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TORSION THEORIES OF SIMPLICIAL GROUPS WITH TRUNCATED MOORE COMPLEX

Guillermo LÓPEZ CAFAGGI

Résumé. Nous introduisons un treillis linéaire $\mu(Grp)$ de théories de torsion dans la catégorie des groupes simpliciaux. On définit les théories de torsion où les catégories de torsion et sans torsion sont données, respectivement, par les groupes simpliciaux dont le complexe est tronqué supérieurement et inférieurement. Ces théories de torsion étendent naturellement les théories de torsion dans les groupoides internes dans les groupes. Nous relions ces résultats aux groupes d'homotopie, en montrant en particulier que les groupes d'homotopie sont calculés comme des quotients des groupes de torsion.

Abstract. We introduce a linearly ordered lattice $\mu(Grp)$ of torsion theories in simplicial groups. The torsion theories are defined where the torsion and torsion-free subcategories are given by the simplicial groups with bounded above or below Moore complex, respectively. These torsion theories extend naturally the torsion theories in internal groupoids in groups. Connections of this lattice with the homotopy groups are established since the homotopy groups of a simplicial group can be calculated as the quotients of torsion subojects.

Keywords. semi-abelian category, torsion theories, preradicals, simplicial groups, homotopy groups

Mathematics Subject Classification (2010). 18E13, 18E40, 18N50, 55Q05, 18G35

1. Introduction

The notion of semi-abelian category [17] allows a categorical and unified treatment of the categories of groups, rings, Lie-algebras and other non-abelian categories in a similar way as abelian categories generalise abelian groups and categories of modules. Torsion theories were originally introduced for abelian categories by Dickson, and have been generalized by several authors to different non-abelian categories as for example in [4], [7] and [19].

For a torsion theory in a semi-abelian category $\mathbb{X}$ we mean a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories such that:

1. any morphism $f : T \to F$ with T in $\mathcal{T}$ and F in $\mathcal{F}$ is the zero morphism;
2. for any object X in $\mathbb{X}$ there is a short exact sequence

   $$0 \longrightarrow T_X \longrightarrow X \longrightarrow F_X \longrightarrow 0$$

   with T_X in $\mathcal{T}$ and F_X in $\mathcal{F}$.

An internal groupoid X in $\mathbb{X}$ is a diagram:

$$\begin{tikzcd}
X_2 \arrow[r, shift left=2, "p_2"] \arrow[r, "p_1"] \arrow[r, shift right=2, "p_0"] & X_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_1"] & X_0 \arrow[l, "s_0"']
\end{tikzcd}$$

where

$$\begin{tikzcd}
X_2 \arrow[r, "p_1"] \arrow[d, "p_0"] & X_1 \arrow[d, "d_0"] \\
X_1 \arrow[r, "d_1"] & X_0
\end{tikzcd}$$

is a pullback square. The objects X_0 and X_1 are called the 'object of objects' and the 'object of arrows' of X, the morphisms d_0, d_1 are called 'domain' and 'codomain' and the morphisms $p_0, p_1, p_2, d_0, d_0, s_0$ satisfy the usual equations that determine a category. The category $Grpd(\mathbb{X})$ of internal groupoids in a semi-abelian category $\mathbb{X}$, which is itself semi-abelian, exhibits two examples of non-abelian torsion theories. The first is given by the pair

$(Ab(\mathbb{X}), Eq(\mathbb{X}))$ where $Ab(\mathbb{X})$ is the category of internal abelian obejcts in $\mathbb{X}$ and $Eq(\mathbb{X})$ is the category of equivalence relations, i.e. internal groupoids where the induced morphism $(d_0, d_1) : X_1 \to X_0^2$ is monic [4]. The second example is given by $(Conn(Grpd(\mathbb{X})), Dis(\mathbb{X}))$, where $Conn(Grpd(\mathbb{X}))$ is the category of connected internal groupoids and $Dis(\mathbb{X})$ is the category of discrete groupoids. Since for an internal groupoid X the nerve $\mathcal{N}(X)$ is a simplicial object in $\mathbb{X}$ it is natural to ask if there are torsion theories in simplicial objects such that they expand or generalize those of internal groupoids.

In section 2, we recall the basics theory of torsion theories in semi-abelian categories. Section 3 and 4 introduce two different families of torsion theories, $\mathcal{COK}_n$ and $\mathcal{KER}_n$, in the category of proper chain complexes and we exhibits some connections with the homological aspects of chain complexes. Section 5 and 6 introduce the torsion theories $\mu_{n\geq}$ and $\mu_{\geq n}$ in simplicial groups whose associated Moore complex behave as those in proper chains. Section 7 studies the homotopy groups of the simplicial groups defined by the torsion theories of the lattice $\mu(Grp)$. In particular, the homotopy groups of a simplicial group X can be studied using torsion subobjects.

2. Torsion theories in semi-abelian categories

2.1. *Notation.* By a regular category $\mathbb{X}$ we mean a finitely complete category with coequalizers of kernel pairs with the property that any morphism $f : X \to Y$ in $\mathbb{X}$ factors as a regular epimorphism e_f followed by a monomorphism m_f:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e_f"'] & & Y \\ & f(X) \arrow[ur, "m_f"'] & \end{tikzcd}$$

and these factorizations are pullback stable. As usual, we will call the subobject represented by m_f the *image* of f. A category $\mathbb{X}$ is pointed if it has a zero object 0, i.e. an object which is both initial and terminal. For any pair of objects X, Y in $\mathbb{X}$ the unique morphism $X \to Y$ that factors through the zero object will be denoted by 0.

A regular category $\mathbb{X}$ is called (Barr-)exact if any equivalence relation is a kernel pair $Eq(f)$ for some morphism f in $\mathbb{X}$ [1].

Definition 2.2. *[17] A category $\mathbb{X}$ is called semi-abelian if it is pointed, (Barr-)exact, protomodular in the sense of Bourn ([3]) and has binary coproducts.*

In a semi-abelian category, a short exact sequence is a pair of composable morphisms (k, p), as in the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "k"] & X \arrow[r, "p"] & Y \arrow[r] & 0 \end{tikzcd}$$

such that $k = ker(p)$ is the kernel of p and $p = cok(k)$ is the cokernel of k. In such a short exact sequence the object Y will be denoted as X/K. Recall that in a semi-abelian category $\mathbb{X}$ regular epimorphisms are normal epimorphisms, that is cokernels of some morphisms in $\mathbb{X}$.

We will need the following results.

Lemma 2.3. *[2] Let $\mathbb{X}$ be a semi-abelian category. Given two normal subobjects $k : K \to A$ and $l : L \to A$ such that $k \leq l$, i.e. k factors through l, then there is a short exact sequence:*

$$\begin{tikzcd} 0 \arrow[r] & L/K \arrow[r] & A/K \arrow[r] & A/L \arrow[r] & 0 \end{tikzcd} .$$

Proposition 2.4. *[17] Let $\mathbb{X}$ be a semi-abelian category. Given a commutative diagram in $\mathbb{X}$*

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, "p"] & B \arrow[d, "q"] \\ C \arrow[r, "n"] & D \end{tikzcd}$$

with p and q normal epimorphisms, m a normal monomorphism and n a monomorphism, then n is a normal monomorphism.

Torsion theories can be defined in a more general context, but in this article we will restrict to semi-abelian categories.

Definition 2.5. *Let $\mathbb{X}$ be a semi-abelian category. A torsion theory in $\mathbb{X}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full and replete subcategories of $\mathbb{X}$ such that:*

TT1 A morphism $F : T \to F$ with T in $\mathcal{T}$ and F in $\mathcal{F}$ is a zero morphism.

TT2 For any object X in $\mathbb{X}$ there is a short exact sequence:

$$0 \longrightarrow T_X \xrightarrow{\epsilon_X} X \xrightarrow{\eta_X} F_X \longrightarrow 0 \tag{1}$$

with T_X in $\mathcal{T}$ and F_X in $\mathcal{F}$ (which is necessarily unique up to isomorphism).

In a torsion theory $(\mathcal{T}, \mathcal{F})$, $\mathcal{T}$ is the torsion category whose objects are called torsion objects, and similarly $\mathcal{F}$ is the torsion-free category of the torsion theory. Torsion subcategories are normal mono-coreflective subcategories of $\mathbb{X}$, i.e. coreflective subcategories such that each component ϵ_X of the counit ϵ is a normal monomorphism, while torsion-free subcategories are normal epi-reflective subcategories, so that each component η_X is a normal epimorphism:

$$\begin{tikzcd} \mathcal{T} \arrow[r, bend left, "J"] \arrow[r, phantom, "\perp"] & \mathbb{X} \arrow[l, bend left, "T"] \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{F} \arrow[l, bend left, "I"] \end{tikzcd} . \tag{2}$$

The X-component of the counit ϵ of $J \dashv T$ and of the unit η of $F \dashv I$ both appear in the short exact sequence (1). A subcategory $\mathbb{A}$ of $\mathbb{X}$ is closed under extensions in $\mathbb{X}$ if every time we have a short exact sequence

$$0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0$$

with A and B in $\mathbb{A}$ then X belongs to $\mathbb{A}$. In a torsion theory both $\mathcal{T}$ and $\mathcal{F}$ are closed under extensions in $\mathbb{X}$ [4].

Definition 2.6. *Let $\mathbb{X}$ be a semi-abelian category. A preradical in $\mathbb{X}$ is a normal subfunctor $\sigma : r \to Id_{\mathbb{X}}$ of the identity functor of $\mathbb{X}$, i.e. for all object X we have a normal monomorphism $\sigma_X : r(X) \to X$ and for every morphism $f : X \to Y$ a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ r(X) \arrow[u, "\sigma_X"] \arrow[r, "r(f)"] & r(Y) \arrow[u, "\sigma_Y"'] \end{tikzcd} . \tag{3}$$

Moreover, a preradical r is called:

- *idempotent if* $rr(X) = r(X)$ *for all objects* X*;*
- *a radical if* $r(X/r(X)) = 0$ *for all objects* X*;*
- *hereditary if for every monomorphism* $f : X \to Y$*, diagram (3) is a pullback.*

Given a preradical r we can consider the r-torsion subcategory $\mathcal{T}_r$ and the r-torsion-free subcategory $\mathcal{F}_r$ of $\mathbb{X}$:

$$\mathcal{T}_r = \{X \in \mathbb{X} \mid r(X) \cong X\} \quad \text{and} \quad \mathcal{F}_r = \{X \in \mathbb{X} \mid r(X) \cong 0\}.$$

In general, the pair (T_r, F_r) only satisfies axiom TT1 of a torsion theory. Conversely, a torsion theory $(\mathcal{T}, \mathcal{F})$ defines an idempontent radical

$$t = JT : \ \mathbb{X} \longrightarrow \mathbb{X} \ .$$

In fact, there is a bijection:

$$\{\text{torsion theories in } \mathbb{X}\} \longleftrightarrow \{\text{idempotent radicals in } \mathbb{X}\}.$$

It is easy to see that a hereditary preradical is always idempotent. Conversely, an idempotent preradical is hereditary if and only if the category $\mathcal{T}_r$ is closed under subobjects in $\mathbb{X}$, i.e. for every monomophism $m : X \to Y$ with Y in $\mathcal{T}_r$ then X is in T_r. Furthermore, the previous bijection is restricted to a bijection:

$$\{\text{hereditary torsion theories in } \mathbb{X}\} \longleftrightarrow \{\text{hereditary radicals in } \mathbb{X}\}.$$

A torsion theory $(\mathcal{T}, \mathcal{F})$ is called *hereditary* if $\mathcal{T}$ is closed under subobjects. Similarly, $(\mathcal{T}, \mathcal{F})$ is called *cohereditary* if $\mathcal{F}$ is closed under quotients, i.e. for every normal epimorphism $p : X \to Y$ with X in $\mathcal{F}$ then so is Y in $\mathcal{F}$ (see [7]). It is also useful to recall that in any torsion theory $(\mathcal{T}, \mathcal{F})$, $\mathcal{T}$ is always closed under quotients and $\mathcal{F}$ is closed under subobjects.

In order to characterize torsion-free subcategories among normal epireflective subcategories, it is useful to recall the following result, which was first proved in [4] for homological categories:

Theorem 2.7. *([4], [11], [8]) Let* $\mathbb{X}$ *be a semi-abelian category and* $F \dashv I : \mathbb{X} \to \mathcal{F}$ *a normal epi-reflective subcategory of* $\mathbb{X}$ *with unit* η*, then the following are equivalent:*

1. *$\mathcal{F}$ is a torsion-free subcategory of $\mathbb{X}$;*
2. *the induced radical of $F \dashv I$ is idempotent;*
3. *the reflector $F : \mathbb{X} \to \mathbb{A}$ is semi-left-exact;*
4. *the reflector $F : \mathbb{X} \to \mathbb{A}$ is normal, i.e. $F(ker(\eta_X)) = 0$ for every object X in $\mathbb{X}$.*

Under these conditions the corresponding torsion category of $\mathcal{F}$ is given by the full subcategory $\mathcal{T} = Ker(F) = \{X \mid F(X) \cong 0\}$ and so $(Ker(F), \mathcal{F})$ is a torsion theory in $\mathbb{X}$.

Proof. Equivalences $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ are proved in [11] and $(3) \Leftrightarrow (4)$ is proved in [4]. □

2.8. Given torsion theories $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{S}, \mathcal{G})$ in $\mathbb{X}$ with associated idempotent radicals τ and σ we have that $\mathcal{T} \subseteq \mathcal{S}$ if and only if $\mathcal{G} \subseteq \mathcal{F}$, this allows us to define a partial order in the (possibly big) lattice $\mathbb{X}tors$ of torsion theories in $\mathbb{X}$:

$$(\mathcal{T}, \mathcal{F}) \leq (\mathcal{S}, \mathcal{G}) \quad \text{if and only if} \quad \mathcal{T} \subseteq \mathcal{S}.$$

In this case we have that $\tau \leq \sigma$, so for an object X in $\mathbb{X}$ we have $\tau(X) \leq \sigma(X)$. The lattice $\mathbb{X}tors$ has as bottom and top element the trivial torsion theories denoted as:

$$0 := (0, \mathbb{X}) \quad \text{and} \quad \mathbb{X} := (\mathbb{X}, 0).$$

Given preradicals $\tau \leq \sigma$ we can define the quotient endofunctor as:

$$\sigma/\tau : \mathbb{X} \longrightarrow \mathbb{X} \ , \quad \sigma/\tau(X) = \sigma(X)/\tau(X).$$

and as consequence of Lemma 2.3 for each object X we have a short exact sequence:

$$0 \longrightarrow \sigma(X)/\tau(X) \longrightarrow X/\tau(X) \longrightarrow X/\sigma(X) \longrightarrow 0 \ . \quad (4)$$

For abelian categories the next result is due to P. Gabriel ([26]). A localization of a category $\mathbb{X}$ is a reflective subcategory $L \dashv I : \mathbb{X} \to \mathbb{A}$ such that L preserves finite limits.

Theorem 2.9. *Let $L \dashv I : \mathbb{X} \to \mathbb{A}$ be a localization of a semi-abelian category $\mathbb{X}$ with unit η. The subcategories of $\mathbb{X}$*

$$\mathcal{T}_L = ker(L) = \{X \mid L(X) \cong 0\} = \{X \mid \eta_X = 0\}$$

and

$$\mathcal{F}_L = \{X \mid \eta_X : X \to IL(X) \textit{ is monic}\}$$

define a torsion theory $(\mathcal{T}_L, \mathcal{F}_L)$ in $\mathbb{X}$.

Proof. First, notice that for any object X the morphism $X \to 0$ factors through η_X. Then, if $\eta_X = 0$ the morphism η_X factors through $X \to 0$ and, so $L(X) \cong 0$ since L is a reflection. Hence, for any object X we have that $L(X) = 0$ if and only if $\eta_X = 0$.

TT1) For a morphism $f : X \to Y$ consider the diagram given by the naturality of η:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\ IL(X) \arrow[r, "IL(f)"'] & IL(Y)\,. \end{tikzcd}$$

Now, if $\eta_X = 0$ and η_Y is monic it is clear that f is the zero morphism.

TT2) Consider for an object X the normal epi-mono factorization (p, m) of η_X and the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & ker(\eta_X) \arrow[r, "k"] & X \arrow[r, "p"] \arrow[dr, "\eta_X"'] & \eta_X(X) \arrow[r] \arrow[d, "m"] & 0 \\ & & & IL(X)\,. & \end{tikzcd} \tag{5}$$

To see that $ker(\eta_X)$ is torsion, consider the commutative diagram:

$$\begin{tikzcd} ker(\eta_X) \arrow[r, "k"] \arrow[d, "\eta_{ker(\eta_X)}"'] & X \arrow[d, "\eta_X"] \\ IL(ker(\eta_X)) \arrow[r, "IL(k)"'] & IL(X)\,. \end{tikzcd}$$

Since L preseves finite limits then $IL(k)$ is monic and since $\eta_X k = 0$ this implies that $\eta_{ker(\eta_X)} = 0$. To see that $\eta_X(X)$ is torsion-free consider the diagram:

$$\begin{tikzcd}
X \arrow[rr, "\eta_X"] \arrow[dr, "p"] \arrow[dd, "\eta_X"] & & IL(X) \arrow[dd, "\eta_{IL(X)}"] \\
& \eta_X(X) \arrow[ur, "m"'] \arrow[dd, "\eta_{\eta_X(X)}"] & \\
IL(X) \arrow[rr, "IL(\eta_X)"] \arrow[dr, "IL(p)"'] & & ILIL(X) \\
& IL(\eta_X(X)) . \arrow[ur, "IL(m)"'] &
\end{tikzcd}$$

Notice that since $\mathbb{A}$ is a reflective subcategory then $IL(\eta_X)$ and $\eta_{IL(X)}$ are isomorphisms. Finally, $\eta_{\eta_X(X)}$ is also a monomorphism. □

It is also worth mentioning that, under the assumptions from above, a localization $L : \mathbb{X} \to \mathbb{A}$ induces a preradical on $\mathbb{X}$ as $r = ker(\eta)$, so we have $\mathcal{T}_L = \mathcal{T}_r$.

Corollary 2.10. *The torsion theory $(\mathcal{T}_L, \mathcal{F}_L)$ induced by a localization $L \dashv i$ of a semi-abelian category $\mathbb{X}$ is hereditary.*

Proof. Since L preserves finite limits it preserves monomorphisms so if $m : S \to X$ with $L(X) = 0$ then $L(S) = 0$. □

Lemma 2.11. *Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory in a semi-abelian category $\mathbb{X}$. Then $\mathcal{T}$ is closed under finite limits in $\mathbb{X}$. In particular, $\mathcal{T}$ is closed under kernel pairs of morphisms in $\mathcal{T}$.*

Proof. Since $\mathcal{T}$ is closed under kernels of arrows in $\mathcal{T}$, we only need to prove that $\mathcal{T}$ is closed under pullbacks of morphisms in $\mathcal{T}$. Consider the commutative diagram

$$\begin{tikzcd}
ker(p_1) \arrow[r] \arrow[d, "p_0'"] & P \arrow[r, "p_1"] \arrow[d, "p_0"] & C \arrow[d, "g"] \\
ker(f) \arrow[r] & A \arrow[r, "f"] & B .
\end{tikzcd}$$

with f and g morphisms in $\mathcal{T}$. Since P is a pullback then $ker(p_1) \cong ker(f)$. Now, since $\mathcal{T}$ is closed under subobjects, if A is torsion then $ker(f)$ is also torsion. Consider e, m the normal epi/mono factorization of p_1 and the short exact sequence:

$$0 \longrightarrow ker(p_1) \longrightarrow P \xrightarrow{e} p_1(P) \longrightarrow 0 \ .$$

Then $p_1(P)$ is a subobject of C so it is torsion, and finally, since $\mathcal{T}$ is closed under extension P is torsion. □

A *quasi-hereditary* torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathbb{X}$ is a torsion theory such that $\mathcal{T}$ is closed under regular subobjects, i.e. if $e : X \to T$ is an equalizer with T torsion then X is torsion. In [15] quasi-hereditary torsion theories are studied in homological categories.

Theorem 2.12. *[15] Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in a homological category $\mathbb{X}$. The following are equivalent:*

1. *$(\mathcal{T}, \mathcal{F})$ is quasi-hereditary.*
2. *The associated idempotent radical $t : \mathbb{X} \to \mathbb{X}$ preserves finite limits.*
3. *The associated idempotent radical $t : \mathbb{X} \to \mathbb{X}$ preserves equalizers.*
4. *For every regular subobject $e : E \to A$ in $\mathbb{X}$ then $F(e)$ is a monomorphism in $\mathcal{F}$.*

Lemma 2.13. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in a semi-abelian category $\mathbb{X}$. Then $\mathcal{T}$ is an exact category.*

Proof. We will first prove that an arrow q in $\mathcal{T}$ is a regular epimorphism in $\mathbb{X}$ if and only if it is a regular epimorphism in $\mathcal{T}$. Clearly, if q is a regular epimorphism in $\mathcal{T}$ and the inclusion $J : \mathcal{T} \to \mathbb{X}$ preserves colimits then q is a regular epimorphism in $\mathbb{X}$. Now, if q is a regular epimorphism in $\mathbb{X}$ it is a coequalizer of its kernel pair $Eq(q)$ in $\mathbb{X}$. And since $\mathcal{T}$ is closed under kernel pairs in $\mathbb{X}$ we have the isomorphism $Eq(q) \cong t(Eq(q))$, so q is a regular epimorphism in $\mathcal{T}$.

To prove pullback stability of regular epimorphism consider the pullback diagram in $\mathcal{T}$:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\
A \arrow[r] & B
\end{tikzcd}$$

with p a regular epimorphism in $\mathcal{T}$. Since the inclusion $i : \mathcal{T} \to \mathbb{X}$ preserves pullbacks and quotients, we have that p is a regular epimorphism in $\mathbb{X}$, and so is p' in $\mathcal{T}$.

Finally, since $\mathcal{T}$ is closed under quotients and $\mathbb{X}$ is an exact category, any equivalence relation in $\mathcal{T}$ must be effective and $\mathcal{T}$ is an exact category. □

Theorem 2.14. *Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory in a semi-abelian category $\mathbb{X}$. Then $\mathcal{T}$ is a semi-abelian category.*

Proof. First since $\mathcal{T}$ is coreflective in $\mathbb{X}$ it is complete and cocomplete as well as pointed. By Lemma 2.13 $\mathcal{T}$ is exact. Finally, by Theorem 2.12 the full inclusion $J : \mathcal{T} \to \mathbb{X}$ preserve finite limits and hence short split exact sequences, so if $\mathbb{X}$ is protomodular then so is $\mathcal{T}$. □

This theorem admits a dual version. A Birkhoff subcategory $\mathbb{A}$ of a regular category $\mathbb{X}$, is a full regular epi-reflective subcategory that is closed under subobjects and quotients in $\mathbb{X}$. It is known that if $\mathbb{X}$ semi-abelian then so is any Birkhoff subcategory $\mathbb{A}$.

Corollary 2.15. *Let $(\mathcal{T}, \mathcal{F})$ be a cohereditary torsion theory in a semi-abelian category $\mathbb{X}$. Then $\mathcal{F}$ is a semi-abelian category.*

Proof. If $(\mathcal{T}, \mathcal{F})$ is cohereditary, $\mathcal{F}$ is closed under quotients in $\mathbb{X}$, so $\mathcal{F}$ is a Birkhoff subcategory of $\mathbb{X}$. □

3. Torsion theories in chain complexes

Throughout this section $\mathbb{X}$ will denote a semi-abelian category.

A chain complex M in $\mathbb{X}$ is a family of morphisms $\{\delta_i : M_i \to M_{i-1}\}_{i\in\mathbb{Z}}$ with the condition $\delta_i\delta_{i+1} = 0$ for all i. A morphism of chain complexes $f : M \to N$ is a family of morphisms $f_i : M_i \to N_i$ such that $f_{i-1}\delta_i = \delta_i f_i$

for all i. For a chain complex M and for each i we will write e_i and m_i for the normal epi/mono factorization of each δ_i:

$$\begin{tikzcd}
M_i \arrow[rr, "\delta_i"] \arrow[dr, "e_i"'] & & M_{i-1} \\
& \delta_i(M_i) \arrow[ur, "m_i"'] &
\end{tikzcd}$$

And we call a chain complex M *proper* if each δ_i is a proper morphism i.e. each m_i is a normal monomorphism for each i. We will write $ch(\mathbb{X})$ for the category of chain complexes in $\mathbb{X}$ and $pch(\mathbb{X})$ for the subcategory of proper chain complexes. In [12] it is noticed that since $\mathbb{X}$ is a semi-abelian category then $ch(\mathbb{X})$ is also semi-abelian, but this is not the case for $pch(\mathbb{X})$, since it may not have kernels. However, $pch(\mathbb{X})$ does have cokernels as follows.

Lemma 3.1. *The category $pch(\mathbb{X})$ has cokernels and they are computed as in $ch(\mathbb{X})$.*

Proof. For a morphism $f : M \to N$ of proper chain complexes (M, d), (N, δ) consider the commutative diagram for each i

$$\begin{tikzcd}
M_i \arrow[r, "f_i"] \arrow[dd, "d_i"] & N_i \arrow[rr, "p"] \arrow[dd, "\delta_i"] \arrow[dr, "e_i"] & & cok(f_i) \arrow[dd, "\delta_i'"] \arrow[dr, "e_i'"] & \\
& & \delta_i(N_i) \arrow[rr, "q'"] \arrow[dl, "m_i"] & & \delta_i'(cok(f_i)) \arrow[dl, "m_i'"] \\
M_{i-1} \arrow[r, "f_{i-1}"] & N_{i-1} \arrow[rr, "q"] & & cok(f_{i-1}) &
\end{tikzcd}$$

where δ_i' is induced by universal property of the cokernel p and m_i, e_i and m_i', e_i' are the normal epi-mono image factorizations of δ_i and δ_i' respectively. Now, since taking images is functorial we have q' such that $q'e_i = e_i'p$ and $m_i'q' = qm_i$, then q' is a normal epimorphism since p and e_i' are also normal epimorphisms. Finally, by Proposition 2.4 if m_i is a normal monomorphism and m_i' is a monomorphism, then m_i' is a normal monomorphism and so δ_i' is a proper morphism and it is the cokernel of f in $pch(\mathbb{X})$. $\square$

By a short exact sequence in $pch(\mathbb{X})$ we mean a short exact sequence in $ch(\mathbb{X})$ such that every object is a proper chain complex. We will introduce well-known functors in very different settings of algebraic topology that still makes sense in our context. We will follow the terminology of [5].

3.2. Let $ch(\mathbb{X})_{n\geq}$ be the category of n-truncated (above) chain complexes, i.e. chain complexes defined for degrees $n \geq i$ for a fixed $n \in \mathbb{Z}$. We can identify $ch(\mathbb{X})_{n\geq}$ with the full subcategory of $ch(\mathbb{X})$ of chain complexes with $M_i = 0$ for $i > n$. Actually, we have the functors:

- $\mathbf{tr}_n : \ ch(\mathbb{X}) \longrightarrow ch(\mathbb{X})_{n\geq}$ is the canonical (above) truncation:

$$\mathbf{tr}_n(M) = \ M_n \longrightarrow M_{n-1} \longrightarrow M_{n-2} \longrightarrow \dots$$

- $\mathbf{sk}_n : \ ch(\mathbb{X})_{n\geq} \longrightarrow ch(\mathbb{X})$ is the canonical inclusion or skeleton functor:

$$\mathbf{sk}_n(M) = \ \dots \longrightarrow 0 \longrightarrow 0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \dots$$

- $\mathbf{cosk}_n : \ ch(\mathbb{X})_{n\geq} \longrightarrow ch(\mathbb{X})$ the coskeleton functor is given by:

$$\mathbf{cosk}_n(M) = \ \dots \longrightarrow 0 \longrightarrow Ker(\delta_n) \xrightarrow{ker(\delta_n)} M_n \xrightarrow{\delta_n} M_{n-1} \dots$$

- $\mathbf{cot}_n : \ ch(\mathbb{X}) \longrightarrow ch(\mathbb{X})_{n\geq}$ the (above) cotruncation functor:

$$\mathbf{cot}_n(M) = \ Cok(\delta_{n+1}) \xrightarrow{\delta'_n} M_{n-1} \longrightarrow M_{n-2} \longrightarrow \dots$$

  where δ'_n is induced by $\delta_n : M_n \to M_{n-1}$ and the universal property of $cok(\delta_{n+1})$.

This functors give a string of adjunctions:

$$\mathbf{cot}_n \dashv \mathbf{sk}_n \dashv \mathbf{tr}_n \dashv \mathbf{cosk}_n : \begin{tikzcd} ch(\mathbb{X}) \arrow[d, bend right=80] \arrow[d, phantom, bend right=55, "\dashv"] \arrow[d, phantom, "\dashv"] \arrow[d, phantom, bend left=55, "\dashv"] \arrow[d, bend left=25] \\ ch(\mathbb{X})_{n\geq} \arrow[u, bend left=25] \arrow[u, bend right=80] \end{tikzcd} .$$

We will write $\mathbf{Sk}_n = \mathbf{sk}_n\mathbf{tr}_n$, $\mathbf{Cosk}_n = \mathbf{cosk}_n\mathbf{tr}_n$ and $\mathbf{Cot}_n = \mathbf{sk}_n\mathbf{cot}_n$.

Notice that for abelian categories the cotruncation functor above is exactly L. Illusie's truncation functor in [16].

Lemma 3.3. *The category* $ch(\mathbb{X})_{n-1\geq}$ *is a normal epireflective subcategory of* $ch(\mathbb{X})$ *with the adjunction* $\mathbf{cot}_{n-1} \dashv \mathbf{sk}_{n-1}$. *Moreover, the category* $ch(\mathbb{X})_{n-1\geq}$ *is closed under extensions in* $ch(\mathbb{X})$.

Proof. For a chain complex M the unit η_M of $\mathbf{cot}_{n-1} \dashv \mathbf{sk}_{n-1}$ is given by:

$$\begin{tikzcd}
M = \dots \arrow[r] \arrow[d, "\eta_M"] & M_n \arrow[r, "\delta_n"] \arrow[d] & M_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "cok(\delta_n)"] & M_{n-2} \arrow[r, "\delta_{n-2}"] \arrow[d, "1"] & \dots \\
\mathbf{Cot}_{n-1}(M) = \dots \arrow[r] & 0 \arrow[r] & Cok(\delta_n) \arrow[r, "\delta'_{n-1}"] & M_{n-2} \arrow[r, "\delta_{n-2}"] & \dots
\end{tikzcd}$$

which is a component-wise normal epimorphism, and hence, η_M is a normal epimorphism in $ch(\mathbb{X})$.

Since a short exact sequence in $ch(\mathbb{X})$ is a component-wise short exact sequence then it is clear that $ch(\mathbb{X})_{n-1\geq}$ is closed under extension in $ch(\mathbb{X})$.

