}\Pi_1(\mathrm{c}_2\mathbb{S}^1) \to {\uparrow}\Pi_1(\mathrm{c}\mathbb{I}^\sim)$.

Proof. (a) The functor p_* is bijective on the objects, the flexible points. It is also full, because $p\colon \mathrm{c}_2\mathbb{S}^1 \to \mathrm{c}\mathbb{I}^\sim$ obviously satisfies the path-lifting property II.5.7(i) within c-paths: every c-path $b\colon y \to y'$ in $\mathrm{c}\mathbb{I}^\sim$ has a lifting $a\colon x \to x'$ in $\mathrm{c}\mathbb{I}^\sim$, determined by the starting point $x \in F_y$ (unique in the present case).

The length of b is an integer, equal to the length of a measured in half-circles.

To prove that p_* is faithful we shall show that two c-paths $b, b'\colon y_0 \to y_1$ in $\mathrm{c}\mathbb{I}^\sim$ which are 2-equivalent *have the same length*, so that any pair of their liftings in $\mathrm{c}_2\mathbb{S}^1$ starting at the same point are also 2-equivalent; in other words one can lift along p the 2-equivalence relation – if not the actual 2-paths.

For the sake of simplicity we suppose that $y_0 = 0$, the case $y_0 = 1$ being similar. We use the path spaces $P(\mathbb{I}) = \mathbb{I}^\mathbb{I}$ and $P(\mathbb{I}^2)$ with the compact-open topology, determined by the metric $d(c, c') = \max_t\, d(c(t), c'(t))$ (and the euclidean metric on $\mathbb{I}$ and $\mathbb{I}^2$).

(b) Let P_n be the subspace of $P(\mathbb{I})$ formed of the c-paths $\mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}^\sim$ starting at 0, of length n; let P be their (disjoint) union. We prove now that each P_n is open in P. (This amounts to saying that the length function $P \to \mathbb{N}$ is continuous, which is not obvious as it fails on the whole path space $\mathbb{I}^\mathbb{I}$.)

It will be sufficient to show that any two c-paths $b, b'\colon 0 \to y$ with $d(b, b') < 1/2$ have the same length. If b has length n, it determines a partition of the interval $\mathbb{I}$ in n subintervals

$$\begin{gathered} 0 = t_0 < t_1 < ... < t_n = 1, \\ b(t_0) = 0, \qquad b(t_1) = 1,\ ... \qquad b(t_n) = (1 - (-1)^n)/2, \end{gathered} \tag{12}$$

and is *properly* increasing on $[0, t_1]$, properly decreasing on $[t_1, t_2]$, and so on (by ‘properly’ we mean that it is not constant). There are $n-1$ ‘inversions of monotony’ (each of them occurring on a maximal closed subinterval where b is constant at 1 or 0, alternatively).

The other path b', of length n', has $b'(t_0) = 0$ and $b'(t_1) > 1/2$; because of the form of c-paths in $\mathrm{c}\mathbb{I}^\sim$, it must be properly increasing on some (at least one) subinterval of $[0, t_1]$. It also has $b'(t_2) < 1/2$, and must be properly decreasing on some subinterval of $[t_1, t_2]$; and so on. Finally, it has at least as many inversions of monotony as b, and $n' \geqslant n$. By symmetry, $n = n'$.

(c) Let $K\colon \mathrm{c}\mathbb{I} \times {\uparrow}\mathbb{I} \to \mathrm{c}\mathbb{I}^{\sim}$ be a hybrid 2-path between the c-paths $b, b'\colon 0 \to y$. Proving that they have the same length will achieve the argument.

The family of c-paths

$$u_t\colon \mathrm{c}\mathbb{I} \to \mathrm{c}\mathbb{I}^{\sim} \times {\uparrow}\mathbb{I}, \qquad u_t(s) = (s, t) \qquad\qquad (t \in \mathbb{I}), \tag{13}$$

gives an isometry $u\colon \mathbb{I} \to P(\mathbb{I}^2)$

$$d(u_t, u_{t'}) = \max_s d((s,t),(s,t')) = |t - t'|.$$

Composing u with the map $K_*\colon P(\mathbb{I}^2) \to P(\mathbb{I})$ we get a continuous mapping

$$Ku\colon \mathbb{I} \to P(\mathbb{I}), \qquad t \mapsto K_t = K(-,t)\colon \mathbb{I} \to \mathbb{I}, \tag{14}$$

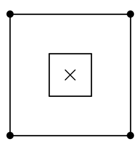
whose values K_t are the intermediate c-paths of K (see II.4.4(a)). They belong to P. Since Ku is defined on a connected space, all of them belong to the same subset P_n, including b and b'. $\square$

2. Analysing obstructions

The analysis of obstructions inside a cubical directed space is a typical problem in concurrency, dealt with in [FGHMR] and many papers (see Part I). It is also studied in [G3], Chapter 3, working with d-spaces. The corresponding problem in rigid c-spaces seems to be far simpler, although it can give a less fine analysis, as shown in 2.3.

2.1 An elementary case

We begin with the 'square annulus' $X \subset \mathrm{c}\mathbb{I}^2$ represented below, namely the compact subspace of the standard c-square which is the complement of the open square $]1/3, 2/3[^2$ (marked with a cross); the latter should be viewed as a single obstruction in an unstoppable process



X

$$\begin{tikzcd} x \arrow[r] & 1 \\ 0 \arrow[u] \arrow[r] \arrow[ur, phantom, "\times"] & y \arrow[u] \end{tikzcd} \qquad {\uparrow}\Pi_1(X) \tag{15}$$

Typically, in the analysis of concurrent processes, the obstruction represents a resource (e.g. a memory storage, an application, a printer) that two (or more) concurrent automata cannot engage at the same time. A path below *or* above the obstruction corresponds to priority of one of them. Modifying the picture, one can represent in a similar way an island in a stream or a one-dimensional obstacle in space-time, as in the Introduction to [G3].

The fundamental category $\uparrow\Pi_1(X)$ is represented in the right diagram-above: it is generated by four arrows forming a non-commutative square, and has two arrows $0 \to 1$ (not drawn in the figure).

Applying Theorem II.5.3(b) one can deduce this fact from the fundamental category of the generated d-space $\hat{X} \subset \uparrow\mathbb{I}^2$, determined in [G3], 3.1.1. But a direct proof is rather simple.

In fact, every c-path $a\colon 0 \to 1$ in X meets the vertical strip

$$S =]1/3, 2/3[\times \mathbb{I}$$

in one connected component of $S \cap X$, either below or above the obstruction. Suppose that a meets the lower component $U =]1/3, 2/3[\times[0, 1/3[$ (open in X). The preimage $a^{-1}(U)$ is an open subinterval of $]0, 1[$ (by continuity and monotony), and we can suppose it is precisely $]1/3, 2/3[$, up to invertible reparametrisation and 2-equivalence. For a second path a' of the same kind and similarly reparametrised, we can suppose that $a(t) \leqslant a'(t)$ for $t \in \mathbb{I}$ (replacing a with $a \wedge a'$).

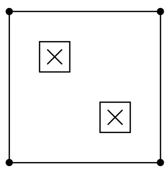
Now the affine interpolation H from a to a' is a hybrid 2-path in $\mathrm{c}\mathbb{I}^2$ and takes the interval $]1/3, 2/3[$ to the rectangle U (by monotony), proving that $a \sim_2 a'$ in X. Similarly, two paths above the obstruction are 2-equivalent in X. Finally, a c-path below the obstruction and another above are not even 2-equivalent in the underlying topological space.

2.2 Two obstructions

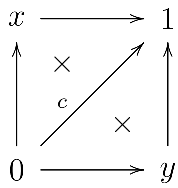
We examine now two subspaces $Y, Z \subset \mathrm{c}\mathbb{I}^2$ which arise from two obstructions, either appearing together (with respect to the generated path order, see I.1.8(c)) or one after the other.

In both cases a direct computation is easy, if more complex than in the previous case; alternatively, one can deduce our results from the fundamental category of the generated d-spaces, described in [G3], 3.9.2 and 3.9.4(b).

(a) *Simultaneous obstructions*. The first case can be modelled with the subspace Y of $\mathrm{c}\mathbb{I}^2$ represented below



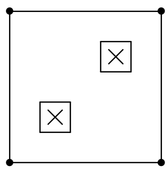
Y



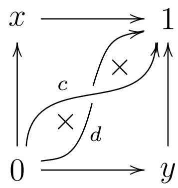
$\uparrow\Pi_1(Y)$ (16)

The fundamental category $\uparrow\Pi_1(Y)$ has again four vertices; from 0 to 1 there are three arrows: $[a]$ (through x), $[b]$ (through y) and $[c]$.

(b) *Consecutive obstructions*. The second case is modelled by $Z \subset \mathrm{c}\mathbb{I}^2$



Z



$\uparrow\Pi_1(Z)$ (17)

In $\uparrow\Pi_1(Z)$ there are now four arrows from 0 to 1: $[a]$ (through x), $[b]$ (through y) and $[c], [d]$.

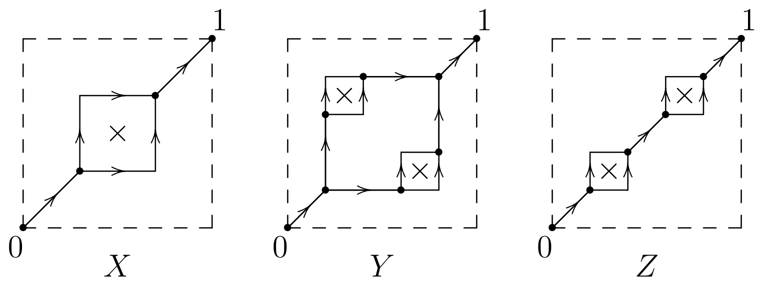
(c) *Comments*. The fundamental category distinguishes these situations, which topology cannot separate: the underlying topological spaces $|Y|$ and $|Z|$ are homeomorphic.

2.3 Obstructions in d-spaces

The d-spaces $\hat{X}, \hat{Y}, \hat{Z}$ generated by the previous c-spaces have the same topological support and the structure induced by the ordered square $\uparrow\mathbb{I}^2$.

Their fundamental category, much more complex than in the previous cases, was studied in [G3], 3.1.1, 3.9.2, 3.9.4(b).

In each case the fundamental category, whose objects are the infinite points of the support, is skeletal and *cannot* be reduced up to equivalence of categories. As analysed in [G3], Section 3.9, it is essentially represented by a 'minimal injective model', future and past equivalent to the given category. Here we get the finite, full subcategories represented below (on 4, 8, 6 objects, respectively), determining the 'branching points' of the process



(18)

A cell marked with a cross is not commutative, while the central cell in $\uparrow\Pi_1(\hat{Y})$ commutes. In $\uparrow\Pi_1(\hat{X})$ there are two arrows $0 \to 1$, in $\uparrow\Pi_1(\hat{Y})$ there are three of them, in $\uparrow\Pi_1(\hat{Z})$ four.

3. Border flexible c-spaces and strict homotopies

We end by examining the relationship of border flexible c-spaces (defined in II.2.1(c)) with strict homotopies (see II.4.3(e)), expressed in Theorem 3.1.

As a consequence, the fundamental category of a border flexible c-space can be simply defined using c-paths up to homotopy with fixed endpoints (see 3.2). Its invariance up to strict homotopies is stated in Theorem 3.3.

The importance of a simple construction, instead of the hybrid construction of Sections II.4 and II.5, is evident – although it does not apply to essential c-spaces like the delayed intervals and the higher c-spheres, which are not border flexible (see II.2.2).

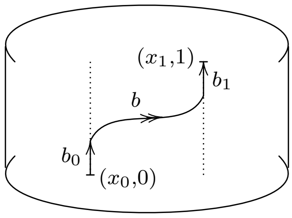
3.1 Theorem (Border flexible c-spaces and homotopies)

Let Y be a border flexible c-space. Every strict homotopy $\varphi\colon X \times \mathrm{c}\mathbb{I} \to Y$ is flexible.

Proof. We are given a c-map $\varphi\colon X \times \mathrm{c}\mathbb{I} \to Y$ which is constant on each fibre $\{x\} \times \mathrm{c}\mathbb{I}$ at a flexible point of X, and we have to prove that φ is also a c-map $X \times {\uparrow}\mathbb{I} \to Y$.

We take a c-path $b = \langle a, h\rangle\colon \mathrm{c}\mathbb{I} \to X \times {\uparrow}\mathbb{I}$, where $a\colon x_0 \to x_1$ is a c-path of X (between flexible points) and $h\colon t_0 \to t_1$ is increasing in ${\uparrow}\mathbb{I}$; we have to prove that φb is controlled in Y.

We insert a path $b_\alpha \colon \mathrm{c}\mathbb{I} \to X \times {\uparrow}\mathbb{I}$ in each fibre of the cylinder at the endpoints x_α (for $\alpha = 0, 1$)



$X \times {\uparrow}\mathbb{I}$ (19)

$$b_0 = \langle e_{x_0}, h_0\rangle \colon \mathrm{c}\mathbb{I} \to X \times {\uparrow}\mathbb{I}, \qquad h_0 \colon 0 \to t_0,$$
$$b_1 = \langle e_{x_1}, h_1\rangle \colon \mathrm{c}\mathbb{I} \to X \times {\uparrow}\mathbb{I}, \qquad h_1 \colon t_1 \to 1,$$

and we get a c-path $b' = \langle a', h'\rangle = b_0 * b * b_1$ in $X \times {\uparrow}\mathbb{I}$ which is controlled in $X \times \mathrm{c}\mathbb{I}$, because h' is an increasing path $0 \to 1$.

Now $\varphi b'$ is controlled in the border flexible c-space Y and each path φb_α is constant (because φ is a strict homotopy). It follows that the middle restriction φb is also controlled in Y. $\square$

3.2 The border flexible case

As a particular case of the previous theorem, if the c-space X is border flexible, a general 2-path $\mathrm{c}\mathbb{I}^2 \to X$ is always a hybrid 2-path $\mathrm{c}\mathbb{I} \times {\uparrow}\mathbb{I} \to X$ (because H is constant on the vertical edges of $\mathrm{c}\mathbb{I}^2$).

Therefore the restricted functor

$$\uparrow\!\Pi_1 \colon \mathrm{c_{bf}}\mathsf{Top} \to \mathsf{Cat}, \tag{20}$$

can be equivalently defined using general 2-paths, based on the standard square $\mathrm{c}\mathbb{I}^2$, instead of hybrid 2-paths based on $\mathrm{c}\mathbb{I} \times {\uparrow}\mathbb{I}$.

The restricted functor is still invariant up to flexible homotopies. But strict homotopies in $\mathrm{c_{bf}}\mathsf{Top}$ are always flexible, giving the following result.

3.3 Theorem (Homotopy invariance, III)

A strict homotopy $\varphi \colon f \to g \colon X \to Y$ of border flexible c-spaces induces the identity of the associated functors

$$f_* = g_* \colon \uparrow\!\Pi_1(X) \to \uparrow\!\Pi_1(Y). \tag{21}$$

Proof. By Theorem 3.1, φ is a strict flexible homotopy. Applying Theorem II.5.4(b), φ_* is the identity of $f_* = g_*$. □

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

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TOPICS IN THE CATEGORICAL ALGEBRA OF CLOSURE SPACES

George JANELIDZE and Manuela SOBRAL

Résumé. Par espace de fermeture nous entendons une paire $(A, \mathcal{C})$, dans laquelle A est un ensemble et $\mathcal{C}$ est un ensemble de sous-ensembles de A fermé sous les intersections arbitraires. Le but de cet article consiste à considérer plusieurs questions qui se posent naturellement dans le cadre de l' algèbre catégorique des espaces de fermeture. Cela inclut l'extensivité (à gauche) de leur catégorie, la description des morphismes de codescente effective, et la description des morphismes de co-revłtement et co-monotones par rapport à une certaine coréflexion des espaces de fermeture dans les ensembles. Cette coréflexion envoie chaque espace de fermeture sur son plus petit sous-ensemble fermé.

Abstract. By a closure space we mean a pair $(A, \mathcal{C})$, in which A is a set and $\mathcal{C}$ a set of subsets of A closed under arbitrary intersections. The purpose of this paper is to consider several questions that naturally arise in the categorical algebra of closure spaces. This includes (left) extensivity of their category, description of effective codescent morphisms, and description of cocovering and comonotone morphisms with respect to a certain coreflection of closure spaces into sets. That coreflection carries a closure space to its smallest closed subset.

Keywords. closure space, left coextensive category, effective codescent morphism, cocovering, comonotone morphism.

Mathematics Subject Classification (2010). 54A05, 18B50, 18E50, 18A40.

1. Introduction

There is a number of types of mathematical structures introduced by various authors as 'generalized topological spaces', and several of them were called closure spaces, the paper [8] being one of many useful references; let us also mention the book [7] and the paper [6] for two important links with category theory, omitting many others. Here we briefly repeat from [10]:

- By a *closure space* we will mean a pair $(A, \mathcal{C})$, in which A is a set and $\mathcal{C}$ a set of subsets of A closed under arbitrary intersections; we will write $A = (A, \mathcal{C}) = (A, \mathcal{C}_A)$ and

  $$\overline{X} = \overline{X}^A = \bigcap_{X \subseteq A' \in \mathcal{C}_A} A'$$

  for a subset X in A. And we say that X is closed in A when $X \in \mathcal{C}_A$, or, equivalently, $\overline{X} = X$.

- A map $f : A' \to A$ of closure spaces is said to be *continuous* if it satisfies (any of) the following three equivalent conditions:

  $$X \in \mathcal{C}_A \Rightarrow f^{-1}(X) \in \mathcal{C}_{A'},$$

  $$X \subseteq A \Rightarrow \overline{f^{-1}(X)} \subseteq f^{-1}(\overline{X}),$$
  $$X' \subseteq A' \Rightarrow f(\overline{X'}) \subseteq \overline{f(X')}.$$

  The category of closure spaces and their continuous maps will be denoted by $\mathbf{CLS}$.

- A continuous map $f : A' \to A$ of closure spaces is said to be *closed* if it satisfies (any of) the following three equivalent conditions:

  $$X' \in \mathcal{C}_{A'} \Rightarrow f(X') \in \mathcal{C}_A,$$

  $$X' \subseteq A' \Rightarrow f(\overline{X'}) \supseteq \overline{f(X')},$$
  $$X' \subseteq A' \Rightarrow f(\overline{X'}) = \overline{f(X')}.$$

- The underlying set functor $U : \mathbf{CLS} \to \mathbf{Sets}$ is *topological* in the sense of categorical topology, and so $\mathbf{CLS}$ is small complete and small cocomplete, and U preserves all existing limits and colimits. In particular, a diagram in $\mathbf{CLS}$ of the form

$$\begin{tikzcd} A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A_2 \arrow[d] \\ A_1 \arrow[r] & B \end{tikzcd}$$

is a pullback diagram in $\mathbf{CLS}$ if and only if its U-image is a pullback diagram in $\mathbf{Sets}$ and $\mathcal{C}_A = \{\pi_1^{-1}(X_1) \cap \pi_2^{-1}(X_2) \mid X_1 \in \mathcal{C}_{A_1} \,\&\, X_2 \in \mathcal{C}_{A_2}\}$. We also have $\overline{X} = \pi_1^{-1}(\overline{\pi_1(X)}) \cap \pi_2^{-1}(\overline{\pi_2(X)})$ for every $X \subseteq A$.

The purpose of this paper is to consider several questions that naturally arise in categorical algebra of closure spaces. They could be asked more generally, replacing $\mathbf{CLS}$ with an abstract *topological category* (as defined, e.g., in the survey paper [2]; see also references therein), or even more generally – and doing that systematically could be an interesting future project. Some of it would even be well known, as, for example, a part of Section 3, or Theorem 4.1 (which can be seen as a special case of the dual of Proposition 9.7 of [13]). But here we only consider specifically the case of $\mathbf{CLS}$.

How *extensive* (in the sense of [4]) is the category $\mathbf{CLS}$? This question is answered in Section 2.

Three adjunctions,

discrete $\dashv$ *underlying set* $\dashv$ *codiscrete* $\dashv$ *smallest closed subset*,

written as $D \dashv U \dashv C \dashv Z$, between $\mathbf{CLS}$ and the category of sets are considered in Section 3. Unlike in the case of topological spaces, it seems that no reasonable 'locally connected' counterpart of D has a left adjoint admitting a nice Galois theory/theory of covering spaces, by which we mean having a nice special case of constructions of [3]. The adjunctions $D \dashv U$ and $U \dashv C$ are also not interesting from this viewpoint since (D and C are fully faithful while) U has both adjoints. It remains to consider the adjunction $C \dashv Z$, which is done in Sections 5 and 6.

In Section 4 we prove that a morphism in $\mathbf{CLS}$ is an effective codescent morphism (=an effective descent morphism in $\mathbf{CLS}^{\mathrm{op}}$) if and only if it is a subspace inclusion (up to an isomorphism). A possible reference to descent theory convenient for our purposes is any of the surveys [12] and [11], although only very preliminary material from there is needed. More precisely, all we will need to have in mind is the dual form of the following well-known fact: *A morphism in a category with pullbacks and coequalizers is an effective descent morphism whenever it is a pullback stable regular epimorphism with the corresponding pullback functor preserving coequalizers.* Note that here the pullback stability already makes the corresponding pullback functor fully faithful, and, in particular, conservative.

In Section 5 we examine dual forms of some notions considered in [3] in the case of $C \dashv Z$. Specifically, we prove that the coreflection Z has stable counits, and characterize cocoverings (which turn out to be the same as trivial cocoverings) and comonotone morphisms. Then, in Section 6, we make immediate conclusions concerning the resulting factorization systems:

- We have the one first constructed for a general reflection in [5], but not the '(colight, comonotone)'-factorization system.
- On the other hand, there is the obvious (dense, closed subspace inclusions)-factorization system, and as shown in Section 5, comonotone morphisms are the same as closed subspace inclusions.

This paper is dedicated to Bill Lawvere, who was the first to see many unusual adjunctions and their roles.

2. Coproducts and non-distributivity

Let $(A_\lambda)_{\lambda\in\Lambda}$ be a family of closure spaces. The coproduct $\sum_{\lambda\in\Lambda} A_\lambda$ is the disjoint union of all A_λ ($\lambda \in \Lambda$), in which a subset X is closed if and only if $X \cap A_\lambda$ is closed in A_λ for each $\lambda \in \Lambda$; writing this we use disjoint union as ordinary union, which we can do here and below without loss of generality. Note that each A_λ is closed in $\sum_{\lambda\in\Lambda} A_\lambda$ if and only if the empty set is closed in each A_λ.

Theorem 2.1. *For every family* $(f_\lambda : A_\lambda \to B_\lambda)_{\lambda\in\Lambda}$ *of morphisms in* CLS*,* *and every $\mu \in \Lambda$, the diagram*

$$\begin{tikzcd} A_\mu \arrow[r] \arrow[d, "f_\mu"'] & \sum_{\lambda\in\Lambda} A_\lambda \arrow[d, "\sum_{\lambda\in\Lambda} f_\lambda"] \\ B_\mu \arrow[r] & \sum_{\lambda\in\Lambda} B_\lambda \end{tikzcd}$$

whose horizontal arrows are coproduct injections, is a pullback diagram.

Proof. Our assertion is true at the level of sets, and all we need to prove is that a subset of A_μ is closed if and only if it is of the form $f_\mu^{-1}(Y) \cap Y'$ for some closed subsets Y of B_μ and Y' of $\sum_{\lambda\in\Lambda} A_\lambda$.

"If": Just note that $f_\mu^{-1}(Y) \cap Y' = f_\mu^{-1}(Y) \cap A_\mu \cap Y'$ and both $f_\mu^{-1}(Y)$ and $A_\mu \cap Y'$ are closed in A_μ, whenever Y is closed in B_μ and Y' is closed in $\sum_{\lambda\in\Lambda} A_\lambda$.

"Only if": For any subset X of A_μ, we have

$$X = A_\mu \cap (X \cup \sum_{\lambda\in\Lambda\setminus\{\mu\}} A_\lambda) = f_\mu^{-1}(B_\mu) \cap (X \cup \sum_{\lambda\in\Lambda\setminus\{\mu\}} A_\lambda),$$

and if X is closed in A_μ, then $X \cup \sum_{\lambda\in\Lambda\setminus\{\mu\}} A_\lambda$ is closed in $\sum_{\lambda\in\Lambda} A_\lambda$. $\square$

Theorem 2.2. *The category* $\mathbf{CLS}$ *is* infinitary left extensive*, that is, for every family* $(A_\lambda)_{\lambda\in\Lambda}$ *of closure spaces, the functor*

$$\Sigma : \prod_{\lambda\in\Lambda} (\mathbf{CLS} \downarrow A_\lambda) \to (\mathbf{CLS} \downarrow \sum_{\lambda\in\Lambda} A_\lambda)$$

is fully faithful.

Proof. This is trivial for empty Λ. For non-empty Λ, just note that taking pullbacks of the form

$$\begin{tikzcd} A_\mu \times_{\sum_{\lambda\in\Lambda} A_\lambda} A \arrow[r] \arrow[d] & A \arrow[d] \\ A_\mu \arrow[r] & \sum_{\lambda\in\Lambda} A_\lambda \end{tikzcd}$$

determines the right adjoint of Σ, and Theorem 2.1 in fact says that the unit of adjunction is an isomorphism. $\square$

However, Theorem 2.2 can also easily be proved directly. Note also that the term “infinitary left extensive” seems to be used here for the first time, although the term “left (co)extensive” was used in [1].

Remark 2.3. Consider Proposition 2.6 and 2.8 in [4]. They say that, in an extensive category, sums (=coproducts) are disjoint and the initial object is strict, respectively. However, the poofs given in [4] show that the same is true in any left extensive category. In particular, these properties hold in $\mathbf{CLS}$. However, in the case of $\mathbf{CLS}$ these properties are obvious anyway.

On the other hand, as the following simple example shows, the category $\mathbf{CLS}$ is not distributive (cf. Proposition 2.2 in [6]), which implies that it is not extensive (and not cartesian closed; and the same applies to the category of finite closure spaces).

Example 2.4. Let $A = \{a, a'\}$, $B = \{b\}$, and $C = \{c\}$ be discrete topological spaces considered as closure spaces (we assume $a \neq a'$ and $b \neq c$ of course). Then we can say that both $A \times (B + C)$ and $(A \times B) + (A \times C)$ have the same underlying set $\{(a, b), (a', b), (a, c), (a', c)\}$, but the set $\{(a, b), (a', c)\}$ is closed in $(A \times B) + (A \times C)$ and not in $A \times (B + C)$.

3. The adjunctions with sets

There are adjunctions

$$\begin{tikzcd}
\mathbf{CLS} \arrow[r, bend left=15, "U"] \arrow[r, bend right=60, "Z"'] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & \mathbf{Sets} \arrow[l, bend right=60, "D"'] \arrow[l, bend left=15, "C"]
\end{tikzcd}$$

where, for a set S and a closure space A, we have:

3.1. $D(S)$ is the discrete topological space (considered as a closure space) with the underlying set S.

3.2. U is the underlying set functor.

3.3. $C(S)$ is what we will call codiscrete S: it has $UC(S) = S$ and $\mathcal{C}_{C(S)} = \{S\}$.

3.4. $Z(A) = \overline{\emptyset}$ is the smallest element of of $\mathcal{C}_A$. When there is no danger of confusion, we will write $CZ(A) = Z(A) = 0_A$.

Note that D has no left adjoint since it does not preserve, say, binary products, and Z has no right adjoint since it does not preserve, say, the coequalizer of

$$\{a\} \overset{f}{\underset{g}{\rightrightarrows}} \{b, c\}\ ,$$

where $Z(\{a\}) = \emptyset$, $Z(\{b, c\}) = \{b, c\}$, $f(a) = b$, and $g(a) = c$.

4. Equalizers, pushouts, and codescent

The equalizer diagram of two parallel morphisms $f, g : A \to B$ in **CLS** can be described as the inclusion map

$$K = \{a \in A \mid f(a) = g(a)\} \to A \text{ with } \mathcal{C}_K = \{X \cap K \mid X \in \mathcal{C}_A\},$$

and easily obtain

Theorem 4.1. *For $B, E \in$ **CLS** with $B \subseteq E$, the following conditions on the inclusion map $(i : B \to E) \in$ **CLS** are equivalent:*

(a) i is a regular monomorphism;

(b) i is a strong monomorphism;

(c) i is a subspace inclusion, that is, a subset of B is closed in it if and only if it can be presented as the intersection of a closed subset of E with B, or equivalently, if and only if

$$\overline{X}^B = \overline{X}^E \cap B$$

for every $X \subseteq B$.

Consider a pushout diagram

$$\begin{tikzcd} B \arrow[r, "i"] \arrow[d, "\alpha"'] & E \arrow[d, "\iota_2"] \\ A \arrow[r, "\iota_1"'] & A +_B E \end{tikzcd}$$

in **CLS**, in which i is as in Theorem 4.1. Assuming $A \cap E = \emptyset$ we can present it as

$$A +_B E = A \cup (E \setminus B) \text{ as sets,}$$

$$\mathcal{C}_{A+_BE} = \{A' \cup E' \mid A' \in \mathcal{C}_A,\ E' \subseteq (E \setminus B),\ \alpha^{-1}(A') \cup E' \in \mathcal{C}_E\},$$

with ι_1 being the inclusion map and ι_2 defined by

$$\iota_2(e) = \begin{cases} \alpha(e), \text{ if } e \in B; \\ e, \text{ if } e \in (E \setminus B). \end{cases}$$

We will informally call this diagram the *standard pushout* of α and i.

