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SOMMAIRE

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

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THE UNIVERSAL OPERAD ACTING ON LOOP SPACES, AND GENERALISATIONS

Eugenia CHENG and Todd TRIMBLE

Résumé. Dans cet article, nous étudions l'opérade globulaire utilisée par Batanin pour définir l'ω-groupoïde fondamental d'un espace. Nous identifions une propriété universelle de cette opérade et nous construisons un cadre catégoriel général des opérades universelles agissant sur une structure donnée. Un exemple motivant est l'opérade universelle agissant sur les espaces de lacets. D'autres exemples comprennent des versions n-dimensionelles de l'ω-groupoïde fondamental—la version à homotopie près, la version enrichie dans les espaces topologiques, ou tout simplement la version tronquée. Identifier la propriété universelle de l'opérade de Batanin nous aide à trouver d'autres opérades convenables à reconnaître les ω-groupoïdes fondamentaux. Nous espérons que ces opérades non-universelles et plus petites nous permettent de démontrer que les ω-groupoïdes définis par les opérades globulaires modélisent les types d'homotopie.

Abstract. In this paper we analyse the globular operad used by Batanin to define the fundamental ω-groupoid of a space. We identify a universal property of this globular operad and give a general categorical framework for universal operads acting on structures. A motivating example is the universal operad acting on loop spaces. Other examples include n-dimensional versions of the fundamental ω-groupoid—up-to-homotopy, enriched in spaces, or simply truncated. Identifying the universal property of Batanin's operad helps us to find other suitable operads for recognising fundamental ω-groupoids. The hope is that these smaller, non-universal operads will enable a proof that

globular operadic ω-groupoids model homotopy types.
Keywords. Operad, globular operad, fundamental ω-groupoid, loop spaces, enriched categories, homotopy types.
Mathematics Subject Classification (2010). 18D50, 18A05, 55P15.

Contents

Introduction

One of the earliest motivations for studying higher-dimensional algebra was Grothendieck's suggestion of modelling homotopy types by "ω-groupoids" [8]. There have been many different approaches to this, as there are many different approaches to defining ω-groupoids. One approach is first to define ω-categories and then to pick out the ω-groupoids among them as those in which every element is "weakly invertible". This is in contrast to the "direct" approach in which non-invertible elements are never considered, for example with the notion of Kan complexes. This and other simplicial approaches (for example [17, 16, 15]) are "non-algebraic" in that they do not *specify* all the operations of the algebraic structure. Some other ways of thinking about this are:

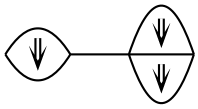
- they demand that composites *exist* rather than specifying them, thus
- they cannot be defined as algebras for a monad or models for an algebraic theory, and
- they can be thought of as nerves of algebraic structures, rather than algebraic structures themselves.

A different family of approaches seeks to specify the operations of an ω-groupoid explicitly, using operads. Operads achieved great success in the study of iterated loop spaces, as a tool for parametrising multiplication of loops [13]. This multiplication is associative and unital only up to homotopy; a similar phenomenon occurs for ω-groupoids but is generalised to all types of composition at every dimension.

One reason that operads are so efficacious for the study of loop spaces is that we can pick combinatorially convenient operads for different situations; the theory tells us how the resulting structures on loop spaces are equivalent. The operads that are useful in practice (for example the little disks operad) often do not have good universal properties, as the universal ones are much too large for practical use.

By contrast, Category Theory tends to seek objects with nice universal properties. One of the aims of this paper is to show that the operads used by Trimble [18] and Batanin [1] in their definitions of n-category have a nice universal property. We take the view that the main purpose of identifying this universal property is to help us find smaller, non-universal operads for practical use. One such "practical use" is the modelling of homotopy types.

There are (at least) two ways of using operads in higher-dimensional algebra. Trimble proceeds inductively, using a classical operad at each dimension. Batanin on the other hand parametrises all dimensions at once, using a more general form of operad called "globular operad", in which the arities of operations are no longer just natural numbers but "globular pasting diagrams" such as



Note that although Trimble's definition, being inductive, can *a priori* only achieve finite n dimensions, a coinductive argument may be used to provide the ω-dimensional version [7].

In both cases, general ω-categories are defined, with the ω-groupoids being identified among them afterwards. This is analogous to the fact that while certain operads may be used to recognise loop spaces, it is not the case that all the algebras for such operads are loop spaces—only those among them that are group-like.

Batanin's definition, or at least, the variant we shall use, says *an ω-category is a globular set equipped with the structure of an algebra for any contractible globular operad*. Then an ω-groupoid is an ω-category in which every cell is weakly invertible (we will recall the precise definitions in Section 4.1). Thus to give the fundamental ω-groupoid of a space X we must

1. identify its underlying globular set UX,
2. find a contractible globular operad that acts on UX, and
3. show that every cell is weakly invertible.

Step (1) is straightforward—the n-cells of UX are found essentially by mapping the topological n-ball B^n into X (with a little care over sources and targets).

Batanin achieves (2) by identifying a particular globular operad K that acts naturally on the underlying globular set of any space. Note that "acting naturally" here means two things—the action is canonical, but also, more technically, the action is natural in X.

Essentially, given an n-pasting diagram α, the operations of K of arity α are the boundary-preserving maps from B^n to the geometric realisation of α, where "boundary-preserving" must be carefully interpreted to take into account all dimensions of boundary. The following facts are then immediately true.

1. For any space X, UX is a K-algebra.

2. Any K-algebra is an ω-category but not necessarily an ω-groupoid.

3. There might be ω-categories that are not K-algebras.

Batanin further shows that for any space X, UX is a K-algebra in which every cell is weakly invertible (in a sense to be made precise). That is, it is an ω-groupoid.

Crucially, there are other globular operads that act naturally on all globular sets UX (that is, naturally in X), and we will prove the following result as an instance of the main theorem.

Theorem 1. *A natural action of a globular operad P on underlying globular sets UX is precisely given by a map $P \longrightarrow K$ of globular operads.*

This result exhibits the action of K as *universal* (in fact terminal) among such operad actions. Another way of saying this is: "Any such natural action factors uniquely through the canonical action of K."

Note that U extends to a functor $\mathsf{Top} \longrightarrow \mathsf{GSet}$ (where we write GSet for the category of globular sets and their morphisms), and the naturality of the actions in question means that in effect we should think of our operads as acting on the *functor* U. In fact we prove the universality result in general for suitable functors $U\colon \mathcal{S} \longrightarrow \mathcal{G}$. Other examples are as follows; here n-GSet denotes the category of n-dimensional globular sets.

1. Loop spaces, using the functor

$$\begin{array}{rccc} \Omega : & \mathsf{Top}_* & \longrightarrow & \mathsf{Top} \\ & X & \longmapsto & \mathsf{Top}_*(S^1, X) \end{array}$$

2. Fundamental n-groupoids, given by the functor

$$\Pi_n : \mathsf{Top} \longrightarrow n\text{-}\mathsf{GSet}$$

which agrees with U at all dimensions less than n, and takes homotopy classes at dimension n (see Section 4.3 for a precise definition).

3. The "incoherent" version of Π_n, a functor

$$U_n : \mathsf{Top} \longrightarrow n\text{-}\mathsf{GSet}$$

which simply truncates UX to n dimensions.

4. The “path space” version of Π_n, a functor
$$\mathcal{P}_n : \mathsf{Top} \longrightarrow \mathsf{Top}\text{-}n\text{-}\mathsf{Gph}.$$
Here Top-n-Gph denotes the category of “n-graphs enriched in Top”; we will treat these as n-graphs in Top where for all $k < n$ the space of k-cells is set-like. Then $\mathcal{P}_n$ agrees with U (and Π_n at all dimensions less than n); for $k = n$ we have a *space* of k-cells of arity α, given by the *space* of boundary-preserving maps from B^k to the geometric realisation of α, where at lower dimensions we took the *set* of such maps.

In each case we have a cartesian monad T on $\mathcal{G}$ giving us a pertinent notion of T-operad, and the main theorem gives us a universal T-operad E_U acting on the functor in question. In (1), T is the free topological monoid monad (giving rise to classical operads in Top); for (2) and (3) we use the free strict n-category monad, and for (4) we use the free topologically-enriched n-category monad.

Note that symmetric operads do not fit into this framework as the free commutative monoid monad is not cartesian; it is however *weakly* cartesian, so symmetric operads do fit into the more general framework of Weber [19]. A direct examination of symmetric operads acting on loop spaces yields a universal symmetric operad analogous to the non-symmetric one, suggesting that the main theorem could be extended to Weber's weakly cartesian framework. However this is beyond the scope of this work.

Once we have identified this simple universal property, we have an obvious method of finding smaller non-universal operads for the given purpose. That is, having constructed the universal operad E_U we just have to look for any T-operad P equipped with an operad map $P \longrightarrow E_U$.

In the case of globular operads we can make use of the work of [4] in which we prove that every Trimble n-category is a Batanin n-category. Part of the proof produces a functor

$$\left\{\begin{array}{l}\text{Classical operads}\\ \text{acting on path spaces}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{Globular operads}\\ \text{acting on } \omega\text{-path spaces}\end{array}\right\}$$

Even the universal operad on the left yields a non-universal operad on the right, and applying the functor to non-universal operads on the left gives further non-universal examples on the right. We will discuss this in Section 4.4.

The structure of this paper is as follows. In Section 1 we prove the main universality theorem in sufficient generality to cover the several key examples to which the remainder of the paper is devoted. In Section 2 we prove some technical results needed to address sizes issues when constructing internal homs; we are mostly working in categories of functors between large categories, the construction of internal homs takes some care. In Section 3 we discuss operads acting on loop spaces. Among other things this serves to emphasise the unwieldy nature of the universal operad in question, and the importance of finding non-universal ones for calculations, as is done in the theory of loop spaces. In Section 4 we discuss the motivating example, globular operads for defining the fundamental ω-groupoids of spaces, together with the various n-dimensional versions described above. We end with a brief discussion of future work.

Note for experts

Experts who wish to read the paper quickly might wish to proceed directly as follows.

1. The definition of E_U, the universal operad acting on a functor U is given in Definition 1.17.

2. The main theorem, giving the universal property of E_U, is Theorem 1.21.

3. The technical theorem addressing size issues is Theorem 2.4.

4. The loop space example is given in Theorem 3.3.

5. The fundamental ω-groupoid example is given in Theorem 4.8.

Terminology and notation

1. Top will denote a category of topological spaces that is complete, cocomplete and cartesian closed, for example the category of compactly generated weakly Hausdorff spaces.

2. $\mathbb{N}$ will denote the natural numbers including 0.

3. By "classical operad" we will always mean non-symmetric operad.

4. We will use the equivalent categories n-GSet and n-Gph more-or-less interchangeably, although technically the former is defined as a presheaf category and the latter by iterated enrichment.

5. We will write T-operads as their underlying collection $(P \longrightarrow T1)$ or as their associated monad P. In other work we refer to these as "$(\mathcal{E}, T)$-operads".

6. We will write n-Pd for the set of n-dimensional globular pasting diagrams (that is, the n-cells of the free strict ω-category on the terminal globular set), and Pd for the set of all globular pasting diagrams.

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1. The main theorem

In this section we will prove the main theorem, exhibiting a universal T-operad acting on a given functor U. There is a slight subtlety involved to ensure that our framework can support all the examples we have in mind, as listed in the introduction. We will need to fix some suitable categories and functors:

- a category $\mathcal{S}$ with initial object; in our examples this will be Top or Top_*;
- $\mathcal{G}$ a cartesian category, for example GSet, n-GSet or Top;
- a functor $U\colon \mathcal{S} \longrightarrow \mathcal{G}$, for example the loop space functor or ω-path space functor;
- a cartesian monad T on $\mathcal{G}$, typically the free ω-category monad or n-dimensional version.

Finally we require that in the slice category $[\mathcal{S}, \mathcal{G}]/\Delta_{T1}$ a particular internal hom defining the endomorphism operad on U exists, where Δ_{T1} denotes the constant functor at $T1$. Some of the results are stated by Leinster for $\mathcal{G}$ a presheaf category, or slightly weaker, for $[\mathcal{S}, \mathcal{G}]$ locally cartesian closed. (Recall that a category $\mathcal{C}$ is called *cartesian* if it has all pullbacks, and *locally cartesian closed* if for all $X \in \mathcal{C}$ the slice category $\mathcal{C}/X$ is cartesian closed.) In fact, either of these requirements is excessive for us, both in the sense of being, abstractly, not necessary and in the sense of excluding the examples we have in mind. We do not need every slice to be cartesian closed, only the one above; in fact we don't even need this slice to be cartesian closed as we are only interested in one particular internal hom, the one defining the endomorphism operad on U. In our examples some care is needed about internal homs because of size issues; this technical issue is resolved in Section 2, and requires the following further conditions:

- The monad $T\colon \mathcal{G} \longrightarrow \mathcal{G}$ is not only cartesian but familially representable (as happens when T is polynomial).
- The functor $U\colon \mathcal{S} \longrightarrow \mathcal{G}$ is a right adjoint from a cocomplete category $\mathcal{S}$ to a presheaf category $\mathcal{G}$, i.e., to a free cocompletion of a small category.

When appropriate, these assumptions are adjusted to fit within an enriched category context; in fact our examples will only be enriched in Set or Top, which helps the technical details go through without many changes.

Our range of examples is summed up in Table 1; for the full definitions see Sections 3, 4.2 and 4.3 for loop spaces, ω-path spaces, and finite-dimensional cases respectively.

1.1 Background

We begin with some background theory which is found in [12], but is simplified here by the fact that we are only considering operads, whereas Leinster provides the more general theory for multicategories.

The content of the following definitions is that, given a cartesian monad S on a cartesian category $\mathcal{E}$, there is a monoidal structure on the category $\mathcal{E}/S1$ of "S-collections", and an S-operad is a monoid in this monoidal category. Moreover, under, suitable conditions $\mathcal{E}$ is enriched in $\mathcal{E}/S1$ and tensored over it, enabling us to define endomorphism operads and use them to express algebra actions.

Table 1: Examples

	U	$\mathcal{S}$	$\mathcal{G}$	T
loop spaces	Ω	Top_*	Top	free Top-monoid
ω-path spaces	U	Top	GSet	free ω-category
n-truncated path spaces	U_n	Top	n-Gph	free n-category
n-homotopy path spaces	Π_n	Top	n-Gph	free n-category
n-topological path spaces	$\mathcal{P}_n$	Top	Top-n-Gph	free Top-n-category

In all that follows, $\mathcal{E}$ is a cartesian category and S is a cartesian monad on it, that is, the functor part preserves pullbacks and the naturality squares for the unit and multiplication are pullbacks. In our examples $\mathcal{E}$ will either be $\mathcal{G}$ or $[\mathcal{S}, \mathcal{G}]$, and S will be the monad T on $\mathcal{G}$ or the induced monad T_* on $[\mathcal{S}, \mathcal{G}]$ respectively.

Definition 1.1. The category of *S-collections* is the slice category $\mathcal{E}/S1$. There is a monoidal structure on $\mathcal{E}/S1$ given as follows. $\begin{array}{c}A\\ \downarrow{\scriptstyle p}\\ S1\end{array} \otimes \begin{array}{c}B\\ \downarrow{\scriptstyle q}\\ S1\end{array}$ is the left-hand edge of the diagram:

$$\begin{tikzcd}[column sep=small]
 & & & \cdot \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\vee"] & \\
 & & SA \arrow[dl, "Sp"'] \arrow[dr, "S!"'] & & B \arrow[dl, "Sq"] \\
 & S^2 1 \arrow[dl, "\mu_1"'] & & S1 & \\
S1 & & & &
\end{tikzcd}$$

Note that we will sometimes write a collection $(P \longrightarrow S1)$ simply as P to simplify the notation. We will sometimes write the tensor product as $(A \otimes B \longrightarrow S1)$ if there is no danger of ambiguity.

Definition 1.2. An *S-operad* is a monoid in the monoidal category $\mathcal{E}/S1$. A morphism of S-operads is a monoid map. S-operads and their morphisms form a category S-Opd.

The following hom and tensor structures are given by the special case $E = 1$ of [12, Prop. 6.4.1].

Definition 1.3. Let S be a cartesian monad on a cartesian category $\mathcal{E}$, where $\mathcal{E}/S1$ is cartesian closed.

1. Given $\begin{array}{c} P \\ \downarrow \\ S1 \end{array} \in \mathcal{E}/S1$ and $A \in \mathcal{E}$ we define $\begin{array}{c} P \\ \downarrow \\ S1 \end{array} \odot A \in \mathcal{E}$ as the vertex of the following pullback:

$$\begin{tikzcd}[column sep=small] & \cdot \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ P \arrow[dr] & & SA \arrow[dl, "S!"] \\ & S1 & \end{tikzcd}$$

This assignation on objects extends to a functor $\mathcal{E}/T1 \times \mathcal{E} \longrightarrow \mathcal{E}$.

2. Given $A, B \in \mathcal{E}$ we define

$$\mathsf{Hom}\,(A,B) = \left[\begin{array}{cc} SA & S1 \times B \\ \downarrow{\scriptstyle S!} \;, & \downarrow{\scriptstyle \pi_1} \\ S1 & S1 \end{array} \right] \in \mathcal{E}/S1$$

where the square brackets denote the exponential in $\mathcal{E}/S1$ where it exists, and π_1 denotes projection onto the first component. This assignation on objects extends to a functor $\mathcal{E}^{\mathrm{op}} \times \mathcal{E} \longrightarrow \mathcal{E}/T1$.

Proposition 1.4. *(Leinster) There is an isomorphism*

$$\mathcal{E} \left(\begin{array}{c} P \\ \downarrow \\ S1 \end{array} \odot A \,,\; B \right) \;\cong\; \mathcal{E}/S1 \left(\begin{array}{c} P \\ \downarrow \\ S1 \end{array} ,\; \mathsf{Hom}\,(A,B) \right)$$

natural in $\begin{array}{c} P \\ \downarrow \\ S1 \end{array}$, *$A$ and B.*

Remark 1.5. Leinster demands that $\mathcal{E}$ be locally cartesian closed but we see that this is excessive for our construction—it does not hold in our examples, but we only need one particular hom in one particular slice to exist.

Leinster uses this result to define the notion of *endomorphism S-operad*; again he does this for S-multicategories, but we only need the operad case.

Proposition 1.6. [12, Proposition 6.4.2] *Let S be a cartesian monad on a cartesian category $\mathcal{E}$. Then given any object $A \in \mathcal{E}$, if the S-collection*

$$\mathsf{End}\,(A) = \mathsf{Hom}\,(A, A)$$

exists then it naturally has the structure of an S-operad.

Definition 1.7. We call End (A) the *endomorphism operad* of A.

Definition 1.8. An *algebra* for an S-operad P is given by an object $A \in \mathcal{E}$ together with a map

$$\begin{array}{c} P \\ \downarrow \\ S1 \end{array} \odot A \longrightarrow A$$

compatible with the operad structure of P; equivalently it is an algebra for the associated monad.

We can equivalently express this using the endomorphism operad.

Proposition 1.9 (Leinster)**.** *let P be an S-operad. If the endomorphism operad on an object $A \in \mathcal{E}$ exists then a P-algebra structure on A is an operad map*

$$\begin{array}{c} P \\ \downarrow \\ S1 \end{array} \longrightarrow \mathsf{End}\,(A).$$

This duality will play a key part in our proof of the main theorem.

1.2 Operad actions

We now seek to abstract the notion of an operad acting on loop spaces, path spaces, or ω-path spaces.

Definition 1.10. Let T be a cartesian monad on a cartesian category $\mathcal{G}$, let P be a T-operad, and let $U\colon \mathcal{S} \longrightarrow \mathcal{G}$ be a functor. An *action of P on U-objects* is given by, for all $X \in \mathcal{S}$ a morphism

$$\alpha_X\colon \begin{array}{c} P \\ \downarrow \\ T1 \end{array} \odot\, UX \longrightarrow UX$$

in $\mathcal{G}$ such that

1. operad compatibility: for all $X \in \mathcal{S}$, α_X exhibits UX as a P-algebra, and

2. naturality: the α_X are the components of a natural transformation

$$\begin{array}{c} P \\ \downarrow \\ T1 \end{array} \odot U(-) \longrightarrow U(-).$$

The aim of this work is to show that under the right hypotheses there is a universal T-operad E_U with such an action, characterised by the following universal property: an action of a T-operad P on U-objects is uniquely and completely determined by a T-operad map $P \longrightarrow E_U$.

The operad E_U will be defined as $\mathsf{ev}_\emptyset(\mathsf{End}\,(U))$ and the universality result holds whenever this definition makes sense. The next few results will build up towards making sense of this formula. Here $\emptyset$ is the initial object of $\mathcal{S}$ if it has one. Now we put $\mathcal{E} = [\mathcal{S}, \mathcal{G}]$ which is cartesian if $\mathcal{G}$ is cartesian, with pullbacks computed pointwise. We use the following monad on $\mathcal{E}$, induced by composition with T.

Lemma 1.11. *Given a cartesian monad T on a cartesian category $\mathcal{G}$ we have a cartesian monad $T_* = T \circ -$ on $[\mathcal{S}, \mathcal{G}]$. Explicitly, given $A\colon \mathcal{S} \longrightarrow \mathcal{G}$ we have*

$$\begin{array}{rccc} T_*A: & \mathcal{S} & \longrightarrow & \mathcal{G} \\ & X & \longmapsto & (TA)X \end{array}$$

Proof. The multiplication and unit for T_* are constructed from those of T and the naturality squares are pullbacks since pullbacks in the functor category are computed pointwise; the pointwise squares are all naturality squares for the multplication and unit of T hence are themselves pullbacks.

Preservation of pullbacks also comes from the fact that pullbacks in the functor category are computed pointwise; the pointwise squares we need to check are all pullbacks in $\mathcal{G}$ with T applied, so are pullbacks since T preserves pullbacks.

□

In our proof of the main theorem we are going to move back and forth between T-operads and T_*-operads using "constant" and "evaluation" functors. Before we define these, it is useful to make a few observations about the structure of the slice category $[\mathcal{S}, \mathcal{G}]/T_*1$.

Remarks 1.12.

1. Given $\begin{array}{c} A \\ \downarrow{\scriptstyle p} \\ T_*1 \end{array}, \begin{array}{c} B \\ \downarrow{\scriptstyle q} \\ T_*1 \end{array} \in [\mathcal{S}, \mathcal{G}]/T_*1$ their tensor product is given by a certain pullback in $[\mathcal{S}, \mathcal{G}]$. This is computed pointwise, and its component at $X \in \mathcal{S}$ is the left-hand edge of

$$\begin{tikzcd}
 & & & \cdot \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
 & & TAX \arrow[dl, "Tp_X"'] \arrow[dr, "T!"'] & & BX \arrow[dl, "q_X"] \\
 & T^2 1 \arrow[dl, "\mu_1"'] & & T1 & \\
T1 & & & &
\end{tikzcd}$$

which we see is the collection $\begin{array}{c} AX \\ \downarrow{\scriptstyle p_X} \\ T1 \end{array} \otimes \begin{array}{c} BX \\ \downarrow{\scriptstyle q_X} \\ T1 \end{array}$.

2. The tensor product $\begin{array}{c} A \\ \downarrow{\scriptstyle p} \\ T_*1 \end{array} \odot B$ is also given by a pullback; this time the component at $X \in \mathcal{S}$ is the vertex of the pullback

$$\begin{tikzcd}
 & \cdot \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
AX \arrow[dr, "p_X"'] & & TBX \arrow[dl, "T!"] \\
 & T1 &
\end{tikzcd}$$

which we see is the tensor product $\begin{array}{c} AX \\ \downarrow{\scriptstyle p_X} \\ T1 \end{array} \odot BX$.

We now define the "constant" and "evaluation" functors.

Lemma 1.13. *Let T be a cartesian monad on a cartesian category $\mathcal{G}$. We have a "constant" functor acting as follows:*

$$\begin{array}{ccc} T\text{-}\mathsf{Opd} & \longrightarrow & T_*\text{-}\mathsf{Opd} \\ P & & \Delta_P \\ \downarrow & \longmapsto & \downarrow \\ T1 & & T_*1 \end{array}$$

where $\Delta_P : \mathcal{S} \longrightarrow \mathcal{G}$ is the constant functor evaluating everywhere at P.

Proof. Given a T-operad P we also write its associated monad (on $\mathcal{G}$) as P. Suppose the operad P is given by the cartesian natural transformation

$$\alpha : P \Longrightarrow T.$$

Then we also have a cartesian natural transformation

$$\alpha_* : P_* \Longrightarrow T_*$$

i.e. a T_*-operad. The components of α_* are given by, for each $A \in [\mathcal{S}, \mathcal{G}]$ the natural transformation

$$\alpha A : PA \Longrightarrow TA.$$

It is easy to check that the naturality squares are pullbacks by examining them pointwise. $\square$

Remark 1.14. It is useful to note that the operad P_* has underlying T_*-collection $(P_*1 \longrightarrow T_*1)$. Here 1 is the terminal object in $[\mathcal{S}, \mathcal{G}]$, so it is the constant functor that sends every object to 1 (the terminal object in $\mathcal{G}$) and every morphism to the identity. Thus, evaluated at X the above collection is just $(P1 \longrightarrow T1)$, that is, the underlying collection of the operad P.

Thus the operad P_* can be thought of as Δ_P, the "constant operad" in $[\mathcal{S}, \mathcal{G}]$ that evaluates everywhere as P. We will sometimes write it in this way, and will often think of it in this way. Similarly it is useful to note that the functor T_*1 is the constant functor Δ_{T1}.

The following corollary is barely more than a matter of notation, but is useful for the proof of the main theorem.

Corollary 1.15. *A natural transformation*

$$\begin{array}{c} P \\ \downarrow \\ T1 \end{array} \odot U(-) \longrightarrow U(-)$$

is precisely a natural transformation

$$\begin{array}{c} P_* \\ \downarrow \\ T_*1 \end{array} \odot U \longrightarrow U.$$

We now define the "evaluation" functor; this is a straightforward generalisation of the evaluation functor used by Leinster [12, Section 9.2].

Definition 1.16. For any $X \in \mathcal{S}$ we have a functor

$$\mathsf{ev}_X : T_*\text{-}\mathsf{Opd} \longrightarrow T\text{-}\mathsf{Opd}$$

which we define in steps as follows.

1. We have a functor
$$\begin{array}{rccc} \mathsf{ev}_X : & [\mathcal{S}, \mathcal{G}] & \longrightarrow & \mathcal{G} \\ & F & \longmapsto & FX \end{array}$$

2. We know $\mathsf{ev}_X(T_*1) = T1$ for all X, so the above functor ev_X extends to a functor
$$\begin{array}{rccc} \mathsf{ev}_X : & [\mathcal{S}, \mathcal{G}]/T_*1 & \longrightarrow & \mathcal{G}/T1 \\ & \begin{array}{c} A \\ \downarrow \\ T_*1 \end{array} & \longmapsto & \begin{array}{c} AX \\ \downarrow \\ T1 \end{array} \end{array}$$

3. It is straightforward to check that this functor is monoidal (this is essentially the content of the first of Remarks 1.12) hence it maps operads to operads. That is, we get a functor
$$\mathsf{ev}_X : T_*\text{-}\mathsf{Opd} \longrightarrow T\text{-}\mathsf{Opd}.$$

We now define our putative universal operad, which is again a generalisation of the one used by Leinster in [12, Section 9.2]; however note that it will take a considerable amount of technical work to show that it exists in the cases of interest.

Definition 1.17. Let T be a cartesian monad on a cartesian category $\mathcal{G}$, $\mathcal{S}$ a category with an initial object $\emptyset$, and U a functor $\mathcal{S} \longrightarrow \mathcal{G}$. If the internal hom

$$\mathsf{Hom}(U,U) = \begin{bmatrix} T_*U & & T_*1 \times U \\ \downarrow & , & \downarrow \\ T_*1 & & T_*1 \end{bmatrix}$$

exists in $[\mathcal{S},\mathcal{G}]/T_*1$ then this gives the endomorphism T_*-operad $\mathsf{End}(U)$. We define the *universal T-operad acting on U-objects* to be the T-operad

$$E_U = \mathsf{ev}_\emptyset(\mathsf{End}(U)).$$

Note that evaluating the collection $\mathsf{Hom}(U,U)$ at $\emptyset$ gives us the underlying collection of E_U, and the operad structure is inherited.

The internal hom certainly exists if $[\mathcal{S},\mathcal{G}]$ is locally cartesian closed, or, more specifically, if the slice $[\mathcal{S},\mathcal{G}]/T_*1$ is cartesian closed. However, in some of our key examples this demand is too stringent, mostly for size reasons; we will address this in Section 2.

The rest of the section will be devoted to identifying the universal property of this operad; the universal property is not studied by Leinster. The next three lemmas show how we can use the initial object of $\mathcal{S}$ to simplify all our calculations. For the rest of this section, T is a cartesian monad on a cartesian category $\mathcal{G}$, P is a T-operad, and $\mathcal{S}$ is a category with an initial object $\emptyset$.

Lemma 1.18. *Let $\mathcal{S}$ have an initial object $\emptyset$ and let $\Delta_V : \mathcal{S} \longrightarrow \mathcal{G}$ denote the constant functor evaluating at $V \in \mathcal{G}$. Then a natural transformation*

$$\alpha\colon \Delta_V \Longrightarrow F$$

is completely determined by its component at $\emptyset$, which has the form

$$\alpha_\emptyset : V \longrightarrow F\emptyset.$$

Proof. Simple diagram chase: the component α_X is determined by the naturality square

$$\begin{tikzcd} V \arrow[r, "\alpha_\emptyset"] \arrow[d, equal] & F\emptyset \arrow[d, "F!"] \\ V \arrow[r, "\alpha_X"'] & FX. \end{tikzcd}$$

□

Lemma 1.19. *With notation as above, a map of T_*-collections*

$$\begin{tikzcd} \Delta_V \arrow[rr, "\alpha"] \arrow[dr] & & F \arrow[dl] \\ & T_*1 & \end{tikzcd}$$

is completely determined by a map of T-collections

$$\begin{tikzcd} V \arrow[rr, "\alpha_\emptyset"] \arrow[dr] & & F\emptyset \arrow[dl] \\ & T1. & \end{tikzcd}$$

Proof. *A priori* a map α of T_*-collections as shown is a natural transformation $\alpha\colon \Delta_V \Longrightarrow F$ making the triangle commute. By Lemma 1.18 the natural transformation α is completely determined by its component at $\emptyset$; it remains to check that the commutativity of the triangle at $\emptyset$ ensures the commutativity of every triangle

$$\begin{tikzcd} V \arrow[rr, "\alpha_X"] \arrow[dr] & & FX \arrow[dl] \\ & T1 & \end{tikzcd}$$

which is accomplished by a simple diagram chase. □

Lemma 1.20. *Let E be a T_*-operad. A map*

$$\begin{tikzcd} P_* \arrow[rr, "\beta"] \arrow[dr] & & E \arrow[dl] \\ & T_*1 & \end{tikzcd}$$

is a map of T_*-operads if and only if applying $\mathsf{ev}_\emptyset$ gives a map of T-operads

$$\begin{tikzcd} P \arrow[rr, "\beta_\emptyset"] \arrow[dr] & & \mathsf{ev}_\emptyset(E) \arrow[dl] \\ & T1. & \end{tikzcd}$$

Proof. We have to check that β respects the monoid structure if $\beta_\emptyset$ does. For multiplication, we check that the following diagram commutes.

$$\begin{tikzcd}
\begin{pmatrix} P_* & & P_* \\ \downarrow & \otimes & \downarrow \\ T_*1 & & T_*1 \end{pmatrix} \arrow[r, "\beta\otimes\beta"] \arrow[d, "\mu"'] & \begin{pmatrix} E & & E \\ \downarrow & \otimes & \downarrow \\ T_*1 & & T_*1 \end{pmatrix} \arrow[d, "\mu"] \\
\begin{pmatrix} P_* \\ \downarrow \\ T_*1 \end{pmatrix} \arrow[r, "\beta"'] & \begin{pmatrix} E \\ \downarrow \\ T_*1 \end{pmatrix}
\end{tikzcd}$$

In principle we have to check that this commutes at every $X \in \mathcal{S}$, which amounts to checking that the following diagram commutes (where by a slight abuse of notation we only write the "variable" part of each collection):

$$\begin{tikzcd}
P \otimes P \arrow[r, "\beta_X \otimes \beta_X"] \arrow[d, "\mu^P"'] & EX \otimes EX \arrow[d, "\mu^E_X"] \\
P \arrow[r, "\beta_X"'] & EX.
\end{tikzcd}$$

However, we see that commutativity of this diagram at any X follows from commutativity at $\emptyset$ using the following diagram:

$$\begin{tikzcd}
P \otimes P \arrow[rr, "\beta_X \otimes \beta_X"] \arrow[d, "\mu^P"'] \arrow[dr, "\beta_\emptyset \otimes \beta_\emptyset"] & & EX \otimes EX \arrow[d, "\mu^E_X"] \\
P \arrow[rr, "\beta_X"] \arrow[ddr, "\beta_\emptyset"'] & E\emptyset \otimes E\emptyset \arrow[ur, "E! \otimes E!"'] \arrow[dd, "\mu^E_\emptyset"] & EX \\
& & \\
& E\emptyset \arrow[uur, "E!"'] &
\end{tikzcd}$$

The top and bottom triangles are naturality "squares" for β (which are triangles as the source functor of β is constant) and the right hand square is a naturality square for μ. The diagrams for the unit work similarly. $\square$

1.3 The universal property

We are finally ready to prove the main theorem.