□

In [19] conditions are given for a normal epireflective subcategory closed under extensions to be a torsion-free category. Here, $ch(\mathbb{X})_{n-1\geq}$ provides a counter-example of this situation, in the sense that $ch(\mathbb{X})_{n-1\geq}$ is a normal epireflective subcategory closed under extensions of $ch(\mathbb{X})$ that is not a torsion-free subcategory. Indeed, the functor $\mathbf{cot}_{n-1}$ is not normal.

For example, we can consider the truncated case of the category $Arr(Grp)$ of arrows in Grp and the cotruncation given by taking cokernel, $\mathbf{cot}_0 = cok : Arr(Grp) \to Grp$. Let $D_4 = \langle a, b \mid a^2 = b^4 = 1, aba = b^{-1} \rangle$ be the diehdral group and $X : \langle a \rangle \to D_4$ be the inclusion morphism. Then η_X is given by the vertical morphisms $(\eta_{X,1}, \eta_{X,0})$ in the diagram

$$\begin{tikzcd}
\langle a \rangle \arrow[r, "X"] \arrow[d, "\eta_{X,1}"] & D_4 \arrow[d, "\eta_{X,0}"] \\
0 \arrow[r] & D_4/\langle a, b^2 \rangle
\end{tikzcd}$$

Here $ker(\eta_X)$ is the inclusion $\langle a \rangle \to \langle a, b^2 \rangle$ which does not have trivial cokernel and so $\mathbf{cot}_0(ker(\eta_X))$ is not trivial.

However, we obtain a torsion theory when restricted to the case of proper chains and also $\mathbf{cot}_{n-1}$ will be a normal functor.

Lemma 3.4. *For each* $n \in \mathbb{Z}$ *the adjunction* $\mathbf{cot}_{n-1} \dashv \mathbf{sk}_{n-1}$ *can be restricted to proper chains:*

$$\begin{tikzcd} \mathbf{cot}_{n-1} \dashv \mathbf{sk}_{n-1} : \; pch(\mathbb{X}) \arrow[r, bend left] \arrow[r, phantom, "\perp"] & pch(\mathbb{X})_{n-1\geq} \arrow[l, bend left] \end{tikzcd} .$$

Proof. It suffices to prove that the $(n-1)$-cotruncation of a proper chain complex M is again proper. Indeed, consider the diagram

$$\begin{tikzcd} \dots \arrow[r] & M_n \arrow[r, "\delta_n"] & M_{n-1} \arrow[rr, "\delta_{n-1}"] \arrow[dr, two heads, "e_{n-1}"] \arrow[dd, two heads, "q_n"] & & M_{n-2} \arrow[r, "\delta_{n-2}"] \arrow[dd, "1"] & \dots \\ & & & \delta_{n-1}(M_{n-1}) \arrow[ur, hook, "m_{n-1}"'] \arrow[dd, "q'_n"] & & \\ \dots \arrow[r] & 0 \arrow[r] & Cok(\delta_n) \arrow[rr, "\delta'_{n-1}"] \arrow[dr, two heads, "e'_{n-1}"'] & & M_{n-2} \arrow[r, "\delta_{n-2}"] & \dots \\ & & & \delta'_{n-1}(Cok(\delta_n)) \arrow[ur, "m'_{n-1}"'] & & \end{tikzcd}$$

where δ'_{n-1} is induced by the cokernel $q_n = cok(\delta_n)$ and, (e_{n-1}, m_{n-1}) and (e'_{n-1}, m'_{n-1}) are the image factorizations of δ_{n-1} and δ'_{n-1}. Since $m_{n-1} = m'_{n-1}q'_n$ is a monomorphism then q'_n is both a monomorphism and a normal epimorphism, hence q'_n is an isomorphism. Then m'_{n-1} is a normal monomorphism. $\square$

Definition 3.5. *We define the full subcategories in* $pch(\mathbb{X})$ *for each* $n \in \mathbb{Z}$*:*

$$\mathcal{EP}_n = \{M \mid \delta_n \textit{ is a normal epi and } M_i = 0 \textit{ for } n-1 > i\}.$$

And, similarly,

$$\mathcal{MN}_n = \{M \mid \delta_n \textit{ is a normal mono and } M_i = 0 \textit{ for } i > n\}.$$

For instance, a proper chain complex M *in* $\mathcal{EP}_n$ *looks like this:*

$$\begin{tikzcd} \dots \arrow[r] & M_{n+1} \arrow[r, "\delta_{n+1}"] & M_n \arrow[r, two heads, "\delta_n"] & M_{n-1} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

with δ_n a normal epimorphism. Similarly, a proper chain complex M in $\mathcal{MN}_n$ looks like this:

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow M_n \xhookrightarrow{\delta_n} M_{n-1} \xrightarrow{\delta_{n-1}} M_{n-2} \longrightarrow \ldots$$

where δ_n is a normal monomorphism.

In addition, we will also consider the full subcategory of $ch(\mathbb{X})$:

$$Ker(\mathbf{cot}_{n-1}) = \{M \mid \mathbf{cot}_{n-1}(M) \cong 0\}.$$

Lemma 3.6. *Let $f : A \to B$ a morphism in $\mathbb{X}$. If f is proper and has trivial cokernel, then f is a normal epimorphism.*

Proof. Consider e, m the normal epi/mono factorization of f and the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "e"'] & B \arrow[r, "q"] & Cok(f) \\
f(A) \arrow[r, "m'"'] \arrow[ur, "m"] & Ker(q) \arrow[u, "k"'] &
\end{tikzcd}$$

where m' is induced by the kernel $Ker(q)$. If f is a proper morphism then m' is an isomorphism since the normal monomorphism m is the kernel of its cokernel $q = cok(f)$. Also, if $Cok(f) = 0$ then k is also an isomorphism. Finally, m is an isomorphism and f is a normal epimorphism. □

Lemma 3.7. *The restriction of the subcategory $Ker(\mathbf{cot}_{n-1})$ to $pch(\mathbb{X})$ is equivalent to $\mathcal{EP}_n$:*

$$Ker(\mathbf{cot}_{n-1}) \cap pch(\mathbb{X}) = \mathcal{EP}_n\,.$$

Proof. A chain complex M in $Ker(\mathbf{cot}_{n-1})$ has all $M_i = 0$ for $n-1 > i$ and the differential δ_n with trivial cokernel, so it follows from lemma 3.6. □

Theorem 3.8. *For each $n \in \mathbb{Z}$ we have:*

1. *The pair $(Ker(\mathbf{cot}_{n-1}), ch(\mathbb{X})_{n-1\geq})$ of subcategories in $ch(\mathbb{X})$ satisfy axiom TT1 of a torsion theory.*
2. *The pair $(\mathcal{EP}_n, pch(\mathbb{X})_{n-1\geq})$ of subcategories in $pch(\mathbb{X})$ is a torsion theory in $pch(\mathbb{X})$.*

Proof. 1) Let $f : M \to N$ a morphism in $ch(\mathbb{X})$ with M in $Ker(\mathbf{cot}_{n-1})$ and N in $ch(\mathbb{X})_{n-1\geq}$:

$$\begin{tikzcd}
\dots \arrow[r] & M_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "f_{n+1}"] & M_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & M_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & 0 \arrow[r] \arrow[d, "f_{n-2}"] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & N_{n-1} \arrow[r, "\delta'_{n-1}"] & N_{n-2} \arrow[r] & \dots
\end{tikzcd}$$

Since $\delta_n : M_n \to M_{n-1}$ has trivial cokernel then $f_{n-1} = 0$, hence $f = 0$.

From 1), we only need to prove axiom TT2 of a torsion theory. For a proper chain complex M the short exact sequence is given by:

$$\begin{tikzcd}
\dots \arrow[r] & M_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "id"] & M_n \arrow[r, two heads, "e_n"] \arrow[d, "id"] & \delta_n(X_n) \arrow[r] \arrow[d, hook, "m_n"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & M_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d] & M_n \arrow[r, "\delta_n"] \arrow[d] & M_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, two heads, "cok(\delta_n)"] & M_{n-2} \arrow[r] \arrow[d, "id"] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & Cok(\delta_n) \arrow[r] & M_{n-2} \arrow[r] & \dots
\end{tikzcd} \tag{6}$$

Notice that since M is a proper chain complex $Cok(\delta_n) \cong M_{n-1}/\delta_n(M_n)$. □

3.9. By duality, let $ch(\mathbb{X})_{\geq n}$ be the category of n-truncated below chain complexes. It is straightforward to define the duals of the functors of 3.2:

- $\mathbf{tr}'_n(M) = \dots \longrightarrow M_{n+2} \longrightarrow M_{n+1} \longrightarrow M_n$
- $\mathbf{sk}'_n(M) = \dots \longrightarrow M_{n+1} \longrightarrow M_n \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$
- $\mathbf{cosk}'_n(M) = \dots \longrightarrow M_{n+1} \longrightarrow M_n \xrightarrow{cok(\delta_{n+1})} cok(\delta_{n+1}) \longrightarrow 0 \dots$
- $\mathbf{cot}'_n(M) = \dots \longrightarrow M_{n+2} \longrightarrow M_{n+1} \xrightarrow{\delta_{n+1}} ker(\delta_n)$

and indeed these also give a string of adjunctions:

$$\mathbf{cosk}'_n \dashv \mathbf{tr}'_n \dashv \mathbf{sk}'_n \dashv \mathbf{cot}'_n : \begin{tikzcd} ch(\mathbb{X})_{\geq n} \arrow[d, bend right=80] \arrow[d, bend left=30] \arrow[d, phantom, bend right=55, "\dashv" description] \arrow[d, phantom, "\dashv" description] \arrow[d, phantom, bend left=55, "\dashv" description] \\ ch(\mathbb{X}) \arrow[u, bend left=30] \arrow[u, bend right=80] \end{tikzcd}.$$

We consider $ch(\mathbb{X})_{n\geq}$ as a subcategory of $ch(\mathbb{X})$ through $\mathbf{sk}'_n$. However, unlike in the case of $ch(\mathbb{X})_{n\geq}$, the category $ch(\mathbb{X})_{\geq n}$ is a torsion category without the need of the restriction to proper chain complexes with $\mathbf{sk}'_n \dashv \mathbf{cot}'_n$ as the coreflection.

Theorem 3.10. *For each $n \in \mathbb{Z}$ we have:*

1. *The adjunction $\mathbf{tr}_{n-1} \dashv \mathbf{cosk}_{n-1} : ch(\mathbb{X}) \to ch(\mathbb{X})_{n-1\geq}$ is a localization and thus by theorem 2.9 it induces a hederitary torsion theory $(\mathcal{T}_{\mathbf{tr}_{n-1}}, \mathcal{F}_{\mathbf{tr}_{n-1}})$ in $ch(\mathbb{X})$.*
2. *The category $\mathcal{T}_{\mathbf{tr}_{n-1}}$ is equivalent to $ch(\mathbb{X})_{\geq n}$.*
3. *The reflector of $ch(\mathbb{X})_{\geq n}$ is given by $\mathbf{cot}'_n$.*
4. *The restriction of $(ch(\mathbb{X})_{\geq n}, \mathcal{F}_{\mathbf{tr}_{n-1}})$ to $pch(\mathbb{X})$ is the torsion theory $(pch(\mathbb{X})_{\geq n}, \mathcal{MN}_n)$ in $pch(\mathbb{X})$.*

Proof. 1) The functor $\mathbf{tr}_{n-1}$ preserves finite limits since it admits a left adjoint, namely $\mathbf{sk}_{n-1}$. 2) is trivial since by definition $\mathcal{T}_{\mathbf{tr}_{n-1}} = Ker(\mathbf{cot}'_n)$. 3) and 4) follow immediately from the associated short exact sequence of $(\mathcal{T}_{\mathbf{tr}_{n-1}}, \mathcal{F}_{\mathbf{tr}_{n-1}})$, which is given by Theorem 2.9.

To be precise, for a chain complex M the component $\eta_M : M \to \mathbf{Cosk}_{n-1}(M)$ the unit η of $\mathbf{tr}_{n-1} \dashv \mathbf{cosk}_{n-1}$ is:

$$\begin{tikzcd} \dots \arrow[r] & M_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d] & M_n \arrow[r, "\delta_n"] \arrow[d, "\delta'_n"] & M_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "1"] & M_{n-2} \arrow[r] \arrow[d, "1"] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & Ker(\delta_{n-1}) \arrow[r, "ker(\delta_{n-1})"] & M_{n-1} \arrow[r, "\delta_{n-1}"] & M_{n-2} \arrow[r] & \dots \end{tikzcd}$$

The normal epi-mono factorization of δ'_n is given by (e_n, m'_n) where m'_n is given by m_n and the universal property of $ker(\delta_{n-1})$:

$$\begin{tikzcd}
M_n \arrow[r, "\delta_n"] \arrow[d, two heads, "e_n"] & M_{n-1} \arrow[r, "\delta_{n-1}"] & M_{n-2} \\
\delta_n(M_n) \arrow[ur, "m_n"] \arrow[r, "m'_n"] & Ker(\delta_{n-1}) \arrow[u, hook, "ker(\delta_{n-1})"']
\end{tikzcd}$$

So the associated short exact sequence for a chain complex M is:

$$\begin{tikzcd}
\dots \arrow[r] & M_{n+1} \arrow[r] \arrow[d, "id"] & ker(\delta_n) \arrow[r] \arrow[d, hook, "ker(\delta_n)"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & M_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d] & M_n \arrow[r, "\delta_n"] \arrow[d, two heads, "e_n"] & M_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "id"] & M_{n-2} \arrow[r] \arrow[d, "id"] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & \delta(M_n) \arrow[r, "m_n"] & M_{n-1} \arrow[r] & M_{n-2} \arrow[r] & \dots
\end{tikzcd} \tag{7}$$

□

Now, we will now restrict ourselves to the case of proper chains.

Definition 3.11. *For each* $n \in \mathbf{Z}$*, the torsion theories in* $pch(\mathbb{X})$ *from theorems 3.8 and 3.10 will be denoted as*

$$\mathcal{COK}_n = (\mathcal{EP}_n, pch(\mathbb{X})_{n-1\geq}) \quad \textit{and} \quad \mathcal{KER}_n = (pch(\mathbb{X})_{\geq n}, \mathcal{MN}_n)$$

with the preradicals $ker_n, cok_n : pch(\mathbb{X}) \to pch(\mathbb{X})$*, respectively.*

We will write $\mathcal{COT}(\mathbb{X})$ *for the set of torsion theories* $\mathcal{COK}_n$ *and* $\mathcal{KER}_n$ *given by the cotruncation functors;*

$$\begin{tikzcd}
\mathcal{COK}_n = \mathcal{EP}_n \arrow[r, bend left] \arrow[r, phantom, "\perp"] & pch(\mathbb{X}) \arrow[l, bend left] \arrow[r, bend left, "\mathbf{cot}_{n-1}"] \arrow[r, phantom, "\perp"] & pch(\mathbb{X})_{n-1\geq} \arrow[l, bend left, "\mathbf{sk}_{n-1}"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{KER}_n = pch(\mathbb{X})_{\geq n} \arrow[r, bend left, "\mathbf{sk}'_n"] \arrow[r, phantom, "\perp"] & pch(\mathbb{X}) \arrow[l, bend left, "\mathbf{cot}'_n"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{MN}_n . \arrow[l, bend left]
\end{tikzcd}$$

The next result asserts that $\mathcal{COT}(\mathbb{X})$ is a linearly order lattice.

Proposition 3.12. *In $pch(\mathbb{X})$ for each $n \in \mathbb{Z}$ we have the embeddings of full subcategories*

$$\ldots \leq pch(\mathbb{X})_{\geq n+1} \leq \mathcal{EP}_{n+1} \leq pch(\mathbb{X})_{\geq n} \leq \mathcal{EP}_n \leq pch(\mathbb{X})_{\geq n-1} \leq \ldots$$

Equivalently,

$$\ldots \geq \mathcal{MN}_{n+1} \geq pch(\mathbb{X})_{n+1\geq} \geq \mathcal{MN}_n \geq pch(\mathbb{X})_{n\geq} \geq \mathcal{MN}_{n-1} \geq \ldots.$$

Moreover, there is a linearly ordered lattice of torsion theories in $pch(\mathbb{X})$:

$$O \leq \ldots \leq \mathcal{KER}_{n+1} \leq \mathcal{COK}_{n+1} \leq \mathcal{KER}_n \leq \mathcal{COK}_n \leq \ldots \leq pch(\mathbb{X})$$

Proof. By definition we have $\mathcal{EP}_n \leq pch(\mathbb{X})_{\geq n-1}$ and since a morphism $M_{n+1} \to 0$ is a normal epimorphism we have $pch(\mathbb{X})_{\geq n} \leq \mathcal{EP}_n$. Recall that the order is reverse for the torsion-free subcategories. □

This construction works with truncated or bounded chains complexes, in particularly we will be interested in the case for $pch(\mathbb{X})_{\geq 0}$, and $pch(\mathbb{X})_{n\geq 0}$, the category of chain complexes bounded above n and below 0 for a fixed n.

Corollary 3.13. *In $pch(\mathbb{X})_{\geq 0}$ there is a linearly ordered lattice of torsion theories given by:*

$$O \leq \ldots \leq \mathcal{KER}_n \leq \mathcal{COK}_n \leq \ldots \leq \mathcal{KER}_1 \leq \mathcal{COK}_1 \leq pch(\mathbb{X})_{\geq 0}$$

Corollary 3.14. *In $pch(\mathbb{X})_{n\geq 0}$ there is a linearly ordered lattice of torsion theories given by:*

$$O \leq \mathcal{KER}_n \leq \mathcal{COK}_n \leq \ldots \leq \mathcal{COK}_2 \leq \mathcal{KER}_1 \leq \mathcal{COK}_1 \leq pch(\mathbb{X})_{n\geq 0}$$

We will write $\mathcal{COT}(pch(\mathbb{X})_{\geq 0})$ and $\mathcal{COT}(pch(\mathbb{X})_{n\geq 0})$ for the lattices corresponding to the bounded cases of $pch(\mathbb{X})_{\geq 0}$ and $pch(\mathbb{X})_{n\geq 0}$.

3.15. *Example*. Consider the bounded case of the lattice $\mathcal{COT}(pch(\mathbb{X})_{2\geq 0})$:

$$O \leq \mathcal{KER}_2 \leq \mathcal{COK}_2 \leq \mathcal{KER}_1 \leq \mathcal{COK}_1 \leq pch(\mathbb{X})_{2\geq 0}\,.$$

This lattice induces a lattice of preradicals $pch(\mathbb{X})_{2\geq 0}$ and hence, for a fixed proper chain complex M, a lattice of torsion subobjects of M:

$$\begin{array}{rc}
M = & M_2 \xrightarrow{\delta_2} M_1 \xrightarrow{\delta_1} M_0 \\
cok_1(M) = & M_2 \xrightarrow{\delta_2} M_1 \overset{e_1}{\twoheadrightarrow} \delta_1(M_1) \\
ker_1(M) = & M_2 \xrightarrow{\delta_2'} ker(\delta_1) \longrightarrow 0 \\
cok_2(M) = & M_2 \overset{e_2}{\twoheadrightarrow} \delta_2(M_2) \longrightarrow 0 \\
ker_2(M) = & ker(\delta_2) \longrightarrow 0 \longrightarrow 0 \\
0 = & 0 \longrightarrow 0 \longrightarrow 0\,.
\end{array}$$

4. Homology

In abelian categories, the nth-homology objects of a chain complex M is usually defined as $H_n(M) = ker(\delta_n)/\delta_{n+1}(M_{n+1})$. We can also consider the dual homology object K_n. In other words, consider the commutative diagram:

$$\begin{tikzcd}
M_{n+1} \arrow[rr, "\delta_{n+1}"] \arrow[d, "e_{n+1}"'] & & M_n \arrow[rr, "\delta_n"] \arrow[dr, "q_{n+1}"'] \arrow[drr, "e_n"] & & M_{n-1} \\
\delta_{n+1}(M_{n+1}) \arrow[r, "m'_{n+1}"'] \arrow[urr, "m_{n+1}"] & Ker(\delta_n) \arrow[ur, "k_n"'] & & Cok(\delta_{n+1}) \arrow[r, "e'_n"'] & \delta_n(M_n) \arrow[u, "m_n"']
\end{tikzcd}$$

where $\delta_n = m_n e_n,\ \delta_{n+1} = m_{n+1}e_{n+1}$ are the normal epi/mono factorizations and m'_{n+1} and e'_n are induced by $Ker(\delta_n)$ and $Cok(\delta_{n+1})$, respectively. Then we have

$$H_n(M) = Cok(M_{n+1} \to Ker(\delta_n)) = Cok(m'_{n+1})$$

and

$$K_n(M) = Ker(Cok(\delta_{n+1} \to M_{n-1})) = Ker(e'_n).$$

It is well-known that in abelian categories the objects $H_n(M)$ and $K_n(M)$ are naturally isomorphic, and in [12] this was proved to be the case also for exact homological categories provided that the chain complex M is proper. The following result provides an alternative proof of this fact as well as showing the connection of the objects $H_n(M), K_n(M)$ with the preradicals in $\mathcal{COT}$.

Lemma 4.1. *Let M be a proper chain complex then the objects $H_n(M)$, $K_n(M)$ are isomorphic and are given by*

$$H_n(M) \cong K_n(M) \cong \mathit{ker}_n(M)/\mathit{cok}_{n+1}(M)$$

where $\mathit{cok}_{n+1}(M)$, $\mathit{ker}_n(M)$ are the torsion subojects of M given by the torsion theories as in Definition 3.11 and where $H_n(M)$, $K_n(M)$ are considered as trivial chain complexes except at the order n that have the objects $H_n(M)$, $K_n(M)$ respectively.

Proof. Since $ch(\mathbb{X})$ is semi-abelian using Lemma 2.3 for the normal subobjects $\mathit{cok}_{n+1}(M) \leq \mathit{ker}_n(M)$ of M we have a short exact sequence in $ch(\mathbb{X})$:

$$\mathit{ker}_n(M)/\mathit{cok}_{n+1}(M) \hookrightarrow M/\mathit{cok}_{n+1}(M) \twoheadrightarrow M/\mathit{ker}_n(M)$$

To be more precise observe that $H_n(M)$ is the cokernel of the inclusion $\mathit{cok}_{n+1}(M) \leq \mathit{ker}_n(M)$:

$$\begin{tikzcd}
0 \arrow[d] & & & & \\
\mathit{cok}_{n+1}(M) = \dots \arrow[r] \arrow[d] & M_{n+1} \arrow[r] \arrow[d] & \delta_{n+1}(M_{n+1}) \arrow[r] \arrow[d, hook, "m'_{n+1}"] & 0 \arrow[r] \arrow[d] & \dots \\
\mathit{ker}_n(M) = \dots \arrow[r] \arrow[d] & M_{n+1} \arrow[r] \arrow[d] & \mathit{ker}(\delta_n) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
H_n(M) = \dots \arrow[r] \arrow[d] & 0 \arrow[r] & H_n(X) \arrow[r] & 0 \arrow[r] & \dots \\
0 & & & &
\end{tikzcd}$$

so $H_n(M) \cong ker_n(M)/cok_{n+1}(M)$; and on the other hand

$$\begin{tikzcd}
0 \arrow[d] & & & & \\
K_n(M) = \dots \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K_n(M) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
M/cok_{n+1}(M) = \dots \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & cok(\delta_{n+1}) \arrow[r] \arrow[d, two heads, "e'_n"] & M_{n-1} \arrow[r] \arrow[d] & \dots \\
M/ker_n(M) = \dots \arrow[r] \arrow[d] & 0 \arrow[r] & \delta_n(M) \arrow[r] & M_{n-1} \arrow[r] & \dots \\
0 & & & &
\end{tikzcd}$$

so $K_n(M) \cong ker(M/cok_{n+1}(M) \to M/ker_n(M))$. Lemma 2.3 yields the isomorphism $H_n(M) \cong K_n(M)$. ☐

Proposition 4.2. *For a proper chain complex M the following are equivalent:*

1. $H_n(M) = 0$ *;*
2. $M/ker_{n+1}(M) \cong \mathbf{Cosk}_n(M)$.

Similarly, the following are equivalent:

1. $H_n(M) = 0$ *;*
2. $cok_n(M) \cong \mathbf{Cosk}'_n(M)$.

where $ker_{n+1}(M)$ and $cok_n(M)$ are the torsion subobjects of M given by the torsion theories in 3.11.

Proof. First, recall from Theorem 3.10 that the unit $M \to \mathbf{Cosk}_n(M)$ factors through the reflection of $\mathcal{MN}_{n+1}$:

$$\begin{tikzcd}
M = \dots \arrow[r] \arrow[d, two heads] & M_{n+2} \arrow[r, "\delta_{n+2}"] \arrow[d] & M_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, two heads, "e_{n+1}"] & M_n \arrow[r, "\delta_n"] \arrow[d] & \dots \\
M/ker_{n+1}(M) = \dots \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \delta_{n+1}(M_{n+1}) \arrow[r, hook, "m_{n+1}"] \arrow[d] & M_n \arrow[r, "\delta_n"] \arrow[d] & \dots \\
\mathbf{Cosk}_n(M) = \dots \arrow[r] & 0 \arrow[r] & ker(\delta_n) \arrow[r, hook, "k(\delta_n)"] & M_n \arrow[r, "\delta_n"] & \dots
\end{tikzcd}$$

And by definition, $\delta_{n+1}(M_{n+1}) \cong ker(\delta_n)$ if and only if $H_n(M) = 0$. The second part is similar, since $cok(\delta_{n+1}) \cong \delta_n(M)$ if and only if $H_n(M) = 0$. □

From [16], it is known that the cotruncation functors $\mathbf{cot}_n$, $\mathbf{cot}'_n$ give truncations in the homology objects. The following lemma generalises these facts.

Lemma 4.3. *Let ker_n, cok_n be the preradicals of the torsion theories in Definition 3.11. For a proper chain complex M we have:*

1. *For all $n > 0$*

$$H_i(cok_n(M)) = H_i(ker_n(M)) = \left\{ \begin{array}{cc} H_i(M) & i \geq n \\ 0 & n > i\,. \end{array} \right.$$

2. *For all $n > 0$*

$$H_i\left(\frac{M}{cok_n(M)}\right) = H_i\left(\frac{M}{ker_n(M)}\right) = \left\{ \begin{array}{cc} 0 & i \geq n \\ H_i(M) & n > i\,. \end{array} \right.$$

3. *For all $n > 0$*

$$H_i\left(\frac{cok_n(M)}{ker_n(M)}\right) = 0 \, \textit{for all } i\,.$$

4. *For $m > n$*

$$H_i\left(\frac{cok_n(M)}{ker_m(M)}\right) = H_i\left(\frac{cok_n(M)}{cok_m(M)}\right) = \left\{ \begin{array}{cc} H_i(M) & m > i \geq n \\ 0 & \textit{otherwise}\,. \end{array} \right.$$

5. *Moreover, for* $m > n$

$$H_i\left(\frac{cok_n(M)}{cok_m(M)}\right) = H_i\left(\frac{ker_n(M)}{cok_m(M)}\right) = H_i\left(\frac{ker_n(M)}{ker_m(M)}\right)\,.$$

6. *In particular, for* $m = n+1$

$$H_i\left(\frac{cok_n(M)}{ker_{n+1}(M)}\right) = \begin{cases} H_i(M) & i = n \\ 0 & i \neq n\,. \end{cases}$$

Proof. It is straightforward to calculate the homology of each chain complex. As an example, for 4) and 5) the following chain complexes

$$\frac{cok_n(M)}{ker_m(M)},\quad \frac{cok_n(M)}{cok_m(M)},\quad \frac{ker_n(M)}{cok_m(M)},\quad \frac{ker_n(M)}{ker_m(M)}$$

have the same homology, as seen from:

$$\ldots \to \delta_m(M_m) \to M_{m-1} \to \ldots \to M_{n+1} \to ker(\delta_n) \longrightarrow 0 \longrightarrow \ldots$$

$$\ldots \longrightarrow 0 \longrightarrow \tfrac{M_{m-1}}{\delta_m(M_m)} \to \ldots \to M_{n+1} \to ker(\delta_n) \longrightarrow 0 \longrightarrow \ldots$$

$$\ldots \to \delta_m(M_m) \to M_{m-1} \to \ldots \to M_{n+1} \longrightarrow M_n \longrightarrow \delta_n(M_n) \to \ldots$$

$$\ldots \longrightarrow 0 \longrightarrow \tfrac{M_{m-1}}{\delta_m(M_m)} \to \ldots \to M_{n+1} \longrightarrow M_n \longrightarrow \delta_n(M_n) \to \ldots$$

□

5. Torsion theories induced by $tr_n \dashv cosk_n$

We define the torsion theories $\mu_{\geq n}$ in simplicial groups. These are simplicial analogues for $\mathcal{MN}_n$ and, as in the case of chain complexes, they are defined by a localization $tr_n \dashv cosk_n$ in simplicial objects. The torsion category of $\mu_{\geq n}$ is the category of simplcial groups such that they are trivial below degree n. First, we recall some basic properties of simplicial objects.

Following [13], the simplicial category Δ has, as objects, finite ordinals $[n] = \{0, 1, \ldots n\}$ and as morphisms monotone functions. In particular, we have the morphisms $\delta_i^n : [n-1] \to [n]$ the injection which does not take the value $i \in [n]$ and $\sigma_i^n : [n+1] \to [n]$ the surjection where $\sigma(i) = \sigma(i+1)$. Any morphism μ in Δ can written uniquely as:

$$\mu = \delta_{i_s}^n \delta_{i_s-1}^{n-1} \ldots \delta_{i_1}^{n-t+1} \sigma_{j_t}^{m-t} \ldots \sigma_{j_2}^{m-2} \sigma_{j_1}^{m-1} .$$

such that $n \geq i_s > \cdots > i_1 \geq 0$, $0 \leq j_t < \cdots < j_1 < m$ and $n = m-t+s$.

The category of simplicial objects in a category $\mathbb{X}$ is the functor category $Simp(\mathbb{X}) = [\Delta^{op}, \mathbb{X}]$. Thus, a simplicial object $X : \Delta^{op} \to \mathbb{X}$ corresponds to a family of objects $\{X_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}$, the *face morphisms* $d_i : X_n \to X_{n-1}$ and the *degeneracies morphisms* $s_i : X_n \to X_{n+1}$ satisfying the *simplicial identities*:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \quad \text{if} \quad i < j \\ s_i s_j &= s_{j+1} s_i \quad \text{if} \quad i \leq j \\ d_i s_j &= \begin{cases} s_{j-1} d_i & \text{if} \quad i < j \\ 1 & \text{if} \quad i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{if} \quad i > j+1 \,. \end{cases} \end{aligned}$$

5.1. Let X be a simplicial object in a pointed category with pullbacks. The Moore normalization functor $N : Simp(\mathbb{X}) \to ch(\mathbb{X})$ is given by $N(X)_0 = X_0$,

$$N(X)_n = \bigcap_{i=0}^{n-1} ker(d_i : X_n \to X_{x-1})$$

and differentials $\delta_n = d_n \circ \cap_i ker(d_i) : N(X)_n \to N(X)_{n-1}$ for $n \geq 1$.