Lemma 4.2. *Regular monomorphisms in* **CLS** *are pushout stable.*

Proof. Consider the standard pushout above. We have to show that the inclusion map $\iota_1 : A \to A +_B E$ is a subspace inclusion. For an arbitrary $X \in \mathcal{C}_A$, we have $\alpha^{-1}(X) \in \mathcal{C}_B$, and so

$$\alpha^{-1}(X) = B \cap \overline{\alpha^{-1}(X)}^E$$

by 4.1(c). We need to find $A' \in \mathcal{C}_A$ and $E' \subseteq (E \setminus B)$ such that inside the disjoint union $A \cup (E \setminus B)$ we have

$$X = (A' \cup E') \cap A \text{ and } \alpha^{-1}(A') \cup E' \in \mathcal{C}_E,$$

and we claim that we can take $A' = X$ and $E' = \overline{\alpha^{-1}(X)}^E \cap (E \setminus B)$. Indeed, the first equality will hold simply because $A' \subseteq A$ and $E' \subseteq (E \setminus B)$, while

$$\alpha^{-1}(A') \cup E' = \alpha^{-1}(X) \cup (\overline{\alpha^{-1}(X)}^E \cap (E \setminus B))$$

$$= (B \cap \overline{\alpha^{-1}(X)}^E) \cup (\overline{\alpha^{-1}(X)}^E \cap (E \setminus B)) = \overline{\alpha^{-1}(X)}^E \in \mathcal{C}_E.$$

$\square$

Theorem 4.3. *A morphism in* $\mathbf{CLS}$ *is an effective codescent morphism if and only if it is a regular monomorphism, or, equivalently, a strong monomorphism.*

Proof. Thanks to (Theorem 4.1 and) Lemma 4.2, it suffices to prove that if $i : B \to E$ satisfies the equivalent conditions of Theorem 4.1, then the associated pushout functor

$$E +_B (-) : (B \downarrow \mathbf{CLS}) \to (E \downarrow \mathbf{CLS})$$

preserves equalizers. But this follows from the fact that the equalizers are preserved at the level of sets, and that any such pushout functor preserves regular monomorphisms, by Lemma 4.2. □

Remark 4.4. The arguments used in the proof above can be copied for the category of topological spaces and the category of topological spaces whose sets of open sets are closed under arbitrary intersections (= Alexandrov-discrete topological spaces, which are in fact nothing but preordered sets). Therefore, in both of these categories, effective codescent morphisms are the same as regular monomorphisms.

5. The coreflection $Z : \mathbf{CLS} \to \mathbf{Sets}$

We will use dual forms of several categorical notions from [5] and [3], such as the one of coreflection with stable counits, dual to reflection with stable units (introduced in [5]):

Theorem 5.1. *The coreflection* $Z : \mathbf{CLS} \to \mathbf{Sets}$ *has stable counits, that is, it preserves colimits of all diagrams* $A \leftarrow B \to A'$ *in which* $B = CZ(B)$*, or, equivalently,* $\mathcal{C}_B = \{B\}$.

Proof. Since Z (obviously) preserves coproducts, it suffices to prove that it preserves coequalizers of all pairs $f, g : B \to A$ in which $B = CZ(B)$. The coequalizer of such a pair can be presented as the canonical map $p : A \to A/R$, where R is the smallest equivalence relation on A containing the set $S = \{(f(b), g(b)) \mid b \in B\}$ and $\mathcal{C}_{A/R}$ is the set of all subsets of A/R whose inverse images under p belong to $\mathcal{C}_A$. We observe:

(i) Since $B = CZ(B) = 0_B$, the images of f and g are subsets of 0_A.

(ii) As follows from (i), $0_A = p^{-1}(p(0_A))$.

(iii) As follows from (ii), $p(0_A)$ belongs to $\mathcal{C}_{A/R}$.

(iv) Since p is surjective and continuous, (ii) and (iii) imply $p(0_A) = 0_{A/R}$.

This makes

$$B \rightrightarrows 0_A \longrightarrow 0_{A/R}$$

a coequalizer diagram in the category of sets. But this diagram is the same as

$$\begin{tikzcd} Z(B) \arrow[r, shift left, "Z(f)"] \arrow[r, shift right, "Z(g)"'] & Z(A) \arrow[r, "Z(p)"] & Z(A/R), \end{tikzcd}$$

which completes the proof. $\square$

As follows from Theorem 5.1, a simplified version of Galois theory [9], recalled in [3], applies to the reflection $Z^{\mathrm{op}} : \mathbf{CLS}^{\mathrm{op}} \to \mathbf{Sets}^{\mathrm{op}}$. And the resulting dualization makes a morphism $\alpha : B \to A$ in $\mathbf{CLS}$:

- a *trivial cocovering*, if the diagram

$$\begin{tikzcd} CZ(B) \arrow[r, "\varepsilon_B"] \arrow[d, "CZ(\alpha)"'] & B \arrow[d, "\alpha"] \\ CZ(A) \arrow[r, "\varepsilon_A"'] & A \end{tikzcd}$$

  in which ε is the counit of the adjunction $C \dashv Z$, is a pushout;

- a *cocovering*, or a *colight morphism*, if there exists a subspace inclusion $i : B \to E$ such that the morphism $\iota_2 : E \to A +_B E$ is a trivial cocovering.

- *covertical* (according to fibration-theoretic terminology), if $Z(\alpha)$ is an isomorphism.

- *comonotone*, if it is a pushout stable covertical morphism.

We are going now to characterize these types of morphisms, except covertical ones, whose definition already characterizes them.

For a closure space A, we will write $\mathrm{int}(A)$ for the set $A \setminus 0_A$ equipped with its induced structure, and call it the *interior* of A, since it is the largest subset of A whose complement is closed. Note that a morphism $\alpha : B \to A$ in **CLS** in general induces only a *partial map* $\mathrm{int}(\alpha) : \mathrm{int}(B) \to \mathrm{int}(A)$; it is a morphism in **CLS** if and only if it is a map, that is, if and only if

$$\alpha^{-1}(0_A) = 0_B.$$

Lemma 5.2. *For any closure space A and a subset A' of $\mathrm{int}(A)$, we have*

$$A' \in \mathcal{C}_{\mathrm{int}(A)} \Leftrightarrow 0_A \cup A' \in \mathcal{C}_A.$$

Proof. "$\Rightarrow$": $A' \in \mathcal{C}_{\mathrm{int}(A)}$ means that $A' = (A \setminus 0_A) \cap A''$ for some $A'' \in \mathcal{C}_A$. Then, since both 0_A and A' are subsets of A'', we have $0_A \cup A' \subseteq A''$. On the other hand, each $a \in A''$ must satisfy one on the following two conditions:

(i) $a \in 0_A$;

(ii) $a \in A \setminus 0_A$, but then a belongs to $(A \setminus 0_A) \cap A'' = A'$.

That is, $0_A \cup A' = A'' \in \mathcal{C}_A$.

The implication "$\Leftarrow$" follows from the equality $A' = (0_A \cup A') \cap (A \setminus 0_A)$. $\square$

Corollary 5.3. *For any closure space A, the assignments*

$$X \mapsto 0_A \cup X \text{ and } Y \mapsto Y \cap (A \setminus 0_A)$$

determine bijections $\mathcal{C}_{\mathrm{int}(A)} \to \mathcal{C}_A$ and $\mathcal{C}_A \to \mathcal{C}_{\mathrm{int}(A)}$ inverse to each other.

Theorem 5.4. *The following conditions on a morphism $\alpha : B \to A$ in* **CLS** *are equivalent:*

(a) *α is a trivial cocovering;*

(b) *$\mathrm{int}(\alpha)$ is an isomorphism (in* **CLS***);*

(c) *α induces bijections*

$$B \setminus 0_B \to A \setminus 0_A \text{ and } \mathcal{C}_A \to \mathcal{C}_B \text{ (where } A' \mapsto \alpha^{-1}(A')).$$

Proof. The morphism $\varepsilon_B : CZ(B) \to B$ is obviously a subspace inclusion. Therefore, and according to the standard pushout construction in Section 4, assuming for simplicity that A and B are disjoint, we can reformulate condition (a) as:

(a′) The map $\tilde{\alpha} : 0_A \cup (B \setminus 0_B) \to A$ defined by

$$\tilde{\alpha}(x) = \begin{cases} x, \text{ if } x \in 0_A; \\ \alpha(x), \text{ if } x \in (B \setminus 0_B). \end{cases}$$

is bijective, and a subset A' of A belongs to $\mathcal{C}_A$ if and only if it is of the form

$$A' = \tilde{\alpha}(0_A \cup B') = 0_A \cup \alpha(B')$$

with $B' \subseteq B \setminus 0_B$ and $0_B \cup B' \in \mathcal{C}_B$, or, equivalently, $B' \in \mathcal{C}_{\mathrm{int}(B)}$.

It is easy to see that the map $\tilde{\alpha}$ defined in (a′) is a bijection if and only if

$$\mathrm{int}(\alpha) : \mathrm{int}(B) \to \mathrm{int}(A)$$

is a morphism in **CLS** that is a bijective map. This allows us to argue as follows.

(a′)⇒(b): Suppose (a′) holds. To prove (b) is to prove that if B' belongs to $\mathcal{C}_{\mathrm{int}(B)}$, then $\mathrm{int}(\alpha)(B')$ belongs to $\mathcal{C}_{\mathrm{int}(A)}$. We have

$$\mathrm{int}(\alpha)(B') = \alpha(B') = (0_A \cup \alpha(B')) \cap (A \setminus 0_A),$$

which belongs to $\mathcal{C}_{\mathrm{int}(A)}$ since, by (a′), $0_A \cup \alpha(B')$ belongs to $\mathcal{C}_A$.

(b)⇒(a′): Applying Corollary 5.3 and then (b), we obtain

$$A' \in \mathcal{C}_A \Leftrightarrow \exists_{X \in \mathcal{C}_{\mathrm{int}(A)}} A' = 0_A \cup X \Leftrightarrow \exists_{B' \in \mathcal{C}_{\mathrm{int}(B)}} A' = 0_A \cup \alpha(B'),$$

which gives (a′).

(b)⇔(c) easily follows from Corollary 5.3. □

Remark 5.5. As also easily follows from Corollary 5.3, the inverse of the bijection $\mathcal{C}_A \to \mathcal{C}_B$ in 5.4(c) is given by $B' \mapsto 0_A \cup \alpha(B')$.

Theorem 5.6. *Every cocovering is a trivial cocovering.*

Proof. Suppose $\alpha : B \to A$ is a cocovering, and let $i : B \to E$ be a subspace inclusion such that $\iota_2 : E \to A +_B E$ is a trivial cocovering. Then ι_2 induces a bijection $E \setminus 0_E \to A +_B E \setminus 0_{A+_BE}$. Using our standard pushout of α and i, we see that since the map $\iota_2 : E \to A \cup (E \setminus B)$ maps B to A via α and maps $E \setminus B$ identically to itself, it induces maps

$$B \setminus 0_E \to A \setminus 0_{A+_BE} \text{ and } (E \setminus B) \setminus 0_E \to (E \setminus B) \setminus 0_{A+_BE},$$

which then both must be bijections. Since

$$A \setminus 0_{A+_BE} = A \setminus A \cap (0_{A+_BE}) = A \setminus 0_A,$$

the first of these bijections is in fact the bijection $B \setminus 0_B \to A \setminus 0_A$ induced by α.

Now, thanks to Theorem 5.4(c)⇒(a), it only remains to prove that the canonical map $\mathcal{C}_A \to \mathcal{C}_B$ is a bijection. For, using the same $i : B \to E$, consider the commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{A+_BE} \arrow[r] \arrow[d] & \mathcal{C}_A \arrow[r, "\subseteq"] \arrow[d] & \{X \subseteq A \mid 0_A \subseteq X\} \arrow[d, "X \mapsto \alpha^{-1}(X)"] \\
\mathcal{C}_E \arrow[r] & \mathcal{C}_B \arrow[r, "\subseteq"'] & \{Y \subseteq B \mid 0_B \subseteq Y\}
\end{tikzcd}$$

of canonical maps. The left-hand square shows that $\mathcal{C}_A \to \mathcal{C}_B$ is surjective, while the right-hand square shows that it is injective. $\square$

Theorem 5.7. *The following conditions on a morphism $\alpha : B \to A$ in* **CLS** *are equivalent:*

(a) α is comonotone;

(b) α is injective and closed.

Proof. (a)⇒(b): Choose any $B_0 \in \mathcal{C}_B$ and consider the pushout

$$\begin{tikzcd}
B \arrow[r, "b \mapsto b"] \arrow[d, "\alpha"'] & B' \arrow[d, "\iota_2"] \\
A \arrow[r, "\iota_1"'] & A'
\end{tikzcd}$$

in which $B' = B$ as sets and $\mathcal{C}'_B = \{Y \in \mathcal{C}_B \mid B_0 \subseteq Y\}$. This allows us to put:

$$A' = A \text{ as sets}, \ \iota_1(a) = a, \ \iota_2(b) = \alpha(b),$$

for all $a \in A$ and $b \in B$, and

$$\mathcal{C}'_A = \{X \in \mathcal{C}_A \mid B_0 \subseteq \alpha^{-1}(X)\}.$$

According to this presentation of our pushout, $0_{A'}$ is the smallest closed subset of A containing $\alpha(0_{B'}) = \alpha(B_0)$. That is,

$$0_{A'} = \overline{\alpha(B_0)}^{A}.$$

Since, by (a), ι_2 is covertical, we conclude that the restriction of α on B_0 must be injective and $\alpha(B_0)$ is closed in A; in particular, taking $B_0 = B$ gives injectivity of α.

(b)⇒(a): Let us change our notation. We can assume, without loss of generality, that α is a closed subspace inclusion and we will rename it as $i : B \to E$. On the other hand, by $\alpha : B \to A$ we will denote now an arbitrary morphism in $\mathbf{CLS}$ (with the same B). We have to show that $\iota_1 : A \to A +_B E$ is covertical, and, in terms of the standard pushout of α and i, this simply means that 0_A is the smallest closed subset in $A +_B E$. For, we recall that a subset of $A +_B E$ is closed if and only if it is of the form $A' \cup E'$ with $A' \in \mathcal{C}_A$, $E' \subseteq (E \setminus B)$, and $\alpha^{-1}(A') \cup E' \in \mathcal{C}_E$. Since every closed subset of B is closed in E, the smallest closed subset of $A +_B E$ is $0_A \cup \emptyset = 0_A$. □

6. Two factorization systems on CLS

As follows from the results of [5], recalled in [3], and Theorem 5.1, the category $\mathbf{CLS}$ admits the *(trivial cocoverings, covertical morphisms)-factorization system* $(\mathbf{E}, \mathbf{M})$, for which:

- $\mathbf{E}$ is the class of all trivial cocoverings, defined as in Section 5;
- $\mathbf{M}$ is the class of all covertical morphisms, defined as in Section 5;

- the $(\mathbf{E}, \mathbf{M})$-factorization of a morphism $\alpha : B \to A$ in $\mathbf{CLS}$ is constructed as $\alpha = \beta\iota_1$ in the diagram

$$\begin{tikzcd}
A & & \\
& B +_{CZ(B)} CZ(A) \arrow[ul, "\beta"'] & CZ(A) \arrow[l, "\iota_2"] \arrow[ull, "\varepsilon_A"'] \\
& B \arrow[u, "\iota_1"] \arrow[uul, "\alpha"] & CZ(B) \arrow[l, "\varepsilon_B"] \arrow[u, "CZ(\alpha)"']
\end{tikzcd}$$

  in which the square part is a pushout and β is induced by α and ε_A.

Note that the diagram above is shaped as diagram (4.2) in [5] and diagram (3.5) in [3], except that the directions of all arrows are opposite since we consider the dual situation.

Now, trying to follow [3], could we (co)localize $\mathbf{E}$ and (co)stabilize $\mathbf{M}$ to obtain '(colight, comonotone)'-factorization system? Obviously not, since every cocovering is trivial (Theorem 5.6) while not every covertical morphism is comonotone (as immediately follows from Theorem 5.7). Nevertheless, we do have a factorization system $(\mathbf{E}', \mathbf{M}')$ on $\mathbf{CLS}$, in which $\mathbf{M}'$ is the class of all comonotone morphisms: we just need to take $\mathbf{E}'$ to be the class of (obviously defined) *dense* morphisms. This can also be seen as a consequence of Theorem 2.4 in [7].

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THE WEAK SUBOBJECT CLASSIFIER AXIOM AND MODULES IN SUP

Ulrich Höhle

Résumé. L'axiome du classificateur faible de sous-objet est introduit de telle manière qu'un schéma de compréhension soit disponible. Cependant, le foncteur d'image inverse, lorsqu'il est restreint aux sous-objets classifiables, n'a pas toujours un adjoint à droite. Pour les quantales unitaires arbitraires, la catégorie des modules à droite (ou à gauche) dans Sup satisfait l'axiome du classificateur faible de sous-objet. Les morphismes caractéristiques sont construits en utilisant les catégories enrichies dans les quantales associées aux modules dans Sup. Si le quantale sous-jacent est commutatif, alors les objets puissance faibles existent également.

Abstract. The weak subobject classifier axiom is introduced in such a way that a comprehension scheme is available. However, the inverse image functor restricted to classifiable subobjects need not have a right adjoint. For arbitrary unital quantales, the category of right (left) modules in Sup satisfies the weak subobject classifier axiom. The characteristic morphisms are constructed using the quantale-enriched categories corresponding with modules in Sup. If the underlying quantale is commutative, then also weak power objects exist.

Keywords. Weak Subobject Classifier, Inverse Image Functor, Quantale, Module in Sup, Enriched Category, Weak Power Object.
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1. Introduction

The subobject classifier axiom is one of the prominent axioms of topos theory. In this paper we weaken this axiom and give up the principle that *every* subobject of a pre-described class of subobjects is classifiable by a unique characteristic morphism (cf. [12, Section 14]). This approach leads to the weak subobject classifier axiom. Since the weak subobject classifier is unique up to an isomorphism, the existence of a weak subobject classifier is always an invariant of a finitely complete category.

Moreover, in any complete epi-mono-category the weak subobject classifier axiom gives rise to a comprehension scheme in the sense of Lawvere (cf. [8]). If additionally a symmetric and monoidal closed structure is imposed, then weak power objects are available. In particular, the weak power object of the unit object is isomorphic to the underlying weak subobject classifier. Since in general neither diagonal arrows nor projections of the tensor product exist, we only focus on the construction of the universal quantifier. If the underlying category is an epi-mono-category and the unique arrow from the unit object to the terminal object is an epimorphism, then the existence of the universal quantifier based on objects follows from the weak subobject classifier axiom. On the other hand the weak subobject classifier axiom does not imply that the restriction of the inverse image functor to classifiable subobjects has in general a right adjoint (cf. Example 4.9). In this sense an analogue of the doctrinal diagram of Kock and Wraith is not available (cf. [6]).

Significant examples of categories satisfying the weak subobject classifier axiom, but not being a topos (resp. quasitopos), appear in the study of modules in the category Sup of complete lattices and join preserving maps. Let $\mathfrak{Q}$ be a unital quantale, then the category of right (left) $\mathfrak{Q}$-modules satisfies the weak subobject classifier axiom. We emphasize that the construction of characteristic morphisms is based here on the underlying $\mathfrak{Q}$-enriched categories associated with right $\mathfrak{Q}$-modules. Moreover, if $\mathfrak{Q}$ is commutative, then there exists a well known symmetric, monoidal closed structure on the category of $\mathfrak{Q}$-modules (cf. [4]). In this context weak power objects exist, and there is again a close relationship between universal quantifiers based on $\mathfrak{Q}$-modules and the respectively associated, $\mathfrak{Q}$-enriched categories (cf. Proposition 5.1).

Finally, as first steps toward categorical logic for $\mathfrak{Q}$-modules, we include the conjunction, the implication, the element relation and the universal quantifier based on $\mathfrak{Q}$-modules as truth arrows.

In order to fix some basic facts we begin with preliminaries on quantales and a survey on modules in Sup.

2. Preliminaries

First we point out that Sup is a symmetric, monoidal, closed category. If X, Y and Z are complete lattices, then a map $X \times Y \xrightarrow{b} Z$ is called a *bimorphim* if b is join-preserving in each variable separately. Due to the universal property of the tensor product $\otimes$ in Sup every bimorphism

$$X \times Y \xrightarrow{b} Z$$

can be identified with a unique join-preserving map $X \otimes Y \xrightarrow{\varphi_b} Z$ making the following diagram commutative:

$$\begin{tikzcd} X \times Y \arrow[r, "\otimes"] \arrow[d, "b"'] & X \otimes Y \arrow[dl, dotted, "\varphi_b"] \\ Z & \end{tikzcd} \tag{2.1}$$

where $X \times Y \xrightarrow{\otimes} X \otimes Y$ is the *universal bimorphism* from $X \times Y$ to $X \otimes Y$. We also call φ_b the unique join-preserving extension of b.

Further, due to the monoidal closedness of Sup for every object Z of Sup the endofunctor $\_ \otimes Z$ has a right adjoint functor $[Z, \_]$, where $[Z, Y]$ is the complete lattice of all join-preserving maps $Z \to Y$ ordered pointwise. Then for each join-preserving map $X \otimes Z \xrightarrow{\varphi} Y$ there exists a unique join-preserving map $X \xrightarrow{\ulcorner\varphi\urcorner} [Z, Y]$ such that the following diagram is commutative:

$$\begin{tikzcd} X \otimes Z \arrow[r, "\ulcorner\varphi\urcorner \otimes 1_Z"] \arrow[dr, "\varphi"'] & {[Z,Y] \otimes Z} \arrow[d, "ev_Y"] \\ & Y \end{tikzcd} \tag{2.2}$$

where ev_Y is the *evaluation arrow* — i.e. the Y-component of the counit of the adjoint situation $Z \otimes \_ \dashv [Z, \_]$. In this context $\ulcorner \varphi \urcorner$ is called the *monoidal adjoint arrow* of φ.

Since Sup has a self-duality determined by the construction of right adjoint maps, we introduce the following notation. The dual lattice of a complete lattice X is denoted by $X^\dagger$ and the corresponding dual order by $\leq^\dagger$. Then the tensor product $X \otimes Y$ has the form $[X, Y^\dagger]^\dagger$ up to an isomorphism (cf. [4]).

A *quantale* is a semigroup $\mathfrak{Q}$ in Sup — i.e. a complete lattice $\mathfrak{Q}$ provided with an associative, binary operation $\mathfrak{Q} \otimes \mathfrak{Q} \xrightarrow{m} \mathfrak{Q}$ in the sense of Sup (cf. [9, (1) on p. 170]). Then the bimorphism $\mathfrak{Q} \times \mathfrak{Q} \xrightarrow{*} \mathfrak{Q}$ determined by m (cf. (2.1)) is a semigroup operation in Set, which is join-preserving in each variable separately. This bimorphism $*$ is also called the *quantale multiplication* of $\mathfrak{Q}$. The right implication $\searrow$ and left implication $\swarrow$ of $*$ are determined by:

$$\alpha \searrow \beta = \bigvee \{\, \gamma \in \mathfrak{Q} \mid \alpha * \gamma \leq \beta \,\} \ \text{ and } \ \beta \swarrow \alpha = \bigvee \{\, \gamma \in \mathfrak{Q} \mid \gamma * \alpha \leq \beta \,\}.$$

Since both types of implications are bimorphisms, they have always unique extensions to join-preserving maps $\mathfrak{Q} \otimes \mathfrak{Q}^\dagger \xrightarrow{\varphi_\searrow} \mathfrak{Q}^\dagger$ and $\mathfrak{Q}^\dagger \otimes \mathfrak{Q} \xrightarrow{\varphi_\swarrow} \mathfrak{Q}^\dagger$, respectively. Finally, the zero element in $\mathfrak{Q}$ coincides with the universal lower bound $\perp$ of $\mathfrak{Q}$.

An element $\alpha \in \mathfrak{Q}$ is *left-sided* (resp. *right-sided*) if $\top * \alpha \leq \alpha$ (resp. $\alpha * \top \leq \alpha$), where $\top$ is the universal upper bound in $\mathfrak{Q}$. An element of $\mathfrak{Q}$ is *two-sided* if it is left- and right-sided.

A *unital quantale* is a monoid in Sup (cf. [9, (1), (2) on p. 170]). The unit $\mathbb{1} \xrightarrow{e} \mathfrak{Q}$ will always be identified with the corresponding element $e \in \mathfrak{Q}$. Typical examples of a unital quantale arise from complete lattices X and are given by the complete lattice $[X, X]$ of all join-preserving self-maps of X provided with the composition as quantale multiplication.

3. Survey on modules in Sup

In this section we begin with a review of some basic properties of modules in Sup. Therefore let $\mathfrak{Q} = (\mathfrak{Q}, *, e)$ be a unital quantale with unit e.

A complete lattice X provided with a left action $\mathfrak{Q} \otimes X \xrightarrow{\ell_X} X$ of $\mathfrak{Q}$ on X (cf. [9, p. 174]) is called a *left $\mathfrak{Q}$-module* in Sup (cf. [4]). Hence ℓ_X

can be identified with its bimorphism $\mathfrak{Q} \times X \xrightarrow{\odot} X$ (cf. (2.1)) satisfying the following additional axioms:

(L1) $\beta \odot (\alpha \odot x) = (\beta * \alpha) \odot x, \qquad \alpha, \beta \in \mathfrak{Q},\ x \in X,$

(L2) $e \odot x = x, \qquad x \in X.$

Let $X = (X, \ell_X)$ and $Y = (Y, \ell_Y)$ be left $\mathfrak{Q}$-modules. A join-preserving map $X \xrightarrow{h} Y$ is a *left $\mathfrak{Q}$-module morphism* if h also preserves the respective left actions — i.e. the commutativity of the following diagram holds:

$$\begin{tikzcd}
\mathfrak{Q} \otimes X \arrow[r, "1_{\mathfrak{Q}} \otimes h"] \arrow[d, "\ell_X"'] & \mathfrak{Q} \otimes Y \arrow[d, "\ell_Y"] \\
X \arrow[r, "h"] & Y
\end{tikzcd}$$

The complete lattice $[X, Y]$ of left $\mathfrak{Q}$-module morphisms $X \to Y$ is ordered pointwise in the sense of Y. Hence joins in $[X, Y]$ are computed pointwise, but not meets.

Obviously left $\mathfrak{Q}$-modules and left $\mathfrak{Q}$-module morphisms form a category denoted by $\mathsf{Mod}_\ell(\mathfrak{Q})$. Referring to [2] $\mathsf{Mod}_\ell(\mathfrak{Q})$ is complete and cocomplete. Further, it is well known that the forgetful functor $\mathcal{U}\colon \mathsf{Mod}_\ell(\mathfrak{Q}) \to \mathsf{Sup}$ has a left adjoint functor $\mathcal{F}\colon \mathsf{Sup} \to \mathsf{Mod}_\ell(\mathfrak{Q})$ acting on objects and morphisms as follows (cf. [9, p. 174]):

$$\mathcal{F}(X) = \mathfrak{Q} \otimes X \quad \text{and} \quad X \xrightarrow{h} Y, \quad \mathfrak{Q} \otimes X \xrightarrow{\mathcal{F}(h) = 1_{\mathfrak{Q}} \otimes h} \mathfrak{Q} \otimes Y.$$

If $\mathcal{M}$ is the monad induced by the adjoint situation $\mathcal{F} \dashv \mathcal{U}$, then $\mathsf{Mod}_\ell(\mathfrak{Q})$ is isomorphic to the category of $\mathcal{M}$-algebras. In particular, all finite limits in $\mathsf{Mod}_\ell(\mathfrak{Q})$ can be computed at the level of Sup.

Moreover, right actions $X \otimes \mathfrak{Q} \xrightarrow{r_X} X$ in Sup are defined similarly and can again be identified with bimorphisms $X \times \mathfrak{Q} \xrightarrow{\boxdot} X$ satisfying now the properties:

(R1) $(x \boxdot \alpha) \boxdot \beta = x \boxdot (\alpha * \beta), \qquad \alpha, \beta \in \mathfrak{Q},\ x \in X,$

(R2) $x \boxdot e = x, \qquad x \in X.$

The respective results corresponding to the previous ones of left $\mathfrak{Q}$-modules holds also for right $\mathfrak{Q}$-modules. In particular, all finite limits in the category $\mathsf{Mod}_r(\mathfrak{Q})$ of right $\mathfrak{Q}$-modules are again computed at the level of Sup.

As a next step we present a fundamental relationship between left and right $\mathfrak{Q}$-modules in Sup (cf. [2, Fact 1 on p. 207]).

Theorem 3.1 *The self-duality of* Sup *determined by the construction of right adjoint maps can be lifted to a contravariant isomorphism between the categories* $\mathsf{Mod}_\ell(\mathfrak{Q})$ *and* $\mathsf{Mod}_r(\mathfrak{Q})$.

Proof. Let (X, ℓ_X) be a left $\mathfrak{Q}$-module and $X^\dagger \xrightarrow{\ell_X^\vdash} (\mathfrak{Q} \otimes X)^\dagger = [\mathfrak{Q}, X^\dagger]$ be the right adjoint map of its left action ℓ_X. Then we introduce a right action $X^\dagger \otimes \mathfrak{Q} \xrightarrow{r_{X^\dagger}} X^\dagger$ of $\mathfrak{Q}$ on $X^\dagger$ by:

$$\begin{tikzcd} X^\dagger \otimes \mathfrak{Q} \arrow[r, "\ell_X^\vdash \otimes 1_\mathfrak{Q}"] \arrow[dr, dotted, "r_{X^\dagger}"'] & {[\mathfrak{Q}, X^\dagger] \otimes \mathfrak{Q}} \arrow[d, "ev_{X^\dagger}"] \\ & X^\dagger \end{tikzcd}$$

The bimorphism $X^\dagger \times \mathfrak{Q} \xrightarrow{\boxdot^\dagger} X^\dagger$ determined by $r_{X^\dagger}$ in the sense of (2.1) has the form (cf. [1, Def. 5.1.2]):

$$x \boxdot^\dagger \alpha = \bigvee \{\, z \in X \mid \alpha \odot z \le x \,\}, \qquad \alpha \in \mathfrak{Q},\ x \in X. \tag{3.1}$$

It is easily seen that $\odot$ satisfies (L1) and (L2) if and only if $\boxdot^\dagger$ satisfies (R1) and (R2). Further, the formation of right adjoint maps determines a contravariant functor $\Gamma\colon \mathsf{Mod}_\ell(\mathfrak{Q}) \to \mathsf{Mod}_r(\mathfrak{Q})$.