Theorem 1.21 (Main theorem). *Let T be a cartesian monad on a cartesian category $\mathcal{G}$, $\mathcal{S}$ a category with an initial object $\emptyset$, and U a functor $\mathcal{S} \longrightarrow \mathcal{G}$. Suppose further that $\mathsf{End}\,(U)$ exists, so we can define*

$$E_U = \mathsf{ev}_\emptyset(\mathsf{End}\,(U)).$$

Let P be a T-operad. Then an action of P on U-objects is precisely a map of T-operads $P \longrightarrow E_U$.

Proof. We write $\mathcal{E} = [\mathcal{S}, \mathcal{G}]$. An action of P on U-objects is by definition a natural transformation

$$\alpha : \begin{array}{c} P_* \\ \downarrow \\ T_*1 \end{array} \odot U \longrightarrow U$$

such that for all X, the component

$$\alpha_X : \begin{array}{c} P \\ \downarrow \\ T1 \end{array} \odot UX \longrightarrow UX$$

exhibits UX as a P-algebra.

By the tensor structure (Proposition 1.4), specifying such an α amounts to specifying a morphism

$$\overline{\alpha} : \begin{array}{c} P_* \\ \downarrow \\ T_*1 \end{array} \longrightarrow \mathsf{Hom}\,(U, U)$$

in $\mathcal{E}/T_*1$ or, writing it out more fully,

$$\begin{tikzcd} P_* \arrow[rr, "\bar{\alpha}"] \arrow[dr] & & \mathsf{Hom}\,(U,U) \arrow[dl] \\ & T_*1. & \end{tikzcd}$$

Now $P_* = \Delta_P$ is a constant functor, so by Lemma 1.19 the natural transformation $\overline{\alpha}$ is completely determined by the component at the initial object $\emptyset$, that is, a map of T-collections

$$\begin{tikzcd} P \arrow[rr, "\bar{\alpha}_\emptyset"] \arrow[dr] & & \mathsf{ev}_\emptyset(\mathsf{Hom}\,(U,U)) \arrow[dl] \\ & T1 & \end{tikzcd}$$

thus our natural transformation α is completely determined by a map of underlying T-collections

$$\overline{\alpha}_\emptyset : \begin{array}{c} P \\ \downarrow \\ T1 \end{array} \longrightarrow \mathsf{ev}_\emptyset(\mathsf{End}\,(U)) = E_U.$$

It remains to show that this is a map of operads if and only if for all $X \in \mathcal{S}$, α_X exhibits UX as a P-algebra. We proceed in steps, by proving that the following are equivalent.

1. For all $X \in \mathcal{S}$, $\alpha_X : \begin{array}{c} P \\ \downarrow \\ T1 \end{array} \odot UX \longrightarrow UX$ exhibits UX as a P-algebra.

2. $\alpha : \begin{array}{c} P_* \\ \downarrow \\ T_*1 \end{array} \odot U \longrightarrow U$ exhibits U as a P_*-algebra.

3. $\overline{\alpha} : \begin{array}{c} P_* \\ \downarrow \\ T_*1 \end{array} \longrightarrow \mathsf{End}\,(U)$ is an operad map.

4. $\overline{\alpha}_\emptyset : \begin{array}{c} P \\ \downarrow \\ T1 \end{array} \longrightarrow \mathsf{ev}_\emptyset(\mathsf{End}\,(U)) = E_U$ is an operad map.

- $1 \iff 2$ is Corollary 1.15.
- $2 \iff 3$ is Proposition 1.9.
- $3 \iff 4$ is Lemma 1.20.

□

Remark 1.22. In this proof it is tempting to use the fact that (1) is equivalent to the assertion that for all $X \in \mathcal{S}$,

$$\overline{\alpha_X} : \begin{array}{c} P \\ \downarrow \\ T1 \end{array} \longrightarrow \mathsf{End}\,(UX)$$

is an operad map; this is true but not helpful at this point, as

$$\mathsf{ev}_X(\mathsf{End}\,(U)) \neq \mathsf{End}\,(UX).$$

1.4 Topological version

The main theorem also holds in a topologically enriched version which we will use for several of our examples. The theorem is essentially the same, provided that the categories and functors involved are interpreted in an enriched sense.

As usual Top will denote a category of topological spaces that is complete, cocomplete and cartesian closed, for example the category of compactly generated weakly Hausdorff spaces. We will be considering $\mathcal{V}$-enriched categories where $\mathcal{V} = \mathsf{Top}$.

The starting point is that we now want the categories $\mathcal{S}$ and $\mathcal{G}$ to be $\mathcal{V}$-categories rather than just plain categories. In our examples, $\mathcal{S}$ will be Top or Top_* and $\mathcal{G}$ will be Top or $[\mathbb{G}^{\text{op}}, \mathsf{Top}]$. We also want the monad T on $\mathcal{G}$ to be enriched: as a functor, T is $\mathcal{V}$-enriched, and the multiplication and unit structures on T should be $\mathcal{V}$-natural transformations. And likewise, the requirement of familial representability of T now means that the functor T is to be an enriched coproduct of (enriched) representable functors.

In the case $\mathcal{V} = \mathsf{Top}$, such adjustments tend to be mild. For example, because the forgetful functor $\mathsf{hom}(1, -)\colon \mathsf{Top} \longrightarrow \mathsf{Set}$ is faithful, $\mathcal{V}$-natural transformations are, in this case, the same as ordinary natural transformations. And in this case, enrichment of functors is a property-like structure: an ordinary functor between the underlying categories of Top-categories, either is or isn't Top-enriched. That is, for a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ the functions $\mathcal{C}(X, Y) \longrightarrow \mathcal{D}(FX, FY)$ giving the action on morphisms either are or are not continuous, and that is the criterion for F to be enriched.

There is an *a priori* distinction between enriched (co)limits and ordinary (co)limits, but under mild assumptions on $\mathcal{S}$ and $\mathcal{G}$ (e.g., they are $\mathcal{V}$-tensored and cotensored; see below), the distinction is one we can ignore entirely. To set the context, we introduce some notation:

- If $\mathcal{C}$ is a $\mathcal{V}$-enriched category, write $\mathcal{C}_0$ for the underlying (plain) category of $\mathcal{C}$, obtained by applying $\mathcal{V}(1, -) : \mathcal{V} \longrightarrow \mathsf{Set}$ to the homs of $\mathcal{C}$ as objects in $\mathcal{V}$, with 1 the monoidal unit of $\mathcal{V}$.

- If D is an ordinary category, we write $\overline{D}$ for the free $\mathcal{V}$-enriched category generated by D. The objects of $\overline{D}$ are those of D, and the enriched homs of D as objects in $\mathcal{V}$ are defined by the formula
$$D(d, d') = \coprod_{f\colon d \longrightarrow d'} 1.$$
In the case $\mathcal{V} = \mathsf{Top}$, it simply means we interpret the hom-sets of D as discrete topological spaces, and eventually by abuse of notation we will write the resulting $\mathcal{V}$-category just as D.

- If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-categories, we write $[\mathcal{A}, \mathcal{B}]$ for the (plain) category of $\mathcal{V}$-functors and $\mathcal{V}$-transformations. We write $\mathsf{Cat}(D, E)$ for the category of functors and transformations between plain categories D, E. The freeness property of $\overline{D}$ then asserts
$$[\overline{D}, \mathcal{C}] \cong \mathsf{Cat}(D, \mathcal{C}_0)$$
for any $\mathcal{V}$-category $\mathcal{C}$.

- If in addition $\mathbb{A}$ is small, there is a canonical way of endowing $[\mathbb{A}, \mathcal{B}]$ with enriched structure, and we write $\mathcal{B}^{\mathbb{A}}$ for this $\mathcal{V}$-category.

Under this notation, for any diagram category D we intend to take (co)limits over, and for any enriched category $\mathcal{C}$, we have a $\mathcal{V}$-enriched $\mathcal{V}$-functor category
$$[\overline{D}, \mathcal{C}]$$
and we have a $\mathcal{V}$-functor $\overline{D} \longrightarrow \overline{1}$, which induces a " diagonal" $\mathcal{V}$-functor

$$\Delta_D = \left(\mathcal{C} \cong [\overline{1}, \mathcal{C}] \longrightarrow [\overline{D}, \mathcal{C}]\right).$$

If Δ_D has an enriched right adjoint $\mathcal{V}\text{-}\lim_D$ (a right adjoint in the 2-category $\mathcal{V}$-Cat), then we say that $\mathcal{C}$ has enriched D-limits. Or, if Δ_D has an enriched left adjoint, then we say that $\mathcal{C}$ has enriched D-colimits.

Enriched D-limits in $\mathcal{C}$, assuming they exist, *are* ordinary D-limits when viewed in $\mathcal{C}_0$. That is to say, if there is an enriched right adjoint

$$\mathcal{V}\text{-}\lim_D \colon [\overline{D}, \mathcal{C}] \longrightarrow \mathcal{C}$$

then applying the forgetful 2-functor $\mathcal{V}$-$\mathsf{Cat} \longrightarrow \mathsf{Cat}$ (which preserves adjunctions as all 2-functors do) we get an ordinary right adjoint in Cat,

$$[\overline{D}, \mathcal{C}]_0 \longrightarrow C_0 \quad \cong \quad \mathsf{Cat}(D, \mathcal{C}_0) \longrightarrow C_0$$

that is right adjoint to an ordinary diagonal functor.

The distinction between enriched D-limits in $\mathcal{C}$ and ordinary D-limits in $\mathcal{C}_0$ is that the former might not exist even if the latter do. The distinction disappears if we know in advance that $\mathcal{C}$ is $\mathcal{V}$-complete, and indeed in our examples $\mathcal{S}$ and $\mathcal{G}$ will be $\mathcal{V}$-complete and $\mathcal{V}$-cocomplete, for straightforward reasons.

Alternatively, if $\mathcal{C}$ is $\mathcal{V}$-tensored, then $\mathcal{C}$ will have enriched D-limits if it has ordinary D-limits [10, p.50]. And if $\mathcal{C}$ is $\mathcal{V}$-cotensored, then $\mathcal{C}$ will have enriched D-colimits if it has ordinary D-colimits. Again, existence of tensors and cotensors is straightforwardly observed in our examples of $\mathcal{S}$ and $\mathcal{G}$.

Under some such niceness assumptions on $\mathcal{S}$ and $\mathcal{G}$ that allow us to forget about distinguishing between enriched and ordinary (co)limits, we may say:

1. Pullbacks in $\mathcal{G}$ are simply pullbacks in $\mathcal{G}_0$.

2. Pullbacks in $[\mathcal{S}, \mathcal{G}]$ are computed pointwise, as in $\mathsf{Cat}(\mathcal{S}_0, \mathcal{G}_0)$.

3. We can consider T as a cartesian monad on $\mathcal{G}_0$ and consider a T-operad P; then the underlying cartesian functor $P\colon \mathcal{G}_0 \longrightarrow \mathcal{G}_0$ turns out to be a $\mathcal{V}$-functor $\mathcal{G} \longrightarrow \mathcal{G}$. This follows from the above, as the action of P can be defined entirely from $P1$ using pullbacks.

This last point means that in effect we do not have to change our definition of T-operads and their maps, but that their underlying functors will all turn out to be $\mathcal{V}$-enriched "for free". This in turn means that the evaluation functor

$$\mathsf{ev}_X : T_*\text{-}\mathsf{Opd} \longrightarrow T\text{-}\mathsf{Opd}$$

still makes sense even though at the level of functors the evaluation functor

$$\mathsf{ev}_X : [\mathcal{S}, \mathcal{G}] \longrightarrow \mathcal{G}_0$$

necessarily now has only the underlying category $\mathcal{G}_0$ as its codomain.

We can now re-state the main theorem as follows.

Theorem 1.23 (**Main theorem, topological version).** *Let $\mathcal{S}$ and $\mathcal{G}$ be* Top*-categories, where $\mathcal{S}$ has an initial object $\emptyset$. Let T be a monad on $\mathcal{G}$ that is cartesian in the* Top*-enriched sense, and let U be a* Top*-functor $\mathcal{S} \longrightarrow \mathcal{G}$. Suppose further that* $\mathsf{End}\,(U)$ *exists, so we can define*

$$E_U = \mathsf{ev}_\emptyset(\mathsf{End}\,(U)).$$

Let P be a T-operad. Then an action of P on U-objects is precisely a map of T-operads $P \longrightarrow E_U$.

Our main theorem gives a universal operad acting on U-objects whenever $\mathsf{End}\,(U)$ exists; in the next section we will show that it does exist in the cases we're interested in. This existence typically involves some fairly technical considerations.

2. Internal hom constructions

In this section we will address some size issues that arise when we construct the endomorphism operad $\mathsf{End}\,(U)$ in practice. These issues arise on account of us seeking an internal hom in the category $[\mathcal{S}, \mathcal{G}]/T_*1$ where the category $\mathcal{S}$ is not small. However, in our examples we are helped by some specific properties of the categories and functors in question, and we will now provide the technical results that make this work.

It may be worth elucidating this issue in the case of one of our motivating examples, $\mathcal{G} = [\mathbb{G}^{\text{op}}, \mathsf{Set}]$. In that case we could invoke equivalences

$$\big[\mathcal{S},\ [\mathbb{G}^{\text{op}}, \mathsf{Set}]\big]/T_*1 \ \simeq\ [\mathcal{S}\times\mathbb{G}^{\text{op}},\ \mathsf{Set}]/T_*1 \ \simeq\ \big[(\mathcal{S}\times\mathbb{G}^{\text{op}})/T_*1,\ \mathsf{Set}\big]$$

and then if $\mathcal{S}$ were small we could use the usual construction of an exponential in a presheaf category $[\mathbb{C}^{\text{op}}, \mathsf{Set}]$:

$$(Y^X)(c) = [\mathbb{C}^{\text{op}}, \mathsf{Set}]\big(\mathbb{C}(-,c) \times X\,,\, Y\big).$$

However, this functor evaluates each object c at a collection of natural transformations, and when the category we're taking presheaves over isn't small, this collection is not guaranteed to be a set. That is, if $\mathcal{S}$ is not small, this construction is not guaranted to produce an object in the required functor category.

In this section we are going to show that this does work in particular circumstances which cover all the examples we have in mind. The main technical result we will use is that we can define internal homs between right adjoints in an enriched functor category of the form

$$\big[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\text{op}}}\big]$$

where $\mathbb{C}$ is small. We then "translate" from our slice category into one of this form via an equivalence

$$\big[\mathcal{S}\,,\, \mathcal{V}^{\mathbb{B}^{\text{op}}}\big]/T_*1 \;\simeq\; \big[\mathcal{S}\,,\, \mathcal{V}^{(\mathbb{B}/T1)^{\text{op}}}\big]$$

and the fact that the functors whose internal hom we are now taking are right adjoints will follow from U being a right adjoint and T being familially representable.

All of the material in this section is developed in the generality of enriched category theory, relying heavily on [10]. Throughout this section $\mathcal{V}$ will be a locally small, complete, cocomplete, cartesian closed category, or a "cartesian cosmos". In fact for our examples we only use the cases Set and Top, a convenient category of small topological spaces.

2.1 Preliminaries on enriched category theory

First we fix our terminology and notation for the enriched setting. Let $\mathbb{C}$ be a small $\mathcal{V}$-category. We write $\mathcal{V}^{\mathbb{C}^{\text{op}}}$ for the $\mathcal{V}$-category of $\mathcal{V}$-presheaves on $\mathbb{C}$, with hom-objects given by the usual end formula:

$$\mathcal{V}^{\mathbb{C}^{\text{op}}}(F,G) = \int_{c:\mathbb{C}} Gx^{Fc}.$$

Note that we use the notation $c : \mathbb{C}$ rather than the more common $c \in C$ in all our (co)end formulae. The exponential here denotes the internal hom in the cartesian closed category V; we might also write internal hom in the form $\mathcal{V}(Fc, Gc)$ when the exponential notation becomes arduous, as there is no ambiguity.

We may refer to the hom-object $\mathcal{V}^{\mathbb{C}^{\mathrm{op}}}(F, G)$ as the enriched hom, to distinguish it from the internal hom: $\mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$ is cartesian closed, with internal hom given by the usual formula

$$\begin{aligned} G^F(c) &= \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}\big(\mathbb{C}(-,c) \times F,\ G\big) \\ &= \int_{d:\mathbb{C}} \mathcal{V}\big(\mathbb{C}(d,c) \times Fd,\ Gd\big) \end{aligned}$$

Now, ultimately we are interested in internal homs in a (plain) category $[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}]$ of $\mathcal{V}$-functors and $\mathcal{V}$-transformations. Given $\mathcal{V}$-functors $F, G\colon \mathcal{S} \longrightarrow \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$, we write their internal hom in $[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}]$ as $[F, G]$. The following thought experiment may elucidate the situation. We could use the equivalence

$$[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}] \simeq [\mathcal{S} \times \mathbb{C}, \mathcal{V}]$$

and attempt to define the internal hom $[F, G] : \mathcal{S} \times \mathbb{C} \longrightarrow \mathcal{V}$ by the end formula:

$$[F,G](s,c) = \int_{t:\mathcal{S}, d:\mathbb{C}} \mathcal{V}\big(\mathcal{S}(s,t) \times \mathbb{C}(d,c) \times F(t,d),\ G(t,d)\big) \tag{2.1}$$

In general this end might not exist as $\mathcal{S}$ is not small, but when it does exist it will be an internal hom as we will carefully verify later in the course of proving Theorem 2.4. This main technical result is to show that this end does exist under some mild conditions on the categories, when F and G are right adjoints. We proceed in two steps.

1. Give circumstances in which the enriched hom between right adjoints in $[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}]$ exists.
2. Show that if F and G are right adjoints in $[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}]$ then the above end is an instance of an enriched hom between (some other) right adjoints, and so the above end exists and gives the internal hom $[F, G]$.

We will then apply this result to the examples we're interested in.

2.2 Internal homs between right adjoints

Our main strategy is to express the end (2.1) as an enriched hom between right adjoints. The following lemma ensures that such an enriched hom will exist. In all that follows when we speak of a right adjoint $\mathcal{S} \longrightarrow \mathcal{V}^{\mathbb{C}^{op}}$ we mean a right adjoint in the 2-category of $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-transformations.

Lemma 2.1. *Suppose $\mathbb{C}$ is small, $\mathcal{S}$ is $\mathcal{V}$-cocomplete, and the functors*

$$\mathcal{S} \underset{G'}{\overset{G}{\rightrightarrows}} \mathcal{V}^{\mathbb{C}^{op}}$$

have respective left adjoints F, F'. Write $\bar{F} = Fy_C$, the restriction of F along the Yoneda embedding $y_C \colon \mathbb{C} \longrightarrow \mathcal{V}^{\mathbb{C}^{op}}$ and similarly $\bar{F}' = F'y_C$. Then the enriched hom $[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{op}}](G, G')$, as an end $\int_{x:\mathcal{S}} \mathcal{V}^{\mathbb{C}^{op}}(Gx, G'x)$, exists and is given by

$$[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{op}}](G, G') \cong \mathcal{S}^{\mathbb{C}}(\bar{F}', \bar{F}).$$

Proof. By [10, Theorem 4.51], G and G' are given by

$$Gx \cong \mathcal{S}(\bar{F}-, x)$$

$$G'x \cong \mathcal{S}(\bar{F}'-, x)$$

We have

$$\begin{aligned}
\mathcal{S}^{\mathbb{C}}(\bar{F}', \bar{F}) &\cong \int_{c:\mathbb{C}} \mathcal{S}(\bar{F}'c, \bar{F}c) && \text{as } \mathbb{C} \text{ is small} \\
&\cong \int_{c:\mathbb{C}} \int_{x:\mathcal{S}} \mathcal{S}(\bar{F}'c, x)^{\mathcal{S}(\bar{F}c, x)} && \text{by enriched Yoneda, [10, 2.31]} \\
&\cong \int_{x:\mathcal{S}} \int_{c:\mathbb{C}} \mathcal{S}(\bar{F}'c, x)^{\mathcal{S}(\bar{F}c, x)} && \text{by the Fubini theorem, [10, 2.8]} \\
&\cong \int_{x:\mathcal{S}} V^{\mathbb{C}^{op}}\big(\mathcal{S}(\bar{F}-, x), \mathcal{S}(\bar{F}'-, x)\big)
\end{aligned}$$

as required. ∎

We now aim to define an internal hom $[F, G] \colon \mathcal{S} \longrightarrow \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$ between right adjoints. It is essentially given by the same formula (2.1) as would be expected if $\mathcal{S}$ were small, except we express it in such a way that the enriched hom we need to invoke is between right adjoints so that Lemma 2.1 ensures it exists. This may give the following constructions an air of over-complication, but the aim is to express something familiar as a composite of right adjoints, which takes a little manœuvring. In the following constructions we use the same hypotheses as in Lemma 2.1.

The following will be the first right adjoint in our enriched hom, derived from F. Given an object $s \in \mathcal{S}$ we write $\mathcal{S}(s,-) \cdot F$ for the following composite; here δ denotes the diagonal, Δ produces the constant functor, and Π denotes the functor taking products.

$$\begin{array}{ccccccccc} \mathcal{S} & \xrightarrow{\delta} & \mathcal{S}\times\mathcal{S} & \xrightarrow{\mathcal{S}(s,-)\times F} & \mathcal{V}\times\mathcal{V}^{\mathbb{C}^{\mathrm{op}}} & \xrightarrow{\Delta\times 1} & \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}\times\mathcal{V}^{\mathbb{C}^{\mathrm{op}}} & \xrightarrow{\Pi} & \mathcal{V}^{\mathbb{C}^{\mathrm{op}}} \\ x & \longmapsto & (x,x) & \longmapsto & \big(\mathcal{S}(s,x),\ F(x)\big) & \longmapsto & \big(\Delta_{\mathcal{S}(s,x)},\ F(x)\big) & \longmapsto & \mathcal{S}(s,x)\times F(x) \end{array}$$

Here $\mathcal{S}(s,x) \times F(x)$ denotes the functor

$$\begin{array}{ccc} C^{\mathrm{op}} & \longrightarrow & V \\ c & \longmapsto & \mathcal{S}(s,x) \times F(x)(c) \end{array}$$

Proposition 2.2. *If F is a right adjoint in $\mathcal{V}$-Cat then so is the above composite.*

Proof. Let I denote the unit $\mathcal{V}$-category. There is an evident $\mathcal{V}$-category $I + I$, and there is a unique $\mathcal{V}$-functor $! : I + I \longrightarrow I$. The diagonal functor $\delta \colon \mathcal{S} \longrightarrow \mathcal{S} \times \mathcal{S}$ may be identified with the functor given by pre-composition with !, which we write as $\mathcal{S}^!$; this has an enriched left adjoint (which, at the underlying category level, is just the coproduct):

$$\begin{tikzcd} \mathcal{S}^I \arrow[r, shift left, "\mathcal{S}^!"] \arrow[r, phantom, "\top"] & \mathcal{S}^{I+I} \arrow[l, shift left, "+ = \mathsf{Lan}_!"] \end{tikzcd}$$

Similarly, we have a unique $\mathcal{V}$-functor $!_{\mathbb{C}} \colon \mathbb{C}^{\mathrm{op}} \longrightarrow I$, and Δ above may be identified with $\mathcal{V}^{!_{\mathbb{C}}}$. This too has an enriched left adjoint:

$$\begin{tikzcd} \mathcal{V}^I \arrow[r, shift left, "\mathcal{V}^{!_{\mathbb{C}}}"] \arrow[r, phantom, "\top"] & \mathcal{V}^{\mathbb{C}^{\mathrm{op}}} \arrow[l, shift left, "\mathsf{colim}_{\mathbb{C}} = \mathsf{Lan}_{!_{\mathbb{C}}}"] \end{tikzcd}$$

The second map has left adjoint $(- \otimes s) \times F'$ where $(- \otimes s) : V \longrightarrow S$ is tensoring with an object s (which is left adjoint to the representable $S(s,-) : S \longrightarrow V$), and F' is the left adjoint of F.

Finally, the enriched left adjoint of Π is the diagonal:

$$\begin{tikzcd} \mathcal{V}^{\mathbb{C}^{\mathrm{op}}} \times \mathcal{V}^{\mathbb{C}^{\mathrm{op}}} \arrow[r, shift left, "\Pi"] \arrow[r, phantom, "\top"] & \mathcal{V}^{\mathbb{C}^{\mathrm{op}}} \arrow[l, shift left, "\delta"] \end{tikzcd}$$

□

The following will be the second right adjoint in our enriched hom, derived from G. For every object $c \in \mathbb{C}$ we write $G^{\mathbb{C}(-,c)}$ for the following composite:

$$\begin{array}{ccccc} \mathcal{S} & \xrightarrow{G} & \mathcal{V}^{\mathbb{C}^{\mathrm{op}}} & \xrightarrow{(-)^{\mathbb{C}(-,c)}} & \mathcal{V}^{\mathbb{C}^{\mathrm{op}}} \\ x & \longmapsto & Gx & \longmapsto & Gx^{\mathbb{C}(-,c)} \end{array}$$

Here $Gx^{\mathbb{C}(-,c)}$ denotes the exponential (in the cartesian closed category $\mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$) of the functors $G(x)$ and the representable $\mathbb{C}(-,c)$.

Proposition 2.3. *If G is a right adjoint in $\mathcal{V}$-*Cat *then so is the above composite.*

Proof. Recall that $\mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$ is cartesian closed in the enriched sense, so that we have a $\mathcal{V}$-natural isomorphism

$$\mathcal{V}^{\mathbb{C}^{\mathrm{op}}}(X \times Y\,,\, Z) \;\cong\; \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}(X\,,\, Z^Y)$$

between enriched functors valued in $\mathcal{V}$. In particular, the enriched functor $(-)^Y$ is an enriched right adjoint; we are using the case $Y = \mathbb{C}(-,c)$. □

We are now ready to prove the main technical theorem we need, constructing internal homs between right adjoints via the above composites.

Theorem 2.4. *Let $\mathcal{V}$ be a locally small, complete, cocomplete cartesian closed category, let $\mathcal{S}$ be a $\mathcal{V}$-cocomplete $\mathcal{V}$-category, and let $\mathbb{C}$ be a small $\mathcal{V}$-category. Let $F, G\colon \mathcal{S} \longrightarrow \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$ be two right adjoints. Then, in the category of $\mathcal{V}$-functors $[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}]$ there is an internal hom $[F,G]$ constructed according to the formula*

$$[F,G](s)(c) \;=\; \big[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}\big]\big(\mathcal{S}(s,-)\cdot F\,,\; G^{\mathbb{C}(-,c)}\big).$$

Proof. First note that this enriched hom may be legitimately formed as an object of $\mathcal{V}$ by Lemma 2.1, since $\mathcal{S}(s,-)\cdot F$ and $G^{\mathbb{C}(-,c)}$ are right adjoints. Thus,

$$\begin{aligned}
[F,G](s)(c) &= \int_{t:\mathcal{S}} \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}\big(\mathcal{S}(s,t)\times F(t),\ G(t)^{\mathbb{C}(-,c)}\big) \\
&\cong \int_{t:\mathcal{S}} \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}\big(\mathcal{S}(s,t)\times F(t)\times \mathbb{C}(-,c),\ G(t)\big) \\
&\cong \int_{t:\mathcal{S}} \int_{d:\mathbb{C}} \mathcal{V}\big(\mathcal{S}(s,t)\times F(t)(d)\times \mathbb{C}(d,c),\ G(t)(d)\big)
\end{aligned}$$

This is the formula expected as in (2.1).

For completeness we now show that $[F,G]$ satisfies the requisite universal property. Thus, suppose we are given a $\mathcal{V}$-functor $X\colon \mathcal{S}\longrightarrow\mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$. We establish a natural bijection between the family of maps of the form $\psi\colon X\longrightarrow[F,G]$ and those of the form $\phi : X\times F\longrightarrow G$.

Let us equivalently regard F,G,X as V-functors $\mathcal{S}\times\mathbb{C}^{\mathrm{op}}\longrightarrow\mathcal{V}$, to avail ourselves of more pleasant notation such as $X(s,c)$. Using the definitions of end and coend, and instances of $\times$–hom adjunctions, we have a natural bijection between natural transformations $\psi\colon X\longrightarrow[F,G]$ and the following extranatural families of maps:

$$\psi(s,c) : X(s,c) \longrightarrow [F,G](s,c) = \int_{t:\mathcal{S}} \int_{d:\mathbb{C}} G(t,d)^{\mathcal{S}(s,t)\times F(t,d)\times \mathbb{C}(d,c)} \quad (1)$$

$$X(s,c) \longrightarrow G(t,d)^{\mathcal{S}(s,t)\times F(t,d)\times \mathbb{C}(d,c)} \quad (2)$$

$$\mathcal{S}(s,t) \times X(s,c) \times \mathbb{C}(d,c) \longrightarrow G(t,d)^{F(t,d)} \quad (3)$$

$$\int^{(s,c)\,:\,\mathcal{S}\times\mathbb{C}^{\mathrm{op}}} \mathcal{S}(s,t) \times X(s,c) \times \mathbb{C}(d,c) \longrightarrow G(t,d)^{F(t,d)} \quad (4)$$

$$X(t,d) \longrightarrow G(t,d)^{F(t,d)} \quad (5)$$

$$X(t,d) \times F(t,d) \longrightarrow G(t,d) \quad (6)$$

$$(X \times F)(t,d) \longrightarrow G(t,d) \quad (7)$$

$$\phi : X \times F \longrightarrow G \quad (8)$$

where line (2) is achieved by definition of extranaturality and coends, line (3) via a $\times$–hom adjunction, (4) by definition of extranaturality and ends, (5) by Yoneda, (6) via a $\times$–hom adjunction, and (7) and (8) by definition. □

Remark 2.5. Note that in practice we will express the formula for the internal hom in the usual format in a functor category, once we know that the end in question exists, that is:

$$\begin{aligned} [F,G](s,c) &= \int_{t:\mathcal{S}} \int_{d:\mathbb{C}} G(t,d)^{\mathcal{S}(s,t)\times F(t,d)\times \mathbb{C}(d,c)} \\ &= [\mathcal{S}\times\mathbb{C}^{\mathrm{op}}, \mathcal{V}](H^{(s,c)} \times F,\ G) \end{aligned}$$

where $H^{(s,c)}$ denotes the appropriate representable $\mathcal{S}(s,-)\times\mathbb{C}(-,c)$.

