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A FUNCTORIAL APPROACH TO DIFFERENTIAL CALCULUS

Wolfgang Bertram, Jérémy Haut

Résumé. Nous montrons que le calcul différentiel (sous sa forme usuelle, ou sous la forme du *calcul différentiel topologique*) admet un plongement plein et fidèle dans une catégorie de foncteurs (des foncteurs d'une petite catégorie dite *catégorie des algèbres tangentes ancrées* vers des ensembles ancrés). Pour préparer cette approche, nous définissons une nouvelle version, plus symétrique, du calcul différentiel, où l'application *ancre* joue un rôle central.

Abstract. We show that differential calculus (in its usual form, or in the general form of *topological differential calculus*) can be fully imdedded into a functor category (functors from a small category of *anchored tangent algebras* to anchored sets). To prepare this approach, we define a new, symmetric, presentation of differential calculus, whose main feature is the central rôle played by the *anchor map*, which we study in detail.

Keywords. differential calculus, functor category, anchor, tangent algebra.

Mathematics Subject Classification (2010). 18A25 , 18B40 , 18D05 , 58A05

Introduction

Differential Calculus is a central ingredient of modern mathematics. While the "working mathematician" takes this tool for granted, thinking about its conceptual foundations remains a potentially important topic. In the present work, we continue the line of research started with [BGN04, Be08, BeS14, Be17], and combine it with what Grothendieck once called the "simple idea of a good functor from rings to sets" (according to W. Lawvere, cf. n-lab)[1]. The "simple idea" mentioned by Grothendieck is currently used in algebraic

[1]Here the quote from the n-lab: "The 1973 Buffalo Colloquium talk by Alexander Grothendieck had as its main theme that the 1960 definition of scheme ... should be abandoned AS the FUNDAMENTAL one and replaced by the *simple idea of a good functor from rings to sets*. The needed restrictions could be more intuitively and more geometrically stated directly in terms of the topos of such functors, and of course the ingredients from the "baggage" could be extracted when needed as auxiliary explanations of already existing objects, rather than being carried always as core elements of the very definition."

geometry, and in Lie Theory, where one often considers a real "space" – for instance, a Lie group $\underline{G}$ – as set of "real points" $G_{\mathbb{R}}$ of a *complex* Lie group $G_{\mathbb{C}}$. This is a kind of non-linear analog of the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ of a real vector space (or of a real Lie algebra). Grothendieck's insight was that this idea of "complexification" should not be limited to *field* extensions, but enlarged to more general *ring* extensions, in order to incorporate operations belonging to *infinitesimal calculus*: a $\mathbb{K}$-Lie group G, or a general $\mathbb{K}$-smooth manifold M, should admit "scalar extensions" $M_{\mathbb{A}}$ akin to a hypothetic tensor product $M \otimes_{\mathbb{K}} \mathbb{A}$, for certain $\mathbb{K}$-algebras $\mathbb{A}$. The simplest example of such an extension is the one by *dual numbers*,

$$\mathbb{K}[\varepsilon] := \mathbb{K}[X]/(X^2) = \mathbb{K} \oplus \varepsilon\mathbb{K} \quad (\varepsilon^2 = 0), \tag{0.1}$$

where the nilpotent element ε is the class $[X]$ modulo (X^2). Grothendieck, following ideas of Weil [We53], realized that the tangent bundle TM of a "space" M, which is "defined over $\mathbb{K}$", could be understood as something like $M \otimes_{\mathbb{K}} \mathbb{K}[\varepsilon]$. This idea has been used by Demazure and Gabriel in their theory of algebraic groups [DG], in differential calculus over general base field and rings [Be08], and in the approach to natural operations in differential geometry via the so-called *Weil functors* ([KMS93], cf. also [BeS14]). The most elaborate and systematic development of these ideas leads to what is called nowadays *synthetic differential geometry* (SDG, see [MR91]). The approach to be presented here pursues the same goals as SDG, but by different means: we keep closer to the idea of generalizing the algebraic tensor product. In a very direct sense, our problem is to generalize the algebraic scalar extension $V_{\mathbb{A}} := V \otimes_{\mathbb{K}} \mathbb{A}$ of a $\mathbb{K}$-module V, to more general spaces M, like, e.g., manifolds – where we face the problem that such an operation won't be possible for *all* $\mathbb{K}$-algebras $\mathbb{A}$, so we have to single out a good class (good category) of algebras for which such an extension is possible. Such a class, called *the category of (anchored) tangent algebras*, will be defined in this paper. It arises naturally, when questioning the very shape of differential calculus, instead of taking it for granted. Let us briefly explain the main ideas.

0.1 Topological differential calculus

In differential calculus we consider maps f whose domain U and codomain U' are *locally linear sets* – by this we mean $U \subset V$ and $U' \subset V'$ are non-empty subsets of linear (or affine, if one prefers) spaces V and V'. In this situation, we may define the *slope* or *difference quotient map*: when $t, s \in \mathbb{K}$ are such that $t - s$ is invertible, we look at the difference quotient

$$f^{[1]}(v_0, v_1; t, s) := f^{[1]}_{(t,s)}(v_0, v_1) := \frac{f(v_0 + tv_1) - f(v_0 + sv_1)}{t - s}. \tag{0.2}$$

To speak of *topological* calculus, we shall assume that V, V' are topological vector spaces or modules over topological fields or rings $\mathbb{K}$, and U, U' are open. For the moment, let's consider the "classical case" $\mathbb{K} = \mathbb{R}$ and $V = \mathbb{R}^n$, $V' = \mathbb{R}^m$. Then the following holds (cf. [BGN04, Be08]): *The map f is of class C^1 if, and only if, the difference quotient map $f^{[1]}$ extends continuously to a map defined on the set*

$$U^{[1]} := \left\{ (v_0, v_1; t, s) \in V^2 \times \mathbb{K}^2 \middle| \begin{array}{l} v_0 + tv_1 \in U \\ v_0 + sv_1 \in U \end{array} \right\}. \tag{0.3}$$

If this is the case, we denote still by $f^{[1]} : U^{[1]} \to U'$ the extended map. Then the differential df of f is given by $f^{[1]}(v_0, v_1; 0, 0) = df(v_0)v_1$. Now, these conditions make perfectly sense for any "good" topological ring $\mathbb{K}$ and for maps defined on open locally linear sets, and thus can serve as *definition* of differentiability over $\mathbb{K}$ – the "topological differential calculus" thus defined has excellent functorial properties allowing to give a "purely algebraic" presentation of certain features of usual calculus (see [BGN04, Be11]). To understand the structure of formulae like (0.2) and (0.3), the following *way of talking* turns out to be useful:

- call $\mathbf{v} = (v_0, v_1)$ "space variables", with v_0 the "foot point" and v_1 the "direction" (in which we differentiate),
- call (t, s) "time variables", and t "target time", and s "source time",
- call (t, s) "regular", or "finite", if $t - s$ is invertible in $\mathbb{K}$, and "singular" or "infinitesimal" else, with $t - s = 0$ being the "most singular value",
- call $v_0 + sv_1$ the "source", and $v_0 + tv_1$ the "target evaluation point",
- for fixed (t, s), call $\alpha\big((v_0, v_1)\big) := v_0 + sv_1$ the "source map", and define the "target map" $\beta\big((v_0, v_1)\big) := v_0 + tv_1$.

The slogan summarizing topological calculus is: *the slope extends continuously (jointly in space and time variables) from finite to singular times*. The notable difference with [BGN04, Be11] is that here we shall use a *pair* of time parameters (t, s), instead of a single parameter t as in loc. cit. Although the expression (0.2) is of course symmetric under switch of target and source time, it will be important to distinguish "target" and "source". The setting of [BGN04, Be11] is gotten by restricting to $s = 0$ (we call this "target calculus"); symmetrically, the theory could also be formulated when letting $t = 0$ ("source calculus"). But now we can take advantage to define a third calculus, the "symmetric calculus", which corresponds to the case $t = -s$: then $v_0 = \frac{v_0 + sv_1 + v_0 + tv_1}{2}$, so the footpoint is the midpoint of target and source evaluation point – see Subsection 2.5.[2]

[2] A price has to be paid: one will have to require that 2 be invertible in $\mathbb{K}$. Analysts won't bother, some algebraists might...

0.2 The underlying algebraic structure: anchor

In the second section we shall carve out the algebraic structures underlying topological differential calculus. As in general groupoid theory, the pair (α, β) given by source and target will be called *anchor map*[3]. We use the same term when considering the pair of time variables (t, s) as a "frozen parameter" (temporarily considered to be fixed); then we write (t, s) as lower index – for instance,

$$U^{[1]}_{(t,s)} := \{(v_0, v_1) \mid (v_0, v_1; t, s) \in U^{[1]}\}. \tag{0.4}$$

For fixed (t, s), we call again *anchor* the (linear) map sending the space variables $\mathbf{v} = (v_0, v_1)$ to the pair of evaluation points:

$$\Upsilon_{(t,s)} : U^{[1]}_{(t,s)} \to U \times U, \qquad \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \mapsto \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & s \\ 1 & t \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} v_0 + sv_1 \\ v_0 + tv_1 \end{pmatrix} = \begin{pmatrix} \alpha(\mathbf{v}) \\ \beta(\mathbf{v}) \end{pmatrix}. \tag{0.5}$$

Of course, a choice is made here: the "first" component of $U \times U$ shall be associated with "source", and the "second" with "target". One of our concerns in the sequel will be to formalize the levels on which such choices are operated. Anyhow, by direct computation, the anchor is seen to be invertible if, and only if, $t - s$ is invertible, and then its inverse is given by

$$\Upsilon^{-1}_{(t,s)} : U \times U \to U^{[1]}_{(t,s)}, \qquad \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \mapsto \frac{1}{t-s} \begin{pmatrix} t & -s \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} \frac{tx_0 - sx_1}{t-s} \\ \frac{x_1 - x_0}{t-s} \end{pmatrix}. \tag{0.6}$$

The first component is an affine combination $v_0 = \frac{s}{s-t} x_1 + \frac{t}{t-s} x_0$, and the second a "difference quotient". From this, comparing with (0.2), we see that $f^{[1]}_{(t,s)}$ is precisely the second component of the map $f^{\{1\}}_{(t,s)} := \Upsilon^{-1}_{(t,s)} \circ (f \times f) \circ \Upsilon_{(t,s)}$, given by

$$f^{\{1\}}_{(t,s)} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} \frac{tf(v_0 + sv_1) - sf(v_0 + tv_1)}{t-s} \\ \frac{f(v_0 + tv_1) - f(v_0 + sv_1)}{t-s} \end{pmatrix}. \tag{0.7}$$

The big advantage is that $f^{\{1\}}_{(t,s)}$ *depends functorially on* f: the "chain rule" simply reads $(g \circ f)^{\{1\}}_{(t,s)} = g^{\{1\}}_{(t,s)} \circ f^{\{1\}}_{(t,s)}$. Now we can reformulate the property of being $C^1_{\mathbb{K}}$ (Lemma 1.2): *The map* $f : U \to U'$ *is of class* $C^1_{\mathbb{K}}$ *if, and only if, for all* $(t, s) \in \mathbb{K}^2$ *there exists a continuous map* $f^{\{1\}}_{(t,s)} : U^{\{1\}}_{(t,s)} \to (U')^{\{1\}}_{(t,s)}$*, jointly continuous also in the parameter* $(t, s) \in \mathbb{K}^2$*, such that*

$$\Upsilon_{(t,s)} \circ f^{\{1\}}_{(t,s)} = (f \times f) \circ \Upsilon_{(t,s)} : \qquad \begin{array}{ccc} U_{(t,s)} & \xrightarrow{f^{\{1\}}_{(t,s)}} & U'_{(t,s)} \\ \Upsilon \downarrow & & \downarrow \Upsilon \\ U \times U & \xrightarrow{f \times f} & U \times U \end{array} \tag{0.8}$$

[3] This map is indeed the anchor map of a groupoid structure, see Subsection 4.2.

In a nutshell, this diagram contains the essential ingredients needed for our approach: our aim is to translate diagram (0.8) into a "categorical" formulation, so that it will make sense in an abstract setting, not requiring topology any more. In a first step, we generalize this diagram at higher order $n \in \mathbb{N}$ (Theorem 1.8): indeed, differentiability at order n is characterized by a diagram of the same kind, replacing $f^{\{1\}}_{(t,s)}$, etc., by higher order maps $f^{\{\mathsf{n}\}}_{(\mathbf{t},\mathbf{s})}$, etc., where $(\mathbf{t},\mathbf{s}) = (t_1,\ldots,t_n;s_1,\ldots,s_n) \in \mathbb{K}^{2n}$. Technically, we work with 2^n-fold direct products, which have to be indexed by elements A of the *n-hypercube* $\mathcal{P}(\mathsf{n})$ (power set of $\mathsf{n} = \{1,\ldots,n\}$).

0.3 The simple idea of a good functor from rings to sets

In order to formalize the idea that the extended domains and maps $(U^{\{\mathsf{n}\}}_{(\mathbf{t},\mathbf{s})}, f^{\{\mathsf{n}\}}_{(\mathbf{t},\mathbf{s})})$ are scalar extensions $(U \otimes_{\mathbb{K}} \mathbb{A}, f \otimes_{\mathbb{K}} \mathbb{A})$, we look at the case $U = \mathbb{K}$. From functoriality, it follows that the spaces $\mathbb{K}^{\{\mathsf{n}\}}_{(\mathbf{t},\mathbf{s})}$ are in fact $\mathbb{K}$-*algebras*, which can easily be identified,

1. in terms of polynomial rings: they are polynomial algebras $\mathbb{K}[X_1,\ldots,X_n]$, quotiented by the relations $(X_i - t_i)(X_i - s_i) = 0$, for $i = 1,\ldots,n$,
2. in terms of tensor products: they are n-fold tensor products of "first order algebras" $\mathbb{K}_{(t_1,s_1)} \otimes \ldots \otimes \mathbb{K}_{(t_n,s_n)}$.

The second item shows that the collection of these algebras $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ forms a *small monoidal category* with respect to the tensor product, where we define morphisms to be given by left or right multiplications coming from the monoid structure. This is the *category* $\mathbf{talg}_{\mathbb{K}}$ *of* $\mathbb{K}$-*tangent algebras*. Every such algebra admits an *anchor morphism* $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} : \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to \mathbb{K}^{\mathcal{P}(\mathsf{n})}$ to the *cube algebra* which is a direct product of copies of $\mathbb{K}$, indexed by the n-hypercube $\mathcal{P}(\mathsf{n})$. We compute an explicit formula describing $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ (Theorem 2.8). This anchor morphism is an isomorphism if, and only if, $(\mathbf{t},\mathbf{s})$ is *regular*, and we give an explicit formula for the inverse morphism (Theorem 2.9).

Now, the "simple idea of a good functor from rings to sets" is to view "$\mathbb{K}$-smooth spaces" as functors $\underline{M}$ from the category $\mathbf{talg}_{\mathbb{K}}$ to the category of sets, satisfying certain conditions specified in Subsection 3.5, and "$\mathbb{K}$-smooth maps" as certain *natural transformations* between functors $\underline{M}$ and $\underline{M}'$, behaving in all respects like a family of "algebraic scalar extensions" $f \otimes_{\mathbb{K}} \mathrm{id}_{\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}}$. Indeed, in the framework of topological differential calculus, for a smooth map $f : M \to M'$, the family $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ satisfies these conditions, and thus "topological calculus" imbeds into "categorical calculus".

In order to fully justify such a functorial approach to differential calculus, one usually requires in SDG that the model be *well-adapted*, that is, that we obtain a *full and faithful* imbedding of a "usual" category of differential calculus into the "functorial" one. We show that, for our setting, this is indeed the case (Theorem 3.11). The proof is much easier than the one of analogs in SDG, because, in essence, the whole setting is designed for such a theorem to hold: it is merely the translation of Theorem 1.8 into a more abstract language.

0.4 Further topics

The aim of this work is to lay the basic framework for a purely categorical approach to calculus over general (commutative) base rings. In Section 4 we briefly indicate further questions and topics to be studied in this context: to study *natural transformations* in the sense of [KMS93], we have to introduce further morphisms in our categories, and in particular those arising via the natural *(higher order) groupoid structure* that exists on the algebras $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$. Very likely, a good understanding requires to understand also the *full* iteration procedure, and not only the restricted one used here, so to include, for instance, also the *simplicial calculus* from [Be13]. Finally, we conjecture that, replacing the usual braiding of tensor products by the braiding defining the *graded tensor product*, the present approach will also turn out to be useful in a categorical approach to *super-calculus*.

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Notation. We write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let $\mathsf{n} = \{1, 2, \ldots, n\}$. Categories are denoted in boldface characters: small letters for small categories, such as $\mathbf{talg}_{\mathbb{K}}$, and capital letters for large categories, such as $\mathbf{Sets}$ (category of sets). The letter $\mathbf{Fn}$ stands for “functor category”, so $\mathbf{Fn}(\mathbf{c}, \mathbf{Sets}) = \mathbf{Sets}^{\mathbf{c}}$ is the category of (covariant) functors from a (small) category $\mathbf{c}$ to $\mathbf{Sets}$. Throughout, $\mathbb{K}$ is a commutative base ring with unit 1.

1. Topological differential calculus

In differential calculus, one usually is mostly interested in the *morphisms*, that is, in *maps of class C^n*. However, let us first say some words about the *objects*:

1.1 Locally linear sets, and the anchor

A *locally linear set* is a pair (U, V), where V is a $\mathbb{K}$-module, and $U \subset V$ a non-empty subset. We define the set $U^{[1]}$ by (0.3), and the *(full) anchor* by

$$\Upsilon : U^{[1]} \to (U \times \mathbb{K})^2, \quad (v_0, v_1; t, s) \mapsto (v_0 + sv_1, s; v_0 + tv_1, t). \tag{1.1}$$

When time parameters $(t,s) \in \mathbb{K}^2$ are fixed, we define $U_{(t,s)} := U^{[1]}_{(t,s)} := U^{\{1\}}_{(t,s)}$ by (0.4), and the *(restricted) anchor*

$$\Upsilon_{(t,s)} := \Upsilon^{\{1\}}_{(t,s)} : U^{[1]}_{(t,s)} \to U \times U \tag{1.2}$$

is given by restricting the map Υ defined above, i.e., it is given by (0.5). Direct computation shows that $\Upsilon_{(t,s)}$ is invertible iff $s - t$ is invertible in $\mathbb{K}$, with inverse given by (0.6). Note that $(U^{[1]}_{(t,s)}, V^2)$ is again a locally linear set, and hence the construction can be iterated, with some new parameter (t_2, s_2), and so on. Explicit formulae, describing this, will be given later (restricted iteration, Def. 1.5).

1.2 The topological setting

In the remainder of this section we assume that $\mathbb{K}$ is a *good topological ring* (i.e., a topological ring whose unit group $\mathbb{K}^\times$ is open and dense, and inversion is a continuous map), that all $\mathbb{K}$-modules are topological modules, and that all locally linear sets (U,V), $(U',V'),\ldots$ are *open* inclusions.

Definition 1.1. *We say that $f : U \to V'$ is* of class $C_1^{\mathbb{K}}$ *if the* slope *given by (0.2) extends to a* continuous *map $f^{[1]} : U^{[1]} \to V'$. We then define, for all $(x,v) \in U \times V$,*

$$df(x)v := \partial_v f(x) := f^{[1]}(x,v;0,0).$$

Remark 1.1. Letting $s = 0$, the preceding definition clearly implies that f is of class $C^1_{\mathbb{K}}$ in the sense of [BGN04] or [Be08]. Conversely, the map denoted here by $f^{[1]}$ can be expressed by the one denoted $f^{[1]}$ in loc. cit., and hence the $C^1_{\mathbb{K}}$-notions used there are equivalent to the one given above. We call the calculus obtained by restricting to $s = 0$ *target calculus*. Recall from [BGN04] that, in the real or complex finite dimensional case this definition is equivalent to all usual ones, and in the infinite dimensional locally convex case it is equivalent to Keller's definition of differentiability.

Lemma 1.2. *For a map $f : U \to U'$, the following are equivalent:*

1. *f is $C^1_{\mathbb{K}}$,*
2. *for all $(t,s) \in \mathbb{K}^2$, there exists a (unique) map $f_{(t,s)} = f^{\{1\}}_{(t,s)} : U_{(t,s)} \to U'_{(t,s)}$, such that*
   - *(a) the map $U^{[1]} \to (U')^{[1]}$, $(x,v;t,s) \mapsto f_{(t,s)}(x,v)$ is continuous,*
   - *(b) for all $(t,s) \in \mathbb{K}^2$,*

$$\Upsilon_{(t,s)} \circ f^{\{1\}}_{(t,s)} = (f \times f) \circ \Upsilon_{(t,s)} : \qquad \begin{array}{ccc} U_{(t,s)} & \xrightarrow{f_{(t,s)}} & U'_{(t,s)} \\ \Upsilon \downarrow & & \downarrow \Upsilon \\ U \times U & \xrightarrow{f \times f} & U' \times U' \end{array}$$

Proof. As we have already seen, when $t-s$ is invertible in $\mathbb{K}$, then $f_{(t,s)}$ is necessarily given by (0.7). Since its second component is the slope $f^{[1]}$, existence of $f_{(t,s)}$, jointly continuous in $(x,v;t,s)$, implies existence of a continuous extension of the slope, whence (2) ⇒ (1). To prove the converse, assume (1) and write $f_{(t,s)}(x,v)=(w_0,w_1)$ with (w_0,w_1) given by (0.7). Assumption (1) means that $w_1=w_1(x,v;t,s)$ admits a continuous extension. Let us show that $w_0=w_0(x,v;t,s)$ also admits a continuous extension. To see this, let $x_0:=f(x+sv)$ and $x_1:=f(x+tv)$. Then $x_0=w_0+sw_1$, $x_1=w_0+tw_1$, whence

$$w_0=x_1-tw_1=f(x+tv)-tf^{[1]}(x,v;t,s),$$

showing that $w_0(x,v)$ extends continuously for all (t,s) if so does $f^{[1]}(x,v;t,s)$. □

Example 1.1. If $f:V\to V'$ is *linear* and continuous, then direct computation using (0.7) shows that $f_{(t,s)}(v_0,v_1)=(f(v_0),f(v_1))$, so f is $C^1_{\mathbb{K}}$.

Remark 1.2. Letting $v_1=0$ in (0.7), we always get $f_{(t,s)}(v_0,0)=(f(v_0),0)$. In diagrammatic form, the map f itself imbeds into $f_{(t,s)}$: we define the imbedding

$$\iota_{(t,s)}:U\to U_{(t,s)},\quad v_0\mapsto(v_0,0) \tag{1.3}$$

then the computation just mentioned shows that $f_{(t,s)}\circ\iota_{(t,s)}=\iota_{(t,s)}\circ f$:

$$\begin{array}{ccc} U_{(t,s)} & \xrightarrow{f_{(t,s)}} & U'_{(t,s)} \\ \iota\uparrow & & \uparrow\iota \\ U & \xrightarrow{f} & U \end{array} \tag{1.4}$$

Note that $\Upsilon\circ\iota$ is the diagonal imbedding $\Delta:U\to U\times U$, $x\mapsto(x,x)$.

In this setting, the usual rules of first order calculus hold (chain rule, product rule, linearity of first differential) – for a systematic exposition we refer to [BGN04, Be08, Be11]. Most important for our purposes is the Chain Rule, which we write in functorial form

$$\forall(t,s)\in\mathbb{K}^2:\quad (g\circ f)_{(t,s)}=g_{(t,s)}\circ f_{(t,s)}. \tag{1.5}$$

This follows easily from Lemma 1.2: for invertible $t-s$, we have functoriality $(g\times g)\circ(f\times f)=(g\circ f)\times(g\circ f)$, and for general (t,s), it follows "by density".

1.3 Full versus restricted iteration

Higher order differentiability is defined by iterating first order differentiability. However, there are various ways of doing so, and it is important to distinguish them. In [BGN04], f is defined to be of class $C^2_{\mathbb{K}}$ if it is C^1 and if $f^{[1]}$ also is C^1, so that we can define $f^{[2]}:=(f^{[1]})^{[1]}$, etc.:

Definition 1.3 (Full iteration)**.** *We say that f is* of class $C^n_{\mathbb{K}}$ *if: f is of class $C^1_{\mathbb{K}}$, and $f^{[1]}$ is of class $C^{n-1}_{\mathbb{K}}$. In this case we let $f^{[n]} := (f^{[1]})^{[n-1]}$.*

Remark 1.3. In the real or complex finite dimensional case this is equivalent to the usual definitions (see [BGN04, Be11]). However, since full iteration repeats the procedure for all variables together, the number of variables explodes, and it is hard to get control over the structure of the maps $f^{[n]}$ (see [Be15b]). To reduce the number of variables, in *restricted iteration* we consider in each step time variables to be frozen, and take difference quotients only with respect to space variables.

Notation. For each $k \in \mathbb{N}$, we denote by an upper index $\{k\}$ a copy of the procedure $\{1\}$ that has been defined above. An upper index $\{i,j\}$ $(i<j)$ indicates a double application of the procedure (first $\{i\}$, then $\{j\}$), etc. E.g., an upper index $\mathsf{n} := \{1,\ldots,n\}$ indicates that we first apply $\{1\}$, then $\{2\}$, etc., and finally $\{n\}$.

To abbreviate, in the sequel, we let $(\mathbf{t},\mathbf{s}) = (t_1,\ldots,t_n;s_1,\ldots,s_n) \in \mathbb{K}^{2n}$.

Definition 1.4 (Restricted iterated domain)**.** *For $U \subset V$, define $U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \subset V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ by*

$$U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} := U^{\{1,\ldots,n\}}_{(\mathbf{t},\mathbf{s})} := (\ldots(U^{\{1\}}_{t_1,s_1})^{\{2\}}_{(t_2,s_2)})\ldots)^{\{n\}}_{(t_n,s_n)} = (U^{\{1\}}_{(t_1,s_1)})^{\{2,\ldots,n\}}_{(t_2,\ldots,t_n,s_2,\ldots,s_n)}.$$

Note that $V_{(t_i,s_i)} \cong V^2$, so $V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \cong V^{(2^n)}$.

Definition 1.5 (Restricted iteration)**.** *A map $f : U \to U'$ is called* of class $C_{\mathbb{K},n}$ *if: it is of class $C^1_{\mathbb{K}}$, and, for all $(t_1,s_1) \in \mathbb{K}^2$, the map $f^{\{1\}}_{(t_1,s_1)}$ is of class $C_{\mathbb{K},n-1}$. In this case we define inductively*

$$f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} := (f^{\{1\}}_{(t_1,s_1)})^{\{2,\ldots,n\}}_{(t_2,\ldots,t_n,s_2,\ldots,s_n)} = (\ldots(f^{\{1\}}_{t_1,s_1})^{\{2\}}_{(t_2,s_2)})\ldots)^{\{n\}}_{(t_n,s_n)} : U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to (U')^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}.$$

We also require that $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ be jointly continuous both in space and in time variables.

Theorem 1.6. *When $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and V is a locally convex topological vector space, then the conditions $C^n_{\mathbb{K}}$ and $C_{\mathbb{K},n}$ are both equivalent to the usual (Keller's) definition of C^n-maps.*

Proof. As already mentioned, $C^n_{\mathbb{K}}$ clearly implies $C_{\mathbb{K},n}$. Equivalence of $C^n_{\mathbb{K}}$ with Keller's definition has been proved in [BGN04]. On the other hand, $C_{\mathbb{K},n}$ obviously implies Keller's C^n-definition, which arises simply by taking $(\mathbf{t},\mathbf{s}) = (0,\ldots,0)$ in the $C_{\mathbb{K},n}$-condition. Thus all three conditions are equivalent. □

Remark 1.4. For general $\mathbb{K}$, properties $C^n_{\mathbb{K}}$ and $C_{\mathbb{K},n}$ cease te be equivalent: in *positive characteristic*, condition $C^n_{\mathbb{K}}$ appears to be strictly stronger than $C_{\mathbb{K},n}$ (cf. the proof of the general Taylor formula in [BGN04, Be11], which really uses *full* iteration; concerning this item, cf. also [Be13]). It would be interesting to have a criterion allowing to decide when $C^n_{\mathbb{K}}$ and $C_{n,\mathbb{K}}$ are equivalent.

Definition 1.7. *For all* $(\mathbf{t}, \mathbf{s}) \in \mathbb{K}^{2n}$*, the* n-th order anchor *of* $U \subset V$ *is defined as follows: for all locally linear sets* $(U, V), (U', V')$*, we consider the map*

$$(U \times U')_{(t,s)} \to U_{(t,s)} \times U'_{(t,s)}, \quad ((v_0, v_0'), (v_1, v_1')) \mapsto ((v_0, v_1), (v_0', v_1'))$$

as identification. Under such identifications, the map $\Upsilon := \Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} :=$

$$(\Upsilon^{\{1\}}_{(t_1,s_1)})^{\{2,\ldots,n\}}_{(t_2,\ldots,t_n,s_2,\ldots,s_n)} : U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to (U^{\{1\}}_{(t_1,s_1)})^{\{2,\ldots,n\}}_{(t_2,\ldots,t_n,s_2,\ldots,s_n)} \times (U^{\{1\}}_{(t_1,s_1)})^{\{2,\ldots,n\}}_{(t_2,\ldots,t_n,s_2,\ldots,s_n)}$$

inductively gives rise to a map $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} : U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to U^{2^n}$ *which we call the* n-fold anchor.

Remark 1.5. In order to fully formalize this definition, we need an explicit labelling of the 2^n copies of U in U^{2^n}. For the moment, this is not needed, and will be taken up later (Def. 2.7). Let us, however, give the result for $n = 2$: space variables have labels $0, 1, 2, 12$ corresponding to the subsets of $\{1, 2\}$, so we write $\mathbf{v} = (v_0, v_1, v_2, v_{12}) \in U^{\{1,2\}}_{(t_1,t_2,s_1,s_2)}$. Then iteration shows that the linear map Υ is given by the (block) matrix (Kronecker product of two first-order anchors)

$$\begin{pmatrix} 1 & s_1 \\ 1 & t_1 \end{pmatrix} \otimes \begin{pmatrix} 1 & s_2 \\ 1 & t_2 \end{pmatrix} = \begin{pmatrix} 1 & s_1 & s_2 & s_1 s_2 \\ 1 & t_1 & s_2 & t_1 s_2 \\ 1 & s_1 & t_2 & s_1 t_2 \\ 1 & t_1 & t_2 & t_1 t_2 \end{pmatrix}, \tag{1.6}$$

so we have four "evaluation points" given by the four lines of the (block) matrix:

$$\begin{aligned} \Upsilon_\emptyset(\mathbf{v}) &= v_\emptyset + s_1 v_1 + s_2 v_2 + s_1 s_2 v_{12}, \\ \Upsilon_1(\mathbf{v}) &= v_\emptyset + t_1 v_1 + s_2 v_2 + t_1 s_2 v_{12}, \\ \Upsilon_2(\mathbf{v}) &= v_\emptyset + s_1 v_1 + t_2 v_2 + s_1 t_2 v_{12}, \\ \Upsilon_{12}(\mathbf{v}) &= v_\emptyset + t_1 v_1 + t_2 v_2 + t_1 t_2 v_{12}. \end{aligned} \tag{1.7}$$

The inverse matrix of (1.6) is the Kronecker product of the inverses of the respective first order anchors (when these are invertible): it is given by

$$\frac{1}{t_1 - s_1} \begin{pmatrix} t_1 & -s_1 \\ -1 & 1 \end{pmatrix} \otimes \frac{1}{t_2 - s_2} \begin{pmatrix} t_2 & -s_2 \\ -1 & 1 \end{pmatrix} = \frac{1}{(\mathbf{t} - \mathbf{s})_2} \begin{pmatrix} t_1 t_2 & -s_1 t_2 & -t_1 s_2 & s_1 s_2 \\ -t_2 & t_2 & s_2 & -s_2 \\ -t_1 & s_1 & t_1 & -s_1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{1.8}$$

where $(\mathbf{t} - \mathbf{s})_2 := (t_1 - s_1)(t_2 - s_2)$. For the general case, see Theorem 2.9.

Theorem 1.8. *For a map* $f : U \to U'$*, the following are equivalent:*

1. f *is* $C_{\mathbb{K},n}$*,*

2. *for all* $(\mathbf{t},\mathbf{s}) \in \mathbb{K}^{2n}$*, there exists a (unique) map* $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} : U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to (U')^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$*, such that*

   (a) $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}(\mathbf{v})$ *is jointly continuous in space and time variables* $(\mathbf{v};\mathbf{t},\mathbf{s})$*,*

   (b) *for all* $(\mathbf{t},\mathbf{s}) \in \mathbb{K}^{2n}$*,* $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \circ f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = f^{2^n} \circ \Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$*:*

$$\begin{array}{ccc} U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} & \overset{f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}}{\longrightarrow} & (U')^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \\ \Upsilon^{\mathsf{n}}_{\mathbf{t},\mathbf{s}} \downarrow & & \downarrow \Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \\ U^{2^n} & \overset{f^{2^n}}{\longrightarrow} & (U')^{2^n}. \end{array}$$

The map $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ *depends functorially on* f*:* $(f \circ g)^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \circ g^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ (Chain Rule).

Proof. By induction: for $n = 1$, this is Lemma 1.2. Assume the claim holds on level $n-1$ and apply it to f replaced by $f^{\{1\}}_{(t_1,s_1)}$. From the inductive definitions, it follows readily that the properties are again equivalent on level n. The (higher order) Chain Rule now also follows by induction. □

Example 1.2. Using Formula (1.8), let us give explicit formulae for $n = 2$:

$$f^{2}_{(t_1,t_2,s_1,s_2)}(\mathbf{v}) = \Upsilon^{-1}\big(f(\Upsilon_\emptyset(\mathbf{v})), f(\Upsilon_1(\mathbf{v})), f(\Upsilon_2(\mathbf{v})), f(\Upsilon_{12}(\mathbf{v}))\big)$$

$$= \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathbf{2}}} \begin{pmatrix} t_1t_2 f(\Upsilon_\emptyset \mathbf{v}) - s_1t_2 f(\Upsilon_1\mathbf{v}) - t_1s_2 f(\Upsilon_2\mathbf{v}) + s_1s_2 f(\Upsilon_{12}\mathbf{v}) \\ -t_2 f(\Upsilon_\emptyset\mathbf{v}) + t_2 f(\Upsilon_1\mathbf{v}) + s_2 f(\Upsilon_2\mathbf{v}) - s_2 f(\Upsilon_{12}\mathbf{v}) \\ -t_1 f(\Upsilon_\emptyset\mathbf{v}) + s_1 f(\Upsilon_1\mathbf{v}) + t_1 f(\Upsilon_2\mathbf{v}) - s_1 f(\Upsilon_{12}\mathbf{v}) \\ f(\Upsilon_\emptyset\mathbf{v}) - f(\Upsilon_1\mathbf{v}) - f(\Upsilon_2\mathbf{v}) + f(\Upsilon_{12}\mathbf{v}) \end{pmatrix} \tag{1.9}$$

Since $(\mathbf{t}-\mathbf{s})_{\mathbf{2}} = (t_1-s_1)(t_2-s_2)$, the first term is in fact an affine combination of values of f at the four evaluation points, whereas the other three terms are "zero-sum combinations" of these values, and hence correspond to "true" difference quotients. In order to state results at arbitrary order, we need some notation:

1.4 Hypercube notation, and formula for higher order slopes

Definition 1.9. *We call* n-hypercube *the power set* $\mathcal{P}(\mathsf{n}) = \mathcal{P}(\{1,\ldots,n\})$*. It serves as index set for space variables, which we write in the form* $\mathbf{v} = (v_A)_{A\in\mathcal{P}(\mathsf{n})}$*. Recall that* $\mathcal{P}(\mathsf{n})$ *is a semigroup for union* $\cup$ *and intersection* $\cap$*, and a group with respect to the* symmetric difference

$$A\Delta B = (A\cup B)\setminus(A\cap B) = (A\cap B^c)\cup(B\cap A^c),$$

where $A^c = \mathsf{n}\setminus A$ *is the complement of* A *in* n*. Recall also that* $A^c\Delta B^c = A\Delta B$*, and that* $A\Delta B^c = (A\Delta B)^c = A^c\Delta B$*, whence* $|A\Delta B^c| = n - |A\Delta B|$.