On the other hand, every right action r_X on X induces a left action $\ell_{X^\dagger}$ on the dual lattice $X^\dagger$ of X as follows. First we compute the monoidal adjoint

$$X \xrightarrow{\ulcorner r_X \urcorner} [\mathfrak{Q}, X]$$

of r_x (cf. (2.2)). Then the left action $\ell_{X^\dagger}$ on $X^\dagger$ is given by the right adjoint map $\mathfrak{Q} \otimes X^\dagger \xrightarrow{\ell_{X^\dagger} = (\ulcorner r_X \urcorner)^\vdash} X^\dagger$. The bimorphism $\mathfrak{Q} \times X^\dagger \xrightarrow{\odot^\dagger} X^\dagger$ determined by $\ell_{X^\dagger}$ has the form

$$\alpha \odot^\dagger x = \bigvee \{\, z \in X \mid z \boxdot \alpha \le x \,\}, \qquad \alpha \in \mathfrak{Q},\ x \in X, \tag{3.2}$$

where $\boxdot$ is the bimorphism corresponding to r_X. Because $\Gamma(X^\dagger, \ell_{X^\dagger}) = (X, r_X)$, Γ is a contravariant isomorphism. $\square$

We illustrate the previous theorem by a simple example.

Example 3.2 Let $\mathfrak{Q} = (\mathfrak{Q}, m, e)$ be a monoid in Sup (which can also be viewed as a unital quantale). It follows from the associativity and unit axiom of monoids that m can be read as left action of $\mathfrak{Q}$ on $\mathfrak{Q}$ or as right action of $\mathfrak{Q}$ on $\mathfrak{Q}$. Obviously, $\mathfrak{Q}$ is the free left (resp. right) $\mathfrak{Q}$-module on a singleton. Hence every monomorphism in $\mathsf{Mod}_\ell(\mathfrak{Q})$ (resp. $\mathsf{Mod}_r(\mathfrak{Q})$) is an injective map.
(a) If we consider m as a left action on $\mathfrak{Q}$, then the bimorphism corresponding to the right action $r_{\mathfrak{Q}^\dagger}$ on $\mathfrak{Q}^\dagger$ induced by m in the sense of Theorem 3.1 has the form (cf. (3.1)):

$$\gamma \boxdot^\dagger \alpha = \bigvee\{\,\beta \in \mathfrak{Q} \mid \alpha * \beta \leq \gamma\,\} = \alpha \searrow \gamma, \qquad \alpha, \gamma \in \mathfrak{Q}. \tag{3.3}$$

Hence the right action $r_{\mathfrak{Q}^\dagger}$ is uniquely determined by the right implication of the quantale multiplication $*$. In particular, if $\mathfrak{Q}^\dagger \otimes \mathfrak{Q} \xrightarrow{c_{\mathfrak{Q}^\dagger\mathfrak{Q}}} \mathfrak{Q} \otimes \mathfrak{Q}^\dagger$ is the relevant component of the symmetry in Sup, then $r_{\mathfrak{Q}^\dagger} = \varphi_\searrow \circ c_{\mathfrak{Q}^\dagger\mathfrak{Q}}$. Moreover, if $\gamma \in \mathfrak{Q}$ is right-sided, then $\alpha \searrow \gamma$ is also right-sided for all $\alpha \in \mathfrak{Q}$. Hence, if $\mathbb{R}(\mathfrak{Q})$ is the subquantale of all right-sided elements of $\mathfrak{Q}$, then $\mathbb{R}(\mathfrak{Q})^\dagger$ is a right $\mathfrak{Q}$-submodule of $(\mathfrak{Q}^\dagger, r_{\mathfrak{Q}^\dagger})$. By abuse of notation we denote the right action in $\mathbb{R}(\mathfrak{Q})^\dagger$ again by $r_{\mathfrak{Q}^\dagger}$.
(b) If we consider m as a right action on $\mathfrak{Q}$, then the left action $\ell_{\mathfrak{Q}^\dagger}$ on $\mathfrak{Q}^\dagger$ induced by m in the sense of Theorem 3.1 (cf. (3.2)) is uniquely determined by the left implication — i.e.

$$\alpha \odot^\dagger \gamma = \bigvee\{\,\beta \in \mathfrak{Q} \mid \beta * \alpha \leq \gamma\,\} = \gamma \swarrow \alpha, \qquad \alpha, \gamma \in \mathfrak{Q}. \tag{3.4}$$

In particular, $\ell_{\mathfrak{Q}^\dagger} = \varphi_\swarrow \circ c_{\mathfrak{Q}\mathfrak{Q}^\dagger}$. Moreover, if $\mathbb{L}(\mathfrak{Q})$ is the subquantale of all left-sided elements of $\mathfrak{Q}$, then by analogy with (a) the complete lattice $\mathbb{L}(\mathfrak{Q})^\dagger$ is a left $\mathfrak{Q}$-submodule of $(\mathfrak{Q}^\dagger, \ell_{\mathfrak{Q}^\dagger})$. By abuse of notation we denote the left action in $\mathbb{L}(\mathfrak{Q})^\dagger$ again by $\ell_{\mathfrak{Q}^\dagger}$.

It follows immediately from Theorem 3.1 and Example 3.2 that every epimorphism in $\mathsf{Mod}_\ell(\mathfrak{Q})$ (resp. $\mathsf{Mod}_r(\mathfrak{Q})$) is surjective. Moreover, every epimorphism is the coequalizer of its kernel pair and every monomorphism is the equalizer of its cokernel pair. In particular, $\mathsf{Mod}_\ell(\mathfrak{Q})$ (resp. $\mathsf{Mod}_r(\mathfrak{Q})$) is an epi-mono-category.

With regard to Section 4 we recall that the terminal and initial objects coincide and form consequently the null object in $\mathsf{Mod}_\ell(\mathfrak{Q})$ (resp. $\mathsf{Mod}_r(\mathfrak{Q})$). Hence $\mathsf{Mod}_\ell(\mathfrak{Q})$ (resp. $\mathsf{Mod}_r(\mathfrak{Q})$) is a pointed category and every left (resp. right) $\mathfrak{Q}$-module X has a unique global point $0 \to X$ and is represented by the universal lower bound of X.

The next proposition is a non-commutative version of [2, Lem. 3.1.27].

Proposition 3.3 *Let (X, r_X) be a right $\mathfrak{Q}$-module, (Y, ℓ_Y) be a left $\mathfrak{Q}$-module and $(Y^\dagger, r_{Y^\dagger})$ be the right $\mathfrak{Q}$-module induced by (Y, ℓ_Y) in the sense of Theorem 3.1. Further, let $\boxdot$ and $\odot$ be the bimorphisms determined by r_X and ℓ_Y respectively. Then a join-reversing map $X \xrightarrow{f} Y$ is a right $\mathfrak{Q}$-module morphism $(X, r_X) \xrightarrow{f} (Y^\dagger, r_{Y^\dagger})$ if and only if the following equivalence holds for all $\alpha \in \mathfrak{Q}$, $x \in X$ and $y \in Y$:*

$$y \le f(x \boxdot \alpha) \iff \alpha \odot y \le f(x).$$

Proof. Let $(X, r_X) \xrightarrow{f} (Y^\dagger, r_{Y^\dagger})$ be a right $\mathfrak{Q}$-module morphism. Then the definition of $\boxdot^\dagger$ (cf. (3.1)) implies that the following chain of equivalences holds:

$$y \le f(x \boxdot \alpha) \iff y \le f(x) \boxdot^\dagger \alpha \iff \alpha \odot y \le f(x).$$

Conversely, if we assume that $y \le f(x \boxdot \alpha)$ if and only if $\alpha \odot y \le f(x)$ for all $\alpha \in \mathfrak{Q}, x \in X, y \in Y$, then we obtain:

$$\begin{aligned} f(x \boxdot \alpha) \le f(x \boxdot \alpha) &\iff \alpha \odot f(x \boxdot \alpha) \le f(x) \\ &\iff f(x \boxdot \alpha) \le f(x) \boxdot^\dagger \alpha. \end{aligned}$$

Further, the definition of $\boxdot^\dagger$ implies that $\alpha \odot (f(x) \boxdot^\dagger \alpha) \le f(x)$. Referring again to the previous equivalence we obtain $f(x) \boxdot^\dagger \alpha \le f(x \boxdot \alpha)$. Hence $X \xrightarrow{f} Y^\dagger$ is a right $\mathfrak{Q}$-module morphism and the assertion follows. $\square$

If, in the previous proposition, we interchange right $\mathfrak{Q}$-modules and left $\mathfrak{Q}$-modules, then we can give a respective characterization of left $\mathfrak{Q}$-module morphisms.

Proposition 3.4 *Let (X, ℓ_X) be a left $\mathfrak{Q}$-module, (Y, r_Y) be a right $\mathfrak{Q}$-module and $(Y^\dagger, \ell_{Y^\dagger})$ be the left $\mathfrak{Q}$-module induced by (Y, r_Y) in the sense of Theorem 3.1. Further, let $\odot$ and $\boxdot$ be the bimorphisms determined by ℓ_X and r_Y respectively. Then a join-reversing map $X \xrightarrow{f} Y$ is a left $\mathfrak{Q}$-module morphism $(X, \ell_X) \xrightarrow{f} (Y^\dagger, \ell_{Y^\dagger})$ if and only if the following equivalence holds for all $\alpha \in \mathfrak{Q}$, $x \in X$ and $y \in Y$:*

$$y \leq f(\alpha \odot x) \iff y \boxdot \alpha \leq f(x).$$

As a second step we point out that the self-duality in Sup also permits to associate a $\mathfrak{Q}$-enriched category with every right $\mathfrak{Q}$-module.

For the convenience of the reader we review the details of this construction. By analogy with the situation in $\mathsf{Mod}_\ell(\mathfrak{Q})$ we first compute the right adjoint map

$$X^\dagger \xrightarrow{r_X^\vdash} (X \otimes \mathfrak{Q})^\dagger = [X, \mathfrak{Q}^\dagger]$$

of the right action $X \otimes \mathfrak{Q} \xrightarrow{r_X} X$, and in a second step we construct a join-preserving map $X^\dagger \otimes X \xrightarrow{\varphi} \mathfrak{Q}^\dagger$ by applying the evaluation arrow:

$$\begin{tikzcd} X^\dagger \otimes X \arrow[r, "r_X^\vdash \otimes 1_X"] \arrow[dr, dotted, "\varphi"'] & {[X, \mathfrak{Q}^\dagger] \otimes X} \arrow[d, "ev_{\mathfrak{Q}^\dagger}"] \\ & \mathfrak{Q}^\dagger \end{tikzcd}$$

The bimorphism $X^\dagger \times X \xrightarrow{\hom_X} \mathfrak{Q}^\dagger$ determined by φ (cf. (2.1)) has the form:

$$\hom_X(x, y) = \bigvee \{ \alpha \in \mathfrak{Q} \mid y \boxdot \alpha \leq x \}, \qquad x, y \in X. \tag{3.5}$$

If we now consider $\hom_X$ as $\mathfrak{Q}$-valued map defined on the cartesian product $X \times X$ in Set and fix the given order on X, then we can reformulate the lattice-theoretic properties of $\hom_X$ as follows:

The map $\hom_X$ is meet-preserving in the first variable and join-reversing in the second variable. (3.6)

Further, we conclude from (R1) and (R2) that $\hom_X$ is a $\mathfrak{Q}$-valued hom-object assignment — i.e. the $\mathfrak{Q}$-enriched composition law and the existence of $\mathfrak{Q}$-enriched identities

$$\hom_X(x,y) * \hom_X(z,x) \le \hom_X(z,y) \text{ and } e \le \hom_X(x,x), \quad x,y \in X,$$

hold in the framework given by the monoidal biclosed category determined by $\mathfrak{Q}$ (cf. [5]).

As a consequence of this construction we now show that $\hom_X$ gives rise to specific module morphisms in Sup, which will play a significant role in Section 4.

For this purpose, let us consider the given right action r_X on X and the right action $r_{\mathfrak{Q}^\dagger}$ on $\mathfrak{Q}^\dagger$ induced by the left action on $\mathfrak{Q}$ (cf. Example 3.2 (a)). If we fix the first variable in $\hom_X$, then it follows immediately from Proposition 3.3, (3.5) and (R1) that

$$(X, r_X) \xrightarrow{\hom_X(x,\_\_)} (\mathfrak{Q}^\dagger, r_{\mathfrak{Q}^\dagger})$$

is a right $\mathfrak{Q}$-module morphism for all $x \in X$.

On the other hand, if we fix the second variable in $\hom_X$, then we consider the respective left actions $\ell_{X^\dagger}$ and $\ell_{\mathfrak{Q}^\dagger}$ on $X^\dagger$ and $\mathfrak{Q}^\dagger$ induced by the respective right actions on X and on $\mathfrak{Q}$ (cf. Example 3.2 (b)) in the sense of Theorem 3.1. Now we refer to (3.2) and obtain:

$$\begin{aligned} \beta \le \hom_X(\gamma \odot^\dagger x, y) &\iff y \boxdot \beta \le \gamma \odot^\dagger x \\ &\iff (y \boxdot \beta) \boxdot \gamma \le x \\ &\iff \beta * \gamma \le \hom_X(x,y). \end{aligned}$$

Hence Proposition 3.4 implies that $(X^\dagger, \ell_{X^\dagger}) \xrightarrow{\hom_X(\_\_,y)} (\mathfrak{Q}^\dagger, \ell_{\mathfrak{Q}^\dagger})$ is a left $\mathfrak{Q}$-module morphism for all $y \in X$.

We can summarize the previous observations in the following formulae:

$$\begin{aligned} \hom_X(\alpha \odot^\dagger x, y) &= \hom_X(x,y) \swarrow \alpha \text{ and} \\ \hom_X(x, y \boxdot \alpha) &= \alpha \searrow \hom_X(x,y). \end{aligned} \tag{3.7}$$

Finally, it can be shown that the $\mathfrak{Q}$-enriched category $(X, \hom_X)$ is skeletal and cocomplete. In this context we recall that $\mathsf{Mod}_r(\mathfrak{Q})$ is isomorphic to the category of cocomplete and skeletal $\mathfrak{Q}$-enriched categories — a result, which has been established by I. Stubbe in 2006 in the more general context of quantaloid-enriched categories (cf. [11] and [2, Sect. 3.3.3]).

4. The weak subobject classifier axiom

In this section we present a weakening of the subobject classifier axiom and explore its first categorical consequences. As a motivating example we reveal the special role of categories of modules in Sup.

Definition 4.1 *Let C be a finitely complete category with terminal object T. Further, let Ω be an object of C and $T \overset{t}{\rightarrowtail} \Omega$ be a global point of Ω.*

(a) A monomorphism $U \overset{\psi}{\rightarrowtail} X$ is called (t,Ω)-classifiable *if there exists a morphism $X \xrightarrow{\varphi} \Omega$ such that*

$$\begin{tikzcd} U \arrow[r, "!_U"] \arrow[d, tail, "\psi"'] & T \arrow[d, tail, "t"] \\ X \arrow[r, dotted, "\varphi"] & \Omega \end{tikzcd} \tag{4.1}$$

is a pullback square. In particular, φ is said to be a classifying morphism *of the monomorphism $U \overset{\psi}{\rightarrowtail} X$.*

(b) The pair (t,Ω) is called a weak subobject classifier *if the following conditions are satisfied:*

(WS1) *If X is an object of C, then every global point $T \rightarrowtail X$ of X is (t,Ω)-classifiable.*

(WS2) *If a monomorphism $U \overset{\psi}{\rightarrowtail} X$ is (t,Ω)-classifiable, then it is uniquely (t,Ω)-classifiable — i.e. φ in the pullback (4.1) is uniquely determined by ψ.*

Obviously every (t,Ω)-classifiable monomorphism is an equalizer (cf. [12, Prop. 14.3]).

Further, it follows immediately from the previous definition that every weak subobject classifier is unique up to an isomorphism. Hence the existence of a weak subobject classifier is an invariant of a finitely complete category. In this context we introduce the following terminology:

The morphism t is called the *arrow true*. Further, a morphism φ with codomain Ω is a called a *characteristic morphism.* If φ is uniquely determined by a monomorphism $U \overset{\psi}{\rightarrowtail} X$ in the sense of the pullback in (4.1), then we write χ_ψ instead of φ.

In particular we have the following:

Theorem 4.2 *Let $\mathfrak{Q} = (\mathfrak{Q}, *, e)$ be a unital quantale with unit e. Then the right $\mathfrak{Q}$-module $(\mathbb{R}(\mathfrak{Q})^\dagger, r_{\mathfrak{Q}^\dagger})$ (cf. Example 3.2 (a)) is the weak subobject classifier in $\mathsf{Mod}_r(\mathfrak{Q})$, and the arrow true is represented by the universal lower bound of $\mathbb{R}(\mathfrak{Q})^\dagger$.*

Proof. First we notice that the universal lower bound in $\mathbb{R}(\mathfrak{Q})^\dagger$ is the universal upper bound $\top$ in $\mathfrak{Q}$.
(a) Let (X, r_X) be a right $\mathfrak{Q}$-module and $\perp$ be the universal lower bound of X. Further, let hom_X be the hom-object assignment of the $\mathfrak{Q}$-enriched category (X, hom_X) associated with (X, r_X) (cf. Section 3). Since $\mathrm{hom}_X(\perp, y)$ is right-sided for all $y \in X$ (cf. (3.5)), the range of $\mathrm{hom}_X(\perp, \_)$ is contained in $\mathbb{R}(\mathfrak{Q})$, and consequently

$$(X, r_X) \xrightarrow{\mathrm{hom}_X(\perp, \_)} (\mathbb{R}(\mathfrak{Q})^\dagger, r_{\mathfrak{Q}^\dagger})$$

is a right $\mathfrak{Q}$-module morphism. The right $\mathfrak{Q}$-submodule determined by the pullback of the arrow true $0 \xrightarrow{t} \mathfrak{Q}^\dagger$ along $\mathrm{hom}_X(\perp, \_)$ is given by

$$\{\, y \in X \mid \mathrm{hom}_X(\perp, y) = \top \,\} = \{\, \perp \,\}.$$

Hence (WS1) is satisfied.
(b) Let $(X, r_X) \xrightarrow{\varphi} (\mathbb{R}(\mathfrak{Q})^\dagger, r_{\mathfrak{Q}^\dagger})$ be a right $\mathfrak{Q}$-module morphism and U be the right $\mathfrak{Q}$-submodule of (X, r_X) determined by the pullback of the arrow true along φ — i.e.

$$U = \{\, x \in X \mid \varphi(x) = \top \,\}. \tag{4.2}$$

Further, let hom_X be the hom-object assignment of the $\mathfrak{Q}$-enriched category (X, hom_X) associated with (X, r_X). Then $y \boxdot \mathrm{hom}_X(x, y) \le x$ holds for all $x, y \in X$. Since φ preserves the respective right actions and is isotone, the previous relation implies $\varphi(y) \boxdot^{\dagger} \hom_X(x,y) \leq^{\dagger} \varphi(x)$, which is equivalent to

$$\varphi(x) \leq \hom_X(x,y) \searrow \varphi(y), \quad \text{i.e. } \hom_X(x,y) * \varphi(x) \leq \varphi(y), \quad x,y \in X,$$

where we have also referred to (3.3). Then φ is a cocontinuous contravariant $\mathfrak{Q}$-presheaf on $(X, \hom_X)$ in the terminology of $\mathfrak{Q}$-enriched category theory (cf. [10, Def. 3.1, p. 21 and 23]). Since $\mathfrak{Q}$ is unital, we conclude from (4.2):

$$\bigvee\{\, \hom_X(x,y) \mid x \in U \,\} \leq \varphi(y), \qquad y \in X. \tag{4.3}$$

Now we recall that $\varphi(y)$ is right-sided for all $y \in X$, hence:

$$\varphi(y \boxdot \varphi(y)) = \varphi(y \boxdot \varphi(y)) * \top = \big(\varphi(y) \searrow \varphi(y)\big) * \top = \top,$$

and so $y \boxdot \varphi(y) \in U$. Thus $\varphi(y) \leq \hom_X(y \boxdot \varphi(y), y)$ follows, and the inequality in (4.3) turns into an equality. In particular, the relation:

$$\varphi(y) = \bigvee_{\varphi(x)=\top} \hom_X(x,y) * \top = \bigvee_{\varphi(x)=\top} \hom_X(x \boxdot \top, y) \tag{4.4}$$

holds for all $y \in X$, and so (WS2) is verified. $\square$

By analogy with Theorem 4.2 the left $\mathfrak{Q}$-module $(\mathbb{L}(\mathfrak{Q}^{\dagger}, \ell_{\mathfrak{Q}^{\dagger}})$ is the weak subobject classifier in $\mathsf{Mod}_{\ell}(\mathfrak{Q})$ (cf. Example 3.2 (b)). Indeed, we have only to observe that the hom-object assignment of a left $\mathfrak{Q}$-module (X, ℓ_X) is the hom-object assignment determined by the dual right $\mathfrak{Q}$-module $(X^{\dagger}, r_{X^{\dagger}})$ of (X, ℓ_X) — i.e.

$$\hom_X(x,y) = \bigvee\{\, \alpha \in \mathfrak{Q} \mid y \boxdot^{\dagger} \alpha \leq^{\dagger} x \,\} = \bigvee\{\, \alpha \in \mathfrak{Q} \mid \alpha \odot x \leq y \,\}$$

for each $x, y \in X$. Consequently the lattice-theoretic properties of $\hom_X$ (cf. (3.6)) are read in $X^{\dagger}$.

Remark 4.3 Let (X, r_X) be a right $\mathfrak{Q}$-module and $(X, \hom_X)$ be the associated $\mathfrak{Q}$-enriched category. Since (4.4) describes the characteristic morphisms of (X, r_X) in the sense of $\mathsf{Mod}_r(\mathfrak{Q})$, it is easily seen that for every right-sided element $\alpha \in \mathfrak{Q}$ and $x \in X$ the map $X \xrightarrow{\hom_X(x \boxdot \alpha, \_)} \mathbb{R}(\mathfrak{Q}^{\dagger})$ is a right $\mathfrak{Q}$-module morphism and the characteristic morphism of the right $\mathfrak{Q}$-submodule $\downarrow(x \boxdot \alpha) = \{\, y \in X \mid y \leq x \boxdot \alpha \,\}$ of X. If $\mathfrak{Q}$ is commutative, then $\mathbb{R}(\mathfrak{Q})$ is the subquantale $\mathbb{I}(\mathfrak{Q})$ of all two-sided elements of $\mathfrak{Q}$ and $\hom_X(x \boxdot \alpha, \_)$ is the elementary tensor $x \otimes_{\mathfrak{Q}} \alpha$ of the tensor product $X \otimes_{\mathfrak{Q}} \mathbb{I}(\mathfrak{Q})$ of $\mathfrak{Q}$-modules, where the action on $\mathbb{I}(\mathfrak{Q})$ is given by the quantale multiplication.

4.1 Classifiable subobjects

Let C be a finitely complete category with terminal object T provided with a weak subobject classifier (t, Ω). We begin with the simple observation that the identity 1_X of an object X is always (t, Ω)-classifiable. The characteristic morphism of 1_X is the composition $X \xrightarrow{!_X} T \overset{t}{\rightarrowtail} \Omega$ and is denoted by true_X.

Further, recall that a subobject of X is an equivalence class of monomorphisms with codomain X. If a representing monomorphism of a subobject S of X is (t, Ω)-classifiable, then it is easily seen that every further representing monomorphism of S is also (t, Ω)-classifiable. Hence, due to the unique classification of (t, Ω)-classifiable monomorphisms with codomain X, every characteristic morphism of X is uniquely determined by its corresponding subobject — i.e. there exists a bijective map between all (t, Ω)-classifiable subobjects of X and all charactertistic morphisms of X.

In a first step we show that the pullback of (t, Ω)-classifiable subobjects is again (t, Ω)-classifiable. In particular, (t, Ω)-classifiable monomorphisms are pullback stable.

Proposition 4.4 *Let $X \xrightarrow{f} Y$ be a morphism and $U \overset{m}{\rightarrowtail} Y$ be a (t, Ω)-classifiable monomorphism. Then the pullback $V \overset{n}{\rightarrowtail} X$ of m along f is again (t, Ω)-classifiable. In particular, if χ_m is the characteristic morphism of m, then $\chi_m \circ f$ is the characteristic morphism of n.*

Proof. We consider the commutative diagram (cf. [12, Proof of Prop. 14.4]):

$$\begin{tikzcd}
V \arrow[r, tail, "n"] \arrow[d] & X \arrow[d, "f"] \\
U \arrow[r, tail, "m"'] \arrow[d, "!_U"'] & Y \arrow[d, "\chi_m"] \\
T \arrow[r, tail, "t"] & \Omega
\end{tikzcd}$$

Hence the result follows from the Pullback Lemma. $\square$

The formulation of the previous proposition can be seen as an extension of the well known result [12, Prop. 14.4] to the area of weak subobject classifiers.

Now we show that the binary intersection of (t,Ω)-classifiable subobjects is again (t,Ω)-classifiable. For this purpose we will apply (WS1). By analogy with topos theory, the characteristic morphism $\Omega\times\Omega \xrightarrow{\chi_\cap} \Omega$ of the global point $T \xrightarrow{\langle t,t\rangle} \Omega\times\Omega$ will be referred as the *conjunction* in C.

Example 4.5 In $\mathsf{Mod}_r(\mathfrak{Q})$ the conjunction $\chi_\cap$ coincides with the binary meet of right-sided elements. In fact, since the hom-object assignment of the weak subobject classifier $\mathbb{R}(\mathfrak{Q}^\dagger, r_{\mathfrak{Q}^\dagger})$ is given by $\hom_{\mathbb{R}(\mathfrak{Q})^\dagger}(\alpha,\beta) = \beta \swarrow \alpha$, the relation (4.4) implies: $\chi_\cap(\beta_1,\beta_2) = (\beta_1 \swarrow \top)\wedge(\beta_2 \swarrow \top) = \beta_1 \wedge \beta_2$ for all $\beta_1,\beta_2 \in \mathbb{R}(\mathfrak{Q})$.

Theorem 4.6 *Let $U_1 \xrightarrow{m_1} X$ and $U_2 \xrightarrow{m_2} X$ be (t,Ω)-classifiable monomorphisms. Then the monomorphism $V \xrightarrow{n} X$ determined by the following pullback square*:

$$\begin{tikzcd} V \arrow[r, tail, "m_1'"] \arrow[d, tail, "m_2'"'] \arrow[dr, dotted, tail, "n"] & U_2 \arrow[d, tail, "m_2"] \\ U_1 \arrow[r, tail, "m_1"'] & X \end{tikzcd} \tag{4.5}$$

is again (t,Ω)-classifiable. Moreover, if χ_{m_i} is the characteristic morphism of $U_i \xrightarrow{m_i} X$ $(i=1,2)$ and χ_n is the characteristic morphism of $V \xrightarrow{n} X$, then the relation $\chi_n = \chi_\cap \circ \langle \chi_{m_1}, \chi_{m_2}\rangle$ holds.

Proof. In the case of a weak subobject classifier axiom (cf. Definition 4.1) we can also follow the same strategy as in topos theory. We consider the characteristic morphisms $X \xrightarrow{\chi_{m_1}} \Omega$ and $X \xrightarrow{\chi_{m_2}} \Omega$ corresponding to m_1

and m_2 and prove that the outer rectangle of the diagram:

$$\begin{tikzcd}
V \arrow[r, tail, "n"] \arrow[d] & X \arrow[d, "\langle \chi_{m_1}, \chi_{m_2} \rangle"] \\
T \arrow[r, tail, "\langle t,t \rangle"] \arrow[d] & \Omega \times \Omega \arrow[d, "\chi_\cap"] \\
T \arrow[r, tail, "t"] & \Omega
\end{tikzcd}$$

is a pullback square. For this purpose it is sufficient to show that the upper square is a pullback. The commutativity of the upper square is evident. Now let us consider a morphism $Z \xrightarrow{\ell} X$ with $\langle t, t\rangle \circ !_Z = \langle \chi_{m_1}, \chi_{m_2}\rangle \circ \ell$. Then the weak subobject classifier axiom implies that there exist morphisms $Z \xrightarrow{\varphi_i} U_i$ $(i = 1, 2)$ such that $m_1 \circ \varphi_1 = \ell = m_2 \circ \varphi_2$. Finally, the pullback square (4.5) guarantees the existence of $Z \xrightarrow{\psi} V$ satisfying $\varphi_1 = m_2' \circ \psi$ and $\varphi_2 = m_1' \circ \psi$. Now we observe $n \circ \psi = m_1 \circ m_2' \circ \psi = m_1 \circ \varphi_1 = \ell$. Hence the assertion is verified.
Finally, the relation $\chi_n = \chi_\cap \circ \langle \chi_{m_1}, \chi_{m_2}\rangle$ follows from the uniqueness of the classification. □

Since the identity 1_Ω of Ω is the characteristic morphism of the arrow true, Theorem 4.6 implies that $\chi_\cap$ is idempotent — i.e. $1_\Omega = \chi_\cap \circ \langle 1_\Omega, 1_\Omega\rangle$. Moreover, the unique classification shows that $(\Omega, \chi_\cap)$ is a commutative monoid in C w.r.t. the monoidal structure determined by the product in C. In particular, the arrow true is the unit of $(\Omega, \chi_\cap)$. Hence $(\Omega, \chi_\cap)$ induces a partial order on the set $\mathrm{HOM}_{\mathsf{C}}(X, \Omega)$ of all characteristic morphisms of (X, r_X) by

$$\chi_1 \leq \chi_2 \iff \chi_1 = \chi_\cap \circ \langle \chi_1, \chi_2\rangle, \qquad \chi_1, \chi_2 \in \mathrm{HOM}_{\mathsf{C}}(X, \Omega).$$

Obviously $(\mathrm{HOM}_{\mathsf{C}}(X, \Omega), \leq)$ is a semilattice. Due to the weak subobject classifier axiom, $(\mathrm{HOM}_{\mathsf{C}}(X, \Omega), \leq)$ is order-isomorphic to the partially ordered set $\mathrm{sub}_{cl}(X)$ of all (t, Ω)-classifiable subobjects of X.