2.3 Application to the endomorphism operad

We now show how to apply Theorem 2.4 to show that $\mathsf{End}\,U$ may be formed in our cases of interest. Recall that in our examples:

- $\mathcal{V}$ is Set or Top.

- $\mathcal{S}$ is Top or Top_*.

- $\mathbb{B}$ is a small (plain) category which we may then regard as a Top-category in which all hom objects are discrete spaces. In our examples $\mathbb{B}$ is the globular category $\mathbb{G}$ or the finite version $\mathbb{G}_n$ (or indeed the terminal category $dir0o$.

- T is a familially representable $\mathcal{V}$-monad on $\mathcal{V}^{\mathbb{B}^{\mathrm{op}}}$ (so in particular is cartesian); we will come back to this definition shortly.

- T_* is the induced cartesian monad

$$\left[\mathcal{S},\ \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}\right] \xrightarrow{T\circ-} \left[\mathcal{S},\ \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}\right]$$

- U is a $\mathcal{V}$-functor $\mathcal{S} \longrightarrow \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}$; all our examples of U are constructed via a functor $\mathbb{B} \xrightarrow{D} \mathcal{S}$ with

$$U(X) = \mathcal{S}(D-, X) \in \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}$$

thus U is a right adjoint; its left adjoint is the left Kan extension of D along the Yoneda embedding

$$\begin{tikzcd} \mathbb{B} \arrow[r, "y_{\mathbb{B}}"] \arrow[d, "D"'] & \mathcal{V}^{\mathbb{B}^{\mathrm{op}}} \arrow[dl, shift right, ""'] \arrow[dl, phantom, "\perp"] \\ \mathcal{S} \arrow[ur, shift right, "U"'] & \end{tikzcd}$$

We seek to construct the endomorphism operad $\mathsf{End}\,U$ as the following internal hom in the (plain) slice category $[\mathcal{S}, \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}]/T_*1$:

$$\mathsf{Hom}\,(U,U) = \begin{bmatrix} T_*U & T_*1 \times U \\ \downarrow \quad , & \downarrow \\ T_*1 & T_*1 \end{bmatrix}$$

We are going to use Theorem 2.4. Our first step here is to re-express the slice category as an equivalent category of the form $[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}]$, and our next step will be to show that under that equivalence, the objects whose internal hom we're taking become right adjoints.

The first step is straightforward for $\mathcal{V} = \mathsf{Set}$ so we cover that case first; it requires a little more effort for $\mathcal{V} = \mathsf{Top}$.

Lemma 2.6. *Let $\mathcal{S}$ be locally small and cocomplete, and $\mathbb{B}$ small, and T a monad on $[\mathbb{B}^{\mathrm{op}}, \mathsf{Set}]$. There are equivalences of categories:*

$$\Big[\mathcal{S},\ [\mathbb{B}^{\mathrm{op}}, \mathsf{Set}]\Big]/T_*1 \ \simeq\ \Big[\mathcal{S},\ [\mathbb{B}^{\mathrm{op}}, \mathsf{Set}]/T1\Big] \ \simeq\ \Big[\mathcal{S},\ \big[(\mathbb{B}/T1)^{\mathrm{op}}, \mathsf{Set}\big]\Big]$$

where $\mathbb{B}/T1$ denotes the category of elements.

Proof. First note that the functor T_*1 is the composite

$$\begin{array}{ccccc} \mathcal{S} & \xrightarrow{1} & [\mathbb{B}^{\mathrm{op}}, \mathsf{Set}] & \xrightarrow{T} & [\mathbb{B}^{\mathrm{op}}, \mathsf{Set}] \\ c & \longmapsto & 1 & \longmapsto & T1 \end{array}$$

so it is the constant functor Δ_{T1}. Thus an object $(F \xrightarrow{\alpha} T_*1)$ in the first category amounts to a cocone $(F\_ \longrightarrow T1)$. This gives the first equivalence. The second follows from the fact that slices of presheaf toposes are equivalent to presheaf toposes as follows:

$$[\mathbb{B}^{\mathrm{op}},\ \mathsf{Set}]/T1 \ \simeq\ \big[(\mathbb{B}/T1)^{\mathrm{op}},\ \mathsf{Set}\big]$$

□

For the case $\mathcal{V} = \mathsf{Top}$ we deal with the two equivalences separately; the first follows easily, with the only subtlety being that $\mathcal{V}^{\mathbb{B}^{\mathrm{op}}}/T1$ is now an enriched slice category. However as we are only considering enrichment in Top this amounts to the same as the ordinary slice but with a topology on the homs, and that topology is inherited. We will express this lemma in simpler terms to emphasise the fact that nothing very special is going on, but what we have in mind here is $\mathcal{G} = \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}$ and $X = T1$.

Lemma 2.7. *Let $\mathcal{V} = \mathsf{Top}$, let $\mathcal{S}$ and $\mathcal{G}$ be $\mathcal{V}$-categories. Consider $X \in \mathcal{G}$ and write $\Delta_X \colon \mathcal{S} \longrightarrow \mathcal{G}$ for the constant functor. Then there is an equivalence of categories*

$$\Big[\mathcal{S},\ \mathcal{G}\Big]/\Delta_X \ \simeq\ \Big[\mathcal{S},\ \mathcal{G}/X\Big]$$

Proof. As in the previous proof, an object $(F \xrightarrow{\alpha} \Delta_X)$ in the first category amounts to a cocone $(F\_ \longrightarrow X)$, that is, an object in the second category. The only extra subtlety here is that the enriched structure of $\mathcal{G}/X$ is inherited from $\mathcal{G}$, so F being a $\mathcal{V}$-functor on the left ensures that the stated corrspondence does produce a $\mathcal{V}$-functor on the right. □

We now deal with the second equivalence. In what follows we will sometimes realise plain categories as Top-categories in which all the hom-spaces are discrete; by abuse of notation we will not change the notation for this.

Lemma 2.8. *Let $\mathcal{V} = \mathsf{Top}$. Let $\mathbb{A}$ be a small (plain) category. Let $F\colon \mathbb{A} \longrightarrow \mathsf{Set}$ be any functor and $i : \mathsf{Set} \longrightarrow \mathsf{Top}$ be the functor each set to the discrete space on it. Then there is a $\mathcal{V}$-equivalence of $\mathcal{V}$-categories:*

$$\mathcal{V}^{\mathbb{A}}/iF \simeq \mathcal{V}^{\mathbb{A}/F}$$

where $\mathbb{A}/F$ denotes the category of elements of F.

Proof. We borrow the standard proof that

$$[\mathbb{A}, \mathsf{Set}]/F \simeq [\mathbb{A}/F, \mathsf{Set}]$$

—we just have to check continuity in a few key places. We know that we have a functor

$$[\mathbb{A}, \mathsf{Set}]/F \xrightarrow{\alpha} [\mathbb{A}/F, \mathsf{Set}]$$

that is full, faithful and essentially surjective on objects. Recall that $\mathbb{A}/F$ is the category of elements of F given as follows.

- Objects are pairs $(a \in \mathbb{A}, x \in Fa)$.
- A morphism $(a, x) \longrightarrow (a', x')$ is a morphism $f\colon a \longrightarrow a' \in \mathbb{A}$ such that $Ff(x) = x'$.

First we recall the action of α. Given an element $(S, \theta) = \begin{array}{c} S \\ \downarrow{\scriptstyle \theta} \\ F \end{array} \in [\mathbb{A}, \mathsf{Set}]/F$ with components $\begin{array}{c} Sa \\ \downarrow{\scriptstyle \theta_a} \\ Fa \end{array}$, we will write $\alpha(S, \theta) = \bar{S} \in [\mathbb{A}/F, \mathsf{Set}]$, and its action is as follows:

- On objects: $(a, x) \in \mathbb{A}/F$ is sent to the set ${\theta_a}^{-1}(x) \subseteq Sa$.

- On morphisms: the morphism

  $$(a,x) \xrightarrow{f} (a',x')$$

  is sent to

  $$\theta_a^{-1}(x) \longrightarrow \theta_{a'}^{-1}(x')$$

  the restriction of Sf to the fibre $\theta_a^{-1}(x)$; this works because of naturality of θ.

The functor α is full, faithful and essentially surjective; it has (pseudo)-inverse β given as follows. Given $R \in [\mathbb{A}/F, \mathsf{Set}]$, the element

$$\beta(R) = \begin{array}{c} \hat{R} \\ \big\downarrow{\scriptstyle \theta^R} \\ F \end{array} \in [\mathbb{A}, \mathsf{Set}]/F$$

is given by

$$\hat{R}a = \coprod_{x \in Fa} R(a,x). \tag{2.2}$$

The map ${\theta^R}_a : R(a,x) \longrightarrow Fa$ sends everything to $x \in Fa$. The rest of the data is induced by the universal property of the coproduct (2.2).

In order to modify this proof for the topological case, we need to check that

1. If S is a Top-functor $\mathbb{A} \longrightarrow \mathsf{Top}$ then $\bar{S}$ becomes a Top-functor $\mathbb{A}/F \longrightarrow \mathsf{Top}$,
2. If R is a Top-functor $\mathbb{A}/F \longrightarrow \mathsf{Top}$ $\hat{R}$ becomes a Top-functor $\mathbb{A} \longrightarrow \mathsf{Top}$,
3. the components of θ^R are continuous, and
4. α and β are themselves Top-functors.

These all follow by making the preimage, coproduct and restriction maps in Top instead of in Set. $\Box$

Corollary 2.9. *Under the usual hypotheses we have the following equivalences of categories:*

$$\left[\mathcal{S}, \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}\right]/T_*1 \;\simeq\; \left[\mathcal{S}, \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}/T1\right] \;\simeq\; \left[\mathcal{S}, \mathcal{V}^{(\mathbb{B}/T1)^{\mathrm{op}}}\right]$$

Our next task is to take the objects whose internal hom we want to calculate in the slice category $\left[\mathcal{S}, \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}\right]/T_*1$, and "translate" them into the functor category $\left[\mathcal{S}, \mathcal{V}^{(\mathbb{B}/T1)^{\mathrm{op}}}\right]$ and show that they are right adjoints so that we may take their internal hom.

We first briefly recall some helpful results and definitions; we will not state these in very great generality.

Lemma 2.10. *Let $\mathcal{V}$ be* Set *or* Top *and keep the usual hypotheses. Consider the canonical morphism $T1 \xrightarrow{!} 1$ in $\mathcal{V}^{\mathbb{B}^{\mathrm{op}}}$. Then there is a $\mathcal{V}$-adjunction which we will write as*

$$\begin{tikzcd} \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}/T1 \arrow[r, shift left, "\Sigma_{T1}"] \arrow[r, phantom, "\perp"] & \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}/1 \simeq \mathcal{V}^{\mathbb{B}^{\mathrm{op}}} \arrow[l, shift left, "(T1)^*"] \end{tikzcd}$$

where $(T1)^*$ is given by pullback along $!$ (so in this case, effectively it is just a product) and Σ_{T1} is $! \circ -$ (sometimes called the dependent sum).

Lemma 2.11. *The $\mathcal{V}$-functor $T : \mathcal{V}^{\mathbb{B}^{\mathrm{op}}} \longrightarrow \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}$ can be canonically factorised as:*

$$\begin{array}{ccccccccc} \mathcal{V}^{\mathbb{B}^{\mathrm{op}}} & \simeq & \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}/1 & \xrightarrow{\hat{T}} & \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}/T1 & \xrightarrow{\Sigma_{T1}} & \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}/1 & \simeq & \mathcal{V}^{\mathbb{B}^{\mathrm{op}}} \\ X & \longmapsto & \begin{matrix} X \\ \downarrow ! \\ 1 \end{matrix} & \longmapsto & \begin{matrix} TX \\ \downarrow T! \\ T1 \end{matrix} & \longmapsto & \begin{matrix} TX \\ \downarrow ! \\ 1 \end{matrix} & \longmapsto & TX \end{array}$$

Definition 2.12. The functor T is called a *parametric right adjoint* (p.r.a.) if $\hat{T}$ is a right adjoint. A monad is called *parametric right adjoint* if its functor part is p.r.a. and its unit and multiplication are cartesian. Any familially representable monad is p.r.a.

We are finally ready to tackle the internal hom in question.

Theorem 2.13. *Under the usual hypotheses, including that T is a parametric right adjoint and that U is a right adjoint, the following internal hom in $[\mathcal{S}, \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}]/T_*1$ exists:*

$$\left[\begin{matrix} T_*U \\ \downarrow T_*1 \\ T_*1 \end{matrix} \quad , \quad \begin{matrix} T_*1 \times U \\ \downarrow p \\ T_*1 \end{matrix} \right]$$

where p denotes projection onto the first component.

Proof. We "translate" each of these objects from the slice category above into the functor category

$$\left[\mathcal{S},\ \mathcal{V}^{(\mathbb{B}/T1)^{\mathrm{op}}}\right]$$

as in Lemma 2.7 and then express them as right adjoints. We then apply Theorem 2.4 with $\mathbb{C} = \mathbb{B}/T1$.

First note that according to the first equivalence of Corollary 2.9, the object $(T_*U \xrightarrow{T_*!} T_*1)$ becomes the cocone $(TU\_ \xrightarrow{T!} T1)$. So we take $\hat{T} \circ U$, the following composite of right adjoints, giving the cocone required:

$$\begin{array}{ccccccc}
\mathcal{S} & \xrightarrow{U} & \mathcal{V}^{\mathbb{B}^{\mathrm{op}}} & \xrightarrow{\hat{T}} & \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}/T1 & \simeq & \mathcal{V}^{(\mathbb{B}/T1)^{\mathrm{op}}} \\
x & \longmapsto & Ux & \longmapsto & \begin{array}{c} TUx \\ \downarrow{\scriptstyle T!} \\ T1 \end{array} & &
\end{array}$$

For the second object of our internal hom, note that according to the first equivalence of Corollary 2.9 the object $(T_*1 \times U \xrightarrow{p} T_*1)$ becomes the cocone $(T1 \times U\_ \xrightarrow{p} T1)$. We consider $(T1)^* \circ U$, the following composite of right adjoints, giving the cocone required:

$$\begin{array}{ccccccc}
\mathcal{S} & \xrightarrow{U} & \mathcal{V}^{\mathbb{B}^{\mathrm{op}}} & \xrightarrow{(T1)^*} & \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}/T1 & \simeq & \mathcal{V}^{(\mathbb{B}/T1)^{\mathrm{op}}} \\
x & \longmapsto & Ux & \longmapsto & \begin{array}{c} T1 \times Ux \\ \downarrow{\scriptstyle p} \\ T1 \end{array} & &
\end{array}$$

Here $(T1)^*$ is a right adjoint with left adjoint given by Σ_{T1}. Theorem 2.4 now applies, and we can compute the internal hom of right adjoints:

$$\left[\hat{T} \circ U, (T1)^* \circ U.\right]$$

□

3. Loop spaces

In this section we discuss the first of our motivating examples—operads acting on loop spaces. Most of the work here is just in unravelling the definitions to show that the universal operad acting on loop spaces is the one we are expecting from [14].

In this example we take

$$\begin{array}{rcl} \mathcal{S} & = & \mathsf{Top}_* \\ \mathcal{G} & = & \mathsf{Top} \\ U & = & \Omega = \mathsf{Top}_*(S,-) \text{ where here } \mathsf{Top}_*(X,Y) \text{ denotes the unbased space of} \\ & & \text{based maps } X \longrightarrow Y \text{, and } S \text{ is the unit circle} \\ T & = & \text{free topological monoid monad on } \mathsf{Top}\text{, thus } TX = \coprod\limits_{k\in\mathbb{N}} X^k \text{ (see [9])} \end{array}$$

Note that the initial object in Top_* is the one-point space; we will still write it as $\emptyset$ although it is not empty.

Thus $T1$ is the space $\mathbb{N}$ with the discrete topology, a T-operad is just a classical (non-Σ) operad, and operads acting on U-objects are just operads acting on loop spaces. We seek to understand the universal operad

$$E_\Omega = \mathsf{ev}_\emptyset(\mathsf{End}\,(U)).$$

We will show that

$$E_\Omega(k) = \mathsf{Top}_*(S, S^{\vee k}),$$

the operad which has been called the "universal operad acting on loop spaces" by Salvatore and others [14, 12, 2].

First we use the results of Section 2 to show that $\mathsf{End}\,(\Omega)$ exists.

Proposition 3.1. *With the definitions as above, we can form* $\mathsf{End}\,(\Omega)$ *as an internal hom in the slice category* $[\mathcal{S},\mathcal{G}]/T_*1$.

Proof. The monad T is familially representable and U is a right adjoint: we identify Top with $\mathsf{Top}^{dir0o^{\mathrm{op}}}$, and note that U can be regarded as being constructed via the functor $dir0o \xrightarrow{S} \mathsf{Top}_*$ picking out the circle S. Then

$$U = \mathsf{Top}_*(S,-) \in \mathsf{Top}$$

is a right adjoint; its left adjoint is the left Kan extension of S along the Yoneda embedding

$$\begin{tikzcd} dir0o \arrow[r, "y_{dir0o}"] \arrow[d, "S"'] & \mathsf{Top}^{dir0o^{\mathrm{op}}} \simeq \mathsf{Top} \arrow[dl, shift right, ""] \arrow[dl, phantom, "\perp"] \\ \mathsf{Top}_* \arrow[ur, shift right, "U"'] & \end{tikzcd}$$

We can thus apply Theorem 2.4 to form the required internal hom as

$$[\hat{T} \circ U, (T1)^* \circ U].$$

$\square$

We now set about unravelling what $E_\Omega = \mathsf{ev}_\emptyset(\mathsf{End}(U))$ is. The first step is to understand the slice category in question, re-expressed as a functor category.

Now note that $T1$ in this case is $\mathbb{N}$, the discrete category made into a Top-category, and $\mathsf{Top}/\mathbb{N} \equiv \mathsf{Top}^{\mathbb{N}}$. So the first equivalence of Corollary 2.9 in this case becomes:

$$\begin{aligned} [\mathsf{Top}_*, \mathsf{Top}]/T_*1 &\simeq [\mathsf{Top}_*, \mathsf{Top}/T1] \\ &\simeq [\mathsf{Top}_*, \mathsf{Top}^{\mathbb{N}}] \\ &\simeq [\mathsf{Top}_* \times \mathbb{N}, \mathsf{Top}] \end{aligned}$$

Example 3.2. An element $\begin{array}{c} S \\ \downarrow \\ T_*1 \end{array}$ in $[\mathsf{Top}_*, \mathsf{Top}]/T_*1$ consists of, for all $X \in \mathsf{Top}_*$ a continuous map $\begin{array}{c} SX \\ \downarrow \\ \mathbb{N} \end{array}$ such that for all $f : X \longrightarrow X'$ the following diagram commutes

$$\begin{tikzcd} SX \arrow[rr, "Sf"] \arrow[dr] & & SX' \arrow[dl] \\ & \mathbb{N}. & \end{tikzcd}$$

Since $\mathbb{N}$ is discrete, we know $SX = \coprod_n S_n X$, say, where each S_n is a functor $\mathsf{Top}_* \longrightarrow \mathsf{Top}$. Thus we have a functor

$$\begin{array}{ccc} \mathsf{Top}_* \times \mathbb{N} & \longrightarrow & \mathsf{Top} \\ (X, n) & \longmapsto & S_n(X). \end{array}$$

Conversely, given a functor $S : \mathsf{Top}_* \times \mathbb{N} \longrightarrow \mathsf{Top}$ we have for all n a functor

$$S_n = S(n, -) : \mathsf{Top}_* \longrightarrow \mathsf{Top}.$$

This corresponds to $\begin{matrix}\bar{S}\\ \downarrow\\ \mathbb{N}\end{matrix}$ by $\bar{S}X = \coprod_n S_n(X)$.

Theorem 3.3. *For all $k \geq 0$ we have $E_\Omega(k) = \mathsf{Top}_*(S, S^{\vee k})$.*

Proof. We write $\mathcal{S} = \mathsf{Top}_*$. We must calculate $\mathsf{ev}_\emptyset(\mathsf{Hom}\,(\Omega,\Omega))$. Recall that $\mathsf{Hom}\,(\Omega,\Omega) \in [\mathcal{S},\mathsf{Top}]/T_*1$ is given by

$$\left[\begin{matrix} T_*\Omega \\ \big\downarrow{\scriptstyle T_*!} \\ T_*1 \end{matrix}\ ,\ \begin{matrix} T_*1\times\Omega \\ \big\downarrow{\scriptstyle \pi_1} \\ T_*1 \end{matrix}\right] = \left[\begin{matrix} A \\ \big\downarrow \\ \Delta_{\mathbb{N}} \end{matrix}\ ,\ \begin{matrix} B \\ \big\downarrow \\ \Delta_{\mathbb{N}} \end{matrix}\right],$$

say, where the square brackets denote the exponential in the slice category $[\mathcal{S},\mathsf{Top}]/T_*1$. To calculate this hom we express it in the equivalent category $[\mathcal{S}\times\mathbb{N},\mathsf{Top}]$; then to evaluate it at $\emptyset$ we evaluate it at $(\emptyset,k)$ for each $k\in\mathbb{N}$.

Now

$$\begin{array}{rclclcl} AX &=& T\Omega X &=& \coprod_k (\Omega X)^k &=& \coprod_k \mathcal{S}(S^{\vee k},X) \\ BX &=& \mathbb{N}\times\Omega X &=& \coprod_k \Omega X &=& \coprod_k \mathcal{S}(S,X) \end{array}$$

so

$$\begin{array}{rcl} \bar{A}(X,k) &=& \mathcal{S}(S^{\vee k},X) \\ \bar{B}(X,k) &=& \mathcal{S}(S,X). \end{array}$$

We now use the formula for the internal hom in $[\mathcal{S}\times\mathbb{N},\mathsf{Top}]$ as in Remark 2.5:

$$\mathsf{Hom}\,(\bar{A},\bar{B})(\emptyset,k) = \big[\mathcal{S}\times\mathbb{N},\mathsf{Top}\big]\big(H^{(\emptyset,k)}\times\bar{A},\bar{B}\big) \tag{3.1}$$

where

$$H^{(\emptyset,k)}\big(Y,m\big) = \big(\mathcal{S}\times\mathbb{N}\big)\big((\emptyset,k),(Y,m)\big) = \begin{cases} 1 & k=m \\ \emptyset & \text{otherwise} \end{cases}$$

where here 1 and $\emptyset$ are terminal and initial respectively in Set.

Now $\mathcal{S}\times\mathbb{N}$ is a coproduct $\coprod_{m\in\mathbb{N}}\mathcal{S}$, so in general

$$[\mathcal{S}\times\mathbb{N},\mathsf{Top}](F,G) \cong \prod_{m\in\mathbb{N}} \big[\mathcal{S},\mathsf{Top}\big]\big(F(-,m),G(-,m)\big).$$

So, using (3.1) above, we have:

$$\begin{aligned}\mathsf{Hom}\,(\bar{A},\bar{B})(\emptyset,k) &= \prod_{m\in\mathbb{N}} [\mathcal{S},\mathsf{Top}]\Big(H^{(\emptyset,k)}(-,m)\times\mathcal{S}(S^{\vee m},-),\mathcal{S}(S,-)\Big)\\ &= [\mathcal{S},\mathsf{Top}]\big(\mathcal{S}(S^{\vee k},-),\mathcal{S}(S,-)\big)\\ &= \mathcal{S}(S,S^{\vee k}) \quad \text{by the enriched Yoneda Lemma.}\end{aligned}$$

□

Remark 3.4. This operad is often thought of as the "coendomorphism operad" on S in Top_*; we now see that it is derived from the *endomorphism* operad in the functor category $[\mathsf{Top}_*,\mathsf{Top}]$, on the representable functor at S.

Example 3.5. Let D be the non-Σ version of the little intervals operad, so $D(k)$ is the space of configurations of k disjoint intervals inside the unit interval. It is well-known that D acts naturally on loop spaces; in fact the action is explicitly defined via the action of the universal operad E. Given an element of $D(k)$ we derive an element of $E(k)$, that is a based continuous map $S \longrightarrow S^{\vee k}$, as follows. We identify the endpoints of the (big) unit interval to make the circle S of the domain; we then map any point outside the little intervals to the basepoint of $S^{\vee k}$, and map the ith little interval to the ith circle in the wedge $S^{\vee k}$. The element of $D(k)$ is then considered to act on loop spaces via the action of this derived element of $E(k)$.

Note that there are many operations of E that do not arise in this way. Broadly these fall into three types:

- maps $S \longrightarrow S^{\vee k}$ that are not surjective, so "omit" some loops,
- maps that involve going "backwards" around a loop, or
- maps that involve going more than once around loops.

In this sense the universal operad is "too big", and the operads that have proved efficacious in loop space theory are much smaller.

Examples 3.6. Let $D(k)$ be the space of continuous, endpoint-preserving maps $[0,1] \longrightarrow [0,k]$. These act on loop spaces naturally as they act on path spaces naturally. Other examples arise as suboperads of this one, for example by using only increasing maps, or piecewise linear increasing maps.

One might wish to restrict further to smooth maps (for example in order to reparametrise cobordisms). This is more complicated as one would have to ensure that composites remained smooth; this is related to some issues tackled using collars in [3].

4. Fundamental ω-groupoids

In this section we discuss our motivating example, the functor

$$U : \mathsf{Top} \longrightarrow \mathsf{GSet}$$

giving the "fundamental globular set" or "ω-path space" of a space. In a sense there is no further calculation to be done as Leinster has already worked out what the globular operad $\mathsf{ev}_\emptyset(\mathsf{End}\,(U))$ is; see [12, Example 9.2.7], in which Leinster writes the operad in question as $P' = (\mathsf{ev}_\emptyset)_*(\mathsf{End}\,(\Pi_\omega))$. Thus, the history of this operad may be summarised as follows.

1. Batanin defines the operad directly [1].
2. Leinster expresses the operad as $\mathsf{ev}_\emptyset(\mathsf{End}\,(U))$ [12].
3. The present work establishes that $\mathsf{End}\,(U)$ does exist, so that this expression for P' makes sense, and exhibits the universal property of P'.

In fact as usual the story is slightly more complicated as we use the "non-algebraic Leinster variant" of contractible globular operads (as used by Cisinski [5]). In this section we will give some of the details of Leinster's calculation and then show how to modify the proof to achieve the n-dimensional versions in Section 4.3.

4.1 Globular theory

We first recall some theory from [1]; for an alternative treatment see [12] or [6].

Definition 4.1. Let T be the free strict ω-category monad on $\mathsf{GSet} = [\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]$; T is familially representable and in particular cartesian.

- A T-operad (in the sense of Definition 1.2) is called a *globular operad*.
- A globular operad is called *contractible* if its underlying T-collection is contractible.
- A T-collection $\begin{array}{c} A \\ \downarrow p \\ T1 \end{array}$ is called *contractible* if

    1. given any 0-cells $a, b \in A$ and a 1-cell $y : pa \longrightarrow pb \in T1$, there exists a 1-cell $x : a \longrightarrow b \in A$ such that $px = y$, and
    2. for all $m \geq 1$, given any m-cells $a, b \in A$ that are "parallel" i.e. $sa = sb$ and $ta = tb$, and an $(m+1)$-cell $y : pa \longrightarrow pb \in T1$, there exists an $(m+1)$-cell $x : a \longrightarrow b \in A$ such that $px = y$.

  Note that for the finite n-dimensional version, we use the free strict n-category monad which we denote T_n, and need an extra condition at the nth dimension as follows: given any parallel n-cells $a, b \in A$ with $pa = pb \in T1$, we have $a = b$.

- A *weak ω-category* is any algebra for any contractible globular operad.
- A *weak ω-groupoid* is a weak ω-category in which every cell is weakly invertible. Batanin defines this via the n-coskeleta of the ω-category—the idea is that to be weakly invertible a k-cell in an ω-category should be weakly invertible in the n-coskeleton (the weak n-category formed by quotienting out by $(n+1)$-cells) for each $n \geq k$; weak invertibility in an n-category for finite n can be defined by induction. Since we will not need to use this definition we refer the reader to [1] for the full details.

The analogy with loop spaces should be clear; where for loop spaces we used operads with an operation of arity k for each $k \in \mathbb{N}$, we now have an operation of arity α for every pasting diagram α.

We wish to exhibit every globular set UX as an ω-groupoid, so first we must find a contractible globular operad that acts on each UX. Batanin proposes the following operad. Essentially the operations of arity $\alpha \in n\text{-}\mathsf{Pd}$ are the continuous, boundary-preserving maps from the topological n-ball to the geometric realisation of α. However we must be careful about exactly what boundary must be preserved. The idea is that the spaces in question should have globular "sources" and "targets" of each lower dimension, and these are the boundaries that should be preserved. This is expressed in Batanin's definition of "coglobular span in Top".

