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SOMMAIRE

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INTEGRAL AND DIFFERENTIAL STRUCTURE ON THE FREE C^∞-RING MODALITY

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Résumé. Les catégories intégrales ont été récemment développées comme homologues aux catégories différentielles. En particulier, les catégories intégrales sont équipées d'un opérateur d'intégration, appelé la transformation intégrale, dont les axiomes généralisent les identités d'intégration de base du calcul comme l'intégration par parties. Cependant, la littérature sur les catégories intégrales ne contient aucun exemple décrivant l'intégration de fonctions lisses arbitraires : les exemples les plus proches impliquent l'intégration de fonctions polynomiales. Cet article comble cette lacune en développant un exemple de catégorie intégrale dont la transformation intégrale agit sur des 1-formes différentielles lisses. De plus, nous fournissons un autre point de vue sur la structure différentielle de cet exemple clé, nous étudions les dérivations et les codéréliction dans ce contexte et nous prouvons que les anneaux C^∞ libres sont des algèbres de Rota-Baxter.

Abstract. Integral categories were recently developed as a counterpart to differential categories. In particular, integral categories come equipped with an integration operator, known as an integral transformation, whose axioms generalize the basic integration identities from calculus such as integration by parts. However, the literature on integral categories contains no example that captures integration of arbitrary smooth functions: the closest are examples involving integration of polynomial functions. This paper fills in this gap

by developing an example of an integral category whose integral transformation operates on smooth 1-forms. We also provide an alternative viewpoint on the differential structure of this key example, investigate derivations and coderelictions in this context, and prove that free C^∞-rings are Rota-Baxter algebras.
Keywords. differential categories; C-infinity rings; Rota-Baxter algebras; monads; algebra modalities; monoidal categories; derivations; Kähler differentials.
Mathematics Subject Classification (2020). 18F40; 18M05; 18C15; 26B12; 13N15; 13N05; 26B20; 03F52.

1. Introduction

One of the most important examples of a differential category [5] captures differentiation of smooth functions by means of (co)differential structure on the free C^∞-ring monad on $\mathbb{R}$-vector spaces; this example was given in [5, §3] as an instance of a more general construction (called the S^∞ construction). It is important for at least three reasons: firstly, it is a differential category based directly on ordinary differential calculus. Secondly, through an analogy with the role of commutative rings in algebraic geometry, C^∞-rings play an important role in the semantics of *synthetic differential geometry* [17, 25] and so provide a key benchmark for the generalization of aspects of commutative algebra in differential categories, including the generalizations of derivations and Kähler differentials in [7]. Thirdly, the free C^∞-ring monad provides a key example of a differential category that does not possess the Seely (also known as storage) isomorphisms, as we discuss in Remark 5.16, because it is well known that the canonical linear map $C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ is not an isomorphism. Differential category structure can be simplified if one assumes the Seely isomorphisms (for more on this, see [4]); this key example shows why it is important to not assume them in general.

A recent addition to the study of categorical calculus is the story of integration and the fundamental theorems of calculus with the discovery of *integral* and *calculus* categories [10] and differential categories with *antiderivatives* [10, 12]. These discoveries show that both halves of calculus can be developed at this abstract categorical level. The first notion of integration in a differential category was introduced by Ehrhard in [12] with the introduction of differential categories with antiderivatives, where one builds an integral structure from the differential structure. Integral categories and calculus categories were then introduced in the second author's Master's thesis [19], under the supervision of Bauer and Cockett. Integral categories have an axiomatization of integration that is independent from differentiation, while the axioms of calculus categories describe compatibility relations between a differential structure and an integral structure via the two fundamental theorems of calculus. In particular, every differential category with antiderivatives is a calculus category. Cockett and the second author also published an extended abstract [9] and then a journal paper [10] which provided the full story of integral categories, calculus categories, and differential categories with antiderivatives.

