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LIMITS OF TOPOLOGICAL SPACES AS ENRICHED CATEGORIES

Derek Scott Cook and Ittay Weiss

Résumé. Le lien entre les espaces métriques et la topologie repose sur certaines propriétés du treillis des réels non négatifs. En raison de la célèbre observation de Lawvere selon laquelle la théorie des espaces métriques est une branche de la théorie des catégories enrichies, il est naturel d'étudier dans quelle mesure le lien avec la topologie survit lors de l'enrichissement dans d'autres petits cosmos. En même temps, l'essor récent des techniques topologiques en science des données soulève la question de savoir quelles propriétés théoriques du treillis des réels non négatifs jouent un rôle vital, dans le but d'axiomatiser ces propriétés afin d'améliorer l'applicabilité des techniques au-delà métrique classique. Nous considérons ces deux motivations comme les faces théorique et applicable d'une même médaille mathématique. Nous identifions une large classe de quantales comme réponse commune aux deux questions, et utilisons les résultats pour présenter une construction de limites d'espaces qui est classiquement équivalente à la construction topologique, mais qui a un potentiel constructif différent.
Abstract. The link between metric spaces and topology relies on various lattice theoretic properties of the non-negative reals. Due to Lawvere's famous observation that metric space theory is a branch of enriched category theory, it is natural to study the extent to which the link with topology survives when enriching in other small cosmoses. At the same time, the recent flourish of topological techniques in data science raises the question of which lattice theoretic properties of the non-negative reals play a vital role, with the aim of axiomatising just those properties in order to enhance the applicability of techniques beyond the classical metric setting. We view these two motiva-

tions as the theoretical and applicable sides of the same mathematical coin. We pinpoint a wide class of quantales as the common answer to the two questions, and use the results to present a construction of limits of spaces that is classically equivalent to the topological one, but has constructively different potential.
Keywords. quantale, quantale enrichment, generalised metric space, constructive complete distributivity, induced topology, topological data analysis.
Mathematics Subject Classification (2010). 18D20, 18F75, 18B35, 54E35.

1. Introduction and motivation

It is well known ([20]) that it is fruitful to view a metric space as a small category enriched in the monoidal category $[0, \infty]$, with an arrow $x \to y$ precisely when $x \geq y$, and monoidal structure provided by addition. Any aspect of metric space theory thus becomes a source of enriched categorical investigation. Our interest is in continuity for a function $f\colon X \to Y$ between two sets, each of which is the set of objects of an enriched category. A suitable category $\mathcal{V}$ to enrich in is a *cosmos*, i.e., a symmetric closed monoidal cocomplete category ([19]). When $\mathcal{V}$ is also small, it is canonically equivalent to a complete lattice. Such a lattice is then precisely a commutative quantale Q. Equivalently, a (commutative) quantale is a (commutative) monoid object in the category $\mathbf{CJLat}$ of complete join lattices with respect to its tensor product (see [17] and [10]). Let $\mathbf{cQnt}$ be the category of commutative quantales with morphisms the join preserving monoidal functors. In more detail, a quantale Q is a complete lattice with joins $\bigvee$, meets $\bigwedge$, bottom element $\bot$, and top element $\top$. It is equipped with a monoidal product, namely an associative operation $\cdot$ with a two-sided unit 1, and it distributes over arbitrary joins, i.e.,

$$x \cdot \bigvee S = \bigvee\{x \cdot s \mid s \in S\} \quad \text{and} \quad \bigvee S \cdot x = \bigvee\{s \cdot x \mid s \in S\}$$

for all $x \in Q$ and $S \subseteq Q$. A morphism $f\colon Q \to Q'$ is a monoidal functor, again necessarily strict, that is also a complete join homomorphism. The quantale is *affine* if its monoidal unit is the top element. It is commutative when its monoidal product is symmetric (necessarily strictly so). Recall from [11] that a complete lattice L is *constructively completely distributive* (CCD) if $\bigvee\colon \mathcal{D}(L) \to L$, as a functor from the lattice of down-closed subsets of L, admits a left adjoint $\Downarrow(-)$. Explicitly, $y \in \Downarrow(x)$ precisely when, for all subsets $S \subseteq Q$, the condition $x \leq \bigvee S$ implies that $y \leq s$ for some $s \in S$. We then say that y is *totally below* x, and write $y \lll x$. The CCD condition then amounts to

$$x = \bigvee \Downarrow(x),$$

for all $x \in L$. We shall use the auxiliary notation

$$\downarrow(x) = \{y \in L \mid y \leq x\}, \quad \downarrow\!\!\!\!\!\!-\,(x) = \{y \in L \mid x < y\}, \quad \text{and} \quad \Uparrow(x) = \{y \in L \mid y > x\}$$

for elements in a lattice.

Let Q_1 and Q_2 be quantales, X a small Q_1-category, Y a small Q_2-category, and $f{:}\,X \to Y$ a function between the underlying sets of objects. We say that f is *continuous* at $x \in X$ if for all $\varepsilon \lll \top$ in Q_2 there exists a $\delta \lll \top$ in Q_1 such that

$$\delta \lll X(x,y) \implies \varepsilon \lll Y(fx, fy).$$

Expectantly, we say that f is *continuous* if f is continuous at all points $x \in X$. It is easily seen that the identity function from a small Q-category to itself is continuous, and that the composition of continuous functions is continuous. Therefore, if Γ is a class of quantales, we obtain the category $\Gamma\mathbf{Cat}_{\text{cont}}$ consisting of all small Q-categories, where Q is of class Γ, with all continuous functions as morphisms. For instance, if $\Gamma = \{[0,\infty]\}$, then $\Gamma\mathbf{Cat}_{\text{cont}}$ is the category of all Lawvere metric spaces with morphisms all functions satisfying the usual cauchy condition of continuity.

A quantale is a *value quantale* ([12]) when it is affine, its underlying lattice is CCD, and $\Downarrow(\top)$ is closed under finite joins. The following result was noted in [32, 7].

Theorem 1.1. *Let* F *be the class of all value quantales. The open ball topology functor* $\mathcal{O}{:}\,\mathsf{F}\mathbf{Cat}_{\text{cont}} \to \mathbf{Top}$ *is an equivalence of categories. In fact, it is the unique equivalence between these categories as concrete categories.*

As a consequence, there arises a translation mechanism between topology and the language of enriched categories. Given any topological concept, one may ask whether it is captured enriched categorically in a natural fashion. For instance, since $\mathbf{Top}$ is complete, so is $\mathsf{F}\mathbf{Cat}_{\text{cont}}$. If X and Y are two objects of $\mathsf{FCat}_{\mathrm{cont}}$, their product may be computed as $\Phi((\mathcal{O}(X) \times \mathcal{O}(Y)))$, where the product is computed in $\mathbf{Top}$ and $\Phi\colon \mathbf{Top} \to \mathsf{FCat}_{\mathrm{cont}}$ is any choice of an equivalency. Of course, this is not what we mean by capturing products enriched categorically. Instead, we insist on treating X and Y as enriched categories and remain firmly within the category $\mathsf{FCat}_{\mathrm{cont}}$, without passing to the equivalent $\mathbf{Top}$. Doing so, suppose that X and Y are enriched, respectively, in the quantales Q_1 and Q_2. Suppose further that a suitable coproduct quantale $Q = Q_1 \coprod Q_2$ exists. It is then natural to define the Q-enriched category $X \times Y$ with $\mathrm{ob}(X \times Y) = \mathrm{ob}(X) \times \mathrm{ob}(Y)$ and

$$(X \times Y)((x,y),(x',y')) = X(x,x') \otimes Y(y,y')$$

where, for elements $x \in Q_1$ and $y \in Q_2$, we write $x \otimes y$ for the element $\iota_1(x) \cdot \iota_2(y) \in Q_1 \coprod Q_2$, namely the monoidal product in the coproduct $Q_1 \coprod Q_2$, of the canonical injections of x and y in it. We will show that this construction is legitimate and that it results in the categorical product of X and Y in $\mathsf{FCat}_{\mathrm{cont}}$. A fortiori, the open ball topology $\mathcal{O}(X \times Y)$ must coincide with the usual product topology of $\mathcal{O}(X)$ and $\mathcal{O}(Y)$. We will treat all small limits in this enriched categorical sense.

1.1 The plan of the article

The rest of the introduction discusses the foundational aspect of the approach above to topological data analysis, and quickly surveys related work. This is a potential application we see to the approach we present, but the presentation itself is of independent interest. Section 2 leads to the identification of topological quantales, namely lattice-theoretic conditions that guarantee that the classical link between metric spaces and topology extends to the quantale enrichment case. Section 3 develops the infrastructure of coproducts of commutative quantales required for the main result. Throughout the paper, and particularly in that section, we pay attention to the constructive validity of the results. Section 4 then presents the construction of all small limits, deliberately by the use of essentially metric techniques. The proof can be seen as a very elementary $\epsilon - \delta$ style proof. However, its correctness rests upon the precise lattice theoretic machinery developed earlier. In particular, the construction of limits of spaces is as constructive as the lattices that are used for metrising each of the ingredient spaces.

1.2 Connection to topological data analysis

In topological data analysis (TDA) the starting point is a point cloud, which is nothing but a finite metric space. Of course, in practice all data is finite, but mathematically the restriction to finite metric spaces is artificial. Dropping it, the starting point of TDA is a metric space. Equivalently, the starting point of TDA is a small $[0, \infty]$-enriched category. The techniques used are then topological. Topology is by design blind to small perturbations in the presentation of a metric space, and this precisely leads to robustness in the analysis of the data to various types of contamination (see [3]). TDA techniques typically result in what is known as a bar code; a combination of the blindness of topology together with the rigidity of the metric presentation of the problem. Stated more formally, the algorithm is performed on the metric space X and not on its open ball topology $\mathcal{O}(X)$. The metric presentation is crucial.

A categorical understanding of TDA, as begun in [4], must handle the tension (see [29]) between the metric presentation of the problem and the topological techniques. In particular, any topological technique that can be used for TDA must allow the scale ε to affect the computation. This is often achieved by converting the given metric space with a chosen scale ε, into purely topological form, and running a computation on that. The work below presents a rather harmonious passage from the metric to topology, primarily without changing the objects of the category. This phenomenon may simplify the interaction between the metric input and the topological processing inherent to TDA.

Going back to the importance of the metric presentation of the point cloud, the current phrasing of TDA is limited to operate only on classical metric spaces. In this work we present an equivalency $\mathtt{TQ}\mathbf{Cat}_{\text{cont}} \simeq \mathbf{Top}$, where the objects of the category on the left-hand side are generalised metric spaces, taking values in lattices more general than $[0, \infty]$. The approach we take singles out such suitable lattices that ensure the results are constructive. In other words, TDA remains applicable for data presented as an object in $\mathtt{TQ}\mathbf{Cat}_{\text{cont}}$. This increases the domain of applicability of TDA and allows greater flexibility when modelling data. We also mention the importance and subtleties of developing algebraic topology constructively. For instance, it is vital for the basics of algebraic topology that a space admits the path joining property (see [21]). Classically, this is not an issue, but in order to lead to executable algorithms, the underlying mathematics must be constructive. For a metric space, rather than just a topological space, the path joining property can be inferred under suitable condition (again, see [21]). By exhibiting $\mathbf{Top}$ as equivalent to $\mathtt{TQ}\mathbf{Cat}_{\mathrm{cont}}$, we open up the possibility of developing algebraic topology for generalized metric spaces, and in a constructive manner.

Already the case of the product of two spaces (alluded to above) demonstrates the potential of our approach. Suppose that X and Y are metric spaces, each thought of as the input for analysis. Applying TDA to $X \times Y$ then represents a case of multidimensional persistence; a well-known problem ([5]). A related scenario is that of multiparameter persistence, requiring sophisticated tools as developed in [14]. A metric in the classical sense for which $\mathcal{O}(X \times Y)$ is the product topology surely exists, e.g., the Euclidean metric or the $\inf$ metric. From a TDA perspective, the choice of which metric is used is paramount. The bar code that will be produced with either of the mentioned metrics does not record features as they occur in X and Y independently. Our approach offers an alternative: a metrisation of $X \times Y$ taking values in $[0, \infty] \otimes [0, \infty]$ as an approach to multiparameter analysis. As mentioned, since our results are constructive, existing TDA techniques are still applicable. Approaches to the foundations of TDA in general, and addressing multidimensionality in particular, that take a similar path to ours are, respectively, [9] and [8], emphasising constructive methods and topos theory.

1.3 Relation to other work

There is plenty of existing literature on quantale enriched categories, a survey of which is not intended. We point out here the references we are aware of that, at least tangentially, touch upon the issues we consider. We start off by mentioning [33], by the second named author. That work provides a comparison between Flagg's value quantales and their precursor concept, namely Kopperman's value semigroups. Some attempts were made there toward a construction of limits, but the background lattice theory was clunky and is much improved in this current work.

[30] discusses extension of functors in the context of quantale enrichment, clearly noting what happens when the quantale is constructively com-

pletely distributive. In particular, in that case, the Pompeiu-Hausdorff metric is obtained as such a functor extension. In this context we mention [1] and [26].

Quantale enrichment in a single quantale, namely $Q\mathbf{Cat}$, are studied in [28] as a rich source of concretely symmetric closed-monoidal topological categories. It is shown, conversely, that such a topological category induces a quantale. That article works toward characterising those categories equivalent to $Q\mathbf{Cat}$. The emphasis there is on a single quantale, and enriched functors as morphisms. In light of the information in the introduction above, it is interesting to extend the question and ask which categories occur, up to equivalence, as $\Gamma\mathbf{Cat}_{\mathrm{cont}}$ for a class Γ of quantales.

Categories enriched in quantales (and quantaloids, see [27]) are well studied in computer science. Here we mention [31], offering a topologically flavoured study, and [25], emphasising a dynamical interpretation. The latter notes that it is the abandonment of the commutativity of the quantale that results in dynamics. It is also primarily concerned with the categorical consequences of the complete distributivity of the quantale. Both aspects appear in our work, as we are careful to trace the role of commutativity, and the effects lattice properties have on the enrichment.

It is interesting that [2], when proving that $\mathbf{Top}^{\mathrm{op}}$ is a quasi-variety, uses complete distributivity, while we require complete distributivity when presenting $\mathbf{Top}$ as a category of enriched categories. Constructive complete distributivity features prominently in [22], which elaborates further on [2].

Finally, in [15] quantaloid enrichment is considered from a topological perspective close to ours. In particular, the authors associate with such an enrichment a closure operator and note simple conditions for the closure operator to land in topological spaces. Our work below addresses the closure operator alongside its interior operator twin, in the case of quantale enrichment. We expect that a similar story unfolds for quantaloid enrichment.

2. Topological quantales

For a metric space X, the closure operator is a monad on $\mathcal{P}(X)$, the interior operator is a monad on $\mathcal{P}(X)^{\mathrm{op}}$, and each monad determines the other via set complementation. The aim of this section is to identify a class of quantales Q for which this phenomenon holds for all small Q-categories X. We do so by examining what holds in general, and how lattice-theoretic properties of Q affect the situation. We start by furnishing such an X with closure and interior operators.

Definition 2.1. *Let Q be a quantale, X a small Q-category, and $S \subseteq X$ a set of objects. We write*

$$X(x, S) = \bigvee \{X(x, s) \mid s \in S\}$$

where $x \in X$ is an arbitrary object. The closure *of S is the set*

$$\mathsf{cl}(S) = \{x \in X \mid X(x, S) = \top\},$$

and its interior *is*

$$\mathsf{in}(S) = \{x \in S \mid \exists \varepsilon \lll \top\colon \quad \varepsilon \lll X(x, y) \implies y \in S\}.$$

For $r \in L$ and $x \in X$, the open ball *of radius r about x is*

$$\mathrm{B}_r(x) = \{y \in X \mid r \lll X(x, y)\},$$

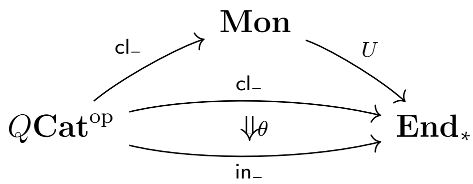
so that $x \in \mathsf{in}(S)$ is equivalently the existence of $\varepsilon \lll \top$ with $\mathrm{B}_\varepsilon(x) \subseteq S$, as usual.

Remark 2.2. When $Q = [0, \infty]$, these concepts attain the usual interpretations in a metric space. Unlike the definition of the closure operator, the interior operator requires justification. The inadequacy of naively using $\mathrm{B}_\varepsilon(x) = \{y \in X \mid r < X(x, y)\}$ instead is gleaned from Theorem 1.1 above — its validity depends on using $\lll$.

Let $\mathbf{End}$ be the category of endofunctors; its objects are a category $\mathscr{C}$ together with a functor $F\colon \mathscr{C} \to \mathscr{C}$, with a typical morphism $(G, \theta)\colon (\mathscr{C}, F) \to (\mathscr{C}', F')$ consisting of a functor $G\colon \mathscr{C} \to \mathscr{C}'$ and a natural transformation $\theta\colon F'G \Rightarrow GF$. Let $\mathbf{End}_*$ be the category of pointed endomorphisms, i.e., (F, η), where $\eta\colon \mathrm{Id}_{\mathscr{C}} \Rightarrow F$ is a natural transformation, and those morphisms (G, θ) that respect the points, in the sense that $G\theta = \theta\eta' G$. There is an evident forgetful functor $\mathbf{Mon} \to \mathbf{End}_*$ from the category of monads.

A consequence of the Axiom of Choice is that in any complete lattice L, if $x \lll \bigvee S$, then $x \lll s$ for some $s \in S$ (see Proposition 3.2 below). This plays an important role in the final part of the following result.

Theorem 2.3. *Let Q be an affine quantale. The assignments $X \mapsto (\mathcal{P}(X), \mathsf{cl}_X)$ and $X \mapsto (\mathcal{P}(X)^{\mathrm{op}}, \mathsf{in}_X)$ are the object parts of the functors*



to the category of pointed endofunctors, each of which acts on a Q-functor $f\colon X \to Y$ by sending f to the inverse image function $f^{\leftarrow}$. The functor cl_- factorises through monads, and the two functors are related by the natural transformation θ carried by the set complementation functor $\neg\colon \mathcal{P}(X) \to \mathcal{P}(X)^{\mathrm{op}}$.

Proof. The claim that cl_X is a functor is that $S \subseteq S' \implies \mathsf{cl}_X(S) \subseteq \mathsf{cl}_X(S')$, which is clear. The claim that it is a pointed functor is that $S \subseteq \mathsf{cl}_X(S)$, which is just as clear. Similarly, and as trivially, in_X is a functor since $S \subseteq S' \implies \mathsf{in}_X(S) \subseteq \mathsf{in}_X(S')$, and it is pointed since $S \supseteq \mathsf{in}_X(S)$. The claim that cl_X is a monad is that $\mathsf{cl}_X^2(S) \subseteq \mathsf{cl}_X(S)$, so let $x \in X$ satisfy $X(x, \mathsf{cl}_X(S)) = \top$, and we must show that $X(x, S) = \top$. It suffices to show, for a given $y \in \mathsf{cl}_X(S)$, that $X(x, y) \leq X(x, S)$. And indeed, using affineness,

$$X(x, y) = X(x, y) \cdot \top = X(x, y) \cdot X(y, S) \leq X(x, S)$$

by the distributivity law in the quantale and the composition inequality in X. Finally, the existence of the natural transformation θ is the claim that $\mathsf{in}_X(\neg S) \subseteq \neg(\mathsf{cl}_X(S))$. To see its validity, assume to the contrary that $x \in \mathsf{in}_X(\neg S) \cap \mathsf{cl}_X(S)$, namely there exists $\varepsilon \lll \top$ with $\varepsilon \lll X(x, y) \implies y \notin S$, and $X(x, S) = \top$. But then $\varepsilon \lll X(x, S)$ and so, by Proposition 3.2 below, it must be that $\varepsilon \lll X(x, s)$ for some $s \in S$, a contradiction. □

Remark 2.4. A situation where $\mathsf{in}\colon \mathcal{P}(X)^{\mathrm{op}} \to \mathcal{P}(X)^{\mathrm{op}}$ fails to be a monad is given in Example 2.11.

Historically, Kuratowski favoured closed sets for the axiomatisation of topology while Sierpiński pioneered open sets. We allow this anecdote to dictate our choice of terminology.

Definition 2.5. *A quantale Q is* sierpiński *if $\varepsilon \lll t$ implies $\varepsilon \lll t \cdot \bigvee \Downarrow(\top)$, for all $t \in Q$.*

Proposition 2.6. *Let Q be a quantale and X a small Q-category. If Q is sierpiński, then $\mathsf{in}(\mathrm{B}_r(x)) = \mathrm{B}_r(x)$, for all $r \in Q$, and $\mathsf{in}\colon \mathcal{P}(X)^{\mathrm{op}} \to \mathcal{P}(X)^{\mathrm{op}}$ is a monad.*

Proof. Fix $x \in X$, $r \in Q$, and $y \in \mathrm{B}_r(x)$, i.e., $r \lll X(x,y)$. We require a $\delta \lll \top$ with $\mathrm{B}_\delta(y) \subseteq \mathrm{B}_r(x)$. Now, since $r \lll X(x,y) \cdot \bigvee \Downarrow(\top) = \bigvee\{X(x,y) \cdot \delta \mid \delta \lll T\}$, a $\delta \lll \top$ exists with $r \lll X(x,y) \cdot \delta$ (again by Proposition 3.2). If $\delta \lll X(y,z)$, then

$$r \lll X(x,y) \cdot \delta \leq X(x,y) \cdot X(y,z) \leq X(x,z)$$

and so $z \in \mathrm{B}_r(x)$, as required. The fact that in is now a monad, namely that $\mathsf{in}(S) \subseteq \mathsf{in}^2(S)$, follows at once. □

In the classical case $Q = [0, \infty]$, the monad cl_X is a kuratowski closure operator, namely its carrier functor $S \mapsto \mathsf{cl}(S)$ preserves finite unions. Similarly, the functor in_X preserves finite intersections. In other words, if $\mathbf{reEnd}_*$ denotes the full subcategory of $\mathbf{End}_*$ spanned by right exact endofunctors, then the functors cl_- and in_- factorise via the inclusion $\mathbf{reEnd}_* \to \mathbf{End}_*$. Neither claim holds generally.

Example 2.7. Consider the quantale $Q = \mathcal{P}(S)$ of all subsets of a set S. Viewed as a closed monoidal category with intersection as monoidal product, its self-enrichment structure yields the Q-category X with $\mathrm{ob}(X) = \mathcal{P}(S)$ and $X(x,y) = \{s \in S \mid s \in x \implies s \in y\} = \neg x \vee y$. For a collection $\mathcal{A} \subseteq X$ we have $X(x, \mathcal{A}) = \neg x \vee \bigvee \mathcal{A}$, and thus $\mathsf{cl}(\mathcal{A}) = \mathcal{P}(\bigvee \mathcal{A})$ — it need not preserve finite joins. Direct computation shows that $\varepsilon \lll \top$ if, and only if, ε is a sub-singleton. Noting that

$$\mathrm{B}_{\{s\}}(x) = \begin{cases} \mathcal{P}(X) & s \notin x \\ \{y \subseteq S \mid s \in y\} & s \in x \end{cases}$$

shows that $\mathsf{in}(\mathcal{A}) = \{a \in \mathcal{A} \mid \exists s \in a\colon \quad s \in y \implies y \in \mathcal{A}\}$ — it need not preserve finite meets.

Definition 2.8. *Let L be a complete lattice. If $\downarrow\!\!\!\!\downarrow(\top)$ is closed under finite joins, then L is* kuratowski. *If $\Downarrow(\top)$ is closed under finite joins, then L is* sierpiński.

We say that a quantale Q is kuratowski if its underlying lattice is. We say that Q is *entirely sierpiński* if both it and its underlying lattice are sierpiński.

Proposition 2.9. *Let Q be an affine quantale and X a small Q-category. If Q is kuratowski, then* cl *is a kuratowski closure operator.*

Proof. We only need to verify preservation of finite unions. For binary unions it suffices to show that $\mathsf{cl}(S \cup S') \subseteq \mathsf{cl}(S) \cup \mathsf{cl}(S')$, which follows at once since $X(x, S \cup S') = X(x, S) \cup X(x, S')$. It remains to see that $\mathsf{cl}(\varnothing) = \varnothing$, and indeed, if $x \in \mathsf{cl}(\varnothing)$, then $X(x, \varnothing) = \top$, but the former is $\bot$, forcing Q to collapse. But then $\downarrow\!\!\!\!\downarrow(\top)$ is not closed under the empty join. □

In agreement with our historical convention, we call the set-theoretic dual of a kuratowski closure operator, namely a comonad $\mathsf{in}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ that preserves finite meets, a *sierpiński interior operator*. Recall from [18] the theory of free monads on (pointed) functors. It is clear that in admits a free monad; its value on S is $\mathsf{in}^{\alpha}(S)$, where α is a sufficiently large ordinal ensuring the stabilisation of the decreasing chain $\{\mathsf{in}^{\beta}(S)\}_{\beta}$, where $\mathsf{in}^{\beta+1} = \mathsf{in}(\mathsf{in}^{\beta}(S))$ and, for a limit ordinal γ, $\mathsf{in}^{\gamma}(S) = \bigcap\{\mathsf{in}^{\beta} \mid \beta < \gamma\}$.

Proposition 2.10. *Let Q be an affine quantale and X a small Q-category. If the underlying lattice of Q is sierpiński, then* in *preserves finite meets, and the free monad on it is a sierpiński interior operator. If Q is entirely sierpiński, then* in *is already a sierpiński interior operator.*

Proof. Assuming the underlying lattice is sierpiński, the equality $\mathsf{in}(S \cap S') = \mathsf{in}(S) \cap \mathsf{in}(S')$ follows at once since if $x \in \mathsf{in}(S) \cap \mathsf{in}(S')$, witnessed by $\varepsilon, \varepsilon' \lll \top$, respectively, then $\varepsilon \vee \varepsilon' \lll \top$ witnesses that $x \in \mathsf{in}(S \cap S')$. In order to show that $\mathsf{in}(X) = X$, note that the only obstruction to that equality is if Q admits no $\varepsilon \lll \top$ at all, which can happen only if Q collapses. But then $\Downarrow(\top)$ is not closed under the empty join.

It is now clear that if Q is entirely sierpiński, then in is a sierpiński interior operator. If we only know that in is a pointed finite-union preserving functor but not necessarily a monad, it is clear that the finite-union preservation survives the free monad construction, thus yielding a sierpiński interior operator. □

In the classical case $Q = [0, \infty]$, it is well known that open and closed sets are dual concepts: S is open/closed if, and only if, $X \setminus S$ is closed/open. Stated differently, set complementation $\neg\colon \mathcal{P}(X) \to \mathcal{P}(X)^{\mathrm{op}}$, since it is a complete lattice isomorphism, induces an isomorphism $\neg\colon \mathsf{Monad}(\mathcal{P}(X)) \to \mathsf{coMonad}(\mathcal{P}(X))$, given by $(\neg F)S = \neg(F(\neg S))$, that restricts to an isomorphism between the kuratowski and sierpiński operators. Thus, when $Q = [0, \infty]$, the natural transformation $\theta\colon \mathsf{in}_{-} \to \mathsf{cl}_{-}$ (which is carried by $\neg$) is a natural isomorphism. Direct verification shows that in the case considered in Example 2.7, one has that $\mathsf{in}_X(\neg S) = \neg\mathsf{cl}_X(S)$, namely the component of θ is an isomorphism. We shall shortly see why that holds true for all small Q-categories X for $Q = \mathcal{P}(S)$. We first observe that the same phenomenon does not hold true for arbitrary quantales.

Example 2.11. Let Q be a complete boolean algebra, viewed as a quantale with operation given by $\wedge$ (since any complete boolean algebra is a frame, this is legitimate). Let X be Q as a Q-category, thus $X(x,y) = \neg x \vee y$, where $\neg$ is the boolean complement operator. Clearly then, for $\mathcal{A} \subseteq X$, $X(x, \mathcal{A}) = \neg x \vee \bigvee \mathcal{A}$, and so $\mathsf{cl}(\mathcal{A}) = \downarrow(\bigvee \mathcal{A})$. Computing the interior operator requires knowledge of the set $\Downarrow(\top)$. Let us consider two extremes: the atomic and atom-less cases. If Q is an atomic complete boolean algebra, then $Q \cong \mathcal{P}(S)$, $\Downarrow(\top)$ is the set of sub-atomic elements, and the situation reduces to that of Example 2.7. If Q is atom-less, then $\Downarrow(\top) = \{\bot\}$, and it follows that

$$\mathrm{B}_{\bot}(x) = \{y \in X \mid \bot \lll \neg x \vee y\} = \begin{cases} X & x < \top \\ \Uparrow(\bot) & x = \top \end{cases}$$

using the simple observation that in any lattice, $\bot \lll x$ holds precisely when $x \neq \bot$. Therefore,

$$\mathsf{in}(\mathcal{A}) = \begin{cases} X & \mathcal{A} = X \\ \{\top\} & \mathcal{A} = \Uparrow(\bot) \\ \varnothing & \text{otherwise} \end{cases}$$

and in particular, $\mathsf{in}^2(\Uparrow(\bot)) \subsetneq \mathsf{in}(\Uparrow(\bot))$. The interior operator is thus not a monad. Since the closure operator is always a monad, it is thus impossible that θ is a natural isomorphism in this case.

The final piece of this section is a lattice-theoretic property under which θ is necessarily a natural isomorphism. The following terminology is explained in Subsection 3.1.

Definition 2.12. *A complete lattice L is* CCD *at* $\top$ *if* $\top = \bigvee \Downarrow(\top)$.

We say that Q is CCD at $\top$ if its underlying lattice is. Obviously, any CCD lattice is CCD at $\top$, and so, certainly, $\mathcal{P}(S)$ is CCD at $\top$ (cf. Example 2.7). A non-atomic complete boolean algebra satisfies $\bigvee \Uparrow(\top) = \bot$, so it is far from being CCD at $\top$ (cf. Example 2.11). The following results clarify much of the mechanics in both examples.

Proposition 2.13. *Let Q be an affine quantale. If Q is* CCD *at* $\top$*, then* $\theta\colon \mathsf{cl}_- \Rightarrow \mathsf{in}_-$ *is a natural isomorphism.*

Proof. Since $\mathsf{in}(\neg S) \subseteq \neg\mathsf{cl}(S)$ always holds, we only need to show the reverse inclusion, so assume $X(x,S) < \top$. Since $\bigvee \Downarrow(\top) = \top$, there exists some $\varepsilon \lll \top$ such that $X(x,S) \ngeq \varepsilon$. But then $\mathrm{B}_\varepsilon(x) \subseteq \neg S$, since $\varepsilon \lll X(x,y)$ together with $y \in S$ leads to the contradiction $X(x,S) \geq X(x,y) \geq \varepsilon$. □

Proposition 2.14. *If an affine quantale Q is* CCD *at* $\top$*, then Q is sierpiński.*

Proof. We need to show that if $\varepsilon \lll t$, then $\varepsilon \lll t \cdot \bigvee \Downarrow(\top)$. But $\bigvee \Downarrow(\top) = \top$, and $\top$ is the quantale unit. □

As a consequence, if Q is an affine quantale that is CCD at $\top$, then in, and not just cl, is guaranteed to be a monad. The next result is slightly less immediate.

Proposition 2.15. *If Q is an affine quantale that is* CCD *at* $\top$*, then Q is kuratowski if, and only if, Q is entirely sierpiński.*

Proof. See [7, Proposition 3]. □

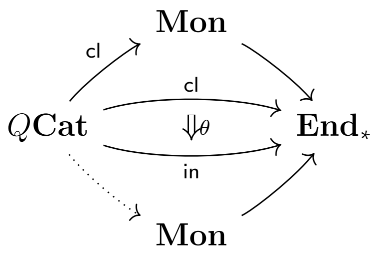
The above considerations highlight certain quantales as foundational in topology.

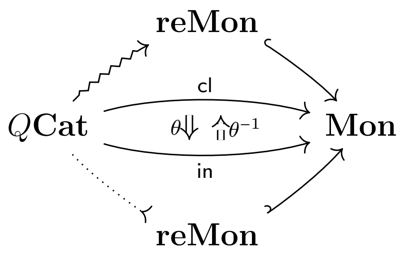
Definition 2.16. A topological quantale *is a commutative affine quantale Q that is* CCD *at* $\top$ *and kuratowski (and thus entirely sierpiński).*

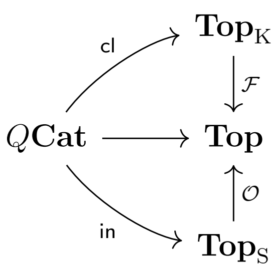
Remark 2.17. The commutativity of Q was not required in any of the result so far. The effect of commutativity is of importance when we come to consider coproducts of quantales below.

The following theorems embody the idea of viewing topological spaces as enriched categories.

Theorem 2.18. *Consider the diagrams*







where Q is an affine quantale. Regarding the functor $\mathsf{cl}\colon Q\mathbf{Cat} \to \mathbf{End}_*$ *factoring over* $\mathbf{Mon}$*, the functor* $\mathsf{in}\colon Q\mathbf{Cat} \to \mathbf{End}_*$*, and the natural transformation* $\mathsf{cl} \Rightarrow \mathsf{in}$ *from Theorem 2.3, we can specify that:*

1. *If Q is sierpiński, then* $\mathsf{in}\colon Q\mathbf{Cat} \to \mathbf{End}_*$ *factors over* $\mathbf{Mon}$.
2. *if Q is kuratowski, then* in *factors over* $\mathbf{reMon}$.
3. *if Q is entirely sierpiński, then* in *factors over* $\mathbf{reMon}$.
4. *if Q is* CCD *at* $\top$*, then* $\mathsf{cl} \Rightarrow \mathsf{in}$ *is a natural isomorphism.*
5. *if Q is kuratowski and* CCD *at* $\top$*, then all the above happens; in other words, both closure and interior operators are topological and specify the same topological space.*

Theorem 2.19. *Let* TQ *be the class of topological quantales and consider the category* $\mathtt{TQ}\mathbf{Cat}_{\mathrm{cont}}$ *whose objects are all small Q-categories where Q*

is a topological quantale, and morphisms the cauchy continuous functions $f: X \to Y$, namely those satisfying the familiar ε–δ condition as described in the introduction. The unique functor $\mathtt{T}\mathcal{Q}\mathbf{Cat}_{\mathrm{cont}} \to \mathbf{Top}$ above, which we denote by $X \mapsto \mathcal{O}(X)$, is the functor that associates with a small Q-category its (unique!) topology, and acts as the identity on morphisms (which is valid since any $\varepsilon - \delta$ continuous function is also continuous with respect to the open ball topology). This functor takes the familiar form where $U \subseteq X$ is declared open precisely when

$$\forall x \in U \quad \exists \varepsilon \lll \top: \quad \mathrm{B}_\varepsilon(x) \subseteq U.$$

These functors, for the various topological quantales Q, patch up together to form the functor

$$\mathcal{O}: \mathtt{T}\mathcal{Q}\mathbf{Cat}_{\mathrm{cont}} \to \mathbf{Top}$$

and this functor is an equivalence of categories.

