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SOMMAIRE

CAHIERS DE TOPOLOGIE ET
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WEAK FUSION 2-CATEGORIES

Thibault D. DECOPPET

Résumé. Nous introduisons un affaiblissement de la notion de 2-catégorie fusion donnée dans [6]. Ensuite, nous établissons un nombre de propriétés des 2-catégories (multi)fusion. En particulier, nous démontrons que les duaux à gauche et à droite d'un objet d'une 2-catégorie fusion coïncident. Finalement, nous décrivons la loi de fusion des 2-catégories fusion associées à certaines catégories fusion tressées qui sont pointées.

Abstract. We introduce a weakening of the notion of fusion 2-category given in [6]. Then, we establish a number of properties of (multi)fusion 2-categories. In particular, we prove that the left and the right duals of an object in a multifusion 2-category coincide. Finally, we describe the fusion rule of the fusion 2-categories associated to certain pointed braided fusion categories.

Keywords. Multifusion 2-Categories, Fusion 2-Categories, Braided Fusion Categories.

Mathematics Subject Classification (2020). 18M15, 18M20, 18N10, 18N25 (Primary), 18M05 (Secondary).

Introduction

We present an algebraic definition of fusion 2-categories that partially fills the gap between the two existing definitions ([6] and [13]). In fact, we show that our definition is essentially equivalent to that of [6]. Further, it is possible to show that the (separable) fusion 2-categories of [13] are examples of the objects we call fusion 2-categories. The proof is not difficult, but requires developing the theory of fusion 2-categories much further, so we postpone it to [5]. However, proving the converse is a much more delicate problem. Namely, it essentially amounts to proving that every fusion 2-category is a fully dualizable object in an appropriate symmetric monoidal 4-category. We hope to return to this point in following work, but, in the meantime, we present some elementary results on the structure of (multi)fusion 2-categories.

Let us briefly describe the content of the different sections of the present article. Section 1 contains results on monoidal 2-categories that we will be needed. We begin by reviewing adjoints. Providing we are working in a monoidal 2-category that has right adjoints, we show how these adjoints can be assembled to give a monoidal 2-functor. We go on to review the definitions of duals and of coherent duals. Further, given a 2-category that has right duals, we explain how they can be put together to define a 2-functor. We conjecture that this 2-functor is in fact monoidal. Finally, we explain some properties of monoidal 2-categories that have right adjoints and right duals.

Fusion 2-categories as defined in [6] are Gray monoids satisfying some properties. In section 2, we generalize their definition by starting with a monoidal 2-category in the sense of [15]. We also give a decomposition result for multifusion 2-categories that is analogous to that of multifusion 1-categories. Then, we show that our definition of a fusion 2-category is essentially equivalent to that of Douglas-Reutter. We continue by deriving some consequences of the existence of duals in multifusion 2-categories. Most notably, we show that right and left duals agree. Next, we show that the connected component of the unit in a fusion 2-category forms a fusion sub-2-category, and we prove that connected fusion 2-categories correspond precisely to braided fusion categories. Finally, we explain how to compute the fusion rule of the connected fusion 2-category associated to certain pointed braided fusion categories, generalizing a result of [8]. We use this to compute the fusion rules of some fusion 2-categories.

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1. Adjoints and Duals in Monoidal 2-Categories

1.1 Adjoints

The following definition is well-known.

Definition 1.1.1. Let $\mathfrak{C}$ be a 2-category and $f : A \to B$ a 1-morphism in $\mathfrak{C}$. A right-adjoint for f is a 1-morphism $f^* : B \to A$ together with 2-morphisms $\epsilon : f \circ f^* \Rightarrow Id_B$ and $\eta : Id_A \Rightarrow f^* \circ f$ satisfying the snake equations

$$(\epsilon \circ f) \cdot (f \circ \eta) = Id_f,$$
$$(f^* \circ \epsilon) \cdot (\eta \circ f^*) = Id_{f^*},$$

in which we have omitted the relevant coherence 2-isomorphisms. One defines left-adjoints dually.

Remark 1.1.2. It is well-known that a right-adjoint for a 1-morphism f is unique up to unique 2-isomorphism.

Notation 1.1.3. Let $\mathfrak{C}$ be a 2-category. We denote by $\mathfrak{C}^{2op}$ the 2-category obtained from $\mathfrak{C}$ by reversing the direction of the 2-morphisms. Analogously, we denote by $\mathfrak{C}^{1op}$ the 2-category obtained by reversing the direction of the 1-morphisms. If $\mathfrak{C}$ is monoidal, then it is clear that so is $\mathfrak{C}^{2op}$. The 2-category $\mathfrak{C}^{1op}$ can also be given a monoidal structure, but this construction is slightly trickier, as one needs to "invert" 1-morphisms. Using the algebraic definition of monoidal 2-category given in [15], this poses no problem.

Lemma 1.1.4. *Let $\mathfrak{C}$ be a 2-category with right-adjoints. There is a 2-functor*

$$(-)^* : \mathfrak{C} \to \mathfrak{C}^{1op;2op}$$

that is the identity on objects and sends a 1-morphism f to the 1-morphism underlying a chosen right adjoint. Dually, if $\mathfrak{C}$ has left-adjoints, there is a 2-functor denoted by ${}^*(-)$ that sends 1-morphisms to their left adjoints. Further, if $\mathfrak{C}$ has left and right-adjoints, ${}^*(-)$ is a pseudo-inverse for $(-)^*$.

Proof. The is well-known, for instance, see [11], or appendix A.1 below. □

If we assume in addition that $\mathfrak{C}$ is monoidal, then the 2-functor $(-)^*$ can be made monoidal.

Lemma 1.1.5. *Let $\mathfrak{C}$ be a monoidal 2-category with right-adjoints. The 2-functor $(-)^*$ of lemma 1.1.4 can be made monoidal. If, in addition, $\mathfrak{C}$ has left-adjoints, then the monoidal 2-functors ${}^*(-)$ is a monoidal pseudo-inverse for $(-)^*$.*

Proof. See appendix A.1 below. □

1.2 Duals

Some care has to be taken with respect to what one calls a dual in a monoidal 2-category $\mathfrak{C}$. For us, a right dual for an object A of $\mathfrak{C}$ consists of an object $A^\sharp$, together with two 1-morphisms $i_A : I \to A^\sharp \Box A$ and $e_A : A \Box A^\sharp \to I$ satisying the snake equations up to 2-isomorphisms. Similarly, one can give a definition of a left dual for A. These definitions have the advantage of being concise and easy to check, but they are not convenient to use in constructions. That is why we shall also need to consider the refinement defined in [14] called a coherent right dual.

Definition 1.2.1. Let A be an object of a monoidal 2-category $\mathfrak{C}$. A coherent right dual for A consists of an object $A^\sharp$ in $\mathfrak{C}$, 1-morphisms $i_A : I \to A^\sharp \Box A$ and $e_A : A \Box A^\sharp \to I$, and 2-isomorphisms[1]

$$C_A : (e_A \Box A) \circ a^\bullet_{A,A^\sharp,A} \circ (A \Box i_A) \Rightarrow Id_A,$$

$$D_A : Id_{A^\sharp} \Rightarrow (A^\sharp \Box e_A) \circ a_{A^\sharp,A,A^\sharp} \circ (i_A \Box A^\sharp),$$

satisfying the two swallowtail equations (depicted in figures 3 and 4 of [14]). We will also say that the data $(A, A^\sharp, i_A, e_A, C_A, D_A)$ is a coherent dual pair, or that $(A, A^\sharp, i_A, e_A, C_A, D_A)$ is a coherent left dual for $A^\sharp$.

[1]The notation $a^\bullet_{A,A^\sharp,A}$ is used in [15] to refer to the chosen ajoint pseudo-inverse of the 1-morphism $a_{A,A^\sharp,A}$.

Remark 1.2.2. In the notation of definition 1.2.1, if we assume that $\mathfrak{C}$ is a strict cubical monoidal 2-category, the swallowtail equations simplify to

$$\left[e_A \circ (C_A \Box A^\sharp)\right] \cdot \left[\phi_{(e_A, Id_I),(Id_{A\Box A^\sharp}, e_A)} \circ (A \Box i_A \Box A^\sharp)\right] \cdot \left[e_A \circ (A \Box D_A)\right] = Id_{e_A},$$

$$\left[(A^\sharp \Box C_A) \circ i_A\right] \cdot \left[(A^\sharp \Box e_A \Box A) \circ \phi_{(i_A, Id_I),(Id_{A^\sharp \Box A}, i_A)}\right] \cdot \left[(D_A \Box A) \circ i_A\right] = Id_{i_A}.$$

It is clear that every coherent right dual is a right dual. Hence, it is natural to ask whether every right dual can be made into a coherent right dual. This question was solved in [14].

Corollary 1.2.3. *[14, Corollary 2.8] Every right dual can be made coherent, and every left dual can be made coherent.*

Definition 1.2.4. Let $\mathfrak{C}$ be a monoidal 2-category. We say that $\mathfrak{C}$ has right duals, resp. left duals, if every object has a right dual, resp. left dual. We say that $\mathfrak{C}$ is rigid if it has right and left duals.

Using the above result from [14], we believe that one can construct a right dual 2-functor on any monoidal 2-category with right duals.

Notation 1.2.5. Given $\mathfrak{C}$ a monoidal 2-category with monoidal product $\Box$, we denote by $\mathfrak{C}^{\Box op}$ the monoidal 2-category with the opposite monoidal product.

Conjecture 1.2.6. *Let $\mathfrak{C}$ be a monoidal 2-category with right duals. There exists a monoidal 2-functor*

$$(-)^\sharp : \mathfrak{C} \to \mathfrak{C}^{\Box op;1op}$$

that sends an object A to the object underlying a right dual for A. If, in addition, $\mathfrak{C}$ has left duals, there is a monoidal 2-functor, which we denote by ${}^\sharp(-)$ that sends an object to its left dual. Further, ${}^\sharp(-)$ is a monoidal pseudo-inverse for $(-)^\sharp$.

Remark 1.2.7. We prove in lemma A.2.2 below that the underlying 2-functor exists. However, checking the coherence axioms for a monoidal 2-category involves making sure that big composites of interchangers agree, and we have not found a satisfactory way to deal with these.

The decategorified version of the next lemma is well-known.

Lemma 1.2.8. *Let $\mathfrak{C}$ be a monoidal 2-category, and A, B, and C be objects of $\mathfrak{C}$.*

1. *If C has a right dual, there are natural equivalences*

$$Hom_{\mathfrak{C}}(A, B\Box C) \simeq Hom_{\mathfrak{C}}(A\Box C^{\sharp}, B),$$
$$Hom_{\mathfrak{C}}(C\Box A, B) \simeq Hom_{\mathfrak{C}}(A, C^{\sharp}\Box B).$$

2. *If C has a left dual, there are natural equivalences*

$$Hom_{\mathfrak{C}}(A\Box C, B) \simeq Hom_{\mathfrak{C}}(A, B\Box^{\sharp}C),$$
$$Hom_{\mathfrak{C}}(A, C\Box B) \simeq Hom_{\mathfrak{C}}(^{\sharp}C\Box A, B).$$

Proof. Without loss of generality, we may assume that $\mathfrak{C}$ is strict cubical (this follows from the coherence theorem of [12]). Let $(C, C^{\sharp}, i_C, e_C, C_C, D_C)$ be a coherent dual pair. Then, the functors

$$\begin{array}{ccc} Hom_{\mathfrak{C}}(A, B\Box C) & \rightleftarrows & Hom_{\mathfrak{C}}(A\Box C^{\sharp}, B) \\ f & \mapsto & (B\Box e_C)\circ(f\Box C^{\sharp}) \\ (g\Box C)\circ(A\Box i_C) & \leftarrow\!\shortmid & g \end{array}$$

form an adjoint equivalence with counit

$$[(A\Box C_C)\circ f]\cdot[(B\Box e_C\Box C)\circ\phi_{(f,Id),(Id,i_C)}]$$

and unit

$$[\phi_{(g,Id),(Id,e_C)}\circ(A\Box i_C\Box C^{\sharp})]\cdot[g\circ(B\Box D_C)].$$

The triangle identites follow from the swallowtail equations. The naturality in A and B is clear from the definition. $\Box$

Remark 1.2.9. The above lemma can be reformulated by saying that certain 2-functors form a 2-adjunction. For instance, if C has a right dual, then $(-)\Box C^{\sharp}$ is left 2-adjoint to $(-)\Box C$.

1.3 Interactions

The monoidal 2-categories we will consider have both adjoints and duals. That is why we now examine the properties of such monoidal 2-categories.

Lemma 1.3.1. *Let $A^\sharp$ be a right dual for A in $\mathfrak{C}$, a monoidal 2-category with right adjoints, with unit i_A and counit e_A. Then $A^\sharp$ has a right dual.*

Proof. Using the monoidal 2-functor $(-)^*$ constructed in lemma 1.1.5, one gets that the image of i_A and e_A under $(-)^*$ witness that $A^\sharp$ is a right dual for A in $\mathfrak{C}^{1op;2op}$. This means that A is a right dual for $A^\sharp$ in $\mathfrak{C}$. $\square$

Remark 1.3.2. Lemma 1.3.1 can be generalized to n-categories, see [1, Lem. 4.1.2].

Corollary 1.3.3. *Let $\mathfrak{C}$ be a monoidal 2-category with right adjoints and right duals. For any object A of $\mathfrak{C}$, $A^{\sharp\sharp}$ is equivalent to A.*

Corollary 1.3.4. *Let $\mathfrak{C}$ be a monoidal 2-category with right adjoints. If $\mathfrak{C}$ has right duals, then it also has left duals.*

2. Fusion 2-Categories

Throughout, we work over a fixed algebraically closed field $\Bbbk$ of characteristic zero.

2.1 (Multi)Fusion 2-Categories

Definition 2.1.1. A multifusion 2-category $\mathfrak{C}$ is a finite semisimple 2-category ([4] definition 2.1.3) equipped with a rigid $\Bbbk$-linear monoidal structure. In particular, it comes equipped with a bilinear 2-functor

$$\square : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C},$$

and a monoidal unit I. A fusion 2-category is a multifusion 2-category whose monoidal unit is simple.

Remark 2.1.2. By their very definition, all the standard results of monoidal 2-category theory (up to linearization) apply to multifusion 2-categories. For instance, by [15], every multifusion 2-category is equivalent to a skeletal multifusion 2-category.

It is well-known that the monoidal unit of a multifusion category splits as a direct sum of non-isomorphic simple objects (see [7] section 4.3). Using the fact that every object of a finite semisimple 2-categories decomposes into a direct sum of simple objects (see proposition 1.4.5 of [6] and lemma 2.1.5 of [4]), a similar result holds for multifusion 2-categories.

Lemma 2.1.3. *Let $\mathfrak{C}$ be a multifusion 2-category. Let X_i, $i = 1, ..n$ be the finitely many simple objects appearing in the decomposition of the monoidal unit as a direct sum of simple objects, i.e.*

$$I \simeq \boxplus_{i=1}^{n} X_i.$$

Then, $Hom_{\mathfrak{C}}(X_i, X_j)$ is non-zero if and only if $i = j$. Moreover, we have that $X_i \Box X_j$ is equivalent to X_i if $i = j$ and to 0 otherwise.

Proof. For any i, we have $X_i \Box I \simeq X_i$. As X_i is simple, there exists precisely one j such that $X_i \Box X_j$ is non-zero. Together with the reverse argument on j, this shows that

$$X_i \simeq X_i \Box X_j \simeq X_j.$$

If $i \neq j$, then $X_i \Box I$ would have $(X_i \Box X_i) \boxplus (X_i \Box X_j)$ as a summand, whence would not be simple. Thus, we must have $i = j$ and $X_i \Box X_i \simeq X_i$. Moreover, this shows that X_i is both a left and a right dual for X_i.

Let i, j be arbitrary. Then, we have:

$$Hom_{\mathfrak{C}}(X_i, X_j) \simeq Hom_{\mathfrak{C}}(X_i \Box ({}^{\sharp}X_j), I) \simeq Hom_{\mathfrak{C}}(\delta_{ij} X_i, I).$$

The last term is non-zero precisely when $i = j$. This finishes the proof. □

Remark 2.1.4. Lemma 2.1.3 can be seen as a generalization of the fact that braided multifusion categories have no non-zero entries away from the diagonal (see lemma 5.3 of [3]).

Let $\mathfrak{C}$ be a multifusion 2-category. We write ${}_i\mathfrak{C}_j$ for the semisimple 2-category $X_i \Box \mathfrak{C} \Box X_j$. The following result is a direct analogue of the usual decomposition of a multifusion category.

Lemma 2.1.5. *The semisimple 2-categories ${}_i\mathfrak{C}_i$ are fusion 2-categories, and the finite semisimple 2-categories ${}_i\mathfrak{C}_j$ are $({}_i\mathfrak{C}_i, {}_j\mathfrak{C}_j)$-bimodule 2-categories. Finally, the following matrix*

$$\begin{pmatrix} {}_1\mathfrak{C}_1 & \cdots & {}_1\mathfrak{C}_n \\ \vdots & \ddots & \vdots \\ {}_n\mathfrak{C}_1 & \cdots & {}_n\mathfrak{C}_n \end{pmatrix}$$

represents the fusion rule of the multifusion 2-category $\mathfrak{C}$.

Example 2.1.6. The 2-category of representations of a finite 2-groupoid $\mathcal{G}$ is semisimple and finite by [6]. It inherits a (symmetric) monoidal structure from the symmetric monoidal structure on $\mathbf{2Vect}$. The monoidal unit is given by the constant 2-functor $\mathcal{G} \to \mathbf{2Vect}$ with value $\mathbf{Vect}$. This 2-representation splits as the direct sum of the simple 2-representations that are constant with value $\mathbf{Vect}$ on exactly one component of $\mathcal{G}$ and 0 on the others.

2.2 Comparison with strict Fusion 2-Categories

Douglas and Reutter have used in [6] the term fusion 2-category to refer to certain Gray monoids; We call such objects strict fusion 2-categories. Now, we want to compare their definition with ours.

Lemma 2.2.1. *Let $\mathfrak{C}$ be a multifusion 2-category. There exists a multifusion 2-category $\mathfrak{D}$, whose underlying monoidal 2-category is a strict cubical $\Bbbk$-linear monoidal 2-category, that is linearly equivalent to $\mathfrak{C}$. Moreover, if $\mathfrak{C}$ is fusion, so is $\mathfrak{D}$.*

Proof. Using a $\Bbbk$-linear version of the coherence theorem of [12], we obtain a strict cubical $\Bbbk$-linear monoidal 2-category $\mathfrak{D}$ that is linearly monoidally equivalent to $\mathfrak{C}$. Observe that the underlying linear equivalence of 2-categories witnesses that the 2-category $\mathfrak{D}$ is a finite semisimple 2-category. Moreover, rigidity is preserved by monoidal equivalences of 2-categories. This proves the first part of the result. The last part follows from the Whitehead theorem for monoidal 2-categories (see [15]). □

Lemma 2.2.2. *There is a bijection between weak fusion 2-categories, whose underlying monoidal 2-category is a strict cubical (or opcubical) $\mathbb{k}$-linear monoidal 2-category and strict fusion 2-categories.*[2]

Proof. The $\mathbb{k}$-linear version of [2] lemma 2.16 shows that there is a bijection between strict cubical $\mathbb{k}$-linear monoidal 2-categories and $\mathbb{k}$-linear Gray monoids with one object. Moreover, the equivalence does not affect the underlying 2-categories, and by lemma 2.1.4 of [4] its finite semisimple in the sense of [6]. Thus, the only thing we have to prove is that this bijection respects the existence of duals. This property follows from the fact that the monoidal product with a fixed 1-morphism is invariant under this bijection by construction. □

Remark 2.2.3. In particular, we may invoke all the results that [6] have proven for strict fusion 2-categories, and apply them to fusion 2-categories.

2.3 Duals in Multifusion 2-Categories

Specializing lemma 1.3.1 to multifusion 2-categories, we obtain the following lemma.

Lemma 2.3.1. *Let $\mathfrak{C}$ be a multifusion 2-category, and let $A^\sharp$ be a right dual for A in $\mathfrak{C}$, then A is a right dual for $A^\sharp$.*

On the one hand, the decategorified analogue of the next result is well-known: it says that left and right duals in a fusion category agree. The proof relies crucially on the category being semisimple. On the other hand, in the context of fusion 2-categories, the proof has a very distinct flavour; it uses lemma 2.3.1, which applies in great generality (see corollary 1.3.3).

Corollary 2.3.2. *Let $\mathfrak{C}$ be a multifusion 2-category, and A an object of $\mathfrak{C}$. Then, $A^{\sharp\sharp}$ is equivalent to A.*

Corollary 2.3.3. *Let $\mathfrak{C}$ be a monoidal finite semisimple 2-category. If $\mathfrak{C}$ has right duals, then it also has left duals, i.e it is multifusion.*

[2]This statement can be made rigorous using a set-theoretic argument. For instance, one could use a bigger universe.

As is the case in any multifusion category, the right dual of a simple object is again a simple object.

Lemma 2.3.4. *In any multifusion 2-category, the left and right duals of a simple object are simple.*

Proof. Let A be a simple object with right dual $A^\sharp$. Observe that the right dual of a non-zero object has to be non-zero. Further, a right dual for a direct sum is given by the direct sum of the right duals. Thus, if $A^\sharp$ were not simple, i.e. had two non-zero summands, then $A^{\sharp\sharp}$ would have two non-zero summands. This contradicts the fact that $A \simeq A^{\sharp\sharp}$ is simple. $\square$

Lemma 2.3.4 implies that the operation of taking the right dual induces a bijection on the set of equivalence classes of simple objects. Now, observe that lemma 1.2.8 also applies to multifusion 2-categories, yielding the following results:

Corollary 2.3.5. *Let $\mathfrak{C}$ be a multifusion 2-category, and A, B two simple objects such that $Hom_{\mathfrak{C}}(A, B)$ is non-trivial. Then, $Hom_{\mathfrak{C}}(A^\sharp, B^\sharp)$ is non trivial.*

Proof. Note that it is enough to prove that $Hom_{\mathfrak{C}}(B^\sharp, A^\sharp)$ is non-trivial. Namely, the 2-functor $(-)^*$ of lemma 1.1.4 provides us with a linear equivalence:

$$Hom_{\mathfrak{C}}(A^\sharp, B^\sharp) \simeq Hom_{\mathfrak{C}}(B^\sharp, A^\sharp).$$

Now, using lemma 1.2.8, there are linear equivalences:

$$Hom_{\mathfrak{C}}(B^\sharp, A^\sharp) \simeq Hom_{\mathfrak{C}}(A\Box B^\sharp, I) \simeq Hom_{\mathfrak{C}}(A, B).$$

This concludes the proof. $\square$

Corollary 2.3.6. *Let $\mathfrak{C}$ be a fusion 2-category, and A, J two simple objects such that J is in the component of the monoidal unit (i.e. $Hom_{\mathfrak{C}}(I, J)$ is non-zero), then $J\Box A$ is a direct sum of simple objects in the connected component of A.*

Proof. Let B be a simple summand of $J\Box A$. There exists a non-trivial 1-morphism between $J\Box A$ and B. Thence, there is a non-trivial 1-morphism between $B\Box A^\sharp$ and J. By proposition 1.2.19 of [6], this implies that there exists a non-trivial 1-morphism between $B\Box A^\sharp$ and I, which is equivalent to saying that A and B are in the same connected component. $\square$

We examine the behaviour of simple objects under the monoidal product and arbitrary 2-functors.

Lemma 2.3.7. *Let $\mathfrak{C}$ be a fusion 2-category and C, D two non-zero objects. Then $C\Box D$ is non-zero.*

Proof. By rigidity, D has a right dual $D^\sharp$. In particular, the decomposition of $D\Box D^\sharp$ into simple objects contains a copy of J, a simple object in the connected component of I. By definition, there exists a non-zero 1-morphism $f: I\rightarrow J$. Thus, we get a map

$$C\Box I \xrightarrow{C\Box f} C\Box J \hookrightarrow C\Box D\Box D^\sharp.$$

On one hand, if $C\Box D$ were equivalent to zero, then $C\Box J\simeq 0$, whence we would have $C\Box f\simeq 0$. On the other hand, f has a left adjoint ${}^*f$, and the 2-functor $C\Box(-)$ preserves adjunctions. As Id_I is a direct summand of ${}^*f\circ f$, we find that $C\Box f\neq 0$. Consequently, $C\Box D$ must be non-zero. □

Proposition 2.3.8. *Let $F:\mathfrak{C}\rightarrow\mathfrak{D}$ be a monoidal 2-functor from a fusion 2-category to a multifusion 2-category. For any non-zero object C of $\mathfrak{C}$, we have that $F(C)$ is non-zero.*

Proof. As F is a monoidal 2-functor, we know that $F(I)$ is non-zero. Now, the evaluation 1-morphism $e_C: C\Box C^\sharp\rightarrow I$ is non-zero, and has a left adjoint. As F preserves adjunctions, we find that $F(e_C)$ is non-zero. Given that $F(C)\Box F(C^\sharp)\simeq F(C\Box C^\sharp)$, this implies that $F(C)$ is non-zero. □

2.4 Connected Fusion 2-Categories

The goal of this section is to study a special class of fusion 2-categories: connected fusion 2-categories. They are ubiquitous both because the 2-category of finite semisimple module categories associated to a braided fusion category is connected, and because every fusion 2-category has a connected fusion 2-category as a full sub-2-category. This is similar to the fact that every topological monoid admits a connected submonoid given by the connected component of the identity.

Definition 2.4.1. A connected fusion 2-category is a fusion 2-category whose underlying finite semisimple 2-category is connected as in definition 1.2.22 of [6].

Remark 2.4.2. By the categorical Schur lemma, i.e. proposition 1.2.19 of [6], in order to show that a fusion 2-category is connected, it is enough to check that the Hom-categories from the monoidal unit to any simple object is non-trivial.

Given a fusion category $\mathcal{C}$, we write $\mathbf{Mod}(\mathcal{C}) := Cau(\mathrm{B}\mathcal{C})$ for the Cauchy completion of $\mathrm{B}\mathcal{C}$ in the sense of [10]. By theorem 3.1.7 of [10], we may also think of $\mathbf{Mod}(\mathcal{C})$ as the 2-category of separable algebras, bimodules, and bimodule maps in $\mathcal{C}$. Thus, $\mathbf{Mod}(\mathcal{C})$ is a finite semisimple 2-category by theorem 1.4.8 of [6], and it is connected by proposition 2.3.5 of [4]. Further, by proposition 1.3.13 of [6], $\mathbf{Mod}(\mathcal{C})$ is equivalent to the 2-category of finite semisimple right $\mathcal{C}$-module categories. The equivalence sends a separable algebra A in $\mathcal{C}$ to $Mod_{\mathcal{C}}(A)$ the finite semisimple category of right A-modules in $\mathcal{C}$. Now, if we equip $\mathcal{C}$ with a braiding, more can be said.

Proposition 2.4.3. *[6, Construction 2.1.19] Let $\mathcal{C}$ be a braided fusion category. Then, $\mathbf{Mod}(\mathcal{C})$ is a connected fusion 2-category, with monoidal product given by $\boxtimes_{\mathcal{C}}$ the balanced Deligne tensor product.*

Proof. Note that the monoidal 2-category $\mathrm{B}\mathcal{C}$ is rigid. Thence, through the proof of theorem 4.1.1 of [10], we find that its Cauchy completion, $\mathbf{Mod}(\mathcal{C})$, is a multifusion 2-category with monoidal product $\Box$. Explicitly, the monoidal structure is as follows: Given two separable algebras A, B in $\mathcal{C}$, representing two right $\mathcal{C}$-module categories $Mod_{\mathcal{C}}(A)$ and $Mod_{\mathcal{C}}(B)$, their product $A \otimes B$ is again a separable algebra in $\mathcal{C}$. The separable algebra $A \otimes B$ represents an object $Mod_{\mathcal{C}}(A \otimes B)$ in $\mathbf{Mod}(\mathcal{C})$, which is, by construction, the monoidal product of $Mod_{\mathcal{C}}(A)$ and $Mod_{\mathcal{C}}(B)$. The result follows from the equivalence of right $\mathcal{C}$-module categories:

$$Mod_{\mathcal{C}}(A) \boxtimes_{\mathcal{C}} Mod_{\mathcal{C}}(B) \simeq Mod_{\mathcal{C}}(A) \boxtimes_{\mathcal{C}} RMod_{\mathcal{C}}(B^{op})$$

$$\simeq Bimod_{\mathcal{C}}(A, B^{op}) \simeq Mod_{\mathcal{C}}(A \otimes B) = Mod_{\mathcal{C}}(A) \Box Mod_{\mathcal{C}}(B),$$

where we have used the equivalence of right $\mathcal{C}$-module categories

$$Mod_{\mathcal{C}}(B) \simeq RMod_{\mathcal{C}}(B^{op})$$

between the category of left B-modules and the category of right B^{op}-modules.

□

Definition 2.4.4. Let $\mathfrak{C}$ be a fusion 2-category with monoidal unit I. We denote by $\mathfrak{C}^0$ the connected component of the identity, i.e. the full additive sub-2-category on the simple objects that admit a non-zero morphism from I.

Proposition 2.4.5. *The 2-category $\mathfrak{C}^0$ is a fusion sub-2-category of $\mathfrak{C}$ that is connected.*

Proof. We begin by proving that $\mathfrak{C}^0$ is finite semisimple. The only property which is not obvious is that $\mathfrak{C}^0$ has all condensates. Let $(A, ...)$ be a 2-condensation monad in $\mathfrak{C}^0$, and let $(A, B, f, g, ...)$ be an extension in $\mathfrak{C}$ of $(A, ...)$ to a 2-condensation. Observe that for every simple summand C in $\mathfrak{C}$ of B, the composite of f with the projection $B \to C$ is a non-zero 1-morphism $A \to C$. (If this 1-morphism was zero, $(A, B, f, g, ...)$ would not be a 2-condensation.) Thus, we find that C is in $\mathfrak{C}^0$. Further, by definition, we have that $\mathfrak{C}^0$ is connected.

By corollary 2.3.6, the monoidal product of $\mathfrak{C}$ restricts to give $\mathfrak{C}^0$ a monoidal structure. Finally, as $I^\sharp \simeq I$, we find by corollary 2.3.5 that this monoidal structure is rigid, and $\mathfrak{C}^0$ is clearly fusion. ☐

Corollary 2.4.6. *Let $\mathfrak{C}$ be a fusion 2-category. Then, there is an equivalence of monoidal 2-categories:*

$$Mod(End_{\mathfrak{C}}(I)) \simeq \mathfrak{C}^0.$$

Proof. As a consequence of the proof of the above proposition, we find that

$$\mathrm{B}End_{\mathfrak{C}}(I) \hookrightarrow \mathfrak{C}^0$$

is a Cauchy completion. Further, this inclusion is monoidal, whence, by the 3-universal property of the Cauchy completion, we get the desired result. ☐

Corollary 2.4.6 shows that the behavior of the monoidal product on the connected component of the identity is completely determined by the braiding on the fusion category $End_{\mathfrak{C}}(I)$.

Proposition 2.4.7. *There is an equivalence between the category of connected fusion 2-categories and equivalence classes of monoidal linear 2-functors, and the category of braided fusion categories and equivalence classes of braided tensor functors.*

Proof. Let us denote by $\mathcal{A}$ the category of connected fusion 2-categories and equivalence classes of monoidal linear 2-functors, and by $\mathcal{B}$ the category of braided fusion categories and equivalence classes of braided tensor functors. Taking the endomorphism category of the monoidal unit yields a functor

$$End_{(-)}(I) : \mathcal{A} \to \mathcal{B},$$

and taking the Cauchy completion of the delooped braided fusion category gives a functor

$$\mathbf{Mod}(-) = Cau(\mathrm{B}(-)) : \mathcal{B} \to \mathcal{A}.$$

Using the 3-universal property of the Cauchy completion, one finds that these functors are pseudo-inverses for one another. $\square$

2.5 Examples given by Pointed Braided Fusion Categories

Let $\mathcal{C}$ be a pointed braided fusion category. By results of [7], we know that this corresponds equivalently to the data of a finite abelian group A equipped with an abelian 3-cocycle (ω, β). We will assume that ω is trivial. (If A has odd order, this can always be done.) We denote the braided fusion category associated to this data by $\mathbf{Vect}_A^\beta$. It is known that finite semisimple indecomposable right module categories over $\mathbf{Vect}_A^\beta$ correspond to pairs (E, ϕ), where E is a subgroup of A and ϕ is 2-cocycle on E with value in $\Bbbk^\times$ (considered up to 2-coboundary). We denote the corresponding right module category by $\mathcal{M}(E, \phi)$. Proposition 3.16 of [9], explains how to compute the relative Deligne tensor product of two right $\mathcal{C}$-modules when β is trivial. We now generalize this result.

Let (E, ϕ), and (F, ψ) be two pairs consisting of a subgroup of A, and an appropriate 2-cocycle. Let $Alt(\phi) : E \times E \to \Bbbk^\times$ and $Alt(\psi) : F \times F \to \Bbbk^\times$ be the corresponding skew-symmetric bilinear forms, i.e.

$$Alt(\phi)(e_1, e_2) := \phi(e_1, e_2)/\phi(e_2, e_1),$$

and similarly for $Alt(\psi)$. We define a skew-symmetric bicharacter b on $E \oplus F$ by

$$b((e_1, f_1), (e_2, f_2)) := Alt(\phi)(e_1, e_2)Alt(\psi)(f_1, f_2)\beta(f_1, e_2)/\beta(f_2, e_1).$$

The group $E \cap F$ embeds in $E \oplus F$ via $e \mapsto (e, -e)$, thus we can consider its orthogonal complement $(E \cap F)^\perp$ under the bicharacter b. Now, let H be the image of $(E\cap F)^{\perp}$ under the canonical map $E\oplus F\rightarrow A$. The restriction of b to $(E\cap F)^{\perp}$ descends to a skew-symmetric bilinear form b' on H, which corresponds to an element of $H^2(H,\Bbbk^{\times})$ represented by a chosen 2-cocycle ρ (see proposition 3.6 of [16]).

Proposition 2.5.1. *Write $\mathcal{C}=\mathbf{Vect}_A^{\beta}$, we have*

$$\mathcal{M}(E,\phi)\boxtimes_{\mathcal{C}}\mathcal{M}(F,\psi)\simeq\boxplus_{i=1}^{m}\mathcal{M}(H,\rho),$$

where

$$m=\frac{|(E\cap F)^{\perp}||(E\cap F)|}{|E||F|}.$$

Proof. Let $A(E,\phi)$ be the algebra in $\mathcal{C}$ whose underlying object is $\Bbbk E$ and whose multiplication is given by ϕ, and similarly for $A(F,\psi)$. By definition, $\mathcal{M}(E,\phi)=Mod_{\mathcal{C}}(A(E,\phi))$, and $\mathcal{M}(F,\psi)=Mod_{\mathcal{C}}(A(F,\psi))$. The relative Deligne tensor product is given by the category of left modules over the algebra $A(E,\phi)\otimes A(F,\psi)$ in $\mathcal{C}$. Note that the multiplication of this algebra is twisted by the braiding β of $\mathcal{C}$. More precisely, the multiplication is given by the 2-cocycle τ on $E\oplus F$ defined by

$$((e_1,f_1),(e_2,f_2))\mapsto\phi(e_1,e_2)\psi(f_1,f_2)\beta(f_1,e_2).$$

In particular, the corresponding skew-symmetric bicharacter on $E\oplus F$ is b as defined above. Finally, using proposition 2.11 of [9], we obtain the desired result. □

Remark 2.5.2. It should be possible to generalize proposition 2.5.1 to the case were ω is not assumed to be trivial. However, the above proof does not immediately generalize because if ω is not trivial, then ϕ and ψ may not be 2-cocycles, and thus $Alt(\phi)$ and $Alt(\psi)$ may not be skew-symmetric bicharacters.