The functor N preserves finite limits. In [12], it was proved that if $\mathbb{X}$ is a semi-abelian category N also preserves regular epimorphisms, and hence, it preserves short exact sequences. Moreover, for a simplicial object X the Moore complex $N(X)$ is a proper chain complex and we can define the n-homology object of a simplicial object X as

$$H_n(X) = H_n(N(X)).$$

The objects $H_n(N(X))$ are internal abelian objects for $n \geq 1$ (see [12]).

This generalises the results proven by Moore for the case of simplicial groups, where the homotopy groups of a simplicial group X can be calculated as $\pi_n(X) = H_n(N(X))$ (see, for example [24]).

5.2. If Δ_n is the full subcategory of Δ with objects $[m]$ for $m \leq n$, an n-truncated simplicial object X in $\mathbb{X}$ is a functor $X : \Delta_n^{op} \to \mathbb{X}$. Let $Simp_n(\mathbb{X})$ be the category of n-truncated simplicial objects, then there is truncation functor:

$$tr_n : \ Simp(\mathbb{X}) \longrightarrow Simp_n(X)$$

which simply forgets the objects X_i and the morphisms s_i, d_i for degrees $i > n$.

It is a standard application of Kan extensions that if X is finitely complete/cocomplete (as is the case if $\mathbb{X}$ is semi-abelian) each functor tr_n admits a left/ right adjoint named the n-skeleton and n-coskeleton, respectively: $sk_n \dashv tr_n \dashv cosk_n$. We will write $Sk_n = sk_n tr_n$ and $Cosk_n = cosk_n tr_n$.

If $\mathbb{X}$ has finite limits, the n-coskeleton of an n-truncated simplicial object X is described as follows. For an n-truncated simplicial object X the *simplicial kernel* of the face morphisms $d_0, \ldots, d_n : X_n \to X_{n-1}$ is an object Δ_{n+1} with morphisms $\pi_0, \ldots, \pi_{n+1} : \Delta_{n+1} \to X_n$ such that $d_i \pi_j = d_{j-1} \pi_i$ for all $i < j$ and it is universal with this property: given a family of morphisms $p_0, \ldots, p_{n+1} : Y \to X_n$ such that $d_i p_j = d_{j-1} p_i$ for all $i < j$ then there is a unique morphism $\alpha : Y \to \Delta_{n+1}$ such that $\pi_i \alpha = p_i$. Moreover, the universal property of the simplicial kernel Δ_{n+1} allows to define degeneracies morphisms $s_i : X_n \to \Delta_{n+1}$. So, the simplicial kernel of X defines an $(n+1)$-truncated simplicial object. Finally, the $cosk(X)$ is given by the iteration of successive simplicial kernels.

For $n = 0$, the 0-coskeleton is known as the indiscrete functor $Ind : \mathbb{X} \to Simp(\mathbb{X})$ given by:

$$\begin{tikzcd} Ind(X) = \ldots \arrow[r, shift left=3, "\pi_4"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "\pi_0"'] & X^4 \arrow[r, shift left=3, "\pi_4"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "\pi_0"'] & X^3 \arrow[r, shift left=3, "\pi_4"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "\pi_0"'] & X^2 \arrow[r, shift left=3, "\pi_1"] \arrow[r, shift right=3, "\pi_0"'] & X . \arrow[l, "s_0" description] \end{tikzcd}$$

where X^n is n-fold product of X and the degeneracies are defined by the product projections.

For $\mathbb{X} = Grps$, the simplicial kernel Δ_{n+1} of a n-truncated simplicial group X can be described as the subgroup of X_n^{n+2} of $(n+2)$-tuples

$(x_0, \ldots, x_{n+1})$ such that $\pi_i(x_j) = \pi_{j-1}(x_i)$ for $i < j$ and where π_i are the product projections.

The following result was first proved for simplicial groups in [9]. The generalization is straighforward.

Theorem 5.3. *Let $\mathbb{X}$ be a pointed category with finite limits. For a simplicial object X with corresponding Moore complex M, then the Moore complex of n-coskeleton $Cosk_n(X)$ satifies:*

- $N(Cosk_n(X))_i = M_i$ *for* $n \leq i$;
- $N(Cosk_n(X))_{n+1} = ker(\delta_n : M_n \to M_{n+1})$;
- $N(Cosk_n(X))_i = 0$ *for* $i > n + 1$.

In other words, the Moore normalization N commutes up to isomorphism with the coskeleton functors for simplicial objects and chain complexes:

$$\begin{tikzcd}
Simp(\mathbb{X}) \arrow[r, "N"] \arrow[d, bend right, "tr_n"'] & ch(\mathbb{X}) \arrow[d, bend right, "\mathbf{tr}_n"'] \\
Simp_n(X) \arrow[r, "N"] \arrow[u, bend right, "cosk_n"'] \arrow[u, phantom, "\dashv"] & ch(\mathbb{X})_{n\geq} \arrow[u, bend right, "\mathbf{cosk}_n"'] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

Definition 5.4. *Let $\mathbb{X}$ be a semi-abelian category. For each $n > 1$ we have that $tr_{n-1} \dashv cosk_{n-1}$ is a localization since tr_{n-1} admits a left adjoint, namely sk_{n-1}. Then, by Theorem 2.9, it defines a hereditary torsion theory in $Simp(\mathbb{X})$ which will be written as:*

$$\mu_{\geq n} := (\mathcal{T}_{tr_{n-1}}, \mathcal{F}_{tr_{n-1}}).$$

We will also write $\mu_{\geq n} : Simp(\mathbb{X}) \to Simp(\mathbb{X})$ for the associated idempotent radical.

By definition, the category $\mathcal{T}_{tr_{n-1}}$ is the full subcategory of simplicial objects X with $X_i = 0$ for $n - 1 \geq i$.

The torsion theory $\mu_{\geq 1} = (\mathcal{T}_{tr_0}, \mathcal{F}_{tr_0})$ naturally extends the torsion theory $(Ab(\mathbb{X}), Eq(\mathbb{X}))$ in $Grpd(\mathbb{X})$, to this end we need to recall a result about Mal'tsev categories.

5.5. A category $\mathbb{X}$ is called a Mal'tsev category if any internal reflexive relation is an equivalence relation. Semi-abelian categories are Mal'tsev categories.

Proposition 5.6. *[10] Let $\mathbb{X}$ be a regular Mal'tsev category. The category $Grpd(\mathbb{X})$ is closed under subobjects in $Simp(\mathbb{X})$.*

Proposition 5.7. *Let $\mathbb{X}$ be a semi-abelian category and, for $n = 0$, consider the torsion theory $\mu_{\geq 1} = (\mathcal{T}_{tr_0}, \mathcal{F}_{tr_0})$ in $Simp(\mathbb{X})$. The torsion-free subcategory $\mathcal{F}_{tr_0}$ is equivalent to $Eq(\mathbb{X})$. On the other hand, $Ab(\mathbb{X}) \subset \mathcal{T}_{tr_0}$.*

Proof. For $n = 0$ we have that $tr_0 \dashv cosk_0 \cong ()_0 \dashv Ind$. The unit of $()_0 \dashv Ind$ for a simplicial object X is given by

$$\begin{tikzcd}
X = \dots \arrow[r, shift left=3, "d_4"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "d_0"'] \arrow[d, "\eta_X"] & X_3 \arrow[r, shift left=3, "d_3"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "d_0"'] \arrow[d, "{(d_0,d_1,d_2,d_3)}"] & X_2 \arrow[r, shift left=3, "d_2"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "d_0"'] \arrow[d, "{(d_0,d_1,d_2)}"] & X_1 \arrow[r, shift left=3, "d_1"] \arrow[r, shift right=3, "d_0"'] \arrow[d, "{(d_0,d_1)}"] & X_0 \arrow[l, "s_0"'] \arrow[d, "1"] \\
Ind(X) = \dots \arrow[r, shift left=3, "\pi_4"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "\pi_0"'] & X_0^4 \arrow[r, shift left=3, "\pi_4"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "\pi_0"'] & X_0^3 \arrow[r, shift left=3, "\pi_4"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "\pi_0"'] & X_0^2 \arrow[r, shift left=3, "\pi_1"] \arrow[r, shift right=3, "\pi_0"'] & X_0 . \arrow[l, "s_0"']
\end{tikzcd}$$

Since $\mathbb{X}$ is a Mal'tsev category and by Proposition 5.6, if X has a monic unit η_X then X is a groupoid since $Ind(X)$ is an equivalence relation. Finally, since $(d_0, d_1) : X_1 \to X_0^2$ is monic then X is an equivalence relation and $\mathcal{F}_{tr_0} \subseteq Eq(\mathbb{X})$. Conversely, an equivalence relation always has a monic unit η_X.

On the other hand, an internal abelian group X has $X_0 = 0$, so $Ab(\mathbb{X}) \subset \mathcal{T}_{tr_0}$. □

Theorem 5.8. *Let $\mathbb{X}$ be a semi-abelian category and X a simplicial object with Moore chain complex M. Then, the normalization functor N maps the short exact sequence of X given by the torsion theory $\mu_{\geq n+1}$ in $Simp(X)$ into the short exact sequence of M given by the torsion theory $\mathcal{KER}_{n+1}$ in $pch(\mathbb{X})$.*

Moreover, N maps the torsion theory $\mu_{\geq n+1}$ into $\mathcal{KER}_{n+1}$:

$$\begin{tikzcd}
\mathcal{T}_{tr_n} \arrow[r, phantom, "\perp"] \arrow[r, bend left] \arrow[d, "N"] & Simp(\mathbb{X}) \arrow[l, bend left] \arrow[r, phantom, "\perp"] \arrow[r, bend left] \arrow[d, "N"] & \mathcal{F}_{tr_n} \arrow[l, bend left] \arrow[d, "N"] \\
pch(\mathbb{X})_{\geq n+1} \arrow[r, phantom, "\perp"] \arrow[r, bend left] & pch(\mathbb{X}) \arrow[l, bend left] \arrow[r, phantom, "\perp"] \arrow[r, bend left] & \mathcal{MN}_n \arrow[l, bend left]
\end{tikzcd}$$

i.e. the subcategory $\mathcal{T}_{tr_n}$ is mapped into $pch(\mathbb{X})_{\geq n+1}$ and $\mathcal{F}_{tr_n}$ into $\mathcal{MN}_n$.

Proof. Since $\mathbb{X}$ is semi-abelian the normalization functor N preserves short exact sequences and also preserves the normal epi/mono factorization of morphisms in $Simp(X)$. Since N commutes (up to isomorphism) with the truncation and coskeleton functors we have that for a simplicial object X and its Moore complex M, the functor N maps the short exact sequence in $Simp(\mathbb{X})$:

$$\begin{tikzcd}
0 \arrow[r] & ker(\eta_X) \arrow[r, "k"] & X \arrow[r, "e"] \arrow[dr, "\eta_X"'] & \eta_X(X) \arrow[r] \arrow[d, "m"] & 0 \\
& & & Cosk_n(X) &
\end{tikzcd}$$

into

$$\begin{tikzcd}
N(ker(\eta_X)) = \dots \arrow[r] \arrow[d, hook, "m(k)"] & M_{n+2} \arrow[r, "\delta_{n+2}"] \arrow[d] & ker(\delta_{n+1}) \arrow[r] \arrow[d, two heads, "e_{n+1}"] & 0 \arrow[r] \arrow[d] & \dots \\
N(X) = \dots \arrow[r] \arrow[d, two heads, "M(e)"] \arrow[dd, bend right, "N(\eta_x)"'] & M_{n+2} \arrow[r, "\delta_{n+2}"] \arrow[d] & M_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, two heads, "e_{n+1}"] & M_n \arrow[r, "\delta_n"] \arrow[d] & \dots \\
N(\eta_X(X)) = \dots \arrow[r] \arrow[d, "M(m)"] & 0 \arrow[r] \arrow[d] & \delta_{n+1}(N_{n+1}) \arrow[r, hook, "m_{n+1}"] \arrow[d] & M_n \arrow[r, "\delta_n"] \arrow[d] & \dots \\
N(Cosk_n(X)) = \dots \arrow[r] & 0 \arrow[r] & ker(\delta_n) \arrow[r, hook, "k(\delta_n)"] & M_n \arrow[r, "\delta_n"] & \dots
\end{tikzcd}$$

Since the short exact sequence of the torsion theories is preserved from $\mu_{\geq n+1}$ to $\mathcal{KER}_{n+1}$, it follows that $N(\mathcal{T}_{tr_n}) \subseteq pch(\mathbb{X})_{\geq n+1}$ and $N(\mathcal{F}_{tr_n}) \subseteq \mathcal{MN}_n$. □

6. Torsion theories of truncated Moore complexes in simplicial groups

In order to define the torsion theories $\mu_{n\geq}$, analogues for the torsion theories $\mathcal{COK}_n$ in simplicial objects, we will restrict ourselves to the case of the category of $\mathbb{X} = Grps$. With this stronger assumption the subcategories $\mathcal{M}_{\geq n}$ and $\mathcal{M}_{n\geq}$ of simplicial groups with trivial Moore complex below/above at order n will appear as torsion/torsion-free subcategories of the torsion theories $\mu_{n\geq}$ and $\mu_{\geq n}$, respectively.

We need to recall some results of D. Conduché [9]. In particular, in a simplicial group X each X_n can be decomposed as successive semi-direct products of the objects of its Moore complex M_i with $i \leq n$.

6.1. (cf. [9]) In order to avoid multiple subscripts we will write $\sigma_i = \bar{i}$ for the degeneracy maps of Δ.

For any object $[n] = \{0 < 1 < \cdots < n\}$ of the simplicial category Δ we will introduce an order in $S(n)$ the set of surjective maps of Δ with domain $[n]$. Any surjective map $\sigma : [n] \to [m]$ is written uniquely as $\sigma = \bar{i}_1\bar{i}_2\ldots\bar{i}_{n-m}$ with $i_1 < i_2 < \cdots < i_{n-m}$. We introduce the inverse lexicographic order in $S(n, m)$ the set of surjective maps form $[n]$ to $[m]$:

$$\bar{i}_1\bar{i}_2\ldots\bar{i}_{n-m} < \bar{j}_1\bar{j}_2\ldots\bar{j}_{n-m} \text{ if } i_{n-1} = j_{n-m}, \ldots, i_{s+1} = j_{s+1}, \text{ and } i_s > j_s.$$

This order extends to $S(n)$ by setting $S(n, m) < S(n, l)$ if $m > l$.

As an example, for $S(4)$ we have:

$$\begin{aligned} id_{[4]} < \bar{3} < \bar{2} < \bar{2}\bar{3} < \bar{1} < \bar{1}\bar{3} < \bar{1}\bar{2} < \bar{1}\bar{2}\bar{3} < \bar{0} < \bar{0}\bar{3} < \bar{0}\bar{2} \\ < \bar{0}\bar{2}\bar{3} < \bar{0}\bar{1} < \bar{0}\bar{1}\bar{3} < \bar{0}\bar{1}\bar{2} < \bar{0}\bar{1}\bar{2}\bar{3} \end{aligned}$$

For a simplicial group X with Moore complex M and a surjective map $\mathbf{i} = \bar{i}_1\bar{i}_2\ldots\bar{i}_r$ we have $s_{\mathbf{i}} = s_{i_r}\ldots s_{i_1}$ and $d_{\mathbf{i}} = d_{i_1}\ldots d_{i_r}$. Using the order of $S(n)$ we have a filtration of X_n by the subgroups

$$G_{n,\mathbf{i}} = \bigcap_{\mathbf{j}\geq\mathbf{i}} ker(d_{\mathbf{j}}).$$

Notice that $G_{n,id} = 0$ and $G_{n,n\bar{-}1} = M(X)_n$. The order $S(n)$ satisfies for a surjective map $\mathbf{i} : [n] \to [r]$ and its successor $\mathbf{j}$ we have the semidirect product

$$G_{n,\mathbf{j}} \cong G_{n,\mathbf{i}} \rtimes_{s_{\mathbf{i}}} M_r.$$

Finally, this implies that X_n decomposes as a sequence of semi-direct products:

$$X_n = (\dots(M_n \rtimes_{s_{n-1}} M_{n-1}) \rtimes_{s_{n-2}} \dots) \rtimes_{s_{p-1}\dots s_0} M_0$$

Corollary 6.2. *For each $n \in \mathbb{N}$, the category $\mathcal{M}_{\geq n+1}$ and $\mathcal{T}_{tr_n}$ are equivalent.*

Moreover, the category $\mathcal{M}_{\geq n+1}$ is a torsion subcategory in $Simp(\mathbb{X})$;

$$(\mathcal{T}_{tr_n}, \mathcal{F}_{tr_n}) \cong (\mathcal{M}_{\geq n+1}, \mathcal{F}_{tr_n}).$$

Proof. It follows immediately from the semidirect decomposition that a simplicial group X has $X_i = 0$ for $n > i$ if and only if $M_i = 0$ for $n > i$. □

The analogue of the cotruncation functor for simplicial groups was introduced by T. Porter as follows.

6.3. (see [25]) There is a cotruncation functor $Cot_n : Simp(Grp) \to Simp(Grp)$ such that

$$\begin{tikzcd} Simp(Grp) \arrow[r, "N"] \arrow[d, "Cot_n"] & chn(Grp) \arrow[d, "\mathbf{Cot}_n"] \\ Simp(Grp) \arrow[r, "N"] & chn(Grp) \end{tikzcd} \tag{8}$$

commutes up to natural isomorphism, where N is the Moore normalization functor. The functor $Cot_n(X)$ is defined as follows:

$$Cot_n(X)_i = X_i \quad \text{for} \quad n > i\,,$$

$$Cot_n(X)_n = X_n/\delta_{n+1}(M_{n+1})\,,$$

and for $i > n$ the object $Cot_n(X)_i$ is obtained by deleting all M_k for $k > n$ and replacing M_n by $M_n/\delta_{n+1}(M_{n+1})$ in the semi-direct decomposition.

We recall some useful properties of this functor.

Proposition 6.4. *(see [25]) Let $\mathcal{M}_{n\geq}$ be the full subcategory of $Simp(Grp)$ defined by those simplicial groups whose Moore complex is trivial for dimensions greater than n. Let $i_n : \mathcal{M}_{n\geq} \to Simp(Grp)$ the inclusion functor then*

1. *Cot_n is left adjoint of i_n;*
2. *the unit $\eta_X : X \to Cot_n(X)$ of the adjunction is a regular epimorphism which induces an isomorphism in $\pi_i(X)$ for $i \leq n$;*
3. *for any simplicial group X, $\pi(Cot_n(X)) = 0$ for $i > n$;*
4. *the inclusion $\mathcal{M}_{n\geq} \to \mathcal{M}_{n+1\geq}$ correspond to a natural epimorphism*

   $$\eta_n : Cot_{n+1} \to Cot_n$$

   and, for a simplicial group X, then $Ker(\eta_n(X))$ is a $K(\pi_{n+1}(X), n+1)$-simplicial group (an Eilenberg-Mac Lane simplicial group).

This cotruncation functor for simplicial groups is normal and thus defines a torsion theory in $Simp(Grp)$ as in Theorem 2.7.

Corollary 6.5. *The subcategory $\mathcal{M}_{n\geq}$ of $Simp(Grp)$ given by the simplicial groups with trivial Moore complex for dimension greater than n is a torsion-free subcategory of $Simp(Grp)$. The torsion theory is given by the pair*

$$\mu_{n\geq} = (Ker(Cot_n), \mathcal{M}_{n\geq}).$$

Proof. By Theorem 2.7 it suffices to prove that the functor Cot_n is normal. Let η be the unit as in 6.4, for a simplicial group X with a Moore complex M. Since taking normalization preserves short exact sequences we have that the Moore complex of $ker(\eta_X)$ is:

$$\begin{tikzcd} \ldots \arrow[r] & M_{n+2} \arrow[r, "d_{n+2}"] & M_{n+1} \arrow[r, two heads, "e_{n+1}"] & d_{n+1}(M_{n+1}) \arrow[r] & 0 \arrow[r] & \ldots \end{tikzcd}$$

which is trivial under the chain cotruncation $\mathbf{cot}_n$. Since the functor $\mathbf{Cot}_n$ and Cot_n commute with the Moore normalization as in 5.3 we have that $Cot_n(ker(\eta_X)) = 0$ for any simplicial group X. □

Definition 6.6. *For each n, we will denote the torsion theory in $Simp(Grp)$ given by the functor Cot_n as:*

$$\mu_{n\geq} = (Ker(Cot_n), \mathcal{M}_{n\geq})$$

and also the associated idempotent radical will be denoted by

$$\mu_{n\geq} : Simp(\mathbb{X}) \to Simp(\mathbb{X}).$$

The category $\mathcal{M}_{0\geq}$ is equivalent to the category $Dis(Grp)$ of discrete simplicial groups, simplicial groups where all degenerecies and face morphisms are the identity. And it is well-know from Loday's article [21] that the category $\mathcal{M}_{1\geq}$ is equivalent to the category of internal grupoids $Grpd(Grp)$.

Corollary 6.7. *The categories $Dis(Grp)$ of discrete simplicial groups and the category $Grpd(Grp)$ of internal grupoids are torsion-free subcategories of $Simp(\mathbb{X})$.*

Theorem 6.8. *Let be X a simplicial group with Moore complex M. The normalization functor N maps the short exact sequence of X given by the torsion theory $\mu_{n\geq}$ into the short exact sequence of M given by $\mathcal{COK}_{n+1}$.*

Moreover, N maps the torsion category $Ker(Cot_n)$ into the torsion category $\mathcal{EP}_{n+1}$ and, respectively, the torsion-free category $\mathcal{M}_{n\geq}$ into the torsion free category $pch(Grp)_{n\geq}$;

$$\begin{tikzcd}
Ker(Cot_n) \arrow[r, bend left] \arrow[r, phantom, "\perp"] \arrow[d, "N"] & Simp(\mathbb{X}) \arrow[l, bend left] \arrow[r, bend left] \arrow[r, phantom, "\perp"] \arrow[d, "N"] & \mathcal{M}_{n\geq} \arrow[l, bend left] \arrow[d, "N"] \\
\mathcal{EP}_{n+1} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & pch(\mathbb{X}) \arrow[l, bend left] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & pch(Grp)_{n\geq} \arrow[l, bend left]
\end{tikzcd}$$

Proof. Since the cotruncation functors commute up to isomorphism with normalization as in diagram (8) and N preserves short exact sequences, the short exact sequence in $Simp(Grp)$:

$$0 \longrightarrow ker(\eta_X) \longrightarrow X \xrightarrow{\eta_X} Cot_n(X) \longrightarrow 0$$

is mapped under N into the short exact sequence (written vertically) in $pch(Grp)$:

$$\begin{tikzcd}
N(ker(\eta_x)) = \dots \arrow[r] \arrow[d] & M_{n+1} \arrow[r, "e_{n+1}"] \arrow[d] & \delta_{n+1}(M_{n+1}) \arrow[r] \arrow[d, "m_{n+1}"] & 0 \arrow[r] \arrow[d] & \dots \\
M = \dots \arrow[r] \arrow[d] & M_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d] & M_n \arrow[r, "\delta_n"] \arrow[d] & M_{n-1} \arrow[r] \arrow[d] & \dots \\
N(Cot_n(X)) = \dots \arrow[r] & 0 \arrow[r] & M_n/\delta_{n+1}(M_{n+1}) \arrow[r, "\delta'_n"] & M_{n-1} \arrow[r] & \dots .
\end{tikzcd}$$

Since the associated short exact sequence of the torsion theory is preserved as above, it follows that $N(Ker(Cot_n)) \subset \mathcal{EP}_{n+1}$ and $N(\mathcal{M}_{n\geq}) \subset pch(Grp)_{n\geq}$. $\square$

Theorem 6.9. *The torsion subcategories of the torsion theories $\mu_{n\geq}$ and $\mu_{\geq n+1}$ in $Simp(Grp)$ are linearly ordered as:*

$$0 \subseteq \dots \subseteq Ker(Cot_{n+1}) \subseteq \mathcal{M}_{\geq n+1} \subseteq Ker(Cot_n) \\ \subseteq \mathcal{M}_{\geq n} \subseteq \dots \subseteq Simp(Grp).$$

Moreover, the torsion theories $\mu_{n\geq}$ and $\mu_{\geq n+1}$ form a linearly ordered lattice $\mu(Grp)$:

$$\dots \leq \mu_{n+1\geq} \leq \mu_{\geq n+1} \leq \mu_{n\geq} \leq \mu_{\geq n} \leq \dots \\ \dots \leq \mu_{\geq 2} \leq \mu_{1\geq} \leq \mu_{\geq 1} \leq \mu_{0\geq} \leq Simp(Grp)\,.$$

Proof. First we will prove $\mathcal{M}_{\geq n+1} \subseteq Ker(Cot_n)$. For a simplicial group X and M its Moore normalization and η the unit as in Proposition 6.4. Then, if $M_i = 0$ for $n \geq i$ then $X_i = 0$ for $n \geq i$ and since η_X is a normal epimorphism then $Cot_n(X)_i = 0$ for $n \geq i$. It follows from the semi-direct decomposition that $Cot_n(X) = 0$.

Now we prove $Ker(Cot_n) \subseteq \mathcal{M}_{\geq n}$. From it is clear that if $Cot_n(X) = 0$ we have $X_i = 0$ for $n - 1 \geq i$ then $M_i = 0$ for $n - 1 \geq i$ and X is in $\mathcal{M}_{\geq n}$. $\square$

Definition 6.10. *We will write $\mu(Grp)$ for the linearly order lattice of torsion theories in $Simp(Grp)$ given by $\mu_{n\geq}$ and $\mu_{\geq n}$.*

Theorem 6.11. *The torsion theory* $\mu_{n\geq}$ *is hereditary and* $\mu_{\geq n}$ *is cohereditary. Moreover, the subcategories* $\mathcal{M}_{n\geq}$ *and* $\mathcal{M}_{\geq n}$ *are semi-abelian.*

Proof. It follows from the fact that N is an exact functor. Then, $\mathcal{M}_{n\geq}$ and $\mathcal{M}_{\geq n}$ are semi-abelian by Theorem 2.14 and Corollary 2.15, respectively. □

Theorem 6.12.

1. *For* $n \geq 1$, *a simplicial group* X *is torsion for* $\mu_{n\geq}$ *(i.e. it belongs to* $Ker(Cot_n)$*) if and only if and* $tr_{n-1}(X) = 0$ *and* η_X *is a normal epimorphism, where* η *is the unit of* $tr_n \dashv cosk_n$.
2. *For* $n = 0$, X *belongs to* $Ker(Cot_0)$ *if and only if* η_X *is a normal epimorphism with* η *the unit of* $tr_0 \dashv cosk_0$.

Proof. 1) Recall that $\mu_{n\geq} \leq \mu_{\geq n}$ we have the inclusion of torsion subcategories $Ker(Cot_n) \subseteq Ker(tr_{n-1})$. Then, a simplicial group X belongs to $Ker(tr_{n-1})$ if and only if its Moore complex M is trivial for degrees $n-1 \geq i$, and moreover X belongs to $Ker(Cot_n)$ if and only if, in addition, $\delta_{n+1} : M_{n+1} \to M_n$ is a normal epimorphism.

Let X belong to $Ker(tr_{n-1})$ and $\eta_X : X \to Cosk_n$ where η is the unit of the adjunction $tr_n \dashv cosk_n$. The normalization of η_X is

$$\begin{tikzcd}
\dots \arrow[r] & M_{n+2} \arrow[r] \arrow[d] & M_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "\delta_{n+1}"] & M_n \arrow[r] \arrow[d, "1"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & M_n \arrow[r, "1"] & M_n \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

Since the normalization functor is conservative we have that δ_{n+1} is a normal epimorphism if and only if η_X is a normal epimorphism.

The proof of 2) is similar to the above. □

Notice, as expected from the torsion theory $(Conn(Grpd), Dis(Grp))$ in $Grpd(Grp)$, that the torsion category $Ker(Cot_0)$ in $Simp(Grp)$ contains the subcategory of connected internal groupoids $Conn(Grpd)$, i.e. internal groupoids X with the condition that $(d_0, d_1) : X_1 \to X_0^2$ is a normal epimorphism.

7. Homotopy groups and torsion subobjects

Definition 7.1. *For $m \geq n$ and the idempotent radicals of the torsion theories of $\mu(Grp)$:*

$$\cdots \leq \quad \mu_{m\geq} \quad \leq \quad \mu_{\geq m} \quad \leq \cdots \leq \quad \mu_{n\geq} \quad \leq \quad \mu_{\geq n} \quad \leq \cdots,$$

we consider the quotients of preradicals of $Simp(Grp)$:

$$\Pi^{\geq n}_{m\geq} := \frac{\mu_{\geq n}}{\mu_{m\geq}}, \quad \Pi^{n\geq}_{\geq m} := \frac{\mu_{n\geq}}{\mu_{\geq m}}, \quad \Pi^{\geq n}_{\geq m} := \frac{\mu_{\geq n}}{\mu_{\geq m}}, \quad \Pi^{n\geq}_{m\geq} := \frac{\mu_{n\geq}}{\mu_{m\geq}};$$

as well as, for all n the trivial quotients:

$$\Pi^{\geq n} := \frac{\mu_{\geq n}}{0} \cong \mu_{\geq n}\,,\ \Pi_{\geq n} := \frac{Id}{\mu_{\geq n}}\,,\ \Pi^{n\geq} := \frac{\mu_{n\geq}}{0} \cong \mu_{n\geq}\,,\ \Pi_{n\geq} := \frac{Id}{\mu_{n\geq}}\,.$$

For a simplicial group X the objects

$$\Pi^{\geq n}_{m\geq}(X), \quad \Pi^{n\geq}_{\geq m}(X), \quad \Pi^{\geq n}_{\geq m}(X), \quad \Pi^{n\geq}_{m\geq}(X),$$
$$\Pi^{\geq n}(X), \quad \Pi_{\geq n}(X), \quad \Pi^{n\geq}(X), \quad \Pi_{n\geq}(X)$$

will be called the fundamental simplicial groups of X. Accordingly, the family of endofunctors:

$$\Pi^{\geq n}_{m\geq}, \quad \Pi^{n\geq}_{\geq m}, \quad \Pi^{\geq n}_{\geq m}, \quad \Pi^{n\geq}_{m\geq}, \quad \Pi^{\geq n}, \quad \Pi_{\geq n}, \quad \Pi^{n\geq}, \quad \Pi_{n\geq}$$

will be called fundamental simplicial functors.