However, an important potential example was missing in those papers: an integral category structure on the free C^∞-ring monad that would be compatible with the known differential structure. Such an example is important for the same reasons as above: it would give an integral category that resembles ordinary calculus, and it would show that it is useful to avoid assuming the Seely isomorphisms for integral categories (noting that, as with differential categories, the assumption of the Seely isomorphisms can simplify some of the structure: for example, see [20, Theorem 3.8]). The journal paper on integral categories [10] presented an integral category of polynomial functions, but it was not at all clear from its definition (and not known) that the formula for its deriving transformation could be generalized to yield an integral category of arbitrary smooth functions.

Developing such an example (namely an integral category structure for the free C^∞-ring monad) is the central goal of this paper. As noted above, the existence of such an example, involving infinitely differentiable functions, demonstrates the relevance and importance of the definition of integral categories. Since this new example most resembles ordinary calculus, it might in retrospect seem unsurprising. But what is surprising about this story is that (1) the initial work on integral categories, in studying structural aspects of integration and antiderivatives, had arrived at an axiomatics that was not initially known to admit a model in smooth functions but instead had a polynomial model that, in the multivariable case, seemed unfamiliar, and that (2) despite this, we now show here that a model in smooth functions exists and is based on familiar notions of vector calculus, namely line integrals.

In considering the integral side of this example, we have also found additional results and ideas for the differential side. In particular, in order to define the integral structure for this example, we have found it helpful to give an alternative presentation of its differential structure and its monad S^∞ on the category of vector spaces over $\mathbb{R}$. The original paper on differential categories [5] did not mention the fact that S^∞ is the free C^∞-ring monad, nor that it is a finitary monad, although it did construct this monad as an instance of a more general construction applicable for certain Lawvere theories carrying differential structure. However, that paper [5] did not define S^∞ by means of the usual recipe through which a finitary monad is obtained from its corresponding Lawvere theory; instead, the endofunctor S^∞ was defined in [5, §3] by associating to each real vector space V a set $S^\infty(V)$ consisting of certain mappings $h : V^* \to \mathbb{R}$ on the algebraic dual V^* of V.

To facilitate our work with this example, we have found it helpful to give an alternative approach, via the theory of finitary monads. Since the monad S^∞ is finitary, we are able to exploit standard results on locally finitely presentable categories and finitary monads to show that the differential structure carried by S^∞ arises by left Kan extension from structure present on the finite-dimensional real vector spaces. Aside from shedding some new light on this important example, this approach enables us to define an integral structure on S^∞ through a similar method of left Kan extension, starting with integration formulae for finite-dimensional spaces.

In addition to providing a key new example of an integral category, this paper also has some further interesting aspects. The first is in its investigation of derivations in this context. A recent paper by Blute, Lucyshyn-Wright, and O'Neill [7] defined derivations for (co)differential categories. Here we show that derivations in this general sense, when applied to the C^∞-ring example that we consider here, correspond precisely to derivations of the Fermat theory of smooth functions as defined by Dubuc and Kock [11]. This provides additional evidence that the Blute/Lucyshyn-Wright/O'Neill definition is the appropriate generalization of derivations in the context of codifferential categories. We also show that while this key example does not possess a *codereliction* (see [4, 5]), it does possess structure sharing many of the key features of a codereliction.

Finally, we conclude with an interesting result on Rota-Baxter algebras.

By definition, an integral category satisfies a certain Rota-Baxter axiom. By showing that the smooth algebra example is an integral category, we get as a corollary that free C^∞-rings are Rota-Baxter algebras (Proposition 6.10), a result that appears to be new.

The paper is organized as follows. In Section 2, we review differential and integral categories, working through the definitions using the standard polynomial example. In Section 3 we review and discuss some aspects of finitary monads that will be useful in relation to our central 'smooth' example, including some results that are known among practitioners but whose statements we have not found to appear in the literature. In Section 4, we review generalities on C^∞-rings, and we define the C^∞-ring monad (and algebra modality) on real vector spaces. In Section 5 we define the differential structure of this example, as well as consider derivations and (co)derelictions in this context. Finally, in Section 6, we establish the integral structure of the central example, and we conclude by proving that free C^∞-rings have Rota-Baxter algebra structure.