If C is a complete category, then Theorem 4.6 holds also for any family of (t, Ω)-classifiable subobjects of X. Hence in this case $\mathrm{sub}_{cl}(X)$ is a complete lattice, and for any subobject with representing monomorphism $U \overset{m}{\rightarrowtail} X$ its (t, Ω)-*classifiable hull* exists — i.e. there exists a (t, Ω)-classifiable monomorphism $\widetilde{U} \overset{\widetilde{m}}{\rightarrowtail} X$ determined by the following properties:

(CL1) There exists a (mono)morphism $U \xrightarrow{j_U} \widetilde{U}$ such that $m = \widetilde{m} \circ j_U$.

(CL2) For every further (t, Ω)-classifiable monomorphism $V \overset{n}{\rightarrowtail} X$ satisfying the condition $m = n \circ j_{VU}$ with $U \overset{j_{VU}}{\rightarrowtail} V$ there exists a morphism $\widetilde{U} \xrightarrow{\widetilde{j_{VU}}} V$ such that $\widetilde{m} = n \circ \widetilde{j_{VU}}$ holds.

Hence $\widetilde{m}$ is uniquely determined by m up to an isomorphism.

Finally, if C is complete, then for every morphism $X \xrightarrow{f} Y$ the inverse image functor $\mathrm{sub}_{cl}(Y) \xrightarrow{f^{-1}} \mathrm{sub}_{cl}(X)$ has a left adjoint. It is an open question whether f^{-1} has a right adjoint.

4.2 Comprehension scheme

In this subsection we do not only assume that the finitely complete category C satisfies the weak subobject classifier axiom, but also that for every subobject its (t, Ω)-*classifiable hull* exists. Referring to [7, 8] the question arises to which extent the weak subobject classifier axiom is a weakening of the comprehension principle. Following Lawvere, we understand a morphism $E \xrightarrow{x} X$ as an element of X "defined over E" and for every monomorphism $U \overset{m}{\rightarrowtail} X$ we say $x \in m$ if there exists $\widetilde{x}$ such that

$$\begin{tikzcd} E \arrow[r, dotted, "\widetilde{x}"] \arrow[dr, "x"'] & U \arrow[d, tail, "m"] \\ & X \end{tikzcd}$$

i.e. $x = m \circ \widetilde{x}$. Further, recall that true_E is the composition $E \xrightarrow{!_E} T \overset{t}{\rightarrowtail} \Omega$, where T is the terminal object and t is the arrow true (cf. [7]). Then the weak subobject classifier axiom says the following. Given any "propositional function" (i.e. characteristic morphism of X) $X \xrightarrow{\chi} \Omega$ there exists a monomorphism $\{X|\chi\}$ with codomain X such that for any $E \xrightarrow{x} X$

$$x \in \{X|\chi\} \iff \chi \circ x = \mathrm{true}_E,$$

and, conversely, for *every* monomorphism with codomain X there exists the *smallest* (t, Ω)-classifiable monomorphism — i.e. its (t, Ω)-classifiable hull, with codomain X which has a ***unique*** "characteristic function" χ. If C is an epi-mono-category, then this relationship can be expressed by an adjoint situation (cf. [8]) — i.e. there exists a functor $\mathcal{F}\colon \mathsf{C}/X \to \mathrm{Hom}(X, \Omega)$, which has a right adjoint.

In fact, $\mathcal{F}$ acts on objects as follows. For $E \xrightarrow{p} X$ we first construct the epi-mono-factorization

$$\begin{tikzcd} E \arrow[rr, "p"] \arrow[dr, two heads, "p^*"'] & & X \\ & U \arrow[ur, tail, "m"'] & \end{tikzcd}$$

and subsequently we consider the (t, Ω)-classifiable hull $\widetilde{U} \overset{\widetilde{m}}{\rightarrowtail} X$ of $U \overset{m}{\rightarrowtail} X$. Then $\mathcal{F}(p)$ is given by the characteristic morphism of $\widetilde{U} \overset{\widetilde{m}}{\rightarrowtail} X$. Further, for every morphism $p_1 \xrightarrow{\pi} p_2$ the epi-mono-factorization leads to the following commutative diagram:

$$\begin{tikzcd} E_1 \arrow[r, two heads, "p_{1*}"] \arrow[dd, "\pi"'] & U_1 \arrow[r, tail, "j_{U_1}"] \arrow[dd, dotted, "\widehat{\pi}"'] \arrow[dr, tail, "m_1"'] & \widetilde{U_1} \arrow[d, "\widetilde{m}_1"] \\ & & X \\ E_2 \arrow[r, two heads, "p_2^*"] & U_2 \arrow[r, tail, "j_{U_2}"'] \arrow[ur, tail, "m_2"] & \widetilde{U}_2 \arrow[u, "\widetilde{m}_2"'] \end{tikzcd}$$

Since $\widetilde{m}_2 \circ j_{U_2} \circ \widehat{\pi} = m_1$, we conclude from the universal property (CL2) that there exists a morphism $\widetilde{U}_1 \xrightarrow{\widetilde{j}_{\widetilde{U}_2\widetilde{U}_1}} \widetilde{U}_2$ such that $\widetilde{m}_1 = \widetilde{m}_2 \circ \widetilde{j}_{\widetilde{U}_2\widetilde{U}_1}$. If χ_i is the characteristic morphism of $\widetilde{m}_i$ ($i = 1, 2$), then $\chi_1 = \chi_\cap(\chi_1, \chi_2)$ follows — i.e. $\mathcal{F}(p_1) \leq \mathcal{F}(p_2)$.

On the other hand there exists a functor $\mathcal{G}\colon \mathrm{Hom}(X, \Omega) \to \mathsf{C}/X$ determined by the pullback diagram:

$$\begin{tikzcd} X \arrow[d, "\chi"'] & \widetilde{U} \arrow[l, tail, "\widetilde{m}"'] \arrow[d] \\ \Omega & T \arrow[l, tail, "t"] \end{tikzcd}$$

Since there is a natural transformation $\eta\colon \mathrm{id}_{\mathsf{C}} \to \mathcal{G}\mathcal{F}$ with the p-component $E \xrightarrow{\eta_p} \mathcal{G}(\mathcal{F}(p))$ defined by $\eta_p = j_U \circ p^*$ with $\widetilde{m} \circ j_U = m$ and $p = m \circ p^*$, it is not difficult to show that $\mathcal{G}$ is right adjoint to $\mathcal{F}$. In this sense a "comprehension scheme" holds in C.

4.3 (t, Ω)-classifiable subobjects in $\mathsf{Mod}_r(\mathfrak{Q})$

Let $(t, \Omega) = (t, (\mathbb{R}(\mathfrak{Q})^\dagger, r_{\mathfrak{Q}^\dagger}))$ be the weak subobject classifier in $\mathsf{Mod}_r(\mathfrak{Q})$ (cf. Theorem 4.2) and (X, r_X) be a right $\mathfrak{Q}$-module. Since in $\mathsf{Mod}_r(\mathfrak{Q})$ the characteristic morphism $\chi_\cap$ is the binary *meet* in $\mathbb{R}(\mathfrak{Q})$ (cf. Example 4.5), the complete lattice $\mathrm{HOM}_{\mathsf{Mod}_r(\mathfrak{Q})}((X, r_X), (\mathbb{R}(\mathfrak{Q})^\dagger, r_{\mathfrak{Q}^\dagger}))$ ($\cong$ $\mathrm{sub}_{cl}(X, r_X)$) is the dual lattice of the complete lattice $[X, \mathbb{R}(\mathfrak{Q})^\dagger]$ of all characteristic morphisms of (X, r_X) ordered pointwise in $\mathbb{R}(\mathfrak{Q})^\dagger$.

As a first step we give a characterization of characteristic morphisms.

Proposition 4.7 *Let (X, r_X) be a right $\mathfrak{Q}$-module and $\hom_X$ be the associated hom-object assignment. Then for every characteristic morphism $X \xrightarrow{\chi} \mathbb{R}(\mathfrak{Q})^\dagger$ there exists a unique element $x \in X$ satisfying the following conditions*

$$x \boxdot \top = x \quad \textit{and} \quad \chi(y) = \textit{hom}_X(x, y), \quad y \in X. \tag{4.6}$$

Proof. The anti-symmetry of the partial order on X implies the uniqueness of the element x in (4.6). In order to confirm the existence of x we proceed as follows:

$$x = \bigvee \{\, z \in X \mid e \leq \chi(z) \,\}.$$

Since $\chi(z)$ is right-sided for all $z \in X$ and χ itself is join-reversing, we obtain $\chi(x) = \top$. Now we observe $\chi(x \boxdot \top) = \top \searrow \top = \top$ — i.e. $x \boxdot \top \leq x$ and consequently $x \boxdot \top = x$. Since every right $\mathfrak{Q}$-module morphism is also a $\mathfrak{Q}$-functor in the sense of $\mathfrak{Q}$-enriched category theory, the relation $\hom_X(x, \_) \leq \chi$ holds. On the other hand we observe $e \leq \chi(y) \searrow \chi(y) = \chi(y \boxdot \chi(y))$. Hence $y \boxdot \chi(y) \leq x$ — i.e. $\chi(y) \leq \hom_X(x, y)$, and the relation (4.6) is verified. $\square$

As an immediate corollary from Remark 4.3 and Proposition 4.7 we obtain that every (t, Ω)-classifiable subobject of (X, r_X) is a right $\mathfrak{Q}$-submodule U having the following form:

$$\exists x \in X \text{ with } x \boxdot \top = x \quad \text{such that} \quad U = \downarrow x = \{\, y \in X \mid y \leq x \,\} \tag{4.7}$$

Hence, for an arbitrary right $\mathfrak{Q}$-submodule U of (X, r_X), its (t, Ω)-classifiable hull $\widetilde{U}$ is given by $\widetilde{U} = \downarrow(\bigvee U)$.

Remark 4.8 Let $\mathfrak{Q}$ be an integral quantale (i.e. the unit is the universal upper bound of $\mathfrak{Q}$), then $\mathfrak{Q}^\dagger$ is the weak subobject classifier in $\mathsf{Mod}_r(\mathfrak{Q})$. Since $\mathfrak{Q}^\dagger$ is a $\mathfrak{Q}$-bimodule, for every right $\mathfrak{Q}$-module (X, r_X) the complete lattice $[X, \mathfrak{Q}^\dagger]$ of all characteristic morphisms on X is a *left* $\mathfrak{Q}$-module with the left action $\ell_{[X,\mathfrak{Q}^\dagger]}$ determined by:

$$(\alpha \odot \chi)(x) = \chi(x) \swarrow \alpha, \qquad \alpha \in \mathfrak{Q},\ x \in X.$$

Since $\mathrm{HOM}_{\mathsf{Mod}_r(\mathfrak{Q})}((X, r_X), (\mathbb{R}(\mathfrak{Q})^\dagger, r_{\mathfrak{Q}^\dagger})) = [X, \mathbb{R}(\mathfrak{Q})^\dagger]^\dagger$, we may conclude from Theorem 3.1 that $\mathrm{HOM}_{\mathsf{Mod}_r(\mathfrak{Q})}((X, r_X), (\mathbb{R}(\mathfrak{Q})^\dagger, r_{\mathfrak{Q}^\dagger}))$ is a right $\mathfrak{Q}$-module and its dual right action of $\ell_{[X,\mathbb{R}(\mathfrak{Q})^\dagger]}$ has the form:

$$f \boxdot^\dagger \alpha = \bigwedge\{\, g \in [X, \mathbb{R}(\mathfrak{Q})^\dagger] \mid f(x) * \alpha \leq g(x) \text{ for all } x \in X \,\}.$$

Further, let us invoke again Theorem 3.1 and consider the dual left $\mathfrak{Q}$-module $(X^\dagger, \ell_{X^\dagger})$ of (X, r_X).Then we conclude from Proposition 4.7 and (3.7) that there exists a left $\mathfrak{Q}$-module isomorphism $X^\dagger \xrightarrow{\eta_X} [X, \mathfrak{Q}^\dagger]$ defined by:

$$\eta_X(x) = \hom_X(x, \_), \qquad x \in X.$$

If $(X, r_X) \xrightarrow{f} (Y, r_Y)$ is a right $\mathfrak{Q}$-module morphism, then the right adjoint *left* $\mathfrak{Q}$-module morphism $(Y^\dagger, \ell_{Y^\dagger}) \xrightarrow{f^\vdash} (X^\dagger, \ell_{X^\dagger})$ satisfies the following chain of equivalences for all $x \in X$ and $y \in Y$:

$$\begin{aligned} \alpha \le \hom_X(y, f(x)) &\iff f(x) \boxdot \alpha \le y \\ &\iff x \boxdot \alpha \le f^\vdash(y) \iff \alpha \le \hom_X(f^\vdash(y), x). \end{aligned}$$

Hence the diagram

$$\begin{tikzcd} (Y^\dagger, \ell_{Y^\dagger}) \arrow[r, "\eta_Y"] \arrow[d, "f^\vdash"'] & {[Y, \mathfrak{Q}^\dagger]} \arrow[d, "\Theta_f"] \\ (X^\dagger, r_{X^\dagger}) \arrow[r, "\eta_X"] & {[X, \mathfrak{Q}^\dagger]} \end{tikzcd}$$

is commutative, where Θ_f is given by $\Theta_f(\chi) = \chi \circ f$ for all $\chi \in [Y, \mathfrak{Q}^\dagger]$ (cf. Proposition 4.4). So we obtain that in the case of integral quantales the restriction of the inverse image functor f^{-1} to (t, Ω)-classifiable subobjects of (Y, r_Y) is equivalent to the right adjoint left $\mathfrak{Q}$-module morphism $f^\vdash$ of f.

Finally, let us consider the case of arbitrary unital quantales. Then we need some more terminology. An element x of a right $\mathfrak{Q}$-module (X, r_X) is *well-sided* if $x \boxdot \top = x$. The set $\mathbb{W}(X)$ of all well-sided elements of X is a complete sublattice of X in the sense of Sup, but not necessarily a right $\mathfrak{Q}$-submodule of X. The inclusion map $\mathbb{W}(X) \hookrightarrow X$ is meet-preserving.

If $(X, r_X) \xrightarrow{f} (Y, r_Y)$ is a right $\mathfrak{Q}$-module morphism, then the right adjoint $f^\vdash$ of f viewed as morphism in Sup factors through $\mathbb{W}(Y)^\dagger$ in the following way:

$$\begin{tikzcd} \mathbb{W}(Y)^\dagger \arrow[r, hook, "\iota_{Y^\dagger}"] \arrow[d, dotted, "f^*"'] & Y^\dagger \arrow[d, "f^\vdash"] \\ \mathbb{W}(X)^\dagger \arrow[r, "\iota_{X^\dagger}"] & X^\dagger \end{tikzcd}$$

Since in this situation $\mathbb{W}(Y)^\dagger \xrightarrow{\eta_Y} [Y, \mathbb{R}(\mathfrak{Q})^\dagger]$ is only an order isomorphism (cf. Proposition 4.7), we refer to Remark 4.8 and conclude that the restriction of f^{-1} to $\mathrm{sub}_{cl}(Y, r_Y)$ has a right adjoint functor $\mathrm{sub}_{cl}(X, r_X) \xrightarrow{\forall_f}$ $\mathrm{sub}_{cl}(Y, r_Y)$ if and only if f^* is meet-preserving in the respective orders of $\mathbb{W}(Y)^\dagger$ and $\mathbb{W}(X)^\dagger$.

As an illustration of this situation we include the following simple example.

Example 4.9 Let $\mathfrak{Q}$ be a unital quantale without zero divisors. Further, let us view $\mathfrak{Q}$ as right $\mathfrak{Q}$-module w.r.t. the right quantale multiplication. Then the complete sublattice $\mathbb{W}(\mathfrak{Q})$ coincides with the subquantale $\mathbb{R}(\mathfrak{Q})$ of all right-sided elements of $\mathfrak{Q}$. In general $\mathbb{R}(\mathfrak{Q})$ is not a right $\mathfrak{Q}$-submodule of $\mathfrak{Q}$.
Further we fix an element $\alpha \in \mathfrak{Q} \setminus \{\perp\}$. The left translation in $\mathfrak{Q}$ by α — i.e.

$$f_\alpha(\gamma) = \alpha * \gamma, \qquad \gamma \in \mathfrak{Q},$$

is a right $\mathfrak{Q}$-module morphism $\mathfrak{Q} \xrightarrow{f_\alpha} \mathfrak{Q}$. Then the right adjoint left $\mathfrak{Q}$-module morphism $\mathfrak{Q}^\dagger \xrightarrow{(f_\alpha)^\vdash} \mathfrak{Q}^\dagger$ of f_α has the form $(f_\alpha)^\vdash(\gamma) = \alpha \searrow \gamma$ with $\gamma \in \mathfrak{Q}$. Since $\mathfrak{Q}$ does not have zero divisors, the relation $(f_\alpha)^\vdash(\perp) = \perp$ follows. Hence the restriction of $(f_\alpha)^\vdash$ to $\mathbb{R}(\mathfrak{Q})^\dagger$ is meet-preserving in $\mathbb{R}(\mathfrak{Q})^\dagger$ (i.e. the restriction of the inverse image functor $(f_\alpha)^{-1}$ to $\mathrm{sub}_{cl}(\mathfrak{Q}, *)$ has a right adjoint) if and only if for all nonempty subsets A of $\mathbb{R}(\mathfrak{Q})$ the following relation holds:

$$\alpha \searrow (\bigvee A) = \bigvee_{\gamma \in A} (\alpha \searrow \gamma). \tag{4.8}$$

There exist unital quantales without zero divisors, in which (4.8) is violated. For example let us consider the idemptotent, non-commutative and unital quantale C_4^r on the 4-chain with $\perp < a < e < \top$, where e is the unit and a satisfies the properties $\top * a = \top$ and $a * \top = a$. Then the tensor product $C_4^r \otimes C_4^r$ in the sense of quantales (cf. [2, p. 92]) is a unital quantale without zero divisors, in which (4.8) is violated for certain non-zero elements of $C_4^r \otimes C_4^r$. The details are as follows. The subquantale $\mathbb{R}(C_4^r \otimes C_4^r)$ of all right-sided elements consists of six elements:

$$\perp,\ \ \delta = a \otimes a,\ \ \alpha = \top \otimes a,\ \ \beta = a \otimes \top,\ \ \gamma = (\top \otimes a) \vee (a \otimes \top),\ \ \top = \top \otimes \top$$

with $\delta = \alpha \wedge \beta$ and $\gamma = \alpha \vee \beta$. Now we consider the left translation f_γ on

$C_4^r \otimes C_4^r$ by γ. Then $\gamma \searrow (\alpha \vee \beta) = \gamma \searrow \gamma = \top$ and

$\gamma \searrow \alpha = (a \otimes \top) \searrow (\top \otimes a) = \bot$ and $\gamma \searrow \beta = (\top \otimes a) \searrow (a \otimes \top) = \bot$,

hence the relation (4.8) is violated, and consequently the restriction of the inverse image functor $(f_\gamma)^{-1}$ to $\mathsf{sub}_{cl}(C_4^r \otimes C_4^r)$ does not have a right adjoint. But on the other hand, if we consider the left translation f_α on $C_4^r \otimes C_4^r$ by α, then the relation (4.8) is satisfied and the restriction of the inverse image functor $(f_\alpha)^{-1}$ to $\mathsf{sub}_{cl}(C_4^r \otimes C_4^r)$ has a right adjoint.

4.4 The implication as truth arrow in $\mathsf{Mod}_r(\mathfrak{Q})$

Let $\mathfrak{Q}^\dagger \times \mathfrak{Q}^\dagger \xrightarrow{\pi_1} \mathfrak{Q}^\dagger$ be the projection onto the first coordinate. By analogy to topos theory we consider the equalizer in $\mathsf{Mod}_r(\mathfrak{Q})$

$$U \rightarrowtail \mathbb{R}(\mathfrak{Q})^\dagger \times \mathbb{R}(\mathfrak{Q})^\dagger \underset{\chi_\cap}{\overset{\pi_1}{\rightrightarrows}} (\mathbb{R}(\mathfrak{Q})^\dagger$$

and observe that the (t, Ω)-classifiable hull of U coincides with $\mathbb{R}(\mathfrak{Q})^\dagger \times \mathbb{R}(\mathfrak{Q})^\dagger$. This is the motivation to avoid the direct product in $\mathsf{Mod}_r(\mathfrak{Q})$ and to lift the tensor product of Sup to $\mathsf{Mod}_r(\mathfrak{Q})$ as follows. Let (X, r_X) be a right $\mathfrak{Q}$-module and Y be a complete lattice. Then on the tensor product $X \otimes Y$ we consider the right action $(X \otimes Y) \otimes \mathfrak{Q} \xrightarrow{r} X \otimes Y$ determined on elementary tensors by:

$$(x \otimes y) \boxdot \alpha = (x \boxdot \alpha) \otimes y, \qquad \alpha \in \mathfrak{Q},\ x \in X,\ y \in Y.$$

Since every tensor is a join of elementary tensors, the corresponding hom-object assignment has the following form:

$$\hom_{X \otimes Y}(f, g) = \bigwedge_{x \otimes y \le g} \{\, \alpha \in \mathfrak{Q} \mid y \le f(x \boxdot \alpha) \,\}, \qquad f, g \in X \otimes Y. \quad (4.9)$$

We apply this situation to the right $\mathfrak{Q}$-module $\mathfrak{Q}$ provided with the right multiplication as right action and the complete lattice $\mathbb{R}(\mathfrak{Q})^\dagger$. After these preparations we now consider the following tensor:

$$f = \bigvee \{\, \mu \otimes \nu \mid \mu \in \mathfrak{Q},\ \nu \in \mathbb{R}(\mathfrak{Q})^\dagger,\ \mu \le \nu \,\},$$

where $\le$ is the order in $\mathfrak{Q}$. We view f as the «tensorial» analogue of the equalizer $\underset{\chi_\cap}{\overset{\pi_1}{\rightrightarrows}}$.

If $\mu \in \mathfrak{Q}$, then $\mu \leq \mu * \top$, and for all $g \in \mathfrak{Q} \otimes \mathbb{R}(\mathfrak{Q})^\dagger$ the following chain of equivalences hold:

$$\mu \otimes (\mu * \top) \leq^\dagger g \iff \mu * \top \leq^\dagger g(\mu) \iff g(\mu) \leq \mu * \top.$$

Thus the explicit form of f is given by $f(\alpha) = \alpha * \top$ for all $\alpha \in \mathfrak{Q}$, and the characteristic morphism of the (t, Ω)-classifiable subobject $\downarrow f$ has the following form:

$$\begin{aligned}\chi(\alpha \otimes \beta) &= \mathrm{hom}_{\mathfrak{Q}\otimes\mathbb{R}(\mathfrak{Q})^\dagger}(f, \alpha \otimes \beta) = \bigvee\{\, \gamma \in \mathfrak{Q} \mid \beta \leq^\dagger f(\alpha * \gamma)\,\} \\ &= \bigvee\{\, \gamma \in \mathfrak{Q} \mid \alpha * \gamma * \top \leq \beta\,\} = \bigvee\{\, \gamma \in \mathfrak{Q} \mid \alpha * \gamma \leq \beta\,\} = \alpha \searrow \beta.\end{aligned}$$

Hence $\chi = \mathrm{hom}_{\mathfrak{Q}\otimes\mathbb{R}(\mathfrak{Q})^\dagger}(f, \_)$ coincides with the join-preserving extension $\varphi_\searrow$ of the right implication $\mathfrak{Q} \times \mathbb{R}(\mathfrak{Q})^\dagger \xrightarrow{\searrow} \mathbb{R}(\mathfrak{Q})^\dagger$ viewed as bimorphism. In this sense we consider the characteristic morphism χ of $\downarrow f$ as the *implication* in $\mathsf{Mod}_r(\mathfrak{Q})$.

Let c be the symmetry in Sup, and let us consider the restriction of the quantale multiplication in its second factor to $\mathbb{R}(\mathfrak{Q})$. Then the right adjoint of the implication $\mathbb{R}(\mathfrak{Q}) \xrightarrow{\chi^\vdash} [\mathfrak{Q}, \mathbb{R}(\mathfrak{Q})]$ coincides with the monoidal adjoint of $\mathbb{R}(\mathfrak{Q}) \otimes \mathfrak{Q} \xrightarrow{c_{\mathbb{R}(\mathfrak{Q})\mathfrak{Q}}} \mathfrak{Q} \otimes \mathbb{R}(\mathfrak{Q}) \xrightarrow{m} \mathbb{R}(\mathfrak{Q})$. This observation underlines the close relationship between the implication in $\mathsf{Mod}_r(\mathfrak{Q})$ and the given quantale multiplication in $\mathfrak{Q}$.

5. Weak power object

If a symmetric and monoidal closed structure is imposed on a finitely complete category C with a weak subobject classifier, then we can always have weak power objects in the following sense. Let $\otimes$ be the tensor product in C, and $[X, \_]$ be the right adjoint functor of $\_ \otimes X$ for every object X. Further, let (t, Ω) be the weak subobject classifier. Now we can pull back the evaluation arrow $[X, \Omega] \otimes X \xrightarrow{ev_\Omega} \Omega$ along the arrow true t and obtain a (t, Ω)-classifiable monomorphism $\epsilon_X \overset{\varepsilon}{\rightarrowtail} [X, \Omega] \otimes X$. Then for every (t, Ω)-classifiable monomorphism $R \overset{r}{\rightarrowtail} Y \otimes X$ there exists a unique morphism $Y \xrightarrow{f_r} [X, \Omega]$

such that the following diagram is a pullback square:

$$\begin{tikzcd} R \arrow[r, tail, "r"] \arrow[d] & Y \otimes X \arrow[d, dotted, "f_r \otimes 1_X"] \\ \epsilon_X \arrow[r, tail, "\varepsilon"] & {[X,\Omega] \otimes X} \end{tikzcd} \tag{5.1}$$

Since $[X, \Omega]$ is uniquely determined up to an isomorphism by the pullback (5.1), we also call $[X, \Omega]$ the *weak power object* of X and the subobject $(\varepsilon_X, \varepsilon)$ the *element relation* in $[X, \Omega] \otimes X$.

Let $\mathbb{1}$ be the unit object of the tensor product $\otimes$. Then it is not difficult to show that the weak power object $[\mathbb{1}, \Omega]$ is isomorphic to the weak subobject classifier Ω.

It is also convenient to recall the concept of *naming arrows* in the context of symmetric monoidal closed categories (cf. [3, page 78]). Let $X \xrightarrow{f} Y$ be an arrow and $\mathbb{1} \otimes X \xrightarrow{\ell_X} X$ be the X-component of the natural isomorphism $\mathbb{1} \otimes \_ \xrightarrow{\ell} \mathrm{id}_{\mathsf{C}}$. Then the monoidal adjoint $\ulcorner f \circ \ell_X \urcorner$ of $f \circ \ell_X$ is called the *name* of f and is denoted by $\ulcorner f \urcorner$.

Further, let T be the terminal object in C. If C is an epi-mono-category and the unique arrow $\mathbb{1} \xrightarrow{!_{\mathbb{1}}} T$ is an epimorphism, then the universal quantifier exists in the following sense. Let $\ulcorner \mathrm{true}_X \urcorner$ be the name of true_X. Then the commutativity of the diagram

$$\begin{tikzcd} \mathbb{1} \otimes X \arrow[d, "!_{\mathbb{1}} \otimes 1_X"'] \arrow[rrd, "\mathrm{true}_X \circ \ell_X"] & & \\ T \otimes X \arrow[r, "!_{T \otimes X}"] & T \arrow[r, tail, "t"] & \Omega \end{tikzcd}$$

implies the decomposition $\ulcorner \mathrm{true}_X \urcorner = \ulcorner (t \circ !_{T \otimes X}) \urcorner \circ !_{\mathbb{1}}$. Hence the image of $\ulcorner \mathrm{true}_X \urcorner$ is the global point $T \xrightarrow{\ulcorner t \circ !_{T \otimes X} \urcorner} [X, \Omega]$ and is thus (t, Ω)-classifiable according to (WS1). The characteristic morphism of the image of $\ulcorner \mathrm{true}_X \urcorner$ is the *universal quantifier* of X, which we denote by $\forall_X$. Since in general the tensor product does not have projections, we leave the construction of the existential quantifier as an open question.

In the following we briefly sketch the situation in the category of modules on a unital quantale $\mathfrak{Q}$. In order to have a symmetric monoidal closed structure we have to assume that $\mathfrak{Q}$ is *commutative* (cf. [4]). The complete lattice $[X, Y]$ of all $\mathfrak{Q}$-module morphisms $(X, \odot) \xrightarrow{f} (Y, \odot)$ is a $\mathfrak{Q}$-module provided with the pointwise defined action. Then for $(x, y) \in X \times Y$ the elementary tensor $x \otimes_{\mathfrak{Q}} y$ is determined by

$$x \otimes_{\mathfrak{Q}} y = \bigwedge\{\, f \in [X, Y^{\dagger}] \mid x \otimes y \le f \,\}$$

where $\otimes$ is the tensor product in Sup. Hence the action on $x \otimes_{\mathfrak{Q}} y$ has the form

$$\alpha \odot (x \otimes_{\mathfrak{Q}} y) = \bigwedge\{\, f \in [X, Y^{\dagger}] \mid x \otimes_{\mathfrak{Q}} y \le \alpha \odot^{\dagger} f \,\}, \qquad \alpha \in \mathfrak{Q},$$

and the well known relation $(\alpha \odot x) \otimes_{\mathfrak{Q}} y = x \otimes_{\mathfrak{Q}} (\alpha \odot y) = \alpha \odot (x \otimes_{\mathfrak{Q}} y)$ follows from Proposition 3.3. In this context, we recall that the category $\mathsf{Mod}(\mathfrak{Q})$ of $\mathfrak{Q}$-modules is symmetric and monoidal closed w.r.t. $\otimes_{\mathfrak{Q}}$ (cf. [4, 2]).

Since $\mathfrak{Q}$ is commutative, the weak subobject classifier in $\mathsf{Mod}(\mathfrak{Q})$ is given by the dual $\mathfrak{Q}$-module $\mathbb{I}(\mathfrak{Q})^{\dagger}$ of all two-sided elements of $\mathfrak{Q}$. Then the weak power object of a $\mathfrak{Q}$-module X is the $\mathfrak{Q}$-module $[X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$ of all characteristic morphisms of X. In this situation we point out that the $\mathfrak{Q}$-module $\mathrm{HOM}(X, \mathbb{I}(\mathfrak{Q}^{\dagger}))$ coincides with the tensor product $X \otimes_{\mathfrak{Q}} \mathbb{I}(\mathfrak{Q})$.

