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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

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A MAL'TSEV GLANCE AT THE FIBRATION $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ OF INTERNAL CATEGORIES

Dominique BOURN

Résumé. Nous montrons comment la fibration $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ des catégories internes à $\mathbb{E}$ est munie de deux types de structures partiellement liées à des concepts protomodulaires et mal'tseviens, et nous en explicitons quelques conséquences. Cela mène, entre autres, à la notion de *catégorie Schreier-spéciale* qui détermine une sous-catégorie protomodulaire de chaque fibre $Cat_Y\mathbb{E}$.

Abstract. We show how the fibration $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ associated with the internal categories in $\mathbb{E}$ is endowed with two kinds of structure which are partially dealing with Mal'tsev and protomodular concepts, and we make explicit some consequences. This leads, inter alia, to the notion of *Schreier special category* which determines a protomodular subcategory of any fiber $Cat_Y\mathbb{E}$.

Keywords. Mal'tsev and protomodular categories; split epimorphisms; internal categories and groupoids; connected, aspherical and affine groupoids, direction of aspherical affine groupoids, internal weak equivalence.

Mathematics Subject Classification (2020). 18A20, 18C10, 18C40, 18E13.

Introduction

Given any finitely complete category $\mathbb{E}$, the fibration $(\ )_0 : Grd\mathbb{E} \to \mathbb{E}$ of internal groupoids in $\mathbb{E}$ is known to have a strong structural property [3]: any fiber $Grd_Y\mathbb{E}$ above the object Y in $\mathbb{E}$ is protomodular [3] and thus a Mal'tsev category, on the model of the fiber $Grd_1\mathbb{E}$ above the terminal object 1 which is nothing but the category $Gp\mathbb{E}$ of internal groups in $\mathbb{E}$. Nothing comparable did exist for the fibration $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ of internal categories whose first structural properties were, strictly speaking, investigated in [2] more than thirty years ago, the notion of internal categories having been initiated, long before, in the pionnering work of C. Ehresmann [16].

But in [12, 13] was introduced a new structural aspect of the category Mon of monoids with the notion of Schreier split epimorphism and the associated notion of partial protomodularity (see Definition 3.1). Since Mon is nothing but the fiber Cat_1, the main aim of this work was to investigate whether it was possible to extend this result to any fiber Cat_Y and $Cat_Y\mathbb{E}$; in other words, this aim was to identify a class Σ_Y of split epimorphisms in the fibers $Cat_Y\mathbb{E}$ which would imply a partial protomodularity inside them. This is done in Section 3.1 with the extension of the notion of Schreier split epimorphism to internal functors, and this leads to the notion of *Schreier special category* (see Definition 4.4) which determines protomodular subcategories of the fibers $Cat_Y\mathbb{E}$.

Unexpectedly, a more global property of the fibration $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ did emerge during this investigation, but, this time, related to the notion of partial mal'tsevness (see Theorem 1.4) from which a spectacular Mal'tsev type consequence is drawn: when $\mathbb{E}$ is regular, given any pullback in $Cat\mathbb{E}$:

$$\begin{array}{ccc} X_\bullet & \xrightarrow{x_\bullet} & X'_\bullet \\ {\scriptstyle f_\bullet}\downarrow\uparrow{\scriptstyle s_\bullet} & & {\scriptstyle f'_\bullet}\downarrow\uparrow{\scriptstyle s'_\bullet} \\ Y_\bullet & \xrightarrow[y_\bullet]{} & Y'_\bullet \end{array}$$

where $y_\bullet$ is a fully faithful functor above a regular epimorphism y_0 in $\mathbb{E}$ and $(f'_\bullet, s'_\bullet)$ is any split epimorphism in a fiber of the fibration $(\ )_0$, then the upward square is necessarily a pushout, see Proposition 1.5.

Now, what is probably the most surprising in the whole process is the following observation: in the fiber $Cat_1 = Mon$, the notion of Schreier epimorphism is not intrinsic to Mon, clearly refereing to a non-homomorphic retraction of a given homomorphism. However, enlarging the definition to the whole category Cat, we get to a notion which is intrinsically bound to Cat, and even more surprisingly, it is intrinsically bound to its 2-*categorical nature*; and this, of course, remains valid for $Cat\mathbb{E}$, see Proposition 3.6.

Finally, in the last section, we shall complete the observations of [6, 8] about the affine groupoids (and their directions) in showing that they are stable under weak equivalences between groupoids; namely, under a specific class of functors which are cartesian with respect to the fibration $(\ )_0$.

The article is organized along the following lines: Section 1) is devoted to recalls about the basics on internal categories and culminates with the first structural observation, Theorem 1.4, from which we draw a short observation about the composition and the permution of some equivalence relations in the (non-regular) context of $Cat\mathbb{E}$. Section 2) introduces the structural concept of Mal'tsev fibration hidden behind this theorem and investigates its consequences. Section 3) is dealing with the partial protomodularity of the fibers $Cat_Y\mathbb{E}$ and culminates with the second structural observation: Theorem 3.5. Section 4) explicitely describes some consequences of this partial protomodularity. In particular, if the protomodular Schreier-core of $Cat_1 = Mon$ is the category Gp of groups, the protomodular Schreier-core of Cat_Y does not consist in the only groupoids with Y as set of objects, see Definition 4.4. All the results of this article as far as this point have been pre-published in [9]. Finally, Section 5) brings some precisions about the fibration $Grd\mathbb{E} \to \mathbb{E}$ relatively to the class of affine groupoids.

1. Internal categories

1.1 Basics

In this article any category $\mathbb{E}$ will be supposed finitely complete, and any pullback of an identity map will be chosen as being an identity map. We shall use a 3-truncated simplicial notation [21] for any internal category (including all the degeneracy maps which do not appear in the following diagram):

$$X_\bullet : \quad X_3 \underset{d_0^{X_\bullet}}{\overset{d_4^{X_\bullet}}{\substack{\longrightarrow \\ \xrightarrow{d_3^{X_\bullet}} \\ \longrightarrow \\ \xrightarrow{d_1^{X_\bullet}} \\ \longrightarrow}}} X_2 \underset{d_0^{X_\bullet}}{\overset{d_2^{X_\bullet}}{\substack{\longrightarrow \\ \xrightarrow{d_1^{X_\bullet}} \\ \longrightarrow}}} X_1 \underset{d_0^{X_\bullet}}{\overset{d_1^{X_\bullet}}{\substack{\longrightarrow \\ \xleftarrow{s_0^{X_\bullet}} \\ \longrightarrow}}} X_0$$

where X_2 (resp. X_3) is obtained by the pullback of d_0 along d_1 (resp. d_0 along d_2), and for any internal functor as well. We denote by $Cat\mathbb{E}$ the category of internal categories in $\mathbb{E}$, and by $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ the forgetful functor associating with any internal category $X_\bullet$ its "object of objects" X_0. The category $Cat\mathbb{E}$ is finitely complete since, by commutation of limits, it is easy to see that the finite limits in $Cat\mathbb{E}$ are built levelwise in $\mathbb{E}$. So, the forgetful functor $(\ )_0$ is left exact.

The functor $(\ )_0$ is actually a fibration whose cartesian maps are the internal fully faithful functors and whose maps in the fibers are the internal functors which are "identities on objects" (ido-functors for short).

It is clear that the fiber $Cat_1\mathbb{E}$ above the terminal object 1 is nothing but the category $Mon\mathbb{E}$ of internal monoids in $\mathbb{E}$. Any fiber $Cat_Y\mathbb{E}$ above an object Y, with $Y \neq 1$, has an initial object with the discrete equivalence relation Δ_Y and a terminal one with the indiscrete one ∇_Y. So, the left exact fully faithful functor $\nabla : \mathbb{E} \to Cat\mathbb{E}$ admits the fibration $(\ )_0$ as left adjoint and makes the pair $((\ )_0, \nabla)$ a *fibered reflection* in the sense of [2]. A functor $f_\bullet$ is then cartesian if and only if the following left hand side square is a pullback in $Cat\mathbb{E}$, or, equivalently the right hand side one is a pullback in $\mathbb{E}$:

$$
\begin{array}{ccc}
X_\bullet & \xrightarrow{f_\bullet} & Y_\bullet \\
\downarrow & & \downarrow \\
\nabla_{X_\bullet} & \xrightarrow[\nabla_{f_\bullet}]{} & \nabla_{Y_\bullet}
\end{array}
\qquad\qquad
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
{\scriptstyle (d_0,d_1)}\downarrow & & \downarrow{\scriptstyle (d_0,d_1)} \\
X_0 \times X_0 & \xrightarrow[f_0 \times f_0]{} & Y_0 \times Y_0
\end{array}
$$

As for any left exact fibration, we get:

Proposition 1.1. *1) The cartesian functors are stable under pullbacks.*

2) Given any commutative square in $Cat\mathbb{E}$ where both $x_\bullet$ and $y_\bullet$ are cartesian functors:

$$
\begin{array}{ccc}
X_\bullet & \xrightarrow{x_\bullet} & X'_\bullet \\
{\scriptstyle f_\bullet}\downarrow & & \downarrow{\scriptstyle f'_\bullet} \\
Y_\bullet & \xrightarrow[y_\bullet]{} & Y'_\bullet
\end{array}
$$

then it is a pullback:
1) if and only if its image by $(\ )_0$ is a pullback
2) in particular when $f_\bullet$ and $f'_\bullet$ are ido-functors.

Given any ido-functor $f_\bullet : X_\bullet \to Y_\bullet$, the following left hand side pullback inside the fiber $Cat_Y\mathbb{E}$ above Y will be called its *kernel*; it only retains the endomorphisms in $X_\bullet$ which are sent on identities in $Y_\bullet$; the pullback on the right hand side, introducing the kernel of the terminal map in the fiber $Cat_Y\mathbb{E}$, only retains what is called the *endosome* $(\mathrm{End}X)_\bullet$ of the internal category $X_\bullet$, namely the internal monoid in the slice category $\mathbb{E}/Y$ consisting in the only endomorphisms of $X_\bullet$:

$$\begin{array}{ccc} (\mathrm{Ker}f)_\bullet & \xrightarrow{(\ker f)_\bullet} & X_\bullet \\ \downarrow\uparrow & & \downarrow f_\bullet \\ \Delta_Y & \xrightarrow[0_{Y_\bullet}]{} & Y_\bullet \end{array} \qquad\qquad \begin{array}{ccc} (\mathrm{End}X)_\bullet & \xrightarrow{(\epsilon X)_\bullet} & X_\bullet \\ (\rho X)_\bullet \downarrow\uparrow (\sigma X)_\bullet & & \downarrow \\ \Delta_Y & \xrightarrow[0_Y]{} & \nabla_Y \end{array}$$

1.2 Natural transformations

We know that $Cat\mathbb{E}$ is actually underlying a 2-category with the notion of internal *natural transformations* between internal functors: in simplicial terms, they are just homotopies between the 3-truncated simplicial morphisms that are the internal functors. The cartesianness of the 1-arrows fits well with the 2-cells: an internal functor $f_\bullet : X_\bullet \to Y_\bullet$ is fully faithful if and only if, given any natural transformation $\gamma : f_\bullet.g_\bullet \Rightarrow f_\bullet.g'_\bullet$, there is a unique natural transformation $\bar{\gamma} : g_\bullet \Rightarrow g'_\bullet$ such that $\gamma_\bullet = f_\bullet.\bar{\gamma}_\bullet$. We have even better:

Proposition 1.2. *A split epimorphism $(f_\bullet, s_\bullet) : X_\bullet \rightleftarrows Y_\bullet$ in $Cat\mathbb{E}$ is cartesian if and only if it is a* strict left inverse equivalence*, namely it is such that there is a natural isomorphism $\gamma_\bullet : 1_{X_\bullet} \Rightarrow s_\bullet.f_\bullet$ satisfying $f_\bullet.\gamma_\bullet = 1_{f_\bullet}$ and $\gamma_\bullet.s_\bullet = 1_{s_\bullet}$ (which implies immediately $f_\bullet.s_\bullet = 1_{Y_\bullet}$):*

$$s_\bullet.f_\bullet \gamma \Leftarrow X_\bullet \underset{f_\bullet}{\overset{s^\bullet}{\rightleftarrows}} Y_\bullet$$

Proof. Suppose $f_\bullet : X_\bullet \to X_\bullet$ cartesian and split by $s_\bullet$. Accordingly, from the identity natural isomorphism between $f_\bullet$ and $f_\bullet = f_\bullet.s_\bullet.f_\bullet$, we get a natural isomorphism $\gamma : 1_{X_\bullet} \Rightarrow s_\bullet.f_\bullet$ such that $1_{f_\bullet} = f_\bullet.\gamma_\bullet$. From that we get $f_\bullet.(\gamma_\bullet.s_\bullet) = f_\bullet.s_\bullet = 1_{Y_\bullet}$, whence: $f_\bullet.s_\bullet = 1_{Y_\bullet}$.

Conversely, suppose we have a left inverse equivalence given by a natural isomorphism $\gamma_\bullet : 1_{X_\bullet} \Rightarrow s_\bullet.f_\bullet$. Starting with any natural transformation $\tau_\bullet : f_\bullet.g_\bullet \Rightarrow f_\bullet.g'_\bullet$, the natural transformation $\bar{\tau}_\bullet = (\gamma_\bullet g'_\bullet)^{-1}.s_\bullet\tau_\bullet.\gamma_\bullet g_\bullet : g_\bullet \Rightarrow g'_\bullet$ is the unique one such that $f_\bullet.\bar{\tau}_\bullet = \tau_\bullet$ □

In a way, the above proposition shows how the cartesian split epimorphisms in $Cat\mathbb{E}$ capture, as soon as level 1, a hidden invertible aspect of the 2-categorical level of the category $Cat\mathbb{E}$.

1.3 The regular context

In this section we shall suppose that $\mathbb{E}$ is a regular category [1]. We shall recall the effect of this further property on the fibration $(\ )_0$, see [2]. The category $Cat\mathbb{E}$ is certainly not itself a regular category; we already know how much characterizing the regular epimorphisms in Cat is complicated. However we can assert about $(\ )_0$ two very interesting and strong properties:

Proposition 1.3. *Let $\mathbb{E}$ be a regular category. Then:*
1) any fiber $Cat_Y\mathbb{E}$ is a regular category;
2) any cartesian functor $f_\bullet : X_\bullet \to Y_\bullet$ above a regular epimorphism f_0 in $\mathbb{E}$ is a pullback stable regular epimorphism in $Cat\mathbb{E}$.

Proof. 1) Since $\mathbb{E}$ is regular, the regular epimorphisms are stable under products in $\mathbb{E}$, and in any slice category $\mathbb{E}/Y$. So, the regular epimorphisms $f_\bullet : X_\bullet \to Y_\bullet$ in the fiber $Cat_Y\mathbb{E}$ are levelwise epimorphisms in $\mathbb{E}$: namely they are such that $f_0 = 1_Y$, and the pair (f_1, f_2) is a pair of regular epimorphisms. So, the fiber $Cat_Y\mathbb{E}$ is immediately a regular one.

2) From the above characterization of cartesian maps, when f_0 is a regular epimorphism in $\mathbb{E}$, so is f_1 as a pullback of the regular epimorphism $f_0 \times f_0$. From that, by commutation of limits, so is f_2. So, again, we get a levelwise regular epimorphism is $\mathbb{E}$. Now, let $h_\bullet : X_\bullet \to Z_\bullet$ be any functor annihilating the kernel equivalence relation $R[f_\bullet]$. This implies that h_0 and h_1 annihilate the kernel equivalence relations $R[f_0]$ and $R[f_1]$. Since f_0 and f_1 are regular epimorphisms in $\mathbb{E}$, we get unique factorizations $\bar{h}_0 : Y_0 \to Z_0$ and $\bar{h}_1 : Y_1 \to Z_1$ such that $\bar{h}_0.f_0 = h_0$ and $\bar{h}_1.f_1 = h_1$. Since f_1 is a regular epimorphism, the pair $(\bar{h}_0, \bar{h}_1)$ produces a morphism between the underlying reflexive graphs. On the other hand, this pair induces a map $\bar{h}_2 : Y_1 \times_0 Y_1 \to Z_1 \times_0 Z_1$. We check that $(\bar{h}_0, \bar{h}_1)$ is actually underlying an internal functor (i.e. internally respects the composition of morphisms) by composition with the regular epimorphism f_2. So, the functor $f_\bullet : X_\bullet \to Y_\bullet$ is a regular epimorphism in $Cat\mathbb{E}$. Clearly, when $\mathbb{E}$ is a regular category, this kind of functor is stable under pullbacks in $Cat\mathbb{E}$. □

1.4 First structural observation

We reached our first (from two) main structural observation (which was already made for the category $Grd\mathbb{E}$ of groupoids in $\mathbb{E}$ (Proposition 1 in [6])):

Theorem 1.4. *Any commutative square of split epimorphisms in $Cat\mathbb{E}$:*

$$\begin{array}{ccc} X_\bullet & \underset{\sigma_\bullet}{\overset{x_\bullet}{\rightleftarrows}} & X'_\bullet \\ {\scriptstyle f_\bullet}\downarrow\uparrow{\scriptstyle s_\bullet} & & {\scriptstyle f'_\bullet}\downarrow\uparrow{\scriptstyle s'_\bullet} \\ Y_\bullet & \underset{\tau_\bullet}{\overset{y_\bullet}{\rightleftarrows}} & Y'_\bullet \end{array}$$

where both $x_\bullet$ and $y_\bullet$ are cartesian and both $f_\bullet$ and $f'_\bullet$ are ido-functors is such that the pair $(s_\bullet, \sigma_\bullet)$ of subobjects of $X_\bullet$ in $Cat\mathbb{E}$ is jointly extremally epic, or, in other words, is such that their supremum as subobjects of $X_\bullet$ is nothing but $1_{X_\bullet}$.

Proof. First observe that, according to Proposition 1.1, this square is necessarily a pullback in $Cat\mathbb{E}$. On the other hand, thanks to the Yoneda embedding, it is enough to check the assertion in Cat. According with the notations of the previous proposition, consider, for any map ϕ, the following commutative diagram in the category $X_\bullet$:

$$\begin{array}{ccc} a & \xrightarrow{\gamma_\bullet(a)} & \sigma_\bullet x_\bullet(a) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \sigma_\bullet x_\bullet(\phi)} \\ b & \xrightarrow[\gamma_\bullet(b)]{} & \sigma_\bullet x_\bullet(b) \end{array}$$

Since the isomorphism $\gamma_\bullet(a)$ comes from $1_{x_\bullet(a)}$ and the pair $(s_\bullet, s'_\bullet)$ is a pair of ido-functors (i.e. the objects of $X_\bullet$ (resp. $X'_\bullet$) and $Y_\bullet$ (resp. $Y'_\bullet$) coincide), this isomorphism is nothing but the image by $s_\bullet$ of the isomorphism $\gamma_\bullet(a)$ in the category $Y_\bullet$. Consequently any subcategory $U_\bullet$ of $X_\bullet$ containing $Y_\bullet$ and $X'_\bullet$ contains $\gamma_\bullet(a)$, $\gamma_\bullet(b)$ and $\sigma_\bullet x_\bullet(\phi)$; so, it contains ϕ. □

Inspired by the knowledge of the Mal'tsev processes, we get the following:

Proposition 1.5. *Let $\mathbb{E}$ be a regular category. Any pullback in $Cat\mathbb{E}$:*

$$\begin{array}{ccc} X_\bullet & \xrightarrow{x_\bullet} & X'_\bullet \\ f_\bullet \downarrow \uparrow s_\bullet & & f'_\bullet \downarrow \uparrow s'_\bullet \\ Y_\bullet & \xrightarrow[y_\bullet]{} & Y'_\bullet \end{array}$$

where $y_\bullet$ is a cartesian regular epimorphism and $(f'_\bullet, s'_\bullet)$ a split epimorphic ido-functor is such that the upward square is a pushout.

Proof. Apply Proposition 2.4 below. □

1.5 A short remark about the composition of relations in $Cat\mathbb{E}$

The category Cat (and a fortiori $Cat\mathbb{E}$ even if $\mathbb{E}$ is regular) is not a regular one; so it is not possible in general to compose internal relations. Let us call *ido* (resp. *cartesian*) equivalence relation any equivalence relation $(d_0, d_1) : R_\bullet \rightrightarrows X_\bullet$ such that d_0 (and thus d_1) is an ido-functor (resp. a cartesian one).

Proposition 1.6. *Given any category $\mathbb{E}$, let $(R_\bullet, S_\bullet)$ be any pair of an ido and a cartesian equivalence relation on $X_\bullet$ in $Cat\mathbb{E}$. Then:*
1) $R_\bullet \cap S_\bullet = \Delta_{X_\bullet}$;
2) $R_\bullet$ and $S_\bullet$ are composable and permute.

Proof. By the Yoneda embedding it is enough to show that in Set. The first point is a consequence of the fact that if a parallel pair $(\phi, \psi) : x \rightrightarrows x' \in X_\bullet$ is in $R_\bullet \cap S_\bullet$, the fact that $d_0^{S_\bullet}$ is cartesian implies $\phi = \psi$. Now consider the square construction $R_\bullet \Box S_\bullet$ given by the largest double equivalence relation on $X_\bullet$ produced from $R_\bullet$ and $S_\bullet$ in $Cat\mathbb{E}$:

$$\begin{array}{ccc} R_\bullet \Box S_\bullet & \rightleftarrows & S_\bullet \\ \downarrow\uparrow\downarrow & & d_0^{S_\bullet} \downarrow\uparrow\downarrow d_1^{S_\bullet} \\ R_\bullet & \underset{d_0^{R_\bullet}}{\overset{d_1^{R_\bullet}}{\rightleftarrows}} & X_\bullet \end{array}$$

Then 1) implies that the canonical factorization $R_\bullet \square S_\bullet \to R_\bullet \times_{X_\bullet} S_\bullet$ to the following pullback is a monomorphism:

$$\begin{array}{ccc} R_\bullet \times_{X_\bullet} S_\bullet & \rightleftarrows & S_\bullet \\ \downarrow\uparrow & & \downarrow\uparrow d_0^{S_\bullet} \\ R_\bullet & \underset{d_1^{R_\bullet}}{\rightleftarrows} & X_\bullet \end{array}$$

So, according to Theorem 1.4, it is an isomorphism, whence 2). □

2. Mal'tsev fibration

In this section, we shall make explicit some formal aspects of the previous property of the fibered reflection $((\ )_0, \nabla)$.

A category $\mathbb{E}$ is said to be a *Mal'tsev* one [14, 15], when any reflexive relation is an equivalence relation. This is a categorical characterization of the Mal'tsev varieties, namely those ones which produce a Mal'tsev term, i.e. a ternary term p satisfying $p(x, y, y) = x = p(y, y, x)$ [19]. In [4] was produced the following characterization:

Theorem 2.1. *For any category $\mathbb{E}$, the following conditions are equivalent:*
1) $\mathbb{E}$ is a Mal'tsev category;
2) given any pullback of split epimorphisms in $\mathbb{E}$:

$$\begin{array}{ccc} X & \underset{\sigma}{\overset{x}{\rightleftarrows}} & X' \\ f\downarrow\uparrow s & & f'\downarrow\uparrow s' \\ Y & \underset{\tau}{\overset{y}{\rightleftarrows}} & Y' \end{array}$$

the pair (s, σ) of subobjects of X is jointly extremally epic.

So, it is legitimate to introduce the following:

Definition 2.2. *A fibration $U : \mathbb{C} \to \mathbb{D}$ is said to be a Mal'tsev fibration when it it is left exact and such that any square of split epimorphism:*

$$\begin{array}{ccc} X & \underset{\sigma}{\overset{x}{\rightleftarrows}} & X' \\ f\downarrow\uparrow s & & f'\downarrow\uparrow s' \\ Y & \underset{\tau}{\overset{y}{\rightleftarrows}} & Y' \end{array}$$

where both x and y are cartesian maps and both f and f' are inside a fiber is such that the pair (s, σ) of subobjects of X is jointly extremally epic,

Now, according to Theorem 1.4, our first structural observation becomes: *the fibered reflection $((\ )_0, \nabla)$ is a Mal'tsev one*. On the model of what happens for Mal'tsev categories we get the following characterization we shall need later on:

Lemma 2.3. *A left exact fibration $U : \mathbb{C} \to \mathbb{D}$ is a Mal'tsev one if and only if, for any square of split epimorphisms where y is cartesian and f' inside a fiber:*

$$\begin{array}{ccc} W & \underset{\check{\sigma}}{\overset{\check{x}}{\rightleftarrows}} & X' \\ {\scriptstyle \check{f}}\downarrow\uparrow{\scriptstyle \check{s}} & & {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} \\ Y & \underset{\tau}{\overset{y}{\rightleftarrows}} & Y' \end{array}$$

the induced factorization $(\check{f}, \check{\sigma}) : W \to X$ is an extremal epimorphism.

We can now easily generalize a well-known Mal'tsev type process with the following:

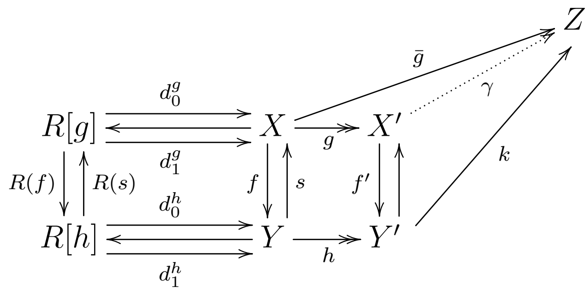
Proposition 2.4. *Let $U : \mathbb{C} \to \mathbb{D}$ be a Mal'tsev fibration. Suppose, in addition, that $\mathbb{D}$ is a regular category and that any cartesian map in $\mathbb{C}$ above a regular epimorphism in $\mathbb{D}$ is a regular epimorphism in $\mathbb{C}$. Then:*
1) this class Θ of regular epimorphisms in $\mathbb{C}$ is stable under pullbacks;
2) given such a regular epimorphism $h : Y \twoheadrightarrow Y'$ and any pullback in $\mathbb{C}$:

$$\begin{array}{ccc} X & \overset{g}{\twoheadrightarrow} & X' \\ {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} & & {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} \\ Y & \underset{h}{\twoheadrightarrow} & Y' \end{array}$$

where (f', s') is a split epimorphism inside a fiber, the upward square is a pushout.

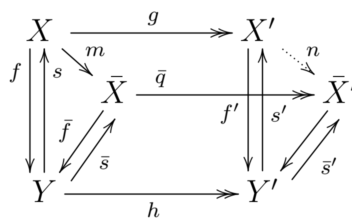
Accordingly, pulling back the split epimorphisms in the fibers along the regular epimorphism h in $\mathbb{C}$ is a fully faithful process.

Proof. The first point is straightforward as soon as the fibration is left exact. Now, since h is a cartesian regular epimorphism in $\mathbb{C}$, so is g. Consider any pair $(\bar{g}, k)$ of maps in $\mathbb{C}$:



such that $h.k = \bar{g}.s$ $(*)$. Complete the diagram with the horizontal kernel equivalence relations. Now, it is clear that we shall get the desired dotted factorization γ if and only if the map $\bar{g}$ coequalizes the pair (d_0^g, d_1^g). The left hand side squares are pullbacks, since so is the right hand side one. Accordingly, the pair $(R(s), s_0^g)$ is jointly extremally epic. So, the coequalization in question can be checked by composition with s_0^g (straightforward) and with $R(s)$, which is a direct consequence of $(*)$.

The pulling back in question is clearly faithful since it is pulling back along pullback stable regular epimorphisms. As for the fullness, consider the following diagram where the two quadrangles are pullbacks of split epimorphisms in the fibers along the cartesian regular epimorphism h in $\mathbb{C}$ and where m is any morphism of split epimorphisms:

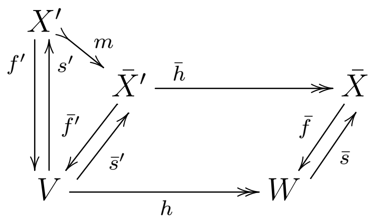


The commutative square of split epimorphims being a pullback, the upward square towards X' is a pushout; so, the map m produces the desired dotted factorization n. □

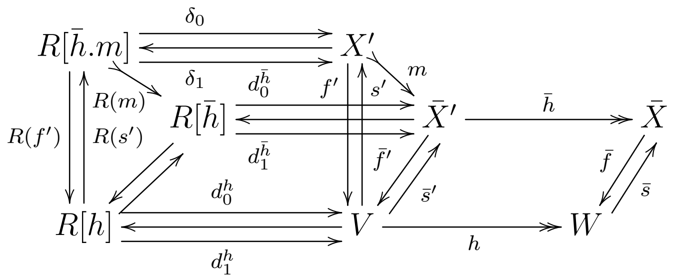
When we have a cartesian split epimorphism the result if even stronger:

Proposition 2.5. *Let $U : \mathbb{C} \to \mathbb{D}$ be a Mal'tsev fibration. Then, given any cartesian split epimorphism $(h, t) : V \rightleftarrows W$ in $\mathbb{C}$, pulling back along it* *the split epimorphisms in the fibers of U is a process which is "saturated on subobject".*

Proof. This means that, given any subobject m of $(\bar{f}', \bar{s}') = h^*(\bar{f}, \bar{s})$ in the fiber above $U(V)$:

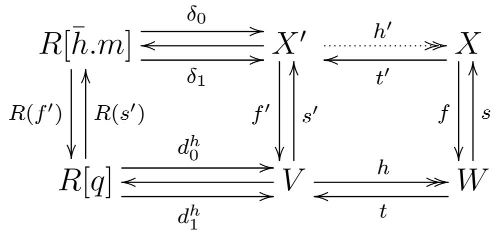


there is a subobject n of $(\bar{f}, \bar{s})$ in the fiber above $U(W)$ such that $h^*(n) = m$. For that, complete the diagram with the kernel equivalence relations $R[h]$, $R[\bar{h}]$ and $R[\bar{h}.m]$. The factorization $R(m)$ between the two last ones is then a monomorphism.



The left hand side quadrangles indexed by 0 and 1 are pullbacks since so is the right hand side one. In the context of a Mal'tsev fibration, the left hand side commutative vertical squares are pullbacks as well: indeed, since $R(m)$ is a monomorphism, it is also the case for the factorization τ of the left hand side vertical square indexed by 0 to the pullback of (f', s') along the split epimorphism (d_0^h, s_0^h); but this τ is an extremal epimorphism as well by Lemma 2.3, since the fibration is a Mal'tsev one; so this factorization τ is an isomorphism, and the vertical left hand side square indexed by 0 is a pullback.

Now, the following downward left hand side diagram is underlying a discrete fibration between equivalence relations:



Then, by Lemma 2.2 in [7], the above right hand side pullback along t gives to t' a retraction h' above h which is the quotient of the equivalence relation $R[\bar{h}.m]$; accordingly, $n = t^*(m)$ produces the monomorphism we were looking for. □

3. Schreier split epimorphims in the fibers $Cat_Y\mathbb{E}$

3.1 Schreier split epimorphims

It is clear that all the previous results concerning the fibered reflection $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ remain valid for its restriction $(\ )_0 : Grd\mathbb{E} \to \mathbb{E}$ to the internal groupoids in $\mathbb{E}$. One main structural fact estalished for this last fibration is the following one: any fiber $Grd_Y\mathbb{E}$ is protomodular [3], as is $Grd_1\mathbb{E} = Gp\mathbb{E}$ the category of internal groups in $\mathbb{E}$. This property does not hold in the fibers $Cat_Y\mathbb{E}$, see $Mon\mathbb{E}$. The aim of this section is to show that, however, any fiber $Cat_Y\mathbb{E}$ keeps one fraction of this property: it is only protomodular with respect to a certain class Σ_Y of split epimorphisms, on the model of what is shown for the categories Mon of monoids in [12] and $Mon\mathbb{E}$ of internal monoids in $\mathbb{E}$ in [13].

In the category Mon of monoids, a split epimorphism $(f, s) : A \rightleftarrows B$ is said to be a *Schreier* one if, for any $b \in B$, the map $\mu_b\colon \mathrm{Ker}\, f \to f^{-1}(b)$ defined by $\mu_b(k) = k \cdot s(b)$ is bijective [20]. This defines a class Σ of split epimorphims in Mon which makes this category Σ-protomodular [12, 10], where:

Definition 3.1. *A pointed category $\mathbb{E}$ is said to be Σ-protomodular, when,* *given any split epimorphism $(f, s) \in \Sigma$, the following pullback:*

$$\begin{array}{ccc} \mathrm{Ker} f & \overset{\ker f}{\rightarrowtail} & X \\ \downarrow\uparrow & & f \downarrow\uparrow s \\ 1 & \underset{0_Y}{\rightarrowtail} & Y \end{array}$$

is such that the pair $(s, \ker f)$ of subobjects of X is jointly extremally epic, namely is such that the supremum of this pair of subobjects is 1_X.

This definition extends the notion of pointed *protomodular category* [3] where the previous property holds for any split epimorphism in $\mathbb{E}$, the major examples of such categories being the category $Gp = Grd_1$ of groups. Recall the following observation which produces a global characterization of the Schreier split epimorphisms in Mon [12]:

Proposition 3.2. *A split epimorphism $(f, s) : X \rightleftarrows Y$ in Mon is a Schreier one if and only if there is a set-theoretical retraction $q : X \to \mathrm{Ker} f$ to the homomorphic inclusion* $\ker f$*:*

$$\begin{array}{ccc} \mathrm{Ker} f & \underset{\ker f}{\rightarrowtail} & X \\ \downarrow\uparrow & & f \downarrow\uparrow s \\ 1 & \underset{0_Y}{\rightarrowtail} & Y \end{array} \qquad q : X \dashrightarrow \mathrm{Ker} f$$

which, in addition, is such that, for all $x \in X$, we get $q(x) \cdot sf(x) = x$ and, for all $(k, y) \in \mathrm{Ker} f \times Y$, we get $q(k \cdot s(y)) = k$.

From that, it is not difficult to extend this definition to any fiber $Cat_Y \mathbb{E}$:

Definition 3.3. *Let $(f_\bullet, s_\bullet) : X_\bullet \rightleftarrows Y_\bullet$ be any split epimorphic ido-functor in the fiber $Cat_Y \mathbb{E}$. It is called a Schreier split epimorphism when there is a retraction $q_1 : X_1 \to (\mathrm{Ker} f)_1$ of $(\ker f)_1$ in $\mathbb{E}$:*

$$\begin{array}{ccc} (\mathrm{Ker} f)_1 & \underset{(\ker f)_1}{\rightarrowtail} & X_1 \\ \downarrow\uparrow & & f_1 \downarrow\uparrow s_1 \\ Y & \underset{s_0^{Y_\bullet}}{\rightarrowtail} & Y_1 \end{array} \qquad q_1 : X_1 \dashrightarrow (\mathrm{Ker} f)_1$$

such that: 1) $d_0^{X_\bullet}.(\ker f)_1.q_1 = d_1^{X_\bullet}$*, 2)* $d_1^{X_\bullet}.(s_1 f_1, q_1) = 1_{X_1}$ *and:*
3) $q_1.d_1^{X_\bullet}.(s_1 \times_Y (\ker f)_1) = p_0^K : (\mathrm{Ker} f)_1 \times_Y Y_1 \to (\mathrm{Ker} f)_1$.

The set-theoretical translations of these three equations are:
1) for any map $\phi : a \to b$ in the category $X_\bullet$, we get an endomap map: $q_1(\phi) : b \to b$ in $(Kerf)_1$ such that $\phi = q_1(\phi).s_1 f_1(\phi)$;
2) for any pair (ψ, α) of arrows in $Y_1 \times (\mathrm{Ker} f)_1$ with $d_1(\psi) = d_0(\alpha) = d_1(\alpha)$, we get: $q_1(\alpha.s_1(\psi)) = \alpha$.

In other words, we get a Schreier split epic ido-functor when, given any pair (a, b) of objects in the category $X_\bullet$, the monoid of endomorphisms on b belonging to the kernel of this functor $f_\bullet$ produces a special kind of action on the subsets of $Hom(a, b)$ whose elements have a same image by $f_\bullet$, an action which will be more precisely understood in Proposition 3.6. We shall denote by Σ_Y the class of Schreier split epimorphisms in the fiber $Cat_Y\mathbb{E}$. This class has good stability properties:

Proposition 3.4. *In* $Cat_Y\mathbb{E}$*, the class* Σ_Y *is stable under pullbacks and under composition. It is "point-congruous": namely, it is stable under products and under finite limits inside the category* $Pt(Cat_Y\mathbb{E})$ *of split epimorphims in* $Cat_Y\mathbb{E}$*.*

Here is now our second structural fact:

Theorem 3.5. *Any fiber* $Cat_Y\mathbb{E}$ *is* Σ_Y*-protomodular, i.e. it is such that, given any Schreier split epic ido-functor* $(f_\bullet, s_\bullet) : X_\bullet \to Y_\bullet$*, the following pullback in* $Cat_Y\mathbb{E}$*:*

$$\begin{array}{ccc} (\mathrm{Ker} f)_\bullet & \overset{(\ker f)_\bullet}{\rightarrowtail} & X_\bullet \\ \downarrow\uparrow & & {\scriptstyle f_\bullet}\downarrow\uparrow{\scriptstyle s_\bullet} \\ \Delta_Y & \underset{0_{Y_\bullet}}{\rightarrowtail} & Y_\bullet \end{array}$$

makes jointly extremally epic the pair $(s_\bullet, (\ker f)_\bullet)$ *of subobjects of* $X_\bullet$*.*

Proof. Thanks to the Yoneda embedding, we are allowed to check it in the set-theoretical environment, and it is quasi-immediate. Consider the following diagram in the category $X_\bullet$ for any map $\phi : a \to b$:

$$\begin{array}{ccc} a & \xrightarrow{s_\bullet f_\bullet(\phi)} & b \\ \| & & \downarrow{\scriptstyle q_1(\phi)} \\ a & \xrightarrow[\phi]{} & b \end{array}$$

It is commutative by the axiom 1, so that any subcategory of $X_\bullet$ containing $(\ker f)_\bullet$ and $s_\bullet$ is equal to $X_\bullet$. □

3.2 Back again to the 2-categorical nature of $Cat\mathbb{E}$

In this section, we shall show how, actually, the notion of Schreier split epimorphisms is, again in a hidden way, related to the 2-categorical dimension of $Cat\mathbb{E}$. More than that, this notion, which is not intrinsic to $Mon\mathbb{E}$ by the presence of the non-homomorphic retraction q, becomes intrinsic, and more precisely intrinsic to this 2-dimensional aspect, when it is contextualized in $Cat\mathbb{E}$.