Following Proposition 6.4, the homotopy groups of

$$\Pi_{n\geq}(X) = Id/\mu_{n\geq}(X) = Cot_n(X)$$

are the same as X for $n \geq i$ and trivial elsewhere. The homotopy groups of the fundamental simplicial groups are the same as X or trivial at some degrees. The following result generalizes 3) and 4) of Proposition 6.4.

Theorem 7.2. *Let be X a simplicial group with Moore complex M. The homotopy groups of the fundamental simplicial group of X are calculated as follows:*

1. *For all* $n \geq 0$
$$\pi_i(\Pi^{n\geq}(X)) = \pi_i(\Pi^{\geq n+1}) = \begin{cases} \pi_i(M) & i \geq n+1 \\ 0 & n+1 > i\,. \end{cases}$$

2. *For all* $n \geq 0$
$$\pi_i(\Pi_{n\geq}(X)) = \pi_i(\Pi^{\geq n+1}(X)) = \begin{cases} 0 & i \geq n+1 \\ \pi_i(X) & n+1 > i\,. \end{cases}$$

3. *For all* $n \geq 0$
$$\pi_i(\Pi^{n\geq}_{\geq n+1}(X)) = 0 \, \textit{for all } i\,.$$

4. *For* $m > n \geq 0$
$$\pi_i(\Pi^{n\geq}_{\geq m+1}(X)) = \begin{cases} \pi_i(X) & m+1 > i \geq n+1 \\ 0 & \textit{otherwise}\,. \end{cases}$$

5. *Moreover, for* $m > n \geq 0$ *and for all* i
$$\pi_i(\Pi^{n\geq}_{\geq m+1}(X)) = \pi_i(\Pi^{n\geq}_{m\geq}(X)) = \pi_i(\Pi^{\geq n+1}_{\geq m+1}(X)) = \pi_i(\Pi^{\geq n+1}_{m\geq}(X))\,.$$

6. *In particular, for* $m = n+1$
$$\pi_i(\Pi^{n\geq}_{\geq n+2}(X)) = \begin{cases} \pi_i(X) & i = n+1 \\ 0 & i \neq n\,. \end{cases}$$

Proof. Since the Moore Normalization preserves short exact sequences and the preradicals $\mu_{n\geq}$ and $\mu_{\geq n}$ are mapped into the preradicals cok_n and ker_n, this follows from the calculations of Lemma 4.3. □

7.3. For an abelian group A, a simplicial group X is an Eilenberg-Mac Lane simplicial group of type $K(A, n)$ or a $K(A, n)$-simplicial group, if it has $\pi_n(X) = A$ and all other homotopy groups trivial.

In particular, the n-th Eilenberg-Mac Lane simplicial group $K(A, n)$ for an abelian group A (in symmetric form) is defined as follows. Consider the $(n+1)$-truncated simplicial group $k(A, n)$:

$$\begin{tikzcd} k(A,n) = & A^{n+1} \arrow[r, shift left=2, "d_{n+1}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "d_0"'] & A \arrow[r, shift left=2, "0"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "0"'] & 0 \arrow[r, shift left=2, "0"] \arrow[r, shift right=2, "0"'] & \cdots \arrow[r, shift left=2, "0"] \arrow[r, shift right=2, "0"'] & 0 \end{tikzcd}$$

where the non-trivial face morphisms are

$$(d_0, d_1, \dots, d_{n+1}) = (p_0, p_0 + p_1, p_1 + p_2, \dots, p_{n-1} + p_n, p_n)\,,$$

where p_i are the product projections. Then, we define

$$K(A,n) = cosk_{n+1}(k(A,n)).$$

Indeed, it is easy to see that the the Moore complex of $K(A,n)$ is:

$$M(K(A,n)) = \ \dots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \dots \longrightarrow 0\ . \tag{9}$$

This construction yields an embedding of the category Ab of abelian groups into the category of simplicial groups $Simp(Grp)$ at any degree $n \geq 1$:

$$K(\_,n):\ Ab \longrightarrow Simp(Grp)\ .$$

At $n = 1$, it correspond with the usual definition of an abelian group as a simplicial group. For $n = 0$, we will also consider the embedding of discrete simplicial groups $Dis : Grp \to Simp(Grp)$.

We introduce the following result without proof. It relies on the equivalence between simplicial groups and the category of hypercrossed modules introduced in [6], a generalized version of the Dold-Kan theorem. Details can be found in [22].

Lemma 7.4. *([22]) Let X be a simplicial group with Moore complex*

$$M = \ \dots \longrightarrow 0 \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$$

then X isomorphic to the Eilenberg-Mac Lane simplicial group $K(A,n)$.

The next corollary generalizes part 4) of Proposition 6.4.

Corollary 7.5. *For $n \geq 0$ and a simplicial group X the simplicial groups:*

$$\Pi^{n\geq}_{\geq n+2}(X)\,, \quad \Pi^{n\geq}_{n+1\geq}(X)\,, \quad \Pi^{\geq n+1}_{\geq n+2}(X)\,, \quad \Pi^{\geq n+1}_{n+1\geq}(X)$$

are $K(\pi_{n+1}(X), n+1)$-simplicial groups.

Moreover, $\Pi^{\geq n+1}_{n+1\geq}(X)$ is isomorphic to $K(\pi_{n+1}(X), n+1)$ the $(n+1)$-th Eilenberg-Mac Lane simplicial group of $\pi_{n+1}(X)$.

Proof. It follows by definition from 6) of theorem 7.2.

Moreover, the Moore complex of $\Pi^{\geq n+1}_{n+1\geq}(X)$ isomorphic to the chain complex $\frac{ker_{n+1}(M)}{cok_{n+2}(M)}$, i.e.:

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow \pi_{n+1}(X) \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

It follows from lemma 7.4 that $\Pi^{\geq n+1}_{n++1\geq}(X) \cong K(\pi_{n+1}(X), n+1)$. □

Corollary 7.6. *1. The fundamental functor $\Pi_{0\geq}$ is naturally isomorphic to the connected component functor π_0 followed by the discrete functor:*

$$\begin{tikzcd}
Simp(Grp) \arrow[rr, "\Pi_{0\geq}"] \arrow[dr, "\pi_0"'] & & Simp(Grp) \\
& Grp \arrow[ur, "Dis"'] &
\end{tikzcd}$$

2. For $n \geq 1$, the fundamental functor $\Pi^{\geq n+1}_{n+1\geq}$ is naturally isomorphic to the homotopy group functor π_n followed by the embedding $K(\_, n)$:

$$\begin{tikzcd}
Simp(Grp) \arrow[rr, "\Pi^{\geq n+1}_{n+1\geq}"] \arrow[dr, "\pi_n"'] & & Simp(Grp) \\
& Ab \arrow[ur, "K(\_{,}n)"'] &
\end{tikzcd}$$

Proof. 1) From Corollary 6.7, the torsion theory $\mu_{0\geq}$ has as torsion-free reflector the functor $\Pi_{0\geq} = Cot_0 = Dis\pi_0$ since $\pi_0(X) = coeq(d_0, d_1) = X_0/\delta_1(M_1)$. 2) It follows from Corollary 7.5. □

Following [13], the *fundamental groupoid* or Poincaré groupoid $\Pi_1(X)$ of a simplicial set X has as objects the set X_0, the vertices of X, and morphisms are generated by the elements of X_1 and their formal inverses and the relations $s_0(x) = 1_x$ if $x \in X_0$ and $(d_0\sigma)(d_2\sigma) = d_1\sigma$ if $\sigma \in X_2$. Recently, in [10] the fundamental groupoid $\Pi_1 : Simp(\mathbb{X}) \to Grpd(\mathbb{X})$ has been studied for simplicial objects in an exact Mal'tsev category $\mathbb{X}$ as the left adjoint of the nerve functor $\mathcal{N} : Grpd(\mathbb{X}) \to Simp(\mathbb{X})$. Indeed, if $\mathbb{X}$ is semi-abelian (in particular the category of groups as in our case) for a simplicial object X, $\Pi_1(X)$ is the unique groupoid strucutre that has

$$\begin{tikzcd} X_1/(d_2(Ker(d_0) \cap Ker(d_1))) \arrow[r, shift left=2] \arrow[r, shift right=2] & X_0 \arrow[l] \end{tikzcd}$$

as the underlying reflexive graph, which for simplicial groups corresponds to the cotruncation functor Cot_1. Thus we have:

Corollary 7.7. *The fundamental functor* $\Pi_{1\geq}$ *is naturally isomorphic to the fundamental groupoid functor as indicated in the diagram:*

$$\begin{tikzcd} Simp(Grp) \arrow[rr, "\Pi_{1\geq}"] \arrow[dr, "\Pi_1"'] & & \mathcal{M}_{1\geq} \\ & Grpd(Grp) \arrow[ur, "\mathcal{N}"'] & \end{tikzcd}$$

8. Further remarks

1. The proof of Lemma 7.4 relies on the notion of Carrasco and Cegarra's hypercrossed modules [6]. A hypercrossed module is a chain complex M with group actions $M_i \to Aut(M_j)$ and binary operations $M_i \times M_j \to M_k$ satisfying some equations. The equivalence of hypercrossed modules and simplicial groups extends the equivalence between crossed modules and internal groupoids. Thus, the torsion theories presented here can be studied in hypercrossed modules using cotruncations as in chain complexes. In particular, in [23] torsion theories in varieties of hypercrossed modules are studied and it is shown to have easier descriptions. These varieties include the category of Ashley's reduced crossed complexes and Conduché's 2-crossed modules.

2. For a simplicial group X, the lattice $\mu(Grp)$ of torsion theories induces a lattice of torsion subgroups of X

$$0 \leq \cdots \leq \mu_{\geq 2}(X) \ \leq \ \mu_{1\geq}(X) \ \leq \ \mu_{\geq 1}(X) \ \leq \ \mu_{0\geq}(X) \ \leq \quad X.$$

Similarly, we can consider the chain sequence of quotients given by the torsion-free quotients

$$\ldots \twoheadrightarrow X/\mu_{1\geq}(X) \twoheadrightarrow X/\mu_{\geq 1}(X) \twoheadrightarrow X/\mu_{0\geq}(X) \twoheadrightarrow 0\,.$$

This sequence of quotients has the property that $X/\mu_{n\geq}(X)$ and $X/\mu_{\geq n+1}(X)$ are n*-truncated*, this means they have trivial homotopy groups above n. These observations present similarities to the notions of Postnikov systems and factorization systems (n-connected, n-truncated) used in different settings of homotopy theory (see for example Chapter 6 [14]). However, a direct connection is still unknown.

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A COMBINATORIAL E_∞-ALGEBRA STRUCTURE ON CUBICAL COCHAINS AND THE CARTAN–SERRE MAP

Ralph M. Kaufmann　　*Anibal M. Medina-Mardones*

Résumé. Les cochaînes cubiques sont munies d'un produit associatif, dual à la diagonale de Serre, relevant la structure commutative graduée en cohomologie. Dans ce travail, nous introduisons par des méthodes combinatoires explicites une extension de ce produit à une structure E_∞. Comme application, nous prouvons que l'application de Cartan–Serre, qui relie les cochaînes singulières cubiques et simpliciales d'espaces, est un quasi-isomorphisme de E_∞-algèbres.

Abstract. Cubical cochains are equipped with an associative product, dual to the Serre diagonal, lifting the graded commutative structure in cohomology. In this work we introduce through explicit combinatorial methods an extension of this product to a full E_∞-structure. As an application we prove that the Cartan–Serre map, which relates the cubical and simplicial singular cochains of spaces, is a quasi-isomorphism of E_∞-algebras.

Keywords. Cubical sets, cochain complex, cup product, Cartan–Serre map, E_∞-algebras, operads.

Mathematics Subject Classification (2020). 55N45, 18M70, 18M85.

1. Introduction

Instead of simplices, in his groundbreaking work on fibered spaces Serre considered cubes as the basic shapes used to define cohomology, stating that:

> Il est en effet evident que ces derniers se pretent mieux que les simplexes a l'etude des produits directs, et, a fortiori, des espaces fibres qui en sont la generalisation. [Ser51, p.431]

Cubical sets, a model for the homotopy category, were considered by Kan [Kan55; Kan56] before introducing simplicial sets, they are central to non-abelian algebraic topology [BHS11], and have become important in Voevodsky's program for univalent foundations and homotopy type theory [KV20; Coh+17]. Other areas that highlight the relevance of cubical methods are applied topology, where cubical complexes are ubiquitous in the study of images [KMM04], condensed matter physics, where models on cubical lattices are central [Bax85], and geometric group theory [Gro87], where fundamental results have been obtained considering actions on certain cube complexes characterized combinatorially [Ago13].

Cubical cochains are equipped with the *Serre algebra structure*, a lift to the cochain level of the graded ring structure in cohomology. Using an acyclic carrier argument it can be shown that this product is commutative up to coherent homotopies in a non-canonical way. The study of such objects, referred to as E_∞-algebras, has a long history, where (co)homology operations [SE62; May70], the recognition of infinite loop spaces [BV73; May72] and complete algebraic models of the p-adic homotopy category [Man01] are key milestones. The goal of this work is to introduce a description of an explicit E_∞-algebra structure naturally extending the Serre algebra structure, and relate it to one on simplicial cochains extending the Alexander–Whitney algebra structure.

We use the combinatorial model of the E_∞-operad $\mathrm{U}(\mathcal{M})$ obtained from the finitely presented prop $\mathcal{M}$ introduced in [Med20a]. The resulting $\mathrm{U}(\mathcal{M})$-algebra structure on cubical cochains is induced from a natural $\mathcal{M}$-bialgebra structure on the chains of representable cubical sets, which is determined by only three linear maps. To our knowledge, this is the first effective construction of an E_∞-algebra structure on cubical cochains. Non-constructively, this result could be obtained using a lifting argument based on the cofibrancy of the reduced version of the operad $\mathrm{U}(\mathcal{M})$ in the model category of operads [Hin97; BM03], but this existence statement is not very useful in concrete situations. To illustrate the advantages of an effective construction let us consider a prime p. The mod p cohomology of spaces is equipped with natural stable endomorphisms, known as Steenrod operations [SE62]. Following an operadic viewpoint developed by May [May70], in [KM21] we exhibited integral elements in $\mathrm{U}(\mathcal{M})$ representing Steenrod operations on the mod p homology of $\mathrm{U}(\mathcal{M})$-algebras. Since, as proven in this article, the cochains of a cubical set are equipped with a $\mathrm{U}(\mathcal{M})$-algebra structure, we obtain natural cochain level multioperations for cubical sets representing Steenrod operation at every p. This cubical cup-(p, i) products are explicit enough to have been implemented in the open source computer algebra system `ComCH` [Med21a].

We now turn to the comparison between cubical and simplicial cochains. In [Ser51, p. 442], Serre described for any topological space $\mathfrak{Z}$ a natural quasi-isomorphism

$$\mathrm{S}^\bullet_\square(\mathfrak{Z}) \to \mathrm{S}^\bullet_\triangle(\mathfrak{Z}) \tag{1}$$

between its cubical and simplicial singular cochains, stating this to be a quasi-isomorphism of algebras with respect to the usual structures. We will consider a well known Quillen equivalence

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\mathcal{U}"'] & \mathsf{cSet} \arrow[l, bend right, "\mathcal{T}"'] \end{tikzcd}$$

between simplicial and cubical sets, and construct a natural chain map

$$\mathrm{N}^\bullet_\square(\mathcal{U}Y) \to \mathrm{N}^\bullet_\triangle(Y) \tag{2}$$

for every simplicial set Y. In [Med20a], a natural $\mathrm{U}(\mathcal{M})$-algebra structure extending the Alexander–Whitney coalgebra structure was constructed on simplicial sets. With respect to it and the one defined here for cubical sets we have the following results after passing to a sub-E_∞-operad of $\mathrm{U}(\mathcal{M})$.

Theorem. *The map presented in Equation (2) is a quasi-isomorphism of E_∞-algebras.*

From this result, stated as Theorem 15, we deduce the following two. The first one concerns the triangulation functor $\mathcal{T}$ and it is stated more precisely as Corollary 16.

Corollary. *There is a natural zig-zag of E_∞-algebra quasi-isomorphisms between the cochains of a cubical set and those of its triangulation.*

The next one concerns the map presented in Equation (1), relating the cubical and simplicial singular cochains of a space, and it is stated more precisely as Corollary 17.

Corollary. *The Cartan–Serre map is a quasi-isomorphism of E_∞-algebras.*

Remark. In this introduction we have used the setting defined by cochains and products since it is more familiar, whereas in the rest of the text we use the more fundamental one defined by chains and coproducts.

Outline

We recall the required notions from homological algebra and category theory in Section 2. The necessary concepts from the theory of operads and props is reviewed in Section 3, including the definition of the prop $\mathcal{M}$. Section 4 contains our main contribution; an explicit natural $\mathcal{M}$-bialgebra structure on the chains of representable cubical sets and, from it, a natural E_∞-coalgebra structure on the chains of cubical sets. The comparison between simplicial and cubical chains is presented in Section 5, where we show that the Cartan–Serre map is a quasi-isomorphism respecting E_∞-structures. We close presenting some future work in Section 6.

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2. Conventions and preliminaries

2.1 Chain complexes

Throughout this article $\Bbbk$ denotes a commutative and unital ring and we work over its associated closed symmetric monoidal category of differential (homologically) graded $\Bbbk$-modules $(\mathsf{Ch}, \otimes, \Bbbk)$. We refer to the objects and morphisms of this category as *chain complexes* and *chain maps* respectively. We denote by $\mathrm{Hom}(C, C')$ the chain complex of $\Bbbk$-linear maps between chain complexes C and C', and refer to the functor $\mathrm{Hom}(-, \Bbbk)$ as *linear duality*.

2.2 Presheaves

Recall that a category is said to be *small* if its objects and morphisms form sets. We denote the category of small categories by Cat. Given categories B and C with B small we denote their associated *functor category* by $\mathsf{Fun}(\mathsf{B}, \mathsf{C})$. A category is said to be *cocomplete* if any functor to it from a small category has a colimit. If A is small and C cocomplete, then the (*left*) *Kan extension of g along f* exists for any pair of functors f and g in the diagram below, and it is the initial object in $\mathsf{Fun}(\mathsf{B}, \mathsf{C})$ making

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "g"] \arrow[d, "f"'] & \mathsf{C} \\
\mathsf{B} \arrow[ur, dashed, bend right] &
\end{tikzcd}$$

commute. A Kan extension along the *Yoneda embedding*, i.e., the functor

$$\mathcal{Y} \colon \mathsf{A} \to \mathsf{Fun}(\mathsf{A}^{\mathrm{op}}, \mathsf{Set})$$

induced by the assignment

$$a \mapsto \big(a' \mapsto \mathsf{A}(a', a)\big),$$

is referred to as a *Yoneda extension*. Abusively we use the same notation for a functor and for its Yoneda extension. We refer to objects of $\mathsf{Fun}(\mathsf{A}^{\mathrm{op}}, \mathsf{Set})$ in the image of the Yoneda embedding as *representable*.

3. Operads, props and E_∞-structures

We now review the definition of the finitely presented prop $\mathcal{M}$ introduced in [Med20a] and whose associated operad is a model of the E_∞-operad. Given its small number of generators and relations, is well suited to explicitly define E_∞-structures. We start by recalling some basic material from the theory of operads and props.

3.1 Symmetric (bi)modules

Let $\mathbb{S}$ be the category whose objects are the non-negative integers $\mathbb{N}$ and whose set of morphisms between n and n' is empty if $n \neq n'$ and is otherwise the symmetric group $\mathbb{S}_n$. A *left* $\mathbb{S}$*-module* (resp. *right* $\mathbb{S}$*-module* or $\mathbb{S}$*-bimodule*) is a functor from $\mathbb{S}$ (resp. $\mathbb{S}^{\mathrm{op}}$ or $\mathbb{S}^{\mathrm{op}} \times \mathbb{S}$) to Ch. In this paper we prioritize left module structures over their right counterparts. As usual, taking inverses makes both perspectives equivalent. We respectively denote by $\mathsf{Mod}_{\mathbb{S}}$ and $\mathsf{biMod}_{\mathbb{S}}$ the categories of left $\mathbb{S}$-modules and of $\mathbb{S}$-bimodules with morphisms given by natural transformations.

Given a chain complex C, we have the following key examples of a left and a right $\mathbb{S}$-module

$$\mathrm{End}^C(n) = \mathrm{Hom}(C, C^{\otimes n}), \qquad \mathrm{End}_C(m) = \mathrm{Hom}(C^{\otimes m}, C),$$

and of an $\mathbb{S}$-bimodule

$$\mathrm{End}_C^C(m, n) = \mathrm{Hom}(C^{\otimes m}, C^{\otimes n}),$$

where the symmetric actions are given by permutation of tensor factors.

The group homomorphisms $\mathbb{S}_n \to \mathbb{S}_1^{\mathrm{op}} \times \mathbb{S}_n$ induce a forgetful functor

$$\mathrm{U} \colon \mathsf{biMod}_{\mathbb{S}} \to \mathsf{Mod}_{\mathbb{S}} \tag{3}$$

defined explicitly on an object $\mathcal{P}$ by $\mathrm{U}(\mathcal{P})(n) = \mathcal{P}(1, n)$ for $n \in \mathbb{N}$. The similarly defined forgetful functor to right $\mathbb{S}$-modules will not be considered.

3.2 Composition structures

Operads and *props* are obtained by enriching $\mathbb{S}$-modules and $\mathbb{S}$-bimodules with certain composition structures. Intuitively, these are obtained by abstracting the composition structure naturally present in the left $\mathbb{S}$-module End^C

(or right $\mathbb{S}$-module End_C), naturally an operad, and the $\mathbb{S}$-bimodule End_C^C, naturally a prop. More explicitly, an operad $\mathcal{O}$ is a left $\mathbb{S}$-module with chain maps

$$\Bbbk \to \mathcal{O}(1),$$
$$\mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_r) \otimes \mathcal{O}(r) \to \mathcal{O}(n_1 + \cdots + n_r),$$

satisfying relations of associativity, equivariance and unitality. Similarly, a prop $\mathcal{P}$ is an $\mathbb{S}$-bimodule together with chain maps

$$\Bbbk \to \mathcal{P}(n,n),$$
$$\mathcal{P}(m,k) \otimes \mathcal{P}(k,n) \to \mathcal{P}(m,n),$$
$$\mathcal{P}(m,n) \otimes \mathcal{P}(m',n') \to \mathcal{P}(m+m',n+n'),$$

satisfying certain natural relations. For a complete presentation of these concepts we refer to Definition 11 and 54 of [Mar08]. We respectively denote the category of operads and props with structure preserving morphisms by Oper and Prop.

Let C be a chain complex, $\mathcal{O}$ an operad, and $\mathcal{P}$ a prop. An *$\mathcal{O}$-coalgebra* (resp. *$\mathcal{O}$-algebra* or *$\mathcal{P}$-bialgebra*) structure on C is a structure preserving morphism $\mathcal{O} \to \mathrm{End}^C$ (resp. $\mathcal{O} \to \mathrm{End}_C$ or $\mathcal{P} \to \mathrm{End}_C^C$). We mention that the linear dual of an $\mathcal{O}$-coalgebra is an $\mathcal{O}$-algebra.

Since the forgetful functor presented in Equation (3) induces a functor

$$\mathrm{U} \colon \mathsf{Prop} \to \mathsf{Oper},$$

any $\mathcal{P}$-bialgebra structure on C

$$\mathcal{P} \to \mathrm{biEnd}_C^C$$

induces a $\mathrm{U}(\mathcal{P})$-coalgebra structure on it

$$\mathrm{U}(\mathcal{P}) \to \mathrm{U}(\mathrm{biEnd}_C^C) \cong \mathrm{coEnd}^C.$$

3.3 E_∞-operads

Recall that a *projective $\mathbb{S}_n$-resolution* of a chain complex C is a quasi-isomorphism $R \to C$ from a chain complex R of projective $\Bbbk[\mathbb{S}_n]$-modules. An $\mathbb{S}$-module M is said to be E_∞ if there exists a morphism of $\mathbb{S}$-modules $M \to \underline{\Bbbk}$ inducing for each $n \in \mathbb{N}$ a free $\mathbb{S}_n$-resolution $M(n) \to \Bbbk$. An operad is said to be an *E_∞-operad* if its underlying $\mathbb{S}$-module is E_∞. A prop $\mathcal{P}$ is said to be an *E_∞-prop* if $\mathrm{U}(\mathcal{P})$ is an E_∞-operad.

3.4 Presentations

The *free prop* construction is the left adjoint to the forgetful functor from props to $\mathbb{S}$-bimodules. Explicitly, the free prop $\mathrm{F}(M)$ generated by an $\mathbb{S}$-bimodule M is constructed using isomorphism classes of directed graphs with no directed loops that are enriched with the following labeling structure. We think of each directed edge as built from two compatibly directed half-edges. For each vertex v of a directed graph Γ, we have the sets $in(v)$ and $out(v)$ of half-edges that are respectively incoming to and outgoing from v. Half-edges that do not belong to $in(v)$ or $out(v)$ for any v are divided into the disjoint sets $in(\Gamma)$ and $out(\Gamma)$ of incoming and outgoing external half-edges. For any positive integer n let $\overline{n} = \{1, \dots, n\}$ and set $\overline{0} = \emptyset$. For any finite set S, denote the cardinality of S by $|S|$. The labeling is given by bijections

$$\overline{|in(\Gamma)|} \to in(\Gamma), \qquad \overline{|out(\Gamma)|} \to out(\Gamma),$$

and

$$\overline{|in(v)|} \to in(v), \qquad \overline{|out(v)|} \to out(v),$$

for every vertex v. We refer to the isomorphism classes of such labeled directed graphs with no directed loops and m incoming and n outgoing half-edges as (m, n)-*graphs*. We denote the set these form by $\mathfrak{G}(m, n)$. We use graphs immersed in the plane to represent elements in $\mathfrak{G}(m, n)$, with the direction implicitly given from top to bottom and the labeling from left to right. Please consult Figure 1 for an example.

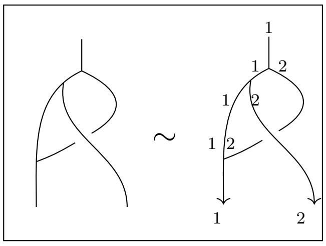


Figure 1: Immersed graph representing a $(1, 2)$-graph.

We consider the right action of $\mathbb{S}_m$ and the left action of $\mathbb{S}_n$ on a (m, n)-graph given respectively by permuting the labels of $in(\Gamma)$ and $out(\Gamma)$. This action defines the $\mathbb{S}$-bimodule structure on the free prop

$$\mathrm{F}(M)(m,n) \;=\; \bigoplus_{\substack{\Gamma \text{ in} \\ \mathfrak{G}(m,n)}} \;\bigotimes_{\substack{v \text{ in} \\ Vert(\Gamma)}} out(v) \otimes_{\mathbb{S}_p} M(p,q) \otimes_{\mathbb{S}_p} in(v), \tag{4}$$

where we have simplified the notation writing p and q for $\overline{|in(v)|}$ and $\overline{|out(v)|}$ respectively. The differential $\partial_{\mathrm{F}(M)}$ is the extension of that of M to the tensor product (4), and the prop structure is induced by the "identity graphs"

$$||\dots|$$

together with (relabeled) grafting and disjoint union.

Let G be an assignment of a set $G(m,n)_d$ to each $m,n \in \mathbb{N}$ and $d \in \mathbb{Z}$. Denote by $\Bbbk[\mathbb{S}^{\mathrm{op}} \times \mathbb{S}]\{G\}$ the $\mathbb{S}$-bimodule mapping (m,n) to the chain complex with trivial differential and degree d part equal to

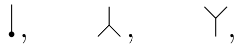
$$\Bbbk[\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n]\big\{G(m,n)_d\big\}.$$

We will denote by $\mathrm{F}(G)$ the free prop generated by this $\mathbb{S}$-bimodule. Let $\partial \colon \Bbbk[\mathbb{S}^{\mathrm{op}} \times \mathbb{S}]\{G\} \to \mathrm{F}(G)$ be a morphism of $\mathbb{S}$-bimodules whose canonical extension $\partial \colon\ \mathrm{F}(G) \to \mathrm{F}(G)$ defines a differential. We denote by $\mathrm{F}_\partial(G)$ the prop obtained by endowing $\mathrm{F}(G)$ with this differential. Let R be a collection of elements in $\mathrm{F}(G)$ and denote by $\langle R \rangle$ the smallest ideal containing R. The prop *generated by G modulo R with boundary ∂* is defined to be $\mathrm{F}_\partial(G)/\langle R \rangle$.

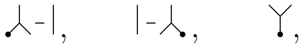
3.5 The prop $\mathcal{M}$

We now recall the E_∞-prop that is central to our constructions.

Definition 1. Let $\mathcal{M}$ be the prop generated by

 (5)

in $(1,0)_0$, $(1,2)_0$ and $(2,1)_1$ respectively, modulo the relations

 (6)

with boundary defined by

∂  $= 0,$ ∂  $= 0,$ ∂  $=$  (7)

Explicitly, any element in $\mathcal{M}(m,n)$ can be written as a linear combination of the (m,n)-graphs generated by those in (5) via grafting, disjoint union and relabeling, modulo the ideal generated by the relations in (6). Its boundary is determined by (7) using (4).

Proposition 2 ([Med20a, Theorem 3.3])**.** *$\mathcal{M}$ is an E_∞-prop.*

Remark. The prop $\mathcal{M}$ is obtained from applying the functor of chains to a prop over the category of cellular spaces [Med21b], a quotient of which is isomorphic to the E_∞-operad of stable arc surfaces [Kau09].

4. An E_∞-structure on cubical chains

In this section we construct a natural $\mathcal{M}$-bialgebra structure on the chains of representable cubical sets. These are determined by three natural linear maps satisfying the relations defining $\mathcal{M}$. A Yoneda extension then provides the chains of any cubical set with a natural $\mathrm{U}(\mathcal{M})$-coalgebra structure. We begin by recalling the basics of cubical topology.

4.1 Cubical sets

The objects of the *cube category* $\square$ are the sets $2^n = \{0,1\}^n$ with $2^0 = \{0\}$ for $n \in \mathbb{N}$, and its morphisms are generated by the *coface* and *codegeneracy* maps

$$\begin{aligned} \delta_i^\varepsilon &= \mathrm{id}_{2^{i-1}} \times \delta^\varepsilon \times \mathrm{id}_{2^{n-1-i}} \colon 2^{n-1} \to 2^n, \\ \sigma_i &= \mathrm{id}_{2^{i-1}} \times \sigma \times \mathrm{id}_{2^{n-i}} \quad \colon 2^n \to 2^{n-1}, \end{aligned}$$

where $\varepsilon \in \{0,1\}$ and the functors

$$\begin{tikzcd} 2^0 \arrow[r, bend left, "\delta^0"] \arrow[r, bend right, "\delta^1"'] & 2^1 \arrow[r, "\sigma"] & 2^0 \end{tikzcd}$$

are defined by

$$\delta^0(0) = 0, \qquad \delta^1(0) = 1, \qquad \sigma(0) = \sigma(1) = 0.$$

More globally, the category $\square$ is the free strict monoidal category with an assigned internal bipointed object. We refer to [GM03] for a more leisurely exposition and variants of this definition.