Since, for commutative quantales, the complete sublattice $\mathbb{W}(X)$ of all well-sided elements of X is a $\mathfrak{Q}$-submodule of X, we can express Proposition 4.7 in this context as follows. Let hom_X be the hom-object assignment associated with X and $\mathbb{W}(X)^{\dagger}$ be the dual $\mathfrak{Q}$-module of $\mathbb{W}(X)$. Then there exists a $\mathfrak{Q}$-module isomorphism $\mathbb{W}(X)^{\dagger} \xrightarrow{\eta_X} [X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$ determined by

$$\eta_X(x) = \mathrm{hom}_X(x, \_), \qquad x \in \mathbb{W}(X).$$

Hence we can also identify $\mathbb{W}(X)^{\dagger}$ with the weak power object of X. In particular $\mathbb{W}(X) \cong X \otimes_{\mathfrak{Q}} \mathbb{I}(X)$.

Moreover, by abuse of notation let us denote the restriction of hom_X to $\mathbb{W}(X)^{\dagger} \times X$ again by hom_X. Since hom_X is a bimorphism (cf. (3.6) and

(3.7)), the evaluation arrow $[X, \mathbb{I}^{\dagger}(\mathfrak{Q})] \otimes X \xrightarrow{ev_{\mathbb{I}(\mathfrak{Q})^{\dagger}}} \mathbb{I}(\mathfrak{Q})^{\dagger}$ is the unique extension of $\hom_X$ to a $\mathfrak{Q}$-module morphism $\mathbb{W}(X)^{\dagger} \otimes_{\mathfrak{Q}} X \xrightarrow{\varphi} \mathbb{I}(\mathfrak{Q})^{\dagger}$ making the following diagram commutative:

$$\begin{tikzcd}
\mathbb{W}(X)^{\dagger} \times X \arrow[r, "\otimes_{\mathfrak{Q}}"] \arrow[dr, "\hom_X"'] & \mathbb{W}(X)^{\dagger} \otimes_{\mathfrak{Q}} X \arrow[d, dotted, "\varphi \cong ev_{\mathbb{I}(\mathfrak{Q})^{\dagger}}"] \\
& \mathbb{I}(\mathfrak{Q})^{\dagger}
\end{tikzcd}$$

Hence the (t, Ω)-classifiable subobject of the element relation ϵ_X is given by

$$\varepsilon_X = \downarrow f \quad \text{where} \quad f = \bigvee \{ x \otimes_{\mathfrak{Q}} y \mid (x, y) \in \mathbb{W}(X)^{\dagger} \times X,\ \hom_X(x, y) = \top \}.$$

Finally, we recall that the underlying, commutative and unital quantale $\mathfrak{Q}$ viewed as $\mathfrak{Q}$-module is the unit object of $\mathsf{Mod}(\mathfrak{Q})$ (cf. [4, 2]). It is easily seen that the unique arrow $\mathfrak{Q} \xrightarrow{!_{\mathfrak{Q}}} T$ is an epimorphism. Hence for every $\mathfrak{Q}$-module the universal quantifier $\forall_X$ exists.

Proposition 5.1 *Let X be a $\mathfrak{Q}$-module and $\mathbb{W}(X)^{\dagger} \xrightarrow{\eta_X} [X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$ be the isomorphism identifying characteristic morphisms with well-sided elements. If $\hom_X$ is the hom-object assignment associated with X, then the universal quantifier of X has the form*:

$$\forall_X(\eta_X(x)) = \hom_X(x, \top), \qquad x \in \mathbb{W}(X)^{\dagger}.$$

Proof. Let us recall that the universal lower bound in $[X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$ is the constant characteristic morphism of X attaining $\top$ for all $x \in X$ — i.e. true_X. Then the image of the name $\ulcorner \mathrm{true}_X \urcorner$ coincides with the unique global point $T \rightarrowtail [X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$. In order to compute the corresponding characteristic morphism of the global point $T \rightarrowtail [X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$ we have to associate the hom-object assignment with the $\mathfrak{Q}$-module $[X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$. Referring to Section 3 it is important to understand that in this context we always have to read a $\mathfrak{Q}$-module as a *right* $\mathfrak{Q}$-module. Therefore, if $\varphi, \psi \in [X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$, the hom-object assignment of $[X, \mathbb{I}(\mathfrak{Q})^{\dagger}]$ is determined by:

$$\hom_{[X, \mathbb{I}(\mathfrak{Q})^{\dagger}]}(\varphi, \psi) = \bigvee \{ \alpha \in \mathfrak{Q} \mid \alpha \searrow \psi \leq^{\dagger} \varphi \} = \bigwedge_{z \in X} \big(\varphi(x) \searrow \psi(x) \big).$$

Now we choose $x \in \mathbb{W}(X)^{\dagger}$. Referring to Remark 4.3 we obtain for $\varphi = \eta_X(x)$:

$$\begin{aligned}\forall_X(\eta_X(x)) &= \bigwedge_{z\in X} (\top \searrow \eta_X(x)(z)) = \bigwedge_{z\in X} \eta_X(x)(z) \\ &= \bigwedge_{z\in X} \hom_X(x,z) = \hom_X(x,\top).\end{aligned}$$

Hence the assertion is verified. □

If we understand the universal upper bound in X as *true*, then the universal quantifier applied to the (t,Ω)-classifiable subobject $\downarrow x$ corresponding to $\eta_X(x)$ can be interpreted as the extent to which x is true.

In this sense there exists a close relationship between hom-object assignments of $\mathfrak{Q}$-modules and truth arrows in $\mathsf{Mod}(\mathfrak{Q})$.

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A CONSTRUCTIVE ACCOUNT OF THE KAN–QUILLEN MODEL STRUCTURE AND OF KAN'S Ex^{∞} FUNCTOR

Simon Henry

Résumé. Nous donnons une preuve constructive de l'existence d'une structure de catégorie de modèles cartésienne fermée et propre sur la catégorie des ensembles simpliciaux, dont les cofibrations génératrices sont les inclusions de bords et les cofibrations triviales génératrices les inclusion de cornets. La différence principale avec l'approche classique est que toutes les inclusions ne sont pas des cofibrations (seulement celles satisfaisant certaines conditions de décidabilités) et tous les objets ne sont pas cofibrants.
La preuve repose sur trois ingrédients principaux: D'abord, l'existence d'une structure de catégorie de modèles faible sur les ensembles simpliciaux, ensuite l'interaction avec la version semi-simpliciale de cette structure et enfin l'utilisation du foncteur Ex^{∞} de Kan, et plus précisement de la preuve directe de S.Moss que l'application $X \to \mathrm{Ex}^{\infty} X$ est une cofibration anodyne, dont nous montrons qu'elle est constructive si on suppose que X est cofibrant.

Abstract. We give a fully constructive proof that there is a proper cartesian ω-combinatorial model structure on the category of simplicial sets, whose generating cofibrations and trivial cofibrations are the usual boundary inclusion and horn inclusion. The main difference with classical mathematics is that constructively not all monomorphisms are cofibrations (only those satisfying some decidability conditions) and not every object is cofibrant.
The proof relies on three main ingredients: First, our construction of a weak model categories on simplicial sets, then the interplay with the semi-simplicial versions of this weak model structure and finally, the use of Kan Ex^{∞}-

functor, and more precisely of S.Moss' direct proof that the natural map $X \to \mathrm{Ex}^{\infty} X$ is an anodyne cofibration, which we show is constructive when X is cofibrant.

Keywords. Model categories, constructive mathematics, simplicial sets, Ex^{∞}-functor.
Mathematics Subject Classification (2020). 03F55, 55U35,55U40,18N50.

Contents

1. Introduction

The goal of this paper is to give a fully constructive proof of the existence of the usual Kan–Quillen model structure on simplicial sets, and of some of its classical properties. "Constructive" here can be taken to mean "Without the axiom of choice and the law of excluded middle", or a bit more precisely as "in the internal logic of an elementary topos with a natural number objects".

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It can also be formalized in Aczel's (CZF) [1] and probably in considerably weaker foundations as well, see Remark 1.6. Our main theorem is:

1.1 Theorem. *There is a proper cartesian Quillen model structure on the category of simplicial sets such that:*

- *The trivial fibrations are the morphisms with the right lifting property against all boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$.*
- *Cofibrations are the monomorphisms $f : A \to B$ which are "level wise complemented" (i.e. for all integers n for each $b \in B([n])$ it is decidable if $b \in A([n])$ or not), and such that for all $b \in B([n]) - A([n])$ it is decidable if b is a degenerate cell or not.*
- *The fibrations are the "Kan fibrations", i.e. they are the morphisms with the right lifting property against the horn inclusion: $\Lambda^k[n] \hookrightarrow \Delta[n]$. Dually trivial cofibrations are the retract of ω-transfinite compositions of pushouts of coproducts of horn inclusions.*

Note that assuming the law of excluded middle the class of cofibrations boils down to the class of all monomorphisms and hence one recovers the usual Kan–Quillen model structure.
After we announced this result, two other proofs, have been found by N. Gambino, C. Sattler and K. Szumilo and appeared in [6].

This theorem is obtained by patching together the following results: Theorem 2.2.9 gives the existence of a model structure with the appropriate cofibrations and trivial fibrations, Proposition 2.2.10 gives left properness, Proposition 3.5.1 shows that the fibrations and trivial cofibrations are indeed as specified here and Proposition 3.5.2 shows that it is also right proper. Cartesianness was already known, but reproved as Proposition 3.2.6.

One can also say a few words about the equivalences of the model structure of Theorem 1.1: they are defined (as Definition 2.2.3) using the forgetful functor to semi-simplicial sets and the weak model structure on semi-simplicial sets constructed in Theorem 5.5.6 of [8]. Concretely, this means that a map between Kan complexes is a weak equivalence if it admit an inverse up to homotopy as a semi-simplicial maps. For general map, we need to first take fibrant replacement and then use the previous definition. Note that Proposition 2.2.2 shows that this notion of equivalence is compatible with the notion we used in [8]. Moreover, Proposition 5.2.6 of [8] shows that weak equivalences admit the usual characterization in terms of homotopy groups, as long as the homotopy groups are defined not as quotient sets but as *setoids*.

As we do not assume the axiom of choice, one needs to make precise some details regarding Theorem 1.1: a "structure of fibration" (resp. trivial fibration) on a map f is the choice of a solution to each lifting problem of a horn inclusion (resp. boundary inclusion) against f. No uniformity condition is required on these lifts. A fibration (resp. trivial fibration) is a morphism which admits at least one structure of fibration (resp. trivial fibration), but the choice of the structure is considered irrelevant.
More generally, we will follow the convention that (unless exceptionally stated otherwise) every statement of the form $\forall a, \exists b$ should be interpreted as the existence of a function that given "a" produces a "b". In particular, when one says that a morphism has the lifting property against some set of arrows it means that one has a function that produces a solution to each lifting problem. We will use the convention constantly in the present paper, i.e. every time we say that "there exists" some x, we mean that one specific x has been chosen for each possible value of the parameters involved in the statement.

As fibrations and trivial fibrations are defined by the right lifting property against a small set of morphisms between finitely presented objects, it is very easy to apply a constructive version of the small object argument to show that one has two weak factorization systems, which will be called as follows:

1.2 Definition.

- The weak factorization system cofibrantly generated by the boundary inclusion $\partial\Delta[n] \hookrightarrow \Delta[n]$ is called "cofibrations/trivial fibrations".
- The weak factorization system cofibrantly generated by the horn inclusion $\Lambda^k[n] \hookrightarrow \Delta[n]$ is called "Anodyne cofibrations/Kan fibrations".

We have discussed the constructive validity of the small object argument in appendix C of [8], though there are probably other references doing this.
Note that anodyne cofibrations will in the end be the trivial cofibrations, and Kan fibrations will be what we have called fibrations in the statement of the main Theorem 1.1, but this will be one of the last results we will prove. In the meantime we will distinguish between Kan fibrations and "strong fibrations" and between anodyne cofibrations and "trivial cofibrations" (these two other concepts being defined as Definition 2.2.3). Simplicial sets whose map to the terminal simplicial set is a Kan fibration will be called either Kan complexes, or fibrant simplicial sets.

1.3 Remark. Before going any further, we should pause here to insist on a very important remark: one of the key differences between what we are doing in the present paper and the usual construction of the Kan–Quillen model structure in classical mathematics is that the cofibrations are no longer exactly the monomorphisms. It can be shown, see for example Proposition 5.1.4 in [8], that the class of cofibrations generated by the boundary inclusion, i.e. the class of arrow which have the left lifting property against all trivial fibration is exactly the class of cofibrations described in the statement of Theorem 1.1. In particular one has:

Not every simplicial set is cofibrant ! A simplicial set X is cofibrant if and only if it is decidable whether a cell of X is degenerate or not.

This introduces some changes compared to the classical situation, for example the left properness of the model structure on simplicial sets is no longer automatic, and the assumption that certain objects need to be cofibrant tends to appear in a lot of results. Compare for example Corollary 3.3.4, Proposition 3.3.5 and Proposition 3.4.1 to their classical counterparts.
One can also show the classical Eilenberg-Zilber lemma, asserting that a cell $x \in X([n])$ can be written uniquely as $\sigma^* y$ for σ a degeneracy and y a non-degenerate cells holds if and only if X is cofibrant. A general constructive version of the Eilenberg-Zilber lemma can be found as Lemma 5.1.2 in [8] and does implies that the statement above holds for cofibrant simplicial sets. The converse (that the validity of the Eilenberg-Zilber lemma implies

cofibrancy of X) is immediate from the decidability of equality between morphisms of the category Δ: if a cell is written $\sigma^* y$ with y non-degenerate one can decide if it is degenerate or not depending on if σ is the identity (an isomorphism) or not.

The general structure of the proof of this theorem (and in fact of the paper) is as follows:

- In Section 2.1 we review the existence of a "weak model structure" on simplicial sets and semi-simplicial sets from [8], which is our starting point.
- In Section 2.2, more precisely in Theorem 2.2.9, we will (up to a technical detail, see the Remark 1.4 below) extend this to a model structure on the category of simplicial sets with cofibrations (and trivial fibrations) as specified above, but we will not show that trivial cofibrations are the same as anodyne cofibrations, or equivalently that the fibrations (called "strong fibrations") are the Kan fibrations. This part is based on the use of semi-simplicial sets.
- Left properness of this model structure also follows from semi-simplicial techniques (see Proposition 2.2.10).
- The overall goal[1] of Section 3 is to introduce Kan's Ex^{∞}-functor. This is done following the work of S. Moss from [15], which can be made constructive at the cost of only minor modification. This will allow us to show that the fibrations of the model structure above are exactly the Kan fibrations (Proposition 3.5.1) and to prove the right properness of this model structure (Proposition 3.5.2), as well as to fix a small gap in constructiveness of Section 2.2 (see the remark below).

1.4 Remark. The gap we are referring too in this last point is that in Section 2.2, the "strong fibrations" (i.e. the fibrations of the model structure on simplicial sets) are defined as the map having the right lifting property against all cofibrations which are equivalences. It is unclear if they can be

[1] We will give a more detailed account of its contents at the beginning of this section.

defined by a lifting property against a small set and hence if trivial cofibration/strong cofibration do form a weak factorization system as a model category structure should require. In Proposition 2.2.7 we give a formal argument that shows it is the case, but it is unlikely that this argument can be made constructive. What definitely solve the problem constructively is the proof in Proposition 3.5.1 that this factorization is actually just the "anodyne cofibrations/Kan fibrations" factorization, but this require all the material of Section 3.
This being said, the reader should note that even before Section 3, it holds constructively that the factorization as an anodyne cofibration followed by a Kan fibration of an arrow with fibrant target is a "trivial cofibration/strong fibration" factorization (because of the third point of Lemma 2.2.6). Hence it holds constructively, even without the results of Section 3, that any arrow with fibrant target admits such a factorization, i.e. one already has something like a right[2]) semi-model category without invoking the properties of Kan Ex^{∞} functor.

1.5 Remark. The fact that we need to invoke the good properties of Kan's Ex^{∞} functor to show that the class of fibration is indeed the class of Kan fibrations of course remind us of D-C.Cisinski's approach to the construction of Kan–Quillen model structure in [3]. We do not really know how deep are the similarities between our proof and D-C.Cisinski's proof. Our initial plan on this problem was actually to try to see if this approach of Cisinski can be made constructive or not. While we definitely do not exclude the possibility that this is the case, it seemed to represent a considerably harder task than what we have achieved here. One of the problems is that Cisinski's theory relies heavily on a set theoretical argument similar to the one we mention in the proof of Proposition 2.2.7, whose constructiveness seems unlikely, and we have not been able to separate his proof that fibrations are the Kan fibrations from this set theoretic argument. The other problem being simply that Cisinski's approach, while very elegant, relies on a considerable amount of machinery whose constructivity would have to be carefully checked, while S. Moss approach, while more technical is considerably more self-contained.

[2]More precisely, we have a right semi-model structure in the sense of Fresse from [4], but not in the sense of Spitzweck from [17].

1.6 Remark. Finally, I only said that "constructive" meant something like internal logic of an elementary topos with a natural number object for simplicity, but everything is actually completely predicative for some, relatively strong, sense of this word. I believe that everything can be formalized within the internal logic of an "Arithmetic universe", i.e. a pretopos with parametrized list objects (see for example [13]). Such a formalization of course requires some modifications: for example it wouldn't make sense to say that a morphism "is a fibration" in the sense that "there exists a structure of fibrations on the morphisms" as the set of all "structure of fibration" on a given morphism cannot be defined, but it would make sense to consider a morphism endowed with a structure of fibration, and to show that given such a pair one can perform some construction.
Although working in such framework in an explicit way forces us to be extremely careful about a huge number of details and makes everything considerably more complicated, and would make the paper considerably longer. For this reason we will not do it explicitly. It seems to me that this is typically the sort of thing that should be done with a proof assistant.
There is one part of this claim that I have not checked carefully: Whether such a weak framework is sufficient to use the case of the small object argument that we need, i.e. construct the cofibration/trivial fibration and the anodyne cofibration/Kan fibration factorization systems (generated respectively by boundary inclusion and horn inclusion) on simplicial set and semi-simplicial sets, though it seems reasonable that a complicated encoding using list object can achieve this. More precisely this should follow from the fact that the initial model theorem for partial horn theories of Vickers and Palmgren in [16] is believed to be provable internally in an arithmetic universe, and the factorization obtained from R.Garner's version of the small object argument (from [7]) are constructed as certain initial structure that can be described using partial horn logic.

1.7 Remark. In a joint paper with Nicola Gambino [5], we will show that this Quillen model structure on simplicial sets admit all the necessary structure to interpret homotopy type theory, with type and context being interpreted as bifibrant objects. This was the main motivation for the present paper and the two papers have been written in close connection. I would also like to thanks Nicola Gambino for the helpful comments he made about earlier versions of the present paper.

1.8 Notation. Δ and Δ_+ denotes the category of finite non-empty ordinal, respectively with non-decreasing map and non-decreasing injection between them. $\widehat{\Delta}$ is the category of simplicial sets and $\widehat{\Delta_+}$ is the category of semi-simplicial sets (see 2.1.2). One denotes by $\Delta[n]$ and $\Delta_+[n]$ the representable simplicial and semi-simplicial sets corresponding to the ordinal $[n] = \{0, \dots, n\}$. Our usual notation for the boundary of the n-simplex and its k-th horn, both for simplicial and semi-simplicial versions are: $\partial\Delta[n] \quad \Lambda^k[n] \quad \partial\Delta_+[n] \quad \Lambda^k_+[n]$
The boundary inclusion map is denotes ∂_n or $\partial[n] : \partial\Delta[n] \to \Delta[n]$, the i-th face map is denoted $\partial^i[n]$ or ∂^i_n or just $\partial^i : \Delta[n-1] \to \Delta[n]$, for the map corresponding to the order preserving injection from $[n-1]$ to $[n]$ which only skip i. The degeneracy $\Delta[n+1] \to \Delta[n]$ that hits i twice is denoted σ^i. Given a simplicial or semi-simplicial sets X, the image of a cell $x \in X_n$ be the i-th face map is denoted $d_i x$.

1.9 Notation. Finally, we will define many different classes of maps between simplicial and semi-simplicial sets. To help the reader navigate this, we list them all here and recall their definition. This is not meant to be read at this point, but used as a reference latter if the reader needs to remember what a certain class of maps is. In particular, many of the claim we make here will be properly justified latter in the paper.
In the category $\widehat{\Delta}$ of simplicial sets, we consider the following classes of maps:

- *Trivial fibrations* are the map with the right lifting property against the boundary inclusions $\partial\Delta[n] \to \Delta[n]$.
- *Cofibrations* are the map with the left lifting property against trivial fibration. They are also the retracts of ω-transfinite compositions of pushouts of coproducts of boundary inclusions. It is shown as proposition 5.1.4 of [8] that cofibrations can be characterized as inclusion satisfying some decidability conditions as stated in Theorem 1.1.
- *Kan fibrations* are the map with the right lifting property against the horn inclusion $\Lambda^k[n] \hookrightarrow \Delta[n]$.
- *Anodyne cofibrations* are the map with the left lifting property against Kan fibrations. Equivalently, they are the retract of ω-transfinite compositions of pushouts of coproducts of horn inclusions.

- *weak equivalences* where defined for arrow between objects that are either fibrant or cofibrant in [8] as maps that are invertible in the homotopy category (the homotopy category being defined using homotopy class of maps between bifibrant objects). In the present paper we extend the definition to general objects by redefining weak equivalences of simplicial sets as the map that are weak equivalences of the underlying semi-simplicial sets. The usual characterization using homotopy groups can also be used as long as homotopy groups are defined as setoids, see proposition 5.2.6 of [8].

- *trivial cofibrations* are the maps that are both cofibrations and weak equivalences. We show that trivial cofibrations and anodyne cofibrations are the same in Proposition 3.5.1, but this is one of the last result of the paper, so almost everywhere in the paper these class are assumed to be potentially different.

- *Strong fibrations* are the map that have the right lifting property agains trivial cofibrations. It also follows from Proposition 3.5.1 that they are the same as Kan fibration.

- *Degeneracy quotient* and *degeneracy detecting maps* is a unique factorization system on $\widehat{\Delta}$ which is studied in Section 3.1. It mostly serves as a technical tool to establish decidability conditions that are central to make the proofs in Section 3.4 constructive.

In the category $\widehat{\Delta_+}$ of semi-ssimplicial sets, we consider the following classes of maps:

- *Trivial fibrations* and *Kan fibrations* are defined as the maps with the right lifting property against respectively the semi-simplicial boundary inclusion $\partial\Delta_+[n] \to \Delta_+[n]$ and the semi-simplicial horn inclusion $\Lambda^k_+[n] \hookrightarrow \Delta_+[n]$.

- *Cofibrations* and *anodyne cofibrations* are defined as the map with the left lifting property against respectively *trivial fibrations* and *Kan fibrations*. They can also be described as the maps that are retracts of ω-transfinite compositions of pushouts of coproducts of respectively the boundary inclusion and the horn inclusion. Semi-simplicial cofibrations have been shown in Section 5 of [8] to be exactly the inclusion that are levelwise complemented (the precise statement is in Theorem 5.5.6 of [8], the proof is the same as for Proposition 5.1.4).

- *Weak equivalences* are the maps that are invertible in the homotopy category of the weak model structure defined by the maps above. Because every semi-simplicial set is cofibrant, the notion makes sense for arbitrary maps (in a weak model category, only objects which are either fibrant or cofibrant have an image in the homotopy category).
- *Trivial cofibrations* are the cofibrations which are weak equivalences. In $\widehat{\Delta}$ trivial cofibration and anodyne cofibrations are not expected to be the same. Trivial cofibrations have the right lifting property against all Kan fibrations between fibrant objects, but not against general Kan fibrations.
- Of course, one could also define the class of *strong fibrations*, as the maps with the right lifting property against all trivial cofibrations, but the notion turn out to serve no purpose in the present paper.

2. Constructing the model structure

2.1 Review of the weak model structures

2.1.1. One of the achievement of [8], which is the starting point of the present paper, is the construction of a "weak model structure" on the category of simplicial sets where fibrations (between fibrant objects) and cofibrations (between cofibrant objects) are as specified above.
More explicitly this means that there is a class of maps called "equivalences[3]" in the category of simplicial sets that are either fibrant or cofibrant (in the sense above) such that:

- Weak equivalences (between objects that are either fibrants or cofibrant) contains isomorphisms, are stable under composition and satisfies 2-out-of-3 (and the stronger 2-out-of-6 property).

[3]In most of the literature this are called weak equivalence, though we can't think of any reasons to keep the adjective "weak" other than history, so we will simply drop it.

- A cofibration between cofibrant objects is a weak equivalence if and only if it has the left lifting properties against all fibrations between fibrant objects (such a map is called a trivial cofibration).
- A fibrations between fibrant objects is a trivial fibration if and only if it is a weak equivalence[4].
- The localization of the category of fibrant or cofibrant objects at the weak equivalences can be described as the category of fibrant and cofibrant objects with homotopy classes of maps between them. Where the homotopy relation is defined as usual, using equivalently a path object or a cylinder object. This localization is called the homotopy category.
- The weak equivalences are exactly the morphisms that are invertible in the homotopy category (which proves the first point immediately).

One can deduce from this various characterization of weak equivalences: for example, a map from a cofibrant object to a fibrant object is a weak equivalence if and only if it can be factored as a trivial cofibration followed by a trivial fibration. Note that at this point it does not makes sense to ask whether a map $X \to Y$ is a weak equivalence if one of X or Y is neither fibrant nor cofibrant.

2.1.2. In [8, theorem 5.5.6] we also showed that a similar "weak model structure" exists on the category of semi-simplicial sets. Semi-simplicial sets are "simplicial sets without degeneracies", i.e. collection of sets $X_0, \dots, X_n, \dots$ with "face maps" satisfying the same relations as the face maps of a simplicial sets. Equivalently they are presheaves on the category Δ_+ of finite non-empty ordinals and injective order preserving maps between them.
The generating cofibrations in the category of semi-simplicial sets are the semi-simplicial boundary inclusion:

$$\partial\Delta_+[n] \hookrightarrow \Delta_+[n],$$

[4]Here we use the fact that that trivial fibrations are characterized by a lifting property against cofibration between cofibrant objects, which might not be the case in a general weak model category.

where $\partial\Delta_+[n]$ and $\Delta_+[n]$ respectively denotes the semi-simplicial subset of non-degenerate cells in $\Delta[n]$ and $\partial\Delta[n]$. Note that the $\Delta_+[n]$ also corresponds to the representable semi-simplicial sets, so that a morphism $\Delta_+[n] \to X$ is the same as an n-cell of X and a morphism $\partial\Delta_+[n] \to X$ is the data of a collection of n cells of dimension $n-1$ with compatible boundary exactly as simplicial morphisms from $\partial\Delta[n]$ to a simplicial sets X. In particular a morphism $f : X \to Y$ of simplicial sets is a trivial fibration if and only if its image by the forgetful functor to semi-simplicial sets is a trivial fibration (in the sense that it has the right lifting property against the generating cofibration).
As there are no degeneracies in $\widehat{\Delta_+}$ the description of cofibrations simplifies to just "levelwise complemented monomorphism" i.e. the class of monomorphism $f : X \to Y$ such that for each n, and for each $y \in Y([n])$ it is decidable whether $y \in X([n])$ or not (this is also discussed in [8, theorem 5.5.6]). In particular, every semi-simplicial set is cofibrant.
Similarly, a morphism of semi-simplicial sets is said to be a Kan fibration when it has the lifting property against the semi-simplicial version of the horn inclusion $\Lambda^k_+[n] \hookrightarrow \Delta_+[n]$, where $\Lambda^k_+[n]$ and $\Delta_+[n]$ respectively denotes respectively the semi-simplicial sets of non-degenerate cells in $\Lambda^k[n]$ and $\Delta[n]$). As above a simplicial morphism between simplicial sets is a Kan fibration if and only if its image by the forgetful functor to simplicial sets is a Kan fibration of semi-simplicial sets.
In this weak model structure on semi-simplicial sets, the cofibrations are as described above, the fibrant objects are the semi-simplicial Kan complexes and the fibrations and trivial fibrations between fibrant objects are the Kan fibrations and trivial fibrations. The big difference with the model structure on simplicial sets is that as every semi-simplicial set is cofibrant, the classes of weak equivalences is defined between arbitrary objects of the category. Note that we do not claim that every trivial cofibration (i.e. cofibration which is an equivalence) is an anodyne cofibration (i.e. a retract of a transfinite composition of pushout of coproducts of semi-simplicial horn inclusion) : the anodyne cofibration have the left lifting property against all Kan fibrations, the trivial cofibration only against Kan fibration between Kan complexes.

2.1.3 Remark. Note that it is well known, even classically, that this model structure cannot be a Quillen model structure. As every object is cofibrant, it can be seen by a combinatorial argument that, at least classically, it is a "right semi-model structure" in the sense of [2]). But for example the codiagonal map $\Delta_+[0] \coprod \Delta_+[0] \to \Delta_+[0]$, where $\Delta_+[0]$ denotes the representable semi-simplicial sets by the ordinal $[0] = \{0\}$ is easily seen to have the lifting property of trivial fibrations (there is no higher cells to lift !) while it is clearly not a weak equivalence.

The forgetful functor from simplicial sets to semi-simplicial sets is very well behaved: we showed in [8, theorem 5.5.6] that it is both a left and right Quillen equivalence, and we will prove as Proposition 2.2.2 that it preserves and detect weak equivalences without any assumption of fibrancy/cofibrancy. As all object in $\widehat{\Delta_+}$ are cofibrant, this will allow to remove some assumption of cofibrancy in various places.