A functor $f_\bullet : X_\bullet \to Y_\bullet$ in Cat is a *cofibration*, when, given any pair $(a, \psi) \in X_0 \times Y_1$ with $d_0(\phi) = f_0(a)$, there is a universal map with domain a above it in X_1, which is called the *cocartesian* map above ψ. It is a *split cofibration* when the choice of these universal maps is enforced. In [22], it is shown that the split (co-)fibrations above $Y_\bullet$ in $Cat\mathbb{E}$ are clearly internally defined, as being the algebras of a left exact monad on the slice category $Cat\mathbb{E}/Y_\bullet$ which explicitely uses the natural transformations (via the notion of "comma category") and, therefore, is wholly based on the 2-categorical nature of $Cat\mathbb{E}$.

Now consider a ido-functor $f_\bullet : X_\bullet \to Y_\bullet$; if, in addition it is a split cofibration in Cat, the choice of the cocartesian maps determines an actual functorial splitting $s_\bullet : Y_\bullet \to X_\bullet$ of $f_\bullet$ which is such that any map $s_1(\psi)$ is the chosen cocartesian above ψ.

Proposition 3.6. *Let $(f_\bullet, s_\bullet) : X_\bullet \rightleftarrows Y_\bullet$ be a split epimorphism in a fiber $Cat_Y\mathbb{E}$. The following conditions are equivalent:*
1) it is a Schreier split epimorphism in $Cat_Y\mathbb{E}$;
2) it is an internal split ido-cofibration in $Cat\mathbb{E}$.

Proof. Let us check it in the set-theoretical environment. Suppose it is a Schreier split epimorphism. Start with a map $\psi : a \to b \in Y_1$; we are going to show that $s_1(\psi) : a \to b$ is a cocartesian map above ψ. So, let $\phi : a \to c$ be a map in X_1 with a factorization $f_1(\phi) = \chi.\psi$ in Y_1. Then, with the map $q_1(\phi).s_1(\chi) : b \to c$, we get the unique map in X_1 such that $\phi = (q_1(\phi).s_1(\chi)).s_1(\psi)$ and $f_1(q_1(\phi).s_1(\chi)) = \chi$.

Conversely suppose it is an internal split ido-cofibration. Since $s_1(\psi)$ is cocartesian, it determines, for any map $\phi : a \to b \in X_1$ above ψ a unique factorization $q_1(\phi) : b \to b$ such that $f_1(q_1(\phi)) = 1_a$ and $q_1(\phi).s_1(\psi) = \phi$; the first equality insuring that $q_1(\phi)$ is in $(\mathrm{Ker} f)_1$ and the second one insuring Axiom 1). Axiom 2) is then straighforward. □

By duality, let us define by Σ_Y^{op} the class of split ido-fibrations in $Cat_Y\mathbb{E}$. It is clear that this class is stable under pullback, point-congruous and makes the fiber $Cat_Y\mathbb{E}$ a Σ_Y^{op}-protomodular category as well.

4. Outcomes of the partial protomodularity of $Cat_Y\mathbb{E}$

So, any fiber $Cat_Y\mathbb{E}$ inherites all the properties of a Σ-protomodular category, see [10]. Here, we shall develop some of them. The first one is the following:

Proposition 4.1 ([10]). *Any fiber $Cat_Y\mathbb{E}$ is a Σ_Y-Mal'tsev category; i.e. when the split epimorphism $(f'_\bullet, s'_\bullet)$ is a Schreier one, any pullback of split epimorphisms in $Cat_Y\mathbb{E}$:*

$$\begin{array}{ccc} X_\bullet & \underset{\sigma_\bullet}{\overset{x_\bullet}{\rightleftarrows}} & X'_\bullet \\ f_\bullet \downarrow \uparrow s_\bullet & & f'_\bullet \downarrow \uparrow s'_\bullet \\ Y_\bullet & \underset{\tau_\bullet}{\overset{y_\bullet}{\rightleftarrows}} & Y'_\bullet \end{array}$$

is such that the pair $(s_\bullet, \sigma_\bullet)$ of subobjects of $X_\bullet$ is jointly extremally epic.

Then let us introduce the following:

Definition 4.2. *Let Σ be any class of split epimorphims in a category $\mathbb{E}$. A reflexive relation R on an object X: $R \underset{d_0^R}{\overset{d_1^R}{\rightrightarrows}} X$ is said to be a Σ-one, when the split epimorphism (d_0^R, s_0^R) is in Σ, a morphism $f : X \to Y$ is*

said to be Σ-special when its kernel equivalence relation $R[f]$ is a Σ-one. An object X is said to be Σ-special when the terminal map $\tau_X : X \to 1$ is Σ-special or, equivalently, when its indiscrete equivalence relation ∇_X is a Σ-one. The same kind of definition can be extended to reflexive graphs, and to internal categories and groupoids.

Warning: a split Σ-special morphism belongs to Σ, but the converse is not necessarilty true: a split epimorphism belonging to Σ is not necessarily Σ-special. In Mon, the natural preorder on the commutative monoid $\mathbb{N}$ of intergers is an emblematic example of a reflexive and transitive Schreier reflexive relation [12]. More generally, any preorder on a group G provides us with an example of a reflexive and transitive Schreier relation in Mon. We shall denote by $SPoMon$ the category of Schreier preordered monoids and order-preserving homomorphisms.

Theorem 4.3 ([10]). *Suppose the class Σ is stable under pullback and $\mathbb{E}$ is a Σ-Mal'tsev category. Then any reflexive (resp. reflexive and symmetric) Σ-relation is transitive (resp. an equivalence relation). When, in addition, Σ is point congruous, the full subcategory SP/Y of the slice category $\mathbb{E}/Y$ whose objects are the Σ-special morphims is a Mal'tsev category. It is the case, in particular of the Σ-core, namely the full subcategory $\Sigma\mathbb{E}_\sharp$ of $\mathbb{E}$ whose objects are Σ-special.*

When $\mathbb{E}$ is Σ-protomodular and Σ is point congruous, then SP/Y and $\Sigma\mathbb{E}_\sharp$ are protomodular.

In Mon, the core associated with the class of Schreier split epimorphisms is the protomodular subcategory Gp of groups [12]. In this section, inter alia, we shall characterize the objects of the protomodular core associated with the class Σ_Y in the fiber $Cat_Y\mathbb{E}$.

First, given any category $Y_\bullet$, its terminal map in the fiber $Y_\bullet \to \nabla_Y$ is a monomorphism if and only if $Y_\bullet$ is a preorder on Y. Since any monomorphism is necessarily Σ_Y-special, then any internal preorder on Y, seen as an internal category in $\mathbb{E}$, lies in the Σ_Y-core.

A reflexive relation R on a category $X_\bullet$ is given by a reflexive relation on each $Hom(a,b)$ which is stable under composition in $X_\bullet$. *It is a Schreier reflexive relation* if and only if for any pair (ψ,χ) of parallel arrows between a and b, we have $\psi R \chi$ if and only if there is a unique map $\phi : b \to b$ such that $1_b R \phi$ and $\chi = \phi.\psi$.

Accordingly, an equivalence relation R on a category $X_\bullet$ is a Schreier one if and only if:
1) for any object a, the class $\bar{1}_a$ is a subgroup of End_a;
2) the left action of the group $\bar{1}_b$ on any $Hom(a,b) \neq \emptyset$ is free and the classes of the equivalence relation R on $Hom(a,b)$ coincide with the orbits of this action. Whence the following definition which characterizes those categories which are in the protomodular Σ_Y-core (resp. the Σ_Y^{op}-core):

Definition 4.4. *An internal category $Y_\bullet$ is called Schreier special (respectively Schreier opspecial) when:*
1) the endosome $(EndY)_1 \rightleftarrows Y$ is a group in $\mathbb{E}/Y$; in other words, any endomorphism in $Y_\bullet$ is an automorphism;
2) the natural left action (resp. right action) of this group on the object $d_1^{Y_\bullet} : Y_1 \to Y$ (resp. $d_0^{Y_\bullet} : Y_1 \to Y$) in the slice category $\mathbb{E}/Y$ is simply transitive in this slice category.

Accordingly, as expected, the groupoids are Schreier-special (respectively Schreier opspecial) categories, but they are not the only ones, since, as we just saw, such is any preorder. On the other hand, the core, being protomodular, is a Mal'tsev category. Accordingly there is an intrinsic notion of *affine object* when this object is endowed with a (unique) internal Mal'tsev operation. A Schreier special category is an affine object in the protomodular core if and only if the group defined in 1) is an abelian one. In this way, any preorder appears as an affine Schreier special category.

We shall now briefly introduce some easy processes to produce Schreier split epimorphisms and Schreier special categories in the set-theoretical context.

Let Y be any set and $Y_\bullet$ any category with Y as set objects. Then, by the Grothendiek construction, any functor $F_\bullet : Y_\bullet \to Mon$ produces a Schreier split epimorphism $(U_\bullet, S_\bullet) : (FY)_\bullet \rightleftarrows Y_\bullet$ where the set objects of $(FY)_0$ is Y, and where a map $a \to b$ in $(FY)_1$ is given by a pair (ψ, α) with $\psi : a \to b \in Y_1$ and $\alpha \in F(b)$ and where the composition in the category $(FY)_\bullet$ is given by: $(\psi', \alpha').(\psi, \alpha) = (\psi'.\psi, \alpha' \cdot F(\psi')(\alpha))$. The identity map on the object a in $(FY)_0 = Y$ is given by $(1_a, 0_{F(a)})$; then we set $U_\bullet(\psi, \alpha) = \psi$ and $S_\bullet(\psi) = (\psi, 0_{F(b)})$.

Now, in the same way, starting with any functor $\bar{F}_\bullet : Y_\bullet \to SPoMon$, and denoting by $F_\bullet : Y_\bullet \to Mon$ the associated functor which forgets the preorders, then $\bar{F}_\bullet$ produces a Schreier preorder on the category $(FY)_\bullet$, by $(\psi,\alpha) \leq (\psi',\alpha')$ if and only if $\psi' = \psi$ and $\alpha \leq \alpha'$. This is the case, in particular, when $\bar{F}$ is chosen as the constant functor on the monoid $\mathbb{N}$.

Proposition 4.5. *Let $Y_\bullet$ be a Schreier special category and $F_\bullet : Y_\bullet \to Gp$ any functor, then the category $(FY)_\bullet$ is a Schreier special category as well. Accordingly, when $\mathbb{T}$ is a preorder on the set Y, any category $(F\mathbb{T})_\bullet$ of this kind is a Schreier special category. It is affine as soon as the functor F takes its values in the category Ab of abelian groups.*

Proof. Since $Y_\bullet$ is a Schreier special category, any $Hom_{Y_\bullet}(a,a)$ is a group. On the other hand, the restriction of the functor $F_\bullet$ to this group produces a group homomorphism $F_a : Hom_{Y_\bullet}(a,a) \to Aut(F(a))$. Then, being nothing but the semi-direct product $F(a) \rtimes Hom_{Y_\bullet}(a,a)$, any $Hom_{(FY)_\bullet}(a,a)$ is a group.

Now let $((\psi,x),(\chi,y))$ be a parallel pair of morphims between a and b in $(FY)_\bullet$. Since $Y_\bullet$ is a Schreier special category, there is a unique inversible map $\alpha : b \to b$ such that $\alpha.\psi = \chi$. Then the map $(\alpha, y \cdot F(\alpha)(x^{-1}))$ in $Hom_{(FY)_\bullet}(b,b)$ is the unique one such that: $(\alpha, y \cdot F(\alpha)(x^{-1})).(\psi,x) = (\chi,y)$. The last assertion is straightforward once recalled that any preorder is an affine Schreier special category. □

In this way, any group homomorphism $h : G \to G'$, seen as a functor: $\{0 \to 1\} \to Gp$ gives rise to a Schreier special category with two objects which is neither a preorder, nor a groupoid. It is an affine object in the protomodular core when both G and G' are abelian.

We shall close this section by just recalling two important consequences of the Σ_Y-protomodularity of the fibers $Cat_Y\mathbb{E}$, and we shall refer to [10] and [5] for the details:
1) any regular epic Σ_Y-special ido-functor $f_\bullet : X_\bullet \twoheadrightarrow Y_\bullet$ in $Cat_Y\mathbb{E}$ is the cokernel of its kernel;
2) there is, in $Cat_Y\mathbb{E}$, an intrinsic notion of *abelian* Σ_Y-equivalence relation R. When, in addition, the ground category $\mathbb{E}$ is exact, we can associate with any Σ_Y-special ido-extension $f_\bullet : X_\bullet \twoheadrightarrow Y_\bullet$ having an abelian kernel equivalence relation an internal abelian group $A_\bullet \rightleftarrows Y_\bullet$ in $(Cat_Y\mathbb{E})/Y_\bullet$ called the *direction* of this extension $f_\bullet$. Furthermore the set $Ext_{A_\bullet}(Y_\bullet)$ of isomorphism classes of such extensions above $Y_\bullet$ with a given direction $A_\bullet \rightleftarrows Y_\bullet$

is canonically endowed with an abelian group structure via a construction generalizing the classical Baer sum.

5. Some p recisions about the fibration $Grd\mathbb{E} \to \mathbb{E}$

When $\mathbb{E}$ is regular, the base-changes of the fibration $Cat\mathbb{E} \to \mathbb{E}$ along a regular epimorphism $f : X \twoheadrightarrow Y$ in $\mathbb{E}$ are fully faithful, and thus conservative, since the cartesian maps above regular epimorphisms in $\mathbb{E}$ are pullback stable regular epimorphisms in $Cat\mathbb{E}$; it is a fortiori the case for the same kind of base-changes of the fibration $Grd\mathbb{E} \to \mathbb{E}$.

In this section we shall show that, for $Grd\mathbb{E}$, the base-changes along a morphism $h : U \to X$ in $\mathbb{E}$ partially keep a conservative aspect, provided that the objects U and X have same support, and the ground category $\mathbb{E}$ is efficiently regular (and a fortiori exact), see Theorem 5.7 and Proposition 5.13 below. For that we need the following:

Definition 5.1 ([8]). *A category $\mathbb{E}$ is said to be efficiently regular when it is regular and such that any equivalence relation T on an object X which is a subobject $i : T \rightarrowtail R[f]$ of an effective equivalence relation by an effective monomorphism (i.e. an equalizer) i is itself effective.*

This notion is clearly stable under slicing and coslicing. The categories $GpTop$ and $AbTop$ of topological groups and topological abelian groups are examples of efficiently regular categories which are not exact ones.

5.1 Connected and aspherical internal groupoids

From now on, in this section, we shall suppose that the category $\mathbb{E}$ is at least regular. In such a category, the support of an object X is the subobject $J \rightarrowtail 1$ determined by the canonical decomposition of the terminal map $\tau_X : X \to 1$. Accordingly, an object X is said to have a global support when the terminal map $\tau_X : X \to 1$ is a regular epimorphism.

Then, since any fiber $Grd_Y\mathbb{E}$ is a regular category as well, any groupoid has a support in its fiber: $X_\bullet \twoheadrightarrow SuppX_\bullet \rightarrowtail \nabla_{X_0}$, and this support is an equivalence relation in $\mathbb{E}$. Let us recall the following:

Definition 5.2 ([6]). *A groupoid $X_\bullet$ is said to be connected when it has a global support in the fiber $Grd_{X_0}\mathbb{E}$; it is said to be aspherical, when, in* *addition, it object of objects X_0 has a global support in $\mathbb{E}$. When the equivalence relation $SuppX_\bullet$ has a quotient map $\gamma_{X_\bullet}$ in $\mathbb{E}$, its codomain, denoted by $\pi_0(X_\bullet)$, is called the internal object of the "connected components" of the groupoid $X_\bullet$. Then clearly $\gamma_{X_\bullet}$ is the coequalizer of the pair $(d_0^{X_\bullet}, d_1^{X_\bullet})$ in $\mathbb{E}$.*

In Set a groupoid $X_\bullet$ is aspherical when $X_0 \neq \emptyset$ and the groupoid $X_\bullet$ is connected in the usual sense. The connected groupoids are stable under the base-changes f^* of the fibration $Grd\mathbb{E} \to \mathbb{E}$. Aspherical ones are stable only when the domain of f has a global support.

5.2 Affine internal groupoids

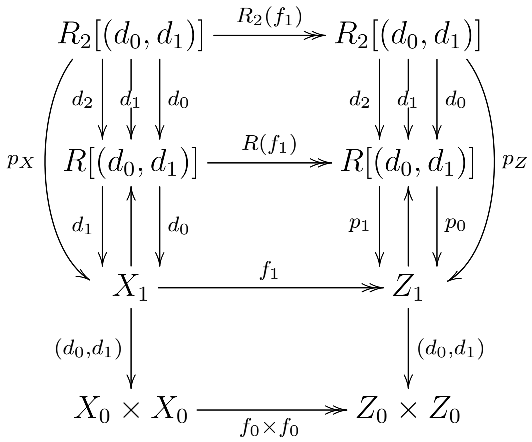
Now, observe that, given any internal groupoid $X_\bullet$ in $\mathbb{E}$, the object $(d_0^{X_\bullet}, d_1^{X_\bullet}) : X_1 \to X_0 \times X_0$ in the slice category $\mathbb{E}/(X_0 \times X_0)$ (which is nothing but the level 1 of the terminal functor $X_\bullet \to \nabla_X$ in the fiber $Grd_{X_0}\mathbb{E}$) is canonically endowed with an *associative Mal'tsev operation* p defined (in set-theoererical terms) by $p(\phi, \chi, \psi) = \phi.\chi^{-1}.\psi$ for any triple of parallel maps in $X_\bullet$. This Mal'tsev operation will be a keypoint in the development below. Observe moreover that: $(*)$ $p(\phi.\beta, \chi.\beta, \psi.\beta) = p(\phi, \chi, \psi).\beta$ and $p(\gamma.\phi, \gamma.\chi, \gamma.\psi) = \gamma.p(\phi, \chi, \psi)$.

Definition 5.3. *The groupoid $X_\bullet$ is said to be* affine *in $\mathbb{E}$ when the ternary operation p is autonomous or, equivalently, when p is underlying an internal functor $p_\bullet : X_\bullet \times_0 X_\bullet \times_0 X_\bullet \to X_\bullet$ in the fiber $Grd_{X_0}\mathbb{E}$.*

The same notion was introduced under the name of *abelian* groupoid in [6], but we do prefer now *affine*, since the above second assertion exactly means that $X_\bullet$ is an affine object in the protomodular (whence Mal'tsev) fiber $Grd_{X_0}\mathbb{E}$. Any equivalence relation is an affine groupoid.

Proposition 5.4. *Given any fully faithful (=cartesian) functor $f_\bullet : X_\bullet \to Z_\bullet$, the groupoid $X_\bullet$ is affine as soon as so is $Z_\bullet$. If, in addition, $f_\bullet$ is split, $X_\bullet$ is affine if and only if so is $Z_\bullet$. When the category $\mathbb{E}$ is regular, the same equivalence holds for any cartesian regular epimorphism $f_\bullet$.*

Proof. The first point is straightforward, and the second too, since when a cartesian functor $f_\bullet$ is split, its splitting $s_\bullet$ is cartesian as well. Let us go to the third one. Let $f_\bullet : X_\bullet \twoheadrightarrow Z_\bullet$ be any cartesian regular epimorphism and $X_\bullet$ an affine groupoid. Then consider the following diagram in $\mathbb{E}$:



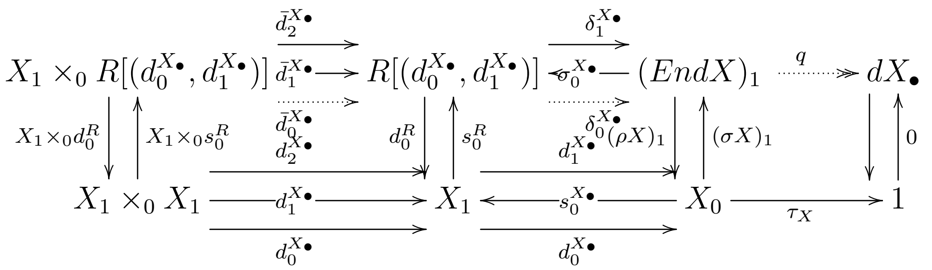
The lower commutative square is a pullback since $f_\bullet$ is cartesian. Accordingly so are the upper ones. Since f_0 is a regular epimorphism, so are f_1 and the factorizations $R(f_1)$ and $R_2(f_1)$. Clearly the Mal'tsev operations p commute with any functor $f_\bullet$. Now, when p_X is autonomous (=$X_\bullet$ affine), so is p_Z since $R_2(f_1)$ is a regular epimorphism; so, $Z_\bullet$ is affine. □

5.3 The direction of an affine aspherical groupoid

In Set, given an affine groupoid $X_\bullet$, all the maps $\phi : x \to x'$ produce the same group homomorphism $\alpha \mapsto \phi.\alpha.\phi^{-1}$ between the groups End_x and $End_{x'}$. When, in addition, this groupoid is aspherical all these groups End_x are isomorphic; so, by the choice of an object x_0 and of a map $\phi_x : x \to x_0$ for all x, we get a canonical equivalence of categories $X_\bullet \simeq End_{x_0}$; in this way, the (=any) abelian group End_{x_0} becomes a meaningful invariant of this affine aspherical groupoid $X_\bullet$.

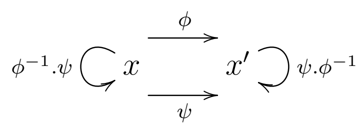
We are now going to recall from [6] how to define this invariant, called the *direction* of the aspherical affine groupoid $X_\bullet$, in an internal way. For that, we shall need the following kind of *anatomical decomposition of what is an internal groupoid* $X_\bullet$ which will consist in showing that *the upper horizontal part of the following diagram is again an internal groupoid* resulting of what we shall call:

the canonical action of the groupoid $X_\bullet$ on its endosome $(EndX)_\bullet$:



Proof. In the central part of this diagram, given any parallel pair $(\phi, \psi) \in R[(d_0^{X_\bullet}, d_1^{X_\bullet})]$ of morphisms of $X_\bullet$, we set $\delta_1^{X_\bullet}(\phi, \psi) = \psi.\phi^{-1}$. It is then straightforward to check that the downward square with horizontal maps indexed by 1 is a pullback, and then that $s_0^{X_\bullet}$ induces the retraction of $\delta_1^{X_\bullet}$ defined by $\sigma_0^{X_\bullet}(\alpha) = (1_{d_0(\alpha)}, \alpha)$. Being a pullback, this square transfers the group structure in the slice category $\mathbb{E}/X_0$ given by the endosome $(EndX)_\bullet$ to a group structure on the object d_0^R in $\mathbb{E}/X_1$ which is given by $(\phi, \psi) \circ (\phi, \chi) = (\phi, p(\psi, \phi, \chi))$.

Now let us set: $\delta_0^{X_\bullet}(\phi, \psi) = \phi^{-1}.\psi$. Then the downward and upward squares with horizontal maps indexed by 0 do commute as well, as shown by the following diagram:

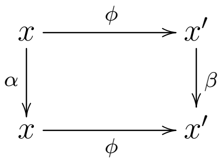


This makes the map $\delta_0^{X_\bullet}$ satisfy the first axiom of an action of the groupoid $X_\bullet$ on the split epimorphism $((\rho X)_1, (\sigma X)_1)$. Furthermore this map $\delta_0^{X_\bullet}$ respects the group laws of the slice categories since:
$\delta_0^{X_\bullet}(\phi, \psi) \circ \delta_0^{X_\bullet}(\phi, \chi) = (\phi^{-1}.\psi) \circ (\phi^{-1}.\chi)) = \phi^{-1}.\psi.\phi^{-1}.\chi = \phi^{-1}.p(\psi, \phi, \chi)$;
while: $\delta_0^{X_\bullet}((\phi, \psi) \circ (\phi, \chi)) = \delta_0^{X_\bullet}(\phi, p(\psi, \phi, \chi)) = \phi^{-1}.p(\psi, \phi, \chi)$.
So, the map $\delta_0^{X_\bullet}$ satisfies the first axiom of an action of the groupoid $X_\bullet$ on the endosome group $(EndX)_\bullet$.

We are going to show now that it satisfies the second axiom of an action, namely that, in the diagram above, we get $\delta_0^{X_\bullet}.\bar{d}_0^{X_\bullet} = \delta_0^{X_\bullet}.\bar{d}_1^{X_\bullet}$, where $\bar{d}_0^{X_\bullet}(\gamma, (\phi, \psi)) = (\gamma, \phi^{-1}.\psi.\gamma)$. So:
$\delta_0^{X_\bullet}.\bar{d}_0^{X_\bullet}(\gamma, (\phi, \psi)) = \delta_0^{X_\bullet}(\gamma, \phi^{-1}.\psi.\gamma) = \gamma^{-1}.\phi^{-1}.\psi.\gamma$;
and: $\delta_0^{X_\bullet}.\bar{d}_1^{X_\bullet}(\gamma, (\phi, \psi)) = \delta_0^{X_\bullet}(\phi.\gamma, \psi.\gamma) = \gamma^{-1}.\phi^{-1}.\psi.\gamma$.

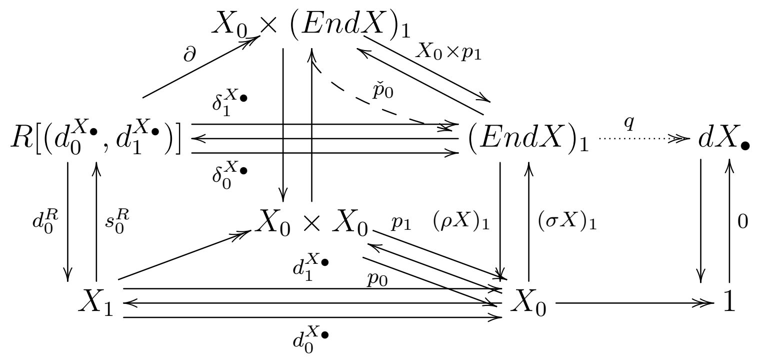
This makes the upper horizontal diagram a groupoid (we shall denote by $\overline{EndX}_\bullet$) since the action in question is produced above a groupoid. So, the vertical downward and upward maps produce a split epic functor. This functorial situation is coherent with the vertical group structure on $(EndX)_\bullet$:
since $\bar{d}_0^{X_\bullet}((\gamma,(\phi,\psi))\circ(\gamma,(\phi,\psi'))) = \bar{d}_0^{X_\bullet}(\gamma,(\phi,p(\psi,\phi,\psi'))$
$= (\gamma,\phi^{-1}.p(\psi,\phi,\psi').\gamma)$;
and: $\bar{d}_0^{X_\bullet}(\gamma,(\phi,\psi))\circ\bar{d}_0^{X_\bullet}(\gamma,(\phi,\psi')) = (\gamma,\phi^{-1}.\psi.\gamma)\circ(\gamma,\phi^{-1}.\psi'.\gamma)$
$= (\gamma,p(\phi^{-1}.\psi.\gamma,\gamma,\phi^{-1}.\psi'.\gamma)) = (\gamma,p(\phi^{-1}.\psi.\gamma,\phi^{-1}.\phi.\gamma,\phi^{-1}.\psi'.\gamma))$.
By the identities $(*)$ given at the beginning of tis section, we get the desired equality between the two previous terms. □

The quickest way to make emerge this anatomical decomposition is to see it as underlying the double category whose double arrows are the commutative squares in $X_\bullet$:



(which can be identified up to isomorphism with the parallel pair $(\phi,\beta.\phi)\in R[(d_0^{X_\bullet},d_1^{X_\bullet})]$) and where the vertical and horizontal compositions are clear. These double arrows are the arrows of the groupoid $\overline{EndX}_\bullet$ having the endomap α as domain and the endomap β as codomain.

Now, let us suppose that, in addition, the groupoid $X_\bullet$ is connected, and let us consider the right hand side quadrangled pullback above the projection $p_1 : X_0\times X_0\to X_0$:



Since, in the lower left hand side part of this diagram, the square indexed by 1 is a pullback the factorization $\partial : R[(d_0^{X_\bullet}, d_1^{X_\bullet})] \to X_0 \times (EndX)_1$ induced by the regular epimorphism $(d_0^{X_\bullet}, d_1^{X_\bullet}) : X_1 \twoheadrightarrow X_0 \times X_0$ is a regular epimorphism as well; as such ∂ is the quotient of its kernel equivalence relation $R[\partial]$. In set-theoretical terms we get $\partial(\phi, \psi) = (d_0(\psi), d_1^{X_\bullet}(\phi, \psi)) = (d_0(\phi), d_1^{X_\bullet}(\phi, \psi))$. So, in categorical terms, we get $\partial = (d_0^{X_\bullet}.d_0^R, \delta_1^{X_\bullet}) = (d_0^{X_\bullet}.d_1^R, \delta_1^{X_\bullet})$.

Lemma 5.5. *In any category $\mathbb{E}$, the kernel equivalence relation $R[\partial]$ is nothing but the Chasles equivalence relation $Ch[p]$ on the object $R[(d_0^{X_\bullet}, d_1^{X_\bullet})]$ associated with the associative Mal'tsev operation $p : R_2[(d_0^{X_\bullet}, d_1^{X_\bullet})] \to X_1$. We get the inclusion $R[\partial] \subset R[\delta_0^{X_\bullet}]$ if and only if the groupoid $X_\bullet$ is affine.*

Proof. Given any associative Mal'tsev operation p on a set X, recall from [5] that it induces an equivalence relation $Ch[p]$ on $X \times X$ defined by $(x, z)Ch[p](x', z')$ if $x' = p(x, z, z')$. Thanks to the Yoneda embedding it is enough to check our assertion in Set. Starting with two pairs (ϕ, ψ) and (ϕ', ψ') of parallel arrows in $X_\bullet$, such that $\partial(\phi, \psi) = \partial(\phi', \psi')$, these pairs have same domain and codomain, and are such that $\psi.\phi^{-1} = \psi'.(\phi')^{-1}$. This last point holds if and only: if $\phi' = \phi.\psi^{-1}.\psi' = p(\phi, \psi, \psi')$; whence our first point.

Two pairs (ϕ, ψ) and (ϕ', ψ') of parallel arrows in the groupoid $X_\bullet$ are such that $\delta_0^{X_\bullet}(\phi, \psi) = \delta_0^{X_\bullet}(\phi', \psi')$ if and only if $\phi^{-1}.\psi = (\phi')^{-1}.\psi'$, namely $\phi' = \psi'.\psi^{-1}.\phi = p(\psi', \psi, \phi)$. So that $R[\partial] \subset R[\delta_0^{X_\bullet}]$ if and only $p(\phi, \psi, \psi')$ $= p(\psi', \psi, \phi)$, namely if and only if the groupoid $X_\bullet$ is affine. □

Proposition 5.6. *Let $\mathbb{E}$ be a regular category and $X_\bullet$ an internal connected affine groupoid. Then, in the above diagram, there is a factorization $\check{p}_0$ above p_0 such that $\check{p}_0.\partial = \delta_0^{X_\bullet}$. The pair $(\check{p}_0, X_0 \times p_1)$ is then underlying the equivalence relation $Supp\overline{End}X_\bullet$, and we get a discrete fibration $Supp\overline{End}X_\bullet \to \nabla_{X_0}$.*

Suppose now $\mathbb{E}$ is efficiently regular, and $X_\bullet$ is aspherical. Then the equivalence relation $Supp\overline{End}X_\bullet$ admits a quotient $dX_\bullet$ which makes the right hand side square a pullback and provides $dX_\bullet$ with an internal abelian group structure. This abelian group $dX_\bullet$ is called the direction *of the aspherical affine groupoid $X_\bullet$.*

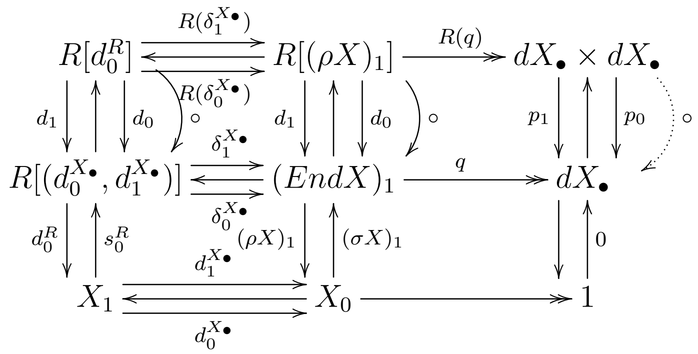
This construction functorially extends to any ido-functor $f_\bullet : X_\bullet \to Z_\bullet$ between aspherical affine groupoids in $\mathbb{E}$ in a way which makes the following *square a pullback:*

$$\begin{array}{ccc} (EndX)_\bullet & \xrightarrow{q_{X_\bullet}} & dX_\bullet \\ {\scriptstyle (Endf)_\bullet}\downarrow & & \downarrow{\scriptstyle df_\bullet} \\ (EndZ)_\bullet & \xrightarrow[q_{Z_\bullet}]{} & dZ_\bullet \end{array}$$

Accordingly, $df_\bullet$ is an isomorphism if and only if so is $(Endf)_\bullet$

Proof. By the previous lemma, as soon as $X_\bullet$ is an affine groupoid, we get $R[\partial] \subset R[\delta_0^{X_\bullet}]$. If $X_\bullet$ is connected, the map ∂ is a regular epimorphism, whence the factorization $\check{p}_0$ in question. Since the quadrangled square indexed by 1 is a pullback and ∇_{X_0} is a relation, so is $(\check{p}_0, X_0 \times p_1)$. Accordingly, this relation is underlying the equivalence relation $Supp\overline{End}X_\bullet$ which, then, is endowed with a discrete fibration $Supp\overline{End}X_\bullet \to \nabla_{X_0}$. Accordingly the inclusion $i : Supp\overline{End}X_\bullet \rightarrowtail ((\rho X)_1)^{-1}(\nabla_{X_0})$ is split in $\mathbb{E}$ and consequently a regular monomorphism in $\mathbb{E}$, so that, when $\mathbb{E}$ is efficiently regular, $Supp\overline{End}X_\bullet$ has a quotient $dX_\bullet$ which, by the Barr-Kock Theorem in regular categories, makes the above right hand square a pullback.

It remains to show that $dX_\bullet$ is endowed with an abelian group structure. First, it is clear that when the groupoid $X_\bullet$ is affine the endosome group $(EndX)_\bullet$ is abelian. Now consider the following extention of the previous diagram by the kernel equivalence relations of the vertical maps:



Since the lower right and side square is a pullback, such are the upper right hand side ones; and consequently the factorization $R(q)$ is a regular epimorphism. We showed that the maps $\delta_0^{X_\bullet}$ and $\delta_1^{X_\bullet}$ did respect the group structures $\circ$. Accordingly they produce the right hand side vertical dotted factorization

$\circ$ which gives the abelian group structure to $dX_\bullet$. The last assertion is then straightforward. □

We can now assert the first result we were aiming to:

Theorem 5.7. *Let $\mathbb{E}$ be an efficiently regular category and $h : U \to X$ a morphism such that U and X have same support in $\mathbb{E}$. Given any ido-functor $f_\bullet : X_\bullet \to Z_\bullet$ between connected affine groupoids, if $h^*(f_\bullet) : h^*(X_\bullet) \to h^*(Z_\bullet)$ is an isomorphism, then so is $f_\bullet$.*

Proof. Let $h = m.\bar{h}$ be the canonical decomposition of h into a regular epimorphism and a monomorphism. Since, by Proposition 1.3.2, the base change $\bar{h}^*$ is certainly conservative, it is enough to prove our assertion when $m : U \rightarrowtail X$ is a monomorphism.

Let us denote by $J \rightarrowtail 1$ the common support of U and X. This J is the common quotient of the equivalence relations ∇_X and ∇_U. In particular the connected groupoids $X_\bullet$ and $Z_\bullet$ in $\mathbb{E}$ become aspherical groupoids in the category $\mathbb{F} = \mathbb{E}/J$ which is efficiently regular as well. All the diagrams in $\mathbb{E}$ involved by our assertion actually lie in $\mathbb{F}$ and they are preserved by the left exact forgetful functor $\mathbb{F} \to \mathbb{E}$ which is obviously conservative and which preserves and reflects the regular epimophisms. So, we can now work without any restriction in the category $\mathbb{F}$.