Definition 4.2. A *coglobular n-span* in a category $\mathcal{C}$ is a commuting diagram of the following shape in $\mathcal{C}$.

$$\begin{tikzcd}
& x & \\
a_{n-1} \arrow[ur] & & b_{n-1} \arrow[ul] \\
a_{n-2} \arrow[u] \arrow[urr] & & b_{n-2} \arrow[u] \arrow[ull] \\
a_{n-3} \arrow[u] \arrow[urr] & & b_{n-3} \arrow[u] \arrow[ull] \\
\vdots \arrow[u] \arrow[urr] & & \vdots \arrow[u] \arrow[ull] \\
a_1 & & b_1 \\
a_0 \arrow[u] \arrow[urr] & & b_0 \arrow[u] \arrow[ull]
\end{tikzcd}$$

Example 4.3. The topological n-ball B^n has maps

$$B^{n-1} \underset{t}{\overset{s}{\rightrightarrows}} B^n$$

given by the inclusions of the north and south hemispheres. This makes B^n into a coglobular n-span in Top as

$$\begin{tikzcd}
& B^n & \\
B^{n-1} \arrow[ur, "s"] & & B^{n-1} \arrow[ul, "t"'] \\
B^{n-2} \arrow[u, "s"] \arrow[urr, "s"'] & & B^{n-2} \arrow[u, "t"'] \arrow[ull, "t"] \\
\vdots \arrow[u, "s"] \arrow[urr, "s"'] & & \vdots \arrow[u, "t"'] \arrow[ull, "t"] \\
B^1 & & B^1 \\
B^0 \arrow[u, "s"] \arrow[urr, "s"'] & & B^0 \arrow[u, "t"'] \arrow[ull, "t"]
\end{tikzcd}$$

Example 4.4. Let α be an n-dimensional pasting diagram with source and target $\partial\alpha$. Then there are inclusions of the geometric realisations

$$\begin{tikzcd} {|\partial\alpha|} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & {|\alpha|} \end{tikzcd}$$

into the "source" and "target". This makes $|\alpha|$ into a coglobular n-span in Top as

$$\begin{tikzcd}
& {|\alpha|} & \\
{|\partial\alpha|} \arrow[ur, "s"] & & {|\partial\alpha|} \arrow[ul, "t"'] \\
{|\partial^2\alpha|} \arrow[u, "s"] \arrow[urr, "s" near start] & & {|\partial^2\alpha|} \arrow[u, "t"'] \arrow[ull, "t"' near start] \\
\vdots \arrow[u, "s"] \arrow[urr, "s" near start] & & \vdots \arrow[u, "t"'] \arrow[ull, "t"' near start] \\
{|\partial^{n-1}\alpha|} & & {|\partial^{n-1}\alpha|} \\
{|\partial^{n}\alpha|} \arrow[u, "s"] \arrow[urr, "s" near start] & & {|\partial^{n}\alpha|} \arrow[u, "t"'] \arrow[ull, "t"' near start]
\end{tikzcd}$$

Definition 4.5. A *map of coglobular n-spans in $\mathcal{C}$*

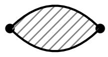
$$\begin{tikzcd}
& x & \\
a_{n-1} \arrow[ur, "s"] & & b_{n-1} \arrow[ul, "t"'] \\
a_{n-2} \arrow[u, "s"] \arrow[urr, "s" near start] & & b_{n-2} \arrow[u, "t"'] \arrow[ull, "t"' near start] \\
\vdots \arrow[u, "s"] \arrow[urr, "s" near start] & & \vdots \arrow[u, "t"'] \arrow[ull, "t"' near start] \\
a_1 & & b_1 \\
a_0 \arrow[u, "s"] \arrow[urr, "s" near start] & & b_0 \arrow[u, "t"'] \arrow[ull, "t"' near start]
\end{tikzcd}
\quad\longrightarrow\quad
\begin{tikzcd}
& x' & \\
a'_{n-1} \arrow[ur, "s"] & & b'_{n-1} \arrow[ul, "t"'] \\
a'_{n-2} \arrow[u, "s"] \arrow[urr, "s" near start] & & b'_{n-2} \arrow[u, "t"'] \arrow[ull, "t"' near start] \\
\vdots \arrow[u, "s"] \arrow[urr, "s" near start] & & \vdots \arrow[u, "t"'] \arrow[ull, "t"' near start] \\
a'_1 & & b'_1 \\
a'_0 \arrow[u, "s"] \arrow[urr, "s" near start] & & b'_0 \arrow[u, "t"'] \arrow[ull, "t"' near start]
\end{tikzcd}$$

is given by a map $v\colon x \longrightarrow x'$ and for all $0 \leq i \leq n-1$ maps

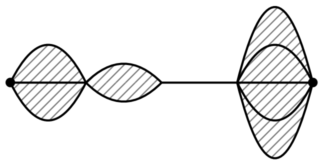
$$\begin{array}{rcl} f_i : a_i & \longrightarrow & a'_i \\ g_i : b_i & \longrightarrow & b'_i \end{array}$$

making everything commute.

Example 4.6. (Informal.) Consider the following coglobular 2-spans in Top: the ball B^2



and the geometric realisation $|\alpha|$



A map of coglobular spans $B^2 \longrightarrow |\alpha|$ is a map of the underlying spaces such that

- the top and bottom boundaries of B^2 are mapped to the top and bottom boundaries respectively of $|\alpha|$, and
- the endpoints of the top and bottom boundaries of B^2 are mapped to the endpoints of the top and bottom boundaries of $|\alpha|$.

4.2 The universal operad acting on ω-path spaces

We now invoke the results of Section 2 to show that $\mathsf{End}\,(U)$ in this case exists. As we are enriching in Set here we will revert to the more usual notation $[\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]$ instead of $\mathsf{Set}^{\mathbb{G}^{\mathrm{op}}}$.

Theorem 4.7. *Let T be the free ω-category monad on $[\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]$, and $U\colon \mathsf{Top} \longrightarrow [\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]$ the ω-path space functor. Then the internal hom*

$$\mathsf{End}\,(U) = \left[\begin{array}{ccc} T_*U & & T_*1 \times U \\ \downarrow & , & \downarrow \\ T_*1 & & T_*1 \end{array}\right]$$

exists in $\big[\mathsf{Top}, [\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]\big]/T^*1$.

Proof. We know that T is familially representable and U is a right adjoint: U is constructed via a functor $\mathbb{G} \xrightarrow{D} \mathsf{Top}$ where $D(n) = B^n$ the topological n-ball. Then

$$U(X) = \mathcal{S}(D-, X) \in [\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]$$

thus U is a right adjoint; its left adjoint is the left Kan extension of D along the Yoneda embedding

$$\begin{tikzcd} \mathbb{G} \arrow[r, "y_{\mathbb{G}}"] \arrow[d, "D"'] & {[\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]} \arrow[dl, shift right] \\ \mathsf{Top} \arrow[ur, shift right, "U"'] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

We can thus apply Theorem 2.4 to form the required internal hom as

$$[\hat{T} \circ U, (T1)^* \circ U].$$

□

We now sketch the calculation of the universal globular operad acting on ω-path spaces. This can be found in [12, Example 9.2.7] but we give some of the details here as we will be modifying the calculation to give the finite-dimensional cases in the next section.

In order to calculate the operad in this case, we need to use the exponential in the slice category $\big[\mathsf{Top}, [\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]\big]/T_*1$, which we will do via Remark 2.5 and the equivalences of Lemma 2.6:

$$\begin{aligned} \Big[\mathsf{Top},\ [\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]\Big]/T_*1 \ &\simeq\ \Big[\mathsf{Top},\ \big[(\mathbb{G}/T1)^{\mathrm{op}}, \mathsf{Set}\big]\Big] \\ &\simeq\ \Big[\mathsf{Top} \times (\mathbb{G}/T1)^{\mathrm{op}},\ \mathsf{Set}\Big] \end{aligned}$$

The rest of the calculation is given by Leinster; we will sketch the main details here.

Theorem 4.8 (Leinster, [12, Example 9.2.7]). *Let $U\colon \mathsf{Top} \longrightarrow \mathsf{GSet}$ be the ω-path space functor. Then the operad $E_U = \mathsf{ev}_\emptyset(\mathsf{End}\,(U))$ has as operations of arity α the maps of coglobular spans $B^n \longrightarrow |\alpha|$. Here α is a pasting diagram of dimension n and $|\alpha|$ is its geometric realisation.*

Proof. (Sketch) Write $\mathcal{S} = \mathsf{Top}$. Now

$$\mathsf{End}\,(U) = \begin{bmatrix} T_*U & T_*1 \times U \\ \downarrow \quad , & \downarrow \\ T_*1 & T_*1 \end{bmatrix} = \begin{bmatrix} A & B \\ \downarrow \ , & \downarrow \\ T_*1 & T_*1 \end{bmatrix},$$

say, where the square brackets denote the exponential in $\big[\mathcal{S}, [\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]\big]/T_*1$, which we know to exist by Theorem 2.4. Here

$$T_*1(X) = T1 : \mathbb{G}^{\mathrm{op}} \longrightarrow \mathsf{Set}$$

for all $X \in \mathcal{S}$, and

$$\overline{T_*1}(X, n) = T1(n) \in \mathsf{Set}.$$

Note that $\mathbb{G}^{\mathrm{op}}/T1$ has as objects pairs $(n \in \mathbb{N}, \alpha \in T1(n) = n\text{-}\mathsf{Pd})$. Now given

$$\begin{array}{c} S \\ \downarrow{\scriptstyle p} \\ T_*1 \end{array} \in \big[\mathcal{S}, [\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]\big]/T_*1,$$

with components

$$\begin{array}{c} SX \\ \downarrow{\scriptstyle p_X} \\ T1 \end{array},$$

we get

$$\bar{S} \in \big[\mathcal{S} \times (\mathbb{G}^{\mathrm{op}}/T1), \mathsf{Set}\big]$$

given by

$$\bar{S}(X, n, \alpha) = {p_X}^{-1}(\alpha) \subseteq SX(n).$$

Conversely given $S \in \big[\mathcal{S} \times (\mathbb{G}^{\mathrm{op}}/T1), \mathsf{Set}\big]$ we have $\begin{array}{c} \hat{S} \\ \downarrow{\scriptstyle p} \\ T_*1 \end{array} \in \big[\mathcal{S}, [\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]\big]/T_*1$

given by

$$\hat{S}X(n) = \coprod_{\alpha \in T1(n)} S(X, n, \alpha)$$

thus the fibre of $\hat{S}X$ over α is $S(X, n, \alpha)$.

So we have

$$\begin{array}{rcl} \bar{A}(X, n, \alpha) & = & \{\text{ pasting diagrams of shape } \alpha \text{ in } UX\,\} \\ & = & \mathcal{S}(|\alpha|, X) \end{array}$$

$$\begin{array}{rcl} \bar{B}(X, n, \alpha) & = & \{\, n\text{-cells in } UX\,\} \\ & = & \mathcal{S}(B^n, X) \end{array}$$

We can now use the usual internal hom formula in the functor category

$$\left[\mathcal{S} \times (\mathbb{G}^{\mathrm{op}}/T1), \mathsf{Set}\right]$$

to get

$$\overline{\mathsf{End}\,(U)} = \left[\mathcal{S} \times (\mathbb{G}^{\mathrm{op}}/T1), \mathsf{Set}\right]\left(H^{\bullet} \times \bar{A}, \bar{B}\right).$$

To find $\mathsf{ev}_{\emptyset}(\mathsf{End}\,(U))$ we can calculate fibre by fibre—the fibre over an n-pasting diagram α is

$$\overline{\mathsf{End}\,(U)}(\emptyset, n, \alpha) = \left[\mathcal{S} \times (\mathbb{G}^{\mathrm{op}}/T1), \mathsf{Set}\right]\left(H^{(\emptyset,n,\alpha)} \times \bar{A}, \bar{B}\right)$$

as a set of natural transformations.

Note that

$$H^{(\emptyset,n,\alpha)}(X, m, \beta) = \begin{cases} 1 & \alpha = \beta \\ \{s, t\} & \beta = \partial\alpha \\ \emptyset & \text{otherwise} \end{cases}$$

Thus a natural transformation as above must have component at (X, m, β) of the form:

- if $\alpha = \beta$, $\mathcal{S}(|\alpha|, X) \longrightarrow \mathcal{S}(B^n, X)$
- if $\beta = \partial\alpha$
$$\{s, t\} \times \mathcal{S}(|\beta|, X) \longrightarrow \mathcal{S}(B^m, X)$$
hence a pair of maps $\mathcal{S}(|\beta|, X) \longrightarrow \mathcal{S}(B^m, X)$,
- otherwise: $\emptyset \longrightarrow \mathcal{S}(B^m, X)$ i.e. the trivial map.

We now examine naturality; as our domain is a product category we can examine naturality in X and (m, β) separately.

- Naturality in X tells us we must have a natural transformation
$$\mathcal{S}(|\alpha|, -) \longrightarrow \mathcal{S}(B^n, -)$$
and for each $0 \leq m < n$ two natural transformations
$$\mathcal{S}(|\partial^{n-m}\alpha|, -) \longrightarrow \mathcal{S}(B^m, -).$$
By Yoneda this is just an element of $\mathcal{S}(B^n, |\alpha|)$ and two elements of $\mathcal{S}(B^m, |\partial^{n-m}\alpha|)$ for each $0 \leq m < n$, that is, the underlying data for a morphism of coglobular spans.

- Naturality in (m, β) tells us that we have the necessary commuting conditions to be a morphism of coglobular spans as required.

$\square$

4.3 Finite-dimensional versions

There are three finite n-dimensional versions of this that follow immediately, one by taking truncations, one by taking homotopy classes at the top dimension, and one by taking path spaces at the top. The analogous result for the truncated version follows immediately, while the other versions follow with a little effort. First we recall the functors in question, described in the introduction. Note that they all agree on the first $(n-1)$ dimensions.

Definition 4.9. We will define the following functors for each $n \geq 0$.

- "n-truncation" $U_n : \mathsf{Top} \longrightarrow n\text{-}\mathsf{GSet} \simeq [\mathbb{G}_n^{\mathrm{op}}, \mathsf{Set}]$
- "fundamental n-groupoid" $\Pi_n : \mathsf{Top} \longrightarrow n\text{-}\mathsf{GSet}$
- "n-path space" $\mathcal{P}_n : \mathsf{Top} \longrightarrow \mathsf{Top}\text{-}n\text{-}\mathsf{Gph} \subset [\mathbb{G}_n^{\mathrm{op}}, \mathsf{Top}]$

U_nX is the n-dimensional truncation of UX. Π_nX agrees with UX for all dimensions up to $n-1$ but $(\Pi_nX)(n)$ is given by homotopy classes of n-cells in UX in the following sense: we identify any parallel n-cells $x, y \in UX(n)$ if there is an $(n+1)$-cell $f : x \longrightarrow y$ in $UX(n+1)$. That is, we apply the functor $q_n : \mathsf{GSet} \longrightarrow n\text{-}\mathsf{GSet}$ which is left adjoint to the functor

$$n\text{-}\mathsf{GSet} \xrightarrow{D_n} \mathsf{GSet}$$

that adds in putative identities at every dimension above n. (Note that in general the description of q_n would require us to generate an equivalence relation from the above relation; however in the case of globular sets of the form UX the above description suffices since we always have reverse and composite homotopies.)

For $\mathcal{P}_n$ we are thinking of a "Top-enriched n-graph" as an n-graph whose n-cells form a space but every lower dimension is just a set. However in order to apply Theorem 2.4 we are going to express these as n-globular spaces, that is, objects X of the enriched presheaf category $\mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}}$ such that for all

$k < n$, $X(k)$ is indiscrete. As with globular sets, given k-cells x, y we also write $X(x, y)$ for the subset (or subspace) of $X(k+1)$ of cells with domain x and codomain y.

Then $\mathcal{P}_0(X) = X$ and for $n > 0$ we have

- $\mathcal{P}_n(X)$ agrees with $\mathcal{P}_{n-1}(X)$ at all dimensions up to $n-2$.
- $\mathcal{P}_n(X)(n-1)$ is the set of points of the space $\mathcal{P}_{n-1}(X)(n-1)$ (more precisely, the indiscrete space on the underlying set of points).
- Given $x, y \in \mathcal{P}_n(X)(n-1)$, we have $\mathcal{P}_n(X)(x, y) = \mathcal{P}_{n-1}(X)(x, y)$ (the path space).

Remarks 4.10.

1. The functor Π_n will be used to find fundamental n-groupoids of spaces, while U_n is used in [7] when constructing ω-categories from "incoherent" n-categories. $\mathcal{P}_n$ can be thought of as an "(∞, n)" version, where algebraic information is extracted up to dimension n, with non-algebraic information remaining in higher dimensions.
2. We could give ω-dimensional versions of these functors, but in fact these would all be the same as U.

Corollary 4.11. *Let T_n be the free strict n-category monad on n-GSet. Then there is a universal n-globular operad (i.e. T_n-operad) acting on U_n given by the n-truncation of E_U.*

Proof. This is immediate, with proof as in the proof of Theorems 4.7 and 4.8. $\square$

Theorem 4.12. *There is a universal n-globular operad acting on Π_n given by $q_n E_U$.*

Proof. We prove this by adapting the proof of Theorem 4.8. For $m < n$ the mth dimension behaves exactly as for U; for the n-cells we must calculate $\Pi_n X$ and $T_n \Pi_n X$ so we must quotient $\mathcal{S}(B^n, X)$ and $\mathcal{S}(|\alpha|, X)$ by the equivalence relation demanded by our definition of Π_n.

It is useful to make this equivalence relation precise. $\Pi_n = q_n U$ so all parallel n-cells x, y of UX are to be identified if there is an $(n+1)$-cell $f\colon x \longrightarrow y$. In UX this means

$$B^n \overset{x}{\underset{y}{\rightrightarrows}} X$$

are identified if they can be expressed as composites

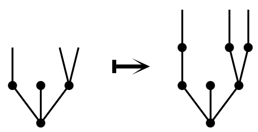
$$B^n \overset{s}{\underset{t}{\rightrightarrows}} B^{n+1} \xrightarrow{f} X \tag{4.1}$$

for some map f.

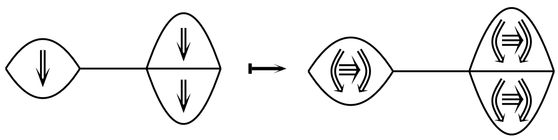
For $T_n q_n U$ we ask when we identify

$$|\alpha| \overset{x}{\underset{y}{\rightrightarrows}} X.$$

Write $\Sigma\alpha$ for the $(n+1)$-pasting diagram given by taking the tree for α and extending each leaf by one level. For example



or in pictures



Then we identify the n-cells x and y if they can be expressed as

$$|\alpha| \overset{|s|}{\underset{|t|}{\rightrightarrows}} |\Sigma\alpha| \xrightarrow{f} X \tag{4.2}$$

for some f. Note that the operation Σ is a form of suspension, and given a coglobular map

$$|\alpha| \longrightarrow |\beta|$$

there is a coglobular map

$$|\Sigma\alpha| \longrightarrow |\Sigma\beta|$$

making the following diagram commute

$$\begin{tikzcd} |\alpha| \arrow[r, shift left, "|s|"] \arrow[r, shift right, "|t|"'] \arrow[d, "p"'] & |\Sigma\alpha| \arrow[d, dashed] \\ |\beta| \arrow[r, shift left, "|s|"] \arrow[r, shift right, "|t|"'] & |\Sigma\beta| \end{tikzcd}$$

This is because the geometric realisation of an n-pasting diagram is homotopy equivalent to the n-ball, and the parallel pair of maps $|s|, |t|$ in each case gives, up to homotopy, the inclusion of the n-sphere (expressed as a pair of n-balls glued along their boundary) into the boundary of the $(n+1)$-ball.

Then we must ask the following questions.

1. Which natural transformations $\mathcal{S}(|\alpha|, -) \longrightarrow \mathcal{S}(B^n, -)$ respect this equivalence relation?

2. Which natural transformations $\mathcal{S}(|\alpha|, -) \longrightarrow \mathcal{S}(B^n, -)$ become the same on equivalence classes?

It is useful to note that the globular operad E_U is contractible (in the sense of 4.1). Now, we know that a natural transformation $\mathcal{S}(|\alpha|, -) \longrightarrow \mathcal{S}(B^n, -)$ is given by precomposition with a map $B^n \xrightarrow{p} |\alpha|$, and the naturality condition in $\mathbb{G}_n^{\text{op}}/T_n 1$ ensures that this will have to be a map of coglobular spans as before. We must check when equivalent elements of $\mathcal{S}(|\alpha|, X)$ are mapped to equivalent elements of $\mathcal{S}(B^n, X)$. In fact this is the case for all $B^n \xrightarrow{p} |\alpha|$ as follows. Writing our equivalent elements of $\mathcal{S}(|\alpha|, X)$ as

$$\begin{tikzcd} |\alpha| \arrow[r, shift left, "|s|"] \arrow[r, shift right, "|t|"'] & |\Sigma\alpha| \arrow[r, "f"] & X \end{tikzcd}$$

we map them to $\mathcal{S}(B^n, X)$ by precomposition with p to give the two maps

$$\begin{tikzcd} B^n \arrow[r, "p"] & |\alpha| \arrow[r, shift left, "|s|"] \arrow[r, shift right, "|t|"'] & |\Sigma\alpha| \arrow[r, "f"] & X \end{tikzcd}$$

which are equivalent via

$$\begin{tikzcd} B^n = |b_n| \arrow[r, shift left, "|s|"] \arrow[r, shift right, "|t|"'] \arrow[d, "p"'] & |\Sigma b_n| = B^{n+1} \arrow[d, dashed] \\ |\alpha| \arrow[r, shift left, "|s|"] \arrow[r, shift right, "|t|"'] & |\Sigma\alpha| \arrow[r, "f"] & X. \end{tikzcd}$$

Here we are writing b_n for the n-pasting diagram consisting of a single n-cell, thus $|b_n| = B^n$ and $|\Sigma b_n| = B^{n+1}$ so we have the maps

$$\begin{tikzcd} B^n \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & B^{n+1} \arrow[r, "f"] & X \end{tikzcd}$$

as in diagram (4.1) as required.

Next we show that all parallel maps $B^n \xrightarrow{p} |\alpha|$ induce the same map on equivalence classes. That is, given

$$\begin{tikzcd} B^n \arrow[r, shift left, "p"] \arrow[r, shift right, "p'"'] & {|\alpha|} \end{tikzcd}$$

agreeing on all boundaries, we show that for any $|\alpha| \xrightarrow{f} X$ the induced maps

$$\begin{tikzcd} B^n \arrow[r, shift left, "p"] \arrow[r, shift right, "p'"'] & {|\alpha|} \arrow[r, "f"] & X \end{tikzcd}$$

are equivalent elements of $\mathcal{S}(B^n, X)$ by expressing them as

$$\begin{tikzcd} {|\alpha|} \arrow[r, shift left, "|s|"] \arrow[r, shift right, "|t|"'] & {|\Sigma\alpha|} \arrow[r, "f"] & X \end{tikzcd}$$

as in diagram (4.2).

In fact since $|\alpha|$ is contractible we have

$$\begin{tikzcd} & B^{n+1} \arrow[d, dashed, "f"] & \\ B^n \arrow[ur, shift left, "s"] \arrow[ur, shift right, "t"'] \arrow[r, shift left, "p"] \arrow[r, shift right, "p'"'] & {|\alpha|} \arrow[r, "f"'] & X \end{tikzcd}$$

commuting serially giving an expression of the form of (4.2) as required. □

We now turn our attention to the more topological case. We use the monad S_n for "free n-categories internal to Top"; this monad is constructed in the same way as the free strict n-category monad (for n-categories internal to Set), except that we take pullbacks in Top instead of in Set. It follows immediately that S_n is p.r.a. We will also call this monad the "free topological n-category monad".

Now note that we can construct $\mathcal{P}_n$ via the usual Kan extension construction as below: we start with a functor $\mathbb{G}_n \xrightarrow{D} \mathsf{Top}$ where $D(n) = B^n$

the topological n-ball, and form the usual induced functor which we will temporarily call V

$$V(X) = \mathcal{S}(D-, X) \in \mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}}$$

and we then post compose with a functor

$$\mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}} \longrightarrow \mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}}$$

which leaves the top dimension the same but at every lower dimension takes the indiscrete space on the underlying set of points. Note that this construction uses the functors O producing the underlying set of points and I producing the indiscrete space:

$$\begin{tikzcd} \mathsf{Top} \arrow[r, shift left, "O"] \arrow[r, phantom, "\perp"] & \mathsf{Set} \arrow[l, shift left, "I"] \end{tikzcd}$$

Given an n-globular space

$$X_n \overset{s}{\underset{t}{\rightrightarrows}} X_{n-1} \overset{s}{\underset{t}{\rightrightarrows}} \quad \cdots \quad \overset{s}{\underset{t}{\rightrightarrows}} X_0$$

we produce the n-globular space

$$X_n \overset{s}{\underset{t}{\rightrightarrows}} OIX_{n-1} \overset{s}{\underset{t}{\rightrightarrows}} \quad \cdots \quad \overset{s}{\underset{t}{\rightrightarrows}} OIX_0$$

and with the source and target maps on n-cells proceeding via the counit of the adjunction $O \dashv I$. We will call this functor $OI_{<n}$.

Thus we have the following situation giving the functor $\mathcal{P}_n$:

$$\begin{tikzcd} \mathbb{G}_n \arrow[r, "y_{\mathbb{G}}"] \arrow[d, "D"'] & \mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}} \arrow[r, "OI_{<n}"] & \mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}} \\ \mathsf{Top} \arrow[ur, "V"] \arrow[urr, dashed, "\mathcal{P}_n"'] & & \end{tikzcd}$$

Lemma 4.13. *The functor $\mathcal{P}_n$ is a right adjoint.*

Proof. As Top is complete, well-powered, and has a cogenerator, it suffices to check that $\mathcal{P}_n$ preserves limits. As limits in $\mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}}$ are computed pointwise it suffices to check that for each object $k \in \mathbb{G}_n^{\mathrm{op}}$ the composite

$$\mathsf{Top} \xrightarrow{\mathcal{P}_n} \mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}} \xrightarrow{\mathsf{eval}_k} \mathsf{Top}$$

preserves limits. When $k = n$ this is just V, which we know is a right adjoint so preserves limits. When $k < n$ this is OI, and this is a composite of right adjoints so preserves limits. ☐

Theorem 4.14. *Write S_n for the free topological n-category monad on $\mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}}$. Then there is a universal S_n-operad acting on $\mathcal{P}_n$ whose m-cells are those of E_U for $m < n$, and whose space of n-cells of arity α is the* space *of coglobular maps $B^n \longrightarrow |\alpha|$.*

Note that $S_n 1$ is discrete at every dimension—it is in fact the same as $T_n 1$, just with each set of k-cells realised as a discrete space.

Proof. As S_n is p.r.a. and $\mathcal{P}_n$ is a right adjoint we may use Theorem 2.4 and compute the internal hom in the slice category

$$[\mathsf{Top}, \mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}}]/S_{n*}1.$$

Note that, under the equivalences of Corollary 2.9 and by Remark 2.5 we can use equivalences

$$\begin{aligned} [\mathsf{Top}, \mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}}]/S_{n*}1 &\simeq [\mathsf{Top}, \mathsf{Top}^{(\mathbb{G}_n^{\mathrm{op}}/S_n 1)}] \\ &\simeq [\mathsf{Top} \times \mathbb{G}_n^{\mathrm{op}}/S_n 1,\ \mathsf{Top}] \end{aligned}$$

Now $\mathcal{P}_n$ agrees with E_U everywhere except dimension n so this is the only dimension we need to consider here. In fact $\mathcal{P}_n$ agrees at dimension n as well if we simply "reinterpret" the notation

$$\mathcal{P}_n(X)(n) = \mathsf{Top}(B^n, X)$$

where this must now mean the *space* of maps $B^n \longrightarrow X$.

Now, following the proof of Theorem 4.8, to find the spaces of n-cells of arity α of $\mathsf{ev}_\emptyset(\mathsf{End}\,(U))$ we must calculate

$$\big[\mathsf{Top} \times (\mathbb{G}_n^{\mathrm{op}}/S_n 1), \mathsf{Top}\big]\big(H^{(\emptyset,n,\alpha)} \times \bar{A}, \bar{B}\big)$$

where

$$\begin{aligned} \bar{A}(X, m, \beta) &= \mathsf{Top}(|\beta|, X) \\ \bar{B}(X, m, \beta) &= \mathsf{Top}(B^m, X) \end{aligned}$$

interpreted as *spaces* of maps. We now have to calculate this as a space of enriched natural transformations using the end formula

$$\int_{(X,m,\beta)} \mathsf{Top}\Big(H^{(\emptyset,n,\alpha)}(X,m,\beta) \times \mathsf{Top}(|\beta|,X)\ ,\ \mathsf{Top}(B^m,X)\ \Big).$$

As before we have

$$H^{(\emptyset,n,\alpha)}(X,m,\beta) = \begin{cases} 1 & n=m, \alpha=\beta \\ \{s,t\} & n>m, \beta=\partial\alpha \\ \emptyset & \text{otherwise} \end{cases}$$

Fixing $\alpha=\beta$ we get

$$\int_{X\in\mathsf{Top}} \mathsf{Top}\ \big(\mathsf{Top}(|\alpha|,X)\ ,\ \mathsf{Top}(B^n,X)\ \big) = \mathsf{Top}(B^n,\alpha)$$

by enriched Yoneda. The rest of the end formula gives the same commuting conditions as before.

Note that evaluating the endomorphism operad $\mathsf{End}\,(\Pi_n)$ at (X,m,β) where $m<n$ gives the same answer as for $\mathsf{End}\,(U)$, expressed as a discrete space, so this internal hom is indeed in our full subcategory as required. $\square$

Example 4.15. Operads acting on path spaces.

Note that the case $n=1$ gives us operads acting on path spaces, but not in the most obvious sense as the operads in question will not be classical operads but S_1-operads.

The monad S_1 is the free topological category monad on what we might call $\mathsf{Gph(Top)}$, the category of graphs in spaces; $S_1 1$ has a single object, and its single hom-space is the discrete space $\mathbb{N}$. Thus an S_1-operad P has as its underlying data

- a set P_0 of objects, and
- for every pair a,b of objects and every arity $k\in\mathbb{N}$ a space of operations.

In particular any classical operad can be expressed as an S_1-operad with a single object; this is the "suspension" operation used in [4].

The functor $\mathcal{P}_1 : \mathsf{Top} \longrightarrow \mathsf{Top}^{\mathbb{G}_1^{\mathrm{op}}}$ takes a space X and produces the globular space with

- objects the indiscrete space on the points of X, and
- the hom-space is the space of paths in X, $\mathsf{Top}(I, X)$.

Note that this is not quite the same as a Top-enriched graph, which, essentially, would treat the hom-space as a disjoint union of individual hom-spaces $X(x, y)$.

The theorem then gives us the universal S_1-operad acting on this sense of path space, and examining the construction in that case shows that it is the suspension of the operad used by Trimble, which has $E(k) \in \mathsf{Top}$ is the space of continuous, endpoint-preserving maps $[0, 1] \longrightarrow [0, k]$. Thus we can say that Trimble's operad has the following universal property: its suspension is the universal S_1-operad acting on path spaces.

Note the notion of "operad acting on path spaces" can be defined directly without going via S_1-operads (see for example [11]), but making an abstract version of that approach is tricky. Trimble [18] uses the category Bip of bipointed spaces; however this is not a straightforward generalisation of the use of Top_* for loop spaces. For loop spaces we have

$$\begin{aligned} \Omega X &= \mathsf{Top}_*(S, X) \\ (\Omega X)^k &= \mathsf{Top}_*(S^{\vee k}, X); \end{aligned}$$

for path spaces we can try to replace the circle S with the interval I regarded as a bipointed space via its endpoints, giving

$$X(x, y) = \mathsf{Bip}\big(\{I, 0, 1\}, \{X, x, y\}\big).$$

However raising this to the power of k does not give us a string of k composable paths as we require. Trimble instead expresses the action on path spaces using "operads in topological profunctors".

4.4 Non-universal examples

In this final section we discuss non-universal versions of the operads studied in the previous sections. One class of non-universal examples comes from applying the work of [4]. Recall that in this work we showed that every Trimble n-category is a Batanin n-category. One part of this takes a classical operad acting on path spaces and iteratively produces from it an n-globular operad acting on n-path spaces for any n.

Applying the construction to the universal operad E (regarded as a classical operad) gives a suboperad $\bar{E}$ of G. Thus, by the main theorem, $\bar{E}$ acts on n-path spaces. Given any non-universal example P we have a canonical morphism $P \longrightarrow E$ giving rise to a morphism $\bar{P} \longrightarrow \bar{E}$ and hence $\bar{P} \longrightarrow \bar{E} \longrightarrow G$. So $\bar{P}$ also acts on n-path spaces.

Thus for example we can apply this to the little intervals operad or other non-universal classical operads and get a smaller operad acting on n-path spaces. However this general method only allows us to control the operations at the lowest dimension. The 1-cells of $\bar{P}$ are formed from the points of $P(k)$, but the 2-cells also involve the paths of $P(k)$ and the 3-cells involve the homotopies between paths, and so on. This approach suffices for some purposes and in future work we will use it to show that doubly degenerate Trimble 3-categories, parametrised by the little intervals operad, give braided monoidal categories in a suitable sense.

