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Directeur de la publication: Andrée C. EHRESMANN,
Faculté des Sciences, Mathématiques LAMFA
33 rue Saint-Leu, F-80039 Amiens.

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXV-2 (2024)

COMPUTADS AND STRING DIAGRAMS FOR N-SESQUICATEGORIES

Manuel Araújo

Résumé. Une n-sesquicatégorie est un ensemble n-globulaire avec des opérations de composition strictement associatives et unitaires, qui ne sont cependant pas tenues de satisfaire les lois d'échange de Godement qui s'appliquent aux n-catégories. Dans [6], nous avons montré comment celles-ci peuvent être définies comme des algèbres sur une monade $T_n^{D^s}$ dont les opérations sont des diagrammes de cordes simples. Dans le présent article, nous donnons une description explicite des polygraphes pour cette monade et nous prouvons que la catégorie associée de computades est une catégorie de préfaisceaux. Nous utilisons ceci pour décrire une notation de diagrammes de cordes pour représenter des composés arbitraires dans des n-sesquicatégories. Ceci est un pas vers une théorie des diagrammes de cordes pour les n-catégories semistrictes.

Abstract. An n-sesquicategory is an n-globular set with strictly associative and unital composition and whiskering operations, which are however not required to satisfy the Godement interchange laws which hold in n-categories. In [6] we showed how these can be defined as algebras over a monad $T_n^{D^s}$ whose operations are simple string diagrams. In the present paper, we give an explicit description of computads for the monad $T_n^{D^s}$ and we prove that the category of computads for this monad is a presheaf category. We use this to describe a string diagram notation for representing arbitrary composites in n-sesquicategories. This is a step towards a theory of string diagrams for semistrict n-categories.

Keywords. String diagrams. Higher categories. Monads. Computads.
Mathematics Subject Classification (2010). 18N20, 18N30.

1. Introduction

The use of string diagram notation as a tool for representing composites in higher categories is becoming ever more widespread. This paper is part of a project which aims to give a definition of semistrict n-category based on a purely algebraic/combinatorial notion of string diagram. In [6] we defined a monad $T_n^{D^s}$ on the category of n-globular sets, whose operations we call simple string diagrams. We give a generators and relations description of $T_n^{D^s}$, which allows us to characterize its algebras, which we call n-sesquicategories, as n-globular sets equipped with strictly associative and unital composition and whiskering operations, which however do not satisfy the Godement interchange laws that hold in a strict n-category. We think of simple string diagrams as analogous to the globular pasting diagrams used in the definition of the monad T_n^{str} whose algebras are strict n-categories ([28]). In the present paper we study computads for the monad $T_n^{D^s}$ and show how morphisms in an n-sesquicategory generated by a computad C can be depicted as general C-labelled string diagrams. We also prove that the category of computads for this monad is equivalent to the category of presheaves on a small category of computadic cell shapes. In future work, we will show how to add coherent weak interchange laws to get a notion of semistrict n-category,

1.1 Results

We now describe the main result in this paper. Denote by Comp_{n+1}^n the category of $(n+1)$-computads for $T_n^{D^s}$, by $\mathbb{1}$ the terminal $(n+1)$-computad and by $F_n(C)$ the free n-sesquicategory generated by an n-computad C. Cells $c \in \mathbb{1}_k$ for $k \leq n+1$ are called **k-cell shapes** and morphisms $d \in F_n(\mathbb{1})_k$ for $k \leq n$ are called **unlabelled k-diagrams**. A morphism $x \in F_n(C)$ is said to have shape d if its image in $F_n(\mathbb{1})$ is d. Given such d, we construct a computad $\hat{d}$ with the property that d-shaped morphisms in a computad D are in canonical bijection with maps $\hat{d} \to D$. Using this, we define a small category Cell_{n+1} whose objects are cell shapes, together with a fully faithful embedding $\widehat{(-)} : \mathrm{Cell}_{n+1} \hookrightarrow \mathrm{Comp}^n_{n+1}$. From this we construct the nerve/realization adjunction

$$|-| : \mathrm{Psh}(\mathrm{Cell}^n_{n+1}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Comp}^n_{n+1} : N \ .$$

Theorem 1.1. *The adjunction* $|-| : \mathrm{Psh}(\mathrm{Cell}_{n+1}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Comp}^n_{n+1} : N$ *is an equivalence of categories.*

We now give an outline of the proof. In [6] we showed that $T_n^{D^s}$ has a presentation with generators $\mathcal{O}_n$ and relations $\mathcal{E}_n$. We can describe morphisms in the free n-sesquicategory $F_n(C)$ generated by an n-computad C as equivalence classes of trees whose internal vertices are labelled by generators in $\mathcal{O}_n$ and whose leaves are labelled by cells in C. The equivalence relation is generated by the relations in $\mathcal{E}_n$. We then prove that each of these trees has a unique **normal form** in its equivalence class. This allows us to show that for an unlabelled diagram d the category $\mathrm{Comp}^n_{n+1}(d)$ of pairs (C, x), where C is an $(n+1)$-computad and x is a morphism of shape d, has an initial object, which we denote $(\hat{d}, \tilde{d})$. This allows us to construct the nerve/relization adjunction as mentioned above and then the proof of the Theorem follows by formal arguments from the fact $(\hat{c}, \tilde{c})$ is initial, for $c \in \mathrm{Cell}_{n+1}$.

Remark 1.2. In fact our proof of the Theorem above applies to any globular operad presented by generators and relations, as long as this presentation admits a theory of normal forms. See Remark 5.16 for details.

The theory of normal forms also provides an **algorithm** to decide whether two morphisms in the free n-sesquicategory $F_n(C)$ generated by C given as composites of generating cells are actually equal.

After we've established this Theorem, we go on to give a description of the diagrammatic interpretation of morphisms in the n-sesquicategory generated by C as C-**labelled string diagrams**. Normal forms are an essential ingredient in this description.

1.2 Related work

The string diagrammatic calculus for monoidal categories and bicategories is by now well established. Generalizations to Gray 3-categories also exist, in the theory of surface diagrams ([11], [27]). Recently there has been a lot of progress in extending this to higher dimensions, with the discovery of the theory of associative n-categories ([19]), later developed into the manifold diagrams of [20]. These manifold diagrams have a combinatorial counterpart, which the authors of [20] call trusses, which are in turn equivalent to the notion of zigzgags introduced in [34] and which forms the basis for an online proof assistant for diagrammatic calculus in higher categories ([1]).

There are two main differences between the approach above and the one followed in this paper. The first is that the input of our theory is the simple **combinatorial** notion of simple string diagram introduced in [6], whereas manifold diagrams start from the geometry and obtain from that a combinatorial description, by passing to exit path posets. The second is that we want to produce an **algebraic** notion of semistrict n-categories, by which we mean that these will be algebras over a certain monad on n-globular sets. One advantage of the manifold diagrams approach to semistrict n-categories is that all coherences are already encoded in the basic cell shapes, whereas we naturally produce a theory of n-sesquicategories, to which we then have to add coherent weak interchange laws. The main advantage of our approach is its simplicity, as in a sense everything follows from the combinatorial notion of simple string diagrams introduced in [6].

Most closely related to our work is [10]. There the authors develop a framework which is the basis for another proof assistant for diagrammatic calculus in higher categories ([9]). The authors have a notion of **signature**, which corresponds exactly to a computad for $T_n^{D^s}$, and a notion of **diagram** over a signature, which corresponds exactly to a morphism in the n-sesquicatery generated by a computad. In this sense, our work can also be seen as providing a mathematical foundation for the kinds of higher categorical structures implemented by this proof assistant.

Our work is also related to questions in the general theory of computads ([36], [37], [33], [14], [12], [32]). If one considers the monad T_n^{str} whose algebras are strict n-categories, then computads consist of presentations for strict n-categories. Cells of dimension $k \leq n$ are generating k-morphisms and $(n+1)$-cells are relations. The cells of the terminal n-computad for T_n^{str} are the most general n-categorical cell shapes and the morphisms in the n-category generated by it can be thought of as general unlabelled pasting diagrams. One would then like to say that the category of computads for T_n^{str} is a category of presheaves on the cell shapes, but this turns out to be false ([31], [17]), essentially because of the Eckman-Hilton argument. This lead to the question of finding conditions on monads or restrictions on allowable cells in the associated computads that guarantee that one obtains a presheaf category ([13],[26],[18]). Our paper can also be seen as a continuation of this line of research, providing a monad on n-globular sets which is related to T_n^{str} and whose category of computads is a presheaf category. On pasting diagrams, see also [21].

Finally, the motivation for developing this theory was to be able to use string diagrams to prove results about higher categories. In [3] we develop a string diagram calculus for strict 4-categories and we use it to prove a result about fibrations of mapping 4-groupoids. In [4] and [5] we use this string diagram calculus to prove coherence results for adjunctions in 3 and 4-categories. In [2], we use a string diagram caculus for strict monoidal 3-categories to prove a coherence result for 3-dualizable objects in strict symmetric monoidal 3-categories.

After the appearance of the present paper on the arXiv, an independent proof of the fact that computads for n-sesquicategories from a presheaf category has appered in [22]. The authors define n-sesquicategories directly by generating operations and relations, so their theory does not mention the combinatorics of simple string diagrams. Their use of rewriting theory to establish the existence of normal forms is a very interesting alternative to our methods in Section 4 of the present paper. To go from normal forms to the main result, they then appeal to Makkai's criterion for presheaf categories. Our approach to this in Section 5 gives a shorter and more direct proof.

1.3 Future work

One can construct a monad T_n^{ss} by adding $(k+1)$-operations (resp. relations) to $T_n^{D^s}$ connecting pairs of simple k-string diagrams that map to the same k-pasting diagram under the map of monads $T_n^{D^s} \to T_n^{str}$, for $k \leq n-1$ (resp. $k = n$). By constrution, this comes with a contractible map $T_n^{ss} \to T_n^{str}$.

In dimensions $\leq n-1$ the monad T_n^{ss} is constructed from $T_n^{D^s}$ by freely adding operations, so the methods in this paper should apply to show that the associated category of n-computads is a presheaf category (although the category of $(n+1)$-computads is not, as relations between n-dimensional operations are added). We can define **semistrict n-categories** as T_n^{ss}-algebras. By construction, they will admit a string diagram calculus. Moreover, a result conjectured in [25, 6.2.3] suggests a possible way of proving that any weak n-category is equivalent to a semistrict n-category in this sense. This is the subject of ongoing research and we will explore it in future papers.

We are also interested in finding finite descriptions of T_n^{ss}. In an upcoming paper, we show how to construct T_3^{ss} by adding a finite set of generators and relations to the monad $T_3^{D^s}$. We then show that its algebras agree with Gray 3-categories. We are working on extending this to dimension 4.

Once the definitions of semistrict 3 and 4-categories are in place, we can extend the coherence results for adjunctions of [4] and [5] to this setting. We will then put this together to extend the coherence result for 3-dualizable objects of [2] to this setting. An extension of this result to the fully weak setting would give a finite presentation of the framed fully extended 3-dimensional bordism category, by the Cobordism Hypothesis ([8],[29],[7],[23]).

2. Background

Denote by gSet_n the category of n-globular sets. Given a finitary monad $T : \mathrm{gSet}_n \to \mathrm{gSet}_n$ one can define categories Comp_k^T of computads for T, for $k = 0, \cdots, n+1$, together with adjunctions

$$F_k : \mathrm{Comp}_k^T \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Alg}_T : V_k.$$

This is done inductively, by defining a k-computad C to be a tuple $(C_k, C_{\leq k-1}, s, t)$ where C_k is a set, which we call the set of k-cells of C, $C_{\leq k-1}$ is a $(k-1)$-computad, and $s, t : C_k \to F_{k-1}(C_{\leq k-1})_{k-1}$ satisfy the globularity relations

$ss = st$ and $ts = tt$. One then defines F_k, for $k \le n$, by the pushout

$$\begin{tikzcd} T_n^{D^s}(C_k \times \partial\theta^{(k)}) \arrow[r] \arrow[d] & T_n^{D^s}(C_k \times \theta^{(k)}) \arrow[d] \\ F_{k-1}(C_{\le k-1}) \arrow[r] & F_k(C), \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

where $\theta^{(k)}$ is the globular set represented by k. For $k = n+1$, we replace the inclusion $\partial\theta^{(k)} \hookrightarrow \theta^{(k)}$ by the collpase $\partial\theta^{(n+1)} \to \theta^{(n)}$. Similarly, one defines V_k by a pullback. See [35] for a detailed exposition of this theory of computads (for the original references, see [36], [37], [33], [14] and [12]).

Remark 2.1. There are incusion maps $\mathrm{Comp}_k^T \hookrightarrow \mathrm{Comp}_{k+1}^T$ for $k \le n$, so we can think of k-computads as $(n+1)$-computads. For this reason, we sometimes write Comp^T instead of Comp_{n+1}^T and use the term computad to refer to an $(n+1)$-computad.

In [6] we introduced a monad $T_n^{D^s}$ on globular sets, based on a notion of **simple string diagram** and we defined an **n-sesquicategory** as an algebra over this monad. The is a map $T_n^{D^s} \to T_n^{str}$ to the monad for strict n-categories, so any strict n-category is an n-sesquicategory. In fact n-sesquicategories are just strict n-categories without the interchange laws.

Notation 2.2. We denote by Sesq_n the category of $T_n^{D^s}$-algebras.

In [6] we gave a presentation of $T_n^{D^s}$ by generators $\mathcal{O}_n$ and relations $\mathcal{E}_n$. There we think of the generators as simple string diagrams, but here we interact with the monad $T_n^{D^s}$ only through this presentation, so we may as well view the generators as symbols. There is a generator $\circ_{i,j}$ for each $i, j = 1, \cdots, n$ and a generator u_i for each $i = 1, \cdots, n$. Given an n-sesquicategory $\mathcal{C}$, the generator $\circ_{i,j}$ induces a map $\circ_{i,j}^{\mathcal{C}} : \mathcal{C}_i \times_{\mathcal{C}_m} \mathcal{C}_j \to \mathcal{C}_M$, where $m = \min\{i, j\}$ and $M = \max\{i, j\}$. We call this **composition** when $i = j$ and **whiskering** when $i \neq j$. The generator u_i induces a map $u_i^{\mathcal{C}} : \mathcal{C}_{i-1} \to \mathcal{C}_i$ and we call $u_i^{\mathcal{C}}(x)$ the **identity** on x. The relations in $\mathcal{E}_n$ essentially express the associativity and unitality of $\circ_{i,j}$ (these relations also appear below Definition 3.8).

Notation 2.3. We denote by $\mathcal{O}_n$ and $\mathcal{E}_n$ the sets of generators and relations for $T_n^{D^s}$ introduced in [6] and described in the preceding paragraph.

Remark 2.4. The monad $T_n^{D^s}$ corresponds to an n-globular operad D_n^s and the presentation by generators and relations corresponds to a presentation of the globular operad in the sense of [24].

We also characterize n-sesquicategories inductively as categories $\mathcal{C}$ equipped with a lift of the Hom functor

$$\begin{tikzcd}
& \mathrm{Sesq}_{n-1} \arrow[dr, "(-)_0"] & \\
\mathcal{C}^{op} \times \mathcal{C} \arrow[ur, dotted, "\underline{\mathrm{Hom}}_{\mathcal{C}}"] \arrow[rr, "\mathrm{Hom}_{\mathcal{C}}"'] & {} \arrow[u, phantom, "="] & \mathrm{Set},
\end{tikzcd}$$

but this will not be relevant in the present paper.

We now briefly review the generators and relations description of $T_n^{D^s}$, which our description of computads in the present paper will build on. This discussion will be informal, see [6] for details.

Definition 2.5. *Let X be an n-graded set. A k-dimensional $(\mathcal{O}_n, X)$-labelled tree is a rooted tree T, together with*

1. *a labelling of its internal vertices $I(T)$ by generators in $\mathcal{O}_n$;*
2. *a labelling of its leaves $L(T)$ by elements in X;*
3. *a bijection between the incoming edges at an internal vertex and the inputs of the associated generator;*

such that

1. *the root label has dimension k;*
2. *the source of each incoming edge at an internal vertex has a label of the appropriate dimension.*

We denote the set of k-dimensional $(\mathcal{O}_n, X)$-labeled trees by $\mathrm{Tree}_n^{\mathcal{O}}(X)(k)$ or $\mathrm{Tree}_n^{\mathcal{O}}(X)_k$.

Terminology 2.6. An $(\mathcal{O}_n, X)$**-labelled subtree** (or simply **subtree** for short) of an $(\mathcal{O}_n, X)$-labelled tree T consists of all vertices (internal and leaves) that can be reached from a chosen internal vertex of T (the root of the subtree) by travelling towards the leaves.

An $\mathcal{O}_n$**-labelled tree** is a tree with a labelling of its vertices by $\mathcal{O}_n$. An $\mathcal{O}_n$**-labelled subtree** of an $(\mathcal{O}_n, X)$-labelled tree T is a subtree of T in the usual sense, containing no leaves and inheriting the $\mathcal{O}_n$-labelling.

When X is an n-globular set, we can define source and target maps $s, t : \mathrm{Tree}_n^{\mathcal{O}}(X)_k \to \mathrm{Tree}_n^{\mathcal{O}}(X)_{k-1}$, although in general they won't satisfy the globularity relation.

Definition 2.7. *An n**-preglobular set** is an n-graded set $X = \coprod_{i=0}^n X_i$ equipped with source and target maps $s, t : X_k \to X_{k-1}$. A **globular relation** on X is a relation $\sim$ such that*

1. *if $x \sim \tilde{x}$ then $s(x) \sim s(\tilde{x})$ and $t(x) \sim t(\tilde{x})$;*
2. *$ss(x) \sim st(x)$ and $ts(x) \sim tt(x)$.*

Note that this means the quotient $X/{\sim}$ is an n-globular set.

So given an n-globular set X, we have an n-preglobular set $\mathrm{Tree}_n^{\mathcal{O}}(X)$.

Definition 2.8. *We define an n-preglobular subset $\mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(X) \subset \mathrm{Tree}_n^{\mathcal{O}}(X)$ of $\stackrel{\epsilon}{=}$-compatible trees, equipped with a preglobular relation $\stackrel{\epsilon}{=}$. The definition is by induction on height. The relation $\stackrel{\epsilon}{=}$ is generated by the relations in $\mathcal{E}_n$. A tree is $\stackrel{\epsilon}{=}$-compatible if for every subtree of the form $x \to \circ_{i,j} \leftarrow y$ we have $s^{i-m+1}(x) \stackrel{\epsilon}{=} t^{j-m+1}(y)$, where $m = \min\{i, j\}$.*

Finally we define $\widetilde{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}}(X) := \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(X)/\stackrel{\epsilon}{=}$ and we show that this defines a monad on n-globular sets. We construct a map of monads

$$\varphi : \widetilde{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}} \to T_n^{D^s}.$$

Each generator in $\mathcal{O}_n$ corresponds to a simple string diagram, so one can use composition of simple string diagrams to produce this map.

Theorem 2.9 ([6])**.** *The map $\varphi : \widetilde{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}} \to T_n^{D^s}$ is an isomorphism of monads.*

3. Computads for $T_n^{D^s}$

We give an explicit description of computads for $T_n^{D^s}$ and of the n-sesquicategories generated by them, which we will later show is equivalent to the notion described in the previous section. We will simply call them computads, leaving the monad $T_n^{D^s}$ implicit.

Definition 3.1. *Given $k \leq n+1$, an (n,k)-**precomputad** (or simply k-**precomputad**, leaving n implicit) C consists of sets C_i for $0 \leq i \leq k$, together with maps $s,t : C_i \to \mathrm{Tree}_n^{\mathcal{O}}(C_{\leq i-1})_{i-1}$ for $1 \leq i \leq k$.*

In the definition below we use the following notation for grafting of trees.

Notation 3.2. Given and $(\mathcal{O}_n, C)$-labelled tree $x \in \mathrm{Tree}_n^{\mathcal{O}}(C)_{i-1}$ we denote by

$$x \to u_i$$

the $(\mathcal{O}_n, C)$-labelled tree obtained by adding a new new vertex to x labelled by u_i and an edge from the root of x to this new vertex. The new vertex now becomes the root of this new tree. Similarly, given $x \in \mathrm{Tree}_n^{\mathcal{O}}(C)_i$ and $y \in \mathrm{Tree}_n^{\mathcal{O}}(C)_j$ we denote by

$$x \to \circ_{i,j} \leftarrow y$$

the $(\mathcal{O}_n, C)$-labelled tree obtained by adding a new root labelled by $\circ_{i,j}$.

Definition 3.3. *Given a k-precomputad C, we define source and target maps $s,t : \mathrm{Tree}_n^{\mathcal{O}}(C)_i \to \mathrm{Tree}_n^{\mathcal{O}}(C)_{i-1}$, for $1 \leq i \leq n$. For trees of height zero, these are the maps $s,t : C_i \to \mathrm{Tree}_n^{\mathcal{O}}(C)_{i-1}$. For trees of nonzero height, we use the following inductive formulas for s, where $j < i$ and x and y are trees with appropriate dimensions in each case. The map t is defined by the same formulas, replacing every instance of s with t.*

$$s(x \to u_i) = x;$$

$$s(x \to \circ_{i,i} \leftarrow y) = s(y);$$

$$s(x \to \circ_{j,i} \leftarrow y) = x \to \circ_{j,i-1} \leftarrow s(y);$$

$$s(x \to \circ_{i,j} \leftarrow y) = s(x) \to \circ_{i-1,j} \leftarrow y.$$

Remark 3.4. Since the source or target of a k-cell may be an arbitrary $(\mathcal{O}_n, C)$-labelled tree, the source and target maps above can increase the height of trees. This is in contrast to the situation of [6], where we considered $\mathrm{Tree}_n^{\mathcal{O}}(X)$ for a globular set X. However, these maps always decrease the dimension of the tree, so the arguments in [6] which relied on induction on the height of the tree can now be replaced by simultaneous induction on both the height and the dimension of the tree, as we will do below in Definition 3.8.

The following definitions refer to each other and should be interpreted by mutual induction.

Definition 3.5. *Given $k \leq n+1$, an (n,k)-**computad** (or simply k-**computad**, leaving n implicit) C consists of sets C_i for $0 \leq i \leq k$, together with maps $s, t : C_i \to \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C_{\leq i-1})_{i-1}$ for $1 \leq i \leq k$, such that $ss(x) \overset{\epsilon}{=} st(x)$ and $ts(x) \overset{\epsilon}{=} tt(x)$ for all $x \in C_i$.*

Terminology 3.6. A **computad** is an (n,k)-computad, where n is usually implicit in the context and $k \leq n+1$ is arbitrary.

Notation 3.7. Let X be an n-graded set. We denote by

$$\tau_{\leq h} \mathrm{Tree}_n^{\mathcal{O}}(X) \subset \mathrm{Tree}_n^{\mathcal{O}}(X)$$

the n-graded subset consisting of trees of height at most h.

The definition that follows is almost identical to the analogous one in [6]. The only difference is the one explained in Remark 3.4.

Definition 3.8. *Let C be a computad. For each k, we define, by induction on h, subsets $\tau_{\leq h} \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k \subset \tau_{\leq h} \mathrm{Tree}_n^{\mathcal{O}}(C)_k$ equipped with a relation $\overset{\epsilon}{=}_h$. Elements in $\tau_{\leq h} \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$ are called $\overset{\epsilon}{=}_{h-1}$-**compatible**. We say that $x \in \mathrm{Tree}_n^{\mathcal{O}}(C)_k$ is $\overset{\epsilon}{=}$-compatible if it is $\overset{\epsilon}{=}_h$-compatible for some h and define $\mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k \subset \mathrm{Tree}_n^{\mathcal{O}}(C)_k$ the set of $\overset{\epsilon}{=}$-compatible elements. Finally, we define the relation $\overset{\epsilon}{=}$ on $\mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$ by declaring $x \overset{\epsilon}{=} \tilde{x}$ when $x \overset{\epsilon}{=}_h \tilde{x}$ for some h. The definition is by overall induction on k and is presented below.*

When $h = 0$, we let $\tau_{\leq 0} \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k := \tau_{\leq 0} \mathrm{Tree}_n^{\mathcal{O}}(C)_k = C_k$ and the relation $\overset{\epsilon}{=}_0$ is $=$.

Now consider $h \geq 1$. Any $x \in \tau_{\leq h} \operatorname{Tree}_n^{\mathcal{O}}(C)_k$ of height zero is $\overset{\epsilon}{=}_{h-1}$-compatible. Let $x \in \tau_{\leq h-1} \operatorname{Tree}_n^{\mathcal{O}}(C)_i$, $y \in \tau_{\leq h-1} \operatorname{Tree}_n^{\mathcal{O}}(C)_j$ and $m = \min\{i, j\}$. Then

$$\begin{tikzcd} & \circ_{i,j} & \\ x \arrow[ur] & & y \arrow[ul] \end{tikzcd}$$

is $\overset{\epsilon}{=}_{h-1}$-compatible if and only if x, y are $\overset{\epsilon}{=}_{h-2}$-compatible and $s^{i-m+1}(x) \overset{\epsilon}{=} t^{j-m+1}(y)$. Moreover, $x \to u_{i+1}$ is $\overset{\epsilon}{=}_{h-1}$-compatible if and only if x is $\overset{\epsilon}{=}_{h-2}$-compatible. Now we must define the globular relation $\overset{\epsilon}{=}_h$ on $\tau_{\leq h} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$.

If $x, y \in \tau_{\leq h} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$ have height zero and $x \overset{\epsilon}{=}_0 y$, then $x \overset{\epsilon}{=}_h y$.

Let $i \leq k$, $x \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_{i-1}$, $y \in \tau_{\leq h-1} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$. If $x \overset{\epsilon}{=} t^{k-i+1}(y)$, then

$$(\lambda_{i,k}) : \quad \begin{tikzcd} & \circ_{i,k} & & & \\ u_i \arrow[ur] & & y \arrow[ul] & \overset{\epsilon}{=}_h & y. \\ x \arrow[u] & & & & \end{tikzcd}$$

Let $i \leq k$, $x \in \tau_{\leq h-1} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$, $y \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_{i-1}$. If $s^{k-i+1}(x) \overset{\epsilon}{=} y$, then

$$(\rho_{k,i}) : \quad \begin{tikzcd} & \circ_{k,i} & & & \\ x \arrow[ur] & & u_i \arrow[ul] & \overset{\epsilon}{=}_h & x. \\ & & y \arrow[u] & & \end{tikzcd}$$

Let $i < k$, $x \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_i$, $y \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_{k-1}$. If $s(x) \overset{\epsilon}{=} t^{k-i}(y)$, then

$$(\rho_{i,k}) : \quad \begin{tikzcd} & \circ_{i,k} & & & & u_k & \\ x \arrow[ur] & & u_k \arrow[ul] & \overset{\epsilon}{=}_h & & \circ_{i,k-1} \arrow[u] & \\ & & y \arrow[u] & & x \arrow[ur] & & y \arrow[ul] \end{tikzcd} .$$

Let $i < k$, $x \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_{k-1}$, $y \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_i$. If $s^{k-i}(x) \overset{\epsilon}{=} t(y)$, then

$$(\lambda_{k,i}): \quad \begin{tikzcd} & \circ_{k,i} & \\ u_k \arrow[ur] & & y \arrow[ul] \\ x \arrow[u] & & \end{tikzcd} \quad \stackrel{\epsilon}{=}_h \quad \begin{tikzcd} & u_k & \\ & \circ_{k-1,i} \arrow[u] & \\ x \arrow[ur] & & y \arrow[ul] \end{tikzcd} \quad .$$

Let $k \geq 1$ and $x, y, z \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$. If $s(x) \stackrel{\epsilon}{=} t(y)$ and $s(y) \stackrel{\epsilon}{=} t(z)$, then

$$(\circ_{k,k,k}): \quad \begin{tikzcd} & & \circ_{k,k} & \\ & \circ_{k,k} \arrow[ur] & & z \arrow[ul] \\ x \arrow[ur] & & y \arrow[ul] & \end{tikzcd} \quad \stackrel{\epsilon}{=}_h \quad \begin{tikzcd} & \circ_{k,k} & & \\ x \arrow[ur] & & \circ_{k,k} \arrow[ul] & \\ & y \arrow[ur] & & z \arrow[ul] \end{tikzcd} \quad .$$

Let $i < k$, $x, y \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_i$, $z \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$. If $s(x) \stackrel{\epsilon}{=} t(y)$ and $s(y) \stackrel{\epsilon}{=} t^{k-i+1}(z)$, then

$$(\circ_{i,i,k}): \quad \begin{tikzcd} & & \circ_{i,k} & \\ & \circ_{i,i} \arrow[ur] & & z \arrow[ul] \\ x \arrow[ur] & & y \arrow[ul] & \end{tikzcd} \quad \stackrel{\epsilon}{=}_h \quad \begin{tikzcd} & \circ_{i,k} & & \\ x \arrow[ur] & & \circ_{i,k} \arrow[ul] & \\ & y \arrow[ur] & & z \arrow[ul] \end{tikzcd} \quad .$$

Let $i < k$, $x, z \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_i$, $y \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$. If $s(x) \stackrel{\epsilon}{=} t^{k-i+1}(y)$ and $s^{k-i+1}(y) \stackrel{\epsilon}{=} t(z)$, then

$$(\circ_{i,k,i}): \quad \begin{tikzcd} & & \circ_{k,i} & \\ & \circ_{i,k} \arrow[ur] & & z \arrow[ul] \\ x \arrow[ur] & & y \arrow[ul] & \end{tikzcd} \quad \stackrel{\epsilon}{=}_h \quad \begin{tikzcd} & \circ_{i,k} & & \\ x \arrow[ur] & & \circ_{k,i} \arrow[ul] & \\ & y \arrow[ur] & & z \arrow[ul] \end{tikzcd} \quad .$$

Let $i < k$, $x \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$, $y, z \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_i$. If $s^{k-i+1}(x) \stackrel{\epsilon}{=} t(y)$ and $s(y) \stackrel{\epsilon}{=} t(z)$, then

$$(\circ_{k,i,i}): \quad \begin{tikzcd} & & \circ_{k,i} & \\ & \circ_{k,i} \arrow[ur] & & z \arrow[ul] \\ x \arrow[ur] & & y \arrow[ul] & \end{tikzcd} \quad \stackrel{\epsilon}{=}_h \quad \begin{tikzcd} & \circ_{k,i} & & \\ x \arrow[ur] & & \circ_{i,i} \arrow[ul] & \\ & y \arrow[ur] & & z \arrow[ul] \end{tikzcd} \quad .$$

Let $i < k$, $x \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_i$, $y, z \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$. If $s(x) \overset{\epsilon}{=} t^{k-i+1}(y)$ and $s(y) \overset{\epsilon}{=} t(z)$, then

$$(\circ_{i,k,k}) : \begin{tikzcd} & \circ_{i,k} & & \\ x \arrow[ur] & & \circ_{k,k} \arrow[ul] & \\ & y \arrow[ur] & & z \arrow[ul] \end{tikzcd} \overset{\epsilon}{=}_h \begin{tikzcd} & & & \circ_{k,k} & & & \\ & \circ_{i,k} \arrow[urr] & & & & \circ_{i,k} \arrow[ull] & \\ x \arrow[ur] & & y \arrow[ul] & & x \arrow[ur] & & z \arrow[ul] \end{tikzcd} .$$

Let $i < k$, $x, y \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$, $z \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_i$. If $s(x) \overset{\epsilon}{=} t(y)$ and $s^{k-i+1}(y) \overset{\epsilon}{=} t(z)$, then

$$(\circ_{k,k,i}) : \begin{tikzcd} & & \circ_{k,i} & \\ & \circ_{k,k} \arrow[ur] & & z \arrow[ul] \\ x \arrow[ur] & & y \arrow[ul] & \end{tikzcd} \overset{\epsilon}{=}_h \begin{tikzcd} & & & \circ_{k,k} & & & \\ & \circ_{k,i} \arrow[urr] & & & & \circ_{k,i} \arrow[ull] & \\ x \arrow[ur] & & z \arrow[ul] & & y \arrow[ur] & & z \arrow[ul] \end{tikzcd} .$$

Let $i < j < k$ and take $x \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_i$, $y \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_j$ and $z \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$. If $s(x) \overset{\epsilon}{=} t^{j-i+1}(y)$ and $s(y) \overset{\epsilon}{=} t^{k-j+1}(z)$, then

$$(\circ_{i,j,k}) : \begin{tikzcd} & \circ_{i,k} & & \\ x \arrow[ur] & & \circ_{j,k} \arrow[ul] & \\ & y \arrow[ur] & & z \arrow[ul] \end{tikzcd} \overset{\epsilon}{=}_h \begin{tikzcd} & & & \circ_{j,k} & & & \\ & \circ_{i,j} \arrow[urr] & & & & \circ_{i,k} \arrow[ull] & \\ x \arrow[ur] & & y \arrow[ul] & & x \arrow[ur] & & z \arrow[ul] \end{tikzcd} .$$

Let $i < j < k$ and take $x \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_i$, $y \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$ and $z \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_j$. If $s(x) \overset{\epsilon}{=} t^{k-i+1}(y)$ and $s^{k-j+1}(y) \overset{\epsilon}{=} t(z)$, then

$$(\circ_{i,k,j}) : \begin{tikzcd} & \circ_{i,k} & & \\ x \arrow[ur] & & \circ_{k,j} \arrow[ul] & \\ & y \arrow[ur] & & z \arrow[ul] \end{tikzcd} \overset{\epsilon}{=}_h \begin{tikzcd} & & & \circ_{k,j} & & & \\ & \circ_{i,k} \arrow[urr] & & & & \circ_{i,j} \arrow[ull] & \\ x \arrow[ur] & & y \arrow[ul] & & x \arrow[ur] & & z \arrow[ul] \end{tikzcd} .$$

Let $i < j < k$ and take $x \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_j$, $y \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$ and $z \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_i$. If $s(x) \overset{\epsilon}{=} t^{k-j+1}(y)$ and $s^{k-i+1}(y) \overset{\epsilon}{=} t(z)$, then

$$(\circ_{j,k,i}) : \quad \begin{tikzcd}[column sep=small] & & \circ_{k,i} & \\ & \circ_{j,k} \arrow[ur] & & z \arrow[ul] \\ x \arrow[ur] & & y \arrow[ul] & \end{tikzcd} \quad \overset{\epsilon}{=}_h \quad \begin{tikzcd}[column sep=small] & & & \circ_{j,k} & & & \\ & \circ_{j,i} \arrow[urr] & & & & \circ_{k,i} \arrow[ull] & \\ x \arrow[ur] & & z \arrow[ul] & & y \arrow[ur] & & z \arrow[ul] \end{tikzcd} .$$

Let $i < j < k$ and take $x \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$, $y \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_j$ and $z \in \tau_{\leq h-2} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_i$. If $s^{k-j+1}(x) \overset{\epsilon}{=} t(y)$ and $s^{j-i+1}(y) \overset{\epsilon}{=} t(z)$, then

$$(\circ_{k,j,i}) : \quad \begin{tikzcd}[column sep=small] & & \circ_{k,i} & \\ & \circ_{k,j} \arrow[ur] & & z \arrow[ul] \\ x \arrow[ur] & & y \arrow[ul] & \end{tikzcd} \quad \overset{\epsilon}{=}_h \quad \begin{tikzcd}[column sep=small] & & & \circ_{k,j} & & & \\ & \circ_{k,i} \arrow[urr] & & & & \circ_{j,i} \arrow[ull] & \\ x \arrow[ur] & & z \arrow[ul] & & y \arrow[ur] & & z \arrow[ul] \end{tikzcd} .$$

Let $x, \tilde{x} \in \tau_{\leq h-1} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_{k-1}$. If $x \overset{\epsilon}{=}_{h-1} \tilde{x}$, then

$$(u_k) : \quad \begin{array}{c} u_k \\ \uparrow \\ x \end{array} \quad \overset{\epsilon}{=}_h \quad \begin{array}{c} u_k \\ \uparrow \\ \tilde{x} \end{array} .$$

Let $x, \tilde{x} \in \tau_{\leq h-1} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_i$, $y, \tilde{y} \in \tau_{\leq h-1} \operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_j$ and $m = \min\{i, j\}$. If $x \overset{\epsilon}{=}_{h-1} \tilde{x}$, $y \overset{\epsilon}{=}_{h-1} \tilde{y}$, $s^{i-m+1}(x) \overset{\epsilon}{=} t^{j-m+1}(y)$ and $s^{i-m+1}(\tilde{x}) \overset{\epsilon}{=} t^{j-m+1}(\tilde{y})$ then

$$(\circ_{i,j}) : \quad \begin{tikzcd}[column sep=small] & \circ_{i,j} & \\ x \arrow[ur] & & y \arrow[ul] \end{tikzcd} \quad \overset{\epsilon}{=}_h \quad \begin{tikzcd}[column sep=small] & \circ_{i,j} & \\ \tilde{x} \arrow[ur] & & \tilde{y} \arrow[ul] \end{tikzcd} .$$

Lemma 3.9. *The construction above defines an n-preglobular subset $\operatorname{Tree}_n^{\mathcal{O},\mathcal{E}}(C) \subset \operatorname{Tree}_n^{\mathcal{O}}(C)$ with a globular relation $\overset{\epsilon}{=}$, meaning we have*

1. *if $x \in \operatorname{Tree}_n^{\mathcal{O}}(C)$ is $\overset{\epsilon}{=}$-compatible, then so are $s(x)$ and $t(x)$;*
2. *if $x \overset{\epsilon}{=} \tilde{x}$ then $s(x) \overset{\epsilon}{=} s(\tilde{x})$ and $t(x) \overset{\epsilon}{=} t(\tilde{x})$;*
3. *if x is $\overset{\epsilon}{=}$-compatible, then $ss(x) \overset{\epsilon}{=} st(x)$ and $ts(x) \overset{\epsilon}{=} tt(x)$.*

Proof. The proof is very similar to the one for the analogous result in [6], the only difference being the one already mentioned in Remark 3.4. $\square$

Definition 3.10. *Let $0 \leq k \leq n$ and let C be a k-computad. We write*

$$\overline{\mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}}(C) := \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)/\overset{\epsilon}{=} .$$

Definition 3.11. *Let C be an $(n+1)$-computad. We define a relation $\overset{\epsilon,C}{=}$ on $\mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)$ by adding the new equation $s(x) \overset{\epsilon,C}{=} t(x)$ for each $x \in C_{n+1}$.*

Lemma 3.12. *Let C be an $(n+1)$-computad. Then $\overset{\epsilon,C}{=}$ is a globular relation on $\mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)$.*

Proof. This is easy to check. □

Definition 3.13. *Let C be an $(n+1)$-computad. We write*

$$\overline{\mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}}(C) := \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)/\overset{\epsilon,C}{=} .$$

Remark 3.14. The fact that $\overset{\epsilon}{=}$ and $\overset{\epsilon,C}{=}$ are globular relations implies $\overline{\mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}}(C)$ is an n-globular set. Using the isomorphism of monads $\widetilde{\mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}} \to T_n^{D^s}$ allows us to define a $T_n^{D^s}$ action on $\overline{\mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}}(C)$ by simply grafting trees. We refer to $\overline{\mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}}(C)$ as the **n-sesquicategory presented by C**. When C is an n-computad this is a **free n-sesquicategory**. When C is an $(n+1)$-computad, this is a quotient of the free n-sesquicaegory generated by $C_{\leq n}$ by the relations in C_{n+1}.