We know that any fiber $Grd_Y\mathbb{F}$ is protomodular [3]. This fiber being regular as well, an ido-functor $f_\bullet : X_\bullet \to Z_\bullet$ between connected groupoids is an isomorphism if and only if the functor $(Endf)_\bullet$ is itself an isomorphism. If, moreover, the two groupoids are aspherical and affine, then, thanks to the previous proposition, this condition is equivalent to: $df_\bullet$ is an isomorphism. So, our assumption is equivalent to: $dm^*(f_\bullet)$ is an isomorphism. Now, consider the following diagram in $\mathbb{F}$, where U and X have a global support:

$$\begin{array}{ccccc}
(End(m^*X))_1 & \overset{\tilde{m}}{\rightarrowtail} & (EndX)_1 & \overset{q}{\twoheadrightarrow} & dX_\bullet \\
{\scriptstyle (\rho(m^*X))_1}\downarrow\ \uparrow{\scriptstyle (\sigma(m^*X))_1} & & {\scriptstyle (\rho X)_1}\downarrow\ \uparrow{\scriptstyle (\sigma X)_1} & & \downarrow\ \uparrow{\scriptstyle 0} \\
U & \overset{m}{\rightarrowtail} & X & \overset{\tau_X}{\twoheadrightarrow} & 1
\end{array}
\qquad \tau_U : U \to 1$$

Since the map $m^*(X_\bullet) \rightarrowtail X_\bullet$ is cartesian, the left hand side square is a pullback. And since U has a global support, we get $dm^*X_\bullet = dX_\bullet$; so, we get $dm^*f_\bullet = df_\bullet$ as well. Accordingly, $df_\bullet$ is an isomorphism, which implies that so is $f_\bullet$. □

5.4 Internal weak equivalences and affine groupoids

Let $f_\bullet : X_\bullet \to Z_\bullet$ be any functor in $Grd\mathbb{E}$. Consider the following left and side pullback in $\mathbb{E}$:

$$\begin{array}{ccccc} Af_0 & \xrightarrow{\phi_0} & Z_1 & \xrightarrow{d_1^{Z_\bullet}} & Z_0 \\ {\scriptstyle \delta_0}\downarrow\uparrow{\scriptstyle \sigma_0} & & {\scriptstyle d_0^{Z_\bullet}}\downarrow\uparrow{\scriptstyle s_0^{Z_\bullet}} & & \\ X_0 & \xrightarrow[f_0]{} & Z_0 & & \end{array}$$

Definition 5.8. *Let $\mathbb{E}$ be a regular category. An internal functor $f_\bullet : X_\bullet \to Z_\bullet$ is said to be essentially surjective when the upper horizontal map $d_1^{Z_\bullet}.\phi_0$ is a regular epimorphism in $\mathbb{E}$. It is said to be a weak equivalence, when, in addition, it is fully faithful (i.e. $(\ )_0$-cartesian).*

Given any essentially surjective functor $f_\bullet : X_\bullet \to Z_\bullet$, the objects X_0 and Z_0 have necessarily same support. The essentially surjective functors (resp. the weak equivalences) are stable under composition; when $g_\bullet.f_\bullet$ is essentially surjective, so is $g_\bullet$.

The 2-category $Cat\mathbb{E}$ is actually a strongly representable 2-category in the sense of [17]: namely, for any internal category $Z_\bullet$, there is a universal natural transformation with codomain $Z_\bullet$:

$$ComZ_\bullet \underset{(\bar{\delta}_1^Z)_\bullet}{\overset{(\bar{\delta}_0^Z)_\bullet}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}}}} Z_\bullet$$

where $(ComZ)_0$ is Z_1 and $(ComZ)_1$ is the internal object of the "commutative squares" in $Z_\bullet$, i.e. it is obtained as the object $R[d_1^{Z_\bullet}]$ determined by the kernel equivalence relation of the map $d_1^{Z_\bullet} : Z_2 \to Z_1$ in $\mathbb{E}$. We get a common section $(\bar{\sigma}_0^Z)_\bullet$ of the pair $((\bar{\delta}_0^Z)_\bullet, (\bar{\delta}_1^Z)_\bullet)$ from the identity natural transformation $1_{Z_\bullet} \Rightarrow 1_{Z_\bullet}$. Internal groupoids are characterized among internal categories by the following:

Lemma 5.9. *[2] An internal category $Z_\bullet$ is groupoid if and only if the split epimorphism $((\bar{\delta}_0^Z)_\bullet, (\bar{\sigma}_0^Z)_\bullet)$ (resp. $((\bar{\delta}_1^Z)_\bullet, (\bar{\sigma}_0^Z)_\bullet)$) is cartesian in $Cat\mathbb{E}$.*

Now consider the following left hand side pullback in $Grd\mathbb{E}$:

$$\begin{array}{ccccc} Af_\bullet & \xrightarrow{\phi_\bullet} & ComZ_\bullet & \xrightarrow{(\bar{\delta}_1^Z)_\bullet} & Z_\bullet \\ {\scriptstyle (\bar{\delta}_0^f)_\bullet}\downarrow\uparrow{\scriptstyle (\bar{\sigma}_0^f)_\bullet} & & {\scriptstyle (\bar{\delta}_0^Z)_\bullet}\downarrow\uparrow{\scriptstyle (\bar{\sigma}_0^Z)_\bullet} & & \\ X_\bullet & \xrightarrow[f_\bullet]{} & Z_\bullet & & \end{array}$$

Straightforward are the following observations:

Lemma 5.10. *1) The split epimorphism $((\bar{\delta}_0^f)_\bullet, (\bar{\sigma}_0^f)_\bullet)$ is cartesian. 2) We get $f_\bullet = ((\bar{\delta}_1^Z)_\bullet.\phi_\bullet).(\bar{\sigma}_0^f)_\bullet$. Accordingly, 3) the functor $f_\bullet$ is cartesian if and only if so is the functor $(\bar{\delta}_1^Z)_\bullet.\phi_\bullet$.*

When the category $\mathbb{E}$ is an exact one, any equivalence relation $SuppX_\bullet$ has a quotient which is nothing but the internal object $\pi_0(X_\bullet)$ of the "connected components" of the groupoid $X_\bullet$. This construction produces a left adjoint to the fully faithful functor $\Delta : \mathbb{E} \to Grd\mathbb{E}$. Basic is the following:

Lemma 5.11. *Let $\mathbb{E}$ be an exact category. Given any parallel pair $(f_\bullet, g_\bullet)$ of functors between groupoids, we get $\pi_0(f_\bullet) = \pi_0(g_\bullet)$ as soon as we have a natural transformation $\alpha : f_\bullet \Rightarrow f'_\bullet$.*

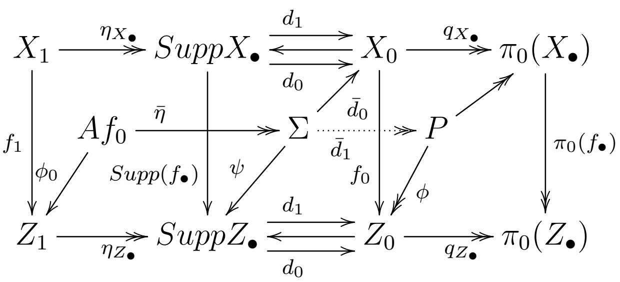
Proof. Given any natural transformation $\alpha : f_\bullet \Rightarrow g_\bullet$, the map $\alpha_0 : X_0 \to Z_1$ underlying this natural transformation is such that $d_0^{Z_\bullet}.\alpha_0 = f_0$ and $d_1^{Z_\bullet}.\alpha_0 = g_0$, so that the coequalizer $\gamma_{Z_\bullet} : Z_0 \twoheadrightarrow \pi_0(Z_\bullet)$ of the pair $(d_0^{Z_\bullet}, d_1^{Z_\bullet})$ coequalizes the pair (f_0, g_0) as well, which implies $\pi_0(f_\bullet) = \pi_0(g_\bullet)$. $\square$

More meaningful are following ones:

Lemma 5.12. *Let $\mathbb{E}$ be an exact category. 1) When the functor $f_\bullet$ is fully faithful (=cartesian), then $\pi_0(f_\bullet)$ is a monomorphism. 2) The functor $f_\bullet$ is essentially surjective if and only if $\pi_0(f_\bullet)$ is a regular epimorphism. So, when $f_\bullet$ is a weak equivalence, then $\pi_0(f_\bullet)$ is an isomorphism.*

Proof. When $f_\bullet$ is fully faithful, then $SuppX_\bullet = f_0^{-1}(SuppZ_\bullet)$, so that $\pi_0(f_\bullet)$ is a monomorphism.

It is clear that, as soon as f_0 is a regular epimorphism, $\pi_0(f_\bullet)$ is a regular epimorphism. Suppose $f_\bullet$ is essentially surjective. The functor $\phi_\bullet : Af_\bullet \to ComZ_\bullet$ determines a natural transformation $\alpha : f_\bullet.(\bar{\delta}_0^f)_\bullet \Rightarrow (\bar{\delta}_1^X)_\bullet.\phi_\bullet$. So, we get $\pi_0(f_\bullet).\pi_0((\bar{\delta}_0^f)_\bullet) = \pi_0((\bar{\delta}_1^Z)_\bullet.\phi_\bullet)$. Now, when $f_\bullet$ is essentially surjective, the map $\pi_0((\bar{\delta}_1^Z)_\bullet.\phi_\bullet)$ is a regular epimorphism since $d_1^{Z_\bullet}.\phi_0$ is a regular epimorphism; accordingly, so is $f_\bullet$. Conversely, suppose that the map $\pi_0(f_\bullet)$ is a regular epimorphism. Then consider the following diagram built from the vertical functor $f_\bullet$:



where ϕ is the pullback of the regular epimorphism $\pi_0(f_\bullet)$ along the regular epimorphism $q_{Z_\bullet}$, so that ϕ is a regular epimorphism; and where ψ is the pullback of f_0 along d_0. These pullbacks produce a factorization $\bar{d}_1 : \Sigma \to P$ above d_1, which by commutation of limits makes the upper upward right hand side quadrangle a pullback as well. Accordingly, since $q_{X_\bullet}$ is a regular epimorphism, so is $\bar{d}_1$. Finally, let ϕ_0 be the pullback of ψ along the regular epimorphism $\eta_{Z_\bullet}$ so that: 1) $\bar{\eta}$ is a regular epimorphism and 2) the map $\bar{d}_0.\bar{\eta}$ is nothing but the map δ_0 of the diagram defining Af_0 in the definition of an essentially surjective functor. Now $d_1^{Z_\bullet}.\phi_0 = d_1.\eta_{Z_\bullet}.\phi_0 = d_1.\psi.\bar{\eta} = \phi.\bar{d}_1.\bar{\eta}$, where these three last maps are regular epimorphisms. So, $d_1^{Z_\bullet}.\phi_0$ is a regular epimorphism, and the functor $f_\bullet$ is essentially surjective. $\square$

Whence the second important result we were aiming to:

Proposition 5.13. *Let $\mathbb{E}$ be an exact category. Consider any weak equivalence $f_\bullet : X_\bullet \to Z_\bullet$. Then $X_\bullet$ is affine if and only if $Z_\bullet$ is affine. In this case these groupoids are both aspherical in the slice category $\mathbb{E}/Q$ where $Q = \pi_0(Z_\bullet)$, and they have same direction in $\mathbb{E}/Q$.*

Accordingly, when $Z_\bullet$ is affine, then the groupoid $Z_\bullet$ is an equivalence relation if and only if so is $X_\bullet$.

Proof. In any category $\mathbb{E}$, when $f_\bullet : X_\bullet \to Z_\bullet$ is fully faithful, then $X_\bullet$ is affine (resp. an equivalence relation) as soon as so is $Z_\bullet$.

Conversely, suppose that $X_\bullet$ is affine. According to Lemma 5.10.1, so is the groupoid $Af_\bullet$. When $f_\bullet$ is fully faithful, so is the functor $\psi_\bullet = (\bar{\delta}_1^Z)_\bullet.\phi_\bullet$. But ψ_0 is a regular epimorphism in $\mathbb{E}$. Then, according to Proposition 5.4, the groupoid $Z_\bullet$ is affine, since so is $Af_\bullet$.

Since $f_\bullet$ is a weak equivalence, $\pi_0(f_\bullet)$ is an isomorphism, and consequently all the diagrams involved by our situation lie in the slice category $\mathbb{E}/\pi_0(Z_\bullet)$ where the groupoids $X_\bullet$ and $Z_\bullet$ become aspherical; so that, being affine, they get a direction in this exact slice category. It remains to show that these directions are the same. For that consider the following diagram:

$$\begin{array}{ccccc}
(EndX)_1 & \xrightarrow{(Endf)_1} & (EndZ)_1 & \xrightarrow{q_Z} & dZ_\bullet \\
{\scriptstyle (\rho X)_1}\downarrow \uparrow{\scriptstyle (\sigma X)_1} & & {\scriptstyle (\rho Z)_1}\downarrow \uparrow{\scriptstyle (\sigma Z)_1} & & \downarrow \uparrow{\scriptstyle 0} \\
X_0 & \xrightarrow{f_0} & Z_0 & \longrightarrow & \pi_0(Z_\bullet)
\end{array}$$

The right hand side square is a pullback by definition. The left hand side one is a pullback since $f_\bullet$ is fully faithful. So the whole rectangle is a pullback; and since the long lower horizontal map is a regular epimorphism, this rectangle defines the direction of the aspherical affine groupoid $X_\bullet$. Accordingly $dX_\bullet = dZ_\bullet$.

Now, suppose $Z_\bullet$ is affine; then $X_\bullet$ is affine. Saying that $X_\bullet$ is an equivalence relation is saying that its direction in the slice category $\mathbb{E}/\pi_0(Z_\bullet)$ is trivial, namely the terminal object. According to the first part of the proposition, so is the direction of $Z_\bullet$, which, in turn, means that $Z_\bullet$ is an equivalence relation. □

The Theorem 5.7 and the last part of the previous proposition are of particular interest in the Mal'tsev and Gumm categories, where any groupoid is affine, see [8], [18] and [11].

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A SKETCH FOR DERIVATORS

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Résumé. Nous montrons d'abord que les dérivateurs peuvent être vus comme des modèles d'une 2-esquisse projective homotopique appropriée. Après avoir discuté de la λ-présentabilité locale homotopique de la 2-catégorie des dérivateurs, pour un certain cardinal régulier approprié λ, comme application nous montrons que les dérivateurs de petite présentation sont des objets λ-présentables homotopiques.
Abstract. We show first that derivators can be seen as models of a suitable homotopy limit 2-sketch. After discussing homotopy local λ-presentability of the 2-category of derivators, for some appropriate regular cardinal λ, as an application we prove that derivators of small presentation are homotopy λ-presentable objects.
Keywords. Derivator, Sketch, Homotopy, Presentation.
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1. Introduction

Derivators were introduced by Grothendieck in his manuscript [22] written between the end of 1990 and the beginning of 1991, though the term first appeared in his letter to Quillen [21] of 1983. Similar notions appeared, independently, in Heller's work [23] of 1988 with the name of homotopy theories, and later, in 1996, in Franke's paper [16] with the name of systems of triangulated diagram categories. Then they were studied, for example, by Heller himself [24], Maltsiniotis [35], Cisinski [9], [11], Cisinski and Neeman [12], Keller [26], Tabuada [44], Groth [18], Groth, Ponto and Shulman [20].

A reason for proposing derivators is to provide a formalism improving that of triangulated categories. In fact, triangulated categories lack a good theory of homotopy limits and homotopy colimits, in the sense that, though they can be defined, they can not be expressed by means of an explicit universal property. An example of this is the non-functoriality of the cone construction. Since in the case of the derived category of an abelian category or the homotopy category of a stable model category or of a stable $(\infty, 1)$-category, these construction can be made functorial, it means that when passing to the homotopy category the information for the construction of homotopy limits and homotopy colimits is lost. A derivator, as opposed to the homotopy (or derived) category, contains enough information to deal in a satisfactory way with homotopy limits and homotopy colimits. The idea in derivators is not only to consider the homotopy (or derived) category, but also to keep track of the homotopy (or derived) categories of diagrams and homotopy Kan extension between them. An advantage of working with derivators is also the possibility of describing them completely by means of the theory of 2-categories.

As proved by Cisinski [9], model categories give rise to derivators, yielding a pseudo-functor between the 2-category of model categories and the 2-category of derivators. Building on this and on Dugger's result [14] about presentation of combinatorial model categories, Renaudin [40] proved that the pseudo-localization of the 2-category of combinatorial model categories at the class of Quillen equivalences is biequivalent to the 2-category of derivators of small presentation. These are defined by imposing, in a suitable sense, relations on a derivator associated to the model category of simplicial presheaves on a small category $\mathcal{C}$, which plays the role of a free derivator on $\mathcal{C}$. In this sense, small presentation of derivators resembles finite presentation of modules over rings or of models of algebraic theories, when given in terms of generators and relations. However, in these last two cases, finite presentation can be characterized also intrinsically: finitely presented modules (or models) are those which represent functors preserving filtered colimits. The search for an analogous intrinsic formulation of small presentation for derivators has been the motivation for this paper.

The main result we have obtained is the construction of a homotopy limit 2-sketch whose homotopy models can be identified with derivators. A (homotopy) limit sketch is a way to describe a theory defined by means of (homotopy) limits. The 2-categories of (homotopy) models of (homotopy) limit 2-sketches are the (homotopy) locally presentable 2-categories. Therefore, the construction of a homotopy limit 2-sketch for derivators, besides providing some kind of algebraic description of derivators, supplies also a framework in which to discuss homotopy presentability. Indeed, as an application, we prove that derivators of small presentation are homotopy λ-presentable models, partially meeting our original motivation.

We summarize the content of the paper and present the results.

In section 2 we recall (right, left) derivators, as they were defined by Grothendieck [22], and we present Cisinski's result mentioned above. In this paper, in order to study presentability, we will assume that the 2-category of diagrams $\mathfrak{Dia}$ on which derivators are defined is small with respect to a fixed Grothendieck universe.

In section 3, we recall the definition of the weighted homotopy limit 2-sketch $\mathfrak{S}$ and of its category of models. We explain, then, how to include pseudo-natural transformations as morphisms between models in a new 2-category of models $\mathfrak{hMod}_{\mathfrak{S}}^{ps}$.

In section 4 we present our main result: we prove that the 2-category $\mathfrak{Der}^r$ of right derivators, cocontinuous pseudo-natural transformations (2.8) and modifications, is the 2-category of models of a weighted homotopy limit 2-sketch, whose construction is explicitly exhibited.

Theorem 4.1. There exists a weighted homotopy limit 2-sketch $\mathfrak{S} = (\mathfrak{G}, \mathcal{P})$ and a biequivalence from the 2-category $\mathfrak{hMod}_{\mathfrak{S}}^{ps}$ to the 2-category $\mathfrak{Der}^r$.

In section 5 we recall the theory of homotopy presentable categories, together with the notion of presentable object in the homotopic sense. We have:

Corollary 5.11. $\mathfrak{hMod}_{\mathfrak{S}}^{ps}$ is a homotopy locally λ-presentable 2-category, where λ is a regular cardinal bounding the size of every category in $\mathfrak{Dia}$.

In section 6, we prove first, in lemma 6.3, after passing to a realized sketch, that representable models correspond to derivators defined by model categories of the form $sSet^{\mathcal{C}^{op}}$, for some small category $\mathcal{C}$. As an application, derivators can be reconstructed by means of homotopy λ-filtered colimits as follows:

Corollary 6.4. Any right derivator is a homotopy λ-filtered colimit in $\mathfrak{Der}^r$ of λ-small homotopy 2-colimits of derivators of the form $\mathcal{F}(\mathcal{C}) = \Phi(sSet^{\mathcal{C}^{op}})$.

Finally, after recalling Renaudin's definitions and result on small presentability, we obtain:

Theorem 6.14. A derivator of small presentation is a homotopy λ-presentable object of $\mathfrak{Der}^r$.

The author would like to thank Kuerak Chung for introducing the topic, Bernhard Keller for bringing this problem to his attention, Steve Lack and John Power for useful suggestions, Georges Maltsiniotis and Mike Shulman for useful comments.

2. Derivators

In this section we recall derivators as introduced by Grothendieck [22, 1]. Derivators of small presentation, defined by Renaudin [40, 3.4], will be recalled instead in section 6. Besides these two references, introductions to derivators are found for instance in [35], [9, 1] or [18, 1].

We fix a Grothendieck universe $\mathcal{U}$ and we denote by $\mathfrak{Cat}$ the 2-category of $\mathcal{U}$-small categories, and by $\mathcal{C}at$ the ordinary category underlying $\mathfrak{Cat}$.

Definition 2.1. *A category of diagrams, which we denote by $\mathfrak{Dia}$, is a full 2-subcategory of $\mathfrak{Cat}$ such that:*

1. *it contains the empty category, the terminal category e and the category $\Delta_1 = \mathbb{2}$ associated to the ordered set $\{0 < 1\}$;*

2. *it is closed under finite coproducts and pullbacks;*

3. *it contains the overcategories $\mathcal{C}/D$ and the undercategories $D\backslash\mathcal{C}$ corresponding to any functor $u : \mathcal{C} \to \mathcal{D}$ and to any object $D \in \mathcal{D}$;*

4. *it is stable under passage to the opposite category.*

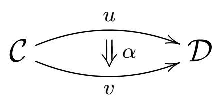
Examples of categories of diagrams are $\mathfrak{Cat}$ itself, the 2-category $\mathfrak{Cat}_f$ of finite categories, the 2-category of partially ordered sets or the 2-category of finite ordered sets.

In this paper we will assume that $\mathfrak{Dia}$ is $\mathcal{U}$-small, because, although the definitions regarding derivators do not depend on this, this hypothesis guarantees that all limits and colimits with which we will be concerned are $\mathcal{U}$-small. So we will assume the existence of a regular cardinal λ such that all the categories in $\mathfrak{Dia}$ are λ-small.

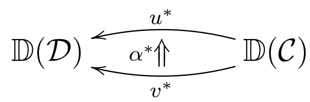
Definition 2.2. *A prederivator of domain* $\mathfrak{Dia}$ *is a strict 2-functor*

$$\mathbb{D} : \mathfrak{Dia}^{coop} \to \mathfrak{Cat}.$$

In other words, applying a prederivator $\mathbb{D}$ to the diagram



yields the diagram



where we have set $u^* = \mathbb{D}(u)$, $v^* = \mathbb{D}(v)$ and $\alpha^* = \mathbb{D}(\alpha)$.

Example 2.3. For any category $\mathcal{C} \in \mathfrak{Dia}$, the representable 2-functor $\mathfrak{Dia}(-^{op}, \mathcal{C})$ is a prederivator of domain $\mathfrak{Dia}$. Actually, any $\mathcal{C} \in \mathfrak{Cat}$ defines a prederivator of domain $\mathfrak{Dia}$.

We present now the definitions of derivator, right derivator and left derivator, as introduced by Grothendieck [22]. There are other variants, which, however, we do not consider in this paper (see, for instance, [12, 1]).

Definition 2.4. *A derivator is a prederivator* $\mathbb{D}$ *satisfying the following axioms.*

1. *For every* $\mathcal{C}_0$ *and* $\mathcal{C}_1$ *in* $\mathfrak{Dia}$*, the functor*

$$\mathbb{D}(\mathcal{C}_0 \amalg \mathcal{C}_1) \longrightarrow \mathbb{D}(\mathcal{C}_0) \times \mathbb{D}(\mathcal{C}_1),$$

induced by the canonical inclusions $\mathcal{C}_i \to \mathcal{C}_0 \amalg \mathcal{C}_1$*, is an equivalence of categories. Moreover,* $\mathbb{D}(\varnothing)$ *is equivalent to the terminal category* e.

2. *A morphism $f : A \to B$ of $\mathbb{D}(\mathcal{C})$ is an isomorphism if and only if, for any object D of $\mathcal{C}$, the morphism in $\mathbb{D}(e)$*

$$c_D^*(f) : c_D^*(A) \longrightarrow c_D^*(B)$$

is an isomorphism, where $c_D : e \to \mathcal{C}$ denotes the constant functor at D.

3. *For every $u : \mathcal{C} \to \mathcal{D}$ in $\mathfrak{Dia}$, the functor*

$$u^* : \mathbb{D}(\mathcal{D}) \longrightarrow \mathbb{D}(\mathcal{C})$$

has both left and right adjoints

$$u_! : \mathbb{D}(\mathcal{C}) \longrightarrow \mathbb{D}(\mathcal{D}) \tag{1}$$
$$u_* : \mathbb{D}(\mathcal{C}) \longrightarrow \mathbb{D}(\mathcal{D}), \tag{2}$$

called homological and cohomological direct image functor respectively.

4. *Consider diagrams in $\mathfrak{Dia}$ of the form*

$$\begin{array}{ccc} D\backslash\mathcal{C} & \xrightarrow{f} & \mathcal{C} \\ {\scriptstyle t}\downarrow & {\scriptstyle\alpha}\nearrow & \downarrow{\scriptstyle u} \\ e & \xrightarrow[c_D]{} & \mathcal{D} \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{C}/D & \xrightarrow{f} & \mathcal{C} \\ {\scriptstyle t}\downarrow & \swarrow{\scriptstyle\beta} & \downarrow{\scriptstyle u} \\ e & \xrightarrow[c_D]{} & \mathcal{D} \end{array}$$

where $D \in \mathcal{D}$, t is the unique functor to the terminal category e, f the obvious forgetful functor, c_D the constant functor at D, α and β the canonical natural transformations. Apply $\mathbb{D}$

$$\begin{array}{ccc} \mathbb{D}(D\backslash\mathcal{C}) & \xleftarrow{f^*} & \mathbb{D}(\mathcal{C}) \\ {\scriptstyle t^*}\uparrow & \swarrow{\scriptstyle\alpha^*} & \uparrow{\scriptstyle u^*} \\ \mathbb{D}(e) & \xleftarrow[c_D^*]{} & \mathbb{D}(\mathcal{D}) \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{D}(\mathcal{C}/D) & \xleftarrow{f^*} & \mathbb{D}(\mathcal{C}) \\ {\scriptstyle t^*}\uparrow & {\scriptstyle\beta^*}\nearrow & \uparrow{\scriptstyle u^*} \\ \mathbb{D}(e) & \xleftarrow[c_D^*]{} & \mathbb{D}(\mathcal{D}) \end{array}$$

and use axiom 3 to construct the Beck-Chevalley transformations

$$\alpha_{bc}^* : t_! f^* \Rightarrow c_D^* u_! \tag{3}$$
$$\beta_{bc}^* : c_D^* u_* \Rightarrow t_* f^*, \tag{4}$$

shown in the diagrams

$$
\begin{array}{ccc}
\mathbb{D}(D\backslash\mathcal{C}) & \xleftarrow{f^*} & \mathbb{D}(\mathcal{C}) \\
{\scriptstyle t_!}\downarrow & \Downarrow \alpha^*_{bc} & \downarrow{\scriptstyle u_!} \\
\mathbb{D}(e) & \xleftarrow[c^*_D]{} & \mathbb{D}(\mathcal{D})
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{D}(\mathcal{C}/D) & \xleftarrow{f^*} & \mathbb{D}(\mathcal{C}) \\
{\scriptstyle t_*}\downarrow & \beta^*_{bc} \Uparrow & \downarrow{\scriptstyle u_*} \\
\mathbb{D}(e) & \xleftarrow[c^*_D]{} & \mathbb{D}(\mathcal{D})
\end{array}
$$

and given respectively by the composites

$$t_! f^* \Rightarrow t_! f^* u^* u_! \Rightarrow t_! t^* c^*_D u_! \Rightarrow c^*_D u_!$$
$$c^*_D u_* \Rightarrow t_* t^* c^*_D u_* \Rightarrow t_* f^* u^* u_* \Rightarrow t_* f^*.$$

Then the natural transformations α^*_{bc} and β^*_{bc} are isomorphisms.

Definition 2.5. *A right derivator is a prederivator such that:*

- *it satisfies axioms 1 and 2;*
- *it admits homological direct image functors $u_!$ for any functor u in $\mathfrak{Dia}$;*
- *every α^*_{bc} as in* (3) *is an isomorphism.*

A left derivator is defined in an analogous way.

Example 2.6. Let $\mathcal{M}$ be a model category and W the class of its weak equivalences. The prederivator $\mathrm{Ho}[-^{op}, \mathcal{M}]$, which on objects $\mathcal{C} \in \mathfrak{Dia}$ is defined as the homotopy category

$$\mathrm{Ho}[\mathcal{C}^{op}, \mathcal{M}] = [\mathcal{C}^{op}, \mathcal{M}][W_{\mathcal{C}}^{-1}],$$

where $W_{\mathcal{C}}$ is the class of objectwise weak equivalences, defines a derivator. Its value on the terminal category e is just the homotopy category $\mathrm{Ho}(\mathcal{M})$ of $\mathcal{M}$. Its complete definition and the proof that it does define a derivator is the subject of [9].

We use pseudo-natural transformations to define 1-morphisms of derivators.

Definition 2.7. *A morphism of prederivators $\theta : \mathbb{D}_1 \to \mathbb{D}_2$ is a pseudo-natural transformation $\theta : \mathbb{D}_1 \Rightarrow \mathbb{D}_2$.*

Explicitly, a pseudo-natural transformation $\theta : \mathbb{D}_1 \Rightarrow \mathbb{D}_2$ consists of the following data:

1. for any $\mathcal{C} \in \mathfrak{Dia}$, a functor
$$\theta_{\mathcal{C}} : \mathbb{D}_1(\mathcal{C}) \longrightarrow \mathbb{D}_2(\mathcal{C});$$

2. for any $\mathcal{C}$, $\mathcal{D}$ and $u : \mathcal{C} \to \mathcal{D}$ in $\mathfrak{Dia}$, an isomorphism
$$\beta^\theta_{\mathcal{C}\mathcal{D}_u} \equiv \beta^\theta_u : u_2^* \circ \theta_{\mathcal{D}} \Rightarrow \theta_{\mathcal{C}} \circ u_1^*,$$
where $u_i^* = \mathbb{D}_i(u)$ for $i = 1, 2$, which is natural in u, that is, for any $\alpha : u \Rightarrow v$ in $\mathfrak{Dia}$ the diagram
$$\begin{array}{ccc} v_2^* \circ \theta_{\mathcal{D}} & \xrightarrow{\beta^\theta_v} & \theta_{\mathcal{C}} \circ v_1^* \\ {\scriptstyle \alpha_2^* * \theta_{\mathcal{D}}}\downarrow & & \downarrow{\scriptstyle \theta_{\mathcal{C}} * \alpha_1^*} \\ u_2^* \circ \theta_{\mathcal{D}} & \xrightarrow[\beta^\theta_u]{} & \theta_{\mathcal{C}} \circ u_1^* \end{array}$$
is commutative;

these data are required to fulfill the following coherence conditions
$$\begin{aligned} &\beta^\theta_{1_{\mathcal{C}}} = 1_{\theta_{\mathcal{C}}} \\ &\beta^\theta_{vu} = (\beta^\theta_u * v_1^*) \circ (u_2^* * \beta^\theta_v) \end{aligned}$$
for any composable u and v.

Definition 2.8. *A morphism of right derivators $\theta : \mathbb{D}_1 \to \mathbb{D}_2$ is cocontinuous if it is compatible with the homological direct image functors, namely, for every u in $\mathfrak{Dia}$ the Beck-Chevalley transform*
$$\beta^\theta_{u_!} : u_{2!} \circ \theta_{\mathcal{C}} \Rightarrow \theta_{\mathcal{D}} \circ u_{1!}$$
is an isomorphism.

Continuous morphisms of (left) derivators are defined in an analogous way.

It remains to define 2-morphisms of derivators.

Definition 2.9. *Given two (pre)derivators $\mathbb{D}_1$ and $\mathbb{D}_2$ and two morphisms θ_1, $\theta_2 : \mathbb{D}_1 \to \mathbb{D}_2$, a 2-morphism $\lambda : \theta_1 \to \theta_2$ is a modification $\lambda : \theta_1 \Rrightarrow \theta_2$ between the underlying pseudo-natural transformations.*

Explicitly, a modification $\lambda : \theta_1 \Rrightarrow \theta_2$ consists of a family of natural transformations

$$\lambda_{\mathcal{C}} : \theta_{1\mathcal{C}} \Rightarrow \theta_{2\mathcal{C}}$$

for any $\mathcal{C} \in \mathfrak{Dia}$, such that for every $u : \mathcal{C} \to \mathcal{D}$ of $\mathfrak{Dia}$ the diagram

$$\begin{array}{ccc} u_2^* \circ \theta_{1\mathcal{C}} & \xrightarrow{\beta_u^{\theta_1}} & \theta_{1\mathcal{D}} \circ u_1^* \\ {\scriptstyle u_2^* * \lambda_{\mathcal{C}}} \downarrow & & \downarrow {\scriptstyle \lambda_{\mathcal{D}} * u_1^*} \\ u_2^* \circ \theta_{2\mathcal{C}} & \xrightarrow[\beta_u^{\theta_2}]{} & \theta_{2\mathcal{D}} \circ u_1^* \end{array} \tag{5}$$

is commutative.

We organize what has been introduced so far into the following 2-categories:

1. $\mathfrak{PDer}$ the 2-category of prederivators, morphisms of prederivators and 2-morphisms,
2. $\mathfrak{Der}^r$ the 2-category of right derivators, cocontinuous morphisms and 2-morphisms,
3. $\mathfrak{Der}^l$ the 2-category of left derivators, continuous morphisms and 2-morphisms,
4. $\mathfrak{Der}^{rl}$ the 2-category of derivators, continuous and cocontinuous morphisms and 2-morphisms,
5. $\mathfrak{Der}_{ad}$ the 2-category of derivators, morphisms of derivators whose components have right adjoints, and modifications.

We conclude this section by telling more about the relationship between derivators and model categories outlined in example 2.6. Let $\mathfrak{ModQ}$ denote the 2-category of model categories, left Quillen functors and natural transformations. Cisinski proved in [9] that the map in example 2.6

$$ob\mathfrak{ModQ} \longrightarrow ob\mathfrak{Der}_{ad}$$
$$\mathcal{M} \longmapsto \mathrm{Ho}[-^{op}, \mathcal{M}]$$

extends to 1-morphisms and 2-morphisms: he showed that a Quillen adjunction $F : \mathcal{M}_1 \rightleftarrows \mathcal{M}_2 : G$ induces for any $\mathcal{C} \in \mathfrak{Dia}$ an adjunction of total derived functors

$$\mathbf{L}\tilde{F} : \mathrm{Ho}[\mathcal{C}^{op}, \mathcal{M}_1] \rightleftarrows \mathrm{Ho}[\mathcal{C}^{op}, \mathcal{M}_2] : \mathbf{R}\tilde{G},$$

where $\tilde{F}$ and $\tilde{G}$ act by composing with F and G respectively, and so it defines a pair of adjoint morphisms between the corresponding derivators.

Theorem 2.10. *The construction above defines a pseudo-functor*

$$\Phi : \mathfrak{ModQ} \to \mathfrak{Der}_{ad}$$

taking Quillen equivalences to equivalences of derivators.

We will use the symbol $\Phi(\mathcal{M})$ for the derivator $\mathrm{Ho}[-^{op}, \mathcal{M}]$ constructed from a model category $\mathcal{M}$.

We will recall other facts about derivators, especially the definition of small presentation, in section 6.

3. Sketches

Sketches, introduced by Ehresmann [15], are a way of presenting a theory which can be defined by means of limits and colimits. It turns out that the categories of models of sketches can be characterized intrinsically as the accessible categories (Lair [32, 1-2]), and, in particular, the categories of models of limit sketches are the locally presentable categories (Gabriel and Ulmer [17]).

Though the underlying idea is the same, there are different types of sketches, depending on the type of limits and colimits which define the theory we want to describe. In this section we recall, in some detail, homotopy limit 2-sketches: in fact, in section 4 we will prove that derivators can be identified, up to equivalence, with the homotopy models of a sketch of this type. The 2-category of homotopy models, pseudo-natural transformations and modifications is then homotopy locally presentable. As an application, in section 6, we will use this framework to study small presentability of derivators.

Homotopy limit sketches were proposed by Rosický [41] with the purpose of extending rigidification results of Badzioch [1] and Bergner [2] to finite limit theories. Lack and Rosický in [31] proved that the $\mathcal{V}$-categories of homotopy models of homotopy limit $\mathcal{V}$-sketches can be characterized as the homotopy locally presentable $\mathcal{V}$-categories.

We will consider only the case $\mathcal{V} = \mathcal{C}at$, since this is the one of derivators. Recall that $\mathcal{C}at$ has a model structure, known as the standard model structure, where weak equivalences are the equivalences of categories, and fibrations are the isofibrations; this model structure is combinatorial, all objects are fibrant and, assuming the axiom of choice, also cofibrant, moreover, $\mathfrak{Cat}$ becomes a monoidal model 2-category (in the sense of [34, A.3.1.2]).