Proof. The proof is essentially due to [12]. Obviously, $\mathcal{O}$ is faithful, and due to open balls being open sets, the standard textbook proof shows $\mathcal{O}$ is full. Flagg utilised the free frame construction $\Omega: \mathbf{Set} \to \mathbf{Frm}$, with frames viewed as quantales. In more detail, $\Omega(X)$ is the collection of all down closed collections of finite subsets of X, ordered by inclusion. Let us show that $\Omega(X)$ is kuratowski, so suppose $a, b < \top$, which means a misses a finite subset F_a, and b misses a finite subset F_b. But then if $a \vee b = \top$, then $F = F_a \cup F_b$ must be there, which would force it into either a or b. Since $F_a \subseteq F$, it cannot belong to a. b is similarly prohibited. To see that $\Omega(X)$ is $\mathtt{CCD}$ at $\top$ it suffices to note that $a \lll \top$ precisely when there exists a finite subset $F_a \subseteq X$ such that a consists only of subsets of F_a. The join of such elements a is thus the entire collection $\top$ of all finite subsets of X. In other words, Ω lands in topological quantales. Now, to show that $\mathcal{O}$ is surjective on objects, given a topology τ on X, let $X(x,y) \in \Omega(\tau)$ be the collection of all finite subsets of $\tau_{x \to y}$, where $\tau_{x \to y} = \{U \in \tau \mid x \in U \implies y \in U\}$. □

3. Coproducts of commutative quantales

The construction of limits in $\mathtt{T}\mathcal{Q}\mathbf{Cat}_{\mathrm{cont}}$ relies on coproducts of commutative quantales, and those rely on colimits of complete join lattices. We are particularly interested in stability properties of topological quantales under coproducts, and so proceed to introduce the relevant notions alongside a study of the totally below relation.

3.1 Constructive complete distributivity

Recall that $\downarrow(-): L \to \mathcal{D}(L)$, where L is a complete lattice and $\mathcal{D}(L)$ is the lattice of its down-closed subsets, has a left adjoint given by $\bigvee$, and that L is `CCD` precisely when $\bigvee$ has a left adjoint $\Downarrow(-)$. The definition of such a left adjoint dictates that

$$\Downarrow(x) = \{y \in L \mid y \lll x\}$$

in the sense of the totally below relation $y \lll x$, namely that for all $S \subseteq L$ with $x \leq \bigvee S$ there exists $s \in S$ with $y \leq s$.

Even if L is not `CCD`, the definition above still yields a functor $\Downarrow(-): L \to \mathcal{D}(L)$. Consider the functor $\bigsqcup: \mathcal{D}(L) \to L$ given by

$$\bigsqcup S = \bigvee\{x \in L \mid \Downarrow(x) \subseteq S\}.$$

Proposition 3.1. *The following conditions for a complete lattice L are equivalent:*

1. *For all $a \lll b$, if $b \leq \bigvee S$, then $a \lll s$ for some $s \in S$.*

2. *The functor $\Downarrow(-)$ is a left adjoint.*

Proof. Assuming the first condition, we show that $\Downarrow(-) \dashv \bigsqcup$. It suffices to demonstrate the unit and counit conditions, namely $x \leq \bigsqcup \Downarrow(x)$ and $\Downarrow(\bigsqcup S) \subseteq S$, of which the former is trivial. For the latter, suppose $x \lll \bigvee\{y \in L \mid \Downarrow(y) \subseteq S\}$, so, by the assumed condition, $x \lll y$ for some y with $\Downarrow(y) \subseteq S$, thus $x \in S$. For the converse, note that if $\Downarrow(-)$ is a left adjoint and $b \leq \bigvee S$, then $\Downarrow(b) \subseteq \Downarrow(\bigvee S) = \bigcup\{\Downarrow(s) \mid s \in S\}$. □

Proposition 3.2. *If the background set theory admits the Axiom of Choice, then, for all complete lattices L, the functor $\Downarrow(-)$ is a left adjoint.*

Proof. We demonstrate the first condition of Proposition 3.1. Proceeding by contradiction, suppose $a \lll b$, $b \leq \bigvee S$, and yet $a \lll s$ holds for not a single $s \in S$. Choose, for each $s \in S$, a set T_s with $s \leq \bigvee T_s$ and so that $a \leq t$ fails for all $t \in T_s$. The set $T = \bigcup\{T_s \mid s \in S\}$ contradicts $a \lll \bigvee S$. □

Remark 3.3. We proceed under the assumption that for all lattices L concerning us, the functor $\Downarrow(-)$ admits a right adjoint. In light of Proposition 3.2, this is automatic if the Axiom of Choice holds. Otherwise, the assumption we are making is that we restrict to those lattices that are sufficiently constructive to admit the required right adjoint.

For any complete lattice L let $\mathtt{CCD}(L) = \{x \in L \mid \bigvee \Downarrow(x) = x\}$, which we call the CCD core of L. It is easily seen that the CCD core of L is a complete join sublattice of it, but it need not itself be CCD. We obtain the following play on words.

Theorem 3.4. *Let L be a complete lattice. The following conditions are equivalent (and define what it means for L to be CCD):*

- $\bigvee$ *has a left adjoint*
- $\bigvee = \bigsqcup$
- $\mathit{CCD}(L) = L$.

3.2 Tensor product of complete join lattices

The category $\mathbf{CJLat}$ of complete join lattices is well known to support a symmetric closed monoidal structure ([17, 24]). The tensor product of complete lattices L_1, L_2 is a function $\beta\colon L_1 \times L_2 \to L_1 \otimes L_2$ that is universal among all functions $L_1 \times L_2 \to L$ that are join preserving in each variable. Much as in ring theory, the tensor product can be constructed as a quotient of a free lattice. Writing $x \otimes y$ for $\beta(x, y)$, and referring to such elements as elementary tensors, every element in $L_1 \otimes L_2$ is a join of elementary tensors, for all $x > \bot$ in L_1 and $y > \bot$ in L_2 we have that $x \otimes y \leq x' \otimes y'$ if, and only if, both $x \leq x'$ and $y \leq y'$, $\bigwedge_i x_i \otimes y_i = \bigwedge_i x_i \otimes \bigwedge_i y_i$, and $x \otimes \bigvee S = \bigvee\{x \otimes s \mid s \in S\}$. Thus meets are computed point-wisely in $L_1 \otimes L_2$. The join of arbitrary elementary tensors, however, does not admit such a simple formula.

The following result is [23, Lemma 37]:

Theorem 3.5. *If L_1 and L_2 are CCD, then so is their tensor product.*

We require a refinement of this result and an analysis of the totally below relation in the tensor product. It is convenient to use the fact ([17]) that

$$L_1 \otimes L_2 \cong \underline{\mathbf{CJLat}}(L_1, L_2^{\mathrm{op}})^{\mathrm{op}},$$

the opposite of the complete lattice of join preserving morphisms $L_1 \to L_2^{\mathrm{op}}$ (see [13] for a detailed description). In this model, the elementary tensors are given by

$$\beta(x,y)(a) = \begin{cases} \top & \text{if } a = \bot, \\ y & \text{if } \bot < a \leq x, \\ \bot & \text{if } a \nleq x, \end{cases}$$

and an arbitrary element $f \in L_1 \otimes L_2$ has the canonical presentation

$$f = \bigvee_{x \in L_1} \beta(x, f(x))$$

as a join of elementary tensors.

It is straightforward that if $a \otimes b \lll x \otimes y$, then both $a \lll x$ and $b \lll y$. The following result will assist in obtaining conditions for the converse implication.

Lemma 3.6. *Let S be a subset of the tensor product*

$$L_1 \otimes L_2 \cong \mathbf{CJLat}(L_1, L_2^{\mathrm{op}})^{\mathrm{op}}$$

of two complete lattices and write s for the point-wise join of S, i.e. $s(x) = \bigvee_{g \in S} g(x)$ computed in L_2 for all $x \in L_1$. Now define

$$f(x) = \bigwedge_{x' \lll x} s(x')$$

for all $x \in L_1$, again computed in L_2. The following properties hold.

1. *f is an upper bound of S in $L_1 \otimes L_2$.*
2. *If h is an upper bound of S, then $f(x) \leq h(x)$ for all $x \in \mathit{CCD}(L_1)$.*
3. *$(\bigvee S)(x) = f(x)$ for all $x \in \mathit{CCD}(L_1)$.*

Proof. Recall that $\Downarrow(-)$ is a left adjoint.

1. Firstly, $f\colon L_1 \to L_2^{\mathrm{op}}$ belongs to $L_1 \otimes L_2$, namely it preserves joins, since

$$f(\bigvee \mathcal{A}) = \bigwedge_{x' \in \Downarrow(\bigvee \mathcal{A})} s(x') = \bigwedge_{a \in A} \bigwedge_{x' \in \Downarrow(a)} s(x') = \bigwedge_{a \in \mathcal{A}} f(a).$$

Now, for any $g \in S$ and $x' \lll x$, clearly, $g(x) \leq g(x') \leq s(x')$, showing that $g(x) \leq f(x)$ point-wisely, and thus f is an upper bound of S.

2. An upper bound h clearly satisfies $s(x') \leq h(x')$. Assuming that $x \in \mathtt{CCD}(L_1)$, we obtain that

$$f(x) \leq \bigwedge_{x' \lll x} h(x') = h(\bigvee_{x' \lll x} x') = h(x).$$

3. Immediate.

□

Proposition 3.7. *Let L_1 and L_2 be complete lattices, $a \lll x$ in L_1, and $b \lll y$ in L_2. If $x \in \mathtt{CCD}(L_1)$, then $a \otimes b \lll x \otimes y$.*

Proof. Suppose that $x \otimes y \leq \bigvee S$ for some $S \subseteq L_1 \otimes L_2$. By Lemma 3.6 and the given conditions we have that

$$b \lll y \leq (\bigvee S)(x) = \bigwedge_{x' \lll x} \bigvee_{g \in S} g(x') \leq \bigvee_{g \in S} g(a),$$

and so $b \leq g_0(a)$ for some $g_0 \in S$. Therefore

$$a \otimes b = \beta(a, b) \leq g_0$$

as can be seen from the expression for $\beta(a, b)$, recalling that g_0 is antitone. □

We summarise as follows.

Theorem 3.8. *For complete lattices L_1 and L_2, if $x \in \mathtt{CCD}(L_1)$ and $y \in \mathtt{CCD}(L_2)$, then $x \otimes y \in \mathtt{CCD}(L_1 \otimes L_2)$ and $t \lll x \otimes y$ if, and only if, $t \leq a \otimes b$ with $a \lll x$ and $b \lll y$. If L_1 and L_2 are CCD at $\top$ and are sierpiński, then $L_1 \otimes L_2$ is sierpiński.*

Proof. The characterisation of $t \lll x \otimes y$ follows from the observation that $x \otimes y = \bigvee\{a \otimes b \mid a \lll x, b \lll y\}$ as soon as $x \in \mathtt{CCD}(L_1)$ and $y \in \mathtt{CCD}(L_2)$, from which $x \otimes y \in \mathtt{CCD}(L_1 \otimes L_2)$ is immediate. With this property, the claim about the sierpiński property follows from $a \otimes b \vee a' \otimes b' \leq (a \vee a') \otimes (b \vee b')$. □

3.3 Coproducts of complete join lattices

Coproducts in $\mathbf{CJLat}$ ([17]) are particularly simple to describe, due to its strong self duality: if $f\colon L_1 \to L_2$ is join preserving, then it has a right adjoint $g\colon L_2 \to L_1$, which, when written as $f^{\mathrm{op}}\colon L_2^{\mathrm{op}} \to L_1^{\mathrm{op}}$, is join preserving, and thus yields an isomorphism $\mathbf{CJLat} \to \mathbf{CJLat}^{\mathrm{op}}$. The product L of lattices $\{L_k\}_{k\in I}$ is given by the usual product of the underlying sets, equipped with component-wise operations. The projections $\pi_k\colon L \to L_k$ preserve meets, and so admit left adjoints $\iota_k\colon L_k \to L$. It then holds that L with these morphisms is the coproduct in $\mathbf{CJLat}$, and $\pi_k \circ \iota_k = \mathrm{Id}_{L_k}$.

Proposition 3.9. *Let $\{L_k\}_{k\in I}$ be a collection of complete lattices. Then $a \lll x$ in the coproduct $\coprod L_k$ if, and only if, there exists $k_0 \in I$ and $\bar{a} \in L_{k_0}$ such that $\bar{a} \lll \pi_{k_0}(x)$ and $a = \iota_{k_0}(\bar{a})$.*

Proof. Clearly, $x = \bigvee\{\iota_k(\pi_k(x)) \mid k \in I\}$, so for $a \lll x$ the existence of k_0 follows from Proposition 3.2. □

3.4 Coproducts of commutative quantales

Since the category of commutative quantales is $\mathsf{cMon}(\mathbf{CJLat})$, it follows from general considerations that it is cocomplete (and complete). We require a concrete enough description of coproducts, sufficient to see that topological quantales admit coproducts.

It is a simple matter that, much as in the case of commutative rings, finite coproducts in $\mathbf{cQnt}$ are given by the tensor product. This follows again from general considerations of commutative monoid objects in a symmetric closed monoidal category. In a nutshell, the multiplication of a commutative quantale Q is a function $Q \times Q \to Q$, preserving joins in each variable, and thus corresponds to a morphism $\lfloor\cdot\rfloor\colon Q \otimes Q \to Q$ from the tensor product of the underlying lattice. If Q_1 and Q_2 are commutative quantales, then one obtains a binary operation on $Q_1 \otimes Q_2$, namely the one corresponding to

$$(Q_1 \otimes Q_2) \otimes (Q_1 \otimes Q_2) \to (Q_1 \otimes Q_1) \otimes (Q_2 \otimes Q_2) \xrightarrow{\lfloor\cdot\rfloor\otimes\lfloor\cdot\rfloor} Q_1 \otimes Q_2$$

utilising the canonical symmetry isomorphism $Q_2 \otimes Q_1 \to Q_1 \otimes Q_2$. It is easily seen that then $Q_1 \otimes Q_2$ is a commutative quantale, and that with the evident morphisms $Q_1 \to Q_1 \otimes Q_2 \leftarrow Q_2$ it is the coproduct in $\mathbf{cQnt}$.

Theorem 3.10. *The category* $\mathbf{tQnt}$ *of topological quantales, as a full subcategory of* $\mathbf{cQnt}$*, is closed under finite coproducts, and* $\varepsilon \lll \top$ *holds in* $Q_1 \otimes \cdots \otimes Q_n$ *if, and only if, there exist* $\varepsilon_k \in Q_k$ *with* $\varepsilon_k \lll \top$ *such that* $\varepsilon \leq \varepsilon_1 \otimes \cdots \otimes \varepsilon_n$.

Proof. The empty coproduct is the quantale $\mathcal{B} = \{\bot < \top\}$ of boolean truth values, and it is clearly topological. The characterisation of $\varepsilon \lll \top$ in this case simply says that all $\varepsilon \in \mathcal{B}$ satisfy $\varepsilon \lll \top$. Suppose that Q_1 and Q_2 are topological quantales. It is clear that $Q = Q_1 \otimes Q_2$ is affine and the rest of the claims follow from the fact that the underlying lattice of Q is the tensor product in $\mathbf{CJLat}$, together with Theorem 3.8. □

This leaves the case of infinite coproducts. As in any category, directed colimits together with finite coproducts suffice to construct all coproducts, as follows. For a set I consider the poset $\mathrm{Fin}(I)$ of all finite subsets of I, under inclusion. For a small collection $\{Q_k\}_{k \in I}$ of commutative quantales indexed by I their coproduct is the colimit

$$\coprod_{k \in I} Q_k = \operatorname*{colim}_{\mathrm{Fin}(I)} (S \mapsto \bigotimes_{k \in S} Q_k).$$

Directed colimits are (again) particularly simple in $\mathbf{cQnt}$, namely they are created by the functor $\mathbf{cQnt} \to \mathbf{CJLat}$ (see [16, C1.1, Lemma 1.1.8], and the discussion surrounding it).

We continue to use the elementary tensor notation $x_1 \otimes \cdots \otimes x_n$ to stand for $\iota_1(x_1) \vee \cdots \vee \iota_n(x_n)$, where ι_k is the canonical injection into the (possibly infinite) coproduct.

Theorem 3.11. *The category* $\mathbf{tQnt}$ *of topological quantales, as a full subcategory of* $\mathbf{cQnt}$*, is closed under infinite coproducts, and* $\varepsilon \lll \top$ *holds in* $\bigotimes_{k \in I} Q_k$ *if, and only if, there exist finitely many* $\varepsilon_k \in Q_k$ *with* $\varepsilon \lll \top$ *such that* $\varepsilon \leq \varepsilon_1 \otimes \cdots \otimes \varepsilon_n$.

Proof. Combine Proposition 3.9 with Theorem 3.8 and the form of the infinite coproduct of commutative quantales. □

To conclude this section we note that coproducts of quantales occur naturally in applications. For instance, recall the quantales $[0, 1]$ with multiplication and $[0, \infty]^{\mathrm{op}}$ with addition. Their coproduct is usually denoted by

Δ, the quantale of distance distribution functions. Small Δ-categories are also known as probabilistic metric spaces. Since the constituent quantales are topological, so is Δ. This quantale was in circulation long before it was realised that it is simply the coproduct of two very naturally occurring quantales. Further, it is clear that $[0,1] \cong [0,\infty]^{\mathrm{op}}$ as quantales, and so, up to isomorphism, Δ is simply the coproduct of $[0,\infty]^{\mathrm{op}}$ with itself. This can be iterated to any cardinality, yielding a transfinite ladder of topological quantales. See [6] for more details.

4. Limits of spaces

In this section we construct limits in $\mathtt{TQ}\mathbf{Cat}_{\mathrm{cont}}$. The novelty, of course, is not in the completeness of the category but in the techniques used. The interest in these techniques is their very existence. Not all formalisms are created equal; while it is fairly straightforward to define the product topology in terms of open sets, doing so in terms of closed sets is not readily achieved. It is thus not a priori clear that an enriched-categorically flavoured construction of products exists.

It suffices to construct all small products and all equalisers. For a Q-category X and a subset $A \subseteq X$ of its objects, the full subcategory on A is the Q-category with $A(x,y) = X(x,y)$.

Theorem 4.1. *The equaliser of $f, g\colon X \to Y$ in $\mathtt{TQ}\mathbf{Cat}_{\mathrm{cont}}$ is the full subcategory of X on $E = \{x \in X \mid f(x) = g(x)\}$.*

Proof. Straightforward. □

We now turn to products, so fix a family $\{X_k\}_{k\in I}$ of objects in $\mathtt{TQ}\mathbf{Cat}_{\mathrm{cont}}$, indexed by a set I. Each X_k is a small Q_k-category where Q_k is a topological quantale. Let $Q = \coprod_{k\in I} Q_k$ be the coproduct in the category of $\mathbf{cQnt}$, equipped with the canonical injections $\iota_k\colon Q_k \to Q$. Let X be the Q-category with

$$\mathrm{ob}(X) = \prod_{k\in I} \mathrm{ob}(X_k) \quad \text{and} \quad X(x,y) = \bigvee_{k\in I} \iota_k(X_k(\pi_k(x), \pi_k(y)))$$

with the join computed in Q. It is easily seen to be a small Q-category, by extending the fact that, for elementary tensors in $Q_1 \otimes Q_2$, $(a \otimes b)\cdot(c\otimes d) = (a\cdot c)\otimes(b\cdot d)$, to the coproduct of quantales.

In the next proof, we use the following observation. A function $f: X \to Y$ is continuous at x precisely when for all $\varepsilon \lll \top$ there exists $\delta \lll \top$ such that $X(x,y) \leq \delta \implies Y(fx, fy) \leq \varepsilon$. The equivalency with the definition of continuity as given above follows from the fact that $\Downarrow(-)$ is a left adjoint.

Theorem 4.2. *With the evident projection functions $X \to X_k$, the Q-category X is the product of $\{X_k\}_{k\in I}$ in $\mathcal{TQ}\mathbf{Cat}_{\mathrm{cont}}$.*

Proof. The quantale Q is topological, and it is clear that the projection functions are continuous. It remains to establish the universal property, so assume continuous functions $f_k: Y \to X_k$ from some small R-category Y are given, where R is some topological quantale. The function $g: Y \to X$ we seek is dictated to be the unique one satisfying $\pi_k \circ g = f_k$, so we only need to show that g is continuous. For that, let $y \in Y$ and $\varepsilon \lll \top$ be given, where ε is chosen in Q. By Theorem 3.11, $\varepsilon \leq \iota_{k_1}(\varepsilon_1) \vee \cdots \vee \iota_{k_n}(\varepsilon_n)$, where $\varepsilon_{k_i} \lll \top$ holds in Q_{k_i}. By the continuity of f_{k_i}, there exists $\delta_{k_i} \lll \top$ in R such that $\delta_{k_i} \leq Y(y,y') \implies \varepsilon_{k_i} \leq X_{k_i}(f_{k_i}(y), f_{k_i}(y'))$. Let $\delta = \delta_{k_1} \vee \cdots \vee \delta_{k_n}$, which satisfies $\delta \lll \top$ since R is sierpiński. We claim that $\delta \leq Y(y,y') \implies \varepsilon \leq X(g(y), g(y'))$, namely that g is continuous at y. Assume $\delta \leq Y(y,y')$, and fix k_i. Then certainly $\delta_{k_i} \leq Y(y,y')$, and thus $\varepsilon_{k_i} \leq X_{k_i}(f_{k_i}(y), f_{k_i}(y')) = X_{k_i}(\pi_{k_i}(g(y)), \pi_{k_i}(g(y')))$. Upon applying ι_{k_i} we obtain that $\iota_{k_i}(\varepsilon_{k_i}) \leq X(g(y), g(y'))$, and as this holds for $k_1, \ldots, k_n$, it follows that $\iota_{k_1}(\varepsilon_{k_1}) \vee \cdots \vee \iota_{k_n}(\varepsilon(k_n)) \leq X(g(y), g(y'))$. By the choice of ε this inequality completes the proof. □

To conclude, let us speculate on the applicability of this last construction in data analysis. Any data analysis endevour starts with recording the data, very often as a point-cloud data structure, i.e., a metric space or, in our terminology, a small $[0,\infty]^{\mathrm{op}}$-category. Often, the data does not naturally appear in metric form and some manipulation, including simplification or arbitrary choice, is required in order to obtain a metric space. Higher dimensional data is often encoded in terms of some metric on $\mathbb{R}^n$, again possibly skewing the data. Having more quantales at hand provides more flexibility. For instance, suppose data is collected coordinate wise as ordinary metric spaces X_i, but the data analysis requires patching the coordinates together. The last theorem provides a canonical metrisation for the entire space of coordinates. It is expected to introduce less bias or distortion into the data, while ensuring the topologicity of the scenario.

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THE DERIVATOR OF SETOIDS

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Résumé. Sans l'axiome du choix, la complétion exacte libre de la catégorie des ensembles (i.e. la catégorie des "sétoides") peut ne pas être complète ou cocomplète. Nous montrerons que, néanmoins, elle peut être enrichie d'un dérivateur : la structure formelle des catégories de diagrammes reliés par des foncteurs d'extension de Kan. De plus, ce dérivateur est la cocomplétion libre d'un point dans une classe de "1-dérivateurs tronqués" (qui se comporte comme une 1-catégorie plutôt que comme une catégorie d'ordre supérieur).

En mathématiques classiques, la cocomplétion libre d'un point par rapport à l'ensemble des dérivateurs est la théorie de l'homotopie des espaces. Ainsi, s'il existe une théorie de l'homotopie dont on peut montrer qu'elle possède cette propriété universelle de manière constructive, sa 1-troncature doit contenir non seulement des ensembles, mais aussi des sétoïdes. Ceci suggère que soit les sétoïdes sont un aspect inévitable de la théorie de l'homotopie constructive, soit on a besoin d'une modification plus radicale de la notion de théorie d'homotopie.

Abstract. Without the axiom of choice, the free exact completion of the category of sets (i.e. the category of setoids) may not be complete or cocomplete. We will show that nevertheless, it can be enhanced to a derivator: the formal structure of categories of diagrams related by Kan extension functors. Moreover, this derivator is the free cocompletion of a point in a class of "1-truncated derivators" (which behave like a 1-category rather than a higher category).

In classical mathematics, the free cocompletion of a point relative to all derivators is the homotopy theory of spaces. Thus, if there is a homotopy theory that can be shown to have this universal property constructively, its 1-truncation must contain not only sets, but also setoids. This suggests that either setoids are an unavoidable aspect of constructive homotopy theory, or

more radical modifications to the notion of homotopy theory are needed.
Keywords. derivator, setoid, exact completion, constructive mathematics, axiom of choice, anafunctor
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Contents

1. Introduction

Can homotopy theory be developed in constructive mathematics, or even in ZF set theory without the axiom of choice? Recently this question has begun to attract more attention, due partly to the rise of interest in Homotopy Type Theory and Univalent Foundations [Uni13]. The latter is a *constructive* type theory whose first model was nevertheless relentlessly *classical*, using the Kan–Quillen model category of simplicial sets [KL19]. Since then, constructive models of homotopy type theory have been found in categories of cubical sets [BCH14, BCH19, CCHM16, ABC$^+$17, ACC$^+$21], and the model category of simplicial sets has been developed constructively [Hen19, GSS19, GH19, GHSS21], though not quite to the point of strictly modeling type theory.

In particular, there are now at least two constructive homotopy theories — the aforementioned simplicial sets and the equivariant cartesian cubical sets of [ACC^{+}21] — that can *classically* be shown to present the homotopy theory of spaces. However, it is not known whether they are *constructively* equivalent to *each other*. Thus one may naturally wonder: if they are not equivalent, which is the "correct" constructive homotopy theory of spaces?[1] Or, perhaps, are they both "incorrect"? What does "correct" even mean?

In fact, both of these homotopy theories have a property that at first may seem peculiar: their 1-truncations (meaning their subcategory of homotopy 0-types) are *not* equivalent to the category of (constructive) sets that we started from. The 1-truncation of simplicial sets appears to be equivalent to the free exact completion $\mathsf{Set}_{\mathsf{ex}}$ of Set [CM82], a.k.a. the category of "setoids" (Simon Henry, personal communication). The 1-truncation of equivariant cartesian cubical sets may not be equivalent to $\mathsf{Set}_{\mathsf{ex}}$ (Andrew Swan, personal communication), but neither is it equivalent to Set. This is a significant departure from both classical mathematics and homotopy type theory, in which sets can be regarded, up to equivalence, as homotopy 0-types. (Note that the inclusion $\mathsf{Set} \hookrightarrow \mathsf{Set}_{\mathsf{ex}}$ is an equivalence if and only if the axiom of choice holds.)

In particular, this means that when homotopy type theory is interpreted in one of these constructive model categories, its internally-defined "sets" will be interpreted in the model as some kind of *setoid* rather than as actual sets. This is somewhat disturbing for the prospect of constructive applications of homotopy type theory and its semantics. At a stretch, one might even regard it as evidence for the *incorrectness* of *both* of these model categories.

In this paper we propose one possible correctness criterion for a constructive homotopy theory of spaces. Moreover, we provide some evidence that, the foregoing remarks notwithstanding, the 1-truncation of *any* theory satisfying this criterion *must* contain at least $\mathsf{Set}_{\mathsf{ex}}$, not just Set. In a moment we will discuss possible interpretations of this fact, but first let us explain the criterion and the evidence.

Classically, the homotopy theory of spaces has a universal property: it is

[1] By "space" we mean some combinatorial notion of ∞-groupoid. It is probably not reasonable to expect a theory of ∞-groupoids to be constructively equivalent to the homotopy theory of *topological* spaces, as continuous functions are much less flexible constructively than classically.

the free cocomplete $(\infty, 1)$-category generated by a point [Lur09, 5.1.5.6], just as Set is the free cocomplete 1-category generated by a point. However, this is somewhat circular as a characterization, since an $(\infty, 1)$-category is defined to have *spaces* as hom-objects.[2] One possible way around this would be to work with *presentations* of $(\infty, 1)$-categories using 1-categorical structures such as Quillen model categories. However, universal properties *of* $(\infty, 1)$-categories (as opposed to universal properties of objects *in* an $(\infty, 1)$-category) are hard to express at this level — indeed, this is one of the main reasons for the recent explicit use of $(\infty, 1)$-categories instead of model categories in applications such as [Lur09]. Moreover, although in classical mathematics most interesting complete and cocomplete $(\infty, 1)$-categories (including all locally presentable ones) can be presented by model categories, we ought not to assume *a priori* that this will still be the case constructively.

Instead, we can work with a 1-categorical *quotient* of an $(\infty, 1)$-category. The ordinary homotopy category, obtained by identifying equivalent pairs of parallel morphisms, is too coarse for this purpose; but an enhancement of it (due to Heller [Hel88], Grothendieck [Gro91], and Franke [Fra96]) turns out to be sufficient. Namely, given a complete and cocomplete $(\infty, 1)$-category $\mathscr{C}$, we consider the homotopy 1-categories of the functor $(\infty, 1)$-categories $\mathscr{C}^A$ for all small 1-categories A, together with the restriction functors relating them and their left and right adjoints (homotopy Kan extensions). This structure is nowadays called a *derivator* (after Grothendieck), and it retains a surprising amount of information about $\mathscr{C}$.

In particular, Heller [Hel88] and Cisinski [Cis06] have shown, in classical mathematics, that the derivator Space of spaces is the free cocompletion of a point. This means that for any other derivator $\mathscr{D}$, the category of cocontinuous morphisms Space $\to \mathscr{D}$ (those that preserve the "formal left Kan extensions" included in the structure of a derivator) is equivalent to the category $\mathscr{D}(\mathbb{1})$ of "diagrams of shape $\mathbb{1}$" in $\mathscr{D}$ (i.e. "objects of $\mathscr{D}$"; here $\mathbb{1}$ denotes the terminal category). Although a derivator is intuitively a homotopical, i.e. $(\infty, 1)$-categorical, object, formally this universal property lives at the same categorical level as the universal property of Set: derivators, like

[2]To be sure, not all definitions of $(\infty, 1)$-category explicitly incorporate hom-spaces. But the question of the correct constructive definition of $(\infty, 1)$-category seems likely to be at least as difficult as that of the correct constructive definition of ∞-groupoid, i.e. homotopy space.

1-categories, form a 2-category, and the universal property is an equivalence involving hom-categories therein. In the words of Cisinski [Cis10a]:

> This provides a first argument that the usual homotopy theory of simplicial sets plays a central role... and for this, we didn't take for granted that homotopy types should be that important: its universal property is formulated with category theory only.... derivators provide a truncated version of higher category theory which gives us the language to characterize higher category theory using only usual category theory, without any emphasis on any particular model (in fact, without assuming we even know any).

Thus, a natural correctness criterion for a constructive homotopy theory of spaces would be that it defines a derivator Space that is the free cocompletion of a point.

Of course, it is not *a priori* clear that such a derivator even exists in constructive mathematics. We will not attempt to construct one in this paper. Instead, we will attempt to understand how Space *would* behave, if it exists, by studying derivators that ought to be *localizations* of it. By this we mean derivators that should be obtained from Space by universally inverting some class of morphisms among cocontinuous morphisms, although in good situations this equivalent to being a reflective subcategory of Space (a *reflective localization*).

Classically, Space has many interesting reflective localizations, such as those that invert some set of prime numbers. More relevantly for us, for all integers $n \geq -2$ it has a reflective localization Space_n consisting of homotopy n-types. In particular, Space_0 is just the category Set of sets (regarded as a derivator), while Space_{-1} is the poset Prop of truth values (which, classically, is the two-element lattice) and Space_{-2} is the terminal derivator. Moreover, each Space_n is the free cocompletion of a point in the world of "$(n+1)$-truncated derivators" — those that behave like $(n+1, 1)$-categories rather than $(\infty, 1)$-categories.[3] In particular, this universal property for Set generalizes its ordinary one, giving it a mapping property into all 1-truncated derivators, not just those that arise from 1-categories.

[3]These "n-truncated derivators" are distinct from the "n-derivators" of [Rap19]. The former are 1-derivators (in the terminology of [Rap19]) that act as if they arose from an $(n, 1)$-category, while the latter generalize the definition of derivator to use n-categories in place of 1-categories.

In this paper we will exhibit, in constructive mathematics, derivators Space_n that have this universal property for $n = 0, -1, -2$. In fact, for $n = 0$ and -1 (and thus presumably for all $n \geq -1$) the notion of "$(n+1)$-truncated derivator" multifurcates constructively into several different notions, with several different corresponding localizations.