Definition 3.15. *Given k-computads C, D a map $f : C \to D$ is a collection of maps $f_i : C_i \to D_i$ such that $s(f_i(x)) \overset{\epsilon}{=} f_{i-1}(s(x))$ and $t(f_i(x)) \overset{\epsilon}{=} f_{i-1}(t(x))$ for all $x \in C_i$, where we have inductively used the map on trees induced by a map of $(k-1)$-computads.*

A map $f : C \to D$ induces a map $f : \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C) \to \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(D)$ by applying f to leaf labels.

Definition 3.16. *For $k \leq n+1$, we denote by Comp_k^n the category of (n,k)-computads and (n,k)-computad maps.*

Remark 3.17. Adding empty sets of cells provides an inclusion map

$$\mathrm{Comp}_k^n \hookrightarrow \mathrm{Comp}_{k+1}^n,$$

for $k \leq n$, so we can think of k-computads as $(n+1)$-computads. For this reason, we sometimes write Comp^n instead of Comp^n_{n+1} and refer to $(n+1)$-computads simply as **computads**. In fact we will ususally denote this category simply by Comp, leaving n implicit.

Lemma 3.18. *Let C be a computad. Then the following diagram is a pushout, for $k \leq n$.*

$$\begin{tikzcd}
T_n^{D^s}(C_k \times \partial\theta^{(k)}) \arrow[r] \arrow[d] & T_n^{D^s}(C_k \times \theta^{(k)}) \arrow[d] \arrow[dl, phantom, "\lrcorner"] \\
\overline{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}}(C_{\leq k-1}) \arrow[r] & \overline{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}}(C_{\leq k})
\end{tikzcd}$$

Proof. We must show that functors

$$\varphi_{\leq k} : \overline{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}}(C_{\leq k}) \to \mathcal{C}$$

correspond to pairs $(\varphi_{\leq k-1}, \varphi_k)$, where $\varphi_{\leq k-1} : \overline{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}}(C_{\leq k-1}) \to \mathcal{C}$ is a functor and $\varphi_k : C_k \to \mathcal{C}_k$ is a map, such that $\varphi_{\leq k-1}(s(x)) = s(\varphi_k(x))$ and $\varphi_{\leq k-1}(t(x)) = t(\varphi_k(x))$ for all $x \in C_k$. This is clear. $\square$

Lemma 3.19. *Let C be a computad. Then the following diagram is a pushout.*

$$\begin{tikzcd}
T_n^{D^s}(C_{n+1} \times \partial\theta^{(n+1)}) \arrow[r] \arrow[d] & T_n^{D^s}(C_{n+1} \times \theta^{(n)}) \arrow[d] \arrow[dl, phantom, "\lrcorner"] \\
\overline{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}}(C_{\leq n}) \arrow[r] & \overline{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}}(C)
\end{tikzcd}$$

Proof. We must show that functors

$$\varphi : \overline{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}}(C) \to \mathcal{C}$$

correspond to functors $\varphi_{\leq n} : \overline{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}}(C_{\leq n}) \to \mathcal{C}$ such that $\varphi_{\leq n}(s(x)) = \varphi_{\leq n}(t(x))$ for all $x \in C_{n+1}$. This is clear. $\square$

Proposition 3.20. *For $k \leq n+1$, the canonical map $\mathrm{Comp}_k^n \to \mathrm{Comp}_k^{T_n^{D^s}}$ is an equivalence of categories, and the following diagram commutes up to canonical natural isomorphism.*

$$\begin{tikzcd}
\mathrm{Comp}_k^n \arrow[rr, "\simeq"] \arrow[dr, "\overline{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}}"'] & & \mathrm{Comp}_k^{T_n^{D^s}} \arrow[dl, "F_k"] \\
& \mathrm{Sesq}_n &
\end{tikzcd}$$

Proof. Using induction on k and Lemmas 3.18 and 3.19 we get a canonical map

$$\mathrm{Comp}_k^n \to \mathrm{Comp}_k^{T_n^{D^s}}$$

such that the above diagram commutes up to canonical natural isomorphism.

To construct an inverse, given a computad $C \in \mathrm{Comp}_k^{T_n^{D^s}}$ and using induction on k one can view its source and target maps as

$$s, t : C_i \to \overline{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}}(C_{\leq i-1})_{i-1}$$

for $i \leq k$. Using the axiom of choice to obtain a section

$$\overline{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}}(C_{\leq i-1})_{i-1} \to \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C_{\leq i-1})_{i-1}$$

of the quotient map, we finally obtain maps $s, t : C_i \to \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C_{\leq i-1})_{i-1}$ as in Definition 3.1. $\square$

Remark 3.21. One can avoid using the axiom of choice by using instead normal forms, which give an explicit unique representative of each equivalence class in $\overline{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}}(C)$.

4. Normal form

In this section, given an n-computad C, we introduce the notion of normal form for elements of $\mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)$. Denoting by $N(C) \subset \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)$ the n-graded subset of elements in normal form, we prove that for any $x \in \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)$ there exists a unique $n(x) \in N(C)$ such that $n(x) \stackrel{\epsilon}{=} x$.

Remark 4.1. The proof below actually gives an algorithm for finding the normal form $n(x)$ associated to any term $x \in \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)$. Thus it gives an algorithm for deciding whether two such terms are equivalent.

Remark 4.2. If C is an $(n+1)$-computad, every term $x \in \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)$ still has a unique normal form $n(x) \overset{\epsilon}{=} x$. However, any nontrivial relation in C_{n+1} will provide terms x, y such that $x \overset{\epsilon,C}{=} y$ and $n(x) \neq n(y)$ (recall Definition 3.11). So normal forms apply most naturally to n-computads.

Notation 4.3. We write $m(\circ_{i,j}) := \min\{i, j\}$. When v is an internal vertex in an $(\mathcal{O}_n, C)$-labelled tree with label $\circ_{i,j}$, we write $m(v) = m(\circ_{i,j})$.

Definition 4.4. *An $x \in \mathrm{Tree}_n^{\mathcal{O}}(C)$ is m-**ordered** if for every edge of the form $v \to w$, where v, w are $\circ$-labelled, we have $m(v) < m(w)$.*

Definition 4.5. *An $\mathcal{O}_n$-labelled tree is m-**constant** if there are no u-labelled vertices and for every edge $v \to w$ we have $m(v) = m(w)$.*

Definition 4.6. *An m-**constant component** of $x \in \mathrm{Tree}_n^{\mathcal{O}}(C)$ is a maximal m-constant $\mathcal{O}_n$-labelled subtree.*

Definition 4.7. *An m-constant $\mathcal{O}_n$-labelled tree is in **normal form** if it is of one of the following forms:*

1. $$\begin{tikzcd}[row sep=tiny, column sep=tiny] \circ_{i,k} & & & \\ & \cdots \arrow[ul] & & \\ & & \circ_{i,k} \arrow[ul] & \\ & & & \circ_{k,i} \arrow[ul] \\ & & \cdots \arrow[ur] & \\ & \circ_{k,i} \arrow[ur] & & \end{tikzcd}$$ *with $i < k$;*

2. $$\begin{tikzcd}[row sep=tiny, column sep=tiny] & & \circ_{k,k} \\ & \cdots \arrow[ur] & \\ \circ_{k,k} \arrow[ur] & & \end{tikzcd}$$.

Definition 4.8. *An $x \in \mathrm{Tree}_n^{\mathcal{O}}(C)$ is in **normal form** if it is m-ordered, it contains no edges of the form $u \to \circ$ and each of its m-constant components is in normal form.*

Definition 4.9. *Let $x \in \mathrm{Tree}_n^{\mathcal{O}}(C)$. Define its **cell dimension** to be the maximum of the dimensions of the cells labelling the leaves of x. Denote this by* $\mathrm{cd}(x)$.

Lemma 4.10. *Let $x, \tilde{x} \in \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)$ and suppose $x \overset{\epsilon}{=} \tilde{x}$. Then* $\mathrm{cd}(x) = \mathrm{cd}(\tilde{x})$.

Proof. This is clear. □

Lemma 4.11. *Given $x \in \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)$, there exists $n(x) \overset{\epsilon}{=} x$ which is in normal form.*

Proof. One uses the defining equations of $\overset{\epsilon}{=}$ to rearrange generators. The (λ) and (ρ) relations allow us eliminate all units u_i for $i \leq \mathrm{cd}(x)$ and push the other units towards the root. Then the $(\circ)$ relations allow us to pass to an m-ordered tree and finally to put each m-constant component in normal form. □

Now we need to show that this normal form is unique.

Proposition 4.12. *Let $x, \tilde{x} \in \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)$ be in normal form and suppose $x \overset{\epsilon}{=} \tilde{x}$. Then $x = \tilde{x}$.*

We will prove this below. First we reduce to diagrams without u-labelled vertices.

Lemma 4.13. *Let $x, \tilde{x} \in \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$ be in normal form and suppose $x \overset{\epsilon}{=} \tilde{x}$. Let $\mathrm{cd} := \mathrm{cd}(x) = \mathrm{cd}(\tilde{x})$. Then $x = (x_\circ \to u_{\mathrm{cd}+1} \to \cdots \to u_k)$ and $\tilde{x} = (\tilde{x}_\circ \to u_{\mathrm{cd}+1} \to \cdots \to u_k)$, where $x_\circ$ and $\tilde{x}_\circ$ are in normal form, have no u-labelled vertices and $x_\circ \overset{\epsilon}{=} \tilde{x}_\circ$.*

Proof. It is obvious that one can decompose elements in normal form into a unit chain and a component containing no units. The only thing that requires proof is the fact that $x_\circ \overset{\epsilon}{=} \tilde{x}_\circ$. This follows from the observation that $x_\circ = s^{k-\mathrm{cd}}(x)$ and $\tilde{x}_\circ = s^{k-\mathrm{cd}}(\tilde{x})$. □

The above Lemma allows us to reduce the proof of Proposition 4.12 to the case where x, $\tilde{x}$ have no units. Now we would like to reduce to the case where one gets from x to $\tilde{x}$ without introducing units along the way.

Definition 4.14. *Define* $\mathrm{Tree}_n^{\mathcal{O}(\circ)}(C) \subset \mathrm{Tree}_n^{\mathcal{O}}(C)$ *to be the preglobular subset consisting of those trees not containing any* u*-labelled vertices. We then define a preglobular subset* $\mathrm{Tree}_n^{\mathcal{O}(\circ),\mathcal{E}(\circ)}(C) \subset \mathrm{Tree}_n^{\mathcal{O}(\circ)}(C)$ *of* $\overset{\circ}{=}$*-compatible trees with a globular relation* $\overset{\circ}{=}$*, in exactly the same way we defined* $\overset{\epsilon}{=}$ *and* $\overset{\epsilon}{=}$*-compatibility, except we omit all equations involving* u*.*

Definition 4.15. *We define the* ***reduction*** $r(x) \in \mathrm{Tree}_n^{\mathcal{O}(\circ)}$ *of* $x \in \mathrm{Tree}_n^{\mathcal{O}}(C)$ *inductively, as follows. We let* $r(x) = x$ *when* x *has height zero. Then, for* $i < k$*, we let* $r(x \to u_k) = \emptyset$ *and*

$$r(x \to \circ_{k,k} \leftarrow y) = \begin{cases} r(y) & r(x) = \emptyset; \\ r(x) & r(y) = \emptyset; \\ r(x) \to \circ_{k,k} \leftarrow r(y) & \textit{otherwise;} \end{cases}$$

$$r(x \to \circ_{i,k} \leftarrow y) = \begin{cases} r(y) & r(x) = \emptyset; \\ \emptyset & r(y) = \emptyset; \\ r(x) \to \circ_{i,k} \leftarrow r(y) & \textit{otherwise;} \end{cases}$$

$$r(x \to \circ_{k,i} \leftarrow y) = \begin{cases} r(x) & r(y) = \emptyset; \\ \emptyset & r(x) = \emptyset; \\ r(x) \to \circ_{k,i} \leftarrow r(y) & \textit{otherwise.} \end{cases}$$

Lemma 4.16. *If* $x \in \mathrm{Tree}_n^{\mathcal{O}(\circ)}$*, then* $r(x) = x$*.*

Proof. This is obvious. ☐

Lemma 4.17. *If* $w \in \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)$ *and* $r(w) = \emptyset$ *then* $s(w) \overset{\epsilon}{=} t(w)$*.*

Proof. The proof is by induction on the height of w. There are four cases, corresponding to the four possible root labels: u_k, $\circ_{k,k}$, $\circ_{i,k}$ and $\circ_{k,i}$, for $i < k$. Each of these follows by a simple argument. ☐

Lemma 4.18. *Given* $w, \tilde{w} \in \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)$*, we have*

1. $r(w) \in \mathrm{Tree}_n^{\mathcal{O}(\circ),\mathcal{E}(\circ)}(C)$*;*
2. *if* $r(w) \neq \emptyset$*, then* $sr(w) \overset{\circ}{=} rs(w)$ *and* $tr(w) \overset{\circ}{=} rt(w)$*;*

3. if $w \overset{\epsilon}{=} \tilde{w}$, then $r(w) \overset{\circ}{=} r(\tilde{w})$.

Proof. The proof is by mutual induction on dimension and height. For *1.* there is a case for each possible root label of w: u_k, $\circ_{k,k}$, $\circ_{i,k}$ and $\circ_{k,i}$ $(i < k)$. Each of these follows from a simple inductive argument.

For *2.* there are cases for root labels u_k, $\circ_{k,k}$, $\circ_{k-1,k}$, $\circ_{k,k-1}$, $\circ_{i,k}$ and $\circ_{k,i}$ $(i < k-1)$. We explain the $\circ_{k,k}$ case and leave the others to the reader. Let $w = (x \to \circ_{k,k} \leftarrow y)$. Now

$$sr(w) = \begin{cases} sr(x) & r(y) = \emptyset \\ sr(y) & r(y) \neq \emptyset \end{cases}$$

and $rs(w) = rs(y)$. If $r(y) \neq \emptyset$, we have $rs(y) \overset{\circ}{=} sr(y)$ by induction, so $rs(w) \overset{\circ}{=} sr(w)$. When $r(y) = \emptyset$, we need to show that $sr(x) \overset{\circ}{=} rs(y)$. We have $r(x) \neq \emptyset$, because $r(w) \neq \emptyset$. Then $sr(x) \overset{\circ}{=} rs(x)$ by induction. We also have $s(y) \overset{\epsilon}{=} t(y)$ by Lemma 4.17. Since w is $\overset{\epsilon}{=}$-compatible, we have $s(x) \overset{\epsilon}{=} t(y)$, so we have $s(x) \overset{\epsilon}{=} s(y)$ and then using *3.* by induction we have $rs(x) \overset{\circ}{=} rs(y)$, so $sr(x) \overset{\circ}{=} rs(y)$.

To prove *3.*, there is one case for each of the defining equations of $\overset{\epsilon}{=}$. We explain the $(\circ_{k,k,k})$ case, leaving the others to the reader. Let w and $\tilde{w}$ be the left and right hand sides of this equation, respectively. If at least one of the trees $r(x)$, $r(y)$, $r(z)$ is empty, then we get $r(w) = r(\tilde{w})$ and we are done. So we may assume they are all nonempty. In this case we get $r(w) \overset{\circ}{=} r(\tilde{w})$ by the same $(\circ_{k,k,k})$ equation, as long as $r(x)$, $r(y)$, $r(z)$ are $\overset{\circ}{=}$-compatible, $sr(x) \overset{\circ}{=} tr(y)$ and $sr(y) \overset{\circ}{=} tr(z)$. This first condition follows from *1.* by induction on height. The second condition follows from *2.* and *3.* by induction on height and dimension. □

Notation 4.19. Given $x \in \mathrm{Tree}_n^{\mathcal{O}}(C)$, we denote by $L(x)$ its set of leaves. Given $\ell \in L(x)$, we denote by $|\ell|$ the dimension of the cell labellng ℓ. We denote by $L_{\geq i}(x) \subset L(x)$ the set of leaves ℓ such that $|\ell| \geq i$.

Definition 4.20. *Given $x \in \mathrm{Tree}_n^{\mathcal{O}(\circ)}(C)$, we define $M(x) = \max\{j : |L_{\geq j}(x)| \geq 2\}$. If x only has one leaf, then $M(x) = -\infty$.*

Lemma 4.21. *If $x \overset{\circ}{=} \tilde{x}$, then $M(x) = M(\tilde{x})$.*

Proof. One just needs to check that this holds for each of the defining equations of $\overset{\circ}{=}$, which is easy. □

Definition 4.22. *Given $w \in \mathrm{Tree}_n^{\mathcal{O}}(C)$, we define a linear ordering of $L(w)$ as follows. When $w = (x \to u_k)$ then $L(w) = L(x)$ and we can just use induction on height. When $w = (x \to \circ_{i,j} \leftarrow y)$ then $L(w) = L(x) \coprod L(y)$ and we define the linear order on $L(w)$ by using the linear orders on $L(x)$ and $L(y)$ provided by induction on height, together with the rule that $\ell_x > \ell_y$ for any $\ell_x \in L(x)$ and $\ell_y \in L(y)$.*

Lemma 4.23. *Let $w, \tilde{w} \in \mathrm{Tree}_n^{\mathcal{O}(\circ),\mathcal{E}(\circ)}(C)$ and suppose $w \overset{\circ}{=} \tilde{w}$. Let $M := M(w) = M(\tilde{w})$. Then there is a (necessarily unique) order preserving isomorphism $L_{\geq M}(w) \to L_{\geq M}(\tilde{w})$.*

Proof. One just needs to check this for each of the equations defining $\overset{\circ}{=}$. The only equations requring some consideration are $(\circ_{i,k,k})$, $(\circ_{k,k,i})$, $(\circ_{i,j,k})$, $(\circ_{i,k,j})$, $(\circ_{j,k,i})$ and $(\circ_{k,j,i})$, which double some of the leaves. In each case, one can see that this doubling does not affect leaves in $L_{\geq M}$. For example, the $(\circ_{i,k,k})$ equation doubles the leaves in the subtree x. But since we have no u-labelled vertices, the subtrees y and z must both have at least one leaf labelled by a k-cell, so that w must have at least two leaves labelled by k-cells, so that $M = k$. Then $i < k$ implies $i < M$, so there are no leaves labelled by cells of dimension $\geq M$ in x. □

Definition 4.24. *Let $w \in \mathrm{Tree}_n^{\mathcal{O}(\circ),\mathcal{E}(\circ)}(C)_k$, let $M(w) \leq M \leq k$, and let $\ell \in L_{\geq M}(w)$. We define $\sigma_\ell^M(w) \in \mathrm{Tree}_n^{\mathcal{O}(\circ)}(C)_{|\ell|}$ by induction on height as follows. If w has height zero, then it consists of a single leaf ℓ, and we let*

$$\sigma_\ell^M(\ell) = \ell.$$

If w has nonzero height, then we have a case for each possible root label. For $p, q \in \{M, k\}$ (with at least one equal to k) and $i < M$, we let

$$\begin{aligned} \sigma_\ell^M(x \to \circ_{p,q} \leftarrow y) &= \begin{cases} \sigma_\ell^M(x), & \ell \in L(x); \\ \sigma_\ell^M(y), & \ell \in L(y); \end{cases} \\ \sigma_\ell^M(x \to \circ_{i,k} \leftarrow y) &= (x \to \circ_{i,|\ell|} \leftarrow \sigma_\ell^M(y)); \end{aligned}$$

$$\sigma_\ell^M(x \to \circ_{k,i} \leftarrow y) \quad = \quad (\sigma_\ell^M(x) \to \circ_{|\ell|,i} \leftarrow y).$$

Lemma 4.25. *Let* $w \in \mathrm{Tree}_n^{\mathcal{O}(\circ),\mathcal{E}(\circ)}(C)_k$*, let* $M(w) \leq M \leq k$ *and let* $\ell \in L_{\geq M}(w)$*. Then*

1. $\sigma_\ell^M(w)$ *is* $\overset{\circ}{=}$*-compatible;*
2. $s^{|\ell|-i+1}(\sigma_\ell^M(w)) \overset{\circ}{=} s^{k-i+1}(w)$ *and* $t^{|\ell|-i+1}(\sigma_\ell^M(w)) \overset{\circ}{=} t^{k-i+1}(w)$ *for* $i < M$*;*
3. *if* $w \overset{\circ}{=} \tilde{w}$ *and* $\tilde{\ell} \in L_{\geq M}(\tilde{w})$ *is the image of* ℓ*, then* $\sigma_\ell^M(w) \overset{\circ}{=} \sigma_{\tilde{\ell}}^M(\tilde{w})$*.*

Proof. The proof is by mutual induction on the height of w. One proves *1.* easily by splitting into the cases wich appear in the definition of σ_ℓ^M and using *2.* on trees of smaller height.

To prove *2.*, we again split into the cases appearing in the definition of σ_ℓ^M. We explain only the case $w = (x \to \circ_{p,q} \leftarrow y)$, as the others are simpler. We also do only s, as t is completely analogous. So we compute

$$\begin{aligned} s^{|\ell|-i+1}(\sigma_\ell^M(x \to \circ_{p,q} \leftarrow y)) \quad &= \begin{cases} s^{|\ell|-i+1}(\sigma_\ell^M(x)), & \ell \in L(x) \\ s^{|\ell|-i+1}(\sigma_\ell^M(y)), & \ell \in L(y) \end{cases} \\ &\overset{\circ}{=} \begin{cases} s^{p-i+1}(x), & \ell \in L(x) \\ s^{q-i+1}(y), & \ell \in L(y), \end{cases} \end{aligned}$$

where we used induction. On the other hand

$$s^{k-i+1}(x \to \circ_{p,q} \leftarrow y) = s^{q-i+1}(y).$$

Now recall that $\overset{\circ}{=}$ is a globular relation, so $ss \overset{\circ}{=} st$. Moreover, we have $s^{p-m+1}(x) \overset{\circ}{=} t^{q-m+1}(y)$, where $m = \min\{p, q\}$, because w is $\overset{\circ}{=}$-compatible. This allows us to compute

$$s^{p-i+1}(x) = s^{m-i}s^{p-m+1}(x) \overset{\circ}{=} s^{m-i}t^{q-m+1}(y) \overset{\circ}{=} s^{m-i}s^{q-m+1}(y) \overset{\circ}{=} s^{q-i+1}(y)$$

so we are done.

For *3.* there is one case for each of the defining equations of $\overset{\circ}{=}$. The arguments are simple in every case, so we leave them to the reader.

□

Definition 4.26. *Let $w \in \mathrm{Tree}_n^{\mathcal{O}(\circ),\mathcal{E}(\circ)}(C)$. We define $\sigma_\ell(w) := \sigma_\ell^{M(w)}(w)$.*

Definition 4.27. *Let $x \in \mathrm{Tree}_n^{\mathcal{O}(\circ)}(C)$. We define*

$$H(x) = |\{m(v) : v \in I(T)\}|.$$

Lemma 4.28. *Let $x, \tilde{x} \in \mathrm{Tree}_n^{\mathcal{O}(\circ),\mathcal{E}(\circ)}(C)$ and suppose $x \overset{\circ}{=} \tilde{x}$. Then*

$$H(x) = H(\tilde{x}).$$

Proof. One checks this is true for each of the equations defining $\overset{\circ}{=}$, which is easy. □

Lemma 4.29. *Let $x, \tilde{x} \in \mathrm{Tree}_n^{\mathcal{O}(\circ),\mathcal{E}(\circ)}(C)$ be in normal form and suppose $x \overset{\circ}{=} \tilde{x}$. Then $x = \tilde{x}$.*

Proof. The proof is by induction on $H := H(x) = H(\tilde{x})$. If $H = 0$, then x, $\tilde{x}$ both have height zero, so they must be equal as the relation $\overset{\circ}{=}$ is just $=$ on elements of height zero.

Now suppose $H \geq 1$ and let $l = |L_{\geq M}(x)| = |L_{\geq M}(\tilde{x})|$. Then x, $\tilde{x}$ each consist of a maximal m-constant component containing the root, which we denote x_0 and $\tilde{x}_0$, to which are grafted trees $x_1, \cdots, x_l$ and $\tilde{x}_1, \cdots, \tilde{x}_l$. Moreover, it is easy to see that $x_i = \sigma_{\ell_i}(x)$ and $\tilde{x}_i = \sigma_{\tilde{\ell}_i}(\tilde{x})$, where $L_{\geq M}(x) = \{\ell_1 < \cdots < \ell_l\}$ and $L_{\geq M}(\tilde{x}) = \{\tilde{\ell}_1 < \cdots < \tilde{\ell}_l\}$. By Lemma 4.25, we must have $\sigma_{\ell_i}(x) \overset{\circ}{=} \sigma_{\tilde{\ell}_i}(\tilde{x})$ and so by induction we have $\sigma_{\ell_i}(x) = \sigma_{\tilde{\ell}_i}(\tilde{x})$.

Now we must show $x_0 = \tilde{x}_0$. If $M = k$ then both must be equal to $\circ_{k,k} \to \cdots \to \circ_{k,k}$, where there are $l-1$ copies of $\circ_{k,k}$. If $M < k$, then only one of the leaves ℓ_i will be labelled by a k-cell, let it be ℓ_p. This also means $\tilde{\ell}_p$ must be the only leaf in $\tilde{x}$ labelled by a k-cell. Then both x_0 and $\tilde{x}_0$ must be equal to $\circ_{k,M} \to \cdots \to \circ_{k,M} \to \circ_{M,k} \to \cdots \to \circ_{M,k}$ where we have $p-1$ copies of $\circ_{k,M}$ and $l-p$ copies of $\circ_{M,k}$. □

Proof of Proposition 4.12. We have $x, \tilde{x} \in \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)_k$, both in normal form, and $x \overset{\epsilon}{=} \tilde{x}$. By Lemma 4.13, we can assume that $x, \tilde{x}$ contain no u-labelled vertices. By Lemmas 4.16 and 4.18, we then have $x \overset{\circ}{=} \tilde{x}$ and we can apply Lemma 4.29 to conclude $x = \tilde{x}$. □

Notation 4.30. We denote by $N(C) \subset \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C)$ the n-graded subset consisting of terms in normal form.

Corollary 4.31. *Let C be an n-computad. Then the map*

$$n : \overline{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}}(C) \to N(C)$$

sending an equivalence class to its unique representative in normal from is an inverse to the map $N(C) \hookrightarrow \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(C) \to \overline{\mathrm{Tree}}_n^{\mathcal{O},\mathcal{E}}(C)$.

Proof. This follows directly from Proposition 4.12. □

We record here two important properties of normal forms.

Lemma 4.32. *If x is in normal form then any subtree of x is also in normal form. If ϕ is a map of computads, then $\phi(x)$ is in normal form if and only if x is.*

Proof. This is clear. □

5. Comp is a presheaf category

We now define a category Cell_{n+1} of computadic cell shapes of dimension $\leq n+1$, with a fully faithful functor $\mathrm{Cell}_{n+1} \hookrightarrow \mathrm{Comp}_{n+1}^n$, which we ususally denote $\mathrm{Cell} \hookrightarrow \mathrm{Comp}$. Then we construct the associated nerve/realization adjunction

$$|-| : \mathrm{Psh}(\mathrm{Cell}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Comp} : N$$

and prove that it is an equivalence of categories.

The main ingredient is Proposition 5.5, which uses normal forms in an essential way. The rest of the section could probably be shortened by appealing to the theory of familial representability ([16],[26]). We choose to present the arguments here for the reader's convenience, as this does not take too much space.

Definition 5.1. *Denote by $\mathbb{1}$ the terminal computad. One can define it inductively by saying that it has exactly one 0-cell and exactly one k-cell x with $s(x) = x_0$ and $t(x) = x_1$ for each ordered pair (x_0, x_1) of parallel $(k-1)$-morphisms in $F_{k-1}(\mathbb{1}_{\leq k-1})$.*

Notation 5.2. For any computad C, we denote by $\sigma : C \to \mathbb{1}$ the unique map of computads.

We think of the cells in $\mathbb{1}$ as **cell shapes** and of the morphisms in the free n-sesquicategory $F_n(\mathbb{1})$ as **unlabelled diagrams**. Now we show how one can associate a computad to each unlabelled diagram.

Definition 5.3. *Let $k \leq n$ and let $d \in F_n(\mathbb{1})$ be an unlabelled k-diagram. We define a category $\mathrm{Comp}(d)$ as follows. Its objects are pairs (C, x), where C is a computad, $x \in F_n(C)$ and $F_n(\sigma)(x) = d$. A morphism $(C, x) \to (D, y)$ is a map of computads $\varphi : C \to D$ such that $F_n(\varphi)(x) = y$.*

Now we show that the category $\mathrm{Comp}(d)$ has an initial object. For this we will need to take colimits in Comp. The following result seems to be well known, but not having found a suitable reference we include a simple proof here.

Lemma 5.4. *Let $T : \mathrm{gSet}_n \to \mathrm{gSet}_n$ be a finitary monad and let $k \leq n+1$. For each $m \leq k$, denote by $[-]_m : \mathrm{Comp}_k^T \to \mathrm{Set}$ the functor taking a k-computad to its set of m-cells. Then the following hold:*

1. *the category Comp_k^T is cocomplete;*
2. *each functor $[-]_m$ is cocontinuous;*
3. *the functors $[-]_m$ for $m = 0, \cdots, k$ jointly reflect isomorphisms.*

Proof. Let $\Gamma : I \to \mathrm{Comp}_k^T$ be a diagram. To prove *1.* and *2.* we may as well assume $m = k$, otherwise we can pass to the underlying diagram of m-computads. We construct, by induction on k, a k-computad C which will be the colimit of this diagram. Define its set of k-cells to be $C_k := \mathrm{colim}_i[\Gamma(i)]_k$ and its underlying $(k-1)$-computad as $C_{\leq k-1} := \mathrm{colim}_i[\Gamma(i)]_{\leq k-1}$. Now define source and target maps

$$s, t : \mathrm{colim}_i[\Gamma(i)]_k \to [F_{k-1}(\mathrm{colim}_i[\Gamma(i)]_{\leq k-1})]_{k-1}$$

by the composite

$$\Gamma(i)_k \to [F_{k-1}(\Gamma(i)_{\leq k-1})]_{k-1} \to \mathrm{colim}_i[F_{k-1}(\Gamma(i)_{\leq k-1})]_{k-1} \to$$

$$\to [\mathrm{colim}_i\, F_{k-1}(\Gamma(i)_{\leq k-1})]_{k-1} = [F_{k-1}(\mathrm{colim}_i\, \Gamma(i)_{\leq k-1})]_{k-1}$$

where the equality comes from the fact that F_{k-1} is left adjoint and the last arrow is induced by the maps

$$[F_{k-1}([\Gamma(i)]_{\leq k-1})]_{k-1} \to [\operatorname{colim}_i F_{k-1}([\Gamma(i)]_{\leq k-1})]_{k-1}$$

on sets of $(k-1)$-morphisms associated to the canonical maps of T-algebras $F_{k-1}([\Gamma(i)]_{\leq k-1}) \to \operatorname{colim}_i F_{k-1}([\Gamma(i)]_{\leq k-1})$. Now one needs to check that s, t satisfy globularity and that the construction has the right universal property. This is straightforward. Point *3.* is easy to prove by induction. $\square$

Proposition 5.5. *For each unlabelled diagram $d \in F_n(\mathbb{1})$ the category* $\mathrm{Comp}(d)$ *has an initial object, which we denote $(\hat{d}, \tilde{d})$.*

Proof. We construct $(\hat{d}, \tilde{d})$ by induction on the dimension of d and on the height of its normal form.

If d is a 0-diagram then it consists of a single 0-cell. Then $\hat{d}$ is the 0-computad with a single 0-cell and $\tilde{d}$ is the diagram consisting of that 0-cell. Now suppose d consists of a single k-cell. By induction on dimension and the fact that $ss(d) = st(d)$ and $ts(d) = tt(d)$, we have the following diagram.

$$\begin{tikzcd}
& \widehat{s(d)} & \\
\widehat{s^2(d)} \arrow[ur, hook] \arrow[dr, hook] & & \widehat{t^2(d)} \arrow[ul, hook'] \arrow[dl, hook'] \\
& \widehat{t(d)} &
\end{tikzcd}$$

We build $\hat{d}$ by taking the colimit of this diagram in Comp and then adding a k-cell $\tilde{d} : \widehat{s(d)} \to \widehat{t(d)}$. It's now easy to see, by induction on dimension, that $(\hat{d}, \tilde{d})$ is an initial object in $\mathrm{Comp}(d)$.

Now suppose d has normal form $x \to \circ_{i,j} \leftarrow y$. Let $m = \min\{i, j\}$ and let $x \cap y = s^{i-m+1}(x) = t^{j-m+1}(y)$. By induction on height and dimension, we have a diagram

$$\begin{tikzcd} \widehat{x \cap y} \arrow[r, hook] \arrow[d, hook] & \hat{x} \\ \hat{y} & \end{tikzcd}.$$

We let $\hat{d}$ be the pushout of this diagram in Comp and

$$\tilde{d} = (\tilde{x} \to \circ_{i,j} \leftarrow \tilde{y}),$$

where we take $\tilde{x}, \tilde{y}$ in normal form. Since d is in normal form, so are x and y, by Lemma 4.32. Therefore, again by Lemma 4.32 and uniqueness of normal forms, $\tilde{x}, \tilde{y}$ map to x, y in $\mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(\mathbb{1})$. Then $(\tilde{x} \to \circ_{i,j} \leftarrow \tilde{y})$ maps to $(x \to \circ_{i,j} \leftarrow y)$ in $\mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(\mathbb{1})$ and therefore it is in normal form, by Lemma 4.32. Given $(C, m) \in \mathrm{Comp}(d)$, a map $(\hat{d}, \tilde{d}) \to (C, m)$ is given by maps $f : \hat{x} \to C$ and $g : \widehat{y} \to C$ such that

$$(f(\tilde{x}) \to \circ_{i,j} \leftarrow g(\tilde{y})) \stackrel{\epsilon}{=} m.$$

Note that the left hand side is already in normal form, by Lemma 4.32. This means the normal form of m must be equal to this, by uniqueness of normal form. This determines $f(\tilde{x}), g(\tilde{y}) \in F_n(C)$ uniquely, because it determines their normal forms as the two evident subtrees of the normal form of m. Then by induction this determines f, g uniquely, so we conclude that $(\hat{d}, \tilde{d})$ is initial.

Finally suppose d has normal form $x \to u_i$. Then take $\hat{d} = \hat{x}$ and $\tilde{d} = \tilde{x} \to u_i$. □

Remark 5.6. The pair $(\hat{d}, \tilde{d})$ corresponds to what is called a **polyplex** in [15] and [26]. In [26], the essential condition for establishing that a certain class of polygraphs forms a presheaf category is the fact that the groups of autmorphisms of polyplexes are trivial.

Remark 5.7. In fact, we don't need normal forms to construct these computads. We only need them to prove that they are initial. One can construct $(\hat{x}, \tilde{x})$ for any term $x \in \mathrm{Tree}_n^{\mathcal{O},\mathcal{E}}(\mathbb{1}_{\leq n})$ by the same inductive procedure used above. When $x \stackrel{\epsilon}{=} y$ is one of the generating equations in $\mathcal{E}_n$, we obtain an isomorphism $\varphi : \hat{x} \to \hat{y}$ such that $\varphi(\tilde{x}) \overset{\epsilon}{=} \tilde{y}$ by the same generating equation. This means $(\hat{d}, \tilde{d})$ is well defined up to isomorphism. If x is in normal form and φ is an automorphism of $\hat{x}$ such that $\varphi(\tilde{x}) \overset{\epsilon}{=} \tilde{x}$, then $\varphi(\tilde{x}) = \tilde{x}$, as they are both in normal form. This implies $\varphi = \mathrm{id}$. So in the presence of normal forms there are no automorphisms, so $(\hat{d}, \tilde{d})$ is well defined up to unique isomorphism and it is initial in $\mathrm{Comp}(d)$.

Example 5.8. It is well known ([31],[17]) that, for $n \geq 2$, the category of 3-computads for the monad T_n^{str} whose algebras are strict n-categories, is not a presehaf category. This example illustrates why the the above Proposition fails in this case. Denote by $\mathbb{1}$ the terminal computad for T_n^{str} and let $s : \mathrm{id}_* \Rightarrow \mathrm{id}_*$ be the unique 2-cell in $\mathbb{1}$ whose source and target are the identity on the unique 0-cell. We construct a diagram $d \in F_2(\mathbb{1})_2$ consisting of the vertical composite $s \circ s$. Consider $(\hat{d}, \tilde{d}) \in \mathrm{Comp}^{T_n^{str}}(d)$ defined by letting $\hat{d}$ be the computad consisting of a 0-cell $*$, together with two 2-cells $\alpha, \beta : \mathrm{id}_* \Rightarrow \mathrm{id}_*$, and $\tilde{d}$ the vertical composite $\alpha \circ \beta$. By the Eckmann-Hilton argument, we have $\alpha \circ \beta = \beta \circ \alpha$, so $\hat{d}$ admits a nontrivial automorphism which maps $\tilde{d} \mapsto \tilde{d}$, namely the one that permutes α and β. If $\mathrm{Comp}(d)$ had an initial object I, then the unique map $I \to (\hat{d}, \tilde{d})$ would be invariant under composition with this automorphism. This would mean that α, β are not in the image of the map, so I contains only 0-cells, which is absurd.

In order to show that Comp is a presheaf category, what we actually need is the fact that $\mathrm{Comp}(c)$ has an initial object when $c \in \mathbb{1}_k$ is a computadic cell shape. This will fail for any 3-cell shape whose source or target is the diagram d above.

Definition 5.9. *Let $c \in \mathbb{1}_{n+1}$ be an $(n+1)$-cell shape. We define* $\mathrm{Comp}(c)$ *to be the category of pais (C, x) where C is a computad and $x \in C_{n+1}$ is an $(n+1)$-cell such that $\sigma(x) = c$.*

Corollary 5.10. *For each $k \leq n+1$ and each k-cell $c \in \mathbb{1}_k$ the category* $\mathrm{Comp}(c)$ *has an initial object, denoted $(\hat{c}, \tilde{c})$.*

Proof. For $k \leq n$, this is just Proposition 5.5. For an $(n+1)$-cell c, we use

Proposition 5.5 to construct the diagram

$$
\begin{tikzcd}
 & \widehat{s(c)} & \\
\widehat{s^2(c)} \arrow[ur, hook] \arrow[dr, hook] & & \widehat{t^2(c)} \arrow[ul, hook'] \arrow[dl, hook'] \\
 & \widehat{t(c)} &
\end{tikzcd}
$$

and then we take the colimit and add an $(n+1)$-cell $\tilde{c} : \widetilde{s(c)} \to \widetilde{t(c)}$. □

Remark 5.11. The pair $(\hat{c}, \tilde{c})$ corresponds to what is called a **plex** in [15] and [26] or a **computope** in [30].