Definition 1.10. *For all* $\mathbf{t}, \mathbf{s} \in \mathbb{K}^n$ *and* $A \in \mathcal{P}(\mathsf{n})$, *we let* $\mathbf{t}_\emptyset = 1 = \mathbf{s}_\emptyset$, *and*

$$\mathbf{t}_A = \prod_{k \in A} t_k, \qquad \mathbf{s}_A = \prod_{k \in A} s_k, \qquad (\mathbf{t} - \mathbf{s})_A = \prod_{k \in A} (t_k - s_k).$$

Call $(\mathbf{t}, \mathbf{s})$ regular, *or* finite, *if,* $\forall i = 1, \ldots, n : (t_i - s_i) \in \mathbb{K}^\times$, *and* singular *if* $\forall i = 1, \ldots, n : (t_i - s_i) \notin \mathbb{K}^\times$, *and* mixed *else.*

Theorem 1.11. *Let* $f : U \to U'$ *be of class* $C_{\mathbb{K},n}$. *Then, for all regular* $(\mathbf{t}, \mathbf{s}) \in \mathbb{K}^{2n}$, *and all* $B \in \mathcal{P}(\mathsf{n})$, *the component* $(f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}(\mathbf{v}))_B$ *is given by*

$$(f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}(\mathbf{v}))_B = \frac{1}{(\mathbf{t} - \mathbf{s})_{\mathsf{n}}} \sum_{A \in \mathcal{P}(\mathsf{n})} (-1)^{|A \Delta B|} \mathbf{s}_{B^c \cap A} \mathbf{t}_{B^c \cap A^c} \, f\Big(\sum_{C \in \mathcal{P}(\mathsf{n})} \mathbf{s}_{C \cap A^c} \mathbf{t}_{C \cap A} v_C \Big).$$

The proof will be given in Subsection 2.4. For $B = \emptyset$, the component is an affine combination of values of f at the 2^n evaluation points, and for all other components it is again a "zero sum combination".

1.5 Categories of locally linear sets and $C_{\mathbb{K},n}$-maps

To summarize, we have defined a *category of locally linear sets and their morphisms*:

Definition 1.12. *We denote by* $\mathbf{Llin}_{\mathbb{K},n}$ *the category whose objects are pairs* (U, V), *where* V *is a topological* $\mathbb{K}$*-module and* $U \subset V$ *a non-empty open subset, and morphisms are* $C_{\mathbb{K},n}$*-maps* $f : U \to U'$. *(For* $n = 0$, *morphisms are continuous maps, and for* $n = \infty$, *these are maps that are* $C_{\mathbb{K},n}$ *for all* $n \in \mathbb{N}$.*)*

Definition 1.13. *For* $m \geq n$ *and* $(\mathbf{t}, \mathbf{s}) \in \mathbb{K}^{2n}$, *the* $(n; \mathbf{t}, \mathbf{s})$-tangent functor *is the functor from* $\mathbf{Llin}_{\mathbb{K},m}$ *to* $\mathbf{Llin}_{\mathbb{K},m-n}$ *given by* $(U, V) \mapsto (U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}, V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})})$ *and* $f \mapsto f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$.

Remark 1.6 (Manifolds). By the usual glueing procedures, one may now define $C_{\mathbb{K},n}$-*manifolds over* $\mathbb{K}$, modelled on locally linear sets – since these methods are independent of the particular form of differential calculus, we do not wish to go here into details (see [Be16] for a formulation of such principles, adapted to most general contexts). The $(n; \mathbf{t}, \mathbf{s})$-tangent functor then carries over to manifolds : for every $\mathbb{K}$-smooth manifold M we have a "generalized higher order tangent bundle" $M^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$, depending functorially on M, and coming with an anchor map $M^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to M^{2^n}$.

2. The rings of calculus: tangent algebras

Our next aim is to understand the $(n; \mathbf{t}, \mathbf{s})$-tangent functor as a *functor of scalar extension*, from $\mathbb{K}$ to a ring denoted by $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$, and which we shall define next.

2.1 The scaloid, and the algebras $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$.

The scaloid is the index set that will be used in the following construction of *tangent algebras*:

Definition 2.1. *We call* scaloid *the* free monoid over $\mathbb{K}^2$*, that is, the disjoint union over* $n \in \mathbb{N}_0$ *of all* $\mathbb{K}^{2n}$*:*

$$\mathbf{scal} := \mathbf{scal}_{\mathbb{K}} := \coprod_{n\in\mathbb{N}_0} \mathbb{K}^{2n}$$

(in the following, we write $(\mathbf{t},\mathbf{s})$ *with* $\mathbf{t},\mathbf{s} \in \mathbb{K}^n$ *for elements of* $\mathbb{K}^{2n}$*), together with its monoid structure given by juxtaposition, and denoted by*

$$(\mathbf{t},\mathbf{s}) \oplus (\mathbf{t}',\mathbf{s}') = (t_1,\ldots,t_n,t'_1,\ldots,t'_m;s_1,\ldots,s_n,s'_1,\ldots,s'_m) = (\mathbf{t}\oplus\mathbf{t}',\mathbf{s}\oplus\mathbf{s}')\ .$$

We denote by $\mathbb{K}[X_1,\ldots,X_n]$ the algebra of polynomials in n variables with coefficients in $\mathbb{K}$. It can be defined inductively by using the isomorphisms, where $\otimes_{\mathbb{K}}$ denotes the tensor product of two associative $\mathbb{K}$-algebas,

$$\mathbb{K}[X_1,X_2] \cong (\mathbb{K}[X_1])[X_2] \cong \mathbb{K}[X_1]\otimes_{\mathbb{K}}\mathbb{K}[X_2], \tag{2.1}$$

so, by induction, we have an iterated tensor product of algebras

$$\mathbb{K}[X_1,\ldots,X_n] \cong \mathbb{K}[X_1]\otimes_{\mathbb{K}}\ldots\otimes_{\mathbb{K}}\mathbb{K}[X_n]. \tag{2.2}$$

Definition 2.2. *For* $(\mathbf{t},\mathbf{s}) \in \mathbb{K}^{2n}$*, we define the* $(\mathbf{t},\mathbf{s})$-tangent algebra

$$\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} := \mathbb{K}[X_1,\ldots,X_n]/((X_i-t_i)(X_i-s_i), i=1,\ldots,n)$$

(quotient by the ideal $I_{(\mathbf{t},\mathbf{s})}$ *generated by all* $(X_i-t_i)(X_i-s_i), i=1,\ldots,n$*).*

Lemma 2.3. *The algebra* $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ *is a free* $\mathbb{K}$*-module of dimension* 2^n*, having a canonical basis indexed by elements* A *of the* n*-cube* $\mathcal{P}(\mathsf{n})$*,*

$$e_A := [X^A], \qquad X^A = \prod_{k\in A} X_k.$$

It is also isomorphic to an n*-fold tensor product of first order tangent algebras* $\mathbb{K}^{\{i\}}_{(t_i,s_i)} = \mathbb{K}[X_i]/((X_i-s_i)(X_i-t_i))$*:*

$$\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = \mathbb{K}^{\{1\}}_{(t_1,s_1)}\otimes\ldots\otimes\mathbb{K}^{\{n\}}_{(t_n,s_n)}.$$

Proof. For $n=1$, the claim is obviously true: a polynomial algebra $\mathbb{K}[X]$ quotiented by the ideal generated by a polynomial of degree 2 is of dimension 2, with $\mathbb{K}$-basis the classes $[1]$ and $[X]$. For $n>1$, the claim follows by induction using (2.1). □

Theorem 2.4. *Assume $\mathbb{K}$ is a good topological ring. Then the structure maps $+$ and $\cdot$ of the ring $\mathbb{K}$ are of class $C_{\mathbb{K},\infty}$, and applying n-fold restricted iteration with parameters $(\mathbf{t},\mathbf{s})$ yields a good topological ring which is canonically isomorphic to $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ (whence in particular is a free $\mathbb{K}$-module of dimension 2^n)*

Proof. The structure maps are continuous and (bi)-linear, hence smooth (both in the full and restricted sense, cf. [BGN04]). By functoriality, and applying, concerning Cartesian products, the convention from Def. 1.7, rings are transformed by the iterated functors into rings. We have to show that the ring structure on the underlying set of $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ is precisely the one defined above. For $n=1$ and regular $(t_1,s_1)=(t,s)$, this follows from the explicit formulae for difference calculus : slightly more general, given a bilinear continuous map $\beta : V\times W \to Y$, thought of as a "product", so let us write $v\bullet w := \beta(v,w)$, we compute

$$\beta^{\{1\}}_{(t,s)} : V_{(t,s)} \times W_{(t,s)} \to Y_{(t,s)}, \quad (\begin{pmatrix} v_0 \\ v_1\end{pmatrix}, \begin{pmatrix} w_0 \\ w_1\end{pmatrix}) \mapsto \begin{pmatrix} v_0 \\ v_1\end{pmatrix} \bullet^{\{1\}}_{(t,s)} \begin{pmatrix} w_0 \\ w_1\end{pmatrix}$$

which by an explicit computation using Formula (0.7) is given by

$$\begin{pmatrix} v_0 \\ v_1\end{pmatrix} \bullet^{\{1\}}_{(t,s)} \begin{pmatrix} w_0 \\ w_1\end{pmatrix} = \begin{pmatrix} v_0\bullet w_0 - st\, v_1\bullet w_1 \\ v_0\bullet w_1 + v_1\bullet w_0 + (s+t)v_1\bullet w_1 \end{pmatrix}. \tag{2.3}$$

Now, decomposing the product of $\mathbb{K}^{\{1\}}_{(t,s)}$ according to the canonical basis $e_0=[1], e_1=[X]$, we get exactly the same formula, whence the claim for $n=1$ and regular (t,s). By density, the claim follows for all (t,s), and by straightforward induction, using Lemma 2.3, it now follows for all elements $(\mathbf{t},\mathbf{s})$ of the scaloid. Finally, by general argments ([Be08, Be11]), the ring $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ is again "good". □

By exactly the same arguments we see also that the structure maps $V\times V\to V$ and $\mathbb{K}\times V\to V$ of a topological $\mathbb{K}$-module are smooth, and give by restricted iteration rise to the corresponding structure maps of the scalar-extended module $V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = V\otimes_{\mathbb{K}} \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$; also, if $f: V\to V'$ is *linear*, then $f^{\mathsf{n}}_{\mathbf{t},\mathbf{s}}$ coincides with the algebraic scalar extension $f\otimes \mathrm{id}_{\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}}$.

2.2 Source and target

Evaluation of a class $[P]\in \mathbb{K}[X]/((X-s)(X-t))$ at elements $x\in\mathbb{K}$ is in general not well-defined, but it is so for $x=s$ and $x=t$. Thus we get two algebra morphisms $\alpha,\beta : \mathbb{K}^{\{1\}}_{(t,s)} \to \mathbb{K}$, called *source* and *target*

$$\alpha([P]) = P(s), \qquad \beta([P]) = P(t). \tag{2.4}$$

(Note that α is coupled with s and β with t, so the order of (s,t) matters.) With respect to the basis $e_0=[1]$, $e_1=[X]$, we have $\alpha(v_0+v_1e_1)=v_0+sv_1$, $\beta(v_0+v_1e_2)=v_0+tv_1$, which is in keeping with the definitions in Subsection 0.2. In Appendix B we describe the structure of $\mathbb{K}^{\{1\}}_{(t,s)}$ in an intrinsic way, via α and β; this may be useful for a further structure theory, but is not directly needed in the sequel.

2.3 The anchor

Putting source and target together, the *first order anchor* is the algebra morphism defined by

$$\Upsilon^{\{1\}}_{(t,s)} : \mathbb{K}_{(t,s)} \to \mathbb{K} \times \mathbb{K}, \quad [P] \mapsto (\alpha(P), \beta(P)) = (P(s), P(t)).$$

Lemma 2.5. *The first order anchor is an isomorphism if, and only if, (t,s) is regular, i.e., iff $t-s \in \mathbb{K}^\times$.*

Proof. The $\mathbb{K}$-linear map $\Upsilon_{(t,s)}$ is bijective iff its determinant $t-s \in \mathbb{K}^\times$, see Subsection 0.2. □

Higher order anchors can be defined in two (equivalent) ways: either by evaluating (classes of) polynomials in several variables on a hypercube of evaluation points, or by tensoring first order anchors. Here we choose the latter approach. For this, we need some definitions and conventions:

Definition 2.6 (Hypercubic spaces and algebras)**.** *Let $N \subset \mathbb{N}$ be a finite subset of cardinal n. The* hypercubic space, based on *N, is by definition the free $\mathbb{K}$-module $\mathbb{K}^{\mathcal{P}(N)}$ of dimension 2^n of functions from $\mathcal{P}(N)$ to $\mathbb{K}$, with its canonical basis*

$$E_A = E^N_A : \mathcal{P}(N) \to \mathbb{K}, \qquad E_A(A) = 1, \ \forall B \neq A : E_A(B) = 0.$$

A hypercubic space carries several important algebra structures. When equipping $\mathbb{K}^{\mathcal{P}(N)}$ with its pointwise algebra structure*, i.e., considering it as the algebra of functions from $\mathcal{P}(N)$ to $\mathbb{K}$, so that the product of the canonical basis elements is*

$$E^N_A \cdot E^N_B = \delta_{A,B} E^N_A ,$$

we say that $\mathbb{K}^{\mathcal{P}(N)}$ is the N-hypercube algebra*. When $N = \mathsf{n} = \{1, \ldots, n\}$, we often omit the upper index n, and just speak of* the n-hypercube algebra*.*

Remark 2.1. See Appendix A for some basic facts about linear algebra on hypercubic spaces (independent of the algebra structure). For induction procedures, the following remark is useful: If N_1 and N_2 are disjoint subsets of $\mathbb{N}$, then

$$\mathcal{P}(N_1) \times \mathcal{P}(N_2) \to \mathcal{P}(N_1 \sqcup N_2), \quad (A, B) \mapsto A \cup B$$

is a bijection, whence we get an isomorphism (of modules, and of cube-algebras)

$$\mathbb{K}^{\mathcal{P}(N_1)} \otimes \mathbb{K}^{\mathcal{P}(N_2)} \cong \mathbb{K}^{\mathcal{P}(N_1 \sqcup N_2)}.$$

In particular, by induction, there is a canonical isomorphism

$$\mathbb{K}^{\mathcal{P}(\{1,\ldots,n\})} \cong \mathbb{K}^{\mathcal{P}(\{1\})} \otimes \ldots \otimes \mathbb{K}^{\mathcal{P}(\{n\})}.$$

Note that the neutral element of $\mathbb{K}^{\mathcal{P}(N)}$ is the function that is 1 everywhere, that is

$$1 = \sum_{A\in\mathcal{P}(N)} E_A^N .$$

Definition 2.7. *The* n-fold anchor *is the tensor product of* n *copies of the first order anchor: it is the algebra morphism*

$$\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} := \otimes_{i=1}^n \Upsilon^{\{i\}}_{(t_i,s_i)} : \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to \mathbb{K}^{\mathcal{P}(\mathsf{n})},$$

where for each $k\in\mathbb{N}$, $\Upsilon^{\{k\}}_{(t_k,s_k)} : \mathbb{K}^{\{k\}}_{(t_k,s_k)} \to \mathbb{K}^{\mathcal{P}(\{k\})}$ *is a copy of the first order anchor. Thus, by definition,*

$$\Upsilon^{\{k\}}_{(t_k,s_k)}(e_\emptyset) = E^k_\emptyset + E^k_k, \qquad \Upsilon^{\{k\}}_{(t_k,s_k)}(e_k) = s_k E^k_\emptyset + t_k E^k_k .$$

For the categorical approach, it is not strictly necessary to have an explicit formula for the higher order anchor; however, such a formula allows to derive the explicit formula for the higher order slopes, and thus makes the whole procedure algorithmic and computable. Recall Formula (1.6) for the matrix of the second order anchor, which is the Kronecker product of two first-order anchors. Note that, when $s_1 = 1 = s_2$, then this matrix is a *symmetric matrix*, whereas for $t_1 = 1 = t_2$, this is not the case. Using notation introduced above, we generalize:

Theorem 2.8. *Fix* $n\in\mathbb{N}$, *and* $(\mathbf{t},\mathbf{s})\in\mathbb{K}^{2n}$. *With respect to the bases* $(e_A)_{A\in\mathcal{P}(\mathsf{n})}$ *in its domain and* $(E_A)_{A\in\mathcal{P}(\mathsf{n})}$ *in its range, the* n*-fold anchor is given by*

$$\Upsilon = \Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = \sum_{(A,B)\in\mathcal{P}(\mathsf{n})^2} \mathbf{t}_{A\cap B}\mathbf{s}_{A\cap B^c}\, e_A^* \otimes E_B .$$

In other terms, it is characterized by the following equivalent conditions:

1. $\Upsilon(e_A) = \sum_{B\in\mathcal{P}(\mathsf{n})} \mathbf{t}_{A\cap B}\mathbf{s}_{A\cap B^c} E_B$,
2. $\Upsilon(\sum_{A\in\mathcal{P}(\mathsf{n})} v_A e_A) = \sum_{B\in\mathcal{P}(\mathsf{n})} \big(\sum_{A\in\mathcal{P}(\mathsf{n})} \mathbf{t}_{A\cap B}\mathbf{s}_{A\cap B^c} v_A\big) E_B$,
3. *the matrix of* Υ *with respect to these bases has coefficients*

$$\Upsilon_{(B,A)} := E_B^*(\Upsilon(e_A)) = \mathbf{t}_{A\cap B}\mathbf{s}_{A\cap B^c}, \qquad (A,B)\in\mathcal{P}(\mathsf{n})^2 .$$

In particular, in the symmetric case $\mathbf{s} = -\mathbf{t}$*, we have* $\Upsilon_{(B,A)} = (-1)^{|A\cap B|}\mathbf{s}_A$*, so*

$$\Upsilon = \Upsilon^{\mathsf{n}}_{(-\mathbf{s},\mathbf{s})} = \sum_{A\in\mathcal{P}(\mathsf{n})} \mathbf{s}_A \sum_{B\in\mathcal{P}(\mathsf{n})} (-1)^{|A\cap B|} e_A^* \otimes E_B.$$

Proof. This is the special case of Theorem A.1 for $\mathbf{a} = 1 = \mathbf{c}$, $\mathbf{b} = \mathbf{s}$, $\mathbf{d} = \mathbf{t}$. □

Next, to compute the inverse of the anchor, in the regular case, recall Formula (1.8) concerning the case $n = 2$. This generalizes as follows:

Theorem 2.9. *Fix* $(\mathbf{t},\mathbf{s}) \in \mathbb{K}^{2n}$*. Recall the notation* $(\mathbf{t}-\mathbf{s})_{\mathsf{n}} = \prod_{k=1}^n (t_k - s_k)$*. The anchor map* $\Upsilon = \Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ *is invertible if, and only if,* $(\mathbf{t},\mathbf{s})$ *is regular, i.e.,* $t_k - s_k$ *is invertible for all* $k = 1,\ldots,n$*, and then its inverse map is given by the formula*

$$\Upsilon^{-1} = \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}} \sum_{(A,B)\in\mathcal{P}(\mathsf{n})^2} (-1)^{|A\Delta B|} \mathbf{s}_{A^c\cap B} \mathbf{t}_{B^c\cap A^c}\; E_A^* \otimes e_B.$$

Equivalently,

1. $\Upsilon^{-1}(E_A) = \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}} \sum_{B\in\mathcal{P}(\mathsf{n})} (-1)^{|A\Delta B|} \mathbf{s}_{A^c\cap B}\mathbf{t}_{B^c\cap A^c}\; e_B$,
2. $\Upsilon^{-1}(\sum_{A\in\mathcal{P}(\mathsf{n})} y_A E_A) = \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}} \sum_{B\in\mathcal{P}(\mathsf{n})} (-1)^{|A\Delta B|} y_A \mathbf{s}_{A^c\cap B}\mathbf{t}_{B^c\cap A^c}\; e_B$.

In particular, in case $\mathbf{s} = -\mathbf{t}$*, we get (using* $(A\Delta B) \sqcup (A^c \cap B^c) = (A\cap B)^c$*)*

$$\Upsilon^{-1}(E_A) = \frac{1}{(-2)^n \mathbf{s}_{\mathsf{n}}} \sum_{B\in\mathcal{P}(\mathsf{n})} (-1)^{|A\cap B|} \mathbf{s}_{B^c}\, e_B.$$

Proof. This is a special case of Theorem A.2. □

2.4 The n-th order restriced slope map

Now we prove the already anounced formula from Theorem 1.11 for $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ when $(\mathbf{t},\mathbf{s})$ is regular. We decompose $\mathbf{v} \in V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = V \otimes_{\mathbb{K}} \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ in the form $\mathbf{v} = \sum_{A\in\mathcal{P}(\mathsf{n})} v_A e_A$, and $\Upsilon(\mathbf{v}) = \sum_{A\in\mathcal{P}(\mathsf{n})} \Upsilon_A(\mathbf{v}) E_A$, with the 2^n *evaluation points* given by

$$\Upsilon_A(\mathbf{v}) = \sum_{C\in\mathcal{P}(\mathsf{n})} \mathbf{s}_{C\cap A^c}\mathbf{t}_{C\cap A} v_C.$$

Then

$$\begin{aligned}
f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}\Big(\sum_{A\in\mathcal{P}(\mathsf{n})} v_A e_A\Big) &= \Upsilon^{-1}\Big(\sum_{A\in\mathcal{P}(\mathsf{n})} f\big(\Upsilon_A(\mathbf{v})\big)\Big) \\
&= \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}} \sum_{B\in\mathcal{P}(\mathsf{n})} e_B \Big(\sum_{A\in\mathcal{P}(\mathsf{n})} (-1)^{|A\Delta B|} \mathbf{t}_{A^c\cap B^c}\mathbf{s}_{B^c\cap A} f\big(\Upsilon_A(\mathbf{v})\big)\Big) \\
&= \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}} \sum_{B\in\mathcal{P}(\mathsf{n})} e_B \Big(\sum_{A\in\mathcal{P}(\mathsf{n})} (-1)^{|A\Delta B|} \mathbf{t}_{A^c\cap B^c}\mathbf{s}_{B^c\cap A} f\big(\sum_{C\in\mathcal{P}(\mathsf{n})} \mathbf{t}_{C\cap A}\mathbf{s}_{C\cap A^c} v_C\big)\Big).
\end{aligned}$$

2.5 Target calculus, source calculus, and symmetric calculus

There are three special cases of calculus, as defined here, that deserve attention:

1. *target calculus*, obtained when $\mathbf{s} = \mathbf{0}$, i.e., $\forall i$, $s_i = 0$;
2. *source calculus*, obtained when $\mathbf{t} = \mathbf{0}$,
3. *symmetric calculus*, obtained when $\mathbf{s} = -\mathbf{t}$, i.e., $\forall i$, $s_i + t_i = 0$.

In these cases, the range of scaloid parameter reduces to $\mathbb{K}^n$ instead of $\mathbb{K}^{2n}$, and the relations satisfied by the canonical basis $(e_A)_{A\in\mathcal{P}(\mathsf{n})}$ are relatively simple:

1. *target calculus*, $e_i^2 = t_i e_i$, whence $e_A^2 = \mathbf{t}_A e_A$ and $e_A e_B = \mathbf{t}_{A\cap B} e_{A\cup B}$,
2. *source calculus*, same, with $\mathbf{s}$ instead of $\mathbf{t}$,
3. *symmetric calculus*, $e_i^2 = 4t_i^2$, so $e_A^2 = 4^{|A|}\mathbf{t}_A^2$, $e_A e_B = 4^{|A\cap B|}\mathbf{t}_{A\cap B}^2 e_{A\Delta B}$.

The "most singular value" is in all cases $\mathbf{t} = \mathbf{0} = \mathbf{s}$, whereas the "unit value" is

1. *target calculus*, "unit" $\mathbf{t} = \mathbf{1} = (1, \ldots, 1)$, $\mathbf{s} = \mathbf{0}$,
2. *source calculus*, "unit" $\mathbf{t} = \mathbf{0}$, $\mathbf{s} = \mathbf{1}$,
3. *symmetric calculus*, "unit" $\mathbf{t} = \mathbf{1}, \mathbf{s} = -\mathbf{1} = (-1, \ldots, -1)$ (another convention would be to divide this by 2, if 2 is invertible in $\mathbb{K}$).

Thus, taking for $(\mathbf{t}, \mathbf{s})$ the unit value, the algebra $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ with its canonical basis,

1. in *target calculus*, is the *semigroup algebra of the monoid* $(\mathcal{P}(\mathsf{n}), \cup)$,
2. idem in *source calculus*,
3. in *symmetric calculus*, after normalizing by division by 2, is the *group algebra of the group* $(\mathcal{P}(\mathsf{n}), \Delta)$ *with group law given by the symmetric difference* Δ.

In all three cases, the anchor, being a morphism to the multiplicative algebra of functions on $\mathcal{P}(\mathsf{n})$, plays the rôle of a *Fourier transform.* Namely, for $A \in \mathcal{P}(\mathsf{n})$, the linear form $E_A^* : \mathbb{K}^{\mathcal{P}(\mathsf{n})} \to \mathbb{K}$ is the A-projection, which is a *character*, i.e., an algebra morphism into the base ring. Thus the 2^n components of Υ,

$$\Upsilon_A := E_A^* \circ \Upsilon : \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to \mathbb{K}, \quad x \mapsto \sum_{C\in\mathcal{P}(\mathsf{n})} \mathbf{s}_{C\cap A^c}\mathbf{t}_{C\cap A} x_C$$

also are characters (for $n = 1$, these are just the source and target projections; for $n \geq 1$, they can be considered as higher order versions of source and target maps). For instance, when $\mathbf{t} = -\mathbf{s}$ is constant $\frac{1}{2}$, then from the explicit formula above we get all 2^n characters of the group $(\mathcal{P}(\mathsf{n}), \Delta)$ (for $A \in \mathcal{P}(\mathsf{n})$),

$$\Upsilon_A = \chi_A : \mathcal{P}(\mathsf{n}) \to \{\pm 1\}, \quad B \mapsto \chi_A(B) = (-1)^{|A \Delta B|} \tag{2.5}$$

Thus the matrix of Υ is the character table of the abelian group $(\mathcal{P}(\mathsf{n}), \Delta)$, which is also the matrix of the Fourier transform when identifying this group with its dual group.

3. The categorical approach

In the preceding section we have described how to define, starting with a $\mathbb{K}$-smooth function f, a family of functions $(f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})})_{(\mathbf{t},\mathbf{s};n) \in \mathbf{scal}_{\mathbb{K}}}$, behaving well with tangent algebras, anchors, and their corresponding scalar extensions. In the present section, we describe an abstract, categorical setting capturing the main features of these constructions. The procedure is very much like the classical one, starting from polynomial functions, to define abstract polynomial rings. In general, one cannot recover all abstract polynomials by polynomial functions; for this we need assumptions on $\mathbb{K}$ (e.g., of topological nature).

3.1 The small monoidal categories in question

Let $\mathbf{c}$ be a monoid, with "product" denoted by $\oplus$ and neutral element 0. It gives rise to a small category that shall also be denoted by $\mathbf{c}$: its objects are elements $t \in \mathbf{c}$, and morphisms are given by compositions of left- and right multiplications in the monoid, i.e., of the form

$$t \to t_1 \oplus t \oplus t_2, \quad t, t_1, t_2 \in \mathbf{c}.$$

The monoids we are interested in will all be *left and right cancellative*, that is, $t \oplus s = t' \oplus s \Rightarrow t = t'$ and $t \oplus s = t \oplus s' \Rightarrow s = s'$; thus the small category $\mathbf{c}$ is *skeletal* in the sense of [CWM], p. 93: two objects are isomorphic iff they are equal. Now, here are the cases we are interested in:

1. The monoid $\mathbb{N}_0$ with its usual addition, and neutral element 0.

2. Recall from Definition 2.1 that objects of the scaloid $\mathbf{scal}_{\mathbb{K}}$ are elements $(\mathbf{t}, \mathbf{s})$ of the free monoid over $\mathbb{K}^2$. The neutral element is the empty word. Morphisms are now defined as above.

3. The *small category of* $\mathbb{K}$*-tangent algebras* $\mathbf{talg}_{\mathbb{K}}$ has objects the algebras $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ defined in Def. 2.2, together with their label $(\mathbf{t}, \mathbf{s})$. The monoidal structure is given by the tensor product of associative $\mathbb{K}$-algebras, which now serves to define also the morphisms in this category. The neutral element is $\mathbb{K}$, labelled by the empty word.

Lemma 3.1. *The small monoidal categories* $\mathbf{talg}_{\mathbb{K}}$ *and* $\mathbf{scal}_{\mathbb{K}}$ *are isomorphic (in the sense defined in [CWM], p. 92): under this bijection,* $\mathbb{K}^{\mathfrak{n}}_{(\mathbf{t},\mathbf{s})}$ *corresponds to* $(\mathbf{t},\mathbf{s})$.

Proof. By the definitions given above, the map $\mathbf{talg}_{\mathbb{K}} \to \mathbf{scal}_{\mathbb{K}}$ is well-defined, its inverse map is $(\mathbf{t},\mathbf{s}) \mapsto \mathbb{K}^{\mathfrak{n}}_{(\mathbf{t},\mathbf{s})}$. As we have seen in Lemma 2.3, this bijection then is an isomorphism of monoids. □

Lemma 3.2. *The "length" or "degree" map* $\ell : \mathbf{scal}_{\mathbb{K}} \to \mathbb{N}_0$*, associating to each word its length, is a monoid morphism, and defines a functor of monoidal categories.*

Proof. Obviously, ℓ is a morphism, and by routine computation such a morphism induces a morphism (functor) of the corresponding monoidal categories. □

3.2 Functor categories

Next we consider *functor categories*. We mostly follow notations and conventions from [CWM, MM92]. Thus, we denote by $\mathbf{Sets}$ the (large) category of sets and set-maps, and (following notation from [MM92], p. 25) by $\mathbf{Sets}^{\mathbf{2}}$ the (large) category of *anchored sets*, that is, objects (M,γ,M') are maps $\gamma : M \to M'$, where morphisms are *anchor-compatible pairs of maps* $\Phi : M \to N$, $\Phi' : M' \to N'$, i.e. $\gamma_N \circ \Phi = \Phi' \circ \gamma_M$.

Functors from a category C to a category B, together with their natural transformations, form a *functor category* $\mathbf{Fn}(C,B) = B^C$ (see e.g. [CWM], II.4, or [MM92]). Specifically, for $\mathbf{c}$ one of the small monoidal categories mentioned above, we are interested in functor categories $\mathbf{Fn}(\mathbf{c},\mathbf{Sets}) = \mathbf{Sets}^{\mathbf{c}}$ or $\mathbf{Fn}(\mathbf{c},\mathbf{Sets}^2)$. If $\underline{M} : \mathbf{c} \to \mathbf{Sets}$ is a functor, then for every object $a \in \mathbf{c}$ we write $M_a := \underline{M}(a)$ (the set obtained by applying $\underline{M}$ to a), and for every morphism $\phi : a \to b$ of $\mathbf{c}$, we write $M_\phi : M_a \to M_b$ for the induced set-map. Likewise, for each natural transformation $\underline{f} : \underline{M} \to \underline{N}$, we write $f_a : M_a \to N_a$ for the corresponding set-map from $\underline{M}(a)$ to $\underline{N}(a)$. The compatibility condition then is

$$\forall \phi : a \to b, \forall \underline{f} : \quad N_\phi \circ f_a = f_b \circ M_\phi .$$

Composition of natural transformations is defined "pointwise", i.e., for two laws $\underline{f} : \underline{M} \to \underline{N}$, $g : \underline{N} \to \underline{P}$ and all objects a of $\mathbf{c}$, we have $(\underline{g} \circ \underline{f})_a := g_a \circ f_a : M_a \to P_a$.

Definition 3.3. *For each object* a *of* $\mathbf{c}$*,* evaluation at level a*, defined by*

$$\mathrm{ev}_a : \underline{M} \mapsto M_a, \ \underline{f} \mapsto f_a,$$

is a functor from $\mathbf{Fn}(\mathbf{c},\mathbf{Sets})$ *to* $\mathbf{Sets}$*. In particular, when* $\mathbf{c}$ *is monoidal with neutral element* 0*, we call simply* evaluation *the evaluation* ev_0 *at* 0.

In the following, our concern will be to define ("extension") functors that go in the direction opposite to $\mathrm{ev}_0 : \mathbf{Fn}(\mathbf{c},\mathbf{Sets}) \to \mathbf{Sets}$.

3.3 Cubic extensions of sets.

For each set M and $n \in \mathbb{N}$, we have a hypercube of sets $M^{\mathcal{P}(\mathsf{n})} \cong M^{2^n}$. This gives rise to a "cubic extension functor":

Lemma 3.4. *Let us define*

$$\iota : \mathbf{Sets} \to \mathbf{Sets}^{\mathbb{N}_0}, \qquad \begin{array}{rcl} M \mapsto \underline{\underline{M}} & := & (0 \mapsto M,\, n \mapsto M^{\mathcal{P}(\mathsf{n})}) \\ f \mapsto \underline{\underline{f}} & := & (0 \mapsto f,\, n \mapsto f^{\mathcal{P}(\mathsf{n})}) \end{array}.$$

Then $\underline{\underline{M}} : \mathbb{N}_0 \to \mathbf{Sets}$ *is a functor, and (for* $f : M \to N$*),* $\underline{\underline{f}} : \underline{\underline{M}} \to \underline{\underline{N}}$ *is a natural transformation, and* ι *is a functor from* $\mathbf{Sets}$ *to* $\mathbf{Fn}(\mathbb{N}_0, \mathbf{Sets})$ *such that* $\mathrm{ev}_0 \circ \iota = I_{\mathbf{Sets}}$ *is the identity functor on* $\mathbf{Sets}$.

Proof. The main point is to see that $\underline{\underline{M}}$ is a functor. Indeed, this follows from the identifications $(M^A)^B = M^{A \times B}$ together with $\mathcal{P}(\mathsf{n}+\mathsf{m}) = \mathcal{P}(\mathsf{n}) \times \mathcal{P}(\mathsf{m})$:

$$M^{\mathcal{P}(\mathsf{n}+\mathsf{m})} = M^{\mathcal{P}(\mathsf{n}) \times \mathcal{P}(\mathsf{m})} = (M^{\mathcal{P}(\mathsf{n})})^{\mathcal{P}(\mathsf{m})}.$$

(In particular, for $n = 0$, this means that $M = M_0 \to M^{\mathcal{P}(\mathsf{m})}$ is the diagonal imbedding: an element $x \in M$ corresponds to the constant function $x : \mathcal{P}(\mathsf{m}) \to M$ having value x.) Next, the properties of a natural transformation for $\underline{\underline{f}}$ are easily checked, as are those saying that ι is a functor. Finally, by definition, for the neutral element, $\underline{\underline{M}}_0 = M$, whence $\mathrm{ev}_0 \circ \iota(M) = M$. □

Definition 3.5. *Let us call* cubic set *the realisation* $\underline{\underline{M}}$ *of a set* M *as a functor described by the lemma, and denote by* $\mathbf{CubeSet}$ *the image of* ι*, the* cubic realisation of the category $\mathbf{Sets}$.