2. Background on differential and integral categories

This section reviews the central structures of the paper: (co)differential categories, (co-)integral categories, and (co-)calculus categories [5, 10]. Throughout this section, we will highlight the particular example of the category of $\mathbb{R}$-vector spaces with polynomial differentiation and integration [5, Proposition 2.9]. While much of this material is standard, we have included it here to set a consistent notation and to clarify precisely which definitions we are using (for example, the definition of (co)differential category changed from [5] to [6]).

We should first explain the intuition behind codifferential categories, as compared to differential categories. Differential categories were introduced to provide the categorical semantics of differential linear logic [12]. Briefly, a differential category comes equipped with a coalgebra modality, which in particular is a comonad !, and a natural transformation $\mathsf{d}_A : !A \otimes A \to !A$ which axiomatizes the basic properties of differentiation. The coKleisli morphisms $f : !A \to B$ are to be thought of as *smooth* maps $A \to B$, that is, the maps that are infinitely differentiable. Indeed, the derivative of a coKleisli morphism is the morphism $\mathsf{D}[f] : !A \otimes A \to !A$ defined as $\mathsf{D}[f] = \mathsf{d}_A f$.

Codifferential categories are the dual of differential categories. Therefore, a codifferential category comes equipped with an algebra modality, which is a monad S such that every SC comes equipped with a natural commutative monoid structure, and this time a natural transformation $\mathsf{d}_C : SC \to SC \otimes C$ which again axiomatizes the basic properties of differentiation such as the product rule and the chain rule. The intuition here is that SC may be thought of as a space of smooth (or differentiable) scalar-valued functions on C. Indeed, thinking of Kleisli morphisms $f : B \to SC$ as smooth maps $C \to B$ and writing the monoidal unit as k, we are led to regard the 'elements' $f : k \to SC$ of SC as smooth functions $C \to k$. The Kleisli composition for the monad S then may be seen as composition of smooth maps, while the monoid structure on SC corresponds to multiplication of smooth maps. The natural transformation d_C is then a differential operator that sends a smooth function on C to its derivative. For more details on these intuitions, see the original paper [5].

We now recall the various elements of the definition of codifferential categories, beginning first with the monoidal and additive structure. Here we use the term *additive category* to refer to any category enriched in commutative monoids, while this term is more often used for categories that are enriched in abelian groups and have finite biproducts.

Definition 2.1. *An **additive symmetric monoidal category** consists of a symmetric monoidal category $(\mathcal{C}, \otimes, k, \sigma)$ such that $\mathcal{C}$ is enriched over commutative monoids and $\otimes$ preserves the commutative monoid structure in each variable separately.*

Example 2.2. The category of vector spaces over $\mathbb{R}$ and $\mathbb{R}$-linear maps between them, $\mathbb{R}$-Vec, is an additive symmetric monoidal category with the structure given by the standard tensor product and the standard additive enrichment of vector spaces.

The next requirement for a codifferential category is an algebra modality, which is a monad S for which every free S-algebra comes equipped with a natural commutative monoid structure:

Definition 2.3. *If $(\mathcal{C}, \otimes, k, \sigma)$ is a symmetric monoidal category, an **algebra modality** $(\mathsf{S}, \mathsf{m}, \mathsf{u})$ on $\mathcal{C}$ consists of:*

- *a monad $\mathsf{S} = (S, \mu, \eta)$ on $\mathcal{C}$;*

- *a natural transformation* m*, with components* $\mathsf{m}_C : SC \otimes SC \to SC$ $(C \in \mathcal{C})$*;*
- *a natural transformation* u*, with components* $\mathsf{u}_C : k \to SC \ \ (C \in \mathcal{C})$*;*

such that

- *for each object* C *of* $\mathcal{C}$*,* $(SC, \mathsf{m}_C, \mathsf{u}_C)$ *is a commutative monoid (in the symmetric monoidal category* $\mathcal{C}$*);*
- *each component of* μ *is a monoid morphism (with respect to the obvious monoid structures).*

Such an algebra modality $(\mathsf{S}, \mathsf{m}, \mathsf{u})$ *will also be denoted by* $(S, \mu, \eta, \mathsf{m}, \mathsf{u})$ *or by* S.