For one natural notion of "1-truncated derivator", we find that Set is the free cocompletion of a point. However, there are intuitively "1-categorical" derivators that are not 1-truncated in this sense. Notably, we will show that for any complete category $\mathscr{E}$ having small coproducts preserved by pullback, its exact completion $\mathscr{E}_{\mathsf{ex}}$ can be enhanced to a derivator, which is *not* "1-truncated" in the naive Set-based sense. There is a weaker notion of 1-truncatedness that does encompass these examples, but in this world Set is no longer the free cocompletion of the point: instead that role is taken by $\mathsf{Set}_{\mathsf{ex}}$.[4] There is also an intermediate notion of "1-truncatedness", whose free cocompletion of a point is a derivator version of $\mathsf{Set}_{\mathsf{reg}}$, the free regular completion of Set. We will refer to these three notions of 1-truncatedness as being Set-*local*, $\mathsf{Set}_{\mathsf{ex}}$-*local*, and $\mathsf{Set}_{\mathsf{reg}}$-*local* respectively.

A similar thing happens one dimension down: in addition to the lattice Prop, we have a derivator version of $\mathsf{Set}_{\mathsf{pos}}$, the preorder reflection of Set. Each of them is the free cocompletion of a point in its corresponding world of local derivators.

The class of $\mathsf{Set}_{\mathsf{ex}}$-local derivators is broader than that of Set-local ones, and in particular there is a cocontinuous map of derivators $\mathsf{Set}_{\mathsf{ex}} \to \mathsf{Set}$ but not conversely. Thus, if both were realized as reflective subcategories of Space, then $\mathsf{Set}_{\mathsf{ex}}$ would be the larger one. This provides our evidence that if a free cocompletion of a point exists constructively, its 1-truncation must involve $\mathsf{Set}_{\mathsf{ex}}$ and not just Set.[5]

[4] It is unclear exactly how this universal property of the *derivator* $\mathsf{Set}_{\mathsf{ex}}$ is related to the usual universal property of the *category* $\mathsf{Set}_{\mathsf{ex}}$. But it is reminiscent of the result of [Car95, Corollary to Lemma 4.1] that classically, the free exact completion of the small-coproduct completion of a small category is equivalent to its presheaf category, i.e. its free cocompletion. (Note that Set is the free small-coproduct completion of a point, as well as the free cocompletion of a point.)

[5] There is the possibility that this 1-truncation could be something even larger than $\mathsf{Set}_{\mathsf{ex}}$. It is not clear whether $(\mathsf{Set}_{\mathsf{ex}})_{\mathsf{ex}}$ can be made into a derivator at all, but if it could be then it would be one possible candidate. In addition, the 1-truncation of cubical sets may also be larger than $\mathsf{Set}_{\mathsf{ex}}$ (Andrew Swan, personal communication), so it is another possibility.

I can think of at least three responses to this observation. The first is to bite the bullet and accept that the correct homotopy theory of spaces is constructively the "∞-exact completion" of Set, and in particular its 0-truncated objects are setoids rather than sets. Thus, when applying homotopy theory constructively, we would be forced to use setoids, either exclusively or in tandem with sets.

This may be satisfying if our motivations for constructivity are purely philosophical. Indeed, some constructivist schools start from a foundation whose primitive objects are not sets but some kind of "pre-set" or "type" that lacks quotients entirely, such as some formalizations of Bishop's constructive mathematics [BB85] or Martin-Löf's original constructive type theory [ML84]. In this case, if "the category of sets" is to be exact, it *must* be defined as a free exact completion of the category of pre-sets, and so the appearance of an exact completion is entirely unproblematic.[6]

However, if we also care about categorical semantics, the appearance of setoids is troubling. When interpreting constructive mathematics internally in a category, it is the sets, not the setoids, that correspond to objects of that category. If our category of interest happens itself to be an exact completion of some other category, we might be able to interpret our mathematics in the latter, with the former category appearing as the exact completion of the latter. However, although some important categories are exact completions (such as some presheaf toposes and realizability toposes), many are not (such as most sheaf toposes), so this approach cannot work for them. This is related to the problem of constructing "realizability higher toposes" whose underlying 1-topos is an ordinary realizability topos [Uem19, SU19].

Another problem with exact completions is that they destroy impredicativity: even if Set has a subobject classifier, $\mathrm{Set}_{\mathrm{ex}}$ generally will not. Again, a philosophical predicativist may be unbothered by this, but it is disconcerting to *choose* to work with an impredicative category Set and nevertheless be forced into the predicative $\mathrm{Set}_{\mathrm{ex}}$ as soon as we start trying to do homotopy theory.

The second response is to reject our proposed "correctness criterion" for the homotopy theory of spaces. And indeed, there are obvious grounds on which to do so. Namely, our notion of derivator is based on small categories

[6] Relatedly, note that the model category of simplicial objects constructed in [GHSS21] requires only a category with finite limits and extensive countable coproducts.

and *functors* between them; but there are good arguments that in the absence of the axiom of choice, the correct notion of morphism between categories is instead that of an *anafunctor* [Mak96, Bar06, Rob12]. This suggests that we should instead be considering "ana-derivators" defining using anafunctors. In that world, it might be the case that the free cocompletion of a point consists of spaces and anafunctors between them, and has Set as its 1-truncation.

However, there are difficulties involved in making this work. Already for categories, it is impossible to prove even in ZF set theory that the bicategory of categories and anafunctors is locally small, cartesian closed, or complete [aK17]. (There are much weaker axioms than AC that suffice for local smallness and cartesian closure, such as SCSA [Mak96] and WISC [Rob12], but their constructive status is arguable, and it is unclear whether they imply completeness as well.) It seems likely that similar problems would arise in building a derivator out of 1-groupoids and anafunctors, let alone ∞-groupoids and ∞-anafunctors.

It may be more feasible to construct only a *left* derivator of groupoids and anafunctors, which has colimits but not limits. However, there are applications for which this would be insufficient; for instance, defining and constructing stacks requires taking limits over infinite sieves to define categories of descent data.

Finally, the third response is to reject the whole idea of *defining* spaces constructively out of sets, and instead *start* from a foundational theory such as homotopy type theory [Uni13], in which spaces are primitive objects. (Note that "computably" constructive flavors of homotopy type theory are also now available, such as the cubical type theories of [CCHM16, ABC$^+$17].) This allows "sets" to be *defined* as homotopy 0-types, without forcing the appearance of any exact completion. Semantically, this means working with the internal language of an $(\infty, 1)$-topos, within which sits the internal language of a 1-topos. This would be my personal preferred approach; I will comment on it further in Remark 8.7.

Background theory

We work in an informal constructive set theory, assuming neither the axiom of choice nor the law of excluded middle, with one universe to define a size boundary between large and small categories. Most or all of our results could probably be formalized in the internal language of an elementary topos containing a universe [Str05]; or in a membership-based set theory like IZF with a universe (or a weaker variant, since we probably do not need much replacement or collection); or in a dependent type theory with UIP, function extensionality, and quotients, like XTT [SAG19]. The arguments should be predicative, as long as we allow Prop, like Set, to be a large category. Importantly, however, we do require effective quotients, so that our category Set of sets is exact.

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2. The free exact completion

We start by reviewing the free exact completion. Recall that an **exact category** (in the sense of Barr) is a category with finite limits and such that every internal equivalence relation has a pullback-stable quotient of which it is the kernel.

Let $\mathscr{E}$ be a 1-category with finite limits; we recall from [CM82] how to build an exact category $\mathscr{E}_{\mathsf{ex}}$ from it freely.[7] A first thought might be to take the equivalence relations in $\mathscr{E}$ as the objects of $\mathscr{E}_{\mathsf{ex}}$, each such standing in for the quotient of itself. This produces a category in which every equivalence relation coming from $\mathscr{E}$ has an effective quotient (see Example 5.19), but

[7] $\mathscr{E}_{\mathsf{ex}}$ is sometimes written $\mathscr{E}_{\mathsf{ex/lex}}$, to emphasize that we started from a category $\mathscr{E}$ with only finite limits (i.e. one that is left exact, or "lex"). This is to distinguish it from other exact completions such as $\mathscr{E}_{\mathsf{ex/reg}}$, which requires $\mathscr{E}$ to be a regular category, and unlike the ex/lex completion is an idempotent operation.

it also introduces new equivalence relations that do not yet have quotients. Thus, we need something more general, which turns out to be the following.

Definition 2.1. A **pseudo-equivalence relation** in $\mathscr{E}$ consists of:

- Objects X_0 and X_1, with morphisms $s, t : X_1 \rightrightarrows X_0$.
- A morphism $r : X_0 \to X_1$ such that $sr = tr = 1$.
- A morphism $v : X_1 \to X_1$ such that $sv = t$ and $tv = s$.
- A morphism $m : X_1 {}^t\times^s_{X_0} X_1 \to X_1$ such that $sm = s\pi_1$ and $tm = t\pi_2$.

In other words, a pseudo-equivalence relation has the operations of an internal groupoid, but without any axioms. In particular, any object $X \in \mathscr{E}$ induces a "discrete" pseudo-equivalence relation with $X_1 = X_0 = X$; this provides a functor $\mathscr{E} \hookrightarrow \mathscr{E}_{\mathsf{ex}}$ to the category $\mathscr{E}_{\mathsf{ex}}$ defined as follows:

Definition 2.2. The **free exact completion** $\mathscr{E}_{\mathsf{ex}}$ of $\mathscr{E}$ has:

- As objects, pseudo-equivalence relations.
- As morphisms $X \to Y$, equivalence classes of pairs of morphisms $f_0 : X_0 \to Y_0$ and $f_1 : X_1 \to Y_1$ in $\mathscr{E}$ with $sf_1 = f_0 s$ and $tf_1 = f_0 t$, modulo the relation that $(f_0, f_1) \sim (g_0, g_1)$ if there exists a morphism $h : X_0 \to Y_1$ with $sh = f_0$ and $th = g_0$.

We refer to a pair (f_0, f_1) as a **morphism representative**, and an h as a **witness of equality** of two such.

Remark 2.3. A pseudo-equivalence relation can also be defined as an internal *bicategory* in $\mathscr{E}$ such that any two parallel 1-cells are related by a unique 2-cell and all 1-cells are equivalences. The tricategory of such "locally bidiscrete bigroupoids" is "locally tridiscrete", and its homotopy 1-category (obtained by identifying naturally equivalent functors) is $\mathscr{E}_{\mathsf{ex}}$. Our results about $\mathscr{E}_{\mathsf{ex}}$ could be obtained by specializing facts about bicategories and tricategories, but we will give concrete proofs instead.

It is proven in [CM82] that $\mathscr{E}_{\mathsf{ex}}$ is an exact category, and that this construction defines a left pseudo-adjoint to the forgetful 2-functor from exact categories to categories with finite limits. In particular, the inclusion $\mathscr{E} \hookrightarrow \mathscr{E}_{\mathsf{ex}}$ preserves finite limits; but even if $\mathscr{E}$ was already exact, this functor does not in general preserve quotients of equivalence relations. The only exception is if $\mathscr{E}$ is exact and satisfies the "axiom of choice" that regular epimorphisms are split, in which case the inclusion $\mathscr{E} \hookrightarrow \mathscr{E}_{\mathsf{ex}}$ is an equivalence.

We will not repeat the proofs of these facts, but we sketch the following:

Lemma 2.4. $\mathscr{E}_{\mathsf{ex}}$ *has finite limits.*

Proof. The terminal object has $T_0 = T_1 = 1$. For pullbacks, let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be a cospan in $\mathscr{E}_{\mathsf{ex}}$, select representatives (f_0, f_1) and (g_0, g_1) and define

$$P_0 = (X_0 \times Y_0) \times_{(Z_0 \times Z_0)} Z_1$$
$$P_1 = (P_0 \times P_0) \times_{(X_0 \times X_0 \times Y_0 \times Y_0)} (X_1 \times Y_1). \qquad \square$$

The particular objects P_0 and P_1 constructed above depend on the chosen representatives (f_0, f_1) and (g_0, g_1). Thus, in the absence of the axiom of choice (now meaning the usual axiom of choice in Set), $\mathscr{E}_{\mathsf{ex}}$ does not have a *specified* pullback functor $(\mathscr{E}_{\mathsf{ex}})^{\to\leftarrow} \to \mathscr{E}_{\mathsf{ex}}$, even if $\mathscr{E}$ has such a functor. (Although it does have a specified binary *product* functor.) The situation with infinite diagrams is even worse: without choice we have no way to select representatives for all the morphisms in the diagram simultaneously, so even if $\mathscr{E}$ is complete, $\mathscr{E}_{\mathsf{ex}}$ may not be.

Remark 2.5. The category of setoids *is* complete and cocomplete if we regard it as an $\mathcal{E}$-category, i.e. a category enriched over setoids (see e.g. [Ac21]). Indeed, from the perspective of Remark 2.3, the $\mathcal{E}$-category of setoids is a tricategory of certain bicategories, so it can be complete even if its homotopy category is not. We will not pursue this direction; the point of this paper is to observe that setoids arise unavoidably in homotopy theory *even* if we try our best to remain in the world of ordinary categories. See §8 for further discussion.

We can avoid all these problems with limits and colimits by considering a notion of *coherent* diagrams in $\mathscr{E}_{\mathsf{ex}}$.

Definition 2.6. Let A be a small category. A **coherent A-diagram** in $\mathscr{E}_{\mathsf{ex}}$ is:

- For each object $a \in A$, an object $X_a \in \mathscr{E}_{\mathsf{ex}}$.
- For each morphism $\alpha : a \to a'$ in A, a morphism representative $X_\alpha : X_a \to X_{a'}$, consisting of morphisms $X_{\alpha,0} : X_{a,0} \to X_{a',0}$ and $X_{\alpha,1} : X_{a,1} \to X_{a',1}$ in $\mathscr{E}$ with $sX_{\alpha,1} = X_{\alpha,0}s$ and $tX_{\alpha,1} = X_{\alpha,0}t$.
- For each $a \in A$, a morphism $X_r : X_{a,0} \to X_{a,1}$ with $sX_r = 1$ and $tX_r = X_{1_a,0}$ (i.e. a witness that $X_{1_a} \sim 1$).
- For each $\alpha : a \to a'$ and $\alpha' : a' \to a''$, a morphism $X_{\alpha,\alpha'} : X_{a,0} \to X_{a'',1}$ with $sX_{\alpha,\alpha'} = X_{\alpha',0}X_{\alpha,0}$ and $tX_{\alpha,\alpha'} = X_{\alpha'\alpha,0}$ (i.e. a witness that $X_{\alpha'}X_\alpha \sim X_{\alpha'\alpha}$).

For coherent A-diagrams X, Y, a **morphism representative** $f : X \to Y$ is:

- For each $a \in A$, morphisms $f_{a,0} : X_{a,0} \to Y_{a,0}$ and $f_{a,1} : X_{a,1} \to Y_{a,1}$ with $sf_{a,1} = f_{a,0}s$ and $tf_{a,1} = f_{a,0}t$ (i.e. a representative of a morphism $X_a \to Y_a$).
- For each $\alpha : a \to a'$ in A, a morphism $f_\alpha : X_{a,0} \to Y_{a',1}$ with $sf_\alpha = Y_{\alpha,0}f_{a,0}$ and $tf_\alpha = f_{a',0}X_{\alpha,0}$ (i.e. a witness that $Y_\alpha f_a \sim f_{a'}X_\alpha$).

A **witness of equality** between two such representatives is

- a family of morphisms $h_a : X_{a,0} \to Y_{a,1}$ with $sh_a = f_{a,0}$ and $th_a = g_{a,0}$.

The **morphisms** of coherent diagrams are the equivalence classes of morphism representatives, modulo the existence of a witness of equality. This defines **the category of coherent diagrams**, which we denote $\mathscr{E}_{\mathsf{ex}}(A)$.

Lemma 2.7. *If $A = \mathbb{1}$ is the terminal category, then $\mathscr{E}_{\mathsf{ex}}(\mathbb{1}) \simeq \mathscr{E}_{\mathsf{ex}}$.*

Proof. This is not a definitional equality, since an object of $\mathscr{E}_{\mathsf{ex}}(\mathbb{1})$ contains the additional data of an endomorphism representative with witnesses that it is idempotent and equal to the identity. But it is straightforward to see that these additional data are redundant. □

Remark 2.8. The 1-category $\mathscr{E}_{\mathsf{ex}}$ is also the hom-wise quotient of a 1-category of pseudo-equivalence relations and morphism representatives, as studied in [KP14]. But the same is not true of $\mathscr{E}_{\mathsf{ex}}(A)$: its morphism representatives cannot be composed associatively (though they become associative after quotienting by witnesses of equality). From the perspective of Remark 2.3, $\mathscr{E}_{\mathsf{ex}}(A)$ is the homotopy 1-category of a tricategory of trifunctors.

Remark 2.9. If the axiom of choice holds, then because the equivalence relation on morphisms in $\mathscr{E}_{\mathsf{ex}}(A)$ makes no reference to $f_{a,1}$ or f_α, instead of including the latter as data in a morphism we can simply assert that for each a or α such a morphism exists. Similarly, since the definition of morphisms makes no reference to X_r or $X_{\alpha,\alpha'}$, up to equivalence of categories we can simply assert that these exist. The latter assertion then says simply that X is a functor $A \to \mathscr{E}_{\mathsf{ex}}$, and similarly the former says that morphism is just a natural transformation. Thus, the axiom of choice implies that $\mathscr{E}_{\mathsf{ex}}(A) \simeq (\mathscr{E}_{\mathsf{ex}})^A$. Note that this is the axiom of choice for the ambient set theory, not the "axiom of choice" that regular epimorphisms split in $\mathscr{E}$ (though of course the two coincide if $\mathscr{E} = \mathsf{Set}$). In addition, even in the absence of the axiom of choice this holds whenever A is a finite category.

Example 2.10. If $u : A \to B$ is a functor between small categories and $X \in \mathscr{E}_{\mathsf{ex}}(B)$, we have a coherent diagram $u^*X \in \mathscr{E}_{\mathsf{ex}}(A)$ defined by precomposing all the data of X with the action of u on objects and morphisms. This defines a **restriction** functor $u^* : \mathscr{E}_{\mathsf{ex}}(B) \to \mathscr{E}_{\mathsf{ex}}(A)$. In particular, the functor $p_A : A \to \mathbb{1}$ induces for any $X \in \mathscr{E}_{\mathsf{ex}} \simeq \mathscr{E}_{\mathsf{ex}}(\mathbb{1})$ a **constant** coherent diagram $p_A^*X \in \mathscr{E}_{\mathsf{ex}}(A)$.

Theorem 2.11. *Suppose $\mathscr{E}$ is complete, with specified limit functors $\mathscr{E}^A \to \mathscr{E}$ for all small categories A. Then each functor $p_A^* : \mathscr{E}_{\mathsf{ex}} \to \mathscr{E}_{\mathsf{ex}}(A)$ has a right adjoint.*

Proof. We define the "limit" of a coherent diagram $Y \in \mathscr{E}_{\mathsf{ex}}(A)$ as follows. Let L_0 be the equalizer of the following parallel pair in $\mathscr{E}$:

$$\left(\prod_{a\in A} Y_{a,0} \times \prod_{\alpha:a\to a'} Y_{a',1}\right) \rightrightarrows \prod_{\alpha:a\to a'} (Y_{a',0} \times Y_{a',0}).$$

Here the components of the first morphism at $\alpha : a \to a'$ are $Y_{\alpha,0} : Y_{a,0} \to Y_{a',0}$ and $1_{Y_{a',0}}$, while those of the second morphism are $s : Y_{a',1} \to Y_{a',0}$ and $t : Y_{a',1} \to Y_{a',0}$. Then let L_1 be the pullback

$$(L_0 \times L_0) \underset{\prod_{a\in A}(Y_{a,0}\times Y_{a,0})}{\times} \textstyle\prod_{a\in A} Y_{a,1}.$$

Note that Y contains all the necessary data to define these objects, without any choices necessary. It is straightforward to show that L is a pseudo-equivalence relation.

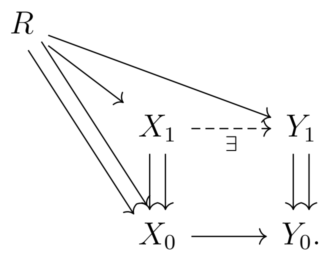
Now we define a counit $p_A^* L \to Y$. For each a, the components $L_0 \to Y_{a,0}$ and $L_1 \to Y_{a,1}$ are just the evident projections; and likewise for the morphisms $L_0 \to Y_{a',1}$ for each $\alpha : a \to a'$.

It remains to show that any morphism $f : p_A^* X \to Y$ factors uniquely through L. Choose a representative of f; then the components $f_{a,0} : X_0 \to Y_{a,0}$ and $f_{a,1} : X_1 \to Y_{a,1}$ and $f_\alpha : X_0 \to Y_{a',1}$ are exactly what is needed to define morphisms $\overline{f}_0 : X_0 \to L_0$ and $\overline{f}_1 : X_1 \to L_1$ with $s\overline{f}_1 = \overline{f}_0 s$ and $t\overline{f}_1 = \overline{f}_0 t$. Moreover, the representatives of the composite $p_A^* X \to p_A^* L \to Y$ are literally equal in $\mathscr{E}$ to those of f, so we can choose $h_a = r f_{a,0}$ to exhibit this composite as equal to f in $\mathscr{E}_{\mathsf{ex}}(A)$.

Finally, suppose $g : X \to L$ is such that the composite $p_A^* X \xrightarrow{g} p_A^* L \to Y$ is equal to f in $\mathscr{E}_{\mathsf{ex}}(A)$. Choosing a representative for g, we obtain components $g_{a,0} : X_0 \to Y_{a,0}$ and $g_\alpha : X_0 \to Y_{a',1}$ and $g_{a,1} : X_1 \to Y_{a,1}$ satisfying the appropriate equations. Choosing a witness of equality to f, we have morphisms $h_a : X_a \to Y_{a,1}$ with $sh_a = f_{a,0}$ and $th_a = g_{a,0}$. But this is exactly what we need to define a witness $h : X_0 \to L_1$ exhibiting $\overline{f} \sim g$ in $\mathscr{E}_{\mathsf{ex}}$. ☐

For the case of colimits, we need $\mathscr{E}$ to admit certain free constructions. Since our eventual interest is mainly in the case $\mathscr{E} = \mathsf{Set}$, we will not worry about the minimum this requires of $\mathscr{E}$, instead merely noting:

Lemma 2.12. *Suppose $\mathscr{E}$ has finite limits, and countable coproducts preserved by pullback. Then for any parallel pair $R \rightrightarrows X_0$, there is a pseudo-equivalence relation $X_1 \rightrightarrows X_0$ with a map $\eta : R \to X_1$ over $X_0 \times X_0$, such that for any pseudo-equivalence relation $Y_1 \rightrightarrows Y_0$ and morphism $f_0 : X_0 \to Y_0$ with $g : R \to Y_1$ over $f_0 \times f_0$, there exists a $f_1 : X_1 \to Y_1$ over $f_0 \times f_0$ such that $f_1 \eta = g$:*



Proof. Define

$$X_1 = \sum_{\substack{n\in\mathbb{N}\\ \varepsilon_1,\dots,\varepsilon_n\in\{+1,-1\}}} R^{\varepsilon_1} \times_{X_0} R^{\varepsilon_2} \times_{X_0} \cdots \times_{X_0} R^{\varepsilon_n}$$

where R^{+1} means the given span $X_0 \leftarrow R \rightarrow X_0$ and R^{-1} means the reversed span. (The summand for the case $n = 0$ is just X_0.) In the internal language of $\mathscr{E}$, X_1 is the object of zigzags such as

$$x_0 \xrightarrow{r_1} x_1 \xleftarrow{r_2} x_2 \xleftarrow{r_3} \cdots \xrightarrow{r_n} x_n$$

in which each arrow is labeled by an element of R, with the two maps $R \rightrightarrows X_0$ regarded as source and target, and each arrow in the zigzag can point in either direction. The resulting $X_1 \rightrightarrows X_0$ is actually the free internal †-category on the directed graph $R \rightrightarrows X_0$.

Finally, given f_0 and g as in the statement, we define f_1 on each summand of X_1 by applying g to each factor of R, then the symmetry operation of Y to each factor with $\varepsilon_k = -1$, and then some bracketing of the transitivity operation of Y to combine all the factors into one (in the case $n = 0$ this means the reflexivity operation of Y). The inclusion η is the summand with $n = 1$ and $\varepsilon_1 = +1$, where no operations are needed other than g, so we have $f_1\eta = g$. □

We refer to $X_1 \rightrightarrows X_0$ as in Lemma 2.12 as the **free pseudo-equivalence relation** generated by $R \rightrightarrows X_0$, although to be precise it is only "weakly free" (the morphism f_1 is not unique).

Theorem 2.13. *If $\mathscr{E}$ has finite limits and small coproducts preserved by pullback, then each functor $p_A^* : \mathscr{E}_{\mathsf{ex}} \to \mathscr{E}_{\mathsf{ex}}(A)$ has a left adjoint.*

Although we only require $\mathscr{E}$ to have co*products*, here A is an arbitrary small category; thus $\mathscr{E}_{\mathsf{ex}}$ has more "colimits" (in this sense) than $\mathscr{E}$ does.

Proof. Given $X \in \mathscr{E}_{\mathsf{ex}}(A)$, let C_0 be the coproduct $\sum_{a\in A} X_{a,0}$, and let C_1 be the pseudo-equivalence relation on C_0 freely generated (Lemma 2.12) by

$$\sum_{\alpha:a\to a'} \Big(X_{a,0} \times_{X_{a',0}} X_{a',1} \Big) \rightrightarrows C_0.$$

Here the pullback is the "object of triples (x, x', ξ)" where $x \in X_{a,0}$, $x' \in X_{a',0}$, and $\xi \in X_{a',1}$ is a witness that $X_{\alpha,0}(x) \sim x'$. The projection to C_0 picks out x and x' in the summands $X_{a,0}$ and $X_{a',0}$. (Note that neither of these is the copy of $X_{a',0}$ that we pull back over; that is $X_{\alpha,0}(x)$.)

Now we define a unit $X \to p_A^* C$. For each a, the component $X_{a,0} \to C_0$ is just the coproduct inclusion. To define the component $X_{a,1} \to C_1$, the idea is to send a witness $\xi \in X_{a,1}$ that $x \sim x'$ to the image under η of the witness that $X_{1_a,0}(x) \sim x \sim x'$ obtained by transitivity from ξ and X_r. And to define the witness $X_{a,0} \to C_1$ of naturality associated to $\alpha : a \to a'$, the idea is to send $x \in X_{a,0}$ to (the image under η of) the reflexivity witness that $X_{\alpha,0}(x) \sim X_{\alpha,0}(x)$.

It remains to show that any morphism $f : X \to p_A^* Y$ factors uniquely through C. Choose a representative of f; then the components $f_{a,0} : X_{a,0} \to Y_0$ define a morphism $C_0 \to Y_0$, while the components $f_{a,1} : X_{a,1} \to Y_1$ and $f_\alpha : X_{a,0} \to Y_1$ can be combined with transitivity, and the freeness of C, to induce a morphism $\overline{f} : C \to Y$. The composite components $X_{a,0} \to (p_A^* C)_{a,0} \to (p_A^* Y)_{a,0} = Y_0$ are then literally equal to $f_{a,0} : X_{a,0} \to Y_0$, so we can use $h_a = r f_{a,0}$ to exhibit this composite as equal to f in $\mathscr{E}_{\mathsf{ex}}(A)$.

Finally, suppose $g : C \to Y$ is such that the composite $X \to p_A^* C \xrightarrow{g} p_A^* Y$ is equal to f in $\mathscr{E}_{\mathsf{ex}}(A)$. Choosing a representative for g, we have components $g_{a,0} : X_{a,0} \to Y_0$ and $g_\alpha : X_{a,0} \to Y_1$ and $g_{a,1} : X_{a,1} \to Y_1$. And choosing a witness of its equality to f, we have morphisms $h_a : X_{a,0} \to Y_1$ with $s h_a = f_{a,0}$ and $t h_a = g_{a,0}$. But this is exactly what we need to define a witness $h : C_0 \to Y_1$ exhibiting $\overline{f} \sim g$ in $\mathscr{E}_{\mathsf{ex}}$. □

Thus, although $\mathscr{E}_{\mathsf{ex}}$ does not have infinite limits or colimits, or specified pullbacks, there is a sense in which it is strongly complete and cocomplete. In §4 we will see that derivators give us a way of making this precise.

Remark 2.14. Combining Remark 2.9 and Theorem 2.11, we see that if the axiom of choice holds and $\mathscr{E}$ is complete, then so is $\mathscr{E}_{\mathsf{ex}}$ (as an ordinary category). This was already observed by [HT96]; in their construction, the axiom of choice enters in the fact that epimorphisms of presheaves are closed under arbitrary products.

Similarly, combining Remark 2.9 and Theorem 2.13, we see that if the axiom of choice holds and $\mathscr{E}$ has small coproducts preserved by pullback, then $\mathscr{E}_{\mathsf{ex}}$ is cocomplete. Related facts were observed by [Men00] and [CV98];

the axiom of choice is hidden because they deal explicitly only with finite coproducts.

3. Derivators

A derivator is an abstraction of the structure possessed by the homotopy categories of diagrams in a complete and cocomplete $(\infty, 1)$-category. Early authors such as [Hel88, Gro91, Fra96] chose slightly different sets of axioms, but nowadays the community seems to have mostly settled on the definition of Grothendieck. As is often the case, we have to rephrase the definition to make it constructively useful. We will also follow [Hel88, Col20] in distinguishing *left* and *right* derivators that have only "colimits" and "limits", respectively.

Let $\mathcal{C}at$ and $\mathcal{C}AT$ be the 2-categories of small and large categories. For $A \in \mathcal{C}at$, let A_0 denote the discrete category on its objects, with inclusion $\iota_A : A_0 \to A$.

Definition 3.1. A **prederivator** is a 2-functor $\mathscr{D} : \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}AT$. A prederivator is a **semiderivator** if:

(Der1) $\mathscr{D} : \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}AT$ preserves products indexed by projective[8] sets. That is, if I is projective, the functor $\mathscr{D}(\sum_{i \in I} A_i) \to \prod_{i \in I} \mathscr{D}(A_i)$ is an equivalence, in the constructive sense that we have a specified quasi-inverse to it.

(Der2) For any $A \in \mathcal{C}at$, the functor $\iota_A^* : \mathscr{D}(A) \to \mathscr{D}(A_0)$ is conservative (that is, isomorphism-reflecting).

A **left derivator** is a semiderivator such that

(Der3L) Each functor $u^* : \mathscr{D}(B) \to \mathscr{D}(A)$ has a specified left adjoint $u_!$.

(Der4L) Given functors $u : A \to C$ and $v : B \to C$ in $\mathcal{C}at$, let (u/v) denote their comma category, with projections $p : (u/v) \to A$ and $q : (u/v) \to B$. If B is a discrete category, then the canonical mate-transformation $q_! p^* \to v^* u_!$ is an isomorphism.

[8] A set I is projective if every surjection $J \twoheadrightarrow I$ has a section. Thus finite sets are always projective, and the axiom of choice is equivalently "all sets are projective".

Dually, a **right derivator** is a semiderivator such that

(Der3R) Each functor $u^* : \mathscr{D}(B) \to \mathscr{D}(A)$ has a specified right adjoint u_*.

(Der4R) Given u and v as in (Der4L), if instead A is a discrete category, then the mate-transformation $u^* v_* \to p_* q^*$ is an isomorphism.

A **derivator** is a semiderivator that is both a left derivator and a right derivator. Finally, a prederivator is **strong** if

(Der5) For any $A \in \mathcal{C}at$, the induced functor $\mathscr{D}(A \times \mathbb{2}) \to \mathscr{D}(A)^{\mathbb{2}}$ is full and essentially surjective, where $\mathbb{2} = (0 \to 1)$ is the interval category.

We immediately record the most basic class of examples.

Example 3.2. Let $\mathscr{C}$ be an ordinary category, and $\mathscr{C}(A) = \mathscr{C}^A$ the functor category, with 2-functorial action by restriction. This 2-functor preserves *all* products, and (Der2) holds because isomorphisms in functor categories are pointwise, while (Der5) is obvious since the functor in question is an isomorphism. Thus $\mathscr{C}$ defines a strong semiderivator, which we call a **representable** semiderivator and abusively denote also by $\mathscr{C}$.

If $\mathscr{C}$ is cocomplete, the restriction functors admit left adjoints given by pointwise Kan extensions; thus (Der3L) holds, and (Der4L) asserts that these Kan extensions are pointwise, so $\mathscr{C}$ is a left derivator. Similarly, if $\mathscr{C}$ is complete, it is a right derivator. In particular, Set is a derivator.

Remark 3.3. The usual definition, as e.g. in [Gro13, Col20], differs in that:

- Axiom (Der1) is asserted for all products, not just projectively indexed ones.[9]
- Axiom (Der2) asserts that the family of functors $a^* : \mathscr{D}(A) \to \mathscr{D}(\mathbb{1})$ are jointly conservative, for all objects $a \in A$. This is equivalent to (Der2) in the presence of the classical (Der1), since $A_0 \cong \sum_{a \in A} \mathbb{1}$.

[9]Although sometimes $\mathcal{C}at$ is replaced in the definition by a smaller 2-category, such as the 2-category of finite categories, finite posets, or finite direct categories. In this case (Der1) is weakened to refer only to the coproducts that exist therein, such as finite ones.

- Axiom (Der4L) requires that B be the terminal category $\mathbb{1}$, and dually for (Der4R). However, by [Gro13, Prop. 1.26], in the presence of the classical (Der1) and (Der2) this implies that the same statements hold without *any* restriction on B (see Lemma 3.6 below), including in particular our (Der4).

Thus, the substantial difference is the weakening of (Der1), which is only weaker in the absence of the axiom of choice.[10] Our weaker version appears to be necessary constructively; for some explanation, see the proof of Lemma 4.2.

Perhaps surprisingly, our definition suffices for most of the theory of derivators; axiom (Der1) is rarely needed, and usually only for finite products. Intuitively, while a classical (pre)derivator has an underlying ordinary category $\mathscr{D}(\mathbb{1})$, one of our (pre)derivators has an underlying Set-*indexed category* consisting of the categories $\mathscr{D}(I)$ where I is a discrete category. We can then reproduce the usual theory by using indexed categories in place of ordinary ones. (Note that a prederivator is, in particular, a $\mathcal{C}at$-indexed category.)

