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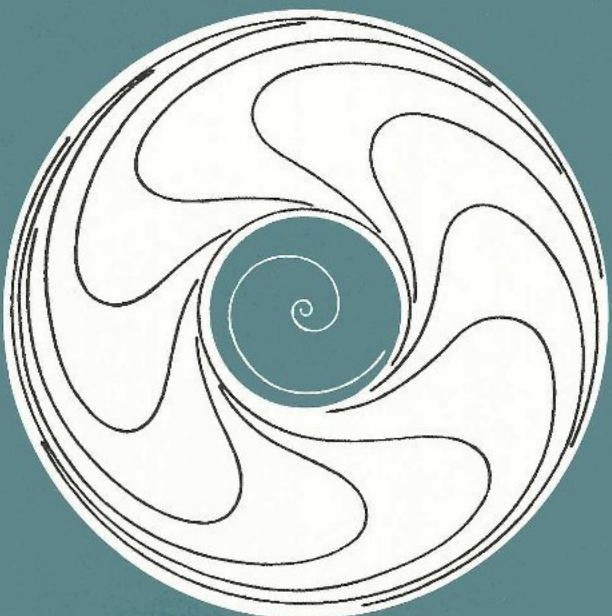

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HERON'S FORMULA, AND VOLUME FORMS

Anders KOCK

Résumé. La formule de Heron pour les aires des triangles (et pour d'autres simplexes) s'applique, dans une variété Riemannienne quelconque, aux simplexes qui sont infinitésimaux en un certain sens précis. Ceci conduit, dans la dimension supérieure, à une description géométrique de la forme volume sur la variété.

Abstract. Heron's formula for areas of triangles (and for other simplices) is applied in any Riemannian manifold, for simplices that are infinitesimal in a certain precise sense. This leads, in the top dimension, to a geometric description of the volume form of the manifold.

Keywords. Volume of simplices. Cayley-Menger determinant. Riemannian metric. Synthetic differential geometry.

Mathematics Subject Classification (2010). 14A25; 51K10; 51M25.

Introduction

The Greek geometers (Heron et al.) discovered a remarkable formula, expressing the area of a triangle in terms of the lengths of the three sides. Here, length and area are seen as non-negative numbers, which involves, in modern terms, formation of *absolute value* and *square root*. To express the notions and results involved without these non-smooth constructions, one can express the Heron Theorem in terms of the *squares* of the quantities in question: if $g(A, B)$ denotes the *square* of the length of the line segment given by A and B, the Heron formula says that the square of the area of the triangle ABC may be calculated by a simple algebraic formula out the three numbers $g(A,B)$, $g(A,C)$, and $g(B,C)$. Explicitly, the formula appears in (1) below. In modern terms, the formula is (except for a combinatorial constant -16^{-1}) the determinant of a certain symmetric 4×4 matrix constructed out of three numbers; see (2) below. This determinant, called the Cayley-Menger determinant, generalizes to simplices of higher dimensions, so that e.g. the square of the volume of a tetrahedron (3-simplex) $(ABCD)$ in space is given (except for a combinatorial constant) by the determinant of a certain 5×5 matrix constructed out of the six square lengths of the edges of the tetrahedron (by a formula already known by Piero della Francesca in the Renaissance).

The Heron formula is symmetric w.r.to permutations of the $k+1$ vertices of a k-simplex. Also, it does not refer to the vector space or affine structure of the ambient space.

The square lengths, square areas, square volumes etc. of the simplices can also be calculated by another well known and simple expression: namely as $(1/k!)^2$ times the Gram determinant of a certain $k \times k$ matrix constructed from the simplex, by choosing one of its vertices as origin. The Gram determinant itself expresses the square volume of the parallelepipedum spanned by k vectors in V that go from the origin to the remaining vertices.

An important difference between the two formulae is the $(k+1)!$-fold symmetry in the Heron formula, where the Gram formula is apriori only $k!$-fold symmetric, because of the special role of the chosen origin.

It is useful to think in terms of the quantities occurring as being quantities whose physical dimension is some power of length (measured in meter m, say), so that length is measured in m, area in m^2, square area in m^4, etc. Tangent vectors are not used in the following; they would have physical dimension of $m \cdot t^{-1}$ (velocity). The word *square-density* is used in any dimension. Square length, square area, and square volume are examples. The theory developed here was also attempted in my [8] (whose basis is the Gram method). I hope that the present account will be less ad hoc.

1. Cayley-Menger matrices

The basic idea for the construction of a square k-volume function goes, for the case $k = 2$, back to Heron of Alexandria (perhaps even to Archimedes); they knew how to express the square of the area of a triangle S (whether located in Euclidean 2-space or in a higher dimensional Euclidean space) in terms of an expression involving only the lengths a, b, c of the three sides:

$$area^2(S) = t \cdot (t - a) \cdot (t - b) \cdot (t - c)$$

where $t = \frac{1}{2}(a+b+c)$. Substituting for t, and multiplying out, one discovers ([3] 1.53) that all terms involving an odd number of any of the variables a, b, c cancel, and we are left with

$$area^2(S) = -16^{-1}(a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2), \qquad (1)$$

an expression that only involves the *squares* a^2, b^2 and c^2 of the lengths of the sides.

The expression in the parenthesis here may be written in terms of the determinant of a 4×4 matrix (described in (2) below). This provides a blueprint for how to generalize from 2-simplices (= triangles) to k-simplices, in terms of determinants of certain $(k + 2) \times (k + 2)$ matrices, "Cayley-Menger matrices/determinants"; they again only involve the square lengths of the $\binom{k+1}{2}$ edges of the simplex.

A k-simplex X in a space M is a $(k + 1)$-tuple of points (vertices) $(x_0, x_1, \ldots, x_k)$ in M. If $g : M \times M \to R$ satisfies $g(x, x) = 0$ and $g(x, y) = g(y, x)$ for all x and y (like a metric $\mathrm{dist}(x, y)$, or its square), one may construct a $(k+2) \times (k+2)$ matrix $C(X)$ by the following recipe:

1) Take the $(k + 1) \times (k + 1)$ *matrix* $c(X)$ *whose* ij*th entry is* $g(x_i, x_j)$.

2) Enlarge this matrix $c(X)$ *to a* $(k+2) \times (k+2)$*- matrix* $C(X)$ *by bordering it with* $(0, 1, \ldots, 1)$ *on the top and on the left.*

Both $c(X)$ and $C(X)$ have 0s down the diagonal and are symmetric, by the two assumption about g. For the case $k = 2$, $C(X)$ is depicted here, writing $g(ij)$ for $g(x_i, x_j)$ for brevity; note $g(01) = g(10)$ etc., so that the matrix is symmetric.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & g(01) & g(02) \\ 1 & g(10) & 0 & g(12) \\ 1 & g(20) & g(21) & 0 \end{bmatrix} \tag{2}$$

(The indices of the rows and columns are most conveniently taken to be $-1, 0, 1, 2$.)

This is the Cayley-Menger[1] matrix $C(X)$ for the simplex X, and its determinant is its Cayley-Menger determinant. Heron's formula then says that the value of this determinant is, modulo the "combinatorial" factor -16^{-1}, the square of the *area* of a triangle with vertices x_0, x_1, x_2, as expressed in terms of squares $g(x_i, x_j)$ of the *distances* between them. Similarly for (square-) volumes of higher dimensional simplices. Note that no coordinates are used in the construction of this matrix/determinant.

The general formula is that the square of the volume of a k-simplex is $-(-2)^{-k} \cdot (k!)^{-2}$ times the determinant of C, e.g. for $k = 1, 2$, and 3, the factors are 2^{-1}, -16^{-1}, and 288^{-1}, respectively.

Proposition 1.1. *The Cayley-Menger determinant for a k-simplex is invariant under the $(k+1)!$ symmetries of the vertices of the simplex.*

Proof. Interchanging the vertices x_i and x_j has the effect of first interchanging the ith and jth column, and then interchanging the ith and jth row of the new matrix. Each of these changes will change the determinant by a factor -1. □

2. Square volumes in coordinates

2.1 Heron's formula

We shall now work in the space R^n, with its standard metric. So the square of the distance between x and y is $\sum_{i=1}^{n}(x_i - y_i)^2$. This is the matrix product $(x-y)^T \cdot (x-y)$, where elements in R^n are identified with $n \times 1$ matrices (column matrices), and where $(-)^T$ denotes transposition of matrices. The

[1] We shall sometimes use the acronym "CM" for "Cayley-Menger".

displayed matrix product is therefore a 1×1 matrix, i.e. an element of R. For a symmetric $n \times n$ matrix G, we may more generally consider the matrix product $(x-y)^T \cdot G \cdot (x-y)$ (if G is the identity $n \times n$ matrix I, we retrieve the standard metric). Then the function $g(x,y) := (x-y)^T \cdot G \cdot (x-y)$ has the two properties $g(x,x) = 0$ and $g(x,y) = g(y,x)$, which was all we needed to describe the Cayley-Menger determinant (the g thus defined may not be a square-metric in any reasonable sense. This would require that G is positive definite; we return to this in Section 5.)

We denote by heron_G the "square volume" for k-simplices in R^n, when calculated using such G for the entries in the Cayley-Menger determinant.

2.2 Gram's formula

For a k-tuple of vectors $(y_1, \ldots, y_k)$ in R^n, one may form the Gram determinant: first form the $n \times k$ matrix Y whose k columns are the y_js. Then form the (symmetric) $k \times k$ matrix obtained as the matrix product $Y^T \cdot Y$, where Y^T denotes the transpose of Y. So the ijth entry in $Y^T \cdot Y$ is the inner product $y_i \cdot y_j$. Let us write

$$\mathrm{Gram}(Y) := \det(Y^T \cdot Y)$$

for the determinant of this $k \times k$ matrix. The significance of this determinant is that it describes the square of the volume of the parallelepipedum spanned by the k vectors y_j. Therefore the square $\mathrm{gram}(Y)$ of the volume of the simplex spanned by these vectors is smaller, it is

$$\mathrm{gram}(Y) = (k!)^{-2} \cdot \mathrm{Gram}(Y).$$

Let Y be as above, and let G be a symmetric $n \times n$ matrix. Then we may instead of $Y^T \cdot Y$ consider the (symmetric) $k \times k$ matrix given by $Y^T \cdot G \cdot Y$, and write

$$\mathrm{Gram}_G(Y) := \det(Y^T \cdot G \cdot Y),$$

thus for the $n \times n$ identity matrix I, $\mathrm{Gram}_I(Y) = \mathrm{Gram}(Y)$.

Proposition 2.1. *If an $n \times n$ matrix G can be written $G = H^T \cdot H$ for an $n \times n$ matrix H, then (G is symmetric and) we have for any $n \times k$ matrix Y that*

$$\mathrm{Gram}_G(Y) = \mathrm{Gram}_I(H \cdot Y),$$

(and hence also $\mathrm{gram}_G(Y) = \mathrm{gram}_I(H \cdot Y)$*).*

Proof. We have for $\mathrm{Gram}_G(Y)$ the following calculation

$$\det(Y^T \cdot G \cdot Y) = \det(Y^T \cdot H^T \cdot H \cdot Y) = \det((H \cdot Y)^T \cdot (H \cdot Y))$$

which is $\mathrm{Gram}(H \cdot Y)$ (i.e. $\mathrm{Gram}_I(H \cdot Y)$). □

Remark. We note that if R denotes the real numbers, then the existence of an invertible matrix H with $H^T \cdot H = G$ is equivalent to G being positive definite in the standard sense, see e.g. Proposition 6 in [10] XI.4.

2.3 Comparison formula

For R^n, it makes sense to compare the values of the Heron and Gram formulas for square volume of a k-simplex $X = (x_0, x_1, \ldots, x_k)$. For $j = 1, \ldots, k$, let y_j denote the vector $x_j - x_0 \in R^n$, and let $Y = (y_1, \ldots, y_k)$ denote the resulting $n \times k$ matrix. Let $C = C(X)$ denote the $(k+2) \times (k+2)$ matrix ((Heron-) Cayley-Menger) arising from the square distances between the vertices, as described above, and let $Y^T \cdot Y$ be the Gram $k \times k$ matrix of the simplex, likewise described above. There is a known relation between their determinants

$$-(-2)^{-k} \det(C) = \det(Y^T \cdot Y). \tag{3}$$

For a proof, see reference [4].

Note that the left hand hand side in (3) does not make use of the algebraic structure of R^n, but only on the (square-) distance function (arising from the inner product). This flexibility will be crucial when we later on consider Riemannian manifolds.

We denote the square volume of a simplex X, as calculated in terms of the Cayley-Menger matrix C, by $\mathrm{heron}(X)$, and denote the square volume of the corresponding parallelepipedum, as calculated by Gram's method, by $\mathrm{Gram}(X)$ So by dividing (3) by $(k!)^2$, we have

$$\mathrm{heron}(X) = \mathrm{gram}(X) \tag{4}$$

We described at the end of Section 2.1 how one may modify the Heron expression using a symmetric $n \times n$ matrix G, so one may ask whether the G-modified $\mathrm{heron}_G(X)$ equals $\mathrm{gram}_G(X)$? This holds if G is the identity $n \times n$ matrix, by (4).

Proposition 2.2. *Let $X = (x_0, \dots, x_k)$ be a k-simplex in R^n. If G is positive definite, in the sense that $G = H^T \cdot H$ for some square matrix H, then* $\mathrm{heron}_G(X) = \mathrm{gram}_G(X)$.

Proof. The submatrix $c(X)$ of $C(X)$ for calculating $\mathrm{heron}_G(X)$ is the $(k+1) \times (k+1)$ matrix whose i, j entry is the G-square distance between x_i and x_j, i.e. it is

$$(x_i - x_j)^T \cdot G \cdot (x_i - x_j) = (x_i - x_j)^T \cdot H^T \cdot H \cdot (x_i - x_j)$$

$$= (H \cdot (x_i - x_j))^T \cdot (H \cdot (x_i - x_j)) = (H \cdot x_i - H \cdot x_j)^T \cdot (H \cdot x_i - H \cdot x_j)$$

which is the i, j entry in the CM matrix for the simplex $H \cdot X$. We conclude that $\mathrm{heron}_G(X) = \mathrm{heron}(H \cdot X)$. By (4), $\mathrm{heron}(H \cdot X) = \mathrm{gram}(H \cdot X)$, which in turn is $\mathrm{gram}_G(X)$ by Proposition 2.1. □

Remark. In terms of physical dimensions alluded to in the Introduction: volume of a k-simplex has dimension m^k, so its square volume has dimension $(m^k)^2$; the entries $g(x_i, x_j)$ in the Cayley-Menger matrix have physical dimension m^2, and expanding its determinant, all terms are products of k copies of these entries. (The entries 0 and 1 in the top line and left column in the matrix are "pure" quantities, i.e. of dimension m^0). So the value of the determinant is of physical dimension $(m^2)^k$. The Heron formula is then meaningful in the sense that it equates quantities of dimension $(m^2)^k$ and $(m^k)^2$.

In particular, the comparison between the square volumes of a k-simplex, as calculated by Heron-Cayley-Menger and by Gram, which is a consequence of (3), is dimensionally meaningful; both have physical dimension m^{2k}.

2.4 The terms in the Cayley-Menger determinant

Given a k-simplex $X = (x_0, \dots, x_k)$ (in a space M, with a "square distance" function $g(x, y)$, as in Section 1). Consider the CM determinant $C(X)$ as described by the recipe in Section 1. The terms of this determinant have $k + 2$ factors; but two of these factors are 1 (expand after top row, and then after leftmost column, and use that the top left entry is 0). So each of the terms of the CM determinant is a product of k factors placed in a k-element pattern S in the $(k+1) \times (k+1)$- matrix $c(X)$; and S has no entries in the diagonal, since the CM matrix has 0s in the diagonal. We want to describe and classify the patterns that occur: To place a set of k chess rooks on an $m \times m$ chess board ($k \leq m$), so that no one of them can beat another one, can be expressed: no two of them are placed in the same row, and no two of them are placed in the same column. Let us for brevity call such a k-element set S of positions in an $m \times m$ matrix a *rook pattern*.

We conclude that the terms in a CM matrix for a k-simplex are named by k-element rook patterns S (containing no diagonal entries) in $\{0 \ldots, k\} \times \{0, \ldots, k\}$; the term named by such S is $\pm \prod_{(i,j) \in S} g(x_i, x_j)$.

Each such pattern S gives rise to an oriented graph with $k+1$ vertices $0, 1, \ldots k$, and with an edge from i to j ($i \neq j$) if $(i, j) \in S$. Hence this graph has k edges. Also, for every vertex i, there is a most one edge with i as domain, and at most one edge with i as codomain.

For such oriented graphs with $k+1$ vertices, there are two alternatives (mutually exclusive): 1) the graph is *singular*, in the sense that there is some closed path in the graph; 2) there is a path of length k, passing through each of the $k+1$ vertices; we call such rook-pattens and their graphs *non-singular*. Note that for a non-singular path, there are exactly two extreme vertices, and $k-1$ intermediate vertices.

Looking at the classical Heron formula (1), the three first terms are named by singular graphs, the three (really six) last terms are named by non-singular graphs.

3. Differential forms and square densities

In this Section, we work in the context of synthetic differential geometry (SDG); this is a category $\mathcal{E}$ (with suitable properties, say a topos, but less will do for the present note), together with a basic commutative ring object $R \in \mathcal{E}$, the "number line", satisfying certain axioms. In such context, one derives a notion of n-dimensional *manifold* M; this means objects which locally are diffeomorphic[2] to R^n. Since we shall only consider local issues,

[2]the maps in the category $\mathcal{E}$ are termed *smooth*, and an isomorphism in $\mathcal{E}$ is therefore termed a *diffeomorphism*

we shall use the term manifold for any object which *admits* a open inclusion $M \to R^n$ (called a *chart*), but no such chart is part of the structure of M.

In such M, one may define, for each $r = 0, 1, 2, \ldots$ a binary (reflexive symmetric) relation $M_{(r)} \subseteq M \times M$. For x and y (generalized[3]) elements of M, we write $x \sim_r y$ if $(x, y) \in M_{(r)} \subseteq M \times M$. For R^n itself, the relation $\sim_r$ may be described in terms of (generalized) elements as follows: ,

$x \sim_r y$ *iff for any* $r + 1$*-linear function* $\phi : R^n \times \ldots \times R^n \to R$*, we have*

$$\phi(x - y, \ldots, x - y) = 0. \tag{5}$$

In particular: if $x \sim_2 y$, then any trilinear $\phi : R^n \times R^n \times R^n \to R$, vanishes on $(x - y, x - y, x - y)$. (Here "linear" means "R-linear").

It can be proved in the context of SDG that the relation $\sim_r$ is preserved and reflected by local diffeomorphisms of R^n and hence, via charts from R^n, $\sim_r$ makes sense for arbitrary n-dimensional manifolds M, but is independent of the choice of chart. (That $\sim_r$ is well defined, independent of of the chart chosen, is a version of Ehresmann's theory of jets, [5].)

One has that $x \sim_r y$ implies $x \sim_{r+1} y$. We are in the present paper only interested in the case $r = 0, 1, 2$ (where $x \sim_0 x$ is equivalent to $x = y$, since on R^n, there are sufficiently many 1-linear $R^n \to R$, e.g. the n projections).

In particular we consider, for a natural number k, the object of r-infinitesimal k-simplices in M, meaning the subobject of $M \times M \times \ldots \times M$ ($k+1$ times) consisting of $k + 1$-tuples $(x_0, x_1, \ldots, x_k)$ of elements of M with $(x_i, x_j) \in M_{(r)}$ for all $i, j = 0, 1, \ldots, k$; such a $k + 1$-tuple, we shall call an r*-infinitesimal* k*-simplex*; the x_is are the *vertices* of the simplex.

For $r = 1$ and $r = 2$, we shall consider certain maps from the object of r-infinitesimal k-simplices $(x_0, \ldots, x_k)$ in M to R, namely (smooth!) maps which have the property that they vanish if $x_i = x_j$ for some $i \neq j$. For $r = 1$, combinatorial differential k forms ω have this property. (In the context of SDG, such maps are automatically alternating with respect to the $(k + 1)!$ permutations of the x_is, see [7] Theorem 3.1.5.)

[3]We use the well known "synthetic" language to express constructions in categories $\mathcal{E}$ with finite limits, in "elementwise" terms. Recall that a generalized element of an object M in a category $\mathcal{E}$ is just an arbitrary map in $\mathcal{E}$ with codomain M; see e.g. [6] II.1, [11] V.5, or [12] 1.4.

For $r = 2$, such maps have not been considered much[4], except for the case where $k = 1$, where 1-square densities g (pseudo-Riemannian metrics), in the combinatorial sense (recalled after Definition 3.3 below), are examples of such maps; for this case, we think of $g(x_0, x_1)$ as the square of the distance between x_0 and x_1. For manifolds M, we have

Proposition 3.1. *Given $g : M_{(2)} \to R$ with $g(x, x) = 0$ for all x. Then g is symmetric iff it vanishes on $M_{(1)} \subseteq M_{(2)}$.*

Proof. In a chart $M \cong R^n$, consider, for fixed x, the degree ≤ 2 part of the Taylor expansion of g around x. Then g is given as

$$g(x, y) = C(x) + \Omega(x; x - y) + (x - y)^T \cdot G(x) \cdot (x - y),$$

where $C(x)$ is a constant, Ω is linear in the argument after the semicolon, and $G(x)$ is a symmetric $n \times n$ matrix . To say that g vanishes on the diagonal $M_{(0)}$ (i.e. $g(x, x) = 0$ for all x) is equivalent to saying that $C(x) = 0$ for all x. We now compare $g(x, y)$ and $g(y, x)$; we claim

$$(x - y)^T \cdot G(x) \cdot (x - y) = (y - x)^T \cdot G(y) \cdot (y - x). \tag{6}$$

For, Taylor expanding from x the $G(y)$ on the right hand side, gives that the difference between the two sides is $(y - x) \cdot dG(x; y - x) \cdot (y - x)$ which is trilinear in $y - x$, and therefore vanishes, since $x \sim_2 y$. So we have that if C vanishes, then g is symmetric; vice versa, if g is symmetric, its restriction to $M_{(1)}$ is likewise symmetric, and (being a differential 1-form), it is alternating, so the Ω-part vanishes, which in coordinate free terms says: $g(x, y) = 0$ for $x \sim_1 y$. □

(For the number line R, $(x_0, x_1) \in R_{(2)}$ iff $(x_0 - x_1)^3 = 0$, and the map g given by $g(x_0, x_1) := (x_0 - x_1)^2$ is a map as described in the Proposition. In fact, it is the restriction of the standard "square-distance" function $R \times R \to R$.)

So we recall, respectively pose, the following definitions, corresponding to $r = 1$ and $r = 2$. Let M be a manifold.

[4]For $r = 2$ and $k = 1$, such things were in [7] 8.1 called "quadratic differential forms".

Definition 3.2. *A (combinatorial)* differential k-form *on M is an R-valued function ω on the object of 1-infinitesimal k-simplices in M, which is alternating with respect to the $(k+1)!$ permutations of the vertices of the simplex.*

So ω vanishes on simplices where two vertices are equal.

For the category $\mathcal{E}$ of affine K-schemes, combinatorial differential forms were studied and applied in [2]; here R is the scheme represented by the algebra of polynomials in one variable over K. The commutative ring representing the objects of 1-infinitesimal simplices is described explicitly. It is a version of the construction of the module of Kaehler differentials.

Definition 3.3. *A* k-square-density *on M is an R-valued function on the object of 2-infinitesimal k-simplices in M, which is symmetric with respect to the $(k+1)!$ permutations of the vertices of the simplex, and which vanishes on simplices where two vertices are equal.*

Note that for $k = 1$, Proposition 3.1 gives that 1-square densities (square lengths) g have the property that they vanish not just on $M_{(0)}$ (the diagonal), but also on $M_{(1)}$: $g(x, y) = 0$ if $x \sim_1 y$.

I apologize for the following proliferation of terminology:
1-square density = differential quadratic form = pseudo-Riemannian metric (where "differential quadratic form" was the term used in [7], Section 8.1).

3.1 k-square-densities heron_g from 1-square-densities g

Given a 1-square-density g. We shall argue that the Cayley-Menger determinants, using this g, for 2-infinitesimal simplices $(x_0, \ldots, x_k)$, define a k-square-density. We already argued (Proposition 1.1) that these determinants are symmetric: the value does not change when interchanging x_i and x_j. We have to argue for the vanishing condition required. If $x_i = x_j$, then $g(x_i, x_m) = g(x_j, x_m)$ for all m, and this implies that the ith and jth rows in the Cayley-Menger matrix are equal, which implies that the determinant is 0.

3.2 k-square-densities from differential k-forms

Essentially this is the process of *squaring* (in R) the values, so it is tempting to denote the square-density which we are aiming for, by ω^2. Precisely: we get a well defined k-square-density out of a differential k-form by a two step procedure: 1) to *extend* the given k form ω to a suitable function $\overline{\omega}$, to allow as inputs not just 1-infinitesimal k-simplices, but also also certain 2-infinitesimal k-configurations; and then 2) *squaring* $\overline{\omega}$ valuewise.

Given a combinatorial k-form ω on M. In a coordinate chart R^n, it may be expressed (as in [7] 3.1) in terms of a function $\Omega : M \times (R^n)^k \to R$ which is k-linear and alternating in the last k arguments,

$$\omega(x_0, x_1, \ldots, x_k) = \Omega(x_0; x_0 - x_1, x_0 - x_2, \ldots, x_0 - x_k). \tag{7}$$

The right hand side is defined without restrictions in the $x_0 - x_i$s. Let us denote it $\overline{\omega}$.

Proposition 3.4. *The valuewise square $\overline{\omega}^2$, when applied to 2-infinitesimal k-simplices, is a k-square density.*

Proof. It clearly vanishes if two vertices are equal, since Ω, hence $\overline{\omega}$, have this property. For the $(k+1)!$-fold symmetry: interchanging x_i and x_j (for $i, j \geq 1$) gives a sign change in the value of $\overline{\omega}$, since Ω is alternating in the last k arguments. So squaring the value gives no change. For interchange of x_0 and x_i for $i \geq 1$, a more delicate argument is needed: We shall only do the case $k = 1$. First, we have by a Taylor expansion from x_0

$$\Omega(x_1; x_0 - x_1) = \Omega(x_0; x_0 - x_1) + d\Omega(x_0; x_1 - x_0; x_0 - x_1)$$

$$+ \text{ a term } d^2\Omega(x_0; \ldots), \text{ trilinear in } x_1 - x_0.$$

The trilinear term vanishes, because $x_1 \sim_2 x_0$. Now we square, and get

$$\Omega(x_1; x_0 - x_1)^2 = \Omega(x_0; x_0 - x_1)^2 + 2 \cdot \Omega(x_0; x_0 - x_1) \cdot d\Omega(x_0; x_1 - x_0, x_0 - x_1)$$

$$+ \text{ a term } (d\Omega(x_0; \ldots))^2, \text{ quadrilinear in } x_1 - x_0.$$

The quadrilinear term vanishes because $x_1 \sim_2 x_0$, but also the term $\Omega \cdot d\Omega$ vanishes, because it is trilinear in $x_1 - x_0$. So we get

$$\Omega(x_1; x_0 - x_1)^2 = \Omega(x_0; x_0 - x_1)^2 = \Omega(x_0; x_1 - x_0)^2,$$

as desired. $\square$

We shall prove that the square density constructed is independent of the choice of chart used for constructing it. The unicity can be formulated without reference to any coordinate chart. To formulate it, let us introduce an auxiliary terminology: a function $\overline{\omega}$ from the set of $(k+1)$-tuples $(x_0, x_1, \ldots, x_k)$ with $x_0 \sim_2 x_i$ for $i = 1, \ldots, k$ we call an *extended form*, if it takes value 0 if two of its arguments are equal. Such an extended form restricts to a function on the set of 1-infinitesimal k-simplices, and hence it makes sense to say that $\overline{\omega}$ extends a given (combinatorial) differential k-form ω. We shall then prove the coordinate free assertion:

Proposition 3.5. *If two extended k-forms $\overline{\omega}$ and $\overline{\omega}'$ extend the same differential k-form ω, then $\overline{\omega}^2 = \overline{\omega}'^2$.*

Proof. We have to prove that

$$\overline{\omega}^2(x_0, x_1, \ldots, x_k) = \overline{\omega}'^2(x_0, x_1, \ldots, x_k),$$

for any 2-infinitesimal k-simplex $(x_0, x_1, \ldots, x_k)$. It suffices prove it for $M = R^n$ and with $x_0 = 0$. In this case $\overline{\omega}$ and $\overline{\omega}'$ are functions Ω and $\Omega' : D_2(n) \times \ldots \times D_2(n) \to R$ (k factors in the product). Here $D_2(n) \subseteq R^n$ has for its (generalized) elements $x \in R^n$ with $x \sim_2 0$. By the basic axiom scheme of SDG, the ring A of functions $D_2(n) \to R$ is of the form $A = A_0 \oplus A_1 \oplus A_2$, with A_0 consisting of the constant functions $R^n \to R$, A_1 of the linear functions $R^n \to R$, and A_2 of the (homogeneous) quadratic functions $R^n \to R$. This A is a graded ring (only non-zero in degrees 0,1 and 2). The ideal of functions vanishing on 0 is $A_1 \oplus A_2 \subseteq A$. So the ideal of functions $(D_2(n))^k \to R$, which vanish if at least one of its arguments is 0, is the k-fold tensor product of $(A_1 \oplus A_2)$,

$$(A_1 \oplus A_2)^{\otimes k} \subseteq A^{\otimes k}. \tag{8}$$

The ring $A^{\otimes k}$ is k-graded, with e.g. the multidegree $(1, \ldots, 1)$ consisting of the k-linear functions $(R^n)^k \to R$

By assumption, both Ω and Ω' belong to the ideal (8). The assumption that both Ω and Ω' restrict to the same differential k-form ω implies that Ω and Ω' agree in their component of multidegree $(1, \ldots, 1)$ (this component being the coordinate expression of ω). Thus $\Omega' = \Omega + \theta$, with θ of multidegree $\geq (1, \ldots, 1)$ and of total degree $\geq k+1$. The required equation is, in these terms, that $(\Omega+\theta)^2 = \Omega^2$, and this is a simple "counting degrees"-argument in the k-graded ring A^k:

$$(\Omega+\theta)^2 = \Omega^2 + 2\Omega\cdot\theta + \theta^2. \tag{9}$$

Here, θ^2 has total degree $\geq 2\cdot(k+1) \geq 2k+1$, which is 0 since A^k is 0 in total degrees $> 2k$; and θ is a linear combination of terms of multidegree of the form $(1,1,\ldots,1+p,\ldots 1)$ for $p \geq 1$, so $\theta\cdot\omega$ is a linear combination of terms of multidegree

$$(1,1,\ldots,1+p,\ldots,1) + (1,1,\ldots,1,\ldots,1) = (2,2,\ldots,2+p,\ldots,2)$$

which is of total degree $2k+p \geq 2k+1$. So the two last terms in (9) are 0, and this proves the Proposition. □

Because of the Proposition, there is a well-defined "squaring" process, leading from differential k-forms to k-square-densities on a manifold M: extend the form ω, and square the result. It is natural to denote this square density by ω^2, with the understanding that it means $\overline{\omega}^2$ for any extended form $\overline{\omega}$, extending ω.