Definition 5.12. *Let* Cell_{n+1} *be the category whose objects are cell shapes* $c \in \mathbb{1}_k$ *for* $k \leq n+1$*, and where a morphism* $c \to d$ *is a map of computads* $\hat{c} \to \hat{d}$*. We usually denote this simply by* Cell*. It comes with a fully faithful functor*

$$\widehat{(-)} : \mathrm{Cell} \hookrightarrow \mathrm{Comp}\,.$$

Definition 5.13. *We define the nerve functor*

$$N : \mathrm{Comp} \to \mathrm{Psh(Cell)}$$

as the composite $\mathrm{Comp} \hookrightarrow \mathrm{Psh(Comp)} \to \mathrm{Psh(Cell)}$ *of the Yoneda embedding with the restriction along* $\widehat{(-)}$*.*

Definition 5.14. *We define the realization functor by the following left Kan extension, which exists because* Comp *is cocomplete.*

$$
\begin{tikzcd}
\mathrm{Cell} \arrow[r, hook, "\widehat{(-)}"] \arrow[d, hook, "\mathcal{Y}"'] & \mathrm{Comp} \\
\mathrm{Psh(Cell)} \arrow[ur, dashed, "|-|"'] &
\end{tikzcd}
$$

We thus obtain the usual nerve/realization adjunction

$$|-|: \mathrm{Psh}(\mathrm{Cell}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Comp} : N \ .$$

Theorem 5.15. *The adjunction*

$$|-|: \mathrm{Psh}(\mathrm{Cell}_{n+1}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Comp}^n_{n+1} : N$$

is an equivalence.

Proof. By [18][Proposition 5.14] it is enough to show that the functors

$$\mathrm{Comp}(\widehat{c}, -) : \mathrm{Comp} \to \mathrm{Set},$$

for $c \in \mathrm{Cell}$, are cocontinuous and jointly reflect isomorphisms. This follows easily from Lemma 5.4 and the fact that for $c \in \mathbb{1}_k$ we have

$$\mathrm{Comp}(\widehat{c}, C) = \{x \in C_k : \sigma(x) = c\},$$

which follows from Corrollary 5.10. □

Remark 5.16. All results in this section hold, with the same proofs, for any n-globular operad given by generators and relations as long as it admits a suitable theory of normal forms. More precisely, given a presentation $(\mathcal{G}, \mathcal{R})$ for an n-globular operad, what we need is an n-graded subset $N(C) \subset \mathrm{Tree}^{\mathcal{G},\mathcal{R}}_n(C)$ of terms in normal form, for each n-computad C, with the following properties:

1. the induced map $N(C) \to \overline{\mathrm{Tree}}^{\mathcal{G},\mathcal{R}}_n(C)$ is an n-graded bijection (i.e. there is a unique term in normal form in each equivalence class);

2. each subtree of a tree in normal from is in normal form;

3. given a map of n-computads $\phi : C \to D$, we have $\phi(x) \in N(D)$ if and only if $x \in N(C)$.

Remark 5.17. Because of condition 3. in the previous Remark, it is enough to define $N(\mathbb{1}_{\leq n})$ for the terminal computad $\mathbb{1}_{\leq n}$ and then let $x \in N(C)$ if and only if $\sigma(x) \in N(\mathbb{1}_{\leq n})$. It is also enough to check condition 2. for the terminal computad. However, it is not enough to check 1. for $\mathbb{1}_{\leq n}$, as one can have $x \neq y$ in $N(C)$ such that $\sigma(x) = \sigma(y)$.

Remark 5.18. We now describe an alternative approach to showing that the category of computads for an n-globular operad presented by generators and relations is a presheaf category. This was inspired by a discussion with Samuel Mimram about the theory of rewriting.

Denote by $\Gamma_n^{\mathcal{G},\mathcal{R}}$ the free groupoid on the graph with vertices the terms $x \in \mathrm{Tree}_n^{\mathcal{G},\mathcal{R}}(\mathbb{1}_{\leq n})$ and edges corresponding to the generating equations in $\mathcal{R}$. In Remark 5.7, we defined a functor $(\widehat{-}, \widetilde{-})$ from $\Gamma_n^{\mathcal{G},\mathcal{R}}$ to the groupoid whose objects are pairs (C, x) where C is a computad and $x \in \mathrm{Tree}_n^{\mathcal{G},\mathcal{R}}(C)$ is a term and whose morphisms $(C, x) \to (D, y)$ are isomorphisms $C \to D$ such that $\varphi(x) \overset{\mathcal{R}}{=} y$. This functor is easily seen to be full, so the group of automorphisms of $(\hat{x}, \tilde{x})$ is a quotient of the group of automorphisms of x. In order to show that the category of computads for this n-globular operad is a presheaf category, it is enough to show that the former is trivial. In Remark 5.7 we showed how one can use normal forms to do this.

On the other hand, one can consider the relation on the arrows of $\Gamma_n^{\mathcal{G},\mathcal{R}}$ obtained by declaring that the application of a generating equation at any location in a tree commutes with the application of another equation in a disjoint location. The functor $(\widehat{-}, \widetilde{-})$ respects this relation. If the automorphism groups of the quotient of $\Gamma_n^{\mathcal{G},\mathcal{R}}$ by this relation are trivial, then the associated category of computads is a preseheaf category. It is an interesting question whether this is true for n-sesquicategories.

More generally, if one can describe a relation on the arrows of $\Gamma_n^{\mathcal{G},\mathcal{R}}$ which is preserved by $(\widehat{-}, \widetilde{-})$ and such that the automorphism groups of the quotient of $\Gamma_n^{\mathcal{G},\mathcal{R}}$ by this relation are trivial, this proves that the associated category of computads is a preseheaf category.

6. String diagrams for n-sesquicategories

Let C be an n-computad for $T_n^{D^s}$. In this section we explain how to associate a C**-labelled string diagram** to each morphism in $F_n(C)$. This is an extremely useful graphical notation for describing composites and performing computations in n-sesquicategories. We will in the future extend this to semistrict n-categories by adding interchangers, which will allow us to apply in this more general context the techniques used in [2],[3], [4] and [5] (used there in the context of strict 3 and 4-categories).

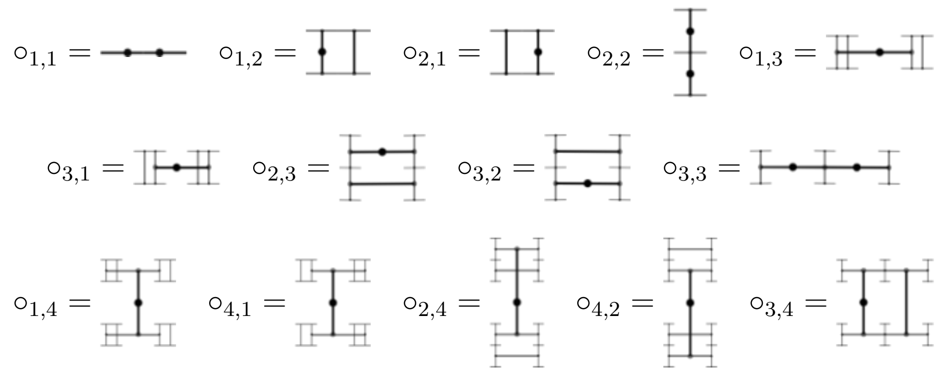
The essential ingredient here is the theory of normal forms, which will allow us to describe the graphical notation for a term by induction on the $(\mathcal{O}_n, C)$-labelled tree corresponding to its normal form.

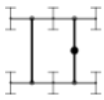
It is enough to decribe the unlabelled diagrams corresponding to morphisms in $F_n(\mathbb{1})$, since C-labelled diagrams are then obtained by simply adding labels at appropriate places.

So let $w \in F_n(\mathbb{1})_k$. We proceed by induction on k and the height of the normal form of w. When $k = 0$, the morphism w simply consists of the unique 0-cell in $\mathbb{1}$. The associated diagram is just a point. In general, when k is odd (resp. even) we depict a generating k-cell $w \in \mathbb{1}_k$ by drawing the $(k-1)$-diagram corresponding to its source on the left (resp. top), the one corresponding to its target on the right (resp. bottom) and then forming a double cone on this disjoint union. We denote this double cone by drawing the cone point in the middle and curves connecting each cell in the source and target diagrams to the cone point. We need to distinguish lines which correspond to cells of different codimension, which we can do by using different thickness, transparency, dashing or any other method.

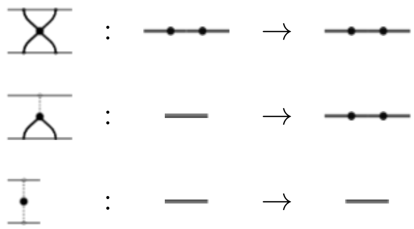
Now suppose $w = (x \to \circ_{i,j} \leftarrow y)$ is in normal form. By induction, we already know how to draw the diagrams associated to x and y and the diagram $\circ_{i,j}$ determines how we should compose these two pictures to obtain the picture for w. Finally, if $w = (x \to u_k)$ in normal form, then we draw two copies of x and we draw lines connecting generators, again using a different notation for generators of different codimension.

We now give some examples. First it is useful to recall fom [6] the graphical notation for the generators $\circ_{i,j}$. We include here the pictures for $i, j \leq 4$.



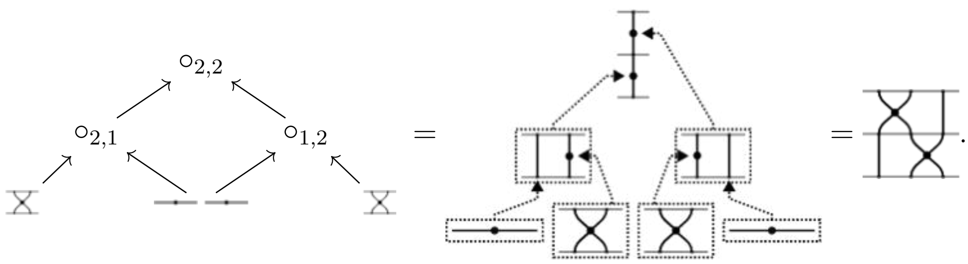
$\circ_{4,3} =$  $\circ_{4,4} =$ 

Now we can move on to examples of unlabelled cells and diagrams. The unique 1-cell in $\mathbb{1}$ is denoted . The diagram $* \to u_1$ is denoted . The diagram $\to \circ_{1,1} \leftarrow$ is denoted . Here are some 2-cells, with their source and target 1-diagrams.

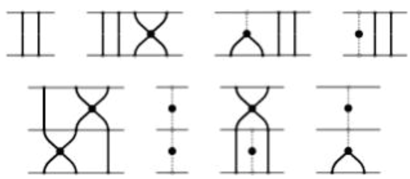


The 2-diagrams $* \to u_1 \to u_2$ and  $\to u_2$ are denoted

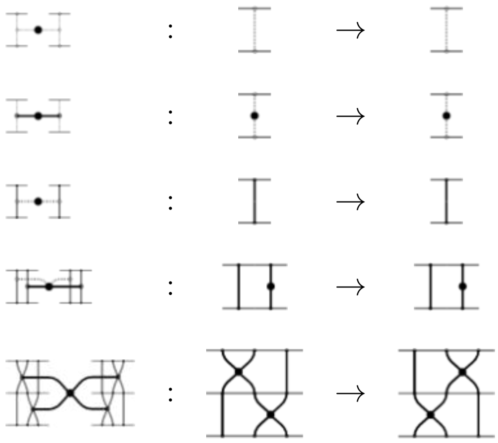
Here is a 2-diagram in normal form, then its normal form where we replace each generator by its picture and finally the picture of the diagram itself.



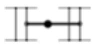
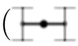
Here are some other 2-diagrams.



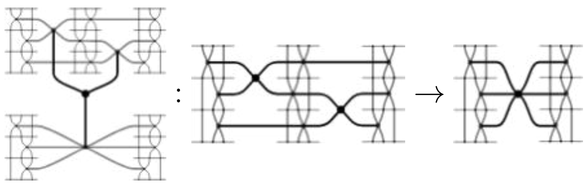
Here are some 3-cells with their source and target 2-diagrams.



Notice how the notation distinguishes the 3-cell  above from the diagram

 $= ($  $\rightarrow \circ_{3,1} \leftarrow$ $)$.

Here is a 4-cell with its source and target 3-diagrams.



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Manuel Araújo
manuel.araujo@ist.utl.pt

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXV-2 (2024)

MACNEILLE COMPLETIONS OF SUBORDINATION ALGEBRAS

M. Abbadini, G. Bezhanishvili, L. Carai

Résumé. Les algèbres de subordination S5 sont une généralisation naturelle des algèbres de de Vries. Il a été prouvé récemment que la catégorie $\mathsf{SubS5^S}$ des algèbres de subordination S5 et des relations de subordination compatibles est équivalente à la catégorie des espaces compacts de Hausdorff et des relations fermées. Nous généralisons la complétion de MacNeille des algèbres de Boole au cadre des algèbres de subordination S5, et utilisons le caractère relationnel des morphismes de $\mathsf{SubS5^S}$ pour prouver que le foncteur de complétion de MacNeille établit une équivalence entre $\mathsf{SubS5^S}$ et sa sous-catégorie pleine des algèbres de de Vries. De plus, nous montrons que le foncteur qui associe à chaque algèbre de subordination S5 le *frame* de ses idéaux ronds établit une dualité entre $\mathsf{SubS5^S}$ et la catégorie des *frames* compacts réguliers et des homomorphismes de *preframes*. Nos résultats n'utilisent pas l'axiome du choix et fournissent un éclairage supplémentaire sur les dualités de type Stone pour les espaces compacts de Hausdorff avec différents types de morphismes. En particulier, nous montrons comment elles se restreignent aux sous-catégories amples de $\mathsf{SubS5^S}$ correspondant aux relations continues et aux fonctions continues entre espaces compacts de Hausdorff.

Abstract. S5-subordination algebras are a natural generalization of de Vries algebras. Recently it was proved that the category $\mathsf{SubS5^S}$ of S5-subordination algebras and compatible subordination relations between them is equivalent to the category of compact Hausdorff spaces and closed relations. We generalize MacNeille completions of boolean algebras to the setting of S5-subordination algebras, and utilize the relational nature of the morphisms in $\mathsf{SubS5^S}$ to prove that the MacNeille completion functor establishes an equiv-

alence between $\mathsf{SubS5^S}$ and its full subcategory consisting of de Vries algebras. We also show that the functor that associates to each $\mathsf{S5}$-subordination algebra the frame of its round ideals establishes a dual equivalence between $\mathsf{SubS5^S}$ and the category of compact regular frames and preframe homomorphisms. Our results are choice-free and provide further insight into Stone-like dualities for compact Hausdorff spaces with various morphisms between them. In particular, we show how they restrict to the wide subcategories of $\mathsf{SubS5^S}$ corresponding to continuous relations and continuous functions between compact Hausdorff spaces.
Keywords. Compact Hausdorff space, Gleason cover, closed relation, continuous relation, de Vries algebra, subordination relation, proximity, MacNeille completion, ideal completion, compact regular frame.
Mathematics Subject Classification (2020). 18F70, 54E05, 06D22, 06E15, 54D30, 54G05.

Contents

1. Introduction

With each compact Hausdorff space X, we can associate numerous algebraic structures that determine X up to homeomorphism. This yields various dualities for the category KHaus of compact Hausdorff spaces and continuous functions. In this paper we are interested in two dualities for KHaus from pointfree topology. By Isbell duality [Isb72], KHaus is dually equivalent to the category KRFrm of compact regular frames and frame homomorphisms; and by de Vries duality [dV62], KHaus is dually equivalent to the category DeV of de Vries algebras and de Vries morphisms.

Isbell duality is established by working with the contravariant functor $\mathcal{O}\colon \mathsf{KHaus} \to \mathsf{KRFrm}$ which associates with each compact Hausdorff space X the compact regular frame $\mathcal{O}(X)$ of open subsets of X, and with each continuous function $f\colon X \to Y$ the frame homomorphism $f^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$. De Vries duality is established by working with the contravariant functor $\mathcal{RO}\colon \mathsf{KHaus} \to \mathsf{DeV}$. Writing int for the interior and cl for the closure, $\mathcal{RO}$ associates with each $X \in \mathsf{KHaus}$ the de Vries algebra $(\mathcal{RO}(X), \prec)$ of regular open subsets of X, where $U \prec V$ iff $\mathsf{cl}(U) \subseteq V$, and with each continuous function $f\colon X \to Y$ the de Vries morphism $\mathcal{RO}(f)\colon \mathcal{RO}(Y) \to \mathcal{RO}(X)$ given by $\mathcal{RO}(f)(V) = \mathsf{int}(\mathsf{cl} f^{-1}[V])$ for each $V \in \mathcal{RO}(Y)$.

As a consequence of Isbell and de Vries dualities, KRFrm is equivalent to DeV. This equivalence can be obtained directly, without first passing to KHaus [Bez12]. We thus arrive at the following diagram, where the horizontal arrow represents an equivalence and the slanted arrows with the letter d on top represent dual equivalences.

$$\begin{tikzcd} & \mathsf{KHaus} \arrow[dl, leftrightarrow, "d"'] \arrow[dr, leftrightarrow, "d"] & \\ \mathsf{KRFrm} \arrow[rr, leftrightarrow] & & \mathsf{DeV} \end{tikzcd}$$

Several authors have considered generalizations of KHaus where functions are replaced by relations. A relation R between two compact Hausdorff spaces X and Y is *closed* if R is a closed subset of $X \times Y$ and it is *continuous* if in addition the R-preimage of each open subset of Y is open in X. A function between compact Hausdorff spaces is closed iff it is continuous. But for relations this results in two different categories $\mathsf{KHaus}^{\mathsf{R}}$ and $\mathsf{KHaus}^{\mathsf{C}}$. In the former, morphisms are closed relations; and in the latter, they are continuous relations. Clearly KHaus is a wide subcategory of $\mathsf{KHaus}^{\mathsf{C}}$, which in turn is a wide subcategory of $\mathsf{KHaus}^{\mathsf{R}}$.

In [BGHJ19] KRFrm was generalized to $\mathsf{KRFrm}^{\mathsf{C}}$, DeV to $\mathsf{DeV}^{\mathsf{C}}$ (see Section 2 for the definitions of these categories), and it was shown that the commutative diagram above extends to the following commutative diagram.

$$\begin{tikzcd}
 & \mathsf{KHaus}^{\mathsf{C}} \arrow[dl, leftrightarrow, "d"'] \arrow[dr, leftrightarrow, "d"] & \\
\mathsf{KRFrm}^{\mathsf{C}} \arrow[rr, leftrightarrow] & & \mathsf{DeV}^{\mathsf{C}}
\end{tikzcd}$$

On the other hand, in [Tow96, JKM01] the category KRFrm was generalized to $\mathsf{KRFrm}^{\mathsf{P}}$, where morphisms are preframe homomorphisms (that is, they preserve finite meets and directed joins), and it was shown that $\mathsf{KRFrm}^{\mathsf{P}}$ is dually equivalent to $\mathsf{KHaus}^{\mathsf{R}}$. In a recent paper [ABC23] we introduced the category $\mathsf{DeV}^{\mathsf{S}}$ whose objects are de Vries algebras and whose morphisms are compatible subordination relations. We proved that $\mathsf{DeV}^{\mathsf{S}}$ is equivalent to $\mathsf{KHaus}^{\mathsf{R}}$ and hence dually equivalent to $\mathsf{KRFrm}^{\mathsf{P}}$. Thus, we arrive at the following commutative diagram that extends the two diagrams above.

$$\begin{tikzcd}
 & \mathsf{KHaus}^{\mathsf{R}} \arrow[dl, leftrightarrow, "d"'] \arrow[dr, leftrightarrow] & \\
\mathsf{KRFrm}^{\mathsf{P}} \arrow[rr, leftrightarrow, "d"] & & \mathsf{DeV}^{\mathsf{S}}
\end{tikzcd}$$

Our aim here is to give a direct choice-free proof of the duality between $\mathsf{KRFrm}^{\mathsf{P}}$ and $\mathsf{DeV}^{\mathsf{S}}$. From this we derive a direct choice-free proof of the equivalence between $\mathsf{KRFrm}^{\mathsf{C}}$ and $\mathsf{DeV}^{\mathsf{C}}$, as well as an alternative choice-free proof of the equivalence between KRFrm and DeV.

Our main tool is the category $\mathsf{SubS5}^{\mathsf{S}}$ of S5-subordination algebras introduced in [ABC23]. Objects of $\mathsf{SubS5}^{\mathsf{S}}$ were already considered by Meenakshi [Mee66], who studied proximity relations on an arbitrary boolean algebra. In [ABC23] we used a generalization of Stone duality to closed relations [Cel18, KMJ23] and the machinery of allegories [FS90] to show that $\mathsf{SubS5}^{\mathsf{S}}$ is equivalent to the category $\mathsf{StoneE}^{\mathsf{R}}$ whose objects are Stone spaces equipped with a closed equivalence relation and whose morphisms are special closed relations (see Definition 2.13(1)). Since DeV^S is a full subcategory of $\mathsf{SubS5}^\mathsf{S}$, restricting this equivalence yields an equivalence between DeV^S and the full subcategory Gle^R of $\mathsf{StoneE}^\mathsf{R}$ consisting of Gleason spaces. It turns out that these four categories are equivalent to $\mathsf{KHaus}^\mathsf{R}$. Consequently, DeV^S is equivalent to $\mathsf{SubS5}^\mathsf{S}$, but the proof goes through $\mathsf{KHaus}^\mathsf{R}$ and hence uses the axiom of choice.

In this paper we generalize MacNeille completions of boolean algebras to S5-subordination algebras and give a direct choice-free proof of the equivalence between $\mathsf{SubS5}^\mathsf{S}$ and DeV^S. We also specialize the notion of a round ideal of a proximity lattice [War74] to our setting to obtain a contravariant functor from $\mathsf{SubS5}^\mathsf{S}$ to $\mathsf{KRFrm}^\mathsf{P}$, yielding a choice-free proof that $\mathsf{SubS5}^\mathsf{S}$ is dually equivalent to $\mathsf{KRFrm}^\mathsf{P}$. We thus arrive at the following commutative diagram.

$$\begin{tikzcd}
& \mathsf{SubS5}^\mathsf{S} \arrow[dl, leftrightarrow, "d"'] \arrow[dr, leftrightarrow] & \\
\mathsf{KRFrm}^\mathsf{P} \arrow[rr, leftrightarrow, "d"] & & \mathsf{DeV}^\mathsf{S}
\end{tikzcd}$$

We also study the wide subcategories of these categories whose morphisms encode continuous relations and continuous functions between compact Hausdorff spaces.

The paper is organized as follows. In Section 2 we recall the existing dualities for compact Hausdorff spaces that are relevant for our purposes. In Section 3 we describe the round ideal functor from $\mathsf{SubS5}^\mathsf{S}$ to $\mathsf{KRFrm}^\mathsf{P}$. In Section 4 we define MacNeille completions of S5-subordination algebras and prove that the resulting functor yields an equivalence between $\mathsf{SubS5}^\mathsf{S}$ and DeV^S. We then use this result to show that the round ideal functor from $\mathsf{SubS5}^\mathsf{S}$ to $\mathsf{KRFrm}^\mathsf{P}$ is a dual equivalence. In Section 5 we study the wide subcategories of these categories whose morphisms encode continuous relations between compact Hausdorff spaces. In Section 6 we further restrict our attention to the morphisms that encode continuous functions between compact Hausdorff spaces. Finally, in Section 7 we give dual descriptions of the round ideal and MacNeille completions of S5-subordination algebras.

All the categories considered in this paper are listed in Tables 1 to 4 and all the equivalences and dual equivalences in Fig. 2 at the end of Section 6.

2. Preliminaries

In this section we briefly recall Isbell duality, de Vries duality, and their generalizations. We start by recalling some basic definitions from pointfree topology (see, e.g., [PP12]). A *frame* or *locale* is a complete lattice L satisfying the join-infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}.$$

Each $a \in L$ has the *pseudocomplement* given by $a^* = \bigvee \{x \in L \mid a \wedge x = 0\}$. We say that a is *compact* if $a \leq \bigvee S$ implies $a \leq \bigvee T$ for some finite $T \subseteq S$, and that a is *well-inside* b (written $a \prec b$) if $a^* \vee b = 1$. A frame L is *compact* if 1 is compact and it is *regular* if $a = \bigvee \{x \in L \mid x \prec a\}$ for each $a \in L$.

A *frame homomorphism* between two frames is a map that preserves arbitrary joins and finite meets. We recall from the introduction that KRFrm is the category of compact regular frames and frame homomorphisms and that KHaus is the category of compact Hausdorff spaces and continuous functions.

Theorem 2.1 (Isbell duality). KRFrm *is dually equivalent to* KHaus.

A *preframe homomorphism* between two frames is a map that preserves directed joins and finite meets. We let $\mathsf{KRFrm}^{\mathsf{P}}$ be the category of compact regular frames and preframe homomorphisms. Clearly KRFrm is a wide subcategory of $\mathsf{KRFrm}^{\mathsf{P}}$.

We recall that a relation $R \subseteq X \times Y$ between compact Hausdorff spaces is *closed* if R is a closed subset of $X \times Y$. As usual, for $x \in X$ and $y \in Y$, we write

$$R[x] = \{y \in Y \mid x \mathrel{R} y\} \quad \text{and} \quad R^{-1}[y] = \{x \in X \mid x \mathrel{R} y\}.$$

Also, for $F \subseteq X$ and $G \subseteq Y$, we write

$$R[F] = \bigcup \{R[x] \mid x \in F\} \quad \text{and} \quad R^{-1}[G] = \bigcup \{R^{-1}[y] \mid y \in G\}.$$

Then R is closed iff $R[F]$ is closed for each closed $F \subseteq X$ and $R^{-1}[G]$ is closed for each closed $G \subseteq Y$ (see, e.g., [BBSV17, Lem. 2.12]). We let $\mathsf{KHaus}^{\mathsf{R}}$ be the category of compact Hausdorff spaces and closed relations, where identities are identity relations and composition is relation composition. We recall that for two relations $R_1 \subseteq X_1 \times X_2$ and $R_2 \subseteq X_2 \times X_3$ the relation composition $R_2 \circ R_1 \subseteq X_1 \times X_3$ is defined by

$$x_1 \ (R_2 \circ R_1) \ x_3 \iff \exists x_2 \in X_2 : x_1 \ R_1 \ x_2 \text{ and } x_2 \ R_2 \ x_3.$$

The category $\mathsf{KHaus}^{\mathsf{R}}$ is a full subcategory of the category of stably compact spaces and closed relations introduced and studied in [JKM01]. It is symmetric in that if R is a closed relation, then its converse $R^{\smile}\colon X_2 \to X_1$ (defined by $y \ R^{\smile} \ x$ iff $x \ R \ y$) is also closed. This defines a dagger on $\mathsf{KHaus}^{\mathsf{R}}$ with which $\mathsf{KHaus}^{\mathsf{R}}$ forms an allegory (see, e.g., [ABC23, Lem. 3.6]). The following theorem generalizes Isbell duality:

Theorem 2.2 ([Tow96, JKM01])**.** $\mathsf{KRFrm}^{\mathsf{P}}$ *is dually equivalent to* $\mathsf{KHaus}^{\mathsf{R}}$.

A closed relation $R \subseteq X \times Y$ between compact Hausdorff spaces is *continuous* if V open in Y implies $R^{-1}[V]$ is open in X. Let $\mathsf{KHaus}^{\mathsf{C}}$ be the wide subcategory of $\mathsf{KHaus}^{\mathsf{R}}$ whose morphisms are continuous relations.

In [BGHJ19, Def. 4.3], motivated by Johnstone's construction of the Vietoris frame of a compact regular frame [Joh82, Sec. III.4], a preframe homomorphism $\Box\colon L \to M$ between compact regular frames is called *continuous* or a c-*morphism* if there is a join-preserving $\Diamond\colon L \to M$ such that

$$\Box(a \vee b) \leq \Box a \vee \Diamond b \quad \text{and} \quad \Box a \wedge \Diamond b \leq \Diamond(a \wedge b).$$

Let $\mathsf{KRFrm}^{\mathsf{C}}$ be the wide subcategory of $\mathsf{KRFrm}^{\mathsf{P}}$ whose morphisms are c-morphisms. The duality of Theorem 2.2 then restricts to the following generalization of Isbell duality:

Theorem 2.3 ([BGHJ19, Thm. 4.8])**.** *The categories* $\mathsf{KRFrm}^{\mathsf{C}}$ *and* $\mathsf{KHaus}^{\mathsf{C}}$ *are dually equivalent.*

Letting $\Diamond = \Box$, we can identify KRFrm with a wide subcategory of $\mathsf{KRFrm}^{\mathsf{C}}$. Thus, we arrive at the following diagram, where the hook arrows represent inclusions of wide subcategories and the horizontal arrows dual equivalences.

$$\begin{tikzcd}
\mathsf{KRFrm}^{\mathsf{P}} \arrow[r, leftrightarrow, "d"] & \mathsf{KHaus}^{\mathsf{R}} \\
\mathsf{KRFrm}^{\mathsf{C}} \arrow[r, leftrightarrow, "d"] \arrow[u, hook] & \mathsf{KHaus}^{\mathsf{C}} \arrow[u, hook] \\
\mathsf{KRFrm} \arrow[r, leftrightarrow, "d"] \arrow[u, hook] & \mathsf{KHaus} \arrow[u, hook]
\end{tikzcd}$$

Definition 2.4. [ABC23, Def. 2.4] Let A, B be boolean algebras. A relation $S \subseteq A \times B$ is a *subordination* if S satisfies the following conditions for all $a, b \in A$ and $c, d \in B$:

(S1) $0 \; S \; 0$ and $1 \; S \; 1$;

(S2) $a, b \; S \; c$ implies $(a \vee b) \; S \; c$;

(S3) $a \; S \; c, d$ implies $a \; S \; (c \wedge d)$;

(S4) $a \leq b \; S \; c \leq d$ implies $a \; S \; d$.

Remark 2.5. The axioms (S1)–(S4) are equivalent to saying that S is a bounded sublattice of $A \times B$ satisfying (S4).

When $A = B$, we say that S is a *subordination on* A. These were introduced in [BBSV17] as a counterpart of quasi-modal operators [Cel01] and precontact relations [DV06, DV07]. As follows from [BBSV17, Thm. 2.22], subordinations on A correspond to closed relations R on the Stone space of A. By [Cel01, DV07] (see also [BBSV17, Lem. 4.6]), we can characterize reflexivity, symmetry, and transitivity of R by the following axioms, where we write $\neg a$ for the complement of a in A.

(S5) $a \; S \; b$ implies $a \leq b$;

(S6) $a \; S \; b$ implies $\neg b \; S \; \neg a$;

(S7) $a \; S \; b$ implies there is $c \in A$ with $a \; S \; c$ and $c \; S \; b$.

Following the modal logic nomenclature, the pairs (B, S) where B is a boolean algebra and S is a subordination on B satisfying (S5)–(S7) were called S5-*subordination* algebras in [ABC23].

These algebras were first introduced in [Mee66], where the notion of a proximity on a set was generalized to an arbitrary boolean algebra. Further generalizations include proximity lattices [War74, Smy92], proximity algebras [GK81], and proximity frames [BH14]. We point out that S5-subordination algebras are exactly the proximity algebras of [GK81] where the underlying Heyting algebra is a boolean algebra.

Definition 2.6. Let $\mathbf{B} = (B, S)$ be an S5-subordination algebra.

1. [dV62, Def. 1.1.1] We call $\mathbf{B}$ a *compingent algebra* if S satisfies the following axiom:

   (S8) If $a \neq 0$, then there is $b \neq 0$ with $b \; S \; a$.

2. [Bez10, Def. 3.2] We call $\mathbf{B}$ a *de Vries algebra* if $\mathbf{B}$ is a compingent algebra and B is a complete boolean algebra.

Remark 2.7. As was pointed out in [BH14, Prop. 7.4], de Vries algebras are exactly those proximity frames where the frame is boolean.

A *de Vries morphism* between de Vries algebras is a map $f\colon B_1 \to B_2$ satisfying the following conditions:

(M1) $f(0) = 0$;

(M2) $f(a \wedge b) = f(a) \wedge f(b)$;

(M3) $a \; S_1 \; b$ implies $\neg f(\neg a) \; S_2 \; f(b)$;

(M4) $f(a) = \bigvee\{f(b) \mid b \; S_1 \; a\}$.

The composition of two de Vries morphisms $f\colon B_1 \to B_2$ and $g\colon B_2 \to B_3$ is the de Vries morphism $g * f\colon B_1 \to B_3$ given by

$$(g * f)(a) = \bigvee\{gf(b) \mid b \; S_1 \; a\}$$

for each $a \in B_1$. Let DeV be the category of de Vries algebras and de Vries morphisms, where identity morphisms are identity functions and composition is defined as above.

Theorem 2.8 (de Vries duality)**.** DeV *is dually equivalent to* KHaus.

In [BGHJ19] de Vries duality was generalized to a duality for $\mathsf{KHaus}^{\mathsf{C}}$. For this, the notion of a de Vries additive map from [BBH15] was utilized. We will instead work with the equivalent notion of a de Vries multiplicative map.

Definition 2.9. A map $\Box\colon B_1 \to B_2$ between de Vries algebras is *de Vries multiplicative* if $\Box 1 = 1$ and for all $a, b, c, d \in B_1$, we have

$$a \; S_1 \; b \text{ and } c \; S_1 \; d \text{ imply } (\Box a \wedge \Box c) \; S_2 \; \Box(b \wedge d).$$

We call $\Box$ *lower continuous* if in addition

$$\Box a = \bigvee \{\Box b \mid b \; S_1 \; a\}$$

for each $a \in B_1$. The composition of two such maps $\Box_1$ and $\Box_2$ is given by

$$(\Box_2 * \Box_1)a = \bigvee \{\Box_2 \Box_1 b \mid b \; S_1 \; a\}.$$

Let $\mathsf{DeV}^{\mathsf{C}}$ be the category of de Vries algebras and lower continuous de Vries multiplicative maps, where identity morphisms are identity functions and composition is defined as above.

Remark 2.10.

1. The results of [BGHJ19] are stated using de Vries additive maps that are lower continuous, where we recall that $\Diamond\colon B_1 \to B_2$ is de Vries additive if $\Diamond 0 = 0$ and $a \; S_1 \; b$ and $c \; S_1 \; d$ imply $\Diamond(a \vee c) \; S_2 \; (\Diamond b \vee \Diamond d)$ for all $a, b, c, d \in B_1$, and it is lower continuous if $\Diamond a = \bigvee\{\Diamond b \mid b \; S_1 \; a\}$ for all $a \in B_1$. To simplify proofs (see, e.g., Lemma 5.12), we will work with $\Box$ instead of $\Diamond$.

2. As observed in [BGHJ19, Rem. 4.11], working with lower continuous de Vries additive maps is equivalent to working with de Vries multiplicative maps that are upper continuous, i.e. maps $\Box$ that satisfy $\Box a = \bigwedge\{\Box b \mid a \; S \; b\}$. Analogously, working with de Vries multiplicative lower continuous maps is equivalent to working with de Vries additive maps that are upper continuous.

3. By a slight adjustment of the proofs of [BBH15, Thms. 4.21, 4.22] it is not difficult to show that the category of de Vries algebras and de Vries additive upper continuous maps between them is equivalent to the category of de Vries algebras and de Vries additive lower continuous maps between them. Similarly, one can show that $\mathsf{DeV}^{\mathsf{C}}$ is equivalent to the category of de Vries algebras and upper continuous de Vries multiplicative maps between them, and hence to the category of de Vries algebras and lower continuous de Vries additive maps between them. Thus, the results of [BGHJ19] apply to our setting.

Theorem 2.11 ([BGHJ19, Thm. 4.14])**.** *The categories* $\mathsf{DeV}^{\mathsf{C}}$ *and* $\mathsf{KHaus}^{\mathsf{C}}$ *are dually equivalent.*

In [BGHJ19] obtaining a de Vries like duality for $\mathsf{KHaus}^{\mathsf{R}}$ was left open. This question was resolved in [ABC23] by working with special subordination relations between de Vries algebras. To introduce them, we require the following definition of compatibility.

Definition 2.12. For $i = 1, 2$ let R_i be a binary relation on a set X_i. We call a relation $T\colon X_1 \to X_2$ *compatible* if $R_2 \circ T = T = T \circ R_1$.

$$\begin{tikzcd} X_1 \arrow[r, "T"] \arrow[d, "R_1"'] \arrow[dr, "T"] & X_2 \arrow[d, "R_2"] \\ X_1 \arrow[r, "T"'] & X_2 \end{tikzcd}$$

Let $\mathsf{SubS5}^{\mathsf{S}}$ be the category of S5-subordination algebras and compatible subordinations between them, where the composition of morphisms is the usual composition of relations, and the identity morphism on an S5-subordination algebra (B, S) is the relation S. Let $\mathsf{DeV}^{\mathsf{S}}$ be the full subcategory of $\mathsf{SubS5}^{\mathsf{S}}$ consisting of de Vries algebras.

To connect $\mathsf{KHaus}^{\mathsf{R}}$ with $\mathsf{SubS5}^{\mathsf{S}}$, it is convenient to first obtain a Stone-like representation of S5-subordination algebras.

Definition 2.13.

1. An S5-*subordination space* is a pair (X, E) where X is a Stone space and E is a closed equivalence relation on X. We let $\mathsf{StoneE}^{\mathsf{R}}$ be the category whose objects are S5-subordination spaces and whose morphisms are compatible closed relations between them.

2. A *Gleason space* is an $\mathsf{S5}$-subordination space (X, E) such that X is *extremally disconnected* (i.e., the closure of an open set is open) and E is *irreducible* (i.e., if F is a proper closed subset of X, then so is $E[F]$). We let Gle^R be the full subcategory of $\mathsf{StoneE}^\mathsf{R}$ whose objects are Gleason spaces.

Theorem 2.14 ([ABC23, Cors. 3.14, 4.7])**.** $\mathsf{KHaus}^\mathsf{R}$*,* $\mathsf{StoneE}^\mathsf{R}$*,* Gle^R*,* $\mathsf{SubS5}^\mathsf{S}$*, and* DeV^S *are equivalent categories.*

$$\begin{tikzcd}
 & \mathsf{StoneE}^\mathsf{R} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] & \mathsf{SubS5}^\mathsf{S} \arrow[d, leftrightarrow] \\
\mathsf{KHaus}^\mathsf{R} \arrow[dr, leftrightarrow] & & \\
 & \mathsf{Gle}^\mathsf{R} \arrow[r, leftrightarrow] & \mathsf{DeV}^\mathsf{S}
\end{tikzcd}$$

To make the paper self-contained, we briefly describe the functors yielding some of the equivalences of Theorem 2.14.

Remark 2.15.