For instance, (Der3L) implies that any left derivator admits "colimit" functors given by $(p_A)_!$ for the functor $p_A : A \to \mathbb{1}$, left adjoint to the "constant diagram" functor $(p_A)^*$, and dually for right derivators and limits. The standard (Der4) axioms then says that the general "Kan extension" functors u_* and $u_!$ can be computed in terms of these, by the usual formula [ML98, Theorem X.3.1]. Our (Der4) says the same in "indexed" or "internal" language, referring not only to "global elements" $c : 1 \to C_0$ but to arbitrary "generalized elements" $v : I \to C_0$, where I is a set.

We now give some examples of how such "indexed reasoning" can be used to reproduce some of the basic results about derivators from the cited references.

[10]The assertion of (Der1) for all projective sets is admittedly a fairly transparent trick for forcing the definition to collapse to the classical one in the presence of the axiom of choice, only slightly less blatant than starting with "if the axiom of choice holds, then. . . ". Probably more natural constructively would be to assert (Der1) only for *finite* products.

Definition 3.4. For a left derivator $\mathscr{D}$, a square 2-cell in $\mathcal{C}at$:

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ {\scriptstyle q}\downarrow & \swarrow & \downarrow{\scriptstyle u} \\ C & \xrightarrow[v]{} & D \end{array}$$

is **$\mathscr{D}$-exact** if the induced map $q_! p^* \to v^* u_!$ is an isomorphism in $\mathscr{D}$. Dually, if $\mathscr{D}$ is a right derivator, such a square is $\mathscr{D}$-exact if the map $u^* v_* \to p_* q^*$ is an isomorphism. (If $\mathscr{D}$ is a derivator, then these two maps are adjunction conjugates, hence the two conditions are equivalent.)

A square is **left** (resp. **right**) **homotopy exact** if it is $\mathscr{D}$-exact for all left (resp. right) derivators $\mathscr{D}$, and **homotopy exact** if it is $\mathscr{D}$-exact for all derivators $\mathscr{D}$.

Note that left and right homotopy exactness are stronger than homotopy exactness, oppositely to how being a derivator is stronger than being a left or right derivator. The functoriality property of mates (e.g. [KS74]) imply that horizontal and vertical pasting preserves (left and right) homotopy exact squares.

Observe that for a set I, an I-indexed family of small categories $A : I \to \mathcal{C}at$ can equivalently be regarded as a category A equipped with a functor $A \to I$, where I denotes also the corresponding discrete category. That is, $\mathcal{C}at^I \simeq \mathcal{C}at/I$. Moreover, if $f, g : A \to B$ are functors between two objects of $\mathcal{C}at/I$, any natural transformation $f \Rightarrow g$ in $\mathcal{C}at$ must in fact lie in $\mathcal{C}at/I$, since I is discrete. In particular, a morphism in $\mathcal{C}at/I$ has a left or right adjoint in $\mathcal{C}at/I$ if and only if it does so in $\mathcal{C}at$.

Lemma 3.5 (cf. [Gro13, Proposition 1.18])**.** *For a set I, let $r : A \to B$ be a right adjoint in $\mathcal{C}at/I$. Then the identity 2-cell is left homotopy exact:*

$$\begin{array}{ccc} A & \xrightarrow{r} & B \\ {\scriptstyle ur}\downarrow & \swarrow & \downarrow{\scriptstyle u} \\ I & = & I \end{array}$$

Proof. If ℓ is the left adjoint of r, then the map $(ur)_! r^* \to u_!$ is conjugate to $u^* \to \ell^*(ur)^*$, which is an identity since the entire adjunction lies over I; hence it is also an isomorphism. □

Lemma 3.6 (cf. [Gro13, Proposition 1.26])**.** *Any comma square is left and right homotopy exact:*

$$\begin{array}{ccc} (u/v) & \xrightarrow{p} & A \\ {\scriptstyle q}\big\downarrow & \swArrow & \big\downarrow{\scriptstyle u} \\ B & \xrightarrow[v]{} & C \end{array}$$

Proof. We prove the left case. By (Der2) and (Der4), it suffices to prove that the pasted rectangle on the left below is homotopy exact, in which the left-hand square is also a comma:

$$\begin{array}{ccccc} (q/\iota) & \longrightarrow & (u/v) & \xrightarrow{p} & A \\ \big\downarrow & \swArrow & {\scriptstyle q}\big\downarrow & \swArrow & \big\downarrow{\scriptstyle u} \\ B_0 & \xrightarrow[\iota]{} & B & \xrightarrow[v]{} & C \end{array} \quad = \quad \begin{array}{ccccc} (q/\iota) & \longrightarrow & (u/v\iota) & \xrightarrow{p} & A \\ \big\downarrow & \swArrow & {\scriptstyle q}\big\downarrow & \swArrow & \big\downarrow{\scriptstyle u} \\ B_0 & = & B_0 & \xrightarrow[v\iota]{} & C \end{array}$$

But this is equal to the pasted rectangle on the right above, where the right-hand square is a comma and the left-hand square is an identity. And the induced functor $(q/\iota) \to (u/v\iota)$ is a right adjoint, so by Lemma 3.5 and (Der4) both of these squares are homotopy exact. □

Lemma 3.7 (cf. [Gro13, Proposition 1.24])**.** *If u is a cloven Grothendieck opfibration, then the identity in any pullback square is left homotopy exact:*

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ {\scriptstyle q}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle u} \\ C & \xrightarrow[v]{} & D \end{array}$$

Dually, if v is a cloven Grothendieck fibration, such a pullback square is right homotopy exact.

Proof. We prove the left case. Let $\iota_D^*(B)$ be the pullback

$$\begin{array}{ccc} \iota_D^*(B) & \longrightarrow & B \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle u} \\ D_0 & \xrightarrow[\iota_D]{} & D \end{array}$$

Then there is an induced functor $\iota_D^*(B) \to (u/\iota_D)$, and the cleaving of u supplies a left adjoint to it over D_0. Similarly, since q is also a cloven opfibration, the induced functor $\iota_C^*(A) \to (q/\iota_C)$ is a right adjoint over C_0. Therefore, by (Der2) and (Der4) and Lemma 3.5, it suffices to prove that the following pasting is homotopy exact:

$$\begin{array}{ccccccc}
\iota_C^*(A) & \longrightarrow & (q/\iota_C) & \longrightarrow & A & \xrightarrow{p} & B \\
\downarrow & \swArrow & \downarrow & \swArrow & {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\
C_0 & = & C_0 & \xrightarrow[\iota_C]{} & C & \xrightarrow[v]{} & D.
\end{array}$$

But this factors as

$$\begin{array}{ccccccc}
\iota_C^*(A) & \longrightarrow & \iota_D^*(B) & \longrightarrow & (u/\iota_D) & \longrightarrow & B \\
\downarrow & \lrcorner & \downarrow & \swArrow & \downarrow & \swArrow & \downarrow{\scriptstyle u} \\
C_0 & \xrightarrow[v_0]{} & D_0 & = & D_0 & \xrightarrow[\iota_D]{} & D.
\end{array}$$

Here the left- and right-hand squares are homotopy exact by (Der4), while the middle square is homotopy exact by Lemma 3.5. □

Definition 3.8. A **morphism** of prederivators is a pseudonatural transformation, and a **transformation** is a modification. We say a morphism $G : \mathscr{D} \to \mathscr{D}'$ of left derivators is **cocontinuous** if for any functor $u : A \to B$, the canonical mate-transformation

$$\begin{array}{ccc}
\mathscr{D}(A) & \xrightarrow{G} & \mathscr{D}'(A) \\
{\scriptstyle u_!}\downarrow & \swArrow & \downarrow{\scriptstyle u_!} \\
\mathscr{D}(B) & \xrightarrow[G]{} & \mathscr{D}'(B)
\end{array}$$

is an isomorphism. We denote the category of morphisms and transformations by $\mathsf{Hom}(\mathscr{D}, \mathscr{D}')$, and its full subcategory of cocontinuous morphisms by $\mathsf{Hom}_{\mathsf{cc}}(\mathscr{D}, \mathscr{D}')$.[11]

Lemma 3.9. *A morphism $G : \mathscr{D} \to \mathscr{D}'$ is cocontinuous if and only if the above condition holds when B is discrete.*

[11] Sometimes the notation $\mathsf{Hom}_!$ is used, but I find this insufficiently visually distinctive.

Proof. By functoriality of mates, combined with (Der2) and (Der4), we can deduce the condition for arbitrary $u : A \to B$ from the condition for $q : (u/\iota_B) \to B_0$. □

Theorem 3.10 (in classical mathematics)**.** *Every Quillen model category* $\mathcal{M}$ *induces a derivator* $\mathrm{Ho}(\mathcal{M})$. *If* sSet *denotes the Kan–Quillen model category of simplicial sets, then* $\mathsf{Space} = \mathrm{Ho}(\mathrm{sSet})$ *is the free cocompletion of a point: there is an object* $* \in \mathsf{Space}(\mathbb{1})$ *such that for any derivator* $\mathscr{D}$*, the induced functor*

$$\mathsf{Hom}_{\mathsf{cc}}(\mathsf{Space}, \mathscr{D}) \to \mathscr{D}(\mathbb{1})$$

is an equivalence of categories.

Proof. Essentially due to Heller [Hel88] and Cisinski [Cis06, Cis04]. □

We will also need two-variable morphisms of derivators, as in [GPS14].

Lemma 3.11 (cf. [GPS14, Theorem 3.11])**.** *For prederivators* $\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3$*, to give a morphism* $\mathscr{D}_1 \times \mathscr{D}_2 \to \mathscr{D}_3$ *is equivalent to giving a family of functors*

$$\mathscr{D}_1(A) \times \mathscr{D}_2(B) \to \mathscr{D}_3(A \times B)$$

varying pseudonaturally over $\mathcal{C}at^{\mathrm{op}} \times \mathcal{C}at^{\mathrm{op}}$. □

If $\circledast$ is such a two-variable morphism, we write $\circledast_A : \mathscr{D}_1(A) \times \mathscr{D}_2(A) \to \mathscr{D}_3(A)$ for its components in the ordinary (or "internal") sense, and $\circledast : \mathscr{D}_1(A) \times \mathscr{D}_2(B) \to \mathscr{D}_3(A \times B)$ for the above equivalent "external" components. The relationship is that $M \circledast_A N \cong \Delta_A^*(M \circledast N)$ while $M \circledast N \cong \pi_1^* M \circledast_{A\times B} \pi_2^* N$.

Definition 3.12. A morphism $\circledast : \mathscr{D}_1 \times \mathscr{D}_2 \to \mathscr{D}_3$ of left derivators is **cocontinuous in its first variable** if for any $u : A \to B$ and $M \in \mathscr{D}_1(A)$ and $N \in \mathscr{D}_2(C)$, the following mate-transformation is an isomorphism in $\mathscr{D}_3(B \times C)$:

$$(u \times 1)_!\,(M \circledast N) \longrightarrow (u_!\,M) \circledast N.$$

See [GPS14, Warning 3.6] for why this has to be formulated with the external product rather than the internal one. There is a dual notion of cocontinuity in the second variable, and an analogue of Lemma 3.9 for two-variable morphisms.

Finally, since $\mathscr{D}(A) \to \mathscr{D}(A) \times \mathscr{D}(A)$ is equivalent to $\nabla^* : \mathscr{D}(A) \to \mathscr{D}(A+A)$ (this uses (Der1) for finite coproducts), in a right derivator the former functor also has a right adjoint. Thus any right derivator $\mathscr{D}$ is "cartesian monoidal", with a product morphism $\times : \mathscr{D} \times \mathscr{D} \to \mathscr{D}$.

Definition 3.13. We say a derivator $\mathscr{D}$ is **distributive** if this $\times$ is cocontinuous in both variables.[12]

For example, a complete and cocomplete category regarded as a derivator as in Example 3.2 is distributive if binary products preserve colimits in each variable, in the usual sense. In particular, Set is distributive.

4. The derivator of setoids

Let $\mathscr{E}$ be, to start with, a category with finite limits.

Lemma 4.1. $\mathscr{E}_{\mathsf{ex}} : \mathbb{C}at^{\mathrm{op}} \to \mathbb{C}AT$ *is a 2-functor.*

Proof. First, the restriction functors $u^* : \mathscr{E}_{\mathsf{ex}}(B) \to \mathscr{E}_{\mathsf{ex}}(A)$ are strictly functorial, being given by simple composition with the data of u. Second, given a natural transformation $\mu : u \Rightarrow v : A \to B$ with components $\mu_a : ua \to va$, for any $X \in \mathscr{E}_{\mathsf{ex}}(B)$ we have an induced family of morphisms $X_{\mu_a,0} : X_{ua,0} \to X_{va,0}$ and $X_{\mu_a,1} : X_{ua,1} \to X_{va,1}$. Third, for $\alpha : a \to a'$, by applying the pseudo-transitivity m to $X_{u\alpha,\mu_{a'}}$ and $X_{\mu_a,v\alpha}$, we have morphisms $X_{ua,0} \to X_{va',1}$ exhibiting naturality. Thus, we obtain a morphism of coherent diagrams $u^*X \to v^*X$. The 2-functoriality axioms follow straightforwardly. □

Lemma 4.2. $\mathscr{E}_{\mathsf{ex}}$ *satisfies (Der1).*

Proof. For any coproduct of categories, the functor $\mathscr{E}_{\mathsf{ex}}(\sum_i A_i) \to \prod_i \mathscr{E}_{\mathsf{ex}}(A_i)$ is bijective on objects. To show that it is full, we must select representatives for a family of morphisms in each $\mathscr{E}_{\mathsf{ex}}(A_i)$ to assemble them into a representative for a morphism in $\mathscr{E}_{\mathsf{ex}}(\sum_i A_i)$; this is possible when I is projective. Similarly, to show that it is faithful, we must select witnesses of equality in each $\mathscr{E}_{\mathsf{ex}}(A_i)$ to assemble into such a witness in $\mathscr{E}_{\mathsf{ex}}(\sum_i A_i)$, which is also possible when I is projective. □

[12] Technically this definition does not require $\mathscr{D}$ to be a full derivator, only a "left derivator with binary products", but we will have no use for that generality.

Lemma 4.3. *$\mathscr{E}_{\mathsf{ex}}$ satisfies (Der2).*

Proof. Let $f : X \to Y$ be a (representative of a) morphism in $\mathscr{E}_{\mathsf{ex}}(A)$, with components $f_{a,0}$, $f_{a,1}$, and f_α. If it is invertible in $\mathscr{E}_{\mathsf{ex}}(A_0)$, then we have families of morphisms $g_{a,0} : Y_{a,0} \to X_{a,0}$ and $g_{a,1} : Y_{a,1} \to X_{a,1}$ representing morphisms of pseudo-equivalence relations $Y_a \to X_a$, and such that $gf \sim 1$ and $fg \sim 1$ in $\mathscr{E}_{\mathsf{ex}}(A_0)$. The latter mean that there exist $h_a : X_{a,0} \to Y_{a,1}$ with $sh_a = g_{a,0}f_{a,0}$ and $th_a = 1$, and also $k_a : Y_{a,0} \to X_{a,1}$ with $sk_a = f_{a,0}g_{a,0}$ and $tk_a = 1$. Using a chosen such h and k, we can define (copying the usual proof that a pointwise invertible natural transformation is invertible in the functor category) for each $\alpha : a \to a'$ a morphism $g_\alpha : Y_{a,0} \to X_{a',1}$ making g a representative of a morphism in $\mathscr{E}_{\mathsf{ex}}(A)$. The same h and k then witness that $gf = 1$ and $fg = 1$ in $\mathscr{E}_{\mathsf{ex}}(A)$. □

Lemma 4.4. *If $\mathscr{E}$ is complete, then $\mathscr{E}_{\mathsf{ex}}$ is a right derivator. If $\mathscr{E}$ has pullback-stable coproducts, then $\mathscr{E}_{\mathsf{ex}}$ is a left derivator.*

Proof. We can use the classical construction of pointwise Kan extensions [ML98, Theorem X.3.1] essentially verbatim, due to the fact that the constructions in Theorems 2.11 and 2.13 are not just adjoints, but have a constructive universal property with respect to *representatives* of morphisms and *witnesses* of equality. That is, there is a function which, given a representative for a morphism $(p_A)^*X \to Y$ in $\mathscr{E}_{\mathsf{ex}}(A)$, produces a representative for the corresponding morphism $X \to L$, where L is the limit constructed in Theorem 2.11; and similarly for witnesses of equality between morphisms, and for colimits. The construction of these functions is essentially contained in the proofs of Theorems 2.11 and 2.13.

Consider the case of limits; the case of colimits is analogous. Given $u : A \to B$, for any $b \in B$ we have the comma category (b/u) with projection $q_b : (b/u) \to A$. For $X \in \mathscr{E}_{\mathsf{ex}}(A)$ and $b \in B$, define $(u_*X)_b = (p_{b/u})_*q_b^*X$, with the limit functor $(p_{b/u})_*$ constructed as in Theorem 2.11. For a morphism $\beta : b \to b'$ in B, the above remark implies that we can give a morphism representative $(u_*X)_b \to (u_*X)_{b'}$ by giving a morphism representative $(p_{b'/u})^*(p_{b/u})_*q_b^*X \to q_{b'}^*X$, consisting of morphism representatives $(p_{b/u})_*q_b^*X \to X_a$ for all morphisms $\beta' : b' \to ua$, with compatibility witnesses. These latter representatives can be given by the projections from $(p_{b/u})_*q_b^*X$ corresponding to the composite $\beta'\beta : b \to ua$, and similarly for the compatibility witnesses. Likewise, the same principles yield witnesses of functoriality and a universal property of u_* as a right adjoint of u^*. Thus (Der3R) holds. To prove (Der4R), in a comma square with A discrete:

$$\begin{array}{ccc} (u/v) & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & \Swarrow & \downarrow{\scriptstyle u} \\ B & \xrightarrow[v]{} & C, \end{array}$$

the construction above shows that $(u^* v_* X)_a$ and $(p_* q^* X)_a$ are limits (as in Theorem 2.11) of the restrictions of X to a pair of isomorphic categories (ua/v) and (a/p). Thus, these limits are isomorphic, in a constructive way that can be done simultaneously for all $a \in A$. □

Lemma 4.5. *$\mathscr{E}_{\mathsf{ex}}$ satisfies (Der5).*

Proof. Analogously to Remark 2.9, since $\mathbb{2}$ is finite, the functor in question is actually an equivalence. □

Theorem 4.6. *For any complete category $\mathscr{E}$ with small coproducts preserved by pullback, $\mathscr{E}_{\mathsf{ex}}$ is a strong distributive derivator.*

Proof. We have verified all the strong derivator axioms in Lemmas 4.2 to 4.5, so it remains only to prove distributivity. For this, we note that if in Theorem 2.11 A is discrete, we can replace the construction given there by the simpler $L_0 = \prod_a Y_{a,0}$ and $L_1 = \prod_a Y_{a,1}$. Now since the "colimits" in Theorem 2.13 are constructed out of pullbacks and coproducts, and both of these are preserved in each variable by finite products, it follows that the derivator products in $\mathscr{E}_{\mathsf{ex}}$ preserve its left Kan extensions in each variable. □

Corollary 4.7. $\mathsf{Set}_{\mathsf{ex}}$ *is a strong distributive derivator.* □

Remark 4.8. The free exact completion is not in general idempotent. In particular, we can have $(\mathsf{Set}_{\mathsf{ex}})_{\mathsf{ex}} \not\simeq \mathsf{Set}_{\mathsf{ex}}$. However, since $\mathsf{Set}_{\mathsf{ex}}$ is not complete or cocomplete as a *category*, Theorem 4.6 does not imply that $(\mathsf{Set}_{\mathsf{ex}})_{\mathsf{ex}}$ is a derivator. It is unclear whether there is a notion of "exact completion of a derivator".

5. Equivalences and locality

As suggested in the introduction, we are interested in derivators that satisfy a relative version of Theorem 3.10, being a free cocompletion of a point in a world of "1-categorical derivators". Thus, we may start by asking what it is that makes a derivator 1-categorical. Intuitively, an $(\infty, 1)$-category "is" a 1-category if all its hom-spaces are 0-truncated; but a derivator does not have explicit hom-spaces.

However, we can detect the same information using limits and colimits of constant diagrams. For instance, for any object M of an $(\infty, 1)$-category, the limit of the constant diagram

$$M \rightrightarrows M$$

is the *free loop space object* LM of M, which is equivalent to M just when M is 0-truncated. Similarly, one dimension down, the product $M \times M$ is equivalent to M just when M is (-1)-truncated, i.e. subterminal. Thus, the "1-categorical" or "0-categorical" nature of an $(\infty, 1)$-category is detected by limits of constant diagrams of this shape.

More generally, in any derivator there is the following relative notion.

Definition 5.1. Let $u : A \to B$ and $v : B \to I$ be functors, where I is a discrete set. We say u is a $\mathscr{D}$**-equivalence over** I, for a prederivator $\mathscr{D}$, if u^* is fully faithful on the image of v^*.

Lemma 5.2. *If $\mathscr{D}$ is a left derivator, then u is a $\mathscr{D}$-equivalence over I if and only if the map $(vu)_! \, (vu)^* \to v_! \, v^*$ is an isomorphism. Dually, if $\mathscr{D}$ is a right derivator, then u is a $\mathscr{D}$-equivalence over I if and only if the map $v_* \, v^* \to (vu)_* \, (vu)^*$ is an isomorphism.*

Proof. By the Yoneda lemma, the stated condition for left derivators is equivalent to saying that

$$\mathscr{D}(I)(v_! \, v^* X, Y) \to \mathscr{D}(I)((vu)_! \, (vu)^* X, Y)$$

is a bijection for all $X, Y \in \mathscr{D}(I)$. But this map is isomorphic to

$$\mathscr{D}(I)(v^* X, v^* Y) \to \mathscr{D}(I)(v^* u^* X, v^* u^* Y),$$

and this being a bijection for all X, Y is Definition 5.1. $\square$

The above considerations might lead us to say that a prederivator $\mathscr{D}$ is 1-categorical if the functor $(\cdot \rightrightarrows \cdot) \to \mathbb{1}$ is a $\mathscr{D}$-equivalence, and 0-categorical if the functor $(\mathbb{1} + \mathbb{1}) \to \mathbb{1}$ is a $\mathscr{D}$-equivalence. However, as we will see, things are a bit more subtle than this. We begin by recording some basic properties of the $\mathscr{D}$-equivalences.

Lemma 5.3. *If $f : I \to J$ is a function between discrete sets and $u : A \to B$ is a $\mathscr{D}$-equivalence over J, for a left or right derivator $\mathscr{D}$, then the pullback $f^*(u)$ is a $\mathscr{D}$-equivalence over I:*

$$\begin{array}{ccccc} f^*A & \xrightarrow{f^*u} & f^*B & \xrightarrow{f^*v} & I \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ A & \xrightarrow[u]{} & B & \xrightarrow[v]{} & J \end{array}$$

Proof. We prove the left case. Any functor with discrete codomain is a cloven opfibration, so by Lemma 3.7 f^* transforms $v_!$ and $(vu)_!$ into $(f^*v)_!$ and $((f^*v)(f^*u))_!$. Since it also commutes with v^* and $(vu)^*$ by functoriality, it preserves the property in Lemma 5.2. □

Lemma 5.4. *Let $I = \sum_{j \in J} I_j$ be a coproduct of sets, with injections $g_j : I_j \to I$, such that the indexing set J is projective. If $u : A \to B$ is a functor over I such that each $g_j^*(u)$ is a $\mathscr{D}$-equivalence over I_j for a left or right derivator $\mathscr{D}$, then u is a $\mathscr{D}$-equivalence over I.*

Proof. By (Der1), isomorphisms in $\mathscr{D}(I)$ are detected in each $\mathscr{D}(I_j)$, and restriction along g_j commutes with the relevant functors as in Lemma 5.3. □

Corollary 5.5. *Assuming the axiom of choice, $u : A \to B$ is a $\mathscr{D}$-equivalence over I if and only if its fiber $u_i : A_i \to B_i$ over each $i \in I$ is a $\mathscr{D}$-equivalence over $\mathbb{1}$.* □

Corollary 5.5 explains why in classical mathematics, $\mathscr{D}$-equivalences are defined without reference to an indexing set I. Note also that for any $f : I \to J$, a $\mathscr{D}$-equivalence over I is also a $\mathscr{D}$-equivalence over J. In particular, any $\mathscr{D}$-equivalence over I is also a $\mathscr{D}$-equivalence over $\mathbb{1}$. Dually, for any functor $u : A \to B$ there is a strongest sort of $\mathscr{D}$-equivalence that it can be, namely over the set $I = \pi_0(B)$ of connected components of B.

Lemma 5.6. *For any prederivator $\mathscr{D}$, the $\mathscr{D}$-equivalences are saturated, in the sense that if a morphism u in $\mathcal{C}at/I$ becomes an isomorphism in $(\mathcal{C}at/I)[(\mathcal{W}_I^{\mathscr{D}})^{-1}]$, where $\mathcal{W}_I^{\mathscr{D}}$ denotes the $\mathscr{D}$-equivalences over I, then u is a $\mathscr{D}$-equivalence. Therefore, the $\mathscr{D}$-equivalences satisfy the 2-out-of-3 property, the 2-out-of-6 property, and are closed under retracts.*

Proof. For fixed $X, Y \in \mathscr{D}(I)$, there is a functor $\Phi_{X,Y} : \mathcal{C}at/I \to \mathsf{Set}^{\mathrm{op}}$ sending $v : A \to I$ to $\mathscr{D}(A)(v^*X, v^*Y)$. Since $\Phi_{X,Y}$ inverts all $\mathscr{D}$-equivalences, it factors through $(\mathcal{C}at/I)[(\mathcal{W}_I^{\mathscr{D}})^{-1}]$; and therefore, if u becomes an isomorphism in $(\mathcal{C}at/I)[(\mathcal{W}_I^{\mathscr{D}})^{-1}]$, it is inverted by $\Phi_{X,Y}$. But if u is inverted by $\Phi_{X,Y}$ for all X, Y, then it is a $\mathscr{D}$-equivalence by definition. □

We now give some examples of $\mathscr{D}$-equivalences.

Proposition 5.7. *For any complete or cocomplete category $\mathscr{C}$, regarded as a derivator, a functor $u : A \to B$ is a $\mathscr{C}$-equivalence over I if:*

- *For each $i \in I$, the functor on fibers $u_i : A_i \to B_i$ induces a bijection on sets of connected components, $\pi_0(u_i) : \pi_0(A_i) \cong \pi_0(B_i)$.*

The converse holds for $\mathscr{C} = \mathsf{Set}$.

Proof. In the cocomplete case, we observe that for $v : B \to I$ where I is discrete, and $X \in \mathscr{C}^I$, we have $(v_! v^* X)_i = \pi_0(B_i) \cdot X_i$, the copower of X_i by the set $\pi_0(B_i)$. Thus the map $(vu)_! (vu)^* \to v_! v^*$ consists of copowers by $\pi_0(u_i)$, so it is an isomorphism if these functions are bijections. The converse when $\mathscr{C} = \mathsf{Set}$ follows by taking $X_i = 1$. □

In particular, the functor $(\cdot \rightrightarrows \cdot) \to \mathbb{1}$ above is a $\mathscr{C}$-equivalence for any such $\mathscr{C}$.

Definition 5.8. If $\mathscr{T}$ and $\mathscr{D}$ are prederivators and every $\mathscr{T}$-equivalence is a $\mathscr{D}$-equivalence, we say that $\mathscr{D}$ is **$\mathscr{T}$-local**.

Thus Proposition 5.7 says that *any complete or cocomplete category $\mathscr{C}$ is* Set*-local*. For many such $\mathscr{C}$ the converse also holds (i.e. Set is $\mathscr{C}$-local), but not all.

Proposition 5.9. *If $\mathscr{C}$ is a complete lattice, regarded as a derivator, then $u : A \to B$ is a $\mathscr{C}$-equivalence over I if:*

- *For each $i \in I$, if B_i is inhabited then so is A_i.*

The converse holds when $\mathscr{C} = \mathsf{Prop}$ is the poset of truth values. Thus, every complete lattice is Prop*-local.*

Proof. For $v : B \to I$ with I discrete, and $X \in \mathscr{C}^I$, we have $(v_! v^* X)_i = \bigvee_{b \in B_i} X_i$, and the join of a constant family (a copower in a lattice) depends only on the support of the indexing set. The converse when $\mathscr{C} = \mathsf{Prop}$ follows by taking $X_i = \top$. □

Remark 5.10. The condition in Proposition 5.9 is equivalent to saying that u_i induces an isomorphism of *supports* $\pi_{-1}(A_i) \cong \pi_{-1}(B_i)$, where $\pi_{-1}(C)$ is the subterminal set corresponding to the proposition "C is inhabited".

Remark 5.11. A functor $u : A \to B$ is an Set-equivalence over I if and only if it is a Set-equivalence over $\mathbb{1}$, since $\pi_0(A) \cong \sum_i \pi_0(A_i)$. However, this is not the case for Prop-equivalences.

Moving down one more categorical dimension, we have the trivial case:

Proposition 5.12. *If* Contr *denotes the terminal derivator, every functor is a* Contr*-equivalence.* □

The subtlety mentioned above is that our derivators of exact completions, though intuitively "1-categorical", are nevertheless not Set-local.

Proposition 5.13. *Let $\mathscr{E}$ be a complete category with small coproducts preserved by pullback. Then $u : A \to B$ is an $\mathscr{E}_{\mathsf{ex}}$-equivalence over I if the following hold:*

- *There is a function $s : B_0 \to A_0$.*
- *There is a function sending any $\beta : b \to b'$ to a zigzag in A from sb to sb' (and hence similarly for any zigzag in B).*
- *There is a function sending each $b \in B$ to a zigzag in B from b to usb.*
- *There is a function sending each $a \in A$ to a zigzag in A from a to sua.*

The converse holds if $\mathscr{E} = \mathsf{Set}$. Thus, every $\mathscr{E}_{\mathsf{ex}}$ is $\mathsf{Set}_{\mathsf{ex}}$-local.

Note that the existence of the zigzags, plus discreteness of I, ensures that s must also be a map over I, i.e. consist of functions $(B_i)_0 \to (A_i)_0$.

Proof. Let $u : A \to B$ satisfy the stated conditions and $v : B \to I$ a functor with I discrete. Let $X, Y \in \mathscr{E}_{\mathsf{ex}}(I)$, consisting essentially of an I-indexed family of pseudo-equivalence relations. We must show that u^* is fully faithful on morphisms between v^*X and v^*Y.

For faithfulness, suppose $f, g : v^*X \to v^*Y$ are morphism representatives and we have a witness of equality consisting of maps $h_a : X_{vua,0} \to Y_{vua,1}$. Then h_{sb}, for $b \in B$, witness that f and g are equal at objects of the form usb. And since v^*X and v^*Y both act as the identity on all morphisms of B, equality of components of f and g transfers, constructively, across all naturality squares. Thus, the assumed zigzags in B can be used to construct a witness that $f \sim g$.

For fullness, suppose $f : u^*v^*X \to u^*v^*Y$ is a morphism representative. Given $b \in B$, we obtain components $g_{b,0} = f_{sb,0}$ and $g_{b,1} = f_{sb,1}$ representing a morphism $g_b : (v^*X)_b \to (v^*Y)_b$. For any $\beta : b \to b'$ in B, by assumption we have a zigzag from sb to sb'; composing naturality squares along this zigzag we can construct a witness g_β making g a morphism representative $v^*X \to v^*Y$. Finally, for any $a \in A$, the assumption yields a zigzag from a to sua, which we can use to construct a witness that $u^*(g) \sim f$.

For the converse, suppose $u : A \to B$ is a $\mathsf{Set}_{\mathsf{ex}}$-equivalence over I, and let $X \in \mathsf{Set}_{\mathsf{ex}}(I)$ be constant at the terminal pseudo-equivalence relation. Then by the construction in Theorem 2.13, $(v_! \, v^*X)_i$ is the pseudo-equivalence relation on the set $(B_i)_0$ of objects of B_i freely generated by reflexivities and the arrows of B_i. Thus, its relations are essentially bracketed zigzags in B_i. Similarly, $((vu)_! \, (vu)^*X)_i$ is the set $(A_i)_0$ with relations being bracketed zigzags in A_i. The stated conditions are then (modulo adding and removing brackets, which is trivial) precisely what it means for these induced maps to be an isomorphism in $\mathsf{Set}_{\mathsf{ex}}(I)$. □

Remark 5.14. For $v : A \to \mathbb{1}$ and $T \in \mathsf{Set}_{\mathsf{ex}}(\mathbb{1})$ the terminal object, the pseudo-equivalence relation $v_! \, v^*T$ described above can be regarded as the *setoid of connected components* of A, so we might naturally denote it $\pi_0^{\mathsf{ex}}(A)$. Thus, the conditions of Proposition 5.13 are equivalent to saying that u induces an equivalence of such setoids, $\pi_0^{\mathsf{ex}}(u) : \pi_0^{\mathsf{ex}}(A) \cong \pi_0^{\mathsf{ex}}(B)$.

Note that the conditions in Proposition 5.13 are *stronger* than those in Proposition 5.7. Thus Set is $\mathsf{Set}_{\mathsf{ex}}$-local, but $\mathsf{Set}_{\mathsf{ex}}$ is not Set-local. Moreover, in the absence of choice, this inequality is strict;

Proposition 5.15. $\mathsf{Set}_{\mathsf{ex}}$ *is* Set*-local if and only if the axiom of choice holds.*

Proof. Let $p : E \to B$ be a surjection of sets. Regard B as a discrete groupoid, and make E a groupoid such that p is fully faithful (i.e. equip it with the kernel pair of p, regarded as an equivalence relation). Then $\pi_0(E) \cong \pi_0(B) = B$, so p is an Set-equivalence. But if it is a $\mathsf{Set}_{\mathsf{ex}}$-equivalence, then p is split. □

However, the functor $(\cdot \rightrightarrows \cdot) \to \mathbb{1}$ is a $\mathsf{Set}_{\mathsf{ex}}$-equivalence, so $\mathsf{Set}_{\mathsf{ex}}$ is still intuitively "1-categorical". Two more examples will help to clarify the situation.