However for more general results more control over the higher dimensions of the globular operad may be desirable. This cannot be done automatically using the machinery of [4], but the present work gives us a first step in the direction of being able to construct more tractable non-universal operads suitable for proving results about weak n-categories.

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A NEW PROOF OF THE JOYAL-TIERNEY THEOREM

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Résumé. Nous donnons une preuve alternative du théorème bien connu de Joyal-Tierney dans la théorie des locales en utilisant la dualité de Priestley pour les cadres.

Abstract. We give an alternative proof of the well-known Joyal-Tierney Theorem in locale theory by utilizing Priestley duality for frames.

Keywords. Frame, localic map, Joyal-Tierney Theorem, Priestley duality

Mathematics Subject Classification (2010). 18F70; 06D22; 06D20; 06D50.

1. Introduction and Preliminaries

A well-known result in locale theory, known as the Joyal-Tierney Theorem, states that a localic map $f\colon M \to L$ is open iff its left adjoint $f^*\colon L \to M$ is a complete Heyting homomorphism (see, e.g., [8, Prop. III.7.2]). In addition, if L is subfit, then f is open iff f^* is a complete lattice homomorphism (see, e.g., [8, Prop. V.1.8]). Our aim is to give another proof of this result utilizing the language of Priestley spaces.

Priestley duality [9, 10] establishes a dual equivalence between the categories of bounded distributive lattices and Priestley spaces. We recall that a *Priestley space* is a Stone space X equipped with a partial order $\leq$ such that $x \not\leq y$ implies the existence of a clopen upset U such that $x \in U$ and $y \notin U$. A *Priestley morphism* is a continuous order-preserving map.

Pultr and Sichler [12] showed how to restrict Priestley duality to the category of frames. We recall (see, e.g., [8, p. 10]) that a *frame* is a complete lattice L satisfying the infinite distributive law $a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\}$ for each $a \in L$ and $S \subseteq L$. A map $h \colon L \to M$ between frames is a *frame homomorphism* if h preserves finite meets and arbitrary joins. Let **Frm** be the category of frames and frame homomorphisms.

Definition 1.1.

1. A Priestley space X is a *localic space*, or simply an *L-space*, provided the closure of an open upset is a clopen upset.
2. A Priestley morphism $f \colon X \to Y$ between L-spaces is an *L-morphism* provided $\mathsf{cl} f^{-1}U = f^{-1}\mathsf{cl}U$ for each open upset U of Y.
3. Let **LPries** be the category of L-spaces and L-morphisms.

Proposition 1.2. [12, p. 198] **Frm** *is dually equivalent to* **LPries**.

Remark 1.3. Since frames are exactly complete Heyting algebras (see, e.g., [6, Prop. 1.5.4]), every L-space is an Esakia space, where we recall that a Priestley space X is an *Esakia space* provided $\downarrow U$ is clopen for each clopen $U \subseteq X$ (equivalently, the closure of an open upset is an upset).

Remark 1.4. The contravariant functors establishing Pultr-Sichler duality are the restrictions of the contravariant functors establishing Priestley duality. They are described as follows.

For an L-space X, let $\mathsf{ClopUp}(X)$ be the frame of clopen upsets of X. The functor $\mathsf{ClopUp} \colon \mathbf{LPries} \to \mathbf{Frm}$ sends $X \in \mathbf{LPries}$ to the frame $\mathsf{ClopUp}(X)$ and an **LPries**-morphism $f \colon X \to Y$ to the **Frm**-morphism $f^{-1} \colon \mathsf{ClopUp}(Y) \to \mathsf{ClopUp}(X)$.

For $L \in \mathbf{Frm}$ let X_L be the set of prime filters of L ordered by inclusion and equipped with the topology whose basis is $\{\phi(a) \setminus \phi(b) : a, b \in L\}$, where $\phi \colon L \to \wp(X_L)$ is the *Stone map* $\phi(a) = \{x \in X_L : a \in x\}$. Then X_L is an L-space and the functor $\mathsf{pf} \colon \mathbf{Frm} \to \mathbf{LPries}$ sends $L \in \mathbf{Frm}$ to X_L and a **Frm**-morphism $h \colon L \to M$ to the **LPries**-morphism $h^{-1} \colon X_M \to X_L$.

Let L, M be frames. Every frame homomorphism $h \colon L \to M$ has a right adjoint $r = h_* \colon M \to L$, called a *localic map*. It is given by

$$r(b) = \bigvee\{a \in L : h(a) \leq b\}.$$

The following provides a characterization of localic maps:

Proposition 1.5. [8, Prop. II.2.3] *A map $r\colon M \to L$ between frames is a localic map iff*

(1) r *preserves all meets* (*so has a left adjoint* $h = r^*$);

(2) $r(a) = 1$ *implies* $a = 1$;

(3) $r(h(a) \to b) = a \to r(b)$.

Let **Loc** be the category of frames and localic maps. The following is obvious from Propositions 1.2 and 1.5:

Proposition 1.6. **Loc** *is dually isomorphic to* **Frm**, *and hence equivalent to* **LPries**.

To define open localic maps, we recall the notion of a sublocale which generalizes that of a subspace. Let L be a frame. A subset S of L is a *sublocale* of L if S is closed under arbitrary meets and $x \to s \in S$ for each $x \in L$ and $s \in S$. Sublocales correspond to nuclei, where we recall (see, e.g., [8, Sec. III.5.3]) that a *nucleus* on L is a map $\nu\colon L \to L$ satisfying

1. $a \leq \nu a$;
2. $\nu\nu a \leq \nu a$;
3. $\nu(a \wedge b) = \nu a \wedge \nu b$.

We can go back and forth between nuclei and sublocales as follows. If ν is a nucleus on L, then $S_\nu := \nu[L]$ is a sublocale of L. Conversely, if S is a sublocale of L, then $\nu_S\colon L \to L$ is a nucleus on L, where ν_S is given by $\nu_S(a) = \bigwedge\{s \in S : a \leq s\}$. This correspondence is one-to-one (see, e.g., [8, Prop. III.5.3.2]).

If $a \in L$, then $\mathfrak{o}(a) := \{a \to x : x \in L\}$ is a sublocale of L, called an *open sublocale* of L, whose corresponding nucleus ν_a is given by $\nu_a(x) = a \to x$ (see, e.g., [8, pp. 33, 35]).

Definition 1.7. [8, p. 37] A localic map $r\colon M \to L$ is *open* if for each open sublocale S of M, the image $r[S]$ is an open sublocale of L.

2. The Joyal-Tierney Theorem

The Joyal-Tierney Theorem provides a characterization of open localic maps (see, e.g., [7, Prop. 7.3] or [8, pp. 37–38]):

Theorem 2.1 (Joyal-Tierney)**.** *Let $r\colon M \to L$ be a localic map between frames with left adjoint h. The following are equivalent*:

(1) *r is open.*

(2) *h is a complete Heyting homomorphism.*

(3) *h has a left adjoint $\ell = h^*$ satisfying the Frobenius condition*

$$\ell(a \wedge h(b)) = \ell(a) \wedge b$$

for each $a \in M$ and $b \in L$.

Our aim is to give an alternative proof of this result using Priestley duality for frames. For this we need to translate the algebraic conditions of Theorem 2.1 into geometric conditions about Priestley spaces. We will freely use the following well-known lemma. For parts (1) and (2) see [4, Lems. 11.21, 11.22]; for part (3) see [11, Prop. 2.6]; and part (4) is a consequence of Esakia's lemma (see [6, Lem. 3.3.12]).

Lemma 2.2.

(1) *For a Priestley space X, the set $\{U \setminus V : U, V \in \mathsf{ClopUp}(X)\}$ is a basis of open sets of X.*

(2) *Let X be a Priestley space. If F, G are disjoint closed subsets of X, with F an upset and G a downset, then there is a clopen upset U of X such that $F \subseteq U$ and $G \cap U = \varnothing$. In particular, every open upset is a union and every closed upset is an intersection of clopen upsets.*

(3) *If F is a closed subset of a Priestley space, then ${\uparrow}F$ and ${\downarrow}F$ are closed.*

(4) *Let $f\colon X \to Y$ be a continuous map between Priestley spaces. For each $x \in X$ we have*

$$f\left[\bigcap\{U \in \mathsf{ClopUp}(X) : x \in U\}\right] = \bigcap\{f[U] : x \in U \in \mathsf{ClopUp}(X)\}.$$

We recall (see, e.g., [4, p. 265]) that if $h\colon L \to M$ is a frame homomorphism and $f\colon X_M \to X_L$ is its Priestley dual, then

$$f^{-1}\phi(a) = \phi h(a). \tag{a}$$

We also recall that if $r\colon M \to L$ is a localic map and S is a sublocale of M, then $r[S]$ is a sublocale of L (see, e.g., [8, Prop. III.4.1]).

Lemma 2.3. *Let $r\colon M \to L$ be a localic map with left adjoint h. If S is a sublocale of M, then $\nu_{r[S]} = r\nu_S h$.*

Proof. Let $a \in L$. We have

$$\begin{aligned}\nu_{r[S]}(a) &= \bigwedge\{r(s) : s \in S,\, a \le r(s)\}\\ &= \bigwedge\{r(s) : s \in S,\, h(a) \le s\}\\ &= r\left(\bigwedge\{s \in S : h(a) \le s\}\right)\\ &= r\nu_S h(a).\end{aligned}$$

Therefore, $\nu_{r[S]} = r\nu_S h$. □

We thus see that a localic map $r\colon M \to L$, with left adjoint h, is open iff for each $a \in M$ there is $b \in L$ with $r\nu_a h = \nu_b$. We use this observation in the proof of the following lemma.

Lemma 2.4. *Let $r\colon M \to L$ be a localic map, h the left adjoint of r, and $f\colon X_M \to X_L$ the Priestley dual of h. The following are equivalent*:

(1) *r is open.*

(2) *If U is a clopen upset of X_M, then $f[U]$ is a clopen upset of X_L.*

Proof. We start by showing that if $a \in M$ and $b, c \in L$, then

$$b \le (r\nu_a h)(c) \iff \phi(b) \cap f[\phi(a)] \subseteq \phi(c). \tag{b}$$

To see this,

$$\begin{aligned} b \le (r\nu_a h)(c) &\iff b \le r(a \to h(c)) \iff h(b) \le a \to h(c)\\ &\iff h(b) \wedge a \le h(c).\end{aligned}$$

Therefore, since $f[f^{-1}(B) \cap A] = B \cap f[A]$ for each A, B, by (a) we have

$$\begin{aligned} b \leq (r\nu_a h)(c) &\iff \phi h(b) \cap \phi(a) \subseteq \phi h(c) \\ &\iff f^{-1}\phi(b) \cap \phi(a) \subseteq f^{-1}\phi(c) \\ &\iff f[f^{-1}\phi(b) \cap \phi(a)] \subseteq \phi(c) \\ &\iff \phi(b) \cap f[\phi(a)] \subseteq \phi(c). \end{aligned}$$

(1)⇒(2). Let $U \in \mathsf{ClopUp}(X_M)$. Then $U = \phi(a)$ for some $a \in M$. By (1) and Lemma 2.3, there is $b \in L$ with $r\nu_a h = \nu_b$. Since $1 = \nu_b(b)$, we have $1 \leq (r\nu_a h)(b)$, so $\phi(1) \cap f[U] \subseteq \phi(b)$ by (b). Therefore, $f[U] \subseteq \phi(b)$. For the reverse inclusion, let $y \in \phi(b)$. If $y \notin f[U]$, then since $f[U]$ is closed in X_L, there is a clopen set containing y and missing $f[U]$. By Lemma 2.2(1), there are $c, d \in L$ with $y \in \phi(c) \setminus \phi(d)$ and $f[U] \cap (\phi(c) \setminus \phi(d)) = \varnothing$. Thus, $f[U] \cap \phi(c) \subseteq \phi(d)$, so $c \leq (r\nu_a h)(d) = \nu_b(d) = b \to d$ by (b). This gives $b \wedge c \leq d$, and hence $\phi(b) \cap \phi(c) \subseteq \phi(d)$, a contradiction since $y \in \phi(b) \cap \phi(c)$ but $y \notin \phi(d)$. Therefore, $y \in f[U]$, and so $\phi(b) \subseteq f[U]$. Consequently, $f[U] = \phi(b)$, and so $f[U] \in \mathsf{ClopUp}(X_L)$.

(2)⇒(1). Let $a \in M$ and set $U = \phi(a)$. Then $U \in \mathsf{ClopUp}(X_M)$, so $f[U] \in \mathsf{ClopUp}(X_L)$ by (2). Therefore, there is $b \in L$ with $\phi(b) = f[U]$. If $c, d \in L$, then by (b),

$$\begin{aligned} c \leq (r\nu_a h)(d) &\iff \phi(c) \cap f[U] \subseteq \phi(d) \\ &\iff \phi(c) \cap \phi(b) \subseteq \phi(d) \\ &\iff c \wedge b \leq d \\ &\iff c \leq b \to d \\ &\iff c \leq \nu_b(d). \end{aligned}$$

Thus, $r\nu_a h = \nu_b$, and hence r is open. □

We next give a dual characterization of when a frame homomorphism has a left adjoint. Let X be a Priestley space. Then we have two additional topologies on X, the topology of open upsets and the topology of open downsets. If cl_i and int_i are the corresponding closure and interior operators ($i = 1, 2$), then it is well known (see, e.g., [3, Lem. 6.5]) that for $A \subseteq X$ we have:

$$\begin{aligned} \mathsf{cl}_1 A = {\downarrow}\mathsf{cl} A \quad &\text{and} \quad \mathsf{int}_1(A) = X \setminus {\downarrow}(X \setminus \mathsf{int} A); \\ \mathsf{cl}_2 A = {\uparrow}\mathsf{cl} A \quad &\text{and} \quad \mathsf{int}_2(A) = X \setminus {\uparrow}(X \setminus \mathsf{int} A). \end{aligned}$$

Let L be a frame and let $a = \bigwedge S$ for $a \in L$ and $S \subseteq L$. Then

$$\phi(a) = \mathsf{int}_1 \bigcap \{\phi(s) : s \in S\} \tag{c}$$

(see, e.g., [2, Lem. 2.3]). This will be used in the following lemma.

Lemma 2.5. *Let $h\colon L \to M$ be a frame homomorphism and $f\colon X_M \to X_L$ its Priestley dual. The following are equivalent*:

(1) *h has a left adjoint.*

(2) *h preserves all meets.*

(3) *$f^{-1}\mathsf{int}_1 F = \mathsf{int}_1 f^{-1} F$ for each closed upset $F \subseteq X_L$.*

(4) *$\uparrow f[U]$ is clopen for each clopen upset $U \subseteq X_M$.*

Proof. (1)⇔(2). This is well known (see, e.g., [4, Prop. 7.34]).

(2)⇒(3). Let F be a closed upset of X_L. By Lemma 2.2(2), we may write $F = \bigcap\{\phi(s) : s \in S\}$ for some $S \subseteq L$. By (a),

$$\begin{aligned} f^{-1}(F) = f^{-1}\left(\bigcap\{\phi(s) : s \in S\}\right) &= \bigcap\{f^{-1}\phi(s) : s \in S\} \\ &= \bigcap\{\phi h(s) : s \in S\}, \end{aligned}$$

so

$$\mathsf{int}_1 f^{-1}(F) = \mathsf{int}_1 \bigcap\{\phi h(s) : s \in S\} = \phi\left(\bigwedge h[S]\right).$$

On the other hand, by (c) we have

$$\mathsf{int}_1 F = \mathsf{int}_1 \bigcap\{\phi(s) : s \in S\} = \phi\left(\bigwedge S\right).$$

Therefore, using (a) again yields

$$f^{-1}(\mathsf{int}_1 F) = f^{-1}\phi\left(\bigwedge S\right) = \phi h\left(\bigwedge S\right).$$

Thus, by (2) we have

$$\mathsf{int}_1 f^{-1}(F) = \phi\left(\bigwedge h[S]\right) = \phi h\left(\bigwedge S\right) = f^{-1}(\mathsf{int}_1 F).$$

(3)⇒(4). Let $U \in \mathsf{ClopUp}(X_M)$ and set $F = {\uparrow} f[U]$. By Lemma 2.2(3), F is a closed upset of Y. By (3),

$$U \subseteq \mathsf{int}_1 f^{-1}(f[U]) \subseteq \mathsf{int}_1 f^{-1} F = f^{-1}\mathsf{int}_1 F,$$

so $f[U] \subseteq \mathsf{int}_1 F$, and hence ${\uparrow} f[U] \subseteq \mathsf{int}_1 F = \mathsf{int}_1 {\uparrow} f[U]$. Thus, ${\uparrow} f[U]$ is clopen.

(4)⇒(1). Let $a \in M$. By (4), ${\uparrow} f[\phi(a)] \in \mathsf{ClopUp}(X_L)$. Therefore, there is a unique $b \in L$ such that $\phi(b) = {\uparrow} f[\phi(a)]$. Letting $\ell(a) = b$ defines a function $\ell \colon M \to L$ such that

$$\phi\ell(a) = {\uparrow} f[\phi(a)]. \tag{d}$$

To see that ℓ is left adjoint to h, let $c \in L$. Since $\phi(c)$ is an upset, by (a) we have

$$\begin{aligned} \ell(a) \le c \iff \phi\ell(a) \subseteq \phi(c) \iff {\uparrow} f[\phi(a)] \subseteq \phi(c) \iff f[\phi(a)] \subseteq \phi(c) \\ \iff \phi(a) \subseteq f^{-1}\phi(c) \iff \phi(a) \subseteq \phi h(c) \iff a \le h(c). \end{aligned}$$

□

We recall (see, e.g., [6, p. 9]) that a map $f \colon X \to Y$ between posets is a *bounded morphism* or a *p-morphism* if ${\downarrow} f^{-1}(y) = f^{-1}({\downarrow} y)$ for each $y \in Y$. Let $h \colon L \to M$ be a frame homomorphism between frames and $f \colon X_M \to X_L$ its Priestley dual. Then f is an L-morphism. It follows from Esakia duality for Heyting algebras [5, 6] that h preserves $\to$ iff f is a p-morphism. This together with Lemma 2.5 yields:

Lemma 2.6. *Let $h \colon L \to M$ be a frame homomorphism and $f \colon X_M \to X_L$ its dual L-morphism. Then h is a complete Heyting homomorphism iff f is a p-morphism and ${\uparrow} f[U]$ is clopen for each clopen upset U of X_M.*

We next provide a dual characterization of the Frobenius condition

$$\ell(a \wedge h(b)) = \ell(a) \wedge b$$

for each $a \in M$ and $b \in L$.

Lemma 2.7. *Let $h \colon L \to M$ be a frame homomorphism with Priestley dual $f \colon X_M \to X_L$. The following are equivalent*:

(1) *h has a left adjoint ℓ and $\ell(a \wedge h(b)) = \ell(a) \wedge b$ for all $a \in M$ and $b \in L$.*

(2) *$\uparrow f[U]$ is clopen and $\uparrow(f[U] \cap V) = \uparrow f[U] \cap V$ for all $U \in \mathsf{ClopUp}(X_M)$ and $V \in \mathsf{ClopUp}(X_L)$.*

Proof. By Lemma 2.5, h has a left adjoint ℓ iff $\uparrow f[U]$ is clopen for each $U \in \mathsf{ClopUp}(X_M)$. It is left to show that $\ell(a \wedge h(b)) = \ell(a) \wedge b$ for each $a \in M$ and $b \in L$ iff $\uparrow(f[U] \cap V) = \uparrow f[U] \cap V$ for each $U \in \mathsf{ClopUp}(X_M)$ and $V \in \mathsf{ClopUp}(X_L)$. Letting $U = \phi(a)$ and $V = \phi(b)$, since $\uparrow f[U] = \phi\ell(a)$ by (d), we have

$$\phi(\ell(a) \wedge b) = \phi\ell(a) \cap \phi(b) = \uparrow f[U] \cap V.$$

On the other hand, since $f[U \cap f^{-1}(V)] = f[U] \cap V$, by (a) we have

$$\begin{aligned}\phi\ell(a \wedge h(b)) &= \uparrow f[\phi(a \wedge h(b))] = \uparrow f[\phi(a) \cap \phi h(b)] \\ &= \uparrow f[\phi(a) \cap f^{-1}\phi(b)] = \uparrow f[U \cap f^{-1}(V)] \\ &= \uparrow(f[U] \cap V).\end{aligned}$$

Thus,

$$\begin{aligned}\ell(a \wedge h(b)) = \ell(a) \wedge b &\iff \phi\ell(a \wedge h(b)) = \phi(\ell(a) \wedge b) \\ &\iff \uparrow(f[U] \cap V) = \uparrow f[U] \cap V.\end{aligned}$$

□

We thus have translated the three conditions of Theorem 2.1 into the dual conditions in the language of Priestley spaces. We next prove that the translated conditions are equivalent.

Theorem 2.8. *Let $f\colon X \to Y$ be a Priestley morphism between L-spaces. The following are equivalent*:

(1) *If $U \in \mathsf{ClopUp}(X)$, then $f[U] \in \mathsf{ClopUp}(Y)$.*

(2) *f is a p-morphism and $\uparrow f[U]$ is clopen for all $U \in \mathsf{ClopUp}(X)$.*

(3) *$\uparrow f[U]$ is clopen and $\uparrow(f[U] \cap V) = \uparrow f[U] \cap V$ for all $U \in \mathsf{ClopUp}(X)$ and $V \in \mathsf{ClopUp}(Y)$.*

Proof. (1)⇒(2). Let $U \in \mathsf{ClopUp}(X)$. By (1), $f[U]$ is an upset of Y, so $\uparrow f[U] = f[U]$. Therefore, $\uparrow f[U]$ is clopen in Y by (1). It is left to prove that f is a p-morphism. For this it suffices to show that $f(\uparrow x)$ is an upset for each $x \in X$ (see, e.g, [6, Prop 1.4.12]). By Lemma 2.2(2),

$$\uparrow x = \bigcap \{U \in \mathsf{ClopUp}(X) : x \in U\},$$

so by Lemma 2.2(4),

$$\begin{aligned} f[\uparrow x] &= f\left[\bigcap \{U \in \mathsf{ClopUp}(X) : x \in U\}\right] \\ &= \bigcap \{f[U] : x \in U \in \mathsf{ClopUp}(X)\}. \end{aligned}$$

Thus, $f[\uparrow x]$ is an upset by (1).

(2)⇒(3). It is sufficient to show that $\uparrow(f[U] \cap V) = \uparrow f[U] \cap V$ for each $U \in \mathsf{ClopUp}(X)$ and $V \in \mathsf{ClopUp}(Y)$. But since f is a p-morphism, $\uparrow f[U] = f[U]$, so $\uparrow f[U] \cap V = f[U] \cap V = \uparrow(f[U] \cap V)$ because $f[U] \cap V$ is an upset.

(3)⇒(1). It suffices to show that $f[U]$ is an upset. If not, then there exist $x \in U$ and $y \in Y$ with $f(x) \leq y$ but $y \notin f[U]$. This yields $y \notin \downarrow(\downarrow y \cap f[U])$, so there is a clopen upset V of Y such that $y \in V$ and $V \cap \downarrow y \cap f[U] = \varnothing$ (see Lemma 2.2(2)). Therefore, $y \notin \uparrow(f[U] \cap V)$ but $y \in \uparrow f[U] \cap V$, a contradiction to (3). Thus, $f[U]$ is an upset. □

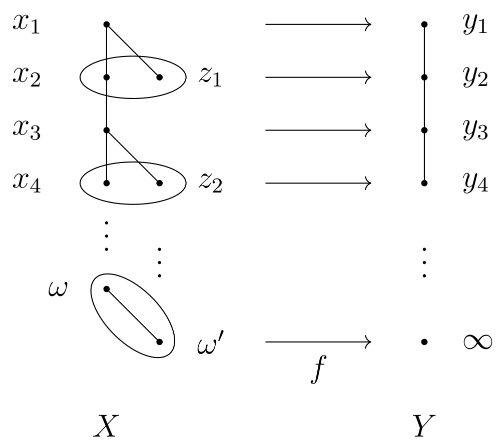
By Lemmas 2.4, 2.6 and 2.7, the three conditions of Theorem 2.8 are equivalent to the corresponding three conditions of Theorem 2.1. Hence, the Joyal-Tierney Theorem is a consequence of Theorem 2.8. We conclude this section with the following observation.

Remark 2.9. Condition (1) of Theorem 2.8 is equivalent to:

(1′) If U is an open upset of X, then $f[U]$ is an open upset of Y.

Clearly (1′) implies (1) since if U is clopen, then $f[U]$ is closed, hence a clopen upset of Y by (1′). Conversely, if U is an open upset, then $U = \bigcup\{V \in \mathsf{ClopUp}(X) : V \subseteq U\}$ by Lemma 2.2(2). Therefore, $f[U] = \bigcup\{f[V] : V \in \mathsf{ClopUp}(X),\ V \subseteq U\}$ is a union of clopen upsets of Y by (1). Thus, $f[U]$ is an open upset of Y. Consequently, (1) is equivalent to f being an open map with respect to the open upset topologies.

On the other hand, this does not imply that f is an open map with respect to the Stone topologies. To see this, we use the space defined in [1, p. 32]. Let X be the 2-point compactification of the discrete space $\{x_n, z_n : n \geq 1\}$ with ω the limit point of $\{x_n : n \geq 1\}$ and ω' the limit point of $\{z_n : n \geq 1\}$. Let Y be the 1-point compactification of the discrete space $\{y_n : n \geq 1\}$. We order X and Y and define the map $f\colon X \to Y$ as shown in the diagram below.



It is straightforward to see that X and Y are L-spaces and f is an L-morphism such that $f[U]$ is a clopen upset of Y for each clopen upset U of X. However, f is not an open map since $U := \{z_n : n \geq 1\} \cup \{\omega'\}$ is an open subset of X whose image $\{y_{2n} : n \geq 1\} \cup \{\infty\}$ is not an open subset of Y.

3. The subfit case

As was shown in [8, Prop. V.1.8], if in the Joyal-Tierney Theorem we assume that L is subfit, then the localic map $r\colon M \to L$ is open iff its left adjoint $h\colon L \to M$ is a complete lattice homomorphism (so h being a Heyting homomorphism becomes redundant). We will give an alternative proof of this result in the language of Priestley spaces.

We recall that a frame L is *subfit* if for all $a, b \in L$ we have

$$a \not\leq b \Longrightarrow (\exists c \in L)(a \vee c = 1 \text{ and } b \vee c \neq 1).$$

We next give a dual characterization of when L is subfit. As usual, for a poset X we write $\min X$ for the set of minimal points of X.

Lemma 3.1. *Let L be a frame and X_L its Priestley space. Then L is subfit iff $\min X_L$ is dense in X_L.*

Proof. First suppose that $\min X_L$ is dense in X_L. To see that L is subfit, let $a, b \in L$ with $a \not\leq b$. Then $\phi(a) \not\subseteq \phi(b)$, so $\phi(a) \setminus \phi(b)$ is a nonempty clopen subset of X. Therefore, there is $x \in (\phi(a) \setminus \phi(b)) \cap \min X_L$. Let $U = X_L \setminus \{x\}$. Then U is an open upset of X_L. Since $\phi(a) \cup U = X_L$ and U is a union of clopen upsets (see Lemma 2.2(2)), compactness of X_L implies that there is a clopen upset $U' \subseteq U$ with $\phi(a) \cup U' = X_L$. Because $U' = \phi(c)$ for some $c \in L$, we have $a \vee c = 1$. On the other hand, since $x \notin \phi(b) \cup U' = \phi(b \vee c)$, it follows that $b \vee c \neq 1$. Thus, L is subfit.

Conversely, suppose that $\min X_L$ is not dense in X_L. Then there is a nonempty clopen subset A of X_L such that $A \cap \min X_L = \varnothing$. We may assume that $A = U \setminus V$, where $U \not\subseteq V$ are clopen upsets of X_L (see Lemma 2.2(1)). From $A \cap \min X_L = \varnothing$ it follows that $U \cap \min X_L \subseteq V$. Let $a, b \in L$ be such that $U = \phi(a)$ and $V = \phi(b)$. Since $U \not\subseteq V$, we have $a \not\leq b$. Suppose $c \in L$ is such that $a \vee c = 1$. Let $W = \phi(c)$. Then $U \cup W = X_L$, so $\min X_L \subseteq U \cup W$. Because $U \cap \min X_L \subseteq V$, this yields $\min X_L \subseteq V \cup W$, which forces $V \cup W = X_L$ because ${\uparrow} \min X_L = X_L$ (see, e.g., [6, Thm. 3.2.1]). Thus, $b \vee c = 1$, and hence L is not subfit. $\square$

Lemma 3.2. *Let $f \colon X \to Y$ be a Priestley morphism between L-spaces. If $\min Y$ is dense in Y and ${\uparrow} f[U]$ is clopen for each $U \in \mathsf{ClopUp}(X)$, then f is a p-morphism.*

Proof. It is sufficient to show that Condition (1) of Theorem 2.8 holds, which amounts to showing that $f[U]$ is an upset for each $U \in \mathsf{ClopUp}(X)$. If not, then ${\uparrow} f[U] \setminus f[U] \neq \varnothing$ for some $U \in \mathsf{ClopUp}(X)$. Let $V = {\uparrow} f[U] \setminus f[U]$. Since ${\uparrow} f[U]$ is open and $f[U]$ is closed, V is a nonempty open subset of Y. Thus, $V \cap \min Y \neq \varnothing$ because $\min Y$ is dense in Y. On the other hand,

$$V \cap \min Y \subseteq {\uparrow} f[U] \cap \min Y = f[U] \cap \min Y.$$

This is a contradiction since $V \cap f[U] = \varnothing$. Consequently, $f[U]$ is an upset. $\square$

As an immediate consequence of Lemma 3.2, we obtain:

Theorem 3.3. *Let $f\colon X \to Y$ be a Priestley morphism between L-spaces. If $\min Y$ is dense in Y, then Condition (2) in Theorem* 2.8 *is equivalent to*

$(2')$ $\uparrow f[U]$ *is clopen for each* $U \in \mathsf{ClopUp}(X)$.

Theorems 2.8 and 3.3 together with Lemmas 2.4 and 2.5 yield the following version of the Joyal-Tierney Theorem:

Corollary 3.4. [8, Prop. V.1.8] *Let $r\colon M \to L$ be a localic map with left adjoint h. If L is subfit, then r is open iff h is a complete lattice homomorphism.*

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

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CATEGORICAL MODELS OF UNSTABLE *G*-GLOBAL HOMOTOPY THEORY

Tobias LENZ

Résumé. Nous prouvons que la catégorie ***G*-Cat** des petites catégories avec G-action forme un modèle de la théorie de l'homotopie instable G-globale pour tout groupe discret G, généralisant la structure de modèle global de Schwede sur **Cat**. En conséquence, nous prouvons que ***G*-Cat** modélise la théorie de l'homotopie équivariante G appropriée, non seulement lorsque nous testons les équivalences faibles sur les points fixes, mais aussi lorsque nous les testons sur les points fixes *d'homotopie* catégorielle.

Abstract. We prove that the category ***G*-Cat** of small categories with G-action forms a model of unstable G-global homotopy theory for every discrete group G, generalizing Schwede's global model structure on **Cat**. As a consequence, we prove that ***G*-Cat** models proper G-equivariant homotopy theory not only when we test weak equivalences on fixed points, but also when we test them on categorical *homotopy* fixed points.

Keywords. Equivariant homotopy theory, nerve functor, model categories, Thomason model structure.