As discussed above, SC may be thought of as a space of smooth maps $C \to k$, and then $\mathsf{m}_C : SC \otimes SC \to SC$ may be interpreted as multiplication of smooth functions, while the unit $\mathsf{u}_C : k \to SC$ picks out the multiplicative identity element, seen as a constant smooth function. Following this interpretation further, the monad unit $\eta_C : C \to SC$ picks out the linear maps, while Kleisli composition effects the composition of smooth maps. Some of these intuitions are illustrated in the following example, and also in a further example that we shall consider in detail in Section 4.

Example 2.4. $\mathbb{R}$-Vec has an algebra modality Sym, which sends a vector space V to the symmetric algebra on V (over $\mathbb{R}$),

$$\mathsf{Sym}(V) := \bigoplus_{n=0}^{\infty} \mathsf{Sym}^n(V)$$

where $\mathsf{Sym}^0(V) := \mathbb{R}, \mathsf{Sym}^1(V) := V$, and for $n \geq 2$, $\mathsf{Sym}^n(V)$ is the quotient of the tensor product of V with itself n times by the equations

$$v_1 \otimes \ldots \otimes v_i \otimes \ldots v_n = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(i)} \otimes \ldots \otimes v_{\sigma(n)}$$

associated to permutations σ of $\{1, 2, \ldots n\}$. It turns out that $\mathsf{Sym}(V)$ is the free commutative $\mathbb{R}$-algebra on the $\mathbb{R}$-vector space V. It is a standard result that $\mathsf{Sym}(V)$ can also be identified with a polynomial ring: if $X = \{x_i \mid i \in I\}$ is a basis for V, then

$$\mathsf{Sym}(V) \cong \mathbb{R}[X].$$

When $V = \mathbb{R}^n$, this makes it clear that $\mathsf{Sym}(V)$ is indeed a ring of smooth functions on V, namely the polynomial functions. For example, if $V = \mathbb{R}^3$ then $\mathsf{Sym}(V) \cong \mathbb{R}[x, y, z]$, which is the space of polynomial functions $\mathbb{R}^3 \to \mathbb{R}$.

We now review the main ingredient of codifferential structure.

Definition 2.5. *If $(\mathcal{C}, \otimes, k, \sigma)$ is an additive symmetric monoidal category with an algebra modality $(S, \mu, \eta, \mathsf{m}, \mathsf{u})$, then a **deriving transformation** on $\mathcal{C}$ is a natural transformation d, with components*

$$\mathsf{d}_C : SC \to SC \otimes C \qquad (C \in \mathcal{C})$$

such that[1]

[d.1] ***Derivative of a constant**:* $\mathsf{ud} = 0$*;*

[d.2] ***Leibniz/product rule**:* $\mathsf{md} = [(1\otimes\mathsf{d})(\mathsf{m}\otimes 1)] + [(\mathsf{d}\otimes 1)(1\otimes\sigma)(\mathsf{m}\otimes 1)]$*;*

[d.3] ***Derivative of a linear function**:* $\eta\mathsf{d} = \mathsf{u} \otimes 1$*;*

[d.4] ***Chain rule**:* $\mu\mathsf{d} = \mathsf{d}(\mu \otimes \mathsf{d})(\mathsf{m} \otimes 1)$*;*

[d.5] ***Interchange**[2]:* $\mathsf{d}(\mathsf{d} \otimes 1) = \mathsf{d}(\mathsf{d} \otimes 1)(1 \otimes \sigma)$.