4. Variable metric tensor

We consider a manifold M. A finite sequence of points

$$\tilde{x} = (x_0, x_1, \ldots, x_k)$$

in M, which are consecutive 2-neigbours, i.e. $x_i \sim_2 x_{i+1}$ for $i = 0, \ldots k-1$, we shall for simplicity call a *path* of length k. We call $x_0, \ldots, x_k$ a *closed* path if $x_0 = x_k$. If M is provided with a pseudo-Riemannian metric g, we shall, for a path $\tilde{x}$ of length k, be interested in products of the form

$$g(\tilde{x}) := g(x_0,x_1)\cdot g(x_1,x_2)\cdot\ldots\cdot g(x_{k-1},x_k). \tag{10}$$

For any chart $M \subseteq R^n$, the metric g is described by a variable "metric tensor" G: i.e. by a family of (symmetric) $n\times n$ matrices $G(x)$, for $x \in M$, and varying smoothly with x; more precisely, G is a map in $\mathcal{E}$ from M to the (finite dimensional) vector space W of $n \times n$ matrices over R. And g is expressed in terms of G: for $x \sim_2 y$ in M

$$g(x,y) = (x-y)^T \cdot G(x) \cdot (x-y) \tag{11}$$

(which equals $(y-x)^T \cdot G(y) \cdot (y-x)$ by (6)). The displayed product (10) is then calculated as the iterated matrix product

$$\begin{aligned} g(\tilde{x}) &= G(\tilde{x}) = \\ &= [(x_0 - x_1) \cdot G(x_0) \cdot (x_0 - x_1)] \cdot [(x_1 - x_2)^T \cdot G(x_1) \cdot (x_1 - x_2)] \cdot \\ &\qquad \cdot \ldots \cdot [(x_{k-1} - x_k)^T \cdot G(x_{k-1}) \cdot (x_{k-1} - x_k)]. \end{aligned} \tag{12}$$

(The square brackets are only inserted for readability; mathematically, they are redundant, by associativity of matrix multiplication.)

Lemma 4.1. *If $\tilde{x}$ is a closed path, then $g(\tilde{x}) = 0$.*

Proof. It suffices to prove this in a chart. In a given chart, g is represented by symmetric matrices $G(x)$, as described above. Using the charts, let a_i be the vector $x_i - x_{i-1}$ for $i = 1, \ldots, k$. Since $x_k = x_0$, we have $a_1 + \ldots + a_k = 0$, so a_k is a linear combination of the a_is, for $i < k$. Then the last factor $[a_k^T \cdot G(x_{k-1}) \cdot a_k]$ in the above product is a linear combination of terms $a_i^T \cdot G(x_{k-1}) \cdot a_j$ with $i < k$ and $j < k$. But among the remaining factors in the product for $G(\tilde{x})$, we have $a_i \cdot G(x_{i-1}) \cdot a_i$, so altogether, a_i appears trilinearly in the corresponding term, and so vanishes since $a_i \sim_2 0$. $\square$

We shall derive some further properties for products of the form (10). With notation as in the previous proof, the product (12) takes the form

$$g(\tilde{x}) = [a_1^T \cdot G(x_0) \cdot a_1] \cdot \ldots \cdot [a_k^T \cdot G(x_{k-1}) \cdot a_k].$$

We write $\overline{G}(\tilde{x})$ for the similar expression, but with all the $G(x_i)$s replaced by $G(x_0)$ where x_0 is the *first* vertex of the path $\tilde{x}$. If $\tilde{x}$ is a path of length k, we get a path of length $k-1$ by omitting the first of the vertex of the path. Let us denote this truncated path by $|\tilde{x}$. Thus in $\overline{G}(|\tilde{x})$, the constant matrix used is $G(x_1)$ because the first vertex of $|\tilde{x}$ is x_1.

Lemma 4.2. *For any path $\tilde{x}$, $g(\tilde{x}) = \overline{G}(\tilde{x})$.*

Proof. By induction of the length k of the path. The assertion is clearly true for $k = 1$. Assume that it holds for $k - 1$. We use the expression (12) for $g(\tilde{x})$. Then

$$g(\tilde{x}) = (x_0 - x_1)^T \cdot G(x_0) \cdot (x_0 - x_1) \cdot \overline{G}(|\tilde{x}), \tag{13}$$

by the induction assumption, used for the path $|\tilde{x}$. So $\overline{G}(|\tilde{x})$ is a matrix product containg the matrix $G(x_1)$ as a factor $k - 1$ times. By Taylor expansion of the function $G : M \to W$, from x_0 in the direction $x_1 - x_0$, we get

$$G(x_1) = G(x_0) + dG(x_0; x_1 - x_0) + \tfrac{1}{2} d^2 G(x_0; x_1 - x_0, x_1 - x_0)$$

in the vector space W of $n \times n$ matrices. In each of these factors in $\overline{G}(|\tilde{x})$, we substitute the Taylor expansion exhibited; and then multiply by $(x_0 - x_1)^T \cdot G(x_0) \cdot (x_0 - x_1)$ to arrive at (13). However, this latter factor is quadratic in $(x_0 - x_1)$. Since $x_0 \sim_2 x_1$, all terms in $\overline{G}(|\tilde{x})$ containing a factor linear or quadratic in $x_1 - x_0$, like $dG(x_0; x_1 - x_0)$, get annihilated by being multiplied by $(x_0 - x_1)^T \cdot G(x_0) \cdot (x_0 - x_1)$, since then in whole product, $x_1 - x_0$ appears in a trilnear way and $x_0 \sim_2 x_1$. So now the matrices $G(x_1)$ have been replaced by $G(x_0)$, and then we have $\overline{G}(\tilde{x})$. This proves the Lemma. □

The same argument, using truncation in the other end (the vertex x_k) of the path, gives that one may also uniformly use $G(x_k)$ instead of the varying $G(x_i)$s. In fact, we may equally well pick any fixed x_j instead of either $G(x_0)$ or $G(x_k)$. For, split the path into two paths $(x_0, \ldots, x_j)$ and $(x_j, \ldots, x_k)$, and pick for the first of these two paths its end vertex x_j, and for the second of the two paths, pick its initial vertex x_j; then use the Lemma 4.2 for paths of length j and of length $k - j$, respectively. Summarizing, the Lemma may be strengthened, and formulated in a more complete way:

Lemma 4.3. *For any path $\tilde{x}$ of length k, for any fixed $j = 1, \ldots, k$, and for any chart, we have $G(\tilde{x}) = \overline{G}(\tilde{x})$, where the $G(x)$ is the expression for g in the chart, and where $\overline{G}$ uses $G(x_j)$ uniformly, (i.e. in (12), all the $G(x_i)$ are replaced by $G(x_j)$).*

Remark. The argument simplifies for the case of "restricted" 2-infinitesimal k-simplices, as considered by [1], since there one has that each of the individual $g(x_i, x_j)$ in a simplex $(x_0, \ldots, x_k)$ may be calculated by using $G(x_0)$.

In preliminary versions of the present note, I only considered the restricted simplices; but the value of the Heron-Cayley-Menger formula on such simplices is probably not enough for characterizing the volume form, which is our aim.

Now recall from Subsection 2.4 that the terms in the CM determinant $C(X)$ for a k-simplex $X = (x_0, \ldots, x_k)$ are named by k-element rook patterns S in the $(k+1) \times (k+1)$ matrix $c(X)$ of square-distances $g(x_i, x_j)$, and that each of these rook patterns gives rise to a graph. Now we concentrate on 2-infinitesimal k-simplices. Given a rook pattern whose graph is singular, i.e. contains a closed path. Then the corresponding product of the $g(x_i, x_j)$s is 0, by Lemma 4.1. So we need only be interested in rook patterns S whose corresponding graph is a non-singular. The corresponding product of k terms is, possibly after renumbering, of the form as displayed in (10). And for such, Lemma 4.2 implies that we, in a chart, may calculate $g(\tilde{x}) = G(\tilde{x})$ by using instead $\overline{G}(\tilde{x})$, that is, we may uniformly use $G(x_0)$ (or any other fixed $G(x_j)$) instead of the varying $G(x_i)$s.

So in the CM determinant for a 2-infinitesimal k-simplex, the terms are named by non-singular rook patterns and so each of the terms may be calculated by expressions (10) for paths; and in any chart, this expression my be calculated as asserted in Lemma 4.3, by picking arbitrarily any x_j in the path. But each non-singular paths of length k in a k simplex passes through all the vertices of the simplex, say x_0.

We conclude that for calculating the square volume of a 2-infinitesimal k simplex $X = (x_0, \ldots, x_k)$, all the factors $g(x_i, x_j)$ in all the terms in the CM determinant may, in any chart, may be replaced by $(x_i - x_j)^T \cdot G(x_0) \cdot (x_i - x_j)$.

From the Lemma, we conclude, for any variable metric tensor G:

Proposition 4.4. *Given a 2-infinitesimal k-simplex* $X = (x_0, \ldots, x_k)$. *Then* $(\mathrm{heron}_g(X) =) \, \mathrm{heron}_G(X) = \mathrm{heron}_{G(x_0)}(X)$.

For, any non-singular path of length k contains all the $k+1$ vertices, in particular they all contain x_0 (although not necessarily as first or last vertex), so we may, by Lemma 4.3, for each non-singular path, pick the constant matrix $G(x_0)$ for the calculation.

Combining with the comparison in (3), we get

Proposition 4.5. *Given a coordinate patch $M \subseteq R^n$ and a 2-infinitesimal k-simplex $X = (x_0, x_1, \ldots, x_k)$ in M. Then*

$$\mathrm{heron}_g(X) = \mathrm{gram}_{G(x_0)}(X).$$

5. Volume form

The volume form is a differential n-form that may be defined on an n-dimensional manifold M equipped with a *positive definite* metric g. (Since we only consider here open subspaces $M \subseteq R^n$, we need not mention the usual orientability requirement for M.) We take "positive definite" in the sense, which for individual symmetric matrices was described in the Remark at the end of Subsection 2.2; but now, for variable metric tensor $G : M \to W$, we require there exists another $H : M \to W$ in $\mathcal{E}$ with $G(x) = H^T(x) \cdot H(x)$ for all $x \in M$. (This is a property of g which does not depend on the chart.) However, the smoothness of such H, which implicitly is assumed here, is for the real C^∞-case with positive definite G, probably not automatic.)

Recall from the last lines of Section 3 the notation ω^2 for the square k-volume constructed out of a differential k-form ω:

Theorem 5.1. *Assume that g is a Riemannian metric on an n-dimensional manifold $M \subseteq R^n$. Then there exists on M a differential n-form ω such that* heron_g *and ω^2 agree on all 2-infinitesimal n-simplices; such ω deserves the name a* volume form *for g.*

Proof. Since the data and assertions in the statement do not depend on the choice of a coordinate chart, it suffices to prove the assertion in an arbitrary chart. So assume that M is identified with an open subspace of R^n and that G is given in terms of the positive definite $n \times n$ matrices $G(x)$ (for $x \in M$) (i.e. $G : M \to W$), with $G(x) = H(x)^T \cdot H(x)$ for all $x \in M$, with $H : M \to W$ smooth. Now consider the extended n-form $\overline{\omega}$, whose value on a 2-infinitesimal n-simplex $X = (x_0, \ldots, x_n)$ is given by the by the formula

$$\overline{\omega}(X) := \frac{\det(H(x_0))}{n!} \cdot \det(Y) \tag{14}$$

where Y denote the $n \times n$ matrix with $y_i := x_i - x_0$ as its ith column. Now $\det G(x_0) = \det(H(x_0))^2$ by the product and transposition rules for determinants. Therefore squaring the defining equality (14) for $\overline{\omega}$ gives

$$\overline{\omega}^2(X) = \frac{\det G(x_0)}{n!^2} \cdot (\det Y)^2 = \frac{1}{n!^2} \det(Y^T \cdot G(x_0) \cdot Y) \qquad (15)$$

for any 2-infinitesimal n-simplex $X = (x_0, \ldots, x_n)$ using again the product rule and transposition rules for determinants. By definition of Gram, the equation continues

$$= \frac{1}{n!^2} \mathrm{Gram}_{G(x_0)}(X) = \mathrm{heron}_{G(x_0)}(X) = \mathrm{heron}_G(X),$$

using the Heron-Gram comparison Proposition 2.2 and Proposition 4.5. This proves the existence of the claimed differential n-form. □

Since $\det G(x_0) = \det(H(x_0)^T \cdot H(x_0)) = \det(H(x_0))^2$ by the product rule for determinants, $\det H(x_0)$ is a square root of $\det G(x_0)$, so except for the ambiguity of square roots, the formula derived here for "the" volume form may be written in the familiar form

$$\frac{\sqrt{\det(G(x_0))}}{n!} \cdot \det(y_1, \ldots, y_n).$$

We would like to have a uniqueness statement for volume form. This requires more structure or assumptions on the basic ring R, namely a positivity notion such that an invertible element in R is positive iff it is a square iff it is a square of a positive element.

Also, one would require that M is oriented, in the sense that there is given an n-form δ on M such that every other n-form on M is of the form $f \cdot \delta$ for a unique $f : M \to R$; this is redundant with the simplifying assumption we have made that M is an open subspace of R^n (where determinant formation provides the desired δ).

Under these circumstances, one may prove that there among the volume forms on M, there is a unique one of the form $f \cdot \delta$ with $f : M \to R$ positive, (meaning that f has only positive values).

References

[1] F. Bár, Affine Connections and second-order affine structures, Cahiers de Top. et Géom. Diff. Cat. 63, 35-58 (2022).

[2] L. Breen and W. Messing, Combinatorial differential forms, Advances in Mathematics 164, 203-282 (2001).

[3] H.S.M. Coxeter, *Introduction to Geometry*, 2nd ed. Wiley 1969.

[4] M. Cretin, Déterminants de Cayley-Menger, `http://math.univ-lyon1.fr/~cretin/OralAlgebre/Cayley-Menger.pdf` (accessed Nov. 2021).

[5] C. Ehresmann, Les prolongements d'une variété différentiable, in *Atti IV Cong. Uni. Mat. Italiana*, 1951, or in *Oeuvres Complètes et Commentées*, Part I (Amiens 1984).

[6] A. Kock, *Synthetic Differential Geometry*, 2nd ed. , London Math. Soc. Lecture Note Series 333, Cambridge University Press 2006 (or 1st ed. 1981).

[7] A. Kock, *Synthetic Geometry of Manifolds*, Cambridge Tracts in Math. 180, Cambridge University Press 2010.

[8] A. Kock, Volume form as volume of infinitesimal simplices, arXiv:CT/0006008 (2000).

[9] A. Kumpera and D. Spencer, *Lie Equations* vol. I, Annals of Math. Studies 73, Priceton University Press 1972.

[10] S. Mac Lane and G. Birkhoff, *Algebra*, Macmillan 1967.

[11] S. Mac Lane and I. Moerdijk, *Sheaves in Geometry and Logic*, Springer Universitext, 1992.

[12] C. McLarty, *Elementary Categories, Elementary Toposes*, Oxford Logic Guides 21, 1995.

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BI-INITIAL OBJECTS AND BI-REPRESENTATIONS ARE NOT SO DIFFERENT

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Résumé. Nous introduisons un foncteur $\mathcal{V}\colon \mathsf{DblCat}_{\mathsf{h,nps}} \to \mathsf{2Cat}_{\mathsf{h,nps}}$ qui extrait d'une double catégorie une 2-catégorie dont les objets et les morphismes sont respectivement les morphismes verticaux et les carrés. Nous donnons une caractérisation des bi-représentations d'un pseudo-foncteur normalisé $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ comme objets doubles bi-initiaux dans la double catégorie $\mathbb{el}(F)$ des éléments de F, ou de manière équivalente comme objets bi-initiaux d'une certaine forme dans la 2-catégorie $\mathcal{V}\mathbb{el}(F)$ des *morphismes* de F. Bien que cela ne soit pas vrai en général, dans le cas d'une 2-catégorie $\mathbf{C}$ qui admet les tenseurs par la catégorie $\mathbb{2} = \{0 \to 1\}$ et d'un foncteur F qui préserve ces tenseurs, nous montrons qu'une bi-représentation de F est précisément un objet bi-initial dans la 2-catégorie $\mathbf{el}(F)$ des *éléments* de F. Nous appliquons ces résultats aux bi-adjonctions et aux bi-limites pondérées.

Abstract. We introduce a functor $\mathcal{V}\colon \mathsf{DblCat}_{\mathsf{h,nps}} \to \mathsf{2Cat}_{\mathsf{h,nps}}$ which extracts from a double category a 2-category whose objects and morphisms are the vertical morphisms and squares. We give a characterisation of bi-representations of a normal pseudo-functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ in terms of double bi-initial objects in the double category $\mathbb{el}(F)$ of elements of F, or equivalently as bi-initial objects of a special form in the 2-category $\mathcal{V}\mathbb{el}(F)$ of *morphisms* of F. Although not true in general, in the special case where the

2-category $\mathbf{C}$ has tensors by the category $\mathbb{2} = \{0 \to 1\}$ and F preserves those tensors, we show that a bi-representation of F is then precisely a bi-initial object in the 2-category $\mathbf{el}(F)$ of *elements* of F. We give applications of this theory to bi-adjunctions and weighted bi-limits.

Keywords. Bi-representations, (double) bi-initial objects, bi-adjunctions, weighted bi-limits, pseudo-commas.

Mathematics Subject Classification (2020). 18N10, 18A05, 18A30, 18A40, 18A25

1. Introduction

In ordinary category theory, properties of a categorical object are often formulated as questions of *representability* of a *presheaf*. By a presheaf, we mean a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$, where $\mathcal{C}$ is a category and Set is the category of sets and functions. A representation of a presheaf comprises the data of an object $I \in \mathcal{C}$ together with a natural isomorphism $\mathcal{C}(-, I) \overset{\cong}{\Rightarrow} F$. In particular, this gives isomorphisms of sets $\mathcal{C}(C, I) \cong FC$, for each object $C \in \mathcal{C}$. A classical theorem, which we shall refer to as the "Representation Theorem", establishes that a presheaf F has a representation precisely when its *category of elements* has an initial object; see for example [13, Proposition III.2.2] or [15, Proposition 2.4.8][1]. This category of elements of F is defined as the slice category $\{*\} \downarrow F$, where $\{*\}$ denotes the singleton set.

Two main examples of properties that can be rephrased in terms of representability are the existence of limits and adjoints for a functor. Indeed, asking that a functor $F\colon \mathcal{I} \to \mathcal{C}$ admits a limit amounts to asking whether the presheaf $[\mathcal{I}, \mathcal{C}](\Delta(-), F)\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ has a representation. Therefore, by the Representation Theorem, this is equivalent to requiring the presence of a terminal object in the slice category $\Delta \downarrow F$ of cones over F. Similarly, the existence of a right adjoint to a functor $L\colon \mathcal{C} \to \mathcal{D}$ may equivalently be reformulated as the existence of a representation of the presheaf $\mathcal{C}(L-, D)\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$, for each object $D \in \mathcal{D}$, or equivalently of the existence of a terminal object in the slice category $L \downarrow D$, for each object $D \in \mathcal{D}$.

[1]Riehl defines the category of elements by hand along with a projection functor to $\mathcal{C}$, rather than $\mathcal{C}^{\mathrm{op}}$; therefore, representations correspond to terminal objects in this setting.

In passing from ordinary categories to 2-categories, we may seek to elevate discussions of representations of ordinary presheaves to their 2-dimensional counter-parts. By a 2-dimensional presheaf, we mean a normal pseudo-functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$, where $\mathbf{C}$ is a 2-category and $\mathbf{Cat}$ is the 2-category of categories, functors, and natural transformations. The data of a 2-dimensional representation is once more an object $I \in \mathbf{C}$, but when it comes to comparing the *categories* $\mathbf{C}(C, I)$ and FC we may retain the idea that this comparison is mediated by an isomorphism of categories, or we may require only the presence of an equivalence of categories. The former choice leads to the notion of a 2*-representation*, while the latter leads to the more general notion of a *bi-representation*.

Recall that an object I in a category $\mathcal{C}$ is initial if we have an isomorphism of sets $\mathcal{C}(I, C) \cong \{*\}$ for all objects $C \in \mathcal{C}$. If we wish to formulate the 2-dimensional definition analogously for an object I in a 2-category $\mathbf{C}$, as before we now have the option of retaining the idea that the universal property should be governed by an isomorphism of categories $\mathbf{C}(I, C) \cong \mathbb{1}$, for all objects $C \in \mathbf{C}$, where $\mathbb{1}$ is the terminal category, or instead asking that the universal property is governed by an equivalence of categories. The former requirement leads to the notion of a 2*-initial object*, while the latter leads to the more general notion of a *bi-initial object*.

Now that the players are ready the game is afoot. The question underpinning the most general 2-dimensional version of the Representation Theorem is this:

Question. Can bi-representations of a normal pseudo-functor F be characterised as *certain* bi-initial objects in *some* 2-category?

As a first guess, based on the Representation Theorem, we might expect that the 2*-category of elements* $\mathbf{el}(F)$ *of* F would be the correct setting for an affirmative answer. The 2-category $\mathbf{el}(F)$ is defined as the *pseudo-slice* 2-category $\mathbb{1} \downarrow F$, where by pseudo-slice we mean a relaxation of the slice 2-category where the triangles of morphisms commute up to a general 2-isomorphism, rather than an identity.

Although we hate to disappoint the reader, to see that this is not the case we will turn our interest to a specific kind of bi-representation. Generalising ordinary limits, bi-representations of the 2-presheaf $[\mathbf{I}, \mathbf{C}](\Delta(-), F)$, known as *bi-limits*, were first introduced by Street in [18, 19] and further studied by

Kelly in [9]. The comparatively stronger special case – 2-representations of the above 2-presheaf, known as *2-limits* – had previously been introduced, independently, by Auderset [1] and Borceux-Kelly [2], and was further developed by Street [17], Kelly [8,9] and Lack in [11]. As $\mathbf{el}([\mathbf{I}, \mathbf{C}](\Delta(-), F))$ is the opposite of the pseudo-slice 2-category $\Delta \downarrow F$ of cones over F, the question now becomes whether bi-limits may be characterised as bi-terminal objects in the pseudo-slice 2-category of cones.

Unfortunately, as the authors show in [3], such a characterisation is not possible in general. The failure stems from the fact that the data of a bi-limit is not wholly captured by a bi-terminal object in the pseudo-slice 2-category of cones (see [3, §5]). In fact, such a failure is actually illustrated by an example of a 2-terminal object in a slice 2-category of cones which is not a 2-limit; see [3, Counter-example 2.12].

We have thus eliminated our first guess from the possible affirmative answers to our question above. To cast further doubt on any positive resolution, correct characterisations of 2-limits as some form of 2-dimensional terminal objects that are present in the literature are all phrased in the language of *double categories*; such results are explored by Grandis [4], Grandis-Paré [5,6], and Verity [20]. These results may all be seen to share the following approach: the slice 2-category of cones does not capture enough data to successfully characterise 2-limits, and so instead more data must be necessarily added in the form of a slice double category of cones. Indeed, in [6] Grandis and Paré write:

> "On the other hand, there seems to be no natural way of expressing the 2-dimensional universal property of weighted (strict or pseudo) limits by terminality in a 2-category."

The state of the art thus seems to suggest that our question admits no positive answer in general. However, our main contribution in response to Grandis and Paré above, is a successful and purely 2-categorical characterisation of 2-limits as certain 2-terminal objects in a "shifted" slice 2-category of cones. In fact, we obtain this result as an application of a purely 2-categorical formulation of a generalisation of the Representation Theorem in the case of bi-representations. To do this, we extend the results of Grandis, Paré, and Verity to general bi-representations of normal pseudo-functors, and obtain in this fashion a double-categorical characterisation of bi-representations in terms of "bi-type" double-initial objects. From this work and some new methods we are able to extract our results.

Let us explore these results in greater detail. Recall that a double category has two sorts of morphisms between objects – the *horizontal* and *vertical* morphisms – and 2-dimensional morphisms called *squares*. So as to distinguish double categories from 2-categories, we will always be careful to name the former by double-struck letters $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, ... whereas the latter will always appear as named by bold letters $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, ...

A 2-category $\mathbf{A}$ can always be seen as a horizontal double category $\mathbb{H}\mathbf{A}$ with only trivial vertical morphisms. This construction extends functorially to an assignment on normal pseudo-functors $F \mapsto \mathbb{H}F$. Associated to each such normal pseudo-functor is a *double category of elements* $\mathbb{el}(F)$ given by the pseudo-slice double category $\mathbb{1} \downarrow\downarrow \mathbb{H}F$, where these pseudo-slices are double-categorical analogues of pseudo-slice 2-categories. Furthermore, we introduce a new notion of *double bi-initial* objects I in a double category $\mathbb{A}$; objects $I \in \mathbb{A}$ for which the projection $I{\downarrow\downarrow}\mathbb{A} \to \mathbb{A}$ is given by an appropriate equivalence of double categories. Note that, in the case of *double-initial* objects as defined by Grandis and Paré in [5, §1.8], the projection is required to be only an isomorphism. With these notions in hand, we are able to formulate the following double categorical characterisation of bi-representations. This appears as the first part of our main theorem, Theorem 6.8.

Theorem A. *Let* $\mathbf{C}$ *be a* 2*-category, and* $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ *be a normal pseudo-functor. The following statements are equivalent.*

(i) The normal pseudo-functor F *has a bi-representation* (I, ρ)*.*

(ii) There is an object $I \in \mathbf{C}$ *together with an object* $i \in FI$ *such that* (I, i) *is double bi-initial in* $\mathbb{el}(F)$*.*

By applying this result to the 2-presheaf $[\mathbf{I}, \mathbf{C}](\Delta(-), F)$ for a given normal pseudo-functor $F\colon \mathbf{I} \to \mathbf{C}$, we derive a generalisation in Corollary 7.22 of results by Grandis, Paré, and Verity, characterising bi-limits as double bi-terminal objects in the pseudo-slice double category $\Delta \downarrow\downarrow F$. This follows from the fact that the double category $\mathbb{el}([\mathbf{I}, \mathbf{C}](\Delta(-), F))$ is isomorphic to the horizontal opposite of the pseudo-slice double category $\Delta \downarrow\downarrow F$.

We now aim to extract a fully 2-categorical statement from Theorem A above. For this, it is enough to characterise double bi-initial objects in a double category $\mathbb{A}$ as *certain* bi-initial objects in *some* 2-category. A first guess for the 2-category in question would be given by the underlying horizontal 2-category $\mathbf{H}\mathbb{A}$ of objects, horizontal morphisms, and squares with trivial vertical boundaries of $\mathbb{A}$. However, the general vertical structure of the double category $\mathbb{A}$ is not captured by this operation, and therefore the 2-category $\mathbf{H}\mathbb{A}$ alone does not suffice for our purposes. To remedy this issue, we introduce a functor $\mathcal{V}$ which extracts from a double category $\mathbb{A}$ a 2-category $\mathcal{V}\mathbb{A}$ whose objects and morphisms are the vertical morphisms and squares of $\mathbb{A}$, respectively. This captures precisely the additional data that was lacking in $\mathbf{H}\mathbb{A}$ in our application, and allows us to prove the below result. In fact, the functor $\mathbf{H}$ is as a retract of $\mathcal{V}$ and so we may leverage $\mathcal{V}$ alone to characterise double bi-initial objects. The below appears as Theorem 5.8 in the paper.

Theorem B. *Let $\mathbb{A}$ be a double category, and $I \in \mathbb{A}$ be an object. The following statements are equivalent.*

(i) The object I is double bi-initial in $\mathbb{A}$.

(ii) The object I is bi-initial in $\mathbf{H}\mathbb{A}$ and the vertical identity e_I is bi-initial in $\mathcal{V}\mathbb{A}$.

(iii) The vertical identity e_I is bi-initial in $\mathcal{V}\mathbb{A}$.

As a direct application of this result to the double category of elements $\mathbb{el}(F)$ of a normal pseudo-functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$, we obtain our fully 2-categorical characterisation of bi-representations. Note that the underlying horizontal 2-category of $\mathbb{el}(F)$ is precisely the 2-category of elements $\mathbf{el}(F)$ of F, but new here is $\mathcal{V}\mathbb{el}(F)$ which we refer to as the *2-category of morphisms of F*, denoted by $\mathbf{mor}(F)$. Indeed, while the objects in $\mathbf{el}(F)$ are pairs (C, x) of an object $C \in \mathbf{C}$ and an object $x \in FC$, the objects of $\mathbf{mor}(F)$ are pairs (C, α) of an object $C \in \mathbf{C}$ and a morphism $\alpha\colon x \to y$ in FC, which justifies the terminology. The following result extends Theorem A and appears as the second part of our main theorem, Theorem 6.8.

Theorem C. *Let $\mathbf{C}$ be a 2-category, and $F\colon \mathbf{C}^{\mathrm{op}} \to$ **Cat** be a normal pseudo-functor. The following statements are equivalent.*

(i) The normal pseudo-functor F has a bi-representation (I, ρ).

(ii) *There is an object $I \in \mathbf{C}$ together with an object $i \in FI$ such that (I, i) is bi-initial in $\mathbf{el}(F)$ and (I, id_i) is bi-initial in $\mathbf{mor}(F)$.*

(iii) *There is an object $I \in \mathbf{C}$ together with an object $i \in FI$ such that (I, id_i) is bi-initial in $\mathbf{mor}(F)$.*

The equivalence of (i) and (iii) in the above theorem gives a satisfying answer to our original question. In particular, to respond Grandis and Paré, we specialise the above theorem to the case of bi-limits to see that bi-limits are equivalently certain types of bi-terminal objects in a 2-category whose objects are given by the *morphisms* of cones – known as modifications – as we will see in Corollary 7.22. Thus the counter-examples of [3] for bi-limits show the presence of $\mathbf{mor}(F)$ in (ii) is necessary in general.

Although the correct characterisation of bi-limits in a 2-category $\mathbf{C}$ above depends on taking morphisms of cones as objects, in the presence of *tensors* in $\mathbf{C}$ these can be simply seen as cones whose summit is a tensor by the category $\mathbb{2} = \{0 \to 1\}$. In the case of 2-limits, Kelly observed in [9, §3] that the presence of tensors by $\mathbb{2}$ causes the 1-dimensional aspect of the universal property of a 2-limit to imply the 2-dimensional aspect. As a consequence, we showed in [3, Proposition 2.13] that a 2-limit is precisely a 2-terminal object in the slice 2-category of cones under such a hypothesis.

This result is part of a far more general framework and we shall approach this in parts. A double categorical analogue of tensors by $\mathbb{2}$ is given by the notion of *tabulators*[2] of vertical morphisms; these are defined by Grandis and Paré in [5, §5.3] as double limits of vertical morphisms seen as double functors. By Theorem A, bi-representations correspond to double bi-initial objects in a certain double category, and it is at this level that we seek a simplification of Theorem B in the presence of tabulators. This is the content of Theorem 5.11.

Theorem D. *Let $\mathbb{A}$ be a double category with tabulators, and $I \in \mathbb{A}$ be an object. Then the following statements are equivalent.*

(i) *The object I is double bi-initial in $\mathbb{A}$.*

[2] In fact these notions are somehow dual, but our applications all involve the horizontal double category associated to the opposite of a 2-category and so tabulators there coincide with tensors by $\mathbb{2}$.

(ii) The object I is bi-initial in $\mathbf{H}\mathbb{A}$.

We now aim to simplify the characterisation of bi-representations given in Theorem C when the 2-category $\mathbf{C}$ has tensors by $\mathbb{2}$, which can be seen as tabulators in the double category $\mathbb{H}\mathbf{C}^{\mathrm{op}}$. For this simplification, we further need the normal pseudo-functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ to preserve these tensors so that overall the double category of elements $\mathbb{el}(F)$ admits tabulators. Although we were not able to use the 2-category of elements $\mathbf{el}(F)$ to give an answer to our question in general, in this special case we may apply Theorem D to recover the following verbatim translation of the Representation Theorem to the 2-categorical setting, which appears as Theorem 6.15.

Theorem E. *Let $\mathbf{C}$ be a 2-category with tensors by $\mathbb{2}$, and $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a normal pseudo-functor which preserves these tensors. The following statements are equivalent.*

(i) The normal pseudo-functor F has a bi-representation (I, ρ).

(ii) There is an object $I \in \mathbf{C}$ together with an object $i \in FI$ such that (I, i) is bi-initial in $\mathbf{el}(F)$.

This applies to the case of bi-limits, and we formulate in Corollary 7.25 a more general version of [3, Proposition 2.13]: a bi-limit is precisely a bi-terminal object in the pseudo-slice 2-category of cones when the ambient 2-category admits tensors by $\mathbb{2}$. This application provides the promised proof of [3, Proposition 5.5].