We denote by $\mathrm{Dgn}(2^m, 2^n)$ the subset of morphism in $\square(2^m, 2^n)$ of the form $\sigma_i \circ \tau$ with $\tau \in \square(2^m, 2^{n+1})$.

The category of *cubical sets* $\mathsf{Fun}(\square^{\mathrm{op}}, \mathsf{Set})$ is denoted by cSet and the representable cubical set $\mathcal{Y}(2^n)$ by $\square^n$. For any cubical set X we write, as usual, X_n instead of $X(2^n)$.

4.2 Cubical topology

Consider the topological n-cube

$$\mathbb{I}^n = \big\{(x_1, \dots, x_n) \mid x_i \in [0,1]\big\}.$$

The assignment $2^n \to \mathbb{I}^n$ defines a functor $\square \to \mathsf{Top}$ with

$$\begin{aligned} \delta_i^\varepsilon(x_1, \dots, x_n) &= (x_1, \dots, x_i, \varepsilon, x_{i+1}, \dots x_n), \\ \sigma_i(x_1, \dots, x_n) &= (x_1, \dots, \widehat{x}_i, \dots, x_n). \end{aligned}$$

Its Yoneda extension is known as *geometric realization*. It has a right adjoint $\mathrm{Sing}^\square \colon \mathsf{Top} \to \mathsf{cSet}$ referred to as the *cubical singular complex* satisfying

$$\mathrm{Sing}^\square(\mathfrak{Z})_n = \mathsf{Top}(\mathbb{I}^n, \mathfrak{Z})$$

for any topological space $\mathfrak{Z}$.

4.3 Cubical chains

The functor of (*normalized*) *chains* $\mathrm{N} \colon \mathsf{cSet} \to \mathsf{Ch}$ is the Yoneda extension of the functor $\square \to \mathsf{Ch}$ defined next. It assigns to an object 2^n the chain complex having in degree m the module

$$\frac{\mathbb{k}\{\square(2^m, 2^n)\}}{\mathbb{k}\{\mathrm{Dgn}(2^m, 2^n)\}}$$

and differential induced by

$$\partial(\mathsf{id}_{2^n}) = \sum_{i=1}^{n} (-1)^i \left(\delta_i^1 - \delta_i^0\right).$$

To a morphism $\tau\colon 2^n \to 2^{n'}$ it assigns the chain map

$$\begin{aligned} \mathrm{N}(\square^n)_m &\longrightarrow \mathrm{N}(\square^{n'})_m \\ \big(2^m \to 2^n\big) &\longmapsto \big(2^m \to 2^n \xrightarrow{\tau} 2^{n'}\big). \end{aligned}$$

The chain complex $\mathrm{N}(\square^n)$ is isomorphic to both: $\mathrm{N}(\square^1)^{\otimes n}$ and the cellular chains on the topological n-cube with its standard CW structure $\mathrm{C}(\mathbb{I}^n)$. We use the isomorphism $\mathrm{N}(\square^n) \cong \mathrm{C}(\mathbb{I}^1)^{\otimes n}$ when denoting the elements in the basis of $\mathrm{N}(\square^n)$ by $x_1 \otimes \cdots \otimes x_n$ with $x_i \in \{[0], [0,1], [1]\}$.

For a topological space $\mathfrak{Z}$, the chain complex $\mathrm{N}(\mathrm{Sing}^{\square}\, \mathfrak{Z})$ is referred to as the *cubical singular chains* of $\mathfrak{Z}$.

4.4 Serre coalgebra

We now recall the *Serre coalgebra structure*, a natural (counital and coassociative) coalgebra structure on cubical chains.

By a Yoneda extension, to define this structure it suffices to describe it on the chains of representable cubical sets $\mathrm{N}(\square^n)$. For $\mathrm{N}(\square^1)$ we have

$$\begin{aligned} \epsilon\big([0]\big) &= 1, & \Delta\big([0]\big) &= [0] \otimes [0], \\ \epsilon\big([1]\big) &= 1, & \Delta\big([1]\big) &= [1] \otimes [1], \\ \epsilon\big([0,1]\big) &= 0, & \Delta\big([0,1]\big) &= [0] \otimes [0,1] + [0,1] \otimes [1]. \end{aligned}$$

The Serre coalgebra structure on a general $\mathrm{N}(\square^n)$ is define using the isomorphism $\mathrm{N}(\square^n) \cong \mathrm{N}(\square^1)^{\otimes n}$ and the monoidal structure on the category of coalgebras. Explicitly, the structure maps are given by the compositions

$$\epsilon\colon\ \mathrm{N}(\square^1)^{\otimes n} \xrightarrow{\epsilon^{\otimes n}} \Bbbk^{\otimes n} \to \Bbbk$$

and

$$\Delta\colon\ \mathrm{N}(\square^1)^{\otimes n} \xrightarrow{\Delta^{\otimes n}} \big(\mathrm{N}(\square^1)^{\otimes 2}\big)^{\otimes n} \xrightarrow{\sigma_{2n}^{-1}} \big(\mathrm{N}(\square^1)^{\otimes n}\big)^{\otimes 2},$$

where σ_{2n} in $\mathbb{S}_{2n}$ is the (n,n)-shuffle mapping the first and second "decks" to odd and even values respectively. An explicit description of σ_{2n} is presented in Equation (13).

Remark. Similarly to how the Alexander–Whitney coalgebra can be interpreted geometrically as the sum of all complementary pairs of front and back faces of a simplex, this coproduct is, up to signs, also given by the sum of complementary pairs of front and back faces of a cube.

For later reference we record a useful description of the value of Δ on the top dimensional basis element of $\mathrm{N}(\square^n)$.

Lemma 3. *For any* $n \in \mathbb{N}$,

$$\Delta\big([0,1]^{\otimes n}\big) = \sum_{\lambda \in \Lambda} (-1)^{\operatorname{ind} \lambda} \left(x_1^{(\lambda)} \otimes \cdots \otimes x_n^{(\lambda)}\right) \otimes \left(y_1^{(\lambda)} \otimes \cdots \otimes y_n^{(\lambda)}\right),$$

where each λ *in* Λ *is a map* $\lambda \colon \{1, \ldots, n\} \to \{0,1\}$ *with* $\lambda(i)$ *interpreted as*

$$\begin{aligned} 0: \ x_i^{(\lambda)} &= [0,1], & \qquad 1: \ x_i^{(\lambda)} &= [0], \\ y_i^{(\lambda)} &= [1], & y_i^{(\lambda)} &= [0,1], \end{aligned}$$

and $\operatorname{ind} \lambda$ *is the cardinality of* $\{i < j \mid \lambda(i) > \lambda(j)\}$.

4.5 Degree 1 product

Let $n \in \mathbb{N}$. For $x = x_1 \otimes \cdots \otimes x_n$ a basis element of $\mathrm{N}(\square^n)$ and $\ell \in \{1, \ldots, n\}$ we write

$$\begin{aligned} x_{<\ell} &= x_1 \otimes \cdots \otimes x_{\ell-1}, \\ x_{>\ell} &= x_{\ell+1} \otimes \cdots \otimes x_n, \end{aligned}$$

with the convention

$$x_{<1} = x_{>n} = 1 \in \mathbb{Z}.$$

We define the *product* $* \colon \mathrm{N}(\square^n)^{\otimes 2} \to \mathrm{N}(\square^n)$ by

$$(x_1 \otimes \cdots \otimes x_n) * (y_1 \otimes \cdots \otimes y_n) = (-1)^{|x|} \sum_{i=1}^{n} x_{<i}\, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i})\, y_{>i},$$

where the only non-zero values of $x_i * y_i$ are

$$[0] * [1] = [0,1], \qquad [1] * [0] = -[0,1].$$

Example. Since in $\mathrm{N}(\square^3)$ we have that

$$\partial\big([0]\otimes[0]\otimes[0]\big)=\partial\big([1]\otimes[1]\otimes[1]\big)=0$$

and

$$\begin{aligned}&\partial\big([0]\otimes[0]\otimes[0]*[1]\otimes[1]\otimes[1]\big)\\ &\qquad=\partial\big([0,1]\otimes[1]\otimes[1]+[0]\otimes[0,1]\otimes[1]+[0]\otimes[0]\otimes[0,1]\big)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad=[1]\otimes[1]\otimes[1]-[0]\otimes[0]\otimes[0],\end{aligned}$$

we conclude that in general $*$ is not a cycle in the appropriate Hom complex, so it does not descend to homology. This product should be understood as an algebraic version of a consistent choice of path between points in a cube. In our case, as illustrated in Figure 2, the chosen path is given by the union of segments parallel to edges of the cube.

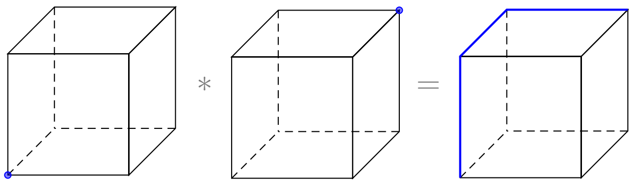
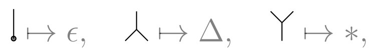


Figure 2: Geometric representation of $\big([0]\otimes[0]\otimes[0]*[1]\otimes[1]\otimes[1]\big)$ where we are using the width-depth-height order.

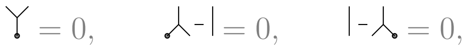
4.6 $\mathcal{M}$-bialgebra on representable cubical sets

Lemma 4. *The assignment*



induces a natural $\mathcal{M}$*-bialgebra structure on* $\mathrm{N}(\square^n)$ *for every* $n\in\mathbb{N}$.

Proof. We need to show that this assignment is compatible with the relations



and

$$\partial \,\text{⫰} = 0, \qquad \partial \,\text{⅄} = 0, \qquad \partial \,\text{Y} = \text{⫰}\,|\,\text{-}\,|\,\text{⫰}.$$

For the rest of this proof let us consider two basis elements of $\mathrm{N}(\square^n)$

$$x = x_1 \otimes \cdots \otimes x_n \qquad \text{and} \qquad y = y_1 \otimes \cdots \otimes y_n.$$

Since the degree of $*$ is 1 and $\epsilon([0,1]) = 0$, we can verify the first relation easily:

$$\varepsilon(x * y) = \sum (-1)^{|x|} \epsilon(y_{<i}) \epsilon(x_{<i}) \otimes \epsilon(x_i * y_i) \otimes \epsilon(x_{>i}) \epsilon(y_{>i}) = 0.$$

For the second relation we want to show that $(\epsilon \otimes \mathsf{id}) \circ \Delta = \mathsf{id}$. Since

$$\begin{aligned}
(\epsilon \otimes \mathsf{id}) \circ \Delta([0]) &= \epsilon([0]) \otimes [0] = [0], \\
(\epsilon \otimes \mathsf{id}) \circ \Delta([1]) &= \epsilon([1]) \otimes [1] = [1], \\
(\epsilon \otimes \mathsf{id}) \circ \Delta([0,1]) &= \epsilon([0]) \otimes [0,1] + \epsilon([0,1]) \otimes [1] = [0,1],
\end{aligned}$$

we have

$$\begin{aligned}
&(\epsilon \otimes \mathsf{id}) \circ \Delta(x_1 \otimes \cdots \otimes x_n) = \\
&\qquad \sum \pm \left(\epsilon\big(x_1^{(1)}\big) \otimes \cdots \otimes \epsilon\big(x_n^{(1)}\big) \right) \otimes \left(x_1^{(2)} \otimes \cdots \otimes x_n^{(2)} \right) \\
&\qquad\qquad = x_1 \otimes \cdots \otimes x_n,
\end{aligned}$$

where the sign is obtained by noticing that the only non-zero term occurs when each factor $x_i^{(0)}$ is of degree 0. The third relation is verified analogously. The fourth and fifth are precisely the well known facts that ϵ and Δ are chain maps. To verify the sixth and final relation we need to show that

$$\partial(x * y) \ + \ \partial\, x * y \ + \ (-1)^{|x|} x * \partial\, y \ = \ \epsilon(x) y \ - \ \epsilon(y) x.$$

We have

$$x * y = \sum (-1)^{|x|} x_{<i}\, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i})\, y_{>i}$$

and

$$\begin{aligned}
\partial(x * y) &= \sum (-1)^{|x|} \ \partial\, x_{<i}\, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i})\, y_{>i} \\
&\quad + \sum (-1)^{|x| + |x_{<i}|} \, x_{<i}\, \epsilon(y_{<i}) \otimes \partial(x_i * y_i) \otimes \epsilon(x_{>i})\, y_{>i} \\
&\quad - \sum (-1)^{|x| + |x_{<i}|} \, x_{<i}\, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i}) \ \partial\, y_{>i}.
\end{aligned}$$

Since $|x| = |x_{<i}| + |x_i| + |x_{>i}|$ and $\epsilon(x_{>i}) \neq 0 \Leftrightarrow |x_{>i}| = 0$ as well as $\partial(x_i * y_i) \neq 0 \Rightarrow |x_i| = 0$ we have

$$\begin{aligned} \partial(x * y) = & \sum (-1)^{|x|}\, \partial\, x_{<i}\, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i})\, y_{>i} \\ & + \sum x_{<i}\, \epsilon(y_{<i}) \otimes \partial(x_i * y_i) \otimes \epsilon(x_{>i})\, y_{>i} \\ & - \sum x_{<i}\, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i})\, \partial\, y_{>i}. \end{aligned} \tag{8}$$

We also have

$$\begin{aligned} \partial\, x * y = & \sum (-1)^{|x|-1}\, \partial\, x_{<i}\, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i})\, y_{>i} \\ & + \sum (-1)^{|x|-1+|x_{<i}|}\, x_{<i}\, \epsilon(y_{<i}) \otimes \partial\, x_i * y_i \otimes \epsilon(x_{>i})\, y_{>i} \\ & + \sum (-1)^{|x|-1+|x_{<i}|}\, x_{<i}\, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(\partial\, x_{>i})\, y_{>i}. \end{aligned}$$

Since

$$\epsilon(\partial\, x_{>i}) = 0, \qquad \partial\, x_i \neq 0 \Leftrightarrow |x_i| = 1,$$

we have

$$\begin{aligned} \partial\, x * y = & \sum (-1)^{|x|-1}\, \partial\, x_{<i}\, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i})\, y_{>i} \\ & + \sum x_{<i}\, \epsilon(y_{<i}) \otimes \partial\, x_i * y_i \otimes \epsilon(x_{>i})\, y_{>i}. \end{aligned} \tag{9}$$

We also have

$$\begin{aligned} (-1)^{|x|}\, x * \partial\, y = & \sum x_{<i}\, \epsilon(\partial\, y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i})\, y_{>i} \\ & + \sum (-1)^{|y_{<i}|}\, x_{<i}\, \epsilon(y_{<i}) \otimes x_i * \partial\, y_i \otimes \epsilon(x_{>i})\, y_{>i} \\ & + \sum (-1)^{|y_{<i}|+|y_i|}\, x_{<i}\, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i})\, \partial\, y_{>i}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} (-1)^{|x|}\, x * \partial\, y = & \sum x_{<i}\, \epsilon(y_{<i}) \otimes x_i * \partial\, y_i \otimes \epsilon(x_{>i})\, y_{>i} \\ & + \sum x_{<i}\, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i})\, \partial\, y_{>i}. \end{aligned} \tag{10}$$

Putting identities (8), (9) and (10) together, we get

$$\begin{aligned} & \partial(x \otimes y) \;+\; \partial\, x * y \;+\; (-1)^{|x|} x * \partial\, y \\ & \quad = \sum \epsilon(y_{<i})\, x_{<i} \otimes \big(\partial(x_i * y_i) + \partial\, x_i * y_i + x_i * \partial\, y_i\big) \otimes \epsilon(x_{>i})\, y_{>i}. \end{aligned}$$

Since

$$\partial(x_i * y_i) \;+\; \partial\, x_i * y_i \;+\; x_i * \partial\, y_i = \epsilon(x_i) y_i \;-\; \epsilon(y_i) x_i,$$

we have

$$\begin{aligned} &\partial(x * y) \;+\; \partial\, x * y \;+\; (-1)^{|x|} x * \partial\, y = \\ &\qquad \sum \epsilon(y_{<i})\, x_{<i} \otimes \epsilon(x_{\geq i}) y_{\geq i} \;-\; \epsilon(y_{\leq i})\, x_{\leq i} \otimes \epsilon(x_{>i}) y_{>i} \\ &\qquad\qquad = \epsilon(x) y - \epsilon(y) x, \end{aligned}$$

as desired, where the last equality follows from a telescopic sum argument. □

4.7 E_∞-coalgebra on cubical chains

Lemma 4 defines a functor from the cube category to that of $\mathcal{M}$-bialgebras. This category is not cocomplete so we do not expect to have an $\mathcal{M}$-bialgebra structure on arbitrary cubical sets. For example, consider the chains on the cubical set X whose only non-degenerate simplices are $v, w \in X_0$. By degree reasons $v * w = 0$ for any degree 1 product $*$ in $\mathrm{N}(X)$. The third relation in $\mathcal{M}$ would then imply the contradiction $0 = w - v$. Since categories of coalgebras over operads are cocomplete we have the following.

Theorem 5. *The Yoneda extension of the composition of the functor* $\square \to \mathsf{biAlg}_{\mathcal{M}}$ *defined in Lemma 4 with the forgetful functor* $\mathsf{biAlg}_{\mathcal{M}} \to \mathsf{coAlg}_{\mathrm{U}(\mathcal{M})}$ *endows the chains of a cubical set with a natural* E_∞*-coalgebra extension of the Serre coalgebra structure.*

4.8 Cohomology operations

In [Ste47], Steenrod introduced natural operations on the mod 2 cohomology of spaces, the celebrated *Steenrod squares*

$$\begin{aligned} \mathrm{Sq}^k \colon\; \mathrm{H}^{-n} &\longrightarrow \mathrm{H}^{-n-k} \\ [\alpha] &\longmapsto \big[(\alpha \otimes \alpha)\Delta_{n-k}\big], \end{aligned}$$

via an explicit construction of natural linear maps $\Delta_i \colon \mathrm{N}(X) \to \mathrm{N}(X) \otimes \mathrm{N}(X)$ for any simplicial set X, satisfying up to signs the following homological relations

$$\partial \circ \Delta_i + \Delta_i \circ \partial = (1+T)\Delta_{i-1}, \tag{11}$$

with the convention $\Delta_{-1} = 0$. These so-called *cup-i coproducts* appear to be fundamental. We mention two results supporting this claim. In higher category theory they define the nerve of n-categories [Med20b] as introduced by Street [Str87]; and, in connection with K- and L-theory, the Ranicki–Weiss assembly [RW90] can be used to show that chain complex valued presheaves over a simplicial complex X can be fully faithfully modeled by comodules over the symmetric coalgebra structure they define on $\mathrm{N}(X)$ [Med22b].

In the cubical case, cup-i coproducts were defined in [Kad99] and [KP16]. The formulas used by these authors are similar to those introduced in [Med23] for the simplicial case, a dual yet equivalent version of Steenrod's original. A new description of cubical cup-i coproducts can be deduced from our E_∞-structure. We first present it in a recursive form

$$\begin{aligned} \Delta_0 &= \Delta, \\ \Delta_i &= (* \otimes \mathsf{id}) \circ (23)(\Delta_{i-1} \otimes \mathsf{id}) \circ \Delta. \end{aligned} \tag{12}$$

A closed form formula for Δ_i uses the $\left(\left\lceil \frac{i+2}{2} \right\rceil, \left\lfloor \frac{i+2}{2} \right\rfloor\right)$-shuffle permutation $\sigma_{i+2} \in \mathbb{S}_{i+2}$ mapping the first and second "decks" to odd and even integers respectively. Explicitly, this shuffle permutation is defined by

$$\sigma_{i+2}(\ell) = \begin{cases} 2\ell - 1 & \ell \leq \left\lceil \frac{i+2}{2} \right\rceil, \\ 2\left(\ell - \left\lceil \frac{i+2}{2} \right\rceil\right) & \ell > \left\lceil \frac{i+2}{2} \right\rceil. \end{cases} \tag{13}$$

Let $\Delta^0 = *^0 = \mathsf{id}$ and define for any $k \in \mathbb{N}$

$$\begin{aligned} *^{k+1} &= * \circ (*^k \otimes \mathsf{id}), \\ \Delta^{k+1} &= (\Delta^k \otimes \mathsf{id}) \circ \Delta. \end{aligned} \tag{14}$$

With this notation it can be checked that Equation (12) is equivalent to

$$\Delta_i = \left(*^{\left\lceil \frac{i+2}{2} \right\rceil} \otimes *^{\left\lfloor \frac{i+2}{2} \right\rfloor} \right) \circ \sigma_{i+2}^{-1}\, \Delta^{i+1}. \tag{15}$$

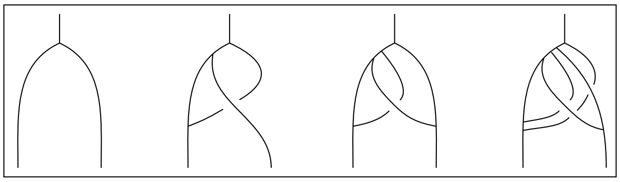


Figure 3: Graphs representing cup-i coproducts.

The first four cup-i coproducts are the images in the endomorphism operad of cubical (and simplicial) chains of the elements $\mathrm{U}(\mathcal{M})$ represented by the graphs in Figure 3.

It is not known if the cup-i coproducts defined in Equation (15) agree with those previously constructed, for which a comparison is also missing. This highlights the value of a potential axiomatic characterization of cubical cup-i coproducts as it exists in the simplicial case [Med22a].

As already mentioned, cup-i coproducts represent the Steenrod squares at the chain level, which are primary operations in mod 2 cohomology. To obtain secondary cohomology operations one studies the cohomological relations these operations satisfy, for example the Cartan and Adem relations [SE62]. To do this at the cubical cochain level, as it was done in [Med20c; BMM21] for the simplicial case, the operadic viewpoint is important, so our E_∞-structure on cubical cochains invites the construction of cochain representatives for secondary operations in the cubical case.

For p an odd prime, Steenrod also introduced operations on the mod p cohomology of spaces using the homology of symmetric groups [Ste52; Ste53]. Using the operadic framework of May [May70], we described in [KM21] elements in $\mathrm{U}(\mathcal{M})$ representing multicooperations defining Steenrod operations at any prime. In particular, as proven in this work, these so-called *cup-(p, i) coproducts* are defined on cubical chains and are expressible, similarly to Equation (15), in terms of Δ, the permutations of factors, and $*$. The aforementioned construction of cubical cup-(p, i) coproducts has been implemented in the open source computer algebra system `ComCH` [Med21a].

5. The Cartan–Serre map

Let us consider, with their usual CW structures, the topological simplex $\mathbb{\Delta}^n$ and the topological cube $\mathbb{I}^n$. In [Ser51, p. 442], Serre described a quasi-isomorphism of coalgebras between the simplicial and cubical singular chains of a topological space. It is given by precomposing with a canonical cellular map $\mathfrak{cs}\colon \mathbb{I}^n \to \mathbb{\Delta}^n$ also considered in [EM53, p.199] where it is attributed to Cartan.

The goal of this section is to deduce from a more general categorical statement that this comparison map between singular chains of a space is a quasi-isomorphism of E_∞-coalgebras.

5.1 Simplicial sets

We denote the *simplex category* by $\triangle$, the category $\mathsf{Fun}(\triangle^{\mathrm{op}}, \mathsf{Set})$ of *simplicial sets* by sSet, and the representable simplicial set $\mathcal{Y}\big([n]\big)$ by $\triangle^n$. As usual, we denote an element in $\triangle^n_m$ by a non-decreasing tuple $[v_0, \dots, v_m]$ with $v_i \in \{0, \dots, n\}$. The *Cartesian product* of simplicial sets is defined by the product of functors. The *simplicial n-cube* $(\triangle^1)^{\times n}$ is the n^{th}-fold Cartesian product of $\triangle^1$ with itself.

We will use the following model of the topological n-simplex:

$$\mathbb{\Delta}^n = \big\{(y_1, \dots, y_n) \in \mathbb{I}^n \mid i \leq j \Rightarrow y_i \geq y_j\big\},$$

whose cell structure associates $[v_0, \dots, v_m]$ with the subset

$$\Big\{\big(\underbrace{1, \dots, 1}_{v_0}, \underbrace{y_1, \dots y_1}_{v_1 - v_0}, \dots, \underbrace{y_m, \dots y_m}_{v_m - v_{m-1}}, \underbrace{0, \dots, 0}_{n - v_m}\big) \mid y_1 \geq \dots \geq y_m\Big\}. \tag{16}$$

The spaces $\mathbb{\Delta}^n$ define a functor $\triangle \to \mathsf{CW}$ with

$$\begin{aligned}
\sigma_i(x_1, \dots, x_n) &= (x_1, \dots, \widehat{x}_i, \dots, x_n)\\
\delta_0(x_1, \dots, x_n) &= (1, x_1, \dots, x_n),\\
\delta_i(x_1, \dots, x_n) &= (x_1, \dots, x_i, x_i, \dots, x_n),\\
\delta_n(x_1, \dots, x_n) &= (x_1, \dots, x_n, 0).
\end{aligned}$$

Its Yoneda extension is the *geometric realization* functor. It has a right adjoint $\mathrm{Sing}^{\triangle}\colon \mathsf{Top} \to \mathsf{sSet}$ referred to as the *simplicial singular complex* satisfying

$$\mathrm{Sing}^{\triangle}(\mathfrak{Z})_n = \mathsf{Top}(\mathbb{\Delta}^n, \mathfrak{Z})$$

for any topological space $\mathfrak{Z}$.

The functor of (*normalized*) *chains* $\mathrm{N}^{\triangle}\colon \mathsf{sSet} \to \mathsf{Ch}$ is the composition of the geometric realization functor and that of cellular chains. We denote the composition $\mathrm{N}^{\triangle} \circ \mathrm{Sing}^{\triangle}$ by $\mathrm{S}^{\triangle}$ and omit the superscript $\triangle$ if no confusion may result from doing so. For any $n \in \mathbb{N}$, the *Alexander–Whitney coalgebra structure* on $\mathrm{N}(\triangle^n)$ is given by

$$\Delta\big([v_0, \dots, v_m]\big) = \sum_{i=0}^{m} [v_0, \dots, v_i] \otimes [v_i, \dots, v_m],$$

and

$$\epsilon\big([v_0, \dots, v_m]\big) = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases}$$

The degree 1 product $*\colon\ \mathrm{N}(\triangle^n)^{\otimes 2} \to \mathrm{N}(\triangle^n)$ is defined by

$$\begin{aligned} &[v_0, \dots, v_p] * [v_{p+1}, \dots, v_m] = \\ &\qquad\qquad \begin{cases} (-1)^{p+|\sigma|}\, \big[v_{\sigma(0)}, \dots, v_{\sigma(m)}\big] & \text{if } v_i \neq v_j \text{ for } i \neq j, \\ 0 & \text{if not,} \end{cases} \end{aligned}$$

where σ is the permutation that orders the totally ordered set of vertices and $(-1)^{|\sigma|}$ is its sign. As shown in [Med20a, Theorem 4.2] the assignment

$$\text{!} \mapsto \epsilon, \quad \curlywedge \mapsto \Delta, \quad \curlyvee \mapsto *,$$

defines a natural $\mathcal{M}$-bialgebra on the chains of representable simplicial sets, and, by forgetting structure, also a natural $\mathrm{U}(\mathcal{M})$-coalgebra. For any simplicial set, a natural $\mathrm{U}(\mathcal{M})$-coalgebra structure on its chains is defined by a Yoneda extension.

5.2 The Eilenberg–Zilber maps

For any permutation $\sigma \in \mathbb{S}_n$ let

$$\mathfrak{i}_\sigma\colon \mathbb{\Delta}^n \to \mathbb{I}^n$$

be the inclusion defined by $(x_1, \dots, x_n) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. If e is the identity permutation, we denote $\mathfrak{i}_e$ simply as $\mathfrak{i}$. The maps $\{\mathfrak{i}_\sigma\}_{\sigma \in \mathbb{S}_n}$ define a subdivision of $\mathbb{I}^n$ making it isomorphic to $\left|(\triangle^1)^{\times n}\right|$ in CW. Using this identification, the identity map induces a cellular map

$$\mathfrak{ez}\colon \mathbb{I}^n \to \left|(\triangle^1)^{\times n}\right|.$$

We denote the induced chain map by

$$\mathrm{EZ}\colon \; \mathrm{N}(\square^n) \to \mathrm{N}\left((\triangle^1)^{\times n}\right).$$

For any topological space $\mathfrak{Z}$, the cubical map

$$\mathcal{U}\,\mathrm{Sing}^{\triangle}(\mathfrak{Z}) \to \mathrm{Sing}^{\square}(\mathfrak{Z})$$

is defined, using the adjunction isomorphism

$$\mathsf{sSet}\big((\triangle^1)^{\times n}, \mathrm{Sing}^{\triangle}(\mathfrak{Z})\big) \cong \mathsf{Top}\big(|(\triangle^1)^{\times n}|, \mathfrak{Z}\big),$$

by the assignment

$$\big(|(\triangle^1)^{\times n}| \xrightarrow{f} \mathfrak{Z}\big) \mapsto \big(\mathbb{I}^n \xrightarrow{\mathfrak{ez}} |(\triangle^1)^{\times n}| \xrightarrow{f} \mathfrak{Z}\big).$$

We denote the induced chain map by

$$\mathrm{EZ}_{\mathrm{S}(\mathfrak{Z})}\colon \; \mathrm{N}^{\square}\big(\mathcal{U}\,\mathrm{Sing}^{\triangle}(\mathfrak{Z})\big) \to \mathrm{S}^{\square}(\mathfrak{Z}).$$

5.3 The Cartan–Serre maps

The cellular map

$$\mathfrak{cs}\colon \mathbb{I}^n \to \mathbb{\Delta}^n$$

is defined by

$$\mathfrak{cs}(x_1, \dots, x_n) = (x_1,\ x_1x_2,\ \dots\ ,\ x_1x_2 \cdots x_n).$$

We denote its induced chain map by

$$\mathrm{CS}\colon \; \mathrm{N}(\square^n) \to \mathrm{N}(\triangle^n).$$

The chain map

$$\mathrm{CS}_{\mathrm{S}(\mathfrak{Z})}\colon \; \mathrm{S}^{\triangle}(\mathfrak{Z}) \to \mathrm{S}^{\square}(\mathfrak{Z})$$

between the singular chain complexes of a topological space $\mathfrak{Z}$ is defined by

$$\mathrm{CS}_{\mathrm{S}(\mathfrak{Z})}(\mathbb{\Delta}^n \to \mathfrak{Z}) = (\mathbb{I}^n \xrightarrow{\mathfrak{cs}} \mathbb{\Delta}^n \to \mathfrak{Z}).$$

These maps were considered in [Ser51, p. 442] where it was stated that $\mathrm{CS}_{\mathrm{S}(\mathfrak{Z})}$ is a natural quasi-isomorphisms of coalgebras. We will prove this in §5.6 showing in fact that it is a quasi-isomorphism of E_∞-coalgebras.