*Such a $\mathcal{C}$ equipped with a deriving transformation d is called a **codifferential category**.*

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[1]Note that here, and throughout, we denote diagrammatic (left-to-right) composition by juxtaposition, whereas we denote right-to-left, non-diagrammatic composition by $\circ$, and functions f are applied on the left, parenthesized as in $f(x)$; however, we write composition of functors in the right-to-left, non-diagrammatic order, and functors F are applied on the left, as in FX. We suppress the use of the monoidal category associator and unitor isomorphisms, and we omit subscripts and whiskering on the right.

[2]This rule was not in the original paper [5], but was later formally introduced in [6], and is used in [10]. It represents the independence of order of partial differentiation.

The intuition here is that the deriving transformation $\mathsf{d}_C : SC \to SC \otimes C$ is a derivation (in the algebraic sense, recalled in Section 5.1) that maps a smooth function on C to its derivative (or differential). We shall consider examples in which C is a finite-dimensional vector space and $SC \otimes C$ is a space of smooth 1-forms on C, equivalently, scalar-valued functions on $C \times C$ that are smooth in their first argument and linear in their second. The first axiom **[d.1]** states that the derivative of a constant function is zero. The second axiom **[d.2]** is the Leibniz (or product) rule which describes how to differentiate the product of two smooth functions. The third axiom rule **[d.3]** says that the derivative of a linear map is constant in its first argument. The fourth axiom **[d.4]** is the chain rule, describing how to differentiate the composition of smooth maps. The last axiom, the interchange rule **[d.5]**, is the independence of order of differentiation.

Example 2.6. $\mathbb{R}$-Vec is a codifferential category with respect to the deriving transformation $\mathsf{d}_V : \mathsf{Sym}(V) \to \mathsf{Sym}(V) \otimes V$ defined on pure tensors by

$$\mathsf{d}_V(v_1 \otimes \ldots \otimes v_n) := \sum_{i=1}^{n} (v_1 \otimes \ldots \otimes v_{i-1} \otimes v_{i+1} \otimes \ldots \otimes v_n) \otimes v_i$$

where $v_1, ..., v_n \in V$. If V has a basis X, then with respect to the isomorphism $\mathsf{Sym}(V) \cong \mathbb{R}[X]$, the map $\mathsf{d}_V : \mathbb{R}[X] \to \mathbb{R}[X] \otimes V$ is given by taking a sum involving the partial derivatives:

$$\mathsf{d}_V(x_1^{n_1} \ldots x_k^{n_k}) = \sum_{i=1}^{k} n_i \cdot x_1^{n_1} \ldots x_i^{n_i - 1} \ldots x_k^{n_k} \otimes x_i.$$

For example, if $V = \mathbb{R}^3$, then $SV = \mathbb{R}[x, y, z]$ and $SV \otimes V = \mathbb{R}[x, y, z] \otimes \mathbb{R}^3$ is a free $\mathbb{R}[x, y, z]$-module on three generators $\mathsf{d}x = 1 \otimes x$, $\mathsf{d}y = 1 \otimes y$, $\mathsf{d}z = 1 \otimes z$, so that the elements of $SV \otimes V$ are 1-forms $f\,\mathsf{d}x + g\,\mathsf{d}y + h\,\mathsf{d}z$ on $\mathbb{R}^3$ with polynomial coefficients $f, g, h \in \mathbb{R}[x, y, z]$. For the polynomial $p(x, y, z) = x^2y^3 + z^5x + 1$ we compute that

$$\mathsf{d}_{\mathbb{R}^3}(x^2y^3 + xz^4 + 1) = 2xy^3 \otimes x + 3x^2y^2 \otimes y + z^4 \otimes x + 5xz^4 \otimes z\ .$$

We now turn to the integral side of this theory, as introduced in [10].