While we have only mentioned the case of bi-limits so far, in this paper the different theorems characterising bi-representations are first specialised to the case of *weighted bi-limits*, which were introduced by Street [17] and Kelly [9]. The cone of a weighted limit is of a special shape, determined by the *weight* – a normal pseudo-functor W taking values in $\mathbf{Cat}$ – and a bi-limit can be seen as a weighted limit with *conical* weight $W = \Delta\mathbb{1}$, i.e., a constant weight at the terminal category. More still, when the weight is conical the pseudo-slice of cones is isomorphic to the opposite of the pseudo-slice of *weighted cones*. Since weighted bi-limits can also be seen as bi-representations of a normal pseudo-functor of a special kind, we also obtain characterisations in Theorems 7.19 and 7.21 of weighted bi-limits in terms of double bi-initial and bi-initial objects. From these we extract the characterisations of bi-limits in terms of double bi-terminal and bi-terminal objects mentioned above.

Another application of the Representation Theorem is to the existence of a right adjoint to a given functor. Going one dimension up, we can define an analogous notion of *bi-adjunction* between 2-categories $\mathbf{C}$ and $\mathbf{D}$. A bi-adjunction comprises the data of a pair of opposed normal pseudo-functors $L\colon \mathbf{C} \to \mathbf{D}$ and $R\colon \mathbf{D} \to \mathbf{C}$ together with a pseudo-natural equivalence $\mathbf{D}(L-,-) \overset{\simeq}{\Rightarrow} \mathbf{C}(-,R-)$. In order to apply our main results to the existence of a right bi-adjoint to a given normal pseudo-functor, there is first the delicate matter of reformulating such a question in terms of bi-representations.

Theorem 7.3 states that a normal pseudo-functor $L\colon \mathbf{C} \to \mathbf{D}$ has a right bi-adjoint if and only if there is a bi-representation of the normal pseudo-functor $\mathbf{D}(L-, D)$ for each object $D \in \mathbf{D}$. This shows that the pseudo-naturality of $\mathbf{D}(L-,-) \overset{\simeq}{\Rightarrow} \mathbf{C}(-,R-)$ in one of the variables is superfluous data, and may always be recovered from merely object-wise information – in analogy with the corresponding result for ordinary adjunctions and representations. Although this result about bi-adjunctions is known and expected, we were unable to find even a statement of this theorem in the literature. Capitalising on this gap we provide a proof in Section 7.1 using some cool 2-dimensional Yoneda tricks rather than a direct construction.

This formulation of the existence of a right bi-adjoint is then amenable to our theorems about bi-representations above and we prove in Theorem 7.11 that L has a right bi-adjoint if and only if there is a double bi-initial object in the pseudo-slice double category $L \downarrow\!\!\downarrow D$ for each object $D \in \mathbf{D}$. As before we derive a purely 2-categorical statement by applying the functors $\mathbf{H}$ and $\mathcal{V}$ to $L \downarrow\!\!\downarrow D$. The resulting 2-categories are isomorphic to the pseudo-slice 2-category $L \downarrow D$ and a "shifted" pseudo-slice 2-category $\mathrm{Ar}_* L \downarrow D$, whose objects are 2-morphisms between LC and D. Finally bi-adjunctions also benefit from the presence of tensors and we prove in Theorem 7.15 that, if the 2-category $\mathbf{C}$ has tensors by $\mathbb{2}$ which are preserved by L[3], then L has a right bi-adjoint if and only if there is a bi-initial object in the pseudo-slice 2-category $L \downarrow D$ for each object $D \in \mathbf{D}$. This special case gives a straightforward 2-categorical version of the characterisation of the existence

[3]Note that tensors by $\mathbb{2}$ are weighted colimits and that left bi-adjoints preserve those (LBAPWBC).

of a right adjoint to an ordinary functor.

As we saw throughout the introduction, there are also 2-type versions of the bi-type notions considered here. All of the theorems given in this paper may also be proven in this stronger setting; the proofs are predictably less involved as there are less coherence conditions to check here. For example, Theorem A in this stronger setting can be formulated as follows: there is a 2-representation of a normal pseudo-functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathsf{Cat}$ if and only if there is a double-initial object in the double category of elements of F, defined here as the *strict* slice double category $\mathbb{1} \downarrow\downarrow \mathbb{H}F$. Similarly, Theorem C for the 2-type case would concern 2-initial objects and stricter versions of $\mathbf{el}(F)$ and $\mathbf{mor}(F)$.

1.1 Outline

The paper is organised as follows. Sections 2 and 3 present the setting of 2-categories and double categories in which we will be working and the functors relating these two settings. We recall notions of trivial fibrations of 2-categories and double categories, which we later use in the definition of bi-initiality.

Then Sections 4 and 5 introduce pseudo-comma 2-categories and double categories, as well as (double) bi-initial objects. We then compare all these 2-categorical notions to their double categorical analogues. After establishing these comparisons, the remaining work to prove the main result is to characterise bi-representations in terms of double bi-initial objects, which we do in Section 6. Finally Section 7 studies applications of our main theorem to bi-adjunctions and (weighted) bi-limits.

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2. Background on 2-categories and double categories

To state and prove our main result, Theorem 6.8 below, we will make use of the languages of 2-categories and of double categories. In particular we will employ the notions of normal pseudo-functor, pseudo-natural transformation, modification, as well as horizontal double categorical counterparts to these notions – double functors and horizontal natural transformations which exhibit pseudo-type behaviour in the horizontal direction. To cement terminology and familiarise ourselves with these notions we will briefly recall the 2-categorical and double categorical concepts at issue in Sections 2.1 and 2.2 below. Readers comfortable with these definitions should skip ahead to Section 3.

2.1 2-categories

Recall that a 2-category is a category enriched in categories. It comprises the data of objects and hom-categories between each pair of objects, together with a *horizontal* composition operation. The objects of the hom-categories are called *morphisms*, the morphisms therein are called 2-*morphisms*, and the composition operation therein is called *vertical* composition of 2-morphisms.

Morphisms between 2-categories which preserve all the 2-categorical structure strictly are called 2-functors. However, in this paper, we consider the more general notion of morphisms of 2-categories, namely *normal pseudo-functors*.

Definition 2.1. Given 2-categories $\mathbf{A}$ and $\mathbf{B}$, a **pseudo-functor** from $\mathbf{A}$ to $\mathbf{B}$ $(F,\phi)\colon \mathbf{A}\to\mathbf{B}$ comprises the data of

(i) an assignment on objects $A\in\mathbf{A}\mapsto FA\in\mathbf{B}$,

(ii) functors $F_{A,A'}\colon \mathbf{A}(A,A')\to\mathbf{B}(FA,FA')$ for each pair of objects $A,A'\in\mathbf{A}$,

(iii) 2-isomorphisms $\phi_A\colon \mathrm{id}_{FA}\overset{\cong}{\Rightarrow} F\,\mathrm{id}_A$ in $\mathbf{B}$ for each object $A\in\mathbf{A}$, called *unitors*,

(iv) 2-isomorphisms $\phi_{a,a'}\colon (Fa')(Fa) \Rightarrow F(a'a)$ in $\mathbf{B}$ for each pair of composable morphisms $a\colon A \to A'$ and $a'\colon A' \to A''$ in $\mathbf{A}$, called *compositors*,

such that these data satisfy naturality, associativity, and unitality conditions. For details see, for example, [7, Definition 4.1.2].

If, for all $A \in \mathbf{A}$, the unitor ϕ_A is given by the identity 2-morphism $\mathrm{id}_{\mathrm{id}_{FA}}$, we say that the pseudo-functor (F, ϕ) is **normal**.

As every pseudo-functor is appropriately isomorphic to a normal one by [12, Proposition 5.2], we choose to simplify our arguments by forgoing the extra data and coherence associated to the former class by working only with normal pseudo-functors.

Notation 2.2. We denote by $\mathsf{2Cat}_{\mathsf{nps}}$ the category of 2-categories and normal pseudo-functors.

We now define a 2-category whose objects are the normal pseudo-functors. For this, we first define its morphisms and 2-morphisms.

Definition 2.3. Given pseudo-functors $F, G\colon \mathbf{A} \to \mathbf{B}$, a **pseudo-natural transformation** $\alpha\colon F \Rightarrow G$ comprises the data of

(i) morphisms $\alpha_A\colon FA \to GA$ in $\mathbf{B}$ for each object $A \in \mathbf{A}$,

(ii) 2-isomorphisms $\alpha_a\colon (Ga)\alpha_A \overset{\cong}{\Rrightarrow} \alpha_{A'}(Fa)$ in $\mathbf{B}$ for each morphism $a\colon A \to A'$ in $\mathbf{A}$ as depicted below.

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ {\scriptstyle Fa}\downarrow & \cong \swArrow_{\alpha_a} & \downarrow{\scriptstyle Ga} \\ FA' & \xrightarrow[\alpha_{A'}]{} & GA' \end{array}$$

such that the 2-morphisms α_a above are natural with respect to 2-morphisms in $\mathbf{A}$, and compatible with the compositors and unitors of F and G. For details see, for example, [7, Definition 4.2.1].

If, for all morphisms $a\colon A \to A'$ in $\mathbf{A}$, the 2-isomorphism component α_a is an identity, i.e., $(Ga)\alpha_A = \alpha_{A'}(Fa)$, then we say that α is 2-**natural**.

Remark 2.4. As a consequence of the compatibility with the unitors above, a pseudo-natural transformation $\alpha\colon F \Rightarrow G$ for which F and G are both normal automatically satisfies $\alpha_{\mathrm{id}_A} = \mathrm{id}_{\alpha_A}$ for all $A \in \mathbf{A}$.

Definition 2.5. Given pseudo-natural transformations $\alpha, \beta \colon F \Rightarrow G$, a **modification** $\Gamma \colon \alpha \Rrightarrow \beta$ comprises the data of 2-morphisms $\Gamma_A \colon \alpha_A \Rightarrow \beta_A$ in $\mathbf{B}$ for each object $A \in \mathbf{A}$ which are compatible with the 2-isomorphism components of α and β. For details see, for example, [7, Definition 4.4.1].

Definition 2.6. Let $\mathbf{A}$ and $\mathbf{B}$ be 2-categories. We define the 2-category $\mathbf{Ps}(\mathbf{A}, \mathbf{B})$ whose objects are normal pseudo-functors from $\mathbf{A}$ to $\mathbf{B}$, morphisms are pseudo-natural transformations, and 2-morphisms are modifications.

2.2 Double categories

In addition to the 2-dimensional concepts above, we will make much use of the possibly less familiar notions of double categories and their morphisms. To prepare for this, we invite the reader to join us in recalling some of the early definitions.

Definition 2.7. A **double category** $\mathbb{A}$ comprises the data of

(i) objects $A, A', B, B', \ldots$,

(ii) horizontal morphisms $a \colon A \to A'$,

(iii) vertical morphisms $u \colon A \mathrel{+\!\!\!\rightarrow} B$,

(iv) squares α with both horizontal and vertical sources and targets, written inline as $\alpha \colon (u \, {}^{a}_{b} \, u')$ or drawn as

$$\begin{array}{ccc} A & \xrightarrow{a} & A' \\ u \downarrow & \alpha & \downarrow u' \\ B & \xrightarrow[b]{} & B' \end{array},$$

(v) horizontal and vertical identity morphisms for each object A, written $\mathrm{id}_A \colon A = A$ and $\mathrm{e}_A \colon A \mathrel{+\!\!\!=} A$ respectively,

(vi) a horizontal identity square for each vertical morphism u and a vertical identity square for each horizontal morphism f, written respectively as

$$\begin{array}{ccc} A & \overset{\mathrm{id}_A}{=\!=} & A \\ {\scriptstyle u}\downarrow & \mathrm{id}_u & \downarrow{\scriptstyle u} \\ B & \underset{\mathrm{id}_B}{=\!=} & B \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{a} & A' \\ {\scriptstyle \mathrm{e}_A}\| & \mathrm{e}_a & \|{\scriptstyle \mathrm{e}_{A'}} \\ A & \xrightarrow[a]{} & A' \end{array}$$

(vii) a horizontal composition operation on horizontal morphisms and squares along a shared vertical boundary,

(viii) a vertical composition operation on vertical morphisms and squares along a shared horizontal boundary,

such that the composition operations are all appropriately associative and unital, and such that horizontal and vertical composition of squares obeys the interchange law. We direct the reader to [4, Definition 3.1.1] for details.

Remark 2.8. Note that a 2-category $\mathbf{A}$ can be seen as a *horizontal* double category $\mathbb{H}\mathbf{A}$, with only trivial vertical morphisms; see Definition 3.1. Dually, we can also see a 2-category $\mathbf{A}$ as a *vertical* double category $\mathbb{V}\mathbf{A}$.

The horizontal embedding is preferred in this document, as it corresponds to the inclusion of 2-categories seen as internal categories to Cat whose category of objects is discrete, into general internal categories to Cat, which are precisely the double categories. This inclusion itself agrees with the inclusion $\mathsf{Cat} = \mathsf{IntCat}(\mathsf{Set}) \rightarrowtail \mathsf{IntCat}(\mathsf{Cat}) = \mathsf{DblCat}$ arising from $\mathsf{Set} \rightarrowtail \mathsf{Cat}$, when categories are seen as 2-categories with only trivial 2-morphisms.

Much as in the case of 2-categories above, we will be interested not in (strict) double functors, which preserve the double categorical structure strictly, but in certain pseudo-type ones. As we choose here to see 2-categories as horizontal double categories, in order to extend this assignment on objects to a functor from $\mathsf{2Cat}_{\mathsf{nps}}$, we need to require that our pseudo-double functors are pseudo in the horizontal direction.

Definition 2.9. Given double categories $\mathbb{A}$ and $\mathbb{B}$, a **(horizontally) pseudo-double functor** $(F, \phi)\colon \mathbb{A} \to \mathbb{B}$ comprises the data of

(i) assignments sending respectively objects A, horizontal morphisms $a\colon A \to A'$, vertical morphisms $u\colon A \nrightarrow B$, and squares $\alpha\colon (u \,{}^{a}_{b}\, u')$ in $\mathbb{A}$ to objects FA, horizontal morphisms $Fa\colon FA \to FA'$, vertical morphisms $Fu\colon FA \nrightarrow FB$, and squares $F\alpha\colon (Fu \,{}^{Fa}_{Fb}\, Fu')$ in $\mathbb{B}$,

(ii) for each $A \in \mathbb{A}$, a vertically invertible unitor square of $\mathbb{B}$ of the form

$$\begin{array}{ccc} FA & \overset{\mathrm{id}_{FA}}{=\!=} & FA \\ \| & \phi_A \; \text{\rotatebox{90}{$\cong$}} & \| \\ FA & \underset{F\,\mathrm{id}_A}{\longrightarrow} & FA \end{array} \quad ,$$

(iii) for each pair of composable horizontal morphisms $a\colon A \to A'$ and $a'\colon A' \to A''$ in $\mathbb{A}$, a vertically invertible compositor square of $\mathbb{B}$ of the form

$$\begin{array}{ccccc} FA & \overset{Fa}{\longrightarrow} & FA' & \overset{Fa'}{\longrightarrow} & FA'' \\ \| & & \phi_{a,a'} \; \text{\rotatebox{90}{$\cong$}} & & \| \\ FA & & \underset{F(a'a)}{\xrightarrow{\qquad\qquad}} & & FA'' \end{array} \quad ,$$

such that

1. vertical compositions of vertical morphisms and squares, as well as vertical identities, are preserved strictly,

2. the unitor squares are natural with respect to vertical morphisms of $\mathbb{A}$,

3. the compositor squares are natural with respect to vertical composition by squares of $\mathbb{A}$,

4. the compositor squares are associative and unital with respect to the unitor squares

If, for all $A \in \mathbb{A}$, the unitor square ϕ_A is given by the vertical identity square $e_{\mathrm{id}_{FA}}$, we say that the pseudo-double functor (F, ϕ) is **normal**.

Notation 2.10. We denote by $\mathsf{DblCat}_{\mathsf{h,nps}}$ the category of double categories and normal pseudo-double functors.

We direct the reader to [4, Definition 3.5.1] for a full elaboration of these conditions for lax-type double functors – though we have interchanged the vertical and horizontal directions by comparison.

We also construct a double category whose objects are the normal pseudo-double functors. We now define its horizontal morphisms which are *horizontal pseudo-natural transformations*. Its vertical morphisms and squares are the *vertical natural transformations* and *modifications*, but we will not have use of the details of these notions, and refer the curious reader to [4, Definitions 3.8.1 and 3.8.3] for these. As ever, we caution the reader that our horizontal and vertical directions are interchanged.

Definition 2.11. Given pseudo-double functors $F, G\colon \mathbb{A} \to \mathbb{B}$, a **horizontal pseudo-natural transformation** $\alpha\colon F \Rightarrow G$ comprises the data of

(i) horizontal morphisms $\alpha_A\colon FA \to GA$ for each $A \in \mathbb{A}$,

(ii) squares $\alpha_u\colon (Fu \,{}^{\alpha_A}_{\alpha_B}\, Gu)$ for each vertical morphism $u\colon A \nrightarrow B$ of $\mathbb{A}$,

(iii) vertically invertible squares

$$\begin{array}{ccccc} FA & \xrightarrow{\alpha_A} & GA & \xrightarrow{Ga} & GA' \\ \Vert & & \alpha_a \ \Vert\wr & & \Vert \\ FA & \xrightarrow[Fa]{} & FA' & \xrightarrow[\alpha_{A'}]{} & GA' \end{array}$$

for each horizontal morphism $a\colon A \to A'$,

such that the squares α_u are coherent with respect to vertical composition and identities, and together with the squares α_a and the compositors and unitors of F and G satisfy horizontal conditions of naturality and unitality. For a full expansion of these conditions, see [4, Definition 3.8.2] – though note again that our horizontal and vertical directions have been interchanged.

If, for all horizontal morphisms a, the square component α_a is an identity, we call α a **horizontal natural transformation**.

Remark 2.12. As a consequence of the axioms, a horizontal pseudo-natural transformation $\alpha\colon F \Rightarrow G$ for which F and G are normal is such that the vertically invertible square α_{id_A} is given by the vertical identity square e_{α_A} for all $A \in \mathbb{A}$.

Definition 2.13. Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. We define the double category $\mathbb{P}\mathrm{s}(\mathbb{A},\mathbb{B})$ in $\mathsf{DblCat}_{\mathsf{h,nps}}$ whose objects are normal pseudo-double functors from $\mathbb{A}$ to $\mathbb{B}$, horizontal morphisms are horizontal pseudo-natural transformations, vertical morphisms are vertical (strict-)natural transformations, and squares are modifications.

3. The functor $\mathcal{V}$ and trivial fibrations

A 2-category $\mathbf{A}$ can be seen as a horizontal double category $\mathbb{H}\mathbf{A}$ with only trivial vertical morphisms. This construction has a right adjoint, which extracts from a double category $\mathbb{A}$ its underlying horizontal 2-category $\mathbf{H}\mathbb{A}$ of objects, horizontal morphisms, and squares with trivial vertical boundaries. Another 2-category $\mathcal{V}\mathbb{A}$ that can be extracted from a double category $\mathbb{A}$ has as objects the vertical morphisms of $\mathbb{A}$, as morphisms the squares of $\mathbb{A}$, and 2-morphisms as described in Definition 3.4. These 2-categories $\mathbf{H}\mathbb{A}$ and $\mathcal{V}\mathbb{A}$ allow one to retrieve most of the structure of the double category $\mathbb{A}$, except for composition of vertical morphisms.

We explore these constructions as a means of comparing a weak notion of initial objects in a double category $\mathbb{A}$ with bi-initial objects in the 2-categories $\mathbf{H}\mathbb{A}$ and $\mathcal{V}\mathbb{A}$. In Section 5.2 these notions of initiality for objects are defined, in analogy with the 1-dimensional case, by requiring the projection from the slice over the considered object to be an appropriate equivalence – in the case of 2-categories, a bi-equivalence. Both in the 2-categorical and double categorical case, these projection morphisms are always strict functors and appear as fibrations in certain model structures. As a consequence, initiality can be equivalently defined by requiring the projection to be a *trivial fibration* – a special case of a bi-equivalence or its double categorical analogue.

With this motivation, in Section 3.2 we recall the definition of a trivial fibration in Lack's model structure on 2-categories and 2-functors [10]. An analogous notion of *double trivial fibrations* is introduced in [14] by the second-named author, Sarazola, and Verdugo as the trivial fibrations in a model structure on double categories and double functors. While the double trivial fibrations are defined as those double functors whose images under $\mathbf{H}$ and $\mathcal{V}$ are trivial fibrations of 2-categories, we show in Theorem 3.8 that double trivial fibrations are precisely those double functors whose image under $\mathcal{V}$ alone is a trivial fibration. This theorem is the essential content of our main result, and is responsible for allowing us to formulate the universal property of bi-representations in terms of bi-initial objects in the corresponding "2-category of morphisms".

3.1 The functors $\mathbb{H}$, H, and $\mathcal{V}$

Let us first introduce the horizontal full embedding functor from $\mathsf{2Cat}_{\mathsf{nps}}$ to $\mathsf{DblCat}_{\mathsf{h,nps}}$.

Definition 3.1. We define a functor $\mathbb{H}\colon \mathsf{2Cat}_{\mathsf{nps}} \to \mathsf{DblCat}_{\mathsf{h,nps}}$ which sends a 2-category $\mathbf{A}$ to the horizontal double category $\mathbb{H}\mathbf{A}$ with the same objects as $\mathbf{A}$, horizontal morphisms given by the morphisms of $\mathbf{A}$, only trivial vertical morphisms, and squares $\alpha\colon (\mathrm{e}_A \;{}^{a}_{b}\; \mathrm{e}_{A'})$ given by the 2-morphisms $\alpha\colon a \Rightarrow b$ of $\mathbf{A}$.

Given a normal pseudo-functor $F\colon \mathbf{A} \to \mathbf{B}$, the induced normal pseudo-double functor $\mathbb{H}F\colon \mathbb{H}\mathbf{A} \to \mathbb{H}\mathbf{B}$ acts as F does on the corresponding data, and respects vertical identities. The compositor vertically invertible squares of $\mathbb{H}F$ are the ones corresponding to the compositor 2-isomorphisms of F.

The functor $\mathbb{H}$ has a right adjoint, given by the following functor.

Definition 3.2. The functor $\mathbf{H}\colon \mathsf{DblCat}_{\mathsf{h,nps}} \to \mathsf{2Cat}_{\mathsf{nps}}$ sends a double category $\mathbb{A}$ to its **underlying horizontal 2-category** $\mathbf{H}\mathbb{A}$ with the same objects as $\mathbb{A}$, morphisms given by the horizontal morphisms of $\mathbb{A}$, and 2-morphisms $\alpha\colon a \Rightarrow b$ given by the squares in $\mathbb{A}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{a} & A' \\ \| & \alpha & \| \\ A & \xrightarrow[b]{} & A' \end{array} .$$

Given a normal pseudo-double functor $F\colon \mathbb{A} \to \mathbb{B}$, the induced normal pseudo-functor $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ acts as F does on the corresponding data, and the data of its compositor 2-isomorphisms are given by the compositor squares of F.

Proposition 3.3. *The functor* $\mathbb{H}\colon \mathsf{2Cat}_{\mathsf{nps}} \to \mathsf{DblCat}_{\mathsf{h,nps}}$ *and the functor* $\mathbf{H}\colon \mathsf{DblCat}_{\mathsf{h,nps}} \to \mathsf{2Cat}_{\mathsf{nps}}$ *form an adjunction* $\mathbb{H} \dashv \mathbf{H}$ *such that the unit* $\eta\colon \mathrm{id}_{\mathsf{2Cat}_{\mathsf{nps}}} \Rightarrow \mathbf{H}\mathbb{H}$ *is an identity.*

Proof. Let $\mathbf{C}$ be a 2-category, and $\mathbb{A}$ be a double category. By specialising Definition 2.9 to the case where the source is a double category with only trivial vertical morphisms, we see that normal pseudo-double functors $\mathbb{H}\mathbf{C} \to \mathbb{A}$ correspond precisely to normal pseudo-functors $\mathbf{C} \to \mathbf{H}\mathbb{A}$, i.e., we have an isomorphism of sets

$$\mathsf{DblCat}_{\mathsf{h,nps}}(\mathbb{H}\mathbf{C}, \mathbb{A}) \cong \mathsf{2Cat}_{\mathsf{nps}}(\mathbf{C}, \mathbf{H}\mathbb{A}) ,$$

natural in $\mathbf{C}$ and $\mathbb{A}$. Moreover, a straightforward computation shows that $\mathbf{H}\mathbb{H}\mathbf{C} = \mathbf{C}$. $\square$

We want to extract another 2-category from a double category which contains the data of all vertical morphisms and squares, and this is done through the following functor $\mathsf{DblCat}_{\mathsf{h,nps}} \to \mathsf{2Cat}_{\mathsf{nps}}$. In [14, Definition 2.10], the second-named author, Sarazola, and Verdugo give a similar definition but in a setting where the morphisms of 2-categories and double categories are *strict*. Under the inclusion of the appropriate subcategories into our weaker setting, our functor may be seen to restrict to theirs.

Although this functor appeared chronologically prior in the work of [14], the original motivation to isolate and treat its definition was our Theorem 5.8 below. Indeed, as we shall see in Remark 5.9, from this context the below definition naturally emerges.

Definition 3.4. Let $\mathbb{V}\mathbb{2}$ denote the free double category on a vertical morphism. We define the functor $\mathcal{V}\colon \mathsf{DblCat}_{\mathsf{h,nps}} \to \mathsf{2Cat}_{\mathsf{nps}}$ to be the composite

$$\mathsf{DblCat}_{\mathsf{h,nps}} \xrightarrow{\mathbb{P}\mathrm{s}(\mathbb{V}\mathbb{2},-)} \mathsf{DblCat}_{\mathsf{h,nps}} \xrightarrow{\mathbf{H}} \mathsf{2Cat}_{\mathsf{nps}} .$$

In particular, it sends a double category $\mathbb{A}$ to the 2-category $\mathcal{V}\mathbb{A}$ whose

(i) objects are the vertical morphisms of $\mathbb{A}$,

(ii) morphisms $\alpha\colon u \to u'$ are squares in $\mathbb{A}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{a} & A' \\ {\scriptstyle u}\big\downarrow & \alpha & \big\downarrow{\scriptstyle u'} \\ B & \xrightarrow[b]{} & B' \end{array} ,$$

(iii) 2-morphisms $(\sigma_0, \sigma_1)\colon \alpha \Rightarrow \alpha'$ are pairs of squares $\sigma_0\colon (\mathrm{e}_A \,{}^{a}_{a'}\, \mathrm{e}_{A'})$ and $\sigma_1\colon (\mathrm{e}_B \,{}^{b}_{b'}\, \mathrm{e}_{B'})$ satisfying the following pasting equality.

$$\begin{array}{ccc} A & \xrightarrow{a} & A' \\ \| & \sigma_0 & \| \\ A & \xrightarrow{a'} & A' \\ {\scriptstyle u}\downarrow & \alpha' & \downarrow{\scriptstyle u'} \\ B & \xrightarrow[b']{} & B' \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{a} & A' \\ {\scriptstyle u}\downarrow & \alpha & \downarrow{\scriptstyle u'} \\ B & \xrightarrow{b} & B' \\ \| & \sigma_1 & \| \\ B & \xrightarrow[b']{} & B' \end{array}$$

Remark 3.5. The functor $\mathcal{V}\colon \mathsf{DblCat}_{\mathsf{h,nps}} \to \mathsf{2Cat}_{\mathsf{nps}}$ is also a right adjoint since it is the composite of two right adjoints $\mathbf{H}$ and $\mathbb{P}\mathrm{s}(\mathbb{V}2, -)$, and its left adjoint is given by $\mathbb{H}(-) \times \mathbb{V}2$.

3.2 Trivial fibrations

We now recall the definition of the *trivial fibrations* in Lack's model structure on the category of 2-categories and 2-functors; see [10].

Definition 3.6. Let $\mathbf{A}$ and $\mathbf{B}$ be 2-categories. Then a 2-functor $F\colon \mathbf{A} \to \mathbf{B}$ is a **trivial fibration** if

(i) for every object $B \in \mathbf{B}$, there is an object $A \in \mathbf{A}$ such that $FA = B$,

(ii) for every pair of object $A, A' \in \mathbf{A}$ and every morphism $b\colon FA \to FA'$ in $\mathbf{B}$, there is a morphism $a\colon A \to A'$ in $\mathbf{A}$ such that $Fa = b$,

(iii) for every pair of parallel morphisms a, b in $\mathbf{A}$ and every 2-morphism $\beta\colon Fa \Rightarrow Fb$ in $\mathbf{B}$, there is a unique 2-morphism $\alpha\colon a \Rightarrow b$ in $\mathbf{A}$ such that $F\alpha = \beta$.

Similarly, we recall double trivial fibrations which were defined as the trivial fibrations in the model structure on double categories and double functors constructed by the second-named author, Sarazola, and Verdugo in [14].

Definition 3.7. Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. Then a double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a **double trivial fibration** if

(i) for every object $B \in \mathbb{B}$, there is an object $A \in \mathbb{A}$ such that $FA = B$,

(ii) for every pair of objects $A, A' \in \mathbb{A}$ and every horizontal morphism $b\colon FA \to FA'$ in $\mathbb{B}$, there is a horizontal morphism $a\colon A \to A'$ such that $Fa = b$,

(iii) for every vertical morphism $v\colon B \nrightarrow D$ in $\mathbb{B}$, there is a vertical morphism $u\colon A \nrightarrow C$ in $\mathbb{A}$ such that $Fu = v$,

(iv) for every pair of horizontal morphisms $a\colon A \to A'$, $c\colon C \to C'$ in $\mathbb{A}$, every pair of vertical morphisms $u\colon A \nrightarrow C$, $u'\colon A' \nrightarrow C'$ in $\mathbb{A}$ and every square β in $\mathbb{B}$ of the form

$$\begin{array}{ccc} FA & \xrightarrow{Fa} & FA' \\ {\scriptstyle Fu}\big\downarrow & \beta & \big\downarrow{\scriptstyle Fu'} \\ FC & \xrightarrow[Fc]{} & FC' \end{array} \quad ,$$

there is a unique square $\alpha\colon (u \,{}^{a}_{c}\, u')$ in $\mathbb{A}$ such that $F\alpha = \beta$.

Since fibrations and weak equivalences in the model structure on double categories are defined to be the double functors whose images under $\mathbf{H}$ and $\mathcal{V}$ are fibrations and weak equivalences in Lack's model structure (see [14, Theorem 3.19]), it is possible to characterise double trivial fibrations in terms of trivial fibrations of 2-categories. However, even more is true: $\mathcal{V}$ alone captures enough data to detect the entire model structure on double categories, and in particular the double trivial fibrations.

Theorem 3.8. *Let $\mathbb{A}$ and $\mathbb{B}$ be double categories and $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. The following statements are equivalent.*

(i) The double functor F is a double trivial fibration.

(ii) The 2-functors $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ and $\mathcal{V}F\colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ are trivial fibrations.

(iii) The 2-functor $\mathcal{V}F\colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ is a trivial fibration.

Proof. The equivalence of (i) and (ii) follows from [14, Corollary 3.14]. The equivalence of (ii) and (iii) is a consequence of the facts that $\mathbf{H}F$ is a retract of $\mathcal{V}F$ and that trivial fibrations are closed under retracts. □

4. Pseudo-comma 2-dimensional categories

We now introduce pseudo-comma double and 2-categories, and show that they are related through the functors $\mathbf{H}, \mathcal{V}\colon \mathsf{DblCat}_{\mathsf{h,nps}} \to \mathsf{2Cat}_{\mathsf{nps}}$. We treat these objects in general so that we may later variously specialise the theory to pseudo-slices both over and under objects. We will use these results for the purposes of comparing double bi-initial objects in a double category with bi-initial objects in the induced 2-categories obtained by applying $\mathbf{H}$ and $\mathcal{V}$ in Section 5, as well as for computing the double categories of elements in the case of bi-adjunctions and weighted bi-limits in Section 7.

Let us first define the pseudo-comma double category of a cospan of normal pseudo-double functors. With an eye to our applications of this theory in Sections 5 and 7, we then give a more explicit description of the data in a *pseudo-slice* double category: a pseudo-comma where one of the double categories involved is terminal.