3.4 Scalar extensions of modules

On the category $\mathbf{Mod}_{\mathbb{K}}$ of $\mathbb{K}$-modules with $\mathbb{K}$-linear maps, we also have the "usual" algebraic scalar extension functor:

Lemma 3.6. *Let us define*

$$\tau : \mathbf{Mod}_{\mathbb{K}} \to \mathbf{Sets}^{\mathbf{scal}_{\mathbb{K}}}, \qquad \begin{array}{rcl} V \mapsto \underline{V} & := & (n, \mathbf{t}, \mathbf{s}) \mapsto V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = V \otimes_{\mathbb{K}} \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \\ f \mapsto \underline{\underline{f}} & := & (n, \mathbf{t}, \mathbf{s}) \mapsto f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = f \otimes_{\mathbb{K}} \mathrm{id}_{\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}} \end{array}.$$

This defines a functor from the category $\mathbf{Mod}_{\mathbb{K}}$ *to* $\mathbf{Fn}(\mathbf{scal}_{\mathbb{K}}, \mathbf{Sets})$ *such that* $\mathrm{ev}_0 \circ \tau$ *is the identity functor on* $\mathbf{Mod}_{\mathbb{K}}$.

Proof. All of this is clear from properties of algebraic scalar extensions, along with the isomorphism of categories $\mathbf{scal}_{\mathbb{K}} \cong \mathbf{talg}_{\mathbb{K}}$. (As in the preceding proof, the main point is that $\underline{V}$ is a functor. In the present case, this holds more generally for general ring morphisms, and not only those coming from the monoidal structure of $\mathbf{scal}_{\mathbb{K}} \cong \mathbf{talg}_{\mathbb{K}}$.) □

Remark 3.1. Clearly, as morphisms in $\mathbf{Mod}_{\mathbb{K}}$ one could also use affine maps instead of linear ones. More generally, following N. Roby [Ro63], one could replace linear maps f by polynomial morphisms, corresponding to "polynomial laws" as defined in loc. cit.

3.5 $\mathbb{K}$-space laws

Now we define a functor category $\mathbf{Space}_{\mathbb{K}}$ of *smooth* $\mathbb{K}$*-space laws*. One could do so for each fixed $n \in \mathbb{N}$, defining $\mathbb{K}$*-space laws of class* C^n, but it is quicker and clearer to do this for all $n \in \mathbb{N}_0$ together.

Definition 3.7. *Objects of* $\mathbf{Space}_{\mathbb{K}}$ *are pairs* $(\underline{M}, \underline{\Upsilon})$*, where* $\underline{M} : \mathbf{scal}_{\mathbb{K}} \to \mathbf{Sets}$ *is a functor and* $\underline{\Upsilon} : \underline{M} \to \underline{\underline{M_0}}$ *is a natural transformation, and morphisms of* $\mathbf{Space}_{\mathbb{K}}$ *are natural transformations* $\underline{f} : \underline{M} \to \underline{M}'$ *commuting with anchors in the sense that*

$$\underline{\Upsilon}' \circ \underline{f} = \underline{\underline{f_0}} \circ \underline{\Upsilon} : \ \underline{M} \to \underline{\underline{M_0'}}.$$

We require that $\mathbf{Mod}_{\mathbb{K}}$ *is a subcategory of* $\mathbf{Space}_{\mathbb{K}}$*, in the sense that on* $\mathbf{Mod}_{\mathbb{K}}$ *the extensions coincide with algebraic scalar extensions coming from the corresponding ring extensions: when* V *is a* $\mathbb{K}$*-module, then* $\underline{\Upsilon} : \underline{V} \to \underline{\underline{V_0}}$ *is, for each* $(n, \mathbf{t}, \mathbf{s}) \in \mathbf{scal}_{\mathbb{K}}$*, given by the anchor of scalar extensions* $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} : V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to V^{\mathcal{P}(\mathsf{n})}$.

Equivalently, a $\mathbb{K}$-space law $(\underline{M}, \Upsilon_M)$ could also be defined as a functor from $\mathbf{scal}_{\mathbb{K}}$ to $\mathbf{Sets}^{\mathbf{2}}$, the category of "anchored sets", satisfying certain properties. The present formulation features the anchor as a kind of "underlying morphism" of functor categories $\mathbf{Space}_{\mathbb{K}} \to \mathbf{CubeSet} \cong \mathbf{Sets}$. At this point, the situation is quite similar to the one given by abstract polynomials $P \in \mathbb{K}[X]$, to which we can associate, by evaluation on $\mathbb{K}$, an underlying set-map $\tilde{P} : \mathbb{K} \to \mathbb{K}$. In order to define a functor in the other direction, we need assumptions.

3.6 The topological case

Let's return to the topological case, and assume that $\mathbb{K}$ is a good topological ring. Recall from Definition 1.12 the category $\mathbf{Llin}_{\mathbb{K},n}$ of locally linear sets with $C_{\mathbb{K},n}$-maps as morphisms ($n \in \mathbb{N}$, or $n = \infty$).

Definition 3.8 (Prolongation functor)**.** *We define a* prolongation functor

$$\iota : \mathbf{Llin}_{\mathbb{K},\infty} \to \mathbf{Fn}(\mathbf{talg}_{\mathbb{K}}, \mathbf{Sets})$$

by associating to an object (U,V) (i.e., U open in a topological $\mathbb{K}$-module V) the functor $\underline{U}$ defined by $(\mathbf{t},\mathbf{s}) \mapsto U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ (Def. 1.5), and to a $C_{\mathbb{K},n}$-map $f : U \to U'$ the natural transformation defined by restricted iteration (Def. 1.5)

$$\underline{f} : \qquad f_{\mathbb{K}} = f, \quad f_{\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}} = f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}.$$

Lemma 3.9. *The correspondence ι defined above defines a $\mathbb{K}$-space, it is indeed a functor, and*

$$\mathrm{ev}_0 \circ \iota = \mathrm{id}_{\mathbf{Llin}_{\mathbb{K},\infty}}.$$

Proof. First of all, $\underline{U}$ defines a $\mathbb{K}$-space: there is an anchor having the required properties; it is a functor: it is compatible with left and right tensoring, and similarly, $C_{\mathbb{K}}$-maps indeed induce natural transformations. Finally, the evaluation functor clearly gives us back the original objects and morphisms, $U_{\mathbb{K}} = U$, $f_{\mathbb{K}} = f$. □

For the moment, the composition $\iota \circ \mathrm{ev}_{\mathbb{K}}$ is not even defined, since the evaluation $\mathrm{ev}_0(\underline{f})$ has no reason to be a *smooth* function. Thus our concern will be to define a subcategory where this is the case. Since the local linear structure plays a decisive role here, we restrict our attention to this situation, allowing us to state the result even as an *isomorphism of categories*.

Definition 3.10. *We define the functor category* $\mathbf{CSpace}_{\mathbb{K}}$ *of* continuous $\mathbb{K}$-space laws *to be the subcategory of* $\mathbf{Space}_{\mathbb{K}}$ *defined as follows:*

1. *categories* $\mathbf{Sets}$ *and* $\mathbf{Sets^2}$ *are replaced by* $\mathbf{Toplin}$ *and* $\mathbf{Toplin^2}$ *(open sets in topological $\mathbb{K}$-modules, and the corresponding continuous anchors and continuous morphisms, meaning that all $\Upsilon_{\mathbb{A}}$ and $f_{\mathbb{A}}$ are continuous maps),*
2. *morphisms $\underline{f}$ are moreover* jointly continuous in the scaloid, *i.e.: for all locally linear sets $(\underline{U},V)$ and morphisms $\underline{f}$, the following map is continuous (where $V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \cong V^{2^n}$ via the e-basis, and likewise for W^{2^n}):*

$$\mathbb{K}^{2n} \times V^{2^n} \supset \{(\mathbf{t},\mathbf{s};\mathbf{v}) \mid \mathbf{v} \in U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}\} \to W^{2^n}, \quad (\mathbf{t},\mathbf{s};\mathbf{v}) \mapsto f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}(\mathbf{v}).$$

Inclusions of (non-empty) open sets in topological $\mathbb{K}$-modules, $U \subset V$, then induce morphisms $\underline{U} \to \underline{V}$, which again will be called "inclusions". The whole set-up of our theory is designed such that the following result becomes essentially a tautology:

Theorem 3.11. *We have two well-defined and mutually inverse functors* ev *and* ι*, defining an isomorphism of categories*

$$\mathbf{Llin}_{\mathbb{K},\infty} \cong \mathbf{CSpace}_{\mathbb{K}}.$$

In particular, ι *defines a full and faithful imbedding of* $\mathbf{Llin}_{\mathbb{K}}$ *into a functor category.*

Proof. Note that in the present case we can speak of equality of objects on both sides in question, and hence the notion of "isomorphism" of these categories makes sense (cf. [CWM], p. 92-93).

Starting with a $C_{\mathbb{K},n}$-function f, it follows from Lemma 3.9 that f can be identified with evaluation at level 0 of the natural transformation $\underline{f}$ defined by f.

To prove the converse, let $\underline{f} : \underline{U} \to \underline{U}'$ be a continuous morphism of laws. We have to show that $\underline{f}$ is induced by a map of class $C_{\mathbb{K},n}$; more precisely, we show that the underlying map $f = \underline{f}_0 : U_{\mathbb{K}} = U \to U' = (U')_{\mathbb{K}}$ is of class $C_{\mathbb{K},n}$, and that it induces $\underline{f}$. As required in Definition 3.7, the anchor of $V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ is given by $\mathrm{id}_V \otimes \Upsilon_{\mathbb{K}}$, and via inclusions, the anchor of $\underline{U}$ is given by restricting the anchor of $\underline{V}$. Since $\underline{f}$ is a morphism, it commutes with the anchor in the sense that

$$\Upsilon \circ f_{\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}} = f_{\mathbb{K}^{\mathcal{P}(\mathsf{n})}} \circ \Upsilon.$$

By the continuity property (2) from Definition 3.10, these maps are continuous and jointly continuous also in $(\mathbf{t},\mathbf{s})$, whence satisfy the condition from Theorem 1.8, showing that the base map $f = f_{\mathbb{K}}$ is of class $C_{\mathbb{K},\infty}$, with the components of $\underline{f}$ given by the construction from topological differential calculus; thus $f_{\mathbb{K}}$ induces the natural transformation $\underline{f}$. □

Remark 3.2. As usual for "tautological" results, the main work lies in the preceding definitions and auxiliary results. To make this yet more plain, let's write G for the monoid $\mathbf{talg}_{\mathbb{K}} \cong \mathbf{scal}_{\mathbb{K}}$ (Lemma 3.1) and C for some subcategory of $\mathbf{Sets^2}$. Assuming C to be small, we may consider the set C^G of all functions from G to C. Clearly, evaluation at the neutral element $o \in G$ defines a map $\mathrm{ev}_o : C^G \to C$. The natural candidate for a map in the other direction is sending C to the "constants" $C \to C^G$, $f \mapsto (g \mapsto f)$. The problem is that the meaning of "constants" has to be carefully defined in a categorical context.

Remark 3.3 (Infinitesimal vs. local and global). A remark on comparison with the case of *Weil laws* as defined in [Be14] is in order here. Taking for $\mathbf{c}_{\mathbb{K}}$ the category of *Weil algebras*, instead of our tangent algebras, we get a formally quite similar theory. However, the anchor becomes "invisible" (for a Weil algebra, it degenerates to a single character), and one may say that Weil algebras are by nature *infinitesimal objects* (because of the nilpotency condition). Thus the link with the local and global theory is not encoded by algebra (as in our approach), and in order to get a *well-adapted* model one has to use more analytic tools (so it is not clear how far these can be generalized beyond the case of real or complex base field) – see [Du79, MR91]. Nevertheless, it might be interesting to look for a category of algebras comprising both Weil algebras and our tangent algebras – in order to prepare the ground, in Appendix B, we describe some algebraic structures that might be useful for such an approach.

4. Further directions

With Theorem 3.11, we have shown that the functor category $\mathbf{Space}_{\mathbb{K}}$ can be considered as a "well adapted model" for general differential calculus. In subsequent work, we will develop the theory further: on the one hand, comparing with SDG, we will investigate categorical questions, on the other hand, by enriching the structure of our category of algebras, the theory naturally offers links with *higher algebra* and with *super-calculus*. We give some short comments on these items.

4.1 Natural transformations, morphisms

In the preceding formulation, we have limited morphisms in the monoidal categories $\mathbf{scal}_{\mathbb{K}}$, resp. $\mathbf{talg}_{\mathbb{K}}$, to the strict minimum necessary to state the general form of the theory. However, in differential geometry, other algebra morphisms play a rôle by inducing *natural transformations*, as explained by the theory of Weil-functors (see [KMS93]). These algebra morphisms appear already on the level of difference calculus: for instance, the automorhism κ (inversion, see Theorem B.1) corresponds to the *exchange automorphism* on the level of $\mathbb{K}^{\mathcal{P}(1)} \cong \mathbb{K}^2$, inducing a global automorphism on the level of the functor categories. Likewise, our monoidal categories are moreover *symmetric braided monoidal*, via the usual braiding $\mathbb{A}\otimes\mathbb{B} \cong \mathbb{B}\otimes\mathbb{A}$ of associative algebras: again, this gives rise to globally defined morphisms (Schwarz's Theorem, and the "canonical flip" of higher tangent bundles) which together with the inversions, generate at n-th order level an automorphism group which is a *hyperoctahedral group* (automorphism group of a hypercube).

4.2 Groupoids, and higher algebra

In topological calculus, the extended domains $U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ carry a natural structure of *n-fold groupoid* (by iteration from Item (5) of Theorem B.1; see [Be15a, Be15b, Be17], for the case of target calculus). This is related to the preceding item: indeed, one can show that the groupoid structure on $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ is internal to the category of algebras, i.e., all structure maps of the groupoid are algebra morphisms. However, in order to "categorify" this, one needs to enlarge our small category of algebras so that it becomes stable under more general operations than just tensor products, such as *fiber products*. This will be taken up in subsequent work.

4.3 Graded calculus

We insist on the importance of the monoidal structure of the categories $\mathbf{talg}_{\mathbb{K}}$ and $\mathbf{scal}_{\mathbb{K}}$, with the aim to adapt the present approach for giving a functorial approach to *super*-calculus. In principle, it seems that the basic structure outlined in Remark 3.2 can be transposed to the monoidal category of *graded algebras and graded tensor products* generated by $\Upsilon_{t,s}$. It remains to understand the precise relation of such a graded categorical calculus with supercalculus, as it is currently presented. To do this, on should concentrate on symmetric calculus ($\mathbf{t} = -\mathbf{s}$), since in this case the groupoid inversion κ (which becomes the grading automorphism of superalgebras) is given by the simple formula $\kappa(v_0 + ev_1) = v_0 - ev_1$ (cf. Theorem B.1).

4.4 Full iteration, and simplicial calculus

As mentioned in Remark 1.3, *full iteration* leads to higher order "tangent maps" $f^{\{1,\ldots,n\}}$ having a very complicated structure. In principle, this structure can also be interpreted in terms of higher groupoids (see [Be15b]). In this setting, the analog of the tangent algebra category $\mathbf{talg}_{\mathbb{K}}$ will be some small higher order category, whose structure remains to be understood yet. Restricting again variables to certain subspaces, one can obtain a sufficiently simple calculus, called *simplicial* in [Be13], and corresponding to the classical concept of *divided differences*. It is certainly possible to put this simplicial calculus into a categorical form, essentially as done in this work for restricted iteration. The advantage should be a better compatibility of calculus with algebra in *positive* characteristic, but the drawback is that the close link with the tensor product, featured in the present approach, gets lost: iteration is no longer given by subsequent tensor products.

A. Hypercubic linear algebra

In this appendix, "linear spaces" are modules over a commutative ring $\mathbb{K}$. Recall Definition 2.6 of a *hypercubic space based on* $N \in \mathcal{P}(\mathbb{N})$. Changing slightly our viewpoint, every free $\mathbb{K}$-module V with basis indexed by $\mathcal{P}(N)$ is isomorphic to $\mathbb{K}^{\mathcal{P}(N)}$ and hence will also be called *hypercubic space*.

When $f : V \to W$ is linear, for bases $(b_j)_{j\in J}$ in V and $(c_i)_{i\in I}$ in W, we denote by $f_{i,j} := c_i^*(f(b_j))$ its *matrix coefficients* (where $(c_i^*)_{i\in I}$ is the dual basis of c). We write also $(\phi \otimes v)(x) = \phi(x) \cdot v$. Then

$$f = \sum_{(i,j)\in I\times J} f_{i,j}\, b_j^* \otimes c_i, \qquad f(b_k) = \sum_i f_{i,k} c_k.$$

When writing a matrix in the usual way as rectangular number array, we use the natural *total order* on the index set – that is, the *lexicographic order*; for instance,

$$\mathcal{P}(\{1,2\}) = (\emptyset, \{1\}, \{2\}, \{1,2\}).$$

In the following, for an n-tuple $\mathbf{a} = (a_i)_{i\in N} \in \mathbb{K}^n$, we use the notation $\mathbf{a}_N := \prod_{i\in N} a_i$, in the same way as we do for $\mathbf{t}, \mathbf{s} \in \mathbb{K}^n$ in the main text. When N is considered to be fixed, and $A \subset N$, we denote by $A^c = N \setminus A$ its complement.

The following result allows to put hands on induction procedures using iterated tensor products, cf. Remark 2.1.

Theorem A.1. *Let $N = \{k_1, \dots, k_n\}$ and $f_i : \mathbb{K}^{\mathcal{P}(\{k_i\})} \to \mathbb{K}^{\mathcal{P}(\{k_i\})}$ linear, with matrix*

$$f_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} : \qquad E^i_\emptyset \mapsto a_i E^i_\emptyset + c_i E^i_i, \quad E^i_i \mapsto b_i E^i_\emptyset + d_i E^i_i.$$

Then the matrix of the linear map $f := \otimes_{i=1}^n f_i : \mathbb{K}^{\mathcal{P}(N)} \to \mathbb{K}^{\mathcal{P}(N)}$ is given by the matrix coefficients, for $(A,B) \in \mathcal{P}(N)^2$,

$$f_{A,B} = E^*_A\big(f(E_B)\big) = \mathbf{a}_{A^c\cap B^c} \cdot \mathbf{b}_{A^c\cap B} \cdot \mathbf{c}_{A\cap B^c} \cdot \mathbf{d}_{A\cap B}.$$

In other terms, $f(E^N_B) = \sum_{A\in\mathcal{P}(N)} \mathbf{a}_{A^c\cap B^c} \cdot \mathbf{b}_{A^c\cap B} \cdot \mathbf{c}_{A\cap B^c} \cdot \mathbf{d}_{A\cap B}\ E^N_A$, or

$$f = \sum_{(A,B)\in\mathcal{P}(N)^2} \mathbf{a}_{A^c\cap B^c} \cdot \mathbf{b}_{A^c\cap B} \cdot \mathbf{c}_{A\cap B^c} \cdot \mathbf{d}_{A\cap B}\ (E^N_B)^* \otimes E^N_A.$$

Proof. When the cardinality n of N is equal to one, then the claim is true, directly by definition of the matrix coefficients. For $n = 2$, the matrix of $f_1 \otimes f_2$ is

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \otimes \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & b_1a_2 & a_1b_2 & b_1b_2 \\ c_1a_2 & d_1a_2 & c_1b_2 & d_1b_2 \\ a_1c_2 & b_1c_2 & a_1d_2 & b_1d_2 \\ c_1c_2 & d_1c_2 & c_1d_2 & d_1d_2 \end{pmatrix}$$

("Kronecker product"). For instance, when $B = \emptyset$, so $B^c = \{1,2\}$,

$$f(E^{\{12\}}_\emptyset) = a_{12}E_\emptyset + c_1a_2E_1 + a_1c_2E_2 + c_{12}E_{12},$$

in keeping with the claim. In the general case, we expand the expression

$$f = \otimes_i f_i = \otimes_i \Big(a_i (E^i_\emptyset)^* \otimes E^i_\emptyset + b_i (E^i_\emptyset)^* \otimes E^i_i + c_i (E^i_i)^* \otimes E^i_\emptyset + d_i (E^i_i)^* \otimes E^i_i \Big)$$

by distributivity: we get a sum of 4^n terms, which correspond exactly to the 4^n terms in the last formula of the claim. (E.g., for $n = 2$, there are 16 terms, corresponding to expanding the product $(a_1+b_1+c_1+d_1)(a_2+b_2+c_2+d_2)$ by distributivity, giving the 16 matrix coefficients shown above. The first column contains the 4 terms from expanding $(a_1+c_1)(a_2+c_2)$, etc.) □

To memorise the formula: for 2×2-matrices and indices, the correspondence is

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \quad : \quad \begin{pmatrix} A^c \cap B^c & A^c \cap B \\ A \cap B^c & A \cap B \end{pmatrix}.$$

Next, we give a formula for the *inverse* of f, when its determinant is invertible. From well-known properties of the Kronecker product it follows that

$$\det(f) = \det(\otimes_{i=1}^n f_i) = (\prod_{i=1}^n \det(f_i))^{2^{n-1}},$$

whence the first statement of the following theorem:

Theorem A.2. *Let N and $f = \otimes_{i=1}^n f_i$ be as in the preceding theorem. Then f is invertible if, and only if, all f_i are invertible, and then its inverse is given by the matrix coeffients, for $(A,B) \in \mathcal{P}(N)^2$ (recall $A\Delta B$ is the symmetric difference)*

$$(f^{-1})_{A,B} = \frac{(-1)^{|A\Delta B|}}{\prod_{i=1}^n \det(f_i)} f_{B^c,A^c} = \frac{(-1)^{|A\Delta B|}}{\prod_{i=1}^n \det(f_i)} \mathbf{a}_{A\cap B} \cdot \mathbf{b}_{A\cap B^c} \cdot \mathbf{c}_{A^c\cap B} \cdot \mathbf{d}_{A^c\cap B^c}.$$

Proof. Assume each f_i is invertible. For $n = 1$, $N = \{k\}$, the inverse is

$$\begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}^{-1} = \frac{1}{(a_k d_k - b_k c_k)} \begin{pmatrix} d_k & -b_k \\ -c_k & a_k \end{pmatrix}. \tag{A.1}$$

For $n = 2$, the matrix of the inverse is the Kronecker product of the inverses

$$\frac{1}{\det(f_1)\det(f_2)} \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} \otimes \begin{pmatrix} d_2 & -b_2 \\ -c_2 & a_2 \end{pmatrix} =$$

$$\frac{1}{\det(f_1)\det(f_2)} \begin{pmatrix} d_1 d_2 & -b_1 d_2 & -d_1 b_2 & b_1 b_2 \\ -c_1 d_2 & a_1 d_2 & c_1 b_2 & -a_1 b_2 \\ -d_1 c_2 & b_1 c_2 & d_1 a_2 & -b_1 a_2 \\ c_1 c_2 & -a_1 c_2 & -c_1 a_2 & a_1 a_2 \end{pmatrix}$$

which is in keeping with the formula announced in the claim. To put this computation into a conceptual framework, note that the inverse in (A.1) is obtained by first taking the adjugate matrix, and then dividing by the determinant. The adjugate $X^\sharp$ of a 2×2-matrix X, in turn, is given by

$$X^\sharp = J X^\top J^{-1},$$

where $X^\top$ is the transposed matrix, $(X^\top)_{(A,B)} = X_{(B,A)}$, and

$$I := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{A.2}$$

i.e., J sends $E_\emptyset \mapsto E_1,\ E_1 \mapsto -E_\emptyset$ (so $X^\sharp$ is the adjoint of X with respect to the canonical symplectic form on $\mathbb{K}^2$; call it "symplectic adjoint"). For each 2×2-matrix M let

$$M_{\mathsf{n}} = \otimes_{i=1}^n M : \mathbb{K}^{\mathcal{P}(\mathsf{n})} \to \mathbb{K}^{\mathcal{P}(\mathsf{n})}.$$

Then, for the matrices I, J, K defined by (A.2), the effect on E_A is

$$I_{\mathsf{n}}(E_A) = (-1)^{|A|} E_A, \quad K_{\mathsf{n}}(E_A) = E_{A^c}, \quad J_{\mathsf{n}}(E_A) = (-1)^{|A^c|} E_{A^c}, \tag{A.3}$$

The inverse of J_{n} is $J_{\mathsf{n}}^{-1}(E_A) = K_{\mathsf{n}} I_{\mathsf{n}}(E_A) = (-1)^{|A|} E_{A^c} = (-1)^n J_{\mathsf{n}}(E_A)$. Using this, we compute

$$\begin{aligned} f^\sharp(E_A) &= J_{\mathsf{n}} \circ f^\top \circ J_{\mathsf{n}}^{-1}(E_A) = (-1)^{|A|} J_{\mathsf{n}} \circ f^\top(E_{A^c}) \\ &= (-1)^{|A|} J_n \sum_B f^\top_{A^c,B} E_B \\ &= (-1)^{|A|} \sum_B f_{B,A^c} (-1)^{|B^c|} E_{B^c} = (-1)^{|A|} \sum_B f_{B^c,A^c} (-1)^{|B|} E_B \\ &= \sum_B (-1)^{|A|} (-1)^{|B|} \mathbf{a}_{A\cap B} \cdot \mathbf{b}_{A\cap B^c} \cdot \mathbf{c}_{A^c\cap B} \cdot \mathbf{d}_{A^c \cap B^c} E_B \end{aligned}$$

which together with $|A| + |B| \equiv |A\Delta B| \mod (2)$, so $(-1)^{|A|}(-1)^{|B|} = (-1)^{|A\Delta B|}$, gives us the adjugate and the claim. □

Remark A.1. In the same way, it follows that, even if f is not invertible, we have

$$f \circ J_{\mathsf{n}} \circ f^\top \circ J_{\mathsf{n}}^{-1} = \prod_{i=1}^n \det(f_i) \cdot \mathrm{id}.$$

B. On the structure of tangent algebras

One may be interested in defining a class of algebras, generalizing the by now classical *Weil algebras* (see [KMS93, MR91]), and the *bundle algebras* from [Be14], incorporating also algebras arising from difference calculus. The following structure theorem might help to select structural features that could be used for defining such a category. We use notation defined in Subsection 2.2.

Theorem B.1 (Structure of the first order tangent algebra $\mathbb{K}^{\{1\}}_{(t,s)}$)**.**

1. *The ideals* $\ker(\alpha)$ *and* $\ker(\beta)$ *satisfy* $\ker(\alpha) \cdot \ker(\beta) = 0$.

2. *The product of* $w, v \in \mathbb{K}^{\{1\}}_{(t,s)}$ *is given by the "fundamental relation"*

$$w \cdot v = \alpha(w) v - \alpha(w)\beta(v) + \beta(v) w.$$

3. *The map*
$$\kappa : \mathbb{K}^{\{1\}}_{(t,s)} \to \mathbb{K}^{\{1\}}_{(t,s)}, \quad v \mapsto (\alpha+\beta)(v)\cdot 1 - v$$
is an algebra automorphism of order 2 *such that* $\alpha\circ\kappa = \beta$*. Moreover,*
$$\forall v \in \mathbb{K}^{\{1\}}_{(t,s)} : \quad v\cdot\kappa(v) = \alpha(v)\beta(v)1.$$

4. *An element* v *is invertible in* $\mathbb{K}^{\{1\}}_{(t,s)}$ *if, and only if,* $\alpha(v)\beta(v) \in \mathbb{K}^\times$*, and then the inverse is*
$$v^{-1} = \frac{1}{\alpha(v)\beta(v)}\kappa(v) = (\frac{1}{\alpha(v)} + \frac{1}{\beta(v)})1 - \frac{v}{\alpha(v)\beta(v)}.$$

5. *The set* $\mathbb{K}^{\{1\}}_{(t,s)}$*, equipped with the following product* $*$ *(for* (u,w) *such* $\alpha(u)=\beta(w)$*), inversion* κ*, and units* $\lambda 1$ *(*$\lambda\in\mathbb{K}$*), is a groupoid:*
$$u * w = u - \alpha(u)1 + w.$$

Proof. (1) $\ker(\alpha) = \mathbb{K}(e-s)$ and $\ker(\beta) = \mathbb{K}(e-t)$, and, by the defining relation of the algebra, $(e-s)(e-t) = [(X-t)(X-s)] = 0$.

(2) Since $\alpha(v-\alpha(v)1) = 0$ and $\beta(w-\beta(w)1) = 0$, the preceding item implies
$$0 = (v-\alpha(v))(w-\beta(w)) = vw - \alpha(v)w - \beta(w)v + \alpha(v)\beta(w).$$

(3) Note that $\kappa(1) = 1+1-1 = 1$ and $\kappa(e) = s+t-e$, whence $\kappa(\kappa(e)) = s+t-(s+t-e) = e$, so $\kappa^2 = \mathrm{id}$. Next, $\alpha(\kappa(v)) = (\alpha+\beta)(v) - \alpha(v) = \beta(v)$. To prove that κ is an automorphism, since $\kappa(1) = 1$, it suffices to show that $\kappa(e^2) = \kappa(e)^2$. Indeed, $\kappa(e)^2 = (t+s)^2 - 2(t+s)e + e^2 = (t+s)^2 - ts - (t+s)e$ and $\kappa(e^2) = \kappa(-ts+(t+s)e) = -ts + (t+s)\kappa(e) = -ts + (t+s)^2 - (t+s)e$. Finally,
$$v\cdot\kappa(v) = \alpha(v)\kappa(v) - \alpha(v)\beta(\kappa v) + \beta(\kappa v)v = \alpha(v)\beta(v)1.$$

(4) If v is invertible, then applying the morphisms α and β, it follows that both $\alpha(v)$ and $\beta(v)$ are invertible. Conversely, the last formula from (3) shows that under this condition v has an inverse given by v^{-1} as in the claim.

(5) The defining properties of a groupoid are easily checked by direct computation, cf. [Be15a, Be17]. □

It is then true, moreover, that the groupoid law $*$ is an algebra morphism from the fiber product algebra $\mathbb{K}_{(t,s)} \times_{\alpha,\beta} \mathbb{K}_{(t,s)}$ to $\mathbb{K}_{(t,s)}$, and thus is "internal" to a certain category of algebras.

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UNE CLASSE D'EXEMPLES D'∞-CATÉGORIES FAIBLES AU SENS DE BATANIN

Jacques PENON

Résumé. Grâce à la pureté de la monade $\mathbb{B}$ de Batanin (établie dans [17]), nous montrons que toute pseudo-algèbre stable (voir la définition 1.3) pour la monade $\hat{\omega}$ des ∞-catégories strictes (étendue à la 2-catégorie des catégories globulaires) est munie d'une structure d'algèbre sur $\hat{\mathbb{B}}$ (l'extension de $\mathbb{B}$ à la 2-catégorie des catégories globulaires).
Abstract. Thanks to the purity of the Batanin's monad $\mathbb{B}$ (established in [17]), we prove that any stable pseudo-algebra (see the definition 1.3) for the monad $\hat{\omega}$ of the strict ∞-categories (extended to the 2-category of globular categories) is equipped with a structure of algebra on $\hat{\mathbb{B}}$ (which is the extension of $\mathbb{B}$ for the 2-category of globular categories).
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Introduction

Lorsqu'il y a une vingtaine d'années A.Burroni nous avait initié aux multi-spans (voir [9], [12], [1] et [2]) il nous avait affirmé qu'ils devaient former une ∞-catégorie faible. Seulement, à l'époque, on n'avait pas encore à notre disposition de définition précise d'∞-catégories faibles. Plus tard, après en avoir proposé une (voir [16]), on s'est tout naturellement demandé si les multi-spans n'en constituaient pas un exemple significatif. Comme ces ∞-catégories faibles (appelées prolixes) sont des algèbres sur une monade construite avec des techniques syntaxiques, la question se posait de savoir si on ne pouvait pas appliquer ce nouvel outillage pour démontrer la "conjecture d'A.Burroni". Finalement une preuve de ce type a pu être mise au point. Mais celle-ci s'appuyait sur le fait que la monade $\mathbb{P}$ des prolixes (dans sa version non réflexive - voir [3] et [13]) devait vérifier une propriété très forte dite de "pureté" (voir [17] ou encore la section 1). Hélas ! un peu plus tard, il s'est avéré qu'il n'en était rien (voir [17], deuxième partie, section 3). Il restait alors la solution de fabriquer une nouvelle monade, sur le modèle de $\mathbb{P}$, toujours avec des techniques syntaxiques, qui puisse être pure. Une telle monade $\mathbb{B}$ fut donc construite et on s'est aperçu ensuite qu'elle n'était autre que la monade de Batanin (voir [17], deuxième partie, section 5). Il ne nous restait plus alors qu'à adapter l'ancienne preuve, pour $\mathbb{P}$, à la nouvelle monade, pour obtenir le résultat escompté. Dans l'énoncé du théorème obtenu, on formule en fait une hypothèse beaucoup plus générale en remplaçant les catégories de multi-spans par des pseudo-algèbres stables sur la monade $\hat{\omega}$ (i.e. le prolongement, à la 2-catégorie $\mathbb{C}G$ des catégories globulaires, de la monade ω sur $\mathbb{G}lob$ des ∞-catégories strictes). On sait en effet que les multi-spans forment une pseudo-algèbre sur $\hat{\omega}$ (voir [2], [22]). Enfin la "stabilité" d'une catégorie globulaire est une propriété générale (voir la section

1) que satisfait sans problème l'exemple formé par les multi-spans.

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1. Position du problème

1.1 L'exemple des multi-spans

• Rappelons qu'un ensemble globulaire est un préfaisceaux sur la catégorie $\underline{Gl}$ (i.e. un foncteur $\underline{Gl}^{op} \to \mathbb{E}ns$) où $\underline{Gl}$ a pour objets les entiers naturels et où les morphismes :

$$0 \underset{d_0^1}{\overset{d_0^0}{\rightrightarrows}} 1 \underset{d_1^1}{\overset{d_1^0}{\rightrightarrows}} 2 \underset{d_2^1}{\overset{d_2^0}{\rightrightarrows}} 3 \rightrightarrows ....$$

...vérifient les équations :

$$\forall k \in [2] = \{0,1\}, \forall j \in [n] = \{0,...,n-1\},\ d_{j+1}^0.d_j^k = d_{j+1}^1.d_j^k.$$

Remarque 1.1. : On a une équivalence de catégorie $[\underline{Gl}^{op}, \mathbb{E}ns] \simeq \mathbb{G}lob$ (où $\mathbb{G}lob$ est défini dans [17] et dans cet article au 2.1). Dans la suite de cet article nous identifierons ces deux catégories.

En termes élémentaires, un ensemble globulaire $\mathbb{G}$ consiste en une suite d'ensembles G_n ainsi que des fonctions sources et buts comme suit :

$$G_0 \underset{\partial_0^1}{\overset{\partial_0^0}{\leftleftarrows}} G_1 \underset{\partial_1^1}{\overset{\partial_1^0}{\leftleftarrows}} G_2 \underset{\partial_2^1}{\overset{\partial_2^0}{\leftleftarrows}} G_3 \leftleftarrows \ldots.$$

...satisfaisant les relations de globularité $\partial_j^k.\partial_{j+1}^0 = \partial_j^k.\partial_{j+1}^1$.Une catégorie globulaire est un foncteur $\underline{Gl}^{op} \to \mathbb{C}at$. La 2-catégorie $[\underline{Gl}^{op}, \mathbb{C}at]$ des catégories globulaires peut être identifiée à la 2-catégorie des catégories internes à la catégorie des ensembles globulaires $[\underline{Gl}^{op}, \mathbb{E}ns]$.

• Il y a deux monades $\omega = (\omega, \eta, \mu)$ et $\mathbb{B} = (B, , \eta, \mu)$ sur $[\underline{Gl}^{op}, \mathbb{E}ns]$ (voir [2]). Les algèbres sur ω sont les ∞-catégories strictes et les algèbres sur $\mathbb{B}$ sont les ∞-catégories faibles telles que définies dans [2]. En outre, il y a un morphisme de monade $b : \mathbb{B} \to \omega$.

Étant donné que ces monades sont cartésiennes on peut les appliquer aux catégories internes donnant ainsi des 2-monades $\hat{\omega}$ et $\hat{\mathbb{B}}$ sur $[\underline{Gl}^{op}, \mathbb{C}at]$ et, de la même manière, b peut être étendu à $\hat{b} : \hat{\mathbb{B}} \to \hat{\omega}$. Notez que les algèbres strictes de $\hat{\omega}$ (resp.$\hat{\mathbb{B}}$) peuvent être idenfiées aux catégories internes dans les algèbres de ω (resp.$\mathbb{B}$).