1. The functor $\mathcal{Q}\colon \mathsf{StoneE}^\mathsf{R} \to \mathsf{KHaus}^\mathsf{R}$ maps an object (X, E) to the quotient space X/E, and a morphism $R\colon (X_1, E_1) \to (X_2, E_2)$ to the morphism $\mathcal{Q}(R)\colon \mathcal{Q}(X_1, E_1) \to \mathcal{Q}(X_2, E_2)$ given by

$$[x]_{E_1}\ \mathcal{Q}(R)\ [y]_{E_2} \iff x\ R\ y$$

(i.e., $\mathcal{Q}(R) = \pi_2 \circ R \circ \pi_1{}^\smile$, where π_1 and π_2 are the quotient maps).

$$\begin{tikzcd}
X_1 \arrow[r, "R"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\
X_1/E_1 \arrow[r, "\mathcal{Q}(R)"'] & X_2/E_2
\end{tikzcd}$$

2. A quasi-inverse of the functor $\mathcal{Q}$ is given by the Gleason cover functor $\mathcal{G}\colon \mathsf{KHaus}^\mathsf{R} \to \mathsf{StoneE}^\mathsf{R}$ which associates to each compact Hausdorff space X the pair $\mathcal{G}(X) = (\widehat{X}, E)$ where $g\colon \widehat{X} \to X$ is the Gleason cover of X and $x\ E\ y$ iff $g(x) = g(y)$ (for Gleason covers see, e.g.,

[Joh82, Sec. III.3.10]). It also maps a closed relation $R\colon X_1 \to X_2$ to the relation $\mathcal{G}(R)\colon \mathcal{G}(X_1) \to \mathcal{G}(X_2)$ given by

$$x\ \mathcal{G}(R)\ y \iff g_1(x)\ R\ g_2(y)$$

(i.e., $\mathcal{G}(R) = g_2{}^{\smile} \circ R \circ g_1$).

$$\begin{tikzcd}
\widehat{X_1} \arrow[r, "\mathcal{G}(R)"] \arrow[d, "g_1"'] & \widehat{X_2} \arrow[d, "g_2"] \\
X_1 \arrow[r, "R"'] & X_2
\end{tikzcd}$$

3. The functor $\mathcal{G}$ is also a quasi-inverse of the restriction of the functor $\mathcal{Q}$ to $\mathsf{Gle}^{\mathsf{R}}$.

4. The inclusion of $\mathsf{Gle}^{\mathsf{R}}$ into $\mathsf{StoneE}^{\mathsf{R}}$ is an equivalence whose quasi-inverse is the composition $\mathcal{G} \circ \mathcal{Q}$.

5. The functor $\mathsf{Clop}\colon \mathsf{StoneE}^{\mathsf{R}} \to \mathsf{SubS5}^{\mathsf{S}}$ maps an object (X, E) to (B, S_E), where B is the boolean algebra of clopen subsets of X and S_E is the binary relation on B given by $U\ S_E\ V$ iff $E[U] \subseteq V$. Also, Clop maps a morphism $R\colon (X_1, E_1) \to (X_2, E_2)$ to the compatible subordination relation $S_R\colon \mathsf{Clop}(X_1, E_1) \to \mathsf{Clop}(X_2, E_2)$ given by $U\ S_R\ V$ iff $R[U] \subseteq V$.

6. A quasi-inverse of the functor Clop is given by the ultrafilter functor $\mathsf{Ult}\colon \mathsf{SubS5}^{\mathsf{S}} \to \mathsf{StoneE}^{\mathsf{R}}$ which associates to each object (B, S) the pair $\mathsf{Ult}(B, S) = (X, R_S)$ where X is the Stone space of ultrafilters of B and $x\ R_S\ y$ iff $S[x] \subseteq y$. We call (X, R_S) the $\mathsf{S5}$-*subordination space* of (B, S). A morphism $T\colon (B_1, S_1) \to (B_2, S_2)$ is mapped by Ult to the morphism $R_T\colon \mathsf{Ult}(B_1, S_1) \to \mathsf{Ult}(B_2, S_2)$ given by $x\ R_T\ y$ iff $T[x] \subseteq y$.

7. The restrictions $\mathsf{Clop}\colon \mathsf{Gle}^{\mathsf{R}} \to \mathsf{DeV}^{\mathsf{S}}$ and $\mathsf{Ult}\colon \mathsf{DeV}^{\mathsf{S}} \to \mathsf{Gle}^{\mathsf{R}}$ are also quasi-inverses of each other.

It follows from Theorems 2.2 and 2.14 that $\mathsf{SubS5}^{\mathsf{S}}$ is dually equivalent to $\mathsf{KRFrm}^{\mathsf{P}}$ and equivalent to $\mathsf{DeV}^{\mathsf{S}}$. The main contribution of this paper is to give direct choice-free proofs of these results by generalizing ideal and MacNeille completions of boolean algebras to the setting of S5-subordination algebras, to fill in the empty boxes of the following diagram, and to show that it commutes up to natural isomorphism. The unlabeled horizontal arrows in the diagram represent equivalences of categories while the ones labeled with the letter d represent dual equivalences. The vertical arrows are inclusions of wide subcategories.

$$
\begin{tikzcd}
\mathsf{SubS5}^{\mathsf{S}} \arrow[r, leftrightarrow] & \mathsf{DeV}^{\mathsf{S}} \arrow[rr, leftrightarrow, "d"] & & \mathsf{KRFrm}^{\mathsf{P}} \arrow[r, leftrightarrow, "d"] & \mathsf{KHaus}^{\mathsf{R}} \arrow[r, leftrightarrow] & \mathsf{StoneE}^{\mathsf{R}} \arrow[r, leftrightarrow] & \mathsf{Gle}^{\mathsf{R}} \\
\boxed{} \arrow[u, hook] \arrow[r, leftrightarrow] & \boxed{} \arrow[u, hook] \arrow[r, leftrightarrow, "d"] & \mathsf{DeV}^{\mathsf{C}} \arrow[r, leftrightarrow] & \mathsf{KRFrm}^{\mathsf{C}} \arrow[u, hook] \arrow[r, leftrightarrow, "d"] & \mathsf{KHaus}^{\mathsf{C}} \arrow[u, hook] \arrow[r, leftrightarrow] & \boxed{} \arrow[u, hook] \arrow[r, leftrightarrow] & \boxed{} \arrow[u, hook] \\
\boxed{} \arrow[u, hook] \arrow[r, leftrightarrow] & \boxed{} \arrow[u, hook] \arrow[r, leftrightarrow, "d"] & \mathsf{DeV} \arrow[u, hook] \arrow[r, leftrightarrow] & \mathsf{KRFrm} \arrow[u, hook] \arrow[r, leftrightarrow, "d"] & \mathsf{KHaus} \arrow[u, hook] \arrow[r, leftrightarrow] & \boxed{} \arrow[u, hook] \arrow[r, leftrightarrow] & \boxed{} \arrow[u, hook]
\end{tikzcd}
$$

Figure 1

3. Round ideals of S5-subordination algebras

For a boolean algebra B, let $\mathcal{I}(B)$ be the set of ideals of B ordered by inclusion. It is well known that $\mathcal{I}(B)$ is a frame, where $I \wedge J = I \cap J$ and $\bigvee I_\alpha$ is the ideal generated by $\bigcup I_\alpha$. Moreover, the compact elements of $\mathcal{I}(B)$ are the principal ideals. This in particular implies that $\mathcal{I}(B)$ is compact and regular.[1] In this section we generalize these results to the frame of round ideals of an S5-subordination algebra.

Round ideals have been extensively studied in pointfree topology and domain theory. In particular, it follows from [War74, Smy92] that the round ideals of a proximity lattice form a stably compact frame. As we pointed out in the previous section, S5-subordination algebras (B, S) are exactly the proximity algebras of [GK81] where the algebra B is a boolean algebra. This additional feature allows us to show that the round ideals of (B, S) form

[1]The frame $\mathcal{I}(B)$ is even zero-dimensional because every element in $\mathcal{I}(B)$ is a join of complemented elements (see [Ban89]).

a compact regular frame. Moreover, associating with each $\mathsf{S5}$-subordination algebra its frame of round ideals defines a contravariant functor from $\mathsf{SubS5^S}$ to $\mathsf{KRFrm^P}$. In Section 4 we will show that this functor is in fact a dual equivalence.

Definition 3.1. Let $\mathbf{B} = (B, S)$ be an $\mathsf{S5}$-subordination algebra. We call an ideal I of B a *round ideal* if $a \in I$ implies $a \ S \ b$ for some $b \in I$. Let $\mathcal{RI}(\mathbf{B})$ be the set of round ideals of $\mathbf{B}$ ordered by inclusion.

Remark 3.2.

1. It is straightforward to see that an ideal I is round iff $I = S^{-1}[I]$, and that if I is an ideal of B, then $S^{-1}[I]$ is a round ideal of $\mathbf{B}$.
2. The notion of a round filter is dual to that of a round ideal. Therefore, a filter F is round iff $F = S[F]$, and if F is a filter of B, then $S[F]$ is a round filter of $\mathbf{B}$.

Let B be a boolean algebra and $X \subseteq B$. We denote by $U(X)$ the set of upper bounds of X, by $L(X)$ the set of lower bounds of X, and by $\neg X$ the set $\{\neg x \mid x \in X\}$. It is well known that $U(X)$ is a filter, $L(X)$ is an ideal, $\neg\neg X = X$, and X is a filter iff $\neg X$ is an ideal. Moreover, $\neg U(X) = L(\neg X)$ and $\neg L(X) = U(\neg X)$.

Lemma 3.3. *Let B be a boolean algebra and S an $\mathsf{S5}$-subordination on B. If $X \subseteq B$, then $\neg S[X] = S^{-1}[\neg X]$.*

Proof. We have that $a \in \neg S[X]$ iff there is $x \in X$ such that $x \ S \ \neg a$. By (S6) this is equivalent to the existence of $x \in X$ such that $a \ S \ \neg x$, which means that $a \in S^{-1}[\neg X]$. $\square$

Theorem 3.4. *Let $\mathbf{B}$ be an $\mathsf{S5}$-subordination algebra.*

(1) *$\mathcal{RI}(\mathbf{B})$ is a subframe of $\mathcal{I}(\mathbf{B})$.*

(2) *If $I \in \mathcal{RI}(\mathbf{B})$, then $I^* = S^{-1}[\neg U(I)] = \neg S[U(I)]$.*

(3) *The well-inside relation on $\mathcal{RI}(\mathbf{B})$ is given by $I \prec J$ iff $U(I) \cap J \neq \varnothing$.*

(4) *$\mathcal{RI}(\mathbf{B})$ is compact and regular.*

Proof. (1). This follows from [War74, Thm. 3] (see also [Smy92, Thm. 1]).

(2). The first equality follows from [War74, Thm. 3] and the second from Lemma 3.3.

(3). By definition, $I \prec J$ iff $I^* \vee J = B$. By item (2), this is equivalent to $\neg S[U(I)] \vee J = B$, which holds iff there are $a \in S[U(I)]$ and $b \in J$ such that $\neg a \vee b = 1$. Since B is a boolean algebra, $\neg a \vee b = 1$ iff $a \leq b$. Because $S[U(I)]$ is a filter (see Remark 3.2(2)), the existence of $a \in S[U(I)]$ with $a \leq b$ is equivalent to $b \in S[U(I)]$. Thus, $I \prec J$ iff $S[U(I)] \cap J \neq \varnothing$. We have that $S[U(I)] \cap J \neq \varnothing$ iff $U(I) \cap S^{-1}[J] \neq \varnothing$. Since J is a round ideal, this is equivalent to $U(I) \cap J \neq \varnothing$.

(4). That $\mathcal{RI}(\mathbf{B})$ is compact follows from item (1). It follows from [War74, Thm. 3] that the relation on $\mathcal{RI}(\mathbf{B})$ given by $U(I) \cap J \neq \varnothing$ is approximating. Thus, item (3) implies that the well-inside relation is approximating, and hence $\mathcal{RI}(\mathbf{B})$ is regular. □

Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be S5-subordination algebras and $T\colon \mathbf{B}_1 \to \mathbf{B}_2$ a compatible subordination. We define $\mathcal{RI}(T)\colon \mathcal{RI}(\mathbf{B}_2) \to \mathcal{RI}(\mathbf{B}_1)$ by setting $\mathcal{RI}(T)(I) = T^{-1}[I]$ for each round ideal I of $\mathbf{B}_2$.

Theorem 3.5. *$\mathcal{RI}\colon \mathsf{SubS5}^{\mathsf{S}} \to \mathsf{KRFrm}^{\mathsf{P}}$ is a well-defined contravariant functor.*

Proof. That $\mathcal{RI}$ is well defined on objects follows from Theorem 3.4(4). We show that it is well defined on morphisms. Let T be a compatible subordination from $\mathbf{B}_1 = (B_1, S_1)$ to $\mathbf{B}_2 = (B_2, S_2)$. Let $I \in \mathcal{RI}(\mathbf{B}_2)$. Since T is a subordination, it is straightforward to see that $T^{-1}[I]$ is an ideal. Because T is compatible, $S_1^{-1}T^{-1}[I] = (T \circ S_1)^{-1}[I] = T^{-1}[I]$, and hence $T^{-1}[I]$ is a round ideal. Thus, $\mathcal{RI}(T)$ is well defined. To show that $\mathcal{RI}(T)$ is a preframe homomorphism, we need to prove that it preserves directed joins and finite meets. That it preserves directed joins is straightforward because directed joins are set-theoretic unions in $\mathcal{I}(\mathbf{B}_1)$ and $\mathcal{I}(\mathbf{B}_2)$, and hence also in their subframes $\mathcal{RI}(\mathbf{B}_1)$ and $\mathcal{RI}(\mathbf{B}_2)$. Moreover, we have that $T^{-1}[B_2] = B_1$ because $a\ T\ 1$ for each $a \in B_1$. Thus, it remains to show that $\mathcal{RI}(T)$ preserves binary meets. Let $I, J \in \mathcal{RI}(\mathbf{B}_2)$. Clearly $T^{-1}[I \cap J] \subseteq T^{-1}[I] \cap T^{-1}[J]$. For the other inclusion, let $a \in T^{-1}[I] \cap T^{-1}[J]$. Then there are $b \in I$, $c \in J$ such that $a\ T\ b$ and $a\ T\ c$. Therefore, $a\ T\ (b \wedge c) \in I \cap J$ by (S3), and hence $a \in T^{-1}[I \cap J]$.

It is straightforward to show that $\mathcal{RI}$ preserves identities and reverses compositions. Thus, $\mathcal{RI}\colon \mathsf{SubS5^S} \to \mathsf{KRFrm^P}$ is a well-defined contravariant functor. $\square$

In the next section we will show that $\mathcal{RI}$ is a dual equivalence.

4. MacNeille completions of S5-subordination algebras

In [ABC23] we showed that the categories $\mathsf{SubS5^S}$ and $\mathsf{DeV^S}$ are equivalent. This was done by observing that each of these categories is equivalent to $\mathsf{KHaus^R}$. In this section we show that the equivalence can be obtained directly by generalizing the theory of MacNeille completions of boolean algebras to S5-subordination algebras.

For a frame L, we recall (see, e.g., [BP96]) that the *booleanization* of L is

$$\mathfrak{B}L = \{a \in L \mid a = a^{**}\},$$

and that $(\mathfrak{B}L, \sqcap, \bigsqcup)$ is a boolean frame (complete boolean algebra), where

$$a \sqcap b = a \wedge b \quad \text{and} \quad \bigsqcup S = \left(\bigvee S\right)^{**}.$$

If L is compact regular, then $(\mathfrak{B}L, \prec)$ is a de Vries algebra, where $\prec$ is the restriction of the well-inside relation on L to $\mathfrak{B}L$. As was shown in [Bez12], this correspondence extends to a covariant functor $\mathfrak{B}\colon \mathsf{KRFrm} \to \mathsf{DeV}$ which is an equivalence. In the more general setting of $\mathsf{KRFrm^P}$ and $\mathsf{DeV^S}$, this correspondence extends to a contravariant functor as follows.

Let $\square\colon L \to M$ be a preframe homomorphism. Define the relation $\mathfrak{B}(\square)\colon \mathfrak{B}M \to \mathfrak{B}L$ by

$$b\, \mathfrak{B}(\square)\, a \iff b \prec \square a.$$

Lemma 4.1. *If $\square\colon L \to M$ is a preframe homomorphism, then the relation $\mathfrak{B}(\square)\colon \mathfrak{B}M \to \mathfrak{B}L$ is a compatible subordination.*

Proof. Let $T = \mathfrak{B}(\square)$. It is straightforward to check that T is a subordination. We only verify (S3). Suppose $b \mathrel{T} a, c$. Then $b \prec \square a$ and $b \prec \square c$. Since $\square$ is a preframe homomorphism, we have $b \prec \square a \wedge \square c = \square(a \wedge c)$.

Thus, T satisfies (S3). We next prove that T is compatible. Let $a \in \mathfrak{B}L$ and $b \in \mathfrak{B}M$. We show that $b\ T\ a$ iff there is $d \in \mathfrak{B}M$ such that $b \prec d\ T\ a$. First suppose that $b\ T\ a$, so $b \prec \Box a$. Since M is compact regular, there is $d \in \mathfrak{B}M$ such that $b \prec d \prec \Box a$ (see, e.g., [Bez12, Rem. 3.2]). Therefore, $b \prec d\ T\ a$. Conversely, suppose that $b \prec d\ T\ a$. Then $b \prec d \prec \Box a$. Thus, $b \prec \Box a$, and so $b\ T\ a$.

It remains to show that $b\ T\ a$ iff there is $c \in \mathfrak{B}L$ such that $b\ T\ c \prec a$. For the right-to-left implication, we have that $c \prec a$ implies $c \leq a$, and hence $\Box c \leq \Box a$ because $\Box$ is order-preserving. Since $b \prec \Box c$, it follows that $b \prec \Box a$, and so $b\ T\ a$. For the left-to-right implication, since L is a regular frame, a is the directed join of $\{c \in \mathfrak{B}L \mid c \prec a\}$. Therefore, since $\Box$ preserves directed joins, $\Box a = \bigvee\{\Box c \mid c \in \mathfrak{B}L,\ c \prec a\}$. Thus, from $b \prec \Box a$, using compactness, we find $c \in \mathfrak{B}L$ such that $c \prec a$ and $b \prec \Box c$. □

We thus define $\mathfrak{B} \colon \mathsf{KRFrm}^{\mathsf{P}} \to \mathsf{DeV}^{\mathsf{S}}$ by sending each compact regular frame L to $(\mathfrak{B}L, \prec)$ and each preframe homomorphism $\Box \colon L \to M$ to $\mathfrak{B}(\Box)$.

Proposition 4.2. $\mathfrak{B} \colon \mathsf{KRFrm}^{\mathsf{P}} \to \mathsf{DeV}^{\mathsf{S}}$ *is a contravariant functor.*

Proof. That $\mathfrak{B}$ is well defined on objects follows from [Bez12, Lem. 3.1] and that it is well defined on morphisms from Lemma 4.1. Let L be a compact regular frame. If $\Box$ is the identity on L, then $\mathfrak{B}(\Box)$ coincides with $\prec$ which is the identity on $(\mathfrak{B}L, \prec)$. Let $\Box_1 \colon L \to M$ and $\Box_2 \colon M \to N$ be two preframe homomorphisms between compact regular frames. We show that $\mathfrak{B}(\Box_2 \circ \Box_1) = \mathfrak{B}(\Box_1) \circ \mathfrak{B}(\Box_2)$. Let $T_1 = \mathfrak{B}(\Box_1)$ and $T_2 = \mathfrak{B}(\Box_2)$. For $a \in \mathfrak{B}L$ and $c \in \mathfrak{B}N$, if $c\ (T_1 \circ T_2)\ a$, then there is $b \in \mathfrak{B}M$ such that $c\ T_2\ b$ and $b\ T_1\ a$. Thus, $c \prec \Box_2 b$ and $b \prec \Box_1 a$. Since $b \prec \Box_1 a$ and $\Box_2$ is order-preserving, we have $\Box_2 b \leq \Box_2 \Box_1 a$. Therefore, $c \prec \Box_2 \Box_1 a$ which means that $c\ \mathfrak{B}(\Box_2 \circ \Box_1)\ a$. Suppose next that $c\ \mathfrak{B}(\Box_2 \circ \Box_1)\ a$. Therefore, $c \prec \Box_2 \Box_1 a$. By arguing as at the end of the proof of Lemma 4.1, there is $b \in \mathfrak{B}M$ such that $c\ T_2\ b$ and $b \prec \Box_1 a$. Thus, $c\ T_2\ b$ and $b\ T_1\ a$ which means that $c\ (T_1 \circ T_2)\ a$. □

Definition 4.3. Let $\mathcal{NI} = \mathfrak{B} \circ \mathcal{RI}$.

$$\begin{tikzcd} \mathsf{SubS5^S} \arrow[r, "\mathcal{RI}"'] \arrow[rr, bend left, "\mathcal{NI}"] & \mathsf{KRFrm^P} \arrow[r, "\mathfrak{B}"'] & \mathsf{DeV^S} \end{tikzcd}$$

By Theorem 3.5 $\mathcal{RI}\colon \mathsf{SubS5^S} \to \mathsf{KRFrm^P}$ is a contravariant functor, and by Proposition 4.2 $\mathfrak{B}\colon \mathsf{KRFrm^P} \to \mathsf{DeV^S}$ is a contravariant functor. Thus, $\mathcal{NI}\colon \mathsf{SubS5^S} \to \mathsf{DeV^S}$ is a covariant functor. In particular, we have

Proposition 4.4. *If $\mathbf{B}$ is an $\mathsf{S5}$-subordination algebra, then $\mathcal{NI}(\mathbf{B})$ is a de Vries algebra.*

Remark 4.5. Since $\prec$ on $\mathcal{NI}(\mathbf{B})$ is the restriction of $\prec$ on $\mathcal{RI}(\mathbf{B})$, by Theorem 3.4(3) we have that $I \prec J$ iff $U(I) \cap J \neq \varnothing$ for all $I, J \in \mathcal{NI}(\mathbf{B})$.

Definition 4.6. Let $\mathbf{B} = (B, S)$ be an $\mathsf{S5}$-subordination algebra. We call $\mathcal{NI}(\mathbf{B})$ the *MacNeille completion* of $\mathbf{B}$. We say that a round ideal I of $\mathbf{B}$ is *normal* if $I \in \mathcal{NI}(\mathbf{B})$.

The next theorem provides a characterization of normal round ideals.

Theorem 4.7. *Let $I \in \mathcal{RI}(\mathbf{B})$. We have*

$$I \in \mathcal{NI}(\mathbf{B}) \iff I = S^{-1}[L(S[U(I)])].$$

Proof. By Lemma 3.3 and Theorem 3.4(2),

$$\begin{aligned} I^{**} &= \neg S[U(\neg S[U(I)])] = \neg S[\neg L(S[U(I)])] \\ &= \neg\neg S^{-1}[L(S[U(I)])] = S^{-1}[L(S[U(I)])]. \end{aligned}$$

Since $I \in \mathcal{NI}(\mathbf{B})$ iff $I = I^{**}$, the result follows. □

Remark 4.8. We recall (see, e.g., [Grä78, p. 98]) that an ideal I of a boolean algebra B is *normal* if $LU(I) = I$, and that the *MacNeille completion* of B is constructed as the complete boolean algebra of normal ideals of B. Definition 4.6 and Theorem 4.7 are an obvious generalization of this. Indeed, if S is the partial ordering of B, then $I \in \mathcal{NI}(\mathbf{B})$ iff I is a normal ideal of B. For further connection, see Proposition 4.14.

An important feature of the MacNeille completion of an S5-subordination algebra $\mathbf{B}$ is that it is isomorphic to $\mathbf{B}$ in $\mathsf{SubS5^S}$ (which happens because morphisms in $\mathsf{SubS5^S}$ are not structure-preserving bijections; see [ABC23, Rem. 3.15(4)]). To see this, we need the following lemma. We freely use the fact that if $I, J \in \mathcal{RI}(\mathbf{B})$, then

$$I \prec J \implies I^{**} \prec J, \tag{1}$$

which is a consequence of $I^{***} = I^*$.

Lemma 4.9. *Let $a \in \mathbf{B}$ and $J \in \mathcal{RI}(\mathbf{B})$. Then $a \in J$ iff there is $I \in \mathcal{NI}(\mathbf{B})$ such that $a \in I \prec J$.*

Proof. For the right-to-left implication, if $a \in I \prec J$, then $a \in I \subseteq J$, and hence $a \in J$. For the left-to-right implication, since J is a round ideal, there is $b \in J$ such that $a\ S\ b$. We have $a \in S^{-1}[b]$ and $b \in U(S^{-1}[b])$. Thus, $S^{-1}[b] \prec J$ by Theorem 3.4(3). Let $I = (S^{-1}[b])^{**}$. Then $I \in \mathcal{NI}(\mathbf{B})$ and $a \in S^{-1}[b] \subseteq I$. Moreover, by (1), $S^{-1}[b] \prec J$ implies $I \prec J$. Consequently, $a \in I \prec J$. $\square$

Let $Q_{\mathbf{B}} \colon \mathbf{B} \to \mathcal{NI}(\mathbf{B})$ be the relation defined by

$$a\, Q_{\mathbf{B}}\, I \iff a \in I.$$

Lemma 4.10. *$Q_{\mathbf{B}}$ is a morphism in* $\mathsf{SubS5^S}$.

Proof. It is easy to see that $Q_{\mathbf{B}}$ is a subordination relation. The equality $Q_{\mathbf{B}} = Q_{\mathbf{B}} \circ S$ follows from $I = S^{-1}[I]$, and the equality $\prec \circ\, Q_{\mathbf{B}} = Q_{\mathbf{B}}$ from Lemma 4.9. $\square$

If $T \colon \mathbf{B}_1 \to \mathbf{B}_2$ is a morphism in $\mathsf{SubS5^S}$, define $\widehat{T} \colon \mathbf{B}_2 \to \mathbf{B}_1$ by

$$b\, \widehat{T}\, a \iff \neg a\, T\, \neg b. \tag{2}$$

Then the relation $\widehat{T}$ is a morphism in $\mathsf{SubS5^S}$ (see the paragraph before [ABC23, Thm. 3.10]).

Lemma 4.11. *$Q_{\mathbf{B}} \colon \mathbf{B} \to \mathcal{NI}(\mathbf{B})$ is an isomorphism.*

Proof. Let $T = \widehat{Q_{\mathbf{B}}} \colon \mathcal{NI}(\mathbf{B}) \to \mathbf{B}$. By (2) and Theorem 3.4(2),

$$I \; T \; a \iff \neg a \; Q_{\mathbf{B}} \; I^* \iff \neg a \in \neg S[U(I)] \iff a \in S[U(I)]. \tag{3}$$

We show that $Q_{\mathbf{B}}$ and T are inverses of each other. For this we need to prove that $T \circ Q_{\mathbf{B}} = S$ and $Q_{\mathbf{B}} \circ T = \prec$.

We first show that $T \circ Q_{\mathbf{B}} = S$. For the inclusion $\subseteq$, let $a, b \in B$, $I \in \mathcal{NI}(\mathbf{B})$, and $a \; Q_{\mathbf{B}} \; I \; T \; b$. Then $a \in I$ and $b \in S[U(I)]$ by (3). Thus, $a \; S \; b$. For the inclusion $\supseteq$, let $a, b \in B$ with $a \; S \; b$. Then $a \in S^{-1}[b]$ and Lemma 4.9 implies that there is $I \in \mathcal{NI}(\mathbf{B})$ such that $a \in I \prec S^{-1}[b]$. By Remark 4.5 and (3),

$$I \prec S^{-1}[b] \iff U(I) \cap S^{-1}[b] \neq \varnothing \iff b \in S[U(I)] \iff I \; T \; b.$$

Thus, $a \; Q_{\mathbf{B}} \; I \; T \; b$.

We next show that $Q_{\mathbf{B}} \circ T = \prec$. Let $I, J \in \mathcal{NI}(\mathbf{B})$. By Remark 4.5 and (3),

$$\begin{aligned} I \prec J &\iff U(I) \cap J \neq \varnothing \iff U(I) \cap S^{-1}[J] \neq \varnothing \\ &\iff S[U(I)] \cap J \neq \varnothing \iff \exists a \in S[U(I)] \cap J \\ &\iff \exists a \in B : I \; T \; a \; Q_{\mathbf{B}} \; J \iff I \; (Q_{\mathbf{B}} \circ T) \; J. \end{aligned}$$

Thus, $Q_{\mathbf{B}} \colon \mathbf{B} \to \mathcal{NI}(\mathbf{B})$ is an isomorphism. $\square$

Proposition 4.12. *Let $\Delta \colon \mathsf{DeV}^{\mathsf{S}} \to \mathsf{SubS5}^{\mathsf{S}}$ be the inclusion functor. Then $Q \colon 1_{\mathsf{SubS5}^{\mathsf{S}}} \to \Delta \circ \mathcal{NI}$ is a natural isomorphism.*

Proof. Let $T \colon \mathbf{B}_1 \to \mathbf{B}_2$ be a morphism in $\mathsf{SubS5}^{\mathsf{S}}$. By Lemma 4.11, it is sufficient to show that $\mathcal{NI}(T) \circ Q_{\mathbf{B}_1} = Q_{\mathbf{B}_2} \circ T$. (Since Δ is the inclusion functor, we omit it from the diagram.)

$$\begin{tikzcd} \mathbf{B}_1 \arrow[r, "Q_{\mathbf{B}_1}"] \arrow[d, "T"'] & \mathcal{NI}(\mathbf{B}_1) \arrow[d, "\mathcal{NI}(T)"] \\ \mathbf{B}_2 \arrow[r, "Q_{\mathbf{B}_2}"] & \mathcal{NI}(\mathbf{B}_2) \end{tikzcd}$$

Let $a \in B_1$ and $I \in \mathcal{NI}(\mathbf{B}_2)$. We have

$$a \; (\mathcal{NI}(T) \circ Q_{\mathbf{B}_1}) \; I \iff \exists J \in \mathcal{NI}(\mathbf{B}_1) : a \in J \text{ and } J \prec T^{-1}[I],$$

and

$$a\ (Q_{\mathbf{B}_2} \circ T)\ I \iff \exists b \in B_2 : a\ T\ b \text{ and } b \in I \iff a \in T^{-1}[I].$$

The two conditions are equivalent by Lemma 4.9. $\square$

Theorem 4.13. *$\mathcal{NI}\colon \mathsf{SubS5^S} \to \mathsf{DeV^S}$ and $\Delta\colon \mathsf{DeV^S} \to \mathsf{SubS5^S}$ are quasi-inverses of each other. Thus, $\mathsf{SubS5^S}$ and $\mathsf{DeV^S}$ are equivalent.*

Proof. By Proposition 4.12, $Q\colon 1_{\mathsf{SubS5^S}} \to \Delta \circ \mathcal{NI}$ is a natural isomorphism. For the same reason, we have a natural isomorphism $Q'\colon 1_{\mathsf{DeV^S}} \to \mathcal{NI} \circ \Delta$ whose component on $\mathbf{B} \in \mathsf{DeV^S}$ is $Q_{\mathbf{B}}$. Thus, $\Delta\colon \mathsf{DeV^S} \to \mathsf{SubS5^S}$ is a quasi-inverse of $\mathcal{NI}$. $\square$

Theorem 4.13 gives a direct choice-free proof that $\mathsf{SubS5^S}$ is equivalent to $\mathsf{DeV^S}$. We next show that when restricted to compingent algebras, $\mathcal{NI}$ yields the usual MacNeille completion.

Proposition 4.14. *Let $\mathbf{B} = (B, S)$ be an $\mathsf{S5}$-subordination algebra.*

(1) *If $\mathbf{B}$ is a compingent algebra, then there is a boolean isomorphism between $\mathcal{NI}(\mathbf{B})$ and the usual MacNeille completion $\overline{B}$ of B.*

(2) *If $\mathbf{B}$ is a de Vries algebra, then there is a structure-preserving bijection between $\mathbf{B}$ and $\mathcal{NI}(\mathbf{B})$.*

Proof. (1). Since $\mathbf{B}$ is a compingent algebra, from [dV62, Thm. 1.1.4] it follows that each $b \in B$ is the supremum of $S^{-1}[b]$. We use this fact to prove that

$$U(S^{-1}[I]) = U(I) \tag{4}$$

for each ideal I of B. Since $S^{-1}[I] \subseteq I$, we have $U(I) \subseteq U(S^{-1}[I])$. For the reverse inclusion, let $a \in U(S^{-1}[I])$. We show that $a \in U(I)$. Let $b \in I$. Then $S^{-1}[b] \subseteq S^{-1}[I]$. Therefore, $a \in U(S^{-1}[b])$, so $a \geq \bigvee S^{-1}[b] = b$. Thus, $a \in U(I)$. This proves (4). A similar argument proves that

$$L(S[F]) = L(F) \tag{5}$$

for each filter F of B. By (4) and (5), for every normal ideal I of B, we have

$$L(S[U(S^{-1}[I])]) = L(S[U(I)]) = L(U(I)) = I.$$

Thus, applying S^{-1} to both sides yields

$$S^{-1}[L(S[U(S^{-1}[I])])] = S^{-1}[I].$$

This shows, by Theorem 4.7, that $S^{-1}[I] \in \mathcal{NI}(\mathbf{B})$ for every normal ideal I of B. This defines an order-preserving map $\alpha\colon \overline{B} \to \mathcal{NI}(\mathbf{B})$.

Conversely, for every $I \in \mathcal{NI}(\mathbf{B})$, we have that $L(U(I))$ is a normal ideal of B. This defines an order-preserving map $\beta\colon \mathcal{NI}(\mathbf{B}) \to \overline{B}$. By (4), for a normal ideal I of B, we have

$$L(U(S^{-1}[I])) = L(U(I)) = I.$$

For a normal round ideal I, by (5) and Theorem 4.7, we have

$$S^{-1}[L(U(I))] = S^{-1}[L(S[U(I)])) = I.$$

Thus, α and β are order-isomorphisms, hence boolean isomorphisms.

(2). It is well known (see, e.g., [GH09, Thm. 22]) that sending b to the downset ${\downarrow}b \coloneqq \{a \in B \mid a \leq b\}$ gives a boolean embedding of B into $\overline{B}$, which is an isomorphism iff B is complete. Composing with α yields the boolean embedding $\iota\colon B \to \mathcal{NI}(\mathbf{B})$ given by $\iota(b) = S^{-1}[b]$. If $\mathbf{B}$ is a de Vries algebra, then ι becomes a boolean isomorphism by item (1). It is left to prove that $a \mathrel{S} b$ iff $\iota(a) \prec \iota(b)$. If $a \mathrel{S} b$, then $a \in U(\iota(a)) \cap \iota(b)$, and so $\iota(a) \prec \iota(b)$ by Remark 4.5. Conversely, suppose that $\iota(a) \prec \iota(b)$. Then $U(\iota(a)) \cap \iota(b) \neq \varnothing$, so there exists $c \in U(\iota(a)) \cap \iota(b)$. Since a is the supremum of $\iota(a) = S^{-1}[a]$, we have that $a \leq c \mathrel{S} b$, and hence $a \mathrel{S} b$. Thus, ι is a structure-preserving bijection between $\mathbf{B}$ and $\mathcal{NI}(\mathbf{B})$. □

Remark 4.15. Let $\mathbf{B} = (B, S)$ be a compingent algebra and $\overline{B}$ the MacNeille completion of B. By [BBSV19, Rem. 5.11], $(\overline{B}, \lhd)$ is a de Vries algebra, where

$$I \lhd J \iff U(I) \cap S^{-1}[J] \neq \varnothing.$$

A straightforward verification shows that the boolean isomorphism of Proposition 4.14(1) is an isomorphism of de Vries algebras between $\mathcal{NI}(\mathbf{B})$ and $(\overline{B}, \lhd)$.

Remark 4.16. Let B be a compingent algebra. Then $Q_{\mathbf{B}}\colon \mathbf{B} \to \mathcal{NI}(\mathbf{B})$ and $\iota\colon \mathbf{B} \to \mathcal{NI}(\mathbf{B})$ are related as follows:

$$a \mathrel{Q_{\mathbf{B}}} I \iff \iota(a) \prec I$$

for each $a \in B$ and $I \in \mathcal{NI}(\mathbf{B})$. Indeed, since $\mathbf{B}$ is a compingent algebra, $a = \bigvee S^{-1}[a]$, so $\uparrow a = U(S^{-1}[a])$, and hence

$$\begin{aligned} a \mathrel{Q_{\mathbf{B}}} I &\iff a \in I \iff \uparrow a \cap I \neq \varnothing \\ &\iff U(S^{-1}[a]) \cap I \neq \varnothing \iff \iota(a) \prec I. \end{aligned}$$

We finish the section by proving that both $\mathsf{SubS5^S}$ and $\mathsf{DeV^S}$ are dually equivalent to $\mathsf{KRFrm^P}$. Let $L \in \mathsf{KRFrm^P}$. By [Bez12, Rem. 3.10], the map $f_L \colon L \to \mathcal{RI}(\mathfrak{B}L)$ given by

$$f_L(a) = \{b \in \mathfrak{B}L \mid b \prec a\}$$

is an isomorphism of frames.

Proposition 4.17. *$f \colon 1_{\mathsf{KRFrm^P}} \to \mathcal{RI} \circ \Delta \circ \mathfrak{B}$ is a natural isomorphism.*

Proof. Let $\Box \colon L \to M$ be a preframe homomorphism. Set $T = \mathfrak{B}(\Box)$. Because each f_L is an isomorphism, it is enough to show that $\mathcal{RI}(T) \circ f_L = f_M \circ \Box$. (Since Δ is the inclusion functor, we omit it from the diagram.)

$$\begin{tikzcd} L \arrow[r, "f_L"] \arrow[d, "\Box"'] & \mathcal{RI}(\mathfrak{B}L) \arrow[d, "\mathcal{RI}(T)"] \\ M \arrow[r, "f_M"] & \mathcal{RI}(\mathfrak{B}M) \end{tikzcd}$$

Let $a \in L$. We have

$$\begin{aligned} \mathcal{RI}(T)(f_L(a)) = T^{-1}[f_L(a)] &= \{b \in \mathfrak{B}M \mid \exists c \in \mathfrak{B}L : b \mathrel{T} c,\ c \prec a\} \\ &= \{b \in \mathfrak{B}M \mid \exists c \in \mathfrak{B}L : b \prec \Box c,\ c \prec a\}, \end{aligned}$$

and $f_M(\Box a) = \{b \in \mathfrak{B}M \mid b \prec \Box a\}$. An argument similar to the last paragraph of the proof of Lemma 4.1 yields

$$\{b \in \mathfrak{B}M \mid \exists c \in \mathfrak{B}L : b \prec \Box c,\ c \prec a\} = \{b \in \mathfrak{B}M \mid b \prec \Box a\},$$

completing the proof. $\square$

Theorem 4.18.

(1) *$\mathcal{RI}$ and $\Delta \circ \mathfrak{B}$ form a dual equivalence between* $\mathsf{SubS5^S}$ *and* $\mathsf{KRFrm^P}$.

(2) *$\mathcal{RI} \circ \Delta$ and $\mathfrak{B}$ form a dual equivalence between* $\mathsf{DeV^S}$ *and* $\mathsf{KRFrm^P}$.

We thus obtain the following diagram of equivalences and dual equivalences that commutes up to natural isomorphism.

$$\begin{tikzcd}
& \mathsf{SubS5^S} \arrow[dl, "\mathcal{RI}"'] \arrow[dr, shift left, "\mathcal{NI}"] & \\
\mathsf{KRFrm^P} \arrow[rr, "\mathfrak{B}"] & & \mathsf{DeV^S} \arrow[ul, shift left, "\Delta"]
\end{tikzcd}$$

Proof. (1). By definition of $\mathcal{NI}$, we have $\Delta \circ \mathfrak{B} \circ \mathcal{RI} = \Delta \circ \mathcal{NI}$. Therefore, $Q\colon 1_{\mathsf{SubS5^S}} \to \Delta \circ \mathfrak{B} \circ \mathcal{RI}$ is a natural isomorphism by Proposition 4.12. Moreover, $f\colon 1_{\mathsf{KRFrm^P}} \to \mathcal{RI} \circ \Delta \circ \mathfrak{B}$ is a natural isomorphism by Proposition 4.17. Thus, $\Delta \circ \mathfrak{B}\colon \mathsf{KRFrm^P} \to \mathsf{SubS5^S}$ is a quasi-inverse of $\mathcal{RI}$.