Example 2.5.3. Recall the notation of example 2.1.12 of [4]. It is well-known that the fusion category $\mathbf{Vect}_{\mathbb{Z}/p\mathbb{Z}}$ admits p distinct braided structures (up to braided monoidal automorphisms of $\mathbf{Vect}_{\mathbb{Z}/p\mathbb{Z}}$ that are the identity on objects). A braiding b on $\mathbf{Vect}_{\mathbb{Z}/p\mathbb{Z}}$ is determined by its value $b_{\Bbbk_1,\Bbbk_1}=e^{\frac{2\pi ik}{p}}$, for any $0\leq k<p$. If we allow arbitrary braided monoidal automorphisms,

there are two distinct braided structures if $p = 2$, and three otherwise. The symmetric or trivial braiding is specified by $k = 0$, and we denote the corresponding braided fusion category by $\mathbf{Vect}^{triv}_{\mathbb{Z}/p\mathbb{Z}}$. The other ones correspond to the cases where $0 < k < p$ is either a quadratic residue or not. For simplicity, we only treat the case where k is a quadratic residue, for which we may assume $k = 1$, and denote the corresponding braided fusion category by $\mathbf{Vect}^{\beta}_{\mathbb{Z}/p\mathbb{Z}}$. The other case is entirely analogous.

Let us begin by examining the monoidal product on the finite semisimple 2-category of finite semisimple right modules categories over $\mathcal{C} := \mathbf{Vect}^{triv}_{\mathbb{Z}/p\mathbb{Z}}$. As $\mathcal{C}$ is the monoidal unit for the induced monoidal structure on $\mathbf{Mod}(\mathcal{C})$, we only have to determine $\mathbf{Vect} \boxtimes_{\mathcal{C}} \mathbf{Vect}$. Recall that as right $\mathbf{Vect}_{\mathbb{Z}/p\mathbb{Z}}$-module categories, we have $\mathbf{Vect} \simeq \mathcal{M}(\mathbb{Z}/p\mathbb{Z}, triv)$ in the notations used above. Now, a straightforward computation using proposition 2.5.1 shows:

$$\mathbf{Vect} \boxtimes_{\mathcal{C}} \mathbf{Vect} \simeq \boxplus_{i=1}^{p} \mathbf{Vect},$$

as right $\mathcal{C}$-module categories.

We now turn our attention to the case $\mathcal{D} := \mathbf{Vect}^{\beta}_{\mathbb{Z}/p\mathbb{Z}}$. As above, $\mathcal{D}$ is the monoidal unit of the induced monoidal structure on $\mathbf{Mod}(\mathcal{D})$, whence we only have to determine $\mathbf{Vect} \boxtimes_{\mathcal{D}} \mathbf{Vect}$. In order to use proposition 2.5.1, we compute that $\langle (1, -1)\rangle^{\perp} \subseteq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ is precisely $\langle (1, -1)\rangle$. This gives $m = 1$, and thus

$$\mathbf{Vect} \boxtimes_{\mathcal{D}} \mathbf{Vect} \simeq \mathcal{M}(0, triv) \simeq \mathcal{D}.$$

This examples shows that the braiding we put on a fusion category can have a big impact on the fusion rule of the associated fusion 2-category.

A. Two 2-Functors

A.1 The Adjoints Monoidal 2-Functor

Notation A.1.1. Let $\mathfrak{C}$ be a (monoidal) 2-category. Given $f : A \to B$ a 1-morphism in $\mathfrak{C}$, we denote by $\{f\}^{1op;2op} : B \to A$ the corresponding 1-morphism in $\mathfrak{C}^{1op;2op}$. We write $\circ^{op}$ for the composition of 1-morphisms in $\mathfrak{C}^{1op;2op}$. Given $\alpha : f \Rightarrow g$ a 2-morphism in $\mathfrak{C}$, we denote by $\{\alpha\}^{1op;2op} : \{g\}^{1op;2op} \Rightarrow \{f\}^{1op;2op}$ the corresponding 2-morphism in $\mathfrak{C}^{1op;2op}$.

Proof of lemma 1.1.4. This is well-known (for instance, see [11]), but let us indicate briefly how to proceed. Thanks to the coherence theorem for 2-categories, we can omit the coherence 2-isomorphisms for $\mathfrak{C}$. The 2-functor $(-)^*$ is defined as follows:

It sends the obejct C of $\mathfrak{C}$ to itself. Given a 1-morphism f, we set $f^* := \{f'\}^{1op;2op}$, where f' is a fixed right adjoint for f with unit η_f and counit ϵ_f. Given a 2-morphism $\alpha : f \Rightarrow g$, we define

$$\alpha^* := \{(f' \circ \epsilon_g) \cdot (f' \circ \alpha \circ g') \cdot (\eta_f \circ g')\}^{1op;2op}.$$

Given two 1-morphisms $f : A \to B, g : B \to C$, the structure 2-isomorphism $g^* \circ^{op} f^* \Rightarrow (g \circ f)^*$ in $\mathfrak{C}^{1op;2op}$ witnessing that $(-)^*$ respects the composition of 1-morphisms is given by the 2-isomorphisms

$$\{(f' \circ g' \circ \epsilon_{g\circ f}) \cdot (f' \circ \eta_g \circ f \circ (g \circ f)') \cdot (\eta_f \circ (f \circ g)')\}^{1op;2op}.$$

Similarly, the unitors are given by $\{\eta^{-1}_{Id_C}\}^{1op;2op} : Id^*_C \Rightarrow Id_C$ for every object C of $\mathfrak{C}$. This data specifies the 2-functor $(-)^*$. The associativity of composition follows from the triangle identities, and the unitality of composition is clear.

Further, if $\mathfrak{C}$ also has left-adjoints, we can dually define a 2-functor ${}^*(-)$. In particular, for any given 1-morphism f, we fix a left adjoint ${}'f$ with unit ξ_f and counit κ_f. It is not hard to see that ${}^*(-)$ is a pseudo-inverse for $(-)^*$ (see the footnote on page 3). For instance, a 2-natural equivalence $\theta : ({}^*(-))^* \Rightarrow Id$ is given by the identity 1-morphism on objects, and on a 1-morphism f by the 2-isomorphism $(\epsilon_{('f)} \circ f) \cdot (('f)' \circ \xi_f)$. □

Lemma A.1.2. *Let $F : \mathfrak{C} \to \mathfrak{D}$ be a 2-functor. There is a 2-natural equivalence e that fits into the following diagram:*

$$\begin{tikzcd}
\mathfrak{C} \arrow[r, "F"] \arrow[d, "(-)^*"'] & \mathfrak{D} \arrow[d, "(-)^*"] \\
\mathfrak{C}^{1op;2op} \arrow[r, "F^{1op;2op}"'] \arrow[ur, Rightarrow, "e"] & \mathfrak{D}^{1op;2op}.
\end{tikzcd}$$

Proof. As above, we omit the coherence 2-isomorphisms of $\mathfrak{C}$, and $\mathfrak{D}$. Given an object C of $\mathfrak{C}$, we set $e_C := Id_{F(C)}$. On the 1-morphism $f : C \to D$, we define e_f as the following 2-isomorphism:

$$\{(\eta_{F(f)} \circ F(f')) \cdot (F(f)' \circ F^{-1}_{f',f}) \cdot (F(f)' \circ F(\epsilon_f))\}^{1op;2op},$$

where $F_{f',f} : F(f') \circ F(f) \Rightarrow F(f' \circ f)$ is the coherence 2-isomorphism supplied by F. It is not hard to check that this defines a 2-natural equivalence. $\square$

Proof of lemma 1.1.5. Thanks to the coherence theorem for monoidal 2-categories, there is an equivalence of monoidal 2-categories $F : \mathfrak{C} \to \mathfrak{D}$ such that $\mathfrak{D}$ is strict cubical. Below, we will endow the adjoint 2-functor $(-)^*$ of $\mathfrak{D}$ with a monoidal structure. Through the natural 2-equivalence of lemma A.1.2, this shows that the adjoint 2-functor $(-)^*$ of $\mathfrak{C}$ is equivalent as a 2-functor to a monoidal one. Hence, it is monoidal itself.

In order to specify a monoidal structure on the 2-functor $(-)^*$ on $\mathfrak{D}$, we need to give some data. To make this more digestible, we use the notations of [15]. We begin by defining the 2-natural equivalence

$$\chi : (-)^*\Box(-)^* \Rightarrow ((-)\Box(-))^*.$$

Given two objects A, B we let $\chi_{A,B} := Id_{A\Box B}$. Given two 1-morphisms $f : A \to B$ and $g : C \to D$ we let the 2-isomorphism $\chi_{f,g} : f^*\Box g^* \Rightarrow (f\Box g)^*$ be given by

$$\{\big((f'\Box g') \circ \epsilon_{f\Box g}\big) \cdot \big(\phi^{-1}_{(f',g'),(f,g)} \circ (f\Box g)'\big) \cdot \big((\eta_f\Box\eta_g) \circ (f\Box g)'\big)\}^{1op;2op}.$$

It is not hard to see that χ is a 2-natural transformation. Further, it is clearly an isomorphism; and we pick $\chi^\bullet$ to be its inverse.

We choose the 1-equivalence ι to be the identity 1-morphism on the monoidal unit. The modifications ω, γ, δ are uniquely specified by the universal property of right adjoints. Namely, we let γ_C be the 2-isomorphism $\{\eta_{Id_C}\}^{1op;2op}$ in $\mathfrak{C}$ and δ_C be the identity 2-morphism of $\{Id_C\}^{1op;2op}$, for every object C of $\mathfrak{C}$. Further, for every A, B, C in $\mathfrak{C}$, the modification $\omega_{A,B,C}$ is given by the 2-isomorphism $\{\epsilon_{Id_{A\Box B\Box C}}\}^{1op;2op}$.

The commutativity of the coherence diagrams can be checked using the uniqueness up to unique isomorphism of right adjoints. Similarly, one can endow the 2-functor ${}^*(-)$ with a monoidal structure.

Finally, we need to construct monoidal 2-natural equivalences witnessing that $(-)^*$ and ${}^*(-)$ are pseudo-inverse monoidal 2-functors. We construct the monoidal 2-natural equivalence $({}^*(-))^* \Rightarrow Id$, the other one can be constructed analogously. Observe that using the argument at the beginning of this proof, it is enough to construct this monoidal 2-natural equivalence on $\mathfrak{D}$. As its underlying 2-natural equivalence we take the 2-natural equivalence θ defined in the proof of lemma 1.1.4 above. Then, in the notation of [15], the 2-isomorphism M is given by the identity 2-morphism on Id_I, and the modification Π is specified on A, B in $\mathfrak{C}$ by the 2-isomorphism $\{\eta_{Id_{A\Box B}}\}^{1op;2op}$. □

A.2 The Duals 2-Functor

Notation A.2.1. Let $\mathfrak{C}$ be a 2-category. Given $f : A \to B$ a 1-morphism in $\mathfrak{C}$, we denote by $\{f\}^{1op} : B \to A$ the corresponding 1-morphism in $\mathfrak{C}^{1op}$. We write $\circ^{op}$ for the composition of 1-morphisms in $\mathfrak{C}^{1op}$. Given $\alpha : f \Rightarrow g$ a 2-morphism in $\mathfrak{C}$, we denote by $\{\alpha\}^{1op} : \{f\}^{1op} \Rightarrow \{g\}^{1op}$ the corresponding 2-morphism in $\mathfrak{C}^{1op}$.

Lemma A.2.2. *Let $\mathfrak{C}$ be a monoidal 2-category that has right duals. Then, there is a 2-functor*

$$(-)^\sharp : \mathfrak{C} \to \mathfrak{C}^{1op}$$

that sends an object A in $\mathfrak{C}$ to the object underlying a right dual for A.

Proof. We may assume that the monoidal 2-category $\mathfrak{C}$ is strict cubical. For every object A in $\mathfrak{C}$, choose a coherent right dual $(A, A^\sharp, i_A, e_A, C_A, D_A)$. These choices allow us to define the following assignments towards defining the 2-functor $(-)^\sharp$. An object A in $\mathfrak{C}$ is sent to $A^\sharp$. A 1-morphism $f : A \to B$ is sent to $f^\sharp := \{f'\}^{1op}$, where

$$f' := (A^\sharp \Box e_B) \circ (A^\sharp \Box f \Box B^\sharp) \circ (i_A \Box B^\sharp) : B^\sharp \to A^\sharp.$$

A 2-morphism $\alpha : f \Rightarrow g : A \to B$ is sent to $\alpha^\sharp := \{\alpha'\}^{1op}$, where

$$\alpha' := (A^\sharp \Box e_B) \circ (A^\sharp \Box \alpha \Box B^\sharp) \circ (i_A \Box B^\sharp) : f' \Rightarrow g'.$$

Given two 1-morphisms $f : A \to B$, $g : B \to C$ in $\mathfrak{C}$, then $\{-\}^{1op}$ of the following 2-isomorphism in $\mathfrak{C}$:

$$\begin{array}{c} f' \circ g' \\ \Big\Downarrow{\scriptstyle \phi} \\ \Big(A^\sharp\Box\big(e_C \circ (g\Box C^\sharp)\big)\Big) \circ \Big(A^\sharp\Box\big((e_B\Box B) \circ (A\Box i_B)\big)\Box C^\sharp\Big) \circ \Big(\big((A^\sharp\Box f) \circ i_A\big)\Box C^\sharp\Big) \\ \Big\Downarrow{\scriptstyle D_B^{-1}} \\ (g \circ f)' \end{array}$$

serves as the structure 2-isomorphism in $\mathfrak{C}^{1op}$ witnessing that $(-)^\sharp$ respects the composition of 1-morphisms. The unitor on A is provided by $\{D_A\}^{1op}$. Using naturality of the interchangers, it is not hard to check the coherence axioms, and so defines a 2-functor. $\square$

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A PRETORSION THEORY FOR THE CATEGORY OF ALL CATEGORIES

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Résumé. Une théorie de prétorsion pour la catégorie de toutes les catégories est présentée. Les prénoyaux et préconoyaux associés sont calculés pour chaque foncteur.
Abstract. A pretorsion theory for the category of all categories is presented. The associated prekernels and precokernels are calculated for every functor.
Keywords. Category of all categories, Torsion theory.
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1. Introduction

In [3] it was shown that, in the category $Preord$ of preordered sets, there is a natural notion of pretorsion theory, in which the partially ordered sets are the torsion-free objects and the sets endowed with an equivalence relation are the torsion objects. The notion of pretorsion theory given in [3] generalizes the notion of torsion theories for pointed categories given in [4].

In this paper we give what can be seen as an extension of the pretorsion theory for preordered sets to categories. A torsion-free object is now a category whose image by the well known reflection $Cat \to Preord$ (cf. [6]) is a partially ordered set. A torsion object is in turn the category whose image by the same functor is an equivalence relation. They are called below respectively the antisymmetric and the symmetric categories (see the beginning of

Section 3).

In this way, the trivial objects which were sets for $Preord$ are now the collections of monoids for the category of all categories Cat.

To assert that in fact this is a pretorsion theory, it was necessary to characterize prekernels and precokernels (corresponding to kernels and cokernels in torsion theories). The crucial result in this paper being the construction of precokernels in Cat (see Proposition 4.2).

2. Pretorsion theory for a general category

Consider two full replete[1] subcategories $\mathcal{T}$ and $\mathcal{F}$ of $\mathcal{C}$. They are called respectively the *torsion* and the *torsion-free subcategories*.

Set $\mathcal{Z} = \mathcal{T} \bigcap \mathcal{F}$, the full subcategory of $\mathcal{C}$ determined by the objects which are both in $\mathcal{T}$ and in $\mathcal{F}$. These objects are called *trivial*.

Let $Triv_{\mathcal{Z}}(X, Y)$ be the set of all morphisms $X \to Y$ in $\mathcal{C}$ that factor through an object of $\mathcal{Z}$. Such morphisms will be called *$\mathcal{Z}$-trivial* (or simply *trivial*, if there is no doubt about the subcategories considered). Notice that these trivial morphisms form an *ideal of morphisms* in the sense of [1].

The following definitions, proposition and example are to be considered in the context of the data just presented.

Definition 2.1. *A morphism $k : X \to A$ is a $\mathcal{Z}$-prekernel (or simply, prekernel) of a morphism $f : A \to A'$ if the following two conditions are satisfied:*

1. *the composite $f \circ k$ is a trivial morphism;*
2. *whenever $\lambda : Y \to A$ is a morphism in $\mathcal{C}$ and $f \circ \lambda$ is trivial, then there exists a unique morphism $\lambda' : Y \to X$ in $\mathcal{C}$ such that $\lambda = k \circ \lambda'$.*

Dually, one obtains the notion of ($\mathcal{Z}$-)precokernel.

[1] I.e., whose objects are closed under isomorphisms.

Proposition 2.2. *Suppose that $\mathcal{Z}$-prekernels and $\mathcal{Z}$-precokernels exist in $\mathcal{C}$. Then, given any morphism f in $\mathcal{C}$,*

$$preker(precoker(preker f)) \cong preker f$$

$$and$$

$$precoker(preker(precoker f)) \cong precoker f,$$

where $preker f$ stands for the $\mathcal{Z}$-prekernel of f, and $precoker f$ stands for the $\mathcal{Z}$-precokernel of f.

Proof. This result is a consequence of the obvious Galois connections associated to each object in $\mathcal{C}$ (cf. [5, §VIII.1], for the classic and similar case of kernel and cokernel).

□

Definition 2.3. *It is said that*

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

is a short $\mathcal{Z}$-preexact sequence (or simply, short preexact sequence) in $\mathcal{C}$ if f is a ($\mathcal{Z}$-)prekernel of g and g is a ($\mathcal{Z}$-)precokernel of f.

Remark 2.4. The Proposition 2.2 gives canonical ways of constructing short $\mathcal{Z}$-preexact sequences, in a category with $\mathcal{Z}$-prekernels and $\mathcal{Z}$-precokernels (cf. [5, §VIII.1], for the classic and similar case of kernel and cokernel).

Definition 2.5. *The pair $(\mathcal{T}, \mathcal{F})$ is a pretorsion theory provided the following two conditions are satisfied:*

1. *$Hom_{\mathcal{C}}(T, F) = Triv_{\mathcal{Z}}(T, F)$, for every object $T \in \mathcal{T}$ and every object $F \in \mathcal{F}$;*

2. *for any object B of $\mathcal{C}$, there is a short $\mathcal{Z}$-preexact sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

with $A \in \mathcal{T}$ and $C \in \mathcal{F}$.

Remark 2.6. The short $\mathcal{Z}$-preexact sequence, given in condition (2) in Definition 2.5 just above, is uniquely determined up to isomorphism (cf. Proposition 3.1 in [2]).

Example 2.7. The pair $(Equiv, Ord)$ is a pretorsion theory for the category $Preord$, whose objects are the reflexive and transitive relations, and the morphisms are the monotone maps. $Equiv$ denotes the subcategory of equivalence relations, and Ord the subcategory of partial orders, so that $Equiv \bigcap Ord = Set$ is the category of sets. Check [3] for details.

3. Symmetric and antisymmetric categories

Cat is the category whose objects are the small categories, and whose morphisms are the functors.

$CatEquiv$ will denote the full subcategory of Cat determined by the *symmetric categories* $\mathbb{T}$, meaning that for any $T, T' \in \mathbb{T}$, if $Hom_{\mathbb{T}}(T, T') \neq \emptyset$ then $Hom_{\mathbb{T}}(T', T) \neq \emptyset$.

$CatOrd$ will denote the full subcategory of Cat determined by the *antisymmetric categories* $\mathbb{F}$: for any $F, F' \in \mathbb{F}$, if $Hom_{\mathbb{F}}(F, F') \neq \emptyset$ and $Hom_{\mathbb{F}}(F', F) \neq \emptyset$, then $F = F'$.

Therefore, $CatMon = CatEquiv \bigcap CatOrd$ is the full (and replete) subcategory of Cat whose objects are classes of monoids.

The trivial functors in $Triv_{CatMon}(\mathbb{A}, \mathbb{B})$ (see the beginning of section 2), are going to be characterized in the following Lemma 3.1.

Lemma 3.1. *The functor $F : \mathbb{A} \to \mathbb{B}$ is trivial if and only if, for every $A, A' \in \mathbb{A}$, if $Hom_{\mathbb{A}}(A, A') \neq \emptyset$ then $F(A) = F(A')$.*

Proof. If $F : \mathbb{A} \to \mathbb{B}$ is trivial, by definition F factors through some $\mathbb{C} \in CatMon$:

$$\mathbb{A} \xrightarrow{G} \mathbb{C} \xrightarrow{H} \mathbb{B}.$$

Then, as $\mathbb{C}$ does not have morphisms between distinct objects, it follows trivially that if there exists a morphism $f : A \to A'$ then $G(A) = G(A')$ and $F(A) = H(G(A)) = H(G(A')) = F(A')$.

Conversely, supposing that

$$\forall_{A, A' \in \mathbb{A}} Hom_{\mathbb{A}}(A, A') \neq \emptyset \Rightarrow F(A) = F(A'),$$

it is obvious that
$$F = H \circ G : \mathbb{A} \xrightarrow{G} \mathbb{C} \xrightarrow{H} \mathbb{B},$$
with $obj(\mathbb{C}) = obj(\mathbb{B})$ ($\mathbb{C}$ and $\mathbb{B}$ have the same objects) and
$$Hom_{\mathbb{C}}(B, B') = \begin{cases} Hom_{\mathbb{B}}(B, B') \; if \; B = B' \\ \\ \emptyset \quad otherwise, \end{cases}$$
H being the inclusion functor, and G the restriction of the functor F to the codomain $\mathbb{C}$. Finally, notice that $\mathbb{C} \in CatMon$. □

The following Proposition 3.2 asserts condition (1) in Definition 2.5. Notice that, in order to show that $(CatEquiv, CatOrd)$ is a pretorsion theory for Cat, it only remains to check condition (2) in Definition 2.5.

Proposition 3.2.
$$Hom_{Cat}(\mathbb{T}, \mathbb{F}) = Triv_{CatMon}(\mathbb{T}, \mathbb{F}),$$
for every $\mathbb{T} \in CatEquiv$ and every $\mathbb{F} \in CatOrd$.

Proof. The proof follows immediately from the respective definitions of symmetrical and antisymmetrical categories $\mathbb{T}$ and $\mathbb{F}$, and from Lemma 3.1. □

4. CatMon-Prekernels and CatMon-Precokernels

Proposition 4.1. *Let $F : \mathbb{A} \to \mathbb{A}'$ be any functor in Cat.*
Then, the functor $K : \mathbb{X} \to \mathbb{A}$ is a prekernel of F, where $obj(\mathbb{X}) = obj(\mathbb{A})$ ($\mathbb{X}$ and $\mathbb{A}$ have the same objects),
$$Hom_{\mathbb{X}}(A, A') = \begin{cases} Hom_{\mathbb{A}}(A, A') \; if \; F(A) = F(A') \\ \\ \emptyset \quad otherwise \; (F(A) \neq F(A')) \end{cases}$$
for every $A, A' \in \mathbb{A}$,
and K is the inclusion functor.

Proof. First, one has to establish that $\mathbb{X}$ is a category:

- $1_A \in Hom_{\mathbb{A}}(A,A) = Hom_{\mathbb{X}}(A,A)$, hence the identity of each object in $\mathbb{A}$ is also in $\mathbb{X}$;
- consider in $\mathbb{X}$ the composable morphisms $A \xrightarrow{f} A' \xrightarrow{g} A''$,

  then, necessarily, by the definition of $\mathbb{X}$,

  $F(A) = F(A') = F(A'') \Rightarrow F(A) = F(A'')$

  $\Rightarrow g \circ f \in Hom_{\mathbb{A}}(A, A'') = Hom_{\mathbb{X}}(A, A'')$,

  hence the composition of any two morphisms of $\mathbb{X}$ is also in $\mathbb{X}$.

Secondly, one has to show that $F \circ K$ is a trivial functor (cf. Lemma 3.1): consider in $\mathbb{X}$ a morphism $f : A \to A'$ with $A \neq A'$; one wants to show that $F \circ K(A) = F \circ K(A')$; being K the inclusion functor, $F \circ K(f) = F(f) : F(A) \to F(A')$, and $F(A) = F(A')$ by the construction of $\mathbb{X}$, that is $F \circ K(A) = F \circ K(A')$.

Thirdly and finally, one has to check the universal property given in Definition 2.1: suppose $\Lambda : \mathbb{Y} \to \mathbb{A}$ is a functor such that $F \circ \Lambda$ is trivial; since K is the inclusion functor, one has to show that $\Lambda(\mathbb{Y}) \subseteq K(\mathbb{X})$; suppose by "reductio ad absurdum" that there is in $\mathbb{Y}$ a morphism $g : Y \to Y'$ such that $\Lambda(g) : \Lambda(Y) \to \Lambda(Y')$ is not in $K(\mathbb{X})$; then, by the construction of $\mathbb{X}$, $F(\Lambda(Y)) \neq F(\Lambda(Y'))$, which contradicts the assumption that $F \circ \Lambda$ is trivial (cf. Lemma 3.1).

$\square$

In the following Proposition 4.2, a construction of a precokernel of any functor is given.

Proposition 4.2. *Let $F : \mathbb{A} \to \mathbb{A}'$ be a functor in Cat.*

Consider the well-known adjunction $(G, U, \eta) : Graph \to Cat$, where G is the left-adjoint of the forgetful functor U from Cat into the category of graphs, and $\eta : 1_{Graph} \to UG$ is the unit of the adjunction (see [5, II.7]).

Let ζ_F be the equivalence relation on the set of objects of $\mathbb{A}'$, $obj(\mathbb{A}')$, generated by $\{(F(A_1), F(A_2)) | Hom_{\mathbb{A}}(A_1, A_2) \neq \emptyset; A_1, A_2 \in \mathbb{A}\}$.

Consider the graph morphism $(1_{mor(\mathbb{A}')}, \pi_{\zeta_F}) : U\mathbb{A}' \to \mathbb{P}$ from the underlying graph of $\mathbb{A}'$ into the graph $\mathbb{P} = (mor(\mathbb{A}'), obj(\mathbb{A}')/\zeta_F)$, where $1_{mor(\mathbb{A}')}$ is the identity on the arrows and π_{ζ_F} is the canonical projection of the set of nodes $obj(\mathbb{A}')$ into its equivalence classes.

Consider the unit morphism of $\mathbb{P}$, $\eta_{\mathbb{P}} : \mathbb{P} \to UG\mathbb{P}$, in the adjunction $G \dashv U$.

Consider finally the canonical functor $\Pi : G\mathbb{P} \to G\mathbb{P}/\equiv$, where $\equiv$ stands for the least congruence (in the sense of [5, II.8]) on $G\mathbb{P}$ which makes the graph morphism

$$U\pi = U\Pi \circ \eta_{\mathbb{P}} \circ (1_{mor(\mathbb{A}')}, \pi_{\zeta_F}) : U\mathbb{A}' \to U(G\mathbb{P}/\equiv)$$

a functor from $\mathbb{A}'$ into $G\mathbb{P}/\equiv$.

Then, $\pi : \mathbb{A}' \to G\mathbb{P}/\equiv$ is a precokernel of $F : \mathbb{A} \to \mathbb{A}'$.

Proof. Let $F' : \mathbb{A}' \to \mathbb{B}$ be a functor such that the composite $F' \circ F$ is trivial. One has to show that there is one and only one functor $H' : G\mathbb{P}/\equiv \to \mathbb{B}$ such that $H' \circ \pi = F'$.

Since $F' \circ F$ is trivial, there is one and only one morphism of graphs φ such that $\varphi \circ (1_{mor(\mathbb{A}')}, \pi_{\zeta_F}) = UF' : U\mathbb{A}' \to U\mathbb{B}$. In fact, define $\varphi(g : [A_1'] \to [A_2']) = F'g : F'(A_1') \to F'(A_2')$. It is well defined since, for instance, if $A_0' \in [A_1']$ then there exists a sequence of morphisms ("zigzag") $A_0 \to \cdots \leftarrow A_n$ such that $F(A_0) = A_0'$ and $F(A_n) = A_1'$, which implies that $F'(A_0') = F'(A_1')$ because $F' \circ F$ is trivial (cf. Lemma 3.1).

There is also only one functor $H : G(\mathbb{P}) \to \mathbb{B}$ such that $\varphi = UH \circ \eta_{\mathbb{P}} : \mathbb{P} \to U\mathbb{B}$, being $\eta_{\mathbb{P}}$ the unit morphism of the adjunction $U \vdash G : Graph \to Cat$ (cf. Theorem 1 in [5, II.7]).

There is also a unique functor H' from the quotient category $G\mathbb{P}/\equiv$ into $\mathbb{B}$ such that $H' \circ \Pi = H$; in order to prove so, one has to show that (cf. [5, II.8]) H identifies $\eta_{\mathbb{P}} \circ (1_{mor(\mathbb{A}')}, \pi_{\zeta_F})(1_{A'}) =< 1_{A'} >$ and $1_{[A']}$, for every $A' \in \mathbb{A}'$, and that H identifies $\eta_{\mathbb{P}}((1_{mor(\mathbb{A}')}, \pi_{\zeta_F})(g)) \circ \eta_{\mathbb{P}}((1_{mor(\mathbb{A}')}, \pi_{\zeta_F})(f)) = \eta_{\mathbb{P}}(g) \circ \eta_{\mathbb{P}}(f) =< f, g >$ with $\eta_{\mathbb{P}}((1_{mor(\mathbb{A}')}, \pi_{\zeta_F})(g \circ f)) = \eta_{\mathbb{P}}(g \circ f) =< g \circ f >$, for every pair $(g : A_2' \to A_3', f : A_1' \to A_2')$ of composable morphisms in $\mathbb{A}'$; this is obvious since $UH \circ \eta_{\mathbb{P}} \circ (1_{mor(\mathbb{A}')}, \pi_{\zeta_F}) = UF'$ and F' is a functor.

It was proved just above that there is a functor H' such that $H' \circ \pi = F'$. It remains to check that such functor is the unique which satisfies $H' \circ \pi = F'$.

Suppose that S' is a functor such that $S' \circ \pi = F'$, then there is a functor S such that $S = S' \circ \Pi$, and then there is a graph morphism σ such that $US \circ \eta_{\mathbb{P}} = \sigma$, with $\sigma \circ (1_{mor(\mathbb{A}')}, \pi_{\zeta_F}) = UF'$, which implies that $\sigma = \varphi$ as defined above, and so $H' = S'$ going backwards.

$\square$

5. Short CatMon-preexact sequences

Let $\mathbb{A}'$ be any category, and let $\mathbb{A}$ be a category with the same objects, $obj(\mathbb{A}) = obj(\mathbb{A}')$, and such that, for any objects $A, B \in \mathbb{A}'$, if $Hom_{\mathbb{A}}(A, B) \neq \emptyset$ then $Hom_{\mathbb{A}'}(A, B) \neq \emptyset$ and $Hom_{\mathbb{A}'}(B, A) \neq \emptyset$ and $Hom_{\mathbb{A}}(A, B) = Hom_{\mathbb{A}'}(A, B)$ and $Hom_{\mathbb{A}}(B, A) \neq \emptyset$.

Let $F : \mathbb{A} \to \mathbb{A}'$ be the inclusion functor of $\mathbb{A}$ in $\mathbb{A}'$, and $\pi : \mathbb{A}' \to G\mathbb{P}/\equiv$ the precokernel of F constructed as in Proposition 4.2.

It is an immediate consequence of the characterization of $CatMon$-prekernel in Proposition 4.1 that F is the prekernel of π, so that we have constructed a short preexact sequence

$$\mathbb{A} \xrightarrow{F} \mathbb{A}' \xrightarrow{\pi} \mathbb{GP}/\equiv$$

for each $\mathbb{A}' \in Cat$, with $\mathbb{A} \in CatEquiv$.

Suppose that, one requires the category $\mathbb{A}$ just defined to satisfy in addition: for every $A, B \in \mathbb{A}'$, if $Hom_{\mathbb{A}'}(A, B) \neq \emptyset$ and $Hom_{\mathbb{A}'}(B, A) \neq \emptyset$ then $Hom_{\mathbb{A}}(A, B) \neq \emptyset$. Then, since $obj(G\mathbb{P}/\equiv) = obj(\mathbb{A}')/\zeta_F$ and by the nature of $\eta_{\mathbb{P}}$ and Π (cf. Proposition 4.2 and [5, II.7,8]), it is clear that $G\mathbb{P}/\equiv \in CatOrd$.

It was proved that, for every category $\mathbb{A}' \in Cat$, there is a short $CatMon$-preexact sequence

$$\mathbb{A} \to \mathbb{A}' \to \mathbb{A}''$$

with $\mathbb{A} \in CatEquiv$ and $\mathbb{A}'' \in CatOrd$.

Hence, the following Theorem can be stated.

Theorem 5.1. *The pair $(CatEquiv, CatOrd)$ is a pretorsion theory for the category of all categories Cat.*

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AFFINE CONNECTIONS AND SECOND-ORDER AFFINE STRUCTURES

Filip BÁR

Dedicated to my good friend Tom Rewwer on the occasion of his 35th birthday.

Résumé. Les variétés lisses ont toujours été intuitivement perçues comme étant des espaces munis d'une géométrie affine à l'échelle infinitésimale. Nous précisons cette notion en Géométrie Différentielle Synthétique en montrant que chaque variété est naturellement munie d'une structure d'espace infinitésimal affine, que nous interprétons comme l'action du clone des combinaisons affines sur une structure infinitésimale du premier ordre construite à partir du premier voisinage de la diagonale. Nous définissons une structure infinitésimale du second ordre basée sur le second voisinage de la diagonale et montrons que sur toute variété une connexion affine symétrique s'étend en une structure infinitésimale affine du second ordre en utilisant la bijection log-exp induite par la connexion.

Abstract. Smooth manifolds have been always understood intuitively as spaces with an affine geometry on the infinitesimal scale. We make this notion precise within Synthetic Differential Geometry by showing that every manifold carries a natural structure of an infinitesimally affine space, which we interpret as the action of the clone of affine combinations on a first-order infinitesimal structure constructed from the first neighbourhood of the diagonal. We define a second-order infinitesimal structure based on the second neighbourhood of the diagonal and show that on any manifold a symmetric affine connection extends to a second-order infinitesimally affine structure using the $\log$-$\exp$ bijection induced by the connection.

Keywords. Infinitesimally affine space, higher-order infinitesimal structure, affine connection, Synthetic Differential Geometry.
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1. Introduction

A deeply rooted intuition about smooth manifolds is that of spaces that become linear spaces in the infinitesimal neighbourhood of each point. On the infinitesimal scale the geometry underlying a manifold is thus affine geometry. To make this intuition precise requires a good theory of infinitesimals as well as defining precisely what it means for two points on a manifold to be infinitesimally close. As regards infinitesimals we make use of *Synthetic Differential Geometry* (SDG) and adopt the neighbourhoods of the diagonal from Algebraic Geometry to define when two points are infinitesimally close. The key observations on how to proceed have been made by Kock in [5]: 1) The first neighbourhood of the diagonal exists on formal manifolds and can be understood as a symmetric, reflexive relation on points, saying when two points are infinitesimal neighbours, and 2) we can form affine combinations of points that are mutual neighbours.

It remains to make precise in which sense a manifold becomes a model of the theory of affine spaces. This has been done in [1]. Firstly, one abstracts from Kock's infinitesimal simplices of mutual infinitesimally neighbouring points to what is called an *infinitesimal structure*. (See also section 2 for a definition.) An infinitesimal structure serves then as the domain of definition for the operations of affine combinations. A space together with an infinitesimal structure (i-structure) and an action of the clone of affine operations on that infinitesimal structure is called an *infinitesimally affine space* (i-affine space).

Formal manifolds and affine schemes (considered as either duals of commutative rings, or C^∞-rings) are examples of i-affine spaces. The i-structures are generated by the first neighbourhood of the diagonal. In this paper we shall construct an i-structure from the second-order neighbourhood of the diagonal on R^n for a ring R satisfying the Kock-Lawvere axioms for second-order infinitesimals. The definition of this i-structure is guided by the requirement that it is preserved by all maps $f : R^n \to R^m$ (hence can be defined on formal manifolds as well) and that the affine structure of R^n restricts to an i-affine space on the second-order i-structure. Both of these hold true for the i-structure generated by the first neighbourhood of the diagonal. In contrast to the first neighbourhood of the diagonal the i-affine structure on the second-order neighbourhood is not preserved by all maps anymore. Therefore, whereas a manifold carries a second-order i-structure, an i-affine structure has to be imposed as an additional piece of data.