5.4 No-go results

Since CS is shown to be a coalgebra map in Lemma 8 and EZ is well known to be one, one may hope for higher structures to be preserved by these maps. We now provide some examples constraining the scope of these expectations.

Example. We will show that EZ does not preserve $\mathrm{U}(\mathcal{M})$-structures. More specifically, that in general

$$\mathrm{EZ}^{\otimes 2} \circ \Delta_1 \neq \Delta_1 \circ \mathrm{EZ}$$

where

$$\Delta_1 = (* \otimes \mathsf{id}) \circ (\mathsf{id} \otimes (12)\Delta) \circ \Delta$$

is the cup-1 coproduct presented in Equation (15). Up to signs, on one hand we have

$$\begin{aligned} \Delta_1\big([01][01]\big) &= [01][01] \otimes [1][01] \ + \ [01][1] \otimes [01][01] \\ &\quad + \ [0][01] \otimes [01][01] \ + \ [01][01] \otimes [01][0]. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{EZ}^{\otimes 2} \circ \Delta_1\big([01][01]\big) &= \big(011 \times 001 + 001 \times 011\big) \otimes 11 \times 01 \\ &\quad + \ 01 \times 11 \otimes \big(011 \times 001 + 001 \times 011\big) \\ &\quad + \ 00 \times 01 \otimes \big(011 \times 001 + 001 \times 011\big) \\ &\quad + \ \big(011 \times 001 + 001 \times 011\big) \otimes 01 \times 00. \end{aligned}$$

On the other hand, we have

$$\Delta_1[0,1,2] = [0,1,2] \otimes [0,1] + [0,2] \otimes [0,1,2] + [0,1,2] \otimes [1,2].$$

Therefore,

$$\begin{aligned} \Delta_1 \circ \mathrm{EZ}\ \big([01][01]\big) &= \Delta_1\big(011 \times 001 + 001 \times 011\big) \\ &= 011 \times 001 \otimes 01 \times 00 \ + \ 01 \times 01 \otimes 011 \times 001 \\ &\quad + \ 011 \times 001 \otimes 11 \times 01 \ + \ 001 \times 011 \otimes 00 \times 01 \\ &\quad + \ 01 \times 01 \otimes 001 \times 011 \ + \ 001 \times 011 \otimes 01 \times 11. \end{aligned}$$

We conclude that

$$\mathrm{EZ}^{\otimes 2}\circ\Delta_1\big([01][01]\big)\neq\Delta_1\circ\mathrm{EZ}\,\big([01][01]\big)$$

since, for example, the basis element $01\times 11\otimes 011\times 001$ appears in the left sum but not in the right one.

Example. We will show that the Cartan–Serre map does not preserve $\mathcal{M}$-structures. More specifically, that in general

$$\mathrm{CS}(x*y)\neq\mathrm{CS}(x)*\mathrm{CS}(y).$$

Consider $x=[1][1]$ and $y=[0][01]$. On one hand we have that

$$\mathrm{CS}\,\big([1][1]\big)*\mathrm{CS}\,\big([0][01]\big)=0$$

since $\mathrm{CS}\,\big([0][01]\big)=0$. On the other hand we have, up to a signs, that

$$\mathrm{CS}\,\Big(([1][1])*([0][01])\Big)=\mathrm{CS}\,\Big([01][01]\Big)=[012],$$

which establishes the claim.

The reason for this incompatibility is that $*$ in the simplicial context is commutative, which is not the case in the cubical one.

Example. We will show that the Cartan–Serre map does not preserve $\mathrm{U}(\mathcal{M})$-structures. More specifically, that in general

$$\mathrm{CS}\circ\widetilde{\Delta}_1\neq\widetilde{\Delta}\circ\mathrm{CS}$$

where

$$\widetilde{\Delta}_1=(*\otimes\mathsf{id})\circ(12)(\mathsf{id}\otimes(12)\Delta)\circ\Delta.$$

On one hand we have that

$$\mathrm{CS}\,\Big(\widetilde{\Delta}_1\big([01][01]\big)\Big)=T\Delta_1\big([012]\big),$$

and on the other that

$$\widetilde{\Delta}_1\circ\mathrm{CS}\,\big([01][01]\big)=\Delta_1\big([012]\big),$$

which establishes the claim.

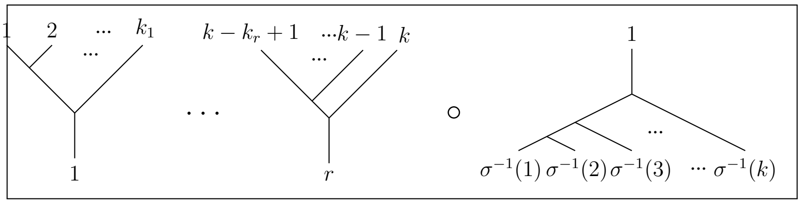
In § 5.6 we will show that CS is a morphism of E_∞-coalgebras. To do so we now introduce an E_∞-suboperad of $\mathrm{U}(\mathcal{M})$ where the incompatibility resulting from the lack of commutativity of $*$ in the cubical context is dealt with.

5.5 Shuffle graphs

Consider $k = k_1 + \cdots + k_r$. A $(k_1, \ldots, k_r)$-shuffle σ is a permutation in $\mathbb{S}_k$ satisfying

$$\begin{aligned} &\sigma(1) < \cdots < \sigma(k_1), \\ &\sigma(k_1+1) < \cdots < \sigma(k_1+k_2), \\ &\qquad \vdots \\ &\sigma(k-k_r+1) < \cdots < \sigma(k). \end{aligned}$$

The (*left comb*) *shuffle graph* associated to such σ is the $(1,k)$-graph



presented as a composition of (left comb) self-graftings of the generators Y and ⅄. With the notation introduced in Equation (14), the $\mathrm{U}(\mathcal{M})$-coalgebra sends the shuffle graph associated to σ to

$$(*^{k_1} \otimes \cdots \otimes *^{k_r}) \circ \sigma^{-1} \Delta^{k-1}.$$

Example. All the graphs in Figure 3 are shuffle graphs. In fact, all the cup-i coproducts presented in Equation (15) are induced from shuffle graphs, whereas

$$\begin{aligned} \widetilde{\Delta}_1 &= (* \otimes \mathsf{id}) \circ (12)(\mathsf{id} \otimes (12)\Delta) \circ \Delta \\ &= (* \otimes \mathsf{id}) \circ (123)\Delta^2, \end{aligned}$$

used in the previous section to probe the limits of the structure preserving properties of CS, is not.

The operad $\mathrm{U}(\mathcal{M}_{sh})$ is defined as the suboperad of $\mathrm{U}(\mathcal{M})$ (freely) generated by shuffle graphs. Explicitly, any element in $\mathrm{U}(\mathcal{M}_{sh})(r)$ is represented by a linear combination of $(1,r)$-graphs obtained by grafting these. The same proof used in [Med20a, p.5] to show that $\mathrm{U}(\mathcal{M})$ is an E_∞-operad can be used to prove the same for $\mathrm{U}(\mathcal{M}_{sh})$.

5.6 E_∞-coalgebra preservation

We devote this subsection to the proof of the following key result.

Theorem 6. *The chain map* $\mathrm{CS}\colon\ \mathrm{N}(\square^n) \to \mathrm{N}(\triangle^n)$ *is a quasi-isomorphism of* $\mathrm{U}(\mathcal{M}_{sh})$*-coalgebras.*

We start by stating an alternative description of the CS map.

Lemma 7. *Let* $x = x_1 \otimes \cdots \otimes x_n \in \mathrm{N}(\square^n)_m$ *be a basis element with* $x_{q_i} = [0,1]$ *for all* $\{q_1 < \cdots < q_m\}$*. If there is* $x_\ell = [0]$ *with* $\ell < q_m$ *then* $\mathrm{CS}(x) = 0$*, otherwise*

$$\mathrm{CS}(x) = \big[q_1 - 1,\ \ldots\ ,\ q_m - 1,\ p(x) - 1\big]$$

where $p(x) = \min\big\{\ell \mid x_\ell = [0]\big\}$ *or* $p(x) = n + 1$ *if this set is empty.*

Proof. This can be directly verified using the cell structure of $\mathbb{A}^n$ described in Equation (16). □

Lemma 8. *The chain map* $\mathrm{CS}\colon\ \mathrm{N}(\square^n) \to \mathrm{N}(\triangle^n)$ *is a quasi-isomorphism of coalgebras.*

Proof. The chain map CS is a quasi-isomorphism compatible with the counit since it is induced from a cellular map between contractible spaces. We need to show it preserves coproducts. By naturality it suffices to verify this on $[0,1]^{\otimes n}$. Recall from Lemma 3 that

$$\Delta\big([0,1]^{\otimes n}\big) = \sum_{\lambda \in \Lambda} (-1)^{\operatorname{ind}\lambda} \left(x_1^{(\lambda)} \otimes \cdots \otimes x_n^{(\lambda)}\right) \otimes \left(y_1^{(\lambda)} \otimes \cdots \otimes y_n^{(\lambda)}\right),$$

where the sum is over all choices for each $i \in \{1, \ldots, n\}$ of

$$\begin{aligned} x_i^{(\lambda)} &= [0,1], & \qquad \text{or} \qquad & & x_i^{(\lambda)} &= [0], \\ y_i^{(\lambda)} &= [1], & & & y_i^{(\lambda)} &= [0,1]. \end{aligned}$$

By Lemma 7, the summands above not sent to 0 by $\mathrm{CS} \otimes \mathrm{CS}$ are those basis elements for which $x_i^{(\lambda)} = [0]$ implies $x_j^{(\lambda)} = [0]$ for all $i < j$. For any one such summand, its sign is positive and its image by $\mathrm{CS} \otimes \mathrm{CS}$ is $[0, \ldots, k] \otimes [k, \ldots, n]$ where $k + 1 = \min\big\{i \mid x_i^{(\lambda)} = [0]\big\}$ or $k = n$ if this set is empty. The summands $[0, \ldots, k] \otimes [k, \ldots, n]$ are precisely those appearing when applying the Alexander–Whitney coproduct to $\mathrm{CS}\left([0,1]^{\otimes n}\right) = [0, \ldots, n]$. This concludes the proof. □

We will consider the basis of $\mathrm{N}(\square^n)$ as a poset with

$$(x_1 \otimes \cdots \otimes x_n) \leq (y_1 \otimes \cdots \otimes y_n)$$

if and only if $x_\ell \leq y_\ell$ for each $\ell \in \{1, \ldots, n\}$ with respect to

$$[0] < [0,1] < [1].$$

As we prove next, an example of ordered elements are the tensor factors of each summand in the iterated Serre diagonal.

Lemma 9. *Writing*

$$\Delta^{k-1}\big([0,1]^{\otimes n}\big) = \sum \pm\, x^{(1)} \otimes \cdots \otimes x^{(k)}$$

with each $x^{(\ell)}$ a basis element of $\mathrm{N}(\square^n)$, we have

$$x^{(1)} \leq \cdots \leq x^{(k)}$$

for every summand.

Proof. This can be proven using a straightforward induction argument whose base case follows from inspecting Lemma 3. □

Lemma 10. *Let x, y and z be basis elements of $\mathrm{N}(\square^n)$. If both $x \leq z$ and $y \leq z$ then either $(x * y) = 0$ or every summand in $(x * y)$ is $\leq z$.*

Proof. Recall that

$$(x_1 \otimes \cdots \otimes x_n) * (y_1 \otimes \cdots \otimes y_n) = (-1)^{|x|} \sum_{\ell=1}^{n} x_{<\ell}\, \epsilon(y_{<\ell}) \otimes x_\ell * y_\ell \otimes \epsilon(x_{>\ell})\, y_{>\ell}.$$

By assumption $x_{<\ell} \leq z_{<\ell}$ and $y_{>\ell} \leq z_{>\ell}$ for every $\ell \in \{1, \ldots, n\}$. If $x_\ell * y_\ell \neq 0$ then $x_\ell * y_\ell = [0,1]$ and either $x_\ell = [1]$ or $y_\ell = [1]$ which implies $z_\ell = [1]$ as well, so $x_\ell * y_\ell \leq z_\ell$. □

Lemma 11. *If x and y are basis elements of $\mathrm{N}(\square^n)$ satisfying $x \leq y$ then*

$$\mathrm{CS}(x * y) = \mathrm{CS}(x) * \mathrm{CS}(y). \tag{17}$$

Proof. We present this proof in the form of three claims. We use Lemma 7, the assumption $x \leq y$, and the fact that the join of basis elements in $\mathrm{N}(\triangle^n)$ sharing a vertex is 0 without explicit mention.

Claim 1. If $\mathrm{CS}(x) = 0$ or $\mathrm{CS}(y) = 0$ then for every $i \in \{1, \dots, n\}$

$$\mathrm{CS}\left(x_{<i}\, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i})\, y_{>i}\right) = 0. \tag{18}$$

Assume $\mathrm{CS}(x) = 0$, that is, there exists a pair $p < q$ such that $x_p = [0]$ and $x_q = [0, 1]$, then (18) holds since:

1. If $i > q$, then x_p and x_q are part of $x_{<i}$.
2. If $i = q$, then $x_q * y_q = 0$ for any y_q.
3. If $i < q$, then $\epsilon(x_{>i}) = 0$.

Similarly, if there is a pair $p < q$ such that $y_p = [0]$ and $y_q = [0, 1]$, then (18) holds since:

1. If $i < p$, then y_p and y_q are part of $y_{>i}$.
2. If $i = p$, then $x_i = [0]$ and $x_i * y_i = 0$.
3. If $i > p$, then either $x_i * y_i = 0$ or $x_i * y_i = [0, 1]$ and $x_p = [0]$.

This proves the first claim and identity (17) under its hypothesis.

Claim 2. If $\mathrm{CS}(x) \neq 0$ and $\mathrm{CS}(y) \neq 0$ then

$$\mathrm{CS}(x * y) = \mathrm{CS}\left(x_{<p_x} \epsilon(y_{<p_x}) \otimes\ x_{p_x} * y_{p_x} \otimes \epsilon(x_{>p_x})\, y_{>p_x}\right)$$

if $p_x = \min\big\{i \mid x_i = [0]\big\}$ is well-defined and $x * y = 0$ if not.

Assume p_x is not well-defined, i.e., $x_i \neq [0]$ for all $i \in \{1, \dots, n\}$. Given that $x \leq y$ we have that $[0] < x_i$ implies $x_i * y_i = 0$, and the claim follows in this case.

Assume p_x is well-defined. We will show that for all $i \in \{1, \dots, n\}$ with the possible exception of $i = p_x$ we have

$$\mathrm{CS}\left(x_{<i}\, \epsilon(y_{<i}) \otimes\ x_i * y_i \otimes \epsilon(x_{>i})\, y_{>i}\right) = 0 \tag{19}$$

This follows from:

1. If $i < p_x$ and $x_i = [1]$ then $y_i = [1]$ and $x_i * y_i = 0$.
2. If $i < p_x$ and $x_i = [0,1]$ then $x_i * y_i = 0$ for any y_i.
3. If $i > p_x$ then Lemma 7 implies the claim since $x_{p_x} = [0]$ and $x_i * y_i \neq 0$ iff $x_i * y_i = [0,1]$.

Claim 3. If $\mathrm{CS}(x) \neq 0$ and $\mathrm{CS}(y) \neq 0$ then (17) holds.

Let us assume that $\{i \mid x_i = [0]\}$ is empty, which implies the analogous statement for y since $x \leq y$. Since neither of x nor y have a factor $[0]$ in them, Lemma 7 implies that the vertex $[n]$ is in both $\mathrm{CS}(x)$ and $\mathrm{CS}(y)$, which implies $\mathrm{CS}(x) * \mathrm{CS}(y) = 0$ as claimed.

Assume now that $p_x = \{i \mid x_i = [0]\}$ is well defined, and let $\{q_1 < \cdots < q_m\}$ with $x_{q_i} = [0,1]$ for $i \in \{1, \dots, m\}$. Since $\mathrm{CS}(x) \neq 0$ Lemma 7 implies that $p_x > q_m$, so $\epsilon(x_{>p_x}) = 1$ and Claim 2 implies

$$\mathrm{CS}(x * y) = \mathrm{CS}\left(x_{<p_x} \epsilon(y_{<p_x}) \otimes \, x_{p_x} * y_{p_x} \otimes y_{>p_x}\right).$$

We have the following cases:

1. If $\epsilon(y_{<p_x}) = 0$ then there is q_i such that $y_{q_i} = [0,1]$ so $[q_i - 1]$ is in both $\mathrm{CS}(x)$ and $\mathrm{CS}(y)$.
2. If $\epsilon(y_{p_x}) \neq 0$ and $y_{p_x} \in \{[0],[0,1]\}$ then $x_{p_x} * y_{p_x} = 0$ and $[p_x - 1]$ is in both $\mathrm{CS}(x)$ and $\mathrm{CS}(y)$.
3. If $\epsilon(y_{p_x}) \neq 0$ and $y_{p_x} = [1]$ let $\{\ell_1 < \cdots < \ell_k\}$ be such that $y_{\ell_j} = [0,1]$ and let $p_y > \ell_k$ be either $n+1$ or $\min\{j \mid y_j = \{0\}\}$ then

$$\begin{aligned}\mathrm{CS}(x * y) &= \mathrm{CS}\left(x_{<p_x} \otimes x_{p_x} * y_{p_x} \otimes y_{>p_y}\right) \\ &= [q_1 - 1, \dots, q_m - 1, p_x - 1, \ell_1 - 1, \dots, \ell_k - 1, p_y - 1] \\ &= \mathrm{CS}(x) * \mathrm{CS}(y).\end{aligned}$$

This concludes the proof. □

Combining the previous two lemmas we obtain the following.

Lemma 12. *Let $x^{(1)} \leq \cdots \leq x^{(k)}$ be basis elements of $\mathrm{N}(\square^n)$. Then,*

$$\mathrm{CS} \circ *^{k-1}\left(x^{(1)} \otimes \cdots \otimes x^{(k)}\right) = *^{k-1} \circ \mathrm{CS}^{\otimes k}\left(x^{(1)} \otimes \cdots \otimes x^{(k)}\right).$$

We are now ready to present the argument establishing that CS is an E_∞-coalgebra map.

Proof of Theorem 6. Since $\mathrm{U}(\mathcal{M}_{sh})$ is generated by elements represented by shuffle graphs, we only need to show that for any $(k_1, \dots, k_r)$-shuffle σ with $k = k_1 + \dots + k_r$ the following holds

$$\mathrm{CS}^{\otimes r}(*^{k_1} \otimes \cdots \otimes *^{k_r}) \circ \sigma^{-1}\Delta^{k-1} = (*^{k_1} \otimes \cdots \otimes *^{k_r}) \circ \sigma^{-1}\Delta^{k-1} \circ \mathrm{CS}\,.$$

By naturality, it suffices to prove this identity for $[0,1]^{\otimes n}$. According to Lemma 9

$$x^{(1)} \leq \cdots \leq x^{(k)}$$

for every summand in

$$\Delta^{k-1}\big([0,1]^{\otimes n}\big) = \sum \pm\, x^{(1)} \otimes \cdots \otimes x^{(k)}.$$

Since σ is a shuffle permutation, Lemma 12 implies that

$$\begin{aligned}\mathrm{CS}^{\otimes r}&(*^{k_1} \otimes \cdots \otimes *^{k_r}) \circ \sigma^{-1}\Delta^{k-1}\big([0,1]^{\otimes n}\big) \\ &= (*^{k_1} \otimes \cdots \otimes *^{k_r}) \circ \sigma^{-1}\,\mathrm{CS}^{\otimes k} \circ \Delta^{k-1}\big([0,1]^{\otimes n}\big).\end{aligned}$$

As proven in Lemma 8, CS is a coalgebra map, which concludes the proof. □

5.7 Categorical reformulation

The assignment $2^n \mapsto (\triangle^1)^{\times n}$ defines a functor $\square \to \mathsf{sSet}$ with

$$\begin{aligned}\delta_i^\varepsilon &\colon (\triangle^1)^{\times n} \to (\triangle^1)^{\times (n+1)} \\ \sigma_i &\colon (\triangle^1)^{\times (n+1)} \to (\triangle^1)^{\times n}\end{aligned}$$

given by inserting $[\varepsilon, \dots, \varepsilon]$ as the i^{th} factor and removing the i^{th} factor respectively. Its Yoneda extension, referred to as triangulation functor, is denoted by

$$\mathcal{T}\colon \mathsf{cSet} \to \mathsf{sSet}.$$

This functor admits a right adjoint

$$\mathcal{U}\colon \mathsf{sSet} \to \mathsf{cSet}$$

defined, as usual, by the expression

$$\mathcal{U}(Y)(2^n) = \mathsf{sSet}\big((\triangle^1)^{\times n},\, Y\big).$$

We mention that, as proven in [Cis06, § 8.4.30], the pair $(\mathcal{T}, \mathcal{U})$ defines a Quillen equivalence when sSet and cSet are considered as model categories.

Definition 13. The simplicial map $\mathsf{cs}\colon\ (\triangle^1)^{\times n} \to \triangle^n$ is defined by

$$[\varepsilon_0^1, \dots, \varepsilon_m^1] \times \cdots \times [\varepsilon_0^n, \dots, \varepsilon_m^n] \mapsto [v_0, \dots, v_m]$$

where $v_i = \varepsilon_i^1 + \varepsilon_i^1\varepsilon_i^2 + \cdots + \varepsilon_i^1 \cdots \varepsilon_i^n$.

Please observe that the maps $\mathfrak{cs}$ and $|\mathsf{cs}| \circ \mathfrak{ez}$ agree.

Definition 14. Let Y be a simplicial set. The map

$$\mathrm{CS}_Y\colon\ \mathrm{N}^{\triangle}(Y) \to \mathrm{N}^{\square}(\mathcal{U}\,Y)$$

is the linear map induced by sending a simplex $y \in Y_n$ to the composition

$$(\triangle^1)^{\times n} \xrightarrow{\mathsf{cs}} \triangle^n \xrightarrow{\xi_y} Y$$

where $\xi_y\colon\ \triangle^n \to Y$ is the simplicial map determined by $\xi_y\big([n]\big) = y$.

Theorem 15. *For any simplicial set Y the map $\mathrm{CS}_Y\colon\ \mathrm{N}^{\triangle}(Y) \to \mathrm{N}^{\square}(\mathcal{U}\,Y)$ is a quasi-isomorphism of $\mathrm{U}(\mathcal{M}_{sh})$-coalgebras which extend respectively the Alexander–Whitney and Serre coalgebra structures.*

Proof. This is a direct consequence of Theorem 6 following from a standard category theory argument, which we now present. Consider the isomorphism

$$\mathrm{N}(\mathcal{U}\,Y) \cong \bigoplus_{n \in \mathbb{N}} \mathrm{N}(\square^n) \otimes \Bbbk\Big\{\mathsf{sSet}\big((\triangle^1)^{\times n}, \triangle^n\big)\Big\}\Big/ \sim$$

and the canonical linear inclusions:

$$\begin{aligned} \mathrm{N}(\square^n) &\longrightarrow \bigoplus_{m \in \mathbb{N}} \mathrm{Hom}\big(\mathrm{N}(\square^m),\, \mathrm{N}(\square^n)\big) \\ (2^m \xrightarrow{\delta} 2^n) &\longmapsto \big(\mathrm{N}(\square^m) \xrightarrow{\mathrm{N}(\delta)} \mathrm{N}(\square^n)\big) \end{aligned}$$

and

$$\begin{aligned}\bigoplus_{n\in\mathbb{N}} \Bbbk\Big\{\mathsf{sSet}\big((\triangle^1)^{\times n}, \triangle^n\big)\Big\} &\longrightarrow \bigoplus_{n\in\mathbb{N}} \mathrm{Hom}\Big(\mathrm{N}\left((\triangle^1)^{\times n}\right), \mathrm{N}(\triangle^n)\Big)\\ \big((\triangle^1)^{\times n} \xrightarrow{f} \triangle^n\big) &\longmapsto \Big(\mathrm{N}\left((\triangle^1)^{\times n}\right) \xrightarrow{\mathrm{N}(f)} \mathrm{N}(\triangle^n)\Big).\end{aligned}$$

We can use these and the naturality of EZ to construct the following chain map which is an isomorphism onto its image.

$$\begin{aligned}\mathrm{N}(\mathcal{U}Y) &\longrightarrow \bigoplus_{n\in\mathbb{N}} \mathrm{Hom}\big(\mathrm{N}(\square^n), \mathrm{N}(Y)\big)\\ (\delta\otimes f) &\longmapsto \big(\mathrm{N}(f)\circ \mathrm{EZ}\circ \mathrm{N}(\delta)\big).\end{aligned}$$

Let Γ be an element in $\mathrm{U}(\mathcal{M}_{sh})(r)$ and denote by $\Gamma^{\square}\colon \mathrm{N}(\mathcal{U}Y)\to \mathrm{N}(\mathcal{U}Y)^{\otimes r}$ and $\Gamma^{\triangle}\colon \mathrm{N}(Y)\to \mathrm{N}(Y)^{\otimes r}$ its image in the respective endomorphism operads. Using the naturality of $\Gamma^{\square}$, we have that $\Gamma^{\square}(\delta\otimes f)$ corresponds to $(\mathrm{N}(f)\circ\mathrm{EZ})^{\otimes r}\circ\Gamma^{\square}\circ\mathrm{N}(\delta)$. On the other hand, the map CS_Y corresponds to

$$\begin{aligned}\mathrm{N}(Y)_n &\longrightarrow \mathrm{N}(\mathcal{U}Y)_n\\ y &\longmapsto \big(\mathrm{N}(\xi_y)\circ\mathrm{CS}\big)\end{aligned}$$

where $\xi_y\colon \triangle^n\to Y$ is determined by $\xi_y\big([n]\big)=y$, and we used that $\mathrm{CS}=\mathrm{N}(\mathsf{cs})\circ\mathrm{EZ}$ to ensure the above assignment is well defined. The image of $\Gamma^{\triangle}(y)$ corresponds to $\mathrm{N}(\xi_y)^{\otimes r}\circ\Gamma^{\triangle}\circ\mathrm{CS}$. So the claim follows from the identity

$$\begin{aligned}\Gamma^{\square}\big(2^n\otimes(\xi_y\circ\mathsf{cs})\big) &= (\mathrm{N}(\xi_y\circ\mathsf{cs})\circ\mathrm{EZ})^{\otimes r}\circ\Gamma^{\square}\\ &= \mathrm{N}(\xi_y)^{\otimes r}\circ\mathrm{CS}^{\otimes r}\circ\Gamma^{\square}\\ &= \mathrm{N}(\xi_y)^{\otimes r}\circ\Gamma^{\triangle}\circ\mathrm{CS}\end{aligned}$$

where we used that $\mathrm{CS}^{\otimes r}\circ\Gamma^{\square}=\Gamma^{\triangle}\circ\mathrm{CS}$ as proven in Theorem 6. □

Corollary 16. *For any cubical set X*

$$\mathrm{N}^{\square}(X)\xrightarrow{\mathrm{N}^{\square}(\xi_X)}\mathrm{N}^{\square}(\mathcal{U}\mathcal{T}X)\xleftarrow{\mathrm{CS}_{\mathcal{T}X}}\mathrm{N}^{\triangle}(\mathcal{T}X),$$

where ξ is the unit of adjunction, is a natural zig-zag of quasi-isomorphisms of $\mathrm{U}(\mathcal{M}_{sh})$*-coalgebras which extend respectively the Serre and Alexander–Whitney coalgebra structures.*

Proof. The map $\mathrm{CS}_{\mathcal{T}X}$ is a quasi-isomorphism of $\mathrm{U}(\mathcal{M}_{sh})$-coalgebras by Theorem 15, whereas $\mathrm{N}^\square(\xi_X)$ is also one since it is induced from a cubical map that is a weak-equivalence. $\square$

Corollary 17. *The singular simplicial and cubical chains of a topological space* $\mathfrak{Z}$ *are quasi-isomorphic as* $\mathrm{U}(\mathcal{M}_{sh})$*-coalgebras which extend respectively the Alexander–Whitney and Serre coalgebra structures. More specifically, the map*

$$\mathrm{CS}_{\mathrm{S}(\mathfrak{Z})} \colon \ \mathrm{S}^\triangle(\mathfrak{Z}) \to \mathrm{S}^\square(\mathfrak{Z})$$

is a quasi-isomorphism of $\mathrm{U}(\mathcal{M}_{sh})$*-coalgebras.*

Proof. It can be verified using that $\mathfrak{cs} = |\mathsf{cs}| \circ \mathfrak{e}_{\mathfrak{Z}}$ that this map factors as

$$\mathrm{CS}_{\mathrm{S}(\mathfrak{Z})} \colon \ \mathrm{S}^\triangle(\mathfrak{Z}) \xrightarrow{\mathrm{CS}_{\mathrm{Sing}^\triangle(\mathfrak{Z})}} \mathrm{N}^\square\big(\mathcal{U}\,\mathrm{Sing}^\triangle(\mathfrak{Z})\big) \xrightarrow{\mathrm{EZ}_{\mathrm{S}(\mathfrak{Z})}} \mathrm{S}^\square(\mathfrak{Z})$$

where the first map was proven in Theorem 15 to be a quasi-isomorphism of $\mathrm{U}(\mathcal{M}_{sh})$-coalgebras, and the second, introduced in § 5.2, is also one since it is induced from a cubical map whose geometric realization is a homeomorphism. $\square$

6. Future work

In the fifties, Adams introduced in [Ada56] a comparison map

$$\boldsymbol{\Omega}\,\mathrm{S}^\triangle(\mathfrak{Z}, z) \to \mathrm{S}^\square(\Omega_z\mathfrak{Z})$$

from his cobar construction on the simplicial singular chains of a pointed space $(\mathfrak{Z}, z)$ to the cubical singular chains on its based loop space $\Omega_z\mathfrak{Z}$. This comparison map is a quasi-isomorphism of algebras, which was shown by Baues [Bau98] to be one of bialgebras by considering Serre's cubical coproduct. In [MR21] the E_∞-coalgebra structure defined here is used to generalize Baues' result, by showing that Adams' comparison map is a quasi-isomorphism of E_∞-bialgebras or, more precisely, of monoids in the category of $\mathrm{U}(\mathcal{M})$-coalgebras.

For a closed smooth manifold M, in [FMS21] a canonical vector field was used to compare multiplicatively two models of ordinary cohomology. On one hand, a cochain complex generated by manifolds with corners over M, with partially defined intersection; on the other, the cubical cochains of a cubulation of M with the Serre product. With the explicit description introduced here of an E_∞-structure on cubical cochains, we expect to build on this multiplicative comparison and enhance geometric cochains [FMS22] with compatible representations of further derived structure.