Mathematics Subject Classification (2010). 55P91, 18G55

Introduction

It is an observation going back to Quillen [Ill72, VI.3] that every topological space is weakly equivalent to the classifying space of a small category, and that in fact taking classifying spaces yields an equivalence between the homotopy category of small categories (formed with respect to those functors that induce homotopy equivalences of classifying spaces) and the usual unstable homotopy category. This comparison was later lifted to a model categorical statement by Thomason [Tho80] who constructed a model structure on the category **Cat** of small categories with the above weak equivalences and proved that it is Quillen equivalent to the usual Kan-Quillen model structure on simplicial sets.

In more recent years, several generalizations and refinements of Thomason's and Quillen's results have been established:

In [BMO⁺15], Bohmann, Mazur, Osorno, Ozornova, Ponto, and Yarnall constructed for any discrete group G a model structure on the category $\boldsymbol{G}$**-Cat** of small G-categories in which a map is a weak equivalence if and only if it induces weak equivalences on all fixed points. Moreover, they proved that $\boldsymbol{G}$**-Cat** is Quillen equivalent to the usual G-equivariant model structure on G-simplicial sets, thereby establishing $\boldsymbol{G}$**-Cat** as a model of unstable *G-equivariant homotopy theory*. This result was strengthened by May, Stephan, and Zakharevich [MZS17] who showed that already the full subcategory of G-posets models the same homotopy theory, generalizing a non-equivariant result due to Raptis [Rap10].

On the other hand, we can consider *global homotopy theory* [Sch18] which, roughly speaking, studies equivariant phenomena that exist uniformly across suitable families of groups, like all finite groups or all compact Lie groups. In this setting, Schwede [Sch19] refined Thomason's result by constructing the so-called *global model structure* on **Cat** and proving that it is Quillen equivalent to the orbispace model of unstable global homotopy theory with respect to finite groups.

In the present paper, we generalize Schwede's result by establishing $\boldsymbol{G}$**-Cat** for every discrete group G as a model of unstable *G-global homotopy theory* in the sense of [Len20, Chapter 1]. G-global homotopy theory arises for example naturally in the study of global infinite loop spaces [Len20, Chapter 2] or in the form of various 'Galois-global' phenomena [Sch22b].

For every G, G-global homotopy theory admits a Bousfield localization to *proper* G-equivariant homotopy theory—i.e. equivariant homotopy theory where we only consider the fixed points for *finite* subgroups—and we make the localization functor explicit for the model constructed in [BMO⁺15].

As a consequence of this comparison, we obtain a new model structure on ***G*-Cat** that still models proper *G*-equivariant homotopy theory, but whose weak equivalences are now tested on *categorical homotopy fixed points* (i.e. homotopy fixed points formed with respect to the underlying equivalences of categories) as opposed to ordinary fixed points. There is then a Quillen equivalence between this new model structure and the one of [BMO+15], whose right adjoint is given by

$$\mathrm{Fun}(EG,-)\colon \boldsymbol{G}\text{-}\mathbf{Cat}_{\text{homotopy fixed points}} \to \boldsymbol{G}\text{-}\mathbf{Cat}_{\text{fixed points}} \qquad (*)$$

where EG is the contractible groupoid with object set G, equipped with the evident G-action.

Our interest in the above model structure and the Quillen equivalence $(*)$ comes from *equivariant algebraic K-theory* as studied in [GM17, Mer17]. Namely, as observed in [GMMO20, §3], $(*)$ lifts to a functor from the category of small symmetric monoidal categories with G-action to the category of so-called *genuine symmetric monoidal G-categories*. Equivariant algebraic K-theory in the sense of [GM17] is defined in terms of the latter, and it is only this lift of $(*)$ that allows to define the G-equivariant algebraic K-theory of a plain symmetric monoidal category with G-action.

In the sequel [Len22], we will prove that also this lift induces an equivalence of homotopy theories with respect to the above notions of weak equivalences; in particular, from the point of view of algebraic K-theory, there is no harm in just working with ordinary symmetric monoidal categories with G-action. While the argument we will give in [Len22] will be formally mostly independent of the results of the present paper, the equivalence $(*)$ provided much of the original motivation for [Len22]. Moreover, the proof we give here requires much less machinery than its symmetric monoidal counterpart, and we think it is actually instructive to have a direct argument available in this case.

Organization

In Section 1 we recall some basic facts about Thomason's model structure on **Cat** as well as our reference model of unstable G-global homotopy theory in terms of simplicial sets equipped with an action of a specific simplicial monoid.

Section 2 is devoted to establishing a general criterion for the existence of transferred model structures, which we then employ in Section 3 to construct (under a mild technical assumption) a model structure on the category of small categories with the action of a given categorical monoid and to compare it to its simplicial counterpart, partially generalizing [BMO+15]. Using this, we establish a categorical analogue of our reference model of G-global homotopy theory and prove that these two models are Quillen equivalent.

Finally, we construct the desired G-global model structure on $\boldsymbol{G}$**-Cat** in Section 4 and compare it to our previous models of G-global homotopy theory as well as the proper G-equivariant model structure on $\boldsymbol{G}$**-Cat**.

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1. Preliminaries

1.1 *G*-global homotopy theory

[Len20, Chapter 1] introduces several equivalent models of unstable G-global homotopy theory and studies their relation to G-equivariant and global homotopy theory. Here we will recall one of these models, which is based on a specific monoid $\mathcal{M}$ that we call the *universal finite group*:

Definition 1.1. We write $\omega = \{0, 1, \dots\}$ for the set of natural numbers and we denote by $\mathcal{M}$ the monoid (under composition) of all injections $\omega \to \omega$.

Remark 1.2. In addition to their role in (G-)global homotopy theory [Len20, Sch22a], which we will detail below, $\mathcal{M}$-actions have been studied in various places in the literature, for example in relation to the homotopy groups of symmetric spectra [Sch08] or in the study of E_∞-monoids [SS21]. In several of these applications, one imposes an additional *tameness* condition on the $\mathcal{M}$-action, demanding that the action on any given element x only depend on the values of an injection $u \in \mathcal{M}$ on a suitable *finite* set $\mathrm{supp}(x) \subset \omega$. On the other hand, sets with an action of the maximal subgroup core $\mathcal{M}$ (i.e. the group of bijective self-maps of ω) satisfying an analogous notion of tameness have been studied in logic and theoretical computer science under the name *nominal sets* [Pit13], and together with the equivariant maps they form a topos, called the *Schanuel topos*. It is not hard to prove, and also follows by combining [SS21, Proposition 5.6] with [Pit13, Theorem 6.8], that a *tame* $\mathcal{M}$-action is already uniquely determined by the action of the invertible elements, i.e. the Schanuel topos is equivalent to the category of tame $\mathcal{M}$-sets via the forgetful functor.

In contrast to that, we will work with general $\mathcal{M}$-actions throughout the present paper (which are *not* determined by the action of the maximal subgroup), and we instead refer the reader e.g. to [Len20, Sections 1.3 and 2.1] or [Len22, Section 4] for the role of tameness in (G-)global homotopy theory.

Definition 1.3. A finite subgroup $H \subset \mathcal{M}$ is called *universal* if ω with the restriction of the tautological $\mathcal{M}$-action is a complete H-set universe.

Here we call a countable H-set $\mathcal{U}$ a *complete H-set universe* if every other countable H-set embeds into $\mathcal{U}$ equivariantly.

It is in fact not hard to show that every finite group H admits an injective homomorphism $i\colon H \to \mathcal{M}$ such that $i(H)$ is universal, and that any two such homomorphisms differ only by conjugation with an invertible element of $\mathcal{M}$ [Len20, Lemma 1.2.8]. In particular, if X is any simplicial set with an $\mathcal{M}$-action, then we can associate to this an H-fixed point space for any abstract finite group H by picking such a homomorphism $i\colon H \to \mathcal{M}$ and taking $i(H)$-fixed points. However, while this space is independent of the chosen homomorphism i up to isomorphism, this isomorphism itself is *not* canonical, even up to homotopy. One way to solve this is to pass to a certain simplicial extension of the monoid $\mathcal{M}$, which relies on the following construction:

Construction 1.4. Recall that the functor **Cat** $\to$ **Set** sending a small category to its set of objects admits a right adjoint E (usually called the 'chaotic' or 'indiscrete' category functor). Explicitly, EX is the category with object set X and precisely one morphism $x \to y$ for any $x, y \in X$. Composition in EX is defined in the unique possible way, and if $f\colon X \to Y$ is a map of sets, then Ef is the unique functor which is given on objects by f.

Likewise, the functor **SSet** $\to$ **Set** sending a simplicial set Y to its set of 0-simplices admits a right adjoint E. Explicitly, E is given on objects by $(EX)_n = \prod_{i=0}^n X \cong \mathrm{maps}(\{0,\dots,n\}, X)$ with the evident functoriality in n and X.

Note that the nerve of the category EX is indeed canonically isomorphic to the simplicial set of the same name, justifying that we won't distinguish between them notationally.

As E is a right adjoint, it in particular preserves products so that the category or simplicial set $E\mathcal{M}$ inherits a natural monoid structure from $\mathcal{M}$.

Warning 1.5. If G is a discrete group acting on a set X, one can form the *translation category* $X/\!/G$, whose objects are given by the set X and with $\mathrm{Hom}(x, y) = \{g \in G : g.x = y\}$ for any $x, y \in X$. If $X = G$ with its usual action, this agrees with the indiscrete category EG, see e.g. [GMM17, Proposition 1.8], and accordingly some sources like [EM06, Section 10] refer to EG as the translation category (while still using the above notation). Beware however that while the construction of the translation category makes perfect sense for any monoid action, the result for the monoid $\mathcal{M}$ would be different from the indiscrete category $E\mathcal{M}$ (in particular, the translation category $\mathcal{M}/\!/\mathcal{M}$ contains nontrivial endomorphisms).

Definition 1.6. We write $\boldsymbol{E\mathcal{M}}$**-**$\boldsymbol{G}$**-SSet** for the category whose objects are the simplicial sets equipped with an action of the simplicial monoid $E\mathcal{M} \times G$, and whose morphisms are the $(E\mathcal{M} \times G)$-equivariant maps. A map $f\colon X \to Y$ in $\boldsymbol{E\mathcal{M}}$**-**$\boldsymbol{G}$**-SSet** is called a *$G$-global weak equivalence* if f^φ is a weak equivalence for every universal subgroup $H \subset \mathcal{M}$ and every homomorphism $\varphi\colon H \to G$; here we write $(-)^\varphi$ for the fixed points with respect to the *graph subgroup* $\Gamma_{H,\varphi} := \{(h, \varphi(h)) : h \in H\} \subset \mathcal{M} \times G$.

Next, we want to recall the *G-global model structure* on $\boldsymbol{E\mathcal{M}}$**-**$\boldsymbol{G}$**-SSet**. This is actually just a particular instance of the following proposition, which generalizes the usual equivariant model structures for group actions, see e.g. [Ste16, Example 2.14], to monoid actions:

Proposition 1.7. *Let M be a simplicial monoid and let $\mathcal{F}$ be a collection of finite subgroups of M_0. Then there exists a unique model structure on* $\boldsymbol{M}$-**SSet** *in which a map f is a weak equivalence or fibration if and only if f^H is a weak equivalence or fibration, respectively, in the usual Kan-Quillen model structure on* **SSet** *for each $H \in \mathcal{F}$. We will refer to this as the $\mathcal{F}$-*model structure *and to its weak equivalences as the $\mathcal{F}$-*weak equivalences.

The $\mathcal{F}$-model structure is combinatorial with generating cofibrations

$$\{M/H \times \partial\Delta^n \hookrightarrow M/H \times \Delta^n : H \in \mathcal{F}, n \geq 0\}$$

and generating acyclic cofibrations

$$\{M/H \times \Lambda^n_k \hookrightarrow M/H \times \Delta^n : H \in \mathcal{F}, 0 \leq k \leq n\}.$$

Moreover it is simplicial (for the obvious enrichment), proper, and a commutative square is a homotopy pushout or pullback if and only if the induced square on H-fixed points is a homotopy pushout or pullback, respectively, in **SSet** *for every $H \in \mathcal{F}$. Pushouts along underlying cofibrations are homotopy pushouts.*

Finally, the $\mathcal{F}$-weak equivalences are stable under filtered colimits.

Proof. [Len20, Proposition 1.1.2] shows all of these except for the characterizations of homotopy pushouts and pullbacks. The statement about homotopy pullbacks is obvious, while the ones about homotopy pushouts are instances of [Len20, Proposition 1.1.6 and Lemma 1.1.14]. □

Specializing to our situation we get, also see [Len20, Corollary 1.2.34]:

Corollary 1.8. *There is a unique model structure on* $\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**SSet** *in which a map f is a weak equivalence or fibration if and only if f^φ is a weak equivalence or fibration, respectively, in the usual Kan-Quillen model structure on* **SSet**. □

In particular, the weak equivalences of this model structure are precisely the G-global weak equivalences, and accordingly we refer to this as the *G-global model structure*. Proposition 1.7 also provides us with explicit sets of generating (acyclic) cofibrations: writing $E\mathcal{M} \times_\varphi G$ for $(E\mathcal{M} \times G)/\Gamma_{H,\varphi}$ these are given by

$$\{E\mathcal{M} \times_\varphi G \times (\partial\Delta^n \hookrightarrow \Delta^n) : n \geq 0,\ H \subset \mathcal{M} \text{ universal}, \varphi\colon H \to G\}$$
$$\{E\mathcal{M} \times_\varphi G \times (\Lambda^n_k \hookrightarrow \Delta^n) : 0 \leq k \leq n,\ H \subset \mathcal{M} \text{ universal}, \varphi\colon H \to G\}.$$

Finally, we come to the relation between G-global and *proper G-equivariant homotopy theory*, i.e. G-equivariant homotopy theory with respect to the collection of all *finite* subgroups $H \subset G$.

Theorem 1.9. *The functor* $\mathrm{triv}_{E\mathcal{M}}\colon \boldsymbol{G}\textbf{-SSet}_{\text{proper}} \to \boldsymbol{E\mathcal{M}}\textbf{-}\boldsymbol{G}\textbf{-SSet}_{G\text{-global}}$ *equipping a G-simplicial set with the trivial $E\mathcal{M}$-action is homotopical. The induced functor* $\mathrm{Ho}(\mathrm{triv}_{E\mathcal{M}})\colon \mathrm{Ho}(\boldsymbol{G}\textbf{-SSet}) \to \mathrm{Ho}(\boldsymbol{E\mathcal{M}}\textbf{-}\boldsymbol{G}\textbf{-SSet})$ *on homotopy categories fits into a sequence of four adjoints suggestively denoted by*

$$\mathbf{L}(-/E\mathcal{M}) \dashv \mathrm{Ho}(\mathrm{triv}_{E\mathcal{M}}) \dashv (-)^{\mathbf{R}E\mathcal{M}} \dashv \mathcal{R}.$$

Moreover, $(-)^{\mathbf{R}E\mathcal{M}}$ *is a (Bousfield) localization at the $\mathcal{E}$-weak equivalences, where $\mathcal{E}$ denotes the collection of all subgroups $\Gamma_{H,\varphi} \subset \mathcal{M} \times G$ with universal $H \subset \mathcal{M}$ for which φ is* injective.

In particular, $\mathrm{triv}_{E\mathcal{M}} : \boldsymbol{G}\textbf{-SSet}_{\text{proper}} \to \boldsymbol{E\mathcal{M}}\textbf{-}\boldsymbol{G}\textbf{-SSet}_{\mathcal{E}\text{-w.e.}}$ *descends to an equivalence of homotopy categories.*

Proof. See [Len20, Theorem 1.2.92]. □

Remark 1.10. In fact, *loc. cit.* establishes the above result on the level of ∞-categorical localizations. For simplicity, we will stick to the formulation in terms of classical homotopy categories in the present paper.

1.2 The Thomason model structure

We close this section by recalling Thomason's model structure on **Cat** that models the ordinary homotopy theory of spaces. While the usual nerve functor N induces an equivalence of homotopy categories by [Ill72, Corollaire 3.3.1], it can't be part of a Quillen equivalence to the Kan-Quillen model structure as its left adjoint h (sending a simplicial set to its *homotopy category*) is not homotopically well-behaved. Thomason's crucial insight was that we can avoid this issue by using Kan's Sd ⊣ Ex-adjunction [Kan57, §7] to replace the nerve by a weakly equivalent functor:

Theorem 1.11 (Thomason)**.** *There is a unique model structure on* **Cat** *in which a functor $f\colon C \to D$ is a weak equivalence if and only if* $\mathrm{N}(f)$ *is a weak homotopy equivalence of simplicial sets and a fibration if and only if* $\mathrm{Ex}^2\,\mathrm{N}(f)$ *is a Kan fibration. This model structure is combinatorial with generating cofibrations*

$$\{\mathrm{h}\,\mathrm{Sd}^2\,\partial\Delta^n \hookrightarrow \mathrm{h}\,\mathrm{Sd}^2\,\Delta^n \colon n \geq 0\}$$

and generating acyclic cofibrations

$$\{\mathrm{h}\,\mathrm{Sd}^2\,\Lambda^n_k \hookrightarrow \mathrm{h}\,\mathrm{Sd}^2\,\Delta^n \colon 0 \leq k \leq n\}.$$

Moreover, with respect to this model structure the adjunction

$$\mathrm{h}\,\mathrm{Sd}^2\colon \mathbf{SSet}_{\text{Kan-Quillen}} \rightleftarrows \mathbf{Cat} : \mathrm{Ex}^2\,\mathrm{N} \tag{1.1}$$

is a Quillen equivalence.

Proof. The existence of the model structure together with the above choices of generating (acyclic) cofibrations is [Tho80, Theorem 4.9]; as **Cat** is locally presentable, this is then a combinatorial model structure.

It is obvious that (1.1) is a Quillen adjunction. Moreover, the right adjoint is homotopical as Ex is weakly equivalent to the identity functor [Kan57, Lemma 7.4], while the left adjoint is so by Ken Brown's Lemma. Thus, for (1.1) to be a Quillen equivalence one has to show that the ordinary unit and counit are weak equivalences, for which Thomason refers to Fritsch and Latch [FL81, Example 4.12-(v)]. □

Thomason's proof of the above theorem crucially relies on a careful analysis of the (generating) cofibrations. As we will need some of their properties later, we briefly recall them here for easy reference.

Definition 1.12. A sieve $i\colon C \to D$ is called a *Dwyer map* if it can be factored as $i = jf$ such that the following holds:

1. j is a cosieve.

2. f admits a right adjoint.

Proposition 1.13. *Let $i\colon K \hookrightarrow L$ be a cofibration of simplicial sets. Then* $\mathrm{h}\,\mathrm{Sd}^2(i)$ *is a Dwyer map.*

Proof. See [Tho80, Proposition 4.2]. □

Dwyer maps are extremely useful since pushouts along them admit a very explicit description, which we will recall later in Construction 3.5. For now we only record one important consequence of this:

Proposition 1.14. *Let*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \big\downarrow & & \big\downarrow \\ C & \longrightarrow & D \end{array}$$

be a pushout in **Cat** *such that i is a Dwyer map. Then the induced map*

$$\mathrm{N}B \amalg_{\mathrm{N}A} \mathrm{N}C \to \mathrm{N}D$$

is a weak homotopy equivalence.

Proof. This is [Tho80, Proposition 4.3]. □

Finally, let us state Schwede's global refinement of the Thomason model structure, which we will later generalize to the G-global setting:

Theorem 1.15 (Schwede)**.** *There is a unique model structure on* **Cat** *in which a functor $f\colon C \to D$ is a weak equivalence or fibration if and only if* $\mathrm{Fun}(BH,f)$ *is a weak equivalence or fibration, respectively, in the Thomason model structure for every finite group H.*

We call this the global model structure *on* **Cat**. *It is proper and combinatorial with generating cofibrations*

$$I = \{BH \times \mathrm{h}\,\mathrm{Sd}^2\,\partial\Delta^n \hookrightarrow BH \times \mathrm{h}\,\mathrm{Sd}^2\,\Delta^n : n \geq 0, H \textit{ a finite group}\}$$

and generating acyclic cofibrations

$$J = \{BH \times \mathrm{h}\,\mathrm{Sd}^2\,\Lambda^n_k \hookrightarrow BH \times \mathrm{h}\,\mathrm{Sd}^2\,\Delta^n : 0 \leq k \leq n, H \textit{ a finite group}\}.$$

Strictly speaking, I and J are not sets as there are too many finite groups. However, this can be easily cured by restricting to a system of representatives of isomorphism classes of finite groups, which we will tacitly assume below.

Proof. Specializing [Sch19, Theorem 1.12] to the collection $\{BH : H \text{ finite group}\}$ shows that this model structure exists and is proper, also see [Sch19, Theorem 3.3]. Moreover, Schwede's proof explicitly identifies I and J as set of generating cofibrations and generating acyclic cofibrations, respectively. □

The following lemma is crucial to Schwede's proof of the above theorem and will also be instrumental later in establishing our G-global generalization:

Lemma 1.16. *Let X be a small category such that for every $x, y \in X$ there exists both a morphism $x \to y$ as well as $y \to x$. Let moreover*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

be a pushout in **Cat** *where i is a Dwyer map. Then also the induced square*

$$\begin{array}{ccc} \mathrm{Fun}(X,A) & \xrightarrow{\mathrm{Fun}(X,i)} & \mathrm{Fun}(X,B) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(X,C) & \longrightarrow & \mathrm{Fun}(X,D) \end{array}$$

is a pushout.

Proof. This is the first half of [Sch19, Theorem 1.5]. □

2. Transferring model structures

Just as the usual equivariant model structures on ***G*-Cat** or ***G*-SSet**, the G-global model structures we discuss in this paper will be obtained as *transferred model structures*:

Definition 2.1. Let $\mathscr{C}$ be a model category, let $\mathscr{D}$ be a complete and cocomplete category, and let

$$F \colon \mathscr{C} \rightleftarrows \mathscr{D} : U$$

be an adjunction. The *model structure transferred along $F \dashv U$* is the (unique if it exists) model structure on $\mathscr{D}$ in which a morphism f is a weak equivalence or fibration if and only if Uf is a weak equivalence or fibration, respectively, in $\mathscr{C}$.

We now give a criterion for the existence of transferred model structures that we will use for all our constructions later.

Proposition 2.2. *Let $\mathscr{C}$ be a left proper cofibrantly generated model category such that filtered colimits in $\mathscr{C}$ are homotopical, and let I, J be sets of generating (acyclic) cofibrations. Moreover, let $\mathscr{D}$ be a locally presentable category together with an adjunction $F\colon \mathscr{C} \rightleftarrows \mathscr{D} : U$, and assume the following:*

1. *For each $j \in J$, the map UFj is a weak equivalence.*

2. *U sends any pushout square*

$$\begin{array}{ccc} FA & \xrightarrow{Fi} & FB \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

   in $\mathscr{D}$, where $i \in I$ is a generating cofibration of $\mathscr{C}$, to a homotopy pushout in $\mathscr{C}$.

3. *U preserves filtered colimits up to weak equivalence, i.e. for each filtered poset P and each diagram $X_\bullet\colon P \to \mathscr{D}$ the natural comparison map $\operatorname{colim}_P(U \circ X_\bullet) \to U(\operatorname{colim}_P X_\bullet)$ is a weak equivalence.*

Then the transferred model structure on $\mathscr{D}$ exists and it is combinatorial with set of generating cofibrations FI and set of generating acyclic cofibrations FJ. This model structure is left proper with homotopy pushouts created by U, and filtered colimits in $\mathscr{D}$ are homotopical; if $\mathscr{C}$ is right proper, then so is $\mathscr{D}$, and U also creates homotopy pullbacks.

Moreover, in the presence of (2) *and* (3) *the first condition is implied by*

(1′) J consists of maps between cofibrant objects. Moreover, the unit $\eta_\varnothing$ is a weak equivalence, and for each generating cofibration $(X \to Y) \in I$ both η_X and η_Y are weak equivalences.

Finally, under this stronger assumption the adjunction $F \dashv U$ is a Quillen equivalence.

Proof. Assume first that (1)–(3) hold. To see that the transferred model structure exists and is cofibrantly generated by FI and FJ it suffices to verify the assumptions of the usual transfer criterion for cofibrantly generated model categories [Hir03, Theorem 11.3.2]. As $\mathscr{D}$ is locally presentable, the smallness assumption is automatically satisfied, so we only have to show that relative FJ-cell complexes are weak equivalences. By Condition (3) it is then enough to verify that pushouts of maps of the form Fj with $j \in J$ are weak equivalences in $\mathscr{D}$, i.e. sent under U to weak equivalences in $\mathscr{C}$.

Let us consider the class $\mathscr{H}$ of those maps $i' : A' \to B'$ in $\mathscr{D}$ such that U sends pushouts along them to homotopy pushouts, i.e. those maps such that the analogue of Condition (2) holds for them. [Len20, Proposition A.2.7] then shows that $\mathscr{H}$ is closed under pushouts, transfinite compositions, and retracts.

As F preserves pushouts, transfinite compositions, and retracts (being a left adjoint functor), it follows that also $F^{-1}(\mathscr{H})$ is closed under all of these. As it contains all $i \in I$ by assumption, it follows by the characterizations of cofibrations in a cofibrantly generated model category that $F^{-1}(\mathscr{H})$ contains all cofibrations of $\mathscr{C}$; in particular it contains J. Hence if $(j\colon A \to B) \in J$ is a generating acyclic cofibration and we have any pushout square

$$\begin{tikzcd} FA \arrow[r, "Fj"] \arrow[d] & FB \arrow[d] \\ C \arrow[r, "k"'] & D, \end{tikzcd} \tag{2.1}$$

then applying U to this yields a homotopy pushout in $\mathscr{D}$. But UFj is a weak equivalence by Condition (1). It follows that Uk is a weak equivalence, and hence by definition so is k. Altogether, we conclude that the transferred model structure exists and is cofibrantly generated by FI and FJ (hence combinatorial).

But with this established we conclude by the same argument (this time applied in $\mathscr{D}$) from the closure properties of $\mathscr{H}$ that U sends pushouts along cofibrations in $\mathscr{D}$ to homotopy pushouts. Thus, [Len20, Lemma A.2.15] shows that $\mathscr{D}$ is left proper with homotopy pushouts created by U. The statements about filtered colimits and homotopy pullbacks are trivial, finishing the proof of the first half of the proposition.

Now assume that (1′), (2), and (3) hold. We first observe:

Claim. The unit η_X is a weak equivalence for each cofibrant $X \in \mathscr{C}$.

Proof. This is a standard cell induction argument. By Quillen's Retract Argument, any cofibrant object is a retract of an I-cell complex; as weak equivalences are closed under retracts, it therefore suffices to prove the claim for every I-cell complex X.

To this end, we write X as a transfinite composition $\varnothing = X_0 \to X_1 \to \cdots X_\alpha = X$ of pushouts of maps in I for some ordinal α. We will now prove by transfinite induction that η_{X_β} is a weak equivalence for every $\beta \le \alpha$.

For $\beta = 0$ this is part of Condition (1′). If $\beta = \gamma + 1$ is a successor ordinal, then we exhibit $X_\gamma \to X_\beta$ as a pushout of some generating cofibration $i\colon A \to B$ and consider the induced commutative cube

$$\begin{tikzcd}[row sep=small, column sep=small] & UFA \arrow[rr, "UFi"] \arrow[dd] & & UFB \arrow[dd] \\ A \arrow[ur] \arrow[rr, crossing over, "i" near start] \arrow[dd] & & B \arrow[ur] & \\ & UFX_\gamma \arrow[rr] & & UFX_\beta \\ X_\gamma \arrow[ur] \arrow[rr] & & X_\beta \arrow[ur] \arrow[from=uu, crossing over] & \end{tikzcd} \tag{2.2}$$

where all front-to-back maps are given by η. The front square is a homotopy pushout as $\mathscr{C}$ is left proper, and so is the back square by Condition (2) and since F preserves pushouts.

In (2.2), the upper front-to-back maps are weak equivalences by Condition (1′), and so is the lower left one by the induction hypothesis. Thus, also η_{X_β} is a weak equivalence as desired.

Finally, if β is a limit ordinal, then we consider the commutative square

$$\begin{tikzcd} \operatorname{colim}_{\gamma<\beta} X_\gamma \arrow[r] \arrow[d, "\operatorname{colim}_{\gamma<\beta}\eta"'] & X_\beta \arrow[d, "\eta"] \\ \operatorname{colim}_{\gamma<\beta} UFX_\gamma \arrow[r] & UFX_\beta \end{tikzcd}$$

where the horizontal maps are induced by the structure maps $X_\gamma \to X_\beta$; in particular, the upper map is an isomorphism and the lower map is a weak equivalence by Condition (3). On the other hand, the left hand map is a filtered colimit of weak equivalences by the induction hypothesis, hence a weak equivalence by assumption on $\mathscr{C}$. Thus, also the right hand vertical map is a weak equivalence by 2-out-of-3, which completes the proof of the claim. △

If now $j: X \to Y$ is one of the chosen generating acyclic cofibrations, then X and Y are cofibrant by assumption, so η_X and η_Y are weak equivalences by the above. Thus, also UFj is a weak equivalence by 2-out-of-3, proving (1), and in particular supplying the desired model structure. To show that $F \dashv U$ is a Quillen equivalence, we observe that U creates weak equivalences by definition, so that it suffices that $\eta: X \to UFX$ is a weak equivalence for each cofibrant X, which was verified above. □

3. Categories with monoid actions

In this section we want to prove the analogue of Proposition 1.7 for suitable strict monoids in **Cat** as well as a comparison between the simplicial and categorical approaches.

3.1 Equivariant Dwyer maps for groups

Dwyer maps are central to Thomason's treatment of his model structure on **Cat**, and it should come as no surprise that we will need an equivariant version of this. For this it will be convenient to consider the case of ordinary groups first (as some arguments will only work in this setting), so let us fix a not necessarily finite discrete group G.

Definition 3.1. A G-equivariant functor $i: C \to D$ of small G-categories is called a *G-equivariant Dwyer map* if it is a sieve and it admits a factorization $i = jf$ into G-equivariant functors $f: C \to X$, $j: X \to D$ with the following properties:

1. j is a cosieve.

2. f admits a G-equivariant right adjoint, i.e. there exists a G-equivariant functor $r\colon X \to C$ together with G-equivariant natural transformations $\eta\colon \mathrm{id} \Rightarrow ri$ and $\epsilon\colon ir \Rightarrow \mathrm{id}$ satisfying the usual triangle identities.

Remark 3.2. For $G = 1$ the above is equivalent to i being an ordinary (i.e. non-equivariant) Dwyer map, see Definition 1.12. Conversely, if i is a non-equivariant Dwyer map, then it is a G-equivariant Dwyer map with respect to the trivial G-actions on source and target for any discrete group G.

Remark 3.3. May, Stephan, and Zakharevich [MZS17, Definition 4.1] additionally require the map η in (2) to be the identity transformation; however, this can always be arranged, so that the above definition agrees with their notion of a *Dwyer G-map*:

The first condition guarantees that f is again fully faithful. It follows then formally that for any right adjoint $\tilde{r}$ the unit transformation $\tilde{\eta}\colon \mathrm{id} \Rightarrow \tilde{r}f$ is a natural isomorphism. As f is obviously injective on objects, it is well-known in the non-equivariant setting that we may massage $\tilde{r}$ to another right adjoint r of f such that the unit η is actually the identity.