We show that any second-order i-affine structure on a manifold induces a symmetric affine connection, and, conversely, any symmetric affine connection extends to a second-order i-affine structure in such a way that the latter is of the same affine-algebraic form as the canonical connection on an affine space. The second-order i-affine structure is constructed by using the second-order $\log$-$\exp$ bijection induced by the connection as introduced by Kock in [5, chap. 8.2]. With the help of the $\log$-$\exp$ bijection the *infinitesimally linear* (i-linear) structure on the tangent space can be transported to the formal manifold. For affine combinations this is independent of the chosen base point and thus defines an i-affine structure on the second-order i-structure. The $\log$-$\exp$ bijection yields also a natural geometric interpretation of an affine combination as the geometric addition of geodesic line segments extending the familiar vector parallelogram construction from the affine plane to curved space.

2. Infinitesimally affine and linear spaces

We shall work mostly within naive axiomatic SDG, as it is done in [5], for example. Let A be a space. An *i-structure* on A amounts to give an n-ary relation $A\langle n\rangle$ for each $n \in \mathbb{N}$ that defines which n points in A are considered as being 'infinitesimally close' to each other.

Definition 2.1 (i-structure)**.** *Let A be a space. An* **i-structure** *on A is an $\mathbb{N}$-indexed family $n \mapsto A\langle n\rangle \subseteq A^n$ such that*

(1) $A\langle 1\rangle = A$, $A\langle 0\rangle = A^0 = 1$ (the 'one point' space, or terminal object)

(2) For every map $h : m \to n$ of finite sets and every $(P_1, \dots, P_n) \in A\langle n\rangle$ we have $(P_{h(1)}, \dots, P_{h(m)}) \in A\langle m\rangle$

The first condition is a normalisation condition. The second condition makes sure that the relations are compatible: if we have a family of points that are infinitesimally close to each other, then so is any subfamily of these points, or any family created from repetitions. In particular, we obtain that the $A\langle n\rangle$ are symmetric and reflexive relations. An n-tuple $(P_1, \dots, P_n) \in A^n$ that lies in $A\langle n\rangle$ will be denoted by $\langle P_1, \dots, P_n\rangle$ and we shall refer to these as *i-n-tuples*. A map $f : A \to X$ that maps i-n-tuples to i-n-tuples for each $n \in \mathbb{N}$, i.e. $f^n(A\langle n\rangle) \subseteq X\langle n\rangle$, is called an *i-morphism*.

Two trivial examples of i-structures on A are the discrete and the indiscrete i-structure obtained by taking $A\langle n\rangle$ to be the diagonal Δ_n, respectively the whole A^n. The i-structures that are of main interest in SDG are the i-structures generated by the first neighbourhood of the diagonal (as relations). We call them *nil-square i-structures*. For example, let R be a ring[1]. Recall that

$$D(n) = \{(d_1, \dots, d_n) \in R^n \mid d_i d_j = 0,\ 1 \leq i, j \leq n\}$$

On R^n the first neighbourhood of the diagonal is given by

$$\{(P_1, P_2) \mid P_2 - P_1 \in D(n)\}$$

This is a symmetric and reflexive relation and we can construct an i-structure from it: take the first neighbourhood of the diagonal as $R^n\langle 2\rangle$ and define the *nil-square i-structure* on R^n by

$$R^n\langle m\rangle = \{(P_1, \dots, P_m) \mid (P_i, P_j) \in R^n\langle 2\rangle,\ 1 \leq i, j \leq m\}$$

This i-structure is thus *generated* by $R^n\langle 2\rangle$. Not all i-structures $A\langle -\rangle$ of interest need to be generated by $A\langle 2\rangle$. The second-order i-structure defined in section 3 is not, for example.

If the ring R satisfies the *Kock-Lawvere axiom*, that is for every $n \in \mathbb{N}$ and every map $t : D(n) \to R$ there are unique $a_0, \dots, a_n \in R$ such that

$$t(d_1, \dots, d_n) = a_0 + \sum_{j=1}^{n} a_j d_j, \qquad (d_1, \dots, d_n) \in D(n),$$

[1]All rings are assumed to be commutative.

then every map $f : R^n \to R^m$ is an i-morphism of the nil-square i-structures. This is due to the following two facts: linear maps $R^n \to R^m$ map $D(n)$ to $D(m)$, and for $P_2 - P_1 \in D(n)$

$$f(P_2) - f(P_1) = \partial f(P_1)[P_2 - P_1] \tag{1}$$

where $\partial f(P_1)$ denotes the *derivative of* f at P_1. The stated property of linear maps can be checked by direct computation; the existence and uniqueness of the linear map $\partial f(P_1)$ are both a consequence of the Kock-Lawvere axiom.

The nil-square i-structure induces i-structures on subspaces $U \hookrightarrow R^n$ by restriction. For *formally open subspaces* $U \hookrightarrow R^n$, which are stable under infinitesimal perturbations at each point (see [4, I.17] or [1, def. 3.2.5] for a definition), each map $f : U \to R^m$ has a derivative; hence every map $f : U \to V$ between formally open subspaces is an i-morphism. Furthermore, it is possible to glue the i-structures on formally open subspaces together to get an i-structure on a formal manifold and show that every map between formal manifolds is an i-morphism. (See [4, prop. I.17.5] and [1, thm. 3.2.8] for proofs.)

Definition 2.2 (i-affine space)**.** *Let* $A\langle - \rangle$ *be an* i-structure *on* A*. Set* $\mathcal{A}(n) = \{(\lambda_1, ..., \lambda_n) \in R^n \mid \sum_{j=1}^{n} \lambda_j = 1\}$*. The space* A *is said to be an* **i-affine space** *(over* R*), if for every* $n \in \mathbb{N}$ *there are operations*

$$\mathcal{A}(n) \times A\langle n \rangle \to A, \qquad ((\lambda_1, \dots, \lambda_n), \langle P_1, \dots, P_n \rangle) \mapsto \sum_{j=1}^{n} \lambda_j P_j$$

satisfying the axioms

- *(*Neighbourhood*) Let* $\lambda^k \in \mathcal{A}(n)$*,* $1 \le k \le m$*. If* $\langle P_1, \dots, P_n \rangle \in A\langle n \rangle$ *then*
$$\Big(\sum_{j=1}^{n} \lambda_j^1 P_j, \dots, \sum_{j=1}^{n} \lambda_j^m P_j \Big) \in A\langle m \rangle$$
- *(*Associativity*) Let* $\lambda^k \in \mathcal{A}(n)$*,* $1 \le k \le m$*,* $\mu \in \mathcal{A}(m)$ *and* $\langle P_1, \dots, P_n \rangle \in A\langle n \rangle$*. We have*
$$\sum_{k=1}^{m} \mu_k \Big(\sum_{j=1}^{n} \lambda_j^k P_j \Big) = \sum_{j=1}^{n} \Big(\sum_{k=1}^{m} \mu_k \lambda_j^k \Big) P_j$$

(Note that the left-hand side is well-defined due to the neighbourhood axiom.)

- *(*Projection*) Let $n \geq 1$ and let $e_k^n \in R^n$ denote the kth standard basis vector for $1 \leq k \leq n$. For every $\langle P_1, \ldots, P_n \rangle \in A\langle n \rangle$ it holds*

$$\sum_{j=1}^{n} (e_k^n)_j P_j = P_k$$

In particular, we have for $n = 1$ that $1P = P$, $P \in A$.

The neighbourhood axiom makes sure that we can compose affine combinations as we are used to, provided we are working over a fixed i-tuple. The associativity and projection axioms make sure the algebra of affine combinations follows the same rules as in all the R^n. A consequence of the neighbourhood axiom is that every i-tuple generates an affine space over R. This makes precise the statement that the geometry of the space A is affine on the infinitesimal scale.

It is not difficult to show by direct calculation that the affine space R^n satisfies the neighbourhood axiom for the nil-square i-structure making it an i-affine space[2]. Moreover, due to (1) it follows that every map $f : R^n \to R^m$ preserves not only the nil-square i-structure but the i-affine combinations as well. Each map f is an *i-affine map*.

The nil-square structure of R^n restricts to its formally open subspaces. Due to (1) all maps between formally open subspaces become i-affine maps for these i-structures. Like with the i-structures also the i-affine structures on formally open subspaces can be glued together to an i-affine structure on a formal manifold. All maps between formal manifolds become i-affine maps for these i-affine structures [1, thm. 3.2.8]. Any manifold in the sense of classical differential geometry is a formal manifold[3], so any manifold is an i-affine space and any smooth map between manifolds is i-affine.

[2]This is also a consequence of the more general [1, cor. 3.1.6 and 2.3.3].

[3]This is to be understood in the context of well-adapted models of SDG [3], where we have a fully faithful embedding of the category of smooth manifolds into a Grothendieck topos that admits a model of the Kock-Lawvere axioms. This embedding maps the real line $\mathbb{R}$ to R, analytical derivatives to derivatives in SDG and it maps open covers to covers by formally open spaces [3], [4, III.3].

Affine schemes (considered as either duals of commutative rings, or C^∞-rings) become examples of i-affine spaces over their respective nil-square i-structure [1, cor. 2.3.3 and 3.1.6]. Every morphism of affine schemes becomes an i-morphism. Affine C^∞-schemes, for example, form a category of spaces generalising smooth manifolds. Besides manifolds the category fully faithfully embeds locally closed subsets of Euclidean space with smooth maps between them [7, prop. 1.5]. This provides us with a wealth of examples of i-affine spaces. Furthermore, i-affine spaces are surprisingly well-behaved under taking colimits of the underlying spaces [1, chap. 2.6], [2]. This and their algebraic nature makes them a suitable type of space to study geometric notions based on infinitesimals.

Besides i-affine spaces we shall also consider *i-linear spaces*. The definition is almost identical to that of i-affine spaces; the main difference being that an i-linear space has a constant, the zero vector 0.

Definition 2.3 (i-linear space). *Let $V\langle - \rangle$ be an* i-structure *on V. Set $\mathcal{L}(n) = R^n$, $n \in \mathbb{N}$. The space V is said to be an* **i-linear space** *(over R), if for every $n \in \mathbb{N}$ there are operations*

$$\mathcal{L}(n) \times V\langle n \rangle \to V, \qquad ((\lambda_1, \dots, \lambda_n), \langle v_1, \dots, v_n \rangle) \mapsto \sum_{j=1}^{n} \lambda_j v_j$$

where we denote the constant $\mathcal{L}(0) \times V\langle 0 \rangle \cong 1 \to V$ by 0. These operations satisfy the axioms

- *(*Neighbourhood*) Let $\lambda^k \in \mathcal{L}(n)$, $1 \leq k \leq m$. If $\langle v_1, \dots, v_n \rangle \in V\langle n \rangle$ then*

$$\big(\sum_{j=1}^{n} \lambda_j^1 v_j, \dots, \sum_{j=1}^{n} \lambda_j^m v_j \big) \in V\langle m \rangle$$

- *(*Associativity*) Let $\lambda^k \in \mathcal{L}(n)$, $1 \leq k \leq m$, $\mu \in \mathcal{L}(m)$ and $\langle v_1, \dots, v_n \rangle \in V\langle n \rangle$. We have*

$$\sum_{k=1}^{m} \mu_k \big(\sum_{j=1}^{n} \lambda_j^k v_j \big) = \sum_{j=1}^{n} \big(\sum_{k=1}^{m} \mu_k \lambda_j^k \big) v_j$$

and for $0 \in \mathcal{L}(n)$ and $0 \in V$

$$\sum_{j=1}^{n} 0\, v_j = 0$$

- *(*Projection*) Let $n \geq 1$ and let $e_k^n \in R^n$ denote the kth standard basis vector for $1 \leq k \leq n$. For every $\langle v_1, \ldots, v_n \rangle \in A\langle n \rangle$ it holds*

$$\sum_{j=1}^{n} (e_k^n)_j v_j = v_k$$

In particular, we have for $n = 1$ that $1v = v$, $v \in V$.

The existence of the constant $0 \in V$ implies that for any $\langle v_1, \ldots, v_n \rangle \in V\langle n \rangle$ we have $\langle 0, v_1, \ldots, v_n \rangle \in V\langle n+1 \rangle$. This follows from combining the associativity axiom

$$\sum_{j=1}^{n} 0\, v_j = 0$$

with the projection and neighbourhood axioms. In particular, 0 has to be infinitesimally close to any other vector $v \in V$, which has a major implication on the size of V.

An example of an i-linear space is $D(n) \subset R^n$ with the restriction (= pullback) of the nil-square i-structure and the R-linear structure on R^n. More generally, for any *KL vector space* V the space

$$D(V) = \{v \in V \mid \phi[v]^2 = 0 \text{ for any bilinear map } \phi : V^2 \to R\}$$

becomes an i-linear space with the i-structure and R-linear structure induced by V. Writing $\phi[v]^\ell$ for an ℓ-linear map ϕ means that we evaluate ϕ on the ℓ-tuple $(v, \ldots, v)$. Indeed, recall that an R-vector space V is called KL if it satisfies the Kock-Lawvere axiom [4] for all maps $t : D(n) \to V$ and $n \in \mathbb{N}$. For the case of $V = R^n$ we have $D(V) = D(n)$ [5, prop. 1.2.2] and, like R^n, each KL vector space V carries a nil-square i-structure generated by

$$\{(v_1, v_2) \mid v_2 - v_1 \in D(V)\}$$

A KL vector space is called *finite-dimensional* if $V \cong R^n$ for some $n \in \mathbb{N}$. It follows from (1) that any map $f : V \to W$ between finite-dimensional KL vector spaces satisfying $f(0) = 0$ is an i-morphism preserving the linear combinations and thus restricts to an *i-linear map* $D(V) \to D(W)$.

[4] As with R, a KL vector space maybe required to satisfy more axioms from the Kock-Lawvere axiom scheme based on the context. See [5, chap. 1.3], for example.

An important class of examples of i-linear spaces is given by the subsequent general construction. For an i-affine space A and $P \in A$ we define the *monad* around P

$$\mathfrak{M}(P) = \{Q \in A \mid \langle P, Q\rangle \in A\langle 2\rangle\}$$

as the set of points infinitesimally close to P. It carries a natural i-structure

$$\mathfrak{M}(P)\langle n\rangle = \{(Q_1, \ldots, Q_n\rangle) \in \mathfrak{M}(P)^n \mid \langle P, Q_1, \ldots, Q_n\rangle \in A\langle n+1\rangle\}$$

Using the i-affine structure of A we can define a natural action of $\mathcal{L}(n)$ for each $n \in \mathbb{N}$

$$\begin{aligned}&\mathcal{L}(n) \times \mathfrak{M}(P)\langle n\rangle \to \mathfrak{M}(P),\\ &\qquad((\lambda_1, \ldots, \lambda_n), \langle Q_1, \ldots, Q_n\rangle) \mapsto \big(1 - \sum_{j=1}^{n} \lambda_j\big)P + \sum_{j=1}^{n} \lambda_j Q_j\end{aligned}$$

making $\mathfrak{M}(P)$ into an i-linear space with P the zero vector. Any i-affine map f induces an i-linear map

$$f : \mathfrak{M}(P) \to \mathfrak{M}(f(P))$$

This is just the familiar construction of a vector space from an affine space for a given base point P re-phrased in infinitesimal algebra. Indeed, in the case of A being an affine space with the indiscrete i-structure we have $\mathfrak{M}(P) = A$, i-affine maps are precisely the affine maps and the base-point dependency of this construction disappears. We shall denote the action of $\lambda \in \mathcal{L}(n)$ on $\langle Q_1, \ldots, Q_n\rangle \in \mathfrak{M}(P)\langle n\rangle$ by

$$P + \sum_{j=1}^{n} \lambda_j (Q_j - P)$$

As $D(V) = \mathfrak{M}(0)$ for a KL vector space V equipped with the nil-square i-structure the monad construction subsumes the first class of examples.

3. Second-order infinitesimal structures

The important examples of i-structures so far have all been the nil-square i-structures, which are constructed from the first neighbourhood of the diagonal. In this section we wish to define an i-structure $A_2 = A_2\langle - \rangle$ on $A = R^n$ such that $A_2\langle 2 \rangle$ is the second neighbourhood of the diagonal

$$\{(P_1, P_2) \mid P_2 - P_1 \in D_2(n)\}$$

where $D_2(n)$ is the space of second-order infinitesimals

$$D_2(n) = \{(d_1, \ldots, d_n) \in R^n \mid \text{any product of three } d_j \text{ vanishes}\}$$

The i-structure $A_2\langle - \rangle$ shall satisfy

1) All maps $f : R^n \to R^m$ become i-morphisms for the respective second-order i-structures on R^n and R^m

2) The affine space $A = R^n$ becomes an i-affine space over $A_2\langle - \rangle$.

To be able to study 1) we assume henceforth that R is a $\mathbb{Q}$-algebra that satisfies the Kock-Lawvere axiom for $D_2(n)$ with $n \geq 1$.[5] This amounts to say that each map $t : D_2(n) \to R$ is a polynomial function for a uniquely determined polynomial in $R[X_1, \ldots, X_n]$ of total degree ≤ 2, i.e.

$$t(d_1, \ldots, d_n) = a_0 + \sum_{j=1}^{n} a_j d_j + \sum_{1 \leq j \leq k \leq n} a_{jk} d_j d_k$$

for uniquely determined $a_j \in R$ and $a_{jk} \in R$. An important consequence is that every map $f : A \to R^m$ has a *Taylor representation*

$$f(P) - f(Q) = \partial f(Q)[P - Q] + \frac{1}{2}\partial^2 f(Q)[P - Q]^2$$

for $P - Q \in D_2(n)$. Here $\partial^2 f(Q)$ stands for the second derivative of f at Q, which is a symmetric bilinear map $(R^n)^2 \to R^m$. The following characterisation of $D_2(n)$ in [5, prop. 1.2.2] will be useful

$$D_2(n) = \{d \in R^n \mid \phi[d]^3 = 0 \text{ for all trilinear } \phi : (R^n)^3 \to R\}$$

[5] This requirement is not an overly restrictive one. For example, in a well-adapted model, where R is taken to be the embedding of the smooth manifold $\mathbb{R}$, the $\mathbb{R}$-algebra R satisfies the whole Kock-Lawvere axiom scheme [3, thm. 4.5], [7, prop. V.7.2].

It allows us to define $D_2(V)$ for any R-linear space V. Let $V \cong R^n$ be a finite-dimensional KL vector space[6]. We define $\mathrm{DN}_2(V)$ to be the space

$$\begin{aligned}\mathrm{DN}_2(V) = \{(v_1, v_2, v_3) \in \mathrm{D}_2(V)^3 \ | \\ \text{For any trilinear map } \phi : V^3 \to R,\ \phi[v_1, v_2, v_3] = 0\}\end{aligned}$$

In the subsequent definition and discussion we will use $A = V$ to mean the (affine) space induced by the R-linear space V.

Definition 3.1 (Second-order i-structure on R^n)**.** *Let $A = V \cong R^n$ be a finite-dimensional KL vector space. We define the* **second-order i-structure** *A_2 on A by*

(1) $A_2\langle 1\rangle = A$, $A_2\langle 0\rangle = A^0 = 1$

(2) For $m \geq 2$

$$\begin{aligned}A_2\langle m\rangle = \{(P_1, \dots, P_m) \in A^m \ | \\ (P_{i_1} - P_{j_1}, P_{i_2} - P_{j_2}, P_{i_3} - P_{j_3}) \in \mathrm{DN}_2(V), \\ \textit{for all } i_\ell, j_\ell \in \{1, \dots, m\}, 1 \leq \ell \leq 3\}\end{aligned}$$

From the definition it follows readily that A_2 is indeed an i-structure and that

$$A_2\langle 2\rangle = \{(P_1, P_2) \in A^2 \ | \ P_2 - P_1 \in D_2(n)\}$$

is the second neighbourhood of the diagonal, as desired.

The following two results show that the second-order i-structure A_2 is natural and makes any finite-dimensional KL vector space V into an i-affine space.

Theorem 3.2. *Every map $f : V \to W$ between two finite-dimensional KL vector spaces is an i-morphism for the respective second-order i-structures.*

Proof. Let $\langle P_1, \dots, P_n\rangle \in V_2\langle n\rangle$ for an index $n \geq 2$. We have to show

$$\langle f(P_1), \dots, f(P_n)\rangle \in W_2\langle n\rangle$$

[6]Note that V is also assumed to satisfy the respective V-valued Kock-Lawvere axiom for all $D_2(n)$.

By definition this amounts to show

$$\phi[f(P_{i_1}) - f(P_{j_1}), f(P_{i_2}) - f(P_{j_2}), f(P_{i_3}) - f(P_{j_3})] = 0$$

for all $i_\ell, j_\ell \in \{1, \ldots, n\}$, $1 \le \ell \le 3$ and any trilinear form ϕ on W. Since each $P_{i_\ell} - P_{j_\ell} \in D_2(V)$ we can apply Taylor expansion

$$f(P_{i_\ell}) - f(P_{j_\ell}) = \partial f(P_{j_\ell})[P_{i_\ell} - P_{j_\ell}] + \frac{1}{2}\partial^2 f(P_{j_\ell})[P_{i_\ell} - P_{j_\ell}]^2$$

Substituting each $f(P_{i_\ell}) - f(P_{j_\ell})$ with its respective Taylor expansion in ϕ and applying multilinearity to expand the three sums yields a sum of multilinear forms on V of the order 3 or higher with arguments being combinations of $P_{i_\ell} - P_{j_\ell}$ for $i_\ell, j_\ell \in \{1, \ldots, n\}$, $1 \le \ell \le 3$. Because of $\langle P_1, \ldots, P_n\rangle \in A_2\langle n\rangle$ each such multilinear form evaluates to 0, hence does the sum. This shows that

$$\phi[f(P_{i_1}) - f(P_{j_1}), f(P_{i_2}) - f(P_{j_2}), f(P_{i_3}) - f(P_{j_3})] = 0$$

as required. We conclude that f is an i-morphism as claimed. □

Theorem 3.3. *The affine structure on the KL vector space $A = V$ restricts to the second-order i-structure A_2, making A_2 an i-affine subspace of the affine space A (equipped with the indiscrete i-structure).*

Proof. To show A_2 an i-affine subspace of A it suffices to show that the affine operations on A satisfy the neighbourhood axiom for A_2.

Let $\lambda^i \in \mathcal{A}(n)$ for $1 \le i \le m$ and $\langle P_1, \ldots, P_n\rangle \in A_2\langle n\rangle$. We have to show

$$\big\langle \sum_{j=1}^{n} \lambda^1_j P_j, \ldots, \sum_{j=1}^{n} \lambda^m_j P_j\big\rangle \in A_2\langle m\rangle$$

Let ϕ be a trilinear form on V and $i_\ell, j_\ell \in \{1, \ldots, m\}$ for all $1 \le \ell \le 3$. Using $\sum_{j=1}^n \lambda^i_j = 1$ for all $1 \le i \le m$ yields

$$\begin{aligned}
&\phi\big[\sum_{i=1}^{n} \lambda^{i_1}_i P_i - \sum_{j=1}^{n} \lambda^{j_1}_j P_j, \sum_{i=1}^{n} \lambda^{i_2}_i P_i - \sum_{j=1}^{n} \lambda^{j_2}_j P_j, \sum_{i=1}^{n} \lambda^{i_3}_i P_i - \sum_{j=1}^{n} \lambda^{j_3}_j P_j\big] \\
&= \phi\big[\sum_{i,j=1}^{n} \lambda^{i_1}_i \lambda^{j_1}_j (P_i - P_j), \sum_{i,j=1}^{n} \lambda^{i_2}_i \lambda^{j_2}_j (P_i - P_j), \sum_{i,j=1}^{n} \lambda^{i_3}_i \lambda^{j_3}_j (P_i - P_j)\big]
\end{aligned}$$

Applying the trilinearity of ϕ yields a sum of trilinear forms with arguments being combinations of $P_{i_\ell} - P_{j_\ell}$ for $i_\ell, j_\ell \in \{1, \ldots, n\}$, $1 \leq \ell \leq 3$, which all evaluate to zero by assumption. We conclude

$$\langle \sum_{j=1}^{n} \lambda_j^1 P_j, \ldots, \sum_{j=1}^{n} \lambda_j^m P_j \rangle \in A_2\langle m \rangle$$

as required. □

The definitions of the second-order i-structure A_2 together with theorems 3.2 and 3.3 can be generalised to a formally open subspace A of R^n directly. This allows us to glue together the second-order i-structures to a second-order i-structure on a formal manifold and all maps between formal manifolds will preserve that structure.

Theorem 3.4. *Let A be a formal manifold.*

(i) A carries a unique i-structure A_2 with the universal property that any map $f : A \to M$ is an i-morphism, if and only if it is an i-morphism on the charts of A.

(ii) All maps between formal manifolds become i-morphisms for the respective second-order i-structures.

Proof. (i) (Essentially, this part is theorem 2.6.19 in [1] applied to the i-structure only. See also [2].) We consider a *chart* $\iota : U \hookrightarrow A$ of A, i.e. a formally open subspace of A that is also a formally open subspace of R^n. Pulling back the second-order i-structure on R^n yields a second-order i-structure U_2. For each $n \geq 1$ we define $A_2\langle n \rangle$ as the join of the images of $U_2\langle n \rangle$ over all the charts. It is easy to see that this yields an i-structure on A with the desired universal property.

(ii) Let $f : A \to M$ be a map between two formal manifolds equipped with the second-order i-structure as defined in (i) and $\langle P_1, \ldots, P_n \rangle \in A_2\langle n \rangle$. By construction there is an A-chart $\iota : U \hookrightarrow A$, $\phi : U \hookrightarrow R^n$, and $\langle x_1, \ldots, x_n \rangle \in U_2\langle n \rangle$ such that $\iota(x_\ell) = P_\ell$, $1 \leq \ell \leq n$.

We also find an M-chart $j : V \hookrightarrow M$ containing $f(P_1)$. Pulling back j along f yields a formally open subspace $f^*j : f^{-1}(V) \hookrightarrow M$, which becomes a chart after taking the intersection with ι

$$\iota^* f^* j : U \cap f^{-1}(V) \hookrightarrow A, \quad (\iota^* f^* j)^* \phi : U \cap f^{-1}(V) \hookrightarrow R^n$$

(Recall that formally open subspaces are stable under pullback.) Let $W = U \cap f^{-1}(V)$. The restriction of $f : W \to V$ is a map between formally open subspaces of R^n and R^m, respectively, and thus an i-morphism by theorem 3.2 and the constructions of W_2 and V_2. Since $x_1 \in W \subset U$ and W is a formally open subspace of U, we find $\langle x_1, \ldots, x_n \rangle \in W_2\langle n \rangle$ and hence $\langle f(x_1), \ldots, f(x_n) \rangle \in V_2\langle n \rangle$; but this implies that

$$\langle f(P_1), \ldots, f(P_n) \rangle = \langle j(f(x_1)), \ldots, j(f(x_n)) \rangle \in M_2\langle n \rangle$$

and that f is an i-morphism as claimed.

$\square$

Remark 3.5. Instead of forming the union over all charts in the construction of A_2 in the proof of part (i), it is sufficient to consider the union over a covering family, i.e. an *atlas* of A. Moreover, f is an i-morphism if and only if all its restriction to the charts of the atlas are i-morphisms.

Indeed, any chart of $\iota : U \hookrightarrow A$ can be covered by restrictions of charts of the chosen atlas, which are formally open subspaces of both A and some R^n. The same argument as presented in the proof of (ii) above shows that ι is an i-morphism when applied to U and the charts of the atlas.

Note that theorem 3.4 does not extend to the i-affine structures. Maps are not going to preserve the i-affine structure on U_2 for a formally open subspace $U \hookrightarrow R^n$, in general. Only special classes of maps will have that property. Indeed, the Taylor expansion of a map f to second order contains quadratic terms, in general, hence can only preserve affine combinations up to quadratic terms. Therefore, unlike R^n a formal manifold does *not* carry a canonical i-affine structure on its canonical second-order i-structure.

This is in contrast to the nil-square i-structure on R^n, where the i-affine structure is preserved by all maps and therefore induces a canonical i-affine structure on the canonical nil-square i-structure of a formal manifold [1, thm. 3.2.8].

4. Affine connections and second-order i-affine structures

In differential geometry affine connections on a manifold come in three equivalent notions: a geometric notion of parallel transport of tangent vectors along paths, and two algebraic notions; that of a covariant derivative on vector fields and the horizontal subbundle of the iterated tangent bundle. In SDG we can study these notions from the infinitesimal viewpoint by either using tangent vectors [6], [5], which are 'infinitesimal paths' $t : D \to A$ in SDG, or using points [5].

Befitting our consideration of the infinitesimal algebra of points we shall consider Kock's affine connection *for points* as defined in [5, chap. 2.3][7]. It is based on the idea of completing three points P, Q, S to a parallelogram $PQRS$. Here $\langle P, Q\rangle$ and $\langle P, S\rangle$ are first-order neighbours, but Q and S don't need to be. The resulting point R is a first-order neighbour of Q and of S, hence it is a second-order neighbour of P. If we follow [5] and denote the point R by $\lambda(P, Q, S)$ then an *affine connection* (on points) λ is a map mapping a triple (P, Q, S) with $\langle P, Q\rangle, \langle P, S\rangle \in A\langle 2\rangle$ to a point $\lambda(P, Q, S)$ such that

$$\lambda(P, Q, P) = Q$$
$$\lambda(P, P, S) = S$$

These properties are sufficient to derive the other nil-square neighbourhood relationships [5, chap. 2.3]. An affine connection is called *symmetric*, if

$$\lambda(P, Q, S) = \lambda(P, S, Q)$$

For $A = R^n$ a symmetric affine connection is induced by its affine structure

$$\lambda(P, Q, S) = Q + S - P$$

Geometrically, this corresponds to the addition of vectors using parallel transport to construct a vector parallelogram at P. In fact, any i-affine structure on A_2 induces a symmetric affine connection in this way.

[7] Note that all the other notions of affine connection can be derived from that of a pointwise affine connection.

Proposition 4.1. *Let A be a formal manifold that admits an i-affine structure on A_2, then A admits a symmetric affine connection on points.*

Proof. We wish to define the symmetric affine connection λ by

$$\lambda(P, Q, S) := Q + S - P$$

where the right hand side denotes the i-affine combination in A_2. For this to be well-defined we need to show $\langle P, Q, S\rangle \in A_2\langle 3\rangle$. We work in a chart. First note that $Q - P, S - P, Q - S \in D_2(n)$. Let ϕ be a trilinear map. We find

$$\phi[Q-P, S-P, Q-S] = \phi[Q-P, S-P, Q-P] - \phi[Q-P, S-P, S-P] = 0$$

as the two trilinear maps on the right hand side are quadratic in $Q - P \in D(n)$, respectively in $S - P \in D(n)$. This is sufficient to show $\langle P, Q, S\rangle \in A_2\langle 3\rangle$. The defining properties showing λ an affine connection are immediate consequence of the algebra of affine combinations. $\square$

We wish to show the converse, i.e. that any symmetric affine connection λ on a formal manifold A extends to a second-order i-affine structure. Our strategy is to construct the i-affine structure on A_2 by transporting the second-order i-affine structure from the tangent space

$$T_P A = \{t \in A^D \mid t(0) = P\}$$

to the manifold A using the second-order $\log\text{-}\exp$ bijection induced by λ as defined in [5, chap. 8.2].

To begin with note that each tangent vector $t \in T_P A$ is an i-morphism and hence factors through the monad $\mathfrak{M}(P)$ induced by the nil-square i-structure on A. Moreover, any n tangent vectors $t_1, \ldots, t_n \in T_P A$ satisfy

$$\langle t_1(d), \ldots, t_n(d)\rangle \in \mathfrak{M}(P)\langle n\rangle, \qquad \forall d \in D$$

(See [5, chap. 4.2] or [1, chap. 3.3.2] for the details for $n = 2$, which implies the general case.) The R-linear structure on $T_P A$ is obtained pointwise from the i-linear structure on $\mathfrak{M}(P)$ making $T_P A$ a finite-dimensional KL vector space over R [5, chap. 4.2].

Using the i-linear structure on $\mathfrak{M}(P)$ for each $Q \in \mathfrak{M}(P)$ we can define a tangent vector $\log_P(Q) \in T_P A$ by $\log_P(Q)(d) = P + d\,(Q - P)$. This yields an i-linear map

$$\log_P : \mathfrak{M}(P) \to D(T_P A), \qquad Q \mapsto \log_P(Q)$$

The map $\log_P$ has the *exponential map* $\exp_P$ as an inverse [5, thm. 4.3.2], which is given in a chart $U \hookrightarrow A$ as

$$\exp_P(t) = P + v$$

Here $v \in D(n) \subset R^n$ is the *principal part* of $t \in D(T_P A)$ considered in U; that is the unique vector v satisfying $t(d) = P + d\,v$, for all $d \in D$.

As a finite-dimensional KL vector space $T_P A$ is an i-affine space for the second-order i-structure by theorem 3.3. This makes $D_2(T_p A)$, which is the monad for the second-order i-structure at the zero vector, into an i-linear space. On the other hand, the second-order i-structure on A induces a monad $\mathfrak{M}_2(P)$. Kock has shown that using the symmetric affine connection λ we can extend the log-exp bijection to a bijection

$$\log_P : \mathfrak{M}_2(P) \to D_2(T_P A), \qquad \exp_P : D_2(T_P A) \to \mathfrak{M}_2(P)$$

This bijection can now be used to transport the i-linear structure to $\mathfrak{M}_2(P)$. Since this can be done for any point $P \in A$, we can use it to define an action of $\mathcal{A}(n)$ on A_2. We state our main result:

Theorem 4.2. *Let A be a formal manifold and λ a symmetric affine connection on A.*

(1) The second-order log-exp *bijection induced by the connection λ defines an i-affine structure on A_2 by*

$$\mu \cdot \langle P_1, \ldots, P_n \rangle = \exp_{P_1}\Big(\sum_{j=1}^{n} \mu_j \, \log_{P_1}(P_j)\Big) \tag{2}$$

where $\langle P_1, \ldots, P_n \rangle \in A_2\langle n \rangle$ and $\mu \in \mathcal{A}(n)$.

(2) The i-affine structure on A_2 as defined in (1) is an extension of λ in the sense that

$$\lambda(P, Q, S) = (-1, 1, 1)\langle P, Q, S \rangle$$

for all $(P, Q, S) \in A^3$ such that $\langle P, Q \rangle, \langle P, S \rangle \in A\langle 2 \rangle$.

Proof. (1) We shall proceed in two steps. First we note that $\log_P$ (and hence $\exp_P$) are i-morphisms. Indeed, any chart mapping P to $0 \in R^n$ induces a bijection $\mathfrak{M}_2(P) \cong D_2(n)$. Since $T_P A$ is a finite-dimensional KL vector space any map $D_2(n) \to T_P A$ has a unique extension to a map $R^n \to T_P A$; but any such map must be an i-morphism by theorem 3.2. Due to the construction of A_2 from charts in theorem 3.4(i), as well as the definition of the induced i-structures on the respective monads, $\log_P$ must be an i-morphism.

With this and the $\log$-$\exp$ bijection we can see that the i-structure $\mathfrak{M}_2(P)\langle - \rangle$ becomes an i-linear space for each $P \in A$. Moreover, the action of each $\mathcal{L}(n)$ is given by the same formula as in (2). In the next step we need to show that the action as defined in (2) is independent of the base point P.

Lemma 4.3. *Let $\mu \in \mathcal{A}(n)$ and $\langle Q, P, P_1, \ldots, P_n \rangle \in A_2\langle n+2 \rangle$, then*

$$\exp_P\Big(\sum_{j=1}^{n} \mu_j \log_P(P_j)\Big) = \exp_Q\Big(\sum_{j=1}^{n} \mu_j \log_Q(P_j)\Big)$$

With the base point independence all the three axioms of an i-affine structure follow from the respective axioms of the i-linear structure on $\mathfrak{M}_2(P)$ for a suitably chosen $P \in A$ (for example, choosing the first point in the respective i-tuple). To conclude the proof of (1) it remains to show lemma 4.3.