Example 5.16. Let $\mathscr{E}$ be a category with small products and coproducts. For a small category A, let $\mathscr{E}_{\mathsf{pos}}(A)$ denote the following category:

- An object consists of an object $X_a \in \mathscr{E}$ for all $a \in A$, together with a morphism $X_\alpha : X_a \to X_{a'}$ for all $\alpha : a \to a'$ in A.
- A morphism representative $f : X \to Y$ consists of a morphism $f_a : X_a \to Y_a$ for all $a \in A$. Any two such representatives are equivalent.

Thus $\mathscr{E}_{\mathsf{pos}}(A)$ is a (large) preorder, and in particular $\mathscr{E}_{\mathsf{pos}}(\mathbb{1})$ is (equivalent to) the preorder reflection of $\mathscr{E}$.

Arguments like those of Lemmas 4.2 and 4.3, but simpler, show that $\mathscr{E}_{\mathsf{pos}}$ satisfies (Der1) and (Der2). The constant diagram functor $(p_A)^* : \mathscr{E}_{\mathsf{pos}}(\mathbb{1}) \to \mathscr{E}_{\mathsf{pos}}(A)$ has a right and left adjoint given by taking products and coproducts respectively. We can then use these to construct pointwise Kan extensions as in Lemma 4.4, showing that $\mathscr{E}_{\mathsf{pos}}$ is a derivator. If binary products in $\mathscr{E}$ preserve coproducts in each variable, then $\mathscr{E}_{\mathsf{pos}}$ is a distributive derivator.

Proposition 5.17. *For $\mathscr{E}$ a category with small products and coproducts, a functor $u : A \to B$ is an $\mathscr{E}_{\mathsf{pos}}$-equivalence over I if:*

- *There is a function $B_0 \to A_0$ over I.*

The converse holds if $\mathscr{E} =$ Set.

Proof. For $X \in \mathscr{E}_{\mathsf{pos}}(I)$, by construction $(v_! \, v^* X)_i$ is the copower $(B_i)_0 \cdot X_i$, and similarly $((vu)_! \, (vu)^* X)_i = (A_i)_0 \cdot X_i$. Thus, the condition given yields a map backwards, hence an isomorphism in $\mathscr{E}_{\mathsf{pos}}(I)$. The converse follows by letting X_i be the terminal object. □

Remark 5.18. For a category A, write $\pi^{\mathsf{ex}}_{-1}(A)$ for A_0 regarded as an object of $\mathsf{Set}_{\mathsf{pos}}$, a sort of "setoid support". Then similarly to Remarks 5.10 and 5.14, the condition of Proposition 5.17 is equivalent to saying that u induces an equivalence of such supports, $\pi^{\mathsf{ex}}_{-1}(u) : \pi^{\mathsf{ex}}_{-1}(A) \cong \pi^{\mathsf{ex}}_{-1}(B)$.

As with the relationship between Set and $\mathsf{Set}_{\mathsf{ex}}$, the condition of Proposition 5.17 is stronger than that of Proposition 5.9. Thus Prop is $\mathsf{Set}_{\mathsf{pos}}$-local, but $\mathsf{Set}_{\mathsf{pos}}$ is not Prop-local. Indeed, $\mathsf{Set}_{\mathsf{pos}}$ is not even Set-local, though it is still "0-categorical" in that the functor $(\mathbb{1} + \mathbb{1}) \to \mathbb{1}$ is a $\mathsf{Set}_{\mathsf{pos}}$-equivalence.

It is true that $\mathsf{Set}_{\mathsf{pos}}$ is $\mathsf{Set}_{\mathsf{ex}}$-local. It is also local for the following intermediate derivator $\mathsf{Set}_{\mathsf{reg}}$:

Example 5.19. For a category $\mathscr{E}$ with finite limits, its reg/lex completion $\mathscr{E}_{\mathsf{reg}}$ is defined to be the full subcategory of $\mathscr{E}_{\mathsf{ex}}$ on the pseudo-equivalence relations that are kernel pairs. Such kernel pairs are, in particular, actual equivalence relations; and if $\mathscr{E}$ is already exact (like Set), then they include all the equivalence relations.

If we define $\mathscr{E}_{\mathsf{reg}}(A)$ as a similar subcategory of $\mathscr{E}_{\mathsf{ex}}(A)$, then it is closed under the limits of Theorem 2.11 but not the colimits of Theorem 2.13. However, the (regular epi, mono) factorization of a pseudo-equivalence relation always yields an equivalence relation. Thus, if $\mathscr{E}$ is exact, then $\mathscr{E}_{\mathsf{reg}}(A)$ is reflective in $\mathscr{E}_{\mathsf{ex}}(A)$; so we can define left Kan extensions in $\mathscr{E}_{\mathsf{reg}}(A)$ by composing the reflection with those of $\mathscr{E}_{\mathsf{ex}}(A)$. Since the reflections commute with the restriction functors, (Der4) holds.

In sum, if $\mathscr{E}$ is complete, exact, and has small coproducts preserved by pullback, then $\mathscr{E}_{\mathsf{reg}}(A)$ is a derivator. Since products preserve image factorizations, $\mathscr{E}_{\mathsf{reg}}(A)$ is also a distributive derivator.

Remark 5.20. Analogously to Remark 2.3, we can view $\mathsf{Set}_{\mathsf{reg}}$ as the homotopy category of the bicategory of "bidiscrete groupoids" (those in which any two parallel arrows are equal). See [KP14, Kin98].

Proposition 5.21. *Let $\mathscr{E}$ be complete, exact, and have small coproducts preserved by pullback. Then $u : A \to B$ is an $\mathscr{E}_{\mathsf{reg}}$-equivalence over I if the following hold:*

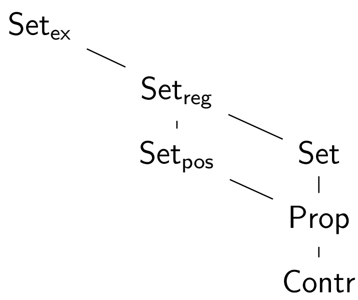


Figure 1: Part of the preorder of relative free cocompletions of a point

- *There is a function* $s : B_0 \to A_0$.
- *For any* $\beta : b \to b'$ *in* B*, there exists a zigzag in* A *from* sb *to* sb' *(and hence likewise for any zigzag in* B*).*
- *For any* $b \in B$*, there exists a zigzag in* B *from* b *to* usb.
- *For any* $a \in A$*, there exists a zigzag in* A *from* a *to* sua.

The converse holds if $\mathscr{E} = \mathsf{Set}$. *Thus, every* $\mathscr{E}_{\mathsf{reg}}$ *is* $\mathsf{Set}_{\mathsf{reg}}$*-local.*

Proof. In $\mathscr{E}_{\mathsf{reg}}$, witnesses of equality are unique when they exist; thus it suffices to assert that they exist rather than specifying them functionally. Hence, we can copy the proof of Proposition 5.13, but without specified zigzags. □

Remark 5.22. Continuing the trend of Remarks 5.10, 5.14 and 5.18, for $v : A \to \mathbb{1}$ and $T \in \mathsf{Set}_{\mathsf{reg}}(\mathbb{1})$ the terminal object, the object $v_! v^* T \in \mathsf{Set}_{\mathsf{reg}}(\mathbb{1})$ is an "equivalence relation of connected components" that we may denote $\pi_0^{\mathsf{reg}}(A)$, and the conditions of Proposition 5.21 are equivalent to saying that u induces an equivalence of such, $\pi_0^{\mathsf{reg}}(u) : \pi_0^{\mathsf{reg}}(A) \cong \pi_0^{\mathsf{reg}}(B)$.

Clearly $\mathsf{Set}_{\mathsf{reg}}$ is $\mathsf{Set}_{\mathsf{ex}}$-local while Set is $\mathsf{Set}_{\mathsf{reg}}$-local. Also, $\mathsf{Set}_{\mathsf{pos}}$ is $\mathsf{Set}_{\mathsf{reg}}$-local. Thus, in the preorder where $\mathscr{D}_1 \le \mathscr{D}_2$ means "$\mathscr{D}_1$ is $\mathscr{D}_2$-local", we have the fragment shown in Figure 1. In §8 we will speculate about extending this upwards.

6. $\mathsf{Set}_{\mathsf{ex}}$ is a relative free cocompletion

We will show each of the derivators in Figure 1 is the free cocompletion of a point in the sub-universe of derivators that are local for it, in the following sense.

Definition 6.1. A left derivator $\mathscr{T}$ is a **relative free cocompletion of a point** if for any $\mathscr{T}$*-local* left derivator $\mathscr{D}$, the "evaluation at the terminal object $* \in \mathscr{T}(\mathbb{1})$" functor

$$\mathsf{Hom}_{\mathsf{cc}}(\mathscr{T}, \mathscr{D}) \to \mathscr{D}(\mathbb{1})$$

is an equivalence of categories.

How do we prove such universal properties? As observed by [Hel88], there is a derivator that can easily be shown to map into any other left derivator, namely the complete and cocomplete category $\mathcal{C}at$. More generally, we have:

Lemma 6.2. *For any left derivator $\mathscr{D}$, there is a morphism $\odot : \mathcal{C}at \times \mathscr{D} \to \mathscr{D}$. Moreover, if $u : E \to F$ is a morphism in $\mathcal{C}at^A$ such that $\sum_a u_a$ is a $\mathscr{D}$-equivalence over A_0, then $u \odot M : E \odot M \to F \odot M$ is an isomorphism in $\mathscr{D}$.*

Proof. As in [GPS14, Theorem 3.11], to define such a two-variable morphism it suffices to give functors $\odot : \mathcal{C}at(A) \times \mathscr{D}(B) \to \mathscr{D}(A \times B)$ that vary pseudonaturally in A and B. The components $\odot_A : (\mathcal{C}at \times \mathscr{D})(A) = \mathcal{C}at(A) \times \mathscr{D}(A) \to \mathscr{D}(A)$ of a pseudonatural transformation are then obtained by composing with restriction along the diagonal $A \to A \times A$.

Given $E \in \mathcal{C}at^A$, let $p_E : \int E \to A$ be its Grothendieck construction, which is a split opfibration. Then we have the following diagram:

$$\begin{array}{ccc} \int E \times B & \xrightarrow{\pi_2} & B \\ {\scriptstyle p_E \times 1}\downarrow & & \\ A \times B & & \end{array}$$

Therefore, given $M \in \mathscr{D}(B)$, we can define

$$E \odot M = (p_E \times 1)_! \, (\pi_2)^*(M) \quad \in \mathscr{D}(A \times B).$$

Pseudonaturality is immediate.

Now suppose $u : E \to F$ is such that $\sum_a u_a$ is a $\mathscr{D}$-equivalence over A_0. To show that $u \odot M$ is an isomorphism, by (Der2) it suffices to restrict it to $A_0 \times B_0$. And since $p_E \times 1$ and $p_F \times 1$ are opfibrations, by Lemma 3.7 the following square is exact, along with the analogous one for F:

$$\begin{array}{ccc} (\sum_a E_a) \times B_0 & \longrightarrow & \int E \times B \\ {\scriptstyle (p_E)_0 \times 1}\downarrow & \lrcorner & \downarrow{\scriptstyle p_E \times 1} \\ A_0 \times B_0 & \longrightarrow & A \times B \end{array}$$

Moreover, the restriction of $M \in \mathscr{D}(B)$ to $(\sum_a E_a) \times B_0$ factors through its restriction to B_0 and also to $A_0 \times B_0$. Now the desired statement simply reduces to the fact that $(\sum_a u_a) \times 1_{B_0}$ is a $\mathscr{D}$-equivalence over $A_0 \times B_0$, which follows from the hypothesis and Lemma 5.3. □

Since left extensions in $\mathscr{D}$ commute with each other, $\odot$ is cocontinuous in its second variable. If it were also cocontinuous in its first variable, defining $E \mapsto E \odot 1$ would give a cocontinuous morphism $\mathcal{C}at \to \mathscr{D}$. This is not generally the case, essentially because $\int E$ is a *oplax* colimit of E rather than a homotopy colimit. However, we can make it true by "localizing $\mathcal{C}at$" in a way that forces such oplax colimits to become "colimits" in a derivator.

Classically, there is a universal way to do this, using the Thomason model structure [Tho80] on $\mathcal{C}at$, which is Quillen equivalent to simplicial sets. This is roughly the approach of [Hel88, Cis06, Cis04]. Model categories for relative free cocompletions of a point can then be obtained by left Bousfield localization. It would be interesting to see whether this approach can be reproduced constructively, but we will not attempt to do that here.

Instead, since Figure 1 contains a maximal element $\mathsf{Set}_{\mathsf{ex}}$, we will just prove explicitly that $\mathsf{Set}_{\mathsf{ex}}$ is a relative free cocompletion of a point, and then deduce the same property for the other derivators in Figure 1. Of course, a more abstract approach will probably be required to extend these results to higher dimensions.

Definition 6.3. For $X \in \mathsf{Set}_{\mathsf{ex}}(\mathbb{1})$, let $\widetilde{X}$ be the category with object set $X_0 + X_1$ and nonidentity arrows $\xi \to s\xi$ and $\xi \to t\xi$ for all $\xi \in X_1$.

Then $\widetilde{X} \odot 1 \in \mathsf{Set}_{\mathsf{ex}}(\mathbb{1})$ is the set $X_0 + X_1$ with pseudo-equivalence relation freely generated by $\xi \sim s\xi$ and $\xi \sim t\xi$.

Lemma 6.4. *$\widetilde{X} \odot 1$ is isomorphic to X in* $\mathsf{Set}_{\mathsf{ex}}$.

Proof. In one direction, we have a map $X \to \widetilde{X} \odot 1$ that is the inclusion of the summand X_0, and sending a witness ξ that $s\xi \sim t\xi$ to the composite witness $s\xi \sim \xi \sim t\xi$. In the other direction, we can act as the identity on X_0 and send $\xi \in X_1$ to $s\xi$ (say), with the generating witnesses of equality $\xi \sim s\xi$ sent to the reflexivity witness for $s\xi$, and the generating witnesses $\xi \sim t\xi$ sent to the witness ξ that $s\xi \sim t\xi$. The composite on X_0 is the identity, while the composite on $\widetilde{X} \odot 1$ is equal to the identity via the witnesses $\xi \sim s\xi$. □

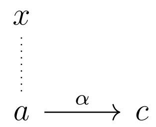
We would like to represent a coherent diagram $X \in \mathsf{Set}_{\mathsf{ex}}(A)$ similarly by an object of $\mathcal{C}at^A$. However, since X is only functorial up to witnesses of equality, a naive pointwise construction does not produce a functor (or even a pseudofunctor) $\widetilde{X} : A \to \mathcal{C}at$. More importantly, the morphisms in $\mathsf{Set}_{\mathsf{ex}}(A)$ are not natural or even pseudonatural for this construction. Thus, we need some kind of strictification.

Remark 6.5. At this point we could attempt to proceed in roughly the same way that derivators are usually constructed in classical homotopy theory (see e.g. [Cis10b] or [Gro13, Proposition 1.30]), by building some kind of *model category* of setoids and morphism representatives whose homotopy category would be $\mathsf{Set}_{\mathsf{ex}}(\mathbb{1})$. We would then lift this model category to a model structure on *strict* A-shaped diagrams and strict natural transformations, whose homotopy category would be equivalent to $\mathsf{Set}_{\mathsf{ex}}(A)$. The machinery of Quillen adjunctions would then give an alternative approach to the construction of the derivator $\mathsf{Set}_{\mathsf{ex}}$, and the strictness of the morphisms in the model category would make it easier to lift the construction $\widetilde{X}$ to diagrams.

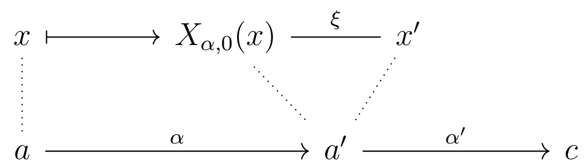
The first step of this approach to $\mathsf{Set}_{\mathsf{ex}}$ was achieved in [Hen20, §4.1] with the construction of a *weak model category* of setoids whose homotopy category is $\mathsf{Set}_{\mathsf{ex}}(\mathbb{1})$. However, the lifting of weak model structures to categories of diagrams does not exist in the literature yet. Rather than develop this machinery here, I have elected to give an explicit construction, which has the additional advantage of being more accessible to a reader without experience in model category theory. But it should be clear that this is only feasible because of the very simple nature of the derivator $\mathsf{Set}_{\mathsf{ex}}$; more complicated examples require more advanced techniques.

Definition 6.6. For $X \in \mathsf{Set}_{\mathsf{ex}}(A)$, let $\widetilde{X} : A \to \mathcal{C}at$ be the following functor.

- For $c \in A$, the category $\widetilde{X}_c$ has two classes of objects:
  (i) Triples (a, α, x) where $\alpha : a \to c$ and $x \in X_{a,0}$, which can be drawn as:



  (ii) Tuples $(a, \alpha, x, a', \alpha', x', \xi)$ where $\alpha : a \to a'$ in A and $x \in X_{a,0}$, while $\alpha' : a' \to c$ in A and $x' \in X_{a',0}$, and $\xi \in X_{a',1}$ satisfies $s\xi = X_{\alpha,0}(x)$ and $t\xi = x'$, as shown:



- The nonidentity morphisms in $\widetilde{X}_c$ are of the form

$$(a, \alpha, x, a', \alpha', x', \xi) \to (a, \alpha'\alpha, x) \qquad \text{and}$$
$$(a, \alpha, x, a', \alpha', x', \xi) \to (a', \alpha', x').$$

- For $\gamma : c \to c'$ in A, the functor $\widetilde{X}_\gamma : \widetilde{X}_c \to \widetilde{X}_{c'}$ is defined on objects by

$$\widetilde{X}_\gamma(a, \alpha, x) = (a, \gamma\alpha, x)$$
$$\widetilde{X}_\gamma(a, \alpha, x, a', \alpha', x', \xi) = (a, \alpha, x, a', \gamma\alpha', x', \xi)$$

For a morphism representative $f : X \to Y$, let $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ be the natural transformation whose component $\widetilde{f}_c : \widetilde{X}_c \to \widetilde{Y}_c$ is defined on objects by

$$\widetilde{f}_c(a, \alpha, x) = (a, \alpha, f_{a,0}(x))$$
$$\widetilde{f}_c(a, \alpha, x, a', \alpha', x', \xi) = (a, \alpha, f_{a,0}(x), a', \alpha', f_{a,0}(x'), m(f_\alpha(x), f_{a,1}(\xi))),$$

where m is the transitivity operation on equality witnesses in $Y_{a'}$.

Lemma 6.7. *For any $X \in \mathsf{Set}_{\mathsf{ex}}(A)$ we have a specified isomorphism $\widetilde{X} \odot * \cong X$, where $* \in \mathsf{Set}_{\mathsf{ex}}(\mathbb{1})$ is the terminal object. Similarly, for any morphism representative $f : X \to Y$ we have a specified witness that the evident square commutes:*

$$\begin{array}{ccc} \widetilde{X} \odot * & \xrightarrow{\cong} & X \\ {\scriptstyle \widetilde{f}\odot *}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \widetilde{Y} \odot * & \xrightarrow[\cong]{} & Y \end{array}$$

Proof. By definition, $\widetilde{X} \odot *$ is the left extension of the constant diagram at $*$ along the functor $p_{\widetilde{X}} : \int \widetilde{X} \to A$. Since this functor is a cloven (indeed, split) opfibration, this extension can be computed using colimits, as in Theorem 2.13, over the fibers. The fiber over $c \in A$ is the category $\widetilde{X}_c$ as defined above. Thus, $(\widetilde{X} \odot *)_c$ has underlying set consisting of the triples (a, α, x) and tuples $(a, \alpha, x, a', \alpha', x', \xi)$, with pseudo-equivalence relation freely generated by witnesses $(a, \alpha'\alpha, x) \sim (a, \alpha, x, a', \alpha', x', \xi)$ and $(a, \alpha, x, a', \alpha', x', \xi) \sim (a', \alpha', x')$.

In one direction, we define a morphism representative $g : \widetilde{X} \odot * \to X$ by

$$\begin{aligned} g_{c,0}(a, \alpha, x) &= X_{\alpha,0}(x) \\ g_{c,0}(a, \alpha, x, a', \alpha', x', \xi) &= X_{\alpha',0}(x') \\ g_{c,1}((a, \alpha'\alpha, x) \sim (a, \alpha, x, a', \alpha', x', \xi)) &= m(X_{\alpha,\alpha'}(x), X_{\alpha',1}(\xi)) \\ g_{c,1}((a, \alpha, x, a', \alpha', x', \xi) \sim (a', \alpha', x')) &= r(X_{\alpha',0}(x')) \\ g_{\gamma}(a, \alpha, x) &= X_{\alpha,gm}(x) \\ g_{\gamma}(a, \alpha, x, a', \alpha', x', \xi) &= X_{\alpha',gm}(x') \end{aligned}$$

(extending to all of $(\widetilde{X} \odot *)_{c,1}$ by freeness). In the other direction, we define a morphism representative $h : X \to \widetilde{X} \odot *$ by

$$\begin{aligned} h_{c,0}(x) &= (c, 1_c, x) \\ h_{c,1}(\xi) &= (c, 1_c, s\xi, c, 1_c, t\xi, m(X_r(s\xi), \xi)) \\ h_{\gamma}(x) &= \big((c, \gamma, x) \sim (c, \gamma, x, c', 1_{c'}, X_{\gamma,0}(x), r(X_{\gamma,0}(x))) \sim (c', 1_{c'}, X_{\gamma,0}(x))\big). \end{aligned}$$

The composite in one direction, $h \circ g$, sends (a, α, x) to $(c, 1_c, X_{\alpha,0}(x))$, for which we have

$$(a,\alpha,x) \sim (a,\alpha,x,c,1_c,X_{\alpha,0}(x),r(X_{\alpha,0}(x))) \sim (c,1_c,X_{\alpha,0}(x)).$$

And it sends $(a,\alpha,x,a',\alpha',x',\xi)$ to $(c,1_c,X_{\alpha',0}(x'))$, for which we have

$$(a,\alpha,x,a',\alpha',x',\xi) \sim (a',\alpha',x')$$

together with a zigzag like that above. And the composite in the other direction, $g \circ h$, sends $x \in X_c$ to $X_{1_c,0}(x)$, which is identified with x by $X_r(x)$. Thus, g and h together represent an isomorphism in $\mathsf{Set}_{\mathsf{ex}}(A)$.

For the second statement, note that $\widetilde{f} \odot 1 : \widetilde{X} \odot * \to \widetilde{Y} \odot *$ sends (a,α,x) to $(a,\alpha,f_{a,0}(x))$. Thus, the composite $X \to \widetilde{X} \odot * \to \widetilde{Y} \odot *$ and $X \to Y \to \widetilde{Y} \odot *$ both send x to $(c,1_c,f_{c,0}(x))$. □

We emphasize, however, that the construction $f \mapsto \widetilde{f}$ does not define any kind of functor yet. Specifically, it is only defined on morphism *representatives*, which do not compose associatively, and the composite of two morphisms of the form $\widetilde{f}$ may no longer be of that form. Thus, we need some way to also detect witnesses of equality at the categorical level. For this we use the following "path space".

Definition 6.8. For $X \in \mathsf{Set}_{\mathsf{ex}}(A)$, let $\wp\widetilde{X} : A \to \mathcal{C}at$ be the following functor.

- For $c \in A$, the category $\wp\widetilde{X}_c$ hos two classes of objects:
  - (i) Triples (a,α,ζ) where $\alpha : a \to c$ and $\zeta \in X_{a,1}$.
  - (ii) Tuples $(a,\alpha,\zeta,a',\alpha',\zeta',\xi,\xi')$ where $\alpha : a \to a'$ and $\zeta \in X_{a,1}$, while $\alpha' : a' \to c$ and $\zeta' \in X_{a',1}$, and $\xi,\xi' \in X_{a',1}$ satisfy $s\xi = X_{\alpha,0}(s\zeta)$, $t\xi = s\zeta'$, $s\xi' = X_{\alpha,0}(t\zeta)$, and $t\xi' = t\zeta'$.
- The nonidentity morphisms in $\wp\widetilde{X}_c$ are of the form:

$$(a,\alpha,\zeta,a',\alpha',\zeta',\xi,\xi') \to (a,\alpha'\alpha,\zeta)$$
$$(a,\alpha,\zeta,a',\alpha',\zeta',\xi,\xi') \to (a',\alpha',\zeta').$$

- For $\gamma : c \to c'$ in A, the functor $\wp\widetilde{X}_\gamma : \wp\widetilde{X}_c \to \wp\widetilde{X}_{c'}$ is defined on objects by

$$\widetilde{X}_\gamma(a, \alpha, \zeta) = (a, \gamma\alpha, \zeta)$$
$$\widetilde{X}_\gamma(a, \alpha, \zeta, a', \alpha', \zeta', \xi, \xi') = (a, \alpha, \zeta, a', \gamma\alpha', \zeta', \xi, \xi').$$

There are two natural transformations $\sigma, \tau : \wp\widetilde{X} \to \widetilde{X}$ defined on objects by

$$\sigma_c(a, \alpha, \zeta) = (a, \alpha, s\zeta)$$
$$\sigma_c(a, \alpha, \zeta, a', \alpha', \zeta', \xi, \xi') = (a, \alpha, s\zeta, a', \alpha', s\zeta', \xi)$$
$$\tau_c(a, \alpha, \zeta) = (a, \alpha, t\zeta)$$
$$\tau_c(a, \alpha, \zeta, a', \alpha', \zeta', \xi, \xi') = (a, \alpha, t\zeta, a', \alpha', t\zeta', \xi').$$

Finally, there is a natural transformation $\rho : \widetilde{X} \to \wp\widetilde{X}$ defined on objects by

$$\rho_c(a, \alpha, x) = (a, \alpha, rx)$$
$$\rho_c(a, \alpha, x, a', \alpha', x', \xi) = (a, \alpha, rx, a', \alpha', rx', \xi, \xi),$$

where r is the witness of reflexivity in X.

Lemma 6.9. *We have $\sigma\rho = \tau\rho = 1_{\widetilde{X}}$, and the functors $\sum_a \rho_a$, $\sum_a \sigma_a$, and $\sum_a \tau_a$ are $\mathsf{Set}_{\mathsf{ex}}$-equivalences over A_0.*

Proof. The first statement is evident. For the second, by the 2-out-of-3 property (Lemma 5.6) it suffices to show that $\sum_a \rho_a$ is a $\mathsf{Set}_{\mathsf{ex}}$-equivalence. Since $(\sum_a \sigma_a) \circ (\sum_a \rho_a) = 1$, it suffices to connect each object of $\wp\widetilde{X}_a$ to its image under $\rho_a\sigma_a$ with a zigzag.

First we need a zigzag between (a, α, ζ) and $(a, \alpha, rs\zeta)$, for which we can use

$$(a, \alpha, rs\zeta) \leftarrow (a, 1_a, \zeta, a, \alpha, rs\zeta, rs\zeta, \zeta) \to (a, \alpha, \zeta).$$

Next we need a zigzag between

$$(a, \alpha, \zeta, a', \alpha', \zeta', \xi, \xi') \quad \text{and} \quad (a, \alpha, rs\zeta, a', \alpha', rs\zeta', \xi, \xi),$$

for which we compose the zigzag constructed as above for (a', α', ζ') with the maps

$$(a, \alpha, rs\zeta, a', \alpha', rs\zeta', \xi, \xi) \to (a', \alpha', rs\zeta') \qquad \text{and}$$
$$(a', \alpha', \zeta') \leftarrow (a, \alpha, \zeta, a', \alpha', \zeta', \xi, \xi'). \qquad \square$$

In homotopical language, Lemma 6.9 says that $\wp\widetilde{X}$ is a "path space" relative to the $\mathsf{Set}_{\mathsf{ex}}$-equivalences.

Definition 6.10. For morphisms $\phi, \psi : \widetilde{X} \to \widetilde{Y}$ in $\mathcal{C}at^A$, a **right homotopy** $\phi \sim \psi$ is a morphism $\theta : \widetilde{X} \to \wp\widetilde{Y}$ such that $\sigma\theta = \phi$ and $\tau\theta = \psi$.

Lemma 6.11. *If $f, g : X \to Y$ are morphism representatives in $\mathsf{Set}_{\mathsf{ex}}(A)$ and $h : f \sim g$ is a witness of equality, then $\widetilde{f}$ and $\widetilde{g}$ are right homotopic.*

Proof. We define $\widetilde{h} : \widetilde{X} \to \wp\widetilde{Y}$ on objects by $\widetilde{h}(a, \alpha, x) = (a, \alpha, h_a(x))$ and

$$\widetilde{h}(a, \alpha, x, a', \alpha', x', \xi) = \\ (a, \alpha, h_a(x), a', \alpha', h_{a'}(x'), m(f_\alpha(x), f_{a,1}(\xi)), m(g_\alpha(x), g_{a,1}(\xi))). \quad \square$$

We can now use this path-space to remedy the problems of functoriality.

Lemma 6.12. *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphism representatives in $\mathsf{Set}_{\mathsf{ex}}(A)$, then $\widetilde{g}\widetilde{f}$ and $\widetilde{gf}$ are right homotopic.*

Proof. By definition, we have

$$\begin{aligned}
\widetilde{g}_c(\widetilde{f}_c(a, \alpha, x)) &= (a, \alpha, g_{a,0}(f_{a,0}(x))) \\
\widetilde{g}_c(\widetilde{f}_c(a, \alpha, x, a', \alpha', x', \xi)) &= (a, \alpha, g_{a,0}(f_{a,0}(x)), a', \alpha', g_{a,0}(f_{a,0}(x')), \\
&\qquad m(g_\alpha(f_{a,0}(x)), g_{a,1}(m(f_\alpha(x), f_{a,1}(\xi))))) \\
\widetilde{gf}_c(a, \alpha, x) &= (a, \alpha, g_{a,0}(f_{a,0}(x))) \\
\widetilde{gf}_c(a, \alpha, x, a', \alpha', x', \xi) &= (a, \alpha, g_{a,0}(f_{a,0}(x)), a', \alpha', g_{a,0}(f_{a,0}(x')), \\
&\qquad m((gf)_\alpha(x), g_{a,1}(f_{a,1}(\xi))))
\end{aligned}$$

where $(gf)_\alpha$ is the composite witness of naturality. Now define $\widetilde{m} : \widetilde{X} \to \wp\widetilde{Z}$ by

$$\begin{aligned}
\widetilde{m}_c(a, \alpha, x) &= (a, \alpha, r(g_{a,0}(f_{a,0}(x)))) \\
\widetilde{m}_c(a, \alpha, x, a', \alpha', x', \xi) &= (a, \alpha, r(g_{a,0}(f_{a,0}(x))), a', \alpha', r(g_{a,0}(f_{a,0}(x'))), \\
&\qquad m(g_\alpha(f_{a,0}(x)), g_{a,1}(m(f_\alpha(x), f_{a,1}(\xi)))), \\
&\qquad\qquad m((gf)_\alpha(x), g_{a,1}(f_{a,1}(\xi)))). \quad \square
\end{aligned}$$

Lemma 6.13. *For $X \in \mathsf{Set}_{\mathsf{ex}}(A)$, the morphisms $\widetilde{1_X}$ and $1_{\widetilde{X}}$ are right homotopic.*

Proof. We can define $\widetilde{i} : \widetilde{X} \to \wp\widetilde{X}$ by

$$\widetilde{i}_c(a, \alpha, x) = (a, \alpha, rx)$$
$$\widetilde{i}_c(a, \alpha, x, a', \alpha', x', \xi) = (a, \alpha, rx, a', \alpha', rx', m((1_X)_\alpha(x), \xi), \xi). \qquad \square$$

Now we show that right homotopies are inverted in $\mathsf{Set}_{\mathsf{ex}}$-local derivators.

Lemma 6.14. *Let $\mathscr{D}$ be a $\mathsf{Set}_{\mathsf{ex}}$-local left derivator. For any $X \in \mathsf{Set}_{\mathsf{ex}}(A)$ and $M \in \mathscr{D}(B)$, we have*

$$\sigma_X \odot 1_M = \tau_X \odot 1_M$$

as morphisms $\wp\widetilde{X} \odot M \to \widetilde{X} \odot M$ in $\mathscr{D}(A \times B)$. Therefore, if $\phi, \psi : \widetilde{X} \to \widetilde{Y}$ are right homotopic, then $\phi \odot \ell = \psi \odot \ell$ for any ℓ.

Proof. By functoriality of $\odot$, we have

$$(\sigma \odot 1_M) \circ (\rho \odot 1_M) = (\tau \odot 1_M) \circ (\rho \odot 1_M).$$

However, by Lemma 6.9, $\sum_a \rho_a$ is a $\mathsf{Set}_{\mathsf{ex}}$-equivalence over A_0, and hence also a $\mathscr{D}$-equivalence since $\mathscr{D}$ is $\mathsf{Set}_{\mathsf{ex}}$-local. Therefore, by Lemma 6.2, $\rho \odot 1_M$ is an isomorphism, and thus cancellable. So $\sigma \odot 1_M = \tau \odot 1_M$.

For the last statement, a right homotopy is a θ with $\sigma\theta = \phi$ and $\tau\theta = \psi$. Thus, $\sigma \odot 1_M = \tau \odot 1_M$ implies $\phi \odot \ell = \psi \odot \ell$ by functoriality. $\square$

This implies that $\odot$ descends from $\mathcal{C}at$ to $\mathsf{Set}_{\mathsf{ex}}$ via $\widetilde{(-)}$.

Definition 6.15. For $X \in \mathsf{Set}_{\mathsf{ex}}(A)$ and $M \in \mathscr{D}(B)$, define $X \mathbin{\widetilde{\odot}} M = \widetilde{X} \odot M$. Similarly, for $f : X \to Y$ in $\mathsf{Set}_{\mathsf{ex}}(A)$ and $\ell : M \to N$ in $\mathscr{D}(B)$, we choose a representative of f and define $f \mathbin{\widetilde{\odot}} \ell = \widetilde{f} \odot \ell$.