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CAHIERS DE TOPOLOGIE ET
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SOME TOPOSES OVER WHICH ESSENTIAL IMPLIES LOCALLY CONNECTED

Jens HEMELAER

Résumé. Nous introduisons la notion de topos EILC: un topos $\mathcal{E}$ tel que tout morphisme géometrique essentiel de but $\mathcal{E}$ est localement connexe. Nous démontrons alors que le topos de faisceaux sur un espace topologique X est EILC si X est Hausdorff (ou plus généralement, si X est Jacobson). Ensuite, nous introduisons les espaces Jacobson est les étendues Jacobson sur un topos de base élémentaire quelconque, et montrons que ceux-ci sont EILC lorsque le topos de base est EILC, sous l'hypothèse de l'existence d'un objet de nombres naturels. Autres examples de topos de Grothendieck qui sont EILC, sont les étendues booléennes et les topos classifiants des groupes compacts. Puis, nous introduisons la notion plus faible de topos CILC: un topos $\mathcal{E}$ tel que tout morphisme géometrique $f : \mathcal{F} \to \mathcal{E}$ avec f^* cartésien fermé, est localement connexe. Nous donnons quelques examples des espaces topologiques resp. catégories petites tel que $\mathbf{Sh}(X)$ resp. $\mathbf{PSh}(\mathcal{C})$ sont CILC. Enfin, nous démontrons que chaque topos élémentaire booléen est CILC.

Abstract. We introduce the notion of an EILC topos: a topos $\mathcal{E}$ such that every essential geometric morphism with codomain $\mathcal{E}$ is locally connected. We then show that the topos of sheaves on a topological space X is EILC if X is Hausdorff (or more generally, if X is Jacobson). We then introduce Jacobson spaces and Jacobson étendues over an arbitrary elementary base topos, and show that these are EILC whenever the base topos is EILC, assuming the existence of a natural numbers object. Further examples of Grothendieck toposes that are EILC are Boolean étendues and classifying toposes of compact groups. Next, we introduce the weaker notion of CILC topos: a topos $\mathcal{E}$ such that any geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is locally connected, as soon as f^* is cartesian closed. We give some examples of topological spaces X and small categories $\mathcal{C}$ such that $\mathbf{Sh}(X)$ resp. $\mathbf{PSh}(\mathcal{C})$ are CILC. Finally, we show that any Boolean elementary topos is CILC.

Keywords. Topos, essential, locally connected, molecular, Jacobson, cartesian closed, EILC, CILC, Beck–Chevalley.
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1. Introduction

For elementary toposes $\mathcal{E}$ and $\mathcal{F}$, a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is called **essential** if the inverse image functor f^* has a left adjoint, that is then usually written as $f_!$. Moreover, we say that f is **locally connected** (or molecular) if f^* has an $\mathcal{E}$-indexed left adjoint, or equivalently, if f is essential and the natural morphism

$$f_!(X \times_{f^*B} f^*A) \to f_!(X) \times_B A \tag{1}$$

is an isomorphism, for all morphisms $A \to B$ in $\mathcal{E}$ and $X \to f^*B$ in $\mathcal{F}$, see [BP80]. The notion of a locally connected geometric morphism is more natural from a geometric point of view; in particular, locally connected geometric morphisms are stable under base change, while essential geometric morphisms are not.

In this article, we will follow an idea of Matías Menni, formulated in his message "Essential vs Molecular" on the category theory mailing list (May 3, 2017), where he mentions the problem of characterizing the toposes $\mathcal{E}$ such that any essential geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is locally connected. Elementary toposes with this property will here be called EILC ("Essential Implies Locally Connected"). Since an additional left adjoint is always $\mathbf{Sets}$-indexed [BP80, §5], the topos of sets is EILC. On the other hand, a typical example of a topos that is not EILC is the topos of sheaves on the Sierpiński space. In [Men21] it was suggested that the family of EILC toposes might coincide with the family of Boolean toposes or the (smaller) family of toposes satisfying the internal axiom of choice. In this paper, we will show that there are also non-Boolean toposes that are EILC, including for example the topos of sheaves $\mathbf{Sh}(X)$ for X an arbitrary Hausdorff topological space. Within the family of Boolean toposes, we show that Boolean étendues and classifying toposes of compact topological groups are EILC. It remains an open problem whether every Boolean topos is EILC.

The notion of an EILC topos has applications to the study of levels of an elementary topos $\mathcal{E}$. A **level** of $\mathcal{E}$, as introduced by Lawvere, is by definition a subtopos $\mathcal{E}'$ of $\mathcal{E}$ such that the inclusion geometric morphism $i : \mathcal{E}' \to \mathcal{E}$ is essential. For example, each open subtopos of $\mathcal{E}$ defines a level of $\mathcal{E}$. Conversely, if $\mathcal{E}$ is an EILC topos, then for any level $\mathcal{E}'$, the inclusion $i : \mathcal{E}' \to \mathcal{E}$ must be locally connected, and because locally connected geometric morphisms are open, we find that any level of $\mathcal{E}$ is given by an open subtopos. So for EILC toposes, the structure of the levels is completely known.

Another situation where EILC toposes are relevant, and the original motivation for this paper, is in the study of precohesive geometric morphisms. In [Law07], Lawvere introduced an axiomatic setting for when a category $\mathcal{E}$ can be seen as a "category of spaces" over a base category $\mathcal{S}$, with both $\mathcal{E}$ and $\mathcal{S}$ cartesian closed and extensive. A first requirement is that there is a string of adjoint functors

$$\begin{tikzcd}
\mathcal{E} \arrow[r, bend left=50, "f_!"] \arrow[r, "f_*"] & \mathcal{S} \arrow[l, bend right=25, "f^*"'] \arrow[l, bend left=25, "f^!"]
\end{tikzcd} \qquad f_! \dashv f^* \dashv f_* \dashv f^!$$

between $\mathcal{E}$ and $\mathcal{S}$. Here f^* is thought of as the functor that sends an object in $\mathcal{S}$ to its associated discrete space object in $\mathcal{E}$. Then for X in $\mathcal{E}$, $f_!(X)$ has an interpretation as the object of connected components (or "pieces") of X, and $f_*(X)$ can be thought of as the object of points of X.

Further relevant axioms in this setting are that f^* (or equivalently, $f^!$) is fully faithful, that $f_!$ preserves finite products, and that the natural map $f_* \to f_!$ is an epimorphism. Lawvere calls this last condition the Nullstellensatz: it expresses that each component has at least one point. If all the conditions above are satisfied, then $\mathcal{E}$ is said to be **precohesive** over $\mathcal{S}$, see the work of Lawvere and Menni [LM15, Definition 2.4].

A particular case of interest is when the string of adjoint functors $f_! \dashv f^* \dashv f_* \dashv f^!$ arises from a geometric morphism $f : \mathcal{E} \to \mathcal{S}$ between elementary toposes (with f^* the inverse image functor). Because f^* has a left adjoint $f_!$, the geometric morphism f is necessarily essential. Moreover, recall that by definition a geometric morphism f is **local** if and only if f^* is fully faithful and f_* has a further right adjoint $f^!$. Finally, the Nullstellensatz holds whenever f is hyperconnected [LM15, Lemma 3.1]. As a result, $\mathcal{E}$ is precohesive over $\mathcal{S}$ if and only if f is hyperconnected, essential and local, with $f_!$ preserving finite products (in particular, f^* is cartesian closed). If this is the case, then the geometric morphism f is itself called precohesive.

A natural question is now whether precohesive geometric morphisms $f : \mathcal{E} \to \mathcal{S}$ are stable under étale base change, or in other words whether for an object X in $\mathcal{S}$, the induced geometric morphism on slice toposes

$$f/X : \mathcal{E}/f^*(X) \to \mathcal{S}/X$$

is again precohesive. It was shown in [LM15, Corollary 10.4] that this is the case whenever f is locally connected. A question that was left open in [LM15] is then whether already every precohesive geometric morphism is locally connected. An affirmative answer would be useful in practice: it is often difficult to verify explicitly whether the map (1) is an isomorphism. The question remains open, although some progress on the question was made in [HR21] and [GS21]. The results in this paper might be useful in attempts to settle the problem, since any precohesive geometric morphism is automatically locally connected whenever the codomain topos is EILC or CILC.

The main goal of the present article is to show that the topos of sheaves $\mathbf{Sh}(X)$ on a topological space is EILC if X is Jacobson. Here we say that a topological space is **Jacobson** if two open subsets are equal whenever they contain the same closed points, see e.g. [Sta22, Section 005T]. For T_1 topological spaces (in particular, Hausdorff topological spaces) this condition is automatically satisfied, because in this case all points are closed. Further, the spectrum $\mathrm{Spec}(R)$ of a commutative ring R is Jacobson (for the Zariski topology) if and only if R is a Jacobson ring. As a result, there are many examples of Jacobson spaces that are not Hausdorff, for example $\mathrm{Spec}(\mathbb{Z})$ or $\mathrm{Spec}(\mathbb{C}[x,y])$.

The notion of Jacobson space can be generalized to an arbitrary base topos $\mathcal{S}$ as follows. Let $e : \mathcal{E} \to \mathcal{S}$ be a geometric morphism. Then we say that e is a **Jacobson space** if it is localic and the family of closed points $p : \mathcal{S} \to \mathcal{E}$ (i.e. closed geometric morphisms p with $ep \simeq 1$) is jointly surjective.

This agrees with the classical notion of Jacobson topological space in the case $\mathcal{S} = \mathbf{Sets}$. More generally, we say that $e : \mathcal{E} \to \mathcal{S}$ is a **Jacobson étendue** if there is a well-supported object E in $\mathcal{E}$ such that $\mathcal{E}/E \to \mathcal{S}$ is a Jacobson space. We show that a Jacobson étendue $\mathcal{E}$ over an elementary base topos $\mathcal{S}$ is EILC as soon as $\mathcal{S}$ is EILC and $\mathcal{E}$ has a natural numbers object.

In order to give a more comprehensive list of EILC toposes, we also show in Section 4 that a Grothendieck topos is EILC if it is a Boolean étendue, or if it is a classifying topos of a compact topological group. In particular, the petit étale topos of a field is EILC, because it coincides with the classifying topos of the absolute Galois group of the field (which is compact). An intriguing problem that is left open is whether the petit étale topos of a Jacobson ring is EILC.

In the last section, we introduce the more general class of CILC toposes ("Cartesian closed Implies Locally Connected"). These are the elementary toposes $\mathcal{E}$ such that any geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is locally connected, as soon as f^* is cartesian closed (i.e. preserves exponential objects). We then introduce a notion of weakly Jacobson geometric morphism, and we show that if $f : \mathcal{E} \to \mathcal{S}$ is weakly Jacobson and $\mathcal{S}$ is EILC, then $\mathcal{E}$ is CILC (under the assumption that $\mathcal{E}$ has a natural number object). Further, we give a characterization of topological spaces X and small categories $\mathcal{C}$ such that $\mathbf{Sh}(X)$ resp. $\mathbf{PSh}(\mathcal{C})$ are weakly Jacobson over the topos of sets. Finally, we show that all Boolean elementary toposes are CILC, extending an earlier result by Matías Menni, who showed that if $\mathcal{S}$ is a Boolean topos and $f : \mathcal{E} \to \mathcal{S}$ is a connected essential geometric morphism with $f_!$ preserving products, then f is locally connected [Men21].

2. Background on Beck–Chevalley conditions

If $f : \mathcal{F} \to \mathcal{E}$ is a locally connected geometric morphism, then in particular the fiber of f in a point $p : \mathbf{Sets} \to \mathcal{E}$ (defined as the pullback of f along p) is locally connected. If f is merely essential, then can we still conclude that its fibers are locally connected? And if the fibers are locally connected, under what assumptions can we conclude that f is itself locally connected? In this section, we can formulate some partial answers to these questions by studying Beck–Chevalley conditions. The technical results from this section will be used later in the paper to prove that certain toposes are EILC or CILC.

Definition 2.1. *We write* $g \rhd_p^q f$ *if there is a commutative diagram*

$$\begin{tikzcd}
\mathcal{F}' \arrow[r, "q"] \arrow[d, "g"'] & \mathcal{F} \arrow[d, "f"] \\
\mathcal{E}' \arrow[r, "p"] & \mathcal{E}
\end{tikzcd} \tag{2}$$

such that the natural map

$$f^*p_* \to q_*g^*$$

is an isomorphism (the Beck–Chevalley condition). Further, we write $g \rhd_p f$ *if there exists a morphism* q *with* $g \rhd_p^q f$*, and* $g \unrhd_p f$ *if moreover* q *can be chosen such that* (2) *is a pullback diagram.*

In order for pullbacks of elementary toposes to exist, we need some technical conditions. Recall that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is **bounded** if there is an object B in $\mathcal{F}$ such that for every object X in $\mathcal{F}$, there is an object I in $\mathcal{E}$ such that X is a subquotient of $B \times p^*(I)$, see [Joh02, Definition B3.1.7]. The pullback of two geometric morphisms $f : \mathcal{F} \to \mathcal{E}$ and $p : \mathcal{E}' \to \mathcal{E}$ exists if either f or p is bounded [Joh02, Proposition B3.3.6]. All localic geometric morphisms are bounded [Joh02, Examples B3.1.8(a)]. In particular, inclusions and étale geometric morphisms are bounded.

When discussing Beck–Chevalley conditions in topos theory, the notion of a tidy geometric morphism is relevant:

Definition 2.2. *Let* $p : \mathcal{E}' \to \mathcal{E}$ *be a geometric morphism. Then we say that* p *is* ***tidy*** *if* p_* *preserves filtered* $\mathcal{E}$*-indexed colimits.*

For an extensive treatment of tidy geometric morphisms, see e.g. Moerdijk and Vermeulen [MV00, Chapter III] or Johnstone [Joh02, C3.4]

We recall some of the history behind this concept, following the introduction of [MV00]. The concept of a tidy geometric morphism was first studied by Edwards in her PhD thesis [Edw80], in the special case where the codomain topos is $\mathbf{Sets}$. Later, the concept was introduced for an arbitrary codomain topos by Tierney, and developed by Lindgren in his PhD thesis [Lin84]. Lindgren referred to these geometric morphisms as being “proper”. Moerdijk and Vermeulen later used the name “tidy” instead, to distinguish the concept from a notion of properness as introduced by Johnstone.

In practice, it might be difficult to check whether a given geometric morphism is tidy. However, every closed inclusion is tidy [MV00, Chapter III, Corollary 5.8], so this gives a large family of concrete examples.

We recall the following properties from the literature.

Proposition 2.3 (See [Joh02])**.** *Consider a pullback diagram*

$$\begin{tikzcd}
\mathcal{F}' \arrow[r, "q"] \arrow[d, "g"'] & \mathcal{F} \arrow[d, "f"] \\
\mathcal{E}' \arrow[r, "p"] & \mathcal{E}
\end{tikzcd}.$$

1. *If f is locally connected and p is bounded, then $g \trianglerighteq_p f$.*
2. *If f is bounded and locally connected, then $g \trianglerighteq_p f$.*
3. *If p is bounded and tidy, and $\mathcal{E}$ has a natural number object, then $g \trianglerighteq_p f$.*
4. *If p is bounded and tidy and f is bounded, then $g \trianglerighteq_p f$.*

A proof for (1) and (2) is given in [Joh02, Theorem C3.3.15]. Further, (3) corresponds to [Joh02, Theorem C3.4.7] and (4) corresponds to [Joh02, Theorem C3.4.10]. See also [MV00, Chapter III, Theorem 4.8], where the boundedness assumption is implicit. If we restrict to Grothendieck toposes, then all geometric morphisms are automatically bounded, and moreover every Grothendieck topos has a natural object. In this setting, (3) and (4) coincide and are attributed to Lindgren [Lin84].

Beck–Chevalley squares can be pasted in the following way:

Proposition 2.4 (Transitivity)**.** *If $h \triangleright^{q'}_{p'} g$ and $g \triangleright^{q}_{p} f$, then $h \triangleright^{qq'}_{pp'} f$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{F}'' \arrow[r, "q'"] \arrow[d, "h"'] & \mathcal{F}' \arrow[r, "q"] \arrow[d, "g"] & \mathcal{F} \arrow[d, "f"] \\
\mathcal{E}'' \arrow[r, "p'"] & \mathcal{E}' \arrow[r, "p"] & \mathcal{E}
\end{tikzcd}$$

Then it follows that $f^* p_* p'_* \simeq q_* g^* p'_* \simeq q_* q'_* h^*$. □

We now introduce the following definition:

Definition 2.5. *Let $f : \mathcal{F} \to \mathcal{E}$ and $p : \mathcal{E}' \to \mathcal{E}$ be geometric morphisms. We say that f is* ***locally connected at*** *p if there is a locally connected geometric morphism g such that $g \rhd_p f$.*

Proposition 2.6 (Descent)**.** *Let $f : \mathcal{F} \to \mathcal{E}$ be an essential geometric morphism, and let $\{p_i : \mathcal{E}_i \to \mathcal{E}\}_{i \in I}$ be a jointly surjective family of geometric morphisms. If f is locally connected at p_i for each $i \in I$, then f is locally connected.*

Proof. It is enough to show that the map

$$\vartheta : f_!(X \times_{f^*B} f^*A) \to f_!(X) \times_B A$$

is an isomorphism, for each X in $\mathcal{F}$ and each diagram $f_!(X) \to B \leftarrow A$ in $\mathcal{E}$. Because the family $\{p_i\}_{i \in I}$ is jointly surjective, it is enough to prove that each $p_i^*(\vartheta)$ is an isomorphism.

Take g and q such that $fq = p_i g$, with g locally connected, and such that the natural map $f^* p_{i,*} \to q_* g^*$ is an isomorphism. Because f and g are essential, there is also a natural isomorphism $g_! q^* \to p_i^* f_!$. We compute:

$$\begin{aligned} p_i^* f_!(X \times_{f^*B} f^*A) &\simeq g_!(q^*X \times_{q^*f^*B} q^*f^*(A)) \\ &\simeq g_!(q^*X \times_{g^*p_i^*B} g^*p_i^*A) \\ &\simeq g_!(q^*X) \times_{p_i^*B} p_i^*A \\ &\simeq p_i^* f_!(X) \times_{p_i^*B} p_i^*A \\ &\simeq p_i^*(f_!(X) \times_B A) \end{aligned}$$

where in the third isomorphism we use that g is locally connected. □

Proposition 2.7 (Stability)**.** *Suppose that $g \trianglerighteq_p f$ with p an inclusion. If f is essential, then g is essential as well.*

Proof. We write q for the pullback of p along f, so we have a pullback diagram of the form

$$\begin{tikzcd} \mathcal{F}' \arrow[r, "q"] \arrow[d, "g"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{E}' \arrow[r, "p"] & \mathcal{E} \end{tikzcd} \tag{3}$$

We claim that $p^* f_! q_*$ is a left adjoint for g^*. We compute:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{E}'}(p^* f_! q_* X, Y) &\simeq \mathrm{Hom}_{\mathcal{F}}(q_* X, f^* p_* Y)\\ &\simeq \mathrm{Hom}_{\mathcal{F}}(q_* X, q_* g^* Y)\\ &\simeq \mathrm{Hom}_{\mathcal{F}'}(X, g^* Y)\end{aligned}$$

where in the second natural bijection we use the Beck–Chevalley condition, and in the third natural bijection we use that q is an inclusion (as pullback of the inclusion p). It follows that $p^* f_! q_*$ is the left adjoint of g^*, so g is essential. □

We will also need the following well-known characterization of cartesian closedness for the inverse image functor.

Proposition 2.8 (Cartesian closedness). *Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. Then the following are equivalent:*

1. *f^* is cartesian closed;*
2. *$(f/E) \trianglerighteq_{\pi_E} f$ for every object E in $\mathcal{E}$, with $\pi_E : \mathcal{E}/E \to \mathcal{E}$ the étale geometric morphism corresponding to E and f/E the pullback of f along π_E.*

Proof. For E in $\mathcal{E}$, consider the pullback diagram

$$\begin{tikzcd}\mathcal{F}/f^*(E) \arrow[r, "\tilde{\pi}_E"] \arrow[d, "f/E"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{E}/E \arrow[r, "\pi_E"] & \mathcal{E}\end{tikzcd} .$$

Both f and f/E are essential, so the Beck–Chevalley isomorphism is in this case given by

$$(f/E)_! \tilde{\pi}_E^* \simeq \pi_E^* f_!.$$

This amounts to the condition that the morphism

$$\vartheta_{F,E} : f_!(F \times f^*(E)) \to f_!(F) \times E$$

is an isomorphism, for each F in $\mathcal{F}$. This is precisely the Frobenius map, and f^* is cartesian closed if and only if $\vartheta_{F,E}$ is an isomorphism for all F and E (see [Joh02, Lemma A1.5.8]). We conclude that f^* is cartesian closed if and only if $(f/E) \trianglerighteq_{\pi_E} f$ for each object E in $\mathcal{E}$. □

3. Jacobson topological spaces and Jacobson étendues

An elementary topos $\mathcal{E}$ will be called **EILC** if any essential geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is locally connected. We will first show that $\mathbf{Sh}(X)$ is EILC for any Jacobson topological space X.

Definition 3.1. *Let X be a topological space, and let $X_0 \subseteq X$ be its subspace of closed points. We then say that X is* ***Jacobson*** *if $U \cap X_0 = V \cap X_0$ implies $U = V$, for all open subsets $U, V \subseteq X$.*

Equivalently, X is Jacobson if and only if for every closed subset $Z \subseteq X$, the subset $Z \cap X_0 \subseteq Z$ is dense, see [Sta22, Section 005T]. However, Definition 3.1 is more natural from a topos-theoretic point of view: it says precisely that the closed points of X define a jointly surjective family of points for $\mathbf{Sh}(X)$.

If X is the spectrum of a commutative ring R, with the Zariski topology, then X is Jacobson if and only if R is a Jacobson ring, in the sense that each prime ideal is an intersection of maximal ideals [Sta22, Lemma 00G3].

We can generalize the notion of Jacobson topological space over an arbitrary base elementary topos $\mathcal{S}$ as follows: we say that a localic geometric morphism $e : \mathcal{E} \to \mathcal{S}$ is a **Jacobson space** if e is localic and the family of closed points $p : \mathcal{S} \to \mathcal{E}$ is jointly surjective (points are by definition sections of e, i.e. $ep \simeq 1$). If $e : \mathcal{E} \to \mathcal{S}$ is a Jacobson space over $\mathcal{S}$, then in particular $\mathcal{E}$ has enough points over $\mathcal{S}$. Note that if e is localic, then any point $p : \mathcal{S} \to \mathcal{E}$ is an inclusion. Indeed, we have a pullback of the form

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "p"] \arrow[d, "p"'] & \mathcal{E} \arrow[d, "{(pe,1_{\mathcal{E}})}"] \\
\mathcal{E} \arrow[r, "\Delta"] & \mathcal{E} \times_{\mathcal{S}} \mathcal{E}
\end{tikzcd}$$

and because e is localic, the diagonal morphism Δ is an inclusion [Joh02, Proposition B3.3.8(ii)]. It then follows that its pullback p is an inclusion as well. So, in this setting, $p : \mathcal{S} \to \mathcal{E}$ is closed as geometric morphism (in the sense of [Joh02, C3.2, p.629]) if and only if p defines a closed subtopos.

We start by describing the idea behind the proof. Let X be a Jacobson topological space. We want to show that any essential geometric morphism $f : \mathcal{F} \to \mathbf{Sh}(X)$ is locally connected. The intuition is that f is locally connected if and only if its fibers are locally connected and moreover "the way in which the fibers are locally connected, varies continuously over the different fibers". Our proof will consist of two parts. First, we show that because f is essential, its fibers over closed points are locally connected. Second, we claim that if f has locally connected fibers over each closed point, then it follows that f is itself locally connected. So the "continuity condition" for the fibers already follows from f being essential. Both steps depend on Beck–Chevalley conditions, and these are the reason we restrict to points that are closed. For the second step to work, we use that $\mathbf{Sh}(X)$ has "enough closed points", i.e. that the closed points form a jointly surjective family (by definition of Jacobson space).

The next lemma gives an abstraction of the ideas described above.

Lemma 3.2. *Let $\mathcal{E}$ be an elementary topos with a natural number object, and let $\{p_i : \mathcal{E}_i \to \mathcal{E}\}_{i \in I}$ be a jointly surjective family, with each p_i a closed inclusion and with each $\mathcal{E}_i$ EILC. Then $\mathcal{E}$ is EILC as well.*

Proof. Take an essential geometric morphism $f : \mathcal{F} \to \mathcal{E}$. We will show that f is locally connected. For each p_i, we consider the pullback diagram

$$\begin{tikzcd}
\mathcal{F}_i \arrow[r, "q_i"] \arrow[d, "f_i"'] & \mathcal{F} \arrow[d, "f"] \\
\mathcal{E}_i \arrow[r, "p_i"] & \mathcal{E}
\end{tikzcd} .$$

Because p_i is a closed inclusion, it is in particular tidy [MV00, Chapter III, Corollary 5.8]. Further, any inclusion is bounded. So by Proposition 2.3.(3) we have $f_i \trianglerighteq_{p_i} f$. It now follows from Proposition 2.7 that f_i is essential. Because $\mathcal{E}_i$ is by assumption EILC, it follows that f_i is locally connected. As a result, f is locally connected at p_i, for each $i \in I$. Using Proposition 2.6 and the fact that the family $\{p_i\}_{i \in I}$ is jointly surjective, we can then conclude that f is locally connected. $\square$

Theorem 3.3. *Let X be a Jacobson topological space. Then $\mathbf{Sh}(X)$ is EILC.*

Proof. Let X be a Jacobson topological space, and let $X_0 \subseteq X$ be the subset of closed points. Then the family $\{p_x : \mathbf{Sets} \to \mathbf{Sh}(X)\}_{x \in X_0}$ is a jointly surjective family of closed inclusions, where p_x denotes the closed inclusion corresponding to the closed subset $\{x\} \subseteq X$. Further, $\mathbf{Sets}$ is EILC, so from Lemma 3.2 we deduce that $\mathbf{Sh}(X)$ is EILC as well. $\square$

The proof above generalizes to a Jacobson space $f : \mathcal{E} \to \mathcal{S}$, for $\mathcal{S}$ an EILC base topos, under the assumption that $\mathcal{E}$ has a natural numbers object. In this case, by definition the family of closed points $\mathcal{S} \to \mathcal{E}$ is jointly surjective, so as soon as $\mathcal{S}$ is EILC we can apply Lemma 3.2.

We can also generalize the notion of Jacobson space in another direction as follows. Recall that an object E of a topos $\mathcal{E}$ is called **well-supported** if the unique morphism $E \to 1$ is an epimorphism. Further, an **étendue** is a topos $\mathcal{E}$ such that there is a well-supported object E in $\mathcal{E}$ such that $\mathcal{E}/E$ is localic (over the base topos $\mathcal{S}$).

Definition 3.4. *Fix an elementary topos $\mathcal{S}$. A geometric morphism $e : \mathcal{E} \to \mathcal{S}$ will be called a **Jacobson étendue** if there is a well-supported object E in $\mathcal{E}$ such that the composition*

$$\mathcal{E}/E \xrightarrow{\pi} \mathcal{E} \xrightarrow{e} \mathcal{S}$$

is a Jacobson space.

If $e : \mathcal{E} \to \mathcal{S}$ is a Jacobson space, then it is also a Jacobson étendue; in this case we can take $E = 1$.

Example 3.5. We give two examples of Grothendieck toposes that are Jacobson étendues (over $\mathbf{Sets}$).

1. $\mathbf{PSh}(G)$ for G a group is a Jacobson étendue. Indeed, we can take $E = G$ with its standard right G-action, and then $\mathbf{PSh}(G)/G \simeq \mathbf{Sets}$.
2. The Jónsson–Tarski topos $\mathcal{J}$ is a Jacobson étendue. Here we can take E to be the free Jónsson-Tarski algebra on one generator, and then $\mathcal{J}/E \simeq \mathbf{Sh}(X)$, for X the Cantor space, see [BF06, Proposition 8.5.2]. The Cantor space is Hausdorff, so it is in particular Jacobson.

Lemma 3.6. *Let $\mathcal{E}$ be an elementary topos and let E be a well-supported object of $\mathcal{E}$. If $\mathcal{E}/E$ is EILC, then $\mathcal{E}$ is EILC as well.*

Proof. Let $f : \mathcal{F} \to \mathcal{E}$ be an essential geometric morphism. The slice

$$f/E : \mathcal{F}/f^*E \to \mathcal{E}/E$$

is then also essential [LM15, Lemma 5.2]. Because $\mathcal{E}/E$ is EILC, we see that f/E is locally connected. Note that f/E is the base change of f

along the étale geometric morphism $\pi : \mathcal{E}/E \to \mathcal{E}$. Because E is well-supported, the étale geometric morphism is a surjection. It follows that f is locally connected as well, because local connectedness can be checked after base change along an étale surjection, see for example [Joh02, Corollary C3.3.2(iv)]. $\square$

Let $\mathcal{E}$ be an elementary topos with a natural number object. If $\mathcal{E}$ is a Jacobson étendue over an EILC base topos $\mathcal{S}$, then we can take a well-supported object E in $\mathcal{E}$ such that $\mathcal{E}/E$ is a Jacobson space over $\mathcal{S}$. By Theorem 3.3, it follows that $\mathcal{E}/E$ is EILC. So by applying Lemma 3.6, we find that $\mathcal{E}$ is EILC. In summary:

Corollary 3.7. *Let $\mathcal{E}$ be an elementary topos with a natural numbers object. If $\mathcal{E}$ is a Jacobson étendue over an EILC base topos $\mathcal{S}$, then $\mathcal{E}$ is EILC as well.*

In particular, $\mathbf{PSh}(G)$ is EILC for any group G, and the Jónsson–Tarski topos is EILC.

4. Boolean étendues and compact topological groups

In this section we restrict to Grothendieck toposes, i.e. toposes bounded over the topos of sets. We will show that both Boolean étendues and classifying toposes of compact topological groups are EILC. Afterwards, we show that a presheaf topos $\mathbf{PSh}(\mathcal{C})$ is EILC if and only if $\mathcal{C}$ is a groupoid.

For both Boolean étendues and classifying toposes of compact topological groups, the argument can be simplified using the following lemma:

Lemma 4.1. *Let $\mathcal{A}$ be a family of toposes with the following properties:*

1. *if $\mathcal{E}$ is in $\mathcal{A}$, then also $\mathcal{E}/E$ is in $\mathcal{A}$, for any object E in $\mathcal{E}$;*
2. *for any essential geometric morphism $f : \mathcal{F} \to \mathcal{E}$, with $\mathcal{E}$ in $\mathcal{A}$, the inverse image functor f^* is cartesian closed.*

Then all toposes in $\mathcal{A}$ are EILC.