Proof. (Lemma) It is sufficient to show this in a chart U. In U we find

$$\lambda(P, Q, S) = Q + S - P + \Gamma_P[Q - P, S - P]$$

for a symmetric bilinear map Γ_P [5, chapter 2.3], which we will refer to as the *connection symbol* of the connection λ as it is done in [5][8]. The second-order extensions $\log_P$ and $\exp_P$ have the following local

[8]Note that Γ_P is the *negative* of the classically defined connection symbol of a covariant derivative. In [5] the connection symbol is referred to as the *Christoffel symbol*.

representations [5, chap. 8.2]

$$\log_P(Q)(d) = P + d\big((Q-P) - \frac{1}{2}\Gamma_P[Q-P]^2\big)$$
$$\exp_P(t) = P + v + \frac{1}{2}\Gamma_P[v]^2$$

for $Q \in \mathfrak{M}_2(P) \subset U$ and $t \in D_2(T_P U)$ with principal part $v \in D_2(n)$. We use this to derive a local formula for the second-order i-affine combination

$$\begin{aligned}
&\exp_P\Big(\sum_{j=1}^{n} \mu_j \log_P(P_j)\Big) \\
&\quad = P + \sum_{j=1}^{n} \mu_j \big((P_j - P) - \frac{1}{2}\Gamma_P[P_j - P]^2\big) \\
&\qquad + \frac{1}{2}\Gamma_P\Big[\sum_{j=1}^{n} \mu_j \big((P_j - P) - \frac{1}{2}\Gamma_P[P_j - P]^2\big)\Big]^2
\end{aligned}$$

In the next step we expand the last connection symbol. Since $\langle P, P_1, \ldots, P_n\rangle \in U\langle n+1\rangle$ all the multilinear occurrences of order three and four in the $P_j - P$ vanish. Using $\sum_{j=1}^{n} \mu_j = 1$ the above expression thus simplifies to the local representation

$$\begin{aligned}
&\exp_P\Big(\sum_{j=1}^{n} \mu_j \log_P(P_j)\Big) \\
&\quad = \sum_{j=1}^{n} \mu_j P_j + \frac{1}{2}\Big(\Gamma_P\big[\sum_{j=1}^{n} \mu_j P_j - P\big]^2 - \sum_{j=1}^{n} \mu_j \Gamma_P[P_j - P]^2\Big)
\end{aligned} \tag{3}$$

The respective local representation for the base point Q is obtained by replacing P with Q in the above equation.

The base point independence is equivalent to the identity

$$\begin{aligned}
&\Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j - P\big]^2 - \sum_{j=1}^{n}\mu_j\Gamma_P[P_j - P]^2 \\
&\quad = \Gamma_Q\big[\sum_{j=1}^{n}\mu_j\,P_j - Q\big]^2 - \sum_{j=1}^{n}\mu_j\Gamma_Q[P_j - Q]^2
\end{aligned} \tag{4}$$

We use that $Q - P \in D_2(n)$ and represent $\Gamma_Q = \Gamma_{P+(Q-P)}$ using a Taylor expansion of order two

$$\begin{aligned}
\Gamma_Q\big[\sum_{j=1}^{n}\mu_j\,P_j - Q\big]^2 = {}&\Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j - Q\big]^2 \\
&+ \partial\Gamma_P\big[Q-P\big]\,\big[\sum_{j=1}^{n}\mu_j\,P_j - Q\big]^2 \\
&+ \frac{1}{2}\partial^2\Gamma_P\big[Q-P\big]^2\big[\sum_{j=1}^{n}\mu_j\,P_j - Q\big]^2
\end{aligned}$$

Due to $\langle Q, P, P_1, \ldots, P_n\rangle \in U_2\langle n+2\rangle$ this simplifies to

$$\Gamma_Q\big[\sum_{j=1}^{n}\mu_j\,P_j - Q\big]^2 = \Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j - Q\big]^2 \tag{5}$$

In the same vein we obtain

$$\sum_{j=1}^{n}\mu_j\Gamma_Q\big[P_j - Q\big]^2 = \sum_{j=1}^{n}\mu_j\Gamma_P\big[P_j - Q\big]^2 \tag{6}$$

Expanding

$$\Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j - Q\big]^2 = \Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j - P + (P-Q)\big]^2$$

yields

$$
\begin{aligned}
\Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j - Q\big]^2 &= \Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j - P\big]^2 + \Gamma_P\big[P-Q\big]^2 \\
&\quad + 2\,\Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j - P,\ P-Q\big] \\
&= \Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j - P\big]^2 + \Gamma_P\big[P-Q\big]^2 \\
&\quad + 2\sum_{j=1}^{n}\mu_j\Gamma_P\big[P_j - P,\ P-Q\big],
\end{aligned}
$$

where we have used $\sum_{j=1}^{n}\mu_j = 1$ in the last step. Expanding

$$
\Gamma_P\big[P_j - Q\big]^2 = \Gamma_P\big[P_j - P + (P-Q)\big]^2
$$

yields

$$
\begin{aligned}
\Gamma_P\big[P_j - Q\big]^2 &= \Gamma_P\big[P_j - P\big]^2 + \Gamma_P\big[P-Q\big]^2 \\
&\quad + 2\,\Gamma_P\big[P_j - P,\ P-Q\big]
\end{aligned}
$$

and thus

$$
\begin{aligned}
\sum_{j=1}^{n}\mu_j\Gamma_P\big[P_j - Q\big]^2 &= \sum_{j=1}^{n}\mu_j\Gamma_P\big[P_j - P\big]^2 + \Gamma_P\big[P-Q\big]^2 \\
&\quad + 2\sum_{j=1}^{n}\mu_j\Gamma_P\big[P_j - P,\ P-Q\big]
\end{aligned}
$$

Combining equations (5) and (6) with the above expansions yields equation (4) and thus establishes the independence of (2) from the chosen base point. □

(2) It remains to show that λ agrees with the affine combination of the second-order i-affine structure as given by (2) for $\mu = (-1, 1, 1)$. As

shown in proposition 4.1 we have $\langle P, Q, S\rangle \in A_2\langle 3\rangle$. We consider everything in a chart U. Combining equations (2) and (3) we get

$$\begin{aligned}(-1,1,1)\cdot\langle P,Q,S\rangle = -P+Q+S+\frac{1}{2}\big(&\Gamma_P[Q-P+S-P]^2\\ &-\Gamma_P[Q-P]^2-\Gamma_P[S-P]^2\big)\end{aligned}$$

Expanding the symmetric bilinear map Γ_P thus results in

$$(-1,1,1)\cdot\langle P,Q,S\rangle = \lambda(P,Q,S)$$

as claimed.

□

5. Conclusion

An action of (the clone of) affine combinations on an i-structure is an algebraic model that makes precise the long-standing idea of differential geometry and of calculus that a (smooth) space has a geometry that is affine at the infinitesimal scale. These algebraic structures have been extracted by the author from Kock's work [4], [5]. The author has then generalised and studied them as infinitesimal models of algebraic theories in [1].

Within the framework of Synthetic Differential Geometry, in particular within the algebraic and well-adapted models of SDG there is a wealth of examples of i-affine spaces besides that of smooth and formal manifolds. This means that *the same* infinitesimal constructs and *the same* algebra of infinitesimals can be applied much more widely and beyond the context of (smooth) manifolds. However, so far (almost) all these examples have been based on the nil-square i-structure only[9].

In this paper we have shown that besides the canonical nil-square i-structure, a formal manifold carries a natural second-order i-structure (theorem 3.4). The affine structure on R^n induces an i-affine structure on the second-order i-structure (theorem 3.3). In contrast to the nil-square i-affine structure the second-order i-affine structure is not preserved by all maps

[9]The only exception has been the pointwise i-affine structure on function spaces studied in [1, chap. 3.3.2].

$R^n \to R^m$, and is hence not natural anymore. However, as we have shown for formal manifolds, there is a correspondence between symmetric affine connections (on points) and second-order i-affine structures (theorem 4.2). This provides us with a first example that a higher-order i-affine structure can be obtained from the data of a higher-order geometric structure on a formal manifold. Moreover, the $\log$-$\exp$ bijection yields also a natural geometric interpretation of an affine combination as the geometric addition of geodesic line segments extending the familiar vector parallelogram construction from the affine plane to curved space.

Does a manifold admit 3rd and higher-order i-affine structures? Firstly, it is possible and straight forward to generalise the construction of a second-order i-structure on R^n to kth-order i-structures that are preserved by all maps $R^n \to R^m$; the idea being that any $(k+1)$-linear occurrences of difference vectors formed from an i-n-tuple has to vanish. Due to the general gluing theorems in [1] it is then possible to show that any formal manifold carries a natural kth-order i-structure. Theorem 3.3 generalises to kth-order i-structures, too, but like with the second-order i-affine structure, kth-order i-affine structures are not preserved by maps $R^n \to R^m$.

As regards the construction of a 3rd-order i-affine structure on formal manifolds the author has obtained two results pointing towards the problem being more intricate than anticipated. Firstly, assuming the existence of a 3rd-order $\log$-$\exp$ bijection theorem 4.2 does not seem to generalise to 3rd-order i-affine structures. However, assuming that the formal manifold is a retract of a formally open subset of some R^n it seems possible to project the 3rd-order i-affine structure of R^n to the manifold. Understanding this discrepancy as well as the geometric obstruction responsible for the failure of the $\log$-$\exp$ bijection in order three is subject to current research.

Does a symmetric affine connection determine an i-affine structure uniquely? We have shown that a symmetric affine connection extends to a second-order i-affine structure on formal manifolds. What we have not addressed is the question whether the second-order i-affine structure is uniquely determined by the connection, or, if not, what structure parametrises the possible freedom of choice.

The author was able to show that in a *well-adapted model* each smooth manifold A carries *only one* i-affine structure on the first-order i-structure A_1. Studying the uniqueness of second-order i-affine structures is current work in progress.

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MOBI SPACES AND GEODESICS FOR THE N-SPHERE

Jorge P. FATELO and Nelson MARTINS-FERREIRA

Résumé. Nous introduisons une structure algébrique, appelée *mobi space*, qui peut être utilisée comme modèle pour les espaces où il existe des chemins géodésiques entre deux points quelconques. Cette nouvelle structure est semblable aux modules sur un anneau. Nous présentons des exemples diversifiés et montrons que la formule d'interpolation linéaire sphérique, qui reproduit les géodésiques sur les n-sphères, est un exemple de *mobi space*.

Abstract. We introduce an algebraic system, called mobi space, which can be used as a model for spaces with geodesic paths between any two of their points. This new algebraic structure is similar to modules over a ring. We present various examples and show that the formula for spherical linear interpolation, which gives geodesics on the n-sphere, is an example of a mobi space.

Keywords. Mobility algebra, mobi algebra, mobi space, affine space, affine mobi space, unit interval, ternary operation, geodesic path, geodesics, sphere, n-sphere, Slerp, damped harmonic oscillator, projectiles.

Mathematics Subject Classification (2010). 08A99, 03G99, 20N99, 08C15.

1. Introduction

The purpose of this work is to introduce an algebraic system which can be used to model spaces with geodesics. The main idea stems from the interplay between algebra and geometry. In affine geometry the notion of affine space is well suited for this purpose. Indeed, in an affine space we have scalar multiplication, addition and subtraction and so it is possible to parametrize, for any instant $t \in [0,1]$, a straight line between points x and y with the formula $(1-t)x + ty$. Such a line is clearly a geodesic path from x to y. In general terms, we may use an operation $q = q(x,t,y)$ to indicate the position, at an instant t, of a particle moving in a space X from a point x to a point y. If the particle is moving along a geodesic path then this operation must certainly verify some conditions. The aim of this project is to present an algebraic structure, (X,q), with axioms that are verified by any operation q representing a geodesic path in a space between any two of its points.

First results concerning this investigation were presented in [5] where a binary operation, obtained by fixing t to a value that positions the particle at half way from x to y, is studied. The whole movement of a particle on a geodesic path is captured when the variable t is allowed to range over a set of values, of which the unit interval is the most natural choice. The investigation of those structures and properties relevant to our study led us to the discovery of a new algebraic structure that was called mobi algebra [6]. A mobi algebra (or mobility algebra), besides being a suitable algebraic model for the unit interval, offers an interesting comparison with rings. A slogan may be used to illustrate that comparison: *a mobi algebra is to the unit interval in the same way as a ring is to the set of reals*. A mobi algebra is an algebraic system $(A, p, 0, ½, 1)$ consisting of a set together with a ternary operation p and three constants (see Definition 2.1). Every ring in which 2 is invertible has a mobi algebra structure, while a mobi algebra in which the element ½ is invertible has a ring structure [6].

Following the analogy with rings, and extending it to modules over a ring, we have arrived at a new structure, called mobility space (mobi space for short). If A is a mobi algebra then a mobi space, say (X,q), is defined over a mobi algebra in the sense that $q = q(x,t,y)$ operates on $x, y \in X$ and $t \in A$. The axioms defining a mobi space (Definition 2.2) are similar to the ones defining a mobi algebra. Several significant examples illustrate the strength of these axioms.

Every space with unique geodesics can be given a mobi space structure [9]. However, when geodesics are not unique (for instance when connecting antipodal points on the sphere) it is still possible to define a mobi space structure on that space. This is done by making appropriate choices and it is illustrated in the last section of this paper. The example of the sphere is considered with spherical linear interpolation (Slerp) whose formula gives rise to a mobi space structure.

2. Mobi space

In this section we give the definition of a mobi space over a mobi algebra. Its main purpose is to serve as a model for spaces with a geodesic path connecting any two points. It is similar to a module over a ring in the sense that it has an associated mobi algebra which behaves as the set of scalars. In [8], we show that the particular case of an affine mobi space is indeed the same as a module over a ring when the mobi algebra is a ring. In the last section, we present examples of geodesics on the n-sphere and on an hyperbolic n-space as mobi spaces over the unit interval.

Let us begin by briefly recalling the notion of a mobi algebra, introduced in [6].

Definition 2.1. *A mobi algebra is a system* $(A, p, 0, \sfrac{1}{2}, 1)$*, in which* A *is a set,* p *is a ternary operation and* 0*,* $\sfrac{1}{2}$ *and* 1 *are elements of A, that satisfies the following axioms:*

(A1) $p(1, \sfrac{1}{2}, 0) = \sfrac{1}{2}$

(A2) $p(0, a, 1) = a$

(A3) $p(a, b, a) = a$

(A4) $p(a, 0, b) = a$

(A5) $p(a, 1, b) = b$

(A6) $p(a, \sfrac{1}{2}, b_1) = p(a, \sfrac{1}{2}, b_2) \implies b_1 = b_2$

(A7) $p(a, p(c_1, c_2, c_3), b) = p(p(a, c_1, b), c_2, p(a, c_3, b))$

(A8) $p(p(a_1, c, b_1), \sfrac{1}{2}, p(a_2, c, b_2)) = p(p(a_1, \sfrac{1}{2}, a_2), c, p(b_1, \sfrac{1}{2}, b_2))$.

In this paper, the structure $(A, p, 0, \frac{1}{2}, 1)$ with $A = [0, 1]$ and

$$p(a, b, c) = a + b(c - a), \tag{1}$$

is called the *canonical mobi algebra*. Note that ½ is used to denote an element in an arbitrary mobi algebra, whereas $\frac{1}{2}$ is the real number in $[0, 1]$.

A mobi space is defined over a mobi algebra as follows.

Definition 2.2. *Let* $(A, p, 0, 1/2, 1)$ *be a mobi algebra. An* A*-mobi space* (X, q)*, consists of a set* X *and a map* $q\colon X \times A \times X \to X$ *such that:*

(X1) $q(x, 0, y) = x$

(X2) $q(x, 1, y) = y$

(X3) $q(x, a, x) = x$

(X4) $q(x, 1/2, y_1) = q(x, 1/2, y_2) \implies y_1 = y_2$

(X5) $q(q(x, a, y), b, q(x, c, y)) = q(x, p(a, b, c), y)$

The axioms **(X1)** to **(X5)** are the natural generalizations of axioms **(A3)** to **(A7)** of a mobi algebra. A natural generalization of **(A8)** is

$$q(q(x_1, a, y_1), 1/2, q(x_2, a, y_2)) = q(q(x_1, 1/2, x_2), a, q(y_1, 1/2, y_2)). \tag{2}$$

This condition, however, is too restrictive and is not in general verified by geodesic paths. That is the reason why we do not include it. When condition (2) is satisfied for all $x_1, x_2, y_1, y_2 \in X$ and $a \in A$, we call the A-mobi space (X, q) *affine* and speak of an A-mobi affine space (see Subsection 3.4 for examples and counterexamples, see also [8]).

If we write $x \oplus y$ instead of $q(x, 1/2, y)$ and consider the special case of equation (2) when $a = 1/2$ then we get the usual medial law

$$(x_1 \oplus y_1) \oplus (x_2 \oplus y_2) = (x_1 \oplus x_2) \oplus (y_1 \oplus y_2).$$

As an illustration of the fact that the medial law does not hold true in general for geodesic paths, let us consider the example of the unit sphere. The midpoint $a \oplus b$ of two points a and b on the equator is again on the equator. Midpoint of North Pole n and any point c on equator is on the 45^{th} parallel. But the geodesic midpoint of two points on the 45^{th} parallel does not live on the 45^{th} parallel, but somewhat to the North of it; the 45^{th} parallel is not a geodesic. So $(a \oplus b) \oplus (n \oplus n)$ is on the 45^{th} parallel, but $(a \oplus n) \oplus (b \oplus n)$

is not, but is north of it. This phenomenon is an aspect of the Gaussian curvature of the sphere.

In a mobi algebra $(A, p, 0, ½, 1)$, $\overline{a} \in A$ is defined for each element $a \in A$ as $\overline{a} = p(1, a, 0)$, which in the canonical case corresponds to $\overline{a} = 1-a$. Note that $½ \in A$ is the unique element with $\overline{½} = ½$. As an immediate consequence of the axioms of a mobi space, we get:

$$q(x, \overline{a}, y) = q(y, a, x). \tag{3}$$

Other properties of mobi spaces can be found in [7].

With the purpose of finding a general procedure to construct mobi spaces, let us consider a simple example with the variables $x, y \in \mathbb{R}$ and $t \in [0, 1]$. First, let us define a map

$$q(x, t, y) = x \cos(t) + y \sin(t)$$

and observe that it satisfies $q(x, 0, y) = x$ but not $q(x, 1, y) = y$. If we put

$$q(x, t, y) = x \cos\left(t\frac{\pi}{2}\right) + y \sin\left(t\frac{\pi}{2}\right)$$

then we have $q(x, 0, y) = x$ and $q(x, 1, y) = y$ but axiom **(X3)**, namely $q(x, t, x) = x$, fails for all values other than $x = 0$ or $t = 0, 1$.

If we change the map q to be

$$q(x, t, y) = x \cos^2\left(t\frac{\pi}{2}\right) + y \sin^2\left(t\frac{\pi}{2}\right) \tag{4}$$

then we get axioms **(X3)** and **(X4)** but the axiom **(X5)** is not verified. Take for example, $x = 1$, $y = 0$, $r = t = \frac{1}{3}$ and $s = 1$ and observe that

$$q(x, r + t(s - r), y) = \cos^2\left(\frac{5\pi}{18}\right)$$

while

$$q(q(x, r, y), t, q(x, s, y)) = \cos^4\left(\frac{\pi}{6}\right)$$

and they are not equal.

One might expect that in order to fix this problem it would be sufficient to find a map θ such that

$$q(x, \theta(r + t(s - r)), y) = q(q(x, \theta(r), y), \theta(t), q(x, \theta(s), y)).$$

However, this is not so simple. Indeed, perhaps a first guess would be to consider the map $\theta(t) = \frac{2}{\pi}\arcsin(t)$. With this modification, (4) would become

$$q(x, \theta(t), y) = x + (y - x)t^2.$$

This new formula, however, still does not satisfy axiom **(X5)**.

There is, nevertheless, a general procedure that leads to a mobi space out of the formula $h(x, t, y) = x + (y - x)t^2$, but it involves some extra work. We have to introduce one extra dimension, while solving a certain system of equations (see Proposition 2.3 below).

First we solve the system of two equations

$$\begin{cases} A + (B - A)\, r^2 &= x \\ A + (B - A)\, s^2 &= y \end{cases}$$

which has a unique solution for every $x, y \in \mathbb{R}$, $r, s \in \mathbb{R}^+$ and $s \neq r$, namely

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{s^2 - r^2} \begin{pmatrix} s^2 & -r^2 \\ -(1 - s^2) & 1 - r^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{5}$$

The mobi space on the set $\mathbb{R} \times \mathbb{R}^+$ (over the unit interval) is thus given by the formula

$$q((x, r), t, (y, s)) = (h(A, r + t(s - r), B), r + t(s - r))$$

with $s \neq r$ and A, B obtained from equation (5). When $s = r$ we put

$$q((x, r), t, (y, s)) = (x + t(y - x), r).$$

We end up with the operation $q : (\mathbb{R} \times \mathbb{R}^+) \times [0, 1] \times (\mathbb{R} \times \mathbb{R}^+) \to (\mathbb{R} \times \mathbb{R}^+)$ defined by

$$q\big((x, r), t, (y, s)\big) = \begin{cases} \Big(x + (y - x)\frac{2rt + (s-r)t^2}{r+s}, r + t(s - r)\Big) & , if \quad s \neq r \\ \big(x + t(y - x), r\big) & , if \quad s = r \end{cases}, \tag{6}$$

turning $(\mathbb{R} \times \mathbb{R}^+, q)$ into a mobi space over the unit interval canonical mobi algebra. This procedure, which provides a way to construct examples of mobi spaces, is detailed in the next proposition and generalized in Section 4. Further examples are given in the next section. In particular, example 3.3(1) gives a physical intuition on this construction.

Proposition 2.3. *Let $([0,1], p, 0, \frac{1}{2}, 0)$ be the canonical mobi algebra. Consider two real valued functions f and g, of one real variable, and let (I, τ) be a mobi space such that:*

1. *$I \subseteq \mathbb{R}$ is an interval of the real numbers;*

2. *the map $\tau : I \times [0,1] \times I \to I$ is defined as*

$$\tau(s, a, t) = s + a(t - s);$$

3. *for any $s, t \in I$, with $t \neq s$,*

$$f(s)\, g(t) \neq f(t)\, g(s). \tag{7}$$

For any real vector space V and any function $K \colon I \to V$, the structure $(V \times I, q)$ is a mobi space where $q \colon (V \times I) \times [0,1] \times (V \times I) \to V \times I$ is defined when $t \neq s$ by the formula ($\tau_a = \tau(s, a, t)$)

$$q((x, s), a, (y, t)) = (f(\tau_a)A + g(\tau_a)B - K(\tau_a), \tau_a), \tag{8}$$

with $A, B \in V$ the unique solutions of the system of equations

$$\begin{cases} f(s)A + g(s)B &= x + K(s) \\ f(t)A + g(t)B &= y + K(t) \end{cases}, \tag{9}$$

whereas

$$q((x, s), a, (y, s)) = ((1 - a)x + ay, s). \tag{10}$$

Proof. This proposition is a particular case of Theorem 4.1 with $U = V = X$ a real vector space and $h(A, t, B) = Af(t) + Bg(t) - K(t)$. The unique solution (A,B) to the system

$$\begin{cases} h(A, s, B) &= x \\ h(A, t, B) &= y \end{cases},$$

for every $x, y \in V$ and $s, t \in I, s \neq t$, is guaranteed by condition (7). $\square$

Several applications of Proposition 2.3 are presented in Subsections 3.2 and 3.3.

3. Examples

In the following list of examples, the underlying mobi algebra structure is the canonical mobi algebra $([0,1], p, 0, \frac{1}{2}, 1)$ with p as in (1). In each case, we present a set X and a ternary operation $q(x,a,y) \in X$, for all $x, y \in X$, and $a \in [0,1]$, verifying the axioms of Definition 2.2. We also explain how to obtain some examples as an instance of a more general construction.

3.1 The canonical mobi space

1. Vector spaces provide examples. For instance:

$$X = \mathbb{R}^n \quad (n \in \mathbb{N})$$

and

$$q(x,a,y) = (1-a)x + a\,y.$$

2. The well known technique of transporting the structure provides us with other ways of presenting the canonical structure. For every bijective map $F\colon X \to X'$, with $X' \subseteq \mathbb{R}^n$ a convex set, we get a mobi space (X,q) with

$$q(x,a,y) = F^{-1}((1-a)F(x) + a\,F(y)).$$

For instance, in the case of dimension one:

(a) If $F(x) = \log x$, $X = \mathbb{R}^+$ and $X' = \mathbb{R}$ then we get the mobi space (X,q), with
$$q(x,a,y) = x^{1-a}y^a.$$

(b) If $F(x) = \frac{1}{x}$, then $(\mathbb{R}^+, q)$ is a mobi space with

$$q(x,a,y) = \frac{xy}{ax + (1-a)y}.$$

3.2 Examples obtained directly from Proposition 2.3

To apply Proposition 2.3, we need three functions f, g and K such that $f(s)g(t) \neq f(t)g(s)$ for all $s,t \in I, s \neq t$. If g is a non-zero constant function, this condition just imply the injectivity of f. Let us begin with $K = 0$, $g = 1$ and a function f injective in I.

1. With $f\colon \mathbb{R}^+ \to \mathbb{R};\ x \mapsto x^2$, we obtain

$$X = \mathbb{R} \times \mathbb{R}^+$$

with the formula

$$q((x,s),a,(y,t)) = \left(x + (y-x)\frac{2\,s\,a + (t-s)a^2}{t+s}, s + a(t-s)\right).$$

This is, in fact, the example displayed in equation (6). Note that, since $r, s \in \mathbb{R}^+$, the second branch in (6) is not necessary.

2. With $f\colon \mathbb{R}^+ \to \mathbb{R};\ x \mapsto \frac{1}{x}$, we get the set

$$X = \mathbb{R} \times \mathbb{R}^+$$

with the formula

$$q((x,s),a,(y,t)) = \left(x + (y-x)\frac{a\,t}{(1-a)s + a\,t}, s + a(t-s)\right).$$

3. With $f\colon \mathbb{R} \to \mathbb{R};\ x \mapsto x^3$, we can consider the set

$$X = \mathbb{R}^2$$

and the formula

$$q((x,s),a,(y,t)) =$$
$$\left(x + (y-x)\frac{3\,s^2\,a + 3\,s(t-s)a^2 + (t-s)^2\,a^3}{s^2 + s\,t + t^2}, s + a(t-s)\right),$$

if $(s,t) \neq (0,0)$, and

$$q((x,0),a,(y,0)) = (x + a\,(y-x), 0)\,.$$

4. In general, applying Proposition 2.3 with $g = 1$, $K = 0$ and f an injective real function of one variable, we get a mobi space in any set $X \subseteq \mathbb{R}^2$ for which the formula

$$q((x,s),a,(y,t)) = \left(x + (y-x)\frac{f(s+(t-s)a) - f(s)}{f(t) - f(s)}, s + a(t-s)\right), \tag{11}$$

if $s \neq t$, and

$$q((x,s),a,(y,s)) = (x + a\,(y-x), s)\,,$$

defines a map $q\colon X \times [0,1] \times X \to X$, as it is the case when X is a convex set.

Let us confirm, with a direct proof, that this operation q verify property (3) of mobi spaces. For $t \neq s$, (11) implies

$$\begin{aligned} & q((x,s),1-a,(y,t)) \\ = & \left(x + (y-x)\frac{f(t+a(s-t)) - f(s)}{f(t) - f(s)}, t + a(s-t)\right) \\ = & \left(y + (x-y)\frac{f(t+a(s-t)) - f(t)}{f(s) - f(t)}, t + a(s-t)\right) \\ = & q((y,t),a,(x,s)). \end{aligned}$$

We will now see some examples obtained from physics.

3.3 Examples with physical interpretation

The following examples, from classical mechanics, can be viewed as an application of Proposition 2.3 with specific expressions for f, g and K.

1. Consider a constant acceleration motion, with $x \in \mathbb{R}^n$, and the following position equation

$$x(t) = x_0 + v_0\,t - k\,t^2.$$

We can think, for instance, of a projectile motion in the plane $\mathbb{R}^2$ where k would be $(0, \frac{g}{2})$ with g being the gravitational acceleration near the

Earth's surface. The constants x_0 and v_0 correspond to the usual initial conditions $x(0) = x_0$ and $x'(0) = v_0$. Imposing boundary conditions like $x(0) = x_0$ and $x(1) = x_1$, lead to:

$$x(t) = x_0 + (x_1 - x_0)\, t + k\, t\,(1 - t).$$

Note that the operation q defined as $q(x_0, t, x_1) = x(t)$ is not a mobi operation: in particular, idempotency $q(x, t, x) = x$ is not verified because a body could go up vertically and then down back to the same place; axiom **(X5)** is not verified either. The way to obtain a mobi space in this context is to let the variable t flow freely in an extra dimension with boundary conditions like $x(t_0) = x_0$ and $x(t_1) = x_1$. These conditions lead to:

$$x(t_0 + a(t_1 - t_0)) = x_0 + a(x_1 - x_0) + k\, a\,(1 - a)(t_1 - t_0)^2.$$

In the scope of Proposition 2.3, we could say that $f(t) = t$, $g(t) = 1$ and $K(t) = k\, t^2$. For any $k \in \mathbb{R}^n$, we have then a mobi space (X, q) over the canonical mobi algebra by taking the set

$$X = \mathbb{R}^{n+1}$$

with the formula

$$\begin{aligned} &q((x, s), a, (y, t)) = \\ &(x + a(y - x) + k\, a\,(1 - a)(t - s)^2, s + a\,(t - s))\,. \end{aligned} \tag{12}$$

Remarks:

(a) This example could be generalized to Special Relativity [4]. In this case, however, the operation q is a partial operation because, in Minkowski spacetime, not every two points can be reached from one another if one point is not inside the *light cone* of the other.

(b) The result (12) may also be obtained from the geodesic equations in coordinates (x, t), given by $\ddot{x} = -2k\, \dot{t}^2$ and $\ddot{t} = 0$, in a space where the invariant square of an infinitesimal line element is

$$dx^2 + 4\, k\, t\, dx\, dt + (4k^2t^2 - c^2)dt^2,$$

for any constant $c \neq 0$.

2. The solutions for the one-dimension motion of the well-known damped harmonic oscillator are of the form

$$Af(t) + Bg(t) - K(t),$$

where A and B are real parameters. If the oscillator is not driven, K(t)=0. Depending on the circumstances, we can have

(a) overdamping: $f(t) = e^{\alpha t},\ g(t) = e^{\beta t},\ \alpha \neq \beta$;

(b) critical damping: $f(t) = e^{\alpha t},\ g(t) = t\, e^{\alpha t}$;

(c) underdamping: $f(t) = e^{\alpha t}\sin(\beta t)$, $g(t) = e^{\alpha t}\cos(\beta t)$, $\beta \neq 0$;

(d) no damping: $f(t) = \sin(\beta t)$, $g(t) = \cos(\beta t)$, $\beta \neq 0$,

where $\alpha, \beta \in \mathbb{R}$ depend on the oscillatory system. For the first two cases, the determinant of the matrix (7) is non-zero for all $s, t \in \mathbb{R}$, with $t \neq s$ and therefore we can apply Proposition 2.3.

(a) In the overdamping case and for any $\alpha, \beta \in \mathbb{R}, \alpha \neq \beta$, we obtain the mobi space $(\mathbb{R}^2, q)$ over the canonical mobi algebra where q is defined, for $t \neq s$, by

$$q((x,s),a,(y,t)) = \Big(\frac{e^{\alpha(1-a)(s-t)} - e^{\beta(1-a)(s-t)}}{e^{\alpha s+\beta t} - e^{\alpha t+\beta s}}\, e^{(\alpha+\beta)t}\, x$$
$$+\frac{e^{\beta a(t-s)} - e^{\alpha a(t-s)}}{e^{\alpha s+\beta t} - e^{\alpha t+\beta s}}\, e^{(\alpha+\beta)s}\, y, s + a(t-s)\Big),$$

and, for $t = s$, by $q((x,s),a,(y,s)) = (x + a(y-x), s)$.

(b) In the critical damping case and for any $\alpha \in \mathbb{R}$, we obtain a mobi space $(\mathbb{R}^2, q)$ over the canonical mobi algebra with the formula

$$q((x,s),a,(y,t)) =$$
$$\left((1-a)\, x\, e^{\alpha a(t-s)} + a\, y\, e^{\alpha(1-a)(s-t)}, s + a(t-s)\right).$$

(c) For the case of underdamping, we can still apply Proposition 2.3 if we restrict the possible values of s and t to, for instance, $I = [0, \pi[$. The case $\alpha = 0$ and $\beta = 1$ is presented in the next item.

(d) Consider the case $f(t) = \sin(t)$ and $g(t) = \cos(t)$. Then, we obtain the mobi space $(\mathbb{R} \times [0, \pi[, q)$ with q defined, for $t \neq s$, by

$$q((x,s), a, (y,t)) =$$
$$\left(\frac{\sin[(1-a)(t-s)]}{\sin(t-s)}\, x + \frac{\sin[a(t-s)]}{\sin(t-s)}\, y, s + a(t-s) \right).$$

The similarity between this formula and the mobi operation (31) of Section 5 will be analysed in a future work.

3.4 Affineness of the examples

We end this section with some comments on whether the examples presented verify the affine condition (2) or not. Examples like those corresponding to example 3.2.4 are, in general, not affine in the sense that they don't verify

$$q\left(q[(x_1,s_1), a, (y_1,t_1)], \frac{1}{2}, q[(x_2,s_2), a, (y_2,t_2)] \right)$$
$$= q\left(q[(x_1,s_1), \frac{1}{2}, (x_2,s_2)], a, q[(y_1,t_1), \frac{1}{2}, (y_2,t_2)] \right).$$

Indeed, in example 3.2.1 for instance, we have that

$$q\left(q[(0,0), \frac{1}{3}, (0,1)], \frac{1}{2}, q[(1,1), \frac{1}{3}, (0,0)] \right) = \left(\frac{5}{27}, \frac{1}{2} \right)$$

but

$$q\left(q[(0,0), \frac{1}{2}, (1,1)], \frac{1}{3}, q[(0,1), \frac{1}{2}, (0,0)] \right) = \left(\frac{1}{6}, \frac{1}{2} \right).$$

Similarly, in example 3.2.3, we have for instance:

$$q\left(q[(0,0), \frac{1}{3}, (0,1)], \frac{1}{2}, q[(1,1), \frac{1}{3}, (0,0)] \right) = \left(\frac{19}{189}, \frac{1}{2} \right)$$

while

$$q\left(q[(0,0), \frac{1}{2}, (1,1)], \frac{1}{3}, q[(0,1), \frac{1}{2}, (0,0)] \right) = \left(\frac{1}{12}, \frac{1}{2} \right).$$

Of course, the canonical mobi spaces 3.1 are affine. Examples 3.2.2, 3.3.1 and 3.3.2b correspond also to affine mobi spaces while 3.3.2a does not. Indeed, for 3.3.2a, we have for instance that

$$q\left(q[(0,0),\tfrac{1}{3},(0,1)],\tfrac{1}{6},q[(1,1),\tfrac{1}{3},(0,0)]\right)$$
$$=\left(\frac{(e^{\alpha/18}-e^{\beta/18})(e^{\alpha/3}+e^{\beta/3})}{e^{\alpha}-e^{\beta}},\frac{7}{18}\right)$$

but

$$q\left(q[(0,0),\tfrac{1}{6},(1,1)],\tfrac{1}{3},q[(0,1),\tfrac{1}{6},(0,0)]\right)$$
$$=\left(e^{2(\alpha+\beta)/3}\frac{(e^{-4\alpha/9}-e^{-4\beta/9})(e^{\beta/6}-e^{\alpha/6})}{(e^{2\alpha/3}-e^{2\beta/3})(e^{\alpha}-e^{\beta})},\frac{7}{18}\right).$$

The two results are different if $\alpha \neq \beta$. However, in the limit situation when $\beta \to \alpha$, the critical case is recovered and the two results are naturally equal. The example 3.3.2c is not affine either.

In the following section we will thoroughly analyse a procedure to construct examples of mobi spaces which in general are not affine mobi spaces. In a sequel to this work we will investigate the case of spaces with geodesics and how to construct mobi spaces out of them.

4. General construction for mobi spaces

We present here a general result from which Proposition 2.3 can be deduced. In general, the examples that are obtained in this way are not affine.