Proposition 6.16. *If $\mathscr{D}$ is $\mathsf{Set}_{\mathsf{ex}}$-local, the definition of $f \mathbin{\widetilde{\odot}} g$ is independent of the choice of representative for f, and defines a functor*

$$\widetilde{\odot} : \mathsf{Set}_{\mathsf{ex}}(A) \times \mathscr{D}(B) \to \mathscr{D}(A \times B).$$

Proof. By Lemma 6.11, any witness of equality $h : f \sim g$ between two morphism representatives yields a right homotopy $\widetilde{f} \sim \widetilde{g}$. Thus, by Lemma 6.14, we have $f \mathbin{\widetilde{\odot}} \ell = \widetilde{f} \odot \ell = \widetilde{g} \odot \ell = g \mathbin{\widetilde{\odot}} \ell$. Functoriality on $\mathsf{Set}_{\mathsf{ex}}(A)$ follows similarly from Lemmas 6.12 and 6.13. $\square$

Now we must show that these functors vary pseudonaturally in A and B.

Definition 6.17. For $X \in \mathsf{Set}_{\mathsf{ex}}(B)$ and $u : A \to B$, let $\omega_{X,u} : \widetilde{u^*X} \to u^*\widetilde{X}$ be the natural transformation defined on objects by

$$\omega_{X,u}(a, \alpha, x) = (ua, u\alpha, x)$$
$$\omega_{X,u}(a, \alpha, x, a', \alpha', x', \xi) = (ua, u\alpha, x, ua', u\alpha', x', \xi).$$

Lemma 6.18. *Let $X, Y \in \mathsf{Set}_{\mathsf{ex}}(C)$ and $A \xrightarrow{v} B \xrightarrow{u} C$, and $f : X \to Y$ a morphism representative. Then the map $\omega_{X,1_A} : \widetilde{X} \to \widetilde{X}$ is equal to $1_{\widetilde{X}}$, and the following diagrams commute:*

$$\begin{array}{ccc} \widetilde{v^*u^*X} & \xrightarrow{\omega_{u^*X,v}} & v^*(\widetilde{u^*X}) \\ & \searrow^{\omega_{X,uv}} & \downarrow{\scriptstyle v^*\omega_{X,u}} \\ & & v^*u^*\widetilde{X} \end{array} \qquad \begin{array}{ccc} \widetilde{u^*X} & \xrightarrow{\widetilde{u^*f}} & \widetilde{u^*Y} \\ {\scriptstyle \omega_{u^*X,u}}\downarrow & & \downarrow{\scriptstyle \omega_{Y,u}} \\ u^*\widetilde{X} & \xrightarrow[u^*\widetilde{f}]{} & u^*\widetilde{Y}. \end{array}$$

Proof. By inspection of the definitions. $\square$

Lemma 6.19. *The functor $\sum_a \omega_{X,u,a}$ is a $\mathsf{Set}_{\mathsf{ex}}$-equivalence over A_0.*

Proof. First, we must define $s : u^*\widetilde{X}_0 \to \widetilde{u^*X}_0$. The first kind of object of $(u^*\widetilde{X})_c$ is (b, β, x) for $\beta : b \to uc$ and $x \in X_{b,0}$. We send this to $(c, 1_c, X_{\beta,0}(x))$ in $(\widetilde{u^*X})_c$. The second kind of object of $(u^*\widetilde{X})_c$ is

$$(b, \beta, x, b', \beta', x', \xi)$$

for $\beta : b \to b'$, $x \in X_{b,0}$, $\beta' : b' \to uc$, $x' \in X_{b',0}$, and $\xi \in X_{b',1}$ a witness that $X_{\beta,0}(x) \sim x'$. We send this to

$$(c, 1_c, X_{\beta'\beta,0}(x), c, 1_c, X_{\beta',0}(x'), m(X_{\beta,\beta'}(x), X_{\beta',1}(\xi)))$$

in $(\widetilde{u^*X})_c$, where $X_{\beta,\beta'}(x)$ is a functoriality witness of X.

Second, we must send morphisms in $(u^*\widetilde{X})_c$ to zigzags in $(\widetilde{u^*X})_c$. We send a morphism $(b,\beta,x,b',\beta',x',\xi) \to (b,\beta'\beta,x)$ to the one-morphism zigzag

$$(c,1_c,X_{\beta'\beta,0}(x),c,1_c,X_{\beta',0}(x'),X_{\beta',1}(\xi)) \to (c,1_c,X_{\beta'\beta,0}(x)),$$

and similarly we send a morphism $(b,\beta,x,b',\beta',x',\xi) \to (b',\beta',x')$ to the one-morphism zigzag

$$(c,1_c,X_{\beta',0}(X_{\beta,0}(x)),c,1_c,X_{\beta',0}(x'),X_{\beta',1}(\xi)) \to (c,1_c,X_{\beta',0}(x'))$$

Third, we must relate each object of $(u^*\widetilde{X})_c$ by a zigzag to its roundtrip image. For (b,β,x), we have

$$(b,\beta,x) \leftarrow (b,\beta,x,uc,1_{uc},X_{\beta,0}(x),r(X_{\beta,0}(x))) \to (uc,1_{uc},X_{\beta,0}(x)),$$

while for $(b,\beta,x,b',\beta',x',\xi)$ we have

$$\begin{aligned}(b,\beta,x,b',\beta',x',\xi) \to (b,\beta'\beta,x) \leftarrow \bullet \to (uc,1_{uc},X_{\beta'\beta,0}(x))\\ \leftarrow (uc,1_{uc},X_{\beta'\beta,0}(x),uc,1_{uc},X_{\beta',0}(x'),X_{\beta',1}(\xi))\end{aligned}$$

where the middle zigzag is as above.

Fourth and finally, we must relate each object of $(\widetilde{u^*X})_c$ by a zigzag to its roundtrip image. For (a,α,x) we have

$$(a,\alpha,x) \leftarrow (a,\alpha,x,c,1_c,X_{u\alpha,0}(x),r(X_{u\alpha,0}(x))) \to (c,1_c,X_{u\alpha,0}(x)),$$

while for $(a,\alpha,x,a',\alpha',x',\xi)$ we have

$$\begin{aligned}(a,\alpha,x,a',\alpha',x',\xi) \to (a,\alpha'\alpha,x) \leftarrow \bullet \to (c,1_c,X_{u(\alpha'\alpha),0}(x))\\ \leftarrow (c,1_c,X_{u(\alpha'\alpha),0}(x),c,1_c,X_{u\alpha',0}(x'),m(X_{u\alpha,u\alpha'}(x),X_{u\alpha',1}(\xi)))\end{aligned}$$

where again the middle zigzag is as above. □

Proposition 6.20. *For any $\mathsf{Set}_{\mathsf{ex}}$-local left derivator $\mathscr{D}$, the functors $\widetilde{\odot}$ of Proposition 6.16 vary pseudonaturally in $A, B \in \mathcal{C}at$. Therefore, they define a morphism of derivators*

$$\widetilde{\odot} : \mathsf{Set}_{\mathsf{ex}} \times \mathscr{D} \to \mathscr{D}.$$

Proof. For $u : A \to A'$ and $v : B \to B'$, the pseudonaturality constraint is

$$\begin{aligned} u^*X \mathbin{\widetilde{\odot}} v^*M = (\widetilde{u^*X} \odot v^*M) &\xrightarrow[\sim]{\omega} (u^*\widetilde{X} \odot v^*M) \\ &\cong (u^* \times v)(\widetilde{X} \odot M) = (u^* \times v)(X \mathbin{\widetilde{\odot}} M). \end{aligned}$$

The map induced by $\omega_{X,u}$ is an isomorphism by Lemma 6.19, while the second isomorphism is the pseudofunctoriality of $\odot$. The axioms for a pseudonatural transformation follow from those of $\odot$ and Lemma 6.18. □

Proposition 6.21. *The above-defined* $\widetilde{\odot}$ *is cocontinuous in both variables.*

Proof. Cocontinuity in the second argument follows from that of $\odot$. For cocontinuity in the first argument, by (the two-variable version of) Lemma 3.9 it suffices to show that for $u : A \to I$ in $\mathcal{C}at$, with I discrete, and $X \in \mathsf{Set}_{\mathsf{ex}}(A)$ and $M \in \mathscr{D}(B)$, the transformation $(u \times 1)_!(X \mathbin{\widetilde{\odot}} M) \to u_!X \mathbin{\widetilde{\odot}} M$ is an isomorphism.

Since I is discrete, we can let $(u_!X)_i$ be the colimit of X restricted to A_i as constructed in Theorem 2.13, and put these together into a coherent diagram $u_!X$. We then have the adjunction unit $\eta : X \to u^*u_!X$, consisting of the injections into these colimits. The map we must show to be an isomorphism is the composite

$$\begin{aligned} (u \times 1)_!(\widetilde{X} \odot M) &\xrightarrow{\widetilde{\eta}} (u \times 1)_!(\widetilde{u^*u_!X} \odot M) \\ &\xrightarrow{\omega} (u \times 1)_!(u^*(\widetilde{u_!X}) \odot M) \\ &\xrightarrow{\sim} (u \times 1)_!\,(u \times 1)^*(\widetilde{u_!X} \odot M) \\ &\to \widetilde{u_!X} \odot M. \end{aligned}$$

Now, the composite $\omega\widetilde{\eta}$ induces a map $\int\omega\widetilde{\eta}$ on Grothendieck constructions:

$$\begin{array}{ccc} \int\widetilde{X} & \xrightarrow{\int\omega\widetilde{\eta}} & \int\widetilde{u_!X} \\ {\scriptstyle p_{\widetilde{X}}}\downarrow & & \downarrow{\scriptstyle p_{\widetilde{u_!X}}} \\ A & \xrightarrow[u]{} & I, \end{array}$$

and the desired map can then be identified with

$$(u\times 1)_!\,(p_{\widetilde{X}}\times 1)_!\,(\pi_2)^*M \xrightarrow{\sim} (p_{\widetilde{u_!X}}\times 1)_!\,(\textstyle\int\omega\widetilde{\eta})_!\,(\pi_2)^*M \to (p_{\widetilde{u_!X}}\times 1)_!\,(\pi_2)^*M.$$

where both projections $\int \widetilde{X} \times B \to B$ and $\int \widetilde{u_! X} \times B \to B$ are denoted π_2. Therefore, as in the proof of Lemma 6.2, it will suffice to show that $\int \omega\widetilde{\eta} : \int \widetilde{X} \to \int \widetilde{u_! X}$ is a $\mathsf{Set}_{\mathsf{ex}}$-equivalence over I.[13]

The objects of $\int \widetilde{X}$ are those of $\widetilde{X}_c$ for all $c \in A$, hence of the two forms (a, α, x) and $(a, \alpha, x, a', \alpha', x', \xi)$ as usual. But its morphisms incorporate the morphisms of A, by the Grothendieck construction; thus we have

$$(a, \gamma\alpha'\alpha, x) \leftarrow (a, \alpha, x, a', \alpha', x', \xi) \to (a', \gamma\alpha', x') \tag{6.22}$$

for any $\gamma : c \to c'$.

Since I is discrete, $\int \widetilde{u_! X}$ is essentially (up to an inessential modification by X_r witnesses) the simple construction of Definition 6.3 applied to $u_! X$. Thus, as objects it has both elements of $(u_! X)_0$, which are pairs (a, x) with $x \in X_{a,0}$, and elements of $(u_! X)_1$. By construction of $u_! X$, the latter sort of element is a sequence

$$\Xi = (a_0, x_0, \alpha_1, \xi_1, a_1, x_1, \alpha_2, \xi_2, \ldots, \alpha_n, \xi_n, a_n, x_n),$$

where each $x_k \in X_{a_k,0}$, and for each k either

- $\alpha_k : a_{k-1} \to a_k$ and ξ_k is a witness that $X_{\alpha_k,0}(x_{k-1}) \sim x_k$, or
- $\alpha_k : a_k \to a_{k-1}$ and ξ_k is a witness that $X_{\alpha_k,0}(x_k) \sim x_{k-1}$.

Such a sequence then comes with morphisms to both (a_0, x_0) and (a_n, x_n).

The functor $\int \omega\widetilde{\eta}$ is defined on objects by

$$\int \omega\widetilde{\eta}(a, \alpha, x) = (a, x)$$
$$\int \omega\widetilde{\eta}(a, \alpha, x, a', \alpha', x', \xi) = (a, x, \alpha, \xi, a', x').$$

As always, we use the characterization of Proposition 5.13.

First, to define a function $s : (\int \widetilde{u_! X})_0 \to (\int \widetilde{X})_0$, we send (a, x) to $(a, 1_a, x)$, and a zigzag sequence Ξ as above to $(a_0, 1_{a_0}, x_0)$.

Second, we can send the morphism $\Xi \to (a_0, x_0)$ to the identity. Before deciding what to do with the morphism $\Xi \to (a_n, x_n)$, note that given α :

[13] This explains our earlier comment that the failure of $\odot$ to be cocontinuous in its first variable is due to $\int$ being an oplax colimit rather than a homotopy colimit.

$a \to a'$ and ξ a witness that $X_{\alpha,0}(x) \sim x'$, we have a zigzag

$$\begin{aligned}(a, 1_a, x) \leftarrow (a, 1_a, x, a, 1_a, x, r(x)) \to (a, \alpha, x)\\ \leftarrow (a, \alpha, x, a', 1_{a'}, x', \xi) \to (a', 1_{a'}, x')\end{aligned}$$

in which the second morphism uses the extra flexibility of (6.22), with $\gamma = \alpha$. Now by concatenating these zigzags, possibly reversed as necessary, we obtain a zigzag from $(a_0, 1_{a_0}, x_0)$ to $(a_n, 1_{a_n}, x_n)$ from any Ξ, which is what we needed.

Third, we need to relate any object of $\int \widetilde{u_! X}$ to its roundtrip image by a zigzag. But an object of the form (a, x) is equal to its roundtrip image, while Ξ comes with a basic morpism to its roundtrip image (a_0, x_0).

Fourth and finally, we must relate any object of $\int \widetilde{X}$ to its roundtrip image. The roundtrip image of (a, α, x) is $(a, 1_a, x)$, for which we have

$$(a, 1_a, x) \leftarrow (a, 1_a, x, a, 1_a, x, r(x)) \to (a, \alpha, x).$$

And the roundtrip image of $(a, \alpha, x, a', \alpha', x', \xi)$ is $(a, 1_a, x)$, for which we have the previous zigzag together with

$$(a, \alpha, x) \leftarrow (a, 1_a, x, a, \alpha, x, r(x)) \to (a, \alpha'\alpha, x) \leftarrow (a, \alpha, x, a', \alpha', x', \xi)$$

Here the middle map uses the extra flexibility of (6.22) with $\gamma = \alpha'$. □

Corollary 6.23. *For any* $\mathsf{Set}_{\mathsf{ex}}$*-local left derivator* $\mathscr{D}$ *and any* $M \in \mathscr{D}(\mathbb{1})$*, there is a cocontinuous morphism* $(- \mathbin{\widetilde{\odot}} M) : \mathsf{Set}_{\mathsf{ex}} \to \mathscr{D}$ *such that* $* \mathbin{\widetilde{\odot}} M \cong M$*, where* $* \in \mathsf{Set}_{\mathsf{ex}}(\mathbb{1})$ *is the terminal object.*

Proof. It remains to show that $* \mathbin{\widetilde{\odot}} M \cong M$. By definition, $* \mathbin{\widetilde{\odot}} M = \widetilde{*} \odot M$, where $\widetilde{*}$ is $(\cdot \rightrightarrows \cdot)$. But the functor $\widetilde{*} \to \mathbb{1}$ is, as noted previously, a $\mathsf{Set}_{\mathsf{ex}}$-equivalence. Thus the induced map $\widetilde{*} \odot M \to M$ is an isomorphism, since $\mathscr{D}$ is $\mathsf{Set}_{\mathsf{ex}}$-local. □

Theorem 6.24. *If* $\mathscr{D}$ *is a* $\mathsf{Set}_{\mathsf{ex}}$*-local left derivator, then the functor*

$$\mathsf{Hom}_{\mathsf{cc}}(\mathsf{Set}_{\mathsf{ex}}, \mathscr{D}) \to \mathscr{D}(\mathbb{1}),$$

induced by evaluation at $* \in \mathsf{Set}_{\mathsf{ex}}(\mathbb{1})$*, is an equivalence of categories. In other words,* $\mathsf{Set}_{\mathsf{ex}}$ *is a relative free cocompletion of a point.*

Proof. The construction of Corollary 6.23 is functorial and the isomorphism is natural. Thus, it suffices to construct, for any cocontinuous $G : \mathsf{Set}_{\mathsf{ex}} \to \mathscr{D}$, an isomorphism $GX \cong X \mathbin{\widetilde{\odot}} G(*)$, natural in G and in $X \in \mathsf{Set}_{\mathsf{ex}}(A)$. For this we have

$$\begin{aligned} GX &\cong G(\widetilde{X} \odot *) \\ &= G((p_{\widetilde{X}} \times 1)_! \, (\pi_2)^*(*)) \\ &\cong (p_{\widetilde{X}} \times 1)_! \, (\pi_2)^* G(*) \\ &= \widetilde{X} \odot G(*) \\ &= X \mathbin{\widetilde{\odot}} G(*), \end{aligned}$$

where the first isomorphism is Lemma 6.7, and the second is because G is cocontinuous. Naturality in G is evident, while naturality in X follows from the second part of Lemma 6.7. □

7. Other relative free cocompletions

Once we have one relative free cocompletion — in our case, $\mathsf{Set}_{\mathsf{ex}}$— it is much easier to construct other $\mathsf{Set}_{\mathsf{ex}}$-local ones. First we note that if $\mathscr{D}$ is distributive (Definition 3.13), then the whole two-variable morphism $\widetilde{\odot}$: $\mathsf{Set}_{\mathsf{ex}} \times \mathscr{D} \to \mathscr{D}$ is determined by the functor $L : \mathsf{Set}_{\mathsf{ex}} \to \mathscr{D}$ defined by

$$LX = X \mathbin{\widetilde{\odot}} *.$$

Lemma 7.1. *If $\mathscr{D}$ is distributive and $\mathsf{Set}_{\mathsf{ex}}$-local, we have a natural isomorphism*

$$X \mathbin{\widetilde{\odot}} M \cong LX \times M$$

for $X \in \mathsf{Set}_{\mathsf{ex}}(A)$ and $M \in \mathscr{D}(B)$.

Here $\times$ on the right-hand side denotes the functor $\mathscr{D}(A) \times \mathscr{D}(B) \to \mathscr{D}(A \times B)$ induced by the cartesian product of $\mathscr{D}$.

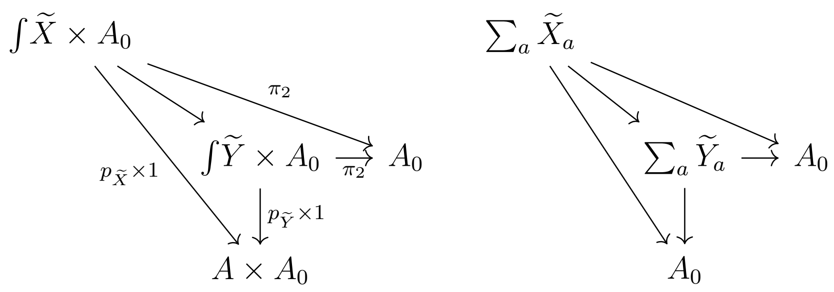


Figure 2: Diagrams for the proof of Corollary 7.2

Proof. By definition,

$$\begin{aligned} X \mathbin{\widetilde{\odot}} M &= (p_{\widetilde{X}} \times 1)_! \, \pi_2^*(M) \\ &\cong (p_{\widetilde{X}} \times 1)_! \, \pi_2^*(* \times M) \\ &\cong (p_{\widetilde{X}} \times 1)_! \, \pi_2^*(*) \times M \qquad \text{(by distributivity)} \\ &= (X \mathbin{\widetilde{\odot}} *) \times M \\ &= LX \times M. \end{aligned}$$ □

Corollary 7.2. *If $\mathscr{D}$ is distributive and* $\mathsf{Set}_{\mathsf{ex}}$*-local, and $f : X \to Y$ is a morphism representative in* $\mathsf{Set}_{\mathsf{ex}}(A)$ *such that Lf is an isomorphism, then $\sum_a \widetilde{f}_a$ is a $\mathscr{D}$-equivalence over A_0.*

Proof. By Lemma 7.1, the assumption implies that $f \mathbin{\widetilde{\odot}} M$ is an isomorphism for any $M \in \mathscr{D}(B)$. In particular, for $M \in \mathscr{D}(A_0)$ the induced map

$$(p_{\widetilde{X}} \times 1)_! \, \pi_2^*(M) \to (p_{\widetilde{Y}} \times 1)_! \, \pi_2^*(M)$$

is an isomorphism, where the functors fit into the diagram on the left of Figure 2.

The two functors $p_{\widetilde{X}} \times 1$ and $p_{\widetilde{Y}} \times 1$ are split opfibrations, and the pullback of $\widetilde{f} \times 1$ along $(\iota_A \times 1_{A_0}) : A_0 \to A \times A_0$ is $\sum_a \widetilde{f}_a$. Thus, the corresponding map for the diagram on the right of Figure 2 is also an isomorphism; but this is precisely to say that $\sum_a \widetilde{f}_a$ is a $\mathscr{D}$-equivalence over A_0. □

Theorem 7.3. *If $\mathscr{T}$ is $\mathsf{Set}_{\mathsf{ex}}$-local and distributive, and $L : \mathsf{Set}_{\mathsf{ex}} \to \mathscr{T}$ has a right adjoint with invertible counit, then $\mathscr{T}$ is a relative free cocompletion of a point.*

Proof. Let $\mathscr{D}$ be a $\mathscr{T}$-local left derivator; we must show that the precomposition functor $(-\circ L) : \mathsf{Hom}_{\mathsf{cc}}(\mathscr{T}, \mathscr{D}) \to \mathsf{Hom}_{\mathsf{cc}}(\mathsf{Set}_{\mathsf{ex}}, \mathscr{D})$ is an equivalence. We have a commutative square

$$\begin{array}{ccc} \mathsf{Hom}_{\mathsf{cc}}(\mathscr{T}, \mathscr{D}) & \xrightarrow{(-\circ L)} & \mathsf{Hom}_{\mathsf{cc}}(\mathsf{Set}_{\mathsf{ex}}, \mathscr{D}) \\ \downarrow & & \downarrow \\ \mathsf{Hom}(\mathscr{T}, \mathscr{D}) & \xrightarrow[(-\circ L)]{} & \mathsf{Hom}(\mathsf{Set}_{\mathsf{ex}}, \mathscr{D}) \end{array}$$

in which the vertical functors are fully faithful. But the bottom functor has a left adjoint $(- \circ R)$, where R is the right adjoint of L, with invertible unit, and hence is also fully faithful. Thus the top functor is also fully faithful. So it suffices to show it is split essentially surjective, i.e. that any cocontinuous $G : \mathsf{Set}_{\mathsf{ex}} \to \mathscr{D}$ factors through L, up to isomorphism, by a specified cocontinuous morphism.

To start with, we have a canonical morphism $GR : \mathscr{T} \to \mathscr{D}$. We also have a unit map $\eta : 1_{\mathsf{Set}_{\mathsf{ex}}} \to RL$, and since the counit of the adjunction is invertible, $L\eta$ is an isomorphism. Thus, by Corollary 7.2, for any $X \in \mathsf{Set}_{\mathsf{ex}}(A)$, if we choose a representative for η_X, then $\sum_a (\widetilde{\eta_X})_a$ is a $\mathscr{T}$-equivalence over A_0. Since $\mathscr{D}$ is $\mathscr{T}$-local, this means it is also a $\mathscr{D}$-equivalence. And since G is of the form $(- \mathbin{\widetilde{\odot}} M)$ for some $M \in \mathscr{D}(\mathbb{1})$, by Theorem 6.24, it follows that G also inverts η_X. In other words, $G\eta$ is an isomorphism $G \cong GRL$.

It remains to show that GR is cocontinuous. This means to show that the mate $u_!GR \to GRu_!$ of the isomorphism $GRu^* \cong u^*GR$ is again an isomorphism. The latter isomorphism is the pasting composite of the following squares:

$$\begin{array}{ccccccc} \mathscr{T}(A) & \xrightarrow{R} & \mathsf{Set}_{\mathsf{ex}}(A) & = & \mathsf{Set}_{\mathsf{ex}}(A) & \xrightarrow{G} & \mathscr{D} \\ {\scriptstyle u^*}\uparrow & \cong & \uparrow{\scriptstyle Ru^*} & \cong & \uparrow{\scriptstyle u^*} & \cong & \uparrow{\scriptstyle u^*} \\ \mathscr{T}(B) & = & \mathscr{T}(B) & \xrightarrow[R]{} & \mathsf{Set}_{\mathsf{ex}}(B) & \xrightarrow[G]{} & \mathscr{D} \end{array}$$

Therefore, by the functoriality of mates, its mate is the pasting composite of the following squares:

$$\begin{array}{ccccccc}
\mathscr{T}(A) & \xrightarrow{R} & \mathsf{Set}_{\mathsf{ex}}(A) & = & \mathsf{Set}_{\mathsf{ex}}(A) & \xrightarrow{G} & \mathscr{D} \\
\downarrow u_! & \swArrow & \downarrow u_! L & \swArrow & \downarrow u_! & \swArrow & \downarrow u_! \\
\mathscr{T}(B) & = & \mathscr{T}(B) & \xrightarrow[R]{} & \mathsf{Set}_{\mathsf{ex}}(B) & \xrightarrow[G]{} & \mathscr{D}
\end{array}$$

The left-hand square is the counit $LR \to 1_{\mathscr{T}}$, which is an isomorphism by assumption. The right-hand square is an isomorphism since G is cocontinuous. Finally, the middle square is the unit $1_{\mathsf{Set}_{\mathsf{ex}}} \to RL$, which as we just showed is inverted by G. Thus, the pasting composite is also an isomorphism, so GR is cocontinuous. □

Remark 7.4. If we omit the hypothesis of distributivity in Theorem 7.3, the same argument implies that $\mathscr{T}$ is a *localization* of $\mathsf{Set}_{\mathsf{ex}}$, in the sense that the precomposition functor $\mathsf{Hom}_{\mathsf{cc}}(\mathscr{T}, \mathscr{D}) \to \mathsf{Hom}_{\mathsf{cc}}(\mathsf{Set}_{\mathsf{ex}}, \mathscr{D})$ is fully faithful, and its full image consists of the morphisms $\mathsf{Set}_{\mathsf{ex}} \to \mathscr{D}$ that invert the same morphisms that are inverted by $L : \mathsf{Set}_{\mathsf{ex}} \to \mathscr{T}$. (More abstractly, this can be expressed as a *coinverter* in the 2-category of derivators: a 2-categorical colimit that universally forces some 2-cell to become invertible.) Distributivity enables us to reexpress this as $\mathscr{T}$ being a relative free cocompletion of a point, without explicit reference to $L : \mathsf{Set}_{\mathsf{ex}} \to \mathscr{T}$.

We have already observed that all the derivators in Figure 1 are $\mathsf{Set}_{\mathsf{ex}}$-local and distributive. Thus, it suffices to show that their L-functors all have right adjoints.

Example 7.5. For $\mathscr{T} = \mathsf{Set}$, L computes the quotient of each pseudo-equivalence relation in a coherent diagram, obtaining an ordinary diagram of sets. This has a right adjoint that assigns to any set the identity (pseudo-)equivalence relation on it, of which it is the quotient. Thus, Set is a relative free cocompletion of a point.

Example 7.6. For $\mathscr{T} = \mathsf{Prop}$, L computes the support $\pi_{-1}(X_0)$ of each pseudo-equivalence relation in a coherent diagram. Since the quotient of a pseudo-equivalence relation is inhabited if and only if X_0 is, this factors through Set via the standard support functor $\mathsf{Set} \to \mathsf{Prop}$. The latter has a right adjoint assigning to each proposition the corresponding subsingleton, which is its own support. Thus, Prop is a relative free cocompletion of a point.

We leave the trivial case $\mathscr{T} = \mathsf{Contr}$ to the reader.

Example 7.7. For $\mathscr{T} = \mathsf{Set}_{\mathsf{pos}}$, L sends each pseudo-equivalence relation to $X_0 + X_1$, which is isomorphic in $\mathsf{Set}_{\mathsf{pos}}$ to X_0. This has a right adjoint that sends each object X of $\mathsf{Set}_{\mathsf{pos}}$ to the *full* (pseudo-)-equivalence relation on it, i.e. $X_0 = X$ and $X_1 = X \times X$. The counit is evidently an isomorphism, so $\mathsf{Set}_{\mathsf{pos}}$ is a free cocompletion of a point.

Example 7.8. Finally, for $\mathscr{T} = \mathsf{Set}_{\mathsf{reg}}$, L sends each pseudo-equivalence in a coherent diagram to its image, which is an actual equivalence relation. This has a right adjoint that sends each equivalence relation to itself as a pseudo-equivalence relation. Thus, $\mathsf{Set}_{\mathsf{reg}}$ is also a free cocompletion of a point.

8. Conclusions and speculations

We have constructed three different relative free cocompletions of a point, Set, $\mathsf{Set}_{\mathsf{reg}}$, and $\mathsf{Set}_{\mathsf{ex}}$, which are nevertheless all intuitively "1-categorical". Similarly, both Prop and $\mathsf{Set}_{\mathsf{pos}}$ are intuitively "0-categorical" (i.e. posetal). Thus we may reasonably wonder, what happens in higher dimensions? The obvious candidate for a 2-categorical (or, more precisely, $(2, 1)$-categorical) relative free cocompletion of a point is a derivator of groupoids; but we have multiple notions of groupoid.

On the one hand, we have the standard notion of groupoid, with hom-sets. These should yield a derivator Gpd: the objects of $\mathsf{Gpd}(A)$ are pseudo-functors $A \to \mathsf{Gpd}$, and its morphisms are isomorphism classes of pseudo-natural transformations. In particular, the isomorphisms in the derivator Gpd would be the *equivalences* of groupoids, in the usual constructive sense with a specified pseudo-inverse functor.

On another hand, we can consider $\mathcal{E}$-*groupoids*, "groupoids enriched over setoids" (see e.g. [HS98, BD08] for $\mathcal{E}$-categories). These should yield a derivator EGpd. And there is a third notion in between, of groupoids enriched over equivalence relations, which should yield a derivator RGpd. It seems likely that we should have an analogous three notions of n-groupoid for all finite n, where the top level is enriched either over Set, $\mathsf{Set}_{\mathsf{reg}}$, or $\mathsf{Set}_{\mathsf{ex}}$. But in the limit $n \to \infty$, where there is no longer a "top level", it seems reasonable to expect the difference to disappear, so that there would be only one absolute free cocompletion of a point Space.

Conjecture 8.1. *One can constructively define an absolute free cocompletion of a point using some kind of cubical sets, simplicial sets, or semisimplicial sets, along with three reflective localizations of it for each finite n, consisting of the n-groupoids enriched over sets, setoids, and equivalence relations at the top dimension.*

However, something funny happens with the locality preorder at dimension 2. Just as the Set-equivalences are the functors inducing an isomorphism under the reflection π_0 of categories into sets, we expect the Gpd-equivalences should be the functors inducing an equivalence under the reflection Π_1 of categories into groupoids. But since $\Pi_1(A)$ has the same set of objects as A, if $f : A \to B$ is a Gpd-equivalence then we have an actual function $B_0 \to A_0$, suggesting that a Gpd-equivalence should also be not just a Set-equivalence but a $\mathsf{Set}_{\mathsf{reg}}$-equivalence. Thus $\mathsf{Set}_{\mathsf{reg}}$ should be Gpd-local, and similarly we expect $\mathsf{Set}_{\mathsf{ex}}$ to be RGpd-local, leading to the placements of Gpd, RGpd, and EGpd in the extension of Figure 1 shown in Figure 3.

The diagonal rows[14] of this diagram are at constant "categorical dimension" while moving vertically downwards passes to the subcategory of truncated objects. That is, the categories of subterminal objects in Set and $\mathsf{Set}_{\mathsf{reg}}$ are equivalent to Prop and $\mathsf{Set}_{\mathsf{pos}}$ respectively, and we expect the categories of 0-truncated objects in Gpd and RGpd to be equivalent to $\mathsf{Set}_{\mathsf{reg}}$ and $\mathsf{Set}_{\mathsf{ex}}$ respectively. Since $\mathsf{Set}_{\mathsf{pos}}$ is also the category of subterminal objects in $\mathsf{Set}_{\mathsf{ex}}$, and $\mathsf{Set}_{\mathsf{ex}}$ should also be the category of 0-truncated objects in EGpd, it is natural to extend the diagram further to the left in a way that "stabilizes" after a certain number of steps, as we have done in gray. One can thus view "exact completion" as adding an additional dimension to the Baez–Dolan "periodic table of n-categories" [BD95], which stabilizes along the n-categorical row at the $(n + 2)^{\mathrm{nd}}$ stage.

It is worth noting that although the derivators in the "middle" of this diagram are, like all the others, relative free cocompletions of a point, they are not as well-endowed with exactness properties. For instance, Set and $\mathsf{Set}_{\mathsf{ex}}$

[14]They are diagonal rather than horizontal, of course, so that the picture is still a sort of "Hasse diagram" of the locality relation (although we do not mean to exclude the possible existence of further intermediate objects not drawn).