Sketch of proof of 2.1.1. We finish this section by presenting the main steps of the argument given in [8] of the existence of the weak model structure on simplicial sets, i.e. all the claims made in 2.1.1. The details of this can be found in [8], but we hope the following summary will be of help to the reader. The proof for semi-simplicial sets is similar.
The first (and essentially only) important technical step is the proof of the so-called "pushout-product" or "corner-product" conditions for the simplicial generating cofibrations and trivial cofibrations. This follows from a completely constructive results of Joyal (theorem 3.2.2 of [11]), in [8] it corresponds to Lemma 5.2.2 (and how it is used in the proof of Theorem 5.2.1 in Corollary 5.2.3). In the present paper we also reproduce a different proof of this claim as Proposition 3.2.6, which is due to S. Moss (see [15, 2.12]).
From the corner-product condition one deduces formally[5] all the usual properties of stability of cofibrations, anodyne cofibrations, fibrations, and trivial fibrations under product and exponential expected in a cartesian model category (see Proposition 3.2.6 and the comment directly below it).
This allows to construct nicely behaved cylinder objects as $\Delta[1] \times X$ and path objects as $X^{\Delta[1]}$, whose legs are appropriately (trivial) (co)fibrations as soon as X is (co)fibrant. More generally, one can construct relative path objects for any fibration $X \twoheadrightarrow Y$ and relative cylinder objects for any cofibration $A \hookrightarrow Y$. Having such relative cylinders and path objects is the definition

[5] using the so-called "Joyal-Tierney calculus" presented in the appendix of [10], though this types of manipulation were known before, maybe in a less elegant or general way.

of weak model structure that we gave in section 2 of [8]. The precise observation that one gets a weak model structure from such a tensor product satisfying the corner-product condition is essentially the construction done in section 3 of [8], summarized by theorem 3.2 there.
Then all the claims made in 2.1.1 follows from the general theory of weak model structure developed in section 2.1 and 2.2 of [8]. We sketch the general strategy here, though at this point we recommend looking directly at subsection 2.1 and 2.2 of [8] which are mostly self contained.
One uses these cylinders and path objects to define the homotopy relation between maps from a cofibrant object to a fibrant object. Using the lifting property one show that the homotopy relation with respect to any cylinder object is equivalent to the homotopy relation with respect to any path object and that these define an equivalence relation compatible to pre-composition and post-composition. The proof is essentially the same as in a full Quillen model structure: the definition of weak model structure is exactly tailored so that the usual proof of these claims can be applied.
This allows to give a first definition of the homotopy category as the category whose objects are the fibrant-cofibrant objects and the maps are the homotopy class of maps. One then proves formally that this homotopy category is equivalent to various localization (see Theorem 2.2.6 in [8]), the last one being the localization of the category of simplicial sets that are either fibrant or cofibrant at all trivial cofibrations with cofibrant domains and all trivial fibrations with fibrant targets. One can then define weak equivalences as the arrow that are invertible in this localization, and one automatically have 2-out-of-6 and all the other good properties of weak equivalences. The fact that trivial fibrations with fibrant target are exactly the fibration (with fibrant targets) that are equivalence is a little harder and use again the properties of the relative path objects (see proposition 2.2.10 in [8]), and similarly for cofibrations. □

2.2 The simplicial model structure

To obtain that simplicial sets form a full Quillen model structure we first need to extend the meaning of "equivalences" so that it makes sense also for arrows between objects that are neither fibrant nor cofibrant. We will do this by exploiting the forgetful functor from the category of simplicial sets to the

category $\widehat{\Delta_+}$ of semi-simplicial sets. As in the category of semi-simplicial sets every object is cofibrant the notion of weak equivalence there is defined for arbitrary arrows, and we will show it is reasonable to define equivalences of simplicial sets as arrows that are equivalences of the underlying semi-simplicial sets.
We start by the following observation:

2.2.1 Lemma.

1. *If $f : X \to Y$ is an anodyne cofibration in $\widehat{\Delta}$, then its image in $\widehat{\Delta_+}$ is also an anodyne cofibration, and in particular is an equivalence.*

2. *Let $f : X \twoheadrightarrow Y$ be a trivial fibration in $\widehat{\Delta}$. Then the image of f in $\widehat{\Delta_+}$ is an equivalence.*

Note that in the second case, it is obvious that f is a trivial fibration in $\widehat{\Delta_+}$, but this is not enough to deduce that is is an equivalence in general, unless its target is fibrant, as $\widehat{\Delta_+}$ only has a weak model structure.

Proof.

- This is corollary 5.5.15.(ii) of [8].

- We first assume that X is cofibrant. In this case one can construct a strong cylinder object for X using the cartesian structure of simplicial sets:

  $$X \coprod X \hookrightarrow \Delta[1] \times X \overset{\sim}{\to} X$$

  with the two maps $X \hookrightarrow \Delta[1] \times X$ being anodyne cofibrations (this follows from the fact that X is cofibrant and the corner-product conditions). Because of point 2, this produces a strong cylinder object for the underlying semi-simplicial set of X in the category of semi-simplicial sets.

  In $\widehat{\Delta_+}$, every object is cofibrant, and the arrow $f : X \to Y$ is still a trivial fibration, so one can find some dotted lifting for the following two squares in $\widehat{\Delta_+}$:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] & X \arrow[d, two heads] \\ Y \arrow[r] \arrow[ur, dotted, "s"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \coprod X \arrow[rr, "(Id_X, sf)"] \arrow[d, hook] & & X \arrow[d, two heads] \\ \Delta[1] \times X \arrow[r] \arrow[urr, dotted, "h"] & X \arrow[r] & Y \end{tikzcd}$$

In particular, s is a section of f, i.e. $fs = Id_Y$, and h a homotopy between Id_X and sf. Hence s is an inverse of f in the homotopy category of $\widehat{\Delta_+}$, which makes f an equivalence in $\widehat{\Delta_+}$.

In the general case (when we do not assume that X is cofibrant), one takes a cofibrant replacement (with a trivial cofibration) $X^c \overset{\sim}{\twoheadrightarrow} X$ and the result above applies to both the trivial fibration $X^c \overset{\sim}{\twoheadrightarrow} X$ and the composite trivial fibration $X^c \overset{\sim}{\twoheadrightarrow} Y$. By 2-out-of-3 for weak equivalences in $\widehat{\Delta_+}$ this implies that the map $X \overset{\sim}{\twoheadrightarrow} Y$ is indeed an equivalence in $\widehat{\Delta_+}$.

□

2.2.2 Proposition. *For a morphism $f : X \to Y$ between simplicial sets that are either fibrant or cofibrant the following are equivalent:*

- *f is an equivalence for the weak model structure in $\widehat{\Delta}$.*
- *The image of f in $\widehat{\Delta_+}$ is an equivalence for the weak model structure on $\widehat{\Delta_+}$*

Proof. If Y is cofibrant, then one can take a fibrant replacement $Y \overset{\sim}{\hookrightarrow} Y^f$. The map $Y \overset{\sim}{\hookrightarrow} Y^f$ is an equivalence both in $\widehat{\Delta}$ and $\widehat{\Delta_+}$, so in both categories f is an equivalence if and only if the composite $X \to Y^f$ is an equivalence, so it is enough to prove the result when Y is fibrant. A similar argument using a cofibrant replacement allows to assume that X is cofibrant.
Assuming both X cofibrant and Y fibrant, one factors f as an anodyne cofibration (with cofibrant domain) followed by a Kan fibration (with fibrant target). The anodyne cofibration is an equivalence in both categories, hence (in both categories) f is an equivalence if and only if the Kan fibration part

is a trivial fibration. But for a map in $\widehat{\Delta}$, being a trivial fibration in $\widehat{\Delta}$ and in $\widehat{\Delta_+}$ are the exact same condition (the lifting property only involves face operations, no degeneracies).

□

This last proposition makes the following definition very reasonable:

2.2.3 Definition.

- An arrow in $\widehat{\Delta}$ is said to be an *equivalence* if its image by the forgetful functor to $\widehat{\Delta_+}$ is an equivalence for the semi-simplicial version of the Kan–Quillen weak model structure mentioned in 2.1.2.

- A *trivial cofibration* is a cofibration which is also an equivalence.

- A *strong fibration* is an arrow that has the right lifting property against all trivial cofibrations.

We remind that the reader, that we will prove in Proposition 3.5.1 that these notions of strong fibrations and trivial cofibrations are equivalent to the usual notion of Kan fibrations and anodyne cofibrations.

2.2.4 Remark. With this definition it is immediate that:

- Isomorphisms are equivalences, and equivalences are stable under composition, satisfies the 2-out-of-3 and even the 2-out-of-6 properties.

- Anodyne cofibrations are trivial cofibrations. Indeed they are cofibrations by definition and they are equivalences in the sense of Definition 2.2.3 by point 1 of Lemma 2.2.1.

- As a consequence, strong fibrations are Kan fibrations.

- Trivial fibrations, defined by the right lifting property against boundary inclusions, are both strong fibrations because they have the right lifting property against all cofibrations, and equivalences because of point 2 of Lemma 2.2.1.

- A Kan fibration (or strong fibrations) with fibrant target is a trivial fibration if and only if it is an equivalence (this follows from proposition Proposition 2.2.2 and the fact that this fact holds in weak model categories).

Maybe it is a good point to recall the following very classical lemma that we will use constantly in this paper:

2.2.5 Lemma. *Assume that a map f is factored as $f = pi$. If i has the left lifting property against f, then f is a retract of p. If p has the right lifting property against f then f is a retract of i.*

Proof. We only prove the first half of the claim, the second is just the dual statement. One form a morphism h as the dotted diagonal filler in first square below (obtained by the lifting property of i against f), which can then be used to form a retract diagram:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "i"'] & A \arrow[d, "f"] \\ B \arrow[r, "p"'] \arrow[ur, dotted, "h"] & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] \arrow[rr, bend left, equal] & B \arrow[r, "h"] \arrow[d, "p"] & A \arrow[d, "f"] \\ C \arrow[r, equal] & C \arrow[r, equal] & C \end{tikzcd}$$

□

2.2.6 Lemma.

(i) A cofibration is a trivial cofibration if and only if it has the left lifting property against all Kan fibrations between Kan complexes.

(ii) An arrow whose target is a Kan complex is a trivial cofibration if and only if it is anodyne.

(iii) An arrow whose target is a Kan complex is a strong fibration if and only if it is a Kan fibration.

(iv) A map is a trivial fibration if and only if it is a strong fibration and an equivalence.

Because of the third point it is equivalent for a simplicial set X that $X \to 1$ is a a Kan fibration (i.e. X is a Kan complex) and that $X \to 1$ is a strong fibration. One will simply say that X is fibrant.

Proof.

(i) Let $f : A \hookrightarrow B$ be a cofibration that is also an equivalence, and we consider a lifting problem of f against a Kan fibration between Kan complexes:

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, hook, "f"] & X \arrow[d, two heads, "p"] \\ B \arrow[r, "u"] & Y \end{tikzcd}$$

In the special case where both u and v are equivalences, then by 2-out-of-3, the map p is also an equivalence. As it is a Kan fibration between Kan complexes it is also a trivial fibration, and hence the lifting problem has a solution because f is a cofibration. We will now show that one can bring back the general case to this situation:

One can factor u as an anodyne cofibration followed by a Kan fibration: $B \overset{\sim}{\hookrightarrow} Y' \twoheadrightarrow Y$ and complete the diagram above by forming the pullback $P = Y' \times_Y X$:

$$\begin{tikzcd} A \arrow[r, "v'"] \arrow[d, hook, "f"] & P \arrow[r, two heads] \arrow[d, two heads] & X \arrow[d, two heads] \\ B \arrow[r, hook, "\sim"] & Y' \arrow[r, two heads] & Y \end{tikzcd}$$

The map v' can be factorized as an anodyne cofibration followed by a Kan fibration:

$$
\begin{tikzcd}
A \arrow[r, "\sim"] \arrow[d, hook, "f"] & P' \arrow[r, two heads] & P \arrow[r, two heads] \arrow[d, two heads] & X \arrow[d, two heads] \\
B \arrow[rr, hook, "\sim"] & & Y' \arrow[r, two heads] & Y
\end{tikzcd}
$$

The case treated above, where the two horizontal maps are equivalences, allows to produce a dotted diagonal lifting of the form:

$$
\begin{tikzcd}
A \arrow[r, "\sim"] \arrow[d, hook, "f"] & P' \arrow[r, two heads] \arrow[dr, two heads] & P \arrow[r, two heads] \arrow[d, two heads] & X \arrow[d, two heads] \\
B \arrow[rr, hook, "\sim"] \arrow[ur, dotted] & & Y' \arrow[r, two heads] & Y
\end{tikzcd}
$$

and this concludes the proof in the general case.

Conversely, assume $i : A \hookrightarrow B$ is a cofibration that has the left lifting property against all Kan fibrations between Kan complexes. One needs to show that i is an equivalence. By taking an anodyne cofibration $B \overset{\sim}{\hookrightarrow} B^f$ to a fibrant object the composite $A \hookrightarrow B^f$ still has the announced lifting property so one can freely assume that B is fibrant in order to show that i is an equivalence. Under that assumption one factors i as an anodyne cofibration followed by a Kan fibration, the Kan fibration has a fibrant target so it has the right lifting property against i. Hence by the retract lemma 2.2.5, i is a retract of the anodyne cofibration part of the factorization, hence it is an anodyne cofibration itself, so that we can conclude that i is an equivalence.

(ii) As mentioned in Remark 2.2.4, anodyne cofibrations are trivial cofibrations. So we only need to show the converse. In the proof of point (1), we have shown in the first part that a trivial cofibration have the left lifting property against fibrations between fibrant objects and in the second that any map with this lifting property and whose target is a Kan complex is an anodyne cofibration. Together this indeed shows that a trivial cofibration whose target is a Kan complexe is an anodyne cofibration.

(iii) We have mentioned already that strong fibrations are Kan fibrations, and (i) shows that Kan fibrations between Kan complexes are strong fibrations.

(iv) Trivial fibration have the right lifting property against all cofibrations, in particular against trivial cofibrations hence they are strong fibrations, and point 2 of Lemma 2.2.1 shows they are equivalences. For the other direction, the proof is essentially the dual the proof of (i). Let p be a strong fibration that is also a weak equivalence, and consider a lifting problem of p against a cofibration:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X \arrow[d, two heads, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

By factoring the map $A \to X$ into a cofibration $A \to A'$ followed by a trivial fibrations and taking the pushout of $A \hookrightarrow B$ along this map $A \to A'$ one reduces the problem to the case where the top map is an equivalence. One can then factor the bottom map as a cofibration followed a trivial fibration:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & A' \arrow[rr, "\sim"] \arrow[d, hook] & & X \arrow[d, two heads, "p"] \\ B \arrow[r, hook] \arrow[ur, phantom, "\ulcorner"] & B' \arrow[r, hook] & Y' \arrow[r, two heads, "\sim"] \arrow[ur, dotted] & Y \end{tikzcd}$$

where the dotted arrow exists because the composed cofibration $A \hookrightarrow Y'$ is a weak equivalence by the 2-out-of-3 properties, and hence has the left lifting property against p. This provides a dotted filling for the initial square.

□

In order to conclude that one has a model structure on simplicial sets, one needs one more proposition.

2.2.7 Proposition. *Any morphism can be factored as a trivial cofibration followed by a strong fibration.*

Again, we will show in Proposition 3.5.1 that this factorization system is actually the same as the anodyne cofibrations/Kan fibrations factorization system, i.e. that trivial cofibrations are the same anodyne cofibrations and that strong fibration are the same as Kan fibrations. Note that at this point it is immediate that anodyne cofibrations are trivial cofibrations, and hence that fibrations are Kan fibrations.

Proof. We will give two proofs of this claim. The first one follows from [14], more precisely its Theorem 3.2, which is not known to be constructive but allows to give a simple and direct proof of the present proposition.
In order to fix the issue with constructivity one gives a second, considerably less direct proof: as mentioned above in Proposition 3.5.1 we will prove independently of the present proposition that trivial cofibrations are the same as anodyne cofibrations, hence showing that the weak factorization mentioned in the proposition exists and is simply the anodyne cofibration/Kan fibration weak factorization system (whose existence follows from the small object arguments).
We still give the first proof as we believe it is interesting on its own as it allows to construct the model structure on simplicial sets without needing to invoke Kan Ex^{∞}-functor.
Theorem 3.2 of [14] claims that the 2-category of presentable categories endowed with a class of cellular morphisms generated by a set of morphisms is closed under pseudo-pullback, and that these pullback are constructed explicitly: the underlying category is the pullback of categories, and the class of cellular morphisms are the morphisms whose image in each component are in the specified classes. We apply this to the following square:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\text{Kan-Cplx}, \text{TrivFib}) \arrow[d] \\ (\widehat{\Delta}, Cof) \arrow[r] & (\text{Kan-Cplx}, \text{All arrows}) \end{tikzcd}$$

here "Cof" denotes the class of cofibration in $\widehat{\Delta}$ which is generated by a set. Kan-Cplx denotes the category of "algebraic Kan complexes", i.e. simplicial set endowed with chosen lifting against horn inclusion and of morphisms compatible to these choices of lifting. The functor $\widehat{\Delta} \to$ Kan-Cplx sends any simplicial set to the "free algebraic Kan complexes it generates",i.e. the left adjoint to the forgetful functor from algebraic Kan complex to simplicial set, or equivalently the functor sending a simplicial set to its canonical fibrant replacement as produced by R.Garner version of the small object argument. The class TrivFib is the left class of the weak factorization on Kan-Cplx cofibrantly generated by the image of the horn inclusion in $\widehat{\Delta}$. The right class of the weak factorization system is hence exactly the class of morphisms whose image by the forgetful functor to $\widehat{\Delta}$ are Kan fibrations. It follows that the morphism in $\widehat{\Delta}$ which are sent to "trivial cofibrations" in Kan-Cplx are exactly the arrows that have the left lifting property against all Kan fibration between Kan complexes. Hence in this case the pullback is the category of simplicial sets with as set of cellular morphisms the maps that are both cofibrations and have the left lifting property against Kan fibration between Kan complexes, i.e. the "trivial cofibrations" as defined above, hence this class of arrow is generated by a set, and hence by the small object argument it is one half of a weak factorization system. □

2.2.8 Remark. After writting this paper, the non-constructive argument used in the proof of Proposition 2.2.7 have been considerably generalized in section 4 of [9], leading to the general notion of "left and right saturation" of a combinatorial or accessible pre-model category. This is a special case of left saturation of a combinatorial pre-model category.

2.2.9 Theorem. *There is a model structure on the category of simplicial sets such that:*

- *The equivalences are as in Definition 2.2.3.*
- *The cofibrations and trivial fibrations are the same as in Theorem 1.1.*
- *The fibrations are the strong fibration of Definition 2.2.3.*

Proof. We have two weak factorization systems, trivial cofibrations have been defined as the cofibrations that are equivalences, and it was shown in

Lemma 2.2.6 that trivial fibrations are the (strong) fibrations that are equivalences. Equivalences are stable by composition, satisfies 2-out-of-6 and contains isomorphisms by definition, so this concludes the proof. □

2.2.10 Proposition. *The model structure of Theorem 2.2.9 is left proper, i.e. the pushout of a weak equivalence along a cofibration is a weak equivalence.*

Proof. Given a pushout square in the category of simplicial sets:

$$\begin{tikzcd} A \arrow[r, "\sim"] \arrow[d, hook] & C \arrow[d, hook] \\ B \arrow[r, "f"] & D \end{tikzcd}$$

then as the forgetful functor to semi-simplicial sets preserves all colimits, this square is again a pushout in the category of semi-simplicial sets. In this category every object is cofibrant, and pushout along a cofibration between cofibrant objects is a left Quillen functor hence preserves equivalences between cofibrant objects, hence f is an equivalence in the category of semi-simplicial sets, and hence is an equivalence in $\widehat{\Delta}$ by Definition 2.2.3. □

3. Kan Ex^{∞}-functor

The goal of this section is to introduce Kan's Ex and Ex^{∞} functors and to use them in Section 3.5 to prove the remaining claim concerning the simplicial model structure. Most of the results here were (in their classical form) originally proved by Kan in [12] (often with quite different proof than the ones we will provide here), but we will mostly follow the approach of S.Moss in [15] which we will make constructive by only adjusting some details.

Section 3.1 is a preliminary section that is of some independent interest but which will have only a very marginal role in the paper: it will only be used to prove some decidability conditions (more precisely Lemma 3.4.4, which will be an easy consequence of Lemma 3.1.8 and Proposition 3.1.10). As such it can be easily ignored by the reader.

Section 3.2 review the notion of "P-structure" introduced by S.Moss, which is mostly a language to talk more conveniently about "Strongly anodyne cofibrations", i.e. transfinite composition of pushouts of coproducts of horn inclusion. This is a key tool to structure the proof of the main results of Section 3.4.
Section 3.3 introduce Kan's barycentric subdivision functor SD, its right adjoint EX and Kan's Ex^∞ functor and proves some of their basic properties. This is very classical material that we reproduce here mostly for completeness and to discuss some constructive aspect.
Section 3.4 reproduces (with some modifications to make it constructive) S.Moss' proof in [15] that the natural transformation $X \to \text{Ex}^\infty X$ is an anodyne cofibration. Constructively this only works when X is cofibrant. We also noted that S.Moss proof can be used to obtain a result which apparently was not known even classically: for any morphisms $f : X \to Y$ (with X cofibrant) the natural morphism:

$$X \to \text{Ex}^\infty X \times_{\text{Ex}^\infty Y} Y$$

is anodyne. This was known classically when Y is terminal, or when $X \to Y$ is a fibration, and we will actually only use it in these two special cases, but it appears that they can be proved at the same time using S. Moss' argument.
Finally Section 3.5 uses the properties of this functor to conclude that all Kan fibrations are strong fibrations (Proposition 3.5.1) and that the model structure on simplicial sets is indeed right proper (Proposition 3.5.2).

3.1 Degeneracy quotient and questions of decidability

In this section we establish some general results about a notion of "degeneracy quotient" that we will introduce. While the notion might have some interest on its own in other context its only use in the present paper is to prove some decidability results, which will follow from Lemma 3.1.8 below. In fact, the only uses of this section in the present paper is in the proof of the decidability conditions of Lemma 3.4.4. Proposition 3.1.11 is not useful for the present paper, but will serve in some future work, in particular in [5] and it was more natural to include its proof here.

3.1.1 Definition. A morphism $f : X \to Y$ between simplicial sets is said to be *degeneracy detecting* if:

$$\forall x \in X, \quad f(x) \text{ is a degenerate cell} \Rightarrow x \text{ is a degenerate cell}$$

Of course the converse implication is true for any simplicial map, so one has that x is a degenerate cell if and only if $f(x)$ is. One says that a cell $x \in X_n$ is σ-degenerate for some degeneracy $\sigma : [n] \to [m]$ if $x = \sigma^* y$ for some y.

3.1.2 Lemma. *Let $\sigma : [n] \to [m]$ be any degeneracy and $x \in X_n$ any cell. The following are equivalent:*

(i) x is σ-degenerate.

(ii) For all face maps $i : [k] \to [n]$ such that the composite σi is non-injective, the cell $i^* x$ is degenerate.

Proof. If $x = \sigma^* y$ then for any such i, $i^* x = (\sigma i)^* y$ which is degenerate if σi is non-injective, so $(i) \Rightarrow (ii)$.
Conversely, let x satisfy (ii). If σ is the identity the result is trivial. If σ is not injective, then x is in particular a degenerate cell, i.e. there exist a non-trivial degeneracy $s : [n] \to [k]$ such that $x = s^* y$. Note that $y = d^* x$ for $d : [k] \to [n]$ any section of s. If for all section d of s, σd is injective, then Lemma 3.1.3 below shows that s factors as $j\sigma$ for some degeneracy $j : [m] \to [k]$ and $x = s^* y = \sigma^* j^* y$ is indeed σ-degenerate. If now σd is non-injective for some section d of s, then $y = d^* x$ is a degenerate cell by assumptions, hence one can write $x = s'^* y'$ for y' of lower dimension than x and start the argument above again, an induction on the dimension concludes the proof. $\square$

3.1.3 Lemma. *Let $\sigma : [n] \to [m]$ and $s : [n] \to [k]$ be two degeneracy, assume that for all $d : [k] \to [n]$ a section of s, σd is injective, then there exists a (unique) $j : [m] \to [k]$ such that $s = j\sigma$.*

One easily see it is also a necessary condition.

Proof. One needs to show that, under the assumption of the lemma, for any two elements $i, j \in [n]$ if $\sigma i = \sigma j$ then $si = sj$. If $si \neq sj$, then we can find a section d of s such that $dsi = i$ and $dsj = j$, indeed, in order to get a section of s, we just need to chose for each k the value $d(k) \in s^{-1}\{k\}$.

So as long as $si \neq sj$, we can chose $d(si) = i$ and $d(sj) = j$, and for any other $k \in [m]$, we can, for example, take for $d(k)$ the smalest element of the fiber $s^{-1}\{k\}$. Of course, all this is constructively possible because $[m]$ is a finite decidable set. Given such a section d, we have $\sigma j = (\sigma d)(sj)$ and $\sigma i = (\sigma d)(si)$, so the injectivity of σd implies that $\sigma i \neq \sigma j$. As equality in $[n]$ is decidable one can take the contrapositive and concludes the proof. □

3.1.4 Proposition. *Let $f : X \to Y$ be a map between simplicial sets, then the followings conditions are equivalent:*

(i) f is degeneracy detecting.

(ii) If $f(x)$ is σ-degenerate for some degeneracy σ then x is σ-degenerate as well.

(iii) f has the (unique) right lifting property against all the degeneracy map $\Delta[n] \to \Delta[m]$.

Proof. (ii) clearly implies (i) and the converse is immediate from Lemma 3.1.2. The lifting in (iii) is automatically unique as degeneracy are epimorphisms in the presheaf category and this lifting property is a reformulation of (ii). □

Given a simplicial set X, $x \in X([n])$ and $\sigma : [n] \to [m]$ a degeneracy, one defines $X[(x,\sigma)]$ as the pushout:

$$\begin{tikzcd} \Delta[n] \arrow[r, "x"] \arrow[d, "\sigma"] & X \arrow[d] \\ \Delta[m] \arrow[r] & X[(x,\sigma)] \end{tikzcd}$$

$X[(x,\sigma)]$ is the universal for map $X \to Y$ making x "σ-degenerate", i.e. given a morphism $f : X \to Y$, it factors as $X \to X[(x,\sigma)]$ if and only if $f(x) = \sigma^* y$ for some $y \in Y([m])$, and such a factorization is unique when it exists.

More generally, given a collection $(x_i \in X([n_i]))_{i\in I}$ and $\sigma_i : [n_i] \to [m_i]$ one can define an object $X[(x_i,\sigma_i)]$ as the pushout of a coproduct of degeneracy maps, which has the following universal property: a morphism $f : X \to Y$ factors (uniquely) through $X \to X[(x_i,\sigma_i)]$ if and only if for all $i \in I$, $f(x_i)$ is a σ_i-degenerate cell.

3.1.5 Definition. A morphisms is said to be a degeneracy quotient if it is obtained as $X \to X[(x_i, \sigma_i)]$ for some collection of $x_i \in X([n_i])$ and $\sigma_i : [n_i] \twoheadrightarrow [m_i]$ as above.

3.1.6 Proposition. *Degeneracy quotient and degeneracy detecting maps form an orthogonal factorization system.*
More precisely, for any morphism $f : X \to Y$ its factorization is obtained as:

$$X \to X[(x_i, \sigma_i)] \to Y$$

where (x_i, σ_i) is the collection of all x_i and σ_i such that $f(x_i)$ is a σ_i-degenerate cell.

Note that this is essentially nothing more than the small object argument, though it is notable that in this case it converges in a single step.

Proof. It is clear from the universal property of $X[(x_i, \sigma_i)]$ that one has a factorization as in the lemma, and the first map is by definition a degeneracy quotient. The map $X[(x_i, \sigma_i)] \to Y$ is degeneracy detecting: given $x \in X[(x_i, \sigma_i)]$, it is the image of a $x_0 \in X$, if the image of x is a degenerate cell in Y, one has $f(x_0) = \sigma^* y$, hence (x_0, σ) appears in the definition of $X[(x_i, \sigma_i)]$, which forces the image of x_0, i.e. x, to be degenerate.
The orthogonality of the two class is relatively immediate as well. Given a lifting problem:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & A \arrow[d] \\
X[(x_i, \sigma_i) \arrow[r] \arrow[ur, dotted] & B
\end{tikzcd}$$

where the right map is degeneracy detecting, then a diagonal filling exists if and only the image of the x_i in A satisfies the appropriate degeneracy conditions. As their images in B satisfies them because of the existence of the square, and as the map $A \to B$ is degeneracy detecting, this is immediate. $\square$

The following is more or less a reformulation of what is a degeneracy quotient that will be convenient:

3.1.7 Lemma. *An epimorphism of simplicial sets $p : A \to B$ is a degeneracy quotient if and only if for any map $f : A \to X$, the map f factors through p if and only if the following condition holds:*

$$\forall a \in A([n]) \quad p(a) \textit{ is a degenerate cell } \Rightarrow f(a) \textit{ is a degenerate cell.} \tag{D}$$

Note that if such a factorization exists then condition (D) holds without any assumption on p, so that if p is a degeneracy quotient then a factorization exists if and only condition (D) holds.