Definition 4.1. Let $G\colon \mathbb{C} \to \mathbb{A}$ and $F\colon \mathbb{B} \to \mathbb{A}$ be normal pseudo-double functors. The **pseudo-comma double category** $G{\downarrow\downarrow}F$ of G and F is defined as the following pullback in $\mathsf{DblCat}_{\mathsf{h,nps}}$,

$$\begin{array}{ccc} G \downarrow\downarrow F & \longrightarrow & \mathbb{P}\mathrm{s}(\mathbb{H}\mathbb{2}, \mathbb{A}) \\ {\scriptstyle \Pi}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle (s,t)} \\ \mathbb{C} \times \mathbb{B} & \xrightarrow[(G,F)]{} & \mathbb{A} \times \mathbb{A} \end{array}$$

where $\mathbb{H}\mathbb{2}$ denotes the free double category on a horizontal morphism and $\mathbb{P}\mathrm{s}(-,-)$ is the double category described in Definition 2.13.

Note that $\Pi\colon G \downarrow\downarrow F \to \mathbb{C} \times \mathbb{B}$ is a *strict* double functor.

Remark 4.2. We give an explicit description of the pseudo-comma double category in the special case where $\mathbb{C} = \mathbb{1}$ is the terminal category and $G = I\colon \mathbb{1} \to \mathbb{A}$ is an object in $\mathbb{A}$. This is the double category $I \downarrow\downarrow F$, called **pseudo-slice double category**, whose

(i) objects are pairs (B, f) of an object $B \in \mathbb{B}$ and a horizontal morphism $f\colon I \to FB$ in $\mathbb{A}$,

(ii) horizontal morphisms $(b,\psi)\colon (B,f)\to(B',f')$ comprise the data of a horizontal morphism $b\colon B\to B'$ in $\mathbb{B}$ and a vertically invertible square ψ in $\mathbb{A}$ of the form

$$\begin{array}{ccccc} I & & \xrightarrow{\quad f' \quad} & & FB' \\ \| & & \psi\ \cong & & \| \\ I & \xrightarrow[f]{} & FB & \xrightarrow[Fb]{} & FB' \end{array} ,$$

(iii) vertical morphisms $(u,\gamma)\colon (B,f)\nrightarrow(C,g)$ comprise the data of a vertical morphism $u\colon B\nrightarrow C$ in $\mathbb{B}$ and a square γ in $\mathbb{A}$ of the form

$$\begin{array}{ccc} I & \xrightarrow{f} & FB \\ \| & \gamma & \big\downarrow Fu \\ I & \xrightarrow[g]{} & FC \end{array} ,$$

(iv) squares

$$\begin{array}{ccc} (B,f) & \xrightarrow{(b,\psi)} & (B',f') \\ (u,\gamma)\big\downarrow & \beta & \big\downarrow(u',\gamma') \\ (C,g) & \xrightarrow[(c,\varphi)]{} & (C',g') \end{array}$$

comprise the data of a square $\beta\colon (u\ {}^{b}_{c}\ u')$ in $\mathbb{B}$ such that the following pasting equality holds in $\mathbb{A}$.

$$\begin{array}{ccccc} I & & \xrightarrow{\quad f' \quad} & & FB' \\ \| & & \psi\ \cong & & \| \\ I & \xrightarrow{f} & FB & \xrightarrow{Fb} & FB' \\ \| & \gamma & Fu\big\downarrow & \beta & \big\downarrow Fu' \\ I & \xrightarrow[g]{} & FC & \xrightarrow[Fc]{} & FC' \end{array} = \begin{array}{ccccc} I & & \xrightarrow{\quad f' \quad} & & FB' \\ \| & & & & \big\downarrow Fu' \\ I & & \xrightarrow{\quad g' \quad} & & FC' \\ \| & & \gamma' \quad \varphi\ \cong & & \| \\ I & \xrightarrow[g]{} & FC & \xrightarrow[Fc]{} & FC' \end{array}$$

The double functor $\Pi\colon I \downarrow\downarrow F \to \mathbb{1} \times \mathbb{B} \cong \mathbb{B}$ is the projection onto the $\mathbb{B}$-component.

If $\mathbb{B} = \mathbb{A}$ and $F = \mathrm{id}_{\mathbb{A}}$, we write $I \downarrow\downarrow \mathbb{A} := I \downarrow\downarrow \mathrm{id}_{\mathbb{A}}$.

We now define the pseudo-comma 2-category of a cospan of normal pseudo-functors, and also give an explicit description of the special case of a pseudo-slice 2-category.

Definition 4.3. Let $G\colon \mathbf{C} \to \mathbf{A}$ and $F\colon \mathbf{B} \to \mathbf{A}$ be normal pseudo-functors. The **pseudo-comma** 2**-category** $G \downarrow F$ is defined as the following pullback in $\mathsf{2Cat}_{\mathsf{nps}}$.

$$\begin{array}{ccc} G \downarrow F & \longrightarrow & \mathbf{Ps}(\mathbb{2}, \mathbf{A}) \\ {\scriptstyle \pi}\big\downarrow \quad \lrcorner & & \big\downarrow{\scriptstyle (s,t)} \\ \mathbf{C} \times \mathbf{B} & \underset{(G,F)}{\longrightarrow} & \mathbf{A} \times \mathbf{A} \end{array}$$

where $\mathbb{2}$ is the free 2-category on a morphism and $\mathbf{Ps}(-,-)$ is the 2-category described in Definition 2.6.

Note that $\pi\colon G \downarrow F \to \mathbf{C} \times \mathbf{B}$ is a *strict* 2-functor.

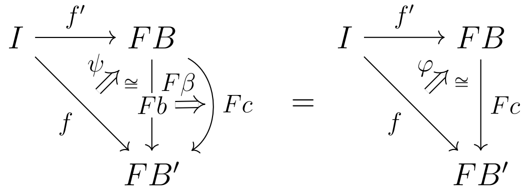
Remark 4.4. We give an explicit description of the pseudo-comma 2-category in the case where $\mathbf{C} = \mathbb{1}$ is the terminal category and $G = I\colon \mathbb{1} \to \mathbf{A}$ is an object in $\mathbf{A}$. This is the 2-category $I \downarrow F$, called a **pseudo-slice** 2**-category**, whose

(i) objects are pairs (B, f) of an object $B \in \mathbf{B}$ together with a morphism $f\colon I \to FB$ in $\mathbf{A}$,

(ii) morphisms $(b, \psi)\colon (B, f) \to (B', f')$ comprise the data of a morphism $b\colon B \to B'$ in $\mathbf{B}$ and a 2-isomorphism ψ in $\mathbf{A}$ of the form

$$\begin{array}{ccc} I & \xrightarrow{\;f\;} & FB \\ & {\scriptstyle f'}\searrow \;\; \overset{\psi}{\nearrow\!\!\!\nearrow}{\scriptstyle \cong} & \big\downarrow{\scriptstyle Fb} \\ & & FB' \end{array} \quad ,$$

(iii) 2-morphisms $\beta\colon (b, \psi) \Rightarrow (c, \varphi)$ comprise the data of a 2-morphism $\beta\colon b \Rightarrow c$ in $\mathbf{B}$ such that the following pasting equality holds in $\mathbf{A}$.



The 2-functor $\pi\colon I \downarrow F \ \to\ \mathbb{1} \times \mathbf{B} \cong \mathbf{B}$ is the projection onto the $\mathbf{B}$-component.

If $\mathbf{B} = \mathbf{A}$ and $F = \mathrm{id}_{\mathbf{A}}$, we write $I \downarrow \mathbf{A} := I \downarrow \mathrm{id}_{\mathbf{A}}$ with $I \in \mathbf{A}$.

Remark 4.5. Given the explications of Remarks 4.2 and 4.4, we wish to draw the reader's attention to an important disparity between the double categories $I \downarrow\downarrow \mathbb{H}F$ and $\mathbb{H}(I \downarrow F)$, for a 2-functor $F\colon \mathbf{B} \to \mathbf{A}$ and an object $I \in \mathbf{A}$. While the latter double category has only trivial vertical morphisms, the former has all 2-morphisms of $\mathbf{A}$ of the form $\gamma\colon f \Rightarrow g$ for $f, g\colon I \to FB$ as vertical morphisms – a far richer stock of information. This is symptomatic of a broader truth: the double category $\mathbb{P}\mathrm{s}(\mathbb{H}\mathbb{2}, \mathbb{H}\mathbf{A})$ has all 2-morphisms of $\mathbf{A}$ as vertical morphisms, while $\mathbb{H}\,\mathbf{Ps}(\mathbb{2}, \mathbf{A})$ is the double category associated to its underlying horizontal 2-category $\mathbf{Ps}(\mathbb{2}, \mathbf{A}) = \mathbf{H}\,\mathbb{P}\mathrm{s}(\mathbb{H}\mathbb{2}, \mathbb{H}\mathbf{A})$ and therefore has only trivial vertical morphisms.

While $\mathbb{H}$ thus does not preserve pseudo-comma objects, the main result of this section says that the functors $\mathbf{H}$ and $\mathcal{V}$ do preserve pseudo-comma objects, in the sense that the 2-category obtained by applying $\mathbf{H}$ or $\mathcal{V}$ to the pseudo-comma double category associated to a cospan is isomorphic to the pseudo-comma 2-category of the image under $\mathbf{H}$ or $\mathcal{V}$ of the original cospan. This is the content of the following proposition and the rest of the section will be devoted to its proof.

Proposition 4.6. *Let $G\colon \mathbb{C} \to \mathbb{A}$ and $F\colon \mathbb{B} \to \mathbb{A}$ be normal pseudo-double functors. Then there are canonical isomorphisms of 2-categories as in the following commutative squares.*

$$\begin{array}{ccc} \mathbf{H}(G \downarrow\downarrow F) & \xrightarrow{\cong} & \mathbf{H}G \downarrow \mathbf{H}F \\ {\scriptstyle \mathbf{H}\Pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbf{H}(\mathbb{C} \times \mathbb{B}) & \xrightarrow[\cong]{} & \mathbf{H}\mathbb{C} \times \mathbf{H}\mathbb{B} \end{array} \qquad \begin{array}{ccc} \mathcal{V}(G \downarrow\downarrow F) & \xrightarrow{\cong} & \mathcal{V}G \downarrow \mathcal{V}F \\ {\scriptstyle \mathcal{V}\Pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{V}(\mathbb{C} \times \mathbb{B}) & \xrightarrow[\cong]{} & \mathcal{V}\mathbb{C} \times \mathcal{V}\mathbb{B} \end{array}$$

To prove this we first show that the functors $\mathbf{H}$ and $\mathcal{V}$ behave well with respect to $\mathbf{Ps}$ and $\mathbb{P}\mathrm{s}$.

Lemma 4.7. *For every* 2*-category* $\mathbf{C}$ *and every double category* $\mathbb{A}$*, there is an isomorphism of* 2*-categories*

$$\mathbf{Ps}(\mathbf{C}, \mathbf{H}\mathbb{A}) \cong \mathbf{H}\,\mathbb{P}\mathrm{s}(\mathbb{H}\mathbf{C}, \mathbb{A})$$

natural in $\mathbf{C}$ *and* $\mathbb{A}$*.*

Proof. Recall from Proposition 3.3 that $\mathbb{H} \dashv \mathbf{H}$ form an adjunction between $\mathsf{2Cat}_{\mathsf{nps}}$ and $\mathsf{DblCat}_{\mathsf{h,nps}}$, so that we have an isomorphism at the level of objects which is natural in $\mathbf{C}$ and $\mathbb{A}$. By unpacking the definition of a horizontal pseudo-natural transformation (see Definition 2.11) between normal pseudo-double functors $\mathbb{H}\mathbf{C} \to \mathbb{A}$, we can see that such data corresponds to that of a pseudo-natural transformation (see Definition 2.3) between normal pseudo-functors $\mathbf{C} \to \mathbf{H}\mathbb{A}$ as the squares in Definition 2.9 (ii) are all trivial given that all vertical morphisms of $\mathbb{H}\mathbf{C}$ are identities. Similarly, one can check that, using Definition 2.5 and [4, Definition 3.8.3], the 2-morphisms of these 2-categories coincide. □

In order to give the next result, an analogous statement for $\mathcal{V}$, we will make use of the following technical lemma.

Lemma 4.8. *For every* 2*-category* $\mathbf{C}$ *and every double category* $\mathbb{A}$*, there are isomorphisms of* 2*-categories*

$$\mathbf{H}\,\mathbb{P}\mathrm{s}(\mathbb{H}\mathbf{C}, \mathbb{P}\mathrm{s}(\mathbb{V}\mathbb{2}, \mathbb{A})) \cong \mathbf{H}\,\mathbb{P}\mathrm{s}(\mathbb{V}\mathbb{2}, \mathbb{P}\mathrm{s}(\mathbb{H}\mathbf{C}, \mathbb{A}))$$

natural in $\mathbf{C}$ *and* $\mathbb{A}$*.*

Proof. That this result holds is due to the appearance of $\mathbb{H}\mathbf{C}$ and $\mathbb{V}\mathbb{2}$ in its statement. Should we replace $\mathbb{V}\mathbb{2}$ by a more general double category with non-trivial horizontal morphisms, or replace $\mathbb{H}\mathbf{C}$ by a more general double category with non-trivial vertical morphisms, the result would fail to hold in general.

We show that we have an isomorphism on objects. A normal pseudo-double functor $\nu\colon \mathbb{H}\mathbf{C} \to \mathbb{P}\mathrm{s}(\mathbb{V}\mathbb{2}, \mathbb{A})$ assigns to each object of $C \in \mathbf{C}$ a vertical morphism $\nu\colon FC \to GC$ in $\mathbb{A}$, to each morphism $c\colon C \to C'$ in $\mathbf{C}$ a square $\nu_c\colon (\nu_C \,{}^{Fc}_{Gc}\, \nu_{C'})$ in $\mathbb{A}$, to each 2-morphism $\gamma\colon c \Rightarrow c'$ in $\mathbf{C}$ two squares $F\gamma$ and $G\gamma$ in $\mathbb{A}$ which satisfy the following pasting equality,

$$
\begin{array}{ccc}
FC & \xrightarrow{Fc} & FC' \\
\| & F\gamma & \| \\
FC & \xrightarrow{Fc'} & FC' \\
{\scriptstyle \nu_C}\downarrow & \nu_{c'} & \downarrow{\scriptstyle \nu_{C'}} \\
GC & \xrightarrow[Gc']{} & GC'
\end{array}
\quad = \quad
\begin{array}{ccc}
FC & \xrightarrow{Fc} & FC' \\
{\scriptstyle \nu_C}\downarrow & \nu_c & \downarrow{\scriptstyle \nu_{C'}} \\
GC & \xrightarrow{Gc} & GC' \\
\| & G\gamma & \| \\
GC & \xrightarrow[Gc']{} & GC'
\end{array}
$$

and to each pair of composable morphisms $c\colon C \to C'$ and $c'\colon C \to C''$ in $\mathbf{C}$, two vertically invertible squares $\phi_{c,c'}$ and $\psi_{c,c'}$ in $\mathbb{A}$ which satisfy the following pasting equality.

$$
\begin{array}{ccccc}
FC & \xrightarrow{Fc} & FC' & \xrightarrow{Fc'} & FC'' \\
{\scriptstyle \nu_C}\downarrow & \nu_c & {\scriptstyle \nu_{C'}}\downarrow & \nu_{c'} & \downarrow{\scriptstyle \nu_{C''}} \\
GC & \xrightarrow{Gc} & GC' & \xrightarrow{Gc'} & GC'' \\
\| & & \psi_{c,c'}\ \cong & & \| \\
GC & & \xrightarrow[G(c'c)]{} & & GC''
\end{array}
\quad = \quad
\begin{array}{ccccc}
FC & \xrightarrow{Fc} & FC' & \xrightarrow{Fc'} & FC'' \\
\| & & \phi_{c,c'}\ \cong & & \| \\
FC & & \xrightarrow{F(c'c)} & & FC'' \\
{\scriptstyle \nu_C}\downarrow & & \nu_{c'c} & & \downarrow{\scriptstyle \nu_{C''}} \\
GC & & \xrightarrow[G(c'c)]{} & & GC''
\end{array}
$$

One can check that not only is this data precisely the underlying data of two normal pseudo-double functors $F, G\colon \mathbb{H}\mathbf{C} \to \mathbb{A}$ together with a vertical natural transformation $\nu\colon F \Rightarrow G$ between them, but also that the various laws governing the compositors of F, G hold. The two diagrams above already demonstrate that ν is natural.

We leave it to the reader to check that the morphisms and 2-morphisms of the 2-categories considered also coincide under this identification. $\square$

Lemma 4.9. *For every* 2*-category* $\mathbf{C}$ *and every double category* $\mathbb{A}$*, there is an isomorphism of* 2*-categories*

$$\mathbf{Ps}(\mathbf{C}, \mathcal{V}\mathbb{A}) \cong \mathcal{V}\,\mathbb{P}\mathrm{s}(\mathbb{H}\mathbf{C}, \mathbb{A})$$

natural in $\mathbf{C}$ *and* $\mathbb{A}$.

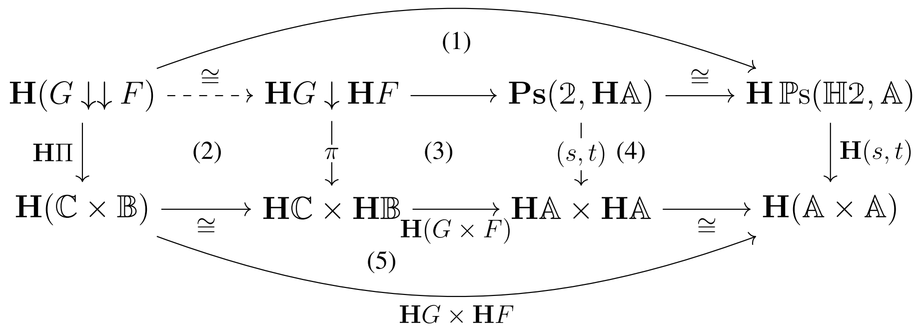
Proof. We have the following isomorphisms

$$
\begin{aligned}
\mathbf{Ps}(\mathbf{C}, \mathcal{V}\mathbb{A}) &= \mathbf{Ps}(\mathbf{C}, \mathbf{H}\,\mathbb{P}\mathrm{s}(\mathbb{V}2, \mathbb{A})) && \text{(definition of } \mathcal{V}\text{)}\\
&\cong \mathbf{H}\,\mathbb{P}\mathrm{s}(\mathbb{H}\mathbf{C}, \mathbb{P}\mathrm{s}(\mathbb{V}2, \mathbb{A})) && \text{(Lemma 4.7)}\\
&\cong \mathbf{H}\,\mathbb{P}\mathrm{s}(\mathbb{V}2, \mathbb{P}\mathrm{s}(\mathbb{H}\mathbf{C}, \mathbb{A})) && \text{(Lemma 4.8)}\\
&= \mathcal{V}\,\mathbb{P}\mathrm{s}(\mathbb{H}\mathbf{C}, \mathbb{A}). && \text{(definition of } \mathcal{V}\text{)}
\end{aligned}
$$

natural in $\mathbf{C}$ and $\mathbb{A}$. ☐

The proof of Proposition 4.6 now follows from these results and the fact that $\mathbf{H}$ and $\mathcal{V}$ are right adjoints, and therefore preserve limits.

Proof (Proposition 4.6). Let us consider the following diagram.



First note that $\mathbf{H}G \downarrow \mathbf{H}F$ is a pullback of the commutative square (3), and, since $\mathbf{H}$ preserves pullbacks, $\mathbf{H}(G \downarrow\downarrow F)$ is a pullback of the outer commutative square. The commutative square (4) is obtained in two steps. First, apply Lemma 4.7 to the 2-categories $\mathbb{1} \sqcup \mathbb{1}$ and 2, respectively, and to the double category $\mathbb{A}$, and use the naturality of these isomorphisms with respect to the 2-functor $\mathbb{1} \sqcup \mathbb{1} \to 2$ given by the inclusion at the two endpoints. Second, apply the isomorphisms

$$\mathbf{Ps}(\mathbb{1} \sqcup \mathbb{1}, \mathbf{H}\mathbb{A}) \cong \mathbf{H}\mathbb{A} \times \mathbf{H}\mathbb{A} \;\text{ and }\; \mathbf{H}\,\mathbb{P}\mathrm{s}(\mathbb{1} \sqcup \mathbb{1}, \mathbf{H}\mathbb{A}) \cong \mathbf{H}(\mathbb{A} \times \mathbb{A}).$$

Note that the bottom isomorphism $\mathbf{H}\mathbb{A} \times \mathbf{H}\mathbb{A} \cong \mathbf{H}(\mathbb{A} \times \mathbb{A})$ of the square (4) is the canonical one coming from the fact that $\mathbf{H}$ preserves products. Similarly, we have a canonical isomorphism $\mathbf{H}\mathbb{B} \times \mathbf{H}\mathbb{C} \cong \mathbf{H}(\mathbb{B} \times \mathbb{C})$ and the diagram (5) commutes. By the universal property of pullbacks, we get an isomorphism $\mathbf{H}(F \downarrow\downarrow G) \cong \mathbf{H}F \downarrow \mathbf{H}G$ such that the diagrams (1) and (2) commute.

The argument is similar for the case of $\mathcal{V}$ since this functor also preserves pullbacks and products, and Lemma 4.9 holds. □

5. 2-dimensional initiality

In Section 5.1 we introduce the new notion of a double bi-initial object in a double category, which we aim to compare with that of a bi-initial object in a 2-category in Section 5.2.

Double bi-initial objects are defined by requiring that the projection double functor from the pseudo-slice double category under this object is a double trivial fibration. While this notion might seem to involve a lot of data *a priori*, in fact we show that there is a straightforward characterisation of double bi-initial objects: an object I is double bi-initial if and only if there is a horizontal morphism to every object, and all square boundaries whose left vertical morphism is e_I have a unique filler. We then show that a similar result holds for bi-initial objects in a 2-category: an object I is bi-initial if and only if there is a morphism to every object, and all parallel such morphisms $I \rightrightarrows C$ have a unique 2-morphism filler which must therefore be invertible.

The main result then says that an object of a double category $\mathbb{A}$ is double bi-initial if and only if its images in the 2-category $\mathbf{H}\mathbb{A}$ and $\mathcal{V}\mathbb{A}$ are bi-initial. In fact, we improve upon this by showing that an object I is bi-initial in $\mathbf{H}\mathbb{A}$ if its vertical identity e_I is bi-initial in $\mathcal{V}\mathbb{A}$. That is, we show that double bi-initial objects in $\mathbb{A}$ may be successfully detected purely 2-categorically as bi-initial objects of a suitable form in $\mathcal{V}\mathbb{A}$.

Finally we show that in the presence of double limits of vertical morphisms in $\mathbb{A}$, called *tabulators*, the reverse implication also holds: the object e_I is bi-initial in $\mathcal{V}\mathbb{A}$ when I is bi-initial in $\mathbf{H}\mathbb{A}$. Taken together these results show that, in the presence of tabulators, the characterisation of double bi-initial objects is now as good as one could hope for: a double bi-initial object in a double category $\mathbb{A}$ is precisely a bi-initial object in the underlying horizontal 2-category $\mathbf{H}\mathbb{A}$.

Both 2-categories and double categories have several duals, but of interest is the opposite $\mathbf{C}^{\mathrm{op}}$ of a 2-category $\mathbf{C}$ and the horizontal opposite $\mathbb{A}^{\mathrm{op}}$ of a double category $\mathbb{A}$. These operations agree with one another under applications of the functors $\mathbb{H}$, $\mathbf{H}$, and $\mathcal{V}$. In particular, later we will have interest in (double) *bi-terminal* objects, which are simply (double) bi-initial objects in the (horizontal) opposite. Correspondingly, all the results of this section dualise to the setting of (double) bi-terminal objects.

5.1 Double bi-initial objects

Let us first give the definition of a double bi-initial object. Recall Remark 4.2, where we described explicitly pseudo-slice double categories.

Definition 5.1. Let $\mathbb{A}$ be a double category. An object I in $\mathbb{A}$ is **double bi-initial** if the projection double functor $\Pi\colon I \downarrow\!\!\downarrow \mathbb{A} \to \mathbb{A}$ is a double trivial fibration.

Although there appears to be a lot of data in this definition, in fact we will show that there is a simpler characterisation of double bi-initial objects.

Proposition 5.2. *Let $\mathbb{A}$ be a double category. An object I in $\mathbb{A}$ is double bi-initial if and only if the following conditions hold:*

(i) for every object $A \in \mathbb{A}$, there is a horizontal morphism $f\colon I \to A$ in $\mathbb{A}$,

(ii') for every vertical morphism $u\colon A \nrightarrow B$ and every pair of horizontal morphisms $f\colon I \to A$ and $g\colon I \to B$ in $\mathbb{A}$, there is a unique square γ in $\mathbb{A}$ of the form

$$\begin{array}{ccc} I & \xrightarrow{f} & A \\ \Vert & \gamma & \downarrow u \\ I & \xrightarrow[g]{} & B \end{array} \quad .$$

In order to prove this, we first elaborate the content of Definition 5.1.

Remark 5.3. By expanding the definition, we get that the projection double functor $\Pi\colon I \downarrow\!\!\downarrow \mathbb{A} \to \mathbb{A}$ being a double trivial fibration is equivalent to the following conditions:

(i) for every object $A \in \mathbb{A}$, there is a horizontal morphism $f\colon I \to A$ in $\mathbb{A}$,

(ii) for every tuple of horizontal morphisms $f\colon I \to A$, $f'\colon I \to A'$, and $a\colon A \to A'$ in $\mathbb{A}$, there is a vertically invertible square ψ in $\mathbb{A}$

$$\begin{array}{ccccc} I & & \xrightarrow{\quad f' \quad} & & A' \\ \| & & \psi \; \begin{smallmatrix}\|\\ \wr\end{smallmatrix} & & \| \\ I & \xrightarrow[f]{} & A & \xrightarrow[a]{} & A' \end{array} ,$$

(iii) for every vertical morphism $u\colon A \nrightarrow B$ in $\mathbb{A}$, there is a square γ in $\mathbb{A}$

$$\begin{array}{ccc} I & \xrightarrow{f} & A \\ \| & \gamma & \downarrow{\scriptstyle u} \\ I & \xrightarrow[g]{} & B \end{array} ,$$

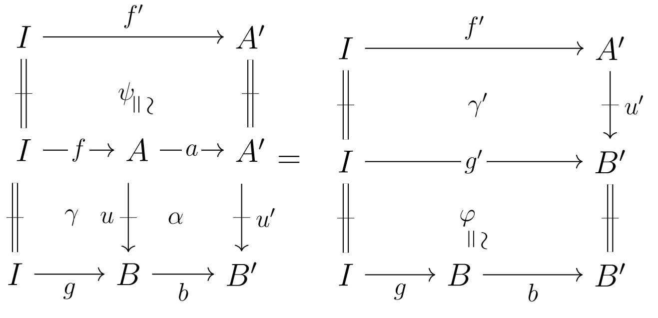
(iv) for every tuple of squares γ, γ', and α in $\mathbb{A}$

$$\begin{array}{ccc} I & \xrightarrow{f} & A \\ \| & \gamma & \downarrow{\scriptstyle u} \\ I & \xrightarrow[g]{} & B \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{f'} & A' \\ \| & \gamma' & \downarrow{\scriptstyle u'} \\ I & \xrightarrow[g']{} & B' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{a} & A' \\ {\scriptstyle u}\downarrow & \alpha & \downarrow{\scriptstyle u'} \\ B & \xrightarrow[b]{} & B' \end{array}$$

and for every pair of vertically invertible squares in $\mathbb{A}$

$$\begin{array}{ccccc} I & & \xrightarrow{\quad f' \quad} & & A' \\ \| & & \psi \; \begin{smallmatrix}\|\\ \wr\end{smallmatrix} & & \| \\ I & \xrightarrow[f]{} & A & \xrightarrow[a]{} & A' \end{array} \qquad \begin{array}{ccccc} I & & \xrightarrow{\quad g' \quad} & & B' \\ \| & & \varphi \; \begin{smallmatrix}\|\\ \wr\end{smallmatrix} & & \| \\ I & \xrightarrow[g]{} & B & \xrightarrow[b]{} & B' \end{array} ,$$

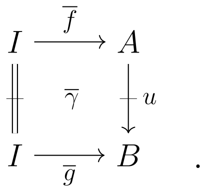
the following pasting equality holds in $\mathbb{A}$.



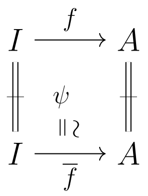
Now we are ready to prove the simpler characterisation of double bi-initial objects.

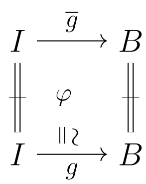
Proof (Proposition 5.2). We first prove that if I is a double bi-initial object in $\mathbb{A}$, then conditions (i) and (ii') hold. It is clear that (i) holds by Remark 5.3 (i). We prove (ii').

Let $f\colon I \to A$ and $g\colon I \to B$ be two horizontal morphisms in $\mathbb{A}$, and let $u\colon A \nrightarrow B$ be a vertical morphism in $\mathbb{A}$. By Remark 5.3 (iii), there is a square $\overline{\gamma}$ in $\mathbb{A}$



By Remark 5.3 (ii) applied to $(\overline{f}, f, \mathrm{id}_A)$ and $(g, \overline{g}, \mathrm{id}_B)$, there are vertically invertible squares ψ and φ in $\mathbb{A}$ as depicted below.





Then we set γ to be the following composite

$$
\begin{array}{ccc}
I & \xrightarrow{f} & A \\
\Vert & \gamma & \downarrow u \\
I & \xrightarrow[g]{} & B
\end{array}
\quad = \quad
\begin{array}{ccc}
I & \xrightarrow{f} & A \\
\Vert & \psi \; \Vert\wr & \Vert \\
I & \xrightarrow{\overline{f}} & A \\
\Vert & \overline{\gamma} & \downarrow u \\
I & \xrightarrow{\overline{g}} & B \\
\Vert & \varphi \; \Vert\wr & \Vert \\
I & \xrightarrow[g]{} & B
\end{array}
$$

which proves the existence.

Now suppose γ' is another such square in $\mathbb{A}$

$$
\begin{array}{ccc}
I & \xrightarrow{f} & A \\
\Vert & \gamma' & \downarrow u \\
I & \xrightarrow[g]{} & B
\end{array}
\quad .
$$

By applying Remark 5.3 (iv) to the squares $(\gamma, \gamma', \mathrm{id}_u)$ and to the vertically invertible squares $(\mathrm{e}_f, \mathrm{e}_g)$, we directly get that $\gamma = \gamma'$ from the pasting equality.

Now, conversely, it is straightforward to see that Condition (ii') implies Remarks 5.3 (ii), 5.3 (iii) and 5.3 (iv), since there is a unique square with each given boundary whose left vertical morphism is e_I. Note that the square in Remark 5.3 (ii) is indeed vertically invertible since every square from e_I to another vertical identity is vertically invertible by (ii'). □

5.2 Double bi-initial objects vs bi-initial objects

We now want to compare the notion of double bi-initial objects in a double category with the notion of bi-initial objects in related 2-categories. We first recall the definition of a bi-initial object, which uses the notion of a pseudo-slice 2-category as elaborated in Remark 4.4.

Definition 5.4. Let $\mathbf{A}$ be a 2-category. An object $I \in \mathbf{A}$ is **bi-initial** if the projection 2-functor $\pi\colon I \downarrow \mathbf{A} \to \mathbf{A}$ is a trivial fibration.

Proposition 5.5. *Let $\mathbf{A}$ be a 2-category, and $I \in \mathbf{A}$ be an object. Then the object I is bi-initial in $\mathbf{A}$ if and only if, for every object $A \in \mathbf{A}$, the unique functor $\mathbf{A}(I, A) \xrightarrow{\simeq} \mathbb{1}$ is part of an equivalence of categories. More precisely, this means that*

(i) for every object $A \in \mathbf{A}$, there is a morphism $f\colon I \to A$ in $\mathbf{A}$,

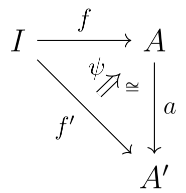
(ii') for every pair of morphisms $f\colon I \to A$ and $f'\colon I \to A$, there is a unique 2-morphism $\alpha\colon f' \Rightarrow f$.

In order to prove this, we first elaborate the content of Definition 5.4.