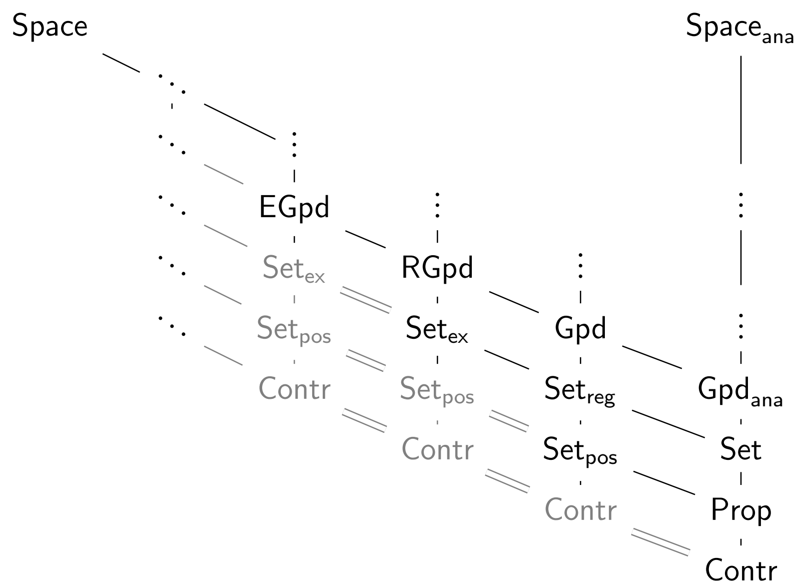


Figure 3: A conjectural enlargement of Figure 1

are both exact, but $\mathsf{Set_{reg}}$ is not: an internal equivalence relation in $\mathsf{Set_{reg}}$ is a *pseudo*-equivalence relation in Set, but it can only be effective in $\mathsf{Set_{reg}}$ if it is an actual equivalence relation. Similarly, but perhaps more surprisingly, Gpd is not exact as a $(2, 1)$-category (in a sense like that of [Str82]): for if it were, its subcategory of 0-truncated objects would be exact as a 1-category, but this subcategory is $\mathsf{Set_{reg}}$.

I expect RGpd to also fail to be $(2, 1)$-exact, though less obviously since its subcategory of 0-truncated objects should be $\mathsf{Set_{ex}}$, which is 1-exact. But $\mathsf{Set_{ex}}$ should also be the subcategory of 0-truncated objects in EGpd, which should be $(2, 1)$-exact. This is analogous to how $\mathsf{Set_{pos}}$ is the subcategory of subterminal objects in both $\mathsf{Set_{ex}}$ and $\mathsf{Set_{reg}}$, though only the former is 1-exact.

Is there a different 2-dimensional relative free cocompletion of a point whose category of 0-truncated objects is Set? To guess what this might be, note that in the parts of Figure 3 that we understand precisely so far, moving to the right can be achieved by passing to a localization. For instance, if we localize $\mathsf{Set_{pos}}$ by inverting the surjections, we obtain Prop. Similarly, if in $\mathsf{Set_{ex}}$ we invert the morphisms $f : X \to Y$ that reflect equality (in the sense that if there exists a witness that $f_0(x) \sim f_0(x')$ then there exists a witness that $x \sim x'$) and such that f_0 is split surjective, we obtain $\mathsf{Set_{reg}}$. If we further invert the morphisms that reflect equality and such that f_0 is merely surjective, we obtain Set.

Analogously, it is natural to guess that RGpd should be obtainable from EGpd by inverting functors that are split-surjective on objects, split-full on morphisms, and reflect equality of parallel morphisms; while Gpd should be similarly obtainable from RGpd by inverting functors that are split-surjective on objects, merely full on morphisms, and reflect equality of parallel morphisms. This suggests that the “missing link” should be obtained from Gpd by inverting the functors that are fully faithful and merely surjective on objects. This is equivalent to inverting the *weak equivalences*: functors that are fully faithful and essentially surjective.[15] The morphisms in this localization are *anafunctors* [Mak96, Bar06, Rob12, Rob18], so we denote it $\mathsf{Gpd_{ana}}$.

Similarly, if we present Set as a localization of $\mathsf{Set_{ex}}$, we could call its morphisms *anafunctions* and write $\mathsf{Set} \simeq (\mathsf{Set_{ex}})_{\mathsf{ana}}$. Equivalently, we can

[15] Recall that every weak equivalence is an equivalence if and only if the axiom of choice holds.

observe that since Set is already exact, it is equivalent to its own exact completion *as a regular category*, i.e. $\mathsf{Set} \simeq \mathsf{Set}_{\mathsf{ex/reg}}$; in general we can present the ex/reg completion as consisting of setoids or equivalence relations with anafunctions between them ("total and functional relations"). This suggests that the missing link $\mathsf{Gpd}_{\mathsf{ana}}$ should be the "$(2,1)$-exact completion of Set as a regular category". This makes sense because the definition of $\mathsf{Gpd}_{\mathsf{ana}}$, unlike that of Gpd, incorporates some information about the regular structure of Set, i.e. the surjective functions of sets.

There are, however, issues with actually performing the localization leading to the hypothetical $\mathsf{Gpd}_{\mathsf{ana}}$. In particular, unlike RGpd and Gpd, it is not a *reflective* localization of EGpd. Worse, even in ZF set theory, with excluded middle but no choice, it is impossible to prove that $\mathsf{Gpd}_{\mathsf{ana}}$ is locally small, cartesian closed, or complete [aK17], and hence it seems unlikely to be a derivator. (This also implies that it cannot be presented by any sort of model category, although weaker structures like a fibration or cofibration category are a possibility.) However, it may be easier to obtain at least a *left* derivator of this sort, with colimits but not necessarily limits.

Conjecture 8.2. *There is a left derivator* $\mathsf{Gpd}_{\mathsf{ana}}$ *composed of groupoids and anafunctors. Moreover:*

- $\mathsf{Gpd}_{\mathsf{ana}}$ *is a relative free cocompletion of a point, and is "$(2,1)$-exact".*
- *Every weak equivalence of categories is a* $\mathsf{Gpd}_{\mathsf{ana}}$*-equivalence.*
- Set *is* $\mathsf{Gpd}_{\mathsf{ana}}$*-local, but* $\mathsf{Set}_{\mathsf{reg}}$ *and* $\mathsf{Set}_{\mathsf{pos}}$ *are not.*
- *The subcategory of 0-truncated objects in* $\mathsf{Gpd}_{\mathsf{ana}}$ *is* Set.

Of course, we can ask analogous questions about n-groupoids for $2 \leq n \leq \infty$.

Conjecture 8.3. *There is a left derivator* $\mathsf{Space}_{\mathsf{ana}}$ *composed of "∞-groupoids and anafunctors". Moreover:*

- $\mathsf{Space}_{\mathsf{ana}}$ *is a relative free cocompletion of a point, and is "$(\infty,1)$-exact".*
- *Every weak equivalence of categories is an* $\mathsf{Space}_{\mathsf{ana}}$*-equivalence.*

- Set *and* $\mathsf{Gpd}_{\mathsf{ana}}$ *are* $\mathsf{Space}_{\mathsf{ana}}$*-local, but* $\mathsf{Set}_{\mathsf{pos}}$, $\mathsf{Set}_{\mathsf{reg}}$*, and* Gpd *are not.*
- *The subcategory of 1-truncated objects in* $\mathsf{Space}_{\mathsf{ana}}$ *is* $\mathsf{Gpd}_{\mathsf{ana}}$.

These conjectural derivators $\mathsf{Gpd}_{\mathsf{ana}}$ and $\mathsf{Space}_{\mathsf{ana}}$ are closely related to the issue raised in §1 that perhaps our definition of derivator is wrong: maybe we should use $\mathcal{C}at_{\mathsf{ana}}$ instead of $\mathcal{C}at$.[16] Since $\mathcal{C}at_{\mathsf{ana}}$ is equivalent to the bicategory obtained by inverting the weak equivalence functors in $\mathcal{C}at$, a natural definition of **ana-derivator** would be simply as a derivator such that $u^* : \mathscr{D}(B) \to \mathscr{D}(A)$ is a (perhaps weak) equivalence whenever $u : A \to B$ is a weak equivalence.

Of the derivators considered in this paper, Set and Prop are ana-derivators, while it seems that the others are not (though I do not have a formal proof). For instance, let $u : A \to B$ be a weak equivalence functor with B discrete, and $X \in \mathsf{Set}_{\mathsf{pos}}(B)$. Then $(u_* u^* X)_b$ is the power $X_b^{u^{-1}(b)_0}$ of the set X_b by the objects in the u-preimage of b. The adjunction unit $X \to u_* u^* X$ consists of the diagonals $X_b \to X_b^{u^{-1}(b)_0}$, but there seems no way to define a family of functions in the other direction without choosing elements of the fibers to give factors to project onto.

Conjecture 8.4. $\mathsf{Gpd}_{\mathsf{ana}}$ *and* $\mathsf{Space}_{\mathsf{ana}}$ *are left ana-derivators. Moreover,* $\mathsf{Space}_{\mathsf{ana}}$ *is the free cocompletion of a point among ana-derivators, while* $\mathsf{Gpd}_{\mathsf{ana}}$ *is a relative free cocompletion of a point therein.*

Remark 8.5. It is natural to wonder, if the right-hand column in Figure 3 has its "own notion of derivator" (the above-defined ana-derivators), why is that not the case for the other columns? In fact, there are other ways to vary the notion of derivator. The notion of derivator we have worked with corresponds roughly to the second column from the right; but one could also replace the 2-categories $\mathcal{C}at$ and/or $\mathcal{C}AT$ by $\mathcal{E}$-2-categories of $\mathcal{E}$-categories,

[16]It seems that replacing $\mathcal{C}AT$ by $\mathcal{C}AT_{\mathsf{ana}}$ makes less of a difference. Since functors are in particular anafunctors, all our examples such as $\mathsf{Set}_{\mathsf{ex}}$ are still derivators with this generalized definition. And as long as *all* the functors $u^*, u_!, u_*$ in the target $\mathscr{D}$, and the components of derivator morphisms, are generalized to anafunctors simultaneously, I would expect essentially the same arguments for their universality to go through.

or $\mathcal{R}$-2-categories of $\mathcal{R}$-categories.[17] I have not pursued this direction; the goal of this paper was to show that *even* if we try as hard as possible to take sets and set-based categories as our basic notions, we seem to be led, ineluctably, either to setoids and $\mathcal{E}$-groupoids, or to anafunctors.

The next question is, if only the right-hand column of Figure 3 consists of $\mathcal{C}at_{\mathsf{ana}}$-derivators, why does the *whole* figure consist of $\mathcal{C}at$-derivators, rather than just the two right-hand columns? In fact, I would expect that if we define $\mathsf{Set}_{\mathsf{ex}}$ (for instance) as an $\mathcal{E}\mathcal{C}at$-derivator, it would *not* be a "$\mathcal{C}at$-derivator" in the sense that u^* is an equivalence for any $\mathcal{E}$-functor u that is inverted by the reflection of $\mathcal{E}\mathcal{C}at$ into $\mathcal{C}at$. The difference is that $\mathcal{C}at$ is a *reflective* localization of $\mathcal{E}\mathcal{C}at$, so that we can make the $\mathcal{E}\mathcal{C}at$-derivator $\mathsf{Set}_{\mathsf{ex}}$ into a $\mathcal{C}at$-derivator in a different way by simply *restricting* its domain to the sub-2-category $\mathcal{C}at$ of $\mathcal{E}\mathcal{C}at$. The latter restriction is the derivator we have called $\mathsf{Set}_{\mathsf{ex}}$ in this paper.

Remark 8.6. The referee has pointed out another interesting question: can the equivalences in the derivators $\mathscr{D}$ of Figure 3 be characterized using isomorphisms of "homotopy groups"? We have seen in Proposition 5.7 and Remarks 5.10, 5.14, 5.18 and 5.22 that the $\mathscr{D}$-equivalences in the cases we've studied can all be characterized as "π_n-equivalences" for a notion of π_n that varies with the column as well as the row. In the next dimension, we expect an equivalence of groupoids to be a functor inducing an isomorphism of π_0 and isomorphisms of π_1 at all basepoints; but each such "homotopy group" could be a set, an equivalence relation, or a pseudo-equivalence relation. Presumably the EGpd-equivalences involve setoids π_0^{ex} and π_1^{ex}, while the $\mathsf{Gpd}_{\mathsf{ana}}$-equivalences involve sets π_0 and π_1, and the others are in between. Relatedly, note that by [Hen20, Proposition 5.2.6], the equivalences of fibrant simplicial sets (a possible model for Space) are characterized constructively by isomorphims of *setoid* homotopy groups.

A positive solution to the above conjectures would, I believe, give a systematic explanation of many confusing aspects of homotopy theory in set-based constructive mathematics. However, it is not clear whether it would conclusively answer the question of what the "correct" constructive theory

[17]To continue getting new notions beyond the fourth column, one would need to generalize to "n-derivators" in the sense of [Rap19], with the domain $\mathcal{C}at$ replaced by some version of $(n, 1)$-$\mathcal{C}at$. That is, the notion of derivator can vary not only with the column but also with the row.

of spaces is, since both candidates Space and $\mathsf{Space}_{\mathsf{ana}}$ have drawbacks: the former truncates to $\mathsf{Set}_{\mathsf{ex}}$ rather than Set, while the latter is not locally small, cartesian closed, or complete.

Of course, such bifurcations of classical notions are not uncommon in constructive mathematics. However, in this case there is more to be said: if we are willing to modify the background theory (while still keeping it "constructive" in at least some sense), we can make $\mathsf{Gpd}_{\mathsf{ana}}$ and $\mathsf{Space}_{\mathsf{ana}}$ much better-behaved.

It is known that local smallness and cartesian closure of $\mathsf{Gpd}_{\mathsf{ana}}$ (and also, presumably, $\mathsf{Space}_{\mathsf{ana}}$) requires much less than the full axiom of choice: it suffices to assume SCSA [Mak96] or WISC [Rob12] (a.k.a. AMC [vdB12]). These weak choice axioms have at least some claim to being constructive, as they often hold in large classes of models of constructive mathematics, such as Grothendieck toposes, realizability toposes, and exact completions. I do not know whether these axioms make $\mathsf{Gpd}_{\mathsf{ana}}$ complete, but there is another axiom that should do so: the Axiom of Stack Completions [BH11], which implies that $\mathsf{Gpd}_{\mathsf{ana}}$ is equivalent to a *reflective* localization of Gpd (hence also of EGpd), whose objects are the "intrinsic stacks" relative to surjections of sets. The constructive nature of ASC is perhaps debatable, but at least it holds in all Grothendieck toposes [JT91].

Another approach is to choose instead to do constructive homotopy theory based on a foundational system in which spaces are primitive objects, such as homotopy type theory. This is my preferred solution, so I will conclude with some remarks about its advantages.

Remark 8.7. As noted in [Lum20], the diagonals of Figure 3 bear a strong resemblance to the hierarchy of *saturation* or *univalence* conditions on higher-categorical structures defined in homotopy type theory [AKS15, ANST21]. When a groupoid is presented by a diagram on an inverse-category signature as in [Mak95, ANST21], it has three ranks of type dependency, corresponding to the objects, morphisms, and equalities. Roughly speaking, $\mathcal{E}$-groupoids correspond to unrestricted categories of this sort, while $\mathcal{R}$-groupoids are univalent at the top rank (equalities), and ordinary groupoids are univalent at the top two ranks (equalities and morphisms).

In a set-based foundation, it is not possible to be more univalent than this; but in homotopy type theory, we can also impose univalence conditions at the bottom rank of objects. The resulting homotopy theory UGpd of *univalent groupoids* is a *reflective* localization of Gpd[18] at the weak equivalences, closely related to the category of "intrinsic stacks" mentioned above in connection with ASC. Hence, UGpd plays a similar role to $\mathsf{Gpd}_{\mathsf{ana}}$, but without the attendant disadvantages. In particular, it is locally small, cartesian closed, exact, and has limits as well as colimits, while its subcategory of 0-truncated objects is Set. Similarly, the category of univalent ∞-groupoids (spaces) plays the expected role of $\mathsf{Space}_{\mathsf{ana}}$.

In fact, a "univalent groupoid" is equivalently just a type with the property of being a 1-type, while a "univalent space" is simply a type with no restrictions. That is, in homotopy type theory the primitive objects are the objects of $\mathsf{Space}_{\mathsf{ana}}$ rather than those of Set, so that none of the elaborate work involved in defining higher groupoids and homotopy spaces is necessary. (The related notions of higher *category*, however, are still nontrivial.)

I expect that the primitive spaces in homotopy type theory form a derivator (although proving this may require an enhanced theory such as [ACK17]). It is unclear whether the resulting derivator of univalent spaces would be a free cocompletion of a point; the answer might depend on how univalent the 1-categories in $\mathcal{C}at$ are assumed to be, and/or on strong classicality axioms such as $\mathsf{AC}_{\infty,-1}$ from [Uni13, Exercise 7.8]. (In particular, since univalent 1-categories are now a reflective localization of non-univalent ones, it seems likely that all the other derivators in Figure 3 will still exist even if we replace $\mathcal{C}at$ by $\mathcal{UC}at$. Thus $\mathsf{Space}_{\mathsf{ana}}$ may not be a free cocompletion of a point unless there is a classicality axiom to collapse the columns.)

However, the "correctness criterion" advanced in this paper for a homotopy theory of spaces is not justified for homotopy type theory anyway. This criterion seeks to characterize the homotopy theory of spaces in terms of sets (or at most 1-categories); thus it makes sense in a world whose primitive objects are sets, but not in a world where spaces are already present as primitive objects.

[18]For the expert, note that here we interpret "groupoids" as particular *precategories* in the sense of [AKS15, Uni13], with no dimension restriction on their type of objects.

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXIV-1 (2023)

A LOGICAL ANALYSIS OF FIXPOINT THEOREMS

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Résumé. Nous démontrons un théorème du point fixe pour des contractions sur des catégories Cauchy-complètes enrichies dans un quantale. Il est valable pour tout quantale dont le treillis sous-jacent est continu, et s'applique à des contractions dont la fonction de contrôle est séquentiellement semi-continue inférieurement. Des conditions suffisantes pour l'unicité du point fixe sont établies. Les exemples comprennent des théorèmes du point fixe connus et nouveaux pour les espaces métriques, les ordres flous, et les espaces métriques aléatoires.
Abstract. We prove a fixpoint theorem for contractions on Cauchy-complete quantale-enriched categories. It holds for any quantale whose underlying lattice is continuous, and applies to contractions whose control function is sequentially lower-semicontinuous. Sufficient conditions for the uniqueness of the fixpoint are established. Examples include known and new fixpoint theorems for metric spaces, fuzzy orders, and probabilistic metric spaces.
Keywords. Quantale, Enriched Category, Fixpoint Theorem.
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Introduction

A beautiful and important result in metric space theory, is Banach's fixpoint theorem [2] from 1922: "Every contraction on a non-empty complete metric space admits a unique fixpoint." The gist of the proof is wonderfully simple: take any element x of the space (X, d) and, iterating the contraction

$f\colon X \to X$, prove that the sequence $(f^n x)_{n\in\mathbb{N}}$ is Cauchy. In the complete space (X, d) this sequence converges, and one then shows that it does so to a (necessarily unique) fixpoint of f. Many generalizations and applications of Banach's theorem have been, and are still, studied.

In 1972, Lawvere [17] famously showed that metric spaces are a particular instance of enriched categories. More impressively still, Lawvere also showed how convergence of Cauchy sequences can adequately be understood via representability of left adjoint distributors, thus lifting the very concept of Cauchy completeness to the level of enriched categories. In his words, "specializing the constructions and theorems of general category theory we can deduce a large part of general metric space theory."

It is thus natural to investigate whether fixpoint theorems still make sense in the vast context of enriched categories. This is precisely the subject of this paper.

More precisely, we shall take quantale-enriched categories as generalization of metric spaces. That is to say, we fix a quantale Q, and work with categories, functors and distributors enriched in Q. Our contribution shows that fixpoint theorems for Q-categories depend on the interplay between three essential parameters. Indeed, a given contraction must be "strong enough" (we shall measure its strength by means of a control function); the space on which it acts must be "complete enough" for the Picard iteration to converge to a fixpoint (we shall take this to be Cauchy-completeness in the sense of Lawvere); but we also need sufficiently strong algebraic properties of the underlying quantale Q to allow for the formulation of precisely that convergence.

In concreto, we shall prove a fixpoint theorem for Cauchy-complete Q-categories[1] that holds for any quantale Q whose underlying complete lattice is continuous and for a specific notion of contraction. Besides, we make plain when and why such a fixpoint is unique (up to isomorphism). As examples we find the classical Banach fixpoint theorem for metric spaces, and Boyd and Wong's [3] generalization thereof (taking the underlying quantale

[1] To stay faithful to Banach's theorem in the metric case, we have chosen to study fixpoints for contractions on *Cauchy-complete* Q-categories. Let us mention, though, that other authors have studied other kinds of completeness, e.g. Wagner [27] chooses liminf-complete Q-categories, whereas Ackerman [1] works with spherically complete Q-categories (and both use a commutative quantale Q).

to be the positive real numbers); but we also formulate new results for fuzzy ordered sets (when working over a left-continuous t-norm) and for probabilistic metric spaces (now the quantale is the tensor product of the positive reals with a left-continuous t-norm).

In Section 1 we shall provide all the necessary notions from quantale-enriched category theory to make this paper reasonably self-contained; we follow [25] for the general theory, and [13] specifically for the comparison between categorical and sequential Cauchy-completeness. In Section 2 we first introduce the contractions that we are interested in, then we show how these contractions determine Cauchy distributors under the appropriate algebraic condition on the quantale Q, and finally we formulate the resulting fixpoint theorem for Cauchy-complete Q-categories. The examples in Section 3 show how our fixpoint theorem generalizes known results from the literature, and provides for new results too. We end with a short conclusion in Section 4.

1. Quantale-enriched categories

1.1 Q-enriched categories, functors and distributors

In this section we recall some key notions from [25] on quantale-enriched categories[2]; we encourage the reader to go back-and-forth to Subsection 1.2 for the relevant examples.

Throughout, we fix a quantale $Q = (Q, \bigvee, \circ, 1)$: it is a complete sup-lattice $(Q, \bigvee)$ endowed with a monoid[3] structure $(Q, \circ, 1)$ such that the product distributes over arbitrary suprema:

$$s \circ (\bigvee_i t_i) = \bigvee_i (s \circ t_i) \quad \text{and} \quad (\bigvee_i s_i) \circ t = \bigvee_i (s_i \circ t).$$

In other words, but more abstractly, a quantale is a monoid in the symmetric monoidal closed category Sup of complete lattices and supremum-preserving morphisms.

[2]That reference actually treats the more general *quantaloid*-enriched category theory, but the reader will easily convert those results to the simpler quantale-enriched case. See also [26] for a gentle introduction to the subject.

[3]We do *not* assume that 1, the unit of the monoid, is the top element of the lattice.

A *Q-enriched category* $\mathbb{C}$ (or *Q-category* $\mathbb{C}$ for short) consists of a set $\mathbb{C}_0$ (of "objects") together with a Q-valued ("hom") predicate

$$\mathbb{C}\colon \mathbb{C}_0 \times \mathbb{C}_0 \to Q\colon (x,y) \mapsto \mathbb{C}(x,y)$$

satisfying, for all $x,y,z \in \mathbb{C}_0$, the following ("composition" and "identity") conditions:

$$\mathbb{C}(x,y) \circ \mathbb{C}(y,z) \leq \mathbb{C}(x,z) \quad \text{and} \quad 1 \leq \mathbb{C}(x,x).$$

A *Q-functor* $F\colon \mathbb{C} \to \mathbb{D}$ between two Q-categories is a function

$$F\colon \mathbb{C}_0 \to \mathbb{D}_0\colon x \mapsto Fx$$

satisfying, for all $x,x' \in \mathbb{C}_0$, the ("functoriality") condition

$$\mathbb{C}(x',x) \leq \mathbb{D}(Fx',Fx).$$

Two such Q-functors $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{B} \to \mathbb{C}$ can be composed in the obvious way to produce a new functor $G \circ F\colon \mathbb{A} \to \mathbb{C}$, and the identity function on $\mathbb{A}_0$ provides for the identity functor $1_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}$. Thus Q-categories and Q-functors are the objects and morphisms of a (large) category $\mathsf{Cat}(Q)$.

A *Q-distributor* (also called *bimodule* or *profunctor*) $\Phi\colon \mathbb{C} \looparrowright \mathbb{D}$ between two Q-categories is a Q-valued predicate

$$\Phi\colon \mathbb{D}_0 \times \mathbb{C}_0 \to Q\colon (y,x) \mapsto \Phi(y,x)$$

satisfying, for all $x,x' \in \mathbb{C}_0$ and $y,y' \in \mathbb{C}_0$, the ("action") condition

$$\mathbb{D}(y',y) \circ \Phi(y,x) \circ \mathbb{C}(x,x') \leq \Phi(y',x').$$

Two such distributors, say $\Phi\colon \mathbb{A} \looparrowright \mathbb{B}$ and $\Psi\colon \mathbb{B} \looparrowright \mathbb{C}$, compose as

$$(\Psi \circ \Phi)\colon \mathbb{C}_0 \times \mathbb{A}_0 \mapsto Q\colon (z,x) \mapsto \bigvee_{y \in \mathbb{B}_0} \Psi(z,y) \circ \Phi(y,x).$$

The identity distributor on $\mathbb{C}$ is the "hom" predicate $\mathbb{C}\colon \mathbb{C}_0 \times \mathbb{C}_0 \to Q$ itself, and so Q-categories and Q-distributors form a (large) category $\mathsf{Dist}(Q)$.

However, there is more: the elementwise ordering of distributors makes $\mathsf{Dist}(Q)$ a 2-category[4].

Applying general 2-categorical algebra, we may now say that two Q-distributors $\Phi\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{B}$ and $\Psi\colon \mathbb{B} \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{A}$ are *(left/right) adjoint*, denoted as $\Phi \dashv \Psi$, if

$$\mathbb{A} \leq \Psi \circ \Phi \quad \text{and} \quad \Phi \circ \Psi \leq \mathbb{B}.$$

Every functor $F\colon \mathbb{A} \to \mathbb{B}$ *represents* an adjoint pair of distributors $F_* \dashv F^*$ defined by

$$F_*(b, a) = \mathbb{B}(b, Fa) \quad \text{and} \quad F^*(a, b) = \mathbb{B}(Fa, b).$$

With this, the inclusion functor

$$\mathsf{Cat}(Q) \to \mathsf{Dist}(Q)\colon \Big(F\colon \mathbb{A} \to \mathbb{B}\Big) \mapsto \Big(F_*\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{B}\Big)$$

naturally makes $\mathsf{Cat}(Q)$ a *locally ordered category* by defining, for any $F, G \in \mathsf{Cat}(Q)$,

$$F \leq G \overset{\text{def}}{\iff} F_* \leq G_*.$$

Whenever $F \leq G$ and $G \leq F$, we write $F \cong G$ and say that these functors are *isomorphic*.

For a fixed Q-category $\mathbb{C}$, we may consider, for any other Q-category $\mathbb{A}$, the map which assigns to any functor $F\colon \mathbb{A} \to \mathbb{C}$ the left adjoint distributor $F_*\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{C}$:

$$\mathsf{Cat}(Q)(\mathbb{A}, \mathbb{C}) \to \mathsf{LAdjDist}(Q)(\mathbb{A}, \mathbb{C})\colon F \mapsto F_*.$$

This map is (by definition of the local order in $\mathsf{Cat}(Q)$) order-preserving and order-reflecting. If, for each $\mathbb{A}$, this maps is also surjective (in words: every left adjoint distributor into $\mathbb{C}$ is representable by a functor), then we say that $\mathbb{C}$ is *Cauchy-complete*.

Let $\mathbb{1}$ be the Q-category defined by $\mathbb{1}_0 = \{*\}$ and $\mathbb{1}(*, *) = 1$. A distributor $\phi\colon \mathbb{1} \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{C}$ is called a (contravariant) *presheaf* on $\mathbb{C}$. There is a natural bijection between Q-functors $\mathbb{1} \to \mathbb{C}$ and elements of $\mathbb{C}_0$. In particular, for any $c \in \mathbb{C}_0$ there is a Q-functor $\Delta_c\colon \mathbb{1} \to \mathbb{C}\colon * \mapsto c$ which represents the

[4] Much better still: $\mathsf{Dist}(Q)$ is a *quantaloid*, i.e. a category enriched in Sup. Since we do not need this very rich structure in this paper, we shall not dwell on it here.

left adjoint presheaf $(\Delta_c)_* \colon \mathbb{C}_0 \times \mathbb{1}_0 \to Q \colon (x, *) \mapsto \mathbb{C}(x, c)$. Therefore, by putting

$$c \leq c' \overset{\text{def}}{\iff} \Delta_c \leq \Delta_{c'} \iff \mathbb{C}(-, c) \leq \mathbb{C}(-, c') \iff 1 \leq \mathbb{C}(c, c')$$

the set $\mathbb{C}_0$ becomes an order $(\mathbb{C}_0, \leq)$. If both $c \leq c'$ and $c' \leq c$ hold, then we write $c \cong c'$ and we say that these objects of $\mathbb{C}$ are *isomorphic*. It is furthermore a result in Q-category theory (which holds in greater generality too) that $\mathbb{C}$ is Cauchy-complete if and only if

$$\mathsf{Cat}(Q)(\mathbb{1}, \mathbb{C}) \to \mathsf{LAdjDist}(Q)(\mathbb{1}, \mathbb{C})$$

is surjective; in words, $\mathbb{C}$ is Cauchy-complete if and only if each left adjoint presheaf on $\mathbb{C}$ is representable.

The importance of Cauchy-complete Q-categories was made very clear in Lawvere's seminal paper [17] on the subject, via its relation to Cauchy sequences. We shall briefly recall a small portion of this, using Hofmann and Reis [13, Section 4.3] as reference.

Given a sequence $x = (x_n)_{n \in \mathbb{N}}$ in a Q-category $\mathbb{C}$, we define

$$C_x := \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \bigwedge_{m \geq N} \mathbb{C}(x_n, x_m).$$

and say that $x = (x_n)_{n \in \mathbb{N}}$ is a *Cauchy sequence* if $C_x \geq 1$. On the other had, we also define

$$\phi_x \colon \mathbb{C}_0 \to Q \colon y \mapsto \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(y, x_n)$$
$$\psi_x \colon \mathbb{C}_0 \to Q \colon y \mapsto \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(x_n, y)$$

and then have for these Q-valued predicates that:

Proposition 1.1.1 *For any sequence $x = (x_n)_{n \in \mathbb{N}}$ of objects in a Q-category $\mathbb{C}$, both ϕ_x and ψ_x are Q-enriched distributors. Furthermore, the sequence $x = (x_n)_{n \in \mathbb{N}}$ is Cauchy (i.e. $C_x \geq 1$) if and only if $\phi_x \dashv \psi_x$.*

Thus it makes perfect sense to speak of (convergence of) Cauchy sequences in any Q-category $\mathbb{C}$, via the representability of the associated adjoint pair of distributors, which is exactly what we shall need to do in the proof of Proposition 2.2.3 further on.

1.2 Examples of Q-enriched categories

In the rest of the paper, our examples of Q-enriched categories will be:

Example 1.2.1 (Ordered sets) The simplest non-trivial example of a quantale is the two-element Boolean algebra $Q = (\{0,1\}, \vee, \wedge, 1)$. In this case, a Q-category is an ordered set $(P, \leq)$, seen as a set P equipped with a binary relation $\leq$ whose characteristic function $P \times P \to \{0,1\}\colon (x,y) \mapsto [\![x \leq y]\!]$ satisfies the following axioms:

(1) $[\![x \leq y]\!] \wedge [\![y \leq z]\!] \leq [\![x \leq z]\!]$,

(2) $1 \leq [\![x \leq x]\!]$.

(This order-relation need not be anti-symmetric; some call this a "preorder".) A Q-functor between such Q-categories is a monotone map between ordered sets. It is well-known (and easy to verify) that every ordered set is, viewed as an enriched category, Cauchy-complete.

Example 1.2.2 (Metric spaces) Let $Q = ([0,\infty], \bigwedge, +, 0)$ be Lawvere's quantale of extended positive real numbers, i.e. it is the segment $[0,\infty]$ (with ∞ included) with the converse (!) of the natural (linear) order, and with the sum as binary operation. As pointed out by Lawvere [17], a Q-category is precisely a *generalised metric space* (X, d), that is, a set X together with a distance function $d\colon X \times X \to [0,\infty]$ such that

(1) $d(x,y) + d(y,z) \geq d(x,y)$,

(2) $0 \geq d(x,x)$.

The adjective "generalized" here indicates that such a metric need not be finitary (so $d(x,y) = \infty$ is allowed) nor symmetric (so $d(x,y) \neq d(y,x)$ is allowed), nor separated (so $d(x,y) = 0 = d(y,x)$ for $x \neq y$ is allowed). A Q-functor between such Q-categories is a non-expanding map between (generalized) metric spaces. Lawvere [17] famously showed that a metric space is Cauchy-complete as enriched category if and only if all Cauchy sequences (in the usual sense for metric spaces) converge.

Example 1.2.3 (Fuzzy orders) A so-called *left-continuous t-norm* is precisely a commutative and integral quantale whose underlying (linear) suplattice is $([0,1],\bigvee)$ (see e.g. [15, 26]); the multiplication of such a quantale is then typically written as $x*y$. Examples include $x*y=xy$ (the "product t-norm"), $x*y=\min\{x,y\}$ (the "minimum t-norm") and $x*y=\max\{x+y-1,0\}$ (the "Lukasiewicz t-norm"); in fact, every *continuous* t-norm (meaning that the multiplication is a continuous function) is in a precise sense an amalgamation of these three (see e.g. [11]). These quantales are the corner stone of "fuzzy" logic: the truth values in this logic can vary between 0 and 1, conjunction is computed with $*$, and implication is computed with the adjoint to multiplication. A category enriched in a left-continuous t-norm $([0,1],\bigvee,*,1)$ thus consists of a set P together with a map $P\times P\to[0,1]\colon(x,y)\mapsto[\![x\leq y]\!]$ satisfying

(1) $[\![x\leq y]\!]*[\![y\leq z]\!]\leq[\![x\leq z]\!]$,

(2) $1\leq[\![x\leq x]\!]$.

Following [28, 20, 5, 18], we call this a *fuzzy (pre)order*: the truth value $[\![x\leq y]\!]\in[0,1]$ is interpreted as "the extent to which $x\leq y$ holds in P". A Q-functor between such Q-categories is a map between fuzzy preorders that does not decrease the value of the "fuzzy" order. By Theorem 4.19 of [13] (and the definition of Cauchy sequence in a Q-category recalled above) it follows that a fuzzy order is categorically Cauchy-complete if and only if all Cauchy sequences (in the usual sense for fuzzy orders, see Definition 4.1 in [5]) converge.