Proof. For $\mathcal{E}$ in $\mathcal{A}$, we have to show that any essential geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is locally connected. This follows from the following characterization of local connectedness: f is locally connected if and only if its slice $f/E : \mathcal{F}/f^*E \to \mathcal{E}/E$ has cartesian closed inverse image functor, for all objects E in $\mathcal{E}$ [Joh02, Proposition C3.3.1]. If f is essential, then each slice f/E is again essential [LM15, Lemma 5.2], and by (1) its codomain $\mathcal{E}/E$ is in $\mathcal{A}$. So by (2) f/E has cartesian closed inverse image functor. □

A **Boolean étendue** is a topos that is both Boolean and an étendue. Note that if E is a well-supported object of a topos $\mathcal{E}$, then $\mathcal{E}$ is Boolean if and only if $\mathcal{E}/E$ is Boolean. So we can alternatively define a Grothendieck topos $\mathcal{E}$ to be a Boolean étendue if there is a well-supported object E in $\mathcal{E}$ and a Boolean locale Y with $\mathcal{E}/E \simeq \mathbf{Sh}(Y)$.

We now show that any Grothendieck topos that is a Boolean étendues, is EILC. The proof is inspired by a related argument by Matías Menni, in his proof that for an arbitrary Boolean topos $\mathcal{E}$, a connected, essential geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is locally connected as soon as $f_!$ preserves finite products.

Proposition 4.2. *Let $\mathcal{E}$ be a Grothendieck topos. If $\mathcal{E}$ is a Boolean étendue, then it is EILC.*

Proof. By Lemma 3.6 it is enough to prove that localic Boolean Grothendieck toposes are EILC. Further, by applying Lemma 4.1 for $\mathcal{A}$ the family of localic Boolean Grothendieck toposes, it is enough to show that any essential geometric morphism $f : \mathcal{F} \to \mathcal{E}$ has cartesian closed inverse image functor, for $\mathcal{E}$ a localic Boolean Grothendieck topos. This is equivalent to showing that for any objects X in $\mathcal{F}$ and A in $\mathcal{E}$ the natural map

$$\vartheta_{X,A} : f_!(X \times f^*A) \to f_!(X) \times A$$

is an isomorphism. Because $\mathcal{E}$ is a localic Grothendieck topos, A can be written as a colimit of subterminal objects. So it is enough to prove that $\vartheta_{X,A}$ is an isomorphism in the special case that A is subterminal (colimits are preserved by $f_!$ and f^* and are stable under pullbacks).

Now take the complement A' of A, so $1 = A \sqcup A'$. Since $\vartheta_{X,1}$ is trivially an isomorphism, its restrictions $\vartheta_{X,A}$ and $\vartheta_{X,A'}$ are isomorphisms as well.

Alternatively, we can argue that in the pullback diagram

$$\begin{tikzcd}
\mathcal{F}/f^*A \arrow[r, "\tilde{\pi}"] \arrow[d, "f/A"'] & \mathcal{F} \arrow[d, "f"] \\
\mathcal{E}/A \arrow[r, "\pi"'] & \mathcal{E}
\end{tikzcd}$$

the Beck–Chevalley condition holds, because $\pi : \mathcal{E}/A \to \mathcal{E}$ is a closed inclusion, in particular bounded and tidy, so Proposition 2.3(3) applies. The Beck–Chevalley condition in this case says precisely that $\vartheta_{X,A}$ is an isomorphism for each object X in $\mathcal{F}$. $\square$

For a topological group G, its classifying topos $\mathbf{Cont}(G)$ is the topos of sets equipped with a continuous action of G (where the sets are seen as topological spaces with the discrete topology). We will now show that $\mathbf{Cont}(G)$ is EILC if the topological group G is compact. We would not gain any generality by considering compact *localic* groups, because over $\mathbf{Sets}$ any compact localic group has enough points [Joh02, Remarks 5.3.14(b)].

We will simplify our argument by applying Lemma 4.1. In order for this to work, we need to consider more generally toposes of the form $\bigsqcup_{i\in I}\mathbf{Cont}(G_i)$, for $(G_i)_{i\in I}$ a family of compact topological groups (the disjoint union is computed in the category of Grothendieck toposes). An object in $\bigsqcup_{i\in I}\mathbf{Cont}(G_i)$ is a family $(A_i)_{i\in I}$ with each A_i an object in $\mathbf{Cont}(G_i)$. We claim that if $\mathcal{E}$ is of the form $\bigsqcup_{i\in I}\mathbf{Cont}(G_i)$ for some family of compact topological groups $(G_i)_{i\in I}$, then $\mathcal{E}/A$ is again of the same form, for each object A in $\mathcal{E}$.

Indeed, if $A = (A_i)_{i\in I}$ is an object in $\mathcal{E} \simeq \bigsqcup_{i\in I}\mathbf{Cont}(G_i)$, then

$$\mathcal{E}/A \simeq \bigsqcup_{i\in I}\mathbf{Cont}(G_i)/A_i.$$

We can write each A_i as a disjoint union of orbits $A_i \cong \bigsqcup_{j\in J_i} G_i/H_{ij}$, with $H_{ij} \subseteq G_i$ an open subgroup, for each $j \in J_i$. Now using the equivalence $\mathbf{Cont}(G_i)/(G_i/H_{ij}) \simeq \mathbf{Cont}(H_{ij})$, we find that

$$\mathcal{E}/A \simeq \bigsqcup_{i\in I}\bigsqcup_{j\in J_i}\mathbf{Cont}(H_{ij}).$$

Note that each group H_{ij} is again compact, because it is an open subgroup of the compact topological group G_i (and open subgroups are closed). So $\mathcal{E}/A$ is of the same form.

For an object $A = (A_i)_{i \in I}$ in $\mathcal{E}$, we would now like to determine when the corresponding étale geometric morphism $\mathcal{E}/A \to \mathcal{E}$ is tidy. Each topos $\mathbf{Cont}(G_i)$ has a canonical point $\mathbf{Sets} \to \mathbf{Cont}(G_i)$ which is an open surjection, so taking the disjoint union of these points gives an open surjection of the form

$$\xi \;:\; \bigsqcup_{i \in I} \mathbf{Sets} \longrightarrow \mathcal{E}.$$

The inverse image functor ξ^* is the forgetful functor, sending a family $(A_i)_{i \in I}$ to the same family $(A_i)_{i \in I}$, but this time each A_i is seen only as a set. The property of being tidy can be checked after base change along the open surjection ξ, so $\mathcal{E}/A \to \mathcal{E}$ is tidy if and only if

$$\bigsqcup_{i \in I} \mathbf{Sets}/A_i \longrightarrow \bigsqcup_{i \in I} \mathbf{Sets}$$

is tidy. We conclude that $\mathcal{E}/A \to \mathcal{E}$ is tidy if and only if the underlying set of A_i is finite, for all $i \in I$. This will be relevant in the next result, because of the relation between tidy geometric morphisms and the Beck–Chevalley condition.

Proposition 4.3. *Let $\{G_i\}_{i \in I}$ be a family of compact topological groups. Then the topos $\bigsqcup_{i \in I} \mathbf{Cont}(G_i)$ is EILC.*

Proof. We write $\mathcal{E} \simeq \bigsqcup_{i \in I} \mathbf{Cont}(G_i)$. Let $f : \mathcal{F} \to \mathcal{E}$ be an essential geometric morphism. We have to show that f is locally connected. Applying Lemma 4.1, it is enough to show that f^* is cartesian closed. In other words, we have to show that the natural map $\vartheta_{X,A} : f_!(X \times f^*A) \to f_!(X) \times A$ is an isomorphism, for X in $\mathcal{F}$ and A in $\mathcal{E}$. Because $\mathcal{E}$ is locally connected, we can write A as a coproduct of connected objects. Coproducts are pullback-stable and preserved by f^* and $f_!$, so we can reduce to the case where A is connected. Note that if A corresponds to the family $(A_i)_{i \in I}$ with each A_i an object in $\mathbf{Cont}(G_i)$, then A being connected implies that there is an index $i_0 \in I$ such that A_{i_0} is a single G_i-orbit, and $A_j = \varnothing$ for all $j \neq i_0$. Using compactness of G_{i_0}, it follows that the underlying set of A_{i_0} is finite. By the discussion above, we then have that the étale geometric morphism $\mathcal{E}/A \to \mathcal{E}$ is tidy. In particular, the pullback square

$$\begin{tikzcd}
\mathcal{F}/f^*A \arrow[r, "\tilde{\pi}"] \arrow[d, "f/A"'] & \mathcal{F} \arrow[d, "f"] \\
\mathcal{E}/A \arrow[r, "\pi"] & \mathcal{E}
\end{tikzcd}$$

satisfies the Beck–Chevalley condition, see Proposition 2.3(3). But this coincides precisely with the statement that the natural map $f_!(X \times f^*A) \to f_!(X) \times A$ is an isomorphism, which is what we wanted to prove. □

For presheaf toposes, we have a jointly surjective family of essential points, and these points are typically not locally connected. As a result, presheaf toposes will usually not be EILC. More precisely:

Proposition 4.4. *Let $\mathcal{C}$ be a small category. Then $\mathbf{PSh}(\mathcal{C})$ is EILC if and only if $\mathcal{C}$ is a groupoid.*

Proof. If $\mathcal{C}$ is a groupoid, then $\mathbf{PSh}(\mathcal{C})$ is a Boolean étendue, so we can use Proposition 4.2 to conclude that $\mathbf{PSh}(\mathcal{C})$ is EILC.

Conversely, suppose that $\mathbf{PSh}(\mathcal{C})$ is EILC. Each object C in $\mathcal{C}$ determines an essential point $p : \mathbf{Sets} \to \mathbf{PSh}(\mathcal{C})$ with $p_!(1) \simeq \mathbf{y}C$, $\mathbf{y}$ the Yoneda embedding. Because of the EILC property, p has to be locally connected. We can then factorize p as a connected, locally connected geometric morphism, followed by an étale geometric morphism. However, because the domain topos is $\mathbf{Sets}$, the connected part is trivial, so p is étale. It follows from $p_!(1) \simeq \mathbf{y}C$ that we then have $\mathbf{PSh}(\mathcal{C}/C) \simeq \mathbf{Sets}$. This is only possible if $\mathcal{C}/C$ has only one object up to isomorphism, or in other words any morphism $D \to C$ is necessarily an isomorphism. Because C was arbitrary, we conclude that $\mathcal{C}$ is a groupoid. □

More generally, we could consider the presheaf toposes $\mathbf{PSh}_{\mathcal{S}}(\mathcal{C})$, for $\mathcal{C}$ an internal category in an arbitrary EILC topos $\mathcal{S}$. However, it is not known at the moment to the present author whether the analogue of Proposition 4.4 would still hold.

5. A weaker property: CILC toposes

We will say that an elementary topos $\mathcal{E}$ is **CILC** if any geometric morphism $f : \mathcal{F} \to \mathcal{E}$ such that f^* is cartesian closed (i.e. preserves exponential objects) is automatically locally connected.

Note that by a result of Barr and Paré [BP80, Theorem 2], if f^* is cartesian closed, then f is essential. I thank Thomas Streicher for pointing out this result to me. So all EILC toposes are in particular CILC. The converse does not hold; we will construct some counterexamples below.

Proposition 5.1. *Let $\mathcal{E}$ be an elementary topos with a natural number object, and suppose that there is a jointly surjective family $\{p_i : \mathcal{E}_i \to \mathcal{E}\}_{i \in I}$, such that each p_i can be factored as a closed inclusion followed by an étale geometric morphism, and such that each $\mathcal{E}_i$ is EILC. Then $\mathcal{E}$ is CILC.*

Proof. Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism with f^* cartesian closed. For each $i \in I$, the geometric morphism p_i factors as

$$\mathcal{E}_i \xrightarrow{j} \mathcal{E}/E \xrightarrow{\pi_E} \mathcal{E}$$

with j a closed inclusion and $\pi_E : \mathcal{E}/E \to \mathcal{E}$ the étale geometric morphism corresponding to an object E in $\mathcal{E}$.

Now consider the corresponding pullback squares

$$\begin{tikzcd}
\mathcal{F}_i \arrow[r, "\tilde{j}"] \arrow[d, "f_i"'] & \mathcal{F}/f^*(E) \arrow[r, "\tilde{\pi}_E"] \arrow[d, "f/E"] & \mathcal{F} \arrow[d, "f"] \\
\mathcal{E}_i \arrow[r, "j"'] & \mathcal{E}/E \arrow[r, "\pi_E"'] & \mathcal{E}
\end{tikzcd} .$$

Note that because j and π_E are localic, they are bounded. Further, since j is a closed inclusion, it is in particular tidy, see [MV00, Chapter III, Corollary 5.8]. So using Proposition 2.3(3), we find $f_i \trianglerighteq_j (f/E)$. Because (f/E) is essential, it then follows from Proposition 2.7 that f_i is essential as well. But then f_i is locally connected, because $\mathcal{E}_i$ is EILC.

Moreover, it follows from cartesian closedness of f^* that $(f/E) \trianglerighteq_{\pi_E} f$, see Proposition 2.8. Using Proposition 2.4 and $p_i \simeq \pi_E \circ j$ we conclude that $f_i \trianglerighteq_{p_i} f$. So for each $i \in I$, we find that f is locally connected at p_i. Because the family $\{p_i : \mathcal{E}_i \to \mathcal{E}\}$ is jointly surjective, we then conclude that f is locally connected, see Proposition 2.6. □

The assumption that $\mathcal{E}$ has a natural number object is relevant when we apply Proposition 2.3(3). Alternatively, we could use Proposition 2.3(4), but then we can only conclude that $\mathcal{E}$ has the property that any *bounded* geometric morphism $f : \mathcal{F} \to \mathcal{E}$, with f^* cartesian closed, is locally connected.

Definition 5.2. *A geometric morphism $f : \mathcal{E} \to \mathcal{S}$ will be called* ***weakly Jacobson*** *if there is a jointly surjective family of points $\{p_i : \mathcal{S} \to \mathcal{E}\}_{i\in I}$, such that each p_i can be factored as a closed inclusion followed by an étale geometric morphism.*

Theorem 5.3. *Let $f : \mathcal{E} \to \mathcal{S}$ be a geometric morphism. Suppose that $\mathcal{E}$ has a natural numbers object. If f is weakly Jacobson and $\mathcal{S}$ is EILC, then $\mathcal{E}$ is CILC.*

Proof. If f is weakly Jacobson, then by definition there is a jointly surjective family of $\{p_i : \mathcal{S} \to \mathcal{E}\}_{i\in I}$ such that each p_i can be factored as a closed inclusion followed by an étale geometric morphism. If moreover $\mathcal{S}$ is EILC, then Proposition 5.1 applies, and we conclude that $\mathcal{E}$ is CILC. □

We will restrict to Grothendieck toposes in the remainder of this section. We first characterize the topological spaces X such that $\mathbf{Sh}(X)$ is weakly Jacobson (over the topos of sets).

Proposition 5.4. *Let X be a topological space, and let $X_{\mathrm{lc}} \subseteq X$ be the subset of locally closed points. Then $\mathbf{Sh}(X)$ is weakly Jacobson if and only if $U \cap X_{\mathrm{lc}} = V \cap X_{\mathrm{lc}}$ implies $U = V$, for all open subsets $U, V \subseteq X$.*

Proof. The locally closed points of X are precisely the points that are open in their closure. So if $x \in X$ is a locally closed point, then there is an open set $U \subseteq X$ such that $U \cap \overline{\{x\}} = \{x\}$. In this situation, x is the only point that can distinguish between the open sets W and $W \cup U$, for $W = X - \overline{\{x\}}$. This implies that a jointly surjective family of points $\{p_i : \mathbf{Sets} \to \mathbf{Sh}(X)\}$ will necessarily contain all points $p_x : \mathbf{Sets} \to \mathbf{Sh}(X)$ corresponding to locally closed points $x \in X_{\mathrm{lc}} \subseteq X$.

In particular, let $\tilde{X}$ be the sobrification of X. Then the elements of X determine a jointly surjective family of points for $\mathbf{Sh}(\tilde{X})$, and by the above this means that all locally closed points of $\tilde{X}$ are also contained in X. So we can assume without loss of generality that X is sober, i.e. that the correspondence between elements of X and topos-theoretic points $\mathbf{Sets} \to \mathbf{Sh}(X)$ (up to isomorphism) is bijective.

Now suppose that $\mathbf{Sh}(X)$ is weakly Jacobson, or in other words that there exists a jointly surjective family of points $\{p_i : \mathbf{Sets} \to \mathbf{Sh}(X)\}_{i\in I}$, such that each p_i can be factored as a closed inclusion followed by an étale geometric morphism. Let $x_i \in X$ be the element corresponding to p_i. The embedding $\{x_i\} \subseteq X$ can then be factored as a closed inclusion $\{x_i\} \subseteq E$ followed by a local homeomorphism $\pi : E \to X$. Take an open set U containing x_i such that the restriction of π defines an homeomorphism from U to the open set $\pi(U) \subseteq X$. Then $\{x_i\} \subseteq X$ factors as a closed inclusion $\{x_i\} \subseteq \pi(U)$ followed by an open inclusion $\pi(U) \subseteq X$. So each x_i is a locally closed point. As a result, the locally closed points of X form a jointly surjective family, i.e. if two open subsets U, V contain the same locally closed points, then $U = V$.

Conversely, suppose that the locally closed points form a jointly surjective family. For each locally closed point $x \in X$, we can write $\{x\} = U \cap V$ with U open and V closed. But then the inclusion $\{x\} \subseteq X$ factorizes as a closed inclusion $\{x\} \subseteq U$ followed by the open inclusion $U \subseteq X$, which is in particular a local homeomorphism. But then $\mathbf{Sh}(X)$ is weakly Jacobson. ☐

Example 5.5. The Sierpinski space is given by $\mathbb{S} = \{m, g\}$ with as open sets $\varnothing$, $\{g\}$ and $\{g, m\}$. Both points are locally closed (g is open and m is closed). It then follows by Proposition 5.4 that $\mathbf{Sh}(\mathbb{S})$ is weakly Jacobson. As a result, $\mathbf{Sh}(\mathbb{S})$ is CILC.

Note that $\mathbf{Sh}(\mathbb{S}) \simeq \mathbf{PSh}(\mathcal{C})$, for $\mathcal{C}$ the category with two objects A and B and a single non-identity morphism $A \to B$. So from Proposition 4.4 it follows that $\mathbf{Sh}(\mathbb{S})$ is not EILC.

Example 5.6. In some topological spaces, none of the points are locally closed. Take for example the set $X \subset \mathcal{P}(\mathbb{N})$ of infinite subsets of natural numbers, with as topology the smallest topology such that the sets

$$U_n = \{V : V \ni n\} \subseteq X$$

are open. Then none of the points of X are locally closed, so $\mathbf{Sh}(X)$ is not weakly Jacobson. However, it is not known to the author whether $\mathbf{Sh}(X)$ is CILC or even EILC.

We also want to give a criterion for when a presheaf topos is weakly Jacobson (over the topos of sets). We first need the following lemma:

Lemma 5.7. *Let $\mathcal{C}$ be a small category with a terminal object. Then $\mathbf{PSh}(\mathcal{C})$ is local. Moreover, its center is a closed inclusion if and only if the terminal object in $\mathcal{C}$ is strict.*

Proof. If $\mathcal{C}$ has a terminal object, then $\mathbf{PSh}(\mathcal{C})$ is local, see [Joh02, Examples C3.6.3(b)]. The center of a local geometric morphism is an inclusion, and in this case it agrees with the essential point $p : \mathbf{Sets} \to \mathbf{PSh}(\mathcal{C})$ corresponding to the terminal object in $\mathcal{C}$.

From [Joh02, Lemma C3.2.4] it then follows that p is closed if and only if every morphism $b : 1 \to C$ in $\mathcal{C}$ admits a right inverse $r : C \to 1$. Whenever such a right inverse r exists, it must also be a left inverse, because rb is an endomorphism of the terminal object. So we find that p is closed if and only if every morphism $1 \to C$ is an isomorphism, or in other words if and only if the terminal object is strict. ☐

Note that the argument in [Joh02, Lemma C3.2.4] is not constructive. So our argument here does not generalize to presheaf toposes over an arbitrary base topos.

Proposition 5.8. *Let $\mathcal{C}$ be a small category. If every morphism in $\mathcal{C}$ admitting a right inverse is an isomorphism, then $\mathbf{PSh}(\mathcal{C})$ is weakly Jacobson. Conversely, if $\mathbf{PSh}(\mathcal{C})$ is weakly Jacobson, then there is a small category $\mathcal{C}'$, with $\mathbf{PSh}(\mathcal{C}) \simeq \mathbf{PSh}(\mathcal{C}')$, such that every morphism in $\mathcal{C}'$ admitting a right inverse is an isomorphism.*

Proof. For every object C, we can consider the corresponding point

$$p_C : \mathbf{Sets} \to \mathbf{PSh}(\mathcal{C}).$$

This point factors as $j : \mathbf{Sets} \to \mathbf{PSh}(\mathcal{C}/C)$ followed by $\pi : \mathbf{PSh}(\mathcal{C}/C) \to \mathbf{PSh}(\mathcal{C})$. If every morphism $f : D \to C$ that admits a right inverse is an isomorphism, then the terminal object in $\mathcal{C}/C$ is strict. But then using Lemma 5.7 we see that j is a closed inclusion. In the definition of weakly Jacobson, we can now take the family $\{p_C : \mathbf{Sets} \to \mathbf{PSh}(\mathcal{C})\}_C$, with C going over the objects of $\mathcal{C}$, to conclude that $\mathbf{PSh}(\mathcal{C})$ is indeed weakly Jacobson.

Conversely, let $\{p_i : \mathbf{Sets} \to \mathbf{PSh}(\mathcal{C})\}_{i\in I}$ be a jointly surjective family, such that each point p_i can be factored as a closed inclusion $j : \mathbf{Sets} \to \mathbf{PSh}(\mathcal{D})$ followed by an étale geometric morphism $\pi : \mathbf{PSh}(\mathcal{D}) \to \mathbf{PSh}(\mathcal{C})$. Because j is a closed inclusion, it must be essential. Indeed, otherwise all essential points would be contained in the complement of the subtopos defined by j, and because the essential points form a jointly surjective family, this means that the complement of j is the full topos $\mathbf{PSh}(\mathcal{D})$, a contradiction. As a result, we know that p_i is essential, for all $i \in I$. The family $\{p_i : \mathbf{Sets} \to \mathbf{PSh}(\mathcal{C})\}_{i\in I}$ is jointly surjective, so we can find a small category $\mathcal{C}'$, with $\mathbf{PSh}(\mathcal{C}') \simeq \mathbf{PSh}(\mathcal{C})$, such that

$$\{p_i : \mathbf{Sets} \to \mathbf{PSh}(\mathcal{C}')\}_{i\in I} = \{p_C : \mathbf{Sets} \to \mathbf{PSh}(\mathcal{C}')\}_{C\in \mathbf{Ob}(\mathcal{C}')},$$

for $p_C : \mathbf{Sets} \to \mathcal{C}'$ the essential geometric morphism associated to C (points are considered up to isomorphism).

For each object C in $\mathcal{C}'$, we can now factorize p_C as a closed inclusion $j : \mathbf{Sets} \to \mathbf{PSh}(\mathcal{D})$ followed by the étale geometric morphism $\pi : \mathbf{PSh}(\mathcal{D}) \to \mathbf{PSh}(\mathcal{C}')$, as above. We further have a different factorization of p_C as an inclusion $j' : \mathbf{Sets} \to \mathbf{PSh}(\mathcal{C}'/C)$ followed by an étale geometric morphism $\pi' : \mathbf{PSh}(\mathcal{C}'/C) \to \mathbf{PSh}(\mathcal{C}')$. In fact, the latter is precisely the (terminal-connected, étale) factorization as described by Caramello in [Car, Section 4.7]. The geometric morphism j' is the center of the local topos $\mathbf{PSh}(\mathcal{C}'/C)$. We want to show that j' is closed, because then we can apply Lemma 5.7.

We apply the (terminal-connected, étale) factorization to the closed inclusion $j : \mathbf{Sets} \to \mathbf{PSh}(\mathcal{D})$. By uniqueness of (terminal-connected, étale) factorizations [Car, Proposition 4.62], this factorization must be given by $j' : \mathbf{Sets} \to \mathbf{PSh}(\mathcal{C}'/C)$ followed by an étale geometric morphism $\pi'' : \mathbf{PSh}(\mathcal{C}'/C) \to \mathbf{PSh}(\mathcal{D})$. Now consider the pullback diagram

$$\begin{tikzcd}
\mathbf{Sets}/A \arrow[r, "\tilde{j}"] \arrow[d, "\gamma"'] & \mathbf{PSh}(\mathcal{C}'/C) \arrow[d, "\pi''"] \\
\mathbf{Sets} \arrow[r, "j"] & \mathbf{PSh}(\mathcal{D})
\end{tikzcd}$$

and note that $j' : \mathbf{Sets} \to \mathbf{PSh}(\mathcal{C}'/C)$ can be written as $j' = \tilde{j} \circ s$, for s a section of γ (see [sga72, Exposé IV, Proposition 5.12]). The geometric morphism $\tilde{j}$ is the pullback of the closed inclusion j, so it is itself a closed inclusion. Further, any section of γ is also a closed inclusion (a discrete topological space has closed points). It follows that j' is a closed inclusion. By Lemma 5.7, this implies that the terminal object in $\mathcal{C}'/C$ is strict. In the above, the object C was arbitrary, so $\mathcal{C}'/C$ has a strict terminal object for all objects C in $\mathcal{C}'$. In other words, if an arbitrary morphism in $\mathcal{C}'$ has a right inverse, then it must be an isomorphism. □

Example 5.9.

1. The topos of directed graphs is the topos of presheaves on a category $\mathcal{C}$ with two objects V and E, and as morphisms the two identity morphisms and $s, t : V \to E$. The only morphisms in $\mathcal{C}$ that admit a right inverse, are the identity morphisms. So the topos of directed graphs is weakly Jacobson, in particular CILC.

2. Let M be a monoid such that every right-invertible element is (two-sided) invertible. Then $\mathbf{PSh}(M)$ is weakly Jacobson, in particular CILC.

3. Let N be the monoid of natural numbers (with zero) under multiplication. Consider the category $\mathcal{C}$ with as objects the left N-set N, with the action given by multiplication, and the terminal left N-set 1. As morphisms, we take the morphisms of left N-sets. The unique morphism $N \to 1$ in $\mathcal{C}$ then admits a right inverse. However, because 1 is a retract of N in $\mathcal{C}$, we have $\mathbf{PSh}(\mathcal{C}) \simeq \mathbf{PSh}(N)$. Moreover, in N there is only one element that admits a right inverse, namely the identity. So $\mathbf{PSh}(\mathcal{C}) \simeq \mathbf{PSh}(N)$ is weakly Jacobson.

4. Let M be a monoid containing a right-invertible element that is not invertible. Suppose that $\mathbf{PSh}(M) \simeq \mathbf{PSh}(M')$ for a different monoid M'. Then M' again contains a right-invertible element that is not invertible, otherwise the Morita equivalence $\mathbf{PSh}(M) \simeq \mathbf{PSh}(M')$ would imply that $M \cong M'$, see for example [Rog, Corollary 7.2(3)]. If, more generally, $\mathcal{C}'$ is a category with $\mathbf{PSh}(M) \simeq \mathbf{PSh}(\mathcal{C}')$, then there is an object in $\mathcal{C}'$ that is a generator, in the sense that its endomorphism monoid M' satisfies $\mathbf{PSh}(M) \simeq \mathbf{PSh}(M')$. We conclude that in $\mathcal{C}'$ there is a right-invertible morphism that is not invertible. It then follows from Proposition 5.8 that $\mathbf{PSh}(M)$ is not weakly Jacobson.

Finally, we also show that every Boolean elementary topos is CILC.

Theorem 5.10. *Every Boolean elementary topos is CILC.*

Proof. Let $\mathcal{E}$ be a Boolean elementary topos, and let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism such that the inverse image functor f^* is cartesian closed. We need to show that for every morphism $\varphi : A \to B$, the associated pullback square

$$\begin{tikzcd}
\mathcal{F}/f^*A \arrow[r, "\tilde{\pi}"] \arrow[d, "f/A"'] & \mathcal{F}/f^*B \arrow[d, "f/B"] \\
\mathcal{E}/A \arrow[r, "\pi"] & \mathcal{E}/B
\end{tikzcd} \tag{4}$$

satisfies the Beck–Chevalley condition $(f/A) \trianglerighteq_{\pi} (f/B)$.

There are two situations in which we know this Beck–Chevalley condition is satisfied. First of all, if φ is a monomorphism, then π is an inclusion, and because $\mathcal{E}/B$ is Boolean, it must be a closed inclusion. So it is bounded and tidy, and then the Beck–Chevalley condition is automatically satisfied, see Proposition 2.3(3). Note however that in order to apply Proposition 2.3(3), we need that $\mathcal{E}$ has a natural numbers object. To avoid this extra assumption, we give an alternative argument. Consider the natural map

$$\vartheta_{X,B,A} \; : \; f_!(X \times_{f^*B} f^*A) \longrightarrow f_!(X) \times_B A.$$

If $\varphi : A \to B$ is an inclusion, then because $\mathcal{E}$ is Boolean, we can take a complement A' of A. The natural map $\vartheta_{X,B,B}$ associated to the identity $B \to B$ is trivially an isomorphism, so its restrictions $\vartheta_{X,B,A}$ and $\vartheta_{X,B,A'}$ are isomorphisms as well. As a result, the diagram (4) satisfies $(f/A) \trianglerighteq_{\pi} (f/B)$ as soon as φ is injective.

A second situation when the Beck–Chevalley condition is satisfied is when $X = 1$, because in this case the Beck–Chevalley condition follows from f^* being cartesian closed, see Proposition 2.8. More generally, if φ is of the form $\pi_B : Y \times B \to B$ (projection on the second component), then the corresponding pullback square is a slice over B of the pullback square for $\varphi : Y \to 1$, so it is again a Beck–Chevalley square, see [Joh02, Lemma A4.1.16].

In general, we can factor $\varphi : Y \to X$ as the inclusion $j : Y \to Y \times X$, $j = (\mathrm{id}_Y, \varphi)$ followed by the projection $\pi_X : Y \times X \to X$. But then the square in (4) satisfies the Beck–Chevalley condition by applying transitivity, see Proposition 2.4. □

Remark 5.11. Theorem 5.10 extends an earlier result by Matías Menni, who showed in [Men21] that a connected, essential geometric morphism $f : \mathcal{F} \to \mathcal{E}$, with $\mathcal{E}$ a Boolean topos, is locally connected as soon as $f_!$ preserves finite products. Note that if f is connected and $f_!$ preserves finite products, then f^* is cartesian closed, see [Joh02, Proposition A4.3.1].

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