Proof. It follows from Lemma 3.1.2, that condition (D) is equivalent to:

$$\forall a \in A([n]) \quad p(a) \text{ is a } \sigma\text{-degenerate cell} \Rightarrow f(a) \text{ is a } \sigma\text{-degenerate cell.} \tag{D'}$$

A factorization of f through p is always unique as p is an epimorphism, so saying that f factors through p if and only if condition (D') (or (D)) holds is equivalent to saying that B (endowed with the map $p : A \to B$) has the universal property of $A[(a_i, \sigma_i)]$ where (a_i, σ_i) are all the pairs of $a_i \in A([n])$ such that $p(a_i)$ is a σ_i-degenerate cell. Hence this indeed holds if and only if $A \to B$ is a degeneracy quotient, as because of Proposition 3.1.6, any degeneracy quotient $p : A \to B$ is isomorphic to $A \to A[(a_i, \sigma_i)]$ where (a_i, σ_i) are all the pairs of $a_i \in A([n])$ such that $p(a_i)$ is a σ_i-degenerate cell. □

This observation has a quite interesting consequence that will be extremely useful to us, and in fact is the unique reason why we are interested in degeneracy quotients in the present paper:

3.1.8 Lemma. *Given $p : A \to B$ a degeneracy quotient of finite decidable simplicial sets, and $f : A \to X$ a morphisms to a cofibrant simplicial set, it is decidable if there exists a diagonal lift:*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \\ B. \arrow[ur, dotted, "?"] & \end{tikzcd}$$

Proof. One can use condition (D) of Lemma 3.1.7 to test whether such a diagonal lift exists. As B is finite and decidable, degeneracy in B is decidable. So for each cell $a \in A$ it is decidable if " $p(a)$ is a degenerate cell $\Rightarrow$ $f(a)$ is a degenerate cell" as both side of the implication are decidable. Moreover this condition is automatically valid for all degenerate cells of A, so it is necessary to test it only on a finite number of cells to know whether f factors through p, which makes the validity of condition (D) decidable and hence the existence of a diagonal lift decidable. $\square$

The following lemma is obvious, but will be a convenient tool to organise the proof that certain maps are degeneracy quotients:

3.1.9 Lemma. *Let $p : A \to B$ be an epimorphism. One considers the equivalence relation $\sim_p$ on A generated by:*

- *If $p(a)$ is a σ-degenerate cell, then $a \sim_p \sigma^* t^* a$ for any section t of σ.*
- *$\sim_p$ is compatible with all the faces and degeneracy maps of A.*

Then p is a degeneracy quotient if and only if any two $a, a' \in A$ such that $pa = pa'$ one has $a \sim_p a'$.

Note that for any morphisms, $a \sim_p a' \Rightarrow pa = pa'$.

Proof. One easily see that $\sim_p$ is exactly the simplicial equivalence relation by which one needs to quotient A to obtain $A[(a_i, \sigma_i)]$ where (a_i, σ_i) is the family of all a_i such that $p(a_i)$ is σ_i degenerate in B. By the second half of Proposition 3.1.6, the map p is a degeneracy quotient if and only if the second maps in the factorization $A \to A[(a_i, \sigma_i)] \to B$ is an isomorphism, which happens if and only if the relation $\sim_p$ is equivalent to $p(a) = p(a')$. $\square$

We continue with a proposition that allows to get many examples of degeneracy quotient (see for example the proof of Lemma 3.4.4).

3.1.10 Proposition. *Let P be a poset with an idempotent order preserving endomorphism π satisfying either $\forall x, \pi x \leqslant x$ or $\forall x, \pi x \geqslant x$. Let $Q = \pi P$. Then the morphism:*

$$N(P) \to N(Q)$$

between the simplicial nerve induced by $\pi : P \to Q$ is a degeneracy quotient.

Proof. We assume that $\pi x \leqslant x$. The other case follows by simply reversing the order relation on P and on all objects of the category Δ. We use Lemma 3.1.9.
Let $p_0 \leqslant p_1 \leqslant \cdots \leqslant p_n$ be an element of $N(P)_n$ and assumes that $p_0, \ldots, p_{i-1} \in Q$, then one forms

$$p_0 \leqslant p_1 \leqslant \cdots \leqslant p_{i-1} \leqslant \pi p_i \leqslant p_i \leqslant \cdots \leqslant p_n$$

It is an element of $N(P)_{n+1}$ whose image in Q is degenerate, as $\sigma^{i*}(\pi p_0 \leqslant \cdots \leqslant \pi p_n)$. This implies that in $N(P)$:

$$(p_0 \leqslant \cdots \leqslant p_n) \sim (p_0 \leqslant \cdots \leqslant p_{i-1} \leqslant \pi p_i \leqslant p_{i+1} \leqslant \cdots \leqslant p_n)$$

In the sense of the equivalence relation of Lemma 3.1.9. Hence using this for all i from 0 to n, one obtains that for any sequence $p_0 \leqslant \cdots \leqslant p_n$ all the

$$(\pi p_0 \leqslant \cdots \leqslant \pi p_{i-1} \leqslant p_i \leqslant \cdots \leqslant p_n)$$

for $i = 0, \ldots, n+1$ are equivalent. In particular any sequence is equivalent to its image by π and finally any two sequences whose image in $N(Q)$ are the same are equivalent.

□

We finish with a proposition that is useful in a related work [5]):

3.1.11 Proposition. *The class of degeneracy quotients is stable under pullback.*

Proof. First we show that given a pullback of the form:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "\phi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta[n] \arrow[d, "\sigma"] \\
\Delta[k] \arrow[r, "f"] & \Delta[m]
\end{tikzcd}$$

where σ is a degeneracy map, the map ϕ is a degeneracy quotient. This is proved using Proposition 3.1.10. Indeed in such a pullback P is nerve of the corresponding pullback of posets, that we will also denote P (because the

nerve functor commutes with pullback). We will show that the map $P \to [k]$ is of the form of Proposition 3.1.10. The map $\sigma : [n] \twoheadrightarrow [m]$ is of this form, with the section $[n] \to [m]$ sending each $i \in [m]$ to the smallest element of the fiber, this gives an order preserving idempotent $\pi : [n] \to [n]$ such that $\pi x \leqslant x$. This induce an idempotent on P sending a pair (i,j) (with $i \in [k]$, $j \in [n]$) to $\pi'(i,j) = (i, \pi j)$. This is still an element of P, $\pi'(i,j) \leqslant (i,j)$ it is idempotent, and its image identifies naturally with $[k]$.
Hence $\phi : P \to \Delta[k]$ is indeed a degeneracy quotient by Proposition 3.1.10.
We now show that given any pullback of the form:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "\phi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta[n] \arrow[d, "\sigma"] \\
X \arrow[r, "f"] & \Delta[m]
\end{tikzcd}$$

for a degeneracy σ, the map ϕ is a degeneracy quotient.
Indeed, one write:

$$X = \operatorname*{Colim}_{\Delta[k] \to X} \Delta[k]$$

Given a $x : \Delta[k] \to X$ one writes P_x the pullback:

$$\begin{tikzcd}
P_x \arrow[r] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[r] \arrow[d, "\phi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta[n] \arrow[d, "\sigma"] \\
\Delta[k] \arrow[r] & X \arrow[r, "f"] & \Delta[m]
\end{tikzcd}$$

All map ϕ_x are degeneracy quotient by the first part of the proof. As the category of simplicial sets is a topos, colimits are universal, hence the morphism ϕ is the colimit of the arrows ϕ_x (in the category of arrows). As the class of degeneracy quotient is the left class of an orthogonal factorization system, the colimit ϕ is also a degeneracy quotient. To give an explicit argument: given a lifting problem of ϕ against a degeneracy detecting map one can construct for each x a lifting:

$$\begin{tikzcd}
P_x \arrow[r] \arrow[d, "\phi_x"] & P \arrow[r] \arrow[d, "\phi"] & A \arrow[d] \\
\Delta[k] \arrow[r] \arrow[urr, dotted] & X \arrow[r] & B
\end{tikzcd}$$

By uniqueness of the lifts, they will all be compatible and produces a morphisms from the colimits to A making the square commutes.

Finally we can prove the claim in the proposition. Given a morphism $f : X \to Y$ any degeneracy map $\Delta[n] \to \Delta[m]$ over Y (i.e with $\delta[m] \to Y$) is sent by the pullback functor $\widehat{\Delta}_{/Y} \to \widehat{\Delta}_{/X}$ to a degeneracy quotient. But a general degeneracy quotient is a pushout of coproduct of degeneracy maps, and this coproduct and pushout are preserved by the pullback functor (because the category of simplicial sets is cartesian closed), and coproduct of pushout of degeneracy quotient are degeneracy quotient so this concludes the proof. □

3.2 P-structures

This section recalls the notion of P-structure introduced in [15] with some minor modification to make it more suitable to the constructive context. A "P-structure" on a morphism $f : A \to B$ is essentially a recipe for constructing it as an iterated pushout of coproduct of horn inclusion $\Lambda^i[n] \hookrightarrow \Delta[n]$. The general idea of this definition is that in such an iterated pushout cells are added by pairs: each pushout by a horn inclusion $\Lambda^i[n] \to \Delta[n]$ adds exactly two non-degenerate cells:

(I) The cell P corresponding to the identity of $\Delta[n]$.

(II) The cell F corresponding to the i-th face $\partial^i[n] : \Delta[n-1] \to \Delta[n]$.

These two cells are connected by $F = d_i P$. So if $A \hookrightarrow B$ is constructed by iterating such pushouts, then one can partition the non-degenerate cells of B that are not in A into "type I" and "type II" and there should be a bijection which associates to any type II cell the type I cell that is added by the same pushout. The formal definition looks like this:

3.2.1 Definition. Let $f : A \to B$ be a cofibration of simplicial sets. A P-structure on f is the data of:

- A (decidable) partition of the set of non-degenerate cells of B which are not in A into:

$$B_{\mathrm{I}} \coprod B_{\mathrm{II}}$$

  called respectively type I cells and type II cells.

- A bijection $P : B_{\mathrm{II}} \overset{\sim}{\to} B_{\mathrm{I}}$.

Such that:

1. For all $x \in B_{\mathrm{II}}$, $dim(Px) = dim(x) + 1$
2. For all $x \in B_{\mathrm{II}}$, there is a unique i such that $d_i(Px) = x$.
3. Every cell of B_{II} has finite P-height (see Definition 3.2.2 and Lemma 3.2.3 below).

Recall that, if $f : A \to B$ is a cofibration, it is decidable whether a cell is in A or not, and for cells not in A it is decidable whether they are degenerate or non-degenerate. So a P-structure gives a partition of the cells of B into for parts: the cells of A, the degenerate cells of B not in A, the type I cells and the type II cells.

In [15], the last condition of this definition was formulated as a well-foundness condition. Well-foundness is a tricky notion constructively so we prefer to avoid it. It should be clear to the reader that the condition we will now explain is equivalent to well-foundness if one assumes classical logic, or if one has a nice enough notion of well-foundness constructively. Intuitively this last condition just asserts that the "recipe" given by the P-structure to construct B from A as an iterated pushout of horn inclusion is indeed well-founded, i.e. can be executed. We will formulate it by introducing for each cell $b \in B$ a set:

$$Ant(b)$$

which corresponds to the set of cells that needs to be constructed before b in the process described by P. In [15] the well-foundness condition is essentially that the order relation generated by $b' \in Ant(b)$ is well-founded. As each $Ant(b)$ is a finite set this is equivalent to the fact that for each b there is an integer k such that when iterating $Ant(b)$ more than k times one has only cells in A. It is this second definition that we will use in our constructive context.

More precisely: Given a cell $b \in B_{\mathrm{II}}$, let i be the unique integer such that $d_i P x = x$, one defines:

$$Ant_0(b) = \{d_j P(b) | j \neq i\}$$

And one defines the set $Ant(b)$ as the union of $Ant_0(b)$ and all (iterated) faces of cells appearing in $Ant_0(b)$.
Similarly, if $b = Pb'$ is type I, one defines:

$$Ant(b) = Ant(b')$$

Finally, if $b \in A$:

$$Ant(b) = \emptyset$$

and if b is not in A but degenerate, then

$$Ant(b) = Ant(b')$$

where b' is the unique non-degenerate cell such that $b = \sigma^* b'$.
One also defines $Ant_{\mathrm{II}}(b)$ to be the set of non-degenerate type II cell in $Ant_0(b)$. Note that in all cases $Ant(b)$ and $Ant_0(b)$ are Kurawtowski-finite[6] sets, and as the subset of type II cell is decidable, $Ant_{\mathrm{II}}(b)$ is also Kurawtowski-finite. One defines $Ant^k(b)$ and $Ant^k_{\mathrm{II}}(b)$ by:

$$Ant^1(b) = Ant(b) \qquad Ant^k(b) = \bigcup_{c \in Ant b} Ant^{k-1} c$$

$$Ant^1_{\mathrm{II}}(b) = Ant_{\mathrm{II}}(b) \qquad Ant^k_{\mathrm{II}}(b) = \bigcup_{c \in Ant_{\mathrm{II}} b} Ant^{k-1}_{\mathrm{II}} c$$

[6] A set X is said to be Kuratowski-finite if $\exists n, \exists x_1, \dots, x_n \in X, \forall x \in X, x = x_1$ or $\dots$ or $x = x_n$.

Note that when applied to a non-degenerate type II cell $b \in B$, all elements of $Ant_{\mathrm{II}}(b)$ (and hence of $Ant^k_{\mathrm{II}}(b)$ as well) are non-degenerate type II cells of the same dimension as b.

3.2.2 Definition.

- One says that b has finite P-height if there exists an integer k such that:

$$Ant^k(b) = \emptyset$$

- One says that b has finite weak P-height if there is an integer k such that:

$$Ant^k_{\mathrm{II}}(b) = \emptyset$$

Note that for each given k and $b \in B$, as the sets $Ant^k(b)$ and $Ant^k_{\mathrm{II}}(b)$ are Kuratowski-finite it is decidable whether or not $Ant^k(b)$ and $Ant^k_{\mathrm{II}}(b)$ are empty. In particular, assuming b has finite (weak) P-height there is smallest integer k, called the (weak) P-height of b, such that $Ant^k_{(\mathrm{II})}(b) = \emptyset$. But in general it might not be decidable whether b has finite (weak) P-height or not.

3.2.3 Lemma. *Let $f : A \hookrightarrow B$ be a cofibration with a P-structure satisfying all the conditions of Definition 3.2.1 but the last. Then the following are equivalent:*

- *Every $b \in B$ has finite P-height.*
- *Every non-degenerate type II cell $b \in B_{II}$ has finite weak P-height.*

Proof. It is clear that $Ant^k_{\mathrm{II}}(b) \subset Ant^k(b)$ hence the first condition implies the second. Conversely, assume that every non-degenerate $b \in B_{\mathrm{II}}$ has finite weak P-height. We will prove by double induction on both the dimension and the weak P-height that all cells of B have finite P-height.
First we assume that all cells of dimension $< n$ have finite P-height. Cells of A have P-height zero. All cells of B of dimension n that are either degenerate or of type I satisfies $Ant(b) = Ant(b')$ for some b' of dimension strictly

less than n, hence for b' of finite P-height by the induction assumption. As $Ant^k(b) = Ant^k(b')$ this implies that b has finite P-height as well.
It remains to show that all non-degenerate n-cells of type II in B have finite P-height. We do that by induction on their weak P-height.
Indeed for a general type II cell b, $Ant(b)$ is constituted of:

- Degenerate or type I cell, that are already known to have finite P-height.
- Faces of cell in $Ant_0(b)$ which are hence of dimension $< n$ and hence are known to be of finite P-height.
- Non-degenerate type II cells that are hence elements of $Ant_{\mathrm{II}}(b)$, but

$$\emptyset = Ant^k_{\mathrm{II}}(b) = \bigcup_{c \in Ant_{\mathrm{II}} b} Ant^{k-1}_{\mathrm{II}} c$$

  hence all $c \in Ant_{\mathrm{II}} b$ have weak P-height at most $k-1$, and hence they all have finite P-height by induction.

So all elements of $Ant(b)$ have finite P-height, let m be the maximum of all these P-height, one has that:

$$Ant^{m+1}(b) = \bigcup_{c \in Ant(b)} Ant^m(b) = \emptyset$$

□

3.2.4 Lemma. *A cofibration with a P-structure is anodyne. More precisely it is a ω-transfinite composition of pushouts of coproducts of horn inclusions.*

A map will be called “strongly anodyne” if it admits a P-structure.

Proof. Let $A \hookrightarrow B$ be a cofibration with a P-structure.
Let $B_k \subset B$ be the subset of B of cells of P-height at most k. One has $B_0 = A$, and B_k is a sub-simplicial set. Indeed, for every cell $b \in B$ all faces of b appear in $Ant(b)$ or are such that $Ant(d_i b) = Ant(b)$ and all degeneracies of b satisfies $Ant(\sigma^* b) = Ant(b)$, hence they all have P-height at most k.

Let U be the set of non-degenerate type II cell of B of P-height exactly k. For each $u \in U$, let i_u be the unique integer such that $d_{i_u}P(u) = u$.
Then the corresponding map $\Delta[n] \overset{Pu}{\to} B_k$ sends $\Lambda^{i_u}[n]$ to B_{k-1} and both u and Pu are in $B_k - B_{k-1}$.
Hence taking the pushout:

$$\begin{tikzcd} \Lambda^{i_u}[n] \arrow[r] \arrow[d] & \Delta[n] \arrow[d] \\ B_{k-1} \arrow[r] & R \end{tikzcd}$$

produces the simplicial set $R \subset B_k$ whose cells are all those of B_{k-1}, u and Pu and all their degeneracy. Taking the pushout by the coproduct of all these horn inclusions for all $u \in U$ gives $B_{k-1} \to B_k$.
Hence $B = \bigcup B_k$ is a ω-transfinite composition of the maps $B_k \to B_{k+1}$ which are all pushouts of coproducts horn inclusions.

□

Classically one also has the converse: any transfinite composition of pushouts of coproducts horn inclusions has a canonical P-structure. Constructively this sort of statement is somehow problematic, mostly because the general notion of "transfinite composition" requires a notion of ordinal to be formulated appropriately, but it works perfectly fine if one restricts to ω-composition:

3.2.5 Proposition. *The class of strongly anodyne cofibration contains all horn inclusion and is stable under pushout and ω-transfinite*[7] *composition. Any morphism can be factored as a strongly anodyne cofibration followed by a Kan fibration, and any anodyne cofibration is a retract of a strongly anodyne cofibration.*

Proof. Horn inclusion have a trivial P-structure with one cell of type I and one cell of type II. It is easy to see that coproduct, pushout and transfinite composition of strongly anodyne cofibration have P-structure induced by the P-structure we start from, for example if $A \hookrightarrow B$ has a P-structure, then

[7] Here the restriction to "ω" is only to avoid the discussion of what is an ordinal constructively.

$C \to B\coprod_A C$ has a P-structure where a cell in $B\coprod_A C$ is type I or II if and only if it is type I or II for the P-structure on $A \hookrightarrow B$ and the map P is the same as the one on B, and similarly for coproducts and transfinite compositions.
It follows that the factorization of the map as an anodyne cofibration followed by a Kan fibration obtained by the small object argument is a strongly anodyne cofibration as it is constructed as a ω-transfinite composition of pushout of coproduct of horn inclusion. Finally any anodyne cofibration j can be factored as a strongly anodyne cofibration followed by a Kan fibration, and the usual retract lemma 2.2.5 shows that j is a retract of the strongly anodyne cofibration part of the factorization.

□

We finish this section by mentioning a very important example where this machinery applies, mostly to serve as an example of how it can be used.
Given two morphisms $f : A \to B$ and $g : X \to Y$ between simplicial sets one defines as usual $f \ulcorner\!\times g$ the cartesian “corner-product” or “pushout-product” of f and g as the morphism:

$$f \ulcorner\!\times g : (A \times Y) \coprod_{A \times X} (B \times X) \to B \times Y,$$

one then has the following well known proposition, which we have referred to in the introduction as the corner-product conditions, and which is a key point in establishing the existence of the weak model structure on simplicial sets. It also corresponds to the fact the model structure on simplicial sets that we are constructing is cartesian.

3.2.6 Proposition. *If i and j are cofibrations, then $i \ulcorner\!\times j$ is a cofibration as well. If one of them is anodyne then $i \ulcorner\!\times j$ is also anodyne.*

As usual (following for example the appendix of [10]) this implies the dual condition, that if $i : A \to B$ is a cofibration and $p : Y \to X$ is a fibration, then the map $[B, Y] \to [B, X] \times_{[A,X]} [A, Y]$ is a fibration (the brackets denotes the cartesian exponential in simplicial sets), and it is a trivial fibration as soon as either i is anodyne or p is a trivial fibration.

Proof. By usual abstract manipulation (see for example the appendix of [10]) it is sufficient to show it when i and j are generating cofibrations/generating

anodyne cofibration. If i and j are generating cofibrations it is very easy to check that $i \ulcorner\!\!\times j$ is a cofibration as defined in the statement of our main theorem 1.1. It remains to check that if i is one of the generating cofibrations, i.e. $\partial\Delta[n] \hookrightarrow \Delta[n]$ for some n, and j is one of the generating anodyne cofibrations, i.e. $\Lambda^k[m] \hookrightarrow \Delta[m]$ for some k, m, then $i \ulcorner\!\!\times j$ is an anodyne cofibration. This is done by constructing an explicit P-structure on $i \ulcorner\!\!\times j$.
The first direct proof of this claim that we know of is in [11] (theorem 3.2.2), here we follow the proof of S.Moss' in 2.12 of [15] to show how P-structures work. We only treat the case $k < m$ for simplicity. The case $k > 0$ can be treated in a completely similar way, by simply reversing the order relation on the $[n]$, which allows to deduce the missing case $k = m$.
A p-cell of $\Delta[n] \times \Delta[m]$ is an order preserving function $[p] \to [n] \times [m]$. It is non-degenerate if and only if it is an injective function. The domain D of $i \ulcorner\!\!\times j$ is:

$$\left(\Delta[n] \times \Lambda^k[m]\right) \coprod_{\partial\Delta[n]\times\Lambda^k[m]} \left(\partial\Delta[n] \times \Delta[m]\right) = \left(\Delta[n] \times \Lambda^k[m]\right) \bigcup \left(\partial\Delta[n] \times \Delta[m]\right)$$

It corresponds to the morphisms $[p] \to [n] \times [m]$ such that either they skip a column or they skip a row other than k, where we consider that $[n] = \{0, \dots, n\}$ numbers the column of $[n] \times [m]$ and $[m] = \{0, \dots, k, \dots, m\}$ numbers the row. So the only non-degenerate cell of $\Delta[n] \times \Delta[m]$ that are not in D are injection $[k] \to [n] \times [m]$ whose first projection takes all possible value, and whose second projection takes all possible values except maybe k.
One says that a cell is type II if either it skip the k^{th} row by going directly from $(a, k-1)$ to $(a+1, k+1)$, in which case one defines Px by adding the intermediate step $(a, k-1), (a+1, k), (a+1, k+1)$, or if the last point where the k^{th} row is reached, is (a, k) followed by $(a+1, k+1)$ in which case Px is defined by inserting the intermediate step: $(a, k), (a, k+1), (a+1, k+1)$.
It is an easy exercise to check that this defines a P-structure.

□

3.3 Kan Ex and Sd functors

Consider the barycentric subdivision functor $\Delta \to \widehat{\Delta}$:

$$\Delta[n] \mapsto \mathrm{S}\textsc{d}\,\Delta[n] := N\mathcal{K}([n])$$

Where $\mathcal{K}([n])$ denotes the set of *finite non-empty decidable* subsets of $[n]$. Functoriality in $[n]$ is given by direct image of subsets on $\mathcal{K}[n])$.
This extends to an adjunction:

$$\mathrm{S}\textsc{d} : \widehat{\Delta} \leftrightarrows \widehat{\Delta} : \mathrm{E}\textsc{x}$$

with:

$$(\mathrm{E}\textsc{x}\,X)_n = \mathrm{Hom}(\mathrm{S}\textsc{d}\,\Delta[n], X) \qquad \mathrm{S}\textsc{d}\,X = \operatorname*{Colim}_{\Delta[n]\to X} \mathrm{S}\textsc{d}\,\Delta[n]$$

The barycentric subdivision construction has a nice expression not just for the $\Delta[n]$, but also for all objects which are in the image of the functor $\widehat{\Delta_+} \to \widehat{\Delta}$, indeed:

3.3.1 Proposition. *The composite:*

$$\widehat{\Delta_+} \to \widehat{\Delta} \overset{\mathrm{S}\textsc{d}}{\to} \widehat{\Delta}$$

is the functor sending a semi-simplicial set X to $N(\Delta_+/X)$.

One can note that as the category Δ_+/X is directed, the nerve $N(\Delta_+/X)$ is itself the image of the semi-simplicial set of its non-degenerate cells. We won't make any use of this remark though.

Proof. This functors $X \mapsto N(\Delta_+/X)$ preserves colimits, because it can be rewritten as:

$$N(\Delta_+/X)_k = \coprod_{F:[k]\to\Delta_+} X(F(k))$$

which is levelwise a coproduct of colimits-preserving functor.
Hence we are comparing to colimits preserving functor, so it is enough to show they are isomorphic when restricted to representable. But $\Delta_+/[n] \simeq \mathcal{K}[n]$ functorially on map of Δ_+ so this concludes the proof.

□

3.3.2 Proposition. $\textsc{Sd}$ *preserves cofibrations and anodyne cofibrations,* $\textsc{Ex}$ *preserves fibrations and trivial fibrations.*

Proof. It is enough to check that the image of the generating cofibrations and generating anodyne cofibrations by $\textsc{Sd}$ are cofibrations and anodyne cofibrations respectively.
In both case one can use Proposition 3.3.1 to compute $\textsc{Sd}$ on the generators as they are image of semi-simplicial maps. This makes the results immediate for cofibrations:

$$\textsc{Sd}\,\partial\Delta[n] \to \textsc{Sd}\,\Delta[n]$$

is the morphism $N(\mathcal{K}[n] - \{[n]\}) \to N(\mathcal{K}[n])$ which is clearly a levelwise complemented monomorphisms between finite decidable, hence cofibrant, simplicial sets.
For the generating anodyne cofibrations,

$$\textsc{Sd}\,\Lambda^i[n] \to \textsc{Sd}\,\Delta[n]$$

is the morphisms $N(\mathcal{K}[n] - \{[n], [n] - \{i\}\}) \to N(\mathcal{K}[n])$. It can then be checked completely explicitly that this is a (strongly) anodyne cofibrations, see Proposition 2.14 of [15] for an explicit description of a P-structure.

□

There is a natural transformation:

$$\textsc{Sd}\,\Delta[n] \to \Delta[n]$$

Which is induced by the order preserving function:

$$\max : \mathcal{K}[n] \to [n]$$

sending each (decidable) subset of $[n]$ to its maximal element. By Kan extension, this gives us natural transformations:

$$\textsc{Sd} \xrightarrow{m} Id \qquad Id \xrightarrow{n} \textsc{Ex}$$

One can hence define a sequences of functors:

$$X \xrightarrow{n_x} \mathrm{Ex}\, X \xrightarrow{n_{\mathrm{Ex}\, X}} \mathrm{Ex}^2\, X \xrightarrow{n_{\mathrm{Ex}^2\, X}} \dots \xrightarrow{n_{\mathrm{Ex}^{k-1}}\, X} \mathrm{Ex}^k\, X \xrightarrow{n_{\mathrm{Ex}^k}\, X} \dots \longrightarrow \mathrm{Ex}^\infty\, X$$

with Ex^∞ the colimit.

3.3.3 Lemma. *For each k, n, there is a (dotted) arrow Ψ_n^k making the following triangle commute.*

$$\begin{tikzcd} \mathrm{SD}^2\, \Lambda^k[n] \arrow[rr, "\mathrm{SD}(m_{\Lambda^k[n]})"] \arrow[dr] & & \mathrm{SD}\, \Lambda^k[n] \\ & \mathrm{SD}^2\, \Delta[n] \arrow[ur, dotted, "\Psi_n^k"] & \end{tikzcd}$$

Proof. The proof given in [3] as proposition 2.1.39 is purely combinatorial and constructive. □

3.3.4 Corollary. *For every cofibrant simplicial set X, $\mathrm{Ex}^\infty\, X$ is a Kan complex.*

The proof that follows essentially comes from [3]. If one does not assume that X is cofibrant it still applies to prove that X has the "existential" right lifting property against horn inclusion, but it does not seem possible to give a uniform choice of solution to all lifting problems without this assumption. Without such a uniform choice of lifting against horn inclusion one cannot construct solution to lifting problems against more complicated anodyne cofibrations that involves an infinite number of pushout of horn inclusion, unless we assume the axiom of choice.

Proof. Lemma 3.3.3 allows to show that given any solid diagram as below, there is a dotted filling:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & \mathrm{Ex}\, X \arrow[d, "n_{\mathrm{Ex}\, X}"] \\ \Delta[n] \arrow[r, dotted] & \mathrm{Ex}^2\, X \end{tikzcd}$$

Indeed, through the adjunction the map $\Lambda^k[n] \to \operatorname{Ex} X$ corresponds to an arrow $\operatorname{SD} \Lambda^k[n] \to X$, which due to Lemma 3.3.3 can be extended in:

$$\begin{tikzcd} \operatorname{SD}^2 \Lambda^k[n] \arrow[r, "\operatorname{SD} m_{\Lambda^k[n]}"] \arrow[d, "\operatorname{SD}^2 -"'] & \operatorname{SD} \Lambda^k[n] \arrow[r] & X \\ \operatorname{SD}^2 \Delta[n] \arrow[ur, "\psi^k_n"'] & & \end{tikzcd}$$

The resulting map $\operatorname{SD}^2 \Delta[n] \to X$ corresponds to a map $\Delta[n] \to \operatorname{Ex}^2 X$ which has exactly the right property to make the square above commute.
Now by smallness of $\Lambda^k[n]$, any map $\Lambda^k[n] \to \operatorname{Ex}^\infty X$ factors in $\operatorname{Ex}^k X$, the observation above produces a canonical filling in $\Delta[n] \to \operatorname{Ex}^{k+1} X$. The choice of the filling, seen as taking values in $\operatorname{Ex}^\infty X$, in general depends on k, but if one further assume that X is cofibrant, than by Lemma 3.4.4, the maps $\operatorname{Ex}^k X \to \operatorname{Ex}^{k+1} X$ are all levelwise decidable inclusion, so there is a smallest k such that the map $\Lambda^k[n] \to \operatorname{Ex}^\infty X$ factors into $\operatorname{Ex}^k X$ and this produces a canonical solution to the lifting problem. □

3.3.5 Proposition. *If $f : X \to Y$ is a fibration (resp. a trivial fibration) with X and Y cofibrant then $\operatorname{Ex}^\infty f : \operatorname{Ex}^\infty X \to \operatorname{Ex}^\infty Y$ is also a fibration (resp. a trivial fibration).*

Similarly to what happen with Corollary 3.3.4, without the assumption that X and Y are cofibrant it is only possible to obtain the "existential" form of the lifting property and no canonical choice of lifting.