Theorem 4.1. *Let (X, q_X) and (I, q_I) be two mobi spaces over a mobi algebra (A, p). Suppose the existence of two sets U, V and a function $h\colon U \times I \times V \to X$ such that the system*

$$\begin{cases} h(\alpha,t_0,\beta) &=& x_0 \\ h(\alpha,t_1,\beta) &=& x_1 \end{cases} \tag{13}$$

has a unique solution for every $x_0, x_1 \in X$ and any $t_0, t_1 \in I$ with $t_1 \neq t_0$, namely

$$\begin{cases} \alpha &=& \alpha(x_0,t_0,x_1,t_1) \\ \beta &=& \beta(x_0,t_0,x_1,t_1) \end{cases}. \tag{14}$$

Then, $(X \times I, q)$ *is a mobi space over the mobi algebra* (A, p) *where*

$$q\colon (X \times I) \times A \times (X \times I) \to (X \times I)$$

is defined using the map χ*, via (14),*

$$\chi(x_0, t_0, a, x_1, t_1) = \begin{cases} h[\alpha, q_I(t_0, a, t_1), \beta] & \textit{if} \quad t_1 \neq t_0 \\ q_X(x_0, a, x_1) & \textit{if} \quad t_1 = t_0 \end{cases} \tag{15}$$

as

$$q\left((x_0, t_0), a, (x_1, t_1)\right) = \left(\chi(x_0, t_0, a, x_1, t_1), q_I(t_0, a, t_1)\right).$$

Proof. The axioms **(X1)**, **(X2)** and **(X3)** are direct consequences of (13) and the fact that q_X and q_I are operations of mobi spaces. To prove **(X4)**, we first observe that

$$q\left[(x_0, t_0), ½, (x_1, t_1)\right] = q\left[(x_0, t_0), ½, (x_1', t_1')\right] \tag{16}$$

implies $q_I(t_0, ½, t_1) = q_I(t_0, ½, t_1')$ and hence $t_1' = t_1$. If $t_1 = t_0$, we also get $q_X(x_0, ½, x_1) = q_X(x_0, ½, x_1')$ and consequently $x_1' = x_1$. When $t_1 \neq t_0$, $t_1' = t_1$ and (16) imply

$$h[\alpha, q_I(t_0, ½, t_1), \beta] = h[\alpha', q_I(t_0, ½, t_1), \beta'] \equiv x_2,$$

where $\alpha' = \alpha(x_0, t_0, x_1', t_1)$ and $\beta' = \beta(x_0, t_0, x_1', t_1)$. Now, because $t_1 \neq t_0 \Rightarrow q_I(t_0, ½, t_1) \neq t_0$, the system

$$\begin{cases} h(\alpha, t_0, \beta) &= x_0 \\ h(\alpha, q_I(t_0, ½, t_1), \beta) &= x_2 \end{cases}$$

has a unique solution, we then conclude that $\alpha = \alpha'$ and $\beta = \beta'$ and consequently that

$$x_1' = h(\alpha', t_1, \beta') = h(\alpha, t_1, \beta) = x_1.$$

Let us now prove **(X5)**. We have to prove that $Q_1 = Q_2$ where:

$$Q_1 \equiv q\left[(x_0, t_0), p(a, b, c), (x_1, t_1)\right]$$

$$Q_2 \equiv q\Big(q\left[(x_0, t_0), a, (x_1, t_1)\right], b, q\left[(x_0, t_0), c, (x_1, t_1)\right]\Big).$$

To simplify the presentation of the proof, the following notations are used:

$$t_a = q_I(t_0, a, t_1), \quad t_c = q_I(t_0, c, t_1),$$

$$\chi_a = h(\alpha, t_a, \beta), \quad \chi_c = h(\alpha, t_c, \beta).$$

- Considering $t_0 \neq t_1$ and $t_a \neq t_c$, we have

$$\begin{aligned} Q_1 &= (h[\alpha, q_I(t_0, p(a,b,c), t_1), \beta], q_I[t_0, p(a,b,c), t_1]) \\ &= (h[\alpha, q_I(t_a, b, t_c), \beta], q_I[t_a, b, t_c]) \end{aligned}$$

and

$$\begin{aligned} Q_2 &= q\Big[(\chi_a, t_a), b, (\chi_c, t_c)\Big] \\ &= (h[\tilde{\alpha}, q_I(t_a, b, t_c), \tilde{\beta}], q_I[t_a, b, t_c]) \end{aligned},$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the unique solutions of the system

$$\begin{cases} h(\tilde{\alpha}, t_a, \tilde{\beta}) = \chi_a \\ h(\tilde{\alpha}, t_c, \tilde{\beta}) = \chi_c \end{cases}$$

which imply that $\tilde{\alpha} = \alpha$ and $\tilde{\beta} = \beta$, by definition of χ_a and χ_c and because $t_a \neq t_c$, therefore $Q_1 = Q_2$.

- Considering $t_0 = t_1$, and hence $t_a = t_c = t_0$, we have

$$\begin{aligned} Q_1 &= \Big(q_X[x_0, p(a,b,c), x_1], q_I[t_0, p(a,b,c), t_1]\Big) \\ &= \Big(q_X[q_X(x_0, a, x_1), b, q_X(x_0, c, x_1)], t_0\Big) \end{aligned}$$

and

$$\begin{aligned} Q_2 &= q\Big((q_X[x_0, a, x_1], t_a), b, (q_X[x_0, c, x_1], t_c)\Big) \\ &= \Big(q_X[q_X(x_0, a, x_1), b, q_X(x_0, c, x_1)], q_I(t_a, b, t_c)\Big) \\ &= \Big(q_X[q_X(x_0, a, x_1), b, q_X(x_0, c, x_1)], t_0\Big) \end{aligned}$$

implying that $Q_1 = Q_2$.

- Considering $t_0 \neq t_1$ and $t_a = t_c$, hence $\chi_a = \chi_c$, we have

$$\begin{aligned} Q_1 &= (h[\alpha, q_I(t_a, b, t_c), \beta], q_I[t_a, b, t_c]) \\ &= (\chi_a, t_a) \end{aligned}$$

and

$$\begin{aligned} Q_2 &= q\Big[(\chi_a, t_a), b, (\chi_c, t_c)\Big] \\ &= \Big(q_X[\chi_a, b, \chi_c], q_I[t_a, b, t_c]\Big)\Big) \\ &= (\chi_a, t_a) \\ &= Q_1. \end{aligned}$$

□

As an example, consider $U = X = V = \mathbb{R}^+$, $I = \mathbb{R}_0^+$, (A, p) the canonical mobi algebra and $h(\alpha, t, \beta) = \alpha\, \beta^t$. Then, for $t_0 \neq t_1$,

$$h[\alpha(x_0, t_0, x_1, t_1), t, \beta(x_0, t_0, x_1, t_1)] = x_0^{\frac{t-t_1}{t_0-t_1}}\, x_1^{\frac{t_0-t}{t_0-t_1}},$$

and if t is $q_I(t_0, a, t_1) = t_0 + a\,(t_1 - t_0)$, we get:

$$q\,[(x_0, t_0), a, (x_1, t_1)] = (x_0^{1-a} x_1^a, t_0 + a\,(t_1 - t_0)).$$

This expression is well-defined even for $t_1 = t_0$. This leaves no option for q_X if we want a continuous operation, as the only possibility is $q_X(x_0, a, x_1) = x_0^{1-a} x_1^a$. But any other mobi operation is allowed when $t_1 = t_0$ and we can write:

$$q\,[(x_0, t_0), a, (x_1, t_0)] = (q_X(x_0, a, x_1), t_0).$$

This example compares with Example 3.1.2a. Note that in Example 3.2.3, the branch corresponding to $(s, t) = (0, 0)$ cannot be obtained by continuity due to the fact that the limit $(s, t) \to (0, 0)$ does not exist. However, the *canonical* expression at $(0, 0)$ is the choice which corresponds to approaching the origin through the path $t = s$.

A useful particular case is when X is a vector space and $h(\alpha, t, \beta) = \alpha f(t) + \beta g(t) - K(t)$ for some scalar maps f, g and vector map K. The following proposition is a slight generalization of Proposition 2.3. Here, the canonical mobi algebra is replaced by an arbitrary one (A, p), the real interval I together with the map τ is replaced by a mobi space (I, q_I) and the real vector space V is replaced by the vector space X over a field F. Moreover q_X may be any mobi operation on X rather than $q_X(x, a, y) = (1 - a)x + ay$ as considered in Proposition 2.3.

Proposition 4.2. *Let (X, q_X) and (I, q_I) be two mobi spaces over a mobi algebra (A, p). Suppose moreover that X is a vector space over a scalar field F and let $f : I \to F$ and $g : I \to F$ be two functions such that, for any $t_0, t_1 \in I$ with $t_0 \neq t_1$, the following inequality holds*

$$f(t_0)\, g(t_1) \neq g(t_0)\, f(t_1). \tag{17}$$

Furthermore, we consider a function $K : I \to X$. Then $(X \times I, q)$ is a mobi space over (A, p) considering that

$$q : (X \times I) \times A \times (X \times I) \to (X \times I)$$

is defined as

$$q\left[(x_0, t_0), a, (x_1, t_1)\right] = \Big(\chi_a(x_0, t_0, x_1, t_1), q_I(t_0, a, t_1)\Big)$$

with

$$\begin{aligned}
&\chi_a\left(x_0, t_0, x_1, t_1\right) \\
&= \frac{g(t_1)\ (x_0 + K(t_0)) - g(t_0)\ (x_1 + K(t_1))}{f(t_0)\, g(t_1) - f(t_1)\, g(t_0)}\, f[q_I(t_0, a, t_1)] \\
&- \frac{f(t_1)\ (x_0 + K(t_0)) - f(t_0)\ (x_1 + K(t_1))}{f(t_0)\, g(t_1) - f(t_1)\, g(t_0)}\, g[q_I(t_0, a, t_1)] \\
&- K[q_I(t_0, a, t_1)],
\end{aligned}$$

when $t_1 \neq t_0$ and $\chi_a\left(x_0, t, x_1, t\right) = q_X(x_0, a, x_1)$ otherwise.

Proof. This is just Theorem 4.1 for the case

$$h(\alpha, t, \beta) = \alpha\, f(t) + \beta\, g(t) - K(t).$$

With $U = V = X$, the system (13) simply reads

$$\begin{pmatrix} f(t_0)\ g(t_0) \\ f(t_1)\ g(t_1) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} x_0 + K(t_0) \\ x_1 + K(t_1) \end{pmatrix}.$$

□

To illustrate this proposition, Example 3.2.4 can be generalized using an arbitrary mobi space I. Consider $h(\alpha, t, \beta) = \alpha f(t) + \beta$, in any set X for which the next formula is well defined. Then, when $t_0 \neq t_1$,

$$q[(x_0, t_0), a, (x_1, t_1)] = \\ (x_0 + (x_1 - x_0)\frac{f(q_I(t_0, a, t_1)) - f(t_0)}{f(t_1) - f(t_0)}, q_I(t_0, a, t_1)).$$

When $t_0 = t_1$, $q[(x_0, t_0), t, (x_1, t_0)] = (q_X(x_0, a, x_1), t_0)$ for any mobi operation q_X.

Even when the system of equations (13) does not have a unique solution, then, in some cases, it is still possible to define a mobi-space. This will be illustrated with the formula for spherical linear interpolation giving geodesics on the n-sphere.

5. Geodesics on the n-sphere

The purpose of this section is to show that a mobi space can be obtained using the geodesic curves on the n-sphere

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle_E = 1\}$$

and on one sheet of the two-sheeted hyperbolic n-space [15], as for instance

$$H^n = \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle_L = -1, x_1 > 0\}.$$

The notations $\langle\, ,\rangle_E$ and $\langle\, ,\rangle_L$ are used for the usual Euclidean and Lorentzian inner products, respectively. For the construction of the mobi operation for both cases at once, it is convenient to consider the family of functions

$$f(a) = \frac{e^{\alpha a} - e^{-\alpha a}}{2\alpha} \text{ and } g(a) = \frac{e^{\alpha a} + e^{-\alpha a}}{2}, \tag{18}$$

where $a \in \mathbb{R}$ and the parameter α is a non-zero complex number. These functions are real functions if and only if α is a real number or a pure imaginary number. In particular, we have that:

- $\alpha = 1 \Rightarrow f(a) = \sinh a$ and $g(a) = \cosh a$;

- $\alpha = i \Rightarrow f(a) = \sin a$ and $g(a) = \cos a$;
- $\alpha \to 0 \Rightarrow f(a) = a$ and $g(a) = 1$.

For our purpose, we want to consider only real functions and therefore, for the rest of this section, it is understood that α is such that f and g are real. In any case, the functions (18) verify the following properties:

$$\begin{aligned} -\alpha^2 f^2(a) + g^2(a) &= 1 && (19)\\ f(a)g(b) + f(b)g(a) &= f(a+b) && (20)\\ \alpha^2 f(a)f(b) + g(a)g(b) &= g(a+b) && (21)\\ f(-a) = -f(a) \quad &, \quad g(-a) = g(a) && (22)\\ f(0) = 0 \quad &, \quad g(0) = 1. && (23) \end{aligned}$$

In general terms, let us consider an interval $I \in \mathbb{R}$ containing 0 where g is injective. Let V be an inner product space, with the inner product denoted by $\langle\, , \rangle$, and a subspace $X \subseteq \{x \in V \mid \langle x, x\rangle = -\alpha^2\}$. Inner product here means a nondegenerate symmetric bilinear form. We are going to show that when there exists a unique function

$$\theta : X \times X \to I$$

such that $-\alpha^2 g[\theta(x, y)] = \langle x, y\rangle$, X can be given the structure of a mobi space. For instance, if X is S^n, θ may be defined as

$$\theta(x, y) = \arccos(\langle x, y\rangle_E) \text{ with } I = [0, \pi]$$

and if $X = H^n$, θ may be defined as

$$\theta(x, y) = \operatorname{arccosh}(-\langle x, y\rangle_L) \text{ with } I = [0, +\infty[.$$

The expressions are similar for any pure imaginary or non-zero real number α. The next two propositions show explicitly how to construct a mobi operation on X using the functions f, g and θ. This construction is based on the spherical linear interpolation (Slerp) used in computer graphics [14]. The first proposition is for the cases where the geodesic between two points is unique which occur when the only zero of f, in I, is zero. This is what happens for H^n but not for S^n because $\sin(\pi) = 0$. Nevertheless, the Proposition 5.1 could still be applied to a portion of the n-sphere which does not contain antipodal points, such as for example $\{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle_E = 1, x_1 > 0\}$.

Proposition 5.1. *Consider the* **real** *functions f and g, of one real variable, verifying the properties* (19) *to* (23) *for some number α. Suppose that g is injective in an interval I containing* 0 *and that, for $a \in I$, we have*

$$f(a) = 0 \Longleftrightarrow a = 0.$$

Let $(V, \langle\ ,\ \rangle)$ be a real inner vector space and consider a subspace

$$X \subseteq \{x \in V \mid \langle x, x\rangle = -\alpha^2\}.$$

If there exists a unique function $\theta : X \times X \to I$ such that

$$\langle x, y\rangle = -\alpha^2 g[\theta(x, y)] \tag{24}$$

and $\theta(x, y) = 0 \Longleftrightarrow y = x$, then (X, q) is a mobi space over the canonical mobi algebra where the ternary operation $q : X \times [0, 1] \times X \to X$ is defined, for $x \neq y$, by

$$q(x, t, y) = \frac{f[\theta(x, y)\,(1-t)]}{f[\theta(x, y)]}\,x + \frac{f[\theta(x, y)\,t]}{f[\theta(x, y)]}\,y, \tag{25}$$

and, otherwise, by $q(x, t, x) = x$.

Proof. To simplify the presentation, we use the notation $\Omega \equiv \theta(x, y)$. First, we have to prove that, when $x, y \in X$, $q(x, t, y)$ is still in X. The case $y = x$ is obvious. For $y \neq x$:

$$\begin{aligned}\langle q(x,t,y), q(x,t,y)\rangle &= \frac{f^2(\Omega(1-t))}{f^2(\Omega)}\langle x, x\rangle + \frac{f^2(\Omega t)}{f^2(\Omega)}\langle y, y\rangle \\ &+ 2\frac{f(\Omega(1-t))f(\Omega t)}{f^2(\Omega)}\langle x, y\rangle\end{aligned}$$

$$\begin{aligned}&= -\frac{\alpha^2}{f^2(\Omega)}(f^2(\Omega(1-t)) + f^2(\Omega t) + 2f(\Omega(1-t))f(\Omega t)g(\Omega)) \\ &= -\frac{\alpha^2}{f^2(\Omega)}(f(\Omega - \Omega t)(f(\Omega)g(\Omega t) + f(\Omega t)g(\Omega)) + f^2(\Omega t)) \\ &= -\frac{\alpha^2}{f^2(\Omega)}(f^2(\Omega)g^2(\Omega t) + f^2(\Omega t)(1 - g^2(\Omega)) \\ &= -\alpha^2(g^2(\Omega t) - \alpha^2 f^2(\Omega t)) = -\alpha^2.\end{aligned}$$

Now, as X is a subspace containing x and y and $q(x,t,y)$ is a linear combination of x and y, we conclude that it is also in X. Axioms **(X1)**, **(X2)**, **(X3)** of a mobi space are a direct consequence of the definition of q. To prove **(X4)** we will use the notation $\Omega' \equiv \theta(x,y')$. If $x \neq y$ and $x \neq y'$, we have that $q\left(x, \frac{1}{2}, y\right) = q\left(x, \frac{1}{2}, y'\right)$ implies

$$\frac{f\left(\frac{\Omega}{2}\right)}{f(\Omega)}(x+y) = \frac{f\left(\frac{\Omega'}{2}\right)}{f(\Omega')}(x+y') \tag{26}$$

Applying the inner product with x in both sides of this equation and using properties (19) to (21) in the form

$$f(\Omega) = 2f\left(\frac{\Omega}{2}\right)g\left(\frac{\Omega}{2}\right) \quad \text{and} \quad 1+g(\Omega) = 2g^2\left(\frac{\Omega}{2}\right),$$

we get

$$\begin{aligned} & \frac{1}{2g\left(\frac{\Omega}{2}\right)}\langle x, x+y\rangle = \frac{1}{2g\left(\frac{\Omega'}{2}\right)}\langle x, x+y'\rangle \\ \Rightarrow \quad & \frac{1}{2g\left(\frac{\Omega}{2}\right)}(-\alpha^2 - \alpha^2 g(\Omega)) = \frac{1}{2g\left(\frac{\Omega'}{2}\right)}(-\alpha^2 - \alpha^2 g(\Omega')) \\ \Rightarrow \quad & g\left(\frac{\Omega}{2}\right) = g\left(\frac{\Omega'}{2}\right) \neq 0. \end{aligned}$$

Going back to (26) with this result, we conclude $y = y'$. If $x = y$ and $x \neq y'$, then $q\left(x, \frac{1}{2}, y\right) \neq q\left(x, \frac{1}{2}, y'\right)$. Indeed, $q\left(x, \frac{1}{2}, x\right) = q\left(x, \frac{1}{2}, y'\right)$ would imply

$$\begin{aligned} & x = \frac{f\left(\frac{\Omega'}{2}\right)}{f(\Omega')}(x+y') \Rightarrow \langle x, x\rangle = \frac{1}{2g\left(\frac{\Omega'}{2}\right)}\langle x, x+y'\rangle \\ & \Rightarrow 1 = \frac{1}{2g\left(\frac{\Omega'}{2}\right)}\left(1+g(\Omega')\right) \Rightarrow g\left(\tfrac{\Omega'}{2}\right) = 1 \Rightarrow y' = x, \end{aligned}$$

in contradiction with the hypothesis. The case $x \neq y$ and $x = y'$ is similar. Obviously **(X4)** is also verified if $x = y$ and $x = y'$. The proof of **(X5)** begin with the observation that:

$$g[\theta(q(x,a,y), q(x,c,y))] = g[\theta(x,y)(c-a)]. \tag{27}$$

Indeed, beginning with the left-hand side of (27), if $y \neq x$:

$$\begin{aligned}
& -\frac{1}{\alpha^2}\langle \frac{f(\Omega(1-a))}{f(\Omega)}x + \frac{f(\Omega a)}{f(\Omega)}y, \frac{f(\Omega(1-c))}{f(\Omega)}x + \frac{f(\Omega c)}{f(\Omega)}y\rangle \\
= \; & \frac{f(\Omega(1-a))f(\Omega(1-c))}{f^2(\Omega)} + \frac{f(\Omega a)f(\Omega c)}{f^2(\Omega)} \\
+ \; & \left(\frac{f(\Omega(1-a))f(\Omega c) + f(\Omega(1-c))f(\Omega a)}{f^2(\Omega)}\right) g(\Omega) \\
= \; & \frac{f^2(\Omega)g(\Omega a)g(\Omega c) - g^2(\Omega)f(\Omega a)f(\Omega c) + f(\Omega a)f(\Omega c)}{f^2(\Omega)} \\
= \; & g(\Omega a)g(\Omega c) - \alpha^2 f(\Omega a)f(\Omega c) \\
= \; & g(\Omega a - \Omega c) = g(\Omega c - \Omega a)
\end{aligned}$$

If $y = x$, then

$$\begin{aligned}
g[\theta(q(x,a,y), q(x,c,y))] &= -\frac{1}{\alpha^2}\langle q(x,a,y), q(x,c,y)\rangle \\
&= -\frac{1}{\alpha^2}\langle x, x\rangle \\
&= 1 = g(0) = g[\theta(x,y)(c-a)].
\end{aligned}$$

Now, because $a, c \in [0,1]$ and $\Omega \in I$ imply $\Omega|c-a| \in I$, we can conclude, since g is injective in I, that

$$|\theta(q(x,a,y), q(x,c,y))| = |\theta(x,y)(c-a)| = |\Omega(c-a)|. \tag{28}$$

By (22), this also imply that, for any $b \in [0,1]$, the following relation is true:

$$\frac{f[\theta(q(x,a,y), q(x,c,y))\, b]}{f[\theta(q(x,a,y), q(x,c,y))]} = \frac{f[\Omega(c-a)\, b]}{f[\Omega(c-a)]}.$$

With these results, we are able to prove **(X5)**. For simplification, we use the notation $\hat{c} \equiv c - a$. First, for $q(x,a,y) \neq q(x,c,y)$ and $x \neq y$:

$$
\begin{aligned}
& q[q(x,a,y),b,q(x,c,y)] \\
= \;& \frac{f[\Omega\,\hat{c}(1-b)]}{f[\Omega\,\hat{c}]}\,q(x,a,y) + \frac{f[\Omega\,\hat{c}\,b]}{f[\Omega\,\hat{c}]}\,q(x,\hat{c}+a,y) \\
= \;& \frac{g[\Omega\,\hat{c}\,b]f[\Omega(1-a)] - f[\Omega\,\hat{c}\,b]g[\Omega(1-a)]}{f(\Omega)}\,x \\
+ \;& \frac{g[\Omega\,\hat{c}\,b]f[\Omega\,a] + f[\Omega\,\hat{c}\,b]g[\Omega\,a]}{f(\Omega)}\,y \\
= \;& \frac{f[\Omega(1-a-\hat{c}\,b)]}{f(\Omega)}\,x + \frac{f[\Omega(a+\hat{c}\,b)]}{f(\Omega)}\,y \\
= \;& q[x,a+(c-a)b,y].
\end{aligned}
$$

If $q(x,a,y) = q(x,c,y)$, then $q(q(x,a,y),b,q(x,c,y)) = q(x,a,y)$. On the other hand, from (27), we conclude that $x = y$ or $c = a$ and in both cases $q(x,a+b(c-a),y) = q(x,a,y)$. $\square$

Before going to Proposition 5.2 that will explain how we can still get a mobi space out of a *Slerp type* formula on the n-sphere despite the fact that geodesics between antipodal points are not unique, let us take a closer look to the formula (25) in the case of S^n. This formula just gives the intersection between S^n and a plane that contains the origin and the points x and y, when x and y are not collinear. Starting at x when $t = 0$, a particle that goes to y on that plane at constant speed will be, at an instant $t \in [0,1]$ at

$$q(x,t,y) = \cos(\Omega\,t)\,x + \sin(\Omega\,t)\,z \tag{29}$$

where

$$\cos(\Omega)\,x + \sin(\Omega)\,z = y. \tag{30}$$

Because $\langle x,x\rangle_E = \langle y,y\rangle_E = 1$ and $\Omega \equiv \theta(x,y) = \arccos\langle x,y\rangle_E$, we have that $\langle x,z\rangle_E = 0$ and $\langle z,z\rangle_E = 1$. When $\sin(\Omega) \neq 0$, we can just solve (30) to obtain z and then (29) reads as expected:

$$q(x,t,y) = \frac{\sin[\Omega\,(1-t)]}{\sin(\Omega)}\,x + \frac{\sin[\Omega\,t]}{\sin(\Omega)}\,y. \tag{31}$$

When $\Omega = 0$, which means $x = y$, there is no journey to make: $q(x,t,x) = x$. When $\Omega = \pi$, which means $y = -x$, we have to choose the plane we wish to travel on. Equivalently, we have to chose the direction $v(x) \in \mathbb{R}^{n+1}$ we want to be playing the role of z. Of course, we still need $\langle x, v(x)\rangle_E = 0$ and $\langle v(x), v(x)\rangle_E = 1$. There is one more condition: to get a mobi space, we also need v to be an even map because, from property (3), $q(x, t, -x) = q(-x, 1-t, x)$ which means that in a round trip, the going and the return must be done on the same path.

Proposition 5.2. *Consider the euclidean n-sphere*

$$X = \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle = 1\},$$

a map $v : X \to X$ such that

$$v(-x) = v(x) \textit{ and } \langle x, v(x)\rangle = 0,$$

and the map $\theta : X \times X \to [0, \pi]$ defined by

$$\theta(x, y) = \arccos(\langle x, y\rangle).$$

With $q : X \times [0, 1] \times X \to X$ defined by

$$q(x, t, y) = \\ \begin{cases} \dfrac{\sin[\theta(x,y)\,(1-t)]}{\sin[\theta(x,y)]}\,x + \dfrac{\sin[\theta(x,y)\,t]}{\sin[\theta(x,y)]}\,y & , if \quad \theta(x,y) \in]0,\pi[\\ \cos[\theta(x,y)\,t]\,x + \sin[\theta(x,y)\,t]\,v(x) & , if \quad \theta(x,y) \in \{0,\pi\} \end{cases},$$

(X, q) is a mobi space over the canonical mobi algebra.

Proof. Most of the proof is the same as the proof of Proposition 5.1. We just have to consider the extra case $y = -x$ corresponding to $\Omega \equiv \theta(x, y) = \pi$. When $\Omega = \pi$, we have $q(x, 0, y) = \cos(0)x + \sin(0)v(x) = x$ and $q(x, 1, y) = \cos(\pi)x + \sin(\pi)v(x) = -x = y$, so Axioms **(X1)**, **(X2)** and **(X3)** of a mobi space are verified. Regarding **(X4)**, when $\Omega = \pi$ and $\Omega' \in]0, \pi[$, we have that $q\left(x, \frac{1}{2}, y\right) \neq q\left(x, \frac{1}{2}, y'\right)$. Indeed $q\left(x, \frac{1}{2}, y\right) = q\left(x, \frac{1}{2}, y'\right)$ implies

$$v(x) = \frac{\sin\left(\frac{\Omega'}{2}\right)}{\sin(\Omega')}(x + y') \Rightarrow 0 = \cos\left(\frac{\Omega'}{2}\right) \Rightarrow \Omega' = \pi,$$

in contradiction with $\Omega' \in]0,\pi[$. The case $\Omega = \pi$ and $\Omega' = 0$ is also incompatible with $q\left(x, \frac{1}{2}, y\right) = q\left(x, \frac{1}{2}, y'\right)$ because $v(x) \neq x$. Interchanging y and y' in the previous situations gives similar results. The case $\Omega = \pi$ and $\Omega' = \pi$ implies $y = -x = y'$, therefore **(X4)** is verified. Regarding **(X5)**, we first observe that (27) is valid for all $x, y \in X$. Indeed, if $y = -x$, then

$$\begin{aligned} & \cos[\theta(q(x,a,y), q(x,c,y))] = \langle q(x,a,y), q(x,c,y)\rangle \\ = \; & \langle \cos(\pi a)\, x + \sin(\pi a)\, v(x), \cos(\pi c)\, x + \sin(\pi c)\, v(x)\rangle \\ = \; & \cos(\pi a)\cos(\pi c) + \sin(\pi a)\sin(\pi c) \\ = \; & \cos[\pi(a-c)]. \end{aligned}$$

So, we have that, $\forall x, y \in X$:

$$\theta(q(x,a,y), q(x,c,y)) = \theta(x,y)|c-a|. \tag{32}$$

From equation (32), we conclude that $\theta(q(x,a,y), q(x,c,y)) = \pi$ if and only if $\Omega = \pi$ and $|c-a| = 1$ and that $\theta(q(x,a,y), q(x,c,y)) = 0$ if and only if $\Omega = 0$ or $c = a$. Therefore, besides the cases already proved in Proposition 5.1, we have to consider the following four situations:

1. $\theta(q(x,a,y), q(x,c,y)) = \pi$, $\Omega = \pi$ and
   (a) $c = 0$, $a = 1$
   (b) $c = 1$, $a = 0$
2. $\theta(q(x,a,y), q(x,c,y)) = 0$, $\Omega = \pi$ and $c = a$
3. $\theta(q(x,a,y), q(x,c,y)) \in]0,\pi[$, $\Omega = \pi$, $c \neq a$ and $|c-a| \neq 1$.

For the situation (1a):

$$\begin{aligned} q[q(x,a,y), b, q(x,c,y)] &= q(-x,b,x) = \cos(\pi b)(-x) + \sin(\pi b)\, v(-x) \\ q[x, a+b(c-a), y] &= q(x, 1-b, -x) \\ &= \cos(\pi - \pi b)\, x + \sin(\pi - \pi b)\, v(x) \\ &= -\cos(\pi b)\, x + \sin(\pi b)\, v(x). \end{aligned}$$

The Axiom **(X5)** is ensured through the hypothesis $v(-x) = v(x)$. For the situation (1b):

$$\begin{aligned} q[q(x,a,y), b, q(x,c,y)] = q(x,b,-x) &= \cos(\pi b)\, x + \sin(\pi b)\, v(x) \\ q[x, a+b(c-a), y] = q(x,b,-x) &= \cos(\pi b)\, x + \sin(\pi b)\, v(x). \end{aligned}$$

In situation (2), $c = a$ and **(X3)** implies **(X5)**. Using $\hat{c} \equiv c - a$, we have for the situation (3):

$$\begin{aligned}
& q[q(x,a,y), b, q(x,c,y)] \\
= \ & \frac{\sin[\pi(c-a)(1-b)]}{\sin[\pi(c-a)]} (\cos(\pi a)\, x + \sin(\pi a)\, v(x)) \\
+ \ & \frac{\sin[\pi(c-a)b]}{\sin[\pi(c-a)]} (\cos(\pi c)\, x + \sin(\pi c)\, v(x)) \\
= \ & \frac{\sin[\pi\hat{c}(1-b)]\cos(\pi a) + \sin[\pi\hat{c}\, b]\cos(\pi(\hat{c}+a))}{\sin[\pi\,\hat{c}]}\, x \\
+ \ & \frac{\sin[\pi\hat{c}(1-b)]\sin(\pi a) + \sin[\pi\hat{c}\, b]\sin(\pi(\hat{c}+a))}{\sin[\pi\,\hat{c}]}\, v(x) \\
= \ & \cos[\pi\,\hat{c}\, b]\cos(\pi a) - \sin[\pi\,\hat{c}\, b]\sin[\pi a]\, x \\
+ \ & \cos[\pi\,\hat{c}\, b]\sin(\pi a) + \sin[\pi\,\hat{c}\, b]\cos[\pi a]\, v(x) \\
= \ & \cos[\pi(a+\hat{c}\, b)]\, x + \sin[\pi(a+\hat{c}\, b)]\, v(x) \\
= \ & q(x, a + b(c-a), y)
\end{aligned}$$

□

To finish this section, we present three examples of the map v used in Proposition 5.2. First, consider the 1-sphere i.e. the circle. We can choose to move between antipodal points in the anticlockwise direction when starting somewhere at the top of the circle and in the clockwise direction when starting at the bottom. More specifically, if $x = (\cos\theta, \sin\theta), \theta \in [0, 2\pi[$, then v is defined as

$$v(x) = \begin{cases} (-\sin\theta, \cos\theta) & if \quad \theta \in [0,\pi[\\ (\sin\theta, -\cos\theta) & if \quad \theta \in [\pi, 2\pi[\end{cases} .$$

Secondly, let us choose to connect two antipodal points on S^2, different from the poles, through the north pole and link the poles (on the z-axis) through the positive x-axis. This gives the following choice for v, considering $x_1^2 + x_2^2 + x_3^2 = 1$:

$$v(x_1, x_2, x_3) = \begin{cases} \dfrac{(-x_1x_3,\ -x_2x_3,\ 1-x_3^2)}{\sqrt{1-x_3^2}} & if \quad x_3 \neq \pm 1 \\ (1,0,0) & if \quad x_3 = \pm 1 \end{cases} .$$

As a third example, consider the 2-sphere parametrized in spherical coordinates as:

$$\{(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi), (\theta,\varphi) \in ([0,2\pi[\times]0,\pi[) \cup (0,0) \cup (0,\pi)\}.$$

A possible map v is the following:

$$\begin{aligned} & v(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \\ = & \begin{cases} (-\sin\theta, \cos\theta, 0) & if \quad \varphi \in \left[0, \frac{\pi}{2}\right[\text{ or } \left(\varphi = \frac{\pi}{2}, \theta \in [0,\pi[\right) \\ (\sin\theta, -\cos\theta, 0) & if \quad \varphi \in \left]\frac{\pi}{2}, \pi\right] \text{ or } \left(\varphi = \frac{\pi}{2}, \theta \in [\pi, 2\pi[\right) \end{cases} . \end{aligned}$$

In this example, v is on the equator in a plane rotated $\frac{\pi}{2}$ around the z-axis from the meridian of x. The choice $\theta = 0$ for the poles connects them through the positive y-axis. The other antipodal points are connected through a path that stays between the parallels of the two points, with an arbitrary choice for antipodal points on the equator.

6. Conclusion

We have introduced a new algebraic structure which captures some features of geodesic paths. Several examples were used to illustrate the difficulty in generating non-trivial examples (other than the affine case) and a general procedure was given in Theorem 4.1. This general construction, however, has the effect of raising an extra dimension. The construction commonly named as Slerp was used to show that mobi spaces include the example of geodesics on the n-sphere. It can be seen as a particular case of our general construction if we fix $t_0 = 0$ and let t_1 be a function of the end points x_0, x_1 as $t_1 = \theta(x_0, x_1)$. In that case there is no need to use the extra dimension, I, because q can be defined through $q(x_0, a, x_1) = h(\alpha, q_I(0, a, \theta(x_0, x_1), \beta)$ with α and β being the solutions to the system of equations (13). This, however, is done at the expense of imposing some tight conditions on the maps h and θ and needs further investigations. Similarly, when h is obtained as a geodesic flow (in a space with unique geodesics) we can take $t_0 = 0$ and $t_1 = 1$ in Theorem 4.1 and observe that (X, q) is a mobi space with $q(x_0, a, x_1) = h(\alpha, a, \beta)$. Once again there is no need for the extra I-dimension but it comes with a cost of imposing extra conditions on the map h. It turns out that when h is a geodesic flow in a space with unique geodesics then the required conditions are satisfied ([9]).

Some lines of future study include the connection with affine geometry [1, 10, 11] or the geometry of geodesics [2, 3]. As well as the study of affine mobi spaces *per se* [8]. The presence of an operation $x \oplus y = q(x, 1/2, y)$ admitting cancellation, together with the property $x \oplus y = y \oplus x$, tells us that the category of mobi spaces is a weakly Mal'tsev category [13, 12]. This was in fact the starting point that originated our investigation on mobi spaces.

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A FUNCTORIAL CHARACTERIZATION OF VON NEUMANN ENTROPY

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Résumé. En utilisant des fibrations convexes de Grothendieck, nous caractérisons l'entropie de von Neumann comme un foncteur des espaces de probabilité non commutatifs de dimension finie et des $*$-homomorphismes préservant l'état aux nombres réels. Nos axiomes reproduisent ceux de Baez, Fritz et Leinster caractérisant la différence d'entropie de Shannon. L'existence de désintégrations pour les espaces de probabilité classiques joue un rôle crucial dans notre caractérisation.