• Il y a une catégorie globulaire particulière $Span : \underline{Gl}^{op} \to \mathbb{C}at$ dont la valeur sur les objets est donnée par $Span_n = [(\underline{Gl}/n)^{op}, \mathbb{E}ns]$ et, considérée comme une catégorie interne dans $[\underline{Gl}^{op}, \mathbb{E}ns]$, a un objet des objets $|Span|$ qui est décrit par $|Span|_n = Ob(Span_n)$. Ainsi $|Span|$ est un ensemble globulaire dont les 0-cellules sont les ensembles, les 1-cellules sont les spans d'ensembles, les 2-cellules sont les spans de spans etc. Via des produits fibrés itérés on obtient une structure
d'$\hat{\omega}$-pseudo-algèbre sur $Span$.(voir [2] et [22]).

• Le but de cet article est de montrer le théorème suivant :

Théorème 1.2. : L'ensemble globulaire $|Span|$ a une structure de $\mathbb{B}$-algèbre.

Pour y parvenir nous aurons besoin d'un "lemme clé" que nous allons essayer de formuler maintenant. Au lieu de nous focaliser sur $Span$, on va élargir le propos et s'interesser à une classe particulière d'$\hat{\omega}$-pseudo-algèbres. Celles qui sont stables. Ce terme s'applique en fait plus généralement aux catégories globulaires.

Définition 1.3. : On dira qu'une catégorie globulaire $\mathbb{G}$ est *stable* si, pour tout $n \in \mathbb{N}$, le foncteur canonique $(\partial_n^1, \partial_n^0) : G_{n+1} \to \bar{\bar{G}}_n$ est isofibrant, où $|\bar{\bar{G}}_0| = |G_0| \times |G_0|$ et, pour $n > 0$,
$|\bar{\bar{G}}_n| = \{(x_1, x_0) \in |G_n|^2 / \forall k \in [2], \partial_{n-1}^k(x_1) = \partial_{n-1}^k(x_0)\}$.
On a la même définition pour les flèches de $\bar{\bar{G}}_n$.

On vérifie que la catégorie globulaire $Span$ est stable. On peut alors étendre le théorème précédent en le formulant ainsi :

Théorème 1.4. : Toute ω-pseudo-algèbre stable a canoniquement une structure de $\mathbb{B}$-algèbre.

Mais on peut encore affiner notre énoncé en précisant ce qu'on entend par "canoniquement".

• Le morphisme de 2-monade $\hat{b} : \hat{\mathbb{B}} \to \hat{\omega}$ produit un foncteur d'oubli ; $U : \hat{\omega}\text{-}Ps\text{-}Alg \longrightarrow \hat{\mathbb{B}}\text{-}Ps\text{-}Alg$. entre les 2-catégories de pseudo-algèbres de $\hat{\omega}$ et $\hat{\mathbb{B}}$. Or on sait déjà (voir [15] et [18] pour le résultat de cohérence général de Power) que la $\hat{\mathbb{B}}$-pseudo-algèbre $U(Span, v, i, a)$ dont la catégorie globulaire sous-jacente est $Span$, est équivalente à une $\hat{\mathbb{B}}$-algèbre stricte. Cependant, la procédure de Power change la catégorie globulaire sous-jacente ce qui est une perte d'information non négligeable.
On se propose ici, pour chaque $\hat{\omega}$-pseudo-algèbre stable $(\mathbb{G}, v, i, a)$ d'effectuer une strictification de $U(\mathbb{G}, v, i, a)$ sans changer la catégorie globulaire sous-jacente. De façon précise :

Lemme 1.5. (lemme clé) $v' : \hat{B}(\mathbb{G}) \to \mathbb{G}$ étant la 1-cellule de structure de la $\hat{\mathbb{B}}$-pseudo-algèbre $U(\mathbb{G}, v, i, a)$, il existe une 2-cellule inversible :

$$r : w \to v' : \hat{B}(\mathbb{G}) \to \mathbb{G}$$

où la flèche w satisfait les axiomes d'une structure de $\hat{\mathbb{B}}$-algèbre stricte et où $(Id_{\mathbb{G}}, r) : U(\mathbb{G}, v, i, a) \to (\mathbb{G}, w, id, id)$ est une flèche de $\hat{\mathbb{B}}\text{-}Ps\text{-}Alg$.

La construction de r se fait par induction. Mais l'ingrédient essentiel qui permet cette induction est la pureté de la monade $\mathbb{B}$. Nous allons donc maintenant expliquer ce que nous entendons par là.

1.2 Prérequis en théorie des monades cartésiennes

(voir [4],[6],[7],[15],[23])

• Plaçons nous tout d'abord dans une catégorie $\mathbb{C}$ à limites à gauche finies, munie d'une monade cartésienne $\mathbb{M} = (M, \eta, \mu)$. On constate que le diagramme suivant :

$$M(1)\ \begin{array}{c}\xleftarrow{\mu_1}\\ \xrightarrow{M\eta_1}\\ \xleftarrow[M!]{}\end{array}\ M^2(1)\ \begin{array}{c}\xleftarrow{\mu_{M1}}\\ \xleftarrow{M\mu_1}\\ \xleftarrow[M^2!]{}\end{array}\ M^3(1)$$

est sous-jacent à une catégorie interne dans $\mathbb{C}$. Dans ce qui suit, cette catégorie interne jouera un rôle central, c'est pourquoi nous la baptisons la catégorie des "décompositions de $\mathbb{M}$" et la notons $Dec(\mathbb{M})$. En fait $Dec(\mathbb{M})$ est sous-jacente à une catégorie interne dans $\mathbb{M}$-Alg. Elle s'écrit :

$$(M(1), \mu_1)\ \begin{array}{c}\xleftarrow{\mu_1}\\ \xrightarrow{M\eta_1}\\ \xleftarrow[M!]{}\end{array}\ (M^2(1), \mu_{M1})\ \begin{array}{c}\xleftarrow{\mu_{M1}}\\ \xleftarrow{M\mu_1}\\ \xleftarrow[M^2!]{}\end{array}\ (M^3(1), \mu_{M^21})$$

On peut aussi la voir comme une une $\hat{\mathbb{M}}$-algèbre stricte. On la notera alors $\hat{Dec}(\mathbb{M})$.

Exemple 1.6. : Lorsque, sur $\mathbb{E}ns$, $\mathbb{M} = \mathbb{M}o$ (la monade des monoïdes) $\hat{Dec}(\mathbb{M})$ est la catégorie simpliciale algébriste.

• Si on désigne par $J_{\hat{\mathbb{M}}} : \hat{\mathbb{M}}\text{-}Alg_s \hookrightarrow \hat{\mathbb{M}}\text{-}Alg_l$ l'inclusion de la 2-catégorie des algèbres strictes et morphismes stricts sur $\hat{\mathbb{M}}$ dans la 2-catégorie des algèbres strictes et des morphismes laxs. Alors $\hat{Dec}(\mathbb{M})$ est l'objet libre, pour $J_{\hat{\mathbb{M}}}$, associé à l'objet terminal $\mathbb{I}$. En appliquant ce résultat à la monade $\mathbb{M}o$, sur $\mathbb{E}ns$, on retrouve le fait que la catégorie simpliciale algébriste est le monoïde classifiant. Dans [5] une proprité universelle analogue a été montrée pour la catégorie globulaire Ω des arbres (où dans ce cas $\mathbb{M} = \omega$).

• Plus généralement, si $\mathbb{K}$ est une 2-catégorie et $\mathbb{S}$ une 2-monade sur $\mathbb{K}$, dans des conditions très faibles (voir [6]) le foncteur inclusion
$\mathbb{S}$-$Alg_s \hookrightarrow \mathbb{S}$-$Alg_l$ admet un 2-adjoint à gauche $(-)^+_{\mathbb{S}}$. Si on note les composantes de l'unité et de la co-unité de cette 2-adjonction sur une $\mathbb{S}$-algèbre stricte A par :

$$p_A : A \to A^+_{\mathbb{S}}\ ,\ q_A : A^+_{\mathbb{S}} \to A$$

(Notons que p_A est un morphisme lax de $\mathbb{S}$-algèbres et q_A est strict) alors on a une adjonction

$$A \underset{p_A}{\overset{q_A}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} A^+_{\mathbb{S}}$$

dans $\mathbb{S}$-Alg_l avec une identité comme co-unité. Ainsi, dans le contexte d'une monade cartésienne sur $\mathbb{M}$, le morphisme $\eta_1 : 1 \to (1)^+_{\hat{\mathbb{M}}}$ est l'adjoint à droite de l'unique morphisme allant dans l'autre sens.

1.3 Monades concrètes syntaxiques

• Plaçons-nous maintenant dans la nouvelle situation suivante :
On se donne une catégorie concrète $(\mathbb{C}, U)$ (c.a.d. $\mathbb{C}$ est une catégorie et $U : \mathbb{C} \to \mathbb{E}ns$ est un foncteur fidèle) et sur $\mathbb{C}$ une monade $\mathbb{M}$.

Définition 1.7. : On dit que $(\mathbb{C}, U, \mathbb{M})$ est une *monade concrète cartésienne* si :
- $(\mathbb{C}, U)$ est une catégorie concrète où $\mathbb{C}$ est à limites à gauche finies,
- $\mathbb{M}$ est une monade cartésienne sur $\mathbb{C}$,
- U, en plus d'être fidèle, préserve les produits fibrés.

Dans ce cas, on note $UDec(\mathbb{M})$ la catégorie donnée par :

$$UM(1) \underset{UM!}{\overset{U\mu_1}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{UM\eta_1}}}}} UM^2(1) \underset{UM^2!}{\overset{U\mu_{M1}}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xleftarrow{UM\mu_1}}}}} UM^3(1)$$

Exemple 1.8. : Supposons que $\mathbb{M}$ soit la monade associée à une opérade globulaire et que U soit défini par

$$U : [\underline{Gl}^{op}, \mathbb{E}ns] \to \mathbb{E}ns, \ X \mapsto \coprod_{n \in \mathbb{N}} X_n$$

alors on obtient ainsi une monade cartésienne concrète et
- un objet de $UDec(\mathbb{M})$ est une opération de l'opérade correspondante,
- un morphisme $\alpha \to \beta$ dans $UDec(\mathbb{M})$ est une décomposition (d'où le choix de la dénomination $UDec(\mathbb{M})$) de l'opération α par le résultat d'une substitution d'une suite d'opérations dans l'opération β.

En reprenant ce qui précède, on obtient une adjonction

$$U(1) \underset{\longrightarrow}{\overset{U(!)}{\underset{\perp}{\longleftarrow}}} UDec(\mathbb{M})$$

dans $\mathbb{C}at$. Remarquons que $U(1)$ est une catégorie discrète. Or, pour une catégorie $\mathbb{C}$, se donner une adjonction $\mathbb{D} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbb{C}$ où $\mathbb{D}$ est une catégorie discrète, cela revient à se donner un choix d'objet final dans chaque composante connexe de $\mathbb{C}$. On peut donc envisager la situation suivante :

Définition 1.9. : 1) Notons $\mathbb{C}at_{ct}$ la catégorie dont les objets sont les catégories équipées d'un choix d'objet final pour chacune de ses composantes connexes et dont les morphismes sont les foncteurs préservant les choix d'objets finaux. Appelons *ct-catégorie* un objet de $\mathbb{C}at_{ct}$ et *ct-foncteur* un morphisme de $\mathbb{C}at_{ct}$.
2) Pour chaque $A \in |\mathbb{C}at_{ct}|$ et $x \in |A|$ on note t_x le choix d'objet final dans la composante connexe de x et par $\tau_x : x \to t_x$ l'unique morphisme.
3) On note $\mathbb{C}CMnd$ la catégorie dont les objets sont les monades cartésiennes concrètes. Un morphisme $(\mathbb{C}, U, \mathbb{M}) \to (\mathbb{C}', U', \mathbb{M}')$ se compose d'un foncteur $F : \mathbb{C} \to \mathbb{C}'$ qui préserve les produits fibrés et satisfait $U'F = U$ et une transformation naturelle $\phi : FM \to M'F$ faisant de (F, ϕ) un op-foncteur entre monades dans le sens de R.Street [19].

Exemple 1.10. : Notons $\mathbb{N}_{\geq}$ l'ensemble partiellement ordonné dont les éléments sont les entiers strictement positifs et dont l'ordre est donné par $\geq$. Alors $1 \in \mathbb{N}$ est l'unique objet final et $\mathbb{N}_{\geq}$ peut être considéré comme une ct-catégorie.

Proposition 1.11. : L'application $(\mathbb{C}, U, \mathbb{M}) \mapsto UDec(\mathbb{M})$ est la composante sur les objets d'un foncteur $\mathbb{C}CMnd \to \mathbb{C}at_{ct}$.

Définition 1.12. : 1) Une monade concrète cartésienne $(\mathbb{C}, U, \mathbb{M})$ est dite *syntaxique* si $UDec(\mathbb{M})$ est un ensemble ordonné et si, en plus, elle est équipée d'un ct-foncteur conservateur $\lambda : UDec(\mathbb{M}) \to \mathbb{N}_{\geq}$.
2) La relation d'ordre sur $UM(1)$ est appelée la relation de *postériorité*. Son ordre opposé (correspondant à $UDec(\mathbb{M})^{op}$) est appelé la relation *d'antériorité*. On la note "$\leq$". Ainsi, pour $t, t' \in UM(1)$, on écrira $t \leq t'$ s'il existe une flèche $t' \to t$ dans $UDec(\mathbb{M})$.

$(\mathbb{C}, U, \mathbb{M}, \lambda)$ étant une monade concrète cartésienne syntaxique, on construit, pour chaque $C \in |\mathbb{C}|$, l'application $L_C : UM(C) \to \mathbb{N}_{\geq}$ définie par $L_C = (UM(C) \xrightarrow{UM!} UM(1) \xrightarrow{\lambda} \mathbb{N}^*)$. On obtient ainsi une transformation naturelle $L : UM \to \bar{\mathbb{N}}$ où $\bar{\mathbb{N}}$ désigne le foncteur constant sur $\mathbb{N}$.

Proposition 1.13. : Sous les hypothèses prècèdentes, on a les propriètès suivantes :
$(MS1) \forall C \in |\mathbb{C}|, \forall x \in U(C),\ L_C.U\eta_C(x) = 1,$
$(MS'1) \forall C \in |\mathbb{C}|, \forall t \in UM(C),\ L_C(t) \leq 1 \ \Rightarrow\ \exists x \in U\eta_C(x) = t,$
$(MS2) \forall C \in |\mathbb{C}|, \forall T \in UM^2(C),\ L_{MC}(T) \leq L_C.U\mu_C(T),$
$(MS'2) \forall C \in |\mathbb{C}|, \forall T \in UM^2(C),\ L_{MC}(T) = L_C.U\mu_C(T) \ \Rightarrow$
$\exists t \in UM(C),\ T = UM\eta_C(t),$
$(MS3)$ Pour tout $C \in |\mathbb{C}|$, l'application suivante est injective :
$(UM!_{MC}, U\mu_C) : UM^2(C) \to UM(1) \times UM(C).$

Preuve : $(MS1)$ On montre que, pour tout $u \in U(1), U\eta_1(u)$ est le plus grand élément de sa composante connexe, qui n'est autre que
$\{\theta \in UM(1)/U!_{M1}(\theta) = u\}$.
$(MS'1)$ On utilise le fait que λ est conservateur et que η est cartésienne.
$(MS2)$ Résulte de la croissance de λ.
$(MS3)$ On utilise la cartésianité de μ.

Remarque 1.14. : On montre dans [17] qu'une monade concrète cartésienne munie d'une transformation naturelle $L : UM \to \bar{\mathbb{N}}$ vérifant les propriétés $(MS1) \to (MS3)$ de la proposition précédente est syntaxique.

Exemples et contre-exemples 1.15. : 1) Les monades $\mathbb{P}$ des prolixes (voir [3] et [13]) et $\mathbb{B}$ de Batanin, sur $\mathbb{G}lob$, munies du foncteur d'oubli canonique
$U : \mathbb{G}lob \to \mathbb{E}ns$ sont des monades concrètes cartésiennes. Elles sont aussi syntaxiques en leur ajoutant l'application $\lambda = L_1$ (voir [17]).
2) La monade $\mathbb{M}o$ des monoïdes, sur $\mathbb{E}ns$, et la monade ω des ∞-catégories strictes, sur $\mathbb{G}lob$, sont cartésiennes. Elles sont aussi concrètes cartésiennes en les munissant de leur foncteur d'oubli évident. Cependant, elles ne peuvent produire de monade concrètes cartésiennes syntaxiques car la flèche

$(M!_{M1}, \mu_1) : M^2(1) \to M(1) \times M(1)$ n'est pas un monomorphisme.

1.4 Monades pures

• Avant de poursuivre notre approche des monades pures, rappelons que, dans une catégorie A quelconque, une flèche $f : x \to y$ est dite *indécomposable* quand, lors d'une factorisation $f = g.h$, g ou h sont une identité.

Définition 1.16. : 1) Soit A une ct-catégorie et $x \in |A|$. On dit que x est *primitif* quand ce n'est pas un choix d'objet final et l'unique morphisme $\tau_x : x \to t_x$ est indécomposable.
2) Une ct-catégorie A est dite *pure* quand, pour tout $x \in |A|$ qui n'est pas un choix d'objet final, le morphisme $\tau_x : x \to t_x$ se factorise à travers un unique objet primitif.
3) Une *monade pure* est une monade concrète cartésienne syntaxique $(\mathbb{C}, U, \mathbb{M}, \lambda)$ telle que la ct-catégorie $UDec(\mathbb{M})$ est pure.

• Fixons maintenant une monade concrète cartésienne syntaxique $\mathcal{M} = (\mathbb{C}, U, \mathbb{M}, \lambda)$.

Proposition 1.17. : Soit $t \in UM(1)$ tel que $\lambda(t) > 1$. Alors t est primitif dans $UDec(\mathbb{M})$ ssi :
$\forall T \in UM^2(1), U\mu_1(T) = t \Rightarrow T = U\eta_{M1}(t)$ ou $T = UM\eta_1(t)$.

<u>*Preuve*</u> : - Si t est primitif, soit $T \in UM^2(1)$ tel que $U\mu_1(T) = t$. Posons $t_0 = U\eta_1.U!_{M1}(t)$ et $t' = UM!_{M1}(T)$. Alors on a la décomposition

$$(t \to t_0) = (t \to t' \to t_0).$$

Comme $t \to t_0$ est indécomposable, on doit avoir $(t \to t') = id_t$, dans $UDec(\mathbb{M})$, ou $(t \to t') = (t \to t_0)$ ce qui s'écrit encore $T = UM\eta_1(t)$ ou $T = U\eta_{M1}(t)$.
Inversement, soit $t' \in UM(1)$ tel que $t_0 \leq t'$ et $t' \leq t$. Soit $T \in UM^2(1)$ tel que $t = U\mu_1(T)$ et $t' = UM!_{M1}(T)$. Alors, par hypothèse, $T = U\eta_{M1}(t)$ (et dans ce cas $t' = t$. Donc $t \to t' = id_t$), ou bien $T = UM\eta_1(t)$(et dans ce cas $t' = t$. Donc $t \to t' = id_t$). Ainsi t est primitif.

Proposition 1.18. : $\mathcal{M}$ est pure ssi :
$\forall t \in UM(1), \lambda(t) > 1 \Rightarrow \exists! \theta \in UM(1), \theta \leq t$ et θ est primitif.

<u>Preuve</u> : : Immédiat.

Définition 1.19. : Supposons $\mathcal{M}$ pure. Pour chaque objet $C \in |\mathbb{C}|$ et $t \in UM(C)$ tel que $L_C(t) > 1$ alors l'unique élément $\theta \in UM(1)$ primitif tel que $UM!_C(t) \leq \theta$ s'appelle la *composante primitive* de t et l'unique $T \in UM^2(C)$ tel que $UM!_{MC}(T) = \theta$ et $U\mu_C(T) = t$ est appelé la *décomposition primitive* de t.

Exemples et contre-exemples 1.20. : (voir [17]) 1) La monade concrète cartésienne syntaxique $(\mathbb{G}lob, U, \mathbb{P}, \lambda)$ n'est pas pure.
2) Par contre $(\mathbb{G}lob, U, \mathbb{B}, \lambda)$ est une monade cartésienne syntaxique pure.

• Nous allons consacrer maintenant le reste de cet article à la démonstration du théorème 1.4 où plus précisément du "lemme clé".

2. Démonstration du Lemme clé

2.1 Quelques rappels

Notation 2.1. : Au lieu de considérer la catégorie $[\underline{Gl}^{op}, \mathbb{E}ns]$, on préfère, dans [17] utiliser une catégorie équivalente. C'est la catégorie $\mathbb{G}lob$ dont :
- les objets $\mathbb{G}$ sont la donnée :
.. d'un ensemble G ,
.. d'une application $dim : G \to \mathbb{N}$ (Pour chaque $n \in \mathbb{N}$, notons $G/n = \{c \in G/ \; dim(c) > n\}$),
.. d'une famille de couples d'applications $(\partial_p^1, \partial_p^0 : G/p \to G)_{p \in \mathbb{N}}$,
vérifiant les propriétés suivantes :
$(EG1)$ $\forall c \in G, \forall p \in \mathbb{N}, \forall k \in [2], \; dim(c) > p \;\Rightarrow\; dim \; \partial_p^k(c) = p$,
$(EG2)$ $\forall c \in G, \forall p, q \in \mathbb{N}, \forall k, k' \in [2]$,
$dim(c) > p > q \;\Rightarrow\; \partial_q^k \partial_p^{k'}(c) = \partial_q^k(c)$.
- les *morphisme* $g : \mathbb{G} \to \mathbb{G}'$ sont des applications $g : G \to G'$ telles que :
$(MEG1)$ $\forall c \in G, \; dim \; g(c) = dim(c)$,
$(MEG2)$ $\forall c \in G, \forall p \in \mathbb{N}, \forall k \in [2], \; dim(c) > p \;\Rightarrow\; \partial_p^k g(c) = g \partial_p^k(c)$.

Remarque 2.2. : En fait, dans [17], les monades $\mathbb{P}$ et $\mathbb{B}$ sont définies sur $\mathbb{G}lob$. Nous continuerons donc, par la suite, à utiliser la catégorie $\mathbb{G}lob$ plutôt que $[\underline{Gl}^{op}, \mathbb{E}ns]$.

• Dans la construction par induction qui va suivre on utilise un matériel déjà donné dans [17]. Résumons succinctement en quoi consiste ce matériel et ses propriétés essentielles. Mais tout d'abord convenons de noter simplement L ce qu' on devrait écrire $L_{\mathbb{G}}$, pour un $\mathbb{G} \in |\mathbb{G}lob|$ quelconque (voir les quelques lignes précédant 1.13).

Notation 2.3. : Soient $\mathbb{G} \in |\mathbb{G}lob|$ et $n, m \in \mathbb{N}$. On pose :

$$B|_n^m(\mathbb{G}) = \{a \in B(\mathbb{G}) /\ \mathrm{L}(a) \leq n,\ \dim(a) \leq m\}.$$

Proposition 2.4. : 1) $\forall \mathbb{G} \in |\mathbb{G}lob|, \forall n, m \in \mathbb{N},\ B|_n^m(\mathbb{G}) \in |\mathbb{G}lob|$.
2) Pour toute flèche $g : \mathbb{G} \to \mathbb{G}'$ de $\mathbb{G}lob$, $B(g)$ se factorise par $B|_n^m(\mathbb{G}) \to B|_n^m(\mathbb{G}')$ dans $\mathbb{G}lob$. En faisant varier g on obtient un sous-endofoncteur, noté $B|_n^m$, de B.

Notation 2.5. : Soient $\mathbb{G} \in |\mathbb{G}lob|$ et $n, m \in \mathbb{N}$. On pose :

$$B^2|_n^m(\mathbb{G}) = \{A \in B^2(\mathbb{G}) /\ \mathrm{L}\mu_{\mathbb{G}}(A) \leq n,\ \dim(A) \leq m\}.$$

Proposition 2.6. : 1) $\forall \mathbb{G} \in |\mathbb{G}lob|, \forall n, m \in \mathbb{N},\ B^2|_n^m(\mathbb{G}) \in |\mathbb{G}lob|$.
2) Pour toute flèche $g : \mathbb{G} \to \mathbb{G}'$ de $\mathbb{G}lob$, $B^2(g)$ se factorise par $B^2|_n^m(\mathbb{G}) \to B^2|_n^m(\mathbb{G}')$ dans $\mathbb{G}lob$. En faisant varier g on obtient un sous-endofoncteur, noté $B^2|_n^m$, de B^2.

Proposition 2.7. : Soient $\mathbb{G} \in |\mathbb{G}lob|$ et $n, m \in \mathbb{N}$. Alors :
1) $\mu_{\mathbb{G}}$ se factorise par $B^2|_n^m(\mathbb{G}) \to B|_n^m(\mathbb{G})$. On la note $\mu_{\mathbb{G}}|_n^m$.
2) $(B^2|_n^m)(\mathbb{G}) \subset (B|_n^m)^2(\mathbb{G})$.

• Prolongeons maintenant ces constructions à la 2-catégorie $\mathbb{C}G = Cat(\mathbb{G}lob)$, ce qui est possible car...

Proposition 2.8. : Pour tout $(m, n) \in \mathbb{N}^2$, $B|_n^m$ et $B^2|_n^m$ commutent aux produits fibrés.

Preuve : - Pour $B|_n^m$: Soit $\mathbb{G}' \xrightarrow{g'} \mathbb{H} \xleftarrow{g} \mathbb{G}$ une paire de flèches de même but dans $\mathbb{G}lob$ et $\mathbb{K}$ son produit fibré. On note $\pi : \mathbb{K} \to \mathbb{G}$ et $\pi' : \mathbb{K} \to \mathbb{G}'$ les projections canoniques et P_n^m le produit fibré de la paire $B|_n^m(\mathbb{G}') \to B|_n^m(\mathbb{H}) \leftarrow B|_n^m(\mathbb{G})$. On montre que, pour tout $a \in B(\mathbb{K})$, on a l'équivalence suivante :
$(B(\pi')(a), B(\pi)(a)) \in P_n^m$ ssi $a \in B|_n^m(\mathbb{K})$. Cela entraine que la flèche canonique $B|_n^m(\mathbb{K}) \to P_n^m$ est un isomorphisme car B commute aux produits fibrés.
- Pour $B^2|_n^m$, on procède de même.

Remarque 2.9. : 1) En conséquence, les endofoncteurs $B|_n^m$ et $B^2|_n^m$ de $\mathbb{G}lob$ induisent deux nouveaux endofoncteurs $\widehat{B|_n^m}$ et $\widehat{B^2|_n^m}$ sur $\mathbb{C}G$ qui sont des sous-endofoncteurs de $\hat{B}$ et $\hat{B}^2$.
2) Comme en 2.7, on constate que, pour tout $\mathbb{G} \in |\mathbb{C}G|$ et tout $(m,n) \in \mathbb{N}^2$,
a) $\hat{\mu}_{\mathbb{G}}$ se factorise par $\widehat{B^2|_n^m}(\mathbb{G}) \to \widehat{B|_n^m}(\mathbb{G})$, on la note $\hat{\mu}_{\mathbb{G}}|_n^m$.
b) $(\widehat{B^2|_n^m})(\mathbb{G})$ est une sous-catégorie globulaire de $(\widehat{B|_n^m})^2(\mathbb{G})$.

Proposition 2.10. : Soient $\mathbb{G} \in |\mathbb{C}G|$ et $w : \widehat{B|_n^m}(\mathbb{G}) \to \mathbb{G}$ une flèche de $\mathbb{C}G$. Alors $\hat{B}(w) : \hat{B}\widehat{B|_n^m}(\mathbb{G}) \to \hat{B}(\mathbb{G})$ se factorise par $\widehat{B^2|_n^m}(\mathbb{G}) \to \widehat{B|_n^m}(\mathbb{G})$. On la note $\hat{B}(w)|_n^m$.

Preuve : Se vérifie facilement.

2.2 Procédure d'induction

Nous pouvons maintenant commencer la preuve du lemme clé. On se fixe une ω-pseudo-algèbre $(\mathbb{G}, v, i, a)$ dans $\mathbb{C}G$, comme il est précisé dans le lemme clé. Il nous faut construire une flèche $w : \hat{\mathbb{B}}(\mathbb{G}) \to \mathbb{G}$ et une 2-cellule inversible $r : w \to v'$ satisfaisant les conditions voulues. En d'autre termes, $v' = v.\hat{b}_{\mathbb{G}}$ et $r : w \to v.\hat{b}_{\mathbb{G}}$ doit faire commuter le diagramme suivant dans

$\mathbb{C}G(\hat{B}^2(\mathbb{G}),\mathbb{G})$:

$$\begin{array}{ccccc}
w.\hat{\mu}_{\mathbb{G}} & \xrightarrow{r.\hat{\mu}_{\mathbb{G}}} & v.\hat{b}_{\mathbb{G}}.\hat{\mu}_{\mathbb{G}} & \xrightarrow{Id} & v.\hat{\mu}_{\mathbb{G}}.\hat{b}^2_{\mathbb{G}} \\
\downarrow{\scriptstyle Id} & & & & \downarrow{\scriptstyle a.\hat{b}^2_{\mathbb{G}}} \\
w.\hat{B}w & & & & v.\hat{\omega}v.\hat{b}^2_{\mathbb{G}} \\
\downarrow{\scriptstyle r.\hat{B}w} & & & & \downarrow{\scriptstyle Id} \\
v.\hat{b}_{\mathbb{G}}.\hat{B}w & & \xrightarrow{v.\hat{b}_{\mathbb{G}}.\hat{B}r} & & v.\hat{b}_{\mathbb{G}}.\hat{B}v.\hat{B}\hat{b}_{\mathbb{G}}
\end{array}$$

Comme on l'a déjà dit, les constructions de w et r se font par induction et pour cela on construit 3 familles d'applications
$(|w_n^m| : B|_n^m(|\mathbb{G}|) \to |\mathbb{G}|)_{(m,n)\in\mathbb{N}^2}, (Fl(w_n^m) : B|_n^m Fl(\mathbb{G}) \to Fl(\mathbb{G}))_{(m,n)\in\mathbb{N}^2}$
et $(r_n^m : B|_n^m(|\mathbb{G}|) \to Fl(\mathbb{G}))_{(m,n)\in\mathbb{N}^2}$, par induction sur $m+n$, de telle sorte qu'elles satisfassent les conditions suivantes :
$(H0)$ Pour tout $m, m', n, n' \in \mathbb{N}$ tels que $m' \leq m$, $n' \leq n$, alors $|w_{n'}^{m'}|$, $Fl(w_{n'}^{m'})$ et $r_{n'}^{m'}$ sont les restrictions de $|w_n^m|$, $Fl(w_n^m)$ et r_n^m.
$(H1)$ Pour tout $m, n \in \mathbb{N}$, $w_n^m = (|w_n^m|, Fl(w_n^m)) : B\widehat{|_n^m}(\mathbb{G}) \to \mathbb{G}$ est un foncteur globulaire.
$(H2)$ Pour tout $m, n \in \mathbb{N}$, $r_n^m : w_n^m \to v.\hat{b}_{\mathbb{G}}.i_n^m$ est une transformation naturelle globulaire (où $i_n^m : B\widehat{|_n^m}(\mathbb{G}) \to \hat{B}(\mathbb{G})$ est l'injection canonique).
$(H3)$ Pour tout $m, n \in \mathbb{N}$, $w_n^m.\hat{B}(w_n^m)|_n^m = w_n^m.(\hat{\mu}_{\mathbb{G}}|_n^m)$.
$(H4)$ Pour tout $m, n \in \mathbb{N}$, le diagramme suivant commute dans
$\mathbb{C}G(B^2\widehat{|_n^m}(\mathbb{G}),\mathbb{G})$:

$$\begin{array}{ccccc}
w_n^m.\hat{\mu}_{\mathbb{G}}|_n^m & \xrightarrow{r_n^m.\hat{\mu}_{\mathbb{G}}|_n^m} & v.\hat{b}_{\mathbb{G}}.i_n^m.\hat{\mu}_{\mathbb{G}}|_n^m & \xrightarrow{Id} & v.\hat{\mu}_{\mathbb{G}}.\hat{b}^2_{\mathbb{G}}.\hat{B}i_n^m.j_n^m \\
\downarrow{\scriptstyle Id} & & & & \downarrow{\scriptstyle a.\hat{b}^2_{\mathbb{G}}.\hat{B}i_n^m.j_n^m} \\
w_n^m.\hat{B}w_n^m|_n^m & & & & v.\hat{\omega}v.\hat{b}^2_{\mathbb{G}}.\hat{B}i_n^m.j_n^m \\
\downarrow{\scriptstyle r_n^m.\hat{B}w_n^m|_n^m} & & & & \downarrow{\scriptstyle Id} \\
v.\hat{b}_{\mathbb{G}}.i_n^m.\hat{B}w_n^m|_n^m & \xrightarrow[Id]{} & v.\hat{b}_{\mathbb{G}}.\hat{B}w_n^m.j_n^m & \xrightarrow[v.\hat{b}_{\mathbb{G}}.\hat{B}r_n^m.j_n^m]{} & v.\hat{b}_{\mathbb{G}}.\hat{B}v.\hat{B}\hat{b}_{\mathbb{G}}.\hat{B}i_n^m.j_n^m
\end{array}$$

où $j_n^m : B^2\widehat{|_n^m}(\mathbb{G}) \to \hat{B}B\widehat{|_n^m}(\mathbb{G})$ est l'injection canonique.
On obtient finalement w et r en recollant les w_n^m et les r_n^m.

2.3 Construction de w_n^m et r_n^m

• Pour le moment, donnons uniquement les définitions de $|w_n^m|, Fl(w_n^m)$ et r_n^m par induction sur $m+n$. Nous renvoyons le lecteur à la sous-section suivante pour la vérification des axiomes $(H_0) \to (H_4)$.

• *Le cas* $n = 1$: Soit $a \in B|_1^m(|\mathbb{G}|)$. Dans ce cas $a = \eta_{|\mathbb{G}|}(c)$ où $c \in U(\mathbb{G})$. On pose alors $|w_1^m|(a) = c$ et
$r_1^m(a) = i(c)^{-1} : c \to |v|.\eta_{|\mathbb{G}|}(c) = |v|.b_{|\mathbb{G}|}(a)$.
Si $a \in B|_1^m Fl(\mathbb{G})$ on a aussi $a = \eta_{Fl\mathbb{G}}(f)$ où $f \in UFl(\mathbb{G})$. On pose alors $Flw_1^m(a) = f$.

• *Le cas* $m = 0$: On remarque que $B|_n^0 = B|_1^0$ et donc, on pose $|w_n^0| = |w_1^0|,\ Fl(w_n^0) = Fl(w_1^0)$ et $r_n^0 = r_1^0$.