If $\mathfrak{C}$ is a small 2-category, then the category underlying $[\mathfrak{C}, \mathfrak{Cat}]$, endowed with the projective model structure, is also a combinatorial model category, whose cofibrant objects can be characterized as follows. Recall that the inclusion

$$i : [\mathfrak{C}, \mathfrak{Cat}] \hookrightarrow \mathcal{P}s(\mathfrak{C}, \mathfrak{Cat})$$

has a left adjoint $\mathcal{Q}$ (see [8, 2.2]), where $\mathcal{P}s(\mathfrak{C}, \mathfrak{Cat})$ denotes the 2-category of 2-functors $\mathfrak{C} \to \mathfrak{Cat}$, pseudo-natural transformations and modifications. Thus, for 2-functors $G, H : \mathfrak{C} \to \mathfrak{Cat}$, there is a natural isomorphism of categories

$$[\mathfrak{C}, \mathfrak{Cat}](\mathcal{Q}G, H) \cong \mathcal{P}s(\mathfrak{C}, \mathfrak{Cat})(G, H). \tag{6}$$

The counit and unit computed at a functor $G : \mathfrak{C} \to \mathfrak{Cat}$ are given by a 2-natural transformation $\varepsilon_G : \mathcal{Q}(G) \to G$ and a pseudo-natural transformation $\eta_G : G \to \mathcal{Q}(G)$ respectively. One of the triangle equations tells us that $\varepsilon_G \circ \eta_G = 1_G$. Since $\eta_G \circ \varepsilon_G \cong 1_G$ (see [4, 4.2]), it follows that $\mathcal{Q}G$ and G are equivalent in $\mathcal{P}s(\mathfrak{C}, \mathfrak{Cat})$. If ε has a section in $[\mathfrak{C}, \mathfrak{Cat}]$, then $\mathcal{Q}G$ and G are equivalent also in $[\mathfrak{C}, \mathfrak{Cat}]$ and G is said to be flexible (see [30, 4.3] and [4, 4.7]). As proved in [29, 4.12], flexible 2-functors are exactly the cofibrant objects of $[\mathfrak{C}, \mathfrak{Cat}]$ with respect to the projective model structure, and $\mathcal{Q}G$ is indeed a cofibrant replacement of G.

Definition 3.1. *Let $\mathfrak{G}$ be a 2-category, $F : \mathfrak{E} \to \mathfrak{G}$ and $G : \mathfrak{E} \to \mathfrak{Cat}$ be 2-functors, where $\mathfrak{E}$ is a small 2-category. Assume G is a cofibrant object of the category $[\mathfrak{E}, \mathfrak{Cat}]$ endowed with the projective model structure. The homotopy 2-limit of F weighted by G exists when there is an object $\{G, F\}_h \in \mathfrak{G}$ and for every object $\mathcal{D}$ of $\mathfrak{G}$ an equivalence of categories*

$$\mathfrak{G}(\mathcal{D}, \{G, F\}_h) \longrightarrow [\mathfrak{E}, \mathfrak{Cat}](G, \mathfrak{G}(\mathcal{D}, F-)) \tag{7}$$

which is 2-natural in $\mathcal{D}$.

In a similar way we define the homotopy 2-colimit $G \star_h F$ of $F : \mathfrak{E} \to \mathfrak{G}$ weighted by $G : \mathfrak{E}^{op} \to \mathfrak{Cat}$ by replacing formula (7) with

$$\mathfrak{G}(G \star_h F, \mathcal{D}) \longrightarrow [\mathfrak{E}, \mathfrak{Cat}](G, \mathfrak{G}(F-, \mathcal{D})).$$

The following definitions are from from [41, 2].

Definition 3.2. *A weighted limit 2-sketch is a pair $\mathfrak{S} = (\mathfrak{G}, \mathcal{P})$ where:*

1. *$\mathfrak{G}$ is a small 2-category;*
2. *$\mathcal{P}$ is a set of 2-cones, that is, quintuples $(\mathfrak{E}, F, G, \mathcal{L}, \gamma)$ where $\mathfrak{E}$ is a small 2-category, the diagram $F : \mathfrak{E} \to \mathfrak{G}$ and the weight $G : \mathfrak{E} \to \mathfrak{Cat}$ are 2-functors, the vertex $\mathcal{L}$ is an object of $\mathfrak{G}$ and $\gamma : G \Rightarrow \mathfrak{G}(\mathcal{L}, F-)$ is a 2-natural transformation.*

A weighted homotopy limit 2-sketch is a weighted limit 2-sketch $\mathfrak{S} = (\mathfrak{G}, \mathcal{P})$ with all weights cofibrant.

Definition 3.3. *A homotopy model of a weighted homotopy limit 2-sketch $\mathfrak{S}$ is a 2-functor $\mathbb{M} : \mathfrak{G} \to \mathfrak{Cat}$ transforming the cones of $\mathcal{P}$ into weighted homotopy 2-limits. We denote by $\mathfrak{hMod}_{\mathfrak{S}}$ the full 2-subcategory of $[\mathfrak{G}, \mathfrak{Cat}]$ spanned by the homotopy models of the weighted homotopy limit 2-sketch $\mathfrak{S}$.*

The 2-categories of the form $\mathfrak{hMod}_{\mathfrak{S}}$ for some weighted homotopy limit 2-sketch $\mathfrak{S}$ are the homotopy locally presentable 2-categories: this fact [31, 9.14(1)] is a consequence of [31, 9.10] (and, actually, holds for a more general $\mathcal{V}$). We will return to these results and to homotopy locally presentable 2-categories in 5.1.

To recover morphisms of derivators, we have to consider pseudo-natural transformations as morphisms between homotopy models. This motivates the following definition.

Definition 3.4. *If $\mathfrak{S}$ is a weighted homotopy limit 2-sketch, we define $\mathfrak{hMod}_{\mathfrak{S}}^{ps}$ to be the full 2-subcategory of $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$ spanned by the homotopy models.*

4. A sketch for derivators

In this section we prove, by giving an explicit construction, that $\mathfrak{Der}^r$ is the 2-category $\mathfrak{hMod}_{\mathfrak{S}}^{ps}$ of homotopy models of a homotopy limit 2-sketch $\mathfrak{S}$. Analogous results hold for $\mathfrak{Der}^l$ and $\mathfrak{Der}^{rl}$, however, here we consider just the case of $\mathfrak{Der}^r$, since this is the one relevant to study of presentability of derivators.

We recall that a biequivalence between 2-categories is a pseudo-functor which is 2-essentially surjective (surjective on objects up to equivalence), and a local equivalence (essentially full on 1-morphisms and full and faithful on 2-morphisms), see [40, 1.1.4-5] and [33, 1.5.13].

Theorem 4.1. *There exists a weighted homotopy limit 2-sketch $\mathfrak{S} = (\mathfrak{G}, \mathcal{P})$ and a biequivalence from the 2-category $\mathfrak{hMod}_{\mathfrak{S}}^{ps}$ to the 2-category $\mathfrak{Der}^r$.*

Since the proof is long, we split it into several parts.

4.1 Idea of the proof

The proof consists of two parts: the first, from subsection 4.2 to 4.7, contains the construction of a homotopy limit 2-sketch $\mathfrak{S} = (\mathfrak{G}, \mathcal{P})$, and the second, in subsection 4.8, the verification that the 2-category $\mathfrak{hMod}_{\mathfrak{S}}^{ps}$ is indeed $\mathfrak{Der}^r$.

The construction of $\mathfrak{S}$ will be carried out as follows. After providing a 2-sketch for prederivators $(\mathfrak{G}, \mathcal{P})$ in subsection 4.2, we will proceed by steps capturing, in subsections 4.3, 4.4, 4.5 and 4.6, each of the four axioms for derivators. More precisely, we will adjoin to $\mathfrak{G}$, at each step, new elements and commutative diagrams, and we will enlarge $\mathcal{P}$ with new cones, in order to express by means of these the axioms for derivators; then, we will redefine $\mathfrak{G}$ as the free 2-category on these data and on the commutativity conditions already in $\mathfrak{G}$ (see remark 4.2 below). Observe that cones in $\mathcal{P}$ are used to capture only axiom 1 and 2.

Remark 4.2. The free construction we use to adjoin new elements to $\mathfrak{G}$ generalizes the analogous construction for ordinary categories (see [5, 5.1]), replacing ordinary graphs with 2-graphs. A 2-graph is a graph "enriched" over the category of small graphs, that is, it is given by a set of vertices and a family of ordinary graphs, one for every pair of vertices (see [33, 1.3.1] for the precise definition). If $2\mathcal{G}r$ denotes the category of 2-graphs and morphisms of 2-graphs, and $2\mathcal{C}at$ the category of 2-categories whose underlying 2-graph belongs to $2\mathcal{G}r$ and 2-functors, then the forgetful functor $2\mathcal{C}at \to 2\mathcal{G}r$ is monadic (see [33, D]).

When a 2-graph contains elements already composable or relations among them, we would like that the free 2-category constructed over it preserves such data. As usual, the idea is to consider, in the given 2-graph, pairs formed by finite sequences of horizontally or vertically composable 2-cells in a prescribed order, sharing horizontal sources and targets, and to require that the components of each pair become equal in the free 2-category. Such pairs, called commutativity conditions, are defined rigorously by Power and Wells [39, 2.5], in terms of labeled pasting schemes, called pasting diagrams in [43]. The proof that pasting 2-cells is well-defined in any 2-category is the subject of [37], of which a brief survey is found in [38, 2]. Denoting by $c2\mathcal{G}r$ the category whose objects are 2-graphs with a set of commutativity conditions and whose morphisms are morphisms of 2-graphs preserving commutativity conditions, a free construction, left adjoint to the forgetful functor $2\mathcal{C}at \to c2\mathcal{G}r$, is provided in Street's paper [43, 5] in terms of "presentations" of 2-categories.

When a 2-graph $\mathfrak{G}$ is built from a 2-category $\mathfrak{C}$ by adjoining new symbols, as in our case, we refer to all the relations among elements of $\mathfrak{C}$ determined by the 2-category structure on $\mathfrak{C}$ as the commutativity conditions defined by $\mathfrak{C}$.

The first step consists in providing a sketch for prederivators.

4.2 Prederivators

Let $\mathfrak{G} = \mathfrak{Dia}^{op}$ and set $\mathcal{P} = \varnothing$. A homotopy model with values in $\mathfrak{Cat}$ is a 2-functor $\mathbb{D} : \mathfrak{G} \to \mathfrak{Cat}$ with domain $\mathfrak{Dia}^{op}$, in other words, a prederivator of domain $\mathfrak{Dia}$. Therefore $\mathfrak{S} = (\mathfrak{G}, \mathcal{P})$ is a homotopy limit 2-sketch whose 2-category $\mathfrak{hMod}_{\mathfrak{S}}^{ps}$ of homotopy models in $\mathfrak{Cat}$ is the 2-category $\mathfrak{PDer}$ of prederivators.

The next steps are concerned with including into the sketch the axioms for derivators.

4.3 Axiom 1

Let $\mathfrak{G} = \mathfrak{Dia}^{op}$ and define $\mathcal{P}$ to be the family of cones in $\mathfrak{Dia}^{op}$ of the form

$$\begin{array}{ccc} & \mathcal{C}_0 \amalg \mathcal{C}_1 & \\ {}^{s_{\mathcal{C}_0}}\swarrow & & \searrow^{s_{\mathcal{C}_1}} \\ \mathcal{C}_0 & & \mathcal{C}_1, \end{array} \tag{8}$$

corresponding to cocones for the coproducts $\mathcal{C}_0 \amalg \mathcal{C}_1$ in $\mathfrak{Dia}$, for any pair of objects $\mathcal{C}_0$ and $\mathcal{C}_1$. Therefore, $s_{\mathcal{C}_0}$ and $s_{\mathcal{C}_1}$ are the arrows in $\mathfrak{Dia}^{op}$ corresponding to the canonical morphisms of the coproduct $\mathcal{C}_0 \amalg \mathcal{C}_1$ taken in $\mathfrak{Dia}$. With the notation of definition 3.2 we can write these cones as

$$(\{0,1\}, F, \delta_e, \mathcal{C}_0 \amalg \mathcal{C}_1, (s_{\mathcal{C}_0}, s_{\mathcal{C}_1})), \tag{9}$$

where $\{0,1\}$ is the discrete 2-category with two objects, $F : \{0,1\} \to \mathfrak{G}$ is the 2-functor mapping i to $\mathcal{C}_i$, for $i = 0, 1$, $\delta_e : \{0,1\} \to \mathfrak{Cat}$ is the constant 2-functor at the terminal category e (which is clearly cofibrant), $\mathcal{C}_0 \amalg \mathcal{C}_1$ denotes the product of $\mathcal{C}_0$ and $\mathcal{C}_1$ in $\mathfrak{Dia}^{op}$ (the coproduct in $\mathfrak{Dia}$) and $s_{\mathcal{C}_i} : \mathcal{C}_0 \amalg \mathcal{C}_1 \to \mathcal{C}_i$ are the canonical projections.

Since models take the product cones (9) to product cones in $\mathfrak{Cat}$, they fulfill the first part of axiom 1. To capture completely axiom 1, we have to include into $\mathcal{P}$ the cone $\varnothing$ with vertex the empty category over the empty diagram, thus forcing $\mathbb{D}(\varnothing) \simeq e$

Observe that $\mathcal{P}$ is a set, as we have assumed that $\mathfrak{Dia}$ is small for the fixed universe $\mathcal{U}$.

4.4 Axiom 2

To capture axiom 2 we need first a reformulation of it in terms of limits. As an intermediate step, we recast it as follows.

Lemma 4.3. *A prederivator $\mathbb{D}$ satisfies axiom 2 if and only if, for any $\mathcal{C} \in \mathfrak{Dia}$, the family of functors $\mathbb{D}(c_D) : \mathbb{D}(\mathcal{C}) \to \mathbb{D}(e)$ induced by the constant functors $c_D : e \to \mathcal{C}$ at $D \in \mathcal{C}$, is jointly conservative, that is, the induced functor*

$$\mathbb{D}(\mathcal{C}) \to \Pi_{D \in \mathcal{C}} \mathbb{D}(e)$$

is conservative.

Conservative functors can be described as follows. Consider a functor $f : A \to B$. Denote by $A^{\mathbb{2}}$ and $B^{\mathbb{2}}$ the categories of arrows of A and B respectively, seen as categories of functors, where $\mathbb{2} = \Delta^1$ is the category corresponding to the ordered set $\{0 < 1\}$. Let $c_A : A \to A^{\mathbb{2}}$ and $c_B : B \to B^{\mathbb{2}}$ denote the canonical inclusions. Let $f^{\mathbb{2}} : A^{\mathbb{2}} \to B^{\mathbb{2}}$ be the functor induced by f via composition. With these data, consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{c_A} & A^{\mathbb{2}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{\mathbb{2}}} \\ B & \xrightarrow[c_B]{} & B^{\mathbb{2}} \end{array} \tag{10}$$

in the 2-category $\mathfrak{Cat}$.

Lemma 4.4. *A functor $f : A \to B$ is conservative if and only if the commutative diagram 10 is a bilimit in $\mathfrak{Cat}$.*

We recall the notion of bilimit: if $F : \mathfrak{E} \to \mathfrak{G}$ and $G : \mathfrak{E} \to \mathfrak{Cat}$ are 2-functors, where $\mathfrak{E}$ is a small 2-category, the bilimit of F weighted by G exists when there is an object $\{G, F\}_b \in \mathfrak{G}$ and for every object $\mathcal{D}$ in $\mathfrak{G}$ an equivalence in $\mathcal{C}\dashv\sqcup$

$$\mathfrak{G}(\mathcal{D}, \{G, F\}_b) \simeq \mathcal{P}s(\mathfrak{E}, \mathfrak{Cat})(G, \mathfrak{G}(\mathcal{D}, F-))$$

natural in $\mathcal{D}$.

Notice, however, that by the isomorphism (6), any bilimit $\{G, F\}_b$ is equivalent to the weighted homotopy limit $\{\mathcal{Q}G, F\}_h$, where $\mathcal{Q}G$ is a cofibrant replacement of G, so that a bilimit is a special case of weighted homotopy limit (definition 3.1).

Proof. The proof of lemma 4.4 is lengthy nevertheless straightforward, so we just outline the idea.

Suppose f is conservative. Observe first that a pseudo-pullback is indeed a bilimit (see [30, 6.12]) and recall its explicit expression (see [5, 7.6.3]): in our case, it is the category whose objects are quintuples (b, w, h, v, g) with $b \in B$, $h \in B^{\mathbb{2}}$, $g \in A^{\mathbb{2}}$, $w : c_B(b) \cong h$, $v : f^{\mathbb{2}}(g) \cong h$, and whose morphisms are triples

$$(x, y, z) : (b, w, h, v, g) \Rightarrow (b', w', h', v', g')$$

with $x : b \to b'$, $y : h \Rightarrow h'$ and $z : g \Rightarrow g'$, such that

$$y \circ w = w' \circ c_B(x)$$
$$y \circ v = v' \circ f^{\mathbb{2}}(z).$$

Denoting by $B \times^{ps}_{B^{\mathbb{2}}} A^{\mathbb{2}}$ the pseudo-pullback of the diagram in figure (10), we have an inclusion of $r : A \to B \times^{ps}_{B^{\mathbb{2}}} A^{\mathbb{2}}$ constructed by means of f. We then define a functor $u : B \times^{ps}_{B^{\mathbb{2}}} A^{\mathbb{2}} \to A$ as follows: on objects (b, w, h, v, g) in $B \times^{ps}_{B^{\mathbb{2}}} A^{\mathbb{2}}$ we set

$$u((b, w, h, v, g)) = g(0),$$

where $0 \in \mathbb{2}$; on morphisms $(x, y, z) : (b, w, h, v, g) \to (b', w', h', v', g')$ we define

$$u((x, y, z)) = z_0,$$

where z_0 denotes the natural transformation z computed at $0 \in \mathbb{2}$. Clearly $ur = 1_A$. That $ru \cong 1_{B \times^{ps}_{B^{\mathbb{2}}} A^{\mathbb{2}}}$, and so that the pair $r : A \rightleftarrows B \times^{ps}_{B^{\mathbb{2}}} A^{\mathbb{2}} : u$ is an equivalence and so A a bilimit, follows from the hypothesis that f is conservative. We omit however this part.

Concerning the converse, observe first that if 10 is a bilimit then (r, u) defined above yields an equivalence $A \simeq B \times^{ps}_{B^{\mathbb{2}}} A^{\mathbb{2}}$. Now, if $n : a \to a'$ is a morphism in A then it defines an object in $A^{\mathbb{2}}$, and, if, in addition, $f(n)$ is also an isomorphism, then it can be extended to an object of $B \times^{ps}_{B^{\mathbb{2}}} A^{\mathbb{2}}$. This finally implies that n is an isomorphism. Again, we omit the details. □

Lemma 4.3 and 4.4 provide a formulation of axiom 2 in terms of limits.

Lemma 4.5. *The functor* $\mathbb{D}(\mathcal{C}) \to \Pi_{D\in\mathcal{C}}\mathbb{D}(e)$ *is conservative if and only if the diagram*

$$\begin{array}{ccc} \mathbb{D}(\mathcal{C}) & \longrightarrow & \mathbb{D}(\mathcal{C})^{\mathbb{2}} \\ \downarrow & & \downarrow \\ \Pi_{D\in\mathcal{C}}\mathbb{D}(e) & \longrightarrow & (\Pi_{D\in\mathcal{C}}\mathbb{D}(e))^{\mathbb{2}} \end{array} \tag{11}$$

is a bilimit, where arrows are as in diagram 10.

Now, as explained in 4.1, we have to add to $\mathcal{P}$ cones, one for each $\mathcal{C} \in \mathfrak{Dia}$, which models will then map to the bilimit (11), thus forcing them to fulfill axiom 2; the weights defining such cones will have to be cofibrant. We proceed as follows.

For every $\mathcal{C} \in \mathfrak{Dia}^{op}$, let $\mathcal{C}'$ denote the category obtained by adjoining an initial object to the discrete category on the objects of $\mathcal{C}$: in other words, $\mathcal{C}'$ is the category whose objects are all those of $\mathcal{C}$ together with a new one $*$ acting as initial object, and whose non-trivial morphisms are just the canonical ones with source the initial object $*$.

Given a derivator $\mathbb{D}$, consider the following functors: a diagram

$$F_{\mathcal{C}} : \mathcal{C}' \to \mathfrak{Cat},$$

which, on objects, maps $*$ to $\mathbb{D}(\mathcal{C})$ and the remaining objects to $\mathbb{D}(e)$, and, on morphisms, sends the morphism $* \to C$, for every object C of $\mathcal{C}$, to the morphism $\mathbb{D}(\mathcal{C}) \to \mathbb{D}(e)$, obtained by applying $\mathbb{D}$ to the functor $c_C : e \to \mathcal{C}$ in $\mathfrak{Dia}$ constant at C in $\mathcal{C}$; a weight

$$G_{\mathcal{C}} : \mathcal{C}' \to \mathfrak{Cat},$$

which, on objects, maps each C of $\mathcal{C}$ to e and $*$ to $\mathbb{2}$, and, on morphisms, takes each $* \to C$ to the canonical morphism $\mathbb{2} \to e$.

We claim that $\{G_{\mathcal{C}}, F_{\mathcal{C}}\}$ is the bilimit (11). This will imply the following form of axiom 2.

Corollary 4.6. *The functor* $\mathbb{D}(\mathcal{C}) \to \Pi_{D\in\mathcal{C}}\mathbb{D}(e)$ *is conservative if and only if* $\mathbb{D}(\mathcal{C}) \cong \{G_{\mathcal{C}}, F_{\mathcal{C}}\}$.

Proof. The claim follows from the observation that a natural transformation $G_{\mathcal{C}} \Rightarrow \mathfrak{Cat}(\{G_{\mathcal{C}}, F_{\mathcal{C}}\}, F_{\mathcal{C}}-)$ consists of:

1. a functor $G_{\mathcal{C}}(*) \to \mathfrak{Cat}(\{G_{\mathcal{C}}, F_{\mathcal{C}}\}, F_{\mathcal{C}}(*))$, that is, a functor $\{G_{\mathcal{C}}, F_{\mathcal{C}}\} \to \mathbb{D}(\mathcal{C})^{\mathbb{2}}$;

2. a functor $G_{\mathcal{C}}(C) \to \mathfrak{Cat}(\{G_{\mathcal{C}}, F_{\mathcal{C}}\}, F_{\mathcal{C}}(C))$ for every object C of $\mathcal{C}$, that is, a functor $\{G_{\mathcal{C}}, F_{\mathcal{C}}\} \to \mathbb{D}(e)$;

3. for every arrow $* \to C$ in $\mathcal{C}'$, with $C \in \mathcal{C}$, a commutative diagram imposing that each composition

$$\{G_{\mathcal{C}}, F_{\mathcal{C}}\} \to \mathbb{D}(e) \to \mathbb{D}(e)^{\mathbb{2}},$$

of the functor in (2) with that induced by $\mathbb{2} \to e$, agrees with the composition

$$\{G_{\mathcal{C}}, F_{\mathcal{C}}\} \to \mathbb{D}(\mathcal{C})^{\mathbb{2}} \to \mathbb{D}(e)^{\mathbb{2}},$$

of the functors in (1) with those induced by $c_C : e \to \mathcal{C}$; of such diagram we display below the part defined by $C \in \mathcal{C}$:

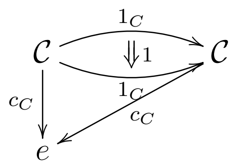
$$\begin{array}{ccc} \{G_{\mathcal{C}}, F_{\mathcal{C}}\} & \longrightarrow & \mathbb{D}(\mathcal{C})^{\mathbb{2}} \\ \downarrow & & \downarrow \\ \mathbb{D}(e) & \longrightarrow & \mathbb{D}(e)^{\mathbb{2}}. \end{array}$$

□

In view of corollary 4.6 we have to impose that the bilimit of diagram (11)), computed by $\{G_{\mathcal{C}}, F_{\mathcal{C}}\}$, is $\mathbb{D}(\mathcal{C})$. To this purpose we consider, for every $\mathcal{C} \in \mathfrak{Dia}^{\mathrm{op}}$, the cone

$$(\mathcal{C}', F'_{\mathcal{C}}, G_{\mathcal{C}}, \mathcal{C}, \gamma), \tag{12}$$

where $\mathcal{C}'$ and $G_{\mathcal{C}}$ have been defined above; $F'_{\mathcal{C}} : \mathcal{C}' \to \mathfrak{G}$ is the functor which, in a way analogous to what $F_{\mathcal{C}}$ does, maps $*$ to $\mathcal{C}$ and the remaining objects to e, and sends the unique morphism $* \to C$, for every object C of $\mathcal{C}$, to the morphism in $\mathfrak{G}$ corresponding to the functor $c_C : e \to \mathcal{C}$ in $\mathfrak{Dia}$ constant at C in $\mathcal{C}$; and γ is a 2-natural transformation $G_{\mathcal{C}} \Rightarrow \mathfrak{G}(\mathcal{C}, F'_{\mathcal{C}}-)$ determined by two identity arrows $\mathcal{C} \to \mathcal{C}$ with the identity 2-morphism between them, and, for each $C \in \mathcal{C}$, by the arrow $c_C : \mathcal{C} \to e$, where the naturality is expressed by the commutativity of the following diagram, of which we display below the part corresponding to $C \in \mathcal{C}$,



Finally, we replace this pseudo-cone by the cone defined by the 2-natural transformation γ' corresponding to γ via the isomorphism (6) which comes after taking a cofibrant replacement $\mathcal{Q}G$ of G. We add to $\mathcal{P}$ all such cones, for every $C \in \mathfrak{Dia}$.

4.5 Axiom 3

If we are constructing a sketch for $\mathfrak{Der}^r$, to capture axiom 3 we freely adjoin to $\mathfrak{G}$ a 1-morphism $u_{(!)} : \mathcal{C} \to \mathcal{D}$ and 2-morphisms $\epsilon_{(u_!)} : u_{(!)}u \Rightarrow 1_{\mathcal{C}}$, $\eta_{(u_!)} : 1_{\mathcal{D}} \Rightarrow uu_{(!)}$, for any 1-morphism $u : \mathcal{D} \to \mathcal{C}$ in $\mathfrak{Dia}^{op}$ which has not already a left adjoint. We impose the following diagrams in $\mathfrak{G}$:

$$\begin{aligned} &(u * \epsilon_{(u_!)}) \circ (\eta_{(u_!)} * u) = 1_u \\ &(\epsilon_{(u_!)} * u_{(!)}) \circ (u_{(!)} * \eta_{(u_!)}) = 1_{u_{(!)}} \end{aligned} \tag{13}$$

These will ensure the existence of a left adjoint to $\mathbb{D}(u)$, for any model $\mathbb{D}$.

We remark that if we are instead interested in a sketch for $\mathfrak{Dia}^l$ then we should adjoin, for any $u : \mathcal{D} \to \mathcal{C}$ in $\mathfrak{Dia}^{op}$ not having a right adjoint, a 1-morphism $u_{(*)} : \mathcal{C} \to \mathcal{D}$ and 2-morphisms $\epsilon_{(u_*)} : uu_{(*)} \Rightarrow 1_{\mathcal{C}}$, $\eta_{(u_*)} : 1_{\mathcal{D}} \Rightarrow u_{(*)}u$, together with diagrams

$$\begin{aligned} &(u_{(u_*)} * \epsilon_{(*)}) \circ (\eta_{(u_*)} * u_{(*)}) = 1_{u_{(*)}} \\ &(\epsilon_{(u_*)} * u) \circ (u * \eta_{(*)}) = 1_u. \end{aligned}$$

If we are constructing a sketch for $\mathfrak{Der}^{rl}$ then all the 1-morphisms, 2-morphisms and relations above should be added.

4.6 Axiom 4

To capture axiom 4 in the sketch for $\mathfrak{Der}^r$, for any diagram in $\mathfrak{Dia}^{op}$ of the form

$$\begin{array}{ccc} D\backslash\mathcal{C} & \xleftarrow{f} & \mathcal{C} \\ {\scriptstyle t}\uparrow & \swArrow_{\alpha} & \uparrow{\scriptstyle u} \\ e & \xleftarrow[d]{} & \mathcal{D} \end{array}$$

(see axiom 4 in definition 2.4 for the meaning of the symbols), we add a 2-morphism $\alpha_{bc}^{-1} : du_{(!)} \Rightarrow t_{(!)}f$ and impose the commutativity conditions

$$\begin{aligned} &[(\epsilon_{(t_!)} * d * u_{(!)}) \circ (t_{(!)} * \alpha * u_{(u_!)}) \circ (t_{(!)} * f * \eta_{(u_!)})] \circ \alpha_{bc}^{-1} = 1_{du_{(!)}} \\ &\alpha_{bc}^{-1} \circ [(\epsilon_{(t_!)} * d * u_{(!)}) \circ (t_{(!)} * \alpha * u_{(u_!)}) \circ (t_{(!)} * f * \eta_{(u_!)})] = 1_{t_{(!)}f}, \end{aligned} \tag{14}$$

provided such a morphism is not already in $\mathfrak{G}$.

If concerned with $\mathfrak{Der}^l$ or $\mathfrak{Der}^{rl}$, we proceed by adapting what done above to the new situation in the obvious way.

4.7 Summary

We summarize the construction of the sketch $\mathfrak{S} = (\mathfrak{G}, \mathcal{P})$ for $\mathfrak{Der}^r$.

4.7.1 Cones

The set $\mathcal{P}$ contains the following cones:

1. $(\{0,1\}, F, \delta_e, \mathcal{C}_0 \amalg \mathcal{C}_1, (s_{\mathcal{C}_0}, s_{\mathcal{C}_1}))$, for any objects $\mathcal{C}_0$ and $\mathcal{C}_1$ of $\mathfrak{Dia}$ (see 4.3);

2. $\varnothing$ the empty cone (see 4.3);

3. $(\mathcal{C}', F'_{\mathcal{C}}, G_{\mathcal{C}}, \mathcal{C}, \gamma')$, for every object $\mathcal{C} \in \mathfrak{Dia}$ (see 12).

4.7.2 $\mathfrak{G}$

The 2-category $\mathfrak{G}$ is the free 2-category on $\mathfrak{Dia}^{op}$ with new symbols and with commutativity conditions adjoined. It is made of the following elements:

1. elements of $\mathfrak{Dia}^{op}$;

2. 1-morphism $u_{(!)}$ and 2-morphisms $\epsilon_{(u_!)}$, $\eta_{(u_!)}$, for every 1-morphism $u \in \mathfrak{Dia}^{op}$ without a left adjoint 4.5;

3. 2-morphism α_{bc}^{-1}, for any 2-morphism $\alpha \in \mathfrak{Dia}^{op}$ as in 4.6;

4. elements obtained as a result of the free construction over the previous elements and the commutativity conditions.

We omit a summary for the sketches for $\mathfrak{Der}^l$ and $\mathfrak{Der}^{rl}$, which can be obtained from the sketch for $\mathfrak{Der}^r$ by making the proper substitutions or additions, as outlined in 4.5 and 4.6.

Remark 4.7. Observe that conservativity can be expressed not only in terms of the bilimit 10, but also by means of the following strict pullback

$$\begin{array}{ccc} A^I & \xrightarrow{b_A} & A^{\mathbb{2}} \\ {\scriptstyle f^I}\downarrow & & \downarrow{\scriptstyle f^{\mathbb{2}}} \\ B^I & \xrightarrow[b_B]{} & B^{\mathbb{2}} \end{array} \tag{15}$$

Since b_B is an isofibration, the pullback above is a homotopy pullback.

If, in order to capture axiom 2, we construct a sketch with cones for each diagram 10, we will have to introduce a new symbol for A^I and a cone to impose what this symbol should be. However, the resulting sketch will be an ordinary 2-sketch, and, since weights are cofibrant, also a homotopy limit 2-sketch.

If considered as an ordinary 2-sketch, to prove biequivalence between models and derivators, since models preserves products strictly while derivators transform coproducts into products up to equivalence, some rigidification will be necessary. This last problem can be faced also by expressing axiom 1 by means of a suitable strict cone, for every $\mathcal{C}_0$ and $\mathcal{C}_1$ in $\mathfrak{Dia}$, and by adjoining an arrow which act as an equivalence between the vertex of such cone and $\mathcal{C}_0 \amalg \mathcal{C}_1$. We could then try to recover 1-morphisms of derivators by restricting to cofibrant models, however, it is not then evident why a cofibrant replacement of a derivator may be identified with some model. Moreover, since the definition of small presentability is up to equivalence, we have preferred a homotopy limit 2-sketch in place of this approach.

4.8 Biequivalence between models and derivators

In this subsection we prove that the 2-category $\mathfrak{Der}^r$ is biequivalent to the 2-category $\mathfrak{hMod}_{\mathfrak{S}}^{ps}$ of models of the homotopy limit 2-sketch $\mathfrak{S}$. If concerned with $\mathfrak{Der}^l$ or $\mathfrak{Der}^{rl}$, the proof is analogous.

We will exhibit a 2-functor

$$\Upsilon : \mathfrak{hMod}_{\mathfrak{S}}^{ps} \longrightarrow \mathfrak{Der}^r,$$

and we will outline why Υ is surjective on objects, full and faithful on both 1-morphisms and 2-morphisms, however, omitting those lenghty verifications which looks nevertheless sufficiently clear for the way the sketch $\mathfrak{S}$ has been constructed.

4.8.1 The 2-functor Υ

Every model $\mathbb{M}$, via the inclusion $\mathfrak{Dia}^{op} \to \mathfrak{S}$, yields a derivator $\Upsilon(\mathbb{M})$.

Given any 1-morphism of models

$$\theta = ((\theta_{\mathcal{X}})_{\mathcal{X}\in\mathfrak{S}}, (\beta_u^{\theta})_{u:\mathcal{X}\to\mathcal{Y}\in\mathfrak{S}}) : \mathbb{M}_1 \to \mathbb{M}_2,$$

consider

$$\Upsilon(\theta) = ((\Upsilon(\theta)_{\mathcal{C}})_{\mathcal{C}\in\mathfrak{Dia}^{op}}, (\beta_u^{\Upsilon(\theta)})_{u:\mathcal{C}\to\mathcal{D}\in\mathfrak{Dia}^{op}}) : \Upsilon(\mathbb{M}_1) \to \Upsilon(\mathbb{M}_2)$$

where

$$\Upsilon(\theta)_{\mathcal{C}} = \theta_{\mathcal{C}}$$

for any $\mathcal{C} \in \mathfrak{Dia}^{op}$, and

$$\beta_u^{\Upsilon(\theta)} = \beta_u^{\theta}$$

for any $u \in \mathfrak{Dia}^{op}$. These data do define a morphism of derivators $\Upsilon(\theta)$: what is left to prove is that $\Upsilon(\theta)$ is cocontinuous, in other words, that, for any $u \in \mathfrak{Dia}^{op}$, the Beck-Chevalley transform $\beta'^{\theta}_{u_{(!)}}$ of β_u^{θ} is an isomorphism; this can be proved directly by showing that $\beta'^{\theta}_{u_{(!)}}$ coincides with $\beta^{\theta}_{u_{(!)}}$ up to isomorphism, however, we omit the lengthy verification.

Concerning Υ on 2-morphisms, a modification $\lambda : \theta_1 \Rrightarrow \theta_2$ in $\mathfrak{hMod}_{\mathfrak{S}}^{ps}$ does define a modification $\Upsilon(\lambda) : \Upsilon(\theta_1) \Rrightarrow \Upsilon(\theta_2)$ in $\mathfrak{Der}^r$, by setting for every $\mathcal{C} \in \mathfrak{Dia}$

$$\Upsilon(\lambda)_{\mathcal{C}} = \lambda_{\mathcal{C}}.$$

It is now straightforward to check that Υ preserves strictly all compositions and identities, and so it is a 2-functor.

4.8.2 Υ is surjective on objects

Any derivator $\mathbb{D}$ can be extended along the canonical functor $\mathfrak{Dia}^{op} \to \mathfrak{G}$ to a model $\Omega(\mathbb{D}) : \mathfrak{G} \to \mathfrak{Cat}$ such that $\Upsilon(\Omega(\mathbb{D})) = \mathbb{D}$. Indeed, it is enough to assign $\Omega(\mathbb{D})$ on the symbols adjoined to $\mathfrak{Dia}^{op}$: by construction of the sketch $\mathfrak{S}$, this assignment is determined by $\mathbb{D}$ itself; for example, $\Omega(\mathbb{D})$ must bring $u_{(!)}$ to a left adjoint $u_!$ to $u^* = \mathbb{D}(u) = \Omega(\mathbb{D})(u)$.

From this we see that two models determining the same derivators are isomorphic.

4.8.3 Υ is full and faithful on 1-morphisms

Consider models $\mathbb{M}_1$ and $\mathbb{M}_2$ and the corresponding derivators $\Upsilon(\mathbb{M}_1)$ and $\Upsilon(\mathbb{M}_2)$. Let $\theta : \Upsilon(\mathbb{M}_1) \to \Upsilon(\mathbb{M}_2)$ be a morphism in $\mathfrak{Der}^r$. We show that we can find a morphism of models $\Omega(\theta) : \mathbb{M}_1 \to \mathbb{M}_2$ such that $\Upsilon(\Omega(\theta)) = \theta$. Let us write

$$\theta = ((\theta_{\mathcal{C}})_{\mathcal{C}\in\mathfrak{Dia}^{op}}, (\beta_u^\theta)_{u:\mathcal{C}\to\mathcal{D}\in\mathfrak{Dia}^{op}}) : \Upsilon(\mathbb{M}_1) \to \Upsilon(\mathbb{M}_2).$$

We start defining

$$\Omega(\theta) = ((\Omega(\theta)_{\mathcal{X}})_{\mathcal{X}\in\mathfrak{G}}, (\beta_u^{\Omega(\theta)})_{u:\mathcal{X}\to\mathcal{Y}\in\mathfrak{G}}) : \mathbb{M}_1 \to \mathbb{M}_2$$

by setting $\Omega(\theta)_{\mathcal{X}} = \theta_X$ for any $X \in \mathfrak{Dia}^{op}$ and $\beta_u^{\Omega(\theta)} = \beta_u^\theta$ for any $u \in \mathfrak{Dia}^{op}$.