Proof. Given a lifting problem:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook, "\sim"] & \operatorname{Ex}^\infty X \arrow[d] \\ \Delta[n] \arrow[r] & \operatorname{Ex}^\infty Y \end{tikzcd}$$

There is an i such that it factors into:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook, "\sim"] & \mathrm{Ex}^i X \arrow[r] \arrow[d] & \mathrm{Ex}^\infty X \arrow[d] \\ \Delta[n] \arrow[r] & \mathrm{Ex}^i Y \arrow[r] & \mathrm{Ex}^\infty Y \end{tikzcd}$$

Moreover, assuming X and Y are cofibrant, Lemma 3.4.4 shows that $\mathrm{Ex}^i X \subset \mathrm{Ex}^{i+1} X$ are levelwise decidable inclusion, so (by finiteness of $\Lambda^k[n]$ and $\Delta[n]$) the set of i such that a factorization as above exists is decidable, and hence there is a smallest such i. Proposition 3.3.2 shows that $\mathrm{Ex}^i f$ is a fibration, so the first square has a diagonal lifting and this concludes the proof. □

3.4 S.Moss' proof that $X \to \mathrm{Ex}\, X$ is an anodyne cofibration

Let $f : X \to Y$ be a simplicial morphism. One has a square:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Ex}^\infty X \arrow[r] & \mathrm{Ex}^\infty Y \end{tikzcd}$$

Our goal in this section is to show that when X is cofibrant the induced map:

$$X \to \mathrm{Ex}^\infty X \underset{Ex^\infty Y}{\times} Y$$

is a strong anodyne cofibration. Note that if $Y = \Delta[0]$ is the terminal object, then $Ex^\infty(Y) = Y$ hence the statement above boils down to the fact that $X \to Ex^\infty X$ is a strong anodyne cofibration. The idea to consider this morphism comes form D.C Cisinski's book [3, Cor 2.1.32], but the proof below follows closely the proof given by S.Moss in [15] that $X \to \mathrm{Ex}^\infty X$ is a strong anodyne cofibration.
Following the argument given in [3, Cor 2.1.32] (reproduced in the proof of Corollary 3.4.7 below), it will be enough to show:

3.4.1 Proposition. *Given $f : X \to Y$ a simplicial morphism, with X cofibrant, then the morphism:*

$$X \to \mathrm{Ex}\, X \underset{\mathrm{Ex}\, Y}{\times} Y$$

is strongly anodyne.

The proof will be concluded in 3.4.6, essentially, we will construct an explicit P-structure on this map. This construction is mostly due to S.Moss in [15]. In addition to the dependency in Y, the main new contributions of this paper in this section is to show that assuming X is cofibrant one can show that sufficiently many decidability conditions can be proved to make S.Moss' argument constructive. In order to do that properly one needs to completely reproduce his argument.
Following, [15] one introduces two functions between the $\mathrm{SD}\,\Delta[n]$.
Let $j_n^k : \mathrm{SD}\,\Delta[n] \to \mathrm{SD}\,\Delta[n]$ and $r_n^k : \mathrm{SD}\,\Delta[n+1] \to \mathrm{SD}\,\Delta[n]$ be the maps defined at the level of posets by:

$$j_n^k\{i\} = \begin{cases} \{i\} & \text{if } i \leqslant k \\ \{0, \dots, i\} & \text{if } i > k \end{cases} \qquad r_n^k\{i\} = \begin{cases} \{i\} & \text{if } i \leqslant k \\ \{0, \dots, i-1\} & \text{if } i = k+1 \\ \{i-1\} & \text{if } i > k+1 \end{cases}$$

Both extended to non-singleton elements as binary join preserving maps. These functions satisfies a certain number of equations, we list here those that we will need, they are all due to S.Moss.

3.4.2 Lemma.

$$j_n^k j_n^h = j_n^h j_n^k = j_n^h \qquad 0 \leqslant h \leqslant k \leqslant n \tag{1}$$
$$Id_{\Delta[n]} = r_n^k \circ \mathrm{SD}\,\partial_{n+1}^{k+1} \qquad 0 \leqslant k \leqslant n \tag{2}$$
$$j_n^k r_n^k = (\mathrm{SD}\,\sigma_n^k) j_{n+1}^k \qquad 0 \leqslant k \leqslant n \tag{3}$$
$$j_n^h r_n^k = j_n^h (\mathrm{SD}\,\sigma_n^k) \qquad 0 \leqslant h < k \leqslant n \tag{4}$$
$$r_n^k j_{n+1}^h = j_n^h r_n^k \qquad 0 \leqslant h \leqslant k \leqslant n \tag{5}$$
$$r_n^k (\mathrm{SD}\,\partial_{n+1}^{i+1}) = (\mathrm{SD}\,\partial_n^i) r_{n-1}^k \qquad 0 \leqslant k < i \leqslant n \tag{6}$$
$$j_n^k r_n^k r_{n+1}^k = j_n^k r_n^k (\mathrm{SD}\,\sigma_{n+1}^{k+1}) \qquad 0 \leqslant k \leqslant n \tag{7}$$
$$j_{n+1}^k (\mathrm{SD}\,\partial_{n+1}^h) j_n^k = j_{n+1}^k (\mathrm{SD}\,\partial_{n+1}^h) \qquad 0 \leqslant k \leqslant n \textit{ and } 0 \leqslant h \leqslant n+1 \tag{8}$$
$$j_n^k r_n^k (\mathrm{SD}\,\partial_{n+1}^i) j_n^{k-1} = j_n^k r_n^k (\mathrm{SD}\,\partial_{n+1}^i) \qquad 0 \leqslant i \leqslant k \leqslant n \tag{9}$$
$$(\mathrm{SD}\,\sigma_n^h) j_{n+1}^k r_{n+1}^k = j_n^{k-1} r_n^{k-1} (\mathrm{SD}\,\sigma_{n+1}^h) \qquad 0 \leqslant h < k \leqslant n+1 \tag{10}$$
$$(\mathrm{SD}\,\sigma_n^h) j_{n+1}^k r_{n+1}^k = j_n^k r_n^k (\mathrm{SD}\,\sigma_{n+1}^{h+1}) \qquad 0 \leqslant k \leqslant h \leqslant n \tag{11}$$

Proof. All the functions involved are nerve of join preserving maps between the $\mathcal{K}[n]$, so it is enough to check the relations at the level of posets and when functions are evaluated at $\{i\}$, where one has explicit formulas for all of them. □

As functions between the $\mathrm{SD}\,\Delta[n]$, j_n^k and r_n^k automatically acts on the cells of $\mathrm{Ex}\,X$. One denotes this action by $x \mapsto x j_n^k$ and $x \mapsto x r_n^k$ which is compatible to the identification of cells of $\mathrm{Ex}\,X$ with functions $\mathrm{SD}\,\Delta[n] \to x$.

By equation (1), the j_n^k are an increasing family of commuting projection whose image defines a series of subsets:

$$X_n = J_n^0 \subset J_n^1 \subset \dots J_n^n = (\mathrm{Ex}\,X)_n$$

where the identifications with $(\mathrm{Ex}\,X)_n$ and X_n comes from the fact that j_n^n is the identity, and $j_n^0 : \mathcal{K}[n] \to \mathcal{K}[n]$ has image isomorphic to $[n]$, with $j_n^0 : \mathcal{K}[n] \to [n]$ being the “Max” function used in the definition of the natural transformation $\mathrm{SD}\,\Delta[n] \to \Delta[n]$.

3.4.3 Notation. For $X \to Y$ any morphism, we define:

$$\mathrm{Ex}_Y(X) = \mathrm{Ex}\,X \underset{\mathrm{Ex}\,Y}{\times} Y.$$

An n-cell in Ex_Y is a morphism $\mathrm{Sd}\,\Delta[n] \to X$ whose image in Y factors through the map $\mathrm{Sd}\,\Delta[n] \to \Delta[n]$. I.e. it is an n-cell of $x \in (\mathrm{Ex}\,X)_n$ which satisfies:

$$fxj_n^0 = fx$$

Note that because of relation (1) and (5), $\mathrm{Ex}_Y\,X$, as a subsimplicial object of $\mathrm{Ex}\,X$, is stable under the action of j_n^k and r_n^k on $\mathrm{Ex}\,X$. We also denote by J_n^k the image of j_n^k in $(\mathrm{Ex}_Y\,X)_n$.
Before going any further, one needs to state some decidability conditions:

3.4.4 Lemma. *If X is a cofibrant simplicial set, then:*

1. *The inclusion $X \subset \mathrm{Ex}_Y\,X$ is levelwise decidable.*
2. *$\mathrm{Ex}_Y\,X$ is cofibrant and $X \to \mathrm{Ex}_Y\,X$ is a cofibration.*
3. *The sets $J_k^n \subset (\mathrm{Ex}_Y\,X)_n$ are decidable.*

Proof. All these decidability problems correspond to the decidability of a factorization of a map $\mathrm{Sd}\,\Delta[n] \to X$ through some epimorphism $\mathrm{Sd}\,\Delta[n] \to K$. In all these cases we will show that the corresponding epimorphism is a degeneracy quotient using Proposition 3.1.10 and conclude about the decidability using Lemma 3.1.8.

1. It corresponds to the map $\mathrm{Sd}\,\Delta[n] \to \Delta[n]$ which is the nerve of the max function $\mathcal{K}[n] \to [n]$, whose section $i \mapsto \{0,\dots,i\}$ satisfies the condition of Proposition 3.1.10.
2. One just needs to check degeneracy are decidable in $\mathrm{Ex}\,X$, so it is about the epimorphism $\mathrm{Sd}(\sigma) : \mathrm{Sd}\,\Delta[n] \to \mathrm{Sd}\,\Delta[m]$ for any degeneracy σ. It is the nerve of $\sigma : \mathcal{K}[n] \to \mathcal{K}[m]$ which has a section satisfying the condition of Proposition 3.1.10 which sends every $P \in \mathcal{K}[m]$ to $\sigma^{-1}P$
3. It corresponds to the map $j_n^k : \mathrm{Sd}\,\Delta[n] \to j_n^k(\mathrm{Sd}\,\Delta[n])$, which is just is the nerve of the projection $j_n^k : \mathcal{K}[n] \to j_n^k\mathcal{K}[n]$ which is already of the form of Proposition 3.1.10.

□

We can now give the definition of the P-structure on $X \hookrightarrow \mathrm{Ex}_Y X$.

- Type I cells are the non-degenerate cells $v \in \mathrm{Ex}_Y(X)$ which are not[8] in X and can be written as yr_n^k with $y \in J_n^k \subset \mathrm{Ex}_Y X$.
- Point 8 of Lemma 3.4.5 will prove that being type I is decidable. Type II cells are just the cells that are not of type I (and which are non-degenerate and not in X).
- For any cell x one defines Px as xr_n^k where k is the smallest integer such that $x \in J_n^k$, i.e. $x \in J_n^k - J_n^{k-1}$. Lemma 3.4.4 shows that the J_n^k are decidable so there is indeed a unique such integer k.

In order to show that being type I is decidable and that P defined this way defines a bijection from type II cells to type I cells, one needs a few technical lemma that we have regrouped in:

3.4.5 Lemma.

1. *If $x \in J_n^k - J_n^{k-1}$, then $d_{k+1}Px = x$.*
2. *$x \in J_n^k$ if and only if $Px \in J_{n+1}^k$*
3. *If $x \in J_n^{k-1}$ then xr_n^k is degenerate.*
4. *P^2x is always a degenerate cell.*
5. *If x is a degenerate or type I cell or in X, then Px is a degenerate cell.*
6. *If $x \in J_n^k - J_n^{k-1}$ then for all $i \leqslant k$, $d_i(Px) \in J_n^{k-1}$.*
7. *If $x \in J_n^k - J_n^{k-1}$ then for all i, with $k+1 < i \leqslant n+1$, $d_i(Px)$ is either of type I or degenerate.*
8. *A non-degenerate cell x in $(\mathrm{Ex}_Y X)_n - X_n$ is type I if and only Px is a degenerate cell.*

[8] It appears that, because of point 2 of Lemma 3.4.5 and the fact that r_n^0 is the same as $\mathrm{SD}\,\sigma_0$, it is actually a consequence from the rest of the definition that type I cells are not in X.

Proof.

1. $d_{k+1}Px$ is $xr_n^k(\text{SD}\,\partial^{k+1})$ which is equal to x by equation (2).

2. Let k is the smallest value such that $xj_n^k = x$, i.e.$Px = xr_n^k$. Equation (5) gives $xr_n^k j_{n+1}^k = xj_n^k r_n^k = xr_n^k$. Hence $Px \in J_{n+1}^k$, in particular $x \in J_n^h \Rightarrow k \leqslant h \Rightarrow Px \in J_{n+1}^h$. Conversely, if $Px \in J_{n+1}^k$ then:

$$\begin{array}{rcll} xj_n^k & = & (Px)(\text{SD}\,\partial^{h+1})j_n^k & (\text{as } x = d_{h+1}Px) \\ & = & (Px)j_{n+1}^k(\text{SD}\,\partial^{h+1})j_n^k & (\text{ as } Px \in J_{n+1}^k) \\ & = & (Px)j_{n+1}^k(\text{SD}\,\partial^{h+1}) & (\text{by equation (8)}) \\ & = & x & (\ Px \in J_{n+1}^k \text{ and } x = d_{h+1}Px) \end{array}$$

   Hence $x \in J_n^k$.

3. $xr_n^k = xj_n^{k-1}r_n^k$ is a degenerate cell because of equation (4)

4. Let k such that $x \in J_n^k - J_n^{k-1}$, then $Px = xr_n^k = xj_n^k r_n^k$ and $Px \in J_{n+1}^k - J_{n+1}^{k-1}$ because of point 2, hence $P^2x = xr_n^k r_{n+1}^k = xj_n^k r_n^k r_{n+1}^k$ which is a degenerate cell because of equation (7).

5. Equations (10) and (11) show that if x is a degenerate cell then Px is a degenerate cell. If $x \in X$, i.e. $x \in J_n^0$ then $Px = xr_n^0$ but $r_n^0 = \text{SD}\,\sigma_0$ so Px is a degenerate cell.

   It follows that if x is of type I, then $x = yr_n^k$ with $y \in J_n^k$ if $y \in J_n^{k-1}$ then x is a degenerate cell because of point 3, hence Px is a degenerate cell because of the first part of the present point, if $y \notin J_n^{k-1}$ then $x = Py$ and hence Px is a degenerate cell because of point 4.

6. This follows immediately from equation (9) as $d_i(Px) = xj_n^k r_n^k(\text{SD}\,\partial^i)$.

7. For $k+1 < i \leqslant n+1$ we have:

$$\begin{array}{rcll} j_n^k r_n^k(\text{SD}\,\partial_{n+1}^i) & = & j_n^k(\text{SD}\,\partial_n^{i-1})r_{n-1}^k & \text{by equation (6)} \\ & = & j_n^k(\text{SD}\,\partial_n^{i-1})j_{n-1}^k r_{n-1}^k & \text{by equation (8)} \end{array}$$

   This equation shows that for $x \in J_n^k$, d_iPx is of the form yr_{n-1}^k for $y \in J_{n-1}^k$, namely $y = x(\text{SD}\,\partial^{i-1})j_{n-1}^k$, hence, if d_iPx is a non-degenerate cell, it is of type I.

8. We have shown in Item 5 that if x is type I then Px is a degenerate cell. Conversely let x be a non-degenerate cell such that Px is a degenerate cell. Let k be such that $x \in J_n^k - J_n^{k-1}$. One has $x = d_{k+1}Px$ by point 1 of the lemma, hence $d_{k+1}Px$ is a non-degenerate cell, which means that Px can only be σ_k-degenerate or σ_{k+1}-degenerate (otherwise $d_{k+1}PX$ would also be a degenerate cell). If Px is σ_k-degenerate then $d_kPx = d_{k+1}Px = x$, but by point 6 of the present lemma $d_kPx \in J_n^{k-1}$ so this is impossible. If Px is σ_{k+1}-degenerate then $d_{k+2}Px = d_{k+1}Px = x$ hence point 7 shows that x is of type I.

□

3.4.6. We are now ready to prove Proposition 3.4.1:

Proof. The goal is to show that the type I cell and the operation P we have defined satisfy the condition of Definition 3.2.1, so that the map is a strongly anodyne cofibration because of Lemma 3.2.4.
Point 8 of Lemma 3.4.5 (combined with Lemma 3.4.4) shows that being a type I cell is decidable. So one can indeed defines type II cells as the cells that are not of type I (and non-degenerate nor in the domain) and get a partition of the non-degenerate cells. It also follows from point 8 that if x is a type II cell then Px is a non-degenerate cell, and it is type I (either by definition or because of point 4). Finally, point 2 shows that P preserve the k such that $x \in J_n^k$, as $X \subset \mathrm{Ex}_Y X$ corresponds to J_n^0 it shows that P never sends cell not in X to cell in X. So P restricts into a function from type II cells to type I cells.
We now show that it is a bijection:
If x is a type I cell than it can be written as yr_n^k with $y \in J_n^k$. By point 3 of Lemma 3.4.5, if $y \in J_n^{k-1}$, then $x = yr_n^k$ is a degenerate cell, hence $y \notin J_n^{k-1}$ and hence $x = Py$. By point 5 of Lemma 3.4.5, if y is a degenerate or type I cell then $x = Py$ is a degenerate cell, hence y is a type II cell. This proves the surjectivity of P.
If x is a type II cell and $y = Px$, then $x = d_{k+1}Px$ (because of point 1 of Lemma 3.4.5) where k can be characterized as the unique integer such that $y \in J_{n+1}^k - J_{n+1}^{k-1}$ (because of point 2 of Lemma 3.4.5). Hence P is injective on type II cell and this concludes the proof that P is a bijection between non-degenerate type II cells and non-degenerate type I cells.

Finally if x is a non-degenerate type II cell, and k is such that $x \in J_n^k - J_n^{k-1}$. Point 1 of Lemma 3.4.5 shows that $d_{k+1}(Px) = x$, while point 6 and 7 show that for all $i \neq k+1$, $d_i Px$ is either in J_n^{k-1} or a type I or degenerate cell, hence always distinct from x. So there is indeed a unique i such that $d_i Px = x$, and it is $k+1$.
It remains to prove the "well-foundedness" or "finite height" condition. It follows from point 6 and 7 of Lemma 3.4.5 that given $x \in J_n^k - J_n^{k-1}$ a non-degenerate type II cell, $Ant_{\mathrm{II}}(x) \subset J_n^{k-1}$. In particular, any cell $x \in J_n^k$ has weak P-height at most k, hence by Lemma 3.2.3 this shows that every cell has finite P-height and hence concludes the proof.

□

3.4.7 Corollary. *For any $f : X \to Y$ with X cofibrant, the morphism:*

$$X \to \mathrm{Ex}^\infty X \underset{\mathrm{Ex}^\infty Y}{\times} Y$$

is a strongly anodyne cofibration.

Proof. Consider $\mathrm{Ex}^k X \times_{\mathrm{Ex}^k Y} Y \to Y$ and apply the functor Ex_Y to it. One obtains:

$$\begin{aligned} \mathrm{Ex}_Y\left(\mathrm{Ex}^k X \times_{\mathrm{Ex}^k Y} Y\right) &= \mathrm{Ex}\left(\mathrm{Ex}^k X \times_{\mathrm{Ex}^k Y} Y\right) \times_{\mathrm{Ex}\, Y} Y \\ &= \left(\mathrm{Ex}^{k+1} X \times_{\mathrm{Ex}^{k+1} Y} \mathrm{Ex}\, Y\right) \times_{\mathrm{Ex}\, Y} Y \end{aligned}$$

in the last terms the map from $\left(\mathrm{Ex}^{k+1} X \times_{\mathrm{Ex}^{k+1} Y} \mathrm{Ex}\, Y\right)$ to $\mathrm{Ex}\, Y$ used in the fiber product is just the second projection, so the fiber product simplifies to:

$$\mathrm{Ex}_Y\left(\mathrm{Ex}^k X \underset{\mathrm{Ex}^k Y}{\times} Y\right) = \mathrm{Ex}^{k+1} X \underset{\mathrm{Ex}^{k+1} Y}{\times} Y$$

And the natural map $\mathrm{Ex}^k X \times_{\mathrm{Ex}^k Y} Y \to \mathrm{Ex}_Y\left(\mathrm{Ex}^k X \times_{\mathrm{Ex}^k Y} Y\right)$ corresponds through this identification to:

$$n_{\mathrm{Ex}^k X} \underset{n_{\mathrm{Ex}^k Y}}{\times} Id_Y : \mathrm{Ex}^k X \underset{\mathrm{Ex}^k Y}{\times} Y \to \mathrm{Ex}^{k+1} X \underset{\mathrm{Ex}^{k+1} Y}{\times} Y$$

It follows by induction that the sequence of maps:

$$X \to \operatorname{Ex} X \underset{\operatorname{Ex} Y}{\times} Y \to \cdots \to \operatorname{Ex}^k X \underset{\operatorname{Ex}^k Y}{\times} Y \to \operatorname{Ex}^{k+1} X \underset{\operatorname{Ex}^{k+1} Y}{\times} Y \to \dots$$

are all strong anodyne cofibrations (and all these objects are cofibrant), and the map $X \to \operatorname{Ex}^\infty X \times_{\operatorname{Ex}^\infty Y} Y$ is their transfinite composite (this last claim can either be observed very explicitly, or formally by commutation of directed colimits with finite limits).

□

3.5 Applications

3.5.1 Proposition. *Kan fibrations are the same as the strong fibrations of Definition 2.2.3. Dually, the trivial cofibrations of Definition 2.2.3 are the same as anodyne cofibrations.*

The proof given here, at least the case of a Kan fibration between cofibrant objects, is essentially the proof proposition 2.1.41 of [3].

Proof. We start with the first half: we observed in Remark 2.2.4 that strong fibrations are Kan fibrations. So we only need to show that any Kan fibration is a strong fibration. We first show this claim for $p : A \twoheadrightarrow B$ a Kan fibration between cofibrant object. One has that $\operatorname{Ex}^\infty(f)$ is a Kan fibration (by Proposition 3.3.2) between fibrant objects (because of Corollary 3.3.4), hence it is a strong fibration (by Lemma 2.2.6.(iii)), in particular any pullback of $\operatorname{Ex}^\infty(f)$ is also a strong fibration. This gives a factorization of p:

$$\begin{tikzcd} A \arrow[r, hook, "\sim"] \arrow[dr, "p"'] & \operatorname{Ex}^\infty(A) \times_{\operatorname{Ex}^\infty(B)} B \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Ex}^\infty A \arrow[d, two heads, "\operatorname{Ex}^\infty p"] \\ & B \arrow[r] & \operatorname{Ex}^\infty B \end{tikzcd}$$

in an anodyne cofibration (by Corollary 3.4.7) followed by strong fibration as a pullback of the strong fibration $\operatorname{Ex}^\infty(p)$. So p is a retract of the strong fibration part by the retract lemma (2.2.5) and hence is itself a strong fibration.

We now move to the case of a general Kan fibration. We first show that a Kan fibration that is also an equivalence is a trivial fibration. Let $p : X \to Y$

be such a Kan fibration and weak equivalence, one needs to show that it has the right lifting property against all boundary inclusion: $\partial\Delta[n] \hookrightarrow \Delta[n]$, consider such a lifting problem:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & X \arrow[d, two heads, "f"] \\ \Delta[n] \arrow[r] & Y \end{tikzcd}$$

One first factors the map $\Delta[n] \to Y$ as a cofibration followed by a trivial fibration and we form a pullback of f along the fibration part to get a diagram:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "u"] \arrow[d, hook] & P \arrow[r, two heads, "\sim"] \arrow[d, two heads, "f'"] & X \arrow[d, two heads, "f"] \\ \Delta[n] \arrow[r, hook] & Z \arrow[r, two heads, "\sim"] & Y \end{tikzcd}$$

By 2-out-of-3 the new fibration f' is again a weak equivalence, but note that now the object Z is cofibrant. One can further factor u in a cofibration followed by a trivial fibration:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, hook] \arrow[d, hook] & K \arrow[r, two heads, "\sim"] \arrow[dr, two heads, "f''"'] & P \arrow[r, two heads, "\sim"] \arrow[d, two heads, "f'"] & X \arrow[d, two heads, "f"] \\ \Delta[n] \arrow[rr, hook] & & Z \arrow[r, two heads, "\sim"] & Y \end{tikzcd}$$

f'' is a Kan fibration between cofibrant objects, hence is a strong fibration by the first part of the proof, moreover it is an equivalence hence it is a trivial fibration by the last point of Lemma 2.2.6, and hence it has the right lifting property against the boundary inclusion which shows that the morphism f is a trivial fibration as well.

One can then conclude the proof by the same argument as used in the proof of the first part of Lemma 2.2.6: Given a lifting problem of a trivial cofibration against a Kan fibration one can, using appropriate factorization, reduce to the case where the top and bottom map of the lifting square are weak equivalences, in which case the Kan fibration is a weak equivalence by 2-out-of-3 and hence is a trivial fibration by the claim we just proved, and hence has the right lifting property against all cofibrations which concludes the proof.
For the second half of the proposition, given a trivial cofibration j one factors it as an anodyne cofibration followed by a Kan fibration. By the first half of the proof, the Kan fibration is a strong fibration and hence has the right lifting property against j. It immediately follows from the retract lemma 2.2.5 that j is a retract of the anodyne cofibration part of the factorization and hence is an anodyne cofibration itself. $\square$

3.5.2 Proposition. *The model structure of Theorem 2.2.9 is right proper, i.e. the pullback of a weak equivalence along a fibration is again a weak equivalence.*

Proof. We consider a square in $\widehat{\Delta}$:

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads, "p"] \\ C \arrow[r, "\sim", "f"'] & A \end{tikzcd}$$

where p is a fibration and f is a weak equivalence, and we need to show that g is a weak equivalence. Using Lemma 3.5.3 below, we can freely assume that A, B and C are cofibrants. This implies that the pullback P is also cofibrant because it is a subobject of the product $B \times C$ which is cofibrant because of the cartesianess of the model structure (Proposition 3.2.6), and the explicit description of cofibrant objects in terms of decidability of degenerateness of cell, immediately shows that a subobject of a cofibrant simplicial set is cofibrant.
In this case when all objects are cofibrant, the result follows immediately from an application of Kan's Ex^{∞} functor: It preserves the pullback square (because it is a right adjoint), it sends each object to a fibrant object, when all the object are fibrant the result is true in any (weak) model category (a constructive argument, valid in weak model category is given as corollary 2.4.4

in [8]), and it detects equivalences between cofibrant objects because the morphism $X \to \mathrm{Ex}^\infty X$ is an anodyne cofibration (hence an equivalence) for X cofibrant.

$\square$

3.5.3 Lemma. *Let $\mathcal{C}$ be a Quillen[9] model category, if for every pullback diagram*

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, two heads, "p"] \\ C \arrow[r, "\sim", "f"'] & A \end{tikzcd}$$

in which A, B and C are cofibrant, p is a fibration, if f is a weak equivalence then so too is g. Then $\mathcal{C}$ is right proper: that is the condition also holds without assuming the A, B and C are cofibrant.

Proof. We consider a pullback as in the lemma, and we need to show that the projection map $P \to B$ is a weak equivalence, but without assuming A, B and C are cofibrant. By assumption, we already know this is the case case when A, B and C are cofibrants. The proof will proceed in three steps, where at each step we relax the cofibrancy assumption on one of the three objects:

First step: We assume that C and A are cofibrant (but not neccessarily B). In this case, we consider a cofibrant replacement $B^c \overset{\sim}{\twoheadrightarrow} B$, and we form the pullback:

$$\begin{tikzcd} Q \arrow[r, "h"] \arrow[d, two heads, "\sim"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B^c \arrow[d, two heads, "\sim"] \\ P \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, two heads, "p"] \\ C \arrow[r, "\sim", "f"'] & A \end{tikzcd}$$

Then $Q \to P$ is a trivial fibration because it is a pullback of a trivial fibration, the outer rectangle is a pullback as the composite of two pullback squares,

[9] It is actually enough to assume that $\mathcal{C}$ is a left semi-model category, as the proof below shows. We will only use it for the Kan-Quillen model structure.

so h is a weak equivalence as the pullback of f along a fibration (with all the object involved cofibrants), hence g is a weak equivalence by 2-out-of-3.
Second step: We only assume that A is cofibrant. We then take a cofibrant replacement $C^c \overset{\sim}{\twoheadrightarrow} C$ of C. and we form the pullback:

$$\begin{tikzcd}
R \arrow[r, two heads, "\sim"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads, "p"] \\
C^c \arrow[r, two heads, "\sim"] & C \arrow[r, "\sim", "f"'] & A
\end{tikzcd}$$

as in the previous case, the map $R \to P$ is a trivial fibration because it is a pullback of a trivial fibration. The composite map $R \to B$ is a pullback along the fibration p of the composite weak equivalence $C^c \to A$, so as C^c and A are cofibrant, we deduce from the first step that the composite $R \to B$ is a weak equivalence. By 2-out-of-3, this shows that g is a weak equivalence and concludes the proof for this case.
Third step: We make no cofibrancy assumption. Then we take a cofibrant replacement $A^c \overset{\sim}{\twoheadrightarrow} A$. We then form a cube

$$\begin{tikzcd}
& P' \arrow[rr] \arrow[dl, two heads, "\sim"'] \arrow[dd] & & B' \arrow[dl, two heads, "\sim"'] \arrow[dd, two heads] \\
P \arrow[rr, crossing over] \arrow[dd] & & B & \\
& C' \arrow[rr] \arrow[dl, two heads, "\sim"'] & & A^c \arrow[dl, two heads, "\sim"'] \\
C \arrow[rr, "\sim"'] & & A \arrow[from=uu, crossing over, two heads] &
\end{tikzcd}$$

where each face is a pullback square. All the diagonal maps are pullback of the trivial fibration $A^c \to A$, and so are trivial fibrations, the map $C' \to A^c$ is a weak equivalence by 2-out-of-3, hence the map $P' \to B'$ is also a weak equivalence as a pullback of a weak equivalence along a fibration (using the fact that A^c is cofibrant and the second step). Hence the map $P \to B$ is a weak equivalence by 2-out-of-3. $\square$

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