Example 1.2.4 (Probabilistic metric space) Fix a *left-continuous t-norm* $([0,1],\bigvee,*,1)$. It was shown by Hofmann and Reis [13], and further explained in [6], that the set

$$\Delta=\{f:[0,\infty]\to[0,1]\mid f(t)=\bigvee_{s<t}f(s)\}$$

of so-called distance distributions[5] is a quantale for pointwise suprema, with

[5]Because domain and codomain are continuous lattices, these are precisely the lower semicontinuous functions, see [8, Proposition II-2.1]; and because domain and codomain are complete linear orders, these are precisely the supremum-preserving maps, see [6, Example 2.1.10].

the convolution product

$$(f * g)(t) = \bigvee_{r+s=t} f(r) * g(s)$$

as binary operation, and

$$e(t) = \begin{cases} 0 \text{ if } t = 0, \\ 1 \text{ else} \end{cases}$$

as two-sided unit. Indeed, it is shown in [9, Examples 2.1.10 and 2.3.36] that the quantale $Q = (\Delta, \bigvee, *, e)$ is the tensor product in the category of suplattices, as well as the coproduct in the category of commutative quantales, of the Lawvere quantale $([0,\infty], \bigwedge, +, 0)$ and the left-continuous t-norm $([0,1], \bigvee, *, 1)$. A Q-category has been called a *(generalized) probabilistic metric space* by some [13, 12], and a *(generalized) fuzzy metric space* by others [16, 7]; it consists of a set X together with a probabilistic distance function $d\colon X \times X \times [0,\infty] \to [0,1]$ such that

(0) $d(x,y,t) = \bigvee_{s<t} d(x,y,s)$,

(1) $d(x,x,t) = 1$ for $t > 0$,

(2) $d(x,y,r) * d(y,z,s) \le d(x,z,r+s)$.

Such an object is often denoted $(X,d,*)$, to stress the importance of the t-norm. The intended meaning of $d(x,y,t)$ is that it expresses "the probability that the distance from x to y is strictly less than t". (Again, we do not insist on finiteness, symmetry or separatedness for such a space, each of which can be expressed suitably; see also [24].) A Q-enriched functor is a map between such spaces that does not decrease such probabilistic distances. Hofmann and Reiss [13] proved that a probabilistic metric space is categorically Cauchy-complete if and only if all Cauchy sequences (as traditionally defined in probabilistic metric spaces, see [4, 13]) converge.

2. Fixpoints for contractions on Q-categories

2.1 Contractions on a Q-enriched category

Let Q be any quantale (and write 0 for its bottom element), and $\mathbb{C}$ any Q-enriched category.

Definition 2.1.1 *If $\varphi : Q \to Q$ and $f : \mathbb{C}_0 \to \mathbb{C}_0$ are maps such that*

1. $\varphi(t) \geq t$ *for all* $t \in Q$,
2. *if* $\varphi(t) = t$ *then* $t = 0$ *or* $1 \leq t$,
3. *for all* $x, y \in \mathbb{C}$, $\mathbb{C}(fx, fy) \geq \varphi(\mathbb{C}(x, y))$,

then we say that f is a φ-contraction, and we say that φ is a control function for f.

A control function φ is thus bigger than the identity function on the whole of Q, and strictly so except possibly in $t = 0$ or $t \geq 1$. Note too that a φ-contraction f is always a Q-functor $f\colon \mathbb{C} \to \mathbb{C}$, but not every Q-functor is φ-contractive for some control function φ.

We wish to investigate the possible fixpoints of such contractions. Let us first make this formal:

Definition 2.1.2 *Let $f\colon \mathbb{C} \to \mathbb{C}$ be a Q-functor. A fixpoint for f is an $u \in \mathbb{C}$ such that $fu \cong u$ in $\mathbb{C}$, that is to say, we have both $1 \leq \mathbb{C}(fu, u)$ and $1 \leq \mathbb{C}(u, fu)$.*

In general, such fixpoints are of course not unique. However, if f is a φ-contraction, and both $fu \cong u$ and $fu' \cong u'$ hold, then it follows from the triangular inequality in $\mathbb{C}$ that

$$\begin{aligned}\mathbb{C}(u, u') &\geq \mathbb{C}(u, fu) \circ \mathbb{C}(fu, fu') \circ \mathbb{C}(fu', u') \\ &\geq 1 \circ \mathbb{C}(fu, fu') \circ 1 \\ &= \mathbb{C}(fu, fu') \\ &\geq \varphi(\mathbb{C}(u, u')) \\ &\geq \mathbb{C}(u, u')\end{aligned}$$

Since $\varphi(t) > t$ for all $0 \neq t \ngeq 1$, we must have $\mathbb{C}(u, u') = 0$ or $\mathbb{C}(u, u') \geq 1$. Exchanging u and u' one sees that also $\mathbb{C}(u', u) = 0$ or $\mathbb{C}(u', u) \geq 1$. Hence there are exactly four possibilities:

$$\left\{\begin{array}{l}\mathbb{C}(u, u') \geq 1 \\ \mathbb{C}(u', u) \geq 1\end{array}\right. \text{ or } \left\{\begin{array}{l}\mathbb{C}(u, u') \geq 1 \\ \mathbb{C}(u', u) = 0\end{array}\right. \text{ or } \left\{\begin{array}{l}\mathbb{C}(u, u') = 0 \\ \mathbb{C}(u', u) \geq 1\end{array}\right. \text{ or } \left\{\begin{array}{l}\mathbb{C}(u, u') = 0 \\ \mathbb{C}(u', u) = 0\end{array}\right.$$

Under mild assumptions on $\mathbb{C}$ we can now formulate uniqueness results for fixpoints.

Proposition 2.1.3 *Let $\mathbb{C}$ be a Q-category all of whose homs are non-zero, and $f\colon \mathbb{C} \to \mathbb{C}$ any φ-contraction. If $fu \cong u$ and $fu' \cong u'$ then $u \cong u'$.*

Proof. In the four possible cases above, only the first is compatible with non-zero homs in $\mathbb{C}$. □

For Q-categories with homs that can be equal to 0, we have a different result.

Proposition 2.1.4 *Let $\mathbb{C}$ be symmetric Q-category (meaning that $\mathbb{C}(x,y) = \mathbb{C}(y,x)$ for all $x, y \in \mathbb{C}$) and $f\colon \mathbb{C} \to \mathbb{C}$ any φ-contraction. If $fu \cong u$ and $fu' \cong u'$ then either $u \cong u'$ or $\mathbb{C}(u,u') = 0$.*

Proof. In the four possible cases above, only the first and the last are compatible with symmetry in $\mathbb{C}$. □

Reckoning that any symmetric Q-category decomposes as a categorical sum of symmetric subcategories, each of which has all homs non-zero, the latter Proposition says that any two distict fixpoints of $f\colon \mathbb{C} \to \mathbb{C}$ must be in different summands of $\mathbb{C}$.

2.2 From contractions to adjoint presheaves

Given any φ-contraction f on a Q-category $\mathbb{C}$ and an object $x \in \mathbb{C}_0$, it follows from Proposition 1.1.1 that the sequence $(f^n x)_{n\in\mathbb{N}}$ determines two distributors,

$$\phi_{x,f}\colon \mathbb{1} \looparrowright \mathbb{C} \quad \text{and} \quad \psi_{x,f}\colon \mathbb{C} \looparrowright \mathbb{1},$$

with elements

$$\phi_{x,f}(y) = \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \mathbb{C}(y, f^n x) \quad \text{and} \quad \psi_{x,f}(y) = \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \mathbb{C}(f^n x, y).$$

We now wish to identify sufficient conditions on Q and $\varphi\colon Q \to Q$ in order to prove an adjunction between these distributors.

To that end, we first recall some pertinent definitions from [8]. Let L be a complete lattice. A subset $D \subseteq L$ is directed if it is non-empty and, for any $x, y \in D$ there exists a $z \in D$ such that $x \vee y \subseteq z$. For two elements $a, b \in L$ we write $a \ll b$, and we say that a is *way below* b, if, for every directed subset $D \subseteq L$, $b \leq \bigvee D$ implies the existence of a $d \in D$ such that $a \leq d$.

Definition 2.2.1 *We say that a complete lattice L is continuous if, for each $a \in L$,*

$$a = \bigvee \{u \in L \mid u \ll a\}.$$

It is well-known that every continuous lattice is meet-continuous—meaning that (binary) meets distribute over (all) directed suprema. Finally, we shall be interested in a weak variant[6] of *lower-semicontinuity*:

Definition 2.2.2 *We say that a function $\varphi : L \to M$ between complete lattices is sequentially lower-semicontinuous if, for any sequence $(t_n)_{n\in\mathbb{N}}$ in L,*

$$\varphi(\bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} t_n) \leq \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \varphi(t_n).$$

Taking inspiration from the "metric" case discussed in [3], we now prove:

Proposition 2.2.3 *Let Q be a quantale whose underlying complete lattice is continuous[7], and let $f : \mathbb{C} \to \mathbb{C}$ be a φ-contraction on a Q-category for which the control function $\varphi\colon Q \to Q$ is sequentially lower-semicontinuous. For any $x \in \mathbb{C}_0$ such that $\mathbb{C}(x, fx) \neq 0 \neq \mathbb{C}(fx, x)$ we have $\phi_{x,f} \dashv \psi_{x,f}$.*

Proof. Putting $C_{x,f} := \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \bigwedge_{m\geq N} \mathbb{C}(f^n x, f^m x) \in Q$, we recall from Proposition 1.1.1 that $\phi_{x,f} \dashv \psi_{x,f}$ if and only if $C_{x,f} \geq 1$. We shall show that $C_{x,f} \ngeq 1$ leads to a contradiction.

(i) Picking an $x \in \mathbb{C}_0$ such that $\mathbb{C}(x, fx) \neq 0 \neq \mathbb{C}(fx, x)$, we put $c_n := \mathbb{C}(f^n x, f^{n+1} x) \in Q$ for all $n \in \mathbb{N}$. By assumption, $0 < c_0 \leq 1$ and the conditions on φ imply that $c_0 \leq \varphi(c_0) \leq c_1$. Repeating the argument we find that $c_n \leq \varphi(c_n) \leq c_{n+1}$, so the sequence is increasing and strictly above 0. Therefore we can compute, using the conditions on φ, that:

$$\bigvee_{N\in\mathbb{N}} c_N = \bigvee_{N\in\mathbb{N}} c_{N+1}$$

[6] A function $f\colon L \to M$ between complete lattices is lower-semicontinuous if the sup-inf condition in Definition 2.2.2 holds for all *nets* in L (i.e. a family of elements indexed by a directed poset).

[7] It is tempting to speak of a *continuous quantale*, yet this terminology is in conflict with that of *continuous t-norm*. Indeed, the underlying lattice of any t-norm is the continuous lattice $[0, 1]$, yet not every t-norm is continuous (as a function in two variables). So we shall stick to the somewhat cumbersome "quantale with underlying continuous lattice".

$$
\begin{aligned}
&= \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} c_{n+1} \\
&\geq \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \varphi(c_n) \\
&\geq \varphi(\bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} c_n) \\
&= \varphi(\bigvee_{N\in\mathbb{N}} c_N) \\
&\geq \bigvee_{N\in\mathbb{N}} c_N
\end{aligned}
$$

We thus find a fixpoint of φ which is not 0, so it must satisfy $1 \leq \bigvee_{N\in\mathbb{N}} c_N$.

(ii) Similarly, the sequence $(a_n := \mathbb{C}(f^{n+1}x, f^n x))_{n\in\mathbb{N}}$ must also satisfy $1 \leq \bigvee_{\in\mathbb{N}} a_n$.

(iii) Next, suppose that $1 \not\leq C_{f,x}$; by continuity of the underlying complete lattice of Q, this means that there exists an $\epsilon \ll 1$ such that $\epsilon \not\leq C_{f,x}$ (and so in particular $\epsilon \neq 0$). Using the definition of $C_{f,x}$ as a sup-inf, we may infer:

$$
\begin{aligned}
&\epsilon \not\leq \bigvee_{k\in\mathbb{N}} \left(\bigwedge_{n\geq k} \bigwedge_{m\geq k} \mathbb{C}(f^n x, f^m x) \right) \\
&\implies \forall k \in \mathbb{N} : \epsilon \not\leq \bigwedge_{n\geq k} \bigwedge_{m\geq k} \mathbb{C}(f^n x, f^m x) \\
&\implies \forall k \in \mathbb{N}, \exists n_k, m_k \geq k : \epsilon \not\leq \mathbb{C}(f^{n_k} x, f^{m_k} x)
\end{aligned}
$$

In the last line above, it cannot be the case that $m_k = n_k$, because otherwise $\mathbb{C}(f^{n_k}x, f^{n_k}x) \geq 1$ (by the "identity" axiom for the Q-category $\mathbb{C}$), which would then also be above $\epsilon \ll 1$. So suppose that $n_k < m_k$, then we can replace m_k by

$$
m'_k := \min\{m > n_k \mid \epsilon \not\leq \mathbb{C}(f^{n_k}x, f^m x)\}
$$

and so we still have $\epsilon \not\leq \mathbb{C}(f^{n_k}x, f^{m'_k}x)$, but now also $\epsilon \leq \mathbb{C}(f^{n_k}x, f^{m_k-1}x)$. Similarly, if $n_k > m_k$ then we may replace n_k by

$$
n'_k := \min\{n > m_k \in \mathbb{N} \mid \epsilon \not\leq \mathbb{C}(f^n x, f^{m_k}x)\}
$$

and we still have $\epsilon \not\leq \mathbb{C}(f^{n'_k}x, f^{m_k}x)$, but now also $\epsilon \leq \mathbb{C}(f^{n'_k-1}x, f^{m_k}x)$. That is to say, we can always pick $n_k, m_k \geq k$ to ensure that

$$\epsilon \not\leq \mathbb{C}(f^{n_k}x, f^{m_k}x) \text{ and } \begin{cases} \text{either} & \mathbb{C}(f^{n_k}x, f^{m_k-1}x) \geq \epsilon \quad (A) \\ \text{or} & \mathbb{C}(f^{n_k-1}x, f^{m_k}x) \geq \epsilon \quad (B) \end{cases}$$

Now denote, for each such pick of $n_k, m_k \geq k \in \mathbb{N}$,

$$d_k := \mathbb{C}(f^{n_k}x, f^{m_k}x);$$

and let us insist that $\epsilon \not\leq d_k$ for all $k \in \mathbb{N}$. In case condition (A) holds for d_k, then in particular $m_k > n_k$ so $m_k \geq 1$, and we can use the “composition” axiom in $\mathbb{C}$ to get

$$\begin{aligned} \epsilon \circ c_{m_k-1} &\leq \mathbb{C}(f^{n_k}x, f^{m_k-1}x) \circ \mathbb{C}(f^{m_k-1}x, f^{m_k}x) \\ &\leq \mathbb{C}(f^{n_k}x, f^{m_k}x) \\ &= d_k \end{aligned}$$

In case condition (B) holds for d_k we can similarly prove that

$$a_{n_k-1} \circ \epsilon \leq d_k.$$

Hence, using in $(*)$ that a continuous lattice is always meet-continuous, and that both sequences

$$\Big(\bigwedge\{d_k \mid k \geq N \text{ and } (A) \text{ holds}\}\Big)_{N\in\mathbb{N}}$$

$$\Big(\bigwedge\{d_k \mid k \geq N \text{ and } (B) \text{ holds}\}\Big)_{N\in\mathbb{N}}$$

are increasing, we may compute that

$$\begin{aligned} \bigvee_{N\in\mathbb{N}} \bigwedge_{k\geq N} d_k &= \bigvee_{N\in\mathbb{N}_0} \bigwedge_{k\geq N} d_k \\ &= \bigvee_{N\in\mathbb{N}_0} \Big(\bigwedge\{d_k \mid k \geq N \text{ and } (A) \text{ holds}\} \\ &\qquad\qquad \wedge \bigwedge\{d_k \mid k \geq N \text{ and } (B) \text{ holds}\}\Big) \end{aligned}$$

$$\begin{aligned}
&\overset{(*)}{=} \left(\bigvee_{N\in\mathbb{N}_0} \bigwedge\{d_k \mid k \geq N \text{ and } (A) \text{ holds}\}\right) \\
&\qquad\qquad \wedge \left(\bigvee_{N\in\mathbb{N}_0} \bigwedge\{d_k \mid k \geq N \text{ and } (B) \text{ holds}\}\right) \\
&\geq \left(\bigvee_{N\in\mathbb{N}_0} \bigwedge\{\epsilon \circ c_{m_k-1} \mid k \geq N \text{ and } (A) \text{ holds}\}\right) \\
&\qquad\qquad \wedge \left(\bigvee_{N\in\mathbb{N}_0} \bigwedge\{a_{n_k-1} \circ \epsilon \mid k \geq N \text{ and } (B) \text{ holds}\}\right) \\
&\geq \left(\bigvee_{N\in\mathbb{N}} \bigwedge_{m\geq N} \epsilon \circ c_m\right) \wedge \left(\bigvee_{N\in\mathbb{N}} \bigwedge_{m\geq N} a_n \circ \epsilon\right) \\
&\geq \left(\epsilon \circ (\bigvee_{N\in\mathbb{N}} \bigwedge_{m\geq N} c_m)\right) \wedge \left((\bigvee_{N\in\mathbb{N}} \bigwedge_{m\geq N} a_n) \circ \epsilon\right) \\
&= \left(\epsilon \circ (\bigvee_{N\in\mathbb{N}} c_N)\right) \wedge \left((\bigvee_{N\in\mathbb{N}} a_N) \circ \epsilon\right) \\
&= (\epsilon \circ 1) \wedge (1 \circ \epsilon) \\
&= \epsilon
\end{aligned}$$

So, even though $\epsilon \not\leq d_k$ (for all $k \in \mathbb{N}$), we do have $0 \neq \epsilon \leq \bigvee_{N\in\mathbb{N}} \bigwedge_{k\geq N} d_k$.

(iv) Using the "composition" axiom in $\mathbb{C}$, we have for every $k \geq N \in \mathbb{N}$ (recall that $n_k, m_k \geq k$ too) that

$$d_k \geq c_{n_k} \circ \mathbb{C}(f^{n_k+1}x, f^{m_k+1}x) \circ a_{m_k} \geq c_{n_k} \circ \varphi(d_k) \circ a_{m_k} \geq c_N \circ \varphi(d_k) \circ a_N$$

and so we may compute that

$$\begin{aligned}
\bigvee_{N\in\mathbb{N}} \bigwedge_{k\geq N} d_k &\geq \bigvee_{N\in\mathbb{N}} \bigwedge_{k\geq N} (c_N \circ \varphi(d_k) \circ a_N) \\
&\geq \bigvee_{N\in\mathbb{N}} \left(c_N \circ (\bigwedge_{k\geq N} \varphi(d_k)) \circ a_N\right) \\
&\overset{(*)}{=} \left(\bigvee_{N\in\mathbb{N}} c_N\right) \circ \left(\bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \varphi(d_k)\right) \circ \left(\bigvee_{N\in\mathbb{N}} a_N\right)
\end{aligned}$$

$$= 1 \circ \left(\bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} \varphi(d_k) \right) \circ 1$$
$$\geq \varphi(\bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} d_k)$$
$$\geq \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} d_k$$

where in $(*)$ we use that the involved sequences are increasing[8]. This means that $\bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} d_k$ is a fixpoint of φ which – as we showed earlier – is not 0, so we must have $1 \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} d_k$.

(v) Since $\epsilon \ll 1 \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{k \geq N} d_k$, and the latter supremum is directed, by continuity of Q there must exist an $N_0 \in \mathbb{N}$ such that $\epsilon \leq \bigwedge_{k \geq N_0} d_k$. Yet, we established earlier that $\epsilon \not\leq d_k$ for all $k \in \mathbb{N}$. This is the announced contradiction. □

2.3 Fixpoint for a contraction on a Cauchy-complete Q-category

In the above Subsection we discovered sufficient conditions for a φ-contraction $f \colon \mathbb{C}_0 \to \mathbb{C}_0$ to determine adjoint distributors. If the Q-category $\mathbb{C}$ is Cauchy-complete, this adjoint pair is represented by an object of $\mathbb{C}$. We will now show that this representing object is a fixpoint for the contraction.

Proposition 2.3.1 *Let Q be any quantale and $f \colon \mathbb{C} \to \mathbb{C}$ any Q-functor on a Cauchy-complete Q-category. If there exists an $x \in \mathbb{C}_0$ such that $\phi_{x,f} \dashv \psi_{x,f}$ then f has a fixpoint.*

Proof. By Cauchy-completeness of $\mathbb{C}$, the presheaves $\phi_{x,f}$ and $\psi_{x,f}$ are representable; so suppose that $\phi_{x,f} = \mathbb{C}(-, u)$ and $\psi_{x,f} = \mathbb{C}(u, -)$ for some $u \in \mathbb{C}_0$. Now we can compute that

$$\mathbb{C}(fu, u) = \phi_{x,f}(fu)$$
$$= \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \mathbb{C}(fu, f^n x)$$

[8]For two sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ of elements in Q, distributivity of product over suprema in Q assures that $(\bigvee_n a_n) \circ (\bigvee_m b_m) = \bigvee_{n,m} (a_n \circ b_m)$. However, when both sequences are increasing, i.e. $n \leq n'$ implies $a_n \leq a_{n'}$ and $b_n \leq b_{n'}$, then this is further equal to $\bigvee_n (a_n \circ b_n)$. The argument obviously extends to three increasing sequences.

$$
\begin{aligned}
&= \bigvee_{N\in\mathbb{N}_0} \bigwedge_{n\geq N} \mathbb{C}(fu, f^n x) \\
&\overset{(*)}{\geq} \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \mathbb{C}(u, f^n x) \\
&= \phi_{x,f}(u) \\
&= \mathbb{C}(u,u) \\
&\geq 1
\end{aligned}
$$

using the "functoriality" axiom for f in $(*)$. Similarly one computes that $\mathbb{C}(u, fu) \geq 1$. Therefore we have both $u \geq fu$ and $fu \geq u$ in (the underlying order of) $\mathbb{C}$, which means that $u \cong fu$, as wanted. □

Putting Propositions 2.2.3 and 2.3.1 together, we arrive at:

Theorem 2.3.2 (Fixpoint theorem) *Let Q be quantale whose underlying lattice is continuous, and let $f\colon \mathbb{C} \to \mathbb{C}$ a φ-contraction on a Cauchy-complete Q-category, for which the control function $\varphi\colon Q \to Q$ is sequentially lower-semicontinuous. If there exists an $x \in \mathbb{C}_0$ such that $\mathbb{C}(x, fx) \neq 0 \neq \mathbb{C}(fx, x)$ then f has a fixpoint, namely the object representing the adjunction $\phi_{x,f} \dashv \psi_{x,f}$.*

In the above Theorem, the obtained fixpoint depends on the element $x \in \mathbb{C}$ chosen such that $\mathbb{C}(x, fx) \neq 0 \neq \mathbb{C}(fx, x)$. However, let us recall that Propositions 2.1.3 and 2.1.4 provide mild conditions on $\mathbb{C}$ to make the fixpoint of a contraction unique.

3. Examples and counterexamples

The examples in this section show how Theorem 2.3.2 generalizes known fixpoint theorems from the literature, and provides new ones too. Also, we mention a counterexample to show that the conditions cannot be weakened unless supplementary conditions are considered.

3.1 Orders

The two-element boolean algebra being a continuous lattice, the quantale $Q = (\{0,1\}, \vee, \wedge, 1)$ satisfies the condition in Theorem 2.3.2, so this Theorem can potentially say something about ordered sets. Note that the functions

$$\varphi_1\colon \{0,1\} \to \{0,1\}\colon \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 1 \end{cases} \quad \text{and} \quad \varphi_2\colon \{0,1\} \to \{0,1\}\colon \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 1 \end{cases}$$

are the only possible control functions (according to Definition 2.1.1). A map $f\colon (P,\leq) \to (P,\leq)$ is a φ_1-contraction if and only if f is monotone; and it is a φ_2-contraction if and only if f is essentially constant ($fx \cong fy$ for all $x, y \in P$). Any non-empty ordered set is Cauchy-complete as a Q-enriched category. It is part of the hypotheses in Theorem 2.3.2 that there exists an $x \in P$ such that $x \leq fx$ and $fx \leq x$; in other words, *by hypothesis* there exists a fixpoint $fx \cong x$. Of course this makes the conclusion of the Theorem (namely, the existence of a fixpoint) trivial! Moreover, the fixpoint that is constructed in the proof (as an object representing a left adjoint presheaf) is in this particular case precisely isomorphic to the fixpoint given as hypothesis. So, for the two-element Boolean algebra, Theorem 2.3.2 does not give any result; the Theorem can thus only be meaningful for "richer" quantales. (We hasten to add that there exist of course very important fixpoint theorems for ordered sets; but these usually require more stringent completeness conditions on the ordered set and/or more stringent continuity conditions on the map. See e.g. [8].)

3.2 Metric spaces

Lawvere's quantale $Q = ([0,\infty], \bigwedge, +, 0)$ is linear, and therefore continuous[9]. It is also an integral quantale: the unit 0 for the monoid structure is the top element of the lattice (note again that the order on $[0,\infty]$ is the reverse of the natural order!). This makes the notion of contraction in Definition 2.1.1 slightly simpler, so by application of Theorem 2.3.2 we can produce the following result:

Corollary 3.2.1 *Let $\varphi\colon [0,\infty] \to [0,\infty]$ be an upper-semicontinuous function so that $\varphi(t) < t$ for any $t \notin \{0,\infty\}$ and $\varphi(0) = 0$. Let $f\colon X \to X$ be a map on a Cauchy-complete generalized metric space (X,d) such that*

[9] Any complete linear lattice L is completely distributive and (therefore) also continuous. In fact, we have $a \ll b$ if and only if either $a = 0$, or $a < b$, or ($a = b$ and $b \neq \bigvee\{x \in L \mid x < b\}$), see [8].

$d(fx, fy) \leq \varphi(d(x, y))$ *for all* $x, y \in X$. *If there is an* $x \in X$ *such that* $d(x, fx) \neq \infty \neq d(fx, x)$ *then the sequence* $(f^n x)_{n\in\mathbb{N}}$ *converges to a fixpoint of* f.

If (X, d) is a *finitary* generalized metric space (i.e. no distance is infinite), then any $x \in X$ will produce a convergent sequence $(f^n x)_{n\in\mathbb{N}}$; and Proposition 2.1.3 implies that all such sequences $(f^n x)_{n\in\mathbb{N}}$ converge to an essentially unique fixpoint of f (unique if the space is also *separated*).

If (X, d) is a *symmetric* generalized metric space, then any $x \in X$ such that $d(x, fx) \neq \infty$ will produce a convergent sequence $(f^n x)_{n\in\mathbb{N}}$; and Proposition 2.1.4 implies that any two fixpoints of f are either isomorphic (equal if the space is also *separated*) or at distance ∞ from each other (i.e. the space (X, d) decomposes as a categorical sum of two non-empty spaces, and the fixpoints are in different summands).

In particular, for ordinary metric spaces we may note:

Example 3.2.2 Let (X, d) be a Cauchy-complete metric space, and suppose that $\varphi\colon [0, \infty] \to [0, \infty]$ is an upper-semicontinuous function that maps 0 to 0 and so that $\varphi(t) < t$ for any $t \notin \{0, \infty\}$. Then any map $f\colon X \to X$ satisfying $d(fx, fy) \leq \varphi(d(x, y))$ for all $x, y \in X$ has a unique fixpoint, and for any $x \in X$ the sequence $(f^n x)_{n\in\mathbb{N}}$ converges to that fixpoint.

Some conditions in this statement can be weakened somewhat. For instance, it is enough to require that φ is (defined on and) upper-semicontinuous on the closure of $\{d(x, y) \mid x, y \in X\}$. Indeed, in the proofs of Propositions 2.1.3, 2.1.4 and 2.2.3, the control function is only applied to (sequences of) elements in that closed set. This is how the above example is formulated by Boyd and Wong [3, Theorem 1] (see also [22]).

On the other hand, the control function defined by $\varphi(t) = k \cdot t$ for $0 < k < 1$ certainly satisfies the conditions in Corollary 3.2.1, so we find the following particular case:

Example 3.2.3 Let $f\colon X \to X$ be a map on a Cauchy-complete generalized metric space (X, d) for which there exists a $0 < k < 1$ such that $d(fx, fy) \leq k \cdot d(x, y)$ for all $x, y \in X$. If there is an $x \in X$ such that $d(x, fx) \neq \infty \neq d(fx, x)$ then the sequence $(f^n x)_{n\in\mathbb{N}}$ converges to a fixpoint of f.

If (X, d) is an ordinary metric space, we find here the well-known Banach Fixpoint Theorem.

Finally, we mention that Ackerman [1] has produced an example of a *non-expansive* contraction – whose control function merely satisfies $\varphi(t) \leq t$ instead of $\varphi(t) < t$ for $t \notin \{0, \infty\}$ – on a Cauchy-complete metric space which does not have a fixpoint. This shows that this condition on the control map cannot be weakened without strenghtening some other conditions in Corollary 3.2.1.

3.3 Fuzzy orders

The quantale $Q = ([0, 1], \bigvee, *, 1)$, where $*$ is a left-continuous t-norm, is linear (thus continuous, see a previous footnote) and integral. Hence, by application of Theorem 2.3.2 we find:

Corollary 3.3.1 *Let $(P, [\![\cdot \leq \cdot]\!])$ be a complete fuzzy preorder. Suppose that $f \colon P \to P$ is a function such that $[\![fx \leq fy]\!] \geq \varphi([\![x \leq y]\!])$ for some lower-semicontinuous function $\varphi \colon [0, 1] \to [0, 1]$ satisfying $\varphi(t) > t$ for all $0 < t < 1$, and $\varphi(1) = 1$. If there is an $x \in P$ such that $[\![x \leq fx]\!] \neq 0 \neq [\![fx \leq x]\!]$, then the sequence $(f^n(x))_{n \in \mathbb{N}}$ converges to a fixpoint of f.*

This is a (straightforward) generalization of Corollary 3.2.1, since Lawvere's quantale $([0, \infty], \bigwedge, +, 0)$ is isomorphic to the product t-norm $([0, 1], \bigvee, \cdot, 1)$ by the order-reversing map $[0, \infty] \to [0, 1] \colon t \mapsto \exp(-t)$.

3.4 Probabilistic metric spaces

The integral quantale $(\Delta, \bigvee, *, e)$ of distance distributions (wrt. a left-continuous t-norm $*$) is completely distributive[10], hence continuous, so we can apply Theorem 2.3.2.

Corollary 3.4.1 *Let $\varphi \colon \Delta \to \Delta$ be a lower-semicontinuous function satisfying $\varphi(u) > u$ for all $0 < u < e$, and $\varphi(e) = e$. Suppose that $f : X \to X$ is a function on a Cauchy-complete generalized probabilistic metric space*

[10] Indeed, the complete distributivity of the underlying suplattices follows from [9, Theorem 2.1.17], who show that the tensor product of completely distributive complete lattices is completely distributive.

$(X, d, *)$ *such that* $d(fx, fy, t) \geq \varphi(d(x, y, t)$ *for all* t. *If there exists an* $x \in X$ *such that* $d(x, fx, t) \neq 0 \neq d(fx, x, t)$ *then* f *has a fixpoint.*

It follows furthermore from Proposition 2.1.3 that, if $d(x, y, \infty) = 1$ for all $x, y \in X$ (i.e. the space is *finitary*), then the fixed point is unique.

There are indeed examples of control functions $\varphi\colon \Delta \to \Delta$ that the above statement asks for, e.g.

$$\varphi(u)(t) := \begin{cases} \frac{1}{2}(u(t) + 1) & \text{if } 0 < t \leq \infty \\ 0 & \text{if } t = 0 \end{cases}$$

Unfortunately though, the "Banach control function" which is appropriate in the setting of probabilistic metric spaces[11],

$$\varphi(u)(t) = u(Kt) \text{ for some } 1 < K < \infty,$$

does not satisfy $\varphi(u) \neq u$ for all $0 \neq u \neq e$ (e.g. the "almost constant" functions, defined by $u(t) = u_0$ for $0 < u_0 < 1$ and $0 < t \leq \infty$, are fixpoints of φ). One possible solution (hinted at by a result in [10]) would be to work with *finitary* probabilistic metric spaces. These can be seen as categories enriched in the subquantale

$$\Delta^+ = \{u \in \Delta \mid u(\infty) = 1\} \cup \{0\}$$

of Δ. Restricted to Δ^+, the Banach control function does not have fixpoints other than 0 and e: if $u \in \Delta^+ \setminus \{0\}$ satisfies $u(t) = u(Kt)$, then for any $0 < t_0 < \infty$,

$$1 = u(\infty) = u(\bigvee_{n \in \mathbb{N}} K^n t_0) = \bigvee_{n \in \mathbb{N}} u(K^n t_0) = u(t_0),$$

so indeed $u = e$. However, we do not know whether Δ^+ is continuous (we conjecture that it is not), so we do not know whether we can apply Theorem 2.3.2 without modifications: this will be a topic of futher investigation.

[11] Contractions with this control function are called *probabilistic q-contractions* in [11].

4. Conclusion and further work

With our study of fixpoint theorems for quantale-enriched categories, we exemplified that such results depend not only on the strength of the contraction and the completeness of the space, but also on the algebraic properties of the underlying quantale: any fixpoint theorem results from an equilibrum between those three aspects.

In future work, we want to investigate how several examples of fixpoint theorems in the literature (see e.g. [5, 10, 11]) fit – or, perhaps, do not fit – with our quantale-enriched approach. This could lead to variants on our Theorem 2.3.2, where different algebraic properties of Q are combined with different conditions on the control functions of contractions, or with different completeness conditions on the Q-categories (see e.g. [27]).

We also intend to study fixpoint theorems for *quantaloid*-enriched categories. This generalization, far from trivial, has the benefit to include in particular the theory of partial metric spaces [26, 14] and of probabilistic partial metric spaces [12], two areas for which only few fixpoint theorems are known [19, 21, 23].

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