We assign now $\Omega(\theta)$ on the symbols adjoined to $\mathfrak{Dia}^{op}$, that is, on $u_{(!)}$, by defining $\beta_{u_{(!)}}^{\Omega(\theta)}$ as the Beck-Chevalley transform of $\beta_u^{\Omega(\theta)}$: with this definition the naturality of $\beta_{u_{(!)}u}^{\Omega(\theta)}$ and of $\beta_{uu_{(!)}}^{\Omega(\theta)}$ with respect to $\epsilon_{u_{(!)}}$ and to $\eta_{u_{(!)}}$ respectively, as well as the coherence conditions, are fulfilled; we skip the verification.

The naturality of $\beta_u^{\Omega(\theta)}$ with respect to 2-morphisms of the form α_{bc}^{-1} is also easily verified.

Therefore, $\Upsilon(\Omega(\theta)) = \theta$, thus proving that Υ is full on 1-morphisms. Since $\beta_{u_{(!)}}^{\Omega(\theta)}$ is completely determined, Υ is also faithful.

4.8.4 Υ is full and faithful on 2-morphisms

Consider a modification $\lambda : \Upsilon(\theta_1) \Rrightarrow \Upsilon(\theta_2)$ in $\mathfrak{Der}^r$, where $\theta_1, \theta_2 : \mathbb{M}_1 \to \mathbb{M}_2$ are 1-morphisms of models. We set

$$\Omega(\lambda)_{\mathcal{C}} = \lambda_{\mathcal{C}}$$

for every object $\mathcal{C}$ in $\mathfrak{Dia}^{op}$.

The commutativity of diagram 5 for $u_{(!)}$ follows from commutativity of diagram 5 for u and the relation between u and $u_{(!)}$ via Beck-Chevalley transforms.

Since $\Omega(\lambda)$ is completely determined by λ, then Υ is full and faithful on 2-morphisms.

5. Homotopy local presentability

5.1 Homotopy locally presentable categories

We recall some definitions and results from [31] regarding homotopy local presentability [31, 9.6] and the characterization [31, 9.13], in the case $\mathcal{V} = \mathcal{C}at$.

We recall the definition of homotopy filtered colimit, by means of which we will introduce homotopy presentability [31, 6.4]. Let λ be a regular cardinal, $\mathfrak{I}$ the free 2-category on an ordinary small λ-filtered category, $F : \mathfrak{I} \to \mathfrak{C}$ a 2-functor, $\delta_e : \mathfrak{I}^{op} \to \mathfrak{Cat}$ the 2-functor constant at the terminal category, $\mathcal{Q}\delta_e$ a cofibrant replacement of δ_e: the homotopy λ-filtered colimit $\mathrm{hocolimF}$ of F is defined as the weighted homotopy colimit $\mathcal{Q}\delta_e \star_h F$. Homotopy filtered colimits are computed up to equivalence by ordinary conical filtered colimits [31, 5.9].

Definition 5.1. *Let $\mathfrak{C}$ be a 2-category. An object C in $\mathfrak{C}$ is homotopy λ-presentable if $\mathfrak{C}(C, -) : \mathfrak{C} \to \mathfrak{Cat}$ preserves homotopy λ-filtered colimits.*

The following is the definition of homotopy locally presentable 2-category [31, 9.6]. Below, a 2-functor $F : \mathfrak{R} \to \mathfrak{S}$ is called a local equivalence if $F_{XX'} : \mathfrak{R}(X, X') \to \mathfrak{S}(F(X), F(X'))$ is an equivalence of categories for every objects X and X' of $\mathfrak{R}$ (see [31, 7] or [40, 1.1.4]).

Definition 5.2. *Let $\mathfrak{C}$ be a 2-category admitting weighted homotopy 2-colimits, $i : \mathfrak{A} \hookrightarrow \mathfrak{C}$ a small full 2-subcategory of homotopy λ-presentable objects. We say that $\mathfrak{A}$ exhibits $\mathfrak{C}$ as strongly homotopy locally λ-presentable if every object of $\mathfrak{C}$ is a homotopy λ-filtered colimit of objects of $\mathfrak{A}$. We say that $\mathfrak{A}$ exhibits $\mathfrak{C}$ as homotopy locally λ-presentable if the induced functor*

$$\mathfrak{C} \xrightarrow{\mathfrak{C}(i,-)} [\mathfrak{A}^{op}, \mathfrak{Cat}] \xrightarrow{\mathcal{Q}} [\mathfrak{A}^{op}, \mathfrak{Cat}]$$

is a local equivalence.

We say that $\mathfrak{C}$ is strongly homotopy locally λ-presentable or homotopy locally λ-presentable if there is some such $\mathfrak{A}$, and that $\mathfrak{C}$ is strongly homotopy locally presentable or homotopy locally presentable if it is so for some λ.

Notice that strongly homotopy local presentability implies homotopy local presentability ([31, 9.7]).

A characterization of homotopy locally presentable 2-categories is [31, 9.13].

Theorem 5.3. *Suppose there exists a combinatorial model 2-category $\mathfrak{D}$ and a biequivalence $\mathfrak{C} \to \mathrm{Int}\mathfrak{D}$, then $\mathfrak{C}$ is strongly homotopy local presentable. Assuming Vopěnka's principle, the converse holds true, and $\mathfrak{D}$ can be taken to be a left Bousfield localization of the 2-category $[\mathfrak{A}^{op}, \mathfrak{Cat}]$, where $\mathfrak{A}$ is as in definition 5.2.*

Note that we will be using only the first part of theorem 5.3 (namely, [31, 9.13]), which does not depend on Vopěnka's principle.

5.2 The 2-category $\mathfrak{hMod}^{ps}_{\mathfrak{S}}$ of homotopy models of $\mathfrak{S}$

We now apply what recalled in 5.1 to $\mathfrak{hMod}^{ps}_{\mathfrak{S}}$. By [31, 9.14(1)] we know that the 2-category of homotopy models of $\mathfrak{S}$ is homotopy locally presentable, however, as we are interested in $\mathfrak{hMod}^{ps}_{\mathfrak{S}}$ where we allow pseudo-natural transformations as 1-morphisms, we show that the same procedure applies also to this case, leading to the same conclusion.

Let $\mathrm{Int}[\mathfrak{G}, \mathfrak{Cat}]$ denote the full 2-subcategory spanned by the flexible 2-functors, that is, the cofibrant objects of $[\mathfrak{G}, \mathfrak{Cat}]$. By means of the cofibrant replacement $\mathcal{Q}$ (see section 3), we have the following result.

Lemma 5.4. *There is a biequivalence $\mathcal{Q} : \mathcal{P}s(\mathfrak{G}, \mathfrak{Cat}) \longrightarrow \mathrm{Int}[\mathfrak{G}, \mathfrak{Cat}]$, provided by the cofibrant replacement functor.*

We soon deduce the following corollary.

Corollary 5.5. *$\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$ is strongly homotopy locally presentable.*

Proof. By [31, 9.8], $\mathrm{Int}[\mathfrak{G}, \mathfrak{Cat}]$ is strongly homotopy locally presentable. The claim now follows now from 5.4 and [31, 9.15]. □

To prove that $\mathfrak{hMod}^{ps}_{\mathfrak{S}}$ is homotopy locally presentable, we show that $\mathfrak{hMod}^{ps}_{\mathfrak{S}}$ is a homotopy orthogonal subcategory of $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$ (see [31, 4.1] for the general definition of homotopy orthogonal). The proof extends the one given in [28, 6.11].

Lemma 5.6. $\mathfrak{hMod}^{ps}_{\mathfrak{S}}$ *is a homotopy orthogonal subcategory of* $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$.

Proof. Consider a cone $(\mathfrak{E}, F, G, \mathcal{L}, \gamma) \in \mathcal{P}$ and the composite, which we denote $i\mathcal{Y}(\gamma)$,

$$G \xrightarrow{\gamma} \mathfrak{G}(\mathcal{L}, F-) \xrightarrow{i\mathcal{Y}_{\mathcal{L},F-}} \mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})(i\mathcal{Y}(F-), i\mathcal{Y}(\mathcal{L})),$$

where $\mathcal{Y}$ indicates the enriched contravariant Yoneda embedding $\mathfrak{G} \to [\mathfrak{G}, \mathfrak{Cat}]$ and i the inclusion $[\mathfrak{G}, \mathfrak{Cat}] \hookrightarrow \mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$. Since $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$ has weighted homotopy 2-colimits (corollary 5.5), $i\mathcal{Y}(\gamma)$ yields a 1-morphism

$$\rho : G \star_h i\mathcal{Y}(F-) \longrightarrow i\mathcal{Y}(\mathcal{L})$$

in $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$.

We prove that a 2-functor $\mathbb{M} : \mathfrak{G} \to \mathfrak{Cat}$ preserves the weighted homotopy 2-limits of $\mathcal{P}$, that is, it is a homotopy model, if and only if, for any $\mathcal{D} \in \mathfrak{Cat}$, the 2-functor $[\mathcal{D}, \mathbb{M}-]$ is homotopy orthogonal in $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$ to the collection of 1-morphisms ρ constructed above from cones of $\mathcal{P}$, namely, the functor $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})(\rho, [\mathcal{D}, \mathbb{M}-])$

$$\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})(i\mathcal{Y}(\mathcal{L}), [\mathcal{D}, \mathbb{M}-]) \longrightarrow \mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})(G \star_h i\mathcal{Y}(F-), [\mathcal{D}, \mathbb{M}-]) \tag{16}$$

is an equivalence of categories.

Since $\mathcal{Y}(\mathcal{L})$ is flexible ([3, 4.6]) and by the enriched Yoneda lemma, we have an equivalence

$$\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})(i\mathcal{Y}(\mathcal{L}), [\mathcal{D}, \mathbb{M}-]) \simeq [\mathcal{D}, \mathbb{M}(\mathcal{L})]. \tag{17}$$

On the other hand, by definition of weighted homotopy 2-colimit, we obtain an equivalence

$$\begin{aligned}\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})(G \star_h i\mathcal{Y}(F-), [\mathcal{D}, \mathbb{M}-]) \simeq \\ \simeq [\mathfrak{G}, \mathfrak{Cat}](G, \mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})(i\mathcal{Y}(F-), [\mathcal{D}, \mathbb{M}-])\end{aligned}$$

and, using again the flexibility of $\mathcal{Y}(\mathcal{L})$ and the enriched Yoneda lemma, an equivalence

$$[\mathfrak{G}, \mathfrak{Cat}](G, \mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})(i\mathcal{Y}(F-), [\mathcal{D}, \mathbb{M}-]) \simeq [\mathfrak{G}, \mathfrak{Cat}](G, [\mathcal{D}, \mathbb{M} \circ F-]. \tag{18}$$

By the equivalences (17) and (18), the functor (16) induces an equivalence

$$[\mathfrak{G}, \mathfrak{Cat}](G, [\mathcal{D}, \mathbb{M} \circ F-] \longrightarrow [\mathcal{D}, \mathbb{M}(\mathcal{L})],$$

or, equivalently, $\mathbb{M}(\mathcal{L}) \simeq \{G, \mathbb{M} \circ F\}_h$, that is, $\mathbb{M}$ takes all the cones of $\mathcal{P}$ to weighted homotopy limit cones. □

Writing Σ for the collection of all morphisms ρ as in lemma 5.6, $\mathfrak{hMod}^{ps}_{\mathfrak{S}}$ can be identified with the homotopy orthogonal subcategory $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})_\Sigma$ of $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$.

Corollary 5.7. $\mathfrak{hMod}^{ps}_{\mathfrak{S}}$ *is strongly homotopy locally presentable, and there are biequivalences*

$$\mathfrak{hMod}^{ps}_{\mathfrak{S}} \longrightarrow \mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})_\Sigma \longrightarrow \mathrm{Int}[\mathfrak{G}, \mathfrak{Cat}]_{\mathcal{Q}(\Sigma)}$$

Proof. By lemma 5.4 and 5.6, the proof follows from proposition [31, 9.9]. □

Observe that $\mathrm{Int}[\mathfrak{G}, \mathfrak{Cat}]$ and $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$ are strongly homotopy locally finitely presentable, as representable functors are homotopy finitely presentables (see [31, 9.8-7.1(3)]). We will prove now that $\mathfrak{hMod}^{ps}_{\mathfrak{S}}$ is strongly homotopy locally λ-presentable, where λ is a regular cardinal which bounds the size of any category in $\mathfrak{Dia}$. First we need a few results summarized in the remark below.

Remark 5.8. (1) By [31, 8.5], $\mathfrak{hMod}^{ps}_{\mathfrak{S}}$ is a homotopy reflective 2-subcategory of $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$. Let j and r denote the inclusion and reflection

$$j : \mathfrak{hMod}^{ps}_{\mathfrak{S}} \rightleftarrows \mathcal{P}s(\mathfrak{G}, \mathfrak{Cat}) : r.$$

Weighted homotopy 2-colimits in $\mathfrak{hMod}^{ps}_{\mathfrak{S}}$ are computed by means of the reflection r from the corresponding weighted homotopy 2-colimit in $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$ (see the proof of [31, 9.9]): if F is a diagram in $\mathfrak{hMod}^{ps}_{\mathfrak{S}}$, then $G \star_h F \simeq r(G \star_h jF)$.

(2) We can now use the biequivalences $\mathcal{Q}$ and i to compute weighted homotopy 2-colimits in $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$: indeed, by [31, 7.1] biequivalences preserve and create weighted homotopy colimits, so if F is a diagram in $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$, then $G \star_h F \simeq G \star_h iQF \simeq i(G \star_h QF)$.

(3) Finally, as explained in the proof of [31, 5.5], weighted homotopy 2-colimits $G \star_h F$ in $\mathrm{Int}[\mathfrak{G}, \mathfrak{Cat}]$ are computed as fibrant replacement of the weighted 2-colimits $G \star F$ in $[\mathfrak{G}, \mathfrak{Cat}]$, so by $G \star F$ itself. The advantage is that weighted 2-colimits in $[\mathfrak{G}, \mathfrak{Cat}]$ are computed pointwise ([28, 3.3]).

(4) It is convenient to replace the weighted homotopy limit 2-sketch $\mathfrak{S} = (\mathfrak{G}, \mathcal{P})$ for derivators with a realized one, that is, whose underlying category has the same objects as $\mathfrak{G}$, whose cones are already homotopy limit cones and whose 2-category of homotopy models is equivalent to that of $\mathfrak{S}$; the proof of the existence of such homotopy limit 2-sketch is analogous to that of [28, 6.21]. We denote this new sketch by $\mathfrak{T}$. In this way, representable 2-functors, which we will write as $\mathfrak{T}(\mathcal{C}, -)$, are automatically homotopy models of $\mathfrak{T}$.

Let λ be a regular cardinal which bounds the size of any category in $\mathfrak{Dia}$.

Lemma 5.9. *Homotopy λ-filtered colimits in $\mathfrak{hMod}^{ps}_{\mathfrak{T}}$ are computed as in $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$, particularly, they are computed pointwise via $\mathcal{Q}$.*

Proof. Let $\mathfrak{I}$ be the free 2-category on an ordinary small λ-filtered category, and $H : \mathfrak{I} \to \mathfrak{hMod}^{ps}_{\mathfrak{T}}$ a 2-functor. We want to prove that the homotopy λ-filtered colimit $\mathrm{hocolim} jH$ in $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$ is indeed the homotopy λ-filtered colimit $\mathrm{hocolim} H$ in $\mathfrak{hMod}^{ps}_{\mathfrak{T}}$, where j denotes the inclusion of $\mathfrak{hMod}^{ps}_{\mathfrak{T}}$ into $\mathcal{P}s(\mathfrak{G}, \mathfrak{Cat})$. To this purpose, we verify that $\mathrm{hocolim} jH$ preserves the weighted homotopy limit cones of $\mathcal{P}$, thus proving that it belongs to $\mathfrak{hMod}^{ps}_{\mathfrak{T}}$.

Notice that, as observed in remark 5.8, $\mathrm{hocolim} jF$ is computed by the pointwise ordinary filtered colimit $\mathrm{colim}\mathcal{Q}jH$ in $[\mathfrak{G}, \mathfrak{Cat}]$.

Since the weighted homotopy limit cones in $\mathcal{P}$ are λ-small, in the sense that they have λ-small diagrams and are weighted by λ-presentable 2-functors, they commute with λ-filtered homotopy colimits ([31, 6.10]). Therefore,

$$\begin{aligned}(\mathrm{colim}\mathcal{Q}jH)(\{G, F\}_h) &\simeq \mathrm{colim}(\mathcal{Q}jH)(\{G, F\}_h)) \\ &\simeq \mathrm{colim}(\{G, \mathcal{Q}jH(F)\}_h) \\ &\simeq \{G, \mathrm{colim}(\mathcal{Q}jH(F))\}_h \\ &\simeq \{G, (\mathrm{colim}(\mathcal{Q}jH)(F)\}_h\end{aligned}$$

□

Finally, the next lemma implies that λ is a degree of homotopy locally presentability for $\mathfrak{hMod}^{ps}_{\mathfrak{T}}$.

Lemma 5.10. *Representable 2-functors on $\mathfrak{T}$ are homotopy λ-presentable objects of $\mathfrak{hMod}^{ps}_{\mathfrak{T}}$. The full 2-subcategory of $\mathfrak{hMod}^{ps}_{\mathfrak{T}}$ spanned by λ-small homotopy 2-colimits of representable models can be taken for the 2-subcategory $\mathfrak{A}$ in definition 5.2.*

Proof. By lemma 5.9 and by the Yoneda lemma for bicategories, representable 2-functors are homotopy λ-presentable objects of $\mathfrak{hMod}^{ps}_{\mathfrak{T}}$.

Since representable models are cofibrant, we can view them as 2-functors in $[\mathfrak{T}, \mathfrak{Cat}]$. From the proof of [31, 9.8], we see that 2-functors which are λ-presentable in $[\mathfrak{T}, \mathfrak{Cat}]$ are homotopy λ-presentable in $\mathrm{Int}[\mathfrak{T}, \mathfrak{Cat}]$. Since $[\mathfrak{T}, \mathfrak{Cat}]$ is locally λ-presentable and representable 2-functors form a set of generators, then every object of $[\mathfrak{T}, \mathfrak{Cat}]$ is a λ-filtered colimit of λ-small colimits of representables. Therefore, by (3) in 5.8, the full 2-subcategory of $\mathrm{Int}[\mathfrak{T}, \mathfrak{Cat}]$ spanned by λ-small homotopy 2-colimits of representable models can be taken as $\mathfrak{A}$ in definition 5.2 for the homotopy λ-presentable 2-category $\mathrm{Int}[\mathfrak{T}, \mathfrak{Cat}]$.

By (1) in remark 5.8 and lemma 5.9, every object of $\mathfrak{hMod}^{ps}_{\mathfrak{T}}$ is a λ-filtered homotopy colimit of λ-small homotopy colimits of representables. □

Corollary 5.11. $\mathfrak{hMod}^{ps}_{\mathfrak{T}}$ *is a homotopy locally λ-presentable 2-category, where λ is a regular cardinal bounding the size of every category in $\mathfrak{Dia}$.*

6. Small presentation

In this section we identify representable models for $\mathfrak{T}$ with a precise type of derivator, and we prove, via the biequivalence in 4.1, that Renaudin's derivators of small presentation are λ-presentable objects.

6.1 Representable models

Denote by $sSet$ the category of simplicial sets with its classical model structure and by $sSet^{\mathcal{C}^{op}}$ the category of simplicial presheaves endowed with the projective model structure. Recall that Φ denotes the pseudo-functor of theorem 2.10. The following result is due to Cisinski (see [11, 3.24]).

Theorem 6.1. *For every right derivator $\mathbb{D}$ and every small category $\mathcal{C}$ in $\mathfrak{Dia}$ there is an equivalence of categories*

$$\mathfrak{Der}^r(\Phi(sSet^{\mathcal{C}^{op}}), \mathbb{D}) \simeq \mathbb{D}(\mathcal{C}).$$

Before outlining how the equivalence in theorem 6.1 is constructed, we rewrite it as follows. Setting $\mathcal{F}(\mathcal{C}) = \Phi(sSet^{\mathcal{C}^{op}})$, we have

$$\Psi : \mathfrak{Der}^r(\mathcal{F}(\mathcal{C}), \mathbb{D}) \simeq \mathbb{D}(\mathcal{C}) : \Xi. \tag{19}$$

Remark 6.2. Consider the morphism of localizers

$$N : (\mathcal{C}at, W_\infty) \longrightarrow (sSet, W_{sSet}),$$

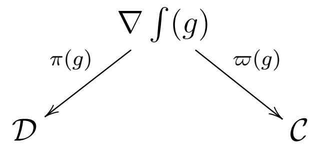
where $N : \mathcal{C}at \to sSet$ is the nerve and W_{sSet} is the class of weak-equivalences of $sSet$ and $W_\infty = N^{-1}W_{sSet}$. This morphism induces an equivalence between the associated derivators, namely, $\mathcal{H}ot_{\mathcal{C}} = [-, \mathcal{C}at^{\mathcal{C}^{op}}][W_\infty^{-1}]$ and $\mathcal{F}(\mathcal{C})$. In view of this, we will use the notation $\mathcal{F}(\mathcal{C})$ also for $\mathcal{H}ot_{\mathcal{C}}$. We refer to [11, 1.1] for more details.

We recall now from [11, 3.18] and [10] how equivalence (19) is constructed. We describe first the functor

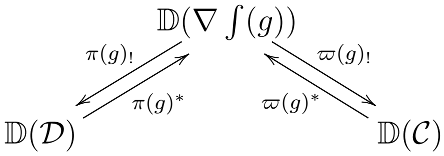
$$\Xi : \mathbb{D}(\mathcal{C}) \longrightarrow \mathfrak{Der}^r(\mathcal{F}(\mathcal{C}), \mathbb{D}).$$

For every $h \in \mathbb{D}(\mathcal{C})$, we indicate how the pseudo-natural transformation $\Xi(h) : \mathcal{F}(\mathcal{C}) \Rightarrow \mathbb{D}$ is defined, by giving the functors $\Xi(h)_{\mathcal{D}}$, for every object

$\mathcal{D} \in \mathfrak{Dia}$, and referring then to [11] for the rest. For $g \in \mathcal{F}(\mathcal{C})(\mathcal{D})$, let ∇g and $\int g$ be the Grothendieck fibration and cofibration associated to $g : \mathcal{D} \times \mathcal{C}^{op} \to \mathcal{C}at$, by fixing $C \in \mathcal{C}^{op}$ and $D \in \mathcal{D}$ respectively. Let $\pi(g) : \nabla \int g \to \mathcal{D}$ and $\varpi(g) : \nabla \int g \to \mathcal{C}^{op}$ be the projections:



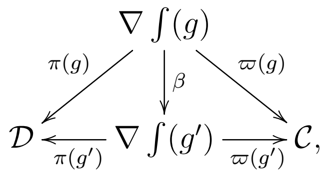
Applying $\mathbb{D}$, we obtain the diagram



The functor $\Xi(h)_{\mathcal{D}} : \mathcal{F}(\mathcal{C})(\mathcal{D}) \to \mathbb{D}(\mathcal{D})$ is defined on objects $g \in \mathcal{F}(\mathcal{C})(\mathcal{D})$ as

$$\pi(g)_! \varpi(g)^*(h). \tag{20}$$

The action of $\Xi(h)_{\mathcal{D}}$ on morphisms is as follows: for $\alpha : g \to g'$ in $\mathcal{F}(\mathcal{C})(\mathcal{D})$, we set $\beta = \nabla \int \alpha$, yielding in $\mathfrak{Dia}$ the commutative diagram



$\Xi(h)_{\mathcal{D}}(\alpha)$ is now defined as the composite

$$\pi(g)_! \varpi(g)^*(h) \cong \pi(g')_! \beta_! \beta^* \varpi(g')^*(h) \longrightarrow \pi(g')_! \varpi(g')^*(h)$$

We refer to [11, 3.19] to complete the definition of $\Xi(h)$.

We now consider the other functor in (19)

$$\Psi : \mathfrak{Der}^r(\mathcal{F}(\mathcal{C}), \mathbb{D}) \longrightarrow \mathbb{D}(\mathcal{C}).$$

As explained in remark 6.2, we can view the Yoneda embedding $\mathcal{Y} : \mathcal{C} \to \mathcal{C}at^{\mathcal{C}^{op}}$ as an object of $\mathcal{F}(\mathcal{C})(\mathcal{C})$. Any 1-morphism of derivators $\theta : \mathcal{F}(\mathcal{C}) \to$

$\mathbb{D}$, when computed at $\mathcal{C}$, yields a functor $\theta_{\mathcal{C}} : \mathcal{F}(\mathcal{C})(\mathcal{C}) \to \mathbb{D}(\mathcal{C})$, whose value $\theta_{\mathcal{C}}(\mathcal{Y})$ at $\mathcal{Y}$ defines $\Psi(\theta)$.

We establish now a correspondence between derivators $\mathcal{F}(\mathcal{C}) = \Phi(sSet^{\mathcal{C}^{op}})$ and representable models $\mathfrak{T}(\mathcal{C}, -)$ of the homotopy limit 2-sketch $\mathfrak{T}$.

Proposition 6.3. *For every $\mathcal{C} \in \mathfrak{Dia}$, the derivator $\Upsilon(\mathfrak{T}(\mathcal{C}, -))$ corresponding to the representable model $\mathfrak{T}(\mathcal{C}, -)$ is equivalent in $\mathfrak{Der}^r$ to $\mathcal{F}(\mathcal{C})$.*

Proof. For the way Υ is defined, the derivator $\Upsilon(\mathfrak{T}(\mathcal{C}, -))$ will be reasonably denoted $\mathfrak{T}(\mathcal{C}, -)$.

On the one hand, equivalence (19) for $\mathbb{D} = \mathfrak{T}(\mathcal{C}, -)$ becomes

$$\Psi : \mathfrak{Der}^r(\mathcal{F}(\mathcal{C}), \mathfrak{T}(\mathcal{C}, -)) \rightleftarrows \mathfrak{T}(\mathcal{C}, \mathcal{C}) : \Xi. \tag{21}$$

Noting that the category $\mathfrak{T}(\mathcal{C}, \mathcal{C})$ has $\mathfrak{Dia}^{op}(\mathcal{C}, \mathcal{C}) = [\mathcal{C}, \mathcal{C}]$ as subcategory, let

$$\varphi : \mathcal{F}(\mathcal{C}) \Rightarrow \mathfrak{T}(\mathcal{C}, -)$$

be the 1-morphism of derivators $\Xi(1_{\mathcal{C}})$.

On the other hand, by the Yoneda lemma for bicategories (see [42, 1.9]) there is an equivalence of categories

$$\Lambda : \mathfrak{hMod}^{ps}_{\mathfrak{T}}(\mathfrak{T}(\mathcal{C}, -), \Omega(\mathcal{F}(\mathcal{C}))) \rightleftarrows \Omega(\mathcal{F}(\mathcal{C}))(\mathcal{C}) : \Pi, \tag{22}$$

where $\Omega(\mathcal{F}(\mathcal{C})$ is any homotopy model such that $\Upsilon\Omega(\mathcal{F}(\mathcal{C})) \simeq \mathcal{F}(\mathcal{C})$ (such models are all equivalent), and, again, we will denote the derivator $\Upsilon\Omega(\mathcal{F}(\mathcal{C}))$ simply as $\Omega(\mathcal{F}(\mathcal{C}))$. Consider the Yoneda embedding $\mathcal{Y} : \mathcal{C} \to \mathcal{C}at^{\mathcal{C}^{op}}$ as an object of $\mathcal{F}(\mathcal{C})(\mathcal{C})$ and, by means of the equivalence above, as an element, which we denote again $\mathcal{Y}$, of $\Omega(\mathcal{F}(\mathcal{C}))$. Let

$$\psi : \mathfrak{T}(\mathcal{C}, -) \Rightarrow \Omega(\mathcal{F}(\mathcal{C}))$$

be the 1-morphism of models $\Pi(\mathcal{Y})$: for $\mathcal{D} \in \mathfrak{G}$ and $g \in \mathfrak{T}(\mathcal{C}, \mathcal{D})$

$$\psi_{\mathcal{D}}(g) = \Omega(\mathcal{F}(\mathcal{C}))(g)(\mathcal{Y}),$$

particularly, when $g : \mathcal{C} \to \mathcal{D}$ is a morphism in $\mathfrak{G}$ corresponding to some $g : \mathcal{D} \to \mathcal{C}$ in $\mathfrak{Dia}$, then $\psi_{\mathcal{D}}(g) = \mathcal{Y} \circ g$. We write ψ also for the morphism of derivators $\Upsilon(\psi)$, and, by the equivalence $\Omega(\mathcal{F}(\mathcal{C}) \simeq \mathcal{F}(\mathcal{C})$, we have $\psi_{\mathcal{D}}(g) \cong \mathcal{Y} \circ g$, for g in $\mathfrak{Dia}$.

To prove the lemma we show there are isomorphic modifications $\varphi \circ \psi \Rrightarrow 1_{\mathfrak{T}(\mathcal{C},-)}$ and $\psi \circ \varphi \Rrightarrow 1_{\mathcal{F}(\mathcal{C})}$.

Formula (20), for $\mathcal{D} \in \mathfrak{G}$ and $g \in \mathcal{F}(\mathcal{C})(\mathcal{D})$, yields

$$\varphi_{\mathcal{D}}(g) = \mathfrak{T}(\mathcal{C}, \pi(g))_! \mathfrak{T}(\mathcal{C}, \varpi(g))(1_{\mathcal{C}}),$$

which we visualize in the diagram

$$\begin{array}{ccccc} & & \mathfrak{T}(\mathcal{C}, \nabla \int(g)) & & \\ & {\scriptstyle \mathfrak{T}(\mathcal{C},\pi(g)_{(!)})} \swarrow \nearrow {\scriptstyle \mathfrak{T}(\mathcal{C},\pi(g))} & & {\scriptstyle \mathfrak{T}(\mathcal{C},\varpi(g))} \nwarrow \searrow {\scriptstyle \mathfrak{T}(\mathcal{C},\varpi(g)_{(!)})} & \\ \mathfrak{T}(\mathcal{C}, \mathcal{D}) & & & & \mathfrak{T}(\mathcal{C}, \mathcal{C}), \end{array}$$

where note that $\mathfrak{T}(\mathcal{C}, \pi(g))_!$ denotes a left adjoint to $\mathfrak{T}(\mathcal{C}, \pi(g))$, and that, viewing $\mathfrak{T}(\mathcal{C}, -)$ as model, $\mathfrak{T}(\mathcal{C}, \pi(g))_!$ equals $\mathfrak{T}(\mathcal{C}, \pi(g)_{(!)})$ up to isomorphism; analogous considerations hold for $\mathfrak{T}(\mathcal{C}, \varpi(g))$ and $\mathfrak{T}(\mathcal{C}, \varpi(g)_{(!)})$. Notice also that $\mathfrak{T}(\mathcal{C}, \varpi(g))$ acts by composing in $\mathfrak{G}$ with the projection $\varpi(g)$, so $\mathfrak{T}(\mathcal{C}, \varpi(g))(1_{\mathcal{C}}) = \varpi(g)$. Similarly $\mathfrak{T}(\mathcal{C}, \pi(g)_{(!)})$ acts by composing in $\mathfrak{G}$ with $\pi(g)_{(!)}$, therefore

$$\varphi_{\mathcal{D}}(g) = \pi(g)_{(!)} \varpi(g).$$

As a consequence we find out that

$$\begin{aligned} \psi \circ \varphi &= \Omega(\mathcal{F}(\mathcal{C}))(\varphi(-))(\mathcal{Y}) \\ &= \Omega(\mathcal{F}(\mathcal{C}))(\pi(-)_{(!)} \varpi(-))(\mathcal{Y}) \\ &= \Omega(\mathcal{F}(\mathcal{C}))(\pi(-)_{(!)}) \Omega(\mathcal{F}(\mathcal{C}))(\varpi(-))(\mathcal{Y}). \end{aligned}$$

So, by the equivalence (19), particularly (20), for $\mathbb{D} = \mathcal{F}(\mathcal{C})$, observing the diagram

$$\begin{array}{ccccc} & & \mathcal{F}(\mathcal{C})(\nabla \int(g)) & & \\ & {\scriptstyle \mathcal{F}(\mathcal{C})(\pi(g))_!} \swarrow \nearrow {\scriptstyle \mathcal{F}(\mathcal{C})(\pi(g))} & & {\scriptstyle \mathcal{F}(\mathcal{C})(\varpi(g))} \nwarrow \searrow {\scriptstyle \mathcal{F}(\mathcal{C})(\varpi(g))_!} & \\ \mathcal{F}(\mathcal{C})(\mathcal{D}) & & & & \mathcal{F}(\mathcal{C})(\mathcal{C}) \end{array}$$

with $g \in \mathcal{F}(\mathcal{C})(\mathcal{D})$, we see that $\psi \circ \varphi$ is isomorphic to $\Xi(\mathcal{Y})$; on the other hand, the image of the identity $1_{\mathcal{F}(\mathcal{C})} \in \mathfrak{Der}^r(\mathcal{F}(\mathcal{C}), \mathcal{F}(\mathcal{C}))$ by Ψ is $\mathcal{Y}$; so

$\psi \circ \varphi$ and $1_{\mathcal{F}(\mathcal{C})}$ are isomorphic in $\mathfrak{Der}^r(\mathcal{F}(\mathcal{C}), \mathcal{F}(\mathcal{C}))$, that is, there is an isomorphic modification $\psi \circ \varphi \Rrightarrow 1_{\mathcal{F}(\mathcal{C})}$.

As to $\varphi \circ \psi$, observe that

$$\begin{aligned}\varphi \circ \psi &= \varphi(\Omega(\mathcal{F}(\mathcal{C}))(-)(\mathcal{Y})) \\ &= \pi(\Omega(\mathcal{F}(\mathcal{C}))(-)(\mathcal{Y}))_{(!)}\varpi(\Omega(\mathcal{F}(\mathcal{C}))(-)(\mathcal{Y})).\end{aligned}$$

The equivalence Λ in (22) maps $\varphi \circ \psi : \mathfrak{T}(\mathcal{C}, -) \Rightarrow \mathfrak{T}(\mathcal{C}, -)$ to the object $\Lambda(\varphi \circ \psi)$ in $\mathfrak{T}(\mathcal{C}, \mathcal{C})$ obtained by computing $\varphi \circ \psi$ at $\mathcal{C}$ and then evaluating at $1_{\mathcal{C}}$:

$$\pi(\Omega(\mathcal{F}(\mathcal{C}))(1_{\mathcal{C}})(\mathcal{Y}))_{(!)}\varpi(\Omega(\mathcal{F}(\mathcal{C}))(1_{\mathcal{C}})(\mathcal{Y})) = \pi(\mathcal{Y})_{(!)}\varpi(\mathcal{Y}).$$

This, by lemma 3.22 in [11], is isomorphic to the identity $1_{\mathcal{C}}$, providing an isomorphic modification $\varphi \circ \psi \Rrightarrow 1_{\mathfrak{T}(\mathcal{C},-)}$. □

As a consequence of lemma 5.10 and proposition 6.3 above we have the following result.

Corollary 6.4. *Any right derivator is a homotopy λ-filtered colimit in $\mathfrak{Der}^r$ of λ-small homotopy 2-colimits of derivators of the form $\mathcal{F}(\mathcal{C}) = \Phi(sSet^{\mathcal{C}^{op}})$.*

6.2 Derivators of small presentation

Let $\mathfrak{ModQ}^c[\mathcal{Q}^{-1}]$ be the pseudo-localization at Quillen equivalences $\mathcal{Q}$ of the 2-category of combinatorial model categories $\mathfrak{ModQ}^c$, as in [40, 2.3]. The following theorem, proved by Renaudin [40, 3.3.2], builds on Dugger's results on universal homotopy theories [13] and on presentations of combinatorial model categories [14].

Theorem 6.5. *The pseudo-functor Φ induces a local equivalence*

$$\tilde{\Phi} : \mathfrak{ModQ}^c[\mathcal{Q}^{-1}] \longrightarrow \mathfrak{Der}_{ad}.$$

Renaudin also describes the essential image of $\tilde{\Phi}$: it is formed by derivators of small presentation. We recall this result and the relevant definitions from [40, 3.4].

Definition 6.6. *Given a prederivator $\mathbb{D}$, a localization of $\mathbb{D}$ is an adjunction $\theta : \mathbb{D} \rightleftarrows \mathbb{D}' : \chi$ such that the counit $\epsilon : \theta \circ \chi \to 1_{\mathbb{D}'}$ is an isomorphism.*

Derivators are invariant under localization, in the sense that a localization of a derivator is again a derivator (see [11, 4.2]).