• *Le cas* $n > 1$ *et* $m > 0$. Soit $a \in B|_n^m(|\mathbb{G}|)$ (resp. $f \in B|_n^m Fl(\mathbb{G})$).
- Si $\mathrm{L}(a) + \dim(a) < n + m$ (resp. $\mathrm{L}(f) + \dim(f) < n + m$), après avoir noté $n' = \mathrm{L}(a)$ et $m' = \dim(a)$ (resp. $n' = \mathrm{L}(f)$ et $m' = \dim(f)$), on pose :
$|w_n^m|(a) = |w_{n'}^{m'}|(a)$ et $r_n^m(a) = r_{n'}^{m'}(a)$ (resp. $Flw_n^m(f) = Flw_{n'}^{m'}(f)$).
- Si $\mathrm{L}(a) + \dim(a) = n + m$ (resp. $\mathrm{L}(f) + \dim(f) = n + m$). Alors $\mathrm{L}(a) = n$ et $\dim(a) = m$ (resp. $\mathrm{L}(f) = n$ et $\dim(f) = m$). Considérons plusieurs cas :
.. *cas où* a *est primitif* : Pour chaque $k \in [2]$, posons $a_k = \partial_{m-1}^k(a)$. On a $a_k \in B|_n^{m-1}|\mathbb{G}|$ et $(r_n^{m-1}(a_1), r_n^{m-1}(a_0)) \in \overline{\overline{UFl\mathbb{G}}}_{m-1}$, mais aussi $\mathrm{d}^1(r_n^{m-1}(a_1), r_n^{m-1}(a_0)) = (\partial_{m-1}^1, \partial_{m-1}^0)(|v|.b_{|\mathbb{G}|}(a))$ (où d^1 et d^0 désignent le but et la source d'une flèche). Alors $\mathbb{G}$ étant stable, on peut poser
$r_n^m(a) = ch(|v|.b_{|\mathbb{G}|}(a), (r_n^{m-1}(a_1), r_n^{m-1}(a_0)))$ (où $ch(-)$ désigne un choix d'isomorphisme provenant de la stabilité de $\mathbb{G}$) et $|w_n^m|(a) = \mathrm{d}^0 r_n^m(a)$.
.. *cas où* f *est primitif* : Pour chaque $k \in [2]$, posons $a^k = \mathrm{d}^k(f)$. Les a^k sont eux-même primitifs et $\mathrm{L}(a^k) = n,\ \dim(a^k) = m$. Alors on peut définir $Fl(w_n^m)(f)$ comme étant le composé suivant (dans $\mathbb{G}$) :

$$|w_n^m|(a^0) \xrightarrow{r_n^m(a^0)} |v|.b_{|\mathbb{G}|}(a^0) \xrightarrow{Flv.b_{Fl\mathbb{G}}(f)} |v|.b_{|\mathbb{G}|}(a^1) \xrightarrow{r_n^m(a^1)^{-1}} |w_n^m|(a^1)$$

.. *cas où* a *n'est pas primitif* : Soit $A \in B^2|\mathbb{G}|$ la décomposition primitive de a (voir 1.19). On voit que $A \in BB|_{n-1}^m|\mathbb{G}|$ et $B|w_{n-1}^m|(A) \in B|_{n-1}^m|\mathbb{G}|$.

On peut alors poser :

$$|w_n^m|(a) = |w_{n-1}^m|.B|w_{n-1}^m|(A)$$

D'un autre côté on voit que $Br_{n-1}^m(A) \in B|_{n-1}^m Fl(\mathbb{G})$ et que $B|v|.Bb_{|\mathbb{G}|}(A)$ est dans $B|_{n-1}^m|\mathbb{G}|$. On peut alors définir $r_n^m(a) : |w_n^m|(a) \to |v|.b_{|\mathbb{G}|}(a)$ comme étant le composé suivant dans $\mathbb{G}$:

$$\begin{array}{ccc}
|w_n^m|(a) & \xrightarrow{\;r_n^m(a)\;} & |v|.b_{|\mathbb{G}|}(a) \\
\downarrow{\scriptstyle Id} & & \uparrow{\scriptstyle Id} \\
|w_{n-1}^m|.B|w_{n-1}^m|(A) & & |v|.\mu_{|\mathbb{G}|}.b^2_{|\mathbb{G}|}(A) \\
{\scriptstyle Flw_{n-1}^m.Br_{n-1}^m(A)}\downarrow & & \uparrow{\scriptstyle (a.b^2_{|\mathbb{G}|}(A))^{-1}} \\
|w_{n-1}^m|.B|v|.Bb_{|\mathbb{G}|}(A) & & |v|.\omega|v|.b^2_{|\mathbb{G}|}(A) \\
& \searrow{\scriptstyle r_{n-1}^m.B|v|.Bb_{|\mathbb{G}|}(A)} & \uparrow{\scriptstyle Id} \\
& & |v|.b_{|\mathbb{G}|}.B|v|.Bb_{|\mathbb{G}|}(A)
\end{array}$$

.. *cas où* f *n'est pas primitif* : De même que pour a, on considère $F \in B^2Fl(\mathbb{G})$ la décomposition primitive de f. On voit encore que $F \in BB|_{n-1}^m Fl(\mathbb{G})$ et que $BFlw_{n-1}^m(F) \in B|_{n-1}^m Fl(\mathbb{G})$. On peut alors poser :

$$Flw_n^m(f) = Flw_{n-1}^m.BFlw_{n-1}^m(F).$$

2.4 Fin de la preuve

• Seule cette partie utilise tout le matériel donné dans [17]. Nous renvoyons donc le lecteur à la consultation de cet article.

• *Le cas* $n = 1$. La vérification des conditions $(H0) \to (H4)$ est longue mais elle se fait sans difficulté (Pour $(H4)$ on utilise un axiome des pseudo-algèbres).

• *Le cas* $m = 0$: Les conditions $(H0) \to (H4)$ ont déjà été vérifiées car $B|_n^0 = B|_1^0$.

• Dans le cas où $n > 1$ et $m > 0$, il nous faut montrer les conditions $(H0) \to (H4)$.

- $(H0)$: Se vérifie sans difficulté.

- $(H1)$: Le cas primitif est long à vérifier mais est sans difficulté particulière. Pour l'autre cas, soient $a \in B|_n^m(|\mathbb{G}|)$, $k \in [2]$, $p \in \mathbb{N}$ tels que $p < \dim(a)$ où a est non-primitif. On considère $A \in B^2|\mathbb{G}|$ la décomposition primitive de a. Alors $\partial_p^k|w_n^m|(a) = \partial_p^k|w_{n-1}^m|.B|w_{n-1}^m|(A) = |w_{n-1}^m|.B|w_{n-1}^m|\partial_p^k(A)$. Mais $\partial_p^k(A) \in BB|_{n-1}^p|\mathbb{G}|$ et $B|w_{n-1}^p|\partial_p^k(A) \in B|_{n-1}^p|\mathbb{G}|$. On en déduit que $|w_{n-1}^m|.B|w_{n-1}^m|\partial_p^k(A) = |w_{n-1}^p|.B|w_{n-1}^p|\partial_p^k(A) = |w_{n-1}^p|.\mu_{|\mathbb{G}|}.\partial_p^k(A) = |w_{n-1}^p|.\partial_p^k.\mu_{|\mathbb{G}|}(A) = |w_{n-1}^p|.\partial_p^k(a) = |w_n^m|.\partial_p^k(a)$. D'où l'identité voulue. De même lorsque $f \in B|_n^m Fl(\mathbb{G})$, on a $\partial_p^k Flw_n^m(f) = Flw_n^m \partial_p^k(f)$. Les autres vérifications se font sans difficulté particulière.

- $(H2)$: Pour le cas primitif, même remarque que pour $(H1)$. Pour l'autre cas, soient $a \in B|_n^m(|\mathbb{G}|)$, $k \in [2]$, $p \in \mathbb{N}$ tels que $p < \dim(a)$, où a est non-primitif. On considère $A \in B^2|\mathbb{G}|$ la décomposition primitive de a. On voit déjà que $\partial_p^k r_n^m(a) =$ $[a^{-1}.b^2_{|\mathbb{G}|}.\partial_p^k(A)] \circ [r_{n-1}^m.B|v|.Bb_{|\mathbb{G}|}.\partial_p^k(A)] \circ [Flw_{n-1}^m.Br_{n-1}^m.\partial_p^k(A)]$. Mais $\partial_p^k(A) \in BB|_{n-1}^p|\mathbb{G}|$, $Br_{n-1}^p.\partial_p^k(A) \in B|_{n-1}^p Fl(\mathbb{G})$ et $B|v|.Bb_{|\mathbb{G}|}.\partial_p^k(A) \in B|_{n-1}^p|\mathbb{G}|$ Donc $\partial_p^k r_n^m(a) =$ $[a^{-1}.b^2_{|\mathbb{G}|}.\partial_p^k(A)] \circ [r_{n-1}^m.B|v|.Bb_{|\mathbb{G}|}.\partial_p^k(A)] \circ [Flw_{n-1}^p.Br_{n-1}^p.\partial_p^k(A)] =$ $r_{n-1}^p.\mu_{|\mathbb{G}|}.\partial_p^k(A) = r_{n-1}^p.\partial_p^k.\mu_{|\mathbb{G}|}(A) = r_{n-1}^p.\partial_p^k(a) = r_{n-1}^m.\partial_p^k(a)$. Les autres vérifications se font sans difficulté particulière.

- $(H3)$: Soit $A \in B^2|_n^m|\mathbb{G}|$, on pose $a = \mu_{|\mathbb{G}|}(A)$. Le cas où $\mathrm{L}(a)+\dim(a) < n+m$ étant immédiat, on peut supposer que $\dim(a) = m$ et $\mathrm{L}(a) = n$.

.. Lorsque $\mathrm{sym}(a) = \square$ où a est primitif, alors $A = \eta_{B|\mathbb{G}|}(a)$ ou $A = B\eta_{|\mathbb{G}|}(a)$. On vérifie alors la condition voulue dans chacun de ces cas.

.. Lorsque a n'est pas primitif posons $\alpha = |A|$. On a $\alpha \leq B!_{|\mathbb{G}|}(a)$. Alors, le cas où $\mathrm{L}(\alpha) = 1$ étant immédiat, on peut supposer que $\mathrm{L}(\alpha) > 1$. Soient α_0 la composante primitive de α et A_0 la décomposition primitive de α. Soit aussi $\mathcal{A} \in B^3|\mathbb{G}|$ tel que $\mu_{B|\mathbb{G}|}(\mathcal{A}) = A$ et $B^2!_{B|\mathbb{G}|}(\mathcal{A}) = A_0$.

On vérifie que $\mathcal{A} \in BB^2|_{n-1}^m|\mathbb{G}|$. Posons aussi $A' = B\mu_{|\mathbb{G}|}(\mathcal{A})$, $A'_0 = B^2|w_{n-1}^m|(\mathcal{A})$ et $a' = B|w_n^m|(A)$. On voit alors que α_0 est la composante primitive de a et A' est la décomposition primitive de a, mais aussi $A', A'_0 \in BB|_{n-1}^m|\mathbb{G}|$ et $a' \in B|_n^m(|\mathbb{G}|)$. De plus α_0 est la composante primitive de a' et A'_0 est la décomposition primitive de a'. On peut maintenant écrire :
$|w_n^m|.\mu_{|\mathbb{G}|}|_n^m(A) = |w_n^m|(a) = |w_{n-1}^m|.B|w_{n-1}^m|(A') =$
$|w_{n-1}^m|.B|w_{n-1}^m|.B\mu_{|\mathbb{G}|}|_{n-1}^m(\mathcal{A}) = |w_{n-1}^m|.B|w_{n-1}^m|.BB|w_{n-1}^m|_{n-1}^m(\mathcal{A}) =$
$|w_{n-1}^m|.B|w_{n-1}^m|(A'_0) = |w_n^m|(a') = |w_n^m|.B|w_n^m|_n^m(A)$. De même, lorsque $F \in B^2|_n^m Fl(\mathbb{G})$, on vérifie que $Flw_n^m.\mu_{Fl\mathbb{G}}|_n^m(F) = Flw_n^m.BFlw_n^m|_n^m(F)$.

- $(H4)$: Soit $A \in B^2|_n^m(|\mathbb{G}|)$. On pose encore $a = \mu_{|\mathbb{G}|}(A)$. Le cas où $\mathrm{L}(a)+\dim(a) < n+m$ étant immédiat, on peut supposer que $\dim(a) = m$ et $\mathrm{L}(a) = n$.

.. Lorsque a est primitif, alors $A = \eta_{B|\mathbb{G}|}(a)$ ou $A = B\eta_{|\mathbb{G}|}(a)$. On vérifie alors la condition voulue dans chacun de ces cas (en utilisant les axiomes des pseudo-algèbres).

.. Lorsque a n'est pas primitif, reprenons les résultats donnés dans $(H3)$. Comme précédemment, on peut supposer que $\mathrm{L}(\alpha) > 1$. Alors on a les identités suivantes :
$[r_{n-1}^m.B(|v|.\omega|v|.b_{|\mathbb{G}|}^2)(\mathcal{A})]^{-1} \circ [a.\omega^2|v|.b_{|\mathbb{G}|}^3(\mathcal{A})] \circ [a.b_{|\mathbb{G}|}^2(A)] \circ [r_n^m.\mu_{|\mathbb{G}|}(A)] =$
$[r_{n-1}^m.B(|v|.\omega|v|.b_{|\mathbb{G}|}^2)(\mathcal{A})]^{-1} \circ [a.\omega^2|v|.b_{|\mathbb{G}|}^3(\mathcal{A})] \circ [a.\mu_{\omega|\mathbb{G}|}.b_{|\mathbb{G}|}^3(\mathcal{A})] \circ r_n^m(a) \underset{*1}{=}$
$[r_{n-1}^m.B(|v|.\omega|v|.b_{|\mathbb{G}|}^2)(\mathcal{A})]^{-1} \circ [Flv.\omega a.b_{|\mathbb{G}|}^3(\mathcal{A})] \circ [a.\omega\mu_{|\mathbb{G}|}.b_{|\mathbb{G}|}^3(\mathcal{A})] \circ r_n^m(a) =$
$[Flw_{n-1}^m.B(a.b_{|\mathbb{G}|}^2)(\mathcal{A})] \circ [r_{n-1}^m.B(|v|.\mu_{|\mathbb{G}|}.b_{|\mathbb{G}|}^2)(\mathcal{A})]^{-1} \circ [a.\omega\mu_{|\mathbb{G}|}.b_{|\mathbb{G}|}^3(\mathcal{A})] \circ$
$r_n^m(a) =$
$[Flw_{n-1}^m.B(a.b_{|\mathbb{G}|}^2)(\mathcal{A})] \circ [r_{n-1}^m.B(|v|.b_{|\mathbb{G}|})(A')]^{-1} \circ [a.b_{|\mathbb{G}|}^2(A')].r_n^m(a) \underset{*2}{=}$
$[Flw_{n-1}^m.B(a.b_{|\mathbb{G}|}^2)(\mathcal{A})] \circ [Flw_{n-1}^m.Br_{n-1}^m(A')] =$
$[Flw_{n-1}^m.B(a.b_{|\mathbb{G}|}^2)(\mathcal{A})] \circ [Flw_{n-1}^m.B(r_{n-1}^m.\mu_{|\mathbb{G}|})(\mathcal{A})] \underset{*3}{=}$
$[Flw_{n-1}^m.Br_{n-1}^m.B^2(|v|.b_{|\mathbb{G}|})(\mathcal{A})] \circ [Flw_{n-1}^m.BFlw_{n-1}^m.B^2r_{n-1}^m(\mathcal{A})] \underset{*4}{=}$
$[r_{n-1}^m.B(|v|.b_{|\mathbb{G}|}).B^2(|v|.b_{|\mathbb{G}|})(\mathcal{A})]^{-1} \circ [a.b_{|\mathbb{G}|}^2.B^2(|v|.b_{|\mathbb{G}|})(\mathcal{A})] \circ$
$[r_n^m.\mu_{|\mathbb{G}|}.B^2(|v|.b_{|\mathbb{G}|})(\mathcal{A})] \circ [Flw_n^m.\mu_{Fl\mathbb{G}}.B^2r_n^m(\mathcal{A})] =$
$[r_{n-1}^m.B(|v|.\omega|v|.b_{|\mathbb{G}|}^2)(\mathcal{A})]^{-1} \circ [a.\omega^2|v|.b_{|\mathbb{G}|}^3(\mathcal{A})] \circ$
$[r_n^m.B(|v|.b_{|\mathbb{G}|})(A)] \circ [Flw_n^m.Br_n^m(A)]$
$(*1)$ Par un axiome des pseudo-algèbres.

$(*2)$ Par définition de r_n^m dans le cas non-primitif car A' est la décomposition primitive de a.
$(*3)$ Par hypothèse d'induction.
$(*4)$ Par définition de r_n^m dans le cas non-primitif car $B^2(|v|.b_{|\mathbb{G}|})(\mathcal{A})$ est la décomposition primitive de $B(|v|.b_{|\mathbb{G}|})(A)$ (lorsque ce dernier est non-primitif).
On en déduit que
$[a.b^2_{|\mathbb{G}|}(A)] \circ [r_n^m.\mu_{|\mathbb{G}|}(A)] = [r_n^m.B|v|.Bb_{|\mathbb{G}|})(A)] \circ [Flw_n^m.Br_n^m(A)] =$
$[Flv.b_{Fl\mathbb{G}}.Br_n^m(A)] \circ [r_n^m.B|w_n^m|(A)]$ qui est l'identité voulue.

• Les trois familles $(|w_n^m|)$, (Flw_n^m) et (r_n^m) ayant été construites et satisfaisant les conditions de $(H0)$ à $(H4)$, on construit facilement un foncteur globulaire $w : \hat{B}(\mathbb{G}) \to \mathbb{G}$ et une transformation naturelle globulaire $r : w \to v.\hat{b}_{\mathbb{G}}$ vérifiant les conditions voulues, ce qui achève la preuve du Lemme clé (voir 1.5).

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A 2CAT-INSPIRED MODEL STRUCTURE FOR DOUBLE CATEGORIES

Lyne Moser, Maru Sarazola, Paula Verdugo

Résumé. On construit une stucture de modèles sur la catégorie DblCat des doubles catégories et doubles foncteurs. Contrairement aux structures de modèles existantes sur les doubles catégories, ces nouvelles structures de modèles recouvrent la structure de modèles de Lack sur les 2-catégories via le plongement horizontal $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$. Ce dernier est à la fois un adjoint de Quillen à gauche et à droite, et est homotopiquement plein et fidèle. De plus, on obtient un enrichissement sur $2\mathrm{Cat}$ de notre structure de modèles sur DblCat, en utilisant une variante du produit tensoriel de Gray.

Sous certaines conditions, on prouve un théorème de Whitehead qui caractérise nos équivalences faibles comme étant les doubles foncteurs qui admettent un pseudo-inverse à équivalence horizontale pseudo-naturelle près.

Abstract. We construct a model structure on the category DblCat of double categories and double functors. Unlike previous model structures for double categories, it recovers the homotopy theory of 2-categories through the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$, which is both left and right Quillen, and homotopically fully faithful. Furthermore, we show that Lack's model structure on $2\mathrm{Cat}$ is both left- and right-induced along $\mathbb{H}$ from our model structure on DblCat. In addition, we obtain a $2\mathrm{Cat}$-enrichment of our model structure on DblCat, by using a variant of the Gray tensor product.

Under certain conditions, we prove a Whitehead theorem, characterizing our weak equivalences as the double functors which admit an inverse pseudo double functor up to horizontal pseudo natural equivalence.

Keywords. Double categories, 2-categories, homotopy theory, enriched

model categories, Whitehead theorem.
Mathematics Subject Classification (2020). 18N10; 18N40; 18D20.

Contents

1. Introduction

In category theory as well as homotopy theory, we strive to find the correct notion of "sameness", often with a specific context or perspective in mind. When working with categories themselves, it is commonly agreed that having an isomorphism between categories is much too strong a requirement, and we instead concur that the right condition to demand is the existence of an *equivalence* of categories.

There are many ways one can justify this in practice, but, at heart, it is due to the fact that the category Cat of categories and functors actually forms a 2-category, with 2-cells given by the natural transformations. Therefore, instead of asking that a functor $F\colon \mathcal{A} \to \mathcal{B}$ has an inverse $G\colon \mathcal{B} \to \mathcal{A}$ such that their composites are *equal* to the identities, it is more natural to ask for the existence of *natural isomorphisms* $\mathrm{id}_{\mathcal{A}} \cong GF$ and $FG \cong \mathrm{id}_{\mathcal{B}}$. In particular, this characterizes F as a functor that is surjective on objects up to isomorphism, and fully faithful on morphisms.

Ever since Quillen's seminal work [21], and even more so in the last two decades, we have come to expect that any reasonable notion of equivalence in a category should lend itself to defining the class of weak equivalences of a model structure. This is in fact the case of the categorical equivalences: the category Cat can be endowed with a model structure, called the *canonical model structure*, in which the weak equivalences are precisely the equivalences of categories.

Going one dimension up and focusing on 2-categories, the 2-functors themselves now form a 2-category, with higher cells given by the pseudo natural transformations, and the so-called modifications between them. We can then define a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ to be a *biequivalence* if it has an inverse $G\colon \mathcal{B} \to \mathcal{A}$ together with pseudo natural equivalences $\mathrm{id}_{\mathcal{A}} \simeq GF$ and $FG \simeq \mathrm{id}_{\mathcal{B}}$, i.e., equivalences in the corresponding 2-categories of 2-dimensional functors. Note that this inverse G is in general a pseudo functor rather than a 2-functor. Furthermore, a Whitehead theorem for 2-categories [14, Theorem 7.4.1] is available, and characterizes the biequivalences as the 2-functors that are surjective on objects up to equivalence, full on morphisms up to invertible 2-cell, and fully faithful on 2-cells.

As in the case of the equivalences of categories, the biequivalences of 2-categories are part of the data of a model structure. Indeed, in [15, 16], Lack defines a model structure on the category $2\mathrm{Cat}$ of 2-categories and 2-functors in which the weak equivalences are precisely the biequivalences; we henceforth refer to it as the *Lack model structure*. In particular, the canonical homotopy theory of categories embeds reflectively in this homotopy theory of 2-categories.

In this paper, we consider another type of 2-dimensional objects, called *double categories*, which have both horizontal and vertical morphisms between pairs of objects, related by 2-dimensional cells called *squares*. These are more structured than 2-categories, in the sense that a 2-category $\mathcal{A}$ can be seen as a horizontal double category $\mathbb{H}\mathcal{A}$ with only trivial vertical morphisms. As a consequence, the study of various notions of 2-category theory benefits from a passage to double categories. For example, a 2-limit of a 2-functor F does not coincide with a 2-terminal object in the slice 2-category of cones, as shown in [2, Counter-example 2.12]. However, by considering the 2-functor F as a horizontal double functor $\mathbb{H}F$, Grandis and Paré prove that a 2-limit of F is precisely a double terminal object in the slice double category of cones over $\mathbb{H}F$; see [9, 11, §4.2] and [8, Theorem 5.6.5].

This horizontal embedding of 2-categories into double categories is fully faithful, and we expect to have a homotopy theory of double categories that contains that of 2-categories; constructing such a homotopy theory is the aim of this paper.

The idea of defining a model structure on the category of double categories is scarcely a new one. In [5], Fiore and Paoli construct a Thomason model structure on the category DblCat of double categories and double functors (more precisely, on the category of n-fold categories), and in [6], Fiore, Paoli, and Pronk construct several categorical model structures on DblCat. However, the horizontal embedding of 2-categories does not induce a Quillen pair between the Lack model structure on $2\mathrm{Cat}$ and any of these model structures on DblCat; this follows from Lemma 8.8. Some intuition is provided by the fact that their categorical model structures on DblCat are constructed from the canonical model structure on Cat. As a result, the weak equivalences in each of these model structures induce two equivalences of categories: one between the categories of objects and horizontal morphisms, and one between the categories of vertical morphisms and squares. However, a biequivalence between 2-categories does not generally induce an equivalence between the underlying categories. Therefore, the horizontal embedding of $2\mathrm{Cat}$ into DblCat will not preserve weak equivalences.

In order to remedy this loss of higher data, we aim to extract from a double category $\mathbb{A}$ two 2-categories whose underlying categories are precisely the ones mentioned above. First, we can promote the underlying category of objects and horizontal morphisms of $\mathbb{A}$ to a 2-category by using the right adjoint to the horizontal embedding $\mathbb{H}$: this is a well-known construction given by the underlying horizontal 2-category $\mathbf{H}\mathbb{A}$, whose 2-cells are given by those squares of $\mathbb{A}$ with trivial vertical boundaries. As shown by Ehresmann and Ehresmann in [4], the category DblCat is cartesian closed, and we denote by $[-,-]$ its internal hom double categories. We can then alternatively describe the underlying horizontal 2-category $\mathbf{H}\mathbb{A}$ as the 2-category $\mathbf{H}[\mathbb{1},\mathbb{A}]$, where $\mathbb{1}$ denotes the terminal category.

From this perspective, the category of vertical morphisms and squares can be seen as the underlying horizontal category of the double category $[\mathbb{V}2, \mathbb{A}]$, where $\mathbb{V}2$ is the free double category on a vertical morphism. To promote this to a 2-category we can simply consider instead the underlying horizontal 2-category $\mathbf{H}[\mathbb{V}2, \mathbb{A}]$; this defines a new functor $\mathcal{V}$ that sends a double category $\mathbb{A}$ to a 2-category $\mathcal{V}\mathbb{A}$ of vertical morphisms, squares, and 2-cells as described in Definition 2.11.

Using these constructions, we introduce a new notion of weak equivalences between double categories, that we suggestively call *double biequivalences*; these are given by the double functors F such that the induced 2-functors $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences in $2\mathrm{Cat}$. This provides a 2-categorical analogue of notions of equivalences between double categories already present in the literature. Notably, double biequivalences are the natural 2-categorical version of equivalences described by Grandis in [8, Theorem 4.4.5 (iv)], which are precisely the double functors inducing equivalences between the categories of objects and horizontal morphisms, and the categories of vertical morphisms and squares.

Since biequivalences can be characterized as the 2-functors which are surjective on objects up to equivalence, full on morphisms up to invertible 2-cell, and fully faithful on 2-cells, our double biequivalences admit a similar description. To give such a description, we introduce new notions of weak invertibility for horizontal morphisms and squares in a double category $\mathbb{A}$; namely, those of *horizontal equivalences* and *weakly horizontally invertible squares*, which correspond to the equivalences in the 2-categories $\mathbf{H}\mathbb{A}$ and $\mathcal{V}\mathbb{A}$, respectively. These notions were independently developed by Grandis and Paré in [10, §2], where the weakly horizontally invertible squares are called *equivalence cells*. Now the double biequivalences can be described as the double functors which are surjective on objects up to horizontal equivalence, full on horizontal morphisms up to vertically invertible square, surjective on vertical morphisms up to weakly horizontally invertible square, and fully faithful on squares.

The double biequivalences are designed in such a way that a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a biequivalence if and only if its associated horizontal double functor $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ is a double biequivalence. This can be seen as a first step towards showing that the homotopy theory of 2-categories sits inside that of double categories. Note that "surjectivity" rather than "fullness" on vertical morphisms is necessary to achieve our goal of defining a model structure on DblCat compatible with the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$. Indeed, as we want $\mathbb{H}$ to preserve weak equivalences, and as the 2-category E_{adj} given by the free-living adjoint equivalence is biequivalent to the terminal category $\mathbb{1}$, the double functor $\mathbb{H}E_{\mathrm{adj}} \to \mathbb{1}$ should be a weak equivalence in DblCat. It is then straightforward to check that such a double functor cannot be full on vertical morphisms, as there is no vertical morphism between the two distinct objects of the horizontal double category $\mathbb{H}E_{\mathrm{adj}}$.

Our first main result, Theorem 3.18, provides the desired model structure on the category of double categories.

Theorem A. *Consider the adjunction*

$$2\mathrm{Cat} \times 2\mathrm{Cat} \underset{(\mathbf{H}, \mathcal{V})}{\overset{\mathbb{H} \sqcup \mathbb{L}}{\rightleftarrows}} \mathrm{DblCat}\,, \quad \bot$$

where each copy of $2\mathrm{Cat}$ *is endowed with the Lack model structure. Then the right-induced model structure on* DblCat *exists. In particular, a double functor is a weak equivalence in this model structure if and only if it is a double biequivalence.*

Since the Lack model structure on $2\mathrm{Cat}$ is cofibrantly generated, so is the model structure on DblCat constructed above. Moreover, every double category is fibrant, since all objects are fibrant in $2\mathrm{Cat}$.

By taking a closer look at the homotopy equivalences in our model structure on DblCat, we identify them as the double functors $F\colon \mathbb{A} \to \mathbb{B}$ such that there is a double functor $G\colon \mathbb{B} \to \mathbb{A}$ and two horizontal pseudo natural equivalences $\mathrm{id}_{\mathbb{A}} \simeq GF$ and $FG \simeq \mathrm{id}_{\mathbb{B}}$. In particular, the usual Whitehead theorem for model structures (see [3, Lemma 4.24]) allows us to identify the double biequivalences between cofibrant double categories as the homotopy equivalences described above.

In fact, we show in Theorem 5.13 that a more lax version of this result, involving a horizontally pseudo double functor G, holds for an even larger class of double categories containing the cofibrant objects; this mirrors the definition of biequivalences in $2\mathrm{Cat}$, which further supports the fact that our double biequivalences provide a good notion of weak equivalences between double categories. As a corollary, we retrieve the Whitehead theorem for 2-categories mentioned above.

Theorem B. *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be double categories such that the underlying vertical category* $U\mathbf{V}\mathbb{B}$ *is a disjoint union of copies of* $\mathbb{1}$ *and* $\mathbb{2}$. *Then a double functor* $F\colon \mathbb{A} \to \mathbb{B}$ *is a double biequivalence if and only if there is a normal horizontally pseudo double functor* $G\colon \mathbb{B} \to \mathbb{A}$, *and horizontal pseudo natural equivalences* $\eta\colon \mathrm{id}_{\mathbb{A}} \simeq GF$ *and* $\epsilon\colon FG \simeq \mathrm{id}_{\mathbb{B}}$.

This Whitehead Theorem is reminiscent of a result by Grandis in [8, Theorem 4.4.5] which characterizes the 1-categorical version of our double biequivalences under a different assumption on the double categories involved; namely, that of horizontal invariance. In [20, Definition 2.10], the authors introduce a notion of *weakly horizontally invariant* double categories, and use them to prove yet another Whitehead Theorem for double biequivalences; see [20, Theorem 8.1]. Moreover, the weakly horizontally invariant double categories are identified as the fibrant objects in a different model structure on DblCat, whose study is the purpose of [20].

We now address our original motivation of constructing a homotopy theory for double categories that contains that of 2-categories through the horizontal embedding. Our model structure on DblCat successfully achieves this goal, and moreover, exhibits the greatest possible compatibility with respect to the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ that one could hope for, as studied in Section 6.

Theorem C. *The adjunctions*

$$2\mathrm{Cat} \;\underset{\mathbf{H}}{\overset{L}{\underset{\perp}{\overset{\perp}{\leftarrow\!\!\xrightarrow{\;\mathbb{H}\;}\!\!\leftarrow}}}}\; \mathrm{DblCat}$$

are both Quillen pairs between the Lack model structure on $2\mathrm{Cat}$ *and the model structure on* DblCat *of Theorem A. Moreover, the functor* $\mathbb{H}$ *is homotopically fully faithful, and the Lack model structure on* $2\mathrm{Cat}$ *is both left- and right-induced from our model structure on* DblCat *along* $\mathbb{H}$.

As a consequence, a 2-functor F is a cofibration, fibration or weak equivalence in $2\mathrm{Cat}$ if and only if the double functor $\mathbb{H}F$ is a cofibration, fibration or weak equivalence in DblCat, respectively.

Having established the exceptional behavior of our model structure with the horizontal embedding, we want to further investigate its relation with the Lack model structure on $2\mathrm{Cat}$. Lack shows in [15] that the model structure on $2\mathrm{Cat}$ is monoidal with respect to the Gray tensor product. In the double categorical setting, there is an analogous monoidal structure on DblCat given by the Gray tensor product constructed by Böhm in [1]. However, this monoidal structure is not compatible with our model structure on DblCat (see Remark 7.3), since it treats the vertical and horizontal directions symmetrically, while our model structure does not. Nevertheless, restricting this Gray tensor product for double categories in one of the variables to $2\mathrm{Cat}$ via $\mathbb{H}$ removes this symmetry and provides an enrichment of DblCat over $2\mathrm{Cat}$ that is compatible with our model structure. More precisely, this enrichment is given by the hom 2-categories of double functors, horizontal pseudo natural transformations, and modifications between them, denoted by $\mathbf{H}[-,-]_{\mathrm{ps}}$.

Theorem D. *The model structure on* DblCat *of Theorem A is a* $2\mathrm{Cat}$*-enriched model structure, where the enrichment is given by* $\mathbf{H}[-,-]_{\mathrm{ps}}$.

The fact that horizontal pseudo natural transformations play a key role was to be expected, since they are the type of transformations that detect our weak equivalences, as established in our version of the Whitehead theorem above.

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2. Double categorical preliminaries

In this section, we recall the basic notions about double categories, and also introduce non-standard definitions and terminology that will be used throughout the paper. The reader familiar with double categories may wish to jump directly to Definition 2.11.

Definition 2.1. A **double category** $\mathbb{A}$ consists of objects, horizontal morphisms, vertical morphisms, and squares, which we denote by

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle u}\big\downarrow\bullet & \alpha & \bullet\big\downarrow{\scriptstyle v} \\ A' & \xrightarrow[b]{} & B' \end{array}$$

with horizontal compositions for horizontal morphisms and squares and vertical compositions for vertical morphisms and squares, which are associative and unital, and such that the horizontal and vertical compositions of squares satisfy the interchange law.

We write id_A and e_A for the horizontal and vertical identity at an object A, e_a for the vertical identity square at a horizontal morphism a, and id_u for the horizontal identity square at a vertical morphism u.

Definition 2.2. Let $\mathbb{A}$, $\mathbb{B}$ be double categories. A **double functor** $F\colon \mathbb{A} \to \mathbb{B}$ consists of maps on objects, horizontal morphisms, vertical morphisms, and squares, which are compatible with domains and codomains and preserve all double categorical compositions and identities strictly.

Notation 2.3. We write DblCat for the category of double categories and double functors.

Proposition 2.4 ([6, Proposition 2.11])**.** *The category* DblCat *is cartesian closed. We denote by* $[\mathbb{A}, \mathbb{B}]$ *the* ***hom double category*** *for* $\mathbb{A}, \mathbb{B} \in \mathrm{DblCat}$. *In particular, for every double category* $\mathbb{A}$*, there is an adjunction*

$$\mathrm{DblCat} \underset{[\mathbb{A}, -]}{\overset{- \times \mathbb{A}}{\rightleftarrows}} \mathrm{DblCat}\,. \quad (\perp)$$

There is another monoidal structure on the category of double categories introduced by Böhm in [1], similar to the Gray tensor product for 2-categories.

Proposition 2.5 ([1, §3])**.** *There is a symmetric monoidal structure on the category* DblCat *given by the Gray tensor product*

$$\otimes_{\mathrm{Gr}} \colon \mathrm{DblCat} \times \mathrm{DblCat} \to \mathrm{DblCat}.$$

Moreover, this monoidal structure is closed and we denote by $[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$ *the* ***pseudo hom double category*** *for* $\mathbb{A}, \mathbb{B} \in \mathrm{DblCat}$. *In particular, for every double category* $\mathbb{A}$*, there is an adjunction*

$$\mathrm{DblCat} \underset{[\mathbb{A}, -]_{\mathrm{ps}}}{\overset{- \otimes_{\mathrm{Gr}} \mathbb{A}}{\rightleftarrows}} \mathrm{DblCat}\,. \quad (\perp)$$

Remark 2.6. Given double categories $\mathbb{A}$ and $\mathbb{B}$, a horizontal morphism in the pseudo hom $[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$ is a **horizontal pseudo natural transformation** $h \colon F \Rightarrow G \colon \mathbb{A} \to \mathbb{B}$. It consists of

(i) a horizontal morphism $h_A \colon FA \to GA$ in $\mathbb{B}$, for each object $A \in \mathbb{A}$,

(ii) a square $h_u \colon (Fu \,{}^{h_A}_{h_{A'}}\, Gu)$ in $\mathbb{B}$, for each vertical morphism $u \colon A \to\!\!\!\!\!\bullet\; A'$ in $\mathbb{A}$, and

(iii) a vertically invertible square $h_a\colon (e_{FA}\ {}^{(Ga)h_A}_{h_B(Fa)}\ e_{GB})$ in $\mathbb{B}$, for each horizontal morphism $a\colon A \to B$ in $\mathbb{A}$, expressing a pseudo naturality condition for horizontal morphisms.

These assignments of squares are functorial with respect to compositions of horizontal and vertical morphisms, and these data satisfy a naturality condition with respect to squares.