(2). By Proposition 4.12, $Q\colon 1_{\mathsf{SubS5^S}} \to \Delta \circ \mathfrak{B} \circ \mathcal{RI}$ is a natural isomorphism. For the same reason, we have a natural isomorphism $Q'\colon 1_{\mathsf{DeV^S}} \to \mathfrak{B} \circ \mathcal{RI} \circ \Delta$ whose component on $\mathbf{B} \in \mathsf{DeV^S}$ is $Q_{\mathbf{B}}$. Thus, $\mathfrak{B}\colon \mathsf{KRFrm^P} \to \mathsf{DeV^S}$ is a quasi-inverse of $\mathcal{RI} \circ \Delta$. $\square$

5. Continuous subordinations

In Section 4 we gave a direct choice-free proof that $\mathsf{SubS5^S}$ is equivalent to $\mathsf{DeV^S}$ and dually equivalent to $\mathsf{KRFrm^P}$. Morphisms of each of these categories encode closed relations between compact Hausdorff spaces. In this section we study the wide subcategories of these categories whose morphisms encode continuous relations between compact Hausdorff spaces.

Recalling from Remark 2.15 the equivalence $\mathcal{Q}\colon \mathsf{StoneE^R} \to \mathsf{KHaus^R}$, we first characterize when $\mathcal{Q}(R)$ is a continuous relation for an arbitrary morphism R in $\mathsf{StoneE^R}$. We then use the equivalence $\mathsf{Clop}\colon \mathsf{StoneE^R} \to \mathsf{SubS5^S}$ to encode this characterization in the language of S5-subordination algebras.

Definition 5.1. Let R be a binary relation on a set X and $U \subseteq X$. Following the standard notation in modal logic, we write $\Box_R U = X \setminus R^{-1}[X \setminus U]$. If R is an equivalence relation, we say that U is *R-saturated* if $R[U] = U$.

Remark 5.2.

1. If R is a closed relation and U is open, then $\Box_R U$ is open.
2. If R is an equivalence relation, then $\Box_R U = X \setminus R[X \setminus U]$ and is the largest R-saturated subset of U. Therefore, U is R-saturated iff $\Box_R U = U$.

Lemma 5.3. *Let $R\colon (X_1, E_1) \to (X_2, E_2)$ be a morphism in $\mathsf{StoneE^R}$. The following are equivalent.*

(1) *The relation $\mathcal{Q}(R)\colon X_1/E_1 \to X_2/E_2$ is a continuous relation.*

(2) *If V is an E_2-saturated open in X_2, then $R^{-1}[V]$ is open in X_1.*

(3) *If $B_1, B_2 \subseteq X_2$ are clopen with $E_2[B_1] \subseteq B_2$, then there is a clopen set $A \subseteq X_1$ such that $R^{-1}[B_1] \subseteq A \subseteq R^{-1}[B_2]$.*

(4) *If $B_1, B_2 \subseteq X_2$ are clopen with $E_2[B_1] \subseteq B_2$, then there is a clopen set $A \subseteq X_1$ such that $A \in \widehat{S_R}[B_1]$ and $\widehat{S_R}[B_2] \subseteq S_{E_1}[A]$.*

Proof. (1)⇔(2). Let $\pi_i\colon X_i \to X_i/E_i$ be the quotient maps for $i = 1, 2$.

$$
\begin{tikzcd}
X_1 \arrow[r, "R"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\
X_1/E_1 \arrow[r, "\mathcal{Q}(R)"'] & X_2/E_2
\end{tikzcd}
$$

Then $\mathcal{Q}(R)^{-1}[U] = \pi_1[R^{-1}[\pi_2^{-1}[U]]]$ for each $U \subseteq X_2/E_2$. The R-inverse image of any subset of X_2 is E_1-saturated by the compatibility of R. Thus, $R^{-1}[\pi_2^{-1}[U]]$ is open iff $\pi_1[R^{-1}[\pi_2^{-1}[U]]]$ is open for each U open of X_2/E_2. Therefore, $\mathcal{Q}(R)$ is continuous iff $R^{-1}[\pi_2^{-1}[U]]$ is open for each U open of X_2/E_2. Since V is an E_2-saturated open in X_2 iff $V = \pi_2^{-1}[U]$ for some U open of X_2/E_2, the equivalence follows.

(2)⇒(3). Suppose $B_1, B_2 \subseteq X_2$ are clopens with $E_2[B_1] \subseteq B_2$. Let $V = \Box_{E_2} B_2$. Then V is an E_2-saturated open. Since $E_2[B_1] \subseteq B_2$, we have that $B_1 \subseteq V$. Therefore, $R^{-1}[B_1] \subseteq R^{-1}[V]$. The set $R^{-1}[B_1]$ is closed and $R^{-1}[V]$ is open by item (2). Thus, there is a clopen set $A \subseteq X_1$ such that $R^{-1}[B_1] \subseteq A \subseteq R^{-1}[V]$. Since $V \subseteq B_2$, we have $R^{-1}[V] \subseteq R^{-1}[B_2]$. Hence, $A \subseteq R^{-1}[B_2]$. This proves item (3).

(3)⇒(2). Let V be an E_2-saturated open subset of X_2. Since $V = \bigcup\{B \in \mathsf{Clop}(X_2) \mid B \subseteq V\}$, we have

$$R^{-1}[V] = \bigcup\{R^{-1}[B] \mid B \in \mathsf{Clop}(X_2), B \subseteq V\}.$$

Thus, it is enough to prove that for every clopen subset B of X_2 contained in V, there is an open subset U_B of X_1 such that $R^{-1}[B] \subseteq U_B \subseteq R^{-1}[V]$ (because then $R^{-1}[V] = \bigcup\{U_B \mid B \in \mathsf{Clop}(X_2), B \subseteq V\}$). Let B be a clopen subset of X_2 contained in V. Since V is E_2-saturated, $E_2[B] \subseteq V$. Because $E_2[B]$ is closed and V is open, there is a clopen subset B' of X_2 such that $E_2[B] \subseteq B' \subseteq V$. By item (3), there is a clopen set $A \subseteq X_1$ such that $R^{-1}[B] \subseteq A \subseteq R^{-1}[B']$. Since $B' \subseteq V$, we have $R^{-1}[B'] \subseteq R^{-1}[V]$, so $A \subseteq R^{-1}[V]$. Therefore, we have found an open subset A of X_1 such that $R^{-1}[B] \subseteq A \subseteq R^{-1}[V]$. Hence, item (2) holds.

(3)⇔(4). This follows from the following two claims.

Claim 5.4. *For clopen sets $A \subseteq X_1$ and $B \subseteq X_2$, we have $R^{-1}[B] \subseteq A$ iff $A \in \widehat{S_R}[B]$.*

Proof of claim. This follows from the equality $\widehat{S_R} = S_{R^\smile}$, shown in the proof of [ABC23, Thm. 2.14]. ☐

Claim 5.5. *For clopen sets $A \subseteq X_1$ and $B \subseteq X_2$, we have $A \subseteq R^{-1}[B]$ iff $\widehat{S_R}[B] \subseteq S_{E_1}[A]$.*

Proof of claim. Let $A \subseteq X_1$ and $B \subseteq X_2$ be clopen sets. Then

$$\begin{aligned}
&\widehat{S_R}[B] \subseteq S_{E_1}[A] \\
&\iff \forall A' \in \mathsf{Clop}(X_1),\ B\ \widehat{S_R}\ A' \text{ implies } A\ S_{E_1}\ A' \\
&\iff \forall A' \in \mathsf{Clop}(X_1),\ R^{-1}[B] \subseteq A' \text{ implies } E_1[A] \subseteq A' \\
&\hspace{20em} \text{(by Claim 5.4)}
\end{aligned}$$

$$\begin{aligned}
&\iff E_1[A] \subseteq \bigcap\{A' \in \mathsf{Clop}(X_1) \mid R^{-1}[B] \subseteq A'\} \\
&\iff E_1[A] \subseteq R^{-1}[B] && \text{(since } R^{-1}[B] \text{ is closed)} \\
&\iff A \subseteq R^{-1}[B] && \text{(since } R^{-1}[B] \text{ is } E_1\text{-saturated).}
\end{aligned}$$

□

This concludes the proof. □

The next definition encodes Lemma 5.3(4) in the language of S5-subordination algebras. By Lemma 5.3(1), this condition is equivalent to the corresponding relation between compact Hausdorff spaces being continuous. Because of this, we call such compatible subordinations continuous.

Definition 5.6. Let $T\colon (B_1, S_1) \to (B_2, S_2)$ be a compatible subordination between S5-subordination algebras. We say that T is *continuous* if the following holds:

$$\forall b_1, b_2 \in B_2 \ \left(b_1 \ S_2 \ b_2 \Rightarrow \exists a \in \widehat{T}[b_1] : \widehat{T}[b_2] \subseteq S_1[a]\right).$$

Lemma 5.7. *Let $T\colon (B_1, S_1) \to (B_2, S_2)$ be a compatible subordination.*

(1) *The following are equivalent:*

  (a) *T is continuous.*

  (b) $\forall b_1, b_2 \in B_2 \ \left(b_1 \ S_2 \ b_2 \Rightarrow \exists a \in \widehat{T}[b_1] : a \in L(\widehat{T}[b_2])\right)$.

  (c) $\forall b_1, b_2 \in B_2 \ \left(b_1 \ S_2 \ b_2 \Rightarrow \exists a \in T^{-1}[b_2] : a \in U(T^{-1}[b_1])\right)$.

(2) *If B_1 is complete, then the following are equivalent:*

  (a) *T is continuous.*

  (b) $\forall b_1, b_2 \in B_2 \ \left(b_1 \ S_2 \ b_2 \Rightarrow b_1 \ \widehat{T} \ (\bigwedge \widehat{T}[b_2])\right)$.

  (c) $\forall b_1, b_2 \in B_2 \ \left(b_1 \ S_2 \ b_2 \Rightarrow (\bigvee T^{-1} \ [b_1]) \ T \ b_2\right)$.

Proof. (1a)⇔(1b). It is enough to prove that $\widehat{T}[b_2] \subseteq S_1[a]$ is equivalent to $a \in L(\widehat{T}[b_2])$. For the left-to-right implication, by (S5) we have $S_1[a] \subseteq U(a)$, and so $\widehat{T}[b_2] \subseteq S_1[a]$ implies $\widehat{T}[b_2] \subseteq U(a)$, which is equivalent to $a \in L(\widehat{T}[b_2])$. For the right-to-left implication, suppose $a \in L(\widehat{T}[b_2])$ and let $a' \in \widehat{T}[b_2]$. Since $\widehat{T}$ is a compatible subordination, there is $a'' \in \widehat{T}[b_2]$ such that $a'' \ S_1 \ a'$. Therefore, $a \leq a'' \ S_1 \ a'$, which implies $a \ S_1 \ a'$, and hence $a' \in S_1[a]$.

(1b)⇔(1c). Suppose that (1b) holds, and let $b_1, b_2 \in B_2$ be such that $b_1 \ S_2 \ b_2$. Then, by (S6), $\neg b_2 \ S_2 \ \neg b_1$. Therefore, by (1b) there is $a \in \widehat{T}[\neg b_2]$ such that $a \in L(\widehat{T}[\neg b_1])$. The condition $a \in \widehat{T}[\neg b_2]$ is equivalent to $\neg a \in T^{-1}[b_2]$. Similarly, the condition $a \in L(\widehat{T}[\neg b_1])$ is equivalent to $\neg a \in U(T^{-1}[b_1])$. Thus, (1b) implies (1c), and the converse is proved similarly.

(2). If B is complete, then (1b)⇔(2b) and (1c)⇔(2c). Thus, the result follows from item (1). □

Lemma 5.8.

(1) *Let (B, S) be an* S5*-subordination algebra. The identity morphism $S\colon (B, S) \to (B, S)$ in* $\mathsf{SubS5^S}$ *is continuous.*

(2) *Let $T_1\colon (B_1, S_1) \to (B_2, S_2)$ and $T_2\colon (B_2, S_2) \to (B_3, S_3)$ be continuous compatible subordinations between* S5*-subordination algebras. Then $T_2 \circ T_1\colon (B_1, S_1) \to (B_3, S_3)$ is a continuous compatible subordination.*

Proof. (1). Since $\widehat{S} = S$, this is immediate from (S7).

(2). It is sufficient to show that $T_2 \circ T_1$ is continuous. Let $c_1, c_2 \in B_3$ be such that $c_1 \ S_3 \ c_2$. By (S7), there is $c \in B_3$ such that $c_1 \ S_3 \ c \ S_3 \ c_2$. Therefore, since T_2 is continuous, there are $b_1 \in \widehat{T}_2[c_1]$ and $b_2 \in \widehat{T}_2[c]$ such that $\widehat{T}_2[c] \subseteq S_2[b_1]$ and $\widehat{T}_2[c_2] \subseteq S_2[b_2]$. We have $b_2 \in \widehat{T}_2[c] \subseteq S_2[b_1]$, and so $b_1 \ S_2 \ b_2$. Thus, since T_1 is continuous, there is $a \in \widehat{T}_1[b_1]$ such that $\widehat{T}_1[b_2] \subseteq S_1[a]$. We have $c_1 \ \widehat{T}_2 \ b_1 \ \widehat{T}_1 \ a$, and hence $a \in (\widehat{T}_1 \circ \widehat{T}_2)[c_1]$. Since $\widehat{T}_1 \circ \widehat{T}_2 = \widehat{T_2 \circ T_1}$, it remains to show that $(\widehat{T}_1 \circ \widehat{T}_2)[c_2] \subseteq S_1[a]$. Let $a' \in (\widehat{T}_1 \circ \widehat{T}_2)[c_2]$. Then there is $b \in B_2$ such that $c_2 \ \widehat{T}_2 \ b \ \widehat{T}_1 \ a'$. We have $b \in \widehat{T}_2[c_2] \subseteq S_2[b_2]$, and thus $b_2 \ S_2 \ b$. From $b_2 \ S_2 \ b \ \widehat{T}_1 \ a'$ we deduce, using the compatibility of $\widehat{T}_1$, that $b_2 \ \widehat{T}_1 \ a'$. Therefore, $a' \in \widehat{T}_1[b_2] \subseteq S_1[a]$, and hence $a' \in S_1[a]$, as desired. □

Definition 5.9. Let $\mathsf{SubS5^{CS}}$ be the wide subcategory of $\mathsf{SubS5^{S}}$ whose morphisms are continuous compatible subordinations, and define $\mathsf{DeV^{CS}}$ similarly.

We next show that Theorem 4.18 restricts to yield the corresponding dual equivalences for $\mathsf{SubS5^{CS}}$ and $\mathsf{DeV^{CS}}$. For this we need the following lemma.

Lemma 5.10. *Let $(B_1, S_1), (B_2, S_2) \in \mathsf{SubS5^{S}}$ and $T \colon B_1 \to B_2$ be a morphism in $\mathsf{SubS5^{S}}$. Let also L_1, L_2 be compact regular frames and $\Box \colon L_1 \to L_2$ a preframe homomorphism.*

(1) *If $T \colon B_1 \to B_2$ is a continuous compatible subordination, then the map $\mathcal{RI}(T) \colon \mathcal{RI}(B_2, S_2) \to \mathcal{RI}(B_1, S_1)$ is a c-morphism.*

(2) *If $\Box \colon L_1 \to L_2$ is a c-morphism, then $\mathfrak{B}(\Box) \colon \mathfrak{B}(L_2) \to \mathfrak{B}(L_1)$ is continuous.*

(3) *If $T \colon B_1 \to B_2$ is an isomorphism in $\mathsf{SubS5^{S}}$, then T is an isomorphism in $\mathsf{SubS5^{CS}}$.*

(4) *If $\Box \colon L_1 \to L_2$ is an isomorphism in $\mathsf{KRFrm^{P}}$, then $\Box$ is an isomorphism in $\mathsf{KRFrm^{C}}$.*

Proof. (1). Let $\Box = \mathcal{RI}(T)$. Then $\Box$ is a preframe homomorphism by Theorem 3.5. We define $\Diamond \colon \mathcal{RI}(B_2, S_2) \to \mathcal{RI}(B_1, S_1)$ by

$$\Diamond I = \{a \in B_1 \mid \exists b \in I : a \in L(\widehat{T}[b])\}.$$

We first show that $\Diamond$ is well defined. It is straightforward to see that $\Diamond I$ is an ideal of B_1. To see that $\Diamond I$ is a round ideal, let $a \in \Diamond I$. Then there is $b \in I$ with $a \in L(\widehat{T}[b])$. Since I is a round ideal, there is $d \in I$ with $b \; S_2 \; d$. Because T is continuous, there is $c \in \widehat{T}[b]$ such that $c \in L(\widehat{T}[d])$ (see Lemma 5.7(1b)). Therefore, $c \in \Diamond I$ since $d \in I$. Because $\widehat{T}$ is compatible, from $b \; \widehat{T} \; c$ it follows that there is $c' \in \widehat{T}[b]$ with $c' \; S_1 \; c$. But then $a \leq c'$ since $a \in L(\widehat{T}[b])$. Thus, $a \leq c' \; S_1 \; c$, so $a \; S_1 \; c$, and hence $\Diamond I$ is a round ideal.

We next show that $\Diamond$ preserves arbitrary joins. It is straightforward to see that $I \subseteq J$ implies $\Diamond I \subseteq \Diamond J$. Therefore, if $\{I_\alpha\} \subseteq \mathcal{RI}(B_2, S_2)$, then $\bigvee \Diamond I_\alpha \subseteq \Diamond \left(\bigvee I_\alpha\right)$. For the reverse inclusion, let $x \in \Diamond \left(\bigvee I_\alpha\right)$. Then there is $b \in \bigvee I_\alpha$ with $x \in L(\widehat{T}[b])$. Since $b \in \bigvee I_\alpha$, there exist $\alpha_1, \dots, \alpha_n$ and $d_i \in I_{\alpha_i}$ for $i = 1, \dots, n$ such that $b \leq d_1 \vee \cdots \vee d_n$. Thus, $x \in L(\widehat{T}[d_1 \vee \cdots \vee d_n])$. Because I_{α_i} is a round ideal for each i, it follows that there exist $e_i \in I_{\alpha_i}$ with $d_i \; S_2 \; e_i$ for each i. By continuity of T, there exist $a_i \in \widehat{T}[d_i]$ with $a_i \in L(\widehat{T}[e_i])$ for each i. So $a_i \in \Diamond I_{\alpha_i}$ for each i and $a_1 \vee \cdots \vee a_n \in \widehat{T}[d_1 \vee \cdots \vee d_n]$. Since $x \in L(\widehat{T}[d_1 \vee \cdots \vee d_n])$, it follows that $x \leq a_1 \vee \cdots \vee a_n$. Consequently, $x \in \bigvee \Diamond I_\alpha$.

It is left to prove that $\Box I \cap \Diamond J \subseteq \Diamond(I \cap J)$ and $\Box(I \vee J) \subseteq \Box I \vee \Diamond J$ for all $I, J \in \mathcal{RI}(B_2, S_2)$. Let $x \in \Box I \cap \Diamond J$. Since $x \in \Box I = T^{-1}[I]$, there is $a \in I$ with $x \; T \; a$. Because $x \in \Diamond J$, there is $b \in J$ with $x \in L(\widehat{T}[b])$. We first show that $x \in L(\widehat{T}[a \wedge b])$. If $e \in \widehat{T}[a \wedge b]$, then $\neg e \; T \; (\neg a \vee \neg b)$. Since $x \; T \; a$, it follows that $(x \wedge \neg e) \; T \; (a \wedge (\neg a \vee \neg b))$. So $(x \wedge \neg e) \; T \; (a \wedge \neg b)$, and hence $(x \wedge \neg e) \; T \; \neg b$. Therefore, $\neg x \vee e \in \widehat{T}[b]$. Because $x \in L(\widehat{T}[b])$, we have $x \leq \neg x \vee e$, and so $x \leq e$. Thus, $x \in L(\widehat{T}[a \wedge b])$. Since $a \wedge b \in I \cap J$, we conclude that $x \in \Diamond(I \cap J)$.

Finally, let $x \in \Box(I \vee J) = T^{-1}[I \vee J]$. Then there is $y \in I \vee J$ with $x \; T \; y$. Thus, there exist $a \in I$, $b \in J$ with $y \leq a \vee b$. Since I and J are round ideals, there exist $a' \in I$, $b' \in J$ with $a \; S_2 \; a'$ and $b \; S_2 \; b'$. Because $\neg a' \; S_2 \; \neg a$ and $b \; S_2 \; b'$, the continuity of T yields that there exist $c \in \widehat{T}[\neg a']$ and $d \in \widehat{T}[b]$ with $c \in L(\widehat{T}[\neg a])$ and $d \in L(\widehat{T}[b'])$. From $c \in \widehat{T}[\neg a']$ it follows that $\neg c \; T \; a'$, so $\neg c \in T^{-1}[I] = \Box I$. Since $d \in L(\widehat{T}[b'])$ and $b' \in J$, we have $d \in \Diamond J$. Therefore, $\neg c \vee d \in \Box I \vee \Diamond J$. We prove that $x \leq \neg c \vee d$, which is equivalent to $c \leq \neg x \vee d$. We have $x \; T \; (a \vee b)$ and $\neg d \; T \; \neg b$ because $d \in \widehat{T}[b]$. Therefore, $(x \wedge \neg d) \; T \; ((a \vee b) \wedge \neg b)$, and so $(x \wedge \neg d) \; T \; (a \wedge \neg b) \leq a$. Thus, $\neg x \vee d \in \widehat{T}[\neg a]$. Since $c \in L(\widehat{T}[\neg a])$, we obtain $c \leq \neg x \vee d$. Consequently, $x \in \Box I \vee \Diamond J$ because $x \leq \neg c \vee d \in \Box I \vee \Diamond J$.

(2). Let $T = \mathfrak{B}(\Box)$. By Lemma 4.1, $T \colon \mathfrak{B}(L_2) \to \mathfrak{B}(L_1)$ is a morphism in $\mathsf{SubS5^S}$. To see that it is continuous, let $b_1, b_2 \in \mathfrak{B}(L_1)$ with $b_1 \prec b_2$. Set $a = \neg \Box \neg b_2$. Then $a \in \mathfrak{B}(L_2)$. We show that $b_1 \; \widehat{T} \; a$ and $a \in L(\widehat{T}[b_2])$. We have $\neg b_2 \prec \neg b_1$, so $\Box \neg b_2 \prec \Box \neg b_1$ since $\Box$ preserves $\prec$ (see [BBH15, Lem. 3.6]). The definition of $\prec$ implies $\neg\neg\Box\neg b_2 \prec \Box \neg b_1$. Therefore, $\neg a \prec \Box \neg b_1$, which gives $\neg a \; T \; \neg b_1$. Thus, $b_1 \; \widehat{T} \; a$. If $x \in \widehat{T}[b_2]$, then $\neg x \; T \; \neg b_2$, so $\neg x \prec \Box \neg b_2$. Therefore, $a = \neg \Box \neg b_2 \prec x$, and hence $a \leq x$. Thus, $a \in L(\widehat{T}[b_2])$, and so T is continuous.

(3). This is a consequence of a stronger result proved in Lemma 6.5(3) below.

(4). Since $\Box$ is an isomorphism in $\mathsf{KRFrm}^{\mathsf{P}}$, it is a poset isomorphism. Defining $\Diamond := \Box$ then yields that $\Box$ is an isomorphism in $\mathsf{KRFrm}^{\mathsf{C}}$. $\Box$

As an immediate consequence of Theorem 4.18 and Lemma 5.10 we obtain:

Theorem 5.11.

(1) *The dual equivalence between* $\mathsf{SubS5}^{\mathsf{S}}$ *and* $\mathsf{KRFrm}^{\mathsf{P}}$ *restricts to a dual equivalence between their wide subcategories* $\mathsf{SubS5}^{\mathsf{CS}}$ *and* $\mathsf{KRFrm}^{\mathsf{C}}$.

(2) *The dual equivalence between* $\mathsf{DeV}^{\mathsf{S}}$ *and* $\mathsf{KRFrm}^{\mathsf{P}}$ *restricts to a dual equivalence between their wide subcategories* $\mathsf{DeV}^{\mathsf{CS}}$ *and* $\mathsf{KRFrm}^{\mathsf{C}}$.

We conclude this section by showing that $\mathsf{DeV}^{\mathsf{CS}}$ is dually isomorphic to $\mathsf{DeV}^{\mathsf{C}}$. Let (B_1, S_1) and (B_2, S_2) be de Vries algebras. If $T\colon B_1 \to B_2$ is a morphism in $\mathsf{DeV}^{\mathsf{CS}}$, we define $\Box_T\colon B_2 \to B_1$ by $\Box_T b = \bigvee T^{-1}[b]$. Also, if $\Box\colon B_2 \to B_1$ is a morphism in $\mathsf{DeV}^{\mathsf{C}}$, we define $T_\Box\colon B_1 \to B_2$ by

$$a \; T_\Box \; b \iff \exists b' \in B_2 \; (a \; S_1 \; \Box b' \text{ and } b' \; S_2 \; b).$$

Lemma 5.12. *Let* (B_1, S_1) *and* (B_2, S_2) *be de Vries algebras.*

(1) *If* $T\colon B_1 \to B_2$ *is a morphism in* $\mathsf{DeV}^{\mathsf{CS}}$*, then* $\Box_T\colon B_2 \to B_1$ *is a morphism in* $\mathsf{DeV}^{\mathsf{C}}$.

(2) *If* $\Box\colon B_2 \to B_1$ *is a morphism in* $\mathsf{DeV}^{\mathsf{C}}$*, then* $T_\Box\colon B_1 \to B_2$ *is a morphism in* $\mathsf{DeV}^{\mathsf{CS}}$.

(3) $\Box_{T_\Box} = \Box$.

(4) $T_{\Box_T} = T$.

Proof. (1). We first show that $\Box_T$ is de Vries multiplicative. It is obvious that $\Box_T 1 = 1$. Let $b_1 \; S_2 \; b_2$ and $d_1 \; S_2 \; d_2$. Since T is continuous and B_1 is complete, by Lemma 5.7(2c)

$$\left(\bigvee T^{-1}\left[b_1\right]\right) T \; b_2 \quad \text{and} \quad \left(\bigvee T^{-1}\left[d_1\right]\right) T \; d_2.$$

Therefore, $(\Box_T b_1 \wedge \Box_T d_1)\ T\ (b_2 \wedge d_2)$. Since T is compatible, there is $x \in B_1$ such that $(\Box_T b_1 \wedge \Box_T d_1)\ S_1\ x\ T\ (b_2 \wedge d_2)$. Thus,

$$(\Box_T b_1 \wedge \Box_T d_1)\ S_1\ x \leq \Box_T(b_2 \wedge d_2),$$

and hence $(\Box_T b_1 \wedge \Box_T d_1)\ S_1\ \Box_T(b_2 \wedge d_2)$. Consequently, $\Box_T$ is de Vries multiplicative. To see that $\Box_T$ is lower continuous, let $x \in T^{-1}[b]$. Since T is compatible, $x\ T\ y\ S_2\ b$ for some $y \in B_2$. Therefore, $x \leq \Box_T y$, and hence $\Box_T b = \bigvee\{\Box_T y \mid y\ S_2\ b\}$. Thus, $\Box_T$ is a morphism in $\mathsf{DeV}^{\mathsf{C}}$.

(2). That $0\ T_\Box\ 0$ is straightforward and that $1\ T_\Box\ 1$ follows from $\Box 1 = 1$. Since $\Box$ is lower continuous, it is order preserving (see [BBH15, Prop. 4.15(2)] and Remark 2.10(2)). Suppose $a, a'\ T_\Box\ b$. Then there exist b_1 and b_2 such that $a\ S_1\ \Box b_1$, $b_1\ S_2\ b$, $a'\ S_1\ \Box b_2$, and $b_2\ S_2\ b$. From $a\ S_1\ \Box b_1$ and $a'\ S_1\ \Box b_2$ it follows that $(a \vee a')\ S_1\ (\Box b_1 \vee \Box b_2) \leq \Box(b_1 \vee b_2)$, and so $(a \vee a')\ S_1\ \Box(b_1 \vee b_2)$. Also, from $b_1\ S_2\ b$ and $b_2\ S_2\ b$ it follows that $(b_1 \vee b_2)\ S_2\ b$. Thus, $(a \vee a')\ T_\Box\ b$. Next suppose $a\ T_\Box\ b, b'$. Then there exist b_1 and b_2 such that $a\ S_1\ \Box b_1$, $b_1\ S_2\ b$, $a\ S_1\ \Box b_2$, and $b_2\ S_2\ b'$. From $a\ S_1\ \Box b_1$ and $a\ S_1\ \Box b_2$ it follows that $a\ S_1\ (\Box b_1 \wedge \Box b_2) = \Box(b_1 \wedge b_2)$ (see [BBH15, Prop. 4.15(2)] and Remark 2.10(2)). Also, from $b_1\ S_2\ b$ and $b_2\ S_2\ b'$ it follows that $(b_1 \wedge b_2)\ S_2\ (b \wedge b')$. Thus, $a\ T_\Box\ (b \wedge b')$. Finally, that $a \leq a'\ T_\Box\ b' \leq b$ implies $a\ T_\Box\ b$ is straightforward. This gives that $T_\Box$ is a subordination.

That $T_\Box \subseteq S_2 \circ T_\Box$ and $T_\Box \subseteq T_\Box \circ S_1$ follow from the fact that S_2 and S_1 satisfy (S7). The reverse inclusions are obvious, so $S_2 \circ T_\Box = T_\Box = T_\Box \circ S_1$. This yields that $T_\Box$ is a compatible subordination.

It is left to prove that $T_\Box$ is continuous. Let $b_1\ S_2\ b_2$. Then there is $y \in B_2$ with $b_1\ S_2\ y\ S_2\ b_2$. Set $a = \Box b_1$. Since $a\ S_1\ \Box y$ and $y\ S_2\ b_2$, we have $a\ T_\Box\ b_2$, so $a \in T_\Box^{-1}[b_2]$. Moreover, if $x\ T_\Box\ b_1$, then there is $z \in B_2$ such that $x\ S_1\ \Box z$ and $z\ S_2\ b_1$. Therefore, $x\ S_1\ \Box b_1$, and so $x\ S_1\ a$. Thus, $a \in U(T_\Box^{-1}[b_1])$ by (S5), and hence $T_\Box$ is continuous by Lemma 5.7(1c). Consequently, $T_\Box$ is a morphism in $\mathsf{DeV}^{\mathsf{CS}}$.

(3). We have

$$\begin{aligned}\Box_{T_\Box} b &= \bigvee T_\Box^{-1}[b] = \bigvee\{a \mid \exists b' \in B_2\ (a\ S_1\ \Box b' \text{ and } b'\ S_2\ b)\}\\ &= \bigvee\{\Box b' \mid b'\ S_2\ b\} = \Box b,\end{aligned}$$

where the second to last equality follows from the facts that S_2 satisfies (S7) and $b' \ S_2 \ b$ implies $\Box b' \ S_1 \ \Box b$, and the last equality from the lower continuity of $\Box$.

(4). We have

$$\begin{aligned} a \ T_{\Box_T} \ b &\iff \exists b' \in B_2 \ \ (a \ S_1 \ \Box_T b' \text{ and } b' \ S_2 \ b) \\ &\iff \exists b' \in B_2 \ \left(a \ S_1 \bigvee T^{-1}[b'] \text{ and } b' \ S_2 \ b\right). \end{aligned}$$

We show that the last condition is equivalent to $a \ T \ b$. Since T is a morphism in $\mathsf{DeV}^{\mathsf{CS}}$ and $b' \ S_2 \ b$, we have $(\bigvee T^{-1}[b']) \ T \ b$ by Lemma 5.7(2c). Therefore, $a \ S_1 \ (\bigvee T^{-1}[b']) \ T \ b$, and so $a \ T \ b$. Conversely, if $a \ T \ b$, there are $a' \in B_1$ and $b' \in B_2$ such that $a \ S_1 \ a' \ T \ b' \ S_2 \ b$. Thus, $a' \leq \bigvee T^{-1}[b']$, and hence $a \ S_1 \bigvee T^{-1}[b']$. $\Box$

As an immediate consequence of Lemma 5.12 we obtain:

Theorem 5.13. $\mathsf{DeV}^{\mathsf{CS}}$ *is dually isomorphic to* $\mathsf{DeV}^{\mathsf{C}}$.

Putting Theorems 5.11 and 5.13 together yields the following analogue of the commutative diagram of equivalences and dual equivalences given at the end of Section 4.

$$\begin{tikzcd}
& \mathsf{SubS5}^{\mathsf{CS}} \arrow[dl, "\mathcal{RI}"'] \arrow[dr, shift left, "\mathcal{NI}"] & & \\
\mathsf{KRFrm}^{\mathsf{C}} \arrow[rr, "\mathfrak{B}"] & & \mathsf{DeV}^{\mathsf{CS}} \arrow[ul, shift left, hook, "\Delta"] \arrow[r, leftrightarrow, "d"] & \mathsf{DeV}^{\mathsf{C}}
\end{tikzcd}$$

Remark 5.14. As we pointed out in Section 2, $\mathsf{KRFrm}^{\mathsf{C}}$ and $\mathsf{DeV}^{\mathsf{C}}$ are dually equivalent to $\mathsf{KHaus}^{\mathsf{C}}$. Hence, $\mathsf{SubS5}^{\mathsf{CS}}$ and $\mathsf{DeV}^{\mathsf{CS}}$ are equivalent to $\mathsf{KHaus}^{\mathsf{C}}$. The wide subcategories of $\mathsf{StoneE}^{\mathsf{R}}$ and $\mathsf{Gle}^{\mathsf{R}}$ that are equivalent to $\mathsf{KHaus}^{\mathsf{C}}$ can be described as follows.

Let (X, E) be an S5-subordination space. A morphism $R \colon X_1 \to X_2$ in $\mathsf{StoneE}^{\mathsf{R}}$ is *continuous* if $R^{-1}[U]$ is open for each E_2-saturated open $U \subseteq X_2$. Let $\mathsf{StoneE}^{\mathsf{C}}$ be the wide subcategory of $\mathsf{StoneE}^{\mathsf{R}}$ whose morphisms are continuous morphisms in $\mathsf{StoneE}^{\mathsf{R}}$ and define $\mathsf{Gle}^{\mathsf{C}}$ similarly. Using Lemma 5.3 it is straightforward to see that the equivalence between $\mathsf{StoneE}^{\mathsf{R}}$ and $\mathsf{Gle}^{\mathsf{R}}$ described in Remark 2.15(4) restricts to an equivalence between

$\mathsf{StoneE^C}$ and $\mathsf{Gle^C}$. By [BGHJ19, Thm. 4.16], $\mathsf{Gle^C}$ is equivalent to $\mathsf{KHaus^C}$. Thus, each of $\mathsf{KHaus^C}$, $\mathsf{StoneE^C}$, and $\mathsf{Gle^C}$ is equivalent or dually equivalent to each of the categories in the diagram above.

6. Functional subordinations

In this section we further restrict our attention to those wide subcategories of $\mathsf{SubS5^S}$ and $\mathsf{KRFrm^P}$ that encode continuous functions between compact Hausdorff spaces. The wide subcategories of $\mathsf{SubS5^S}$ and $\mathsf{StoneE^R}$ equivalent to KHaus were described in [ABC23, Sec. 6], where it was shown that they are equivalent to the categories of maps in the allegories $\mathsf{SubS5^S}$ and $\mathsf{StoneE^R}$. This has resulted in the following notion:

Definition 6.1. [ABC23, Def. 6.4]

1. Call a morphism $T\colon (B_1, S_1) \to (B_2, S_2)$ in $\mathsf{SubS5^S}$ *functional* if
$$\widehat{T} \circ T \subseteq S_1 \quad \text{and} \quad S_2 \subseteq T \circ \widehat{T}.$$
2. Let $\mathsf{SubS5^F}$ be the wide subcategory of $\mathsf{SubS5^S}$ whose morphisms are functional morphisms, and define $\mathsf{DeV^F}$ similarly.

Remark 6.2. If T is functional, then T is continuous. Indeed, let $b_1 \ S_2 \ b_2$. Since T is functional, $S_2 \subseteq T \circ \widehat{T}$, so there exists $a \in B_1$ such that $b_1 \ \widehat{T} \ a$ and $a \ T \ b_2$. Thus, $a \in \widehat{T}[b_1]$. Moreover, if $a' \in \widehat{T}[b_2]$, then $b_2 \ \widehat{T} \ a'$. Therefore, $a \ T \ b_2 \ \widehat{T} \ a'$, so $a \ S_1 \ a'$ because $\widehat{T} \circ T \subseteq S_1$ by the functionality of T. Consequently, T is continuous. Thus, $\mathsf{SubS5^F}$ is a wide subcategory of $\mathsf{SubS5^{CS}}$. Similarly, $\mathsf{DeV^F}$ is a wide subcategory of $\mathsf{DeV^{CS}}$.

We now give a characterization of functional morphisms. For another characterization see [ABC23, Lem. 6.5].

Lemma 6.3. *Let $T\colon (B_1, S_1) \to (B_2, S_2)$ be a morphism in $\mathsf{SubS5^S}$. The following conditions are equivalent.*

(1) *T is functional.*

(2) *The following hold for all $a \in B_1$ and $b_1, b_2, b_1', b_2' \in B_2$:*

(a) *If* $a\ T\ 0$*, then* $a = 0$.

(b) *If* $a\ T\ (b_1 \vee b_2)$*,* $b_1\ S_2\ b'_1$*, and* $b_2\ S_2\ b'_2$*, then there are* $a_1, a_2 \in B_1$ *such that* $a\ S_1\ (a_1 \vee a_2)$*,* $a_1\ T\ b'_1$*, and* $a_2\ T\ b'_2$.

Proof. By [ABC23, Lem. 6.5(1)], $\widehat{T} \circ T \subseteq S_1$ is equivalent to (2a). Therefore, it is sufficient to prove that, under these equivalent conditions, $S_2 \subseteq T \circ \widehat{T}$ is equivalent to (2b).

To prove that $S_2 \subseteq T \circ \widehat{T}$ implies (2b), let $a\ T\ (b_1 \vee b_2)$, $b_1\ S_2\ b'_1$, and $b_2\ S_2\ b'_2$. Since $S_2 \subseteq T \circ \widehat{T}$, from $b_1\ S_2\ b'_1$ and $b_2\ S_2\ b'_2$ it follows that there are $a_1, a_2 \in B_1$ such that $b_1\ \widehat{T}\ a_1\ T\ b'_1$ and $b_2\ \widehat{T}\ a_2\ T\ b'_2$. Therefore, $a\ T\ (b_1 \vee b_2)\ \widehat{T}\ (a_1 \vee a_2)$. Since $\widehat{T} \circ T \subseteq S_1$, it follows that $a\ S_1\ (a_1 \vee a_2)$.