The same proof works in the equivariant setting, but I do not know a reference for this. As this extra condition will become relevant later, let me briefly sketch the argument. We first define $r\colon X \to C$ on objects via

$$r(x) = \begin{cases} c & \text{if } x = f(c) \\ \tilde{r}(x) & \text{if } x \notin \operatorname{im} f. \end{cases}$$

This is well-defined as f is injective on objects, and it is G-equivariant because $\operatorname{im} f$ and hence also $(\operatorname{im} f)^c$ are closed under the action of G; note that this would break down for general monoids. We now define for each $x \in X$ an isomorphism $\varphi_x\colon r(x) \to \tilde{r}(x)$ as follows: if $x = f(c)$, then φ_x is the unit $\tilde{\eta}_c\colon r(x) = c \to \tilde{r}f(c) = \tilde{r}(x)$; otherwise, φ_x is the identity. It is then obvious that this is again compatible with the G-action in the sense that $\varphi_{g.x} = g.\varphi_x$.

There is a unique way to extend r to a functor in such a way that φ becomes a natural isomorphism $r \cong \tilde{r}$, namely $r(\alpha\colon x \to y) = \varphi_y^{-1}\tilde{r}(\alpha)\varphi_x$. It follows then from the above compatibility of φ with the G-action that r is again G-equivariant and that φ is a G-equviariant isomorphism. One then easily checks that $\eta := \mathrm{id} = \varphi^{-1}f \circ \tilde{\eta}\colon \mathrm{id} \Rightarrow rf$ and $\epsilon := \tilde{\epsilon} \circ f\varphi\colon fr \Rightarrow \mathrm{id}$ exhibit r as G-equivariant right adjoint of f as desired.

Below we will need the following closure properties of G-equivariant Dwyer maps, the first one of which can also be found (without proof) as [MZS17, Lemma 4.2].

Lemma 3.4. *Let* $i\colon C \to D$ *be a* G*-equivariant Dwyer map.*

1. *Let* $H \subset G$ *be a subgroup. Then* $i^H\colon C^H \to D^H$ *is a Dwyer map.*
2. *Let* S *be any small* G*-category. Then* $S \times i\colon S \times C \to S \times D$ *is a* G*-equivariant Dwyer map.*
3. *Let* T *be a small right* G*-category. Then also* $\mathrm{Fun}(T, i)\colon \mathrm{Fun}(T, C) \to \mathrm{Fun}(T, D)$ *is a* G*-equivariant Dwyer map.*

Proof. All of these follow the same pattern, so we will only prove the first statement. We pick a factorization $i = jf$ as above and a G-equivariant adjunction $f \dashv r$ with unit η and counit ϵ. It is then easy to check that i^H is again a sieve and that j^H is a cosieve. Moreover, r^H is right adjoint to f^H with unit η^H and counit ϵ^H, so $i^H = j^H f^H$ is the desired factorization. □

In general, pushouts in **Cat** (and hence also in ***G*-Cat**) are very difficult to describe on the level of morphisms. One advantage of ordinary Dwyer maps is that one can give an explicit and tractable description of pushouts along them, see [Sch19, Construction 1.2] which generalizes [BMO$^+$15, Lemma 2.5]:

Construction 3.5. Let

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "c"'] & B \\ C & \end{tikzcd}$$

be a diagram in **Cat** and assume that i is a Dwyer map. Fix a factorization $i = kf$ as in the definition of a Dwyer map and an adjunction $f \dashv r$ such that the unit is the identity; in particular the counit ϵ satisfies $\epsilon f = \mathrm{id}$ by the triangle identities. For simplicity of notation we assume further that i and k are honest inclusions of subcategories; we write X for the source of k. Finally, let us write V for the complement of $\mathrm{Ob}\, A$ in $\mathrm{Ob}\, B$.

We now define a category D as follows: the objects of D are given by the disjoint union $\mathrm{Ob}\, C \amalg V$ and the morphism set between $x, y \in D$ is defined as

$$\mathrm{Hom}_D(x, y) = \begin{cases} \mathrm{Hom}_C(x, y) & \text{if } x, y \in C \\ \mathrm{Hom}_C(x, cr(y)) & \text{if } x \in C \text{ and } y \in V \cap X \\ \mathrm{Hom}_B(x, y) & \text{if } x, y \in V \\ \varnothing & \text{otherwise.} \end{cases} \tag{3.1}$$

Compositions are in such a way that the obvious maps $B \to D$ and $C \to D$ are actual functors; moreover, if $x, y \in C$ and $z \in V \cap X$ then the composition

$$x \xrightarrow{\alpha} y \xrightarrow{\beta} z$$

in D, where α is a morphism $x \to y$ in C and β is a morphism $y \to cr(z)$ in C, is defined as the composition $\beta \circ \alpha$ in C. On the other hand, if $x \in C$, $y, z \in V \cap X$, then the composition $\beta \circ \alpha$ in D, where now $\alpha\colon x \to cr(y)$ is a morphism in C and $\beta\colon y \to z$ is a morphism in $V \cap X \subset B$, is defined as the composition $cr(\beta) \circ \alpha$ in C.

We have a functor $j\colon C \to D$ via the inclusion of C. Moreover, we define $d\colon B \to D$ as follows: on $V \subset B$ the functor d is just given by the inclusion and on $A = V^c$ via c. Finally, if $\beta\colon a \to x$ is a morphism in B, where $a \in A$ and $x \in X \cap V$, then

$$d(\beta) = c(r(\beta))\colon \underbrace{cr(a)}_{=c(a)=d(a)} \to cr(x) \in \mathrm{Hom}_C(c(a), cr(x)) = \mathrm{Hom}_D(d(a), d(x)).$$

We remark that this indeed a complete case distinction as $X \subset B$ is a cosieve (so any morphism starting in $A \subset X$ has to end in $X = (X \cap V) \cup A)$ and $A \subset B$ is a sieve (so any arrow ending in A also has to start in A).

We omit the verification that D is a category and that these are well-defined functors exhibiting D as pushout (which uses that $\epsilon f = \mathrm{id}$), and instead refer the curious reader to [Sch19, Construction 1.2].

Construction 3.6. Now assume that A, B, C are G-categories, i is a G-equivariant Dwyer map, and c is any equivariant functor. By Remark 3.3 we may choose the G-equivariant adjunction $f \dashv r$ such that $\epsilon f = \mathrm{id}$, allowing us to apply the above construction with respect to this data.

We equip D with the following G-action: G acts on the full subcategories C and V in the obvious way; observe that V is indeed preserved by the G-action as its complement is—here we again used that G is a group as opposed to a mere monoid. Finally, if $x \in C$, $y \in V \cap X$ and $\alpha\colon x \to cr(y)$ defines a morphism $x \to y$ in D, then we define the G-action again via the G-action on C; note that this indeed makes sense as both c and r are assumed to be G-equivariant.

We leave the easy verification that this is indeed a G-action and that j and d are G-equivariant to the reader.

As pushouts in ***G*-Cat** are created in **Cat** we immediately get:

Corollary 3.7. *With respect to the above G-action on D,*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "c"'] & B \arrow[d, "d"] \\ C \arrow[r, "j"'] & D \end{tikzcd}$$

becomes a pushout in ***G*-Cat***.* □

Now we can prove that pushouts along G-equivariant Dwyer maps are compatible with passing to fixed points, generalizing [BMO+15, Proposition 2.4].

Proposition 3.8. *Let*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \tag{3.2}$$

be a pushout in ***G*-Cat** *such that i is a G-equivariant Dwyer map. Then for any subgroup $H \subset G$ also the induced square*

$$\begin{tikzcd} A^H \arrow[r, "i^H"] \arrow[d] & B^H \arrow[d] \\ C^H \arrow[r] & D^H \end{tikzcd} \tag{3.3}$$

is a pushout (along a Dwyer map).

Proof. Pick a factorization $i = kf$ together with a right adjoint r of f as in Construction 3.6; we may then assume without loss of generality that (3.2) is the square from this construction.

The map i^H is a Dwyer map by Lemma 3.4-(1); more precisely, by the proof of the lemma the factorization $i^H = k^H f^H$ together with the right adjoint r^H and the natural transformations η^H and ϵ^H exhibits i^H as Dwyer map. Thus it suffices to identify (3.3) with the pushout from Construction 3.5 applied to the ordinary Dwyer map i^H and the above data.

For this we spell out the definitions again: the set of objects of our construction of the pushout (3.2) is $C \amalg V$, where V is the complement of A in B, and the H-fixed points of this is $C^H \amalg V^H$. From the explicit description (3.1) of the Hom-sets we then see that for $x, y \in D^H$

$$\operatorname{Hom}_{D^H}(x,y) = \begin{cases} \operatorname{Hom}_C(x,y)^H & \text{if } x,y \in C \\ \operatorname{Hom}_C(x, cr(y))^H & \text{if } x \in C \text{ and } y \in V \cap X \\ \operatorname{Hom}_B(x,y)^H & \text{if } x,y \in V \\ \varnothing^H & \text{otherwise} \end{cases}$$
$$= \begin{cases} \operatorname{Hom}_{C^H}(x,y) & \text{if } x,y \in C^H \\ \operatorname{Hom}_{C^H}(x, c^H r^H(y)) & \text{if } x \in C^H \text{ and } y \in V^H \cap X^H \\ \operatorname{Hom}_{B^H}(x,y) & \text{if } x,y \in V^H \\ \varnothing & \text{otherwise} \end{cases}$$

As V^H is the complement of A^H in B^H and X^H is the source of f^H, these are literally the objects and morphism sets of the above construction of the pushout of $C^H \leftarrow A^H \rightarrow B^H$. Moreover, one checks by direct inspection that the composition is defined in the same way and that also the structure maps $B^H \rightarrow D^H$ and $C^H \rightarrow D^H$ agree; this finishes the proof. □

Corollary 3.9. *In the situation of the previous corollary, the induced map*

$$\mathrm{N}(B) \amalg_{\mathrm{N}(A)} \mathrm{N}(C) \to \mathrm{N}(D)$$

is an $\mathcal{F}$-weak equivalence for any collection $\mathcal{F}$ of subgroups of G.

Proof. Let $H \subset G$ be any subgroup. We have to show that the induced map on H-fixed points is a weak equivalence. But this map fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{N}(B^H) \amalg_{\mathrm{N}(A^H)} \mathrm{N}(C^H) \arrow[r] \arrow[d] & \mathrm{N}(D^H) \arrow[d] \\
(\mathrm{N}B)^H \amalg_{(\mathrm{N}A)^H} (\mathrm{N}C)^H \arrow[r] \arrow[d] & (\mathrm{N}D)^H \arrow[d, equal] \\
\left(\mathrm{N}B \amalg_{\mathrm{N}A} \mathrm{N}C\right)^H \arrow[r] & (\mathrm{N}D)^H
\end{tikzcd}$$

where all maps are induced by the relevant universal properties of colimits and limits. The two top vertical arrows are isomorphisms as N is a right adjoint, and the lower left vertical arrow is an isomorphism as fixed points commute with pushouts along monomorphisms in **Set** and hence in **SSet**.

But by the previous corollary, the square (3.3) is a pushout along a Dwyer map, hence the top map is a weak equivalence by the classical non-equivariant statement, see Proposition 1.14. The claim follows by 2-out-of-3. □

3.2 Equivariant Dwyer maps for monoids

Let M be a monoid in **Cat**, i.e. a small strict monoidal category.

Definition 3.10. An M-equivariant functor $i\colon C \to D$ is called an *M-equivariant Dwyer map* if it is a core(Ob M)-equivariant Dwyer map in the sense of Definition 3.1, where core(Ob M) denotes the maximal subgroup of the discrete monoid Ob M of objects of M.

Slightly expanding the above definition this means that i is a sieve and that we can find a factorization $i = jf$ *into* core(Ob M)*-equivariant functors* with certain properties. However, all of the results below will just follow formally from the corresponding results for equivariant Dwyer maps with respect to groups established above.

Corollary 3.11. *Let*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

be a pushout in the category $\boldsymbol{M}$-**Cat** *of small categories with M-action, and assume that i is an M-equivariant Dwyer map. Then for any subgroup $H \subset$* $\mathrm{Ob}\, M$ *the induced square*

$$\begin{array}{ccc} A^H & \xrightarrow{i^H} & B^H \\ \downarrow & & \downarrow \\ C^H & \longrightarrow & D^H \end{array}$$

is a pushout along a Dwyer map in **Cat**.

Proof. As pushouts in both $\boldsymbol{M}$-**Cat** as well as **core(Ob $\boldsymbol{M}$)-Cat** are created in **Cat**, this is immediate from Proposition 3.8. □

In order to prove the analogue of Corollary 3.9 we first have to explain how to make $\mathrm{N}(C)$ into an $\mathrm{N}(M)$-simplicial set for a given M-category C:

Construction 3.12. We lift the adjunction $\mathrm{h}\colon \mathbf{SSet} \rightleftarrows \mathbf{Cat} : \mathrm{N}$ to

$$\mathrm{h}_M\colon \mathbf{N}(\boldsymbol{M})\textbf{-SSet} \rightleftarrows \boldsymbol{M}\textbf{-Cat} : \mathrm{N}_M$$

as follows: on underlying categories or simplicial sets h_M and N_M agree with h and N, respectively; in particular, this determines their definition on morphisms. If C is an M-category, then $\mathrm{N}_M(C) = \mathrm{N}(C)$ carries the $\mathrm{N}(M)$-action given by the composition

$$\mathrm{N}(M) \times \mathrm{N}(C) \xrightarrow{\cong} \mathrm{N}(M \times C) \xrightarrow{\mathrm{N(action)}} \mathrm{N}(C)$$

where the left hand map is the inverse of the canonical isomorphism. Similarly, if X is an $\mathrm{N}(M)$-simplicial set, then the M-action on $\mathrm{h}_M(X) = \mathrm{h}X$ is given by

$$M \times \mathrm{h}X \xrightarrow{\cong} \mathrm{hN}(M) \times \mathrm{h}X \xrightarrow{\cong} \mathrm{h}(\mathrm{N}(M) \times X) \xrightarrow{\mathrm{h(action)}} \mathrm{h}X.$$

Here the first map is the inverse of the counit of $\mathrm{h} \dashv \mathrm{N}$ (using that N is fully faithful) and the second map is as above (using that h happens to preserve products).

We omit the easy verification that this is well-defined and that the unit and counit of the original adjunction lift to natural transformations id $\Rightarrow$

$\mathrm{N}_M\mathrm{h}_M$ and $\mathrm{h}_M\mathrm{N}_M \Rightarrow \mathrm{id}$, respectively. It then follows formally that these exhibit h_M as left adjoint of N_M. We moreover observe that for any M-category C and any $m \in \mathrm{Ob}(M)$ the action map $m.\!-\colon \mathrm{N}_M(C) \to \mathrm{N}_M(C)$ agrees with $\mathrm{N}(m.\!-)$ as a map of simplicial sets.

With this terminology we can now formulate the desired generalization to categorical monoids:

Corollary 3.13. *Let*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

be a pushout in $\boldsymbol{M}$-**Cat** *such that i is an M-equivariant Dwyer map. Then the induced map* $\mathrm{N}_M(B) \amalg_{\mathrm{N}_M(A)} \mathrm{N}_M(C) \to \mathrm{N}_M(D)$ *is an* $\mathcal{F}$*-weak equivalence for any collection* $\mathcal{F}$ *of subgroups of* $\mathrm{N}(M)_0$.

Proof. As all the relevant pushouts are created in **Cat** or **SSet**, and since the action of any $H \subset \mathrm{N}(M)_0$ on $\mathrm{N}_M(D)$ is just the one given by functoriality, this follows from Corollary 3.9. □

Corollary 3.14. *In the situation of the previous corollary, the square*

$$\begin{tikzcd} \mathrm{N}_M(A) \arrow[r, "\mathrm{N}_M(i)"] \arrow[d] & \mathrm{N}_M(B) \arrow[d] \\ \mathrm{N}_M(C) \arrow[r] & \mathrm{N}_M(D) \end{tikzcd}$$

is a homotopy pushout in the $\mathcal{F}$*-model structure on* **N(*M*)-SSet** *for any collection* $\mathcal{F}$ *of finite subgroups of* $\mathrm{N}(M)_0$.

Proof. By the previous corollary, the induced map

$$\mathrm{N}_M(B) \amalg_{\mathrm{N}_M(A)} \mathrm{N}_M(C) \to \mathrm{N}_M(D)$$

is a weak equivalence. On the other hand, i is in particular a fully faithful embedding, so that $\mathrm{N}_M(i)$ is an underlying cofibration. Proposition 1.7 then implies that the left hand side already represents the homotopy pushout, finishing the proof. □

Similarly, one generalizes Lemma 3.4 to all categorical monoids. Below, we will freely refer to Lemma 3.4 whenever we actually need the corresponding statement for monoid actions.

3.3 The equivariant Ex-functor

In order to construct the equivariant model structure on $\boldsymbol{M}$-**Cat**, we will need an equivariant generalization of the usual Sd ⊣ Ex adjunction, which turns out to be slightly more subtle than in the case of the adjunction h ⊣ N, also cf. [Sch19, Construction 2.8]:

Construction 3.15. Let N be a simplicial monoid. We define $\mathrm{Ex}_N\colon \boldsymbol{N}\text{-}\mathbf{SSet} \to \boldsymbol{N}\text{-}\mathbf{SSet}$ as follows: on underlying simplicial sets, Ex_N agrees with the usual Ex; in particular, this determines the definition of Ex_N on morphisms.

If now X is any N-simplicial set, then we equip $\mathrm{Ex}_N(X) = \mathrm{Ex}(X)$ with N-action given by

$$N \times \mathrm{Ex}(X) \xrightarrow{e\times\mathrm{Ex}(X)} \mathrm{Ex}(N)\times\mathrm{Ex}(X) \xrightarrow{\cong} \mathrm{Ex}(N\times X) \xrightarrow{\mathrm{Ex}(\text{action})} \mathrm{Ex}(X);$$

here e is the usual natural transformation $\mathrm{id} \Rightarrow \mathrm{Ex}$ [Kan57, Section 3] and the second map is the inverse of the canonical isomorphism. We omit the easy verification that this is well-defined, that the natural transformation e lifts to $e_N\colon \mathrm{id} \Rightarrow \mathrm{Ex}_N$, and that any $n \in N_0$ acts on $\mathrm{Ex}_N(X) = \mathrm{Ex}(X)$ by $\mathrm{Ex}(n.–)$.

Lemma 3.16. *The functor* Ex_N *preserves small limits and filtered colimits.*

Proof. As limits in colimits in $\boldsymbol{N}$-**SSet** are created in **SSet** and as Ex_N agrees with Ex on underlying simplicial sets, this is a immediate consequence of the corresponding statement for Ex. □

The Special Adjoint Functor Theorem implies:

Corollary 3.17. Ex_N *admits a left adjoint* Sd_N. □

Let us fix such an adjunction for the rest of this article.

Construction 3.18. Let $X \in \boldsymbol{N}\text{-}\mathbf{SSet}$ be arbitary. We define $d_N\colon \operatorname{Sd}_N X \to X$ to be the adjunct of $e_N\colon X \to \operatorname{Ex}_N X$, i.e. we have commutative diagrams

$$\begin{tikzcd} \operatorname{Sd}_N X \arrow[r, "\operatorname{Sd}_N e_n"] \arrow[dr, bend right, "d_N"'] & \operatorname{Sd}_N \operatorname{Ex}_N X \arrow[d, "\epsilon"] \\ & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "e_N"'] & \operatorname{Ex}_N \operatorname{Sd}_N X \arrow[d, "\operatorname{Ex}_N d_N"] \\ & \operatorname{Ex}_N X. \end{tikzcd}$$

Obviously, the d_N assemble into a natural transformation $d_N\colon \operatorname{Sd}_N \Rightarrow \mathrm{id}$, and this is by definition the total mate of the square

$$\begin{tikzcd} \boldsymbol{N}\text{-}\mathbf{SSet} & \boldsymbol{N}\text{-}\mathbf{SSet} \arrow[l, "\mathrm{id}"'] \arrow[dl, phantom, "\Leftarrow e_N"] \\ \boldsymbol{N}\text{-}\mathbf{SSet} \arrow[u, "\operatorname{Ex}_N"] & \boldsymbol{N}\text{-}\mathbf{SSet} \arrow[l, "\mathrm{id}"] \arrow[u, "\mathrm{id}"'] \end{tikzcd} \tag{3.4}$$

(picking the trivial adjunctions for all the identity arrows).

We now turn to some properties of the adjunction $\operatorname{Sd}_N \dashv \operatorname{Ex}_N$ as well as the natural transformations d_N and e_N:

Lemma 3.19. *1. The functor* Ex_N *preserves* $\mathcal{F}$*-weak equivalences for any collection* $\mathcal{F}$ *of subgroups of* N_0*.*

2. The natural transformation e_N *is a levelwise weak equivalence.*

Proof. By 2-out-of-3 it suffices to prove the second statement. For this we have to show that for each $H \subset N_0$ and each $X \in \boldsymbol{N}\text{-}\mathbf{SSet}$ the map $(e_N)^H\colon X^H \to \operatorname{Ex}_N(X)^H$ is a weak equivalence. However, we have identified the H-action on $\operatorname{Ex}_N(X) = \operatorname{Ex}(X)$ as the one induced by functoriality. As Ex is a right adjoint, it preserves limits, so we have a canonical isomorphism $\sigma\colon \operatorname{Ex}(X^H) \cong \operatorname{Ex}(X)^H$. Naturality and the universal property of limits then imply that the composition

$$X^H \xrightarrow{e} \operatorname{Ex}(X^H) \xrightarrow{\sigma} \operatorname{Ex}(X)^H$$

(where the left hand map is the *ordinary* $e\colon \mathrm{id} \Rightarrow \operatorname{Ex}$ evaluated at X^H) agrees with $e^H = (e_N)^H$. Hence the claim follows from the fact that the ordinary e is a levelwise weak equivalence [Kan57, Lemma 7.4]. □

Lemma 3.20. *Let $H \subset N_0$ be any subgroup. There is a natural isomorphism τ filling*

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, "N/H\times-"] \arrow[d, "\mathrm{Sd}"'] \arrow[dr, phantom, "\Nearrow\tau"] & N\text{-}\mathbf{SSet} \arrow[d, "\mathrm{Sd}_N"] \\ \mathbf{SSet} \arrow[r, "N/H\times-"'] & N\text{-}\mathbf{SSet}, \end{tikzcd}$$

and moreover τ can be chosen in such a way that for each $K \in \mathbf{SSet}$ the diagram

$$\begin{tikzcd} N/H \times \mathrm{Sd}\, K \arrow[r, "\tau", "\cong"'] \arrow[dr, bend right, "N/H\times d"'] & \mathrm{Sd}_N(N/H \times K) \arrow[d, "d_N"] \\ & N/H \times K \end{tikzcd}$$

commutes (where d as usual denotes the adjunct of e).

Proof. We recall from the proof of the previous lemma that we have a natural isomorphism filling

$$\begin{tikzcd} \mathbf{SSet} & N\text{-}\mathbf{SSet} \arrow[l, "(-)^H"'] \\ \mathbf{SSet} \arrow[u, "\mathrm{Ex}"] \arrow[ur, phantom, "\Swarrow"] & N\text{-}\mathbf{SSet}, \arrow[l, "(-)^H"] \arrow[u, "\mathrm{Ex}_N"'] \end{tikzcd}$$

namely the inverse of the canonical comparison map. We take τ to be the total mate of this, which is a natural isomorphism

$$N/H \times \mathrm{Sd}(-) \cong \mathrm{Sd}_N(N/H \times -).$$

It remains to prove the compatibility of τ with d and d_N. For this we observe that $N/H \times d$ is by definition and the compatibility of mates with pastings the total mate of

$$\begin{tikzcd} \mathbf{SSet} & \mathbf{SSet} \arrow[l, "="'] & N\text{-}\mathbf{SSet} \arrow[l, "(-)^H"'] \\ \mathbf{SSet} \arrow[u, "\mathrm{Ex}"] \arrow[ur, phantom, "\Swarrow_e"] & \mathbf{SSet} \arrow[l, "="] \arrow[u, "="'] \arrow[ur, phantom, "\Swarrow_="] & N\text{-}\mathbf{SSet}. \arrow[l, "(-)^H"] \arrow[u, "="'] \end{tikzcd}$$

But on the other hand this pasting agrees by the proof of the previous lemma with

$$\begin{tikzcd}
\mathbf{SSet} & \boldsymbol{N}\text{-}\mathbf{SSet} \arrow[l, "(-)^H"'] & \boldsymbol{N}\text{-}\mathbf{SSet} \arrow[l, "="'] \\
\mathbf{SSet} \arrow[u, "\mathrm{Ex}"] \arrow[ur, phantom, "\Swarrow_{\mathrm{can}^{-1}}"] & \boldsymbol{N}\text{-}\mathbf{SSet} \arrow[l, "(-)^H"] \arrow[u, "\mathrm{Ex}_N"'] \arrow[ur, phantom, "\Swarrow_{e_N}"] & \boldsymbol{N}\text{-}\mathbf{SSet} \arrow[l, "="] \arrow[u, "="']
\end{tikzcd}$$

whose total mate is (again using compatibility of mates with pasting) precisely $d_N \circ \tau$. This finishes the proof. □

Corollary 3.21. *Let $X \in$ **N-SSet** be isomorphic to $N/H \times K$ for some $K \in$ **SSet** and some subgroup $H \subset N_0$. Then:*

1. *$d_N\colon \mathrm{Sd}_N X \to X$ is an $\mathcal{F}$-weak equivalence for any collection $\mathcal{F}$ of subgroups of N_0.*
2. *$\eta\colon X \to \mathrm{Ex}_N \mathrm{Sd}_N X$ is a weak equivalence.*

Proof. By naturality we may assume without loss of generality that X is actually equal to $N/H \times K$. For the first statement we then simply invoke the previous lemma together with the fact that $d\colon \mathrm{Sd}\,K \to K$ is an ordinary weak equivalence [Kan57, Lemma 7.5].

For the second statement we consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, bend right, "e_N"'] & \mathrm{Ex}_N \mathrm{Sd}_N X \arrow[d, "\mathrm{Ex}_N d_N"] \\
& \mathrm{Ex}_N X
\end{tikzcd}$$

from Construction 3.18. The first part together with Lemma 3.19 implies that both $\mathrm{Ex}_N d_N$ and e_N are weak equivalences; the claim follows by 2-out-of-3. □

3.4 Categories vs. simplicial sets

Let M be a monoid in **Cat** and let $\mathcal{F}$ be a collection of finite subgroups of $\mathrm{Ob}\,M$, which we will confuse with subgroups of $(\mathrm{N}M)_0$. In order to construct the desired model structure on **M-Cat** together with a Quillen equivalence to **M-SSet**, we will need the following mild technical condition:

Definition 3.22. A subgroup $H \subset \mathrm{Ob}(M)$ is called *good* if the right H-action on M given by right multiplication is free. A collection $\mathcal{F}$ of subgroups of $\mathrm{Ob}\, M$ is called *good*, if all $H \in \mathcal{F}$ are good.

Example 3.23. If $\mathrm{Ob}(M) = G$ is a group, then any subgroup $H \subset G$ is good.

Example 3.24. Every subgroup of $\mathrm{Ob}(E\mathcal{M}) = \mathcal{M}$ is good as injections of sets are monomorphisms. More generally, all subgroups of $\mathrm{Ob}(E\mathcal{M} \times G)$ are good.

Theorem 3.25. *Let $\mathcal{F}$ be a good collection of finite subgroups of* $\mathrm{Ob}\, M$*. Then there exists a unique model structure on* ***M*-Cat** *such that a map $f\colon C \to D$ is a weak equivalence or fibration if and only if for each $H \in \mathcal{F}$ the map $f^H\colon C^H \to D^H$ is a weak equivalence or fibration, respectively, in the Thomason model structure.*

This model structure is combinatorial with generating cofibrations

$$\{M/H \times \mathrm{h}\,\mathrm{Sd}^2\, \partial\Delta^n \hookrightarrow M/H \times \mathrm{h}\,\mathrm{Sd}^2\, \Delta^n : n \geq 0,\ H \in \mathcal{F}\}$$

and generating acyclic cofibrations

$$\{M/H \times \mathrm{h}\,\mathrm{Sd}^2\, \Lambda^n_k \hookrightarrow M/H \times \mathrm{h}\,\mathrm{Sd}^2\, \Delta^n : 0 \leq k \leq n,\ H \in \mathcal{F}\}.$$

Moreover, it is proper with homotopy pushouts and pullbacks created by N_M*, and filtered colimits in it are homotopical.*

Finally, the adjunction

$$\mathrm{h}_M\, \mathrm{Sd}^2_{\mathrm{N}(M)}\colon \mathbf{N(\mathit{M})\text{-}SSet} \rightleftarrows \boldsymbol{M}\text{-}\mathbf{Cat} : \mathrm{Ex}^2_{\mathrm{N}(M)}\, \mathrm{N}_M \tag{3.5}$$

is a Quillen equivalence when we equip **N(*M*)-SSet** *with the $\mathcal{F}$-model structure (viewing the elements of $\mathcal{F}$ as subgroups of* $\mathrm{N}(M)_0$ *now).*

If $M = G$ is a discrete group, the above result (without the finiteness condition on $\mathcal{F}$) was proven by Bohmann, Mazur, Osorno, Ozornova, Ponto, and Yarnall [BMO+15, Theorems A and B] although they only explicitly state their result for the collection of *all* subgroups.

Proof. Recall from Proposition 1.7 that the $\mathcal{F}$-model structure on **N(*M*)-SSet** is proper, simplicial, that filtered colimits in it are homotopical, and that it is cofibrantly generated with generating cofibrations

$$I = \{\mathrm{N}(M)/H \times \partial\Delta^n \to \mathrm{N}(M)/H \times \Delta^n : n \geq 0,\ H \in \mathcal{F}\} \tag{3.6}$$

and generating acyclic cofibrations

$$J = \{\mathrm{N}(M)/H \times \Lambda^n_k \to \mathrm{N}(M)/H \times \Delta^n : 0 \le k \le n,\ H \in \mathcal{F}\}.$$

Let us verify the conditions of Proposition 2.2. It is clear that $\boldsymbol{M}$-**Cat** is locally presentable. Moreover, each $N(M)/H$ is evidently cofibrant in **N($\boldsymbol{M}$)-SSet**, and hence so are the sources of the generating acyclic cofibrations as the model structure is simplicial. To complete the verification of Condition (1′) we will prove more generally that the unit is a weak equivalence for each $X = \mathrm{N}(M)/H \times K$ with $H \in \mathcal{F}$ and any $K \in$ **SSet** that can be equipped with the structure of a simplicial complex, i.e. that can be embedded into the nerve of a poset. For this we recall that a standard choice of unit is given by the composition

$$\begin{aligned} X \xrightarrow{\eta} \mathrm{Ex}_{\mathrm{N}(M)} \mathrm{Sd}_{\mathrm{N}(M)} X &\xrightarrow{\mathrm{Ex}_{\mathrm{N}(M)}\,\eta\,\mathrm{Sd}_{\mathrm{N}(M)}} \mathrm{Ex}^2_{\mathrm{N}(M)} \mathrm{Sd}^2_{\mathrm{N}(M)} X \\ &\xrightarrow{\mathrm{Ex}^2_{\mathrm{N}(M)}\,\eta\,\mathrm{Sd}^2_{\mathrm{N}(M)}} \mathrm{Ex}^2_{\mathrm{N}(M)} \mathrm{N}_M \mathrm{h}_M \mathrm{Sd}^2_{\mathrm{N}(M)} X \end{aligned}$$

where the first two maps come from the unit of $\mathrm{Sd}_{\mathrm{N}(M)} \dashv \mathrm{Ex}_{\mathrm{N}(M)}$ and the final one is induced by the unit of $\mathrm{h}_M \dashv \mathrm{N}_M$.