We recall now from [40, 3.4] the concept of presentation in the case of derivators. The motivation comes from Dugger's definitions of homotopically surjective map ([14, 3.1]) and of presentation of a model category ([14, 1] or [13, 6.1]). We will observe an analogy between the definition of a derivator of small presentation (generation) and the definition, by means of the free construction, of a finitely presented (generated) model of an algebraic theory or module over a ring (see for example [6, 3.8.1]). This analogy relies on the use of "generators" and "relations".

Definition 6.7. *A derivator $\mathbb{D}$ has small generation if there is a category $\mathcal{C} \in \mathcal{C}at$ and a localization $\mathcal{F}(\mathcal{C}) \rightleftarrows \mathbb{D}$.*

Definition 6.8. *A derivator $\mathbb{D}$ has small presentation if it has a small generation $\mathcal{F}(\mathcal{C}) \rightleftarrows \mathbb{D}$ and there is a set S of morphisms in $sSet^{\mathcal{C}^{op}}$, such that the S-local equivalences coincide in $\mathcal{F}(\mathcal{C})(e)$ with the inverse image of the isomorphisms in $\mathbb{D}(e)$ by the induced functor $\mathcal{F}(\mathcal{C})(e) \to \mathbb{D}(e)$. In this case, we call the pair $(\mathcal{C}, S)$ a small presentation for $\mathbb{D}$.*

Let $\mathfrak{Der}_{ad}^{fp}$ be the full 2-subcategory of $\mathfrak{Der}_{ad}$ spanned by derivators of small presentation. The next is the main result of [40].

Theorem 6.9. *There is a biequivalence $\mathfrak{ModQ}^c[\mathcal{Q}^{-1}] \to \mathfrak{Der}_{ad}^{fp}$ induced by $\tilde{\Phi}$.*

Proof. See [40, 3.4.4]. □

As a consequence we see that a derivator has small presentation if and only if it is equivalent to a derivator of the form $\Phi(sSet^{\mathcal{C}^{op}}/S)$, where $sSet^{\mathcal{C}^{op}}/S$ denotes the left Bousfield localization of $sSet^{\mathcal{C}^{op}}$ with respect to S.

For algebraic theories, an intrinsic definition of finitely presented model consists in requiring that the model represents a functor which preserves filtered colimits (see proposition [6, 3.8.14]). A similar situation occurs with finitely presented modules over a ring. We would like to see if anything similar holds for derivators of small presentation. To this purpose, we recall from [44, 5.2] the notion of Bousfield localization of derivators, from which we will deduce a reformulation of small presentation.

Definition 6.10. *A derivator $\mathbb{D}$ admits a left Bousfield localization by a subset S of $\mathbb{D}(e)$ if there exists a cocontinuous morphism of derivators*

$$\gamma : \mathbb{D} \longrightarrow L_S\mathbb{D}$$

mapping the elements of S to isomorphisms in $L_S\mathbb{D}(e)$ and such that for any other derivator $\mathbb{D}'$ the morphism γ induces an equivalence of categories

$$\mathfrak{Der}^r(L_S\mathbb{D}, \mathbb{D}') \longrightarrow \mathfrak{Der}^r_S(\mathbb{D}, \mathbb{D}'),$$

where $\mathfrak{Der}^r_S(\mathbb{D}, \mathbb{D}')$ denotes the category of cocontinuous morphisms of derivators which send the elements of S to isomorphisms in $\mathbb{D}'(e)$.

Small presentation is a special case of Bousfield localization.

Proposition 6.11. *If $\mathbb{D}$ is a derivator of small presentation $(\mathcal{C}, S)$, for some category $\mathcal{C}$ and some set S as in definition 6.8, then $\mathbb{D}$ is equivalent to the left Bousfield localization $L_S\mathcal{F}(\mathcal{C})$.*

Proof. This result, due to Cisinski, is [44, 5.4]. □

We would like now to translate the notions introduced above in terms of models by means of the biequivalence $\Upsilon : \mathfrak{hMod}^{ps}_{\mathfrak{T}} \to \mathfrak{Der}^r$. Note, however, that we can not use this biequivalence to transfer the notion of localization from derivators to models: in general, of the two morphisms forming a localization of derivators only one is a morphism in $\mathfrak{Der}^r$. Nevertheless, we can reformulate finite presentation in terms of models by means of proposition 6.11 as it uses only cocontinuous morphisms.

Observe that, as localizations of categories are coinverters, similarly, derivators of small presentation, regarded as Bousfield localizations, can be written as coinverters.

Lemma 6.12. *If $\mathbb{D}$ is a derivator of small presentation $(\mathcal{C}, S)$, then it is equivalent to the coinverter*

$$\mathbb{D} \simeq \operatorname{coinv}\Big(\mathfrak{T}(\tilde{S}, -) \underset{t}{\overset{s}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow\eta}}}} \mathfrak{T}(\mathcal{C}, -)\Big),$$

computed in $\mathfrak{hMod}^{ps}_{\mathfrak{T}}$, where $\tilde{S}$ is the subcategory of the category of arrows of $\mathcal{C}$ spanned by S (s, t and η are defined below in the proof).

Proof. With $\mathbb{D}$ being identified with a homotopy model in $\mathfrak{hMod}^{ps}_{\mathfrak{T}}$, by the Yoneda lemma, the diagram

$$\mathfrak{T}(\tilde{S},-)\ \underset{t}{\overset{s}{\rightrightarrows}}\!\Downarrow\!\eta\ \mathfrak{T}(\mathcal{C},-) \tag{23}$$

corresponds in $\mathfrak{T}$ to the diagram

$$\mathcal{C}\ \underset{v}{\overset{u}{\rightrightarrows}}\!\Downarrow\!\alpha\ \tilde{S},$$

where $\mathfrak{T}(u,-)=t$, $\mathfrak{T}(v,-)=s$ and $\mathfrak{T}(\alpha,-)=\eta$. As $\mathfrak{T}(\mathcal{C},\tilde{S})\simeq \mathrm{Ho}[\tilde{S}^{op}, sSet^{\mathcal{C}^{op}}]$, the coinverter (23) is completely assigned by choosing v and u to be the obvious source and tail functors, and η the canonical natural transformation between them. Since coiverters are PIE-colimits, and so they compute their non-strict counterparts, and by lemma 6.11, it follows that the universal property of the coinverter (23) is just the universal property of the left Bousfield localization of derivators $\mathbb{D}\simeq L_S\mathcal{F}(\mathcal{C})$. □

Theorem 6.13. *If a model of $\mathfrak{T}$ is a Bousfield localization of a representable one, then it is a homotopy λ-presentable object of $\mathfrak{hMod}^{ps}_{\mathfrak{T}}$.*

Proof. Since, by [31, 9.5], λ-small weighted homotopy colimit of homotopy λ-presentable objects are homotopy λ-presentable, the theorem follows from lemma 6.12. □

As a consequence we deduce the following property for small presentation of derivators.

Theorem 6.14. *A derivator of small presentation is a homotopy λ-presentable object of $\mathfrak{Der}^r$.*

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A BASIS THEOREM FOR 2-RIGS AND RIG GEOMETRY

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Résumé. Un semi-anneau unitaire commutatif (ou *rig*, en abrégé) est *intégral* si $1 + x = 1$. Nous montrons que, de même que le classique 'gros topos' de Zariski associé à un corps algébriquement clos, le topos classifiant $\mathcal{Z}$ des rigs intégraux (réellement) locaux est pré-cohésif sur $\mathbf{Set}$. Le problème principal est de montrer que le morphisme géométrique canonique $\mathcal{Z} \to \mathbf{Set}$ est hyperconnexe essentiel et, encore comme dans le cas classique, le problème se réduit à certains résultats purement algébriques. L'hyperconnectivité est liée à une caractérisation inédite des rigs simples due à Schanuel. L'essentialité est un corollaire d'un analogue d'un 'théorème de la base' prouvée ici pour les rigs avec addition idempotente.

Abstract. A commutative unitary semi-ring (or *rig*, for short) is *integral* if $1 + x = 1$. We show that, just as the classical 'gros' Zariski topos associated to an algebraically closed field, the classifying topos $\mathcal{Z}$ of *(really) local* integral rigs is pre-cohesive over $\mathbf{Set}$. The main problem is to show that the canonical geometric morphism $\mathcal{Z} \to \mathbf{Set}$ is hyperconnected essential and, again as in the classical case, the problem reduces to certain purely algebraic results. Hyperconnectedness is related to an unpublished characterization of simple rigs due to Schanuel. Essentiality is a corollary of an analogue of a 'Basis Theorem' for rigs with idempotent addition proved here.

Keywords. Commutative Algebra, Rig Geometry, Axiomatic Cohesion.
Mathematics Subject Classification (2010). 13A99, 14A20, 18B25, 18F10.

Contents

1. Rig geometry

The present work is motivated by the claim (in the second paragraph of [11]) that some semi-combinatorial non-classical examples of cohesion can be handled in ways analogous to Grothendieck's algebraic geometry. More specifically, we are interested in the construction of 'gros' toposes from certain algebraic categories in a way that abstracts the classical construction of the 'gros' Zariski topos and related toposes. To motivate and outline the contents of the paper it is convenient to recall some of the details of that construction and one source of examples. We assume that the reader is familiar with some basic Topos Theory [12, 6], Lattice Theory and Commutative

Algebra. (Incidentally rings are to be understood as in and [2], i.e. commutative, with unit.)

Definition 1.1. A category $\mathcal{C}$ is called *extensive* if it has finite coproducts and the canonical functor $\mathcal{C}/X \times \mathcal{C}/Y \to \mathcal{C}/(X+Y)$ is an equivalence for any X, Y in $\mathcal{C}$.

For instance, every topos is extensive. In contrast, an additive category is extensive if and only if it is degenerate. If $\mathcal{C}$ is extensive, then so is the slice $\mathcal{C}/X$ for any X. See [3] and references therein.

An object X in an extensive category will be called *connected* if it is not initial and, for every coproduct diagram $X_0 \to X \leftarrow X_1$, either X_0 is initial (in which case $X_1 \to X$ is an isomorphism or X_1 is initial (in which case $X_0 \to X$ is an isomorphism). Roughly speaking, an object is connected if it is not empty and has no coproduct decompositions. An object in a topos is connected if and only if it has exactly two complemented subobjects.

A category is called *coextensive* if its opposite is extensive. If $\mathcal{A}$ is coextensive then, trivially by duality, $A/\mathcal{A}$ is coextensive for every A in $\mathcal{A}$, and an object in $\mathcal{A}^{\mathrm{op}}$ is connected if and only if it is directly indecomposable as an object of $\mathcal{A}$.

Let $\mathbf{Ring}$ be the category of rings.

Lemma 1.2. *The category* $\mathbf{Ring}$ *is coextensive. An object in* $\mathbf{Ring}^{\mathrm{op}}$ *is connected if and only if the corresponding ring has exactly two idempotents.*

Proof. This is well-known but let us sketch a proof. A useful characterization [3, Proposition 2.14] states that a category is extensive if and only if coproducts are universal and disjoint. The dual of this characterization may be applied directly to $\mathbf{Ring}$ as soon as we understand (direct) product decompositions there. Recall that if A is a ring and $e \in A$ is idempotent then the span $A[(1-e)^{-1}] \leftarrow A \to A[e^{-1}]$ is a product diagram. Moreover, this construction determines a bijection between direct decompositions (of A) and idempotents (in A). (If $A \cong B \times C$ is a direct product decomposition then the unique element in A corresponding to $(0,1)$ is the associated idempotent.) It easily follows from this description of direct decompositions that products are codisjoint and couniversal (i.e. stable under pushout). □

If $\mathcal{C}$ is an extensive category then the finite families $(X_i \to X \mid i \in I)$ such that the induced $\sum_{i\in I} X_i \to X$ is an isomorphism form the basis of a

Grothendieck topology. If $\mathcal{C}$ is also small then the associated category of sheaves is a topos that we denote by $\mathfrak{G}\mathcal{C}$ and which is sometimes called the 'Gaeta' topos (of $\mathcal{C}$).

Recall that, for any small category $\mathcal{C}$, the Yoneda embedding $\mathcal{C} \to \widehat{\mathcal{C}}$ of $\mathcal{C}$ into the topos of presheaves on $\mathcal{C}$ preserves limits but not colimits. For instance, every representable object in $\widehat{\mathcal{C}}$ is connected so, if $\mathcal{C}$ has finite coproducts then the Yoneda embedding does not preserve them. On the other hand, if $\mathcal{C}$ is extensive then the Gaeta topology is subcanonical and the resulting full embedding $\mathcal{C} \to \mathfrak{G}\mathcal{C}$ preserves finite coproducts [11].

Recall also that a geometric morphism $f : \mathcal{E} \to \mathcal{S}$ is *essential* if its inverse image f^* has a left adjoint usually denoted by $f_! : \mathcal{E} \to \mathcal{S}$. For example, for any small category $\mathcal{C}$, the canonical geometric morphism $\widehat{\mathcal{C}} \to \mathbf{Set}$ is essential. On the other hand, if $\mathcal{C}$ is small and extensive then $\mathfrak{G}\mathcal{C} \to \mathbf{Set}$ need not be essential; although it is in some cases arising in Algebraic Geometry.

If $\mathcal{C}$ is an extensive category then the full subcategory of connected objects will be denoted by $\mathcal{C}_c \to \mathcal{C}$. The existence of finite coproduct decompositions guarantees that the Gaeta topos is essential as the next result shows.

Lemma 1.3. *Let $\mathcal{C}$ be small and extensive. If every object of $\mathcal{C}$ is a finite coproduct of connected objects then the canonical geometric morphism $\mathfrak{G}\mathcal{C} \to \mathbf{Set}$ is essential.*

Proof. If every object in $\mathcal{C}$ is a finite coproduct of connected objects then the Comparison Lemma [6, Theorem C2.2.3] can be applied and it implies that the restriction functor $\widehat{\mathcal{C}} \to \widehat{\mathcal{C}_c}$ restricts itself to an equivalence $\mathfrak{G}\mathcal{C} \to \widehat{\mathcal{C}_c}$. In other words, in this case, the Gaeta topos of $\mathcal{C}$ is a presheaf topos and so the canonical geometric morphism to $\mathbf{Set}$ is essential. □

Let K be a ring and let $K/\mathbf{Ring}$ be the associated coextensive category of K-algebras. Let $(K/\mathbf{Ring})_{fp} \to K/\mathbf{Ring}$ be the full subcategory of finitely presentable K-algebras. The *category of affine K-schemes (of finite type)* is the opposite of the category $(K/\mathbf{Ring})_{fp}$ and, for brevity, it will be denoted by $\mathbf{Aff}_K$. As $(K/\mathbf{Ring})_{fp} \to K/\mathbf{Ring}$ is closed under finite colimits, $\mathbf{Aff}_K$ has finite limits.

Lemma 1.4. *If the ring K is Noetherian then $\mathbf{Aff}_K$ is extensive and every object is a finite coproduct of connected objects.*

Proof. It is enough to check that the subcategory $(K/\mathbf{Ring})_{fp} \to K/\mathbf{Ring}$ is not only closed under closed under finite colimits but also under finite products, so that the domain inherits coextensivity from the codomain.

Finitely generated K-algebras are closed under finite products for arbitrary K but, if K is Noetherian then Hilbert's Basis Theorem implies that finitely generated K-algebras are finitely presented so, in this case, $(K/\mathbf{Ring})_{fp} \to K/\mathbf{Ring}$ is closed under finite products. Also, a Noetherian K-algebra cannot have an infinite product decomposition. (We will review a proof in a more general context later.) □

We stress the role of Noetherianity and Hilbert's Basis Theorem in the proof of Lemma 1.4. We will come back to the issue. We will see that Noetherianity is not necessary to prove extensivity of $\mathbf{Aff}_K$. On the other hand, the finite-coproduct-decomposition property does not hold in general.

The presheaf topos $\widehat{\mathbf{Aff}_K}$ is the classifier of K-algebras. It embeds (via Yoneda) the category of K-affine spaces and every object in $\widehat{\mathbf{Aff}_K}$ is a colimit of affine spaces. In this sense, $\widehat{\mathbf{Aff}_K}$ is a topos of 'K-schemes' but, it does not have the 'right' colimits. In particular, it does not have the right coproducts. Extensivity of $\mathbf{Aff}_K$ permits to solve this problem because we may consider the subtopos $\mathfrak{G}(\mathbf{Aff}_K) \to \widehat{\mathbf{Aff}_K}$ and the finite-coproduct preserving restricted Yoneda embedding $\mathbf{Aff}_K \to \mathfrak{G}(\mathbf{Aff}_K)$, into another topos of 'K-schemes' so to speak, but with better coproducts. (See also [11, Section 5] for a more conceptual discussion on the inexactness of affine schemes.)

Lemma 1.5. *If the ring K is Noetherian then the canonical geometric morphism $\mathfrak{G}(\mathbf{Aff}_K) \to \mathbf{Set}$ is essential.*

Proof. By Lemma 1.4, Lemma 1.3 is applicable to the case $\mathcal{C} = \mathbf{Aff}_K$. □

Let $f : \mathfrak{G}(\mathbf{Aff}_K) \to \mathbf{Set}$ be the canonical geometric morphism. For general reasons, the direct image $f_* : \mathfrak{G}(\mathbf{Aff}_K) \to \mathbf{Set}$ sends X in $\mathfrak{G}(\mathbf{Aff}_K)$ to the set $f_*X = \mathfrak{G}(\mathbf{Aff}_K)(1, X) = X1$ of points of X. More explicitly, in this case, it sends a sheaf $X : (K/\mathbf{Ring})_{fp} \to \mathbf{Set}$ to the set $f_*X = XK$ where K is considered as the initial object of $(K/\mathbf{Ring})_{fp}$. In particular, if X is representable by A in $(K/\mathbf{Ring})_{fp}$ then

$$f_*X = (K/\mathbf{Ring})_{fp}(A, K) = (K/\mathbf{Ring})(A, K)$$

is the set of algebra morphisms from A to the base ring K.

As stressed in [10, Section II], even if we assume that K is a field, the leftmost adjoint $f_! : \mathfrak{G}(\mathbf{Aff}_K) \to \mathbf{Set}$ need not preserve finite products. (See also [16, Example 4.8].) This observation partially motivates the following axiomatization of a topos 'of spaces' over a topos 'of sets'.

Definition 1.6. A geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is called *pre-cohesive* if the adjunction $p^* \dashv p_*$ extends to a string $p_! \dashv p^* \dashv p_* \dashv p^!$ of adjoint functors such that $p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful, $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products and (Nullstellensatz) the canonical transformation $\theta : p_* \to p_!$ is epic.

The intuition is that $\mathcal{E}$ is a 'gros' topos over a topos $\mathcal{S}$ of 'sets'. (See [10], also [16, 14] and references therein.) So the objects of $\mathcal{E}$ are 'spaces' of some kind, $p^* : \mathcal{S} \to \mathcal{E}$ is the full subcategory of discrete spaces and its right adjoint $p_* : \mathcal{E} \to \mathcal{S}$ sends a space X to the set p_*X of points of X. The leftmost adjoint $p_! : \mathcal{E} \to \mathcal{S}$ sends a space X to the set $p_!X$ of 'pieces' of X. The Nullstellensatz condition formulated above captures the idea that 'every piece has a point'. (In the presence of a string $p_! \dashv p^* \dashv p_* \dashv p^!$ with fully faithful $p^*, p^! : \mathcal{S} \to \mathcal{E}$, the Nullstellensatz is equivalent to $p : \mathcal{E} \to \mathcal{S}$ being *hyperconnected*, i.e. that both the unit an counit of $p^* \dashv p_*$ are monic [7].)

Proposition 1.7. *If K is an algebraically closed field then the essential geometric morphism $\mathfrak{G}(\mathbf{Aff}_K) \to \mathbf{Set}$ is pre-cohesive.*

Proof. We already know by Lemma 1.5 that $\mathfrak{G}(\mathbf{Aff}_K) \to \mathbf{Set}$ is essential. In fact, we know it is essential because $\mathfrak{G}(\mathbf{Aff}_K)$ is the topos of presheaves on the category of connected affine K-schemes. So it is enough to apply a characterization of the small categories whose associated presheaf topos is pre-cohesive over $\mathbf{Set}$ [7]: for a small category $\mathcal{D}$ whose idempotents split, the canonical $\widehat{\mathcal{D}} \to \mathbf{Set}$ is pre-cohesive if and only if $\mathcal{D}$ has a terminal object and every object has a point. (See also [16, Proposition 2.10].)

Let $\mathcal{C} = \mathbf{Aff}_K$ be the category of K-affine schemes. Since it has finite limits, idempotents split. Moreover, this property is inherited by the subcategory $\mathcal{C}_c$ of connected objects. As K is a field, it is directly indecomposable. Hence, the terminal object of $\mathbf{Aff}_K$ is connected and so $\mathcal{C}_c$ has a terminal object. Hilbert's Nullstellensatz implies that every object in $\mathcal{C}_c$ has a point. Then, by the result cited in the previous paragraph, $\mathfrak{G}(\mathbf{Aff}_K) = \widehat{\mathcal{C}_c} \to \mathbf{Set}$ is pre-cohesive. $\square$

If K is not algebraically closed then K-affine spaces still induce pre-cohesive geometric morphisms $\mathcal{E} \to \mathcal{S}$, but over a base $\mathcal{S}$ more informative than $\mathbf{Set}$ such as the Galois topos of the base field. See [10] and [16].

The classical 'gros' Zariski topos $\mathcal{Z}_K$ determined by a field K is a (non-presheaf) subtopos of $\mathfrak{G}(\mathbf{Aff}_K)$ and, if K is an algebraically closed field, the canonical geometric morphism $\mathcal{Z}_K \to \mathbf{Set}$ is pre-cohesive, but we need not go into that at this point.

So far we have used extensive categories to sketch some basic constructions in classical algebraic geometry which, in particular, produce a pre-cohesive topos $\mathfrak{G}(\mathbf{Aff}_K)$ over $\mathbf{Set}$ for any algebraically closed field. This sketch will prove useful to recall some of the material in [11] that motivates the original work in the present paper.

Definition 1.8. A *rig* is a set A equipped with two commutative monoid structures $(A, \cdot, 1)$ and $(A, +, 0)$ such that 'product distributes over addition' in the sense that $x \cdot 0 = 0$ and $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ for every $x, y, z \in A$.

The category of rigs and homomorphisms between them will be denoted by $\mathbf{Rig}$. Evidently, the category of rings may be seen as the full subcategory $\mathbf{Ring} \to \mathbf{Rig}$ of those rigs such that the underlying additive structure is a (necessarily Abelian) group. On the other hand, the category of (bounded) distributive lattices appears as the full subcategory $\mathbf{dLat} \to \mathbf{Rig}$ consisting of those rigs such that multiplication is idempotent and the equation $1 + x = 1$ holds [11, Section 8].

It is well-know that many of the constructions among rings have analogues for semi-rings and, in particular, for rigs. For instance, given a multiplicative submonoid $F \subseteq A$ of a rig A it is possible to construct the rig of fractions $A \to A[F^{-1}]$ much as in the case of rings. In particular, for $a \in A$ and $F = \{a^n \mid n \in \mathbb{N}\}$ we will write $A[a^{-1}]$ instead of $A[F^{-1}]$.

Lack of negatives implies that the treatment of idempotents is a little more subtle than in rings. An element b in a rig is called *Boolean* if there is a (necessarily unique) c such that $b + c = 1$ and $bc = 0$. In this case c may be called the *complement* of b. If b is Boolean then it is idempotent.

Proposition 1.9. *The category* $\mathbf{Rig}$ *is coextensive. An object in* $\mathbf{Rig}^{\mathrm{op}}$ *is connected if and only if the corresponding rig has exactly two Boolean elements.*

Proof. Essentially as in rings: direct product decompositions correspond to Boolean elements. More precisely, if A is a rig and $b \in A$ is Boolean (with complement c) then the canonical map $A \to A[b^{-1}] \times A[c^{-1}]$ is an isomorphism. Moreover, every direct product decomposition $A \to B \times C$ is determined as above by a unique Boolean element in A. The argument can be completed as in Lemma 1.2 which then may be seen as a corollary of the present result. □

As in the case of rings we may consider, for a rig K, the coextensive category $K/\mathbf{Rig}$ of K-*rigs* (or K-*algebras*). Notice that for a ring K, the canonical functor $K/\mathbf{Ring} \to K/\mathbf{Rig}$ is an equivalence. On the other hand, of special interest for us is the case of $K = 2$ the distributive lattice with two elements. In this case, $2/\mathbf{Rig} \to \mathbf{Rig}$ may be identified with the full subcategory of rigs with idempotent addition. For any rig K let $(K/\mathbf{Rig})_{fp} \to K/\mathbf{Rig}$ be the full subcategory of finitely presentable K-algebras.

Definition 1.10. The *category of affine K-spaces* is the opposite of the category $(K/\mathbf{Rig})_{fp}$ and it will be denoted by $\mathbf{Aff}_K$.

In Section 2 we prove that $\mathbf{Aff}_K$ is extensive, generalizing one of the two aspects of Lemma 1.4. Sections 3 to 5 culminate in the proof that the second aspect of Lemma 1.4 holds for the case $K = 2$. In other words, we prove that every object in $\mathbf{Aff}_2$ is a finite coproduct of connected objects. This requires the introduction of a suitable notion of Noetherian rig (Section 3) and related 'Basis Theorem' (Section 4) which is probably the main original result of the paper.

Sections 6 proves a Nullstellensatz for 2-rigs (essentially due to Schanuel) which is used in Section 7 to show an analogue of Proposition 1.7 for $K = 2$, namely, that the Gaeta topos of $\mathbf{Aff}_2$ is pre-cohesive over $\mathbf{Set}$. We also give a proof of the folk fact that the Gaeta topos classifies 2-rigs 'without Boolean elements' and that the generic model therein satisfies the Kock-Lawvere axiom for Synthetic Differential Geometry [8].

Our proof of the Nullstellensatz for 2-rigs involves another coextensive variety of rigs that we introduce below.

Definition 1.11. A rig A is *integral* if $1 + x = 1$ for every $x \in A$.

Without a name, integral rigs are briefly considered in [11, Section 8] where there the free integral rig I on one generator x is described as the order $\{0 < \ldots < x^n < \ldots < x^2 < x < x^0 = 1\}$ with the obvious multiplication and it is suggested that the spectrum of I can be visualized as an interval (not a lattice). It is also suggested that I may be viewed as an extended positive line by reading the structure logarithmically, suggesting a connection with tropical geometry.

The coextensive category $\mathbf{iRig}$ of integral rigs and morphisms between them is also studied in [4] where it is shown that, as in the Zariski representation of rings, every integral rig is the algebra of sections of a sheaf of *really local* integral rigs.

Let $\mathbf{iRig}$ be the category of integral rigs. Let $\mathbf{iRig}_{fp} \to \mathbf{iRig}$ be the full subcategory of finitely presentable integral rigs and let **iAff** be the opposite of $\mathbf{iRig}_{fp}$. Using the tools developed for the proof of the Nullstellensatz for 2-rigs we show in Section 8 that **iAff** is extensive and that the associated Gaeta topos is pre-cohesive over $\mathbf{Set}$. We also sketch a proof of the folk result that this topos classifies integral rigs without idempotents.

In Section 9 we recall the definition of really local integral rigs and show that the generic integral rig without idempotents is not really local in the Gaeta topos of **iAff**. In the classical case, the analogous fact that the generic ring without idempotents is not local may be seen as motivating the consideration of the Zariski topos. Section 10 proves that **iAff** has an analogue of the Zariski topology. This topology is proved to be subcanonical in Section 12. It is also proved there that the resulting topos is pre-cohesive and classifies really local integral rigs.

Altogether, the new Basis Theorem and Nullstellensatz for 2-rigs allow us to show that the classifying toposes of certain extensions of the theory of rigs with idempotent addition are 'gros' in the sense of Axiomatic Cohesion. It might be interesting to compare these with the various categories of 'tropical schemes' such as those in [5] and references therein.

As expected, much of the work reported below concerns ideals, so let us quickly recall a couple of basic facts in the context of rigs.

Definition 1.12. An *ideal* of a rig R is an additive submonoid $I \subseteq R$ such that for every $r \in R$ and $y \in I$, $ry \in I$.

If $a \in R$ then the subset $(a) = \{ra \mid r \in R\} \subseteq R$ is a *principal* ideal of the rig R. Ideals in rings coincide with the classical notion.

Every ideal $I \subseteq A$ determines the relation $\approx_I \subseteq A \times A$ defined by $x \approx_I y$ if and only if there are $i, j \in I$ such that $x + i = y + j$. It is straightforward to check that $\approx_I$ is a congruence.

Lemma 1.13. *For any ideal $I \subseteq A$ the quotient $q : A \to A/{\approx_I}$ is the universal map from A sending every element of I to 0. Also, the kernel $q^{-1}0 \subseteq A$ coincides with the ideal $\{x \in A \mid (\exists s \in I)(x + s \in I)\} \subseteq A$.*

Proof. The quotient $A \to A/{\approx_I}$ maps every $t \in I$ to 0. Now let $f : A \to B$ in $\mathbf{Rig}$ be such that $fI = \{0\}$. If $x \approx_I y$ then there are $t, t' \in I$ such that $x + t = y + t'$ in A and so $fx = f(x + t) = f(y + t') = fy$.

Finally, $qx = 0$ if and only if $x \approx_I 0$ in A. This holds if and only if there are $s, s' \in I$ such that $x + s = 0 + s' = s'$. In turn, this is equivalent to the existence of an $s \in I$ such that $x + s \in I$. □

Naturally, the quotient of A by $\approx_I$ will be denoted by $A \to A/I$. Its kernel will be called the *saturation* of I and will be denoted by $\overline{I} \subseteq A$. Of course, $I \subseteq \overline{I} \subseteq A$. The ideal I will be called *saturated* if $I = \overline{I}$ as ideals of A. Notice that, in a ring, every ideal is saturated.

Lemma 1.14. *If $b \in A$ is a Boolean element (with complement c) of the rig A then $A \to A[b^{-1}]$ and $A \to A/(c)$ coincide in the sense that each has the universal property of the other.*

Proof. A Boolean element is invertible if and only if its complement is 0. □

2. The extensive category of affine K-schemes

Fix a rig K. The purpose of the present section is to show that $\mathbf{Aff}_K$ is extensive. We actually show that the subcategory $(K/\mathbf{Rig})_{fp} \to K/\mathbf{Rig}$, which is closed under finite colimits, is also closed under finite products and therefore the domain inherits coextensivity from the codomain. (This may be a folk fact but we have not found it in the literature. It is certainly classical for the case of Noetherian rings K. See Lemma 1.4.)

The full subcategory $(K/\mathbf{Rig})_{fp} \to K/\mathbf{Rig}$ contains the terminal object because it may be presented as $K/(1)$ where (1) is the principal ideal generated by 1. So we are interested in sufficient conditions for the subcategory to be closed under finite products. By [15, Proposition 3.6] it is enough to check that the product of two finitely generated free K-rigs is finitely presented.

The free K-rig on a set S may be identified with the rig of polynomials $K[S]$ with coefficients in K and 'variables' in S. Let S and T be two finite sets. The product $K[S] \times K[T]$ is easily seen to be (finitely) generated by $(1,0)$, $(0,1)$, $(s,0)$ for any $s \in S$ and $(0,t)$ for any $t \in T$. To prove that the product is finitely presented we need to be more detailed so consider the free K-rig $K[S + T + \{\sigma, \tau\}]$. Let $L : K[S + T + \{\sigma, \tau\}] \to K[S]$ be the unique morphism of K-rigs such that $Ls = s$ for every $s \in S$, $Lt = 0$ for every $t \in T$, $L\sigma = 1$ and $L\tau = 0$. The morphism L sends a polynomial $p(S, T, \sigma, \tau) \in K[S + T + \{\sigma, \tau\}]$ to $p(S, 0, 1, 0) \in K[S]$. Similarly, we let $R : K[S + T + \{\sigma, \tau\}] \to K[T]$ be the unique morphism of K-rigs such that $Rs = 0$, $Rt = t$, $R\sigma = 0$ and $R\tau = 1$.

Lemma 2.1. *The map* $\langle L, R\rangle : K[S + T + \{\sigma, \tau\}] \to K[S] \times K[T]$ *is surjective.*

Proof. The map $\langle L, R\rangle$ sends σ to $(1,0)$, τ to $(0,1)$, $s \in S$ to $(s,0)$ and $t \in T$ to $(0,t)$. □

Lemma 2.1 is just another way of saying that finite products of finitely generated free K-rigs are finitely generated. It remains to show that congruence determined by the quotient $\langle L, R\rangle$ is finitely generated.

Lemma 2.2. *The following elements of* $K[S + T + \{\sigma, \tau\}]$

1. st *for every* $s \in S$ *and* $t \in T$,
2. $t\sigma$ *for every* $t \in t$,
3. $s\tau$ *for every* $s \in S$,
4. $\sigma\tau$

are in the kernel of $\langle L, R\rangle$. *Also,* $\langle L, R\rangle(\sigma + \tau) = 1 \in K[S] \times K[T]$.

Proof. Notice that $\langle L, R\rangle(t \cdot \sigma) = (0 \cdot 1, t \cdot 0) = (0, 0) \in K[S] \times K[T]$ for $t \in T$ and $\langle L, R\rangle(\sigma + \tau) = (1 + 0, 0 + 1) = (1, 1)$. We leave the details for the reader. □

Let $\approx$ be the congruence on $K[S + T + \{\sigma, \tau\}]$ generated by the relations

$$st \approx t\sigma \approx s\tau \approx \sigma\tau \approx 0 \quad \sigma + \tau \approx 1$$

for $s \in S$ and $t \in T$. We stress that, as S and T are finite, the congruence $\approx$ is finitely generated. By Lemma 2.2 there exists a unique morphism $\Gamma : K[S + T + \{\sigma, \tau\}]/\approx \to K[S] \times K[T]$ such that the following diagram commutes

$$\begin{array}{ccc} K[S + T + \{\sigma, \tau\}] & \longrightarrow & K[S + T + \{\sigma, \tau\}]/\approx \\ & \searrow^{\langle L,R \rangle} & \downarrow \Gamma \\ & & K[S] \times K[T] \end{array}$$

and Γ is surjective because $\langle L, R\rangle$ is so by Lemma 2.1.

Proposition 2.3. *For any rig K the subcategory $(K/\mathbf{Rig})_{fp} \to K/\mathbf{Rig}$ is closed under finite products and it is therefore coextensive.*

Proof. We continue the argument preceding the statement. It remains to show that Γ is injective. For brevity let $W = K[S + T + \{\sigma, \tau\}]/\approx$.

As σ and τ complement each other in W, they are Boolean and therefore idempotent. Together with the first four items of Lemma 2.2 we deduce that every element of W is of the form $k + p(S) + q(T) + k_\sigma\sigma + k_\tau\tau$ with $p(S) \in K[S]$ and $p(0) = 0$, $q(T) \in K[T]$ and $q(0) = 0$, and $k, k_\sigma, k_\tau \in K$. Moreover, as $k = k(\sigma + \tau) = k\sigma + k\tau$ we conclude that every element of W is of the form

$$p(S) + q(T) + k_\sigma\sigma + k_\tau\tau$$

with $p(S) \in K[S]$ and $p(0) = 0$, $q(T) \in K[T]$ and $q(0) = 0$, and $k_\sigma, k_\tau \in K$.

Let $p'(S) + q'(S) + k'_\sigma\sigma + k'_\tau\tau$ be another element of W in the same 'normal form' and assume that Γ sends them both to the same thing. That is,

$$(p(S) + k_\sigma, q(T) + k_\tau) = (p'(S) + k'_\sigma, q'(T) + k'_\tau)$$

in $K[S] \times K[T]$. Then $p(S) = p'(S)$, $k_\sigma = k'_\sigma$, $q(T) = q'(T)$ and $k_\tau = k'_\tau$. Hence,

$$p(S) + q(T) + k_\sigma\sigma + k_\tau\tau = p'(S) + q'(S) + k'_\sigma\sigma + k'_\tau\tau$$

in W completing the proof that Γ is injective. □

Corollary 2.4. *The category $\mathbf{Aff}_K$ is extensive for any rig K.*

3. Noetherian rigs

In this section we introduce a notion of Noetherianity for rigs involving saturated ideals as defined in Section 1 and which abstracts the standard notion for rings.

Let A be a rig.

Lemma 3.1. *If $I_0 \subseteq I_1 \subseteq \ldots$ is a sequence of saturated ideals of A then so is the union $I = \bigcup_{n\in\mathbb{N}} I_n$.*

Proof. Let $x \in A$ and $s \in I$ be such that $x + s \in I$. Then there are $m, n \in \mathbb{N}$ such that $s \in I_m$ and $x + s \in I_n$. Then $s, x + s \in I_{m+n}$ and, as S_{m+n} is saturated, $x \in I_{m+n} \subseteq I$ by Lemma 1.13. □

For any family $(x_s \in A \mid s \in S)$ there is a least ideal containing the elements in that family. It is called the ideal *generated* by the family. Its elements are those of the form $\sum_{i\in I} a_i x_i$ for some finite subset $I \subseteq S$ and $a_i \in A$ for each $i \in I$. An ideal of A is *finitely generated* if it is generated by finite family. We next introduce something less standard.

Definition 3.2. A saturated ideal is *essentially finitely generated* if it is the saturation of a finitely generated ideal.