To prove that (2b) implies $S_2 \subseteq T \circ \widehat{T}$, let $b_1, b_2 \in B_2$ be such that $b_1\ S_2\ b_2$. By (S7), there is $b \in B_2$ such that $b_1\ S_2\ b\ S_2\ b_2$. We have $1\ T\ (\neg b \vee b)$. By (S6), $b_1\ S_2\ b$ implies $\neg b\ S_2\ \neg b_1$. Thus, by (2b), there are $a_1, a_2 \in B_1$ such that $1\ S_1\ (a_1 \vee a_2)$, $a_1\ T\ \neg b_1$, and $a_2\ T\ b_2$. By (S5), from $1\ S_1\ (a_1 \vee a_2)$ it follows that $1 = a_1 \vee a_2$, so $\neg a_1 \leq a_2$. Since $a_1\ T\ \neg b_1$, we have $b_1\ \widehat{T}\ \neg a_1 \leq a_2$, and hence $b_1\ \widehat{T}\ a_2$. Because $b_1\ \widehat{T}\ a_2\ T\ b_2$, it follows that $b_1\ (T \circ \widehat{T})\ b_2$. Thus, $S_2 \subseteq T \circ \widehat{T}$, completing the proof. $\square$

Our main goal in this section is to show that Theorem 4.18 restricts to yield the corresponding dual equivalences for $\mathsf{SubS5}^{\mathsf{F}}$ and $\mathsf{DeV}^{\mathsf{F}}$. For this we need Lemma 6.5, which requires the following:

Remark 6.4. Let $T\colon (B_1, S_1) \to (B_2, S_2)$ be a morphism in $\mathsf{SubS5}^{\mathsf{S}}$. Since functional morphisms are maps in the allegory $\mathsf{SubS5}^{\mathsf{S}}$ [ABC23, Def. 6.4], it follows from [FS90, p. 199] that T is an isomorphism iff T and $\widehat{T}$ are both functional, in which case $\widehat{T}$ is the inverse of T.

Lemma 6.5. *Let* $(B_1, S_1), (B_2, S_2) \in \mathsf{SubS5}^{\mathsf{S}}$ *and* $T\colon B_1 \to B_2$ *be a morphism in* $\mathsf{SubS5}^{\mathsf{S}}$*. Let also* L_1, L_2 *be compact regular frames and* $\Box\colon L_1 \to L_2$ *a preframe homomorphism.*

(1) *If* $T\colon B_1 \to B_2$ *is functional, then* $\mathcal{RI}(T)\colon \mathcal{RI}(B_2) \to \mathcal{RI}(B_1)$ *is a frame homomorphism.*

(2) *If* $\Box\colon L_1 \to L_2$ *is a frame homomorphism, then* $\mathfrak{B}(\Box)\colon \mathfrak{B}L_2 \to \mathfrak{B}L_1$ *is functional.*

(3) *If $T\colon B_1 \to B_2$ is an isomorphism in* $\mathsf{SubS5^S}$, *then T is an isomorphism in* $\mathsf{SubS5^F}$.

(4) *If $\Box\colon L_1 \to L_2$ is an isomorphism in* $\mathsf{KRFrm^P}$, *then $\Box$ is an isomorphism in* KRFrm.

Proof. (1). Since $\mathcal{RI}(T)$ is a preframe homomorphism (see Theorem 3.5), it is sufficient to prove that it preserves bottom and binary joins. To see that $\mathcal{RI}(T)$ preserves bottom, it is enough to show that $T^{-1}[\{0\}] \subseteq \{0\}$, which follows from Lemma 6.3(2a). To see that $\mathcal{RI}(T)$ preserves binary joins, let I_1, I_2 be round ideals of B_2. It is sufficient to prove that $T^{-1}[I_1 \vee I_2] \subseteq T^{-1}[I_1] \vee T^{-1}[I_2]$. Let $a \in T^{-1}[I_1 \vee I_2]$. Then there are $b_1 \in I_1$, $b_2 \in I_2$ such that $a\ T\ (b_1 \vee b_2)$. Since I_1 and I_2 are round ideals, there are $b_1' \in I_1$ and $b_2' \in I_2$ such that $b_1\ S_2\ b_1'$ and $b_2\ S_2\ b_2'$. By Lemma 6.3(2b), there are $a_1, a_2 \in B_1$ such that $a\ S_1\ (a_1 \vee a_2)$, $a_1\ T\ b_1'$, and $a_2\ T\ b_2'$. Thus, $a \in T^{-1}[I_1] \vee T^{-1}[I_2]$.

(2). We prove that $\mathfrak{B}(\Box)$ satisfies Lemma 6.3(2). To see (2a), let $b \in \mathfrak{B}L_2$ be such that $b\ \mathfrak{B}(\Box)\ 0$, so $b \prec \Box 0$. Since $\Box$ is a frame homomorphism, $\Box 0 = 0$. Therefore, $b \prec 0$, and hence $b = 0$ by (S5). To see (2b), let $b \in \mathfrak{B}L_2$ and $a_1, a_2, a_1', a_2' \in \mathfrak{B}L_1$ be such that $b\ \mathfrak{B}(\Box)\ (a_1 \vee a_2)$, $a_1 \prec a_1'$, and $a_2 \prec a_2'$. Then $b \prec \Box(a_1 \vee a_2)$. But $\Box(a_1 \vee a_2) = \Box a_1 \vee \Box a_2$ because $\Box$ is a frame homomorphism. Therefore, $b \prec \Box a_1 \vee \Box a_2$, and so there is $b' \in \mathfrak{B}(\Box)$ such that $b \prec b' \prec \Box a_1 \vee \Box a_2$. Set $b_1 = b' \wedge \Box a_1$ and $b_2 = b' \wedge \Box a_2$. We have $a_i \prec a_i'$ implies $\Box a_i \prec \Box a_i'$ for $i \in \{1, 2\}$. Thus,

$$b_i = b' \wedge \Box a_i \leq \Box a_i \prec \Box a_i',$$

so $b_i \prec \Box a_i'$, and hence $b_i\ \mathfrak{B}(\Box)\ a_i'$. Moreover, $b \prec b'$ and $b \prec \Box a_1 \vee \Box a_2$ imply that

$$b \prec b' \wedge (\Box a_1 \vee \Box a_2) = (b' \wedge \Box a_1) \vee (b' \wedge \Box a_2) = b_1 \vee b_2.$$

This proves (2b).

(3). This follows from Remark 6.4.

(4). In both $\mathsf{KRFrm^P}$ and KRFrm isomorphisms are order-isomorphisms. $\square$

From Theorem 4.18 and Lemma 6.5 we obtain:

Theorem 6.6.

(1) *The dual equivalence between* $\mathsf{SubS5^S}$ *and* $\mathsf{KRFrm^P}$ *restricts to a dual equivalence between their wide subcategories* $\mathsf{SubS5^F}$ *and* KRFrm.

(2) *The dual equivalence between* $\mathsf{DeV^S}$ *and* $\mathsf{KRFrm^P}$ *restricts to a dual equivalence between their wide subcategories* $\mathsf{DeV^F}$ *and* KRFrm.

In addition, we have:

Theorem 6.7 ([ABC23, Thm. 6.18])**.** DeV *and* $\mathsf{DeV^F}$ *are dually isomorphic.*

Consequently, we arrive at the following analogue of the commutative diagram of equivalences and dual equivalences given at the end of Section 5.

$$\begin{tikzcd}
& \mathsf{SubS5^F} \arrow[dl, "\mathcal{RI}"'] \arrow[dr, shift left, "\mathcal{NI}"] & & \\
\mathsf{KRFrm} \arrow[rr, "\mathfrak{B}"] & & \mathsf{DeV^F} \arrow[ul, shift left, "\Delta"] \arrow[r, leftrightarrow, "d"] & \mathsf{DeV}
\end{tikzcd}$$

Remark 6.8. We recall from [ABC23, Def. 6.1] that $\mathsf{StoneE^F}$ is the wide subcategory of $\mathsf{StoneE^R}$ whose morphisms $R\colon (X_1, E_1) \to (X_2, E_2)$ satisfy $E_1 \subseteq R^{\smile} \circ R$ and $R \circ R^{\smile} \subseteq E_2$. We call such morphisms *functional* and define Gle similarly. By [ABC23, Thm. 6.9], the categories $\mathsf{SubS5^F}$, $\mathsf{DeV^F}$, $\mathsf{StoneE^F}$, Gle, and KHaus are equivalent. Thus, each of these is equivalent or dually equivalent to the categories in the above diagram.

We thus arrive at the following diagram, in which empty boxes of the diagram in Fig. 1 are filled. The number under each double arrow indicates the corresponding statement in the body of the paper.

For the reader's convenience we also list all the categories involved in the diagram.

$$
\begin{tikzcd}
\mathsf{SubS5^S} \arrow[r, leftrightarrow, "4.13"'] & \mathsf{DeV^S} \arrow[rr, leftrightarrow, "d", "4.18"'] & & \mathsf{KRFrm^P} \arrow[r, leftrightarrow, "d", "2.2"'] & \mathsf{KHaus^R} \arrow[r, leftrightarrow, "2.14"'] & \mathsf{StoneE^R} \arrow[r, leftrightarrow, "2.14"'] & \mathsf{Gle^R} \\
\mathsf{SubS5^{CS}} \arrow[u, hook] \arrow[r, leftrightarrow, "5.11"'] & \mathsf{DeV^{CS}} \arrow[u, hook] \arrow[r, leftrightarrow, "d", "5.13"'] & \mathsf{DeV^C} \arrow[r, leftrightarrow, "\substack{2.3\\2.11}"'] & \mathsf{KRFrm^C} \arrow[u, hook] \arrow[r, leftrightarrow, "d", "2.3"'] & \mathsf{KHaus^C} \arrow[u, hook] \arrow[r, leftrightarrow, "5.14"'] & \mathsf{StoneE^C} \arrow[u, hook] \arrow[r, leftrightarrow, "5.14"'] & \mathsf{Gle^C} \arrow[u, hook] \\
\mathsf{SubS5^F} \arrow[u, hook] \arrow[r, leftrightarrow, "6.6"'] & \mathsf{DeV^F} \arrow[u, hook] \arrow[r, leftrightarrow, "d", "6.7"'] & \mathsf{DeV} \arrow[u, hook] \arrow[r, leftrightarrow, "\substack{2.1\\2.8}"'] & \mathsf{KRFrm} \arrow[u, hook] \arrow[r, leftrightarrow, "d", "2.1"'] & \mathsf{KHaus} \arrow[u, hook] \arrow[r, leftrightarrow, "6.8"'] & \mathsf{StoneE^F} \arrow[u, hook] \arrow[r, leftrightarrow, "6.8"'] & \mathsf{Gle} \arrow[u, hook]
\end{tikzcd}
$$

Figure 2

Category	**Objects**	**Morphisms**
$\mathsf{SubS5^S}$	S5-subordination algebras	Compatible subordinations
$\mathsf{SubS5^{CS}}$	S5-subordination algebras	Continuous compatible subordinations
$\mathsf{SubS5^F}$	S5-subordination algebras	Functional compatible subordinations
$\mathsf{DeV^S}$	De Vries algebras	Compatible subordinations
$\mathsf{DeV^{CS}}$	De Vries algebras	Continuous compatible subordinations
$\mathsf{DeV^F}$	De Vries algebras	Functional compatible subordinations
$\mathsf{DeV^C}$	De Vries algebras	Lower continuous de Vries mult. maps
DeV	De Vries algebras	De Vries morphisms

Table 1: Categories of subordination algebras.

Category	**Objects**	**Morphisms**
$\mathsf{KRFrm^P}$	Compact regular frames	Preframe homomorphisms
$\mathsf{KRFrm^C}$	Compact regular frames	Continuous preframe homomorphisms
KRFrm	Compact regular frames	Frame homomorphisms

Table 2: Categories of compact regular frames.

Category	Objects	Morphisms
$\mathsf{KHaus}^{\mathsf{R}}$	Compact Hausdorff spaces	Closed relations
$\mathsf{KHaus}^{\mathsf{C}}$	Compact Hausdorff spaces	Continuous relations
KHaus	Compact Hausdorff spaces	Continuous functions

Table 3: Categories of compact Hausdorff spaces.

Category	Objects	Morphisms
$\mathsf{StoneE}^{\mathsf{R}}$	S5-subordination spaces	Compatible closed relations
$\mathsf{StoneE}^{\mathsf{C}}$	S5-subordination spaces	Continuous compatible closed relations
$\mathsf{StoneE}^{\mathsf{F}}$	S5-subordination spaces	Functional compatible closed relations
$\mathsf{Gle}^{\mathsf{R}}$	Gleason spaces	Compatible closed relations
$\mathsf{Gle}^{\mathsf{C}}$	Gleason spaces	Continuous compatible closed relations
Gle	Gleason spaces	Functional compatible closed relations

Table 4: Categories of subordination spaces.

7. Dual descriptions of the completions

In this final section we give dual descriptions of the round ideal and MacNeille completions of S5-subordination algebras.

Recall that if B is a boolean algebra and X is the Stone space of B, then the isomorphism $\varphi\colon B \to \mathsf{Clop}(X)$ is given by the Stone map

$$\varphi(a) = \{x \in X \mid a \in x\}.$$

This isomorphism induces an order-isomorphism Φ between the frame of ideals of B and the frame of open subsets of X, as well as an order-isomorphism Ψ between the frame of filters of B and the frame of closed subsets of X ordered by reverse inclusion (see, e.g., [GH09, Thm. 33]). The isomorphisms are defined as follows:

$$\Phi(I) = \bigcup\{\varphi(a) \mid a \in I\} \quad \text{and} \quad \Psi(F) = \bigcap\{\varphi(a) \mid a \in F\}.$$

It belongs to folklore that for an ideal I and filter F of B, we have

$$\begin{aligned} \Phi(\neg F) = \Psi(F)^c, \quad \Phi(L(F)) = \mathsf{int}\,(\Psi(F)), \\ \Psi(\neg I) = \Phi(I)^c, \quad \Psi(U(I)) = \mathsf{cl}\,(\Phi(I)). \end{aligned} \tag{6}$$

For the reader's convenience, we give a proof of $\Psi(U(I)) = \mathsf{cl}\,(\Phi(I))$. The other three equalities are proved similarly. Since $b \in U(I)$ iff $\varphi(a) \subseteq \varphi(b)$ for each $a \in I$, we have

$$\Psi(U(I)) = \bigcap\{\varphi(b) \mid b \in U(I)\} = \bigcap\{\varphi(b) \mid \Phi(I) \subseteq \varphi(b)\} = \mathsf{cl}(\Phi(I)),$$

where the last equality follows from the fact that X is a Stone space, hence the closure of a set is the intersection of the clopen sets containing it.

Let $(B, S) \in \mathsf{SubS5^S}$. We recall from Remark 2.15(6) that the $\mathsf{S5}$-subordination space of (B, S) is (X, R_S) where X is the Stone space of B and R_S is given by $x\ R_S\ y$ iff $S[x] \subseteq y$. For simplicity, we write (X, R) instead of (X, R_S).

Lemma 7.1. *Let (B, S) be an $\mathsf{S5}$-subordination algebra and (X, R) its $\mathsf{S5}$-subordination space.*

(1) *If I is an ideal of B, then $\Phi(S^{-1}[I]) = \Box_R\Phi(I)$.*

(2) *If F is a filter of B, then $\Psi(S[F]) = R[\Psi(F)]$.*

Proof. (1). We have

$$\begin{aligned}
\Phi(S^{-1}[I]) &= \bigcup\{\varphi(a) \mid a \in S^{-1}[I]\} = \bigcup\{\varphi(a) \mid \exists\, b \in I : a\ S\ b\} \\
&= \bigcup\{\varphi(a) \mid \exists\, b \in I : R[\varphi(a)] \subseteq \varphi(b)\} \\
&= \bigcup\{\varphi(a) \mid R[\varphi(a)] \subseteq \Phi(I)\} \\
&= \bigcup\{\varphi(a) \mid \varphi(a) \subseteq \Box_R\Phi(I)\} = \Box_R\Phi(I),
\end{aligned}$$

where the third equality follows from the fact that $a\ S\ b$ iff $R[\varphi(a)] \subseteq \varphi(b)$ (see, e.g., [BBSV17, Lem. 2.20]); the fourth from the fact that $R[\varphi(a)]$ is closed, hence compact in X; and the last from the fact that $\Box_R\Phi(I)$ is open and $\{\varphi(a) \mid a \in B\}$ forms a basis for X.

(2). We have:

$$\begin{aligned}
\Psi(S[F]) &= (\Phi(\neg S[F]))^c && \text{(by (6))} \\
&= (\Phi(S^{-1}[\neg F]))^c && \text{(by Lemma 3.3)} \\
&= (\Box_R\Phi(\neg F))^c && \text{(by item (1))} \\
&= (\Box_R(\Psi(F)^c))^c && \text{(by (6))} \\
&= R[\Psi(F)] && \text{(by Remark 5.2(2)).}
\end{aligned}$$

□

We recall from the introduction that $\mathcal{O}(X)$ denotes the frame of open subsets of a topological space X. Since the set of R-saturated open subsets of an S5-subordination space (X, R) forms a subframe of $\mathcal{O}(X)$, it is a frame.

Definition 7.2. For an S5-subordination space $\mathbf{X} = (X, R)$ let $\mathcal{O}_R(\mathbf{X})$ be the frame of R-saturated open subsets of X.

Lemma 7.3. *Let $\mathbf{B} = (B, S)$ be an* S5*-subordination algebra and $\mathbf{X} = (X, R)$ its* S5*-subordination space. An ideal I of B is a round ideal iff $\Phi(I)$ is an R-saturated open subset of X. Therefore, $\mathcal{RI}(\mathbf{B})$ is isomorphic to $\mathcal{O}_R(\mathbf{X})$.*

Proof. We have that I is a round ideal iff $I = S^{-1}[I]$. Since Φ is an isomorphism, Lemma 7.1(1) implies that I is a round ideal iff $\Phi(I) = \Box_R \Phi(I)$. Therefore, I is a round ideal iff $\Phi(I)$ is R-saturated. Thus, the restriction of Φ is an isomorphism from $\mathcal{RI}(\mathbf{B})$ to $\mathcal{O}_R(\mathbf{X})$. □

Let $\mathbf{X} = (X, R)$ be an S5-subordination space and $\pi \colon X \to X/R$ the quotient map given by $\pi(x) = [x]$. It is well known that π lifts to an isomorphism between $\mathcal{O}(X/R)$ and $\mathcal{O}_R(\mathbf{X})$ (see, e.g., [Eng89, Prop. 2.4.3]). This together with Lemma 7.3 yields the following result, which by Isbell duality gives an alternative proof of Theorem 3.4(4).

Theorem 7.4. *Let $\mathbf{B} = (B, S)$ be an* S5*-subordination algebra and $\mathbf{X} = (X, R)$ its subordination space. Then $\mathcal{RI}(\mathbf{B})$ is isomorphic to $\mathcal{O}(X/R)$.*

We recall that the MacNeille completion of a boolean algebra B is isomorphic to $\mathcal{RO}(X)$ where X is the Stone space of B (see, e.g., [GH09, Thm. 40]). We will generalize this result to the setting of S5-subordination algebras. Since regular opens are fixpoints of $\mathsf{int}\,\mathsf{cl} \colon \mathcal{O}(X) \to \mathcal{O}(X)$, we introduce the notion of an R-regular open subset of an S5-subordination space (X, R) by replacing int with $\Box_R\,\mathsf{int}$ and cl with $R\,\mathsf{cl}$.

Definition 7.5. Let $\mathbf{X} = (X, R)$ be an S5-subordination space. We say that an R-saturated open subset of X is *R-regular open* if it is a fixpoint of $\Box_R\,\mathsf{int}\,R\,\mathsf{cl} \colon \mathcal{O}_R(\mathbf{X}) \to \mathcal{O}_R(\mathbf{X})$. Let $\mathcal{RO}_R(\mathbf{X})$ be the poset of R-regular open subsets of X.

Lemma 7.6. *Let $\mathbf{X} = (X, R)$ be an* $\mathsf{S5}$*-subordination space. We equip $\mathcal{RO}_R(\mathbf{X})$ with the relation $\prec$ given by*

$$U \prec V \iff R[\mathsf{cl}(U)] \subseteq V.$$

Then $\mathcal{RO}_R(\mathbf{X})$ is a de Vries algebra isomorphic to $\mathcal{RO}(X/R)$.

Proof. As we pointed out in the paragraph before Theorem 7.4, $\pi \colon X \to X/R$ lifts to an isomorphism $f \colon \mathcal{O}_R(X) \to \mathcal{O}(X/R)$ given by $f(U) = \pi[U]$. We show that for each $U \in \mathcal{O}_R(X)$ we have

$$U \in \mathcal{RO}_R(X) \iff \pi[U] \in \mathcal{RO}(X/R).$$

On the one hand,

$$\begin{aligned} U \in \mathcal{RO}_R(X) &\iff U = \Box_R(\mathsf{int}(R[\mathsf{cl}(U)])) \\ &\iff \pi[U] = \pi[\Box_R(\mathsf{int}(R[\mathsf{cl}(U)]))]. \end{aligned}$$

On the other hand,

$$\pi[U] \in \mathcal{RO}(X/R) \iff \pi[U] = \mathsf{int}(\mathsf{cl}(\pi[U])).$$

Therefore, it is enough to prove that

$$\pi[\Box_R(\mathsf{int}(R[\mathsf{cl}(U)]))] = \mathsf{int}(\mathsf{cl}(\pi[U])).$$

Since $\pi \colon X \to X/R$ is a quotient map and X/R is compact Hausdorff, π is a closed map. Thus, for each R-saturated subset G of X we have

$$\pi[R[\mathsf{cl}(G)]] = \pi[\mathsf{cl}(G)] = \mathsf{cl}(\pi[G]). \tag{7}$$

Moreover, since G is R-saturated,

$$\pi[G^c] = \pi[G]^c. \tag{8}$$

Therefore, if H is an R-saturated subset of X, then

$$\begin{aligned} \pi[\Box_R(\mathsf{int}(H))] &= \pi[R[\mathsf{cl}(H^c)]^c] \\ &= \pi[R[\mathsf{cl}(H^c)]]^c && \text{(by (8))} \\ &= \mathsf{cl}(\pi[H^c])^c && \text{(by (7))} \\ &= \mathsf{int}(\pi[H^c]^c) \\ &= \mathsf{int}(\pi[H]) && \text{(by (8)).} \end{aligned}$$

This equation together with (7) yields

$$\pi[\Box_R(\mathsf{int}(R[\mathsf{cl}(U)]))] = \mathsf{int}(\pi[R[\mathsf{cl}(U)]]) = \mathsf{int}(\mathsf{cl}(\pi[U])).$$

Thus, f restricts to a poset isomorphism and hence a boolean isomorphism between $\mathcal{RO}_R(X)$ and $\mathcal{RO}(X/R)$. By (7), f also preserves and reflects the relation:

$$\begin{aligned} U \prec V &\iff R[\mathsf{cl}(U)] \subseteq V \iff \pi[R[\mathsf{cl}(U)]] \subseteq \pi[V] \\ &\iff \mathsf{cl}(\pi[U]) \subseteq \pi[V] \iff \pi[U] \prec \pi[V]. \end{aligned}$$

Therefore, f is a structure-preserving bijection, hence an isomorphism of de Vries algebras by [dV62, Prop. 1.5.5]. $\square$

Proposition 7.7. *Let $\mathbf{B} = (B, S)$ be an* $\mathsf{S5}$*-subordination algebra and $\mathbf{X} = (X, R)$ its* $\mathsf{S5}$*-subordination space. For a round ideal I of $\mathbf{B}$, we have*:

(1) $\Phi(I^*) = \Box_R\mathsf{int}(\Phi(I)^c)$.

(2) $\Phi(I^{**}) = \Box_R\mathsf{int}(R[\mathsf{cl}\,\Phi(I)])$.

(3) *I is a normal round ideal iff $\Phi(I)$ is an R-regular open subset.*

Consequently, $\mathcal{NI}(\mathbf{B})$ is isomorphic to $\mathcal{RO}_R(\mathbf{X})$.

Proof. (1). We have

$$\begin{aligned} \Phi(I^*) &= \Phi(\neg S[U(I)]) && \text{(by Theorem 3.4(2))} \\ &= (\Psi(S[U(I)]))^c && \text{(by (6))} \\ &= (R[\Psi(U(I))])^c && \text{(by Lemma 7.1(2))} \\ &= (R[\mathsf{cl}\,\Phi(I)])^c && \text{(by (6))} \\ &= \Box_R\mathsf{int}(\Phi(I)^c), \end{aligned}$$

where the last equality follows from the fact that $\mathsf{cl}\,U = (\mathsf{int}(U^c))^c$ for each $U \subseteq X$.

(2). By the proof of item (1), if I is a round ideal, then

$$\Phi(I^*) = (R[\mathsf{cl}\,\Phi(I)])^c = \Box_R\mathsf{int}(\Phi(I)^c).$$

Thus,

$$\Phi(I^{**}) = \Box_R \mathsf{int}(\Phi(I^*)^c) = \Box_R \mathsf{int}(((R[\mathsf{cl}\,\Phi(I)])^c)^c) = \Box_R \mathsf{int}(R[\mathsf{cl}\,\Phi(I)]).$$

(3). Since I is normal iff $I = I^{**}$, this follows from item (2) and Definition 7.5.

Finally, since Φ is an order-isomorphism, its restriction is an isomorphism of the boolean algebras $\mathcal{NI}(\mathbf{B})$ and $\mathcal{RO}_R(\mathbf{X})$. Moreover, if $I, J \in \mathcal{NI}(\mathbf{B})$, then

$$\begin{aligned}
I \prec J &\iff I^* \vee J = B \\
&\iff \Phi(I^* \vee J) = X \\
&\iff \Phi(I^*) \cup \Phi(J) = X \\
&\iff R[\mathsf{cl}\,\Phi(I)]^c \cup \Phi(J) = X \qquad \text{(by the proof of item (1))} \\
&\iff R[\mathsf{cl}\,\Phi(I)] \subseteq \Phi(J) \\
&\iff \Phi(I) \prec \Phi(J).
\end{aligned}$$

Therefore, Φ is an isomorphism of de Vries algebras. $\square$

Combining Lemma 7.6 and Proposition 7.7 yields the following result, which gives an alternative proof of Proposition 4.4.

Theorem 7.8. *Let* $\mathbf{B} = (B, S)$ *be an* $\mathsf{S5}$*-subordination algebra and* $\mathbf{X} = (X, R)$ *its* $\mathsf{S5}$*-subordination space. Then* $\mathcal{NI}(\mathbf{B})$ *is isomorphic to* $\mathcal{RO}(X/R)$.

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Marco Abbadini
School of Computer Science
University of Birmingham
B15 2TT Birmingham (UK)
marco.abbadini.uni@gmail.com

Guram Bezhanishvili
Department of Mathematical Sciences
New Mexico State University
Las Cruces NM 88003 (USA)
guram@nmsu.edu

Luca Carai
Dipartimento di Matematica
Università degli Studi di Milano
20133 Milan (Italy)
luca.carai.uni@gmail.com

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXV-2 (2024)

COVERS IN THE CANONICAL GROTHENDIECK TOPOLOGY

Cynthia Lester

Résumé. Nous explorons la topologie canonique de Grothendieck dans certaines circonstances spécifiques. Tout d'abord, nous utilisons une description de la topologie canonique pour obtenir une variante du théorème de Giraud, qui indique quand une catégorie est équivalente à une catégorie de gerbes sur un site de Grothendieck. Ensuite, nous explorons la topologie canonique de Grothendieck sur les catégories d'ensembles et d'espaces topologiques. Nous donnons une base et une présentation pour la topologie canonique sur la catégorie des ensembles. De plus, puisqu'il existe plusieurs catégories qui peuvent représenter la catégorie des espaces topologiques, nous explorons deux de ces catégories : la catégorie de tous les espaces topologiques, et la catégorie des espaces de Hausdorff faiblement engendrés et compacts. Cette exploration se concentre sur les différences des topologies canoniques de Grothendieck résultantes, ainsi que sur leurs bases et présentations. Troisièmement, nous examinons les topologies canoniques de Grothendieck sur la catégorie des R-modules. Une attention particulière est accordée à la recherche de réductions et à la catégorie des groupes abéliens.

Abstract. We explore the canonical Grothendieck topology in some specific circumstances. First, we use a description of the canonical topology to get a variant of Giraud's Theorem, which indicates when a category is equivalent to a category of sheaves on a Grothendieck site. Second, we explore the canonical Grothendieck topology on the categories of sets and topological spaces. We give a basis and presentation for the canonical topology on the category of sets. Additionally, since there are several categories that can represent "the category of topological spaces," we explore two such categories:

the category of all topological spaces, and the category of compactly generated weakly Hausdorff spaces. This exploration focuses on the differences of the resulting canonical Grothendieck topologies, along with their bases and presentations. Third, we look at the canonical Grothendieck topology on the category of R-modules. A special focus is paid to finding reductions and to the category of abelian groups.

Keywords. Grothendieck topology, Giraud's Theorem, colimit, quotient space, compactly generated weakly Hausdorff, abelian group

Mathematics Subject Classification (2020). 18F10, 18A30, 54B30, 54B15, 20K25, 54B40, 18A20, 16D10.

Contents

1. Introduction

In SGA 4.2.2 Verdier defined the canonical Grothendieck topology as the largest Grothendieck topology where all representable presheaves are sheaves. This paper grew out of an attempt to obtain a precise description of the covers in this Grothendieck topology in the cases of some familiar categories; we investigate the question for sets, abelian groups, R-modules, topological spaces and compactly generated Hausdorff spaces. The category of sets is simple enough that we can give a complete answer, and in the two categories of topological spaces we give a fairly precise description. The question for abelain groups and R-modules seems to be very subtle, though, and we have only been able to obtain partial results. Along the way we prove that the canonical topology has a natural appearance in Giraud's Theorem, which is the source for some of our interest in it.

Sieves will be of particular importance in this paper and so we start with a reminder of its definition; we follow the notation and terminology used by Mac Lane and Moerdijk in [S. Mac Lane and I. Moerdijk, 2012]. For any object X of a category $\mathcal{C}$, we call S a *sieve on* X if S is a collection of morphisms, all of whose codomains are X, that is closed under precomposition, i.e. if $f \in S$ and $f \circ g$ makes sense, then $f \circ g \in S$. In particular, we can view a sieve S on X as a full subcategory of the overcategory $(\mathcal{C} \downarrow X)$.

By work from [C. Lester, 2019], the canonical Grothendieck topology can be characterized in terms of colimits. Specifically, the canonical Grothendieck topology can be described as the collection of all universal colim sieves where:

Definition 1.1. *For a category $\mathcal{C}$, an object X of $\mathcal{C}$ and sieve S on X, we call S a colim sieve if $\varinjlim_S U$ exists and the canonical map $\varinjlim_S U \to X$ is an isomorphism. (Alternatively, S is a colim sieve if X is the universal cocone under the diagram $U\colon S \to \mathcal{C}$.) Moreover, we call S a universal colim sieve if for all arrows $\alpha\colon Y \to X$ in $\mathcal{C}$, α^*S is a colim sieve on Y.*

One use of this presentation is the following variant of Giraud's Theorem:

Proposition 3.14. *If $\mathcal{E}$ is a 'nice' category, then $\mathcal{E}$ is equivalent to the category of sheaves on $\mathcal{E}$ under the canonical topology.*

The universal-colim-sieve presentation also affords us an explicit description of the canonical Grothendieck topology's covers on the category of topological spaces:

Proposition 4.6. *In the category of all topological spaces, $\{A_\alpha \to X\}_{\alpha\in\mathcal{A}}$ is part of a basis for the canonical topology if and only if $\alpha\colon \coprod_{\alpha\in\mathcal{A}} A_\alpha \to X$ is* *a universal quotient map (i.e. α and every pullback of α is a quotient map). Additionally, a sieve S on X is a (universal) colim sieve if and only if there exists some collection $\{A_\alpha \to X\}_{\alpha\in\mathcal{A}} \subset S$ such that $\coprod_{\alpha\in\mathcal{A}} A_\alpha \to X$ is a (universal) quotient map. In particular, $T = \langle\{f : Y \to X\}\rangle$ is a (universal) colim sieve if and only if f is a (universal) quotient map.*

Proposition 4.7. *In the category of compactly generated weakly Hausdorff spaces, $\{A_\alpha \to X\}_{\alpha\in\mathcal{A}}$ is part of the basis for the canonical topology if and only if $\coprod_{\alpha\in\mathcal{A}} A_\alpha \to X$ is a quotient map. In particular, a sieve $S = \langle\{A_\alpha \to X\}_{\alpha\in\mathcal{A}}\rangle$ on X is in the canonical topology if and only if $\coprod_{\alpha\in\mathcal{A}} A_\alpha \to X$ is a quotient map. Moreover, every colim sieve is universal.*

Furthermore, this presentation allows us to more easily compute examples and non-examples in the category of topological spaces; for instance,

Example 4.14/Example 4.15. Take $\mathbb{R}^n \to \mathbb{R}^{n+1}$ be the closed inclusion map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$ and use $\mathbb{R}^\infty$ to denote the direct limit $\varinjlim_{n\in\mathbb{N}} \mathbb{R}^n$ with maps $\iota_n : \mathbb{R}^n \to \mathbb{R}^\infty$. Then the cover generated by $\{\iota_n\}_{n\in\mathbb{N}}$ is not in the canonical topology for the category of all topological spaces but is in the canonical topology for the category of compactly generated weakly Hausdorff spaces.

Additionally, we can use the universal-colim-sieve presentation to get a better idea of the canonical Grothendieck topology's covers on the category of R-modules. For example,

Proposition 5.6. *Let S be the cover generated by $\{f_1 : M_1 \to R, f_2 : M_2 \to R\}$ such that $im(f_i) = a_i R$ for $i = 1, 2$. Then S is in the canonical topology on R-**Mod** if and only if $(a_1, a_2) = R$.*

Proposition 5.8. *Let R be an infinite principal ideal domain. Let S be the cover generated by $\{g_i : R^n \hookrightarrow R^n\}_{i=1}^M \cup \{f_i : R^{m_i} \hookrightarrow R^n \mid m_i < n\}_{i=1}^N$. If S a cover in the canonical topology on R-**Mod**, then $g_1 \oplus \cdots \oplus g_M : R^{nM} \to R^n$ is a surjection.*

Proposition 5.24. *Let S be the cover generated by $\{\mathbb{Z} \xrightarrow{\times a_i} \mathbb{Z}\}_{i=1}^{N}$. Then S is in the canonical topology on $\mathbb{Z}$-**Mod** if and only if gcd$(a_1, \ldots, a_N) = 1$.*

Proposition 5.25. *Let S be the cover generated by $\{\mathbb{Z}^n \xrightarrow{A_i} \mathbb{Z}^n\}_{i=1}^{N}$ where A_i is a diagonal matrix with $\det(A_i) \neq 0$. Then there exists a map $\beta\colon \mathbb{Z} \to \mathbb{Z}^n$ such that $\beta^* S$ is not a colim sieve in $\mathbb{Z}$-**Mod** if and only if gcd$(\det(A_1), \ldots, \det(A_N))$ does not equal 1.*

Organization.

To start this paper we recall some results from [C. Lester, 2019] in Section 2. Then in Section 3 we review Giraud's theorem and prove our Corollary to Giraud's Theorem, i.e. we prove that that every category $\mathcal{C}$, which satisfies some hypotheses, is equivalent to the category of sheaves on $\mathcal{C}$ with the canonical topology.

In Section 4 we briefly discuss the canonical topology on the category of sets before exploring the canonical topology on the category of topological spaces. Specifically, we look at the category of all topological spaces and the category of compactly generated weakly Hausdorff spaces. We are able to refine our description and obtain a basis for the canonical topology; this result reduces the question "Is this in the canonical topology?" to the question "Is a specific map a universal quotient map?" Since universal quotient maps have been studied in-depth (for example by Day and Kelly in [B.J. Day and G.M. Kelly, 1970]), this reduction becomes our most computationally agreeable description of the canonical topology and hence we use it to find some specific examples and non-examples.

Lastly, in Section 5 we investigate the canonical topology on the category of R-modules and the category of abelian groups, where we work towards refining our description by making some reductions and obtaining some exclusionary results. While these reductions and results lead us to some specific examples and non-examples, a basis for the canonical topology remains elusive.

General Notation.

Notation 1.2. For any subcategory S of $(\mathcal{C} \downarrow X)$, we will use U to represent the forgetful functor $S \to \mathcal{C}$. For example, for a sieve S on X, $U(f) =$ domain f.

Notation 1.3. We say that a sieve S on X is *generated* by the morphisms $\{f_\alpha \colon A_\alpha \to X\}_{\alpha \in \mathcal{A}}$ and write $S = \langle \{f_\alpha \colon A_\alpha \to X\}_{\alpha\in\mathcal{A}} \rangle$ if each $f \in S$ factors through one of the f_α, i.e. if $f \in S$ then there exists an $\alpha \in \mathcal{A}$ and morphism g such that $f = f_\alpha \circ g$.

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2. Background

This section contains a review of the results from [C. Lester, 2019] that will be used in this paper.

Lemma 2.1. *Suppose $\mathcal{C}$ is a category with all pullbacks.*
Let $S = \langle \{g_\alpha \colon A_\alpha \to X\}_{\alpha\in\mathfrak{A}} \rangle$ be a sieve on object X of $\mathcal{C}$ and $f \colon Y \to X$ be a morphism in $\mathcal{C}$. Then $f^*S = \langle \{A_\alpha \times_X Y \xrightarrow{\pi_2} Y\}_{\alpha\in\mathfrak{A}} \rangle$.

Proposition 2.2. *Let $\mathcal{C}$ be a cocomplete category. For a sieve in $\mathcal{C}$ on X of the form $S = \langle \{f_\alpha \colon A_\alpha \to X\}_{\alpha\in\mathfrak{A}} \rangle$ such that $A_i \times_X A_j$ exists for all $i, j \in \mathfrak{A}$,*

$$\varinjlim_S U \cong \mathrm{Coeq} \left(\begin{array}{c} \coprod_{(i,j)\in\mathfrak{A}\times\mathfrak{A}} A_i \times_X A_j \\ \downarrow\downarrow \\ \coprod_{k\in\mathfrak{A}} A_k \end{array} \right)$$

where the left and right vertical maps are induced from the projection morphisms $\pi_1 \colon A_i \times_X A_j \to A_i$ and $\pi_2 \colon A_i \times_X A_j \to A_j$.

Lemma 2.3. *Let $\mathcal{C}$ be a category. Then S is a colim sieve on X if and only if f^*S is a colim sieve for any isomorphism $f\colon Y \to X$.*

Recall that a morphism $f\colon Y \to X$ is called an *effective epimorphism* provided $Y \times_X Y$ exists, f is an epimorphism and $c\colon \operatorname{Coeq}(Y \times_X Y \rightrightarrows Y) \to X$ is an isomorphism. Note that this third condition actually implies the second because $f = c \circ g$ where $g\colon Y \to \operatorname{Coeq}(Y \times_X Y \rightrightarrows Y)$ is the canonical map. Indeed, g is an epimorphism by an easy exercise and c is an epimorphism since it is an isomorphism.