Abstract. Using convex Grothendieck fibrations, we characterize the von Neumann entropy as a functor from finite-dimensional non-commutative probability spaces and state-preserving $*$-homomorphisms to real numbers. Our axioms reproduce those of Baez, Fritz, and Leinster characterizing the Shannon entropy difference. The existence of disintegrations for classical probability spaces plays a crucial role in our characterization.

Keywords. Convex category, disintegration, Grothendieck fibration, Landauer's principle, optimal hypothesis, quantum entropy.

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Contents

1 Introduction and outline

In 2011, Baez, Fritz, and Leinster (BFL) characterized the Shannon entropy (difference) of finite probability distributions as the only non-vanishing continuous affine functor $\mathbf{FinProb} \to \mathbb{BR}_{\geq 0}$ from finite probability spaces to non-negative numbers up to an overall non-negative constant [4]. Here, $\mathbf{FinProb}$ is the category of finite sets equipped with probability measures as objects and probability-preserving functions as morphisms. The codomain category, $\mathbb{BR}_{\geq 0}$, is the category consisting of a single object and whose morphisms from that object to itself are all non-negative real numbers equipped with addition as the composition.

A natural follow-up question is whether the von Neumann (or finite-dimensional Segal) entropy can be characterized in a similar manner by replacing $\mathbf{FinProb}$ with $\mathbf{NCFinProb}$, the category of finite quantum (i.e. non-commutative) probability spaces, consisting of unital finite-dimensional C^*-algebras equipped with states as objects and state-preserving unital $*$-homomorphisms as morphisms. Physically, such objects correspond to hybrid classical/quantum systems and the morphisms describe deterministic dynamics, which includes tracing out subsystems. Although this question was partially explored by Baez and Fritz [2], a suitably similar set of axioms was never obtained. The present manuscript accomplishes this task.

There are two difficulties with extending BFL's result to the quantum setting. The first issue is that the difference of von Neumann entropies need not have a fixed sign. There are state-preserving unital $*$-homomorphisms that *increase* the entropy as well as *decrease* the entropy. The sign of the entropy difference is closely related to the fact that Landauer's principle holds for classical systems [25], but could fail for quantum systems [8, 39]. The root of the increase stems from the uncertainty principle and entanglement.

Using our axioms, we show that the existence of disintegrations [36] (called optimal hypotheses in [3]) implies the non-negativity of the entropy difference. Since disintegrations always exist for finite-dimensional classical systems, this proves one of the key assumptions of BFL in their functorial characterization of the Shannon entropy [4].

The second difficulty when attempting to extend BFL's work to quantum systems is that the objects of $\mathbf{NCFinProb}$ are not convex generated by any single object in that category. Note that this occurs for $\mathbf{FinProb}$, where an arbitrary probability space (X, p), with X a finite set and p a probability measure on X, can be decomposed into a convex sum as $(X, p) \cong \bigoplus_{x\in X} p_x\mathbf{1}$, where $\mathbf{1}$ is the (essentially) unique probability space consisting of a single element and p_x is the probability of $x \in X$. In $\mathbf{NCFinProb}$, a quantum probability space such as $(\mathcal{M}_m, \omega)$ cannot be expressed as a convex combination of lower-dimensional probability spaces. Here, $m \in \mathbb{N}$, $\mathcal{M}_m$ is the C^*-algebra of $m \times m$ matrices, and ω is a state on $\mathcal{M}_m$.

In this manuscript, we simultaneously address both these issues and provide a functorial characterization of the von Neumann entropy. This is done by introducing Grothendieck fibrations of convex categories and fibred affine functors. The category $\mathbf{NCFinProb}$ forms a fibration over $\mathbf{fdC^*\text{-}Alg}$, the category of finite-dimensional unital C^*-algebras and unital $*$-homomorphisms, by sending each quantum probability space $(\mathcal{A}, \omega)$ to the underlying C^*-algebra $\mathcal{A}$. The von Neumann entropy (difference) provides a functor

$$\begin{tikzcd} \mathbf{NCFinProb} \arrow[r, "H"] \arrow[d] & \mathbb{BR} \arrow[d] \\ \mathbf{fdC^*\text{-}Alg} \arrow[r] & \underline{\mathbf{1}} \end{tikzcd}, \tag{1.1}$$

where $\underline{\mathbf{1}}$ is the category consisting of a single object and just the identity morphism, $\mathbb{BR}$ is the one-object category whose morphisms consist of all real numbers with composition rule given by addition, and the left vertical arrow is the fibration just mentioned.

The fibres of the left and right fibrations in (1.1) are convex categories. Over each C^*-algebra $\mathcal{A}$ on the left, one has the convex set of states $\mathcal{S}(\mathcal{A})$ on $\mathcal{A}$, which is viewed as a discrete convex category. A morphism $f : \mathcal{B} \to \mathcal{A}$ of C^*-algebras gets lifted to the morphism $\mathcal{S}(f) : \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{B})$ that acts as the pullback of states, sending ω to $\omega \circ f$. On the right, $\mathbb{BR}$ is also a convex category, with convex combinations of real numbers as the convex operation.

This entropy difference functor sends a state $\omega \in \mathcal{S}(\mathcal{A})$ together with a morphism $f : \mathcal{B} \to \mathcal{A}$ to a real number $H_f(\omega)$. Given another state $\xi \in \mathcal{S}(\mathcal{A})$ and a number $\lambda \in [0, 1]$, one obtains the inequality

$$H_f\big(\lambda\omega + (1-\lambda)\xi\big) \geq \lambda H_f(\omega) + (1-\lambda)H_f(\xi), \tag{1.2}$$

which is of fundamental importance in quantum information theory. The non-negativity of the quantity

$$\chi_f(\lambda;\omega,\xi) := H_f\big(\lambda\omega + (1-\lambda)\xi\big) - \lambda H_f(\omega) - (1-\lambda)H_f(\xi) \tag{1.3}$$

is related to the *monotonicity of entropy under partial trace*, which is known to be equivalent to *strong subadditivity* [46]. A special case of this inequality, when $f := !_{\mathcal{A}} : \mathbb{C} \to \mathcal{A}$ is the unique unital $*$-homomorphism into $\mathcal{A}$, leads to the fact that mixing always increases entropy. It is actually only this weaker property that will play a role in our current characterization.

For more general algebras, if ω and ξ have orthogonal supports, and if $f : \mathcal{B} \to \mathcal{A}$ preserves this orthogonality, then equality in (1.2) is obtained. This condition, which we call *orthogonal affinity*, is what replaces the affine assumption of entropy difference made by BFL. However, orthogonal affinity and (1.2) are not enough to guarantee that $H_{\mathcal{A}}(\omega) := H_{!_{\mathcal{A}}}(\omega)$ vanishes on pure states ω. If one imposes this additional assumption, one can show that it is no longer necessary to assume $\chi_f(\lambda;\omega,\xi) \geq 0$ for all inputs. Instead, one can demand the simpler assumption that $H_{\mathcal{A}}(\omega) \geq 0$ for all states ω. In other words, one can replace BFL's non-negativity assumption for classical entropy *difference* with the assumption that $H_{\mathcal{A}}(\omega) \geq 0$ for all states ω on C^*-algebras $\mathcal{A}$, with equality for pure states. The relationships between these assumptions will be made precise in the body of the present manuscript. Our main theorem can then be phrased as follows.

Theorem 1.4 (A functorial characterization of quantum entropy (Theorem 4.26 in body))**.** *Let H : $\mathbf{NCFinProb} \to \mathbb{BR}$ be a continuous and orthogonally affine fibred functor, as in (1.1), for which $H_{\mathcal{A}}(\omega) \geq 0$ for all states $\omega \in \mathcal{S}(\mathcal{A})$, with equality on all pure states, for all finite-dimensional C^*-algebras $\mathcal{A}$. Then there exists a constant $c \geq 0$ such that*

$$H_f(\omega) = c\Big(S(\omega) - S(\omega \circ f)\Big)$$

for all $*$-homomorphisms $\mathcal{B} \xrightarrow{f} \mathcal{A}$ of finite-dimensional C^*-algebras and states $\omega \in \mathcal{S}(\mathcal{A})$.

In this theorem, $S(\omega)$ is the von Neumann entropy of ω, which is given by $S(\omega) = -\mathrm{tr}(\rho \log \rho)$ in the special case when $\omega = \mathrm{tr}(\rho\ \cdot\ )$ is a state on $\mathcal{M}_m$ represented by a unique density matrix ρ, with tr the (un-normalized) trace and $\cdot$ signifying the input of the function, i.e. $\mathcal{M}_m \ni A \mapsto \mathrm{tr}(\rho A)$. More generally, when $\mathcal{A} := \bigoplus_{x \in X} \mathcal{M}_{m_x}$, a state ω on $\mathcal{A}$ can be described by a collection of states $\omega_x \in \mathcal{S}(\mathcal{M}_{m_x})$ and a probability measure p on X such that $\omega(A_x) = p_x\omega_x(A_x)$ for $A_x \in \mathcal{M}_{m_x}$. In this case, the entropy of ω is

$$S(\omega) = -\sum_{x\in X} p_x \log(p_x) - \sum_{x\in X} p_x \mathrm{tr}(\rho_x \log \rho_x). \tag{1.5}$$

Since all finite-dimensional unital C^*-algebras are of this form (up to isomorphism), this specifies the functor H everywhere, since entropy is invariant under isomorphism.

The present manuscript is broken up as follows. We begin by reviewing states, mutual orthogonality, and entropy in Section 2. In particular, we provide translations between some operator-algebraic and physical concepts. Section 3 introduces fiberwise convex structures, fibered functors, and continuity of fibered functors. Section 4 contains our main result and several others of potential interest. In particular, we prove that our axioms imply the non-negativity of $H_f(\omega)$ for commutative C^*-algebras by using the fact that disintegrations exist for morphisms of commutative probability spaces. More generally, we prove that if a disintegration of (f, ω) exists for an arbitrary quantum probability space $(\mathcal{A}, \omega)$, then $H_f(\omega) \geq 0$. We also include a brief historical account of axiomatizations of the von Neumann entropy and how our characterization compares with some of them.

2 States on finite-dimensional C^*-algebras

In this section, we set up notation and compile several standard facts that will be used throughout. All C^*-algebras will be unital and finite-dimensional and all $*$-homomorphisms will be unital unless stated otherwise. We will work in the Heisenberg picture, as will be explained in Example 2.10. Since all of our C^*-algebras will be finite-dimensional, they will always be $*$-isomorphic to direct sums of matrix algebras, so that most of our analysis will involve only linear algebra. An especially suitable reference including more than enough background is Farenick's linear algebra text [12] (see Theorem 5.20 and Proposition 5.26 in [12] for the statement regarding all finite-dimensional C^*-algebras).

Definition 2.1 (Basic definitions)**.** Given a C^*-algebra $\mathcal{A}$, an element $a \in \mathcal{A}$ is ***positive*** iff there exists an $x \in \mathcal{A}$ such that $a = x^*x$. The set of positive elements in $\mathcal{A}$ is denoted by $\mathcal{A}^+$. An element $a \in \mathcal{A}$ is ***self-adjoint*** iff $a^* = a$.

An element $p \in \mathcal{A}$ is a ***projection*** iff $p^*p = p$. The ***orthogonal complement*** of a projection $p \in \mathcal{A}$ is the element $p^\perp := 1_{\mathcal{A}} - p$ (and is also a projection). Positivity defines a partial order on self-adjoint elements and one writes $a \geq a'$ or $a' \leq a$ iff $a - a' \in \mathcal{A}^+$. Given another C^*-algebra $\mathcal{B}$, a ***positive map***[1] $\mathcal{B} \overset{\varphi}{\rightsquigarrow} \mathcal{A}$ is a linear map such that $\varphi(\mathcal{B}^+) \subseteq \mathcal{A}^+$. A ***weight*** on a C^*-algebra $\mathcal{A}$ is a positive map $\mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C}$. A weight is called a ***state*** iff it is unital. The set of states on a C^*-algebra $\mathcal{A}$ are denoted by $\mathcal{S}(\mathcal{A})$.

A ***non-commutative/quantum probability space*** is a pair $(\mathcal{A}, \omega)$ consisting of a C^*-algebra together with a state $\omega \in \mathcal{S}(\mathcal{A})$. A ***state-preserving*** map (a $*$-homomorphism or a positive map) from one non-commutative probability space $(\mathcal{B}, \xi)$ to another $(\mathcal{A}, \omega)$ is a map $\mathcal{B} \overset{f}{\rightsquigarrow} \mathcal{A}$ such that $\xi = \omega \circ f$. A state $\omega \in \mathcal{S}(\mathcal{A})$ is ***pure*** iff it cannot be expressed as a non-trivial convex combination of some pair of distinct states. For the C^*-algebra of $m \times m$ matrices $\mathcal{M}_m$, which is referred to as a ***matrix algebra***, the involution is the conjugate transpose and is denoted by $\dagger$ instead of $*$. If $m = 1$, then $\mathcal{M}_1 \cong \mathbb{C}$ and $\overline{z}$ is used to denote the complex conjugate of $z \in \mathbb{C}$.

Example 2.2 (Density matrices, states, and expectation values)**.** Self-adjointness and positive semidefiniteness of an $m \times m$ matrix coincides with the C^*-algebraic definition of positivity on $\mathcal{M}_m$. Every state ω on $\mathcal{M}_m$ can be expressed as $\omega = \mathrm{tr}(\rho \,\cdot\,)$ for some unique ***density matrix*** $\rho \in \mathcal{M}_m$, which is a positive matrix such that $\mathrm{tr}(\rho) = 1$. Here, and everywhere else in this manuscript, tr denotes the un-normalized trace.

When $\mathcal{A} := \bigoplus_{x \in X} \mathcal{M}_{m_x}$, with X a finite set and $m_x \in \mathbb{N}$, a state ω on $\mathcal{A}$ can be described by a collection of states $\omega_x \in \mathcal{S}(\mathcal{M}_{m_x})$ and a probability measure p on X such that $\omega(A_x) = p_x \omega_x(A_x)$ for $A_x \in \mathcal{M}_{m_x}$ [36, Section 5]. Here, and elsewhere in the manuscript, p_x is used to denote the probability of x with respect to p. Since each state ω_x corresponds to a density matrix $\rho_x \in \mathcal{M}_{m_x}$, ω can equivalently be expressed as $\omega(A_x) = p_x \mathrm{tr}(\rho_x A_x)$ for $A_x \in \mathcal{M}_{m_x}$. We will also use all of the following notations

$$\omega \equiv \sum_{x \in X} p_x \omega_x \equiv \sum_{x \in X} p_x \mathrm{tr}(\rho_x \,\cdot\,)$$

to indicate the same state. In this way, states encode the data of families of

[1] Motivated by stochastic Gelfand–duality [16, 35], $*$-homomorphisms are always drawn with straight arrows $\rightarrow$, while linear maps between algebras are drawn with squiggly arrows $\rightsquigarrow$, in order to distinguish between deterministic maps and stochastic maps.

expectation values. Since every C^*-algebra $\mathcal{A}$ is isomorphic to a finite direct sum of matrix algebras, this is a full description of states on C^*-algebras.

The usefulness of using C^*-algebras as opposed to just matrix algebras is to allow for a combination of classical and quantum setups, such as measurement. Furthermore, direct sums of matrix algebras are used in describing superselection sectors [38, 53], while ensembles, preparations, instruments, etc. are all naturally described by positive maps between certain C^*-algebras that are not just matrix algebras [37, Section 4].

Lemma 2.3 (The support of a weight)**.** *Associated to every weight ω on a C^*-algebra $\mathcal{A}$ is a projection $P_\omega \in \mathcal{A}$ satisfying*

$$\omega(P_\omega A) = \omega(AP_\omega) = \omega(P_\omega A P_\omega) = \omega(A) \qquad \forall\ A \in \mathcal{A}$$

and such that $P_\omega \le Q$ for every other projection Q satisfying this condition (with Q replacing P_ω).

Definition 2.4 (Supports and mutually orthogonal weights)**.** The projection P_ω in Lemma 2.3 is called the ***support*** of ω. Two weights ω, ξ on a finite-dimensional C^*-algebra $\mathcal{A}$ are ***mutually orthogonal***, written $\omega \perp \xi$, iff any of the following equivalent conditions hold.[2]

1. If for any weight χ on $\mathcal{A}$ such that $\chi \le \omega$ and $\chi \le \xi$, then $\chi = 0$.
2. $P_\omega P_\xi = 0$ (which implies $P_\omega P_\xi = P_\xi P_\omega$).

A $*$-homomorphism $\mathcal{B} \xrightarrow{f} \mathcal{A}$ ***preserves the mutual orthogonality*** $\omega \perp \xi$ iff $(\omega \circ f) \perp (\xi \circ f)$.

Lemma 2.5 (The image of a support)**.** *Let $\mathcal{B} \xrightarrow{f} \mathcal{A}$ be a $*$-homomorphism and let $\mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C}$ be a state. Then $f(P_{\omega\circ f}^\perp) \le P_\omega^\perp$ and $f(P_{\omega\circ f}) \ge P_\omega$.*

Proof. The first inequality follows from the fact that f sends projections to projections and $f(\mathcal{N}_{\omega\circ f}) \subseteq \mathcal{N}_\omega$ [36, Section 3], where

$$\mathcal{N}_\xi := \{A \in \mathcal{A} \ :\ \xi(A^*A) = 0\} \tag{2.6}$$

denotes the nullspace associated to a state ξ. The two inequalities are equivalent because

$$\begin{aligned} f(P_{\omega\circ f}) &= f(1_{\mathcal{B}} - P_{\omega\circ f}^\perp) = f(1_{\mathcal{B}}) - f(P_{\omega\circ f}^\perp) \\ &= 1_{\mathcal{A}} - f(P_{\omega\circ f}^\perp) = f(P_{\omega\circ f}^\perp)^\perp \ge P_\omega, \end{aligned} \tag{2.7}$$

[2] For the thermodynamic meaning of mutual orthogonality of states, see [38, Section 2].

where the last inequality used $f(P^{\perp}_{\omega\circ f}) \le P^{\perp}_{\omega}$. A similar calculation shows the converse. ■

Example 2.8 (External convex sums for finite probability spaces)**.** Let X, X', Y, Y' be finite sets, let p and q be probability measures on X and Y, respectively, and let $X \xrightarrow{\phi} X'$ and $Y \xrightarrow{\psi} Y'$ be two functions. Let $\lambda p \oplus (1-\lambda)q$ denote the probability measure on $X \amalg Y$ (the disjoint union) given by

$$(\lambda p \oplus (1-\lambda)q)_z := \begin{cases} \lambda p_z & \text{if } z \in X \\ (1-\lambda)q_z & \text{if } z \in Y \end{cases}.$$

Set $\mathcal{A} := \mathbb{C}^X$ and $\mathcal{B} := \mathbb{C}^Y$ to be the C^*-algebras of functions on X and Y, and similarly $\mathcal{A}' := \mathbb{C}^{X'}$ and $\mathcal{B}' := \mathbb{C}^{Y'}$. Let ω and ξ be the states on $\mathcal{A}$ and $\mathcal{B}$ associated to p and q, i.e. $\omega(A) = \sum_{x\in X} p_x A(x)$ for all $A \in \mathbb{C}^X$ (and similarly for ξ and q). Let $\mathcal{A}' \xrightarrow{f} \mathcal{A}$ and $\mathcal{B}' \xrightarrow{g} \mathcal{B}$ be the $*$-homomorphisms associated to ϕ and ψ via pullback. Namely, if $A' \in \mathbb{C}^{X'}$ is a function on X', then $f(A') := A' \circ \phi$. The disjoint union function $X \amalg Y \xrightarrow{\phi \amalg \psi} X' \amalg Y'$ corresponds to the direct sum $*$-homomorphism

$$\mathbb{C}^{X' \amalg Y'} \cong \mathcal{A}' \oplus \mathcal{B}' \xrightarrow{f \oplus g} \mathcal{A} \oplus \mathcal{B} \cong \mathbb{C}^{X \amalg Y}.$$

Let $\widetilde{\omega}$ and $\widetilde{\xi}$ denote the states on $\mathcal{A} \oplus \mathcal{B}$ given by $\widetilde{\omega}(A \oplus B) := \omega(A)$ and $\widetilde{\xi}(A \oplus B) := \xi(B)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. From these definitions, the state on $\mathcal{A} \oplus \mathcal{B}$ associated to $\lambda p \oplus (1-\lambda)q$ is $\lambda\widetilde{\omega} + (1-\lambda)\widetilde{\xi}$. Furthermore, $\widetilde{\omega} \perp \widetilde{\xi}$ holds and $f \oplus g$ preserves $\widetilde{\omega} \perp \widetilde{\xi}$. This construction of convex sums is one of the main ingredients in BFL's characterization of entropy [4].

Notation 2.9 (Internal direct sum)**.** Let $m \in \mathbb{N}$, Y a finite set, $\{n_y\}_{y\in Y}$ a collection of natural numbers satisfying $m = \sum_{y\in Y} n_y$, and $\{B_y \in \mathcal{M}_{n_y}\}_{y\in Y}$ a collection of matrices. Given an ordering of the elements of Y, set

$$\boxplus_{y\in Y} B_y := \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_{|Y|} \end{bmatrix} \equiv \operatorname{diag}(B_1, \ldots, B_{|Y|}) \in \mathcal{M}_m.$$

This notation will be frequently used, sometimes without explicitly stating that an order has been chosen.[3]

[3]This is not to be confused with the (external) direct sum $\bigoplus_{y\in Y} B_y \in \bigoplus_{y\in Y} \mathcal{M}_{n_y}$, which does not use an ordering on Y and, more importantly, is an element of a different (non-isomorphic) algebra.

Example 2.10 (The partial trace)**.** Working with unital $*$-homomorphisms between C^*-algebras corresponds to the *Heisenberg picture* description of quantum mechanics, as opposed to the more commonly used *Schrödinger picture* in the quantum information theory community. The relationship between the two goes roughly as follows.

If $\mathcal{B} = \mathcal{M}_n$, $\mathcal{A} = \mathcal{M}_m$, and $\mathcal{B} \xrightarrow{f} \mathcal{A}$ is a $*$-homomorphism, then there exists a $p \in \mathbb{N}$ such that $m = pn$ and a unitary $U \in \mathcal{M}_m$ such that $f = \mathrm{Ad}_U \circ g$, where $\mathrm{Ad}_U(A) := UAU^\dagger$ for all $A \in \mathcal{A}$, and where g is

$$\mathcal{B} \ni B \mapsto g(B) := \mathbb{1}_p \otimes B$$

(cf. [1], [52, Lecture 10]). The adjoint, g^*, of g with respect to the ***Hilbert–Schmidt*** or ***Frobenius*** inner product on the vector space of linear maps between $\mathcal{A}$ and $\mathcal{B}$ is given by

$$\mathcal{A} \cong \mathcal{M}_p \otimes \mathcal{M}_n \ni A \otimes B \mapsto g^*(A \otimes B) = \mathrm{tr}(A)B.$$

It is often written as $\mathrm{tr}_{\mathcal{M}_p}$ and is called the ***partial trace*** (see [36, Section 3] or [32, Section 2.4.3] for more details). The adjoint of f is $g^* \circ \mathrm{Ad}_{U^\dagger}$.

Lemma 2.11 (The partial trace on direct sums)**.** *Let $\mathcal{B} := \bigoplus_{y \in Y} \mathcal{M}_{n_y} \xrightarrow{f} \bigoplus_{x \in X} \mathcal{M}_{m_x} =: \mathcal{A}$ be a $*$-homomorphism and let $\omega = \sum_{x \in X} p_x \mathrm{tr}(\rho_x \,\cdot\,)$ be a state on $\mathcal{A}$ (cf. Example 2.2). Then the following facts hold.*

1. *There exists a collection $\{c_{xy}\}$ of non-negative numbers, with c_{xy} called the* **multiplicity** *of the factor $\mathcal{M}_{n_y}$ inside $\mathcal{M}_{m_x}$ associated to f, such that $m_x = \sum_{y \in Y} c_{xy} n_y$ for all $x \in X$.*
2. *There exist unitaries $U_x \in \mathcal{M}_{m_x}$ such that f is of the form*

$$\bigoplus_{y \in Y} \mathcal{M}_{n_y} \ni \bigoplus_{y \in Y} B_y \overset{f}{\mapsto} \bigoplus_{x \in X} U_x \left(\boxplus_{y \in Y} \mathrm{diag}(\overbrace{B_y, \cdots, B_y}^{c_{yx} \text{ times}}) \right) U_x^\dagger.$$

3. *The pullback state $\xi := \omega \circ f$ can be expressed as*

$$\xi = \sum_{y \in Y} q_y \mathrm{tr}(\sigma_y \,\cdot\,), \qquad \textit{where} \qquad q_y \sigma_y = \sum_{x \in X} p_x f_{xy}^*(\rho_x) \qquad \forall\, y \in Y$$

and f_{xy}^* denotes the (Hilbert–Schmidt) adjoint of $f_{xy} : \mathcal{M}_{n_y} \to \mathcal{M}_{m_x}$, which is the component of f mapping between the factors as indicated.

Proof. See [13, Sections 1.1.2 and 1.1.3], [12, Theorem 5.6], and [37, Lemma 6.7]. ■

Lemma 2.12 (∗-isomorphisms preserve mutual orthogonality)**.** *Let $\mathcal{B} \xrightarrow{f} \mathcal{A}$ be a ∗-isomorphism and let ω, ξ be any two states on $\mathcal{A}$. Then $\omega \perp \xi$ implies $(\omega \circ f) \perp (\xi \circ f)$. Furthermore, $\zeta \in \mathcal{S}(\mathcal{A})$ is pure if and only if $\zeta \circ f$ is pure.*

Proof. If P_ω and P_ξ are the supports of ω and ξ, respectively, then the claim will follow if we prove $f^{-1}(P_\omega)$ and $f^{-1}(P_\xi)$ are the supports of $\omega \circ f$ and $\xi \circ f$, respectively, because

$$f^{-1}(P_\omega)f^{-1}(P_\xi) = f^{-1}(P_\omega P_\xi) = f^{-1}(0) = 0. \tag{2.13}$$

It suffices to focus on ω. First, note that $f^{-1}(P_\omega)$ is a projection since f^{-1} is a ∗-homomorphism. Furthermore,

$$(\omega \circ f)\big(f^{-1}(P_\omega)B\big) = \omega\big(P_\omega f(B)\big) = \omega\big(f(B)\big) = (\omega \circ f)(B) \tag{2.14}$$

for all $B \in \mathcal{B}$, which proves that $f^{-1}(P_\omega)$ satisfies the first condition of a support for $\omega \circ f$ in Lemma 2.3. Suppose that Q is another projection satisfying $(\omega \circ f)(QB) = (\omega \circ f)(B)$ for all $B \in \mathcal{B}$. Then $f(Q)$ satisfies

$$\omega\big(f(Q)A\big) = (\omega \circ f)\big(Qf^{-1}(A)\big) = (\omega \circ f)\big(f^{-1}(A)\big) = \omega(A) \tag{2.15}$$

for all $A \in \mathcal{A}$. Hence, since P_ω is the minimal such projection, $P_\omega \le f(Q)$. Since ∗-homomorphisms preserve the $\le$ order structure, $f^{-1}(P_\omega) \le Q$. ■

Example 2.16 (Channels that do not preserve orthogonality)**.** There are many examples of ∗-homomorphisms $\mathcal{B} \to \mathcal{A}$ that do not always preserve mutual orthogonality. A simple example is $!_{\mathbb{C}^2} : \mathbb{C} \to \mathbb{C}^2$, where every pair of mutually orthogonal states gets pulled back to 1. A non-classical example is the ∗-homomorphism $\mathcal{M}_2 \to \mathcal{M}_2 \otimes \mathcal{M}_2$, sending B to $B \otimes \mathbb{1}_2$, and any two density matrices on $\mathbb{C}^2 \otimes \mathbb{C}^2$ corresponding to any two orthogonal Bell states [32, Section 2.3]. In either case, the pullback state is $\frac{1}{2}\mathrm{tr}$.

Lemma 2.17 (Overlapping states remain overlapping under evolution)**.** *Let $\mathcal{B} \xrightarrow{f} \mathcal{A}$ be ∗-homomorphism and let ω, ξ be two states on $\mathcal{A}$ that are* not *mutually orthogonal. Then $\omega \circ f$ and $\xi \circ f$ are also not mutually orthogonal.*

Proof. Suppose, to the contrary, that $P_{\omega\circ f}P_{\xi\circ f} = 0$. Then

$$0 = f(0) = f(P_{\omega\circ f}P_{\xi\circ f}) = f(P_{\omega\circ f})f(P_{\xi\circ f}). \tag{2.18}$$

But, by Lemma 2.5, $f(P_{\omega\circ f}) \geq P_\omega$ and $f(P_{\xi\circ f}) \geq P_\xi$ so that their product cannot vanish by the assumption $P_\omega P_\xi \neq 0$. This is a contradiction. ■

Physics 2.19 (Evolving states with overlapping supports)**.** The interpretation of Lemma 2.17 is that if two states have overlapping supports, then no quantum operation will ever completely separate them. In contrast, Lemma 2.12 says that reversible dynamics (such as unitary evolution) cannot mix states.

Now that we have defined the objects and morphisms of interest, we can define entropy and its generalizations to matrix algebras and C^*-algebras.

Definition 2.20 (Shannon, von Neumann, and Segal entropy)**.** Let ω be a state on $\mathcal{A}$ as in Example 2.2. The ***Segal entropy*** of ω is the non-negative number

$$S_{\mathrm{Se}}(\omega) := S_{\mathrm{Sh}}(p) + \sum_{x\in X} p_x S_{\mathrm{vN}}(\rho_x),$$

where $S_{\mathrm{Sh}}(p) := -\sum_{x\in X} p_x \log(p_x)$ is the ***Shannon entropy*** of a probability measure p on X and $S_{\mathrm{vN}}(\rho) := -\mathrm{tr}\big(\rho\log\rho\big)$ is the ***von Neumann entropy*** of a density matrix ρ on $\mathbb{C}^n$. The convention $0\log 0 := 0$ is used.

On occasion, the letter S will exclusively be used to refer to any of these three entropies, using the input to distinguish which formula should be used. As such, ***entropy*** will refer to any of these three, while ***quantum entropy*** will refer to either S_{Se} or S_{vN}.[4]

We recall the following useful fact about the entropy of convex combinations.

Lemma 2.21 (Concavity inequalities for entropy)**.** *Let $\{\rho_x\}_{x\in X}$ be a collection of density matrices on a Hilbert space indexed by a finite set X. Then*

$$\sum_{x\in X} p_x S_{\mathrm{vN}}(\rho_x) \leq S_{\mathrm{vN}}\left(\sum_{x\in X} p_x\rho_x\right) \leq S_{\mathrm{Sh}}(p) + \sum_{x\in X} p_x S_{\mathrm{vN}}(\rho_x)$$

[4]The Segal entropy was actually defined much more generally for certain infinite-dimensional systems [41]. The Segal entropy also equals $S_{\mathrm{Se}}(\omega) = -\sum_{x\in X}\mathrm{tr}\big(p_x\rho_x\log(p_x\rho_x)\big)$.

for any probability distribution p on X. Furthermore, the second inequality becomes an equality if and only if $\rho_x \perp \rho_{x'}$ for all distinct $x, x' \in X$ such that $p_x \neq 0$ and $p_{x'} \neq 0$.

Proof. The first inequality is the concavity of the von Neumann entropy. Proofs of these claims can be found in [32, Theorem 11.8 (4)] as well [27, Corollary pg 247] and [28, Equation (2.2)]. ■

We now come to our main definition for the entropy change along a morphism.

Definition 2.22 (The entropy change along a morphism)**.** Let $\mathcal{B} \xrightarrow{f} \mathcal{A}$ be a $*$-homomorphism of C^*-algebras and let ω be a state on $\mathcal{A}$. The ***entropy change of*** ω ***along*** f is the number

$$S_f(\omega) := S_{\mathrm{Se}}(\omega) - S_{\mathrm{Se}}(\omega \circ f).$$

The following lemma contains a crucial observation that distinguishes the entropy change along a morphism between commutative versus non-commutative C^*-algebras.

Lemma 2.23 (The entropy change along certain morphisms)**.** *Recall the notation from Definition 2.22.*

1. *If f is a $*$-isomorphism, then $S_f(\omega) = 0$ for all states $\omega \in \mathcal{S}(\mathcal{A})$.*
2. *If $\mathcal{A}$ and $\mathcal{B}$ are commutative C^*-algebras, then $S_f(\omega) \geq 0$ for all states $\omega \in \mathcal{S}(\mathcal{A})$ and $*$-homomorphisms $\mathcal{B} \xrightarrow{f} \mathcal{A}$.*
3. *If $\mathcal{A}$ is not commutative and f is not a $*$-isomorphism, then there exists a state $\omega \in \mathcal{S}(\mathcal{A})$ such that $S_f(\omega) < 0$.*[5]

Proof.

1. Let $\mathcal{A}, \mathcal{B}, \omega, f,$ and ξ be as in Example 2.2. Since f is a $*$-isomorphism, there exists a bijection $X \xrightarrow{\phi} Y$ and a collection of unitaries $U_x \in \mathcal{M}_{m_x}$ such that

$$m_x = n_{\phi(x)} \quad \text{and} \quad p_x U_x \rho_x U_x^\dagger = q_{\phi(x)} \sigma_{\phi(x)} \qquad \forall\, x \in X \qquad (2.24)$$

by Lemma 2.11. The claim $S_f(\omega) = 0$ then follows from the functional calculus and Definition 2.20.

[5] If $\mathcal{B}$ is not commutative, then a $*$-homomorphism $\mathcal{B} \to \mathcal{A}$ does not exist if $\mathcal{A}$ is commutative.

2. Since every commutative finite-dimensional C^*-algebra is isomorphic to functions on a finite set as described in Example 2.8, the Segal entropy becomes the Shannon entropy. If p and q are the probability measures on X and Y corresponding to ω and $\omega \circ f$, respectively, then
$$S_f(\omega) = S_{\mathrm{Se}}(\omega) - S_{\mathrm{Se}}(\omega \circ f) = S_{\mathrm{Sh}}(p) - S_{\mathrm{Sh}}(q), \tag{2.25}$$
which is shown to be non-negative in [4] (see Proposition 4.7 for a more general and abstract proof using disintegrations).
3. If $\mathcal{A}$ is not commutative, then it has some matrix algebra $\mathcal{M}_m$ as a factor with $m > 1$. Let ρ be a rank 1 density matrix in $\mathcal{A}$ with support in $\mathcal{M}_m$ (so that ρ is a pure state). Let A be a self-adjoint $m \times m$ matrix that does not commute with ρ (such a matrix necessarily exists because the center of $\mathcal{M}_m$ consists of multiples of the identity). Let $\sigma(A)$ denote the spectrum of A. Let $\mathcal{B} := \mathbb{C}^{\sigma(A)} \xrightarrow{f} \mathcal{A}$ send e_λ, the function on $\sigma(A)$ whose value at λ is 1 and is 0 elsewhere, to P_λ in $\mathcal{M}_m$, the projection onto the λ-eigenspace. Then $\omega \circ f$ is not a pure state, in the sense that the associated measure on $\sigma(A)$ is not a Dirac measure. Thus, the entropy change is $S_f(\omega) = S_{\mathrm{Se}}(\omega) - S_{\mathrm{Se}}(\omega \circ f) = 0 - S_{\mathrm{Se}}(\omega \circ f) < 0$. ■

Item 2 in Lemma 2.23 was used as an axiom by BFL to characterize the entropy change in the classical setting. Since it fails when one includes non-commutative C^*-algebras, we will have to replace this axiom with one that more accurately reflects the properties of entropy in quantum mechanics.