Remark 5.6. By expanding the definition, we get that the projection 2-functor $\pi\colon I \downarrow \mathbf{A} \to \mathbf{A}$ being a trivial fibration is equivalent to the following conditions:

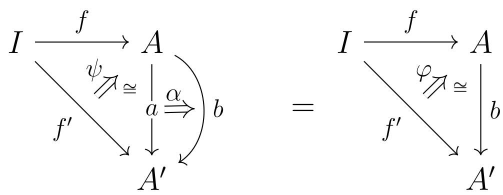
(i) for every object $A \in \mathbf{A}$, there is a morphism $f\colon I \to A$ in $\mathbf{A}$,

(ii) for every tuple of morphisms $f\colon I \to A$, $f'\colon I \to A'$, and $a\colon A \to A'$ in $\mathbf{A}$, there is a 2-isomorphism ψ in $\mathbf{A}$



,

(iii) for every 2-morphism $\alpha\colon a \Rightarrow b$ in $\mathbf{A}$, and every pair of 2-isomorphisms $\psi\colon f \overset{\cong}{\Rightarrow} af'$ and $\varphi\colon f \overset{\cong}{\Rightarrow} bf'$ the following pasting equality holds in $\mathbf{A}$.



Proof (Proposition 5.5). We first prove that if I is a bi-initial object in $\mathbf{A}$ then conditions (i) and (ii') hold. It is clear that (i) holds by Remark 5.6 (i).

We prove (ii'). Given two morphisms $f\colon I \to A$ and $f'\colon I \to A$ in $\mathbf{A}$, by applying Remark 5.6 (ii) to the tuple (f, f', id_A), there is a unique 2-isomorphism $\psi\colon f' \overset{\cong}{\Rightarrow} f$.

Now, conversely, it is straightforward to see that Condition (ii') implies Remarks 5.6 (ii) and 5.6 (iii), since there is a unique 2-morphism between every pair of morphisms from I to any object. Note that every such 2-morphism is in fact invertible by (ii'). □

Since pseudo-slices are special cases of pseudo-commas, Proposition 4.6 may be specialised in this context to give the following result.

Corollary 5.7. *Let $\mathbb{A}$ be a double category, and $I \in \mathbb{A}$ be an object. Then there are canonical isomorphisms of 2-categories as in the following commutative triangles.*

$$\begin{array}{ccc} \mathbf{H}(I \downarrow\!\downarrow \mathbb{A}) & \xrightarrow{\cong} & I \downarrow \mathbf{H}\mathbb{A} \\ & {\scriptstyle \mathbf{H}\Pi} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & \mathbf{H}\mathbb{A} & \end{array} \qquad \begin{array}{ccc} \mathcal{V}(I \downarrow\!\downarrow \mathbb{A}) & \xrightarrow{\cong} & \mathrm{e}_I \downarrow \mathcal{V}\mathbb{A} \\ & {\scriptstyle \mathcal{V}\Pi} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & \mathcal{V}\mathbb{A} & \end{array}$$

Proof. This directly follows from Proposition 4.6, by taking $\mathbb{B} = \mathbb{1}$, $\mathbb{C} = \mathbb{A}$, $F = I\colon \mathbb{1} \to \mathbb{A}$ and $G = \mathrm{id}_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}$. Note that the image of the object $I\colon \mathbb{1} \to \mathbb{A}$ under $\mathcal{V}$ is given by $\mathcal{V}I = \mathrm{e}_I\colon \mathcal{V}(\mathbb{1}) = \mathbb{1} \to \mathcal{V}\mathbb{A}$. □

With this result and the fact that double trivial fibrations are exactly the double functors whose images under $\mathbf{H}$ and $\mathcal{V}$ are trivial fibrations, we may give a 2-categorical characterisation of double bi-initial objects by leveraging this fact as follows.

Theorem 5.8. *Let $\mathbb{A}$ be a double category, and $I \in \mathbb{A}$ be an object. The following statements are equivalent.*

(i) The object $I \in \mathbb{A}$ is double bi-initial.

(ii) The corresponding objects $I \in \mathbf{H}\mathbb{A}$ and $\mathrm{e}_I \in \mathcal{V}\mathbb{A}$ are bi-initial.

(iii) The corresponding object $\mathrm{e}_I \in \mathcal{V}\mathbb{A}$ is bi-initial.

Proof. By definition, an object $I \in \mathbb{A}$ is double bi-initial if and only if the projection double functor $\Pi\colon I \downarrow\!\downarrow \mathbb{A} \to \mathbb{A}$ is a double trivial fibration. By

Theorem 3.8, this is equivalent to saying that the induced 2-functors $\mathbf{H}\Pi$ and $\mathcal{V}\Pi$ are trivial fibrations, or equivalently that $\mathcal{V}\Pi$ alone is a trivial fibration. By Corollary 5.7, this holds if and only if the projection 2-functors $\pi\colon I \downarrow \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{A}$ and $\pi\colon \mathrm{e}_I \downarrow \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{A}$ are trivial fibrations, or equivalently that $\pi\colon \mathrm{e}_I \downarrow \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{A}$ alone is a trivial fibration. By definition of a bi-initial object, this holds if and only if the objects $I \in \mathbf{H}\mathbb{A}$ and $\mathrm{e}_I \in \mathcal{V}\mathbb{A}$ are bi-initial, or equivalently that $\mathrm{e}_I \in \mathcal{V}\mathbb{A}$ is bi-initial. □

Remark 5.9. This theorem served as the initial motivation for the definition of the functor $\mathcal{V}$ whose role is so central in this paper.

Observe that double bi-initial objects in a double category $\mathbb{A}$ have two aspects to their *weak* universal properties: one concerning objects and one concerning vertical morphisms. The former is entirely horizontal in nature and so is completely captured by the underlying horizontal 2-category $\mathbf{H}\mathbb{A}$. The latter, despite concerning vertical morphisms, does not in fact need the full strength of vertical composition in $\mathbb{A}$ to be expressed. Indeed, except for vertical composition by squares with trivial boundary of the form of Definition 3.4 (iii), this aspect of the weak universal property is also somehow horizontal. That is to say, the underlying *horizontal* 2-category $\mathbf{H}\,\mathbb{P}\mathrm{s}(\mathbb{V}2, \mathbb{A})$ is precisely the setting in which to capture this last data as it has vertical morphisms as objects and understands horizontal compositions of general squares. But this 2-category $\mathbf{H}\,\mathbb{P}\mathrm{s}(\mathbb{V}2, \mathbb{A})$ is exactly our $\mathcal{V}\mathbb{A}$!

5.3 Double bi-initial objects and tabulators

We have seen that double bi-initial objects may be detected through purely 2-categorical means. In this section we show that a substantial simplification of the 2-categorical criteria is possible when the double category in question has *tabulators*. These correspond to double limits of vertical morphisms and were introduced by Grandis and Paré in [5, §5.3]. In the presence of tabulators our result is as strong as possible: double bi-initial objects are precisely bi-initial objects in the underlying horizontal 2-category.

Definition 5.10. Let $\mathbb{A}$ be a double category, and $u\colon A \nrightarrow B$ be a vertical morphism in $\mathbb{A}$. A **tabulator** of u is a double limit of the double functor $u\colon \mathbb{V}2 \to \mathbb{A}$, where $\mathbb{V}2$ is the double category free on a vertical morphism. In other words, it is a pair $(\top u, \tau_u)$ of an object $\top u \in \mathbb{A}$ together with a square $\tau_u\colon (\mathrm{e}_{\top u} \, {}^{p}_{q} \, u)$ in $\mathbb{A}$ satisfying the following universal properties.

(i) For every square $\gamma\colon (\mathrm{e}_I \,{}^{f}_{g}\, u)$ in $\mathbb{A}$, there is a unique horizontal morphism $t\colon I \to \top u$ in $\mathbb{A}$ such that the following pasting equality holds.

$$\begin{array}{ccc} I & \xrightarrow{f} & A \\ \| & \gamma & \downarrow u \\ I & \xrightarrow[g]{} & B \end{array} \quad = \quad \begin{array}{ccccc} I & \xrightarrow{t} & \top u & \xrightarrow{p} & A \\ \| & \mathrm{e}_t & \| & \pi & \downarrow u \\ I & \xrightarrow[t]{} & \top u & \xrightarrow[q]{} & B \end{array}$$

(ii) For every tuple of squares $\gamma\colon (\mathrm{e}_I \,{}^{f}_{g}\, u)$, $\gamma'\colon (\mathrm{e}_{I'} \,{}^{f'}_{g'}\, u)$, $\theta_0\colon (v \,{}^{f'}_{f}\, \mathrm{e}_A)$ and $\theta_1\colon (v \,{}^{g'}_{g}\, \mathrm{e}_B)$ in $\mathbb{A}$ satisfying the following pasting equality,

$$\begin{array}{ccc} I' & \xrightarrow{f'} & A \\ v\downarrow & \theta_0 & \| \\ I & \xrightarrow{f} & A \\ \| & \gamma & \downarrow u \\ I & \xrightarrow[g]{} & B \end{array} \quad = \quad \begin{array}{ccc} I' & \xrightarrow{f'} & A \\ \| & \gamma' & \downarrow u \\ I' & \xrightarrow{g'} & B \\ v\downarrow & \theta_1 & \| \\ I & \xrightarrow[g]{} & B \end{array}$$

there is a unique square $\theta\colon (v \,{}^{t'}_{t}\, \mathrm{e}_{\top u})$, where the horizontal morphisms $t\colon I \to \top u$ and $t'\colon I' \to \top u$ are the unique horizontal morphisms given by (i) applied to γ and γ' respectively, such that θ satisfies the following pasting equalities.

$$\begin{array}{ccc} I' & \xrightarrow{f'} & A \\ v\downarrow & \theta_0 & \| \\ I & \xrightarrow[f]{} & A \end{array} \quad = \quad \begin{array}{ccccc} I' & \xrightarrow{t'} & \top u & \xrightarrow{p} & A \\ v\downarrow & \theta & \| & \mathrm{e}_p & \| \\ I & \xrightarrow[t]{} & \top u & \xrightarrow[p]{} & A \end{array}$$

$$\begin{array}{ccc} I' & \xrightarrow{g'} & B \\ v\downarrow & \theta_1 & \| \\ I & \xrightarrow[g]{} & B \end{array} \quad = \quad \begin{array}{ccccc} I' & \xrightarrow{t'} & \top u & \xrightarrow{q} & B \\ v\downarrow & \theta & \| & \mathrm{e}_q & \| \\ I & \xrightarrow[t]{} & \top u & \xrightarrow[q]{} & B \end{array}$$

We say that $\mathbb{A}$ **has tabulators** if there is a tabulator for each vertical morphism u in $\mathbb{A}$.

Theorem 5.11. *Let $\mathbb{A}$ be a double category with tabulators, and $I \in \mathbb{A}$ be an object. Then the following statements are equivalent.*

(i) The object I is double bi-initial in $\mathbb{A}$.

(ii) The object I is bi-initial in $\mathbf{H}\mathbb{A}$.

Proof. If I is double bi-initial in $\mathbb{A}$, then, by Theorem 5.8, I is bi-initial in $\mathbf{H}\mathbb{A}$.

Now suppose that I is bi-initial in $\mathbf{H}\mathbb{A}$. We prove that I satisfies Definition 5.1 (i-iv). First note that Remark 5.6 (i) and (ii) applied to $I \in \mathbf{H}\mathbb{A}$ correspond to Remark 5.3 (i) and (ii) applied to $I \in \mathbb{A}$. Therefore, it remains to show Remark 5.3 (iii) and (iv). Let $u\colon A \nrightarrow B$ be a vertical morphism and let $(\top u, \tau_u)$ be the tabulator of u. By Remark 5.6 (i) applied to the object $\top u$, there is a horizontal morphism $t\colon I \to \top u$. By the first universal property of tabulators, we get a square in $\mathbb{A}$

$$\begin{array}{ccc} I & \xrightarrow{f} & A \\ \Vert & \gamma & \downarrow u \\ I & \xrightarrow[g]{} & B \end{array}$$

as desired. This proves Remark 5.3 (iii).

Now suppose that we have squares in $\mathbb{A}$

$$\begin{array}{ccc} I & \xrightarrow{f} & A \\ \Vert & \gamma & \downarrow u \\ I & \xrightarrow[g]{} & B \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{f'} & A' \\ \Vert & \gamma' & \downarrow u' \\ I & \xrightarrow[g']{} & B' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{a} & A' \\ u \downarrow & \alpha & \downarrow u' \\ B & \xrightarrow[b]{} & B' \end{array}$$

and suppose $(\top u, \tau_u)$ and $(\top u', \tau_{u'})$ are tabulators for u and u' respectively. By the first universal property of tabulators, the squares γ and γ' uniquely correspond to horizontal morphisms $t\colon I \to \top u$ and $t'\colon I \to \top u'$ respectively. Moreover, the square α uniquely corresponds to a horizontal morphism $\top\alpha\colon \top u \to \top u'$. By applying Remark 5.6 (ii) to $t\colon I \to \top u$, $t'\colon I \to \top u'$, and $\top\alpha\colon \top u \to \top u'$, we get a square θ in $\mathbb{A}$ of the form

$$\begin{array}{ccccc} I & \xrightarrow{\quad t' \quad} & & & \top u' \\ \| & & \theta \; \cong & & \| \\ I & \xrightarrow[t]{} & \top u & \xrightarrow[\top \alpha]{} & \top u' \end{array} \quad .$$

By the second universal property of tabulators, this uniquely corresponds to squares $\psi := \theta_0$ and $\varphi := \theta_1$ satisfying the following pasting.

$$\begin{array}{ccccc} I & \xrightarrow{\quad f' \quad} & & & A' \\ \| & & \psi \cong & & \| \\ I & \xrightarrow{f} & A & \xrightarrow{a} & A' \\ \| & \gamma & u \downarrow & \alpha & \downarrow u' \\ I & \xrightarrow[g]{} & B & \xrightarrow[b]{} & B' \end{array} \quad = \quad \begin{array}{ccccc} I & \xrightarrow{\quad f' \quad} & & & A' \\ \| & & \gamma' & & \downarrow u' \\ I & \xrightarrow{\quad g' \quad} & & & B' \\ \| & & \varphi \cong & & \| \\ I & \xrightarrow[g]{} & B & \xrightarrow[b]{} & B' \end{array}$$

Note that ψ and φ are the unique squares for the tuples (f, f', a) and (g, g', b) respectively; see Proposition 5.5 (ii'). This shows Remark 5.3 (iv). □

Corollary 5.12. *Let $\mathbb{A}$ be a double category with tabulators, and $I \in \mathbb{A}$ be an object. Then the object I is bi-initial in $\mathbf{H}\mathbb{A}$ if and only if the corresponding object e_I is bi-initial in $\mathcal{V}\mathbb{A}$.*

Proof. By Theorem 5.11, I is bi-initial in $\mathbf{H}\mathbb{A}$ if and only if I is double bi-initial in $\mathbb{A}$. By Theorem 5.8, this holds if and only if e_I is bi-initial in $\mathcal{V}\mathbb{A}$. □

6. Bi-representations of normal pseudo-functors

In Section 6.1 we state and prove our main result characterising bi-representations of normal pseudo-functors $F\colon \mathrm{C}^{\mathrm{op}} \to$ **Cat** as various sorts of bi-initial objects, where **Cat** is the 2-category of categories, functors, and natural transformations. We give two flavours of such a theorem, one stated in the language of double categories, and the other stated completely in terms of 2-categories. The former predictably relies on the double category of elements of F construction, but in the latter case, we will define from the data of a normal pseudo-functor F not only the 2-category of *elements* of F, but also a 2-category of *morphisms* of F. Moreover, by specialising Theorem 5.8, we see that the latter 2-category subsumes the former for this purpose: bi-representations of a normal pseudo-functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ are precisely bi-initial objects of a particular form in the 2-category of morphisms of F.

We then show in Section 6.2 that, when the 2-category $\mathbf{C}$ has tensors by $\mathbb{2}$ and the normal pseudo-functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ preserves them, the expected characterisation actually holds: bi-representations of F are now precisely bi-initial objects in the 2-category of elements of F.

6.1 The general case

Let us begin by defining the central objects at issue.

Definition 6.1. Let $\mathbf{C}$ be a 2-category, and $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a normal pseudo-functor. A **bi-representation** of F is a pair (I,ρ) of an object $I \in \mathbf{C}$ and a pseudo-natural adjoint equivalence $\rho_-\colon \mathbf{C}(-,I) \overset{\simeq}{\Rightarrow} F$, i.e., an adjoint equivalence in the 2-category $\mathrm{Ps}(\mathbf{C}^{\mathrm{op}},\mathbf{Cat})$.

Remark 6.2. Recall that an equivalence in a 2-category can always be promoted to an *adjoint* equivalence (see, e.g. [16, Lemma 2.1.11]). Therefore, by requiring the pseudo-natural equivalence in Definition 6.1 to be adjoint, we do not lose any generality while simultaneously making the data easier to handle in the forthcoming proofs.

To the data of such a normal pseudo-functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ we will associate a *double category of elements*. This double category will play an analogous role to the classical category of elements in detecting representations.

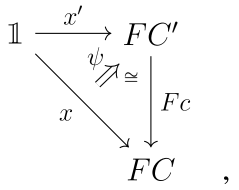
Definition 6.3. Let $\mathbf{C}$ be a 2-category, and $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a normal pseudo-functor. The **double category of elements** $\mathbb{el}(F)$ of F is defined to be the pseudo-slice double category $\mathbb{1} \downarrow\!\downarrow \mathbb{H}F$ induced by the cospan

$$\mathbb{1} \xrightarrow{\mathbb{1}} \mathbb{H}\mathbf{Cat} \xleftarrow{\mathbb{H}F} \mathbb{H}\mathbf{C}^{\mathrm{op}} .$$

More explicitly, it is the double category whose

(i) objects are pairs (C,x) of an object $C \in \mathbf{C}$ and a functor $x\colon \mathbb{1} \to FC$, i.e., an object $x \in FC$,

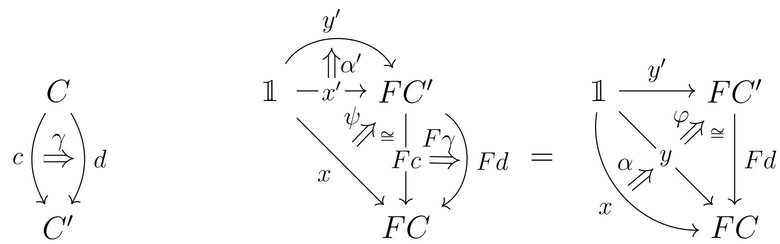
(ii) horizontal morphisms $(c, \psi)\colon (C', x') \to (C, x)$ comprise the data of a morphism $c\colon C \to C'$ of $\mathbf{C}$ and a natural isomorphism ψ of the form



i.e., an isomorphism $\psi\colon x \xrightarrow{\cong} (Fc)x'$ in FC,

(iii) vertical morphisms $\alpha\colon (C, x) \nrightarrow (C, y)$ are natural transformations $\alpha\colon x \Rightarrow y$ of functors $x, y\colon \mathbb{1} \to FC$, i.e., morphisms $\alpha\colon x \to y$ in FC,

(iv) squares $\gamma\colon (\alpha' \,{}^{(c,\psi)}_{(d,\varphi)}\, \alpha)$ comprise the data of a 2-morphism $\gamma\colon c \Rightarrow d$ of $\mathbf{C}$, as displayed below-left, which satisfies the below-right pasting equality,



i.e., the following diagram in FC is commutative.

$$\begin{array}{ccccc} x & & \xrightarrow{\quad\alpha\quad} & & y \\ {\scriptstyle\psi}\downarrow{\scriptstyle\cong} & & & & {\scriptstyle\cong}\downarrow{\scriptstyle\varphi} \\ (Fc)x' & \xrightarrow[(Fc)\alpha']{} & (Fc)y' & \xrightarrow[(F\gamma)_{y'}]{} & (Fd)y' \end{array}$$

Much like in the 1-dimensional case, we are able to construct from a normal pseudo-functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ the 2-category of elements $\mathbf{el}(F)$

of F, but new here is the 2-category of *morphisms* of F. As we shall see, the joint properties of these 2-categories may be leveraged to successfully characterise bi-representations.

Definition 6.4. Let $\mathbf{C}$ be a 2-category, and $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a normal pseudo-functor. We define the following two 2-categories associated to F.

- The 2**-category of elements** $\mathbf{el}(F)$ of F is defined to be $\mathbf{H}\mathbb{el}(F)$.
- The 2**-category of morphisms** $\mathbf{mor}(F)$ of F is defined to be $\mathcal{V}\mathbb{el}(F)$.

The ardently 2-categorical reader may be dismayed by the foray into the realm of double categories to give the above definition. In the coming discussion we will find that we are able to comfortably re-seat these 2-categories as the result of purely 2-categorical considerations.

Observe that exponentiation by the category $\mathbb{2} = \{0 \to 1\}$ gives rise to the classical functor $\mathrm{Ar} := (-)^{\mathbb{2}}\colon \mathsf{Cat} \to \mathsf{Cat}$, the *category of arrows* functors, where Cat is the category of categories and functors.

Definition 6.5. We define the functor $\mathrm{Ar}_*\colon \mathsf{2Cat}_{\mathsf{nps}} \to \mathsf{2Cat}_{\mathsf{nps}}$ as follows. It sends a 2-category $\mathbf{C}$ to the 2-category $\mathrm{Ar}_*\mathbf{C}$ with the same objects as $\mathbf{C}$ and hom-categories $\mathrm{Ar}_*\mathbf{C}(C, C') := \mathrm{Ar}(\mathbf{C}(C, C'))$ for each pair of objects $C, C' \in \mathbf{C}$. That is, a morphism in $\mathrm{Ar}_*\mathbf{C}$ is a 2-morphism of $\mathbf{C}$ and a 2-morphism in $\mathrm{Ar}_*\mathbf{C}$ is a commutative square of vertical composites of 2-morphisms in $\mathbf{C}$.

Given a normal pseudo-functor $F\colon \mathbf{C} \to \mathbf{D}$, the normal pseudo-functor Ar_*F acts as F on objects and as $\mathrm{Ar}\, F$ on hom-categories. The compositors of Ar_*F are given component-wise by the compositors of F.

Remark 6.6. The functor Ar_* is a shadow of our double categorical approach of the previous sections. Indeed, we have the equality of functors $\mathcal{V}\mathbb{H} = \mathrm{Ar}_*$ to complement $\mathbf{H}\mathbb{H} = \mathrm{id}_{\mathsf{2Cat}_{\mathsf{nps}}}$.

Recall that $\mathbb{el}(F)$ is defined as the pseudo-slice double category $\mathbb{1}{\downarrow\downarrow}\mathbb{H}F$. This, coupled with the fact that $\mathbb{H}$ and $\mathcal{V}$ preserve slices allows us to give the following, purely 2-categorical formulations of the 2-categories of elements and morphisms of a normal pseudo-functor.

Remark 6.7. If $\mathbf{C}$ is a 2-category and $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ is a normal pseudo-functor then, by Corollary 5.7 and Remark 6.6, the 2-categories $\mathbf{el}(F)$ and

$\mathbf{mor}(F)$ are isomorphic to the pseudo-slice 2-categories induced by the cospans

$$\mathbb{1} \xrightarrow{\mathbb{1}} \textsf{Cat} \xleftarrow{F} \mathbf{C}^{\mathrm{op}} \text{ and } \mathbb{1} \xrightarrow{\mathbb{1}} \mathrm{Ar}_*\textsf{Cat} \xleftarrow{\mathrm{Ar}_*F} \mathrm{Ar}_*\mathbf{C}^{\mathrm{op}} ,$$

respectively. In particular, $\mathbf{el}(F)$ is the pseudo-type version of the usual 2-category of elements of F.

We are now in a position to give the central result of this paper, a 2-dimensional analogue to the classical relationship between representations and initial objects. The equivalence between (i) and (iii) below gives the promised 2-categorical account of the theorem, and while it may be derived directly, we will find that our work of the previous sections allows for a more efficient approach via (ii).

Theorem 6.8. *Let* $\mathbf{C}$ *be a* 2*-category, and* $(F,\phi)\colon \mathbf{C}^{\mathrm{op}} \to \textsf{Cat}$ *be a normal pseudo-functor. The following statements are equivalent.*

(i) The normal pseudo-functor F *has a bi-representation* (I,ρ)*.*

(ii) There is an object $I \in \mathbf{C}$ *together with an object* $i \in FI$ *such that* (I,i) *is double bi-initial in* $\mathbb{el}(F)$*.*

(iii) There is an object $I \in \mathbf{C}$ *together with an object* $i \in FI$ *such that* (I,i) *is bi-initial in* $\mathbf{el}(F)$ *and* (I,id_i) *is bi-initial in* $\mathbf{mor}(F)$*.*

(iv) There is an object $I \in \mathbf{C}$ *together with an object* $i \in FI$ *such that* (I,id_i) *is bi-initial in* $\mathbf{mor}(F)$*.*

Remark 6.9. Note that conditions (iii) and (iv) are inherently 2-categorical statements. Let us think of an object of a 2-category $\mathbf{A}$ as a normal pseudo-functor $\mathbb{1} \to \mathbf{A}$. Recall that an object $\mathbb{1} \to \mathbf{A}$ is bi-initial if and only if the projection $\mathbf{A} \downarrow I \to \mathbf{A}$ is a trivial fibration, where the slice and its projection are obtained as a certain pullback. Thus condition (iii) instructs us to construct the slice $\mathbf{el}(F) \downarrow (I,i)$ as well as the slice $\mathbf{mor}(F) \downarrow (I,\mathrm{id}_i)$, where the latter is obtained by considering the object given by the composite $\mathbb{1} \to \mathbf{el}(F) \to \mathbf{mor}(F)$, and ensures that both resulting projections are trivial fibrations. Similarly, condition (iv) concerns the latter pullback only, and ensures that its projection functor is a trivial fibration.

First note that the equivalence of conditions (ii), (iii), and (iv) follows directly from Theorem 5.8. The rest of this section will be devoted to the proof of the equivalence between conditions (i) and (ii). For this, we first introduce the following "unique filler" lemma to make efficient the proof of the forward implication.

Lemma 6.10. *Under the assumptions of Theorem 6.8 (i), for every pair of horizontal morphisms* $(f,\psi)\colon (I,i) \to (C,x)$ *and* $(g,\varphi)\colon (I,i) \to (C,y)$*, and every vertical morphism* $\alpha\colon (C,x) \to\!\!\!\!\!+\, (C,y)$ *in* $\mathbb{el}(F)$*, there is a unique square in* $\mathbb{el}(F)$ *of the below form.*

$$\begin{array}{ccc} (I,i) & \xrightarrow{(f,\psi)} & (C,x) \\ \| & \gamma & \big\downarrow{\scriptstyle +}\,\alpha \\ (I,i) & \xrightarrow[(g,\varphi)]{} & (C,y) \end{array} \quad .$$

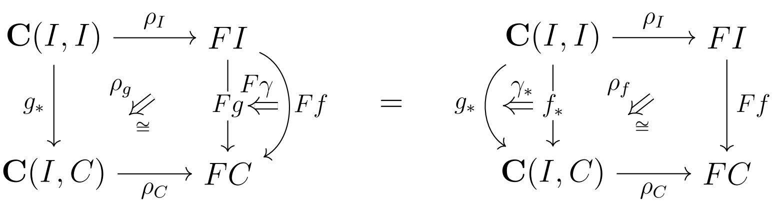
Proof. Suppose that (I,ρ) is a bi-representation of a normal pseudo-functor $(F,\phi)\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$, $(f,\psi)\colon (I,i) \to (C,x)$ and $(g,\varphi)\colon (I,i) \to (C,y)$ are horizontal morphisms in $\mathbb{el}(F)$, and $\alpha\colon (C,x) \to\!\!\!\!\!+\, (C,y)$ is a vertical morphism in $\mathbb{el}(F)$. Define υ to be the unique morphism of FC fitting in the following diagram,

$$\begin{array}{ccc} (Ff)i \xrightarrow{\psi^{-1}} x & \xrightarrow{\alpha} & y \xrightarrow{\varphi} (Fg)i \\ {\scriptstyle (\rho_f)_{\mathrm{id}_I}}\big\downarrow\cong & & \cong\big\downarrow{\scriptstyle (\rho_g)_{\mathrm{id}_I}} \\ \rho_C(f) & \xrightarrow[\upsilon]{\quad\quad\quad} & \rho_C(g) \end{array}$$

where $\rho_f\colon (Ff)\rho_I \overset{\cong}{\Rrightarrow} \rho_C\mathbf{C}(I,f)$ and $\rho_g\colon (Fg)\rho_I \overset{\cong}{\Rrightarrow} \rho_C\mathbf{C}(I,g)$ are the 2-isomorphism components of ρ at f and g. By Definition 6.3 (iv), a square $\gamma\colon (\mathrm{id}_i \,{}^{(f,\psi)}_{(g,\varphi)}\, \alpha)$ in $\mathbb{el}(F)$ is the data of a 2-morphism $\gamma\colon f \Rightarrow g$ of $\mathbf{C}$ such that

$$(Ff)i \xrightarrow{(F\gamma)_i} (Fg)i \quad = \quad (Ff)i \xrightarrow{\psi^{-1}} x \xrightarrow{\alpha} y \xrightarrow{\varphi} (Fg)i.$$

Therefore, we may deduce that this equation holds if and only if we have that $(\rho_g)_{\mathrm{id}_I}(F\gamma)_i = \upsilon(\rho_f)_{\mathrm{id}_I}$, by definition of υ. The left-hand composite of this equality appears as the result of evaluating the below-left pasting at id_I, and this pasting is equal to the below-right pasting by pseudo-naturality of ρ.



We deduce therefore that

$$(\rho_g)_{\mathrm{id}_I}(F\gamma)_i = \upsilon(\rho_f)_{\mathrm{id}_I} \text{ iff } (\rho_C(\gamma))(\rho_f)_{\mathrm{id}_I} = \upsilon(\rho_f)_{\mathrm{id}_I} \text{ iff } \rho_C(\gamma) = \upsilon.$$

All in all then, $\gamma\colon (\mathrm{id}_i \,{}^{(f,\psi)}_{(g,\varphi)}\, \alpha)$ is a square in $\mathbb{el}(F)$ if and only if $\rho_C(\gamma) = \upsilon$. Since $\rho_C\colon \mathbf{C}(I, C) \to FC$ is an equivalence and is therefore fully faithful on morphisms, there is a unique such γ. □

With this lemma established, the proof of the implication (i)⇒(ii) of Theorem 6.8 is readily given.

Proof (Theorem 6.8, (i)⇒(ii)). Suppose (i), that is, we have a specified bi-representation (I, ρ) of F. From this data we will select an object $i \in FI$ and demonstrate that (I, i) is double bi-initial in $\mathbb{el}(F)$. To begin, let us define $i \in FI$ as $i := \rho_I(\mathrm{id}_I)$. We address each of conditions (i-iv) of Definition 5.1 in turn.

Let (C, x) be an object of $\mathbb{el}(F)$. Since $\rho_C\colon \mathbf{C}(C, I) \to FC$ is an equivalence and $x \in FC$, there is a morphism $f\colon C \to I$ in $\mathbf{C}$ together with a isomorphism $\overline{\psi}\colon x \xrightarrow{\cong} \rho_C(f)$ in FC. By post-composing with the inverse of $(\rho_f)_{\mathrm{id}_I}\colon (Ff)i \xrightarrow{\cong} \rho_C(f)$, arising from the 2-isomorphism component $\rho_f\colon (Ff)\rho_I \overset{\cong}{\Rightarrow} \rho_C\mathbf{C}(f, I)$ of ρ at f, we find a horizontal morphism $(f, (\rho_f)^{-1}_{\mathrm{id}_I}\overline{\psi})\colon (I, i) \to (C, x)$ in $\mathbb{el}(F)$, and so we have established Remark 5.3 (i).

The rest of conditions (ii-iv) each follow by applying Lemma 6.10 above, which we elaborate below. First, Remark 5.3 (ii) grants us the existence of a boundary of $\mathbb{el}(F)$ of the form depicted below, and charges us with finding a unique, vertically invertible filler.

$$\begin{array}{ccccc}
(I, i) & & \xrightarrow{(f,\psi)} & & (C, x) \\
\| & & & & \| \\
(I, i) & \xrightarrow[(f',\psi')]{} & (C', x') & \xrightarrow[(c,\varphi)]{} & (C, x)
\end{array}$$

By composing the bottom horizontal morphisms we see that Lemma 6.10 supplies us with a unique filler for this square. That this filler is vertically invertible follows from considering the vertical opposite of the above square combined with further applications of Lemma 6.10.