Definition 2.7. *If* $(\mathcal{C}, \otimes, k, \sigma)$ *is an additive symmetric monoidal category with an algebra modality* $(S, \mu, \eta, \mathsf{m}, \mathsf{u})$*, then an* ***integral transformation*** *on* $\mathcal{C}$ *is a natural transformation* s*, with components*

$$\mathsf{s}_C : SC \otimes C \to SC \qquad (C \in \mathcal{C})$$

such that

[s.1] ***Integral of a constant:*** $(\mathsf{u} \otimes 1)\mathsf{s} = \eta$*;*

[s.2] ***Rota-Baxter rule:*** $(\mathsf{s} \otimes \mathsf{s})\mathsf{m} = [(\mathsf{s} \otimes 1 \otimes 1)(\mathsf{m} \otimes 1)\mathsf{s}] + [(1 \otimes 1 \otimes \mathsf{s})(1 \otimes \sigma)(\mathsf{m} \otimes 1)\mathsf{s}]$*;*

[s.3] ***Interchange:*** $(\mathsf{s} \otimes 1)\mathsf{s} = (1 \otimes \sigma)(\mathsf{s} \otimes 1)\mathsf{s}$.

Such a $\mathcal{C}$ *equipped with an integral transformation* s *is called a* ***co-integral category****.*

The concept of an *integral category* was introduced in [10], and there one can find a discussion of the intuition behind integral categories. In the present paper, we supply some intuition for the dual notion of co-integral category, and we develop an example of a specific co-integral category that confirms this intuition in reasonably full generality. The specific co-integral category that we shall define in Section 6 is one in which $\mathcal{C}$ is the category of real vector spaces and $S(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$ is the space of all real-valued smooth (or C^∞) functions on $\mathbb{R}^n$. In the case where $C = \mathbb{R}^n$, we discuss in Remark 5.5 how the tensor product $SC \otimes C = C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n$ may be identified with the space of smooth differential 1-forms on $\mathbb{R}^n$, equivalently, smooth functions $\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that are linear in their second argument. Using the usual inner product on $\mathbb{R}^n$, smooth 1-forms can be represented also as smooth vector fields $F : \mathbb{R}^n \to \mathbb{R}^n$, as discussed in Remark 5.5. For example, the differential, $\mathsf{d}f$, of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is an example of a smooth 1-form $\mathsf{d}f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, given by $\mathsf{d}f(x, v) = \sum_i \frac{\partial f}{\partial x_i}(x) v_i$, whose associated vector field is the gradient $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ of f (Remark 5.5). In Section 6, we introduce an integral transformation $\mathsf{s}_C : SC \otimes C \to SC$ that sends each 1-form $\omega(\vec{x}, \vec{v})$ to the integral $\int_{C_{\vec{x}}} \omega$ of ω along the straight-line path $C_{\vec{x}}$ from the origin to $\vec{x}$ in $\mathbb{R}^n$—equivalently, the line integral along $C_{\vec{x}}$ of the vector field F corresponding to ω. For example, when $C = \mathbb{R}$, a vector field on $\mathbb{R}$ is simply a smooth function $F : \mathbb{R} \to \mathbb{R}$, with corresponding 1-form $\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $\omega(x, v) = F(x)v$, and $\mathsf{s}_{\mathbb{R}}(\omega) : \mathbb{R} \to \mathbb{R}$ is then the function given by $x \mapsto \int_0^x F(t)\, \mathsf{d}t$.

With this example in mind, we may interpret the first axiom **[s.1]** as stating that the integral of a 1-form that is constant in its first argument is linear. The second axiom **[s.2]** is the Rota-Baxter rule, which roughly is *integration by parts*, but expressed solely in terms of integrals and adapted to 1-forms on $\mathbb{R}^n$. We discuss the Rota-Baxter rule more in Section 6 when discussing Rota-Baxter algebras. Lastly, the third axiom **[s.3]** is the independence of order of integration.