In comparison, the horizontal morphisms in the (strict) hom $[\mathbb{A}, \mathbb{B}]$ are horizontal pseudo natural transformations h such that the vertically invertible squares h_a are identity squares for all a. See [8, §3.2.7] for an explicit description of the data of the hom double category $[\mathbb{A}, \mathbb{B}]$ and [8, §3.8] or [1, §2.2] for the pseudo hom $[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$.

As mentioned in the introduction, there is a full horizontal embedding of the category $2\mathrm{Cat}$ of 2-categories and 2-functors into DblCat.

Definition 2.7. The **horizontal embedding functor** $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ is defined as follows. It takes a 2-category $\mathcal{A}$ to the double category $\mathbb{H}\mathcal{A}$ having the same objects as $\mathcal{A}$, the morphisms of $\mathcal{A}$ as horizontal morphisms, only identities as vertical morphisms, and squares

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \| & \alpha & \| \\ A & \xrightarrow[b]{} & B \end{array}$$

given by the 2-cells $\alpha\colon a \Rightarrow b$ in $\mathcal{A}$. It sends a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ to the double functor $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ that acts as F does on the corresponding data.

The functor $\mathbb{H}$ admits a right adjoint given by the following.

Definition 2.8. We define the functor $\mathbf{H}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$. It takes a double category $\mathbb{A}$ to its **underlying horizontal 2-category $\mathbf{H}\mathbb{A}$**, i.e., the 2-category whose objects are the objects of $\mathbb{A}$, whose morphisms are the horizontal morphisms of $\mathbb{A}$, and whose 2-cells $\alpha\colon a \Rightarrow b$ are given by the squares in $\mathbb{A}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \Vert\!\!\bullet & \alpha & \Vert\!\!\bullet \\ A & \xrightarrow[b]{} & B \end{array}.$$

It sends a double functor $F\colon \mathbb{A} \to \mathbb{B}$ to the 2-functor $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ that acts as F does on the corresponding data.

Proposition 2.9 ([6, Proposition 2.5])**.** *The functors* $\mathbf{H}$ *and* $\mathbb{H}$ *form an adjunction*

$$2\mathrm{Cat} \underset{\mathbf{H}}{\overset{\mathbb{H}}{\rightleftarrows}} \mathrm{DblCat}\,. \quad (\perp)$$

Moreover, the unit $\eta\colon \mathrm{id} \Rightarrow \mathbf{H}\mathbb{H}$ *is the identity.*

Remark 2.10. We can also define a functor $\mathbb{V}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$, sending a 2-category to its associated vertical double category with only trivial horizontal morphisms, and a functor $\mathbf{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$, sending a double category to its underlying vertical 2-category. These form an adjunction $\mathbb{V} \dashv \mathbf{V}$.

We now introduce a new functor between DblCat and $2\mathrm{Cat}$ that extracts, from a double category, a 2-category whose objects and morphisms are the vertical morphisms and squares; this is the functor $\mathcal{V}$ mentioned in the introduction. In order to do this, we use the category $\mathbb{V}\mathbb{2}$, where $\mathbb{2}$ is the (2-)category $\{0 \to 1\}$ free on a morphism. This double category $\mathbb{V}\mathbb{2}$ is therefore the double category free on a vertical morphism.

Definition 2.11. We define the functor $\mathcal{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$ as the composite

$$\mathrm{DblCat} \xrightarrow{[\mathbb{V}\mathbb{2},-]} \mathrm{DblCat} \xrightarrow{\mathbf{H}} 2\mathrm{Cat}.$$

Explicitly, it sends a double category $\mathbb{A}$ to the 2-category $\mathcal{V}\mathbb{A} = \mathbf{H}[\mathbb{V}\mathbb{2}, \mathbb{A}]$ given by the following data.

(i) An object in $\mathcal{V}\mathbb{A}$ is a vertical morphism $u\colon A \,\bullet\!\!\!\to A'$ in $\mathbb{A}$.

(ii) A morphism $(a, b, \alpha)\colon u \to v$ is a square in $\mathbb{A}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle u}\downarrow\!\bullet & \alpha & \bullet\!\downarrow{\scriptstyle v} \\ A' & \xrightarrow[b]{} & B'. \end{array}$$

(iii) A 2-cell $(\sigma_0, \sigma_1)\colon (a, b, \alpha) \Rightarrow (c, d, \beta)$ consists of two squares σ_0 and σ_1 in $\mathbb{A}$ such that the following pasting equality holds.

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \| & \sigma_0 & \| \\ A & \xrightarrow{c} & B \\ {\scriptstyle u}\downarrow\!\bullet & \beta & \bullet\!\downarrow{\scriptstyle v} \\ A' & \xrightarrow[d]{} & B' \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle u}\downarrow\!\bullet & \alpha & \bullet\!\downarrow{\scriptstyle v} \\ A' & \xrightarrow{b} & B' \\ \| & \sigma_1 & \| \\ A' & \xrightarrow[d]{} & B' \end{array}$$

By Propositions 2.4 and 2.9, we obtain the following.

Proposition 2.12. *The functor $\mathcal{V}$ has a left adjoint $\mathbb{L}$*

$$2\mathrm{Cat} \underset{\mathcal{V}}{\overset{\mathbb{L}}{\rightleftarrows}} \mathrm{DblCat} \quad (\mathbb{L} \dashv \mathcal{V})$$

given by $\mathbb{L} = \mathbb{H}(-) \times \mathbb{V}2$.

Notation 2.13. We denote by $\otimes_2\colon 2\mathrm{Cat} \times 2\mathrm{Cat} \to 2\mathrm{Cat}$ the Gray tensor product for 2-categories. It makes $2\mathrm{Cat}$ into a closed symmetric monoidal category with internal homs given by $\mathrm{Ps}[\mathcal{A}, \mathcal{B}]$: the 2-category of 2-functors from $\mathcal{A}$ to $\mathcal{B}$, pseudo natural transformations, and modifications.

The following technical result, which exhibits the behavior of the functors $\mathbb{H}$, $\mathbf{H}$, and $\mathcal{V}$ with respect to pseudo homs, will be of use when we prove the existence of the desired model structure.

Lemma 2.14. *Let $\mathcal{B}$ be a 2-category and $\mathbb{A}$ be a double category. Then there are isomorphisms of 2-categories*

$$\mathbf{H}[\mathbb{H}\mathcal{B},\mathbb{A}]_{\mathrm{ps}} \cong \mathrm{Ps}[\mathcal{B},\mathbf{H}\mathbb{A}] \ \textit{and} \ \mathcal{V}[\mathbb{H}\mathcal{B},\mathbb{A}]_{\mathrm{ps}} \cong \mathrm{Ps}[\mathcal{B},\mathcal{V}\mathbb{A}]$$

natural in $\mathcal{B}$ and $\mathbb{A}$.

Proof. We first consider the isomorphism $\mathbf{H}[\mathbb{H}\mathcal{B},\mathbb{A}]_{\mathrm{ps}} \cong \mathrm{Ps}[\mathcal{B},\mathbf{H}\mathbb{A}]$. On objects, this follows from the adjunction $\mathbb{H} \dashv \mathbf{H}$ given in Proposition 2.9. On morphisms, as there are no non-trivial vertical morphisms in $\mathbb{H}\mathcal{B}$, horizontal pseudo natural transformations out of $\mathbb{H}\mathcal{B}$ are canonically the same as pseudo natural transformations out of $\mathcal{B}$. The argument for 2-morphisms is similar.

For the second isomorphism, first note that $[\mathbb{V}2,\mathbb{A}]_{\mathrm{ps}} = [\mathbb{V}2,\mathbb{A}]$, since there are no non-trivial horizontal morphisms in $\mathbb{V}2$, and therefore horizontal pseudo natural transformations out of $\mathbb{V}2$ correspond to horizontal (strict) natural transformations out of $\mathbb{V}2$. Therefore, we have that

$$\begin{aligned}\mathcal{V}[\mathbb{H}\mathcal{B},\mathbb{A}]_{\mathrm{ps}} &= \mathbf{H}[\mathbb{V}2,[\mathbb{H}\mathcal{B},\mathbb{A}]_{\mathrm{ps}}]_{\mathrm{ps}} \cong \mathbf{H}[\mathbb{H}\mathcal{B},[\mathbb{V}2,\mathbb{A}]_{\mathrm{ps}}]_{\mathrm{ps}} \\ &\cong \mathrm{Ps}[\mathcal{B},\mathbf{H}[\mathbb{V}2,\mathbb{A}]_{\mathrm{ps}}] = \mathrm{Ps}[\mathcal{B},\mathcal{V}\mathbb{A}],\end{aligned}$$

where the first isomorphism follows from the symmetry of the Gray tensor product on DblCat; see Proposition 2.5 below. □

We conclude this section by introducing new notions of weak invertibility for horizontal morphisms and squares in a double category, together with some technical results that will be of use later in the paper. We do not prove these results here, but instead refer the reader to work by the first author [18, Appendix A]. These notions and results were independently developed by Grandis and Paré in [10, §2].

Definition 2.15. A horizontal morphism $a\colon A \to B$ in a double category $\mathbb{A}$ is a **horizontal equivalence** if it is an equivalence in the 2-category $\mathbf{H}\mathbb{A}$.

Definition 2.16. A square $\alpha\colon (u \ {}^{a}_{b} \ v)$ in a double category $\mathbb{A}$ is **weakly horizontally invertible** if it is an equivalence in the 2-category $\mathcal{V}\mathbb{A}$. See [20, Definition 2.5] for a more detailed description.

Remark 2.17. In particular, the horizontal boundaries a and b of a weakly horizontally invertible square α are horizontal equivalences, which we refer to as the *horizontal equivalence data* of α.

Since any equivalence in a 2-category can be promoted to an adjoint equivalence (see, for example, [22, Lemma 2.1.11]), we get the following result.

Lemma 2.18. *Every horizontal equivalence can be promoted to a horizontal* adjoint *equivalence. Similarly, every weakly horizontally invertible square can be promoted to one with horizontal* adjoint *equivalence data.*

Finally, we conclude with a result concerning weakly horizontally invertible squares.

Lemma 2.19 ([18, Lemma A.2.1])**.** *A square whose horizontal boundaries are horizontal equivalences, and whose vertical boundaries are identities, is weakly horizontally invertible if and only if it is vertically invertible.*

Remark 2.20. It follows that, for a 2-category $\mathcal{A}$, a weakly horizontally invertible square in the double category $\mathbb{H}\mathcal{A}$ corresponds to an invertible 2-cell in $\mathcal{A}$.

3. Model structure for double categories

This section contains our first main result, which proves the existence of a model structure on DblCat constructed as a right-induced model structure along the functor $(\mathbf{H}, \mathcal{V})\colon \mathrm{DblCat} \to 2\mathrm{Cat} \times 2\mathrm{Cat}$, where both copies of $2\mathrm{Cat}$ are endowed with the Lack model structure.

An analogue construction could be done for weak double categories and strict double functors, by considering Lack's model structure on bicategories and strict functors. These enjoy the same relations as the ones studied in this paper; we exclude them for expositional purposes.

3.1 Lack model structure on $2\mathrm{Cat}$

We start by recalling the main features of Lack's model structure on $2\mathrm{Cat}$; see [15, 16]. Its class of weak equivalences is given by the *biequivalences*, and we refer to the fibrations in this model structure as *Lack fibrations*.

Definition 3.1. A 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a **biequivalence** if (b1) for every object $B \in \mathcal{B}$, there is an object $A \in \mathcal{A}$ and an equivalence $B \xrightarrow{\simeq} FA$ in $\mathcal{B}$,

(b2) for every pair of objects $A, C \in \mathcal{A}$ and every morphism $b\colon FA \to FC$ in $\mathcal{B}$, there is a morphism $a\colon A \to C$ in $\mathcal{A}$ and an invertible 2-cell $b \cong Fa$ in $\mathcal{B}$, and

(b3) for every pair of morphisms $a, c\colon A \to C$ in $\mathcal{A}$ and every 2-cell $\beta\colon Fa \Rightarrow Fc$ in $\mathcal{B}$, there is a unique 2-cell $\alpha\colon a \Rightarrow c$ in $\mathcal{A}$ such that $F\alpha = \beta$.

Definition 3.2. A 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a **Lack fibration** if

(f1) for every object $C \in \mathcal{A}$ and every equivalence $b\colon B \xrightarrow{\simeq} FC$ in $\mathcal{B}$, there is an equivalence $a\colon A \xrightarrow{\simeq} C$ in $\mathcal{A}$ such that $Fa = b$, and

(f2) for every morphism $c\colon A \to C$ in $\mathcal{A}$ and every invertible 2-cell $\beta\colon b \cong Fc$ in $\mathcal{B}$, there is an invertible 2-cell $\alpha\colon a \cong c$ in $\mathcal{A}$ such that $F\alpha = \beta$.

Theorem 3.3 ([16, Theorem 4])**.** *There is a cofibrantly generated model structure on* $2\mathrm{Cat}$*, called the* Lack model structure*, in which the weak equivalences are the biequivalences and the fibrations are the Lack fibrations.*

Remark 3.4. Note that every 2-category is fibrant in the Lack model structure.

Recall that a *monoidal model category* is a closed monoidal category which admits a model structure compatible with the monoidal structure; see [19, Definition 5.1]. The Lack model structure on $2\mathrm{Cat}$ is monoidal with respect to the Gray tensor product.

Theorem 3.5 ([15, Theorem 7.5])**.** *The category* $2\mathrm{Cat}$ *endowed with the Lack model structure is a monoidal model category with respect to the closed symmetric monoidal structure given by the Gray tensor product.*

3.2 Constructing the model structure for DblCat

We introduce double biequivalences in DblCat inspired by the definition of biequivalences in $2\mathrm{Cat}$. Our convention of regarding 2-categories as horizontal double categories justifies the choice of directions when emulating the definition of biequivalences in the context of double categories.

Definition 3.6. A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a **double biequivalence** if

(db1) for every object $B \in \mathbb{B}$, there is an object $A \in \mathbb{A}$ and a horizontal equivalence $B \xrightarrow{\simeq} FA$ in $\mathbb{B}$,

(db2) for every pair of objects $A, C \in \mathbb{A}$ and every horizontal morphism $b\colon FA \to FC$ in $\mathbb{B}$, there is a horizontal morphism $a\colon A \to C$ in $\mathbb{A}$ and a vertically invertible square in $\mathbb{B}$ of the form

$$\begin{array}{ccc} FA & \xrightarrow{b} & FC \\ \| & \cong & \| \\ FA & \xrightarrow[Fa]{} & FC \end{array},$$

(db3) for every vertical morphism $v\colon B \twoheadrightarrow B'$ in $\mathbb{B}$, there is a vertical morphism $u\colon A \twoheadrightarrow A'$ in $\mathbb{A}$ and a weakly horizontally invertible square in $\mathbb{B}$ of the form

$$\begin{array}{ccc} B & \xrightarrow{\simeq} & FA \\ {\scriptstyle v}\downarrow & \simeq & \downarrow{\scriptstyle Fu} \\ B' & \xrightarrow[\simeq]{} & FA' \end{array},$$

(db4) for every data in $\mathbb{A}$ as below left, and every square in $\mathbb{B}$ as below right,

$$\begin{array}{ccc} A & \xrightarrow{a} & C \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ A' & \xrightarrow[c]{} & C' \end{array} \qquad \begin{array}{ccc} FA & \xrightarrow{Fa} & FC \\ {\scriptstyle Fu}\downarrow & \beta & \downarrow{\scriptstyle Fu'} \\ FA' & \xrightarrow[Fc]{} & FC' \end{array}$$

there is a unique square $\alpha\colon (u \,{}^{a}_{c}\, u')$ in $\mathbb{A}$ such that $F\alpha = \beta$.

Remark 3.7. In $2\mathrm{Cat}$, one can prove that a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a biequivalence if and only if there is a pseudo functor $G\colon \mathcal{B} \to \mathcal{A}$ together with pseudo natural equivalences $\mathrm{id}_{\mathcal{A}} \simeq GF$ and $FG \simeq \mathrm{id}_{\mathcal{B}}$. Under certain hypotheses, we can show a similar characterization of double biequivalences using *horizontal* pseudo natural equivalences. This is done in Section 5.2.

Similarly to the definition of double biequivalence, we take inspiration from the Lack fibrations to define a notion of *double fibrations*.

Definition 3.8. A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a **double fibration** if

(df1) for every object $C \in \mathbb{A}$ and every horizontal equivalence $b\colon B \xrightarrow{\simeq} FC$ in $\mathbb{B}$, there is a horizontal equivalence $a\colon A \xrightarrow{\simeq} C$ in $\mathbb{A}$ such that $Fa = b$,

(df2) for every horizontal morphism $c\colon A \to C$ in $\mathbb{A}$ and for every vertically invertible square $\beta\colon (e_{FA}\ {}^{b}_{Fc}\ e_{FC})$ in $\mathbb{B}$ as depicted below left, there is a vertically invertible square $\alpha\colon (e_A\ {}^{a}_{c}\ e_C)$ in $\mathbb{A}$ as depicted below right such that $F\alpha = \beta$,

$$\begin{array}{ccc} FA & \xrightarrow{b} & FC \\ \| & \beta\ \cong & \| \\ FA & \xrightarrow[Fc]{} & FC \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{a} & C \\ \| & \alpha\ \cong & \| \\ A & \xrightarrow[c]{} & C \end{array}$$

(df3) for every vertical morphism $u'\colon C \twoheadrightarrow C'$ in $\mathbb{A}$ and every weakly horizontally invertible square $\beta\colon (v \overset{\simeq}{\underset{\simeq}{}} Fu')$ in $\mathbb{B}$ as depicted below left, there is a weakly horizontally invertible square $\alpha\colon (u \overset{\simeq}{\underset{\simeq}{}} u')$ in $\mathbb{A}$ as depicted below right such that $F\alpha = \beta$.

$$\begin{array}{ccc} B & \xrightarrow{\simeq} & FC \\ v \downarrow & \beta \simeq & \downarrow Fu' \\ B' & \xrightarrow[\simeq]{} & FC' \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{\simeq} & C \\ u \downarrow & \alpha \simeq & \downarrow u' \\ A' & \xrightarrow[\simeq]{} & C' \end{array}$$

By requiring that a double functor is both a double biequivalence and a double fibration, we get a notion of *double trivial fibration*, which can be described as follows.

Definition 3.9. A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a **double trivial fibration** if it satisfies (db4) of Definition 3.6, and the following conditions:

(dt1) for every object $B \in \mathbb{B}$, there is an object $A \in \mathbb{A}$ such that $B = FA$, (dt2) for every pair of objects $A, C \in \mathbb{A}$ and every horizontal morphism $b\colon FA \to FC$ in $\mathbb{B}$, there is a horizontal morphism $a\colon A \to C$ in $\mathbb{A}$ such that $b = Fa$, and

(dt3) for every vertical morphism $v\colon B \mathrel{\rightarrow\mkern-14mu\bullet} B'$ in $\mathbb{B}$, there is a vertical morphism $u\colon A \mathrel{\rightarrow\mkern-14mu\bullet} A'$ in $\mathbb{A}$ such that $v = Fu$.

Remark 3.10. Note that (dt2) says that a double trivial fibration is *full* on horizontal morphisms, while (dt3) says that a double trivial fibration is only *surjective* on vertical morphisms.

We can use the functors $\mathbf{H}, \mathcal{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$ to characterize double biequivalences and double fibrations through biequivalences and Lack fibrations in $2\mathrm{Cat}$. We state these characterizations here, and defer their proofs to Section 5.1.

Proposition 3.11. *A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a double biequivalence in* DblCat *if and only if the* 2*-functors* $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ *and* $\mathcal{V}F\colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ *are biequivalences in* $2\mathrm{Cat}$.

Proposition 3.12. *A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a double fibration in* DblCat *if and only if the* 2*-functors* $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ *and* $\mathcal{V}F\colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ *are Lack fibrations in* $2\mathrm{Cat}$.

This is intuitively sound, since horizontal equivalences and weakly horizontally invertible squares were defined to be the equivalences in the 2-categories induced by $\mathbf{H}$ and $\mathcal{V}$, respectively.

As a corollary, we get a similar characterization for double trivial fibrations.

Corollary 3.13. *A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a double trivial fibration in* DblCat *if and only if the induced* 2*-functors* $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ *and* $\mathcal{V}F\colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ *are trivial fibrations in the Lack model structure on* $2\mathrm{Cat}$.

To build a model structure on DblCat with these classes of morphisms as its weak equivalences and (trivial) fibrations, we use the notion of *right-induced model structure*. Given a model category $\mathcal{M}$ and an adjunction

$$\mathcal{M} \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathcal{N}, \tag{3.14}$$

we can, under certain conditions, induce a model structure on $\mathcal{N}$ along the right adjoint R, in which a weak equivalence (resp. fibration) is a morphism F in $\mathcal{N}$ such that RF is a weak equivalence (resp. fibration) in $\mathcal{M}$.

Propositions 3.11 and 3.12 suggest that the model structure on DblCat we desire, with double biequivalences as the weak equivalences and double fibrations as the fibrations, corresponds to the right-induced model structure, if it exists, along the adjunction

$$2\mathrm{Cat} \times 2\mathrm{Cat} \underset{(\mathbf{H}, \mathcal{V})}{\overset{\mathbb{H} \sqcup \mathbb{L}}{\rightleftarrows}} \mathrm{DblCat}\,, \quad \bot$$

where each copy of $2\mathrm{Cat}$ is endowed with the Lack model structure. To prove the existence of this model structure, we use results by Garner, Hess, Kędziorek, Riehl, and Shipley in [7, 12]. In particular, we use the following theorem, inspired by the original Quillen Path Object Argument [21].

Theorem 3.15. *Let $\mathcal{M}$ be an accessible model category, and let $\mathcal{N}$ be a locally presentable category. Suppose we have an adjunction $L \dashv R$ between them as in (3.14). Suppose moreover that every object in $\mathcal{M}$ is fibrant and that, for every object $X \in \mathcal{N}$, there is a factorization*

$$X \xrightarrow{W} \mathrm{Path}(X) \xrightarrow{P} X \times X$$

of the diagonal morphism in $\mathcal{N}$ such that RP is a fibration in $\mathcal{M}$ and RW is a weak equivalence in $\mathcal{M}$. Then the right-induced model structure on $\mathcal{N}$ exists.

Proof. This follows directly from [19, Theorem 6.2], which is the dual of [12, Theorem 2.2.1]. Indeed, if every object in $\mathcal{M}$ is fibrant, then the underlying fibrant replacement of conditions (i) and (ii) of [19, Theorem 6.2] are trivially given by the identity. □

Our strategy is then to construct a path object $\mathrm{Path}(\mathbb{A})$ for a double category $\mathbb{A}$ together with double functors W and P factorizing the diagonal morphism $\mathbb{A} \to \mathbb{A} \times \mathbb{A}$, such that their images under $(\mathbf{H}, \mathcal{V})$ give a weak equivalence and a fibration in $2\mathrm{Cat} \times 2\mathrm{Cat}$ respectively.

Definition 3.16. Let $\mathbb{A}$ be a double category. We define a **path object** for $\mathbb{A}$ as the double category $\mathrm{Path}(\mathbb{A}) \coloneqq [\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}}$, where the 2-category E_{adj} is the free-living adjoint equivalence. It comes with a factorization of the diagonal double functor

$$\mathbb{A} \xrightarrow{W} \mathrm{Path}(\mathbb{A}) \xrightarrow{P} \mathbb{A} \times \mathbb{A},$$

where W is the double functor $\mathbb{A} \cong [\mathbb{1}, \mathbb{A}]_{\mathrm{ps}} \to [\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}} = \mathrm{Path}(\mathbb{A})$ induced by the unique map $\mathbb{H}E_{\mathrm{adj}} \to \mathbb{1}$ and P is the double functor $\mathrm{Path}(\mathbb{A}) = [\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}} \to [\mathbb{1} \sqcup \mathbb{1}, \mathbb{A}]_{\mathrm{ps}} \cong \mathbb{A} \times \mathbb{A}$ induced by the inclusion $\mathbb{1} \sqcup \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}}$ at the two endpoints. Note that, since the composite $\mathbb{1} \sqcup \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}} \to \mathbb{1}$ is the unique map, the composite PW is the diagonal double functor $\mathbb{A} \to \mathbb{A} \times \mathbb{A}$.

Proposition 3.17. *For every double category $\mathbb{A}$, the path object of Definition 3.16*

$$\mathbb{A} \xrightarrow{W} \mathrm{Path}(\mathbb{A}) \xrightarrow{P} \mathbb{A} \times \mathbb{A},$$

is such that $(\mathbf{H}, \mathcal{V})W$ is a weak equivalence and $(\mathbf{H}, \mathcal{V})P$ is a fibration in $2\mathrm{Cat} \times 2\mathrm{Cat}$.

Proof. We first prove that $\mathbf{H}W$ and $\mathcal{V}W$ are biequivalences in $2\mathrm{Cat}$. By Lemma 2.14, we have commutative squares

$$\begin{array}{ccc} \mathbf{H}[\mathbb{1}, \mathbb{A}]_{\mathrm{ps}} & \xrightarrow{\mathbf{H}W} & \mathbf{H}[\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}} \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Ps}[\mathbb{1}, \mathbf{H}\mathbb{A}] & \xrightarrow[(\mathbf{H}W)^\sharp]{} & \mathrm{Ps}[E_{\mathrm{adj}}, \mathbf{H}\mathbb{A}] \end{array} \qquad \begin{array}{ccc} \mathcal{V}[\mathbb{1}, \mathbb{A}]_{\mathrm{ps}} & \xrightarrow{\mathcal{V}W} & \mathcal{V}[\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}} \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Ps}[\mathbb{1}, \mathcal{V}\mathbb{A}] & \xrightarrow[(\mathcal{V}W)^\sharp]{} & \mathrm{Ps}[E_{\mathrm{adj}}, \mathcal{V}\mathbb{A}] \end{array}$$

where the 2-functors $(\mathbf{H}W)^\sharp$ and $(\mathcal{V}W)^\sharp$ are induced by the unique map $E_{\mathrm{adj}} \to \mathbb{1}$. As the inclusion $\mathbb{1} \to E_{\mathrm{adj}}$ is a trivial cofibration in $2\mathrm{Cat}$ and $\mathbf{H}\mathbb{A}$ and $\mathcal{V}\mathbb{A}$ are fibrant 2-categories, by monoidality of the Lack model structure, we get that the induced 2-functors

$$R\colon \mathrm{Ps}[E_{\mathrm{adj}}, \mathbf{H}\mathbb{A}] \to \mathrm{Ps}[\mathbb{1}, \mathbf{H}\mathbb{A}] \text{ and } S\colon \mathrm{Ps}[E_{\mathrm{adj}}, \mathcal{V}\mathbb{A}] \to \mathrm{Ps}[\mathbb{1}, \mathcal{V}\mathbb{A}]$$

are trivial fibrations in $2\mathrm{Cat}$. As $R(\mathbf{H}W)^\sharp$ and $S(\mathcal{V}W)^\sharp$ compose to the identity, by 2-out-of-3, we get that $(\mathbf{H}W)^\sharp$ and $(\mathcal{V}W)^\sharp$ are biequivalences.

Again, by 2-out-of-3 applied to the commutative squares above, we conclude that $\mathbf{H}W$ and $\mathcal{V}W$ are biequivalences.

Similarly, one can show that $\mathbf{H}P$ and $\mathcal{V}P$ are Lack fibrations in $2\mathrm{Cat}$, since the 2-functor $\mathbb{1} \sqcup \mathbb{1} \to E_{\mathrm{adj}}$ is a cofibration in $2\mathrm{Cat}$. Therefore, the induced 2-functors

$$\mathrm{Ps}[\mathbb{1} \sqcup \mathbb{1}, \mathbf{H}\mathbb{A}] \to \mathrm{Ps}[E_{\mathrm{adj}}, \mathbf{H}\mathbb{A}] \text{ and } \mathrm{Ps}[\mathbb{1} \sqcup \mathbb{1}, \mathcal{V}\mathbb{A}] \to \mathrm{Ps}[E_{\mathrm{adj}}, \mathcal{V}\mathbb{A}]$$

are fibrations in $2\mathrm{Cat}$, by monoidality of the Lack model structure. □

We are finally ready to prove the existence of the right-induced model structure on DblCat along the adjunction $\mathbb{H} \sqcup \mathbb{L} \dashv (\mathbf{H}, \mathcal{V})$.

Theorem 3.18. *Consider the adjunction*

$$2\mathrm{Cat} \times 2\mathrm{Cat} \underset{(\mathbf{H}, \mathcal{V})}{\overset{\mathbb{H} \sqcup \mathbb{L}}{\rightleftarrows}} \mathrm{DblCat}\,, \quad \bot$$

where each copy of $2\mathrm{Cat}$ *is endowed with the Lack model structure. Then the right-induced model structure on* DblCat *exists. In particular, a double functor is a weak equivalence (resp. fibration) in this model structure if and only if it is a double biequivalence (resp. double fibration).*

Proof. We first describe the weak equivalences and fibrations in the right-induced model structure on DblCat. These are given by the double functors F such that $(\mathbf{H}, \mathcal{V})F$ is a weak equivalence (resp. fibration) in $2\mathrm{Cat} \times 2\mathrm{Cat}$, or equivalently, such that both $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences (resp. Lack fibrations) in $2\mathrm{Cat}$. Then it follows from Propositions 3.11 and 3.12 that the weak equivalences (resp. fibrations) in DblCat are precisely the double biequivalences (resp. double fibrations).

We now prove the existence of the model structure. For this purpose, we want to apply Theorem 3.15 to our setting. First note that $2\mathrm{Cat}$ and DblCat are locally presentable, and that the Lack model structure on $2\mathrm{Cat}$ is cofibrantly generated. In particular, this implies that the product $2\mathrm{Cat} \times 2\mathrm{Cat}$ endowed with two copies of the Lack model structure is combinatorial, hence accessible. Moreover, every pair of 2-categories is fibrant in $2\mathrm{Cat} \times 2\mathrm{Cat}$, since every object is fibrant in the Lack model structure. Finally, for every double category $\mathbb{A}$, Proposition 3.17 gives a factorization

$$\mathbb{A} \xrightarrow{W} \mathrm{Path}(\mathbb{A}) \xrightarrow{P} \mathbb{A} \times \mathbb{A}$$

such that W is a double biequivalence and P is a double fibration. By Theorem 3.15, this proves that the right-induced model structure along $(\mathbf{H}, \mathcal{V})$ on DblCat exists. □

Remark 3.19. Note that every double category is fibrant in this model structure. Indeed, this follows directly from the fact that it is right-induced from a model structure in which every object is fibrant.

4. Generating (trivial) cofibrations and cofibrant objects

In this section, we take a closer look at the (trivial) cofibrations and cofibrant objects in our model structure on DblCat, and we show that the latter is cofibrantly generated.

4.1 Generating sets of (trivial) cofibrations

Recall from Theorem 3.3 that the Lack model structure on $2\mathrm{Cat}$ is cofibrantly generated. As a consequence, our model structure on DblCat is also cofibrantly generated.

Proposition 4.1. *Let $\mathcal{I}_2$ and $\mathcal{J}_2$ denote sets of generating cofibrations and generating trivial cofibrations, respectively, for the Lack model structure on* $2\mathrm{Cat}$. *Then, the sets of morphisms in* DblCat

$$\mathcal{I} = \{\mathbb{H}i,\ \mathbb{H}i \times \mathbb{V}\mathbb{2} \mid i \in \mathcal{I}_2\}, \ \textit{and}\ \mathcal{J} = \{\mathbb{H}j,\ \mathbb{H}j \times \mathbb{V}\mathbb{2} \mid j \in \mathcal{J}_2\}$$

give sets of generating cofibrations and generating trivial cofibrations, respectively, for the model structure on DblCat *of Theorem 3.18.*

Proof. Since the model structure on DblCat is right-induced from two copies of the Lack model structure on $2\mathrm{Cat}$ along the adjunction $\mathbb{H} \sqcup \mathbb{L} \dashv (\mathbf{H}, \mathcal{V})$, sets of generating cofibrations and of generating trivial cofibrations can be given by the images under the left adjoint $\mathbb{H} \sqcup \mathbb{L}$ of the fixed sets of generating cofibrations and generating trivial cofibrations in $2\mathrm{Cat} \times 2\mathrm{Cat}$.

Let i and i' be generating cofibrations in $\mathcal{I}_2$ in $2\mathrm{Cat}$. Then $\mathbb{H}i$ and $\mathbb{L}i = \mathbb{H}i \times \mathbb{V}\mathbb{2}$ are cofibrations in DblCat. To see this apply $\mathbb{H} \sqcup \mathbb{L}$ to the cofibrations $(i, \mathrm{id}_\emptyset)$ and $(\mathrm{id}_\emptyset, i)$, respectively. Similarly, $\mathbb{H}i'$ and $\mathbb{L}i' = \mathbb{H}i' \times \mathbb{V}\mathbb{2}$ are cofibrations in DblCat. Since coproducts of cofibrations are cofibrations, then $(\mathbb{H} \sqcup \mathbb{L})(i, i') = \mathbb{H}i \sqcup \mathbb{L}i'$ can be obtained from $\mathbb{H}i$ and $\mathbb{L}i' = \mathbb{H}i' \times \mathbb{V}\mathbb{2}$. This shows that $\mathcal{I}$ is a set of generating cofibrations of DblCat.

Similarly, we can show that $\mathcal{J}$ is a set of generating trivial cofibrations of DblCat. □

However, we can find sets of generating (trivial) cofibrations, which are both smaller and more descriptive than the ones given above, by using the characterization of fibrations and trivial fibrations in our model structure given in Proposition 3.12 and Corollary 3.13.

Notation 4.2. Let $\mathbb{S}$ be the double category free on a square, $\delta\mathbb{S}$ be its boundary, and $\mathbb{S}_2$ be the double category free on two squares with the same boundary.

$$\mathbb{S} = \begin{array}{ccc} 0 & \longrightarrow & 1 \\ \big\downarrow\!\bullet & \alpha & \big\downarrow\!\bullet \\ 0' & \longrightarrow & 1' \end{array} \; ; \quad \delta\mathbb{S} = \begin{array}{ccc} 0 & \longrightarrow & 1 \\ \big\downarrow\!\bullet & & \big\downarrow\!\bullet \\ 0' & \longrightarrow & 1' \end{array} \; ; \quad \mathbb{S}_2 = \begin{array}{ccc} 0 & \longrightarrow & 1 \\ \big\downarrow\!\bullet & \alpha_0 \; \alpha_1 & \big\downarrow\!\bullet \\ 0' & \longrightarrow & 1' \end{array}$$

We fix notation for the following double functors, which form a set of generating cofibrations for our model structure on DblCat:

- the unique map $I_1 \colon \emptyset \to \mathbb{1}$,
- the inclusion $I_2 \colon \mathbb{1} \sqcup \mathbb{1} \to \mathbb{H}\mathbb{2}$,
- the unique map $I_3 \colon \emptyset \to \mathbb{V}\mathbb{2}$,
- the inclusion $I_4 \colon \delta\mathbb{S} \to \mathbb{S}$, and
- the double functor $I_5 \colon \mathbb{S}_2 \to \mathbb{S}$ sending both squares in $\mathbb{S}_2$ to the non-trivial square of $\mathbb{S}$.

We also fix notation for the following double functors, which form a set of generating trivial cofibrations for our model structure on DblCat:

- the inclusion $J_1\colon \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}}$, where the 2-category E_{adj} is the free-living adjoint equivalence,
- the inclusion $J_2\colon \mathbb{H}\mathbb{2} \to \mathbb{H}C_{\mathrm{inv}}$, where the 2-category C_{inv} is the free-living invertible 2-cell, and
- the inclusion $J_3\colon \mathbb{V}\mathbb{2} \to \mathbb{H}E_{\mathrm{adj}} \times \mathbb{V}\mathbb{2}$; note that the double category $\mathbb{H}E_{\mathrm{adj}} \times \mathbb{V}\mathbb{2}$ is the free-living weakly horizontally invertible square (with horizontal adjoint equivalence data).