Next, Remark 5.3 (iii) grants us a vertical morphism $\alpha\colon (C,x) \nrightarrow (C,y)$ of $\mathbb{el}(F)$ and demands the existence of a square from $(I,i) \,\ensuremath{=\!\!\!|}\, (I,i)$ to α. By our construction of Remark 5.3 (i) above, we may give horizontal morphisms $(f,\psi)\colon (I,i) \to (C,x)$ and $(g,\varphi)\colon (I,i) \to (C,y)$, and thus produce the below boundary in $\mathbb{el}(F)$. An application of Lemma 6.10 shows (iii).

$$\begin{array}{ccc} (I,i) & \xrightarrow{(f,\psi)} & (C,x) \\ \| & & \downarrow \alpha \\ (I,i) & \xrightarrow[(g,\varphi)]{} & (C,y) \end{array}$$

Finally, we must show that Remark 5.3 (iv) holds. That is, we must demonstrate that there is an equality of squares filling a fixed boundary. Fortunately we may apply Lemma 6.10 to this boundary and so conclude the proof. □

We conclude this section by proving the reverse implication.

Proof (Theorem 6.8, (ii)⇒(i)). Suppose (ii), that is, we have a double bi-initial object (I,i) in $\mathbb{el}(F)$. From this data we will construct equivalences $\rho_C\colon \mathbf{C}(C,I) \to FC$ for each $C \in \mathbf{C}$ and then show that they assemble into a pseudo-natural transformation. By a standard result of 2-categories, any equivalence is canonically rectifiable into an adjoint equivalence and so we do not trouble ourselves with additional work after giving ρ.

For a fixed $C \in \mathbf{C}$, let us define the functor $\rho_C\colon \mathbf{C}(C,I) \to FC$ on objects $f\colon C \to I$ as $\rho_C(f) := (Ff)i$ and on morphisms $\gamma\colon f \Rightarrow g$ as $\rho_C(\gamma) := (F\gamma)_i$. As F respects vertical composition of 2-morphisms strictly, it is clear that ρ_C is a functor by construction.

With the functors ρ_C defined, we now show that each of these functors is an equivalence. To that end, let us fix $C \in \mathbf{C}$ and $x \in FC$. Observe that (C,x) is an object of $\mathbb{el}(F)$ so that, since (I,i) is double bi-initial in $\mathbb{el}(F)$, there is a horizontal morphism $(f,\psi)\colon (I,i) \to (C,x)$ in $\mathbb{el}(F)$. This is precisely the data of an object $f \in \mathbf{C}(C,I)$ and an isomorphism

$\psi\colon \rho_C(f) = (Ff)i \xrightarrow{\cong} x$, which shows that ρ_C is essentially surjective on objects. To see that each ρ_C is fully faithful on morphisms, let $f, g\colon C \to I$ be objects in $\mathbf{C}(C, I)$ and $\alpha\colon \rho_C(f) \to \rho_C(g)$ be a morphism between their images in FC. This data is equivalently a pair of horizontal morphisms

$$(f, \mathrm{id}_{\rho_C(f)})\colon (I, i) \to (C, \rho_C(f)) \quad \text{and} \quad (g, \mathrm{id}_{\rho_C(g)})\colon (I, i) \to (C, \rho_C(g)),$$

since $\rho_C(f) = (Ff)i$ and $\rho_C(g) = (Fg)i$ by definition, together with a vertical morphism $\alpha\colon (C, \rho_C(f)) \to\!\!\!\!\!\!\mapstochar\;\; (C, \rho_C(g))$ in $\mathbb{el}(F)$. Since (I, i) is double bi-initial in $\mathbb{el}(F)$, by Proposition 5.2 (ii'), there is a unique square in $\mathbb{el}(F)$ of the form

$$\begin{array}{ccc} (I,i) & \xrightarrow{(f,\mathrm{id}_{\rho_C(f)})} & (C,\rho_C(f)) \\ \| & \gamma & \downarrow \alpha \\ (I,i) & \xrightarrow[(g,\mathrm{id}_{\rho_C(g)})]{} & (C,\rho_C(g)) \end{array} \quad ,$$

that is, a unique 2-morphism $\gamma\colon f \Rightarrow g$ such that $\rho_C(\gamma) = (F\gamma)_i = \alpha$. This shows fully faithfulness of ρ_C.

Now that we have a collection of object-wise equivalences ρ_C we seek to construct the data of the pseudo-naturality comparison natural isomorphisms $\rho_c\colon (Fc)\rho_{C'} \overset{\cong}{\Rightarrow} \rho_C \mathbf{C}(c, I)$ depicted below, for each morphism $c\colon C \to C'$ in $\mathbf{C}$.

$$\begin{array}{ccc} \mathbf{C}(C', I) & \xrightarrow{\rho_{C'}} & FC' \\ {\scriptstyle \mathbf{C}(c,I)}\downarrow & \overset{\rho_c}{\swarrow}\cong & \downarrow {\scriptstyle Fc} \\ \mathbf{C}(C, I) & \xrightarrow[\rho_C]{} & FC \end{array}$$

For $f \in \mathbf{C}(C', I)$ observe that we have $(Fc)\rho_{C'}(f) = (Fc)(Ff)i$ and $\rho_C\mathbf{C}(c, I)(f) = F(fc)i$, so that we can set ρ_c to be $(\phi_{c,-})_i$, the compositor of F at $(c, -)$ evaluated at i. This satisfies all of the required properties of pseudo-naturality. □

In fact, we have additionally proven that bi-representations (I, ρ) of a normal pseudo-functor are determined up to isomorphism by their values on id_I. This conclusion may be seen as a special case of a suitable 2-dimensional Yoneda lemma.

Corollary 6.11. *Let* $\mathbf{C}$ *be a* 2*-category, and* $F\colon \mathbf{C}^{\mathrm{op}} \to \mathsf{Cat}$ *be a normal pseudo-functor. Suppose that* (I, ρ) *is a bi-representation of* F*. Then there is a canonical bi-representation* $(I, \overline{\rho})$ *of* F *given by*

$$\overline{\rho}_C = (F-)(\rho_I(\mathrm{id}_I))\colon \mathbf{C}(C, I) \to FC,$$

for every $C \in \mathbf{C}$*. Moreover, we have that* $\overline{\rho} \cong \rho$*.*

Proof. The construction is given by tracing the proofs above of Theorem 6.8 through (i)⇒(ii) and then (ii)⇒(i). Finally, the isomorphism $\overline{\rho} \cong \rho$ is given by the 2-isomorphism components $(\rho_f)_{\mathrm{id}_I}\colon (Ff)\rho_I(\mathrm{id}_I) \xRightarrow{\cong} \rho_C(f)$ of ρ itself evaluated at id_I, for every $f\colon C \to I$ in $\mathbf{C}$. □

Remark 6.12. In particular, when F is a strict 2-functor, without loss of generality a bi-representation of F may be taken to be a 2-natural adjoint equivalence. Indeed, the bi-representation constructed in Corollary 6.11 is 2-natural.

6.2 The case in presence of tensors by $\mathbb{2}$

Finally we explore a substantial improvement of Theorem 6.8 which is possible when the 2-category $\mathbf{C}$ has *tensors*, defined below, which are preserved by F.

Definition 6.13. Let $\mathbf{C}$ be a 2-category, $C \in \mathbf{C}$ be an object, and $\mathcal{A}$ be a category.

A **power of** C **by** $\mathcal{A}$ is a weighted 2-limit of the 2-functor $C\colon \mathbb{1} \to \mathbf{C}$ by the weight $\mathcal{A}\colon \mathbb{1} \to \mathsf{Cat}$. In other words, it is a pair $(\mathcal{A} \pitchfork C, \lambda)$ of an object $\mathcal{A} \pitchfork C \in \mathbf{C}$ and a functor $\lambda\colon \mathcal{A} \to \mathbf{C}(\mathcal{A} \pitchfork C, C)$ such that, for every object $C' \in \mathbf{C}$, pre-composition by λ induces an isomorphism of categories

$$\lambda_* \circ \mathbf{C}(-, C)\colon \mathbf{C}(C', \mathcal{A} \pitchfork C) \xrightarrow{\cong} \mathsf{Cat}(\mathcal{A}, \mathbf{C}(C', C)).$$

We say that $\mathbf{C}$ **has powers by** $\mathcal{A}$ if there is a power of C by $\mathcal{A}$ for each object $C \in \mathbf{C}$.

Dually, a **tensor of** C **by** $\mathcal{A}$ is a power of C by $\mathcal{A}$ in the opposite 2-category $\mathbf{C}^{\mathrm{op}}$. In other words, it is a pair $(C \otimes \mathcal{A}, \zeta)$ of an object $C \otimes \mathcal{A} \in \mathbf{C}$ and a functor $\zeta\colon \mathcal{A} \to \mathbf{C}(C, C \otimes \mathcal{A})$ such that, for every object $C' \in \mathbf{C}$, pre-composition by ζ induces an isomorphism of categories

$$\zeta^* \circ \mathbf{C}(C, -)\colon \mathbf{C}(C \otimes \mathcal{A}, C') \xrightarrow{\cong} \mathsf{Cat}(\mathcal{A}, \mathbf{C}(C, C'))\ .$$

We say that $\mathbf{C}$ **has tensors by** $\mathcal{A}$ if there is a tensor of C by $\mathcal{A}$ for each object $C \in \mathbf{C}$.

Remark 6.14. Powers may be viewed as a lower-dimensional shadow of the double categorical notion of tabulators seen in Definition 5.10. Indeed, a power of an object $C \in \mathbf{C}$ by the category $\mathbb{2} = \{0 \to 1\}$ in a 2-category $\mathbf{C}$ is precisely a tabulator of the vertical identity e_C in its associated horizontal double category $\mathbb{H}\mathbf{C}$; see [4, Exercise 5.6.2 (c)]. In particular, tabulators in $\mathbb{H}\mathbf{C}^{\mathrm{op}}$ correspond to tensors by $\mathbb{2}$ in $\mathbf{C}$.

From the universal property of powers, one can see that the 2-category **Cat** has powers by any category $\mathcal{A}$ given by $\mathcal{A} \pitchfork \mathcal{C} := \mathbf{Cat}(\mathcal{A}, \mathcal{C})$. Given a 2-category $\mathbf{C}$ with tensors by a category $\mathcal{A}$, we then say that a normal pseudo-functor $F\colon \mathbf{C}^{\mathrm{op}} \to$ **Cat preserves powers by** $\mathcal{A}$ if, for every object $C \in \mathbf{C}$, we have an isomorphism of categories $F(C \otimes \mathcal{A}) \cong \mathbf{Cat}(\mathcal{A}, FC)$ which is natural with respect to the defining cones.

Theorem 6.15. *Let $\mathbf{C}$ be a 2-category which has tensors by $\mathbb{2}$, and let $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a normal pseudo-functor which preserves powers by $\mathbb{2}$. Then the following statements are equivalent.*

(i) The normal pseudo-functor F has a bi-representation (I, ρ).

(ii) There is an object $I \in \mathbf{C}$ together with an object $i \in FI$ such that (I, i) is double bi-initial in $\mathbb{el}(F)$.

(iii) There is an object $I \in \mathbf{C}$ together with an object $i \in FI$ such that (I, i) is bi-initial in $\mathbf{el}(F)$.

In particular, (I, i) is bi-initial in $\mathbf{el}(F)$ if and only if (I, id_i) is bi-initial in $\mathbf{mor}(F)$.

In order to prove this result we will make use of the following lemma.

Lemma 6.16. *Let $\mathbf{C}$ be a 2-category with tensors by $\mathbb{2}$, and $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a normal pseudo-functor which preserves powers by $\mathbb{2}$. Then the double category $\mathbb{el}(F)$ has tabulators.*

Proof. Suppose that $\alpha\colon (C, x) \nrightarrow (C, y)$ is a vertical morphism in $\mathbb{el}(F)$ and $(C \otimes \mathbb{2}, \zeta)$ is a tensor of C by $\mathbb{2}$. Since ζ is a functor $\zeta\colon \mathbb{2} \to \mathbf{C}(C, C \otimes \mathbb{2})$, it corresponds to a 2-morphism

$$C \underset{\zeta_1}{\overset{\zeta_0}{\Downarrow\zeta}} C \otimes \mathbb{2} \quad .$$

Moreover, the morphism $\alpha\colon x \to y$ in FC is equivalently given by a functor $\alpha\colon \mathbb{2} \to FC$ and therefore it corresponds to an object $\overline{\alpha} \in F(C \otimes \mathbb{2})$ as $\mathbf{Cat}(\mathbb{2}, FC) \cong F(C \otimes \mathbb{2})$. We set $\top u := (C \otimes \mathbb{2}, \overline{\alpha}) \in \mathbb{el}(F)$ and τ_u to be the following square in $\mathbb{el}(F)$.

$$\begin{array}{ccc} (C \otimes \mathbb{2}, \overline{\alpha}) & \xrightarrow{(\zeta_0, \mathrm{id}_x)} & (C, x) \\ \Vert & \zeta & \downarrow \alpha \\ (C \otimes \mathbb{2}, \overline{\alpha}) & \xrightarrow[(\zeta_1, \mathrm{id}_y)]{} & (C, y) \end{array}$$

We show that it satisfies the universal properties of tabulators of Definition 5.10. Let

$$\begin{array}{ccc} (C', x') & \xrightarrow{(c, \psi)} & (C, x) \\ \Vert & \gamma & \downarrow \alpha \\ (C', x') & \xrightarrow[(d, \varphi)]{} & (C, y) \end{array}$$

be a square in $\mathbb{el}(F)$. By the universal property of tensors, the 2-morphism $\gamma\colon c \Rightarrow d$ corresponds to a morphism $\overline{\gamma}\colon C \otimes \mathbb{2} \to C'$. Moreover, note that the pair (ψ, φ) gives an isomorphism in $\mathbf{Cat}(\mathbb{2}, FC)$ from α to $(F\gamma)_{x'}$, and since F preserves powers by $\mathbb{2}$, then (ψ, φ) corresponds to an isomorphism $\overline{(\psi, \varphi)}\colon \overline{\alpha} \cong (F\overline{\gamma})_{x'}$ in $F(C \otimes \mathbb{2})$. We get the required horizontal morphism $(\overline{\gamma}, \overline{(\psi, \varphi)})\colon (C', x') \to (C \otimes \mathbb{2}, \overline{\alpha})$ for Definition 5.10 (i).

Similarly, Definition 5.10 (ii) follows from the fact that 2-morphisms in $\mathbf{C}$ of the form

$$C \otimes \mathbb{2} \underset{\overline{\gamma'}}{\overset{\overline{\gamma}}{\Downarrow\overline{\theta}}} C'$$

uniquely correspond to 2-morphisms θ_0, θ_1 between morphisms $C \to C'$ in $\mathbf{C}$ such that $\gamma\theta_0 = \theta_1\gamma'$, by the universal property of tensors. □

Proof (Theorem 6.15). First note that (i) and (ii) are equivalent by Theorem 6.8.

To see that (ii) and (iii) are equivalent consider the following argument. By Lemma 6.16, the double category $\mathbb{el}(F)$ admits tabulators. Thus, by Theorem 5.11, an object (I, i) is double bi-initial in $\mathbb{el}(F)$ if and only if (I, i) is bi-initial in $\mathbf{el}(F)$.

Finally, that (I, i) is bi-initial in $\mathbf{el}(F)$ if and only if (I, id_i) is bi-initial in $\mathbf{mor}(F)$ follows from Corollary 5.12. □

7. Applications to bi-adjunctions and weighted bi-limits

Now that we have satisfied ourselves with the characterisation of Theorem 6.8 of bi-representations of normal pseudo-functor we focus now on two formal applications. In Section 7.1, we will leverage some 2-dimensional arguments to give a characterisation of bi-adjunctions in terms of bi-terminal objects in pseudo-slices. Then, in Section 7.2 we will connect to the counterexamples given in [3] by proving a correct characterisation of bi-limits in terms of bi-terminal objects in pseudo-slices, specialising the supporting theorem in the same section about weighted bi-limits. In both sections we will additionally give improvements on these results by specialising Theorem 6.15 when the 2-categories at issue have tensors by $\mathbb{2}$, which in the case of weighted bi-limits subsumes a known special case.

7.1 Bi-adjunctions

We begin by introducing the notion of a bi-adjunction.

Definition 7.1. Let $\mathbf{C}$ and $\mathbf{D}$ be 2-categories. A **bi-adjunction** between $\mathbf{C}$ and $\mathbf{D}$ comprises the data of normal pseudo-functors $L\colon \mathbf{C} \to \mathbf{D}$ and $R\colon \mathbf{D} \to \mathbf{C}$, and adjoint equivalences of categories

$$\Phi_{C,D}\colon \mathbf{C}(C, RD) \xrightarrow{\simeq} \mathbf{D}(LC, D)$$

pseudo-natural in $C \in \mathbf{C}^{\mathrm{op}}$ and $D \in \mathbf{D}$.

Remark 7.2. We wish to draw the reader's attention to the following relationship between bi-adjunctions and bi-representations. If L and R are embroiled in a bi-adjunction, then in particular for each object $D \in \mathbf{D}$ we may observe that we have a bi-representation

$$\Phi_{-,D}\colon \mathbf{C}(-, RD) \overset{\simeq}{\Rrightarrow} \mathbf{D}(L-, D)$$

of the normal pseudo-functor $\mathbf{D}(L-, D)$. In this sense, bi-adjunctions are "locally" bi-representations.

The major goal of this section is to provide a converse to the above observation. That is, we will concentrate our efforts on establishing that being "locally bi-represented" is, in fact, enough to determine a right bi-adjoint in the sense of Definition 7.1. Such a result is of course expected by analogy to the ordinary categorical version. Giving such a formulation of bi-adjunctions in terms of bi-representations allows us to apply Theorem 6.8 and thereby give a characterisation of bi-adjunctions in terms of bi-terminal objects in pseudo-slices.

Theorem 7.3. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be* 2*-categories, and* $(L, \delta)\colon \mathbf{C} \to \mathbf{D}$ *be a normal pseudo-functor. The following statements are equivalent.*

(i) The normal pseudo-functor $L\colon \mathbf{C} \to \mathbf{D}$ *admits a right bi-adjoint* $R\colon \mathbf{D} \to \mathbf{C}$.

(ii) For all objects $D \in \mathbf{D}$*, there is a bi-representation* (RD, Ψ^D) *of* $\mathbf{D}(L-, D)$*, where* $\Psi^D\colon \mathbf{C}(-, RD) \overset{\simeq}{\Rrightarrow} \mathbf{D}(L-, D)$ *in* $\mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathsf{Cat})$ *is a pseudo-natural adjoint equivalence.*

In order to prove this theorem we will make use of some purely formal results about the nature of bi-representations and bi-adjunctions. These arguments depend crucially upon the apparatus of a 2-dimensional Yoneda lemma – see [7, §8.3] for the bi-categorical account.

Notation 7.4. Let $\mathbf{C}$ be a 2-category. We denote the Yoneda embedding 2-functor by

$$\mathcal{Y}\colon \mathbf{C} \to \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathsf{Cat}) .$$

It sends an object $C \in \mathbf{C}$ to the 2-functor $\mathcal{Y}_C \coloneqq \mathbf{C}(-, C)\colon \mathbf{C}^{\mathrm{op}} \to \mathsf{Cat}$, and acts in the obvious way on hom-categories.

We will make extensive use of the full sub-2-category on the image of the Yoneda 2-functor, but this 2-category is isomorphic to the following.

Definition 7.5. Let $\mathbf{C}$ be a 2-category. We define a 2-category $\mathbf{C}^{\mathcal{Y}}$ with the same objects as $\mathbf{C}$ and whose hom-categories are given by

$$\mathbf{C}^{\mathcal{Y}}(C, C') := \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathsf{\textbf{Cat}})(\mathcal{Y}_C, \mathcal{Y}_{C'})$$

for all $C, C' \in \mathbf{C}$. Composition operations are given by those of the 2-category $\mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathsf{\textbf{Cat}})$.

Remark 7.6. The 2-category $\mathbf{C}^{\mathcal{Y}}$ is isomorphic to the full sub-2-category of $\mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathsf{\textbf{Cat}})$ on the objects of the form $\mathcal{Y}_C$ for $C \in \mathbf{C}$. Observe that we therefore have the following factorisation of 2-functors

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\mathcal{Y}} & \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathsf{\textbf{Cat}}) \\ & {\scriptstyle \overline{\mathcal{Y}}}\searrow \quad \nearrow & \\ & \mathbf{C}^{\mathcal{Y}} & \end{array}$$

where $\overline{\mathcal{Y}}$ is the identity on objects.

We have avoided defining $\mathbf{C}^{\mathcal{Y}}$ as "the full sub-2-category of bi-representables" as this is problematic inasmuch as objects are concerned: we will need to know *which* object is associated to a given bi-representable functor, but a priori any such object is only defined up to equivalence. One way to solve this is to choose, for each bi-representable, a representing object, and the result is precisely our 2-category $\mathbf{C}^{\mathcal{Y}}$ above.

Remark 7.7. Let $\mathbf{C}$ be a 2-category. Then the 2-dimensional Yoneda lemma says that the normal pseudo-functor $\overline{\mathcal{Y}} \colon \mathbf{C} \to \mathbf{C}^{\mathcal{Y}}$ is the identity on objects and induces equivalences between the hom-categories

$$\mathbf{C}(C, C') \simeq \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathsf{\textbf{Cat}})(\mathcal{Y}_C, \mathcal{Y}_{C'}) = \mathbf{C}^{\mathcal{Y}}(C, C'),$$

for all objects $C, C' \in \mathbf{C}$; see [7, Lemma 8.3.12].

Hence one can construct a normal pseudo-functor $\mathcal{E} \colon \mathbf{C}^{\mathcal{Y}} \to \mathbf{C}$ together with pseudo-natural isomorphisms $\eta \colon \mathrm{id}_{\mathbf{C}} \overset{\cong}{\Rightarrow} \mathcal{E}\overline{\mathcal{Y}}$ and $\varepsilon \colon \overline{\mathcal{Y}}\mathcal{E} \overset{\cong}{\Rightarrow} \mathrm{id}_{\mathbf{C}^{\mathcal{Y}}}$. We may see that $\mathcal{E}$ is the identity on objects, and acts on hom-categories $\mathbf{C}^{\mathcal{Y}}(C, C')$ by first taking the C-component of the pseudo-natural transformation or modification, and then evaluating it at $\mathrm{id}_C \in \mathcal{Y}_C(C) = \mathbf{C}(C, C)$. In fact, closer inspection reveals that $\mathrm{id}_{\mathbf{C}} = \mathcal{E}\overline{\mathcal{Y}}$.

Lemma 7.8. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be* 2*-categories, and let* $Q\colon \mathbf{D} \to \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathsf{Cat})$ *be a normal pseudo-functor such that, for all objects* $D \in \mathbf{D}$*,* $QD = \mathcal{Y}_{RD}$ *for an object* $RD \in \mathbf{C}$*. Then there is a normal pseudo-functor* $R\colon \mathbf{D} \to \mathbf{C}$ *and a pseudo-natural isomorphism* $\mathcal{Y}R \cong Q$ *in* $\mathbf{Ps}(\mathbf{D}, \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathsf{Cat}))$.

Proof. First note that the image of $Q\colon \mathbf{D} \to \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathsf{Cat})$ is contained in the full sub-2-category $\mathbf{C}^{\mathcal{Y}}$ of $\mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathsf{Cat})$. That is, we have the following factorisation

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{Q} & \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathsf{Cat}) \\ & {\scriptstyle \overline{Q}}\searrow \quad \nearrow & \\ & \mathbf{C}^{\mathcal{Y}} & \end{array},$$

where $\overline{Q}D = RD$, for all objects $D \in \mathbf{D}$. Using this we define the normal pseudo-functor $R\colon \mathbf{D} \to \mathbf{C}$ to be the composite

$$\mathbf{D} \xrightarrow{\overline{Q}} \mathbf{C}^{\mathcal{Y}} \xrightarrow{\mathcal{E}} \mathbf{C}.$$

By Remark 7.7 we have a pseudo-natural isomorphism $\overline{\mathcal{Y}}R = \overline{\mathcal{Y}}\mathcal{E}\overline{Q} \cong \overline{Q}$ in $\mathbf{Ps}(\mathbf{D}, \mathbf{C}^{\mathcal{Y}})$. By post-composing with the inclusion $\mathbf{C}^{\mathcal{Y}} \hookrightarrow \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathsf{Cat})$ this gives a pseudo-natural isomorphism $\mathcal{Y}R \cong Q$ in $\mathbf{Ps}(\mathbf{D}, \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathsf{Cat}))$. □

Finally we need a technical result relating pseudo-natural isomorphisms in the category $\mathbf{Ps}(\mathbf{D}, \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathsf{Cat}))$ to those in $\mathbf{Ps}(\mathbf{C}^{\mathrm{op}} \times \mathbf{D}, \mathsf{Cat})$. While it is not generally true that these two categories are isomorphic, we may "by hand" show that in a special case pseudo-natural isomorphisms in the former may be recast as occurring in the latter.

Lemma 7.9. *Given normal pseudo-functors* $L\colon \mathbf{C} \to \mathbf{D}$ *and* $R\colon \mathbf{D} \to \mathbf{C}$*, as well as a pseudo-natural isomorphism* $\theta\colon \mathbf{D}(L-,-) \overset{\cong}{\Rightarrow} \mathbf{C}(-,R-)$ *in* $\mathbf{Ps}(\mathbf{D}, \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathsf{Cat}))$*, the data of* θ *gives a pseudo-natural isomorphism* $\gamma\colon \mathbf{D}(L-,-) \overset{\cong}{\Rightarrow} \mathbf{C}(-,R-)$ *in* $\mathbf{Ps}(\mathbf{C}^{\mathrm{op}} \times \mathbf{D}, \mathsf{Cat})$.

Proof. To define γ we must give its value at each pair $(C, D) \in \mathbf{C}^{\mathrm{op}} \times \mathbf{D}$ of objects as well as its 2-morphism component on each morphism of such pairs, and demonstrate that suitable compatibility conditions hold.

Observe that for each $D \in \mathbf{D}$ we have a pseudo-natural isomorphism

$$\theta_D \in \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \textsf{\textbf{Cat}})(\mathbf{D}(L-, D), \mathbf{C}(-, RD)) \,.$$

Thus it makes sense to set the component $\gamma_{(C,D)} \colon \mathbf{D}(LC, D) \to \mathbf{C}(C, RD)$ to be the isomorphism $(\theta_D)_C$.

For each $d \colon D \to D'$ observe that the pseudo-naturality of θ gives the following diagram of pseudo-natural isomorphisms and modifications.

$$\begin{array}{ccc} \mathbf{D}(L-, D) & \overset{\theta_D}{\Longrightarrow} & \mathbf{C}(-, RD) \\ {\scriptstyle \mathbf{D}(L-, d)}\Big\Downarrow & \overset{\cong}{\Lleftarrow}{\scriptstyle \theta_d} & \Big\Downarrow{\scriptstyle \mathbf{C}(-, Rd)} \\ \mathbf{D}(L-, D') & \underset{\theta_{D'}}{\Longrightarrow} & \mathbf{C}(-, RD') \end{array} \tag{7.10}$$

Using this and the normality of L and R, on a morphism $c \colon C' \to C$ of $\mathbf{C}$ we may define $\gamma_{(c,d)} \colon \mathbf{C}(c, Rd)\gamma_{(C,D)} \overset{\cong}{\Rightarrow} \gamma_{(C',D')}\mathbf{D}(Lc, d)$ to be either of the following two, equal pastings.

$$\begin{array}{ccc} \mathbf{D}(LC, D) & \xrightarrow{(\theta_D)_C} & \mathbf{C}(C, RD) \\ {\scriptstyle \mathbf{D}(Lc, D)}\Big\downarrow & {\scriptstyle (\theta_D)_c}\overset{}{\Lleftarrow}{\scriptstyle\cong} & \Big\downarrow{\scriptstyle \mathbf{C}(c, RD)} \\ \mathbf{D}(LC', D) & \xrightarrow{(\theta_D)_{C'}} & \mathbf{C}(C', RD) \\ {\scriptstyle \mathbf{D}(C', d)}\Big\downarrow & {\scriptstyle (\theta_d)_C}\Lleftarrow{\scriptstyle\cong} & \Big\downarrow{\scriptstyle \mathbf{C}(C', Rd)} \\ \mathbf{D}(LC', D') & \underset{(\theta_{D'})_{C'}}{\longrightarrow} & \mathbf{C}(C', RD') \end{array} \quad = \quad \begin{array}{ccc} \mathbf{D}(LC, D) & \xrightarrow{(\theta_D)_C} & \mathbf{C}(C, RD) \\ {\scriptstyle \mathbf{D}(LC, d)}\Big\downarrow & {\scriptstyle (\theta_d)_C}\Lleftarrow{\scriptstyle\cong} & \Big\downarrow{\scriptstyle \mathbf{C}(C, Rd)} \\ \mathbf{D}(LC, D') & \xrightarrow{(\theta_{D'})_C} & \mathbf{C}(C, RD') \\ {\scriptstyle \mathbf{D}(c, D')}\Big\downarrow & {\scriptstyle (\theta_{D'})_c}\Lleftarrow{\scriptstyle\cong} & \Big\downarrow{\scriptstyle \mathbf{C}(c, RD')} \\ \mathbf{D}(LC', D') & \underset{(\theta_{D'})_{C'}}{\longrightarrow} & \mathbf{C}(C', RD') \end{array}$$

This concludes the data of γ, and now we must check that it is pseudo-natural. That is, we must ensure that it is compatible with the compositors of $\mathbf{C}(-, R-)$ and $\mathbf{D}(L-, -)$, that its components on identities are themselves identities, and that it commutes appropriately with 2-cell pasting.

None of these calculations are especially enlightening, and amount to massaging pasting diagrams with the above equality, appealing to the fact that θ_d is a modification as in (7.10), and finally noting that θ_d itself is pseudo-natural. Hence we choose to omit the details for brevity. □

Now we are in a position to give a proof of Theorem 7.3.

Proof (Theorem 7.3). First note that Remark 7.2 directly gives that (i) implies (ii). We show the other implication.

Suppose that (ii) holds, that is, for all objects $D \in \mathbf{D}$, we have a bi-representation (RD, Ψ^D) of $\mathbf{D}(L-, D)$, i.e., a pseudo-natural adjoint equivalence $\Psi^D\colon \mathbf{C}(-, RD) \overset{\simeq}{\Rrightarrow} \mathbf{D}(L-, D)$. We want to construct the data of a normal pseudo-functor $R\colon \mathbf{D} \to \mathbf{C}$ and a pseudo-natural adjoint equivalence $\Phi_{-,-}\colon \mathbf{C}(-, R-) \overset{\simeq}{\Rightarrow} \mathbf{D}(L-, -)$ in $\mathbf{Ps}(\mathbf{C}^{\mathrm{op}} \times \mathbf{D}, \textsf{Cat})$.

In order to do so, we will simultaneously construct a pseudo-functor $(Q, \phi)\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \textsf{Cat}$ such that $Q(C, D) = \mathbf{C}(C, RD) = \mathcal{Y}_{RD}(C)$ for all $(C, D) \in \mathbf{C}^{\mathrm{op}} \times \mathbf{D}$ along with a pseudo-natural adjoint equivalence $\Gamma\colon Q \overset{\simeq}{\Rrightarrow} \mathbf{D}(L-, -)$ in $\mathbf{Ps}(\mathbf{C}^{\mathrm{op}} \times \mathbf{D}, \textsf{Cat})$. Note that while our construction of Q below does not necessarily yield a *normal* pseudo-functor, we may apply a normalisation argument such as [12, Proposition 5.2] to construct a normal pseudo-functor Q^n which agrees with Q on objects and a pseudo-natural isomorphism $\nu\colon Q^n \overset{\cong}{\Rrightarrow} Q$. Note that there is a forgetful functor $\mathbf{Ps}(\mathbf{D}, \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \textsf{Cat})) \to \mathbf{Ps}(\mathbf{C}^{\mathrm{op}} \times \mathbf{D}, \textsf{Cat})$, so that we can see Q^n, Q, and $\mathbf{D}(L-, -)$ as objects in $\mathbf{Ps}(\mathbf{D}, \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \textsf{Cat}))$ and

$$Q^n \underset{\nu}{\overset{\cong}{\Rrightarrow}} Q \underset{\Gamma}{\overset{\simeq}{\Rrightarrow}} \mathbf{D}(L-, -)$$

as an isomorphism in $\mathbf{Ps}(\mathbf{D}, \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \textsf{Cat}))$.