Physics 2.26 (Negative conditional entropy)**.** As another example illustrating the validity of item 3 in Lemma 2.23 using only matrix algebras, take ω on $\mathcal{M}_2 \otimes \mathcal{M}_2 \cong \mathcal{M}_4$ to be a Bell state and let $\mathcal{M}_2 \xrightarrow{f} \mathcal{M}_2 \otimes \mathcal{M}_2$ be the inclusion into one of the factors. Then $S_f(\omega) = -\log(2)$ (cf. Example 2.16). More generally, set $\mathcal{A} := \mathcal{M}_m$, $\mathcal{B} := \mathcal{M}_n$, $\mathcal{A} \xrightarrow{f} \mathcal{A} \otimes \mathcal{B}$ the standard inclusion, and $\omega = \mathrm{tr}(\rho_{\mathcal{AB}}\ \cdot\ )$, where $\rho_{\mathcal{AB}}$ is a density matrix in $\mathcal{A} \otimes \mathcal{B}$ with marginals $\rho_{\mathcal{A}} := \mathrm{tr}_{\mathcal{B}}(\rho_{\mathcal{AB}})$ and $\rho_{\mathcal{B}} := \mathrm{tr}_{\mathcal{A}}(\rho_{\mathcal{AB}})$ (cf. Example 2.10). Then the entropy difference $S_f(\omega) = S_{\mathrm{vN}}(\rho_{\mathcal{AB}}) - S_{\mathrm{vN}}(\rho_{\mathcal{A}})$ is the ***quantum conditional entropy***, which, if negative, necessarily implies that $\rho_{\mathcal{AB}}$ is entangled (see near Equation (21) in [23]). The example we chose in the proof of Lemma 2.23 is meant to illustrate that entanglement is not necessary for $S_f(\omega)$ to be negative.

Physics 2.27 (Information loss or gain and Landauer's principle)**.** In [4], BFL interpreted the non-negative entropy difference between *commutative* algebras as information loss. Indeed, a state-preserving $*$-homomorphism between commutative probability spaces corresponds to a probability-preserving map of finite sets equipped with probabilities. Such a map may identify points in an irreversible manner (in the sense that a probability-preserving inverse need not exist). When two points get identified, the corresponding probabilities add (cf. Definition 4.6) and there is a decrease in entropy. This is closely related to Landauer's principle [25], which states that erasure (information loss) entails the dissipation of energy (in the form of heat) into the environment.

For non-commutative probability spaces, i.e. quantum systems, information and work can be *gained* in certain situations, violating Landauer's principle. The information can be later used for state merging protocols [21, 22] or the corresponding energy can be used to do thermodynamic work [8]. A precise reformulation of the principle has been recently stated and proved in the case of finite-dimensional matrix algebras [39].

We now end this section with a summary of the categories that will be used throughout.

Notation 2.28 (Categories used in this work)**.** In all categories that follow, except the very last one, the composition rule will be function composition.

1. $\mathbf{FinSet}$ is the category whose objects are finite sets and whose morphisms are functions.
2. $\mathbf{FinProb}$ is the category whose objects are ***finite probability spaces***, which are pairs (X, p), with X a finite set and p a probability measure on X. A morphism from (X, p) to (Y, q) is a ***probability-preserving function***, i.e. a function $X \xrightarrow{\phi} Y$ such that $q_y = \sum_{x\in\phi^{-1}(\{y\})} p_x$ for all $y \in Y$, where $\phi^{-1}(\{y\}) := \{x \in X : \phi(x) = y\}$.
3. $\mathbf{fdC^*\text{-}Alg}$ is the category whose objects are (finite-dimensional unital) C^*-algebras and morphisms are (unital) $*$-homomorphisms.
4. $\mathbf{NCFinProb}$ is the category whose objects are (finite-dimensional) non-commutative probability spaces and whose morphisms are state-preserving (unital) $*$-homomorphisms.

5. $\mathbb{BR}$ ($\mathbb{BR}_{\geq 0}$) is the category consisting of a single object and whose morphisms from that object to itself are all real numbers (non-negative real numers) equipped with addition as the composition rule.

Finally, here are some additional categorical notations and terminologies that will be used. Given two categories $\mathcal{C}$ and $\mathcal{D}$, let $\mathcal{C} \times \mathcal{D}$ denote their cartesian product. Let $\mathcal{C} \times \mathcal{D} \xrightarrow{\gamma} \mathcal{D} \times \mathcal{C}$ be the functor that swaps the two inputs. Let $\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \times \mathcal{C}$ be the diagonal functor sending an object x to (x, x) and similarly for morphisms. There are two projection functors, denoted by $\mathcal{C} \times \mathcal{D} \xrightarrow{\pi_1} \mathcal{C}$ and $\mathcal{C} \times \mathcal{D} \xrightarrow{\pi_2} \mathcal{D}$.

3 Fibrations and local convex structures

Fibrations provide a convenient setting to formulate the notion of entropy change as a functor. Non-commutative probability spaces form a discrete fibration over C^*-algebras and the real numbers viewed as a one-object category form an ordinary (Grothendieck) fibration over the trivial category. The fibre over each algebra is the space of states, which has a convex structure. Since real numbers have a convex structure as well, one can make sense of convexity, concavity, or affinity of the functor that computes the entropy change along a morphism of non-commutative probability spaces. The references for fibrations that we follow include [20, 29, 30].

Definition 3.1 (Discrete fibration)**.** A functor $\mathcal{E} \xrightarrow{\pi} \mathcal{X}$ is a ***discrete fibration*** iff for each morphism $x \xrightarrow{f} y$ in $\mathcal{X}$ and for each object v in $\mathcal{E}$ such that $\pi(v) = y$, there exists a unique morphism $u \xrightarrow{\beta} v$ such that $\pi(\beta) = f$. A morphism $u \xrightarrow{\beta} v$ such that $\pi(\beta) = f$ is called a ***lift*** of f.

Example 3.2 (The discrete fibration of non-commutative probability spaces)**.** The functor $\pi : \mathbf{NCFinProb} \to \mathbf{fdC^*\text{-}Alg}$, which sends $(\mathcal{A}, \omega)$ to $\mathcal{A}$ and $(\mathcal{B}, \xi) \xrightarrow{f} (\mathcal{A}, \omega)$ to $\mathcal{B} \xrightarrow{f} \mathcal{A}$, is a discrete fibration. Indeed, given $\omega \in \mathcal{S}(\mathcal{A})$ and $\mathcal{B} \xrightarrow{f} \mathcal{A}$, the unique lift is f itself together with the state on $\mathcal{B}$ given by $\xi = \omega \circ f$. Similarly, the functor $\mathbf{FinProb}^{\mathrm{op}} \to \mathbf{FinSet}^{\mathrm{op}}$ sending a probability space (X, p) to X and a probability-preserving function to the underlying function between sets is a discrete fibration.

Definition 3.3 (Cartesian morphisms and fibrations)**.** Let $\mathcal{E}$ and $\mathcal{X}$ be two categories and let $\mathcal{E} \xrightarrow{\pi} \mathcal{X}$ be a functor. A morphism $u \xrightarrow{\beta} v$ in $\mathcal{E}$ is ***cartesian*** iff for any morphism $x \xrightarrow{f} \pi(u)$ in $\mathcal{X}$ and any morphism $w \xrightarrow{\gamma} v$ in $\mathcal{E}$ such that $\pi(\beta) \circ f = \pi(\gamma)$, there exists a unique morphism $w \xrightarrow{\alpha} u$ in $\mathcal{E}$ such that $\pi(\alpha) = f$ and $\beta \circ \alpha = \gamma$. Let $\mathcal{E}_x$ be the subcategory of $\mathcal{E}$ consisting of the objects u in $\mathcal{E}$ such that $\pi(u) = x$ and $\pi(\beta) = \mathrm{id}_x$ for all morphisms $u \xrightarrow{\beta} v$ with $\pi(u) = x = \pi(v)$. The category $\mathcal{E}_x$ is called the ***fibre*** of π over x and the morphisms in $\mathcal{E}_x$ are called ***vertical morphisms*** of π over x. Given a morphism $x \xrightarrow{f} y$ in $\mathcal{X}$ and an object v in $\mathcal{E}_y$, a ***cartesian lifting of*** f ***with target*** v is a cartesian morphism $u \xrightarrow{\beta} v$ such that $\pi(\beta) = f$. A functor $\pi : \mathcal{E} \to \mathcal{X}$ is a ***fibration*** iff for any morphism $x \xrightarrow{f} y$ in $\mathcal{X}$ and an object v in $\mathcal{E}_y$, a cartesian lifting exists. When π is a fibration, $\mathcal{X}$ is called the ***base***. A fibration for which a cartesian lifting has been chosen for every pair (f, v), with f a morphism in $\mathcal{X}$ and v an object in $\mathcal{E}_y$, is called a ***cloven*** fibration.

Lemma 3.4 (The reindexing functor)**.** *Let $\mathcal{E} \xrightarrow{\pi} \mathcal{X}$ be a cloven fibration and let $f^*(v) \xrightarrow{f_v} v$ be the choice of cartesian lifting of $x \xrightarrow{f} y$ with target v. These data determine a canonical functor $\mathcal{E}_x \xleftarrow{f^*} \mathcal{E}_y$ sending v to $f^*(v)$. For each vertical morphism $w \xrightarrow{\kappa} v$ in $\mathcal{E}_y$, let $f^*(w) \xrightarrow{f^*(\kappa)} f^*(v)$ be the unique morphism in $\mathcal{E}_x$ obtained by the universal property of f_v being cartesian. Then f^* defines a functor, called the* **reindexing functor** *associated to f.*

Proof. This is a standard fact that follows from the uniqueness in the universal property of cartesian morphisms. The details are left as an exercise. ■

To incorporate convex structures on our main examples, we define (strict) convex categories, affine functors, and fibrewise convex structures on fibrations. The following definition of a convex object is an internalization of the algebraic definition of a convex space [14, 15, 18, 19, 31, 42, 43, 50].

Definition 3.5 (Convex category)**.** Given two numbers $\lambda, \mu \in [0, 1]$ set

$$\lambda \llcorner \mu := \lambda\mu \quad \text{and} \quad \lambda \lrcorner \mu := \begin{cases} \frac{\lambda(1-\mu)}{1-\lambda\mu} & \text{if } \lambda\mu \neq 1 \\ \text{arbitrary} & \text{if } \lambda = \mu = 1 \end{cases},$$

where "arbitrary" means that one can assign any value to the quantity. A ***convex category*** (or more generally a ***convex object*** in some cartesian monoidal category) is a category $\mathcal{C}$ (object) together with a family of functors $F_\lambda : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ (morphisms) indexed by $\lambda\in[0,1]$ such that

$$\begin{tikzcd} \mathcal{C}\times\mathcal{C} \arrow[r, bend left, "F_0"] \arrow[r, bend right, "\pi_2"'] & \mathcal{C} \end{tikzcd}\ ,\quad \begin{tikzcd} & \mathcal{C}\times\mathcal{C} \arrow[dr, "F_\lambda"] & \\ \mathcal{C} \arrow[ur, "\Delta"] \arrow[rr, "\mathrm{id}_{\mathcal{C}}"'] & & \mathcal{C} \end{tikzcd}\ ,\quad \begin{tikzcd} \mathcal{C}\times\mathcal{C} \arrow[rr, "\gamma"] \arrow[dr, "F_\lambda"'] & & \mathcal{C}\times\mathcal{C} \arrow[dl, "F_{1-\lambda}"] \\ & \mathcal{C} & \end{tikzcd}\ ,\quad \text{and}$$

$$\begin{tikzcd} \mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[r, "F_\mu\times\mathrm{id}_{\mathcal{C}}"] \arrow[d, "\mathrm{id}_{\mathcal{C}}\times F_{\lambda\lrcorner\mu}"'] & \mathcal{C}\times\mathcal{C} \arrow[d, "F_\lambda"] \\ \mathcal{C}\times\mathcal{C} \arrow[r, "F_{\lambda\llcorner\mu}"'] & \mathcal{C} \end{tikzcd}$$

commute for all $\lambda,\mu\in[0,1]$ (see Definition 2.28 for notation). The notation $\lambda x+(1-\lambda)y := F_\lambda(x,y)$ will be implemented on occasion.

Example 3.6 (Examples of convex categories)**.**

(a) Every convex set is a convex category when viewed as a discrete category. In particular, $\mathcal{S}(\mathcal{A})$, the set of states on a C^*-algebra $\mathcal{A}$, is a convex category.

(b) The convex combination of real numbers turns $\mathbb{B}\mathbb{R}$ into a convex category. If $\mathbb{R}_{\ge 0} := \{r\in\mathbb{R} : r\ge 0\}$, then $\mathbb{B}\mathbb{R}_{\ge 0}$ is also a convex category.

Note, however, that the convex categories of BFL [4] are *not* examples of Definition 3.5 (cf. Remark 3.24).

Definition 3.7 (Affine functors)**.** An ***affine functor*** from one convex category $(\mathcal{C},\{F_\lambda\})$ to another one $(\mathcal{D},\{G_\lambda\})$ is a functor $S:\mathcal{C}\to\mathcal{D}$ such that

$$\begin{tikzcd} \mathcal{C}\times\mathcal{C} \arrow[r, "S\times S"] \arrow[d, "F_\lambda"'] & \mathcal{D}\times\mathcal{D} \arrow[d, "G_\lambda"] \\ \mathcal{C} \arrow[r, "S"'] & \mathcal{D} \end{tikzcd}$$

commutes for all $\lambda\in[0,1]$.

Example 3.8 (The pullback of states is an affine functor)**.** Let $\mathcal{B}\xrightarrow{f}\mathcal{A}$ be a $*$-homomorphism between C^*-algebras. Then the pullback $\mathcal{S}(\mathcal{A})\xrightarrow{\mathcal{S}(f)}\mathcal{S}(\mathcal{B})$, sending ω to $\omega\circ f$, is an affine functor (cf. Example 3.6 (a)) since

$$\big(\lambda\omega+(1-\lambda)\xi\big)\circ f = \lambda(\omega\circ f)+(1-\lambda)(\xi\circ f) \qquad \forall\ \lambda\in[0,1],\ \omega,\xi\in\mathcal{S}(\mathcal{A}).$$

Example 3.9 (Entropy is almost affine)**.** Given $\mathcal{B} \xrightarrow{f} \mathcal{A}$, the assignment $\mathcal{S}(\mathcal{A}) \xrightarrow{S_f} \mathbb{BR}$ sending ω to $S_f(\omega)$ from Definition 2.22 is *not* affine. However, the inequality

$$S_f\big(\lambda\omega + (1-\lambda)\xi\big) \geq \lambda S_f(\omega) + (1-\lambda)S_f(\xi)$$

holds as a corollary of the work of Lieb and Ruskai [26, Theorem 1] and Lindblad [28, Lemma 3]. Nevertheless, and more importantly for our characterization theorem, equality *does* hold when $\omega \perp \xi$ and $(\omega \circ f) \perp (\xi \circ f)$. The proof of this will be given in Proposition 3.19.

Definition 3.10 (Fibrewise convex structures)**.** A ***fibrewise convex structure*** on a fibration $\mathcal{E} \xrightarrow{\pi} \mathcal{X}$ is a cloven fibration where each fibre is a convex category and each reindexing functor $\mathcal{E}_x \xleftarrow{f^*} \mathcal{E}_y$ (as described in Lemma 3.4) is an affine functor. A cloven fibration equipped with a fibrewise convex structure is called a ***fibrewise convex fibration***.

Example 3.11 (Examples of fibrewise convex structures)**.**

(a) The discrete fibration $\mathbf{NCFinProb} \to \mathbf{fdC^*\text{-}Alg}$ has $\mathcal{S}(\mathcal{A})$ as the fibre over each C^*-algebra $\mathcal{A}$. The set of states $\mathcal{S}(\mathcal{A})$ on a C^*-algebra $\mathcal{A}$ has a natural convex structure. Furthermore, each $*$-homomorphism $\mathcal{B} \xrightarrow{f} \mathcal{A}$ has the pullback $\mathcal{S}(\mathcal{B}) \xleftarrow{\mathcal{S}(f)} \mathcal{S}(\mathcal{A})$ as its reindexing functor. This functor is affine, as discussed in Example 3.8.

(b) By a similar argument, $\mathbf{FinProb}^{\mathrm{op}} \to \mathbf{FinSet}^{\mathrm{op}}$ has a natural fibrewise convex structure coming from the convex combination of probability measures and the fact that the pushforward of measures is linear. The fibre over a finite set X is isomorphic to the standard simplex $\Delta^{|X|-1} := \big\{(p_1,\dots,p_{|X|}) \in \mathbb{R}^{|X|}_{\geq 0} \ : \ \sum_{i=1}^{|X|} p_i = 1\big\}$.

(c) The fibration $\mathbb{BR} \to \underline{\mathbf{1}}$ has a convex structure on the only fiber $\mathbb{BR}$ over the single object in the base, as described in Example 3.6.

Definition 3.12 (Morphisms of fibrations)**.** Let $\mathcal{E} \xrightarrow{\pi} \mathcal{X}$ and $\mathcal{F} \xrightarrow{\rho} \mathcal{Y}$ be fibrations. A ***fibred functor***[6] from π to ρ is a pair of functors $\mathcal{E} \xrightarrow{\Phi} \mathcal{F}$ and

[6] Our terminology differs from that of [30], who use 'functor' when the base category is fixed ($\phi = \mathrm{id}$) and '1-cell' for when the base category changes.

$\mathcal{X} \xrightarrow{\phi} \mathcal{Y}$ such that

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\Phi"] \arrow[d, "\pi"'] & \mathcal{F} \arrow[d, "\rho"] \\ \mathcal{X} \arrow[r, "\phi"'] & \mathcal{Y} \end{tikzcd}$$

commutes and such that $\Phi(\beta)$ is cartesian for every cartesian β.

Remark 3.13 (Fibrewise convex structures as internal convex objects)**.** One can equivalently define a fibrewise convex structure as an internal convex object in the category of fibrations over a fixed based, analogous to the fibrewise monoidal structure in [30, Section 3.1].

Briefly, a convex object $\mathcal{E} \xrightarrow{\pi} \mathcal{X}$ in the category of fibrations over a fixed based $\mathcal{X}$ provides the data of a family of fibred functors $F_\lambda : \mathcal{E} \times_\pi \mathcal{E} \to \mathcal{E}$ with a fixed based, where $\mathcal{E} \times_\pi \mathcal{E}$ is the (strict) pullback. The functors F_λ define a convex category structure for every fibre $\mathcal{E}_x$. In addition, they also provide an assignment on morphisms since a pair $(t \xrightarrow{\alpha} u, v \xrightarrow{\beta} w)$ over $x \xrightarrow{f} y$ gets sent to

$$\lambda t + (1-\lambda)v \xrightarrow{\lambda\alpha + (1-\lambda)\beta \equiv F_\lambda(\alpha,\beta)} \lambda u + (1-\lambda)w$$

over $x \xrightarrow{f} y$. This assignment guarantees that the associated reindexing functor $\mathcal{E}_x \xleftarrow{f^*} \mathcal{E}_y$ from Lemma 3.4 can be chosen to be affine as in Definition 3.10. Indeed, if one chooses cartesian liftings $f^*(u) \xrightarrow{f_u} u$ and $f^*(v) \xrightarrow{f_v} v$ of u and v over $x \xrightarrow{f} y$, respectively, then

$$\lambda f^*(u) + (1-\lambda) f^*(v) \xrightarrow{\lambda f_u + (1-\lambda) f_v} \lambda u + (1-\lambda)v$$

can be taken as the lift of $\lambda u + (1-\lambda)v$ over f.

For example, in the fibrewise convex fibration $\mathbf{NCFinProb} \to \mathbf{fdC^*\text{-}Alg}$, if $(\mathcal{B}, \eta) \xrightarrow{g} (\mathcal{A}, \omega)$ and $(\mathcal{B}, \zeta) \xrightarrow{h} (\mathcal{A}, \xi)$ are two morphisms over $\mathcal{B} \xrightarrow{f} \mathcal{A}$, then $g = h = f$ and their convex combination, $\lambda g + (1-\lambda)h$, is just f. In the fibrewise convex fibration $\mathbb{BR} \to \underline{\mathbf{1}}$, the convex combination of objects in the fibre is trivial, while the convex combination of morphisms (elements in $\mathbb{R}$) is the usual convex combination of real numbers.

Definition 3.14 (Convergence in $\mathbf{NCFinProb}$). A sequence $\mathbb{N} \ni n \mapsto \big((\mathcal{B}_n, \xi_n) \xrightarrow{f_n} (\mathcal{A}_n, \omega_n)\big)$ ***converges to*** $(\mathcal{A}, \xi) \xrightarrow{f} (\mathcal{B}, \omega)$ in the category $\mathbf{NCFinProb}$ iff there exists an $N \in \mathbb{N}$ such that $\mathcal{A}_n = \mathcal{A}$, $\mathcal{B}_n = \mathcal{B}$, $f_n = f$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} \omega_n = \omega$, and $\lim_{n\to\infty} \xi_n = \xi$, where the last two limits are with respect to the standard topologies on the state spaces $\mathcal{S}(\mathcal{A})$ and $\mathcal{S}(\mathcal{B})$, respectively.

Remark 3.15 (Justifying the definition of convergence of sequences in $\mathbf{NCFinProb}$). The definition of convergence of a sequence of morphisms in $\mathbf{NCFinProb}$ is motivated by the one in $\mathbf{FinProb}$ from [4, page 4]. However, some justification regarding why the morphisms are assumed to stabilize, i.e. are equal after some $N \in \mathbb{N}$, is needed.

In the case of $\mathbf{FinProb}$, a sequence $(X_n, p_n) \xrightarrow{f_n} (Y_n, q_n)$ ***converges to*** $(X, p) \xrightarrow{f} (Y, q)$ iff the sets X_n, Y_n and the underlying set functions f_n stabilize after a finite natural number in the sequence and $\lim_{n\to\infty} p_n = p$ and $\lim_{n\to\infty} q_n = q$. The sets must stabilize because their associated simplices of probability distributions are distinct and the cardinality of the set dictates which simplex one is using for the space of probability distributions. The functions must stabilize because the set of functions between two finite sets is also a finite set, which has the discrete topology. However, the probability distributions p_n on X and q_n on Y may continue to vary as long as they converge to p and q in the topology associated with the simplices $\Delta^{|X|-1}$ and $\Delta^{|Y|-1}$.

In the case of C^*-algebras, the collection $\hom(\mathcal{B}, \mathcal{A})$ of (unital) $*$-homomorphisms from $\mathcal{B}$ to $\mathcal{A}$ is *not* just a discrete set since the collection of unitary matrices has a non-trivial topology. Nevertheless, one can assume the f_n eventually stabilize. To see this, it suffices to assume $\mathcal{A} = \bigoplus_{x\in X} \mathcal{M}_{m_x}$ and $\mathcal{B} = \bigoplus_{y\in Y} \mathcal{M}_{n_y}$ for some finite sets X and Y and $m_x, n_y \in \mathbb{N}$. In this case, a $*$-homomorphism $\mathcal{B} \xrightarrow{f} \mathcal{A}$ is described by its multiplicities and by a unitary as in Lemma 2.11. The multiplicities entail the constraint $m_x = \sum_{y\in Y} c_{xy} n_y$, but there could be several such multiplicities satisfying these constraints. Indeed, if

$$s_x := \left| \left\{ Y \ni y \mapsto c_{xy} \in \mathbb{Z}_{\geq 0} \ : \ m_x = \sum_{y\in Y} c_{xy} n_y \right\} \right|$$

denotes the number of such solutions, then the number of connected components in $\hom(\mathcal{B}, \mathcal{A})$ is $s := \prod_{x\in X} s_x$ (for example, if $\mathcal{B} = \mathcal{M}_n$ is a matrix algebra, there is only one such component). Hence, a sequence of $*$-homomorphisms converging to another one must necessarily have multiplicities that stabilize. Within such a component, since $\omega \circ f = (\omega \circ \mathrm{Ad}_U) \circ (\mathrm{Ad}_{U^\dagger} \circ f)$ for every unitary U, one can always choose f to be of the form

$$\bigoplus_{y\in Y} \mathcal{M}_{n_y} \ni \bigoplus_{y\in Y} B_y \mapsto \bigoplus_{x\in X} \left(\boxplus_{y\in Y} \mathbb{1}_{c_{yx}} \otimes B_y \right)$$

by conjugating with some appropriate unitary U (cf. Lemma 2.11). This unitary can then be transferred to the state.

Therefore, it suffices to assume the algebras and $*$-homomorphisms stabilize in a convergent sequence, but not necessarily the states.

Definition 3.16 (Continuous fibred functors)**.** A ***continuous fibred functor*** from $\mathbf{NCFinProb} \to$ **fdC*-Alg** to $\mathbb{BR} \to \underline{\mathbf{1}}$ is a fibred functor H such that to every sequence $\mathbb{N} \ni n \mapsto \big((\mathcal{B}_n, \xi_n) \xrightarrow{f_n} (\mathcal{A}_n, \omega_n)\big)$ converging to $(\mathcal{A}, \xi) \xrightarrow{f} (\mathcal{B}, \omega)$ in the category $\mathbf{NCFinProb}$,

$$\lim_{n\to\infty} H\left((\mathcal{B}_n, \xi_n) \xrightarrow{f_n} (\mathcal{A}_n, \omega_n)\right) = H\left((\mathcal{B}, \xi) \xrightarrow{f} (\mathcal{A}, \omega)\right),$$

where the convergence is for a sequence of real numbers.

Notation 3.17 (The function $H_f : \mathcal{S}(\mathcal{A}) \to \mathbb{R}$)**.** For a fibred functor $H : \mathbf{NCFinProb} \to \mathbb{BR}$, set

$$H_f(\omega) := H\left((\mathcal{B}, \xi) \xrightarrow{f} (\mathcal{A}, \omega)\right)$$

for the image of H along a morphism f in $\mathbf{NCFinProb}$. For a fixed $*$-homomorphism $\mathcal{B} \xrightarrow{f} \mathcal{A}$, this defines a function $H_f : \mathcal{S}(\mathcal{A}) \to \mathbb{R}$.

The next definition is the appropriate quantum generalization of the affinity condition used by BFL in their characterization of Shannon entropy [4]. Why this is so will be explained towards the end of this section as well as Proposition 4.13 and Remark 4.20.

Definition 3.18 (Orthogonally affine fibred functor)**.** A fibred functor H from $\mathbf{NCFinProb} \to$ **fdC*-Alg** to $\mathbb{BR} \to \underline{\mathbf{1}}$ is ***orthogonally affine*** iff to each pair of C^*-algebras $\mathcal{B}$ and $\mathcal{A}$, each pair of mutually orthogonal states $\omega, \xi \in \mathcal{S}(\mathcal{A})$, and each $*$-homomorphism $\mathcal{B} \xrightarrow{f} \mathcal{A}$ such that $(\omega \circ f) \perp (\xi \circ f)$,

$$H_f\big(\lambda\omega + (1-\lambda)\xi\big) = \lambda H_f(\omega) + (1-\lambda) H_f(\xi) \qquad \forall\, \lambda \in [0,1].$$

Proposition 3.19 (Entropy difference is continuous and orthogonally affine)**.** *The entropy change functor from Definition 2.22 is a continuous and orthogonally affine fibred functor. In fact, if for any C^*-algebra $\mathcal{A}$ and any pair ω, ξ of mutually orthogonal states on $\mathcal{A}$, a $*$-homomorphism $\mathcal{B} \xrightarrow{f} \mathcal{A}$ preserves the orthogonality $\omega \perp \xi$ if and only if*

$$S_f\big(\lambda\omega + (1-\lambda)\xi\big) = \lambda S_f(\omega) + (1-\lambda)S_f(\xi) \qquad \forall\, \lambda \in [0,1].$$

Before proving this, we introduce a shorthand for the deviation from S_f being affine on the states ω and ξ. The name for this deviation is motivated by [32, Section 12.1.1].

Definition 3.20 (The Holevo information change along a morphism)**.** The ***Holevo information change along*** a $*$-homomorphism $\mathcal{B} \xrightarrow{f} \mathcal{A}$ associated to $\omega, \xi \in \mathcal{S}(\mathcal{A})$ and $\lambda \in [0,1]$ is the number

$$\chi_f(\lambda; \omega, \xi) := S_f\big(\lambda\omega + (1-\lambda)\xi\big) - \lambda S_f(\omega) - (1-\lambda)S_f(\xi).$$

Proposition 3.19 says, in particular, that this deviation vanishes when $\omega \perp \xi$ and $(\omega \circ f) \perp (\xi \circ f)$.

Proof of Proposition 3.19. Continuity of the entropy change follows from continuity of the von Neumann entropy [32, Section 11.3], [11]. To prove the statement regarding orthogonal affinity, suppose $\omega \perp \xi$. Let $\omega' := \omega \circ f$ and $\xi' := \xi \circ f$. If f preserves the mutual orthogonality, then $\omega' \perp \xi'$ and

$$\begin{aligned}\chi_f(\lambda;\omega,\xi) &= S\big(\lambda\omega + (1-\lambda)\xi\big) - S\big(\lambda\omega' + (1-\lambda)\xi'\big)\\ &\qquad - \lambda S_f(\omega) - (1-\lambda)S_f(\xi)\\ &\overset{\text{Lem 2.21}}{=\!=\!=} S(\lambda, 1-\lambda) + \lambda S(\omega) + (1-\lambda)S(\xi)\\ &\qquad\quad - S(\lambda,1-\lambda) - \lambda S(\omega') - (1-\lambda)S(\xi')\\ &\qquad\qquad\qquad - \lambda S_f(\omega) - (1-\lambda)S_f(\xi)\\ &= 0,\end{aligned} \tag{3.21}$$

where $S(\lambda, 1-\lambda)$ is the Shannon entropy of the probability $(\lambda, 1-\lambda)$ on a two element set. Conversely, suppose $\chi_f(\lambda;\omega,\xi) = 0$. Since $\omega \perp \xi$, a similar calculation gives

$$\begin{aligned}0 &= \chi_f(\lambda;\omega,\xi)\\ &\overset{\text{Lem 2.21}}{=\!=\!=} S(\lambda,1-\lambda) + \lambda S(\omega') + (1-\lambda)S(\xi') - S\big(\lambda\omega' + (1-\lambda)\xi'\big),\end{aligned} \tag{3.22}$$

which gives $\omega' \perp \xi'$ by the 'only if' part of Lemma 2.21. ∎

In the last part of this section, we recall the convex combinations and affine functors introduced by BFL [4]. By the next section, we will have enough facts to relate BFL's definition to ours.

Definition 3.23 (An external convex structure on $\mathbf{FinProb}$)**.** For every $\lambda \in [0,1]$, define the convex sum F_λ on objects of $\mathbf{FinProb}$ by

$$\lambda(X,p) \oplus (1-\lambda)(Y,q) := \big(X \amalg Y, \lambda p \oplus (1-\lambda)q\big),$$

where $\lambda p \oplus (1-\lambda)q$ is defined in Example 2.8. The convex sum of morphisms $(X,p) \xrightarrow{\phi} (X',p')$ and $(Y,q) \xrightarrow{\psi} (Y',q')$ is defined to be the disjoint union $\phi \amalg \psi$ as in Example 2.8. The collection of functors $\{F_\lambda\}_{\lambda\in[0,1]}$ is called the ***external convex structure*** on $\mathbf{FinProb}$.

The motivation for calling this an *external* convex structure comes from the distinction between internal and external monoidal fibrations [30, Section 3.1], as will be explained shortly.

Remark 3.24 (The external convex structure on $\mathbf{FinProb}$ does not give a convex category)**.** $\mathbf{FinProb}$ with this family of functors is *not* a convex category in the sense of Definition 3.5. It is, however, a *weak* convex category (called a convex category in [34, Chapter 4]).

A completely analogous definition can be made for the fibration $\mathbf{NCFinProb} \to \mathbf{fdC^*\text{-}Alg}$ using the (external) direct sum of C^*-algebras.

Definition 3.25 (An external convex structure on $\mathbf{NCFinProb}$)**.** For every $\lambda \in [0,1]$, define the convex sum F_λ on objects of $\mathbf{NCFinProb}$ by $\lambda(\mathcal{A},\omega) \oplus (1-\lambda)(\mathcal{B},\xi) := \big(\mathcal{A}\oplus\mathcal{B}, \lambda\omega \oplus (1-\lambda)\xi\big)$, where $\big(\lambda\omega \oplus (1-\lambda)\xi\big)(A \oplus B) := \lambda\omega(A) + (1-\lambda)\xi(B)$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$. The convex sum of morphisms is the direct sum.

This convex structure on $\mathbf{NCFinProb}$ restricts to the one on $\mathbf{FinProb}$ on the subcategory of commutative C^*-algebras since $\mathbb{C}^{X\amalg Y} \cong \mathbb{C}^X \oplus \mathbb{C}^Y$.

Definition 3.26 (Externally affine functor)**.** A functor $H : \mathbf{NCFinProb} \to \mathbb{BR}$ is ***externally affine*** iff

$$H\big(\lambda f \oplus (1-\lambda)g\big) = \lambda H(f) + (1-\lambda)H(g)$$

for all morphisms f,g in $\mathbf{NCFinProb}$ and all $\lambda \in [0,1]$.

Example 3.27 (Examples of externally affine functors)**.**

(a) The difference of Shannon entropies studied by BFL [4] is a continuous externally affine functor $\mathbf{FinProb} \to \mathbb{BR}$. In fact, it is characterized as the unique one whose image always lands in $\mathbb{BR}_{\geq 0}$ (cf. Theorem 3.28).

(b) An example of a continuous externally affine functor $S : \mathbf{NCFinProb} \to \mathbb{BR}$ is the difference of Segal entropies from Definition 2.22.

(c) If $f : (\mathcal{B}, \xi) \xrightarrow{f} (\mathcal{A}, \omega)$ is as in Lemma 2.11, then $K_f(\omega) := S(p) - S(q)$, the difference of the Shannon entropies associated to the probability distributions, defines a continuous externally affine functor $K : \mathbf{NCFinProb} \to \mathbb{BR}$.

Notice that both K and S agree with the Shannon entropy difference on the subcategory of commutative algebras, yet they are *not* proportional.[7]

For reference, we recall BFL's characterization theorem [4].

Theorem 3.28 (BFL's functorial characterization of the Shannon entropy)**.** *If $H : \mathbf{FinProb} \to \mathbb{BR}_{\geq 0}$ is a continuous externally affine functor, then there exists a constant $c \geq 0$ such that $H_\phi(p) = c\big(S(p) - S(q)\big)$ for every probability-preserving function $(X, p) \xrightarrow{\phi} (Y, q)$.*

Without reference to the entropy formulas from Definition 2.22, we will relate internal and external affinity in Proposition 4.13 after developing some general results.

4 Characterizing entropy

This section contains our main result, Theorem 4.26, which is a functorial characterization of the entropy difference in the non-commutative setting. Continuity and orthogonal affinity alone are not quite enough to characterize the von Neumann entropy difference, though they come quite close. By

[7] The existence of these two distinct continuous (externally) affine functors illustrates that continuous affine functors $\mathbf{NCFinProb} \to \mathbb{BR}$ are not characterized by their values on $\mathbf{FinProb}^{\mathrm{op}}$ (when viewed as a subcategory of $\mathbf{NCFinProb}$). In particular, this condition does not characterize the von Neumann entropy difference. This answers a question of John Baez in the negative [2] (see specifically the original post as well as the post on June 7, 2011 at 8:12 AM).

Lemma 2.23, we cannot assume that $S_f(\omega) \geq 0$ for all $*$-homomorphisms f and states ω on the codomain of f, since this inequality fails for non-commutative C^*-algebras.

We propose a close replacement, namely $S_{\mathcal{A}}(\omega) \geq 0$ for all states $\omega \in \mathcal{S}(\mathcal{A})$, with equality on pure states, for all C^*-algebras $\mathcal{A}$. While this may sound quite different, this assumption is a consequence of BFL's assumption $S_f(\omega) \geq 0$ on *commutative* C^*-algebras. Furthermore, in Proposition 4.7, we prove that the non-negativity of entropy difference for *commutative* C^*-algebras is a *consequence* of the fact that state-preserving $*$-homomorphisms between commutative C^*-algebras always have *disintegrations*. More generally, we show that the existence of disintegrations (with non-commutative probability spaces included) implies the non-negativity of entropy difference.

Notation 4.1 ($!_{\mathcal{A}}$ and $H_{\mathcal{A}}$)**.** If $\mathcal{A}$ is a C^*-algebra, then $\mathbb{C} \xrightarrow{!_{\mathcal{A}}} \mathcal{A}$ will always refer to the unique (unital) $*$-homomorphism. If $H : \mathbf{NCFinProb} \to \mathbb{BR}$ is a functor, set $H_{\mathcal{A}} := H_{!_{\mathcal{A}}}$. Also, $\mathbf{FinProb}^{\mathrm{op}}$ will be viewed as the full subcategory of $\mathbf{NCFinProb}$ consisting of commutative probability spaces.