Of course, a finitely generated saturated ideal is essentially finitely generated. In the case of rings the converse holds because ideals of rings are saturated.

Lemma 3.3. *The following are equivalent:*

1. *Every sequence $I_0 \subseteq I_1 \subseteq \ldots$ of saturated ideals of A is stationary; that is, there is an $m \in \mathbb{N}$ such that $I_m = I_n$ for every $n \geq m$.*
2. *Every saturated ideal $I \subseteq A$ is essentially finitely generated.*

Proof. (Just as in the classical case, but taking the necessary precautions to deal with saturation.) Assume that the first item holds and, for the sake of contradiction, let $I \subseteq A$ be a saturated ideal that is not essentially finitely generated. Choose an element $s_0 \in I$, let $S_0 \subseteq A$ be the ideal generated by s_0 and let $\overline{S_0}$ be the saturation which is, of course, essentially finitely generated. Certainly, $\overline{S_0} \subseteq I$ but, as I is not essentially finitely generated, there is an $s_1 \in I$ such that $s_1 \notin \overline{S_0}$. Let $S_1 \subseteq A$ be the ideal generated by s_0, s_1. Then $\overline{S_0} \subset \overline{S_1} \subset I$. Again, there must exist an $s_2 \in I$ such that $s_2 \notin \overline{S_1}$ and continuing with this process we obtain a sequence $\overline{S_0} \subset \overline{S_1} \subset \ldots$ of saturated ideals of A that is not stationary; a contradiction.

Conversely, assume that the second item holds. The union $I = \bigcup_{n\in\mathbb{N}} I_n$ is a saturated ideal by Lemma 3.1 so, by hypothesis, it is essentially finitely generated. Let $(g_s \in I \mid s \in S)$ be a finite family generating an ideal J such that $\overline{J} = I$. Then there are $m_s \in \mathbb{N}$ such that $g_s \in I_{m_s}$. As the set S is finite, $g_s \in I_m$ for $m = \sum_{t\in S} m_t$ and every $s \in S$, so $J \subseteq I_m$. Then $I = \overline{J} \subseteq \overline{I_m} = I_m$ so $I_m = I$. □

Although congruences of rigs are not in bijective correspondence with ideals, the following terminology seems fair.

Definition 3.4. A rig A will be called *Noetherian* if it satisfies the equivalent conditions of Lemma 3.3. Also, a rig is *strongly Noetherian* if every saturated ideal in it is finitely generated.

Of course, strongly Noetherian implies Noetherian; and the converse holds for rings. So a ring is Noetherian in the present 'rig sense' if and only if it is Noetherian in the classical sense.

The following lemmas will be needed later and are simple variations of standard facts about Noetherian rings. The proofs are also variations that take saturation into account. (Recall that, in algebraic categories, regular epimorphisms coincide with surjections.)

Lemma 3.5. *If $A \to B$ is a regular epi in* $\mathbf{Rig}$ *and A is Noetherian then so is B.*

Proof. Let $f : A \to B$ be a map in $\mathbf{Rig}$. For any ideal $I \subseteq B$, the inverse image $f^{-1}I \subseteq A$ is an ideal. Moreover, if I is saturated then so is $f^{-1}I$. Also, if $J \subseteq B$ is another ideal and $I \subseteq J$ then $f^{-1}I \subseteq f^{-1}J$. Hence, every ascending sequence $I_1 \subseteq I_2 \subseteq \ldots$ of saturated ideals of B determines an ascending sequence $f^{-1}I_1 \subseteq f^{-1}I_2 \subseteq \ldots$ of saturated ideals of A. As A is Noetherian, this sequence is stationary. So, to complete the proof, it is enough to prove the following lemma: For $I \subseteq J$ ideals of B such $f^{-1}I = f^{-1}J$, if f is surjective then $I = J$. In turn, it is enough to show that $J \subseteq I$. So let $b \in J$. As f is surjective, $b = fa$ for some $a \in A$. Then $a \in f^{-1}J = f^{-1}I$ so $b = fa \in I$. □

Lemma 3.6. *If the rig A is Noetherian then it is a finite product of directly indecomposable rigs.*

Proof. Assume that A is not a finite direct product of directly indecomposable rigs. Then A is not directly indecomposable so $A = A_0 \times A'_0$ for non-terminal A_0, A'_0. Moreover, either A_0 or A_1 is not a finite direct product of directly indecomposable rigs. Without loss of generality we can assume that A_0 is not. By Lemma 1.14 the projection $A \to A_0$ is the quotient by a saturated ideal $I_0 \subseteq A$. (Indeed, an ideal generated by Boolean element.) Moreover, the ideal is strict because A'_0 is not terminal. By our current assumption, $A_0 = A_1 \times A'_1$ for non-terminal A_1 and A'_1. Again, we may assume that A_1 is not directly indecomposable and let $I_1 \subset A$ be the strict saturated ideal whose quotient is the composite projection $A \to A_0 \to A_1$. Also, $I_0 \subset I_1$ as ideals of A. Repeating the process we obtain a non-stationary sequence $I_0 \subset I_1 \subset \dots$ of saturated ideals of A, contradicting Noetherianity of A. □

4. The lower Basis Theorem

Every commutative monoid determines a pre-order on its underlying set. In particular, addition in a rig induces a pre-order. In more detail, let A be a rig and declare, for every $x, y \in A$, that $x \leq y$ if and only if there is a $d \in A$ such that $x + d = y$. We sometimes call this the 'canonical pre-order' of A. It is easy to check that addition and multiplication are monotone with respect to the canonical pre-order.

An ideal $I \subseteq A$ is called *lower-closed* if $x \leq y \in I$ implies $x \in I$. We stress an obvious corollary of Lemma 1.13: lower-closed implies saturated.

For example, the canonical pre-order of a ring is codiscrete (in the sense that $x \leq y$ for every x, y) so the only lower-closed ideal in a ring is that containing 1. On the other hand, the canonical pre-order of a distributive lattice (considered as a rig) coincides with the lattice.

Fix a rig K.

Lemma 4.1. *If $I \subseteq K[x]$ is a lower closed ideal then every element of I is a sum of monomials in I. Hence, I is generated by the monomials in I.*

Proof. If the polynomial $\sum_{i=0}^{m} k_i x^i$ is in I then, by lower-closedness, I contains $k_i x^i$ for each $0 \leq i \leq m \in \mathbb{N}$. □

In the next auxiliary result the reader may recognize a trick used in the classical proof Hilbert's Basis Theorem. It is no accident.

Lemma 4.2. *If K is such that every lower-closed ideal is finitely generated then for every lower-closed ideal $I \subseteq K[x]$ there is an $n \in \mathbb{N}$ such that I is generated by monomials of degree at most n.*

Proof. By Lemma 4.1 it is enough to check that every monomial in I may be expressed as a linear combination (with coefficients in $K[x]$) of monomials in I of some bounded degree.

Let $L \subseteq K$ be the subset consisting of 0 together with the leading coefficients of polynomials in I. The subset $L \subseteq K$ is clearly an ideal and it is also lower-closed. To see this assume that $a \leq b \in L$. Then there is a polynomial $f = bx^n + (\text{lower terms})$ in I. So $ax^n \leq bx^n \leq f \in I$ and, as I is lower closed, $ax^n \in I$. Hence, $a \in L$ so L is indeed lower-closed.

By hypothesis, there is a finite family $(\kappa_s \in K \mid s \in S)$ spanning L. For each κ_s there exists a polynomial $f_s \in I$ that has κ_s as leading coefficient. Let n be the largest degree of any of the f_s's. Multiplying the polynomials f_s with suitable powers of x we obtain polynomials $g_s \in I$ all of the same degree n and each g_s with leading coefficient κ_s. As I is lower closed, $\kappa_s x^n \in I$ for every $s \in S$.

Let $m \geq n$ and $ax^m \in I$. Then a is a linear combination, with coefficients in K, of $(\kappa_s \mid s \in S)$. So ax^m is a linear combination, with coefficients in $K[x]$, of the polynomials $\kappa_s x^n \in I$. Hence, every monomial in I is a linear combination, with coefficients in $K[x]$, of the monomials in I of degree strictly less than n; as we needed to prove. □

We can now mimic the classical proof of Hilbert's Basis Theorem but using lower-closedness of the ideals involved instead of the existence of negatives.

Theorem 4.3 (The lower Basis Theorem)**.** *If K is such that every lower-closed ideal is finitely generated then every lower-closed ideal of $K[x]$ is finitely generated.*

Proof. Let $I \subseteq K[x]$ be a lower-closed ideal. By Lemma 4.2 there is an $n \in \mathbb{N}$ such that I is generated by the monomials in I of degree at most n. For each $m \leq n$ let $L_m \subseteq K$ be the subset consisting of 0 and all coefficients of monomials of degree m in I. As before, L_m is a lower-closed ideal in K so, by hypothesis, it is generated by a finite family $(\kappa_{m,s} \mid s \in S_m)$. Then every monomial of degree m may be expressed as a linear combination (with coefficients in K, actually) of the monomials $\kappa_{m,s}x^m$. Then every monomial of degree at most n is a linear combination of the finite family of monomials $(\kappa_{m,s}x^m \mid s \in S_m, m \leq n)$. So the same family generates the ideal I. □

5. The 2-Basis Theorem

Let 2 be the initial distributive lattice. For any rig A there is at most one rig morphism $2 \to A$ so the forgetful functor $2/\mathbf{Rig} \to \mathbf{Rig}$ is full as well as faithful. The objects in the subcategory may be identified with the rigs whose addition is idempotent. Of course, from this perspective, the initial object of $2/\mathbf{Rig}$ is 2. Also, idempotence of addition implies that the canonical preorder is anti-symmetric so, for any 2-rig A, we will picture $(A, +, 0)$ as a join-semilattice.

Lemma 5.1. *If A is a 2-rig and $I \subseteq A$ is an ideal then the following hold:*

1. *For every $x, y \in A$, $x \approx_I y$ if and only if there is a $k \in I$ such that $x + k = y + k$.*
2. *The ideal I is saturated if and only if it is lower closed.*

Proof. By the definition of $\approx_I$, $x \approx_I y$ if and only if there are $i, j \in I$ such that $x + i = y + j$. In this case,

$$x + i + j = x + i + i + j = y + j + i + j = y + i + j$$

so we may take $k = i + j$.

Assume that I is saturated. If $x \leq y \in I$ then $x + y = y$ so, by saturation, $x \in I$. On the other hand, if I is lower closed then it is trivially saturated. □

Hence, for 2-rigs, we may reformulate strong Noetherianity as follows.

Proposition 5.2. *A 2-rig is strongly Noetherian if and only if every lower closed ideal is finitely generated.*

Proof. Recall that a rig A is *strongly Noetherian* if every saturated ideal is finitely generated. So the statement follows immediately from the second item of Lemma 5.1. □

Combining Theorem 4.3 and Proposition 5.2 we obtain the following.

Corollary 5.3 (The 2-Basis-Theorem)**.** *If K is a strongly Noetherian 2-rig then so is $K[x]$.*

As in the classical case, a simple induction implies that free 2-rigs on a finite set of generators are strongly Noetherian.

Corollary 5.4. *Finitely generated 2-rigs are Noetherian.*

Proof. Follows from the previous remark and Lemma 3.5. □

Corollary 5.4 and Lemma 3.6 imply the following.

Corollary 5.5. *Every finitely generated 2-rig is a finite product of directly indecomposable finitely generated 2-rigs.*

We don't know if finitely generated implies finitely presentable for 2-rigs so we are forced to state the following separately.

Corollary 5.6. *Every finitely presentable 2-rig is a finite product of directly indecomposable finitely presentable 2-rigs.*

Proof. If A is a finitely presentable 2-rig then it is finitely generated so, by Corollary 5.5, $A = \prod_{s \in S} A_s$ for a finite set S and A_s directly indecomposable for every $s \in S$. By Lemma 1.14, the projection $A \to A_s$ is the quotient by a principal ideal. Hence, as A is finitely presentable, so is A_s. □

We can now deduce an analogue of Lemma 1.4.

Corollary 5.7. *Every object in the extensive* $\mathbf{Aff}_2$ *is a finite coproduct of connected objects.*

6. Integral rigs and a Nullstellensatz for 2-rigs

Section 4 in [11] attributes to Schanuel the result that the only simple rigs are fields and the distributive lattice 2. We prove here a weaker statement with an argument that is more convenient for our purposes.

Let A be a rig and let $F \subseteq A$ be a multiplicative submonoid.

For $x, y \in A$ we write $x \leq_F y$ if there is an $u \in F$ such that $x \leq uy$. In this case we may say that u *witnesses* that $x \leq_F y$. The relation $\leq_F$ on the set A is reflexive because $1 \in F$ and it is transitive because if $u, v \in F$ witness that $x \leq_F y$ and $y \leq_F z$ respectively then uv witnesses that $x \leq_F z$. Hence, $\leq_F$ is a pre-order.

Write $x \approx_F y$ if both $x \leq_F y$ and $y \leq_F x$. As $\leq_F$ is a pre-order, $\approx_F$ is an equivalence relation. We next give a sufficient condition for it to be a congruence.

Lemma 6.1. *If $1 + F \subseteq F \subseteq A$ then $\approx_F$ is a congruence on A. In this case, the quotient $A/{\approx_F}$ is a 2-rig and, it is trivial if and only if A is a ring.*

Proof. We have already seen that the relation $\approx_F$ is an equivalence relation. For $a, b, c, d \in A$ assume that $a \leq_F b$ is witnessed by $u \in F$ and that $c \leq_F d$ is witnessed by $v \in F$. Then uv witnesses that $ac \leq_F bd$.

Assume from now on that $1 + F \subseteq F$. We claim that if $x \in A$ and $a \leq_F b$ then $x + a \leq_F x + b$. By hypothesis there is an $u \in F$ such that $a \leq ub$ so

$$x + a \leq x + ub \leq x + (b + ux) + ub = (x + b) + u(x + b) = (1 + u)(x + b)$$

and hence $x + a \leq_F x + b$, so the claim is proved.

Using the claim one easily shows that if $a \leq_F b$ and $x \leq_F y$ then also $a + x \leq_F b + y$. It follows that $\approx_F$ is a congruence.

Trivially, $1 \leq_F 1 + 1$ and, since $1 + 1 \in F$ by hypothesis, the inequality $1 + 1 \leq (1 + 1)1$ implies $1 + 1 \leq_F 1$. So $1 \approx_F 1 + 1$ and hence the quotient $A/{\approx_F}$ is a 2-rig.

Assume now that $0 = 1$ in the quotient $A/{\approx_F}$. That is, $0 \approx_F 1$ in A. Equivalently, $0 \leq_F 1$ and $1 \leq_F 0$. One of the conjuncts holds trivially and the other is equivalent to $1 \leq 0$. So the quotient is terminal if and only if $1 \leq 0$ in A. □

For instance, we recall Schanuel's construction [17] of the left adjoint to the full inclusion $2/\mathbf{Rig} \to \mathbf{Rig}$. The rig $\mathbb{N}$ of natural numbers with the usual addition and multiplication is initial in $\mathbf{Rig}$. That is, for any rig A there exists a unique $\nabla : \mathbb{N} \to A$ in $\mathbf{Rig}$. The subset $F = \{\nabla n \mid 1 \le n\} \subseteq A$ satisfies the hypotheses of Lemma 6.1. The induced pre-order on A satisfies: $a \le_F b$ if and only if there is an $1 \le n \in \mathbb{N}$ such that $a \le nb$. The quotient by $\approx_F$ is denoted by $\dim : A \to D(A)$ and is universal from A to the inclusion $2/\mathbf{Rig} \to \mathbf{Rig}$. This construction suggests something more general.

Lemma 6.2. *Let $1 + F \subseteq F$ so that $\approx_F$ is a congruence by Lemma 6.1. If $1 \le_F u$ for every $u \in F$, then the quotient $A \to A/{\approx_F}$ is the universal morphism sending $F \subseteq A$ to 1 in the codomain.*

Proof. Trivially $u \le_F 1$ for every $u \in F$. As $1 \le_F u$ by hypothesis, $1 \approx_F u$ for every $u \in F$ so the quotient $A \to A/{\approx_F}$ sends $F \subseteq A$ to the unit 1 in the codomain. Now let $f : A \to B$ in $\mathbf{Rig}$ be such that $fu = 1$ for every $u \in F$. As $1 + 1 \in F$, B is a 2-rig. If $a \le_F b$ then $a \le ub$ for some $u \in F$. Then $fa \le (fu)(fb) = fb$. So, if $a \approx_F b$ then $fa \le fb$ and $fb \le fa$ and, as B is a 2-rig, $fa = fb$. Hence, f factors uniquely through the quotient $A \to A/{\approx_F}$. □

Recall that $\mathbf{iRig} \to \mathbf{Rig}$ is the variety of rigs determined by the equation $1 + x = 1$. We next describe the left adjoint to $\mathbf{iRig} \to \mathbf{Rig}$.

Let $\uparrow 1 \subseteq A$ be the upper-closed multiplicative submonoid of the elements in A above 1. The relation $\approx_{\uparrow 1}$ is a congruence by Lemma 6.1. Denote the associated quotient $A/{\approx_{\uparrow 1}}$ by LA.

Proposition 6.3. *The quotient $A \to LA$ is universal from A to* $\mathbf{iRig} \to \mathbf{Rig}$ *and the resulting left adjoint $L : \mathbf{Rig} \to \mathbf{iRig}$ preserves finite products.*

Proof. Evidently, $1 + x \in \uparrow 1 \subseteq A$ for all x so, by Lemma 6.2, the quotient $A \to LA$ sends $1 + x \in A$ to $1 \in LA$ for every $x \in A$; so LA is integral. To prove that the quotient $A \to LA$ is universal let R be an integral rig and let $f : A \to R$ be a rig homomorphism. Then $fu = 1$ for every $1 \le u \in A$, so f factors through $A \to LA$ by Lemma 6.2.

Let $L : \mathbf{Rig} \to \mathbf{iRig}$ be the resulting left adjoint and denote the unit by η. Let A, B be rigs and let γ be the unique map such that the following diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\eta} & L(A \times B) \\ & \searrow^{\eta\times\eta} & \downarrow \gamma \\ & & LA \times LB \end{array}$$

commutes in $\mathbf{Rig}$. Then γ is surjective so we need only prove that it is monic. Let $(a, b), (a', b') \in A \times B$ and assume that $\gamma(\eta(a, b)) = \gamma(\eta(a', b'))$. Then both $\eta a = \eta a'$ and $\eta b = \eta b'$. Hence $a \approx_{\uparrow 1} a'$ in A and $b \approx_{\uparrow 1} b'$ in B. That is, $a \leq_{\uparrow 1} a'$ and $a' \leq_{\uparrow 1} a$ in A and also $b \leq_{\uparrow 1} b'$ and $b' \leq_{\uparrow 1} b$ in B. Let $1 \leq u \in A$ witness that $a \leq_{\uparrow 1} a'$ and $1 \leq v \in B$ witness that $b \leq_{\uparrow 1} b'$. Then $(1, 1) \leq (u, v) \in A \times B$ and $(a, b) \leq (ua', vb') = (u, v)(a', b')$. Hence $(a, b) \leq_{\uparrow 1} (a', b')$ in $A \times B$. Similarly, $(a', b') \leq_{\uparrow 1} (a, b)$ so $(a, b) \approx_{\uparrow 1} (a', b')$ as we needed to show. □

The inclusion $\mathbf{iRig} \to \mathbf{Rig}$ factors through the right adjoint inclusion $2/\mathbf{Rig} \to \mathbf{Rig}$. The left adjoint to the factorization $\mathbf{iRig} \to 2/\mathbf{Rig}$ is just the restriction of the left adjoint $L : \mathbf{Rig} \to \mathbf{iRig}$. Hence, we may deduce the following result that will be needed later.

Corollary 6.4. *The left adjoint to* $\mathbf{iRig} \to 2/\mathbf{Rig}$ *preserves finite products.*

Combining the integral reflection described above with some of the material in [4] we arrive at the promised weak version of Schanuel's result.

Proposition 6.5 (Nullstellensatz)**.** *For any non-trivial 2-rig A there is a map $A \to 2$.*

Proof. By hypothesis and Lemma 6.1, the codomain of the unit $A \to LA$ is not trivial. Consider now the variety $\mathbf{dLat} \to \mathbf{iRig}$. The left adjoint $L' : \mathbf{iRig} \to \mathbf{dLat}$ is described explicitly in [4, Lemma 4.3] which also implies that the unit $LA \to L'(LA)$ is local (in the sense that it reflects 1) so the distributive lattice $L'(LA)$ is non-trivial. Classical lattice theory then implies the existence of a map $L'(LA) \to 2$, so we have a composite rig morphism $A \to LA \to L'(LA) \to 2$. □

Corollary 6.6 (Nullstellensatz)**.** *Every connected object in* $\mathbf{Aff}_2$ *has a point.*

7. The Gaeta topos of $\mathbf{Aff}_2$

We can now apply standard topos theory to construct a topos 'of spaces' embedding the category of affine 2-spaces in such a way that finite coproducts are preserved.

Theorem 7.1. *The Gaeta topos of* $\mathbf{Aff}_2$ *is pre-cohesive over sets.*

Proof. Exactly as in Proposition 1.7. By Corollary 5.6 the Gaeta topos of $\mathbf{Aff}_2$ is equivalent to the topos of presheaves on the category of connected affine 2-schemes and every every connected affine 2-scheme has a point by Corollary 6.6. □

Theorem 7.1 and the related Proposition 6.5 show that the rig 2 has certain typical properties of algebraically closed fields.

As suggested in [11], standard techniques allow us to give a presentation of the geometric theory classified by the topos of Theorem 7.1. We give details below.

Proposition 7.2. *The Gaeta topos of* $\mathbf{Aff}_2$ *classifies the extension of the theory of 2-rigs presented by the following sequents.*

$$\begin{array}{rcl} 0 = 1 & \vdash & \bot \\ (x + y = 1) \wedge (xy = 0) & \vdash_{x,y} & [(x = 1) \wedge (y = 0)] \vee [(x = 0) \wedge (y = 1)] \end{array}$$

In other words, this $\mathfrak{G}(\mathbf{Aff}_2)$ *classifies 'Boolean-free' 2-rigs.*

Proof. First let us give a dual description of the basis for the Gaeta topology in $(2/\mathbf{Rig})_{fp}$. Our knowledge of products in $2/\mathbf{Rig}$ implies that a Gaeta cocover on an (f.p.) 2-rig A is a finite family $(A \to A[a_i^{-1}] \mid i \in I)$ of maps in $(2/\mathbf{Rig})_{fp}$ such that $a_i a_j = 0$ for every $i, j \in I$ and the ideal $\langle a_i \mid i \in I \rangle$ generated by the a_i's is trivial in the sense that it contains 1. In this case, for brevity, we will also say that the family $(a_i \mid i \in I)$ covers A.

On the other hand, there is a more or less general procedure to exhibit an explicit site for the classifier of Boolean-free 2-rigs. See, for example, [6, Proposition D3.1.10]). Roughly speaking, one first constructs the classifier for the restricted (algebraic) theory presented by the equations and then forces the remaining axioms by imposing a Grothendieck topology. In the present case, the classifier for the theory of 2-rigs may be described as the topos $[(2/\mathbf{Rig})_{fp}, \mathbf{Set}] = \widehat{\mathbf{Aff}_2}$ of functors $(2/\mathbf{Rig})_{fp} \to \mathbf{Set}$; and the classifier of Boolean-free 2-rigs may be obtained as the sheaf topos associated to the least Grothendieck topology on $\mathbf{Aff}_2$ 'forcing' the coherent sequents in the statement. More explicitly, the classifying topos for idempotent 2-rigs may be described as the topos of sheaves on the site $(\mathbf{Aff}_2, J)$ where J is the least Grothendieck topology 'containing' the cocover

$$\begin{array}{ccc} 2[x,y]/(xy=0, x+y=1) & \longrightarrow & 2[x,y]/(x=0, y=1) \cong 2 \\ \big\downarrow & & \\ 2 \cong 2[x,y]/(x=1, y=0) & & \end{array}$$

and the empty cocover on the terminal object. The explicit dual description of the Gaeta topology in the first paragraph implies that the two cocovers generating J are in the basis for the Gaeta topology. So J is included in the Gaeta topology. On the other hand, any binary cocover

$$A/(v) \longleftarrow A \longrightarrow A/(u)$$

with $uv = 0$ and $u + v = 1$ in the Gaeta basis appears as the pushout, along the map $2[(x+y)^{-1}, xy] \to A$ that sends x to u and y to v, of the main coverage generating J. A simple inductive argument as in [12, Lemma VIII.6.2] implies that all the non-empty Gaeta cocovers are in J. Hence, the Gaeta topology is included in J. Altogether, the two topologies are the same. $\square$

It is well-known that for any ring K, the classifier of K-algebras (i.e. the presheaf topos $[(K/\mathbf{Ring})_{fp}, \mathbf{Set}]$) and some of its subtoposes are models of Synthetic Differential Geometry [8, Part III]. Folklore says that this also holds for arbitrary rigs. We end this section with a sketch of the proof that one of the key axioms of SDG holds in the Gaeta topos of 2.

Let $R = (2/\mathbf{Rig})_{fp}(2[x], -)$ in $\mathfrak{G}(\mathbf{Aff}_2)$ be the generic Boolean-free 2-rig. Let the following diagram be a pullback

$$\begin{array}{ccccc} D & \longrightarrow & & \longrightarrow & 1 \\ \big\downarrow & & & & \big\downarrow{\scriptstyle 0} \\ R & \xrightarrow[\Delta]{} & R \times R & \xrightarrow[\cdot]{} & R \end{array}$$

or, alternatively, define $D = \{x \in R \mid x^2 = 0\} \subseteq R$ using the internal language of $\mathfrak{G}(\mathbf{Aff}_2)$. The composite

$$R \times R \times D \xrightarrow{id_R \times \cdot} R \times R \xrightarrow{+} R$$

transposes to a map $R \times R \to R^D$.

Proposition 7.3 (The KL-axiom holds in the Gaeta topos of 2)**.** *The canonical map $R \times R \to R^D$ is an isomorphism in $\mathfrak{G}(\mathbf{Aff}_2)$.*

Proof. For any rig A, the universal morphism $A \to A[\epsilon]$ in $\mathbf{Rig}$ adding an element ϵ of square-zero may be built as usual by taking the additive monoid $A \times A$ equipped with multiplication $(a, a')(b, b') = (ab, ab' + a'b)$ and $\epsilon = (0, 1)$ as selected element of square 0. If we let $a = (a, 0) \in A[\epsilon]$ then every element of $A[\epsilon]$ is of the form $a + b\epsilon$. The object D is representable by $2[\epsilon]$ and the subobject $D \to R$, as a cosieve in $(2/\mathbf{Rig})_{fp}$, is generated by the map $2[x] \to 2[\epsilon]$ sending x to ϵ. (Notice that the pullback defining D could be taken in $\mathbf{Aff}_2$.) The object R^D, as a functor $(2/\mathbf{Rig})_{fp} \to \mathbf{Set}$, sends A in the domain to the underlying set $A \times A$ of $A[\epsilon]$. The canonical map $R \times R \to R^D$, at stage A, sends the ordered pair $(a, b) \in (R \times R)A = A \times A$ to $a + b\epsilon \in (R^D)A = A[\epsilon]$. □

The resulting differential geometry in $\mathfrak{G}(\mathbf{Aff}_2)$ should be an interesting pursuit. See also [11, Section 1].

8. The extensive category of Affine i-schemes

Let $\mathbf{iRig}_{fp} \to \mathbf{iRig}$ be the full subcategory of finitely presentable integral rigs.

Corollary 8.1. *The full subcategory $\mathbf{iRig}_{fp} \to \mathbf{iRig}$ is closed under products and it is therefore coextensive.*

Proof. Let $F[S]$ be the free integral rig generated by the set S. As in Proposition 2.3 we need only show that if S and T are finite then $F[S] \times F[T]$ is finitely presented. By Proposition 2.3 again there are finite sets U, V and a coequalizer

$$2[V] \rightrightarrows 2[U] \longrightarrow 2[S] \times 2[T]$$

in the category $2/\mathbf{Rig}$. The reflection $L : 2/\mathbf{Rig} \to \mathbf{iRig}$ preserves finite products by Corollary 6.4 so it sends the coequalizer above to the coequalizer

$$L(2[V]) \rightrightarrows L(2[U]) \longrightarrow L(2[S]) \times L(2[T])$$

in $\mathbf{iRig}$. As $L(2[W]) = FW$ for any set W, the result follows. □

Naturally, we introduce the following.

Definition 8.2. The *category of affine i-schemes* is the (extensive) opposite of $\mathbf{iRig}_{fp}$ and it will be denoted by **iAff**.

We next show that the Gaeta topos of **iAff** is pre-cohesive using the same techniques that we used for a $\mathbf{Aff}_2$.

Corollary 8.3. *Every finitely generated integral rig is Noetherian.*

Proof. It is clear from the definition of integral rig that $\mathbf{iRig}$ is a variety of 2-rigs so it follows from classical universal algebra that the full subcategory $\mathbf{iRig} \to 2/\mathbf{Rig}$ is regular epireflective and closed under regular quotients and directed unions [1, Corollary 10.21].

By regular epireflectivity every integral rig freely generated by a set of generators is a quotient of free 2-rig freely generated by the same set. If the generating set is finite then the free 2-rig is Noetherian by Lemma 5.4, so the free integral rig is also Noetherian by Lemma 3.5. Lemma 3.5 also implies that finitely generated integral rigs are Noetherian. □

Just as in Corollaries 5.5 and 5.6 we may deduce the next result.

Corollary 8.4. *Every finitely presentable integral rig is a finite product of directly indecomposable finitely presentable integral rigs.*

It is plausible that these finite direct decomposition results may be lifted to other algebraic categories equipped with a suitable functor to $2/\mathbf{Rig}$ or to $\mathbf{iRig}$ such as those discussed in [4], but we will not pursue that here.

Theorem 8.5. *The Gaeta topos of* **iAff** *is pre-cohesive over sets and classifies the extension of the theory of 2-rigs presented by the following sequents.*

$$\begin{array}{rcl} 0 = 1 & \vdash & \bot \\ (x + y = 1) \wedge (xy = 0) & \vdash_{x,y} & [(x = 1) \wedge (y = 0)] \vee [(x = 0) \wedge (y = 1)] \end{array}$$

In other words, this Gaeta topos classifies 'Boolean-free' integral rigs.

Proof. To prove that the topos is pre-cohesive proceed as in Proposition 1.7 (or Theorem 7.1). By Corollary 8.4 the Gaeta topos of **iAff** is equivalent to the topos of presheaves on the category of connected affine i-schemes and every connected affine i-scheme has a point by Corollary 6.6.

Also, the Gaeta topology in **iAff** has the same dual description made explicit in Proposition 7.2. (See [4].) Then the same argument used in 7.2 proves the present result. □

9. Really local integral rigs

A rig A (in a topos, with subobject classifier Ω) is *really local* if the characteristic map $A \to \Omega$ of the subobject of (multiplicatively) invertible elements of A is a rig morphism when Ω is considered equipped with its canonical distributive lattice structure [9].

In an integral rig the unit 1 is the only invertible element. It is then easy to check [4, Lemma 6.2] that an integral rig (in a topos $\mathcal{E}$) is really local if and only if it satisfies the following sequents

$$\begin{array}{rcc} 0 = 1 & \vdash & \bot \\ x + y = 1 & \vdash_{x,y} & (x = 1) \vee (y = 1) \end{array}$$

in the internal logic of $\mathcal{E}$. Notice that this sequents imply those in Theorem 8.5.

Notice also that if R is an integral rig in a topos $\mathcal{E}$ then the sequent

$$(x = 1) \vee (y = 1) \;\; \vdash_{x,y} \;\; x + y = 1$$

holds, but the witnessing inclusion

$$\{(x, y) \mid (x = 1) \vee (y = 1)\} \subseteq \{(x, y) \mid x + y = 1\}$$

of subobjects of $R \times R$ need not be an isomorphism, so R need not be really local. Something similar happens in the classical context: the generic idempotent-free $\mathbb{C}$-algebra is not local (in the classical sense) in the complex Gaeta topos; on the other hand, the same object, as an algebra in the Zariski subtopos, is local; indeed, it is the generic local $\mathbb{C}$-algebra.

Lemma 9.1. *The generic Boolean-free integral rig R is not really local.*

Proof. By Theorem 8.5 the classifier of Boolean-free integral rigs is the Gaeta topos of **iAff** and the generic object therein is the 'affine line' representable by the free integral rig on one generator. To check if R is really local in the Gaeta topos $\mathfrak{G}(\mathbf{iAff})$ it is convenient to present the topos as that of presheaves on connected objects. Let $\mathbf{iRig}_{fpi} \to \mathbf{iRig}_{fp}$ be the full subcategory of directly indecomposable (finitely presentable) integral rigs so that $\mathfrak{G}(\mathbf{iAff})$ may be identified with the functor category $[\mathbf{iRig}_{fpi}, \mathbf{Set}]$.

Let $\mathbf{i} : \mathbf{Set} \to \mathbf{iRig}$ be the left adjoint to the forgetful functor and let $\mathbf{i}[x]$ be the free integral rig on one generator so that the representable object $R = \mathbf{iRig}_{fpi}(\mathbf{i}[x], -)$ in $\mathfrak{G}(\mathbf{iAff})$ is the generic Boolean-free integral rig.

The 'affine plane' $R \times R$ in $\mathfrak{G}(\mathbf{iAff})$ is representable by the free integral rig $\mathbf{i}[x, y]$ on two generators so the subobject $\{(x, 1) \mid x \in R\} \subseteq R \times R$ in $\mathfrak{G}(\mathbf{iAff})$, which is the same thing as the monic $id \times 1 : R \times 1 \to R \times R$, is the cosieve in $\mathbf{iRig}_{fpi}$ generated by the map $\mathbf{i}[x, y] \to \mathbf{i}[x, y, y^{-1}] \cong \mathbf{i}[x]$ that sends x to x, and y to 1. Similarly for $\{(1, y) \mid y \in R\} \subseteq R \times R$. Hence, the subobject $\{(x, y) \mid (x = 1) \vee (y = 1)\} \subseteq R \times R$ is the cosieve in $\mathbf{iRig}_{fpi}$ generated by the span $\mathbf{i}[x] \leftarrow \mathbf{i}[x, y] \to \mathbf{i}[y]$.

On the other hand, the subobject $\{(x, y) \mid x + y = 1\} \subseteq R \times R$ in the topos $\mathfrak{G}(\mathbf{iAff})$ is the cosieve in $\mathbf{iRig}_{fpi}$ generated by the quotient morphism $\mathbf{i}[x, y] \to \mathbf{i}[x, y]/(x + y = 1)$. So it is enough to show that this quotient does not factor through $\mathbf{i}[x, y] \to \mathbf{i}[x]$ or $\mathbf{i}[x, y] \to \mathbf{i}[y]$; but this is easy. $\square$

Loosely speaking, although $\mathfrak{G}(\mathbf{iAff})$ has the 'right' coproducts, the colimit (join)

$$\{(x, 1) \mid x \in R\} \vee \{(1, y) \mid y \in R\}$$

of subobjects of $R \times R$ is not 'right' in $\mathfrak{G}(\mathbf{iAff})$ (or in **iAff**) but we can correct it by a considering a suitable subtopos. Indeed, the least subtopos of $\mathfrak{G}(\mathbf{iAff})$ forcing the inclusion

$$\{(x, y) \mid (x = 1) \vee (y = 1)\} \subseteq \{(x, y) \mid x + y = 1\}$$

to become an isomorphism is the topos of sheaves on **iAff** for the least Grothendieck topology containing the Gaeta coverage and also the sieve (co)generated by the span

$$\mathbf{i}[x] \longleftarrow \mathbf{i}[x, y]/(x + y = 1) \longrightarrow \mathbf{i}[x]$$

in $\mathbf{iAff}^{\mathrm{op}} = \mathbf{iRig}_{fp}$. In the classical case over the complex numbers the analogous construction results in the Zariski topos.

10. The 'Zariski' topos of the theory of integral rigs

Let $\mathcal{C}$ be a category with finite limits and equipped with a distinguished integral rig R. For any finite family $(f_i : X \to R \mid i \in I)$ we denote the composite

$$X \xrightarrow{\langle f_i \mid i \in I\rangle} R^I \xrightarrow{\sum_{i\in I}} R$$

by $\bigoplus_{i\in I} f_i : X \to R$. The family is said to *cocover* X if the diagram below

$$\begin{array}{ccc} X & \longrightarrow & 1 \\ {\scriptstyle \langle f_i \mid i\in I\rangle}\downarrow & \searrow^{\bigoplus_{i\in I} f_i} & \downarrow{\scriptstyle 1} \\ R^I & \xrightarrow[\sum_{i\in I}]{} & R \end{array}$$

commutes.

A finite family $(u_i : U_i \to X \mid i \in I)$ of maps in $\mathcal{C}$ is said to *cover* X if there is a cocovering family $(f_i : X \to R \mid i \in I)$ such that the following diagram is a pullback

$$\begin{array}{ccc} U_i & \longrightarrow & 1 \\ {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle 1} \\ X & \xrightarrow[f_i]{} & R \end{array}$$

for every $i \in I$. Notice that all the maps in a covering family must be monic. One easily sees that isomorphisms cover and that covers are stable under pullback.