Then, by applying Lemma 7.8 to the normal pseudo-functor Q^n, we may extract a normal pseudo-functor $R\colon \mathbf{D} \to \mathbf{C}$ and a pseudo-natural isomorphism $\xi\colon \mathbf{C}(-, R-) \overset{\cong}{\Rrightarrow} Q^n$ in $\mathbf{Ps}(\mathbf{D}, \mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \textsf{Cat}))$. Finally Lemma 7.9 applied to L, R, and the pseudo-natural isomorphism

$$\mathbf{C}(-, R-) \underset{\xi}{\overset{\cong}{\Rrightarrow}} Q^n \underset{\nu}{\overset{\cong}{\Rrightarrow}} Q \underset{\Gamma}{\overset{\simeq}{\Rrightarrow}} \mathbf{D}(L-, -)$$

gives, as desired, a pseudo-natural isomorphism $\mathbf{C}(-, R-) \cong \mathbf{D}(L-, -)$ in $\mathbf{Ps}(\mathbf{C}^{\mathrm{op}} \times \mathbf{D}, \textsf{Cat})$.

It remains to construct the pseudo-functor $Q\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \textsf{Cat}$ and pseudo-natural adjoint equivalence $\Gamma\colon Q \overset{\simeq}{\Rrightarrow} \mathbf{D}(L-, -)$.

On objects $C \in \mathbf{C}$ and $D \in \mathbf{D}$, we define $Q(C, D) \coloneqq \mathbf{C}(C, RD)$ where RD is a representing object which exists by assumption, and we define $\Gamma_{C,D}$ as the adjoint equivalence

$$\Gamma_{C,D} \coloneqq \Psi^D_C\colon Q(C, D) = \mathbf{C}(C, RD) \overset{\simeq}{\Rightarrow} \mathbf{D}(LC, D).$$

On morphisms $c\colon C \to C'$ in $\mathbf{C}$ and $d\colon D \to D'$ in $\mathbf{D}$, we must define $Q(c,d)$ and $\Gamma_{c,d}$ such that they fit in the following square:

$$\begin{array}{ccc} \mathbf{C}(C',RD) & \xrightarrow{\Gamma_{C',D}} & \mathbf{D}(LC',D) \\ {\scriptstyle Q(c,d)}\downarrow & \overset{\Gamma_{c,d}}{\swArrow}\cong & \downarrow{\scriptstyle \mathbf{D}(Lc,d)} \\ \mathbf{C}(C,RD') & \xrightarrow[\Gamma_{C,D'}]{} & \mathbf{D}(LC,D') \end{array} .$$

To do this we, use the equivalence data $(\Psi^D_C, (\Psi^D_C)^{-1}, \eta^D_C, \varepsilon^D_C)$ and set $Q(c,d)$ to be the composite

$$\mathbf{C}(C',RD) \xrightarrow{\Psi^D_{C'}} \mathbf{D}(LC',D) \xrightarrow{\mathbf{D}(Lc,d)} \mathbf{D}(LC,D') \xrightarrow{(\Psi^{D'}_C)^{-1}} \mathbf{C}(C,RD') \quad ,$$

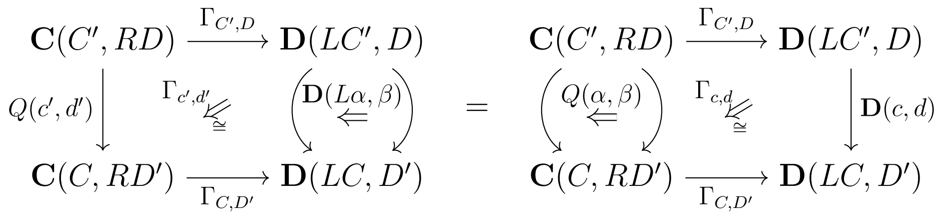
and $\Gamma_{c,d}$ to be the following pasting.

$$\begin{array}{ccc} \mathbf{C}(C',RD) & \xrightarrow{\Psi^D_{C'}} & \mathbf{D}(LC',D) \\ & & \downarrow{\scriptstyle \mathbf{D}(Lc,d)} \\ {\scriptstyle Q(c,d)}\downarrow & & \mathbf{D}(LC,D') \\ & \swarrow{\scriptstyle (\Psi^{D'}_C)^{-1}} \quad \cong \swArrow (\varepsilon^{D'}_C)^{-1} & \| \\ \mathbf{C}(C,RD') & \xrightarrow[\Psi^{D'}_C]{} & \mathbf{D}(LC,D') \end{array} .$$

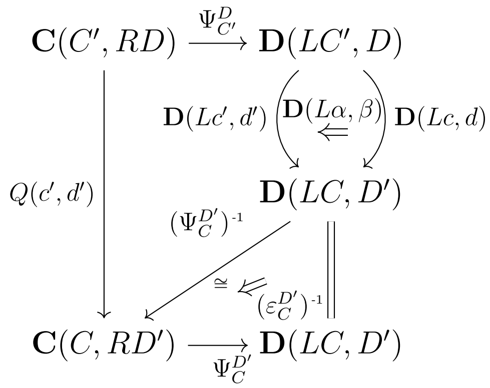
Next, on 2-morphisms $\alpha\colon c \Rightarrow c'$ in $\mathbf{C}$ and $\beta\colon d \Rightarrow d'$ in $\mathbf{D}$, we define $Q(\alpha,\beta)$ to be the following pasting.

$$\mathbf{C}(C',RD) \xrightarrow{\Psi^D_{C'}} \mathbf{D}(LC',D) \underset{\mathbf{D}(Lc',d')}{\overset{\mathbf{D}(Lc,d)}{\Downarrow \mathbf{D}(L\alpha,\beta)}} \mathbf{D}(LC,D') \xrightarrow{(\Psi^{D'}_C)^{-1}} \mathbf{C}(C,RD')$$

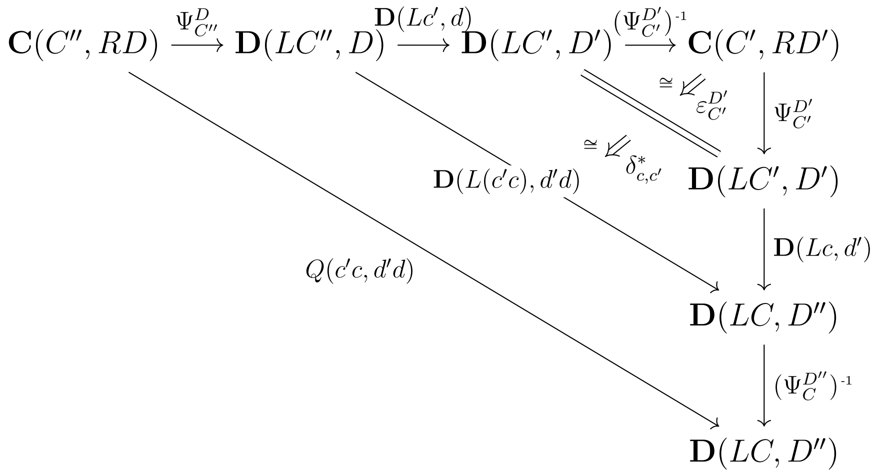
With this definition of Q on 2-morphisms, we can directly check that Γ is natural with respect to this assignment. More precisely, the following pasting equality holds



since both sides are given by the following pasting.

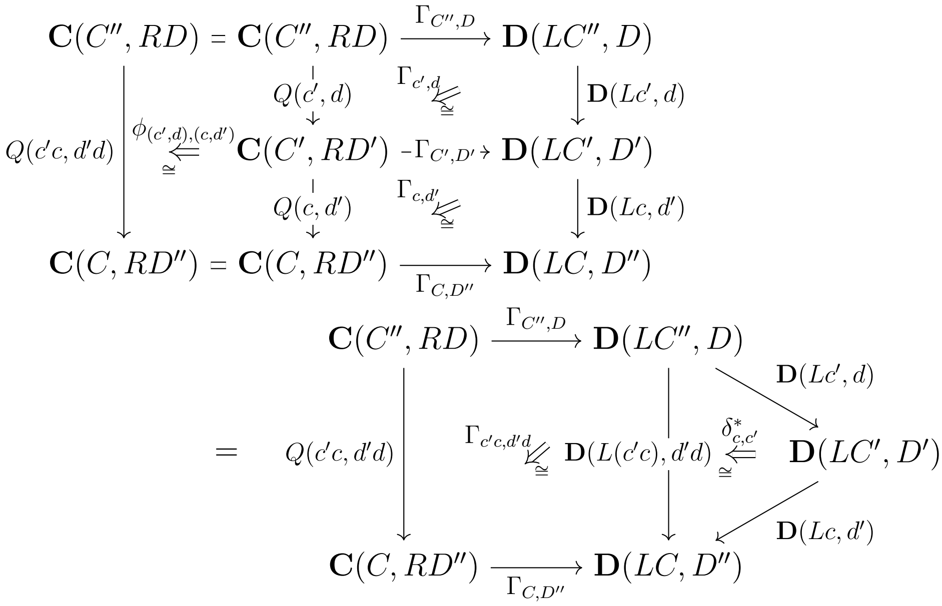


With that achieved, it remains to supply the data of the compositors and unitors of Q and verify the pseudo-naturality conditions of Γ with respect to these. We start with the compositors. For this, let $c\colon C \to C'$, $c'\colon C' \to C''$ be composable morphisms in $\mathbf{C}$ and $d\colon D \to D'$, $d'\colon D' \to D''$ be composable morphisms in $\mathbf{D}$. We define the 2-isomorphism compositor $\phi_{(c',d),(c,d')}\colon Q(c,d')Q(c',d) \overset{\cong}{\Rightarrow} Q(c'c,d'd)$ as the below pasting.

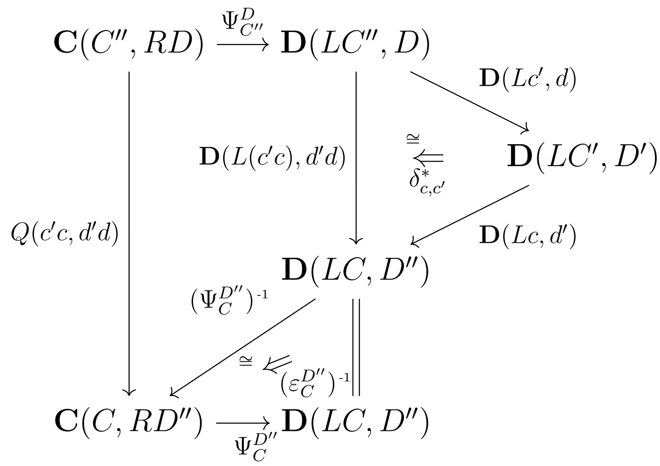


From this definition, the definition of Q on 2-morphisms in terms of L, and the properties of the compositor δ of L, we may directly verify that ϕ is associative and is natural with respect to 2-morphisms.

We need to check that Γ is compatible with the compositors ϕ, namely that the following pasting equality holds.



By direct expansion of definitions, we see that both pastings reduce to the following pasting.



To complete the proof it remains to deal with the unitors. Let $C \in \mathbf{C}$ and $D \in \mathbf{D}$ be objects. Recall that $Q(\mathrm{id}_C, \mathrm{id}_D)$ is given by the following composite

$$\mathbf{C}(C, RD) \xrightarrow{\Psi_C^D} \mathbf{D}(LC, D) \xrightarrow{(\Psi_C^D)^{-1}} \mathbf{C}(C, RD) \quad ,$$

since $\mathbf{D}(L(\mathrm{id}_C), \mathrm{id}_D) = \mathrm{id}_{\mathbf{D}(LC,D)}$ by normality of L. We define the 2-isomorphism unitor $\phi_{(C,D)}$ as the unit

$$\begin{array}{ccc} \mathbf{C}(C,RD) & = & \mathbf{C}(C,RD) \\ & {\scriptstyle \Psi_C^D} \searrow \quad \cong \Downarrow \eta_C^D \quad \nearrow {\scriptstyle (\Psi_C^D)^{-1}} & \\ & \mathbf{D}(LC,D) & \end{array} \quad .$$

From this definition and the triangle equalities for $(\eta_C^D, \varepsilon_C^D)$, we may directly verify that ϕ is satisfies the unitality conditions and that Γ is compatible with the unitor ϕ. This completes the constructions of Q and Γ and proves the theorem. □

We have now successfully shown that bi-adjunctions (L, R, Φ) are equivalently families of bi-representations of $\mathbf{D}(L-, D)$, indexed by the objects $D \in \mathbf{D}$. In the remainder of the section our goal is to combine Theorem 6.8 and Theorem 7.3 in order to obtain the following characterisation of bi-adjunctions in terms of bi-terminal objects in different pseudo-slices. Recall that (double) bi-terminal are defined as (double) bi-initial objects in the (horizontal) opposite.

In the statement of the theorem below, the pseudo-slice double categories $\mathbb{H}L \downarrow\!\downarrow D$ are given by the following cospan in $\mathsf{DblCat}_{\mathsf{h,nps}}$

$$\mathbb{H}\mathbf{C} \xrightarrow{\mathbb{H}L} \mathbb{H}\mathbf{D} \xleftarrow{D} \mathbb{1} \,,$$

and the pseudo-slice 2-categories $L \downarrow D$ and $\mathrm{Ar}_* L \downarrow D$ are given by the following cospans in $\mathsf{2Cat}_{\mathsf{nps}}$

$$\mathbf{C} \xrightarrow{L} \mathbf{D} \xleftarrow{D} \mathbb{1} \text{ and } \mathrm{Ar}_*\mathbf{C} \xrightarrow{\mathrm{Ar}_*L} \mathrm{Ar}_*\mathbf{D} \xleftarrow{D} \mathbb{1} \,,$$

respectively, for objects $D \in \mathbf{D}$.

Theorem 7.11. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be* 2*-categories, and* $L\colon \mathbf{C} \to \mathbf{D}$ *be a normal pseudo-functor. The following statements are equivalent.*

(i) The normal pseudo-functor $L\colon \mathbf{C} \to \mathbf{D}$ *admits a right bi-adjoint* $R\colon \mathbf{D} \to \mathbf{C}$.

(ii) For all objects $D \in \mathbf{D}$*, there is an object* $RD \in \mathbf{C}$ *together with a morphism* $\varepsilon_D\colon LRD \to D$ *in* $\mathbf{D}$ *such that* (RD, ε_D) *is double bi-terminal in* $\mathbb{H}L \downarrow\downarrow D$.

(iii) For all objects $D \in \mathbf{D}$*, there is an object* $RD \in \mathbf{C}$ *together with a morphism* $\varepsilon_D\colon LRD \to D$ *in* $\mathbf{D}$ *such that* (RD, ε_D) *is bi-terminal in* $L \downarrow D$ *and* $(RD, \mathrm{id}_{\varepsilon_D})$ *is bi-terminal in* $\mathrm{Ar}_* L \downarrow D$.

(iv) For all objects $D \in \mathbf{D}$*, there is an object* $RD \in \mathbf{C}$ *together with a morphism* $\varepsilon_D\colon LRD \to D$ *in* $\mathbf{D}$ *such that* $(RD, \mathrm{id}_{\varepsilon_D})$ *is bi-terminal in* $\mathrm{Ar}_* L \downarrow D$.

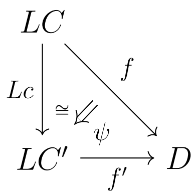
The missing components for the proof of this theorem are canonical isomorphisms of double categories $\mathbb{el}(\mathbf{D}(L-, D)) \cong (\mathbb{H}L \downarrow\downarrow D)^{\mathrm{op}}$, as well as related canonical isomorphisms for the 2-categories $\mathbf{el}(\mathbf{D}(L-, D))$ and $\mathbf{mor}(\mathbf{D}(L-, D))$. This is the content of the following results.

Lemma 7.12. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be* 2*-categories,* $L\colon \mathbf{C} \to \mathbf{D}$ *be a normal pseudo-functor, and* $D \in \mathbf{D}$ *be an object. There is a canonical isomorphism of double categories as in the following commutative triangle.*

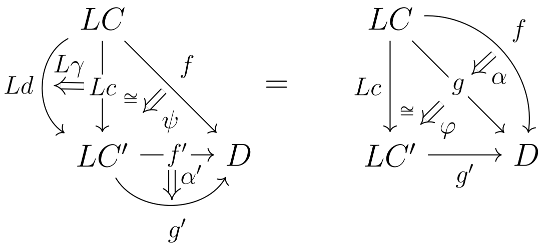
$$\begin{array}{ccc} \mathbb{el}(\mathbf{D}(L-, D)) & \xrightarrow{\cong} & (\mathbb{H}L \downarrow\downarrow D)^{\mathrm{op}} \\ & {\scriptstyle \Pi} \searrow \quad \swarrow {\scriptstyle \Pi^{\mathrm{op}}} & \\ & \mathbb{H}\mathbf{C}^{\mathrm{op}} & \end{array}$$

Proof. We describe the data of the double category $\mathbb{el}(\mathbf{D}(L-, D))$. Then, by a direct comparison with the data in the double category $\mathbb{H}L \downarrow\downarrow D$, which is the dual construction to the double category described in Remark 4.2 with the double functor $F = \mathbb{H}L$ being horizontal, we can see that the isomorphism above canonically holds.

An object in $\mathbb{el}(\mathbf{D}(L-, D))$ is a pair (C, f) of objects $C \in \mathbf{C}$ and $f \in \mathbf{D}(LC, D)$, i.e., a morphism $f\colon LC \to D$ in $\mathbf{D}$. A horizontal morphism $(c, \psi)\colon (C', f') \to (C, f)$ in $\mathbb{el}(\mathbf{D}(L-, D))$ comprises the data of a morphism $c\colon C \to C'$ in $\mathbf{C}$ and an isomorphism $\psi\colon f \xrightarrow{\cong} \mathbf{D}(Lc, D)f'$ in $\mathbf{D}(LC, D)$, i.e., a 2-isomorphism in $\mathbf{D}$

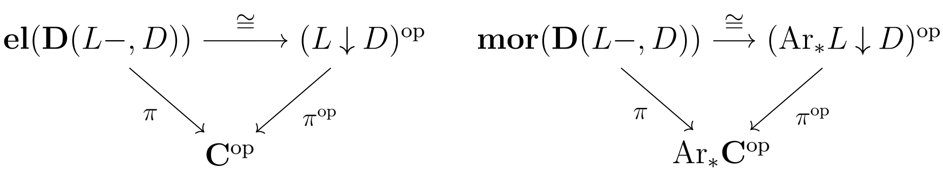


Note that it corresponds to a morphism $(c, \psi)\colon (C, f) \to (C', f')$ in $\mathbb{H}L{\downarrow\downarrow}D$, and it is the reason why we need to take the horizontal opposite $(\mathbb{H}L{\downarrow\downarrow}D)^{\mathrm{op}}$. A vertical morphism $\alpha\colon (C, f) \nrightarrow (C, g)$ in $\mathbb{el}(\mathbf{D}(L-, D))$ is a morphism $\alpha\colon f \to g$ in $\mathbf{D}(LC, D)$, i.e., a 2-morphism $\alpha\colon f \Rightarrow g$ between morphisms $f, g\colon LC \to D$ in $\mathbf{D}$. Finally, a square $\gamma\colon (\alpha' \,{}^{(c,\psi)}_{(d,\varphi)}\, \alpha)$ is a 2-morphism $\gamma\colon c \Rightarrow d$ in $\mathbf{C}$ satisfying the pasting equality in Definition 6.3 (iv), which can be translated into the following pasting equality in $\mathbf{D}$.



□

Corollary 7.13. *Let $\mathbf{C}$ and $\mathbf{D}$ be 2-categories, $L\colon \mathbf{C} \to \mathbf{D}$ be a normal pseudo-functor, and $D \in \mathbf{D}$ be an object. There are canonical isomorphisms of 2-categories as in the following commutative triangles.*



Proof. This follows directly from the definitions of $\mathbf{el}$ and $\mathbf{mor}$, Lemma 7.12 and Proposition 4.6. □

The proof of Theorem 7.11 now follows in a straightforward manner.

Proof (Theorem 7.11). By Theorems 6.8 and 7.3 and Lemma 7.12, we see that (i) and (ii) are equivalent. The equivalences of (ii), (iii), and (iv) follow from Theorem 6.8, Lemma 7.12, and Corollary 7.13. □

Remark 7.14. Although we have proven this result by means of formal arguments involving a reformulation of a 2-dimensional Yoneda lemma (see Remark 7.7), these details are not a necessary feature of the proof of this theorem. For the reader for whom such devices are unfamiliar or otherwise constitute a significant detour, we note here that a pleasingly direct (if somewhat lengthy) proof of this theorem is possible and follows entirely similar lines to the proof of Theorem 6.8.

Much as in the case of Theorem 6.15, we may improve Theorem 7.11 by assuming that $\mathbf{C}$ has tensors by $\mathbb{2}$ and that L preserves them, i.e., for every object $C \in \mathbf{C}$, there is a tensor of LC by $\mathbb{2}$ in $\mathbf{D}$ and we have an isomorphism $L(C \otimes \mathbb{2}) \cong (LC) \otimes \mathbb{2}$ in $\mathbf{D}$ natural with respect to the defining cones.

Theorem 7.15. *Let $\mathbf{C}$ and $\mathbf{D}$ be 2-categories, and $L\colon \mathbf{C} \to \mathbf{D}$ be a normal pseudo-functor. Suppose that $\mathbf{C}$ has tensors by $\mathbb{2}$ and that L preserves them. Then the following statements are equivalent.*

(i) The normal pseudo-functor $L\colon \mathbf{C} \to \mathbf{D}$ admits a right bi-adjoint $R\colon \mathbf{D} \to \mathbf{C}$.

(ii) For all objects $D \in \mathbf{D}$, there is an object $RD \in \mathbf{C}$ together with a morphism $\varepsilon_D\colon LRD \to D$ in $\mathbf{D}$ such that (RD, ε_D) is bi-terminal in $L \downarrow D$.

Remark 7.16. Note that tensors are a special case of a weighted 2-colimit construction. Therefore, if a 2-category $\mathbf{C}$ has tensors by $\mathbb{2}$ and $L\colon \mathbf{C} \to \mathbf{D}$ is a left bi-adjoint, it preserves in particular all tensors by $\mathbb{2}$. In this way, this additional hypothesis on L is entirely anodyne in the following sense: given an L which we suspect to be a left bi-adjoint, in order to apply the above theorem we would need to know that L preserves tensors by $\mathbb{2}$, but this should be part of a "background-check" on L in the first place.

Proof (Theorem 7.15). By Theorems 7.11 and 6.15, it is enough to show that the normal pseudo-functor $\mathbf{D}(L-, D)\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ preserves powers by $\mathbb{2}$. This is indeed the case since it follows from the fact that L preserves tensors by $\mathbb{2}$ that

$$\mathbf{D}(L(C \otimes \mathbb{2}), D) \cong \mathbf{D}((LC) \otimes \mathbb{2}, D) \cong \mathbf{Cat}(\mathbb{2}, \mathbf{D}(LC, D)).$$ □

7.2 Weighted bi-limits

The primary and indeed motivating application of this theory is to the notion of 2-dimensional limits. In [3, Counter-example 2.12], we give an example of a 2-terminal object in the slice 2-category of cones over a 2-functor $F\colon \mathbf{I} \to \mathbf{C}$ that does not give a 2-limit of F. This also gives a counter-example of a bi-terminal object in the pseudo-slice 2-category of cones over F which is not a bi-limit of F, as explained in [3, §5]. However, we show in [3, Proposition 2.13] that when $\mathbf{C}$ has tensors by $\mathbb{2}$, then 2-limits and 2-terminal objects in the slice do correspond precisely. But we deferred the corresponding result for bi-limits to this document.

With this in mind, we now apply Theorem 6.8 to the case of (weighted) bi-limits in order to obtain a correct characterisation in terms of bi-terminal objects. We further prove the deferred results for (weighted) bi-limits involving tensors by $\mathbb{2}$, which are obtained as a direct application of Theorem 6.15.

Let us begin by recalling the definition of a weighted bi-limit.

Definition 7.17. Let $\mathbf{I}$ and $\mathbf{C}$ be 2-categories, and let $F\colon \mathbf{I} \to \mathbf{C}$ and $W\colon \mathbf{I} \to \mathbf{Cat}$ be normal pseudo-functors. A **weighted bi-limit of** F **by** W is a pair (X, λ) of an object $X \in \mathbf{C}$ together with a pseudo-natural transformation $\lambda\colon W \Rightarrow \mathbf{C}(X, F-)$ in $\mathbf{Ps}(\mathbf{I}, \mathbf{Cat})$, such that, for every object $C \in \mathbf{C}$, pre-composition by λ induces an adjoint equivalence of categories

$$\lambda^* \circ \mathbf{C}(-, F)\colon \mathbf{C}(C, X) \xrightarrow{\simeq} \mathbf{Ps}(\mathbf{I}, \mathbf{Cat})(W, \mathbf{C}(C, F-)),$$

where $\mathbf{C}(-, F)\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Ps}(\mathbf{I}, \mathbf{Cat})$ is the normal pseudo-functor sending an object $C \in \mathbf{C}$ to the normal pseudo-functor $\mathbf{C}(C, F-)\colon \mathbf{I} \to \mathbf{Cat}$.

Remark 7.18. Note that a weighted bi-limit (X, λ) induces a 2-natural adjoint equivalence

$$\lambda^* \circ \mathbf{C}(-, F)\colon \mathbf{C}(-, X) \overset{\simeq}{\Rightarrow} \mathbf{Ps}(\mathbf{I}, \mathbf{Cat})(W, \mathbf{C}(-, F)),$$

so that we can see that weighted bi-limits are, in particular, bi-representations of the 2-functor $\mathbf{Ps}(\mathbf{I}, \mathbf{Cat})(W, \mathbf{C}(-, F))\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$. Conversely, if (X, ρ) is a bi-representation, with $\rho\colon \mathbf{C}(-, X) \overset{\simeq}{\Rightarrow} \mathbf{Ps}(\mathbf{I}, \mathbf{Cat})(W, \mathbf{C}(-, F))$ a pseudo-natural adjoint equivalence in $\mathbf{Ps}(\mathbf{C}^{\mathrm{op}}, \mathbf{Cat})$, we may set

$$\lambda := \rho_X(\mathrm{id}_X)\colon W \Rightarrow \mathbf{C}(X, F-).$$

Then by Corollary 6.11, $\overline{\rho} = \lambda^* \circ \mathbf{C}(-,F)$ is a 2-natural adjoint equivalence, that is, a weighted bi-limit of F by W.

We now aim to apply Theorem 6.8 to this setting in order to obtain a characterisation of weighted bi-limits in terms of bi-initial objects in different pseudo-slices.

In the statement of the theorem below, the pseudo-slice double category $W \downarrow\downarrow \mathbb{H}\mathbf{C}(-,F)$ is given by the following cospan in $\mathsf{DblCat}_{\mathsf{h,nps}}$

$$\mathbb{1} \xrightarrow{W} \mathbb{H}\,\mathbf{Ps}(\mathbf{I}, \mathsf{Cat}) \xleftarrow{\mathbb{H}\mathbf{C}(-,F)} \mathbb{H}\mathbf{C}^{\mathrm{op}} ,$$

and the pseudo-slice 2-categories $W \downarrow \mathbf{C}(-,F)$ and $W \downarrow \mathrm{Ar}_*\mathbf{C}(-,F)$ are given by the following cospans in $\mathsf{2Cat}_{\mathsf{nps}}$

$$\mathbb{1} \xrightarrow{W} \mathbf{Ps}(\mathbf{I}, \mathsf{Cat}) \xleftarrow{\mathbf{C}(-,F)} \mathbf{C}^{\mathrm{op}}, \quad \mathbb{1} \xrightarrow{W} \mathrm{Ar}_*\,\mathbf{Ps}(\mathbf{I}, \mathsf{Cat}) \xleftarrow{\mathrm{Ar}_*\mathbf{C}(-,F)} \mathrm{Ar}_*\mathbf{C}^{\mathrm{op}},$$

respectively.

Theorem 7.19. *Let $\mathbf{I}$ and $\mathbf{C}$ be 2-categories, and $F\colon \mathbf{I} \to \mathbf{C}$, $W\colon \mathbf{I} \to \mathsf{Cat}$ be normal pseudo-functors. The following statements are equivalent.*

(i) There is a weighted bi-limit (X, λ) of F by W.

(ii) There is an object $X \in \mathbf{C}$ together with a pseudo-natural transformation $\lambda\colon W \Rightarrow \mathbf{C}(X, F-)$ in $\mathbf{Ps}(\mathbf{I}, \mathsf{Cat})$ such that (X, λ) is double bi-initial in $W \downarrow\downarrow \mathbb{H}\mathbf{C}(-,F)$.

(iii) There is an object $X \in \mathbf{C}$ together with a pseudo-natural transformation $\lambda\colon W \Rightarrow \mathbf{C}(X, F-)$ in $\mathbf{Ps}(\mathbf{I}, \mathsf{Cat})$ such that (X, λ) is bi-initial in $W \downarrow \mathbf{C}(-,F)$ and (X, id_λ) is bi-initial in $W \downarrow \mathrm{Ar}_*\mathbf{C}(-,F)$.

(iv) There is an object $X \in \mathbf{C}$ together with a pseudo-natural transformation $\lambda\colon W \Rightarrow \mathbf{C}(X, F-)$ in $\mathbf{Ps}(\mathbf{I}, \mathsf{Cat})$ such that (X, id_λ) is bi-initial in $W \downarrow \mathrm{Ar}_*\mathbf{C}(-,F)$.

Remark 7.20. At a cursory reading it may surprise readers to learn that weighted *bi-limits* are characterised as somehow *bi-initial* rather than bi-terminal objects. However, such a statement belies their true nature. When we unravel definitions, we see that the double bi-initiality in the pseudo-slice double category $W \downarrow\downarrow \mathbb{H}\mathbf{C}(-,F)$ is expressed over $\mathbb{H}\mathbf{C}^{\mathrm{op}}$, and its presence is indicative of a “mapping in” property for the limiting object in $\mathbf{C}$– precisely as one might expect from bi-limits.

The proof of Theorem 7.19 is deferred to the end of the section, as we need to establish some technical results (Lemma 7.26 and Corollary 7.27) relating the double category $\mathbb{el}(\mathbf{Ps}(\mathbf{I},\mathbf{Cat})(W,\mathbf{C}(-,F)))$ to the pseudo-slice double category $W \downarrow\downarrow \mathbb{H}\mathbf{C}(-,F)$, and similarly so for the 2-categories $\mathbf{el}(\mathbf{Ps}(\mathbf{I},\mathbf{Cat})(W,\mathbf{C}(-,F))$ and $\mathbf{mor}(\mathbf{Ps}(\mathbf{I},\mathbf{Cat})(W,\mathbf{C}(-,F)))$.

Assuming Theorem 7.19, we specialise Theorem 6.15 to the weighted bi-limit case. Here we only need to assume that the 2-category $\mathbf{C}$ has tensors by $\mathbb{2}$ as these are preserved automatically in this special case.