Proposition 4.3. *In the model structure on* DblCat *of Theorem 3.18, a set* $\mathcal{I}'$ *of generating cofibrations is given by*

$$\{I_1\colon \emptyset \to \mathbb{1},\ I_2\colon \mathbb{1} \sqcup \mathbb{1} \to \mathbb{H}\mathbb{2},\ I_3\colon \emptyset \to \mathbb{V}\mathbb{2},\ I_4\colon \delta\mathbb{S} \to \mathbb{S},\ I_5\colon \mathbb{S}_2 \to \mathbb{S}\}$$

and a set $\mathcal{J}'$ *of generating trivial cofibrations is given by*

$$\{J_1\colon \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}},\ J_2\colon \mathbb{H}\mathbb{2} \to \mathbb{H}C_{\mathrm{inv}},\ J_3\colon \mathbb{V}\mathbb{2} \to \mathbb{H}E_{\mathrm{adj}} \times \mathbb{V}\mathbb{2}\}.$$

Proof. It is a routine exercise to check that a double functor is a double trivial fibration as defined in Definition 3.9 if and only if it has the right-lifting property with respect to the cofibrations in $\mathcal{I}'$, and that a double functor is a double fibration as defined in Definition 3.8 if and only if it has the right-lifting property with respect to the trivial cofibrations of $\mathcal{J}'$. This shows that $\mathcal{I}'$ and $\mathcal{J}'$ are sets of generating cofibrations and generating trivial cofibration for DblCat, respectively. □

4.2 Cofibrations and cofibrant double categories

Our next goal is to provide a characterization of the cofibrations in DblCat. In [15, Lemma 4.1], Lack shows that a 2-functor is a cofibration in $2\mathrm{Cat}$ if and only if its underlying functor has the left lifting property with respect to all surjective on objects and full functors. A similar result applies to our model structure.

First, we state a characterization of the functors in Cat which have the left lifting property with respect to all surjective on objects and full (resp. surjective on morphisms) functors, that will be useful to understand the characterization of cofibrations in Proposition 4.7.

Lemma 4.4. *A functor $F\colon \mathcal{A} \to \mathcal{B}$ has the left lifting property with respect to surjective on objects and full (resp. surjective) on morphisms functors if and only if*

(i) the functor F is injective on objects and faithful, and

(ii) there are functors $I\colon \mathcal{B} \to \mathcal{C}$ and $R\colon \mathcal{C} \to \mathcal{B}$ such that $RI = \mathrm{id}_{\mathcal{B}}$, where the category $\mathcal{C}$ is obtained from the image of F by freely adjoining objects and then freely adjoining morphisms between specified objects (resp. by freely adjoining objects and morphisms).

Moreover, a functor $\emptyset \to \mathcal{A}$ has the left lifting property with respect to surjective on objects and full (resp. surjective) on morphisms functors if and only if the category $\mathcal{A}$ is free (resp. a disjoint union of copies of $\mathbb{1}$ and $\mathbb{2}$).

Proof. The statement about "full on morphisms" is proven in [15, Corollary 4.12]. For the "surjective on morphisms" case, the proof is analogous, replacing $\mathbb{1} \sqcup \mathbb{1} \to \mathbb{2}$ by $\emptyset \to \mathbb{2}$.

The second statement about $\emptyset \to \mathcal{A}$ follows from the fact that a retract of a free category is itself free, and similarly for disjoint unions of copies of $\mathbb{1}$ and $\mathbb{2}$. □

Notation 4.5. We write $U\colon 2\mathrm{Cat} \to \mathrm{Cat}$ for the functor that sends a 2-category to its underlying category.

Remark 4.6. The functor $U\mathbf{H}\colon \mathrm{DblCat} \to \mathrm{Cat}$, which sends a double category to its underlying category of objects and horizontal morphisms, has a right adjoint. It is given by the functor $R_h\colon \mathrm{Cat} \to \mathrm{DblCat}$ that sends a category $\mathcal{C}$ to the double category with the same objects as $\mathcal{C}$, horizontal morphisms given by the morphisms of $\mathcal{C}$, a unique vertical morphism between every pair of objects, and a unique square $!\colon (!\ {}^{f}_{g}\ !)$ for every pair of morphisms f, g in $\mathcal{C}$.

Similarly, the functor $U\mathbf{V}\colon \mathrm{DblCat} \to \mathrm{Cat}$ admits a right adjoint R_v.

Proposition 4.7. *A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a cofibration in* DblCat *if and only if*

(i) the underlying horizontal functor $U\mathbf{H}F\colon U\mathbf{H}\mathbb{A} \to U\mathbf{H}\mathbb{B}$ has the left lifting property with respect to all surjective on objects and full functors, and

(ii) the underlying vertical functor $U\mathbf{V}F\colon U\mathbf{V}\mathbb{A} \to U\mathbf{V}\mathbb{B}$ has the left lifting property with respect to all surjective on objects and surjective on morphisms functors.

Proof. Suppose first that $F\colon \mathbb{A} \to \mathbb{B}$ is a cofibration in DblCat, i.e., it has the left lifting property with respect to all double trivial fibrations. In order to show (i), let $P\colon \mathcal{X} \to \mathcal{Y}$ be a surjective on objects and full functor. By the adjunction $U\mathbf{H} \dashv R_h$, saying that $U\mathbf{H}F$ has the left lifting property with respect to P is equivalent to saying that F has the left lifting property with respect to R_hP. We now prove this latter statement.

Note that the double functor $R_hP\colon R_h\mathcal{X} \to R_h\mathcal{Y}$ is surjective on objects and full on horizontal morphisms, since P is so. Moreover, by construction of R_h, there is exactly one vertical morphism and one square for each boundary in both its source and target; therefore R_hP is surjective on vertical morphisms and fully faithful on squares. Hence R_hP is a double trivial fibration, and F has the left lifting property with respect to R_hP since it is a cofibration in DblCat.

Similarly, one can show that (ii) holds, by considering the adjunction $U\mathbf{V} \dashv R_v$ and replacing fullness by surjectivity on morphisms.

Now suppose that $F\colon \mathbb{A} \to \mathbb{B}$ satisfies (i) and (ii). Let $P\colon \mathbb{X} \to \mathbb{Y}$ be a double trivial fibration and consider a commutative square as below left. We want to find a lift $L\colon \mathbb{B} \to \mathbb{X}$ in this square as depicted below. Using (ii), since $U\mathbf{V}P$ is surjective on objects and surjective on morphisms, we have a lift L_v in the below middle diagram. We now wish to find a lift L_h in the diagram below right, that agrees with L_v on objects. Using the characterization of $U\mathbf{H}F$ given in Lemma 4.4, and the fact that $U\mathbf{H}P$ is full, we can extend the given assignment on objects to a functor $L_h\colon U\mathbf{H}\mathbb{B} \to U\mathbf{H}\mathbb{X}$.

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{G} & \mathbb{X} \\ {\scriptstyle F}\downarrow & \overset{L}{\nearrow} & \downarrow{\scriptstyle P} \\ \mathbb{B} & \xrightarrow[Q]{} & \mathbb{Y} \end{array} \qquad \begin{array}{ccc} U\mathbf{V}\mathbb{A} & \xrightarrow{U\mathbf{V}G} & U\mathbf{V}\mathbb{X} \\ {\scriptstyle U\mathbf{V}F}\downarrow & \overset{L_v}{\nearrow} & \downarrow{\scriptstyle U\mathbf{V}P} \\ U\mathbf{V}\mathbb{B} & \xrightarrow[U\mathbf{V}Q]{} & U\mathbf{V}\mathbb{Y} \end{array} \qquad \begin{array}{ccc} U\mathbf{H}\mathbb{A} & \xrightarrow{U\mathbf{H}G} & U\mathbf{H}\mathbb{X} \\ {\scriptstyle U\mathbf{H}F}\downarrow & \overset{L_h}{\nearrow} & \downarrow{\scriptstyle U\mathbf{H}P} \\ U\mathbf{H}\mathbb{B} & \xrightarrow[U\mathbf{H}Q]{} & U\mathbf{H}\mathbb{Y} \end{array}$$

Then, since $P\colon \mathbb{X} \to \mathbb{Y}$ is fully faithful on squares, the assignment on objects, horizontal morphisms, and vertical morphisms given by L_h and L_v

uniquely extend to a double functor $L\colon \mathbb{B} \to \mathbb{Y}$, which gives the desired lift. □

Remark 4.8. From Lemma 4.4 and Proposition 4.7, it is straightforward to see that a cofibration in DblCat is in particular injective on objects, and faithful on horizontal morphisms and vertical morphisms.

Finally, as a corollary of Lemma 4.4 and Proposition 4.7, we obtain a characterization of the cofibrant double categories.

Corollary 4.9. *A double category $\mathbb{A}$ is cofibrant if and only if its underlying horizontal category $U\mathbf{H}\mathbb{A}$ is free and its underlying vertical category $U\mathbf{V}\mathbb{A}$ is a disjoint union of copies of $\mathbb{1}$ and $\mathbb{2}$.*

5. Fibrations, weak equivalences, and Whitehead theorems

The purpose of this section is to describe the weak equivalences and fibrations of our model structure. Section 5.1 provides proofs of Propositions 3.11 and 3.12, which claim that the weak equivalences and fibrations of the right-induced model structure on DblCat of Theorem 3.18 are precisely the double biequivalences and the double fibrations.

In Section 5.2 we turn our attention to another characterization of the weak equivalences, known as the Whitehead theorem. Recall that, in the 2-categorical case, a 2-functor is a biequivalence if and only if it has a pseudo inverse up to pseudo natural equivalence (see [14, Theorem 7.4.1]). A similar statement does not hold in general for double biequivalences, but it does if we assume cofibrancy on the target double category. In particular, we retrieve the usual Whitehead theorem for model categories applied to our setting, and also the characterization of biequivalences stated above. Another version of the Whitehead theorem for double biequivalences is given in [20, Theorem 8.1], which in turn holds for the fibrant objects of the model structure on DblCat defined therein.

5.1 Characterizations of weak equivalences and fibrations

We first prove Proposition 3.11, dealing with weak equivalences. In order to characterize the double functors F such that $(\mathbf{H}, \mathcal{V})F$ is a weak equivalence in $2\mathrm{Cat} \times 2\mathrm{Cat}$, we express what it means for $\mathbf{H}F$ and $\mathcal{V}F$ to be biequivalences in $2\mathrm{Cat}$; this is done by translating (b1-3) of Definition 3.1 for these 2-functors.

Remark 5.1. Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. Then $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ is a biequivalence in $2\mathrm{Cat}$ if and only if F satisfies (db1-2) of Definition 3.6, and the following condition:

(hb3) for every pair of horizontal morphisms $a, c\colon A \to C$ in $\mathbb{A}$ and every square in $\mathbb{B}$ of the form

$$\begin{array}{ccc} FA & \xrightarrow{Fa} & FC \\ \| & \beta & \| \\ FA & \xrightarrow[Fc]{} & FC \end{array},$$

there is a unique square $\alpha\colon (e_A \, {}^{a}_{c} \, e_C)$ in $\mathbb{A}$ such that $F\alpha = \beta$.

Remark 5.2. Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. Then $\mathcal{V}F\colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ is a biequivalence in $2\mathrm{Cat}$ if and only if F satisfies (db3) of Definition 3.6, and the following conditions:

(vb2) for every pair of vertical morphisms $u\colon A \to\!\!\!\!\bullet\; A'$ and $u'\colon C \to\!\!\!\!\bullet\; C'$ in $\mathbb{A}$ and every square $\beta\colon (Fu \, {}^{b}_{d} \, Fu')$ in $\mathbb{B}$, there is a square $\alpha\colon (u \, {}^{a}_{c} \, u')$ in $\mathbb{A}$ and two vertically invertible squares in $\mathbb{B}$ such that the following pasting equality holds,

$$\begin{array}{ccc} FA & \xrightarrow{b} & FC \\ \| & \cong & \| \\ FA & \xrightarrow{Fa} & FC \\ {\scriptstyle Fu}\downarrow & F\alpha & \downarrow{\scriptstyle Fu'} \\ FA' & \xrightarrow[Fc]{} & FC' \end{array} \quad = \quad \begin{array}{ccc} FA & \xrightarrow{b} & FC \\ {\scriptstyle Fu}\downarrow & \beta & \downarrow{\scriptstyle Fu'} \\ FA' & \xrightarrow[d]{} & FC' \\ \| & \cong & \| \\ FA' & \xrightarrow[Fc]{} & FC' \end{array}$$

(vb3) for every tuple of squares $\alpha\colon (u \,{}^{a}_{c}\, u')$ and $\alpha'\colon (u \,{}^{a'}_{c'}\, u')$ in $\mathbb{A}$, and τ_0 and τ_1 in $\mathbb{B}$ as in the pasting equality below left, there are unique squares $\sigma_0\colon (e_A \,{}^{a}_{a'}\, e_C)$ and $\sigma_1\colon (e_{A'} \,{}^{c}_{c'}\, e_{C'})$ in $\mathbb{A}$ satisfying the pasting equality below right, and with the property that $F\sigma_0 = \tau_0$ and $F\sigma_1 = \tau_1$.

$$
\begin{array}{ccc}
FA & \xrightarrow{Fa} & FC \\
\Vert & \tau_0 & \Vert \\
FA & \xrightarrow{Fa'} & FC \\
{\scriptstyle Fu}\downarrow & F\alpha' & \downarrow{\scriptstyle Fu'} \\
FA' & \xrightarrow[Fc']{} & FC'
\end{array}
=
\begin{array}{ccc}
FA & \xrightarrow{Fa} & FC \\
{\scriptstyle Fu}\downarrow & F\alpha & \downarrow{\scriptstyle Fu'} \\
FA' & \xrightarrow[Fc]{} & FC' \\
\Vert & \tau_1 & \Vert \\
FA' & \xrightarrow[Fc']{} & FC'
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
\Vert & \sigma_0 & \Vert \\
A & \xrightarrow{a'} & C \\
{\scriptstyle u}\downarrow & \alpha' & \downarrow{\scriptstyle u'} \\
A' & \xrightarrow[c']{} & C'
\end{array}
=
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
{\scriptstyle u}\downarrow & \alpha & \downarrow{\scriptstyle u'} \\
A' & \xrightarrow[c]{} & C' \\
\Vert & \sigma_1 & \Vert \\
A' & \xrightarrow[c']{} & C'
\end{array}
$$

The reader may have noticed that condition (db4) in Definition 3.6 regarding fully faithfulness on squares has not been mentioned so far, but it is recovered by the conditions (hb3) and (vb2-3) above.

Lemma 5.3. *Suppose that $F\colon \mathbb{A} \to \mathbb{B}$ is a double functor satisfying (hb3) of Remark 5.1, and (vb2-3) of Remark 5.2. Then F satisfies (db4) of Definition 3.6.*

Proof. Suppose $\beta\colon (Fu \,{}^{Fa}_{Fc}\, Fu')$ is a square in $\mathbb{B}$ as in (db4) of Definition 3.6. By (vb2) of Remark 5.2, there is a square $\overline{\alpha}\colon (u \,{}^{\overline{a}}_{\overline{c}}\, u')$ in $\mathbb{A}$ and two vertically invertible squares ψ_0, ψ_1 in $\mathbb{B}$ such that the following pasting equality holds.

$$
\begin{array}{ccc}
FA & \xrightarrow{Fa} & FC \\
\Vert & \psi_0 \,\Vert\!\wr & \Vert \\
FA & \xrightarrow{F\overline{a}} & FC \\
{\scriptstyle Fu}\downarrow & F\overline{\alpha} & \downarrow{\scriptstyle Fu'} \\
FA' & \xrightarrow[F\overline{c}]{} & FC'
\end{array}
=
\begin{array}{ccc}
FA & \xrightarrow{Fa} & FC \\
{\scriptstyle Fu}\downarrow & \beta & \downarrow{\scriptstyle Fu'} \\
FA' & \xrightarrow[Fc]{} & FC' \\
\Vert & \psi_1 \,\Vert\!\wr & \Vert \\
FA' & \xrightarrow[F\overline{c}]{} & FC'
\end{array}
$$

By (hb3) of Remark 5.1 applied to ψ_0 and ψ_1, there are unique squares $\varphi_0\colon (e_A \,{}^{a}_{\overline{a}}\, e_C)$ and $\varphi_1\colon (e_{A'} \,{}^{c}_{\overline{c}}\, e_{C'})$ in $\mathbb{A}$ such that $F\varphi_0 = \psi_0$ and $F\varphi_1 = \psi_1$.

Moreover, the squares φ_0 and φ_1 are vertically invertible by the unicity condition in (hb3). Therefore, the square α given by the following vertical pasting

$$
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
{\scriptstyle u}\big\downarrow & \alpha & \big\downarrow{\scriptstyle u'} \\
A' & \xrightarrow[c]{} & C'
\end{array}
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
\big\| & \varphi_0 \, \wr\!\| & \big\| \\
A & \xrightarrow{\bar{a}} & C \\
{\scriptstyle u}\big\downarrow & \bar{\alpha} & \big\downarrow{\scriptstyle u'} \\
A' & \xrightarrow[\bar{c}]{} & C' \\
\big\| & \varphi_1^{-1} \, \wr\!\| & \big\| \\
A' & \xrightarrow[c]{} & C'
\end{array}
$$

is such that $F\alpha = \beta$. This settles the matter of the existence of the square α. Now suppose there are two squares $\alpha\colon (u \,{}^{a}_{c}\, u')$ and $\alpha'\colon (u \,{}^{a}_{c}\, u')$ in $\mathbb{A}$ such that $F\alpha = \beta = F\alpha'$. Take $\tau_0 = e_{Fa}$ and $\tau_1 = e_{Fc}$ in (vb3) of Remark 5.2. This gives unique squares σ_0 and σ_1 in $\mathbb{A}$ such that the following pasting equality holds

$$
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
\big\| & \sigma_0 & \big\| \\
A & \xrightarrow{a} & C \\
{\scriptstyle u}\big\downarrow & \alpha' & \big\downarrow{\scriptstyle u'} \\
A' & \xrightarrow[c]{} & C'
\end{array}
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
{\scriptstyle u}\big\downarrow & \alpha & \big\downarrow{\scriptstyle u'} \\
A' & \xrightarrow[c]{} & C' \\
\big\| & \sigma_1 & \big\| \\
A' & \xrightarrow[c]{} & C'
\end{array}
$$

and $F\sigma_0 = e_{Fa}$ and $F\sigma_1 = e_{Fc}$. By unicity in (hb3), we must have $\sigma_0 = e_a$ and $\sigma_1 = e_c$. Replacing σ_0 and σ_1 by e_a and e_c in the pasting diagram above implies that $\alpha = \alpha'$. This proves unicity. □

We can now use the above results to obtain the desired characterization of the weak equivalences in our model structure on DblCat.

Proof of Proposition 3.11. Suppose that $F\colon \mathbb{A} \to \mathbb{B}$ is a double functor such that both $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences in $2\mathrm{Cat}$. By Remarks 5.1 and 5.2, we directly have that F satisfies (db1-3) of Definition 3.6. Moreover, by Lemma 5.3, we also have that F satisfies (db4) of Definition 3.6. This shows that F is a double biequivalence.

Now suppose that $F\colon \mathbb{A} \to \mathbb{B}$ is a double biequivalence. We want to show that both $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences in $2\mathrm{Cat}$. To show that $\mathbf{H}F$ is a biequivalence, it suffices to show that (hb3) of Remark 5.1 is satisfied; this follows directly from taking u and u' to be vertical identities in (db4) of Definition 3.6.

It remains to show that $\mathcal{V}F$ is a biequivalence; we do so by proving (vb2-3) of Remark 5.2. To prove (vb2), let $u\colon A \to\!\!\!\!\bullet\ A'$ and $u'\colon C \to\!\!\!\!\bullet\ C'$ be vertical morphisms in $\mathbb{A}$ and β be a square in $\mathbb{B}$ of the form

$$\begin{array}{ccc} FA & \xrightarrow{b} & FC \\ {\scriptstyle Fu}\downarrow\!\bullet & \beta & \bullet\!\downarrow{\scriptstyle Fu'} \\ FA' & \xrightarrow[d]{} & FC' \end{array}\ .$$

By (db2) of Definition 3.6, there are horizontal morphisms $a\colon A \to C$ and $c\colon A' \to C'$ in $\mathbb{A}$ and vertically invertible squares $\varphi_0\colon (e_{FA}\ {}^{b}_{Fa}\ e_{FC})$ and $\varphi_1\colon (e_{FA'}\ {}^{d}_{Fc}\ e_{FC'})$ in $\mathbb{B}$. By (db4) of Definition 3.6, there is a unique square $\alpha\colon (u\ {}^{a}_{c}\ u')$ in $\mathbb{A}$ such that

$$\begin{array}{ccc} FA & \xrightarrow{Fa} & FC \\ {\scriptstyle Fu}\downarrow\!\bullet & F\alpha & \bullet\!\downarrow{\scriptstyle Fu'} \\ FA' & \xrightarrow[Fc]{} & FC' \end{array} = \begin{array}{ccc} FA & \xrightarrow{Fa} & FC \\ \| & \varphi_0^{-1}\,\cong & \| \\ FA & \xrightarrow{b} & FC \\ {\scriptstyle Fu}\downarrow\!\bullet & \beta & \bullet\!\downarrow{\scriptstyle Fu'} \\ FA' & \xrightarrow[d]{} & FC' \\ \| & \varphi_1\,\cong & \| \\ FA' & \xrightarrow[Fc]{} & FC' \end{array},$$

which gives (vb2). Finally, we prove (vb3). Suppose we have a pasting equality in $\mathbb{B}$ as below left.

$$\begin{array}{ccc} FA & \xrightarrow{Fa} & FC \\ \| & \tau_0 & \| \\ FA & \xrightarrow{Fa'} & FC \\ Fu\downarrow & F\alpha' & \downarrow Fu' \\ FA' & \xrightarrow[Fc']{} & FC' \end{array} = \begin{array}{ccc} FA & \xrightarrow{Fa} & FC \\ Fu\downarrow & F\alpha & \downarrow Fu' \\ FA' & \xrightarrow[Fc]{} & FC' \\ \| & \tau_1 & \| \\ FA' & \xrightarrow[Fc']{} & FC' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{a} & C \\ \| & \sigma_0 & \| \\ A & \xrightarrow{a'} & C \\ u\downarrow & \alpha' & \downarrow u' \\ A' & \xrightarrow[c']{} & C' \end{array} = \begin{array}{ccc} A & \xrightarrow{a} & C \\ u\downarrow & \alpha & \downarrow u' \\ A' & \xrightarrow[c]{} & C' \\ \| & \sigma_1 & \| \\ A' & \xrightarrow[c']{} & C' \end{array}$$

By applying (db4) of Definition 3.6 to τ_0 and τ_1, we obtain unique squares $\sigma_0\colon (e_A \, {}^{a}_{a'} \, e_C)$ and $\sigma_1\colon (e_{A'} \, {}^{c}_{c'} \, e_{C'})$ in $\mathbb{A}$ such that $F\sigma_0 = \tau_0$ and $F\sigma_1 = \tau_1$. Moreover, by unicity in (db4) of Definition 3.6, we have the pasting equality above right, since applying F to each vertical composite yields the same squares in $\mathbb{B}$. This proves (vb3), and thus concludes the proof. □

Now we turn our attention to Proposition 3.12, dealing with fibrations. For this, we first translate (f1-2) of Definition 3.2 for $\mathbf{H}F$ and $\mathcal{V}F$.

Remark 5.4. Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. Then $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ is a fibration in $2\mathrm{Cat}$ if and only if F satisfies (df1-2) of Definition 3.8.

Remark 5.5. Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. Then $\mathcal{V}F\colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ is a fibration in $2\mathrm{Cat}$ if and only if F satisfies (df3) of Definition 3.8, and the following condition:

(vf2) for every square $\alpha'\colon (u \, {}^{a'}_{c'} \, u')$ in $\mathbb{A}$ and every square $\beta\colon (Fu \, {}^{b}_{d} \, Fu')$ in $\mathbb{B}$, together with vertically invertible squares τ_0 and τ_1 in $\mathbb{B}$ as in the pasting equality below left, there is a square $\alpha\colon (u \, {}^{a}_{c} \, u')$ in $\mathbb{A}$, together with vertically invertible squares σ_0 and σ_1 in $\mathbb{A}$ as in the pasting equality below right, such that $F\alpha = \beta$, $F\sigma_0 = \tau_0$, $F\sigma_1 = \tau_1$.

$$\begin{array}{ccc} FA & \xrightarrow{b} & FC \\ \| & \tau_0 \cong & \| \\ FA & \xrightarrow{Fa'} & FC \\ Fu\downarrow & F\alpha' & \downarrow Fu' \\ FA' & \xrightarrow[Fc']{} & FC' \end{array} = \begin{array}{ccc} FA & \xrightarrow{b} & FC \\ Fu\downarrow & \beta & \downarrow Fu' \\ FA' & \xrightarrow[d]{} & FC' \\ \| & \tau_1 \cong & \| \\ FA' & \xrightarrow[Fc']{} & FC' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{a} & C \\ \| & \sigma_0 \cong & \| \\ A & \xrightarrow{a'} & C \\ u\downarrow & \alpha' & \downarrow u' \\ A' & \xrightarrow[c']{} & C' \end{array} = \begin{array}{ccc} A & \xrightarrow{a} & C \\ u\downarrow & \alpha & \downarrow u' \\ A' & \xrightarrow[c]{} & C' \\ \| & \sigma_1 \cong & \| \\ A' & \xrightarrow[c']{} & C' \end{array}$$

We can now use the above remarks to show the desired characterization of the fibrations in our model structure.

Proof of Proposition 3.12. It is clear that if a double functor $F\colon \mathbb{A} \to \mathbb{B}$ is such that both $\mathbf{H}F$ and $\mathcal{V}F$ are Lack fibrations in $2\mathrm{Cat}$, then it is a double fibration, by Remarks 5.4 and 5.5.

Suppose now that $F\colon \mathbb{A} \to \mathbb{B}$ is a double fibration. By Remark 5.4, we directly get that $\mathbf{H}F$ is a Lack fibration in $2\mathrm{Cat}$. To show that $\mathcal{V}F$ is also a Lack fibration, it suffices to show that (vf2) of Remark 5.5 is satisfied. Let $\alpha'\colon (u \,{}^{a'}_{c'}\, u')$ be a square in $\mathbb{A}$ and $\beta\colon (Fu \,{}^{b}_{d}\, Fu')$ be a square in $\mathbb{B}$, together with vertically invertible squares τ_0 and τ_1 in $\mathbb{B}$ as in the leftmost pasting equality diagram in (vf2). By (df2) of Definition 3.8, there are vertically invertible squares $\sigma_0\colon (e_A \,{}^{a}_{a'}\, e_C)$ and $\sigma_1\colon (e_{A'} \,{}^{c}_{c'}\, e_{C'})$ in $\mathbb{A}$ such that $F\sigma_0 = \tau_0$ and $F\sigma_1 = \tau_1$. Then the square α given by the vertical composite

$$\begin{array}{ccc} A & \xrightarrow{a} & C \\ u\big\downarrow & \alpha & \big\downarrow u' \\ A' & \xrightarrow[c]{} & C' \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{a} & C \\ \| & \sigma_0 \cong & \| \\ A & \xrightarrow{a'} & C \\ u\big\downarrow & \alpha' & \big\downarrow u' \\ A' & \xrightarrow[c']{} & C' \\ \| & \sigma_1^{-1} \cong & \| \\ A' & \xrightarrow[c]{} & C' \end{array}$$

is such that $F\alpha = \beta$, which proves (vf2). □

5.2 Homotopy equivalences and the Whitehead theorem

Any model category satisfies a Whitehead theorem, stating that the weak equivalences between cofibrant-fibrant objects are precisely the homotopy equivalences; i.e., the morphisms $f\colon X \to Y$ such that there is a morphism $g\colon Y \to X$ with the property that fg and gf are homotopic to the identity. We begin by studying what the notion of homotopy entails in our setting; for this, let us first introduce the notion of horizontal pseudo natural equivalences.

Definition 5.6. Let $F, G\colon \mathbb{A} \to \mathbb{B}$ be double functors. A horizontal pseudo natural transformation $h\colon F \Rightarrow G$ is a **horizontal pseudo natural equivalence** if

(i) the horizontal morphism $h_A\colon FA \to GA$ is a horizontal equivalence in $\mathbb{B}$, for each object $A \in \mathbb{A}$, and

(ii) the square $h_u\colon (Fu \,{}^{h_A}_{h_{A'}}\, Gu)$ is weakly horizontally invertible in $\mathbb{B}$, for each vertical morphism $u\colon A \to\!\!\!\!\!\bullet\;\, A'$ in $\mathbb{A}$.

If the horizontal morphisms $h_A\colon FA \to GA$ are in addition horizontal adjoint equivalences in $\mathbb{B}$, we say that h is a **horizontal pseudo natural adjoint equivalence**.

We write $h\colon F \simeq G$ for such a horizontal pseudo natural transformation.

Remark 5.7. By [18, Lemma A.3.3], a horizontal pseudo natural (adjoint) equivalence as above is precisely an (adjoint) equivalence in the 2-category $\mathbf{H}[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$, or equivalently, a horizontal (adjoint) equivalence in the double category $[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$.

With this definition in hand, we get the following characterization of homotopic double functors.

Proposition 5.8. *Let $F, G\colon \mathbb{A} \to \mathbb{B}$ be double functors. Then F and G are homotopic via the path object $\mathrm{Path}(\mathbb{B})$ of Definition 3.16 if and only if there is a horizontal pseudo natural adjoint equivalence $F \simeq G$.*

Proof. Recall that the path object $\mathrm{Path}(\mathbb{B})$ of Definition 3.16 is given by the pseudo hom double category $[\mathbb{H}E_{\mathrm{adj}}, \mathbb{B}]_{\mathrm{ps}}$, where the 2-category E_{adj} is the free-living adjoint equivalence $\{0 \xrightarrow{\simeq} 1\}$. Therefore, a homotopy between double functors $F, G\colon \mathbb{A} \to \mathbb{B}$ via the path object $\mathrm{Path}(\mathbb{B})$ is a double functor $h\colon \mathbb{A} \to [\mathbb{H}E_{\mathrm{adj}}, \mathbb{B}]_{\mathrm{ps}}$ such that $Ph = (F, G)$ or, equivalently, a double functor

$$\widehat{h}\colon \mathbb{H}E_{\mathrm{adj}} \to [\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$$

such that $\widehat{h}(0) = F$ and $\widehat{h}(1) = G$. This corresponds to a horizontal pseudo natural adjoint equivalence $F \simeq G$ by Remark 5.7. □

Remark 5.9. By the usual Whitehead theorem (see, for example, [3, Lemma 4.24]), a morphism between cofibrant-fibrant objects in a model category is a weak equivalence if and only if it is a homotopy equivalence. Hence, since all double categories are fibrant in the model structure of Theorem 3.18, we can use Proposition 5.8 to characterize double biequivalences between cofibrant objects in DblCat as those double functors which admit an inverse up to horizontal pseudo natural adjoint equivalence, i.e., double functors $F\colon \mathbb{A} \to \mathbb{B}$ such that there is a double functor $G\colon \mathbb{B} \to \mathbb{A}$ together with horizontal pseudo natural adjoint equivalences $\mathrm{id}_{\mathbb{A}} \simeq GF$ and $FG \simeq \mathrm{id}_{\mathbb{B}}$.

In our double categorical setting, we can prove a version of the Whitehead theorem for a wider class of weak equivalences, by only imposing a condition on their target double categories. However, in some cases, the homotopy inverse is not a strict double functor anymore, but it is rather pseudo in the horizontal direction.

Definition 5.10. A **horizontally pseudo double functor** $F\colon \mathbb{A} \to \mathbb{B}$ consists of maps on objects, horizontal morphisms, vertical morphisms, and squares, which are compatible with domains and codomains. These maps preserve identities and compositions of vertical morphisms and of squares strictly, but they preserve identities and compositions of horizontal morphisms only up to vertically invertible squares. These are submitted to associativity, unitality, and naturality conditions. See [8, Definition 3.5.1] for details (note, however, that our definition has reversed the roles of the horizontal and vertical directions).

If F strictly preserves horizontal identities, we say that F is **normal**.

Remark 5.11. Analogously to Remark 2.6 and Definition 5.6, we have notions of horizontal pseudo natural transformations and horizontal pseudo natural equivalences between horizontally pseudo double functors. See [8, §3.8] for precise definitions; note that our definition has reversed the roles of the horizontal and vertical directions.

Our class of double biequivalences contains in particular the double functors that have a horizontally pseudo inverse up to horizontal pseudo natural equivalence.

Proposition 5.12. *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. If there is a normal horizontally pseudo double functor $G\colon \mathbb{B} \to \mathbb{A}$ together with horizontal* *pseudo natural equivalences* $\eta\colon \mathrm{id}_{\mathbb{A}} \simeq GF$ *and* $\epsilon\colon FG \simeq \mathrm{id}_{\mathbb{B}}$*, then* F *is a double biequivalence.*

Proof. Under these assumptions, the double functor F is in particular a horizontal biequivalence as introduced in [20, Definition 8.8]. Therefore F is a double biequivalence by [20, Proposition 8.11]. □

By only requiring that the target of a double biequivalence F does not contain any non-trivial composites of vertical morphisms, we can construct a horizontally pseudo double functor which gives a homotopy inverse of F. As the construction of this homotopy inverse is practically identical to the one in [20, Proposition 8.12], we only specify here the data of the pseudo inverse and of one of the horizontal pseudo natural equivalences, and refer the reader to the proof of [20, Proposition 8.12] for details.

Theorem 5.13. *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be double categories such that the underlying vertical category* $U\mathbf{V}\mathbb{B}$ *is a disjoint union of copies of* $\mathbb{1}$ *and* $\mathbb{2}$*. Then a double functor* $F\colon \mathbb{A} \to \mathbb{B}$ *is a double biequivalence if and only if there is a normal horizontally pseudo double functor* $G\colon \mathbb{B} \to \mathbb{A}$*, and horizontal pseudo natural equivalences* $\eta\colon \mathrm{id}_{\mathbb{A}} \simeq GF$ *and* $\epsilon\colon FG \simeq \mathrm{id}_{\mathbb{B}}$.

Proof. By Proposition 5.12, we directly get the converse implication.

Now suppose that F is a double biequivalence. We highlight the definition of the horizontally pseudo double functor $G\colon \mathbb{B} \to \mathbb{A}$ and the horizontal pseudo natural equivalence $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathbb{B}}$ on objects and vertical morphisms as it is the only part of the construction that differs from [20, Proposition 8.12]. One can easily check that the rest of the proof of [20, Proposition 8.12] does not depend on the weakly horizontally invariant condition that is not required in this statement, and thus can be applied verbatim.

To define G and ϵ on objects and vertical morphisms, we give the values of G and ϵ on each copy of $\mathbb{1}$ and $\mathbb{2}$ in $U\mathbf{V}\mathbb{B}$.

- Given a copy of the form $B\colon \mathbb{1} \to U\mathbf{V}\mathbb{B}$, by (db1) applied to the object $B \in \mathbb{B}$, we get an object $A \in \mathbb{A}$ and a horizontal equivalence $f\colon FA \xrightarrow{\simeq} B$ in $\mathbb{B}$. We set $GB := A$ and $\epsilon_B := f\colon FGB \xrightarrow{\simeq} B$.
- Given a copy of the form $v\colon \mathbb{2} \to U\mathbf{V}\mathbb{B}$, by (db3) applied to the vertical morphism $v\colon B \twoheadrightarrow B'$ in $\mathbb{B}$, we get a vertical morphism $u\colon A \twoheadrightarrow A'$ in $\mathbb{A}$ and a weakly horizontally invertible square β in $\mathbb{B}$ as follows.

$$\begin{array}{ccc} FA & \xrightarrow[\simeq]{f} & B \\ {\scriptstyle Fu}\big\downarrow\!\bullet & \beta \simeq & \bullet\!\big\downarrow{\scriptstyle v} \\ FA' & \xrightarrow[g]{\simeq} & B' \end{array}$$

We set $GB := A$, $GB' := A'$, and $Gv := u$, and we set $\epsilon_B := f\colon FGB \xrightarrow{\simeq} B$, $\epsilon_{B'} := g\colon FGB' \xrightarrow{\simeq} B'$, and $\epsilon_v := \beta\colon (FGv \, {}^{\epsilon_B}_{\epsilon_{B'}} \, v)$.