Different properties of R will determine different properties of covers. Rather than pursuing this idea in the abstract we are going to concentrate on the case $\mathcal{C} = \mathbf{iAff}$ equipped with the integral rig R therein determined by the free integral rig $\mathbf{i}[x]$ on one generator (considered as an object in $\mathbf{iAff}^{\mathrm{op}} = \mathbf{iRig}_{fp}$).

If A is in $\mathbf{iRig}_{fp}$ and X is the corresponding object in $\mathbf{iAff}$ then a map $X \to R$ in $\mathbf{iAff}$ is a map $\mathbf{i}[x] \to A$ in $\mathbf{iRig}_{fp}$; that is, an element in A. So a family $(f_i : X \to R \mid i \in I)$ may be identified with a family $(a_i \mid i \in I)$ of elements in A. The map $\bigoplus_{i\in I} f_i : X \to R$ corresponds to $\sum_{i\in I} a_i \in A$. Hence, the family $(f_i : X \to R \mid i \in I)$ cocovers the object X if and only if $\sum_{i\in I} a_i = 1 \in A$. In this case we say that $(a_i \mid i \in I)$ *cocovers* A.

Lemma 10.1. *A finite family $(a_i \mid i \in I)$ cocovers A if and only if the ideal generated by the family contains $1 \in A$.*

Proof. The generated ideal contains the unit 1 if and only if there is a family $(b_i \mid i \in I)$ such that $\sum_{i\in I} a_i b_i = 1$. In an integral rig this holds if and only if $1 \leq \sum_{i\in I} a_i b_i \leq \sum_{i\in I} a_i$. □

Again, let $\mathbf{i}[x] \to A$ in $\mathbf{iRig}_{fp}$ be the unique map determined by $a \in A$ and let $X \to R$ be the corresponding map in **iAff**. Any pullback in **iAff** as on the left below

$$\begin{array}{ccc} U & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle 1} \\ X & \longrightarrow & R \end{array} \qquad\qquad \begin{array}{ccc} \mathbf{i}[x] & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ A & \longrightarrow & A[a^{-1}] \end{array}$$

corresponds to a pushout in $\mathbf{iRig}_{fp}$ as on the right above, where the top map sends x to 1 and the left map sends x to $a \in A$. Hence, a finite family of $(u_i : U_i \to X \mid i \in I)$ of maps in $\mathcal{C}$ *covers* X if and only if there is a cocover $(a_i \mid i \in I)$ of A such that the map in $\mathbf{iRig}_{fp}$ corresponding to u_i has the universal property of $A \to A[a_i^{-1}]$ for each $i \in I$.

Altogether, already familiar with the (trivial) duality $\mathbf{iAff} = \mathbf{iRig}_{fp}^{\mathrm{op}}$, we may say a *(co)cover* of A in $\mathbf{iRig}_{fp}$ is a finite family of universal maps $(A \to A[a_i^{-1}] \mid i \in I)$ such that $\sum_{i\in I} a_i = 1 \in A$.

In order to continue our study of (co)covers it is convenient to have a concrete construction the universal maps inverting elements in integral rigs.

Let A be an integral rig and $F \subseteq A$ be a multiplicative submonoid. Let $A \to A[F^{-1}]$ be the universal map in **iRig** inverting all the elements of F; in other words, sending all the elements of F to 1. For $x, y \in A$ write $x \mid_F y$ if there is a $w \in F$ such that $wx \leq y$. (Notice the similarity with $\leq_F$ in Section 6; but notice also that, as A is integral, the condition "$1 \leq_F u$ for every $u \in F$" in Lemma 6.2 only holds if F is trivial.) Write $x \equiv_F y$ if $x \mid_F y$ and $y \mid_F x$. Lemma 3.4 in [4] shows that $\equiv_F$ is a congruence and that the quotient $A \to A/{\equiv_F}$ has the universal property of $A \to A[F^{-1}]$.

Lemma 10.2. *The map $A \to A[F^{-1}]$ inverts $a \in A$ if and only if there exists $w \in F$ such that $w \leq a$. Also, the object $A[F^{-1}]$ is terminal if and only if $0 \in F$.*

Proof. The universal map inverts a if and only if $1 \equiv_F a$ if and only if $1 \mid_F a$ and $a \mid_F 1$. One of the conjuncts is trivial and the other is equivalent to the existence of a $w \in F$ such that $w \leq a$.

Also, $A[F^{-1}]$ is terminal if and only if $0 \equiv_F 1$ if and only if $0 \leq_F 1$ and $1 \leq_F 0$. Again, one of the conjuncts is trivial and the other is equivalent to the existence of an $w \in F$ such that $w \leq 0$. $\square$

Taking $a \in A$ and $F = \{a^n \mid n \in \mathbb{N}\} \subseteq A$ we obtain $A \to A[a^{-1}]$. By Lemma 10.2 this map inverts $b \in A$ if and only if there is an $n \in \mathbb{N}$ such that $a^n \leq b$.

We have already observed in the abstract setting that isomorphisms cover and that covers are stable under pullback. So naturally we now concentrate on compositions of (co)covers. In order to carry out the arguments we introduce a small piece of notation. For $a \in A$ we write $\frac{(-)}{a} : A \to A[a^{-1}]$ for the universal map so that, for $b \in A$, the resulting element in $A[a^{-1}]$ is denoted by $\frac{b}{a} \in A[a^{-1}]$. For instance, a straightforward argument using universal properties shows the following.

Lemma 10.3. *For any $a, b \in A$, the composite maps*

$$A \to A[a^{-1}] \to A[a^{-1}][(\frac{b}{a})^{-1}] \quad \text{and} \quad A \to A[b^{-1}] \to A[b^{-1}][(\frac{a}{b})^{-1}]$$

have the universal property of $A \to A[(ab)^{-1}]$ in **iRig** *and the following square is a pushout*

$$\begin{array}{ccc} A & \longrightarrow & A[b^{-1}] \\ \downarrow & & \downarrow \\ A[a^{-1}] & \longrightarrow & A[(ab)^{-1}] \end{array}$$

in **iRig**.

Those familiar with the usual presentation of the Zariski topos will recognize the following auxiliary fact.

Lemma 10.4. *If the family $(\frac{b_j}{a} \mid j \in J)$ covers $A[a^{-1}]$ then there exists a $1 \leq k \in \mathbb{N}$ such that $a^k \leq \sum_{j \in J} ab_j$ in A.*

Proof. By hypothesis we have $\frac{\sum_j b_j}{a} = \sum_j \frac{b_j}{a} = 1$ so, as explained above (Lemma 10.2), there is an $n \in \mathbb{N}$ such that $a^n \leq \sum_j b_j$, so $a^{n+1} \leq \sum_j ab_j$. □

We can now prove that (co)covers compose.

Lemma 10.5. *If $(a_i \mid i \in I)$ covers A and, for each $i \in I$, $(\frac{b_{i,j}}{a_i} \mid j \in J_i)$ covers $A[a_i^{-1}]$ then $(a_i b_{i,j} \mid i \in I, j \in J_i)$ covers A.*

Proof. By hypothesis $\sum_{i\in I} a_i = 1$ and, by Lemma 10.4 above, there is a $1 \leq k_i \in \mathbb{N}$ such that $a_i^{k_i} \leq \sum_j a_i b_{i,j}$ for each $i \in I$. So, by [4, Lemma 4.1],

$$1 = \left(\sum_{i\in I} a_i\right)^{\prod_{i\in I} k_i} \leq \sum_{i\in I} a_i^{k_i} \leq \sum_{i\in I}\sum_{j\in J_i} a_i b_{i,j}$$

as we needed to show. □

We summarize what we have obtained so far in this section.

Proposition 10.6. *The cocovering families in $\mathbf{iRig}_{fp}$ form the basis for a Grothendieck topology on* **iAff** *and the resulting topos of sheaves classifies really local integral rigs.*

Proof. We observed that identities cover and that covers are stable under pullback. Lemma 10.5 proves that covers compose. We therefore have a basis and the resulting topos of sheaves. We occasionally refer to it as the 'Zariski' basis.

An argument analogous to that of Proposition 7.2 (and Theorem 8.5) establishes the classifying role of the topos of sheaves. In more detail one shows that the topology generated be the sequents stated in the beginning of the section coincides with the topology generated by the Zariski basis. To sketch the idea in more detail let $\mathbf{i} : \mathbf{Set} \to \mathbf{iRig}$ be the left adjoint to the forgetful functor. For efficiency we use some familiar notational tricks so, for example we write $\mathbf{i}[(x+y)^{-1}]$ instead of $\mathbf{i}[x,y][(x+y)^{-1}]$. Consider the span

$$\mathbf{i}[x^{-1}, y] \longleftarrow \mathbf{i}[(x+y)^{-1}] \longrightarrow \mathbf{i}[x, y^{-1}]$$

in $\mathbf{iRig}_{fp}$ induced by the sequent $x + y = 1 \vdash_{x,y} (x = 1) \vee (y = 1)$. Clearly, the pair $x, y \in \mathbf{i}[(x+y)^{-1}]$ cocovers. Similarly, the empty family cocovers the terminal algebra. That is, the topology generated by the sequents is included in the Zariski topology. Finally, one checks that these to covers generate the Zariski basis. $\square$

The basis on **iAff** described in this section may be called the 'Zariski' basis. (We stress the evident fact that, as in the classical case over fields, the Zariski basis contains the Gaeta basis.) The topos of sheaves for the Zariski basis on **iAff** will be denoted by $\mathcal{Z}$.

Remark 10.7 (On the representation of integral rigs)**.** Let R be the generic really local integral rig in $\mathcal{Z}$. The results in [4] imply that for any integral rig A there exists a spatial topos $\Gamma : \mathcal{E}_A \to \mathbf{Set}$ and a geometric morphism $\mathcal{O}_A : \mathcal{E}_A \to \mathcal{Z}$ over $\mathbf{Set}$ such that the algebra $\Gamma(\mathcal{O}_A^* R)$ of global sections of the sheaf $\mathcal{O}_A^* R$ of really local integral rigs is isomorphic to A. Compare with the classical Zariski representation of rings.

11. 'Zariski' covers of connected objects

In order to show that the Zariski topos of Section 10 is locally connected (over $\mathbf{Set}$) we will present a locally connected site for it. Local connectedness of the site will follow from the main result of the present section which proves, roughly speaking, that the Zariski basis on **iAff** is well behaved with respect to connectedness. We first need an algebraic result concerning covering families.

Lemma 11.1. *Let A be an integral rig and let the finite family $(a_i \in A \mid i \in I)$ cover A. Then, for any family $(k_i \in \mathbb{N} \mid i \in I)$, $(a_i^{k_i} \in A \mid i \in I)$ covers A.*

Proof. A standard argument using the multinomial theorem. In more detail, if we let $k = I \cdot \max_{i \in I} k_i$ then

$$1 = \left(\sum_{i \in I} a_i \right)^k = \sum_{i \in I} u_i({a_i}^{k_i})$$

for some family $(u_i \mid i \in I)$ of elements of A. Hence, $1 \leq \sum_{i \in I} a_i^{k_i}$. $\square$

The next result has a more geometric flavour.

Lemma 11.2. *Let X be connected in* **iAff** *and let the subobjects $u : U \to X$, $v : V \to X$ form a Zariski cover of X. If u and v are disjoint then either U is initial or V is initial.*

Proof. We argue on the algebraic side. Let $a, b \in A$ cover a directly indecomposable integral rig A. By Lemma 10.3 the cointersection of $A \to A[a^{-1}]$ and $A \to A[b^{-1}]$ is the universal $A \to A[(ab)^{-1}]$.

If the cointersection $A[(ab)^{-1}]$ is terminal then, by Lemma 10.2, there is an $n \in \mathbb{N}$ such that $(ab)^n = a^n b^n = 0$. Also, by Lemma 11.1, $a^n + b^n = 1$. So, as A is directly indecomposable by hypothesis, we may, without loss of generality, assume that $a^n = 1$ and $b^n = 0$. Then $A[b^{-1}]$ is terminal by Lemma 10.2. □

The following variant will be useful.

Lemma 11.3. *Let X be connected in* **iAff** *and let the subobjects $u : U \to X$, $v : V \to X$ form a Zariski cover of X. If U, V are non-initial in* **iAff** *then there is a point in the intersection $u \wedge v$. Equivalently, there are points $1 \to U$ and $1 \to V$ such that the following diagram*

$$\begin{array}{ccc} 1 & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle v} \\ U & \underset{u}{\longrightarrow} & X \end{array}$$

commutes in **iAff**.

Proof. By Lemma 11.2 the intersection is not empty so it is a finite coproduct of connected objects. Hence, a point in the intersection exists by the Nullstellensatz for 2-rigs. □

A subobject $U \to X$ in **iAff** is *basic* if the corresponding map in $\mathbf{iRig}_{fp}$ is of the form $A \to A[a^{-1}]$ for some $a \in A$. The next result shows that finite families of basic subobjects (of a common object) have a kind of 'join'.

Lemma 11.4. *If $(u_i : U_i \to X \mid i \in I)$ is a finite family of basic subobjects in* **iAff** *then there is a basic subobject $u : U \to X$ such that the following hold:*

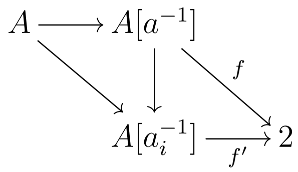
1. *For every $i \in I$, $u_i \le u$ as subobjects of X.*

2. *Every point of U factors through one of the inclusions $U_i \to U$ of the previous item.*

Proof. We argue on the algebraic side. We have a finite family

$$(A \to A[a_i^{-1}] \mid i \in I)$$

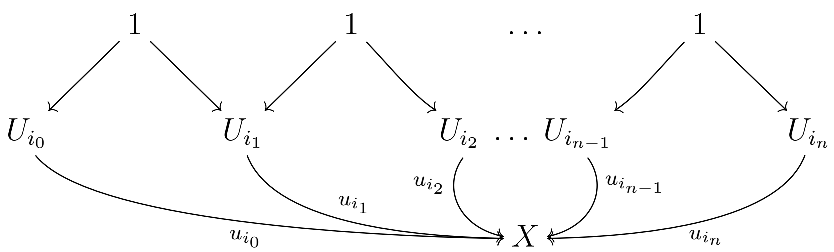
in $\mathbf{iRig}_{fp}$ (corresponding to the family of subobjects in the statement). Let $a = \sum_{i \in I} a_i$ and consider the map $A \to A[a^{-1}]$. Its universal property implies, for each $i \in I$, the existence of a unique map $A[a^{-1}] \to A[a_i^{-1}]$ such that the left triangle below



commutes in $\mathbf{iRig}_{fp}$, so the first item is proved. To prove the second item let f be a map as in the right above. Then $\sum_{i \in I} f a_i = 1 \in 2$ and, as 2 is really local, there is an $i \in I$ such that $f a_i = 1$. So there is an f' such that the right triangle above commutes. □

The next result is a 'Zariski analogue' of a familiar property of open covers of connected topological spaces.

Proposition 11.5. *Let $(u_i : U_i \to X \mid i \in I)$ be a Zariski cover of X in* **iAff** *such that U_i is not initial for each $i \in I$. If X is connected then, for every $k, l \in I$ there exists a sequence $k = i_0, i_1, \ldots, i_n = l \in I$ and a commutative diagram as below.*



Proof. Fix $k \in I$ and let $J \subseteq I$ be the subset of those $l \in I$ such that there is a sequence $k = i_0, i_1, \ldots, i_n = l \in I$ and a diagram as in the statement. Let $u : U \to X$ be the basic subobject determined as in Lemma 11.4 by the family $(u_j : U_j \to X \mid j \in J)$. Similarly, Let $v : V \to X$ be the basic subobject determined by the complement $J' \subseteq I$ of $J \subseteq I$. It is not difficult to check that u, v cover X. Assume for the sake of contradiction that J' is not-empty. Then V is not initial so Lemma 11.3 implies the existence of a point in the intersection of u and v. Lemma 11.4 implies that the same point is in U_j for some $j \in J$ and in $U_{j'}$ for some $j' \in J'$. Then $j' \in J$, which is absurd. Hence J' is empty. □

12. The 'Zariski' topos is pre-cohesive

In Section 10 we equipped **iAff** with the basis of a 'Zariski' topology and showed that the resulting topos $\mathcal{Z}$ of sheaves classifies really local integral rigs. In this section we show that this basis is subcanonical and that the canonical geometric morphism $\mathcal{Z} \to \mathbf{Set}$ is pre-cohesive. (Again, the general strategy is analogous to that of the classical case.)

Lemma 12.1. *Zariski covers in* **iAff** *are jointly epic.*

Proof. We argue on the algebraic side. We prove that if A is a integral rig and the finite family $(a_i \in A \mid i \in I)$ covers A then $(A \to A[a_i^{-1}] \mid i \in I)$ is a jointly monic family of maps.

Let $x, y \in A$ be such that $\frac{x}{a_i} = \frac{y}{a_i}$ in $A[a_i^{-1}]$ for each $i \in I$. Then, there is an $m \in \mathbb{N}$ such that $a_i^m x \leq y$ and $a_i^m y \leq x$ for each $i \in I$. As the family $(a_i \in A \mid i \in I)$ covers, so does $(a_i^m \in A \mid i \in I)$ by Lemma 11.1. That is, $1 = \sum_{i \in I} a_i^m$. Then

$$x = \sum_{i \in I} a_i^m x \leq \sum_{i \in I} y = y$$

and, similarly, $y \leq x$. □

It follows that representable objects in $\widehat{\mathbf{iAff}}$ are separated.

Proposition 12.2. *The 'Zariski' topology on* **iAff** *is subcanonical.*

Proof. Recall that we denote the left adjoint to the forgetful functor by $\mathbf{i} : \mathbf{Set} \to \mathbf{iRig}$ so that the free integral rig on one generator may be denoted by $\mathbf{i}[x]$. We first prove that $R = \mathbf{iRig}_{fp}(\mathbf{i}[x], -) : \mathbf{iRig}_{fp} \to \mathbf{Set}$ is a sheaf. Let $(a_i \mid i \in I)$ cover A. By Lemma 10.3, a family $(\frac{x_i}{a_i} \in A[a_i^{-1}] \mid i \in I)$ is compatible with the cover if, for every $i, j \in I$, $\frac{x_i}{a_i a_j} = \frac{x_j}{a_i a_j} \in A[(a_i a_j)^{-1}]$. By Lemma 10.2 above there is, for each $i, j \in I$, an $m_{i,j} \in \mathbb{N}$ such that $(a_i a_j)^{m_{i,j}} x_i \leq x_j$ and $(a_i a_j)^{m_{i,j}} x_j \leq x_i$. If we let m be the largest of the $m_{i,j}$'s then we get that $(a_i a_j)^m x_i \leq x_j$ and $(a_i a_j)^m x_j \leq x_i$. So, if we let $x = \sum_{i \in I} a_i^m x_i$ then, clearly $a_j^m x_j \leq x$ for every $j \in I$ and also

$$a_j^m x = \sum_{i \in I} (a_i a_j)^m x_i \leq \sum_{i \in I} x_j = x_j$$

so $\frac{x_j}{a_j} = \frac{x}{a_j}$ in $A[a_j^{-1}]$. In other words, the compatible family has an amalgamation. This amalgamation is unique by Lemma 12.1.

Sheaves are closed under finite limits and every object in **iAff** is the equalizer of a parallel pair of maps between finite powers of R. As R is a sheaf, the result follows. □

We next show that the canonical geometric morphism $\mathcal{Z} \to \mathbf{Set}$ is pre-cohesive. It is enough to provide a locally connected site for $\mathcal{Z}$, but the one we have on **iAff** is not. The rest of the section is devoted to find one.

Since we have presented our toposes using bases for Grothendieck topologies it is convenient have a version of the Comparison Lemma in terms of these. The following is surely folklore.

Let $\mathcal{C}$ be a small category equipped with the basis K for a Grothendieck topology. Let $\mathcal{D} \to \mathcal{C}$ be a full subcategory of $\mathcal{C}$. We say that the subcategory is *(K-)dense* if for every C in $\mathcal{C}$ there is a K-cover $(D_i \to C \mid i \in I)$ in $\mathcal{C}$ with D_i in $\mathcal{D}$ for every $i \in I$. For D in $\mathcal{D}$ let $K'D \subseteq KD$ be the set of K-covers $(D_i \to D \mid i \in I)$ such that D_i in $\mathcal{D}$ for every $i \in I$.

Lemma 12.3. *With the notation above, if $\mathcal{D} \to \mathcal{C}$ is K-dense then K' is the basis for a Grothendieck topology on $\mathcal{D}$ and the obvious restriction functor induces an equivalence* $\mathrm{Sh}(\mathcal{C}, K) \to \mathrm{Sh}(\mathcal{D}, K')$.

Proof. It is easy to check that isomorphisms K'-cover and that K'-covers compose. Assume now that $(f_i : D_i \to D \mid i \in I)$ is a K'-cover. So it is a K-cover and then, for any $g : E \to D$ in $\mathcal{D}$, there exists a K-cover $(g_j : C_j \to E \mid j \in J)$ such that for every $j \in J$ there is an $i_j \in I$ such that gg_j factors through f_{i_j}. As $\mathcal{D}$ is K-dense there is, for each $j \in J$, a K-cover $(h_{j,k} : B_{j,k} \to C_j \mid k \in J_j)$ with $B_{k,j}$ in $\mathcal{D}$ for every $k \in J_j$. The composite family $(g_j h_{j,k} : B_{j,k} \to E \mid j \in J, k \in J_j)$ is a K-cover and, as all the domains are in $\mathcal{D}$, it is also a K'-cover. Moreover, for every $j \in J$ and $k \in J_j$ the map $gg_j h_{j,k}$ factors through f_{i_j}. Altogether, we have shown that K' is the basis of a Grothendieck topology.

Let $\overline{K}$ be the Grothendieck topology generated by K. That is, a sieve on C in $\mathcal{C}$ is $\overline{K}$-covering if and only if it contains all the maps in a K-covering family. Density of $\mathcal{D}$ in the 'basis sense' of the statement easily implies that $\mathcal{D}$ is $\overline{K}$-dense in the sense of the Comparison Lemma, so restriction along $\mathcal{D} \to \mathcal{C}$ induces an equivalence $\mathrm{Sh}(\mathcal{C}, \overline{K}) \to \mathrm{Sh}(\mathcal{D}, L)$ where L is the topology on $\mathcal{D}$ induced by $\overline{K}$ (in the sense of the Comparison Lemma). It remains to show that the basis K' generates the topology L.

A sieve S in $\mathcal{D}$ on an object D is L-covering if and only if the generated sieve $\overline{S} = \{fg \mid f : D' \to D \text{ in } S, g : C \to D' \text{ in } \mathcal{C}\}$ in $\mathcal{C}$ is $\overline{K}$-covering. That is, if and only if $\overline{S}$ contains the maps in a K-covering family F on D. Composing F with the special covers provided by density (in the 'basis sense'), as in the paragraph above, we obtain that $\overline{S}$ contains the maps in a K'-cover. In other words, every L-covering sieve is $\overline{K'}$-covering where $\overline{K'}$ is the topology generated by K'. Conversely, if a sieve S on D is $\overline{K'}$-covering then it contains the maps in a K'-covering family. As every K'-covering family is K-covering, $\overline{S}$ is $\overline{K}$-covering and hence, S is L-covering. □

We may now prove the main result of the section.

Theorem 12.4. *The classifier of really local integral rigs is pre-cohesive over sets.*

Proof. By [7, Proposition 1.4] it is enough to provide a connected and locally connected site of definition for $\mathcal{Z}$ such that every object in the site has a point. Let K be the 'Zariski' basis on **iAff** introduced in Section 10. As the Zariski basis contains the Gaeta basis and every object in **iAff** is a finite coproduct of connected objects (Corollary 8.4), the full subcategory $\mathbf{iAff}_c \to \mathbf{iAff}$ of connected objects is K-dense. Lemma 12.3 implies that $\mathcal{Z} = \mathrm{Sh}(\mathbf{iAff}, K)$ is equivalent to $\mathrm{Sh}(\mathbf{iAff}_c, K')$ where K' is the restriction of K. The category $\mathbf{iAff}_c$ has a terminal object (because 2 is directly indecomposable in $\mathbf{iRig}$). That is, the site $(\mathbf{iAff}_c, K')$ is connected. Also, every object has a point by Proposition 6.5. Finally, the site is locally connected by Proposition 11.5. □

Altogether, as in the classical space, the classifier $\mathcal{Z} \to \mathbf{Set}$ of really local integral rigs is pre-cohesive; the Yoneda embedding restricts to a full inclusion $\mathbf{iAff} \to \mathcal{Z}$ that sends Zariski covers to jointly epimorphic families so, in particular, it preserves finite coproducts.

Recent unpublished work on integral rigs by Jipsen and Spada on subdirectly irreducible integral rigs suggests that it is possible to calculate level ϵ of the pre-cohesive toposes $\mathfrak{G}(\mathbf{iAff})$ and $\mathcal{Z}$ as in the classical complex case discussed in [14].

On the other hand, if we let R be the generic really local integral rig then, although the subobject $D = \{x \in R \mid x^2 = 0\} \to R$, is non-trivial, the exponential R^D is not isomorphic to $R \times R$. In other words, the Kock-Lawvere axiom for SDG does not hold. At present it is not clear to the author if this is a drawback or an opportunity for interesting variants of the KL-axiom. Also in contrast with the classical case, the topos of simplicial sets is a subtopos of $\mathcal{Z}$. So there is a full inclusion $\widehat{\Delta} \to \mathcal{Z}$ and, for every X in $\mathcal{Z}$ (an 'i-scheme'), a universal map $X \to SX$ towards a simplicial set. Intuitively, the inverse image $\mathcal{Z} \to \widehat{\Delta}$ is a 'combinatorial realization' analogous to the classical 'geometric realizations' or, perhaps, it is more similar to the 'combinatorial truncations' $\widehat{\Delta} \to \widehat{\Delta_n}$ induced by the inclusions $\Delta_n \to \Delta$ for each $n \in \mathbb{N}$. See Corollary 7.5 in [13].

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

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ORDER CONVERGENCE AND CONVERGENCE IN THE INTERVAL TOPOLOGY IN THE PRESENCE OF COMPACTOIDNESS

Frédéric MYNARD

Résumé. Nous utilisons le concept de filtre compactoid pour obtenir une généralisation avec une preuve simplifiée d'un résultat de van der Zypen qui établissait que si la topologie des intervaux est compacte, alors elle est plus grossière que la topologie de l'ordre. Notre version est localisée et s'applique non seulement aux topologies mais aussi aux convergences.

Abstract. Using the concept of compactoid filter, we obtain a generalized version with a simplified proof of a result of van der Zypen to the effect that whenever the interval topology is compact, it is coarser than the order topology. The present version is localized and applies not only to topologies but also to convergence structures.

Keywords. order topology; order convergence; interval topology; convergence space; pseudotopology; compactoid filter.

Mathematics Subject Classification (2010). 06F30; 54A20.

Dominic van der Zypen has shown [14] that if a topology on a poset is finer than the interval topology and is compact, then it is coarser than the order topology. It is the aim of this short note to provide a stronger and "localized" version of this theorem, with a simpler proof. To localize the compactness hypothesis, I use *compactoid filters* (e.g., [5]). Recall that a filter is *compactoid* if each of its ultrafilters is convergent. Of course, every filter on a compact (topological or convergence) space is compactoid.

Moreover, every convergent filter is compactoid and a space X is compact if and only if $\{X\}$ is compactoid. Hence compactoidness is a common generalization of compactness and convergence. More generally, a filter $\mathcal{F}$ is *compactoid at* $A \subset X$ if each of its ultrafilters has a limit point in A. Such generalization of compactness finds far reaching applications (e.g., [12, 13, 4, 6, 3, 8, 10, 11, 7, 1, 9, 2])

The result concerns various *convergence structures* on a poset $(P, \leq)$. Recall that a *convergence* ξ on a set X is a relation — denoted $x \in \lim_\xi \mathcal{F}$ or $\mathcal{F} \underset{\xi}{\to} x$ whenever x and $\mathcal{F}$ are in relation — between X and the set $\mathbb{F}X$ of filters on X, satisfying

1. $\{x\}^\uparrow \to x$ for every $x \in X$;
2. $\mathcal{F} \geq \mathcal{G} \Longrightarrow \lim \mathcal{F} \supset \lim \mathcal{G}$.

A map $f : (X, \xi) \to (Y, \tau)$ between two convergence spaces is *continuous* if

$$f(\lim_\xi \mathcal{F}) \subset \lim_\tau f[\mathcal{F}],$$

where $f[\mathcal{F}] = \{B \subset Y : f^{-1}(B) \in \mathcal{F}\}$. We refer to [7] for a systematic study of convergence spaces.

The category **CONV** of convergence spaces and continuous maps is a topological cartesian closed and extensional category (hence a quasitopos). The set of convergence structures on a given set is a complete lattice for the order $\xi \leq \tau$ whenever $\mathrm{id} : (X, \tau) \to (X, \xi)$ is continuous. The category **TOP** of topological spaces and continuous maps is a concretely reflective subcategory. Indeed, calling ξ-closed a set containing the ξ-limit of each filter on it, the family of all ξ-closed sets for a convergence is the family of closed sets for a topology, denoted $\mathrm{T}\,\xi$ and called *topological reflection or topological modification of* ξ. It is the finest topology coarser than ξ. A convergence is a *pseudotopology* if $\mathcal{F}$ converges to x whenever every ultrafilter finer than $\mathcal{F}$ does. The subcategory **PSTOP** of **CONV** formed by pseudotopological spaces and continuous maps is concretely reflective, and the *pseudotopological reflection* $\mathrm{S}\,\xi$ of a convergence ξ is given by

$$\lim_{\mathrm{S}\,\xi} \mathcal{F} = \bigcap_{\mathcal{U} \in \beta\mathcal{F}} \lim_\xi \mathcal{U},$$

where $\beta\mathcal{F}$ denotes the set of ultrafilters finer than $\mathcal{F}$.

If $A \subset (P, \leq)$ let $A^u = \bigcap_{a \in A} \uparrow a$ be the set of upper bounds of A and $A^\ell = \bigcap_{a \in A} \downarrow a$ be the set of lower bounds. If now $\mathcal{A} \subset 2^P$, $\mathcal{A}^u = \bigcup_{A \in \mathcal{A}} A^u$ and $\mathcal{A}^\ell = \bigcup_{A \in \mathcal{A}} A^\ell$.

On a poset $(P, \leq)$, I consider the following convergence structures (it is well known and easy to verify that they satisfy the above axioms of convergence):

$$x \in \lim_- \mathcal{F} \text{ if } \bigvee \mathcal{F}^\ell \text{ exists and } x \leq \bigvee \mathcal{F}^\ell; \qquad \text{(lower convergence)}$$

$$x \in \lim_+ \mathcal{F} \text{ if } \bigwedge \mathcal{F}^u \text{ exists and } \bigwedge \mathcal{F}^u \leq x; \qquad \text{(upper convergence)}$$

$$\lim_o \mathcal{F} = \lim_- \mathcal{F} \cap \lim_+ \mathcal{F}. \qquad \text{(convergence in order)}$$

Since $\bigwedge \mathcal{F}^u \geq \bigvee \mathcal{F}^\ell$ ([1]), $x \in \lim_o \mathcal{F}$ if $\bigvee \mathcal{F}^\ell = x = \bigwedge \mathcal{F}^u$. Note also that in complete lattice, $\bigvee \mathcal{F}^\ell = \bigvee_{F \in \mathcal{F}} \bigwedge F$ and $\bigwedge \mathcal{F}^u = \bigwedge_{F \in \mathcal{F}} \bigvee F$.

The *order topology* on $(P, \leq)$ is the topological reflection To of the convergence in order o. Let τ_i^- denote the *lower interval topology,* generated by the complements of lower rays $(x] = \{y : y \leq x\}$. Accordingly, τ_i^+ denotes the *upper interval topology*, generated by complements of upper rays $[x) = \{y : x \leq y\}$, and $\tau_i = \tau_i^- \vee \tau_i^+$ is the *interval topology* on $(P, \leq)$. Note that ([2])

$$\tau_i \leq \mathrm{T}\, o \leq o = + \vee -.$$

Example 1. In the (spatial) frame $\mathcal{O}(X)$ of open subsets of a topological space X, a basic open set for τ_i^- is of the form

$$\mathcal{O}(X) \setminus \{V \in \mathcal{O}(X) : V \subset U_0\} = \{V \in \mathcal{O}(X) : V \cap (U_0)^c \neq \emptyset\},$$

while a basic open set for τ_i^+ is of the form

$$\mathcal{O}(X) \setminus \{V \in \mathcal{O}(X) : U_0 \subset V\} = \{V \in \mathcal{O}(X) : V^c \cap U_0 \neq \emptyset\}.$$

[1] because if $x \in \mathcal{F}^u$ and $y \in \mathcal{F}^\ell$ there is $F_1, F_2 \in \mathcal{F}$ with $x \in F_1^u$ and $y \in F_2^\ell$ so that $y \leq z \leq x$ for every $z \in F_1 \cap F_2$

[2] In fact $+ \geq \tau_i^+$ and $- \geq \tau_i^-$. If $x \in \lim_o \mathcal{F}$, that is, $\bigvee \mathcal{F}^\ell = x = \bigwedge \mathcal{F}^u$, and $x \in P \setminus [t)$ then $P \setminus [t) \in \mathcal{F}$ for otherwise $[t) \in \mathcal{F}^\#$, that is, every $F \in \mathcal{F}$ has an element $t_F \geq t$, so that $t \leq \bigwedge \mathcal{F}^u \leq x$ and $x \in [t)$; a contradiction.

The lower convergence is the counterpart on $\mathcal{O}(X)$ of the upper Kuratowski convergence, that is,

$$\mathcal{F} \underset{-}{\to} U \iff U \subset \bigcup_{F\in\mathcal{F}} \operatorname{int}_X(\bigcap_{O\in F} O),$$

and the upper convergence

$$\mathcal{F} \underset{+}{\to} U \iff \operatorname{int}_X(\bigcap_{F\in\mathcal{F}} \bigcup_{O\in F} O) \subset U.$$

Theorem 2. *Let τ be a convergence on a poset $(P, \leq)$. Then*

1. *Assume that $\tau \geq \tau_i^-$. If a filter $\mathcal{F}$ on P is τ-compactoid at $\{x\} \cup (x]^c$ and $\mathcal{F} \underset{+}{\to} x$, then $\mathcal{F} \underset{\mathrm{S}\tau}{\to} x$;*
2. *Assume that $\tau \geq \tau_i^+$. If a filter $\mathcal{F}$ on P is τ-compactoid at $\{x\} \cup [x)^c$ and $\mathcal{F} \underset{-}{\to} x$, then $\mathcal{F} \underset{\mathrm{S}\tau}{\to} x$;*
3. *Assume that $\tau \geq \tau_i$. If a filter $\mathcal{F}$ on P is τ-compactoid and $\mathcal{F} \underset{o}{\to} x$, then $\mathcal{F} \underset{\mathrm{S}\tau}{\to} x$.*

Proof. (1). Let $\mathcal{U}$ be an ultrafilter finer than $\mathcal{F}$. Since $\mathcal{F}$ is τ-compactoid, $\mathcal{U}$ is τ-convergent to some $y \in \{x\} \cup (x]^c$. Assume $y \neq x$. Then $y \nleq x$ and there exists $z_0 \in \mathcal{U}^u$ such that $y \nleq z_0$, for otherwise, $y \leq \bigwedge \mathcal{U}^u \leq x$ because $\mathcal{U} \underset{+}{\to} x$. Therefore, $(z_0] \in \mathcal{U}$ and $y \in (z_0]^c$. But $(z_0]^c$ is τ_i^--open, hence τ-open because $\tau \geq \tau_i^-$, and therefore belongs to $\mathcal{U}$ because $\mathcal{U} \underset{\tau}{\to} y$; a contradiction. Thus $x = y$ and $\mathcal{U} \underset{\tau}{\to} x$ for every ultrafilter $\mathcal{U}$ of $\mathcal{F}$. In other words, $\mathcal{F} \underset{\mathrm{S}\tau}{\to} x$.

(2) is proved in a similar way.

(3). If $\mathcal{F} \underset{o}{\to} x$, then $\mathcal{F} \underset{+}{\to} x$ and $\mathcal{F} \underset{-}{\to} x$. Moreover, $\mathcal{F}$ is τ-compactoid at $P = \{x\} \cup (x]^c \cup \{x\} \cup [x)^c$. Hence an ultrafilter $\mathcal{U}$ of $\mathcal{F}$ converges to x for τ by either (1) or (2) depending on where the limit point obtained by compactoidness lies. □

The following particular case of (3) extends the main theorem (i.e., Theorem 2.1) of [14] from topologies to convergences.

Corollary 3. *Let τ be a compact convergence on a poset $(P, \leq)$. If $\tau \geq \tau_i$, then $o \geq \mathrm{S}\,\tau$ and therefore $\mathrm{T}\,o \geq \mathrm{T}\,\tau$.*

Proof. If τ is compact, then every filter on P is τ-compactoid. Therefore,

$$\mathcal{U} \underset{o}{\to} x \Longrightarrow \mathcal{U} \underset{\tau}{\to} x$$

for every ultrafilter $\mathcal{U}$, that is, $o \geq \mathrm{S}\,\tau$. □

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