Additionally, $f\colon Y \to X$ is called a *universal effective epimorphism* if f is an effective epimorphism with the additional property that for every pullback diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\pi_g"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"'] & X \end{tikzcd}$$

π_g is also an effective epimorphism.

Corollary 2.4. *Let $\mathcal{C}$ be a cocomplete category with pullbacks. If*

$$S = \langle \{ f\colon Y \to X \} \rangle$$

is a sieve on X, then S is a colim sieve if and only if f is an effective epimorphism. Moreover, S is a universal colim sieve if and only if f is a universal effective epimorphism.

Theorem 2.5. *Let $\mathcal{C}$ be any category. The collection of all universal colim sieves on $\mathcal{C}$ forms a Grothendieck topology.*

Theorem 2.6. *For any (locally small) category $\mathcal{C}$, the collection of all universal colim sieves on $\mathcal{C}$ is the canonical topology.*

Proposition 2.7. *Let $\mathcal{C}$ be a cocomplete category with pullbacks. Futher assume that coproducts and pullbacks commute in $\mathcal{C}$. Then a sieve of the form $S = \langle \{ f_\alpha\colon A_\alpha \to X \}_{\alpha \in \mathcal{A}} \rangle$ is a (universal) colim sieve if and only if the sieve $T = \langle \{ \coprod f_\alpha\colon \coprod_{\alpha \in \mathcal{A}} A_\alpha \to X \} \rangle$ is a (universal) colim sieve.*

Theorem 2.8. *Let $\mathcal{C}$ be a cocomplete category with pullbacks whose coproducts and pullbacks commute. A sieve S on X is a (universal) colim sieve of $\mathcal{C}$ if and only if there exists some $\{A_\alpha \to X\}_{\alpha\in\mathcal{A}} \subset S$ where $\coprod_{\alpha\in\mathcal{A}} A_\alpha \to X$ is a (universal) effective epimorphism.*

Theorem 2.9. *Let $\mathcal{C}$ be a cocomplete category with stable and disjoint coproducts and all pullbacks. For each X in $\mathcal{C}$, define $K(X)$ by*

$$\{A_\alpha \to X\}_{\alpha\in\mathcal{A}} \in K(X) \iff \coprod_{\alpha\in\mathcal{A}} A_\alpha \to X \text{ is a universal effective epimorphism.}$$

Then K is a Grothendieck basis and generates the canonical topology on $\mathcal{C}$.

3. Giraud's Theorem and the Canonical Topology

Giraud's Theorem shows that categories with certain nice properties can be written as sheaves on a Grothendieck site. We show that in fact, modulo universe considerations, one may take this site to be the original category with the canonical topology.

We will use the version of Giraud's Theorem from [S. Mac Lane and I. Moerdijk, 2012]. In fact, the appendix of [S. Mac Lane and I. Moerdijk, 2012] has a thorough discussion of Giraud's theorem and all of the terminology used in it; we will include the basics of this discussion for completeness. We will begin by recalling the definitions used in Mac Lane and Moerdijk's version of Giraud's Theorem.

Throughout this section, let $\mathcal{E}$ be a category with small hom-sets and all finite limits.

DISJOINT AND STABLE COPRODUCTS

Let E_α be a family of objects in $\mathcal{E}$ and $E = \amalg_\alpha E_\alpha$.

Definition 3.1. *The coproduct E is called disjoint if every coproduct inclusion $i_\alpha\colon E_\alpha \to E$ is a monomorphism and, whenever $\alpha \neq \beta$, $E_\alpha \times_E E_\beta$ is the initial object in $\mathcal{E}$.*

Definition 3.2. *The coproduct E is called stable (under pullback) if for every $f\colon D \to E$ in $\mathcal{E}$, the morphisms j_α obtained from the pullback diagrams*

$$\begin{tikzcd} D \times_E E_\alpha \arrow[r] \arrow[d, "j_\alpha"'] & E_\alpha \arrow[d, "i_\alpha"] \\ D \arrow[r, "f"'] & E \end{tikzcd}$$

induce an isomorphism $\coprod_\alpha (D \times_E E_\alpha) \cong D$.

Remark 3.3. If every coproduct in $\mathcal{E}$ is stable, then the pullback operation $- \times_E D$ "commutes" with coproducts, i.e. $(\coprod_\alpha B_\alpha) \times_E D \cong \coprod_\alpha (B_\alpha \times_E D)$.

COEQUALIZER MORPHISMS AND KERNEL PAIRS

Definition 3.4. *We call a morphism* $f \colon Y \to Z$ *in* $\mathcal{E}$ *a coequalizer if there exists some object* X *and morphisms* $\partial_0, \partial_1 \colon X \to Y$ *such that*

$$X \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} Y \xrightarrow{f} Z$$

is a coequalizer diagram.

We remark that every coequalizing morphism is an epimorphism but the converse of this statement is not guaranteed.

Definition 3.5. *The pair of morphisms* $\partial_0, \partial_1 \colon X \to Y$ *are called a kernel pair for* $f \colon Y \to Z$ *if the following is a pullback diagram*

$$\begin{tikzcd} X \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] & Y \arrow[d, "f"] \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

EQUIVALENCE RELATIONS AND QUOTIENTS

Definition 3.6. *An equivalence relation on the object* E *of* $\mathcal{E}$ *is a subobject* R *of* $E \times E$*, represented by the monomorphism* $(\partial_0, \partial_1) \colon R \to E \times E$*, satisfying the following axioms*

1. *(reflexive) the diagonal* $\Delta \colon E \to E \times E$ *factors through* (∂_0, ∂_1)*,*
2. *(symmetric) the map* $(\partial_1, \partial_0) \colon R \to E \times E$ *factors through* (∂_0, ∂_1)*,*

3. (transitivity) if $R \times_E R$ is the pullback

$$\begin{tikzcd} R \times_E R \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & R \arrow[d, "\partial_0"] \\ R \arrow[r, "\partial_1"'] & E \end{tikzcd}$$

then $(\partial_1\pi_1, \partial_0\pi_0) \colon R \times_E R \to E \times E$ factors through R.

Definition 3.7. *If E is an object of $\mathcal{E}$ with equivalence relation R, then the quotient is denoted E/R and is defined to be*

$$\mathrm{Coeq}\left(R \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} E\right)$$

provided that this coequalizer exists.

STABLY EXACT FORKS

A diagram is called a *fork* if it is of the form

$$X \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} Y \xrightarrow{q} Z. \tag{1}$$

Definition 3.8. *The fork (1) is called exact if ∂_0 and ∂_1 are the kernel pair for q, and q is the coequalizer of ∂_0 and ∂_1.*

Definition 3.9. *The fork (1) is called stably exact if the pullback of (1) along any morphism in $\mathcal{E}$ yields an exact fork, i.e. if for any $Z' \to Z$ in $\mathcal{E}$,*

$$X \times_Z Z' \rightrightarrows Y \times_Z Z' \xrightarrow{q \times 1} Z \times_Z Z'$$

is an exact fork.

GENERATING SETS

Definition 3.10. *A set of objects $\{A_i \mid i \in I\}$ of $\mathcal{E}$ is said to generate $\mathcal{E}$ if for every object E of $\mathcal{E}$, $W = \{A_i \to E \mid i \in I\}$ is an epimorphic family (in the sense that for any two parallel arrows $u, v \colon E \to E'$, if every $w \in W$ yields the identity $uw = vw$, then $u = v$).*

GIRAUD'S THEOREM

Theorem 3.11 (Giraud, [S. Mac Lane and I. Moerdijk, 2012]). *A category $\mathcal{E}$ with small hom-sets and all finite limits is a Grothendieck topos if and only if it has the following properties (which we will refer to as Giraud's axioms):*

(i) $\mathcal{E}$ has small coproducts which are disjoint and stable under pullback,

(ii) every epimorphism in $\mathcal{E}$ is a coequalizer,

(iii) every equivalence relation $R \rightrightarrows E$ in $\mathcal{E}$ is a kernel pair and has a quotient,

(iv) every exact fork $R \rightrightarrows E \to Q$ is stably exact,

(v) there is a small set of objects of $\mathcal{E}$ which generate $\mathcal{E}$.

Discussion 3.12. Taken together, Giraud's axioms (ii) and (iv) imply that for each epimorphism $B \xrightarrow{f} A$, the fork $B \times_A B \rightrightarrows B \to A$ is stably exact. The exactness implies f is an effective epimorphism and the stability implies f is a universal effective epimorphism.

Notation 3.13. We use $Sh(\mathcal{E}, J)$ to represent the category of sheaves on the category $\mathcal{E}$ under the topology J.

Suppose the category $\mathcal{E}$ has small hom-sets and all finite limits, satisfies Giraud's axioms, and whose small set of generators (axiom v) is $\mathcal{C}$. In [S. Mac Lane and I. Moerdijk, 2012] Mac Lane and Moerdijk specifically prove $\mathcal{E} \cong Sh(\mathcal{C}, J)$ where J is the Grothendieck topology on $\mathcal{C}$ defined by:

$$S \in J(X) \text{ if and only if } \coprod_{(g\colon D \to X) \in S} D \to X \text{ is an epimorphism in } \mathcal{E}.$$

(In particular, Mac Lane and Moerdijk prove that J is a Grothendieck topology.)

Proposition 3.14. *Suppose the category $\mathcal{E}$ has small hom-sets and all finite limits, satisfies Giraud's axioms, and whose small set of generators (axiom v) is $\mathcal{C}$. Then $\mathcal{E}$ is equivalent to $Sh(\mathcal{C}, C)$ where C is the canonical topology on $\mathcal{C}$.*

Proof. Let J be the topology defined above. Additionally, the above discussion implies that it suffices to show that J is the canonical topology. By Theorem 2.6, we will instead show that every universal colim sieve is in J and that every sieve in J is a universal colim sieve.

By Remark 3.3, coproducts and pullbacks commute and hence for any collection of morphisms $\{A_i \to X\}_{i \in I}$ in $\mathcal{E}$, the diagrams

$$\begin{array}{c} \coprod_{I^2}(A_i \times_X A_j) \\ \downarrow\downarrow \\ \coprod_I A_k \end{array} \quad \text{and} \quad \begin{array}{c} (\coprod_I A_i) \times_X (\coprod_I A_j) \\ \downarrow\downarrow \\ \coprod_I A_k \end{array}$$

are isomorphic. Note: in both diagrams, the two maps down are the obvious ones induced/obtained from a pullback diagram. Thus

$$\operatorname{Coeq}\left(\begin{array}{c} \coprod_{I^2}(A_i \times_X A_j) \\ \downarrow\downarrow \\ \coprod_I A_k \end{array}\right) \cong \operatorname{Coeq}\left(\begin{array}{c} (\coprod_I A_i) \times_X (\coprod_I A_j) \\ \downarrow\downarrow \\ \coprod_I A_k \end{array}\right).$$

But by Proposition 2.2 (which is usable since $\mathcal{E}$ is cocomplete),

$$\operatorname{Coeq}\left(\begin{array}{c} \coprod_{I^2}(A_i \times_X A_j) \\ \downarrow\downarrow \\ \coprod_I A_k \end{array}\right) \cong \varinjlim_{S} U \quad \text{where } S = \langle \{A_i \to X\}_{i \in I} \rangle$$

and

$$\operatorname{Coeq}\left(\begin{array}{c} (\coprod_I A_i) \times_X (\coprod_I A_j) \\ \downarrow\downarrow \\ \coprod_I A_k \end{array}\right) \cong \varinjlim_{T_S} U \quad \text{where } T_S = \left\langle \left\{ \left(\coprod_I A_i \right) \to X \right\} \right\rangle.$$

Hence

$$\underrightarrow{\operatorname{colim}}_{S}\, U \cong \underrightarrow{\operatorname{colim}}_{T_S}\, U$$

$$\text{where } S = \langle \{A_i \to X\}_{i\in I} \rangle \quad \text{and} \quad T_S = \left\langle \left\{ \left(\coprod_I A_i \right) \to X \right\} \right\rangle \tag{2}$$

for any generating set $\{A_i \to X\}_{i\in I}$ of S.

Suppose S is a universal colim sieve. Since S has the some generating set, then by the definition of colim sieve and (2),

$$X \cong \underrightarrow{\operatorname{colim}}_{S}\, U \cong \underrightarrow{\operatorname{colim}}_{T_S}\, U.$$

This implies that T_S is a colim sieve. Hence $\left(\coprod_{(g\colon D\to X)\in S} D\right) \to X$ is an effective epimorphism by Corollary 2.4 and so $S \in J(X)$.

For the converse, suppose that $S \in J(X)$. Thus $p_s\colon \left(\coprod_{(g\colon D\to X)\in S} D\right) \to X$ is an epimorphism, which by Discussion 3.12 is a universal effective epimorphism. Hence by Corollary 2.4, p_s generates a universal colim sieve called T_S. Then by the definition of colim sieve and (2),

$$X \cong \underrightarrow{\operatorname{colim}}_{T_S}\, U \cong \underrightarrow{\operatorname{colim}}_{S}\, U.$$

Therefore S is a colim sieve.

Similar to the last paragraph, we can use (2) to show that f^*S is a colim sieve for any morphism f in $\mathcal{E}$ if we know that T_{f^*S} is a colim sieve. So to finish the proof we will use the fact that T_S is a universal colim sieve to show that T_{f^*S} is a colim sieve. Let $f\colon Y \to X$ be any morphism in $\mathcal{E}$. Then by using S as a generating collection for itself and Lemma 2.1, $f^*S = \langle \{A \times_X Y \to Y \mid A \to X \in S\} \rangle$. Similarly, using Lemma 2.1, $f^*T_S = \left\langle \left\{ \left(\coprod_{(A\to X\in S)} A \right) \times_X Y \to Y \right\} \right\rangle$. Then by Remark 3.3

$$\coprod_{(A\to X)\in S} (A \times_X Y) \cong \left(\coprod_{(A\to X)\in S} A \right) \times_X Y$$

over Y. Therefore,

$$\varinjlim_{T_{f^*S}} U \cong \varinjlim_{f^*T_S} U \cong Y$$

where the first isomorphism is due to the previous few sentences and the second isomorphism is due to the fact that T_S is a universal colim sieve. Thus T_{f^*S} is a colim sieve. $\square$

4. Universal Colim Sieves in the Categories of Sets and Topological Spaces

In this section we examine the canonical topology on the categories of sets, all topological spaces and compactly generated weakly Haudsdorff spaces.

Notation 4.1. We will use **Sets** to denote the category of sets. We will use **Top** to denote the category of all topological spaces, **CG** to denote the category of compactly generated spaces, and **CGWH** to denote the category of compactly generated weakly Hausdorff spaces. When we want to talk about the category of topological spaces without differentiating between **Top** and **CGWH**, then we will use **Spaces**; all results about **Spaces** will hold for both **Top** and **CGWH**.

We will begin with a few reminders about the category of compactly generated weakly Hausdorff spaces based on the references [N.P. Strickland, 2009] and [J.P. May, 1999]. Specifically, there are functors $k\colon$ **Top** $\to$ **CG** and $h\colon$ **CG** $\to$ **CGWH** such that

- For a topological space X with topology τ, a subset Y of X is called k-closed if $u^{-1}(Y)$ is closed in K for every continuous map $u\colon K \to X$ and compact Hausdorff space K. The collection of all k-closed subsets, called $k(\tau)$, is a topology.
- The functor k takes X with topology τ to the set X with topology $k(\tau)$.
- k is right adjoint to the inclusion functor $\iota\colon$ **CG** $\to$ **Top**.
- $h(X)$ is X/E where E is the smallest equivalence relation on X closed in $X \times X$.

- h is left adjoint to the inclusion functor $\iota'\colon$ **CGWH** $\to$ **CG**.
- A limit in **CGWH** is k applied to the limit taken in **Top**, i.e. for a diagram $F\colon I \to$ **CGWH**, the limit of F is $k(\lim_I \iota\iota' F)$.
- A colimit in **CGWH** is h applied to the colimit taken in **Top**, i.e. for a diagram $F\colon I \to$ **CGWH**, the colimit of F is $h(\varinjlim_I \iota\iota' F)$.

Proposition 4.2. *Let S be a sieve on X in either **Sets** or **Top**. Let C be $\varinjlim_S U$. Then the natural map $\varphi\colon C \to X$ is an injection.*

Proof. Suppose $\tilde{y}, \tilde{z} \in C$ and $\varphi(\tilde{y}) = x = \varphi(\tilde{z})$. We can pick a $(Y \to X) \in S$ and a $y \in Y$ that represents $\tilde{y}$, i.e. where $y \mapsto \tilde{y}$ under the natural map $Y \to C$; similarly, we can pick a $(Z \to X) \in S$ and a $z \in Z$ representing $\tilde{z}$. Then the inclusion $i\colon \{x\} \hookrightarrow X$ factors through both Y and Z by $x \mapsto y$ and $x \mapsto z$ respectively. Thus $i \in S$. Hence $\tilde{y} = \tilde{z}$ in C. $\square$

Corollary 4.3. *Let S be a sieve on X in **CGWH**. Then the colimit over S taken in **Top** is in **CGWH**, i.e. $h(\varinjlim_I \iota\iota' U) = \varinjlim_I \iota\iota' U$. Moreover, the natural map $\varphi\colon \varinjlim_S U \to X$ is an injection.*

Proof. We will make use of the following Proposition from [N.P. Strickland, 2009]: if Z is in **CG**, then Z is weakly Hausdorff if and only if the diagonal subspace Δ_Z is closed in $Z \times Z$. Additionally, we remark that colimits of compactly generated spaces computed in **Top** are automatically compactly generated.

Let $C = \varinjlim_S \iota\iota' U$, i.e. C is the colimit over S taken in **Top**. By Proposition 4.2, the natural map $\varphi\colon C \to X$ is an injection; we remark that it is not the statement of Proposition 4.2 that gives this observation since S is not a sieve in Top, instead the proof of Proposition 4.2 holds in this situation since $\{x\}$ is in **CGWH**. Since X is **CGWH**, then Δ_X is closed in $X \times X$. Since φ is a continuous injection, then $(\varphi \times \varphi)^{-1}(\Delta_X) = \Delta_C$ is closed in $C \times C$. $\square$

4.1 Basis and Presentation

The categories **Sets**, **Top** and **CGWH** all satisfy the hypotheses of Theorems 2.9 and 2.8. Thus we have the following corollaries of Theorems 2.9 and 2.8 based on what the universal effective epimorphisms are in each category.

Proposition 4.4. *In* **Sets**, $\{A_\alpha \to X\}_{\alpha\in\mathcal{A}}$ *is part of a basis for the canonical topology if and only if* $\coprod_{\alpha\in\mathcal{A}} A_\alpha \to X$ *is a surjection. In particular, a sieve of the form* $S = \langle\{A_\alpha \to X\}_{\alpha\in\mathcal{A}}\rangle$ *on* X *is in the canonical topology if and only if* $\coprod\limits_{\alpha\in\mathcal{A}} A_\alpha \to X$ *is a surjection. Moreover, every colim sieve is universal.*

Proof. It is easy to see in **Sets** that the effective epimorphisms are precisely the surjections. Since pulling back a surjection yields a surjection, then the universal effective epimorphisms in the category of sets are also the surjections. Lastly, this implies, by Theorem 2.8, that every colim sieve is universal. □

Remark 4.5. Since **Sets** is a Grothendieck topos, we can compare Proposition 4.4 to the proof of Proposition 3.14. Specifically, Proposition 4.4 allows us to determine if a sieve is in the canonical topology by looking only at the sieve's generating set whereas the proof of Proposition 3.14 along with the Grothendieck topology J require us to look at the entire sieve.

Recall that a quotient map f is called *universal* if every pullback of f along a map yields a quotient map.

Proposition 4.6. *In* ***Top***, $\{A_\alpha \to X\}_{\alpha\in\mathcal{A}}$ *is part of a basis for the canonical topology if and only if* $\coprod_{\alpha\in\mathcal{A}} A_\alpha \to X$ *is a universal quotient map. Additionally, a sieve* S *on* X *is a (universal) colim sieve if and only if there exists some collection* $\{A_\alpha \to X\}_{\alpha\in\mathcal{A}} \subset S$ *such that* $\coprod\limits_{\alpha\in\mathcal{A}} A_\alpha \to X$ *is a (universal) quotient map. In particular,* $T = \langle\{f\colon Y \to X\}\rangle$ *is a (universal) colim sieve if and only if* f *is a (universal) quotient map.*

Proof. It is a well-known fact that in **Top** the effective epimorphisms are precisely the quotient maps. □

Proposition 4.7. *In* ***CGWH***, $\{A_\alpha \to X\}_{\alpha\in\mathcal{A}}$ *is part of the basis for the canonical topology if and only if* $\coprod_{\alpha\in\mathcal{A}} A_\alpha \to X$ *is a quotient map. In particular, a sieve* $S = \langle\{A_\alpha \to X\}_{\alpha\in\mathcal{A}}\rangle$ *on* X *is in the canonical topology if and only if* $\coprod\limits_{\alpha\in\mathcal{A}} A_\alpha \to X$ *is a quotient map. Moreover, every colim sieve is universal.*

Proof. This is a consequence of Corollary 2.4, Corollary 4.3, the fact that the universal effective epimorphisms in **Top** are precisely the universal quotient maps, and Proposition 2.36 from [N.P. Strickland, 2009], which states that every quotient map in **CGWH** is universal. $\square$

4.2 Examples in the category of Spaces

In this section we will use our basis to talk about some specific examples; including a special circumstance (when a sieve is generated by one function) and how the canonical topology on the categories **CGWH** and **Top** can differ in this situation.

Definition 4.8. *For a category D, we call $\mathfrak{A} \subset \mathrm{ob(D)}$ a weakly terminal set of D if for every object X in D, there exists some $A \in \mathfrak{A}$ and morphism $X \to A$ in D.*

Additionally, if $F\colon D \to C$ is a functor and D has a weakly terminal set $\mathfrak{A}$, then we call $\{F(A)\}_{A\in\mathfrak{A}}$ a weakly terminal set of F.

For example, if $S = \langle\{A_\alpha \to X\}_{\alpha\in\mathfrak{A}}\rangle$ is a sieve on X then $\{A_\alpha\}_{\alpha\in\mathfrak{A}}$ is the weakly terminal set of U. Or as another example, $\{Y\}$ is the weakly terminal set of the diagram $Y \times_X Y \rightrightarrows Y$. One easy consequence of this in **Top** is a reduction of the colimit topology: V is open in the colimit if and only if the preimage of V is open in each member of the weakly terminal set.

Proposition 4.9. *Let $F\colon D \to$ **Spaces** be a functor where D has a weakly terminal set $\mathfrak{A}$. Suppose $f_A\colon F(A) \to X$ is an open map for all $A \in \mathfrak{A}$, then the induced map $\varphi\colon \varinjlim_D F \to X$ is an open map. Similarly, if the f_A are all closed and $\mathfrak{A}$ is a finite set, then φ is a closed map.*

Proof. Let $C = \varinjlim F$ and $i_A\colon F(A) \to C$ be the natural maps. Both results follow from the easy set equality below for $B \subset C$

$$\varphi(B) = \bigcup_{A\in\mathfrak{A}} f_A(i_A^{-1}(B))$$

since i_A^{-1}, f_A and unions respect open/closed sets in their respective scenarios. $\square$

Corollary 4.10. *Let $S = \langle \{f_\alpha \colon A_\alpha \to X\}_{\alpha \in \mathcal{A}} \rangle$ be a sieve on X in **Spaces** with the induced map $\eta \colon \coprod_{\alpha \in \mathcal{A}} A_\alpha \to X$ a surjection. If all of the f_α are open maps or if $\mathcal{A}$ is a finite collection and all of the f_α are closed maps, then S is a colim sieve.*

Proof. Let $\varphi \colon \varinjlim_S U \to X$ be the natural map. By Proposition 4.2, Corollary 4.3, and the surjectivity of η, φ is a continuous bijection. Then Proposition 4.9 implies that φ is open or closed, depending on the case, and hence an isomorphism. □

This corollary leads us to some nice examples of sieves we would hope are in the canonical topology and actually are!

Example 4.11. Let X be any space and let $\{U_i\}_{i \in I}$ be an open cover of X. Then the inclusion maps $U_i \hookrightarrow X$ generate a universal colim sieve, call it S. Indeed, by Corollary 4.10, S is a colim sieve. Universality is obvious, as the preimage of an open cover is an open cover.

Example 4.12. Let X be any space and let $K_1, \ldots, K_n$ be a closed cover of X. For the exact same reasons as the previous example, the inclusions $K_i \hookrightarrow X$ generate a sieve in the canonical topology.

Before we give our next example, we provide a rephrasing of Theorem 1 from [B.J. Day and G.M. Kelly, 1970], which completely characterizes universal quotient maps in **Top**:

Theorem 4.13 (Day and Kelly, 1970)**.** *Let $f \colon Y \to X$ be a quotient map. Then f is a universal quotient map if and only if for every $x \in X$ and cover $\{G_\alpha\}_{\alpha \in \Lambda}$ of $f^{-1}(x)$ by opens in Y, there is a finite set $\{\alpha_1, \ldots, \alpha_n\} \subset \Lambda$ such that $fG_{\alpha_1} \cup \cdots \cup fG_{\alpha_n}$ is a neighborhood of x.*

Example 4.14. Consider the diagram $B_1 \to B_2 \to B_3 \to \ldots$ and the direct limit $B = \varinjlim B_n$ in **Top**. Let $S = \langle \{\iota_n \colon B_n \to B \,|\, n \in \mathbb{N}\} \rangle$ where ι_n are the natural maps into the colimit. By Proposition 4.6, S is a colim sieve because $\coprod_{n \in \mathbb{N}} B_n \to B$ is obviously a quotient map. However, S is not necessarily in the canonical topology – we can use Proposition 4.6 on specific examples to see when S is and is not in the canonical topology.

For example, suppose there exists an N such that $B_m = B_N$ whenever $m > N$. Then $B = B_N$. Hence it is easy to see by Day and Kelly's condition that the map $\coprod_{n\in\mathbb{N}} B_n \to B$ is a universal quotient map. Therefore, the S from this example is in the canonical topology.

As another example, take $B_n = \mathbb{R}^n$ and let $B_n \to B_{n+1}$ be the closed inclusion map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$. Use $\mathbb{R}^\infty$ to denote the direct limit. We claim that $\coprod_{n\in\mathbb{N}} \mathbb{R}^n \to \mathbb{R}^\infty$ is not a universal quotient map. Indeed, consider Day and Kelly's condition; take $x = 0 \in \mathbb{R}^\infty$ and the open cover in $\coprod_{n\in\mathbb{N}} \mathbb{R}^n$ consisting of open disks $D^n \subset \mathbb{R}^n$ centered at the origin with fixed radius $\epsilon > 0$. Pick any finite collection $D^{n_1}, \ldots, D^{n_k}$ with $n_1 < \cdots < n_k$. Then for $i = 1, \ldots, k$ we can view D^{n_i} as a subset of $\mathbb{R}^{n_k}$. Hence $\cup_{i=1}^k \iota_{n_i}(D^{n_i})$ is $\cup_{i=1}^k \iota_{n_k}(D^{n_i}) \subset \iota_{n_k}(\mathbb{R}^{n_k})$. However, by dimensional considerations, we can see that for all $b \in \mathbb{N}$, $\iota_b(\mathbb{R}^b)$ contains no open sets of $\mathbb{R}^\infty$ and hence $\cup_{i=1}^k \iota_{n_i}(D^{n_i})$ cannot be a neighborhood of x in $\mathbb{R}^\infty$. Remark: To see that $\iota_b(\mathbb{R}^b)$ contains no open sets, suppose to the contrary and call the open set V. Then $\iota_{b+1}^{-1}(V)$ is open in $\mathbb{R}^{b+1}$ and in particular, contains an open ball of dimension $b + 1$. Thus dimensional considerations imply that $\iota_{b+1}^{-1}(V)$ is not contained in the image of $\mathbb{R}^b$ in $\mathbb{R}^{b+1}$. Since each ι_n is an inclusion map, then $\iota_{b+1}\iota_{b+1}^{-1}(V) \not\subset \iota_{b+1}(\mathbb{R}^b)$ and so V is not contained in $\iota_b(\mathbb{R}^b)$, which is our contradiction. Therefore, the S from this example is not in the canonical topology.

Example 4.15. Consider the diagram $B_1 \to B_2 \to B_3 \to \ldots$ and the direct limit $B = \varinjlim B_n$ in **CGWH**. Let $S = \langle\{\iota_n \colon B_n \to B \,|\, n \in \mathbb{N}\}\rangle$ where ι_n are the natural maps into the colimit. Then by Proposition 4.7, S is a universal colim sieve because $\coprod_{n\in\mathbb{N}} B_n \to B$ is a quotient map.

Now we shift our focus to sieves that can be generated by one map, called *monogenic sieves*. There are many reasons one could focus on these kinds of sieves, however by Proposition 2.7, if we fully comprehend when monogenic sieves are in the canonical topology, then we can (in some sense) completely understand the canonical topology. From this point onward, this section will be about monogenic sieves; in other words, by Proposition 4.6 and Proposition 4.7, we will be focusing on (universal) quotient maps.

Remark 4.16. Some examples will talk about the space $\mathbb{R}/\mathbb{Z}$. In this section, this space is not a group quotient but instead is the squashing of the subspace $\mathbb{Z}$ to a point.

Example 4.17. Consider the quotient maps $f\colon S^n \to \mathbb{R}P^n$ and $g\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. There is some subtly, which will depend on the category we are in, in determining if f or g generate universal colim sieves. Throughout the rest of this section we will continue to explore this particular example.

MONOGENIC SIEVES IN **CGWH**

By Proposition 4.7, if X and Y are in **CGWH** and $h\colon Y \to X$, then $\langle\{h\}\rangle$ is in the canonical topology if and only if h is a quotient map. Therefore, we immediately get the following examples:

Example 4.18. Topological manifolds are in **CGWH**. Thus S^n and $\mathbb{R}P^n$ are in **CGWH**. Hence $\langle\{f\colon S^n \to \mathbb{R}P^n\}\rangle$ is in the canonical topology.

Example 4.19. Every CW-complex is in **CGWH**. Thus $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ are in **CGWH**. Hence $\langle\{g\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}\}\rangle$ is in the canonical topology.

MONOGENIC SIEVES IN **TOP**

This section will heavily rely on Theorem 4.13 (the Theorem by Day and Kelly characterizing universal quotient maps in **Top**) because a monogenic sieve generated by f is in the canonical topology if and only if f is a universal quotient map.

Example 4.20. Day and Kelly's theorem implies that every open quotient map is a universal quotient map. Therefore, the quotient map $f\colon S^n \to \mathbb{R}P^n$ is a universal quotient map and $\langle\{f\colon S^n \to \mathbb{R}P^n\}\rangle$ is in the canonical topology.

Example 4.21. The quotient map $g\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is not universal. We will demontrate this in two ways, first by using Day and Kelly's theorem and second by directly showing g is not universal. Note: many sets of $\mathbb{R}/\mathbb{Z}$ will be written as if they are in $\mathbb{R}$ for ease of presentation.

(i) We will look at Day and Kelly's condition for $\mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ with the open cover (in $\mathbb{R}$) $\{G_i := (i-m, i+m)\}_{i\in\mathbb{Z}}$ for a fixed $m \in \left(0, \frac{1}{2}\right)$. For any open set U of $\mathbb{R}/\mathbb{Z}$ containing $\mathbb{Z}$, the quotient topology tells us that $g^{-1}(U)$ is an open neighborhood of $\mathbb{Z} \subset \mathbb{R}$. But for any n, $g^{-1}(\bigcup_{k=1}^{n} gG_{i_k}) =$

$\mathbb{Z} \cup (\bigcup_{k=1}^{n}(i_k - m, i_k + m))$ is not a neighborhood of $\mathbb{Z} \subset \mathbb{R}$. So there cannot be any open set of $\mathbb{R}/\mathbb{Z}$ containing $\mathbb{Z}$ that is contained in $\bigcup_{k=1}^{n} gG_{i_k}$ for any finite collection of the cover.

(ii) To directly show that g is not universal we need to come up with a space and map to $\mathbb{R}/\mathbb{Z}$ where g pulledbacked along this map is not a quotient map. Our candidate is the following: Let $t(\mathbb{R}/\mathbb{Z})$ be the set $\mathbb{R}/\mathbb{Z}$ with the topology where U (written as if it is in $\mathbb{R}$) is said to be open if (a) $\mathbb{Z} \not\subset U$ or (b) U contains $\mathbb{Z}$ and is a neighborhood (in the typical topology) of $(\mathbb{Z} - \{\text{finitely many or no points}\})$. Remark: this topology was used in Day and Kelly's paper (in the proof of their theorem), however they defined the topology using a filter and we have merely rephrased it for convenience.

Define $\kappa \colon t(\mathbb{R}/\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ by the set identity map; this is a continuous map. As a set, the pullback of domain(g) along κ is $\mathbb{R}$ but since it now has the limit topology, we denote the pullback as $t(\mathbb{R})$; in particular, $t(\mathbb{R})$ is $\mathbb{R}$ with the discrete topology. Denote the projection maps as $g' \colon t(\mathbb{R}) \to t(\mathbb{R}/\mathbb{Z})$ and $\kappa' \colon t(\mathbb{R}) \to \mathbb{R}$.

We claim that g' is not a quotient map, i.e. there is some non-open set B in $t(\mathbb{R}/\mathbb{Z})$ with $(g')^{-1}(B)$ open in $t(\mathbb{R})$. Since every $(g')^{-1}(B)$ is open in $t(\mathbb{R})$, then we merely need to find a B that is not open in $t(\mathbb{R}/\mathbb{Z})$; $B = \{\mathbb{Z}\}$ obviously works.

The above example shows us that quotient maps of the form $X \to X/A$ may not generate universal colim sieves. So let's understand these special quotient maps a little better. Specifically, using Day and Kelly's theorem, we can completely state what kinds of subspaces A yield universal quotient maps $X \to X/A$:

Corollary 4.22. *The quotient map $\pi \colon X \to X/A$ is universal if and only if both of the following properties hold:*

1. *If A is not open, then for every open cover $\{G_\alpha\}_{\alpha\in\Lambda}$ of $(\partial A) \cap A$ in X there is a finite collection $\{\alpha_1, \dots, \alpha_n\} \subset \Lambda$ with $A \cup G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$ open in X.*
2. *If A is not closed, then for every open U in X such that $U \cap (\overline{A} - A) \neq \emptyset$, $U \cup A$ is open in X.*

Proof. We will be using Theorem 4.13 in two ways: first by finding the necessary conditions for π to be a universal quotient map (i.e. proving the forward direction) and then second by checking the sufficient conditions in the three cases (i) $x = A$, (ii) $x \in X - \overline{A}$, and (iii) $x \in \overline{A} - A$ (i.e. proving the backward direction).

First suppose that π is a universal quotient map. To see that the first property is necessary, assume that $(\partial A) \cap A \neq \emptyset$, i.e. A is not open, and we have an open cover $\{G_\alpha\}_{\alpha\in\Lambda}$ of $(\partial A)\cap A$. Then we can expand this cover to an open cover of A by adding $Int(A)$ to $\{G_\alpha\}_{\alpha\in\Lambda}$. Now by assumption (using the point A in X/A) there is a finite subcollection $G_{\alpha_1}, \ldots, G_{\alpha_n}, Int(A)$ such that $\pi G_{\alpha_1} \cup \cdots \cup \pi G_{\alpha_n} \cup \pi Int(A)$ is a neighborhood of A in X/A. But $\pi Int(A) \subset \pi G_\alpha$ since $G_\alpha \cap A \neq \emptyset$ and so $Int(A)$ is not necessary in our finite subcollection. Thus $\pi G_{\alpha_1} \cup \cdots \cup \pi G_{\alpha_n}$ is a neighborhood of A; let U be an open subset of $\pi G_{\alpha_1} \cup \cdots \cup \pi G_{\alpha_n}$ containing A. Now by looking at the preimages of U and $\bigcup_{i=1}^n \pi G_{\alpha_i}$ in X, we get that

$$A \subset \pi^{-1}(U) \subset \pi^{-1}(\bigcup_{i=1}^{n} \pi G_{\alpha_i}) = G_{\alpha_1} \cup \cdots \cup G_{\alpha_n} \cup A.$$

Since $\pi^{-1}(U)$ is open, then the above expression implies $A \subset Int(G_{\alpha_1} \cup \cdots \cup G_{\alpha_n} \cup A)$. But since all of the G_α are open, then $G_{\alpha_1} \cup \cdots \cup G_{\alpha_n} \cup A$ is open. Therefore, the first property is necessary.

To see that the second property is necessary, assume that A is not closed and U is any open neighborhood of a fixed $x \in \overline{A} - A$ in X. Since U is an open cover of $\pi^{-1}(\pi(x)) = x$, then by Theorem 4.13, πU is a neighborhood of x; let V be an open subset of πU that contains x. Then by looking at the preimages of V and πU, we see (using that U intersects A nontrivially) that

$$A \subset \pi^{-1}(V) \subset \pi^{-1}(\pi U) = U \cup A.$$

But since $\pi^{-1}(V)$ is open, then $A \subset Int(U \cup A)$, i.e. $U \cup A$ is open. Therefore, the second condition is necessary.

Second let's assume the two conditions hold. We will show π is a universal quotient map by checking that the conditions of Theorem 4.13 hold in all three locations in X/A (i.e. for (i) $x = A$, (ii) $x \in X - \overline{A}$, and (iii) $x \in \overline{A} - A$).

(i) For $A \in X/A$, take any open cover $\{G_\alpha\}_{\alpha\in\Lambda}$ of A in X. If A is open in X, then $\{A\}$ is open in X/A and hence every πG_α is a neighborhood. If A is not open, let Γ be the finite portion of Λ that property 1 guarantees exists, i.e. $A \cup \left(\bigcup_{i\in\Gamma} G_{\alpha_i}\right)$ is open in X and each G_{α_i} intersects A nontrivially. This implies that $\bigcup_{i\in\Gamma} \pi G_{\alpha_i}$ is an open neighborhood of A in X/A (since its preimage is $A \cup \left(\bigcup_{i\in\Gamma} G_{\alpha_i}\right)$).

(ii) Any $x \in X - \overline{A}$ has an open neighborhood $U_x \subset X - \overline{A}$. Notice that π is a homeomorphism on $X - \overline{A}$. Thus for any such x and any open cover W of $\pi^{-1}(x) = x$ in X, πW is a neighborhood of x because the open neighborhood (in X/A) $U_x \cap W$ is contained in πW.

(iii) If A is closed, then this is trivial so assume that A is not closed and let $x \in \overline{A} - A$. For any open cover W of $\pi^{-1}(x) = x$ in X, $\pi^{-1}(\pi W) = W \cup A$, which is open in X by condition 2. Thus πW is an open neighborhood of x in X/A.

Therefore, our two conditions ensure that π satisfies Day and Kelly's universal quotient map condition. □

Corollary 4.22 now gives us a way to produce more examples of sieves in the canonical topology:

Example 4.23. Every quotient of a Hausdorff space by a compact subspace is universal. For example, $\pi\colon D^n \to S^n$ (where $S^n = D^n/\partial D^n$) generates a universal colim sieve.