As there are no composites of vertical morphisms in $\mathbb{B}$, G and ϵ are trivially compatible with vertical morphisms. □

Remark 5.14. If we further require that the double category $\mathbb{B}$ in Theorem 5.13 is cofibrant, we can construct the weak inverse $G\colon \mathbb{B} \to \mathbb{A}$ of F in such a way that it is a strict double functor, since the underlying horizontal category of $\mathbb{B}$ is free. This subsumes the usual Whitehead theorem mentioned in Remark 5.9.

Finally, as a horizontal double category has a discrete underlying vertical category, the result applies in particular to the case where $\mathbb{B}$ is horizontal. We then retrieve the Whitehead theorem for 2-categories, which can be found in [14, Theorem 7.4.1].

Corollary 5.15. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories. Then a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a biequivalence if and only if there is a normal pseudo functor $G\colon \mathcal{B} \to \mathcal{A}$ together with pseudo natural equivalences $\eta\colon \mathrm{id}_{\mathcal{A}} \simeq GF$ and $\epsilon\colon FG \simeq \mathrm{id}_{\mathcal{B}}$.*

Proof. Since F is a biequivalence if and only if $\mathbb{H}F$ is a double biequivalence, as we will see in Theorem 6.5, and $\mathbb{H}\mathcal{B}$ is horizontal, we can apply Theorem 5.13 to $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$. Then $\mathbb{H}F$ is a double biequivalence if and only if there is a normal horizontally pseudo double functor $G'\colon \mathbb{H}\mathcal{B} \to \mathbb{H}\mathcal{A}$ together with horizontal pseudo natural equivalences $\eta'\colon \mathrm{id}_{\mathbb{H}\mathcal{A}} \simeq G'(\mathbb{H}F)$ and $\epsilon'\colon (\mathbb{H}F)G' \simeq \mathrm{id}_{\mathbb{H}\mathcal{B}}$. As normal horizontally pseudo double functors and horizontal pseudo natural equivalence between double categories in the image of $\mathbb{H}$ are equivalently normal pseudo functors and pseudo natural equivalences between their preimages, the data (G', η', ϵ') for $\mathbb{H}F$ uniquely correspond to a data (G, η, ϵ) for F as required. □

6. Quillen pairs between DblCat and $2\mathrm{Cat}$

In this paper, the model structure on DblCat was constructed in such a way as to be compatible with the Lack model structure on $2\mathrm{Cat}$ through the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$. We now study the precise relation between these model structures.

We present here the two Quillen pairs involving the horizontal embedding functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ and its right and left adjoints.

Proposition 6.1. *The adjunction*

$$2\mathrm{Cat} \underset{\mathbf{H}}{\overset{\mathbb{H}}{\rightleftarrows}} \mathrm{DblCat} \quad (\mathbb{H} \dashv \mathbf{H})$$

is a Quillen pair, where $2\mathrm{Cat}$ *is endowed with the Lack model structure and* DblCat *is endowed with the model structure of Theorem 3.18. Moreover, its derived unit is levelwise a biequivalence; in particular, this says that the functor* $\mathbb{H}$ *is homotopically fully faithful.*

Proof. Since the functor $(\mathbf{H}, \mathcal{V})\colon \mathrm{DblCat} \to 2\mathrm{Cat}\times 2\mathrm{Cat}$ and the projection $\mathrm{pr}_1\colon 2\mathrm{Cat} \times 2\mathrm{Cat} \to 2\mathrm{Cat}$ are right Quillen, then so is their composite $\mathbf{H}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$, which proves that $\mathbb{H} \dashv \mathbf{H}$ is a Quillen pair. Moreover, since every object in DblCat is fibrant, the derived unit of the adjunction $\mathbb{H} \dashv \mathbf{H}$ is given by the components of the unit at cofibrant objects, and is therefore levelwise an identity, by Proposition 2.9. □

The functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ is also right Quillen. The existence of its left adjoint is given by the Adjoint Functor Theorem, since $\mathbb{H}$ preserves all limits and colimits between locally presentable categories.

Theorem 6.2. *The adjunction*

$$\mathrm{DblCat} \underset{\mathbb{H}}{\overset{L}{\rightleftarrows}} 2\mathrm{Cat} \quad (L \dashv \mathbb{H})$$

is a Quillen pair, where $2\mathrm{Cat}$ *is endowed with the Lack model structure and* DblCat *is endowed with the model structure of Theorem 3.18.*

Proof. We show that $\mathbb{H}$ is right Quillen, i.e., it preserves fibrations and trivial fibrations.

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a fibration in $2\mathrm{Cat}$; we prove that $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ is a double fibration in DblCat. Since $\mathbf{H}\mathbb{H}F = F$ and F is a fibration, (df1-2) of Definition 3.8 are satisfied. It remains to show (df3) of Definition 3.8. Let us consider a weakly horizontally invertible square in $\mathbb{H}\mathcal{B}$

$$\begin{array}{ccc} B & \xrightarrow[b]{\simeq} & FC \\ \| & \beta \,\wr\| & \| \\ B & \xrightarrow[d]{\simeq} & FC\,. \end{array}$$

Note that its vertical boundaries must be trivial, since all vertical morphisms in $\mathbb{H}\mathcal{B}$ are identities. Then the square β is, in particular, vertically invertible by Lemma 2.19. Since F is a fibration in $2\mathrm{Cat}$, there is an equivalence $c\colon A \xrightarrow{\simeq} C$ such that $Fc = d$, by (f1) of Definition 3.2. Now β can be rewritten as

$$\begin{array}{ccc} FA & \xrightarrow[b]{\simeq} & FC \\ \| & \beta \,\wr\| & \| \\ FA & \xrightarrow[Fc]{\simeq} & FC\,. \end{array}$$

Then β is equivalently an invertible 2-cell $\beta\colon b \cong Fc$ in $\mathcal{B}$. Since F is a fibration in $2\mathrm{Cat}$, there is a morphism $a\colon A \to C$ in $\mathcal{A}$ and an invertible 2-cell $\alpha\colon a \cong c$ in $\mathcal{A}$ such that $F\alpha = \beta$, by (f2) of Definition 3.2. In particular, since c is an equivalence in $\mathcal{A}$, then so is a. This gives a vertically invertible square in $\mathbb{H}\mathcal{A}$ of the form

$$\begin{array}{ccc} A & \xrightarrow[a]{\simeq} & C \\ \| & \alpha \,\wr\| & \| \\ A & \xrightarrow[c]{\simeq} & C \end{array}$$

such that $F\alpha = \beta$; furthermore, by Lemma 2.19, the square α is weakly horizontally invertible. This shows that $\mathbb{H}F$ is a double fibration.

Now let the 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ be a trivial fibration. We show that $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ is a double trivial fibration in DblCat. Since $\mathbf{H}\mathbb{H}F = F$ and F is a trivial fibration, it satisfies (dt1-2) of Definition 3.9. Then (dt3) of Definition 3.9 follows from the fact that F is surjective on objects, since all vertical morphisms are identities. Finally, (dt4) of Definition 3.9 is a direct consequence of F being fully faithful on 2-cells, since all squares in $\mathbb{H}\mathcal{A}$ and $\mathbb{H}\mathcal{B}$ are equivalently 2-cells in $\mathcal{A}$ and $\mathcal{B}$, respectively. This shows that $\mathbb{H}F$ is a double trivial fibration, and concludes the proof. □

Remark 6.3. As we have seen in Proposition 6.1, the functor $\mathbb{H}$ is homotopically fully faithful, and therefore the derived counit of the adjunction $L \dashv \mathbb{H}$ is levelwise a biequivalence.

Remark 6.4. As a consequence of Proposition 6.1 and Theorem 6.2, we see that the functor $\mathbb{H}\colon \mathrm{2Cat} \to \mathrm{DblCat}$ preserves all cofibrations, fibrations, and weak equivalences. Indeed, the fact that it preserves cofibrations and fibrations follows from the fact that $\mathbb{H}$ is both left and right Quillen, while the fact that it preserves weak equivalences is a consequence of Ken Brown's Lemma (see [13, Lemma 1.1.12]), since all objects in $\mathrm{2Cat}$ are fibrant.

In fact, more is true: the horizontal embedding $\mathbb{H}$ also reflects cofibrations, fibrations, and weak equivalences, as we deduce from the following.

Theorem 6.5. *The Lack model structure on* $\mathrm{2Cat}$ *is both left- and right-induced along the adjunctions*

$$\mathrm{2Cat} \; \overset{L}{\underset{\mathbf{H}}{\leftleftarrows}} \!\!\!\! \xrightarrow[\perp]{\;\perp\;\mathbb{H}\;} \mathrm{DblCat}\,,$$

where DblCat *is endowed with the model structure of Theorem 3.18.*

Proof. To show this result, it is enough to prove that a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a biequivalence (resp. Lack fibration, cofibration) in $\mathrm{2Cat}$ if and only if the double functor $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ is a double biequivalence (resp. double fibration, cofibration) in DblCat, as a model structure is uniquely determined by its classes of weak equivalences and fibrations, or alternatively by its classes of weak equivalences and cofibrations.

By Remark 6.4, we have that if F is a biequivalence (resp. Lack fibration, cofibration) in $2\mathrm{Cat}$, then $\mathbb{H}F$ is a double biequivalence (resp. double fibration, cofibration) in DblCat, as $\mathbb{H}$ preserves all of these classes of morphisms.

Conversely, if $\mathbb{H}F$ is a double biequivalence (resp. double fibration), then $\mathbf{H}\mathbb{H}F = F$ is a biequivalence (resp. Lack fibration) by definition of the model structure on DblCat.

It remains to show that if $\mathbb{H}F$ is a cofibration, then so is F. For this, suppose that $\mathbb{H}F$ is a cofibration in DblCat; we show that F has the left lifting property with respect to all trivial fibrations in $2\mathrm{Cat}$. Let $P\colon \mathcal{X} \to \mathcal{Y}$ be a trivial fibration in $2\mathrm{Cat}$ and consider a commutative square as below.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{G} & \mathcal{X} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle P} \\ \mathcal{B} & \xrightarrow[H]{} & \mathcal{Y} \end{array}$$

Since $\mathbb{H}$ preserves trivial fibrations, we have that $\mathbb{H}P$ is a double trivial fibration. Then, as $\mathbb{H}F$ is a cofibration, there is a lift in the diagram below left. By the adjunction $\mathbb{H} \dashv \mathbf{H}$, this corresponds to a lift in the diagram below right, which concludes the proof.

$$\begin{array}{ccc} \mathbb{H}\mathcal{A} & \xrightarrow{\mathbb{H}G} & \mathbb{H}\mathcal{X} \\ {\scriptstyle \mathbb{H}F}\downarrow & \nearrow & \downarrow{\scriptstyle \mathbb{H}P} \\ \mathbb{H}\mathcal{B} & \xrightarrow[\mathbb{H}H]{} & \mathbb{H}\mathcal{Y} \end{array} \qquad \begin{array}{ccc} \mathcal{A} & \xrightarrow{G} & \mathbf{H}\mathbb{H}\mathcal{X} = \mathcal{X} \\ {\scriptstyle F}\downarrow & \nearrow & \downarrow{\scriptstyle \mathbf{H}\mathbb{H}P = P} \\ \mathcal{B} & \xrightarrow[H]{} & \mathbf{H}\mathbb{H}\mathcal{Y} = \mathcal{Y} \end{array}$$

□

We saw that the derived unit (resp. counit) of the adjunction $\mathbb{H} \dashv \mathbf{H}$ (resp. $L \dashv \mathbb{H}$) is levelwise a biequivalence. However, these adjunctions are not expected to be Quillen equivalences, since the homotopy theory of double categories should be richer than that of 2-categories. This is indeed the case, as shown in the following remarks.

Remark 6.6. The components of the derived counit of the adjunction $\mathbb{H} \dashv \mathbf{H}$ are not double biequivalences. To see this, consider the double category $\mathbb{V}\mathbb{2}$ free on a vertical morphism. Since $\mathbf{H}\mathbb{V}\mathbb{2} \cong \mathbb{1} \sqcup \mathbb{1}$ is cofibrant in $2\mathrm{Cat}$, the component of the derived counit at $\mathbb{V}\mathbb{2}$ is given by the component of the counit

$$\epsilon_{\mathbb{V}\mathbb{2}} \colon \mathbb{H}\mathbf{H}(\mathbb{V}\mathbb{2}) \cong \mathbb{1} \sqcup \mathbb{1} \to \mathbb{V}\mathbb{2},$$

which is not a double biequivalence, as it does not satisfy (db3) of Definition 3.6.

Remark 6.7. The components of the derived unit of the adjunction $L \dashv \mathbb{H}$ are not double biequivalences. By Proposition 4.3, the unique map $I_3 \colon \emptyset \to \mathbb{V}\mathbb{2}$ is a generating cofibration in DblCat, so that $\mathbb{V}\mathbb{2}$ is cofibrant. Since all objects in $2\mathrm{Cat}$ are fibrant, the component of the derived unit at $\mathbb{V}\mathbb{2}$ is given by the component of the unit

$$\eta_{\mathbb{V}\mathbb{2}} \colon \mathbb{V}\mathbb{2} \to \mathbb{H}L(\mathbb{V}\mathbb{2}) \cong \mathbb{1},$$

which is not a double biequivalence, as it does not satisfy (db2) of Definition 3.6. Note that the isomorphism above comes from the fact that the left adjoint L collapses the vertical structure and thus $L\mathbb{V}\mathbb{2} \cong \mathbb{1}$.

Remark 6.8. Since we induced the model structure on DblCat along $\mathbb{H} \sqcup \mathbb{L} \dashv (\mathbf{H}, \mathcal{V})$, we also get that the adjunction $\mathbb{L} \dashv \mathcal{V}$ forms a Quillen pair between $2\mathrm{Cat}$ and DblCat. However, note that neither the derived unit nor counit of $\mathbb{L} \dashv \mathcal{V}$ are levelwise weak equivalences.

7. $2\mathrm{Cat}$-enrichment of the model structure on DblCat

The aim of this section is to provide a $2\mathrm{Cat}$-enrichment on DblCat which is compatible with the model structure introduced in Theorem 3.18. Recall that a model category $\mathcal{M}$ is said to be *enriched* over a closed monoidal category $\mathcal{N}$ that is also a model category, if it is a tensored and cotensored $\mathcal{N}$-enriched category and it satisfies the pushout-product axiom (see for example [19, §5] for more details). In particular, the category $\mathcal{N}$ is said to be a *monoidal model category* if its model structure is enriched over itself.

7.1 The model structure on DblCat is not monoidal

In [15, Example 7.2], it is shown that the Lack model structure is not monoidal with respect to the cartesian product. As shown in the remark below, a similar argument also applies in the case of DblCat.

Remark 7.1. By Proposition 4.3, the inclusion $I_2 \colon \mathbb{1} \sqcup \mathbb{1} \to \mathbb{H}\mathbb{2}$ is a generating cofibration in DblCat. However, the pushout product $I_2 \,\square\, I_2$ with respect to the cartesian product is the double functor from the non-commutative square of horizontal morphisms to the commutative square of horizontal morphisms, as in [15, Example 7.2]. Since cofibrations in DblCat are in particular faithful on horizontal morphisms by Remark 4.8, the pushout-product $I_2 \,\square\, I_2$ cannot be a cofibration in DblCat.

As stated in Theorem 3.5, Lack's model structure on $2\mathrm{Cat}$ is monoidal with respect to the Gray tensor product. However, since the cofibrations in DblCat are not as well behaved in the vertical direction as in the horizontal direction; e.g., the underlying vertical category of a cofibrant double category is only a disjoint union of copies of $\mathbb{1}$ and $\mathbb{2}$ rather than a free category, our model structure is not compatible with the Gray tensor product on DblCat (see Proposition 2.5), as we show below.

Notation 7.2. Let $I \colon \mathbb{A} \to \mathbb{B}$ and $J \colon \mathbb{A}' \to \mathbb{B}'$ be double functors in DblCat. We write $I \,\square_{\mathrm{Gr}}\, J$ for their pushout-product

$$I \,\square_{\mathrm{Gr}}\, J \colon \mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{B}' \coprod_{\mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{A}'} \mathbb{B} \otimes_{\mathrm{Gr}} \mathbb{A}' \to \mathbb{B} \otimes_{\mathrm{Gr}} \mathbb{B}'$$

with respect to the Gray tensor product $\otimes_{\mathrm{Gr}}$ on DblCat.

Remark 7.3. The model structure defined in Theorem 3.18 is not compatible with the Gray tensor product $\otimes_{\mathrm{Gr}}$. To see this, recall that $I_3 \colon \emptyset \to \mathbb{V}\mathbb{2}$ is a generating cofibration in DblCat by Proposition 4.3. However the pushout-product

$$I_3 \,\square_{\mathrm{Gr}}\, I_3 \colon \emptyset \to \mathbb{V}\mathbb{2} \otimes_{\mathrm{Gr}} \mathbb{V}\mathbb{2}$$

is not a cofibration, where $\mathbb{V}\mathbb{2} \otimes_{\mathrm{Gr}} \mathbb{V}\mathbb{2}$ is the double category generated by the following data

$$\begin{array}{ccc} 0 & = & 0 \\ \downarrow\!\bullet & & \downarrow\!\bullet \\ 0' & \cong & 1 \\ \downarrow\!\bullet & & \downarrow\!\bullet \\ 1' & = & 1'\,. \end{array}$$

Indeed, since the underlying vertical category of $\mathbb{V}\mathbb{2} \otimes_{\mathrm{Gr}} \mathbb{V}\mathbb{2}$ has non-trivial composites of vertical morphisms, this is not a cofibrant double category by Corollary 4.9.

7.2 $2\mathrm{Cat}$-enrichment of the model structure on DblCat

By restricting the Gray tensor product on DblCat along $\mathbb{H}$ in one of the variables, we get rid of the issue concerning the vertical structure that obstructs the compatibility with the model structure of Theorem 3.18. With this variation, we show that DblCat is a tensored and cotensored $2\mathrm{Cat}$-enriched category, and that the corresponding enrichment is now compatible with our model structure.

Definition 7.4. The tensoring functor $\otimes\colon 2\mathrm{Cat} \times \mathrm{DblCat} \to \mathrm{DblCat}$ is defined to be the composite

$$2\mathrm{Cat} \times \mathrm{DblCat} \xrightarrow{\mathbb{H}\times \mathrm{id}} \mathrm{DblCat} \times \mathrm{DblCat} \xrightarrow{\otimes_{\mathrm{Gr}}} \mathrm{DblCat}.$$

Proposition 7.5. *The category* DblCat *is enriched, tensored, and cotensored over* $2\mathrm{Cat}$*, with*

(i) hom 2*-categories given by* $\mathbf{H}[\mathbb{A},\mathbb{B}]_{\mathrm{ps}}$*, for all* $\mathbb{A},\mathbb{B} \in \mathrm{DblCat}$*,*

(ii) tensors given by $\mathcal{C} \otimes \mathbb{A}$*, for all* $\mathbb{A} \in \mathrm{DblCat}$ *and* $\mathcal{C} \in 2\mathrm{Cat}$*, where* $\otimes$ *is the tensoring functor of Definition 7.4, and*

(iii) cotensors given by $[\mathbb{H}\mathcal{C},\mathbb{B}]_{\mathrm{ps}}$*, for all* $\mathbb{B} \in \mathrm{DblCat}$ *and* $\mathcal{C} \in 2\mathrm{Cat}$*,*

where $[-,-]_{\mathrm{ps}}$ *is the pseudo hom double category of Proposition 2.5.*

Proof. This follows directly from the definition of $\otimes$, and the universal properties of the tensor $\otimes_{\mathrm{Gr}}$ and of the adjunction $\mathbb{H} \dashv \mathbf{H}$. □

We now present the main result of this section.

Theorem 7.6. *The model structure on* DblCat *of Theorem 3.18 is a* $2\mathrm{Cat}$*-enriched model structure, where the enrichment is given by* $\mathbf{H}[-,-]_{\mathrm{ps}}$*.*

The rest of this section is devoted to the proof of this theorem. With that goal, we first prove several auxiliary lemmas.

Notation 7.7. Let $i\colon \mathcal{A}\to\mathcal{B}$ and $j\colon \mathcal{A}'\to\mathcal{B}'$ be 2-functors in $2\mathrm{Cat}$, and let $I\colon \mathbb{A}\to\mathbb{B}$ be a double functor in DblCat. We denote by $i\square_2 j$ the pushout-product

$$i\,\square_2\, j\colon \mathcal{A}\otimes_2\mathcal{B}'\coprod_{\mathcal{A}\otimes_2\mathcal{A}'}\mathcal{B}\otimes_2\mathcal{A}'\to\mathcal{B}\otimes_2\mathcal{B}'$$

with respect to the Gray tensor product $\otimes_2$ on $2\mathrm{Cat}$ (see Notation 2.13), and we denote by $i\,\square\, I$ the pushout-product

$$i\,\square\, I\colon \mathcal{A}\otimes\mathbb{B}\coprod_{\mathcal{A}\otimes\mathbb{A}}\mathcal{B}\otimes\mathbb{A}\to\mathcal{B}\otimes\mathbb{B}$$

with respect to the tensoring functor $\otimes\colon 2\mathrm{Cat}\times\mathrm{DblCat}\to\mathrm{DblCat}$. In particular, we have that $i\,\square\, I=\mathbb{H}i\,\square_{\mathrm{Gr}}\, I$.

Lemma 7.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories. There is an isomorphism of double categories*

$$\mathcal{A}\otimes\mathbb{H}\mathcal{B}\cong\mathbb{H}(\mathcal{A}\otimes_2\mathcal{B}),$$

natural in $\mathcal{A}$ and $\mathcal{B}$.

Proof. By the universal properties of $\otimes$ and $\otimes_2$, the adjunction $\mathbb{H}\dashv\mathbf{H}$, and Lemma 2.14, we have an isomorphism

$$\begin{aligned}\mathrm{DblCat}(\mathcal{A}\otimes\mathbb{H}\mathcal{B},\mathbb{C})&\cong 2\mathrm{Cat}(\mathcal{A},\mathbf{H}[\mathbb{H}\mathcal{B},\mathbb{C}]_{\mathrm{ps}})\cong 2\mathrm{Cat}(\mathcal{A},\mathrm{Ps}[\mathcal{B},\mathbf{H}\mathbb{C}])\\&\cong 2\mathrm{Cat}(\mathcal{A}\otimes_2\mathcal{B},\mathbf{H}\mathbb{C})\cong\mathrm{DblCat}(\mathbb{H}(\mathcal{A}\otimes_2\mathcal{B}),\mathbb{C}),\end{aligned}$$

for every double category $\mathbb{C}$, which is natural in $\mathcal{A}$, $\mathcal{B}$, and $\mathbb{C}$. The result then follows from the Yoneda lemma. □

Remark 7.9. The natural isomorphism $\mathbf{H}[\mathbb{H}(-),-]_{\mathrm{ps}}\cong\mathrm{Ps}[-,\mathbf{H}(-)]$ implies that the adjunction $\mathbb{H}\dashv\mathbf{H}$ is enriched with respect to the $2\mathrm{Cat}$-enrichments $\mathbf{H}[-,-]_{\mathrm{ps}}$ and $\mathrm{Ps}[-,-]$ of DblCat and $2\mathrm{Cat}$, respectively.

Lemma 7.10. *Let $\mathcal{A}$ be a 2-category. There is an isomorphism of double categories*

$$\mathcal{A}\otimes\mathbb{V}\mathbb{2}\cong\mathbb{H}\mathcal{A}\times\mathbb{V}\mathbb{2},$$

natural in $\mathcal{A}$.

Proof. By the universal properties of $\otimes$ and $\times$, and the fact that by the proof of Lemma 2.14 $\mathbf{H}[\mathbb{V}2, \mathbb{B}]_{\mathrm{ps}} = \mathbf{H}[\mathbb{V}2, \mathbb{B}]$ for all $\mathbb{B} \in \mathrm{DblCat}$, we have an isomorphism

$$\begin{aligned}\mathrm{DblCat}(\mathcal{A} \otimes \mathbb{V}2, \mathbb{B}) &\cong 2\mathrm{Cat}(\mathcal{A}, \mathbf{H}[\mathbb{V}2, \mathbb{B}]_{\mathrm{ps}}) = 2\mathrm{Cat}(\mathcal{A}, \mathbf{H}[\mathbb{V}2, \mathbb{B}]) \\ &\cong \mathrm{DblCat}(\mathbb{H}\mathcal{A}, [\mathbb{V}2, \mathbb{B}]) \cong \mathrm{DblCat}(\mathbb{H}\mathcal{A} \times \mathbb{V}2, \mathbb{B}),\end{aligned}$$

for every double category $\mathbb{B}$, which is natural in $\mathcal{A}$ and $\mathbb{B}$. The result then follows from the Yoneda lemma. □

Lemma 7.11. *Let $i\colon \mathcal{A} \to \mathcal{B}$ and $j\colon \mathcal{A}' \to \mathcal{B}'$ be 2-functors in* $2\mathrm{Cat}$. *There are isomorphisms*

$$i \,\square\, \mathbb{H}j \cong \mathbb{H}(i \,\square_2\, j) \quad \textit{and} \quad i \,\square\, (\mathbb{H}j \times \mathbb{V}2) \cong \mathbb{H}(i \,\square_2\, j) \times \mathbb{V}2$$

in the arrow category DblCat^{2}.

Proof. Since $\mathbb{H}$ is a left adjoint, it preserves pushouts and, by Lemma 7.8, we have that it is compatible with the tensors $\otimes$ and $\otimes_2$. Therefore, we have $i \,\square\, \mathbb{H}j \cong \mathbb{H}(i \,\square_2\, j)$. By Lemma 7.10, by associativity of $\otimes_{\mathrm{Gr}}$, and by the first isomorphism, we then get that

$$\begin{aligned}i \,\square\, (\mathbb{H}j \times \mathbb{V}2) &\cong i \,\square\, (j \otimes \mathbb{V}2) \cong (i \,\square\, \mathbb{H}j) \otimes_{\mathrm{Gr}} \mathbb{V}2 \\ &\cong (i \,\square_2\, j) \otimes \mathbb{V}2 \cong \mathbb{H}(i \,\square_2\, j) \times \mathbb{V}2.\end{aligned}$$ □

We are now ready to prove Theorem 7.6.

Proof of Theorem 7.6. Recall from Proposition 4.1 that a set $\mathcal{I}$ of generating cofibrations and a set $\mathcal{J}$ of generating trivial cofibrations for the model structure on DblCat are given by morphisms of the form $\mathbb{H}j$ and $\mathbb{L}j = \mathbb{H}j \times \mathbb{V}2$, where j is a generating cofibration or a generating trivial cofibration in $2\mathrm{Cat}$, respectively.

We show that the pushout-product of a generating cofibration in $\mathcal{I}$ with any (trivial) cofibration in $2\mathrm{Cat}$ is a (trivial) cofibration in DblCat, and that the pushout-product of a generating trivial cofibration in $\mathcal{J}$ with any cofibration in $2\mathrm{Cat}$ is a trivial cofibration in DblCat.

Given cofibrations i and j in $2\mathrm{Cat}$, we know by Lemma 7.11 that

$$i \,\square\, \mathbb{H}j \cong \mathbb{H}(i \,\square_2\, j) \quad \text{and} \quad i \,\square\, (\mathbb{H}j \times \mathbb{V}2) \cong \mathbb{H}(i \,\square_2\, j) \times \mathbb{V}2 = \mathbb{L}(i \,\square_2\, j),$$

and by Theorem 3.5 that $i \square_2 j$ is also a cofibration in $2\mathrm{Cat}$, which is trivial when either i or j is. Since $\mathbb{H}$ and $\mathbb{L}$ preserve (trivial) cofibrations by Proposition 6.1 and Remark 6.8, then $\mathbb{H}(i \square_2 j)$ and $\mathbb{L}(i \square_2 j)$ are cofibrations in DblCat, which are trivial if either i or j is. Taking j to be a generating cofibration or generating trivial cofibration in $2\mathrm{Cat}$, we get the desired results. □

8. Comparison with other model structures on DblCat

In [6], Fiore, Paoli, and Pronk construct several model structures on the category DblCat of double categories. We show in this section that our model structure on DblCat is not related to their model structures in the following sense: the identity adjunction on DblCat is not a Quillen pair between the model structure of Theorem 3.18 and any of the model structures of [6]. This is not surprising, since our model structure was constructed in such a way that the functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ embeds the homotopy theory of $2\mathrm{Cat}$ into that of DblCat, while there seems to be no such relation between their model structures on DblCat and the Lack model structure on $2\mathrm{Cat}$, e.g. see end of Section 9 in [6]. Further evidence is given by the fact that our double biequivalences are 2-categorically induced, while the weak equivalences in the model structures of [6] are rather 1-categorically induced.

We start by recalling the categorical model structures on DblCat constructed in [6]. Since our primary interest is to compare them to our model structure, we only describe the weak equivalences; the curious reader is encouraged to visit their paper for further details.

The first model structure we recall is induced from the canonical model structure on Cat by means of the *vertical nerve*.

Definition 8.1 ([6, Definition 5.1])**.** The **vertical nerve** of double categories is the functor

$$N_v\colon \mathrm{DblCat} \to \mathrm{Cat}^{\Delta^{\mathrm{op}}}$$

sending a double category $\mathbb{A}$ to the simplicial category $N_v\mathbb{A}$ such that $(N_v\mathbb{A})_0$ is the category of objects and horizontal morphisms of $\mathbb{A}$, $(N_v\mathbb{A})_1$ is the category of vertical morphisms and squares of $\mathbb{A}$ and, for $n \geq 2$,

$$(N_v\mathbb{A})_n = (N_v\mathbb{A})_1 \times_{(N_v\mathbb{A})_0} \cdots \times_{(N_v\mathbb{A})_0} (N_v\mathbb{A})_1.$$

Proposition 8.2 ([6, Theorem 7.17])**.** *There is a model structure on* DblCat *in which a double functor F is a weak equivalence if and only if N_vF is levelwise an equivalence of categories.*

The next model structure on DblCat requires a different perspective. For a 2-category $\mathcal{A}$ that admits limits and colimits, there is a model structure on the underlying category $U\mathcal{A}$ in which the weak equivalences are precisely the equivalences of the 2-category $\mathcal{A}$; see [17]. When applying this construction to the 2-category DblCat_h of double categories, double functors, and horizontal natural transformations, one obtains the following model structure on DblCat; see [6, §8.4].

Proposition 8.3. *There is a model structure on* DblCat*, called the* trivial model structure*, in which a double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a weak equivalence if and only if it is an equivalence in the 2-category DblCat_h, i.e., there is a double functor $G\colon \mathbb{B} \to \mathbb{A}$ and two horizontal natural isomorphisms* $\mathrm{id}_{\mathbb{A}} \cong GF$ *and* $FG \cong \mathrm{id}_{\mathbb{B}}$.

Remark 8.4. By comparing this to our Whitehead theorems (see Section 5.2), we see that the weak equivalences in the model structure of Proposition 8.3 require stricter conditions than double biequivalences. Indeed, the units and counits in the statement above are horizontal *strict* natural *isomorphisms*, while in our Whitehead theorems they are horizontal *pseudo* natural *equivalences*. This further supports our claim that the weak equivalences in our model structure are a 2-categorical analogue, and therefore carry more information, than the weak equivalences already present in the literature.

The last model structure is of a more algebraic flavor. Let T be a 2-monad on a 2-category $\mathcal{A}$. In [17], Lack gives a construction of a model structure on the category of T-algebras, in which the weak equivalences are the morphisms of T-algebras whose underlying morphism in $\mathcal{A}$ is an equivalence. In particular, double categories can be seen as the algebras over a 2-monad on the 2-category $\mathrm{Cat}(\mathrm{Graph})$ whose objects are the category objects in graphs; see [6, §9]. This gives the following model structure.

Proposition 8.5. *There is a model structure on* DblCat*, called the* algebra model structure*, in which a double functor F is a weak equivalence if and only if its underlying morphism in the 2-category* $\mathrm{Cat}(\mathrm{Graph})$ *is an equivalence.*

Remark 8.6. In [6, Corollary 8.29 and Theorems 8.52 and 9.1], Fiore, Paoli, and Pronk show that the model structures on DblCat of Propositions 8.2, 8.3 and 8.5 coincide with model structures given by Grothendieck topologies, when double categories are seen as internal categories to Cat. Then, it follows from [6, Propositions 8.24 and 8.38] that a weak equivalence in the algebra model structure is in particular a weak equivalence in the model structure induced by the vertical nerve N_v.

Remark 8.7. At this point, we must mention that [5, 6] define other model structures on DblCat, which are not equivalent to any of the above. However, these are Thomason-like model structures, and are therefore not expected to have any relation to our model structure, which is categorical.

We now proceed to compare these three model structures on DblCat to the one defined in Theorem 3.18. Our strategy will be to find a trivial cofibration in our model structure that is not a weak equivalence in any of the other model structures. Let E_{adj} be the free-living adjoint equivalence 2-category $\{0 \xrightarrow{\simeq} 1\}$. By Proposition 4.3, the inclusion double functor $J_1 \colon \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}}$ at 0 is a generating trivial cofibration in our model structure on DblCat.

Lemma 8.8. *The double functor $J_1 \colon \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}}$ is not a weak equivalence in any of the model structures on* DblCat *of Propositions 8.2, 8.3 and 8.5.*

Proof. We first prove that J_1 is not a weak equivalence in the model structure on DblCat of Proposition 8.2 induced by the vertical nerve. For this, we need to show that

$$N_v(J_1) \colon N_v(\mathbb{1}) = \Delta\mathbb{1} \to N_v(\mathbb{H}E_{\mathrm{adj}})$$

is not a levelwise equivalence of categories. Indeed, the category $N_v(\mathbb{H}E_{\mathrm{adj}})_0$ is the free category generated by $\{0 \rightleftarrows 1\}$ which is not equivalent to $\mathbb{1}$.

By Remark 8.6, a weak equivalence in the algebra model structure on DblCat of Proposition 8.5 is in particular a weak equivalence in the model structure induced by the vertical nerve. Therefore J_1 is not a weak equivalence in the algebra model structure either.

Finally, we show that J_1 is not a weak equivalence in the trivial model structure on DblCat of Proposition 8.3. If J_1 was an equivalence in the 2-category DblCat_h, then its weak inverse would be given by the unique double functor $!\colon \mathbb{H}E_{\text{adj}} \to \mathbb{1}$ and we would have a horizontal natural isomorphism $\mathrm{id}_{\mathbb{H}E_{\text{adj}}} \cong J_1\,!$, where $J_1\,!$ is constant at 0. But such a horizontal natural isomorphism does not exist since 1 is not isomorphic to 0 in $\mathbb{H}E_{\text{adj}}$. Therefore J_1 is not an equivalence. □

Proposition 8.9. *The identity adjunction on* DblCat *is not a Quillen pair between the model structure of Theorem 3.18 and any of the model structures of Propositions 8.2, 8.3 and 8.5.*

Proof. We consider the identity functor $\mathrm{id}\colon \mathrm{DblCat} \to \mathrm{DblCat}$ from the model structure of Theorem 3.18 to any of the other model structures of Propositions 8.2, 8.3 and 8.5, and show that it is neither left nor right Quillen.

Since J_1 is a trivial cofibration in the model structure of Theorem 3.18, but is not a weak equivalence in any of the other model structures as shown in Lemma 8.8, we see that id does not preserve trivial cofibrations; therefore, it is not left Quillen. Moreover, every object is fibrant in the model structure of Theorem 3.18, so that if id was right Quillen, it would preserve all weak equivalences by Ken Brown's Lemma (see [13, Lemma 1.1.12]). However, it does not preserve the weak equivalence J_1, and thus it is not right Quillen. □

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