Lemma 4.2 (H is a coboundary)**.** *Given any $*$-homomorphism $\mathcal{B} \xrightarrow{f} \mathcal{A}$ and a state $\mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C}$, any functor $H : \mathbf{NCFinProb} \to \mathbb{BR}$ satisfies*

$$H_f(\omega) = H_{\mathcal{A}}(\omega) - H_{\mathcal{B}}(\omega \circ f).$$

Proof. This follows from $\mathbb{C}$ being an initial object in $\mathbf{fdC^*\text{-}Alg}$. ■

Lemma 4.3 (Non-negativity of H_f implies vanishing of $H_{\mathcal{A}}$ on pure states)**.** *Let $H : \mathbf{FinProb}^{\mathrm{op}} \to \mathbb{BR}$ be a functor satisfying $H_f(\omega) \geq 0$ for all $\omega \in \mathcal{S}(\mathcal{A})$ and $*$-homomorphisms $\mathcal{B} \xrightarrow{f} \mathcal{A}$ between commutative C^*-algebras.*

1. *If f has a left or right inverse, then $H_f(\omega) = 0$ for all $\omega \in \mathcal{S}(\mathcal{A})$.*
2. *$H_{\mathcal{A}}(\omega) \geq 0$ for all states $\omega \in \mathcal{S}(\mathcal{A})$, with equality on all pure states.*

Proof.

1. Suppose f has a right inverse $\mathcal{A} \xrightarrow{g} \mathcal{B}$. Then functoriality of H implies $0 = H_{\mathrm{id}_{\mathcal{A}}}(\omega) = H_g(\omega \circ f) + H_f(\omega)$ by Lemma 4.2. Since each term is non-negative by assumption, $H_f(\omega) \geq 0$. A similar calculation proves the same inequality if f has a left inverse.

2. First, $H_{\mathcal{A}}(\omega) = H_{!_{\mathcal{A}}}(\omega) \geq 0$ by assumption. By invariance of H under $*$-isomorphisms, it suffices to take $\mathcal{A} = \mathbb{C}^X$, with X a finite set. Any pure state ξ on $\mathbb{C}^X$ is necessarily supported on some $x \in X$. Let $\mathbb{C}^X \xrightarrow{\pi_x} \mathbb{C}$ be the projection onto that component. Then π_x pulls the unique state 1 on $\mathbb{C}$ back to ξ on $\mathbb{C}^X$ and the composite $\mathbb{C} \xrightarrow{!_{\mathbb{C}^X}} \mathbb{C}^X \xrightarrow{\pi_x} \mathbb{C}$ equals $\mathrm{id}_{\mathbb{C}}$. Thus, $H_{\mathbb{C}^X}(\xi) = 0$ by the first item. ■

A partial converse to Lemma 4.3 will illustrate that our axioms for entropy change imply those of BFL. We first prove a lemma about invariance under $*$-isomorphisms given *our* axioms. The proof is quite different from the one in Lemma 4.3, and it uses the convex structure in a crucial way.

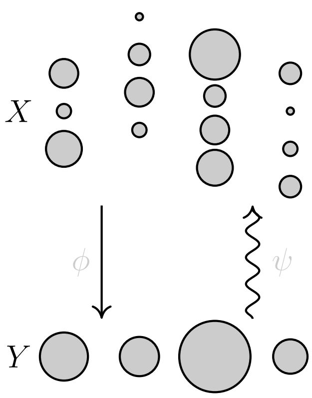
Lemma 4.4 (H is invariant under $*$-isomorphisms)**.** *Suppose* $H : \mathbf{NCFinProb} \to \mathbb{BR}$ *is an orthogonally affine fibred functor for which* $H_{\mathcal{A}}(\xi) = 0$ *for all pure states* ξ *on* $\mathcal{A}$ *and all* C^**-algebras* $\mathcal{A}$*. If* $\mathcal{B} \xrightarrow{f} \mathcal{A}$ *is a* $*$*-isomorphism, then* $H_f(\omega) = 0$ *for all* $\omega \in \mathcal{S}(\mathcal{A})$.

Proof. Let ω be a state on $\mathcal{A}$. Then there exists a convex decomposition $\omega = \sum_{x\in X} p_x\omega_x$ of ω in terms of mutually orthogonal pure states ω_x and a nowhere-vanishing probability measure p on some finite set X. Thus,

$$\begin{aligned} H_f(\omega) &\overset{\text{Lem 2.12}}{\underset{\text{Defn 3.18}}{=\!=\!=}} \sum_{x\in X} p_x H_f(\omega_x) \\ &\overset{\text{Lem 4.2}}{=\!=\!=} \sum_{x\in X} p_x\Big(H_{\mathcal{A}}(\omega_x) - H_{\mathcal{B}}(\omega_x \circ f)\Big) = 0 \end{aligned} \tag{4.5}$$

since $\omega_x \circ f$ is pure by Lemma 2.12. ■

Definition 4.6 (Disintegrations on finite probability spaces)**.** Let (X,p) and (Y,q) be probability spaces and let $\phi : X \to Y$ be a probability-preserving function, i.e. $q = \phi \circ p$. A ***disintegration*** of (ϕ, p, q) (or simply of ϕ if p and q are clear from context) is a stochastic map $Y \overset{\psi}{\rightsquigarrow} X$ such that



$$\begin{tikzcd} & \{\bullet\} \arrow[dl, squiggly, "p"'] \arrow[dr, squiggly, "q"] & \\ X & & Y \arrow[ll, squiggly, "\psi"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X \arrow[dl, "\phi"'] \arrow[d, phantom, "q"] & \\ Y & {} & Y \arrow[ll, "\mathrm{id}_Y"] \arrow[ul, squiggly, "\psi"'] \end{tikzcd},$$

the latter diagram signifying commutativity q-a.e.[8] Here, a ***stochastic map*** $Y \overset{\psi}{\rightsquigarrow} X$ associates to each $y \in Y$ a probability measure ψ_y on X. Composition of stochastic maps is defined via the Chapman–Kolmogorov equation [35, Section 2].

The main fact we will use about disintegrations on finite probability spaces is that they always exist [36, Section 2].

Proposition 4.7 (Positivity of entropy difference on commutative C^*-algebras)**.** *Suppose* $H : \mathbf{NCFinProb} \to \mathbb{BR}$ *is an orthogonally affine fibred functor for which* $H_{\mathcal{A}}(\omega) \geq 0$ *for all states* $\omega \in \mathcal{S}(\mathcal{A})$*, with equality on all pure states, for all* C^**-algebras* $\mathcal{A}$*. Then for* commutative C^**-algebras* $\mathcal{A}$ *and* $\mathcal{B}$*,* $H_f(\omega) \geq 0$ *for all states* $\omega \in \mathcal{A}$ *and all* $*$*-homomorphisms* $\mathcal{B} \xrightarrow{f} \mathcal{A}$*.*

Proof. By invariance of H for $*$-isomorphisms (Lemma 4.4), it suffices to assume $\mathcal{B} = \mathbb{C}^Y$ and $\mathcal{A} = \mathbb{C}^X$ for finite sets X and Y. In this case, let ω be represented by a probability measure p on X, let $X \xrightarrow{\phi} Y$ be the function associated to $\mathcal{B} \xrightarrow{f} \mathcal{A}$, and let $q := \phi \circ p$ be the pushforward measure corresponding to $\omega \circ f =: \xi$ (cf. Example 2.8). Every such probability measure is decomposed as $q = \sum_{y \in Y} q_y \delta_y$, where δ_y is the Dirac delta measure at y defined by $\delta_y(y') \equiv \delta_{yy'}$, which is 1 if $y' = y$ and 0 otherwise. This expresses q as a convex sum of mutually orthogonal measures since $\delta_y \perp \delta_{y'}$ for all $y \neq y'$. Set

$$N_q := \{y \in Y \ : \ q_y = 0\} \tag{4.8}$$

and let $Y \overset{\psi}{\rightsquigarrow} X$ be a disintegration of (ϕ, p, q). Then p also decomposes as

$$p = \sum_{y \in Y} q_y \psi_y \equiv \sum_{y \in Y \setminus N_q} q_y \psi_y, \tag{4.9}$$

where the set of probability measures $\{\psi_y\}_{y \in Y \setminus N_q}$ are mutually orthogonal because ψ_y is a measure supported on $f^{-1}(\{y\})$. Furthermore, ϕ preserves the mutual orthogonality of these measures

$$(\phi \circ \psi_y) \perp (\phi \circ \psi_{y'}) \qquad \forall \, y \neq y' \in Y \setminus N_q, \tag{4.10}$$

[8]The cartoon depicts probability measures as collections of water droplets with total volume 1. The map ϕ combines water droplets and preserves the volume [17], while the disintegration ψ splits the water droplets back into their original sizes.

since $\phi \circ \psi_y = \delta_y$ for all $y \in Y \setminus N_q$. Setting ω_y to be the state corresponding to ψ_y gives

$$
\begin{aligned}
H_f(\omega) &\overset{(4.9)}{=\!=} H_f\left(\sum_{y\in Y\setminus N_q} q_y\omega_y\right) \underset{\text{Defn 3.18}}{\overset{(4.10)}{=\!=\!=}} \sum_{y\in Y\setminus N_q} q_y H_f(\omega_y) \\
&\overset{\text{Lem 4.2}}{=\!=\!=} \sum_{y\in Y\setminus N_q} q_y\Big(H_{\mathcal{A}}(\omega_y) - H_{\mathcal{B}}(\underbrace{\omega_y\circ f}_{\delta_y})\Big) \qquad (4.11)\\
&= \sum_{y\in Y\setminus N_q} q_y H_{\mathcal{A}}(\omega_y) \geq 0.
\end{aligned}
$$

The last line holds because $H_{\mathcal{B}}$ vanishes on pure states and by the assumption that $H_{\mathcal{A}}$ is always non-negative. ■

Proposition 4.7 shows that our axioms *imply* the (seemingly strong) axiom of non-negativity for entropy difference used by BFL in their functorial characterization of Shannon entropy (Theorem 3.28). Combining this fact with Lemma 4.3 suggests that it is reasonable to replace the BFL axiom of non-negativity for entropy difference by non-negativity of $H_{\mathcal{A}}$ and equality to zero on pure states. In fact, a corollary of Proposition 4.7 and BFL's characterization is an alternative functorial characterization of Shannon entropy that does not explicitly use the non-negativity for entropy difference assumption. However, we still need one more important fact to show that our notion for a functor being orthogonally affine is equivalent to BFL's notion of a functor being externally affine on finite probability spaces (Proposition 4.13). We will then use this towards building the final fact used in our characterization theorem.

Lemma 4.12 (Invariance under adjoining zero)**.** *Let $H : \mathbf{NCFinProb} \to \mathbb{BR}$ be an orthogonally affine fibred functor for which $H_{\mathcal{A}}(\omega) \geq 0$ for all states $\omega \in \mathcal{S}(\mathcal{A})$, with equality on all pure states, for all C^*-algebras $\mathcal{A}$. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and let $\pi : \mathcal{A}\oplus\mathcal{B} \twoheadrightarrow \mathcal{A}$ be the projection. Then $H_\pi(\omega) = 0$ for all $\omega \in \mathcal{S}(\mathcal{A})$. In particular, if X and Y are finite sets and $\iota : X \hookrightarrow X \amalg Y$ is the inclusion with associated $*$-homomorphism $\pi : \mathbb{C}^{X\amalg Y} \twoheadrightarrow \mathbb{C}^X$, then $H_\pi(\omega) = 0$ for all states $\omega \in \mathcal{S}(\mathbb{C}^X)$.*

Proof. The proof is similar to that of Lemma 4.4 since $\omega_x \circ \pi$ is pure whenever ω_x is. ■

Proposition 4.13 (External versus orthogonal affinity)**.** *Let H : $\mathbf{FinProb}^{\mathrm{op}} \to \mathbb{BR}$ be a fibred functor for which $H_{\mathcal{A}}(\omega) \geq 0$ for all states $\omega \in \mathcal{S}(\mathcal{A})$, with equality on all pure states, for all commutative C^*-algebras $\mathcal{A}$. Then H is orthogonally affine if and only if H is externally affine.*

Proof. ($\Rightarrow$) Suppose H is orthogonally affine. The external convex sum of $(\mathbb{C}^{X'},\omega') \xrightarrow{f} (\mathbb{C}^X,\omega)$ and $(\mathbb{C}^{Y'},\xi') \xrightarrow{g} (\mathbb{C}^Y,\xi)$ defines a morphism

$$\left(\mathbb{C}^{X'\amalg Y'},\lambda\widetilde{\omega}'+(1-\lambda)\widetilde{\xi}'\right) \xrightarrow{k:=f\oplus g} \left(\mathbb{C}^{X\amalg Y},\lambda\widetilde{\omega}+(1-\lambda)\widetilde{\xi}\right), \tag{4.14}$$

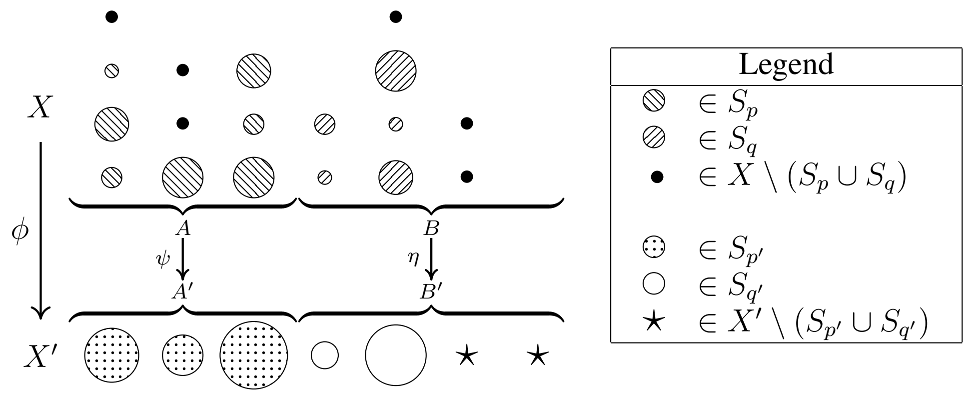
where the tildes denote the states as viewed on the direct sum (cf. Example 2.8). In particular, $(\mathbb{C}^X \oplus \mathbb{C}^Y,\widetilde{\omega}) \xrightarrow{\pi_X} (\mathbb{C}^X,\omega)$ is a morphism in $\mathbf{NCFinProb}$ for example. Furthermore,

$$\widetilde{\omega}\circ k=\widetilde{\omega}', \qquad \widetilde{\xi}\circ k=\widetilde{\xi}', \qquad \widetilde{\omega}\perp\widetilde{\xi}, \qquad \text{and} \qquad \widetilde{\omega}'\perp\widetilde{\xi}', \tag{4.15}$$

which says that $f\oplus g$ preserves the orthogonality of $\widetilde{\omega}$ and $\widetilde{\xi}$. Since H is orthogonally affine,

$$\begin{aligned}
H(k) &\equiv H_{f\oplus g}\left(\lambda\widetilde{\omega}+(1-\lambda)\widetilde{\xi}\right)\\
&\overset{\text{Defn 3.18}}{=\!=} \lambda H_{f\oplus g}(\widetilde{\omega})+(1-\lambda)H_{f\oplus g}(\widetilde{\xi})\\
&\overset{\text{Lem 4.2}}{=\!=} \lambda\Big(H_{\mathbb{C}^{X\amalg Y}}(\widetilde{\omega})-H_{\mathbb{C}^{X'\amalg Y'}}(\widetilde{\omega}')\Big)\\
&\qquad+(1-\lambda)\Big(H_{\mathbb{C}^{X\amalg Y}}(\widetilde{\xi})-H_{\mathbb{C}^{X'\amalg Y'}}(\widetilde{\xi}')\Big)\\
&\overset{\text{Lem 4.12}}{=\!=} \lambda\Big(H_{\mathbb{C}^X}(\omega)-H_{\mathbb{C}^{X'}}(\omega')\Big)+(1-\lambda)\Big(H_{\mathbb{C}^Y}(\xi)-H_{\mathbb{C}^{Y'}}(\xi')\Big)\\
&\overset{\text{Lem 4.2}}{=\!=} \lambda H_f(\omega)+(1-\lambda)H_g(\xi)\equiv\lambda H(f)+(1-\lambda)H(g).
\end{aligned} \tag{4.16}$$

($\Leftarrow$) Suppose H is externally affine. Let p, q be probability measures on X and let p', q' be probability measures on X'. Let $X \xrightarrow{\phi} X'$ be a function that preserves both pairs of probability measures, i.e. $\phi\circ p=p'$ and $\phi\circ q=q'$. Suppose $p\perp q$ as well as $p'\perp q'$. In what follows, we will first show that there exist morphisms $(A,p_{\restriction A}) \xrightarrow{\psi} (A',p'_{\restriction A'})$ and $(B,q_{\restriction B}) \xrightarrow{\eta} (B',q'_{\restriction B'})$ such that $\lambda\psi\oplus(1-\lambda)\eta=\phi$. Let S_r denote the support of $r\in\{p,q,p',q'\}$ (viewed as a subset of X or X' depending on the subscript). By assumption, $S_p\cap S_q=\varnothing$ and $S_{p'}\cap S_{q'}=\varnothing$. Furthermore, ϕ can be visualized as



where the indicated sets are defined by

$$\begin{aligned} A' &:= S_{p'}, & B' &:= S_{q'} \cup \big(X \setminus (S_{p'} \cup S_{q'})\big), \\ A &:= \phi^{-1}(A'), & B &:= \phi^{-1}(B'), \end{aligned} \tag{4.17}$$

and the functions $A \xrightarrow{\psi} A'$ and $B \xrightarrow{\eta} B'$ are defined by restricting ϕ to A and B, respectively. If we also define the probability measures $p_{\restriction A}, q_{\restriction B}, p'_{\restriction A'}$, and $q'_{\restriction B'}$ on A, B, A', and B', respectively, then $(A, p_{\restriction A}) \xrightarrow{\psi} (A', p'_{\restriction A'})$ and $(B, q_{\restriction B}) \xrightarrow{\eta} (B', q'_{\restriction B'})$ are morphisms in **FinProb** and most importantly,

$$\lambda \left(\begin{array}{c} (A, p_{\restriction A}) \\ \big\downarrow \psi \\ (A', p'_{\restriction A'}) \end{array} \right) \oplus (1-\lambda) \left(\begin{array}{c} (B, q_{\restriction B}) \\ \big\downarrow \eta \\ (B', q'_{\restriction B'}) \end{array} \right) = \begin{array}{c} \big(X, \lambda p + (1-\lambda) q\big) \\ \big\downarrow \phi \\ \big(X', \lambda p' + (1-\lambda) q'\big) \end{array} . \tag{4.18}$$

Thus,

$$\begin{aligned} H_\phi\big(\lambda p + (1-\lambda) q\big) &\equiv H\big(\lambda\psi \oplus (1-\lambda)\eta\big) \\ &\overset{\text{Defn 3.26}}{=\!=} \lambda H(\psi) + (1-\lambda) H(\eta) \\ &= \lambda\Big(1H(\psi) + 0H(\eta)\Big) + (1-\lambda)\Big(0H(\psi) + 1H(\eta)\Big) \\ &\overset{\text{Defn 3.26}}{=\!=} \lambda H(1\psi \oplus 0\eta) + (1-\lambda) H(0\psi \oplus 1\eta) \\ &\equiv \lambda H_\phi(p) + (1-\lambda) H_\phi(q), \end{aligned} \tag{4.19}$$

which completes the proof. ■

Remark 4.20 (External affinity ignores the internal structure of quantum states)**.** The objects of $\mathbf{FinProb}$ are convex generated by the single object $\mathbf{1}$, which is the (essentially) unique probability space consisting of a single element. Indeed, an arbitrary finite probability space (X,p) can be decomposed into an external convex sum as $(X,p) \cong \bigoplus_{x\in X} p_x\mathbf{1}$. However, in $\mathbf{NCFinProb}$, a non-commutative probability space such as $(\mathcal{M}_m,\omega)$ cannot be expressed as an external convex combination of lower-dimensional probability spaces. Therefore, the statement "if H is externally affine (on all C^*-algebras), then H is orthogonally affine" is false.[9] Example 3.27 (c) is a counter-example because it is not orthogonally affine. This, together with Proposition 4.13 provides some motivation for our choice of defining convex structures *internally* on the fibres over C^*-algebras.

Corollary 4.21 (Characterizing the Shannon entropy on commutative C^*-algebras)**.** *Suppose $H : \mathbf{NCFinProb} \to \mathbb{BR}$ is a continuous orthogonally affine fibred functor for which $H_{\mathcal{A}}(\omega) \ge 0$ for all states $\omega \in \mathcal{S}(\mathcal{A})$, with equality on all pure states, for all C^*-algebras $\mathcal{A}$. Then there exists a constant $c \ge 0$ such that $H_f = cS_f$ for all $*$-homomorphisms f between* commutative *C^*-algebras.*

Proof. Continuity and functoriality are already assumed. Non-negativity of $H_f(\omega)$ for all states ω and $*$-homomorphisms between commutative C^*-algebras was proved in Proposition 4.7. Finally, the notion of affine orthogonality of H is equivalent to external affinity for commutative C^*-algebras by Proposition 4.13. By BFL's characterization theorem (Theorem 3.28), H is the functor giving the difference of entropies on the subcategory of commutative probability spaces up to an overall non-negative constant. ■

The orthogonally affine assumption for *all* C^*-algebras will provide the last fact needed to prove our characterization theorem.

Lemma 4.22 (Affine orthogonality determines entropy)**.** *Let $H : \mathbf{NCFinProb} \to \mathbb{BR}$ be a continuous and orthogonally affine fibred functor for which $H_{\mathcal{A}}(\omega) \ge 0$ for all states $\omega \in \mathcal{S}(\mathcal{A})$, with equality on all pure states, for all C^*-algebras $\mathcal{A}$. If ω is any state on $\mathcal{A}$, then there exists a constant $c \ge 0$ (independent of the algebras and states) such that $H_{\mathcal{A}}(\omega) = cS(\omega)$.*

[9] Although the converse is still true, as can be seen by a minor modification of the proof of the ($\Rightarrow$) direction in Proposition 4.13.

Proof. By invariance of H under $*$-isomorphisms (Lemma 4.4), it suffices to assume ω is a state as in Example 2.2. Let $N_p := \{x \in X \,:\, p_x = 0\}$ be the nullspace of p. For each $x \in X \setminus N_p$, decompose ω_x into a convex sum $\omega_x = \sum_{y \in Y_x} \psi_{yx}\omega_{yx}$ of pure states $\omega_{yx} \in \mathcal{S}(\mathcal{M}_{m_x})$, where $\{\psi_{yx}\}_{y \in Y_x}$ defines a nowhere-vanishing probability measure on a finite set Y_x whose cardinality equals the rank of the support of ω_x. Thus, $X \setminus N_p \stackrel{\psi}{\rightsquigarrow} \coprod_{x \in X \setminus N_p} Y_x$ defines a stochastic map. Let $P_{yx} \in \mathcal{M}_{m_x}$ denote the one-dimensional projection associated to the pure state ω_{yx}. If P_x denotes the support of ω_x, then $P_x = \sum_{y \in Y_x} P_{yx}$ for all $x \in X \setminus N_p$. Set $\mathcal{B} := \left(\bigoplus_{x \in X \setminus N_p} \mathbb{C}^{Y_x}\right) \oplus \mathbb{C}^{\{\bullet\}}$, where $\mathbb{C}^{\{\bullet\}} \cong \mathbb{C}$, and $\bullet$ merely serves as a label to distinguish it from the rest of the algebra. Define a $*$-homomorphism $\mathcal{B} \xrightarrow{f} \mathcal{A}$ by

$$
\begin{aligned}
\mathbb{C}^{Y_x} \ni e_y &\stackrel{f}{\mapsto} \left(\bigoplus_{x' \in X \setminus \{x\}} 0\right) \oplus P_{yx} \qquad \text{and} \\
\mathbb{C}^{\{\bullet\}} \ni e_\bullet &\stackrel{f}{\mapsto} \left(\bigoplus_{x \in X \setminus N_p} (\mathbb{1}_{m_x} - P_x)\right) \oplus \bigoplus_{x \in N_p} \mathbb{1}_{m_x},
\end{aligned}
\tag{4.23}
$$

where the first case expresses P_{yx} as an element of $\mathcal{B}$ (with 0's on all factors other than $\mathcal{M}_{m_x}$). Then f is a (unital) $*$-homomorphism that preserves the orthogonality of *all* the ω_{yx} states with $y \in Y_x$ and $x \in X \setminus N_p$ (by viewing all the ω_{yx} as states on $\mathcal{A}$ via Lemma 4.12). Therefore,

$$
\begin{aligned}
H_{\mathcal{A}}(\omega) - H_{\mathcal{B}}(\omega \circ f) = H_f(\omega) &= \sum_{x \in X \setminus N} \sum_{y \in Y_x} p_x \psi_{yx} H_f(\omega_{yx}) \\
&= \sum_{x \in X \setminus N} \sum_{y \in Y_x} p_x \psi_{yx} \big(H_{\mathcal{A}}(\omega_{yx}) - H_{\mathcal{B}}(\omega_{yx} \circ f)\big) = 0
\end{aligned}
\tag{4.24}
$$

because ω_{yx} and $\omega_{yx} \circ f$ are pure states.

Consequently,

$$
\begin{aligned}
H_{\mathcal{A}}(\omega) &\overset{(4.24)}{=\!=} H_{\mathcal{B}}(\omega \circ f) \\
&\overset{\text{Cor } 4.21}{=\!=\!=} -c \sum_{x \in X \setminus N_p} \sum_{y \in Y_x} p_x \psi_{yx} \log(p_x \psi_{yx}) \quad \text{for some } c \geq 0 \\
&= -c \sum_{x \in X \setminus N_p} \underbrace{\sum_{y \in Y_x} \psi_{yx}}_{1} p_x \log(p_x) - c \sum_{x \in X \setminus N_p} p_x \sum_{y \in Y_x} \psi_{yx} \log(\psi_{yx}) \qquad (4.25) \\
&= c \left(S(p) + \sum_{x \in X} p_x S(\omega_x) \right),
\end{aligned}
$$

where the last equality follows from the definition of the Shannon entropy for the $S(p)$ term and Lemma 2.21 for the $S(\omega_x)$ term. ∎

Theorem 4.26 (A functorial characterization of quantum entropy)**.** *Let H : $\mathbf{NCFinProb} \to \mathbb{BR}$ be a continuous and orthogonally affine fibred functor for which $H_{\mathcal{A}}(\omega) \geq 0$ for all states $\omega \in \mathcal{S}(\mathcal{A})$, with equality on all pure states, for all C^*-algebras $\mathcal{A}$. Then there exists a constant $c \geq 0$ such that*

$$
H_f(\omega) = c\Big(S(\omega) - S(\omega \circ f)\Big)
$$

for all morphisms $\mathcal{B} \xrightarrow{f} \mathcal{A}$ of C^*-algebras and states $\omega \in \mathcal{S}(\mathcal{A})$.

Proof. Since $H_f(\omega) = H_{\mathcal{A}}(\omega) - H_{\mathcal{B}}(\omega \circ f)$ by Lemma 4.2, Lemmas 4.4 and 4.22 show this equals the entropy difference up to the same constant c. ∎

It is interesting that the notion of a disintegration was used in the proof of Proposition 4.7. Note that in the category of states on (finite-dimensional) C^*-algebras and state-preserving $*$-homomorphisms, disintegrations do not always exist [36]. Nevertheless, when they exist, they imply $H_f(\omega) \geq 0$, as the following proposition shows. Since the definition of a non-commutative disintegration is not needed anywhere else in this work, the reader is referred to [36] for definitions and other facts assumed in the proof.

Proposition 4.27 (If a disintegration for $(f, \omega, \omega \circ f)$ exists, then $S_f(\omega) \geq 0$)**.** *Let $\mathcal{B} \xrightarrow{f} \mathcal{A}$ be a $*$-homomorphism and $\mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C}$ a state on $\mathcal{A}$. If $(f, \omega, \omega \circ f)$ has a disintegration, then $S_f(\omega) \geq 0$.*

Proof. By isomorphism invariance of S, it suffices to consider the case where $\mathcal{A},\mathcal{B},\omega,f$, and ξ are as in Lemma 2.11 (without the unitaries U_x). Let $N_p\subset X$ and $N_q\subset Y$ be the nullspaces of p and q, respectively. Assume that a disintegration of (f,ω,ξ) exists. By the non-commutative disintegration theorem [36], for each $x\in X$ and $y\in Y$ there exist non-negative matrices $\tau_{yx}\in\mathcal{M}_{c_{xy}}$ such that

$$\begin{aligned} \operatorname{tr}\left(\sum_{x\in X}\tau_{yx}\right) &= 1 \qquad \forall\, y\in Y\setminus N_q \\ \text{and}\qquad p_x\rho_x &= \boxplus_{y\in Y}\tau_{yx}\otimes q_y\sigma_y \qquad \forall\, x\in X. \end{aligned} \tag{4.28}$$

One more fact that will be needed is the equality

$$(C\otimes D)\log(C\otimes D) = C\log(C)\otimes D + C\otimes D\log(D) \tag{4.29}$$

for all non-negative square matrices C,D (possibly of different sizes). Computing $S_{\mathcal{A}}(\omega)$ first gives

$$\begin{aligned} S_{\mathcal{A}}(\omega) &\overset{\text{Defn 2.20}}{=\!=} -\sum_{x\in X}\operatorname{tr}\big(p_x\rho_x\log(p_x\rho_x)\big) \\ &\overset{(4.28)}{=\!=} -\sum_{x\in X}\operatorname{tr}\left(\boxplus_{y\in Y\setminus N_q}(\tau_{yx}\otimes q_y\sigma_y)\log\left(\boxplus_{y'\in Y\setminus N_q}\tau_{y'x}\otimes q_{y'}\sigma_{y'}\right)\right) \\ &= -\sum_{x\in X}\sum_{y\in Y\setminus N_q}\operatorname{tr}\big((\tau_{yx}\otimes q_y\sigma_y)\log(\tau_{yx}\otimes q_y\sigma_y)\big) \\ &\overset{(4.29)}{=\!=} -\sum_{x\in X}\sum_{y\in Y\setminus N_q}\operatorname{tr}\big(\tau_{yx}\log(\tau_{yx})\otimes q_y\sigma_y + \tau_{yx}\otimes q_y\sigma_y\log(q_y\sigma_y)\big) \\ &\overset{(4.28)}{=\!=} \sum_{y\in Y\setminus N_q} q_y S\left(\boxplus_{x\in X}\tau_{yx}\right) + S_{\mathcal{B}}(\xi), \end{aligned} \tag{4.30}$$

where $\boxplus_{x\in X}\tau_{yx}$ is viewed as a density matrix on $\mathcal{M}_{s_x}$, where $s_x := \sum_{y\in Y\setminus N_q} c_{yx}$. Thus,

$$S_f(\omega) = S_{\mathcal{A}}(\omega) - S_{\mathcal{B}}(\xi) = \sum_{y\in Y\setminus N_q} q_y S\left(\boxplus_{x\in X}\tau_{yx}\right) \geq 0. \tag{4.31}$$

■

Remark 4.32 (Having a disintegration is not necessary for $S_f(\omega) \geq 0$)**.** If $S_f(\omega) \geq 0$, it is not necessary that a disintegration of $(f, \omega, \omega \circ f)$ exists. A counter-example is the inclusion $\mathcal{M}_2 \to \mathcal{M}_2 \otimes \mathcal{M}_2$, which sends $B \in \mathcal{M}_2$ to $\mathbb{1}_2 \otimes B$, and the density matrix $\rho = \mathrm{diag}(p_1, p_2, p_3, p_4)$, where $p_1, p_2, p_3, p_4 \geq 0$ satisfy $p_1 + p_2 + p_3 + p_4 = 1$, $p_1 + p_3 > 0$, and $p_2 + p_4 > 0$. Then

$$\begin{aligned} S_f(\omega) = &-p_1 \log\left(\frac{p_1}{p_1 + p_3}\right) - p_2 \log\left(\frac{p_2}{p_2 + p_4}\right) \\ &- p_3 \log\left(\frac{p_3}{p_1 + p_3}\right) - p_4 \log\left(\frac{p_4}{p_2 + p_4}\right) \geq 0, \end{aligned}$$

while a disintegration exists if and only if $p_1 p_4 = p_2 p_3$ [36, Section 4].

Remark 4.33 (A brief history and comparison of axiomatizations of quantum entropy)**.** Quantum entropy and its variants were often built upon the classical versions, whose many axiomatizations are reviewed in Csiszar's survey [7]. In 1932, von Neumann obtained a phenomelogical characterization of entropy [49, Chapter V. Section 2]. In 1968, Ingarden and Kossakowski characterized the von Neumann entropy using dimensional partial Boolean rings of projections in Hilbert space [24]. In 1974, Ochs provided a characterization using partial isometric invariance, additivity, subadditivity, and continuity (plus some additional technical axioms) [33]. In 1975, Thirring [44] characterized the von Neumann entropy using axioms closely related to those implemented by Fadeev in his characterization of the Shannon entropy [9, 10], the latter of which was simplified by Renyi [40].[10]

Thirring's characterization is most closely related to ours and it is worth taking the time to spell out his assumptions, which read as follows.

(i) $S(\rho)$ is a continuous function of the eigenvalues of ρ;

(ii) $S(\frac{1}{2}\mathbb{1}_2) = \log 2$;

(iii) If $\mathcal{H} = \bigoplus_{n=1}^{N} \mathcal{H}_n$ is a direct sum of Hilbert spaces and if $\rho = \bigoplus_{n=1}^{N} p_n \rho_n$ is a weighted direct sum of density matrices, where $\{p_n\}_{n \in \{1,\dots,N\}}$ is a probability distribution on $\{1, \dots, N\}$, then $S(\rho) = S(p) + \sum_{n=1}^{N} p_n S(\rho_n)$, where p is viewed as a diagonal matrix on $\mathbb{C}^N$ with entries given by the p_n.

[10] Thirring's statement and proof can be found in [45, (2.2.4) pages 58–61]. However, it seems that the first written account of his proof in English appears in Wehrl's review [51, pages 238–239].

There are actually several implicitly hidden assumptions within these three. For example, the dependence on eigenvalues means $S(\rho) = S(U\rho U^{\dagger})$ for all unitaries U, i.e. $S(\rho)$ is invariant under $*$-isomorphisms. The second item is merely a normalization condition, which we have ignored (it specifies the constant c). The third item is close to our orthogonally affine assumption. However, an implicit assumption is made, which can be expressed as saying that $S(\rho_n)$ is equal to $S(0 \oplus \cdots \oplus \rho_n \oplus \cdots 0)$, i.e. S is invariant under the non-unital inclusion of one matrix algebra into a direct sum. This is closely related to Och's partial isometry invariance assumption. In our characterization, we obtain this property *as well as* invariance under $*$-isomorphisms as a consequence of our axioms.

Two other characterizations of the von Neumann entropy have appeared recently. The first is the topos-theoretic one of Constantin and Döring, which is based on how different commutative subalgebras, called *contexts*, of a fixed C^*-algebra determine its structure [6]. A context may be thought of as probing a quantum system by measurements of an observable and sending any state to the probability measure on the associated set of eigenvalues—in other words, it is a $*$-homomorphism. The collection of all contexts forms a category via inclusion and one can define measures associated to this category via compatible families of probability measures on the contexts *without* defining a state on the embedding algebra. They then classify the quantum entropy by assuming the form of entropy on the subcategory of commutative algebras and minimizing over all contexts. One difference between our assumptions for characterizing the von Neumann entropy is that we do not assume the formula for the Shannon entropy, nor do we assume that commutative algebras play any special role, which are singled out by the existence of disintegrations for *all* state-preserving *-homomorphisms. On the other hand, their characterization emphasizes the physically intuitive operational importance of classical systems for determining the entropy.

Finally, there has also been an abstract characterization of the von Neumann entropy by homological information structures introduced by Baudot and Bennequin (cf. Theorem 3 page 3290 and Theorem 4 page 3313 of [5]), which are further developed by Vigneaux [47, 48]. They seek to understand information quantities more generally. It is not yet clear to us how our methods are related.

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