Theorem 7.21. *Let* $\mathbf{I}$ *and* $\mathbf{C}$ *be* 2*-categories, and* $F\colon \mathbf{I}\to\mathbf{C}$*,* $W\colon \mathbf{I}\to\mathbf{Cat}$ *be normal pseudo-functors. Suppose that* $\mathbf{C}$ *has tensors by* $\mathbb{2}$*. Then the following statements are equivalent.*

(i) There is a weighted bi-limit (X,λ) *of* F *by* W*.*

(ii) There is an object $X\in\mathbf{C}$ *together with a pseudo-natural transformation* $\lambda\colon W\Rightarrow \mathbf{C}(X,F-)$ *in* $\mathbf{Ps}(\mathbf{I},\mathbf{Cat})$ *such that* (X,λ) *is bi-initial in* $W\downarrow\mathbf{C}(-,F)$*.*

Proof. By Theorems 7.19 and 6.15, it is enough to show that the normal pseudo-functor $\mathbf{Ps}(\mathbf{I},\mathbf{Cat})(W,\mathbf{C}(-,F))\colon \mathbf{C}^{\mathrm{op}}\to\mathbf{Cat}$ preserves powers by $\mathbb{2}$. Indeed we have that

$$\begin{aligned}\mathbf{Ps}(\mathbf{I},\mathbf{Cat})(W,\mathbf{C}(C\otimes\mathbb{2},F-)) &\cong \mathbf{Ps}(\mathbf{I},\mathbf{Cat})(W,\mathbf{Cat}(\mathbb{2},\mathbf{C}(C,F-)))\\ &\cong \mathbf{Cat}(\mathbb{2},\mathbf{Ps}(\mathbf{I},\mathbf{Cat})(W,\mathbf{C}(C,F-))\end{aligned}$$

as powers in $\mathbf{Ps}(\mathbf{I},\mathbf{Cat})$ are given by point-wise powers in $\mathbf{Cat}$. □

In the special case where the weight W is constant at the terminal category, i.e., $W=\Delta\mathbb{1}$, the characterisation of weighted bi-limits by $\Delta\mathbb{1}$, called *conical bi-limits*, takes a more familiar form.

In the statement of the corollary below, the pseudo-slice double category $\mathbb{H}\Delta\downarrow\downarrow F$ is given by the following cospan in $\mathsf{DblCat}_{\mathsf{h,nps}}$

$$\mathbb{H}\mathbf{C}\xrightarrow{\mathbb{H}\Delta}\mathbb{H}\,\mathbf{Ps}(\mathbf{I},\mathbf{C})\xleftarrow{F}\mathbb{1}\,,$$

and the pseudo-slice 2-categories $\Delta\downarrow F$ and $\mathrm{Ar}_*\Delta\downarrow F$ are given by the following cospans in $\mathsf{2Cat}_{\mathsf{nps}}$

$$\mathbf{C}\xrightarrow{\Delta}\mathbf{Ps}(\mathbf{I},\mathbf{C})\xleftarrow{F}\mathbb{1}\ \text{ and }\ \mathrm{Ar}_*\mathbf{C}\xrightarrow{\mathrm{Ar}_*\Delta}\mathrm{Ar}_*\,\mathbf{Ps}(\mathbf{I},\mathbf{C})\xleftarrow{F}\mathbb{1}\,,$$

respectively.

Corollary 7.22. *Let $\mathbf{I}$ and $\mathbf{C}$ be 2-categories, and $F\colon \mathbf{I} \to \mathbf{C}$ be a normal pseudo-functor. The following statements are equivalent.*

(i) There is a bi-limit (X, λ) of F.

(ii) There is an object $X \in \mathbf{C}$ together with a pseudo-natural transformation $\lambda\colon \Delta X \Rightarrow F$ in $\mathbf{Ps}(\mathbf{I}, \mathbf{C})$ such that (X, λ) is double bi-terminal in $\mathbb{H}\Delta \downarrow\!\!\downarrow F$.

(iii) There is an object $X \in \mathbf{C}$ together with a pseudo-natural transformation $\lambda\colon \Delta X \Rightarrow F$ in $\mathbf{Ps}(\mathbf{I}, \mathbf{C})$ such that (X, λ) is bi-terminal in $\Delta \downarrow F$ and (X, id_λ) is bi-terminal in $\mathrm{Ar}_*\Delta \downarrow F$.

(iv) There is an object $X \in \mathbf{C}$ together with a pseudo-natural transformation $\lambda\colon \Delta X \Rightarrow F$ in $\mathbf{Ps}(\mathbf{I}, \mathbf{C})$ such that (X, id_λ) is bi-terminal in $\mathrm{Ar}_*\Delta \downarrow F$.

The proof is deferred to the end of the section, where we prove the needed technical results (Lemma 7.28 and Corollary 7.29) relating pseudo-slices of weighted cones for the weight $W = \Delta\mathbb{1}$ to the usual pseudo-slices of cones.

Remark 7.23. As we already mentioned, we show in [3, §5] that the data of a bi-limit of F is not fully captured by a bi-terminal object in the usual pseudo-slice 2-category $\Delta \downarrow F$ of cones. Statement (iv) above shows that by "shifting" the pseudo-slice $\Delta\downarrow F$ to the pseudo-slice $\mathrm{Ar}_*\Delta\downarrow F$ whose *objects* are modifications between cones, we can successfully capture the additional data we require.

In particular, by comparing Corollary 7.22 with the characterisation of bi-adjunctions of Theorem 7.11, we can see bi-limits as a right bi-adjoint. Namely:

Remark 7.24. Let $\mathbf{I}$ and $\mathbf{C}$ be 2-categories. If every normal pseudo-functor $F\colon \mathbf{I} \to \mathbf{C}$ has a bi-limit, then this bi-limit construction extends to a right bi-adjoint to the diagonal 2-functor $\Delta\colon \mathbf{C} \to \mathbf{Ps}(\mathbf{I}, \mathbf{C})$.

By assuming Corollary 7.22 and specialising Theorem 7.21 to the case $W = \Delta\mathbb{1}$, we obtain the promised results of [3, Proposition 5.5].

Corollary 7.25. *Let* $\mathbf{I}$ *and* $\mathbf{C}$ *be* 2*-categories, and* $F\colon \mathbf{I} \to \mathbf{C}$ *be a normal pseudo-functor. Suppose that* $\mathbf{C}$ *has tensors by* $\mathbb{2}$*. Then the following statements are equivalent.*

(i) There is a bi-limit (X, λ) *of* F*.*

(ii) There is an object $X \in \mathbf{C}$ *together with a pseudo-natural transformation* $\lambda\colon \Delta X \Rightarrow F$ *in* $\mathbf{Ps}(\mathbf{I}, \mathbf{C})$ *such that* (X, λ) *is bi-terminal in* $\Delta \downarrow F$*.*

Proof. This follows directly from Corollary 7.22 and Theorem 7.21 applied to $W = \Delta\mathbb{1}$. □

The rest of this section will be devoted to the technical lemmas supporting the proofs of Theorem 7.19 and Corollary 7.22 which give general characterisations of weighted bi-limits and conical bi-limits.

Lemma 7.26. *Let* $\mathbf{I}$ *and* $\mathbf{C}$ *be* 2*-categories, and* $F\colon \mathbf{I} \to \mathbf{C}$*,* $W\colon \mathbf{I} \to \mathbf{Cat}$ *be normal pseudo-functors. There is a canonical isomorphism of double categories as in the following commutative triangle.*

$$\begin{array}{ccc} \mathbb{el}(\mathbf{Ps}(\mathbf{I}, \mathbf{Cat})(W, \mathbf{C}(-, F))) & \xrightarrow{\cong} & W \downarrow\downarrow \mathbb{H}\mathbf{C}(-, F) \\ & {\scriptstyle \Pi}\searrow \quad \swarrow{\scriptstyle \Pi} & \\ & \mathbb{H}\mathbf{C}^{\mathrm{op}} & \end{array}$$

Proof. We give an explicit description of the data of the double category $\mathbb{el}(\mathbf{Ps}(\mathbf{I}, \mathbf{Cat})(W, \mathbf{C}(-, F)))$. By a straightforward comparison with the data of the pseudo-slice double category $W \downarrow\downarrow \mathbb{H}\mathbf{C}(-, F)$ described in Remark 4.2, we will see that the isomorphism above canonically holds.

An object in $\mathbb{el}(\mathbf{Ps}(\mathbf{I}, \mathbf{Cat})(W, \mathbf{C}(-, F)))$ consists of a pair (C, κ) of an object $C \in \mathbf{C}$ and a pseudo-natural transformation $\kappa\colon W \Rightarrow \mathbf{C}(C, F-)$ in $\mathbf{Ps}(\mathbf{I}, \mathbf{Cat})$. Then a horizontal morphism $(c, \Psi)\colon (C', \kappa') \to (C, \kappa)$ in $\mathbb{el}(\mathbf{Ps}(\mathbf{I}, \mathbf{Cat})(W, \mathbf{C}(-, F)))$ consists of a morphism $c\colon C \to C'$ in $\mathbf{C}$ together with an invertible modification Ψ in $\mathbf{Ps}(\mathbf{I}, \mathbf{Cat})$ of the form

$$\begin{array}{ccc} W & \overset{\kappa'}{\Longrightarrow} & \mathbf{C}(C', F-) \\ & {\scriptstyle \kappa}\searrow\!\!\searrow \;\; \overset{\Psi}{\Rrightarrow}\cong & \Big\Downarrow {\scriptstyle \mathbf{C}(c, F-)} \\ & & \mathbf{C}(C, F-) \end{array} \quad .$$

A vertical morphism $\Theta\colon (C,\kappa) \nrightarrow (C,\mu)$ in $\mathbb{el}(\mathbf{Ps}(\mathbf{I},\mathbf{Cat})(W,\mathbf{C}(-,F)))$ is a modification $\Theta\colon \kappa \Rrightarrow \mu$ with $\kappa,\mu\colon W \to \mathbf{C}(C,F-)$ two pseudo-natural transformations in $\mathbf{Ps}(\mathbf{I},\mathbf{Cat})$. Finally, a square $\gamma\colon (\Theta' \,{}^{(c,\Psi)}_{(d,\Phi)}\, \Theta)$ is a 2-morphism $\gamma\colon c \Rightarrow d$ in $\mathbf{C}$ satisfying the pasting equality in Definition 6.3 (iv), which can be translated into a pasting equality for the modification $\mathbf{C}(\gamma,F-)$ in $\mathbf{Ps}(\mathbf{I},\mathbf{Cat})$. □

Corollary 7.27. *Let* $\mathbf{I}$ *and* $\mathbf{C}$ *be* 2*-categories, and let* $F\colon \mathbf{I} \to \mathbf{C}$ *and* $W\colon \mathbf{I} \to \mathbf{Cat}$ *be normal pseudo-functors. There are canonical isomorphisms of* 2*-categories as in the following commutative triangles.*

$$\begin{array}{ccc} \mathbf{el}(\mathbf{Ps}(\mathbf{I},\mathbf{Cat})(W,\mathbf{C}(-,F))) & \xrightarrow{\cong} & W \downarrow \mathbf{C}(-,F) \\ & {}_{\pi}\searrow \quad \swarrow_{\pi} & \\ & \mathbf{C}^{\mathrm{op}} & \end{array}$$

$$\begin{array}{ccc} \mathbf{mor}(\mathbf{Ps}(\mathbf{I},\mathbf{Cat})(W,\mathbf{C}(-,F))) & \xrightarrow{\cong} & W \downarrow \mathrm{Ar}_*\mathbf{C}(-,F) \\ & {}_{\pi}\searrow \quad \swarrow_{\pi} & \\ & \mathrm{Ar}_*\mathbf{C}^{\mathrm{op}} & \end{array}$$

Proof. This follows directly from the definitions of **el** and **mor**, Lemma 7.26 and Proposition 4.6. □

With Lemma 7.26 and Corollary 7.27 above established we may now give a direct proof of Theorem 7.19.

Proof (Theorem 7.19). Recall from Remark 7.18 that a weighted bi-limit of F by W is equivalently given by a bi-representation of the 2-functor $\mathbf{Ps}(\mathbf{I},\mathbf{Cat})(W,\mathbf{C}(-,F))$. Then the result is obtained as a direct application of Theorem 6.8 using Lemma 7.26 and Corollary 7.27. □

In the conical case we may simplify the pseudo-slices above through the below computations to obtain a proof of Corollary 7.22.

Lemma 7.28. *Let* $\mathbf{I}$ *and* $\mathbf{C}$ *be* 2*-categories, and* $F\colon \mathbf{I} \to \mathbf{C}$ *be a normal pseudo-functor. There is a canonical isomorphism of double categories as in the following commutative triangle.*

$$\begin{array}{ccc} \Delta\mathbb{1} \downarrow\downarrow \mathbb{H}\mathbf{C}(-,F) & \xrightarrow{\cong} & (\mathbb{H}\Delta \downarrow\downarrow F)^{\mathrm{op}} \\ & {\scriptstyle \Pi}\searrow \quad \swarrow{\scriptstyle \Pi^{\mathrm{op}}} & \\ & \mathbb{H}\mathbf{C}^{\mathrm{op}} & \end{array}$$

Proof. This follows from the fact that, given an object $C \in \mathbf{C}$, a pseudo-natural transformation $\kappa\colon \Delta\mathbb{1} \Rightarrow \mathbf{C}(C, F-)$ in $\mathbf{Ps}(\mathbf{I}, \mathbf{Cat})$ corresponds to a pseudo-natural transformation $\kappa\colon \Delta C \Rightarrow F$ in $\mathbf{Ps}(\mathbf{I}, \mathbf{C})$. □

Corollary 7.29. *Let* $\mathbf{I}$ *and* $\mathbf{C}$ *be* 2*-categories, and* $F\colon \mathbf{I} \to \mathbf{C}$ *be a normal pseudo-functor. There are canonical isomorphisms of* 2*-categories as in the following commutative triangles.*

$$\begin{array}{ccc} \Delta\mathbb{1} \downarrow \mathbf{C}(-,F) & \xrightarrow{\cong} & (\Delta \downarrow F)^{\mathrm{op}} \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi^{\mathrm{op}}} & \\ & \mathbf{C}^{\mathrm{op}} & \end{array} \qquad \begin{array}{ccc} \Delta\mathbb{1} \downarrow \mathrm{Ar}_*\mathbf{C}(-,F) & \xrightarrow{\cong} & (\mathrm{Ar}_*\Delta \downarrow F)^{\mathrm{op}} \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi^{\mathrm{op}}} & \\ & \mathrm{Ar}_*\mathbf{C}^{\mathrm{op}} & \end{array}$$

Proof. This follows directly from Lemma 7.28 and Proposition 4.6. □

Finally we obtain a straightforward proof of Corollary 7.22.

Proof (Corollary 7.22). This result is obtained by applying Theorem 7.19 to the special case where $W = \Delta\mathbb{1}$ and using Lemma 7.28 and Corollary 7.29. □

References

[1] C. Auderset. Adjonctions et monades au niveau des 2-catégories. *Cahiers Topologie Géom. Différentielle*, 15:3–20, 1974.

[2] Francis Borceux and G. M. Kelly. A notion of limit for enriched categories. *Bull. Austral. Math. Soc.*, 12:49–72, 1975.

[3] tslil clingman and Lyne Moser. 2-limits and 2-terminal objects are too different. Preprint on arXiv:2004.01313, 2020.

[4] Marco Grandis. *Higher dimensional categories*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020. From double to multiple categories.

[5] Marco Grandis and Robert Paré. Limits in double categories. *Cahiers Topologie Géom. Différentielle Catég.*, 40(3):162–220, 1999.

[6] Marco Grandis and Robert Paré. Persistent double limits and flexible weighted limits. `https://www.mscs.dal.ca/~pare/DblPrs2.pdf`, 2019.

[7] Niles Johnson and Donald Yau. 2-dimensional categories. Preprint on arXiv:2002.06055, 2020.

[8] G. M. Kelly. *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge-New York, 1982.

[9] G. M. Kelly. Elementary observations on 2-categorical limits. *Bull. Austral. Math. Soc.*, 39(2):301–317, 1989.

[10] Stephen Lack. A Quillen model structure for 2-categories. *K-Theory*, 26(2):171–205, 2002.

[11] Stephen Lack. A 2-categories companion. In *Towards higher categories*, volume 152 of *IMA Vol. Math. Appl.*, pages 105–191. Springer, New York, 2010.

[12] Stephen Lack and Simona Paoli. 2-nerves for bicategories. *K-Theory*, 38(2):153–175, 2008.

[13] Saunders MacLane. *Categories for the working mathematician*. Springer-Verlag, New York-Berlin, 1971. Graduate Texts in Mathematics, Vol. 5.

[14] Lyne Moser, Maru Sarazola, and Paula Verdugo. A 2Cat-inspired model structure for double categories. Preprint on arXiv:2004.14233, 2020.

[15] Emily Riehl. *Category theory in context*. Aurora Modern Math Originals. Dover Publications, 2016.

[16] Emily Riehl and Dominic Verity. Elements of ∞-category theory. `http://www.math.jhu.edu/~eriehl/elements.pdf`, 2018.

[17] Ross Street. Limits indexed by category-valued 2-functors. *J. Pure Appl. Algebra*, 8(2):149–181, 1976.

[18] Ross Street. Fibrations in bicategories. *Cahiers Topologie Géom. Différentielle*, 21(2):111–160, 1980.

[19] Ross Street. Correction to: "Fibrations in bicategories" [Cahiers Topologie Géom. Différentielle **21** (1980), no. 2, 111–160; MR0574662 (81f:18028)]. *Cahiers Topologie Géom. Différentielle Catég.*, 28(1):53–56, 1987.

[20] Dominic Verity. Enriched categories, internal categories and change of base. *Repr. Theory Appl. Categ.*, (20):1–266, 2011.

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JEAN BÉNABOU (1932–2022):
The man and the mathematician

Francis Borceux

Résumé. Nous esquissons la personnalité de Jean Bénabou, présentons quelques apects importants de son œuvre et dressons une liste de ses publications.
Abstract. We sketch the personality of Jean Bénabou, present some important aspects of his work and provide a list of his publications.
Keywords. Monoidal categories, Bicategories, Distributors, Fibred categories, Empirical sets.
Mathematics Subject Classification (2010). 18CXX, 18C30, 18C35.

Jean Bénabou left this world on Friday, February 11, 2022, at the age of 89. Let us bet that the image that we will keep of him will be that of a highly creative mathematician and an extraordinary speaker.

He was indeed an exceptional speaker. Jean Bénabou was not just going to the blackboard to give a talk; he was entering the scene like a resolute actor "assaulting" the stage to perform a tragedy. He was all the time walking from left to right, from right to left, with a severe, almost aggressive look at his audience. You had the impression that he was not just delivering an interesting mathematical message, he was somehow fighting with you to convince you that he was telling the "'holy mathematical truth". He was having his cigarette holder in the left hand, bringing it regularly to his lips while the cigarette was inevitably extinguished. His talks were masterpieces of clarity. Each of them was prepared up to the last detail; he wanted everyone to follow properly the message that he was delivering. He was so anxious to take all the necessary precautions to be clear, precise and understood, that when the chairman made him observe that his time was over, his last sentence was inevitably something like "I thank you, but unfortunately I did not even have time to tell half of what I wanted to say".

Jean Bénabou was somehow living for mathematics, for category theory. Those who had the chance to visit him in his apartment in Paris or in his country house in La Garde-Freinet, know that he was getting up very early. At five in the morning, he was already doing mathematics, in the quiet atmosphere of an environment where most people were still sleeping. And when yourself were showing up later in the day, you could not escape Jean telling you the ideas on which he had just been working. Mathematics, and particularly categories, were always around, including in the kitchen of his apartment where his white board was hanging, full of proofs and results, carefully calligraphed. He was always ready to share his ideas with you, sorry, to convince you to share his ideas. And the less that one can say, is that a challenge that he often had to face ... was to have too many ideas.

But in the evening, when the moment did arrive to take some good time around a glass of beer or a bottle of wine, Jean Bénabou could become the life and soul of the party. And when Sammy Eilenberg was himself around the table, you had right to a highly pleasant competition of these two, on who would tell the best Jewish jokes. Jean was very generous in his hospitality, very smooth and friendly, another man than the warrior giving talks. And, at Oberwolfach or at La Garde-Freinet, those who could hold a bat could not escape affronting him in some ping-pong match.

But this quick sketch of the personality of Jean Bénabou would not be honest if not mentioning a more controversial aspect of it. As a matter of fact, Jean Bénabou became with the years always more reluctant to publish papers. Not reluctant to write down his ideas: he wrote many long manuscripts which almost never circulated and that his son Roland intends to give now to some university library. In fact Jean Bénabou did often consider that he still had to wait before publishing, because things could yet be improved. Once he also wrote that he was afraid to be one day short of new ideas, and thus found much more exciting to investigate a new idea when it came, than spending time on writing down a paper. But from waiting to waiting, many of his beautiful ideas were never published. And when a good idea is in the air, many people bump into it and independently, via similar or different approaches, provide solutions and publish papers. Jean Bénabou could enter a vehement anger when he was unduly interpreting these works as a kind of theft of his ideas, and even more serious, when he was viewing them as a betrayal of his ideas, just because the point of view adopted by other authors denied his own convictions. Jean was passionate; he could be excessive.

For more details and some anecdotes on Bénabou's personality and life, see Jacques Roubaud's text "Esquisse d'un portrait de Jean Bénabou, catégoricien" [32].

Jean Bénabou is born in Rabat, in Morocco, in 1932. He comes to study in France and enters the *École Normale Supérieure* in 1952. His research is next supported by the *Centre National de la Recherche Scientifique*, from 1956 to 1962. Charles Ehresmann is his supervisor in Paris and both are in those days highly interested in the development of pointless topology, in particular the theory of locales, which becomes the topic of the first publication of Jean Bénabou [1]. He gets also interested in another approach of topological and geometrical problems: the Grothendieck toposes.

In 1963, Bénabou introduces (independently from Mac Lane and his coherence theorems [29]), the notion of a *monoidal category*, under the name of "catégorie avec multiplication". This is presented in a *Note* in the *Compte-Rendus de l'Académie des Sciences de Paris* [3], followed by another one in 1964 [4] and a *Note* on relative categories in 1965 [6]. Disregarding the anteriority of the short Bénabou's *Notes*, written in French, the 1965 *La Jolla* paper of Eilenberg–Kelly on *Closed categories* [28] becomes somehow the standard reference for enriched category theory, a fact which seriously irritates Bénabou.

Jean Bénabou presents his *Doctorat d'État* at the "Université de Paris" in 1966, under the title *Structures algébriques dans les catégories*. It is published in 1968 in the "Cahiers de Topologie et Géométrie Différentielle" [9].

The first part of the thesis generalizes theories of Lawvere and Higgins to define multisorted algebraic structures. For that, Bénabou introduces the notions of types and their associated generic models in relation with Chevalley "ćatégories marquées" (p. 48 of the thesis), sheaves and Grothendieck topologies (p. 49) and implicitly Ehresmann structured categories (remark p. viii).

The second part of the thesis introduces a general notion of "binary systems" and their families of associated categories, with the examples of bimodules and of spans. This frame generalizes the monoidal and relative categories that Bénabou studied earlier.

A very important step in the career of Jean Bénabou is his stay at Chicago, in 1966-67, upon the invitation of Saunders Mac Lane. It is the occasion for him to extend the reflection initiated in his thesis and investigate further pseudo and lax structures. From this, results a well-known paper on *bicategories* and *profunctor/distributors*, published in the Reports of the *Midwest Category Seminar* [7]. This paper is still inspiring and cited today. Various aspects of it clearly influence some of Bénabou's future works on descent and fibered categories, in particular the memoir that he writes with Jacques Roubaud [10].

Back in France, Bénabou organizes a weekly seminar at the *Institut Henri Poincaré*, and later at Jussieu (Université Paris 7). This seminar, together with the Ehresmann seminar (1955–1977), plays a significant role in the development of category theory in France. Toposes and distributors are among the topics most studied in Bénabou's seminar in the early seventies.

Jean Bénabou is also always ready to accept invitations for coming and teaching his ideas in other universities, something that he does excellently well. And from then on, the course notes written by some auditors of his courses, and published as university preprints, become an important source – sometimes the only one – of access to Bénabou's ideas. This is in particular the case in Louvain-la-Neuve [12] to [15] and [31], and in Darmstadt [23].

Jean Bénabou, as already said, had considered Grothendieck toposes as an efficient algebraic tool to handle topological and geometrical problems. When Lawvere and Tierney introduce elementary toposes, Bénabou immediately switches to the study of the internal logical structure of these and to the consideration of the models of (external) algebraic theories in a topos with Natural Number Object [12], [15].

Still largely inspired by the work of Grothendieck, Jean Bénabou starts in the years 1970 to develop what is probably the masterpiece of his mathematical achievement: the theory of fibered categories. He polishes his approach to give it an elegance which reaches beyond what he had done before. In this case, his intense concern of perfection results in his approach being repeatedly improved and almost never published, at the exception of some few aspects [17], [18]. And once more, the publication of the Paré-Schumacher paper [30] on *Indexed categories* becomes somehow the reference to the topic, even if the spirit of the approach is quite different. The less that one can say is that Jean Bénabou is depply affected by this situation and the weak recognition of his own (unpublished) beautiful ideas on fibered categories. For a long time, besides his few early notes, the only access to Bénabou's approach of fibered categories is a beautiful set of notes written (in French) by Jean-Roger Roisin [31], following a course held by Bénabou in 1980 in Louvain-la-Neuve. But Bénabou, in his search of perfection, indefinitely postpones the publication of Roisin's notes, which so never occurs. The author of this bibliographical note has included Bénabou's approach to fibered categories as a chapter in the second volume of his *Handbook of Categorical Algebra*, in 1994.

In the years 2000, another course of Jean Bénabou in Darmstadt results in a collaboration with Thomas Streicher and the publication of preprints [23] and [33], but also a joint paper [24]. Very interesting and original ideas developed by Bénabou in his approach to fibered categories are those of smallness and definability [20]. The almost total ignorance of these beautiful notions in the literature on fibered or indexed categories underlines the fact that Bénabou's ideas on the topic should still remain a source of inspiration. In his permanent search of generalization, Jean Bénabou, around 2012, weakens the notion of fibration to that of foliation [25].

The scope of mathematical interests of Jean Bénabou was very wide, including results on non-standard analysis, on the theory of trees, on the logical foundations of category theory, on the notion of universe in a Grothendieck topos, and so on.

He developed also more philosophically inspired works, like an empiric set theory [19] or an adjunction between *almost* and *very* [26].

A tentative list of Bénabou's Ph.D. students

The topics of the various Ph.D.'s that Bénabou supervised give also evidence of the broadness of his interests. And we suspect that the corresponding tentative list that we provide below is probably not exhaustive.

Brigitte Lesaffre, *Structures algébriques dans les topos elementaires*, thèse de 3ème cycle, Paris VII, 1974.

Jean Celeyrette, *Catégories fibrées et Topoï*, thèse d'état, Paris XIII, 1975.

Yves Diers, *Catégories localisables*, thèse d'état, Paris VI, 1977.

Michel Coste, *Localisation dans les catégories de modèles*, thèse d'état, Paris XIII, 1977.

Marie-Françoise Roy, *Spectre réel d'un anneau et topos étale réel*, thèse d'état, Paris XIII, 1980.

Jacques Penon, *De l'infinitésimal au local*, thèse d'état, Paris VII, 1985.

Dominique Bourn, *La tour de fibrations des n-groupoïdes et la longue suite exacte de cohomologie*, thèse d'état, Paris XIII, 1990.

A tentative list of Bénabou's publications

Our tentative list of papers of Jean Bénabou, together with typescripts of courses that he delivered, is given by the **26 first items in references** below. It is probably not complete.

References

[1] J. Bénabou, Treillis locaux et paratopologies, *Séminaire Ehresmann (Topologie et géométrie différentielle)*, 1, 1957-58, exposé 2, p.1-27.

[2] J. Bénabou, Treillis locaux, *Séminaire Dubreil (Algèbre et théorie des nombres)*, 1957-58, exposé 7, 9p.

[3] J. Bénabou, Catégories avec multiplication, *C. R. Acad. Sc.*, Paris (1963) 256, 1887-1890.

[4] J. Bénabou, Algèbre élémentaire dans les catégories avec multiplication, *C. R. Acad. Sc.*, Paris (1964) 258, 771-774.

[5] J. Bénabou, Critères de représentabilité des foncteurs, *C. R. Acad. Sc.*, Paris (1965) 260, 752-755.

[6] J. Bénabou, Catégories Relatives, *C. R. Acad. Sc.*, Paris (1965) 260, 3824-3827.

[7] J. Bénabou, Introduction to Bicategories, *Reports of the Midwest Category Seminar*, Springer Lecture Notes in Mathematics 47, Springer (1967), pp.1-77.

[8] J. Bénabou, Catégories algébriques et théorie de la descente, *Séminaire Dubreil. Algèbre et théorie des nombres*, tome 21, expos 14 (1967-1968), 1-2.

[9] J. Bénabou, Structures algébriques dans les catégories, *Cahiers de Topologie et Géométrie différentielle*, t. 10, 1968, p. 1-118.

[10] J. Bénabou et J. Roubaud, Monades et descente, *C. R. Acad. Sc.*, Paris, (1970) 270, 96-98.

[11] J. Bénabou et J. Celeyrette, Généralités sur les topos de Lawvere et Tierney, *Séminaire Bénabou*, 1971.

[12] J. Bénabou, Les triples, *Institut de Mathématique pure et appliquée, Université catholique de Louvain*, 1972.

[13] J. Bénabou, Les catégories multiplicatives, *Institut de Mathématique pure et appliquée, Université catholique de Louvain*, 1972.

[14] J. Bénabou, Les distributeurs, Rapport 33 du *Séminaire de Mathématique Pure et Appliquée, Université Catholique de Louvain*, 1973.

[15] J. Bénabou, Problèmes dans les topos, *Institut de Mathématique pure et appliquée, Université catholique de Louvain*, 1973.

[16] J. Bénabou, Théories relatives à un corpus, *C. R. Acad. Sci.*, Paris, (1975) 281, pp.831-834.

[17] J. Bénabou, *Fibrations petites et localement petites*, *C. R. Acad. Sci.*, Paris, (1975) 281, pp.897-900.

[18] J. Bénabou, Fibered Categories and the Foundations of Naive Category Theory, *The Journal of Symbolic Logic*, Vol. 50, No. 1 (Mar., 1985), pp. 10-37.

[19] J. Bénabou, Théorie des ensembles empiriques, Volume 17 et Volume 21 de *Mezura : Cahiers de poétique comparée* : deuxième série : documents de travail, Langues'O, 1988.

[20] J. Bénabou, Definability, Finiteness, Projectivity and Choice, around 1992, unpublished notes.

[21] J.Bénabou and B. Loiseau, Orbits and Monoids in a Topos, *Journal of pure and applied algebra* 92, 1994, pp.29-54.

[22] J. Bénabou, Some geometric aspects of the calculus of fractions, *Applied Categorical Structures*, Vol. 4, 139–165, 1996.

[23] J. Bénabou, Distributors at work, *University of Darmstadt*, june 2000.

[24] J. Bénabou and T. Streicher, Partial toposes, *Theory and Applications of Categories*, Vol. 11, No. 13, 2003, pp. 309–320.

[25] J. Bénabou, Foncteurs représentables et catégories feuilletées, *Conférence à la Journée dédiée à René Guitart, 9/12/2012*, https://www.youtube.com/watch?v=p3vcZDyZ4Yo

[26] J. Bénabou, Le très, le presque et autres articulations logiques du langage, *Conférence Séminaire Mamuphi 4 avril 2015*, vidéo https://www.youtube.com/watch?v=TcuszrPLDsM

References in this notice to authors other than Jean Bénabou.

[27] F. Borceux, *Handbook of categorical Algebra, Vol. 2, Categories and Structures*, Cambridge University Press (1994).

[28] S. Eilenberg and G.M. Kelly, Closed categories, *Proceedings of the Conference on Categorical Algebra*, La Jolla 1965, Springer 1966, 421-562.

[29] S. Mac Lane, Natural associativity and commutativity, *Rice University Studies* 49, 28-46 (1963).

[30] R. Paré and D. Schumacher, Indexed categories and their applications; in "By Peter T. Johnstone et al." LNM Springer 1978.

[31] J.R. Roisin, Les catégories fibrées, d'après un cours de J. Bénabou, *Institut de Mathématique pure et appliquée, Université catholique de Louvain*, 1980.

[32] J. Roubaud, Esquisse d'un portrait de Jean Bénabou, catégoricien, *Images des Mathématiques*, 2011.

[33] T. Streicher, Fibered categories à la Jean Bénabou, *arXiv* 1999 and 2018.

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