By Corollary 3.21 the first map is a weak equivalence. As $\mathrm{Sd}_{\mathrm{N}(M)}(X) \cong \mathrm{N}(M)/H \times \mathrm{Sd}\,K$ (Lemma 3.20) and as $\mathrm{Ex}_{\mathrm{N}(M)}$ is homotopical (Lemma 3.19), the corollary also implies that the second map is a weak equivalence. For the final map we use Lemma 3.20 twice to see $\mathrm{Sd}^2_{\mathrm{N}(M)} X \cong \mathrm{N}(M)/H \times \mathrm{Sd}^2 K$. Now as an ordinary simplicial set $\mathrm{N}(M)/H$ lies in the essential image of N as the nerve preserves *free* quotients, and so does $\mathrm{Sd}^2 K$ by [Tho80, discussion after Proposition 2.5] as K was assumed to admit the structure of a simplicial complex. Since N preserves products, we see that the underlying simplicial set of $\mathrm{Sd}^2_{\mathrm{N}(M)} X$ indeed lies in the essential image of N. But as a map of simplicial sets, the unit $\eta\colon Y \to \mathrm{N}_M\mathrm{h}_M(Y)$ agrees with the usual unit $\eta\colon Y \to \mathrm{Nh}Y$ for any $Y \in \boldsymbol{M}$-**SSet**; as the nerve is fully faithful, we see that this is in fact an isomorphism as soon as the underlying simplicial set of Y lies in the essential image of N, finishing the verification of (1′).

For Condition (2)—i.e. that the right adjoint sends pushouts along generating cofibrations to homotopy pushouts—we show more generally (cf. Lemma 3.4-(2)) that pushouts along M-equivariant Dwyer maps are sent to homotopy pushouts by $\mathrm{Ex}^2_{\mathrm{N}(M)} \circ \mathrm{N}_M$, which is immediate from Corollary 3.14 together with Lemma 3.19.

Finally, $\mathrm{Ex}_{\mathrm{N}(M)}$ preserves filtered colimits by Lemma 3.16, and the same argument as employed there shows that also N_M preserves filtered colimits. Thus, also the composition $\mathrm{Ex}^2_{\mathrm{N}(M)}\,\mathrm{N}_M$ preserves filtered colimits, verifying Condition (3).

Thus, the proposition applies and we see that ***M*-SSet** carries a model structure such that a map f is weak equivalence or fibration if and only if $\mathrm{Ex}^2_{\mathrm{N}(M)}\,\mathrm{N}(f)$ is, i.e. if and only if $(\mathrm{Ex}^2_{\mathrm{N}(M)}\,\mathrm{N}_M(f))^H$ is a weak equivalence or fibration in the Kan Quillen model structure for every $H \in \mathcal{F}$. As both $\mathrm{Ex}_{\mathrm{N}(M)}$ and N_M commute with $(-)^H$ this is indeed equivalent to the condition stated in the theorem.

Moreover, the proposition tells us that (3.5) is a Quillen equivalence, that the model structure obtained this way is proper with homotopy pushouts and pullbacks created by N_M, and that filtered colimits in it are homotopical. Moreover, it shows that the model structure is cofibrantly generated (hence combinatorial) with generating cofibrations $\mathrm{h}_M\,\mathrm{Sd}^2_{\mathrm{N}(M)}(I)$ and generating acyclic cofibrations $\mathrm{h}_M\,\mathrm{Sd}^2_{\mathrm{N}(M)}(J)$. Finally, again using that the nerve preserves free quotients, Lemma 3.20 tells us that also the sets from the theorem form generating cofibrations and generating acyclic cofibrations, respectively. □

Corollary 3.26. *In the situation of the theorem, pushouts along M-equivariant Dwyer maps are homotopy pushouts.*

Proof. We have seen in the above proof that N_M sends such squares to homotopy pushouts. □

Corollary 3.27. *In the above situation, a commutative square is a homotopy pushout if and only if for every $H \in \mathcal{F}$ the induced square on H-fixed points is a homotopy pushout in the Thomason model structure on* **Cat**.

Proof. Using that $\mathrm{N}_M\colon$ ***M*-Cat** $\to$ **N(*M*)-SSet** and $\mathrm{N}\colon$ **Cat** $\to$ **SSet** create homotopy pushouts, this follows from the characterization given in Proposition 1.7. □

Together with Lemma 3.19 we get:

Corollary 3.28. *For any good family $\mathcal{F}$, the homotopical functor*

$$\mathrm{N}_M\colon \boldsymbol{M}\text{-}\mathbf{Cat}_{\mathcal{F}\text{-w.e.}} \to \mathbf{N}(\boldsymbol{M})\text{-}\mathbf{SSet}_{\mathcal{F}\text{-w.e.}}$$

descends to an equivalence of homotopy categories. □

3.5 The G-global model structure

Let us specialize the above to the context of G-global homotopy theory:

Corollary 3.29. *For any discrete group G, there is a unique model structure on* $\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**Cat** *in which a map f is a weak equivalence or fibration if and only if f^φ is a weak equivalence or fibration, respectively, in the Thomason model structure for each universal $H \subset \mathcal{M}$ and each $\varphi\colon H \to G$. We call this the* G-global model structure *and its weak equivalences the* G-global weak equivalences.

This model category is combinatorial with generating cofibrations

$$\{E\mathcal{M}\times_\varphi G\times(\mathrm{h}\,\mathrm{Sd}^2\,\partial\Delta^n \hookrightarrow \mathrm{h}\,\mathrm{Sd}^2\,\Delta^n) : n \geq 0,\ H \subset \mathcal{M} \textit{ universal}, \varphi\colon H \to G\}$$

and generating acyclic cofibrations

$$\{E\mathcal{M}\times_\varphi G\times(\mathrm{h}\,\mathrm{Sd}^2\,\Lambda^n_k \hookrightarrow \mathrm{h}\,\mathrm{Sd}^2\,\Delta^n) : 0 \leq k \leq n,\ H \subset \mathcal{M} \textit{ universal}, \varphi\colon H \to G\}.$$

Moreover, it is proper and filtered colimits in it are homotopical.

Finally, we have a Quillen equivalence

$$\mathrm{h}_{E\mathcal{M}\times G}\,\mathrm{Sd}^2_{E\mathcal{M}\times G}\colon \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet} \rightleftarrows \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{Cat} : \mathrm{Ex}^2_{E\mathcal{M}\times G}\,\mathrm{N}_{E\mathcal{M}\times G}$$

with respect to the model structure from Corollary 1.8 on the left hand side. □

Corollary 3.30. *The homotopical functor*

$$\mathrm{N}_{E\mathcal{M}\times G}\colon \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{Cat}_{G\text{-global}} \to \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}_{G\text{-global}}$$

descends to an equivalence of homotopy categories. □

On the other hand, we can equip $\boldsymbol{G}$-**Cat**—either by Theorem 3.25 above or by [BMO+15, Theorem A]—with the equivariant model structure with respect to the collection of all finite subgroups $H \subset G$; we again call this the *proper G-equivariant model structure*. As in the simplicial case, the G-global and proper G-equivariant model structure are related through a chain of four adjoints:

Proposition 3.31. *The functor* $\mathrm{triv}_{E\mathcal{M}}\colon \boldsymbol{G}\text{-}\mathbf{Cat}_{\mathrm{proper}} \to \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{Cat}_{G\text{-global}}$ *is homotopical and the induced functor* $\mathrm{Ho}(\mathrm{triv}_{E\mathcal{M}})$ *on homotopy categories fits into a sequence of four adjoints suggestively denoted by*

$$\mathbf{L}(-/E\mathcal{M}) \dashv \mathrm{Ho}(\mathrm{triv}_{E\mathcal{M}}) \dashv (-)^{\mathbf{R}E\mathcal{M}} \dashv \mathcal{R}.$$

Moreover, $(-)^{\mathbf{R}E\mathcal{M}}$ *is a (Bousfield) localization at the $\mathcal{E}$-weak equivalences.*

Proof. The diagram of homotopical functors

$$\begin{tikzcd}
\boldsymbol{G}\text{-}\mathbf{Cat}_{\mathrm{proper}} \arrow[r, "\mathrm{triv}_{E\mathcal{M}}"] \arrow[d, "\mathrm{N}_G"'] & \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{Cat}_{G\text{-global}} \arrow[d, "\mathrm{N}_{E\mathcal{M}\times G}"] \\
\boldsymbol{G}\text{-}\mathbf{SSet}_{\mathrm{proper}} \arrow[r, "\mathrm{triv}_{E\mathcal{M}}"'] & \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}_{G\text{-global}}
\end{tikzcd}$$

commutes by direct inspection, and the vertical maps induce equivalences of homotopy categories by Corollary 3.28. Thus, the claim follows from Theorem 1.9 by a straight-forward diagram chase. □

Remark 3.32. Analogously to the case of the simplicial models treated in [Len20, Theorem 1.2.92], the functor $\mathrm{triv}_{E\mathcal{M}}$ is easily seen to be right Quillen with respect to the above model structures, so that its left adjoint $(-)/E\mathcal{M}$ is left Quillen, justifying the above notation $\mathbf{L}(-/E\mathcal{M})$. Moreover, while $\mathrm{triv}_{E\mathcal{M}}$ is *not* left Quillen, we can make it into a left Quillen functor by suitably enlarging the generating cofibrations of the G-global model structure, e.g. using [Len20, Corollary A.2.17]. With respect to this model structure, $(-)^{E\mathcal{M}}$ is then right Quillen and its right derived functor is then right adjoint to $\mathrm{Ho}(\mathrm{triv}_{E\mathcal{M}})$ as before.

4. *G*-categories as models of *G*-global homotopy types

Fix a discrete group G. In this final section, we will prove as our main result that already categories with a mere G-action model unstable G-global homotopy theory. For this we will use the following notion from [Len23, Definition 6.1] which generalizes Schwede's *global equivalences* [Sch19, Definition 3.2]:

Definition 4.1. Let C be a G-category, let H be a finite group, and let $\varphi\colon H \to G$ be a group homomorphism. We define the *φ-'homotopy' fixed points* of C as

$$C^{\text{`}h\text{'}\varphi} := \mathrm{Fun}(EH, \varphi^* C)^H$$

where the H-action on the right is the diagonal of the H-action on C via φ and the one induced by the right regular action on EH.

If f is a G-equivariant functor $f\colon C \to D$, then $f^{\text{`}h\text{'}\varphi}$ is defined analogously. We call $f$ a *$G$-global weak equivalence* if $f^{\text{`}h\text{'}\varphi}$ is a weak homotopy equivalence (i.e. weak equivalence in the Thomason model structure) for every such φ.

Remark 4.2. The above represents the H-homotopy fixed points of $\varphi^* C$ *with respect to the canonical model structure on* **Cat**, i.e. where the weak equivalences are the equivalences of categories. However, as we are generalizing Thomason's model structure (for which homotopy fixed points look quite different), we have decided to put 'homotopy' in quotation marks everywhere.

It is of course crucial that we're taking homotopy fixed points with respect to the 'wrong' model structure here as otherwise being a G-global weak equivalence would be equivalent to being an underlying weak homotopy equivalence.

Theorem 4.3. *There is a unique cofibrantly generated model structure on* ***G*-Cat** *with weak equivalences the G-global weak equivalences and with generating cofibrations given by*

$$\{(E\mathcal{M} \times_\varphi G) \times (\mathrm{h}\,\mathrm{Sd}^2\,\partial\Delta^n \hookrightarrow \mathrm{h}\,\mathrm{Sd}^2\,\Delta^n) : n \geq 0, H \subset \mathcal{M} \textit{ universal}, \\ \varphi\colon H \to G \textit{ homomorphism}\}.$$

This model structure is combinatorial, proper, and filtered colimits in it are homotopical. We call it the thick G-global model structure*. A set of generating acyclic cofibrations is given by*

$$\{(E\mathcal{M}\times_{\varphi}G)\times(\mathrm{h}\,\mathrm{Sd}^2\,\Lambda^n_k\hookrightarrow\mathrm{h}\,\mathrm{Sd}^2\,\Delta^n):0\le k\le n, H\subset\mathcal{M}\textit{ universal},\\ \varphi\colon H\to G\textit{ homomorphism}\}.$$

Moreover, we have a Quillen equivalence

$$\mathrm{forget}\colon \boldsymbol{E\mathcal{M}\text{-}G\text{-}\mathbf{Cat}}_{G\text{-global}}\rightleftarrows \boldsymbol{G\text{-}\mathbf{Cat}}_{\text{thick }G\text{-global}}:\mathrm{Fun}(E\mathcal{M},-)\tag{4.1}$$

(where $\mathrm{Fun}(E\mathcal{M},C)$ *is equipped for every* $C\in$ **G-Cat** *with the left* $E\mathcal{M}$*-action induced by the usual right* $E\mathcal{M}$*-action on itself), and the right adjoint creates homotopy pushouts and homotopy pullbacks.*

The proof will be given below after some preparations.

Lemma 4.4. *Let* $f\colon C\to D$ *be a map of* G*-categories, let* $H\subset\mathcal{M}$ *be any subgroup, and let* $\varphi\colon H\to G$*. Then* $f^{`h'\varphi}$ *is a weak equivalence in the Thomason model structure if and only if* $\mathrm{Fun}(E\mathcal{M},f)^{\varphi}$ *is so. In particular,* f *is a* G*-global weak equivalence in* **G-Cat** *if and only if* $\mathrm{Fun}(E\mathcal{M},f)$ *is a* G*-global weak equivalence in* **$E\mathcal{M}$-G-Cat**.

Proof. This appears as [Len20, Remark 4.1.28]; as we will need similar arguments below, we spell out the proof in detail.

It will be enough to show that the inclusion $H\hookrightarrow\mathcal{M}$ induces a $(G\times H)$-equivariant weak equivalence $\mathrm{Fun}(E\mathcal{M},C)\to\mathrm{Fun}(EH,C)$ for every G-category C. For this we observe that there exists a right H-equivariant map $r\colon\mathcal{M}\to H$ as H acts freely from the right on $\mathcal{M}$. There are then unique isomorphisms $E(i)E(r)=E(ir)\cong\mathrm{id}$ and $E(r)E(i)=E(ri)\cong\mathrm{id}$, and these are automatically H-equivariant. It is then clear that the corresponding restrictions exhibit $\mathrm{Fun}(Er,C)$ as a $(G\times H)$-equivariant quasi-inverse to the map in question; in particular, both are $(G\times H)$-equivariant weak equivalences. □

Definition 4.5. A small $E\mathcal{M}$-G-category C is called *saturated* if the unit $\eta\colon C\to\mathrm{Fun}(E\mathcal{M},\mathrm{forget}\,C)$ induces equivalences on φ-fixed points for all universal $H\subset\mathcal{M}$ and all $\varphi\colon H\to G$.

Remark 4.6. As observed in [Len20, Remark 4.1.24], an $E\mathcal{M}$-G-category C is saturated if and only if the inclusion $C \hookrightarrow \mathrm{Fun}(E\mathcal{M}, C)$ of constant diagrams induces equivalences on φ-fixed points for all φ as above, where the right hand side is now equipped with the diagonal $E\mathcal{M}$-action. This alternative definition is used in [Sch22a, Len20].

Proposition 4.7. *Let $K \subset \mathcal{M}$ be any finite subgroup, let $\psi\colon K \to G$, and let P be any poset (viewed as an $E\mathcal{M}$-G-category with trivial actions). Then $E\mathcal{M} \times_\psi G \times P$ is saturated.*

Proof. As $\mathrm{Fun}(E\mathcal{M}, –)$, the forgetful functor, and $(–)^\varphi$ each preserve products, it suffices to show that both $E\mathcal{M} \times_\psi G$ and P are saturated. But indeed, the first statement is a special case of [Len20, Lemma 4.2.10], and for the second statement we observe that η is even an isomorphism as $E\mathcal{M}$ is a connected groupoid while P has no non-trivial isomorphisms. □

Proof of Theorem 4.3. Obviously, there is at most one such model structure. We will now verify the assumptions (1′), (2), and (3) of Proposition 2.2 for the adjunction (4.1). For this we begin by observing that the images of the standard generating (acyclic) cofibrations of $\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**Cat** are indeed precisely the above sets. Moreover, Lemma 4.4 tells us that the transferred weak equivalences agree with the G-global weak equivalences.

The standard generating acyclic cofibrations of $\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**Cat** have cofibrant sources as the ones of $\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**SSet** have. The remainder of Condition (1′), stating that the unit is a weak equivalence on $\varnothing$ as well as on sources and targets of the standard generating cofibrations, is a special case of Proposition 4.7.

For Condition (2) we consider any pushout

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array} \tag{4.2}$$

in $\boldsymbol{G}$-**Cat** such that i is a G-equivariant Dwyer map; for example, i could be one of the standard generating cofibrations. As all colimits in question are created in **Cat**, Lemma 1.16 shows that applying $\mathrm{Fun}(E\mathcal{M}, –)$ to (4.2) yields a pushout in $\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**Cat**. Moreover, Lemma 3.4-(3) shows that $\mathrm{Fun}(E\mathcal{M}, i)$

is an $(E\mathcal{M} \times G)$-equivariant Dwyer map. Thus, Corollary 3.26 implies that $\mathrm{Fun}(E\mathcal{M}, –)$ sends (4.2) to a homotopy pushout.

It remains to check that $\mathrm{Fun}(E\mathcal{M}, –)$ preserves filtered colimits up to weak equivalence. For this we observe that the functor $\mathrm{Fun}(E\mathcal{M}, –)^\varphi$ is weakly equivalent to $(–)^{`h'\varphi} = \mathrm{Fun}(EH, –)^\varphi \colon \boldsymbol{G}\text{-}\mathbf{Cat} \to \mathbf{Cat}$ by the proof of Lemma 4.4, so it suffices that $(–)^{`h'\varphi}$ preserves filtered colimits up to weak equivalence. But as EH is a finite category and since filtered colimits in **Cat** commute with finite limits, it even preserves filtered colimits up to isomorphism.

Hence Proposition 2.2 implies the existence of the model structure, shows that (4.1) is a Quillen equivalence, and proves all the desired properties. □

We can also construct a variant of the above model structure with fewer cofibrations, which for $G = 1$ recovers Schwede's global model structure on **Cat** recalled in Theorem 1.15:

Proposition 4.8. *There is a unique model structure on* $\boldsymbol{G}$-**Cat** *in which a map is a weak equivalence or fibration if and only if* $f^{`h'\varphi}$ *is a weak equivalence or fibration, respectively, in the Thomason model structure for every universal* $H \subset \mathcal{M}$ *and each* $\varphi\colon H \to G$*. We call this the* G-global model structure. *It is proper (with homotopy pushouts and pullbacks created by the homotopy fixed point functors* $(–)^{`h'\varphi}$ *for varying* φ*) and combinatorial with generating cofibrations*

$$\{EH \times_\varphi G \times (\mathrm{h}\,\mathrm{Sd}^2\,\partial\Delta^n \hookrightarrow \mathrm{h}\,\mathrm{Sd}^2\,\Delta^n) : H \textit{ finite group}, \varphi\colon H \to G, n \geq 0\} \tag{4.3}$$

and generating acyclic cofibrations

$$\{EH \times_\varphi G \times (\mathrm{h}\,\mathrm{Sd}^2\,\Lambda^n_k \hookrightarrow \mathrm{h}\,\mathrm{Sd}^2\,\Delta^n) : H \textit{ finite group}, \varphi\colon H \to G, 0 \leq k \leq n\}. \tag{4.4}$$

Moreover, filtered colimits in this model structure are homotopcial.

Finally, the adjunction

$$\mathrm{id}\colon \boldsymbol{G}\text{-}\mathbf{Cat}_{G\text{-global}} \rightleftarrows \boldsymbol{G}\text{-}\mathbf{Cat}_{G\text{-global thick}} : \mathrm{id}$$

is a Quillen equivalence.

Proof. To construct the model structure and to prove that it has all the properties stated above it suffices to verify the assumptions (1)–(3) of Proposition 2.2 for the adjunction

$$L\colon \prod_{\varphi\colon H\to G} \mathbf{Cat} \rightleftarrows \boldsymbol{G}\text{-}\mathbf{Cat} : \big((-)^{\text{`}h\text{'}\varphi}\big)_\varphi$$

where the product runs over a set of representatives of isomorphism classes of finite groups H and all homomorphisms $\varphi\colon H \to G$. The left adjoint can be calculated as before on objects by $(X_\varphi)_\varphi \mapsto \coprod_\varphi EH \times_\varphi G \times X_\varphi$ and likewise on morphisms. In particular, the above sets agree up to isomorphism with the images of the standard generating (acyclic) cofibrations of $\prod_\varphi \mathbf{Cat}$ under L.

It is clear that the right adjoint sends the maps in (4.4) to weak equivalences. Moreover, the maps in (4.3) are G-equivariant Dwyer maps, so the right adjoint sends pushouts along them to homotopy pushouts by the same argument as in the proof of Theorem 4.3. Finally, the right adjoint clearly preserves filtered colimits.

It is clear that the identity functor $\boldsymbol{G}\text{-}\mathbf{Cat}_{G\text{-global}} \to \boldsymbol{G}\text{-}\mathbf{Cat}_{G\text{-global thick}}$ preserves and reflects weak equivalences, so it only remains to show that it sends generating cofibrations to cofibrations. To this end we fix a finite group H with a homomorphism $\varphi\colon H \to G$, and we pick an injective homomorphism $\iota\colon H \to \mathcal{M}$ with universal image. To finish the proof, it is now enough to show that $EH\times_\varphi G$ is a retract of $E\mathcal{M}\times_{\varphi\iota^{-1}} G$, for which it suffices that H is a right H-equivariant retract of $\mathcal{M}$, where H acts on $\mathcal{M}$ via ι. But indeed, $\iota\colon H \to \mathcal{M}$ is an H-equivariant injection and $\mathcal{M}$ is free, so ι admits an H-equivariant retraction as desired. □

Finally, let us compare the above to the usual proper G-equivariant model structure on $\boldsymbol{G}\text{-}\mathbf{Cat}$:

Definition 4.9. We call a map $f\colon C \to D$ in $\boldsymbol{G}\text{-}\mathbf{Cat}$ a *G-equivariant 'homotopy' weak equivalence* if $f^{\text{`}h\text{'}H} = \mathrm{Fun}(EH,f)^H$ is a weak equivalence for every finite subgroup $H \subset G$.

Theorem 4.10. *The functor*

$$\mathrm{Fun}(EG,-)\colon \boldsymbol{G}\text{-}\mathbf{Cat}_{G\text{-global}} \to \boldsymbol{G}\text{-}\mathbf{Cat}_{\text{proper}} \tag{4.5}$$

is homotopical. The induced functor of homotopy categories is is a (Bousfield) localization at the G-equivariant 'homotopy' weak equivalences, and it is the third term in a sequence of four adjoints.

Proof. We claim that the diagram

$$\begin{tikzcd} \boldsymbol{G}\text{-}\mathbf{Cat}_{G\text{-global}} \arrow[r, "\mathrm{Fun}(EG,-)"] \arrow[d, "\mathrm{Fun}(E\mathcal{M},-)"'] & \boldsymbol{G}\text{-}\mathbf{Cat}_{\text{proper}} \arrow[d, "\mathrm{triv}_{E\mathcal{M}}"] \\ \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{Cat}_{G\text{-global}} \arrow[r, "\mathrm{id}"'] & \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{Cat}_{\mathcal{E}\text{-weak equivalences}} \end{tikzcd} \tag{4.6}$$

of homotopical functors commutes up to a zig-zag of levelwise weak equivalences. Before we prove this, let us show how it implies the theorem: the vertical maps induce equivalences of homotopy categories by Theorem 4.3 and Proposition 3.31, respectively. Thus, the top arrow induces a localization at those maps that are inverted by the functor induced by the lower composite. As the $\mathcal{E}$-weak equivalences are saturated (being the weak equivalences of a model structure), these are precisely those maps f such that $\mathrm{Fun}(E\mathcal{M},f)$ is an $\mathcal{E}$-weak equivalence, which we as before identify with the G-equivariant 'homotopy' weak equivalences. Finally, $(-)^{\mathbf{R}E\mathcal{M}}$ is a localization of $\mathrm{Ho}(\boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{Cat}_{G\text{-global}})$ at the $\mathcal{E}$-weak equivalences by Proposition 3.31, hence equivalent to the functor induced by

$$\mathrm{id}\colon \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{Cat}_{G\text{-global}} \to \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{Cat}_{\mathcal{E}\text{-weak equivalences}}. \tag{4.7}$$

As $(-)^{\mathbf{R}E\mathcal{M}}$ is the third term in a sequence of four adjoints by the aforementioned proposition, so is the functor induced by (4.7), and hence also the one induced by $\mathrm{Fun}(EG,-)$ as desired.

It remains to construct a zig-zag of natural levelwise weak equivalences filling (4.6). For this we will show more generally that for any $E\mathcal{M}$-G-category C the maps

$$\mathrm{Fun}(EG,C) \to \mathrm{Fun}(EG\times E\mathcal{M},C) \leftarrow \mathrm{Fun}(E\mathcal{M},C)$$

induced by the projections $EG \leftarrow EG\times E\mathcal{M} \to E\mathcal{M}$ are $\mathcal{E}$-weak equivalences, for which it is enough that the projections are H-equivariant equivalences for every universal $H\subset\mathcal{M}$ and every *injective* $\varphi\colon H\to G$ (when we let H act on G via φ). This follows as before as both G and $\mathcal{M}$ are free right H-sets with respect to these actions. □

Remark 4.11. Using a similar argument as in the proof of Proposition 4.8 it is not hard to show that (4.5) is right Quillen for the thick G-global model structure whenever the cardinality of G is at most $|\mathcal{M}|$, so that we have a Quillen adjunction

$$EG\times -\colon \boldsymbol{G}\textbf{-Cat}_{\text{proper}} \rightleftarrows \boldsymbol{G}\textbf{-Cat}_{\text{thick }G\text{-global}} : \mathrm{Fun}(EG,-).$$

Remark 4.12. Already an ordinary global space has an *underlying G-space* for every finite group. In [Sch19, Example 3.21], Schwede gives an explicit model of this in terms of a Quillen adjunction

$$EG\times_G -\colon \boldsymbol{G}\textbf{-Cat}_{G\text{-equivariant}} \rightleftarrows \textbf{Cat}_G : \mathrm{Fun}(EG,-) \tag{4.8}$$

with homotopical right adjoint for a certain model structure on **Cat** with the same weak equivalences as the global one, but more cofibrations.

The induced adjunction on homotopy categories is in fact a shadow of Theorem 4.10: for every $\alpha\colon G\to H$, the functor $\alpha^*\colon \boldsymbol{H}\textbf{-Cat}\to \boldsymbol{G}\textbf{-Cat}$ is easily seen to be right Quillen with respect to the thick H-global and thick G-global model structure, respectively. Specializing this to $H=1$ we obtain together with the previous remark a chain of Quillen adjunctions

$$\boldsymbol{G}\textbf{-Cat}_{G\text{-equivariant}} \underset{\mathrm{Fun}(EG,-)}{\overset{EG\times -}{\rightleftarrows}} \boldsymbol{G}\textbf{-Cat}_{\text{thick }G\text{-global}} \underset{\mathrm{triv}_G}{\overset{(-)/G}{\rightleftarrows}} \textbf{Cat}_{\text{thick global}}$$

with homotopical right adjoints, whose composition agrees with (4.8).

As an application of the above comparison, we can now introduce a new model structure on $\boldsymbol{G}$**-Cat** representing proper G-equivariant homotopy theory, whose weak equivalences are tested on 'homotopy' fixed points:

Theorem 4.13. *There is a unique model structure on* $\boldsymbol{G}$**-Cat** *in which a map f is a weak equivalence or fibration if and only if* $\mathrm{Fun}(EG,f)^H$ *is a weak equivalence or fibration, respectively, in the Thomason model structure on* **Cat** *for each finite subgroup $H\subset G$. We call this the* thick G-equivariant 'homotopy' fixed point model structure*; its weak equivalences are precisely the G-equivariant 'homotopy' weak equivalences. This model structure is combinatorial with generating cofibrations*

$$\{EG\times_H G\times(\mathrm{h}\,\mathrm{Sd}^2\,\partial\Delta^n\hookrightarrow \mathrm{h}\,\mathrm{Sd}^2\,\Delta^n) : H\subset G \text{ finite}, n\ge 0\}$$

and generating acyclic cofibrations

$$\{EG \times_H G \times (\mathrm{h}\,\mathrm{Sd}^2\,\Lambda^n_k \hookrightarrow \mathrm{h}\,\mathrm{Sd}^2\,\Delta^n) : H \subset G \text{ finite}, 0 \le k \le n\}$$

and proper; a commutative square is a homotopy pushout or pullback if and only if the induced square on H-'homotopy' fixed points is a homotopy pushout or pullback, respectively, in the Thomason model structure on **Cat** *for each finite $H \subset G$. Moreover, filtered colimits in this model structure are homotopical.*

*Likewise, there is a G-*equivariant 'homotopy' fixed point model structure *on* ***G*-Cat** *in which a map f is a weak equivalence or fibration if and only if $f^{\text{`}h\text{'}H} = \mathrm{Fun}(EH, f)^H$ is a weak equivalence or fibration, respectively, in the Thomason model structure for every finite $H \subset G$. It is again proper with the above characterization of homotopy pushouts and pullbacks, and it is morover combinatorial with generating cofibrations*

$$\{EH \times_H G \times (\mathrm{h}\,\mathrm{Sd}^2\,\partial\Delta^n \hookrightarrow \mathrm{h}\,\mathrm{Sd}^2\,\Delta^n) : H \subset G \text{ finite}, n \ge 0\}$$

and generating acyclic cofibrations

$$\{EH \times_H G \times (\mathrm{h}\,\mathrm{Sd}^2\,\Lambda^n_k \hookrightarrow \mathrm{h}\,\mathrm{Sd}^2\,\Delta^n) : H \subset G \text{ finite}, 0 \le k \le n\}.$$

Finally, the adjunctions

$$\mathrm{id}\colon \boldsymbol{G}\text{-}\mathbf{Cat}_{G\text{-`homotopy'}} \rightleftarrows \boldsymbol{G}\text{-}\mathbf{Cat}_{\text{thick } G\text{-`homotopy'}} : \mathrm{id} \tag{4.9}$$

$$EG \times -\colon \boldsymbol{G}\text{-}\mathbf{Cat}_{\text{proper } G\text{-equivariant}} \rightleftarrows \boldsymbol{G}\text{-}\mathbf{Cat}_{\text{thick } G\text{-`homotopy'}} : \mathrm{Fun}(EG, -) \tag{4.10}$$

are Quillen equivalences.

Proof. To construct the model structures and to establish the above properties, it is enough to verify the assumptions (1)–(3) of Proposition 2.2, which can be done just as in the proof of Proposition 4.8. The same argument as in Lemma 4.4 and the aforementioned proposition then shows that (4.9) is a Quillen adjunction and that both sides have the same weak equivalences, so that it is even a Quillen equivalence.

For the final statement we observe that $\mathrm{Fun}(EG, -)$ preserves (and reflects) weak equivalences as well as fibrations by definition; in particular, (4.10) is a Quillen adjunction. It only remains to show that $\mathrm{Fun}(EG, -)$ descends to an equivalence, which is immediate from Theorem 4.10. □

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