Example 4.24. If A is closed, then $S = \langle\{X \to X/A\}\rangle$ is always a colim sieve. Moreover, it is universal if and only if ∂A is compact. For example, this tells us $\langle\{\mathbb{R} \to \mathbb{R}/[0,\infty)\}\rangle$ is in the canonical topology and reaffirms that $\langle\{\mathbb{R} \to \mathbb{R}/\mathbb{Z}\}\rangle$ is not.

5. Universal Colim Sieves in the Category of R-modules

The category of R-modules does not satisfy the assumptions of Theorem 2.8 or Theorem 2.9. Indeed, coproducts and pullbacks of R-modules do not commute (for example, let $\mathbb{Z}_{(a,b)}$ denote the domain of $\mathbb{Z} \to \mathbb{Z}^2$, $1 \mapsto (a,b)$, then we see that $(\mathbb{Z}_{(1,0)} \oplus \mathbb{Z}_{(0,1)}) \times_{\mathbb{Z}^2} \mathbb{Z}_{(1,1)} \cong \mathbb{Z}$ but $(\mathbb{Z}_{(1,0)} \times_{\mathbb{Z}^2} \mathbb{Z}_{(1,1)}) \oplus (\mathbb{Z}_{(0,1)} \times_{\mathbb{Z}^2} \mathbb{Z}_{(1,1)}) \cong 0$). Thus we do not have basis and presentation results. Instead, we have some smaller results, reductions and examples.

Notation 5.1. Let R be a commutative ring with identity. We will use R-**Mod** for the category of R-modules and **Ab** for the category of abelian groups.

We start with some basic results.

Corollary 5.2. *Any sieve containing a universal effective epimorphism (e.g. a surjection in R-**Mod** or in **Sets**) is a universal colim sieve.*

Proof. This is an immediate consequence of Theorem 2.5 and Corollary 2.4. $\square$

Lemma 5.3. *In R-**Mod**, if a sieve S on X can be generated by at most two morphisms, then the canonical map $c\colon \varinjlim_S U \to X$ is an injection.*

Proof. Suppose $S = \langle\{f\colon Y \to X, g\colon Z \to X\}\rangle$ and $c(x) = 0$. Since every map in S either factors through f or g, then x, as an element of $\bigoplus_{A\to X\in S} A$, is really an element $(y, z) \in Y \oplus Z$ in the colimit. So $c(x) = 0$ implies that $y + z = 0$ in X, i.e. $(y, -z) \in Y \times_X Z$. Thus $y \in Y$ gets identified with $-z \in Z$ in the colimit; hence $(y, z) = (0, z - z) = 0$ in the colimit. Therefore, $x = 0$ in the colimit and the map c is an injection.

Using the fact that $\langle\{A_i \to X\}_\alpha\rangle = \langle\{A_i \to X\}_\alpha \cup \{Z \xrightarrow{0} X\}\rangle$, we can say that any sieve generated by one morphism is also generated by two morphsims. This completes the proof. $\square$

Proposition 5.4. *In R-**Mod**, let*

$$S = \langle\{f\colon Y \to X\}\rangle \qquad \textit{and} \qquad T = \langle\{g\colon U \to X, h\colon V \to X\}\rangle$$

be sieves on X. Then

1. *S is a universal colim sieve if and only if f is a surjection.*
2. *T is a colim sieve if and only if $g \oplus h\colon U \oplus V \to X$ is a surjection.*

Proof. For part 2, Lemma 5.3 tells us that we only need to worry about the surjectivity of $\varinjlim_T U \to X$ but this is exactly what the above condition is.

For part 1, Lemma 5.3 and Lemma 2.1 tell us that we only need worry about the surjectivity of $A \times_X Y \xrightarrow{\pi_1} A$ (the generator of k^*S) for every map $k\colon A \to X$. But $A \times_X Y = \{(a, y) \in A \times Y \mid k(a) = f(y)\}$. Hence π_1 is a surjection for every map k if and only if f is a surjection. $\square$

Lemma 5.5. *In R-**Mod**, suppose $S = \langle \{f_i \colon M_i \to R\}_{i \in I} \rangle$ is a sieve on R such that for every $i \in I$ there exists an $a_i \in R$ with $im(f_i) = a_i R$. If the ideal $(a_i \mid i \in I)$ equals R, then for every R-module homomorphism $g \colon N \to R$, the natural map $\varinjlim_{g^*S} U \to N$ is a surjection.*

Proof. By Proposition 2.2 it suffices to show that $\eta \colon \oplus_i M_i \times_R N \to N$ is a surjection. Let $\pi_i \colon M_i \times_R N \to N$ be the natural map. Fix $x \in N$. Then $a_i g(x) \in a_i R = im(f_i)$ and $a_i g(x) \in im(g)$. Thus $a_i \cdot x \in im(\pi_i) \subset N$ for all $i \in I$. Therefore, $x = 1_R \cdot x$ is in $\oplus_i im(\pi_i) = im(\eta)$ since R is a unital ring and $(a_i \mid i \in I) = R$. $\square$

Proposition 5.6. *Suppose $S = \langle \{f_1 \colon M_1 \to R, f_2 \colon M_2 \to R\} \rangle$ is a sieve on R such that $im(f_i) = a_i R$ for $i = 1, 2$. Then S is in the canonical topology on R-**Mod** if and only if $(a_1, a_2) = R$.*

Proof. If S is in the canonical topology, then S is a colim sieve and hence by Proposition 5.4, $a_1 R \oplus a_2 R = R$.

If $(a_1, a_2) = R$, then by Proposition 5.4, S is a colim sieve. The universality of S follows immediately from Lemma 2.1, Proposition 5.4 and Lemma 5.5. $\square$

Next we include two results that can help us identify when a sieve is not in the canonical topology.

Proposition 5.7. *Let R be any nonzero ring. Let $S = \langle \{f_i \colon A_i \to X\}_{i \in I} \rangle$ be any sieve on X for any nonzero R-module X. If there exists a nonzero $b \in X$ such that $span_R(b) \subset (X - \cup_I Im(f_i)) \cup \{0\}$, then S is not a universal colim sieve.*

Proof. Suppose such a $b \in X$ exists. Define $g \colon R \to X$ by $1 \to b$. Then $Im(g) \cap Im(f_i) = \{0\}$ for all i. Thus for all i, the pullback $R \times_X A_i = ker(g) \times ker(f_i)$ and the image of the natural map $R \times_X A_i \to R$ is $ker(g)$. In particular, $Im\,(\oplus_i R \times_X A_i \to R) = ker(g)$, which by construction is not R. Therefore, $\varinjlim_{g^*S} U \to R$ is not surjective and so g^*S not a colim sieve on R. $\square$

Proposition 5.8. *Let R be an infinite principal ideal domain. Let*

$$S = \langle \{g_i \colon R^n \hookrightarrow R^n\}_{i=1}^M \cup \{f_i \colon R^{m_i} \hookrightarrow R^n \mid m_i < n\}_{i=1}^N \rangle$$

be a sieve on R^n. If S is a universal colim sieve, then $g_1 \oplus \cdots \oplus g_M \colon R^{nM} \to R^n$ is a surjection.

Proof. Let $G = g_1 \oplus \cdots \oplus g_M$. Suppose that G is not a surjection. We will produce a map ϕ that shows S is not universal.

By a change of basis (which is allowable by Lemma 2.3) we may assume that $G = diag(d_1, d_2, \ldots, d_n)$ with $d_i | d_{i+1}$. Because G is not surjectve, then d_n is not a unit. Indeed, if d_n was a unit, then all of the d_i's would also be units and thus G would be surjective. By Lemma 5.9 below, there exists an $x \in R^{n-1}$ so that $span_R\{(x,1)\} \cap Im(f_i) = \{0\}$ for all $i = 1, \ldots, N$. Additionally, since d_n is not a unit, then $(x, 1) \notin Im(G)$.

Define $\phi \colon R \to R^n$ by $1 \mapsto (x, 1)$. We will show that ϕ^*S is not a colim sieve. First we will simplify the generating set of ϕ^*S. By the choice of x, the pullback module of R^{m_i} along ϕ is $\{0\}$ for all $i = 1, \ldots, N$. Therefore, we can write ϕ^*S as $\phi^*S = \langle\{\pi_i \colon R^n \times_{R^n} R \to R\}_{i=1}^{M}\rangle$ where the π_i are the pullbacks of the g_i along ϕ. Since $(x, 1) \notin Im(G)$ and we have the following commutative diagram

$$\begin{array}{ccc} \oplus_{i=1}^{M} R_i^n \times_{R^n} R & \xrightarrow{\oplus_{i=1}^{M}\pi_i} & R \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ \oplus_{i=1}^{M} R_i^n & \xrightarrow[G]{} & R^n \end{array}$$

then $1 \notin Im(\pi_1 \oplus \cdots \oplus \pi_M)$. Therefore, $\eta \colon \varinjlim_{\phi^*S} U \to R$ is not surjective; hence ϕ^*S is not a colim sieve. $\square$

Lastly, for completeness we include the linear algebra result referenced in Proposition 5.8.

Lemma 5.9. *Let R be an infinite principal ideal domain. For any finite collection $V_1, \ldots, V_N$ of submodules of R^n with dim$(V_i) < n$, there exists an $x \in R^{n-1}$ such that $span_R\{(x, 1)\} \cap V_i = \{0\}$ for all i.*

Proof. Let F be the quotient field of R. Let

$$W_i = V_i \otimes_R F.$$

We will use F^{n-1} to refer to the subspace $\{(a_1, \ldots, a_{n-1}, 0) \mid a_i \in F\}$ in F^n. For each $V_i \not\subset F^{n-1}$, fix an element $\nu_i \in V_i$ such that $\nu_i \notin F^{n-1}$ and write $\nu_i = (v_{i1}, \ldots, v_{in})$. Let $\nu_i^0 = (v_{i1}, \ldots, v_{i(n-1)}, 0)$. Lastly, for each $V_i \not\subset F^{n-1}$, define a vector space map $\phi_i \colon W_i \to F^{n-1}$ by $w = (w_1, \ldots, w_n) \mapsto w - \frac{w_n}{v_{in}} \nu_i$

Ideally, we will find an x such that $(x, 1) \notin W_i$ for all i. So first, let's see what kinds of $(z, 1)$ are in W_i by computing $\phi_i(z, 1)$.

$$\begin{aligned} \phi_i(z,1) &= (z,1) - \frac{1}{v_{in}} \nu_i \\ &= z - \frac{1}{v_{in}} \nu_i^0 \end{aligned}$$

Thus

$$z = \phi_i(z,1) + \frac{1}{v_{in}} \nu_i^0.$$

Therefore, if $(z, 1) \in W_i$, then $z = \phi_i(z, 1) + \frac{1}{v_{in}} \nu_i^0$. Based on this result, define $\Gamma_i = im(\phi_i) \oplus span_F\{\nu_i^0\}$. So $(z, 1) \in W_i$ implies $z \in \Gamma_i$.

For each index i exactly one of the following is true:

1. $W_i \subset F^{n-1}$,
2. $W_i \not\subset F^{n-1}$ and $\dim_F(\Gamma_i) < n - 1$,
3. $W_i \not\subset F^{n-1}$ and $\Gamma_i = F^{n-1}$.

For every index j in collection 1, every $x \in R^{n-1}$ satisfies the equation $span_R\{(x, 1)\} \cap V_j = \{0\}$. Thus when picking our x, we only need to consider the indices in collections 2 and 3.

For each index i in collection 2, Γ_i is a proper subspace of F^{n-1}. Since there are only finitely many Γ_i and F is an infinite field, then there exists a $y = (y_1, \ldots, y_{n-1})$ such that $y \neq 0$ and $span_F\{(y, 0)\} \cap \Gamma_i = \{0\}$ for all i in collection 2. By multiplying y by an appropriate $s \in F$ we can clear denominators and so we may assume that $y \in R^{n-1}$. In particular, for all $r \in R$, $ry \notin \Gamma_i$, which implies that $(ry, 1) \notin W_i$. Therefore, for all $r \in R$, $span_R\{(ry, 1)\} \cap V_i = \{0\}$ for all indices in collection 2.

Continuing with the y from the previous paragraph, we now consider the indices k in collection 3 and their corresponding Γ_k. In this situation, $(y, 0) \in \Gamma_k$, i.e. $y = \phi_k(z) + u_k \nu_k^0$ for some $z \in W_k$ and $u_k \in F$. Since R is an infinite ring and collection 3 contains finitely many indices k, we can pick a nonzero $\rho \in R$ such that for all k, $\rho u_k \in R$ and $\rho u_k \neq \frac{1}{v_{kn}}$. Thus $\rho y \neq \phi_k(a) + \frac{1}{v_{kn}}\nu_k^0$ for any $a \in W_k$, which implies that $(\rho y, 1) \notin W_k$. Therefore, $span_R\{(\rho y, 1)\} \cap V_k = \{0\}$ for all indices in collection 3.

We can take $x = \rho y$. $\square$

EXAMPLES

Here we include a few examples and non-examples of sieves in the canonical topology for various rings R.

Example 5.10. In the category of R-modules every surjective map generates a universal colim sieve (see Proposition 5.4). As more specific examples, the sieve $\langle\{\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \mid 1 \mapsto 1\}\rangle$ is in the canonical topology on **Ab**, and in R-**Mod** the sieve $\langle\{R^n \to R \mid (a_1, \ldots, a_n) \mapsto a_1\}\rangle$ is in the canonical topology.

Example 5.11. By Proposition 5.6, $\langle\{R \xrightarrow{a} R, R \xrightarrow{b} R\}\rangle$ is in the canonical topology if and only if $(a, b) = R$. As more specific examples, the sieve $\langle\{\mathbb{Z} \xrightarrow{2} \mathbb{Z}, \mathbb{Z} \xrightarrow{3} \mathbb{Z}\}\rangle$ is in the canonical topology on **Ab**; when the function $\cdot g(x)\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is the map $f(x) \mapsto (g \cdot f)(x)$, then the sieve $\langle\{C^\infty(\mathbb{R}) \xrightarrow{\cdot x} C^\infty(\mathbb{R}), C^\infty(\mathbb{R}) \xrightarrow{\cdot \sin(x)} C^\infty(\mathbb{R})\}\rangle$ is not in the canonical topology on $C^\infty(\mathbb{R})$-modules.

Example 5.12. The sieve $S = \langle\{R \xrightarrow{i_1} R^2, R \xrightarrow{i_2} R^2\}\rangle$ where $i_1(1) = (1, 0)$ and $i_2(1) = (0, 1)$ (in the category of R-modules for nontrivial R) is not in the canonical topology. By Proposition 5.4, S is clearly a colim sieve so to see that S is not universal consider the map $\Delta\colon R \to R^2$, $1 \mapsto (1, 1)$. Then for $k = 1, 2$, i_k pulled back along Δ yields the zero map $z\colon 0 \to R$. Hence Lemma 2.1 says $\Delta^*S = \langle\{z\colon 0 \to R\}\rangle$, which is clearly not a colim sieve.

Similarly $\langle\{R \xrightarrow{i_k} R^n \mid k = 1, \ldots, n\}\rangle$ is a colim sieve but is not in the canonical topology. (This is also a consequence of Proposition 5.7.)

Example 5.13. Let $S = \langle\{f_k\colon \mathbb{Q} \to \mathbb{Q}[t] \mid f_k(1) = 1{+}t{+}\cdots{+}t^k\}_{k=1}^{\infty}\rangle$ in the category of rational vector spaces. This S is not in the canonical topology. (This is a direct consequence of Proposition 5.7 using $b = t$.)

Example 5.14. Let F be an infinite field. In the category of F vector spaces, a sieve of the form $S = \langle \{F^{m_i} \hookrightarrow F^n \mid m_i \leq n\}_{i=1}^{M} \rangle$ is in the canonical topology if and only if $m_i = n$ for some i if and only if S contains an isomorphism. (This is a consequence of Proposition 5.8.)

Proposition 5.15. *Consider the diagram $B_1 \hookrightarrow B_2 \hookrightarrow B_3 \hookrightarrow \ldots$ made with only injective maps and the direct limit $B := \varinjlim B_n$ in R-**mod**. Let the maps $\iota_n \colon B_n \to B$ be the natural maps into the colimit. Then the sieve $\langle \{\iota_n \mid n \in \mathbb{N}\} \rangle$ is a universal colim sieve.*

Proof. Let $\Gamma \colon \mathbb{N} \to S$ by $n \mapsto \iota_n$. Notice that Γ is a final functor; this is easy to see since the injectivity of ι_n and the maps in our diagram imply that $B_i \times_B B_j \cong B_{min(i,j)}$. Thus $\varinjlim_S U$ exists and $\varinjlim_S U \cong \varinjlim_{\mathbb{N}} U\Gamma \cong B$. Therefore, S is a colim sieve.

To see that S is universal, let $f \colon X \to B$ and set $X_i := X \times_B B_i$. For each $n \in \mathbb{N}$, ι_n and $B_n \to B_{n+1}$ are both injective maps; this implies that the natural maps $X_n \to X_{n+1}$ and $X_n \to X$ are also injective maps since the pullback of an injection in R-**Mod** is an injection and $X_i \cong X_{i+1} \times_{B_{i+1}} B_i$. Additionally, it is an easy exercise to see that the direct limit $\varinjlim X_i$ is isomorphic to X. In other words, f^*S is the type of sieve described in the assumptions of this proposition and proved to be a colim sieve in the previous paragraph. □

Example 5.16. Take $B_n = \mathbb{R}^n$ and let $B_n \to B_{n+1}$ be the inclusion map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$. Use $\mathbb{R}^\infty$ to denote the direct limit. Then the above proposition shows that $\langle \{\mathbb{R}^n \hookrightarrow \mathbb{R}^\infty\}_{n \in \mathbb{N}} \rangle$ is in the canonical topology on the category of $\mathbb{R}$ vector spaces. (Compare this to Example 4.14.)

REDUCTIONS

In this part we prove some reductions that allow us to limit our view (of sieve generating sets and the maps universality must be checked over) to the non-full subcategory of free modules with injective maps when R is ‘nice.’ The first reduction will be reducing the types of sieves we need to look at:

Proposition 5.17 (Reduction 1)**.** *In R-**Mod**, let S be a sieve on X. Then the following are equivalent*

1. *S is a universal colim sieve*

2. *f^*S is a universal colim sieve for every surjection $f\colon Y \to X$*

3. *f^*S is a universal colim sieve for some surjection $f\colon Y \to X$*

Proof. It is obvious that 1 implies 2 and 2 implies 3, so it suffices to show 3 implies 1.

Assume f^*S is a universal colim sieve for some fixed surjection $f\colon Y \to X$. Set $T = \langle\{f\colon Y \to X\}\rangle$. By Proposition 5.4, T is a universal colim sieve since f is a surjection. We will now use T together with the Grothendieck topology's transitivity axiom to show that S is a universal colim sieve. Notice that S satisfies the hypotheses of this axiom with respect to T. Indeed, since every $g \in T$ factors as $f \circ k$ for some k, then $g^*S = (fk)^*S = k^*(f^*S)$, which implies that g^*S is a universal colim sieve (as f^*S is universal) for every $g \in T$. Therefore, by the transitivity axiom of a Grothendieck topology, S is a universal colim sieve. □

To rephrase our first reduction: S is a universal colim sieve on X if and only if f^*S is a universal colim on R^n where $f\colon R^n \to X$ is a surjection (note that n is not necessarily assumed to be finite). This reduction means that we can restrict our view to free modules (not necessarily finitely generated). Specifically, we only need to look at sieves on free modules and check the universality condition on free modules. Indeed, S is a universal colim sieve on X if and only if for all $g\colon Y \to X$, g^*S is a universal colim sieve on Y if and only if for all $g\colon Y \to X$, $(gf)^*S$ is a universal colim sieve on R^n for some surjection $f\colon R^n \to Y$.

Proposition 5.18 (Reduction 2)**.** *In R-**Mod** when R is a principal ideal domain, every sieve on R^n equals a sieve of the form*

$$\langle\{g_i\colon R^{m_i} \hookrightarrow R^n\colon m_i \leq n\}_{i\in I}\rangle$$

where the g_i are injections.

Proof. Let $S = \langle\{f_i\colon A_i \to R^n\}_{i\in I}\rangle$ be a sieve on R^n. Set

$$T = \langle\{g_i\colon Im(f_i) \to R^n\}_{i\in I}\rangle$$

where the g_i's are inclusion maps. Since R is a PID and $Im(f_i)$ is a submodule of R^n, then $Im(f_i) \cong R^{m_i}$ for some $m_i \leq n$. Thus T is of the desired form and we will show that $S = T$. First notice that $S \subset T$. To get that T is a subcollection of S, notice that $\tilde{f}_i\colon A_i \to Im(f_i)$ (i.e. f_i with a different codomain) is split because $\tilde{f}_i$ is a surjective map onto a projective module; call the splitting χ_i. Hence $g_i = g_i \circ \tilde{f}_i \circ \chi_i = f_i \circ \chi_i$ implies that $T \subset S$ and completes the proof. $\square$

To rephrase our second reduction: when talking about sieves on R^n, we only need to talk about sieves generated by injections of free modules. Thus we can restrict our view of sieve generating sets to the non-full subcategory of free modules with injective morphisms.

Our next reduction will also assume R is a principal ideal domain. In particular, fix n and a map $f\colon X \to R^n$ for some R-module X. Then since R is a PID, we may write

$$X \cong R^m \oplus K \quad \text{for some } m \leq n, \text{ where}$$

$$R^m \cong Im(f), \quad K = ker(f), \quad f = g + z \quad \text{with}$$

$$g\colon R^m \to R^n \text{ an injection and } z\colon K \to R^n \text{ the zero map.}$$

Proposition 5.19 (Reduction 3)**.** *Let R be a principal ideal domain, S be a sieve on R^n in R-**Mod** and $f\colon X \to R^n$. Then, using the set-up described in the previous paragraph,*

$$\varinjlim_{f^*S} U \cong \left(\varinjlim_{g^*S} U\right) \oplus \left(\varinjlim_{z^*S} U\right).$$

Moreover, z^*S is a universal colim sieve; hence f^*S is a colim sieve if and only if g^*S is a colim sieve.

Sketch of Proof. By Proposition 5.18, we may assume that S can be written in the form $S = \langle \{\eta_i\colon R^{p_i} \hookrightarrow R^n\colon p_i \leq n\}_{i \in I}\rangle$. Consider the diagrams $\mathcal{X}$, $\mathcal{R}$ and $\mathcal{K}$ defined as:

$$\mathcal{X} = \left(\begin{array}{c} \bigoplus_{i\in I}(R^{p_i} \times_{R^n} X) \times_X (R^{p_i} \times_{R^n} X) \\ \downarrow\downarrow \\ \bigoplus_{i\in I}(R^{p_i} \times_{R^n} X) \end{array}\right),$$

$$\mathcal{R} = \left(\begin{array}{c} \bigoplus_{i\in I}(R^{p_i} \times_{R^n} R^m) \times_{R^m} (R^{p_i} \times_{R^n} R^m) \\ \downarrow\downarrow \\ \bigoplus_{i\in I}(R^{p_i} \times_{R^n} R^m) \end{array} \right), \text{ and}$$

$$\mathcal{K} = \left(\begin{array}{c} \bigoplus_{i\in I}(R^{p_i} \times_{R^n} K) \times_{K} (R^{p_i} \times_{R^n} K) \\ \downarrow\downarrow \\ \bigoplus_{i\in I}(R^{p_i} \times_{R^n} K) \end{array} \right)$$

First we look at the objects of $\mathcal{X}$. Since each η_i is injective, then for all i

$$R^{p_i} \times_{R^n} X \cong (R^{p_i} \times_{R^n} R^m) \oplus (R^{p_i} \times_{R^n} K)$$

and for all i, q

$$\begin{aligned}&(R^{p_i} \times_{R^n} X) \times_X (R^{p_q} \times_{R^n} X) \\ &\quad \cong ((R^{p_i} \times_{R^n} R^m) \times_{R^m} (R^{p_q} \times_{R^n} R^m)) \oplus ((R^{p_i} \times_{R^n} K) \times_K (R^{p_q} \times_{R^n} K)).\end{aligned}$$

In other words, $\mathcal{X} \cong \mathcal{R} \oplus \mathcal{K}$. But since colimits "commute" with colimits, then $\mathrm{Coeq}(\mathcal{X}) \cong \mathrm{Coeq}(\mathcal{R}) \oplus \mathrm{Coeq}(\mathcal{K})$. Now by Lemma 2.1 and Proposition 2.2, the first part has been proven, i.e.

$$\varinjlim_{f^*S} U \cong \left(\varinjlim_{g^*S} U \right) \oplus \left(\varinjlim_{z^*S} U \right).$$

Next we notice that z^*S is a universal colim sieve. Indeed, since η_i is an injection and z is the zero map, it easily follows that $z^*S = \langle \{id\colon K \to K\} \rangle$.

To complete the proof, notice that we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Coeq}(\mathcal{X}) \arrow[d, bend right, "\chi"] \cong \mathrm{Coeq}(\mathcal{R}) \arrow[d, "\rho"] \oplus \mathrm{Coeq}(\mathcal{K}) \arrow[d, bend left, "\kappa"] \\ X \cong R^m \oplus K \end{tikzcd}$$

where the vertical maps are the obvious canonical maps. This $\chi = \rho \oplus \kappa$ is an isomorphism if and only if both ρ and κ are isomorphisms. We have already shown that κ is an isomorphism (as z^*S is a universal colim sieve), thus this diagram implies that χ is an isomorphism if and only if ρ is; hence f^*S is colim sieve if and only if g^*S is a colim sieve. $\square$

Lastly, we rephrase our third reduction:

Corollary 5.20. *When R is a PID, a sieve on R^n is a universal colim sieve if and only if f^*S is a colim sieve for every injection $f\colon R^m \to R^n$.*

All together our reductions basically allow us to work in the subcategory of free modules with injective morphisms instead of in R-**Mod**.

5.1 The Category of Abelian Groups

This section will be primarily made up of examples. Additionally, we include a characterization of sieves on $\mathbb{Z}$ and one result for sieves on larger free abelian groups.

Example 5.21. By Corollary 5.6, $\langle\{\mathbb{Z} \xrightarrow{\times a} \mathbb{Z}, \mathbb{Z} \xrightarrow{\times b} \mathbb{Z}\}\rangle$ is a universal colim sieve if and only if a and b are relatively prime.

Example 5.22. The sieve $S = \langle\{\mathbb{Z} \xrightarrow{\times 1} \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z}\}\rangle$ is a universal colim sieve on $\mathbb{Z}/4\mathbb{Z}$ by Corollary 5.2. Additionally, S is not monogenic, i.e. it cannot be written as a sieve generated by one morphism.

Example 5.23. Let $S = \langle\{g\colon \mathbb{Z}^n \hookrightarrow \mathbb{Z}^n\} \cup \{f_i\colon \mathbb{Z}^{m_i} \hookrightarrow \mathbb{Z}^n \mid m_i < n\}_{i=1}^N\rangle$ be a sieve on $\mathbb{Z}^n$. Then S is a universal colim sieve if and only if g is a surjection, i.e. g is an isomorphism. (This is a direct corollary of Proposition 5.8 and Corollary 5.2.)

Ideally, we would like to know a 'nice' basis for the canonical topology on **Ab**, like the bases in Section 4.1; to start moving towards this ideal, we look at the simplest free group, $\mathbb{Z}$. In Example 5.21 we see that a relative prime pair of numbers will generate a universal colim sieve; this is actually true in general, specifically:

Proposition 5.24. *Let $S = \langle\{\mathbb{Z} \xrightarrow{\times a_i} \mathbb{Z}\}_{i=1}^N\rangle$ be a sieve on $\mathbb{Z}$. Then S is a universal colim sieve if and only if gcd$(a_1, \ldots, a_N) = 1$.*

Proof. First assume that S is a universal colim sieve. In particular, the map $\varinjlim_S U \to \mathbb{Z}$ is a surjection, i.e. $\mathbb{Z}^N \to \mathbb{Z}$, $(x_1, \ldots, x_N) \mapsto a_1x_1 + \cdots + a_Nx_N$ is a surjection. Therefore, $(a_1, \ldots, a_N) = \mathbb{Z}$ and this proves the forward direction.

Now assume that $\gcd(a_1, \ldots, a_N) = 1$. We will break the proof that S is a universal colim sieve up into several pieces. First we will reduce the proof to showing that S is a colim sieve. By the reductions (Propositions 5.17, 5.18 and 5.19), universality only needs to be checked along maps of the form $f\colon \mathbb{Z} \xrightarrow{\times k} \mathbb{Z}$ where $k \neq 0$. Fix $k \neq 0$, i.e. fix f, and write $\mathbb{Z}_b$ for the domain of $\mathbb{Z} \xrightarrow{\times b} \mathbb{Z}$. By Lemma 2.1, $f^*S = \langle \{\pi_i\colon \mathbb{Z}_{a_i} \times_{\mathbb{Z}} \mathbb{Z}_k \to \mathbb{Z}_k\}_{i=1}^N \rangle$. Moreover, it is easy to see that the pullback $\mathbb{Z}_{a_i} \times_{\mathbb{Z}} \mathbb{Z}_k \cong \mathbb{Z}$ and π_i must be multiplication by $\frac{a_i}{\gcd(a_i,k)}$. Since $\gcd(a_1, \ldots, a_N)$ equals 1, then $\gcd\left(\frac{a_1}{\gcd(a_1,k)}, \ldots, \frac{a_N}{\gcd(a_N,k)}\right) = 1$ and hence f^*S has the same form as S. Specifically, any argument showing that S is a colim sieve will similarly show that f^*S is a colim sieve. Therefore, it suffices to show that S is a colim sieve.

To see that S is a colim sieve, i.e. to see that the map $\varinjlim_S U \to \mathbb{Z}$ induced by $a_1, \ldots, a_N$ is an isomorphism, let $\alpha = \frac{N(N-1)}{2}$ and notice that

$$\begin{aligned}
\varinjlim_S U &\cong \operatorname{Coeq}\left(\begin{array}{c} \oplus_{i=1}^{\alpha}\mathbb{Z} \\ \downarrow\downarrow \\ \oplus_{i=1}^{N}\mathbb{Z} \end{array}\right) \\
&\cong \text{Cokernel}\left(\phi\colon \mathbb{Z}^\alpha \to \mathbb{Z}^N\right)
\end{aligned}$$

for some map ϕ where the first isomorphism comes from Lemma 2.2 and the last isomorphism comes from the fact that we are working in an abelian category. Now this map ϕ happens to be the third map in the Taylor resolution of $\mathbb{Z}$, i.e. ϕ_1 in [J. Mermini, 2012]. We make two remarks about this previous sentence: (1) we will not prove that our ϕ is [J. Mermini, 2012]'s ϕ_1, although this is easy to observe, and (2) the Taylor resolution in [J. Mermini, 2012] is specifically for polynomial rings, not $\mathbb{Z}$, however, both the definition of the Taylor resolution and the proof that it is in fact a free resolution are analogous. Here is the end of the Taylor resolution:

$$\cdots \to \mathbb{Z}^\alpha \xrightarrow{\phi} \mathbb{Z}^N \xrightarrow{(a_1 \, \dots \, a_N)} \mathbb{Z} \to \mathbb{Z}/(a_1, \dots, a_N)\mathbb{Z} \to 0$$

Since gcd$(a_1, \dots, a_N) = 1$, then it follows that $(a_1 \ \dots \ a_N)$ is a surjection and $\mathbb{Z}/(a_1, \dots, a_N)\mathbb{Z} \cong 0$. Thus we obtain $0 \to Im(\phi) \to \mathbb{Z}^N \to \mathbb{Z} \to 0$, which is an exact sequence and hence implies that the cokernel of ϕ is $\mathbb{Z}$. Additionally, since $(a_1 \ \dots \ a_N)$ induced our map $\varinjlim_S U \to \mathbb{Z}$, then this short exact sequence also says that S is a colim sieve. $\square$

Because of Proposition 5.24, we can now easily determine when a sieve on $\mathbb{Z}$ is in the canonical topology and we can easily come up with examples; for example, $\langle\{\mathbb{Z} \xrightarrow{\times 15} \mathbb{Z}, \mathbb{Z} \xrightarrow{\times 10} \mathbb{Z}, \mathbb{Z} \xrightarrow{\times 12} \mathbb{Z}\}\rangle$ is in the canonical topology whereas the sieve $\langle\{\mathbb{Z} \xrightarrow{\times 15} \mathbb{Z}, \mathbb{Z} \xrightarrow{\times 50} \mathbb{Z}, \mathbb{Z} \xrightarrow{\times 20} \mathbb{Z}\}\rangle$ is not. One may hope for a similar outcome for sieves on $\mathbb{Z}^n$ when $n \geq 2$, however, the Taylor resolution used in the proof of Proposition 5.24 does not seem to generalize in a suitable manner. Instead, we have a proposition that may tell us when a potential sieve is not in the canonical topology.

Proposition 5.25. *Let $S = \langle\{\mathbb{Z}^n \xrightarrow{A_i} \mathbb{Z}^n\}_{i=1}^N\rangle$ where A_i is a diagonal matrix with $\det(A_i) \neq 0$. Then there exists a map $\beta \colon \mathbb{Z} \to \mathbb{Z}^n$ such that $\beta^* S$ is not a colim sieve if and only if gcd$(\det(A_1), \dots, \det(A_N)) \neq 1$.*

Proof. First we set up some notation: Let $A_i = diag(a_{1i}, \dots, a_{ni})$ and $\mathbb{Z}^n_i$ be the domain of A_i.

To prove the backward direction, suppose that gcd$(\det(A_1), \dots, \det(A_N))$ does not equal 1. We can rephrase the assumptions as $a_{ik} \neq 0$ for all k and there exists a prime q such that q divides the product $a_{1i} \dots a_{ni}$ for all i. Set β equal to the diagonal embedding, i.e. $1 \mapsto (1, \dots, 1)$. Then by Lemma 2.1, $\beta^* S = \langle\{f_i \colon \mathbb{Z}^n_i \times_{\mathbb{Z}^n} \mathbb{Z} \to \mathbb{Z}\}_{i=1}^N\rangle$. Let $k_i =$ lcm$(a_{1i}, \dots, a_{ni})$ and $\chi_i \colon \mathbb{Z} \to \mathbb{Z}^n$, $1 \mapsto \left(\frac{k_i}{a_{1i}}, \dots, \frac{k_i}{a_{ni}}\right)$, then

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\chi_i} & \mathbb{Z}^n \\ {\scriptstyle k_i}\big\downarrow & & \big\downarrow{\scriptstyle A_i} \\ \mathbb{Z} & \xrightarrow[\beta]{} & \mathbb{Z}^n \end{array}$$

is a pullback diagram. Moreover, the prime q divides k_i for all i since it divides $a_{1i}\dots a_{ni}$ for all i. Thus $\gcd(k_1,\dots,k_N)\neq 1$. Now by Proposition 5.24, we can see that $\beta^*S=\langle\{\mathbb{Z}\xrightarrow{\times k_i}\mathbb{Z}\}_{i=1}^N\rangle$ is not a universal colim sieve. In particular, the first part of the proof of Proposition 5.24 shows that β^*S is not a colim sieve.

To prove the forward direction, we will prove the contrapositive statement. So suppose that $\gcd(\det(A_1),\dots,\det(A_N))=1$. Let $\beta\colon\mathbb{Z}\to\mathbb{Z}^n$ be given as the matrix $\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}$. To see that $\beta^*S=\langle\{f_i\colon\mathbb{Z}_i^n\times_{\mathbb{Z}^n}\mathbb{Z}\to\mathbb{Z}\}_{i=1}^N\rangle$ is a colim sieve, notice that we have the pullback diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, "k_i"'] & \mathbb{Z}^n \arrow[d, "A_i"] \\ \mathbb{Z} \arrow[r, "\beta"'] & \mathbb{Z}^n \end{tikzcd}$$

where $k_i=\operatorname{lcm}\left(\frac{a_{1i}}{\gcd(a_{1i},b_1)},\dots,\frac{a_{ni}}{\gcd(a_{ni},b_n)}\right)$. Hence, k_i divides $\det(A_i)$. This implies that $\gcd(k_1,\dots,k_n)$ divides $\gcd(\det(A_1),\dots,\det(A_N))$ and hence equals 1. Now by Proposition 5.24, we can see that $\beta^*S=\langle\{\mathbb{Z}\xrightarrow{\times k_i}\mathbb{Z}\}_{i=1}^N\rangle$ is a universal colim sieve. $\square$

Example 5.26. Based on Proposition 5.25 we can automatically say that the sieve $\left\langle\left\{\begin{pmatrix}4&0\\0&14\end{pmatrix},\begin{pmatrix}21&0\\0&2\end{pmatrix},\begin{pmatrix}1&0\\0&49\end{pmatrix}\right\}\right\rangle$ on $\mathbb{Z}^2$ is not in the canonical topology because each matrix has a multiple of 7 somewhere on its diagonal.

Suppose, like in Proposition 5.25, $S=\langle\{\mathbb{Z}^n\xrightarrow{A_i}\mathbb{Z}^n\}_{i=1}^N\rangle$ where each A_i is a diagonal matrix and $\gcd(\det(A_1),\dots,\det(A_N))=1$. In order to determine if S is a universal colim sieve, we (only) need to check if f^*S is a colim sieve for all $f\colon\mathbb{Z}^m\hookrightarrow\mathbb{Z}^n$, $2\le m\le n$. However, this is still a fair amount of work and it would be nice if this process could be simplified further.

Now we finish this section with a few more examples. Note: we will not prove any assertions in these examples, however, they are all basic computations that can be checked using undergraduate linear algebra.

Example 5.27. The sieve $S_1 = \left\langle \left\{ \begin{pmatrix} 7 & 0 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 21 & 0 \\ 1 & 18 \end{pmatrix}, \begin{pmatrix} 24 & 0 \\ 6 & 5 \end{pmatrix} \right\} \right\rangle$ on $\mathbb{Z}^2$ is not in the canonical topology although it is a colim sieve. In particular, S_1 is not universal because f^*S_1 is not a colim sieve for $f\colon \mathbb{Z} \to \mathbb{Z}^2$, $f(1) = (1, 0)$.

If we take the generating set of S_1 and change the 1 in the first matrix to a 0, then we get the following example:

Example 5.28. The sieve $S_2 = \left\langle \left\{ \begin{pmatrix} 7 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 21 & 0 \\ 1 & 18 \end{pmatrix}, \begin{pmatrix} 24 & 0 \\ 6 & 5 \end{pmatrix} \right\} \right\rangle$ on $\mathbb{Z}^2$ is not a colim sieve since $\varinjlim_S U \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$. Therefore, S_2 is also not in the canonical topology.

Finally, if take the generating set of S_2 and change the 18 in the second matrix to a 9, then we get:

Example 5.29. The sieve $S_3 = \left\langle \left\{ \begin{pmatrix} 7 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 21 & 0 \\ 1 & 9 \end{pmatrix}, \begin{pmatrix} 24 & 0 \\ 6 & 5 \end{pmatrix} \right\} \right\rangle$ on $\mathbb{Z}^2$ is a colim sieve, however, whether or not this sieve is in the canonical topology is unknown.

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Cynthia Lester
The College of Wooster
1189 Beall Ave, Wooster, OH, USA 44691
clester@wooster.edu

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Mme Ehresmann, Faculté des Sciences, LAMFA.
33 rue Saint-Leu, F-80039 Amiens. France. ehres@u-picardie.fr

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