The main precursor to the specific co-integral category that we shall define in Section 6 is the following one, which was introduced in [10] and is based instead on *polynomial* functions rather than smooth functions. That paper included the following direct, algebraic definition of an integral transformation for polynomials and did not formulate it in terms of line integration of 1-forms or vector fields, while we shall see in Section 6 that the curious algebraic formula involved actually turns out to be a special case of our integral transformation for smooth 1-forms:

Example 2.8. $\mathbb{R}$-Vec is a co-integral category, with integral transformation $\mathsf{s}_V : \mathsf{Sym}(V) \otimes V \to \mathsf{Sym}(V)$ defined on generators by

$$\mathsf{s}_V((v_1 \otimes \ldots \otimes v_n) \otimes w) := \frac{1}{n+1} \cdot v_1 \otimes \ldots \otimes v_n \otimes w \qquad (v_1, ..., v_n, w \in V).$$

If V has a basis X, then with respect to the isomorphism $\mathsf{Sym}(V) \cong \mathbb{R}[X]$, the integral transformation $\mathsf{s}_V : \mathbb{R}[X] \otimes V \to \mathbb{R}[X]$ is given by

$$\mathsf{s}_V((x_1^{n_1} \ldots x_k^{n_k}) \otimes x_i) = \frac{1}{1 + \sum_{i=1}^{k} n_k} \cdot x_1^{n_1} \ldots x_i^{n_i+1} \ldots x^{n_k}. \qquad (1)$$

For example, if $V = \mathbb{R}^3$, for the polynomial $p(x, y, z) = x^2y^3 + z^5x + 1$ we compute that

$$\mathsf{s}_{\mathbb{R}^3}\left((x^2y^3 + xz^5 + 1) \otimes y\right) = \frac{1}{6}x^2y^4 + \frac{1}{7}xyz^5 + y$$

Note that the form the integral transformation takes in this example is perhaps slightly unexpected: the denominator sums *all* of the exponents in the monomial, not just the exponent on the indeterminate with respect to which integration occurs. As noted in [10], "at first glance this may seem bizarre ... however, [simply taking $n_i + 1$] fails the Rota-Baxter rule for any vector space of dimension greater than one". We shall see in this paper, however, a more abstract reason why this is the right integral transformation on polynomials: it can be recovered from the integral transformation for smooth 1-forms, by restricting to polynomial 1-forms (see Remark 6.8). Thinking about it another way, when the formula (1) was introduced in [10], it was not at all clear how to extend that formula to arbitrary smooth 1-forms; one of the accomplishments of the present work is to cast the formula (1) as a special case of line integration and thereby find this extension.

We now consider categories with differential and integral structure that are compatible (in the sense of the fundamental theorems of calculus).

Definition 2.9. *A* ***co-calculus category*** *[10] is a codifferential category and a co-integral category on the same algebra modality such that the deriving transformation* d *and the integral transformation* s *satisfy the following:*

[c.1] *The* ***Second Fundamental Theorem of Calculus****:* $\mathsf{ds} + S(0) = 1$*;*

[c.2] *The* ***Poincaré condition****: If* $f : B \to S(C) \otimes C$ *is such that*

$$f(\mathsf{d} \otimes 1)(1 \otimes \sigma) = f(\mathsf{d} \otimes 1)$$

then f ***satisfies the First Fundamental Theorem****; that is,* $f\mathsf{sd} = f$.

Remark 2.10. The axioms of a calculus category were first described by Ehrhard in [12] as consequences of his notion of a differential category with antiderivatives.

The axioms of a co-calculus category are based on their namesakes: the fundamental theorems of calculus. The first axiom **[c.1]** is based on a particular case of the Second Fundamental Theorem of Calculus, which in this case states that (in the single-variable case) the integral from 0 to x of the derivative of a function is equal to the difference between the values of that function at the endpoints.

$$\int_0^x f'(t)\ \mathsf{d}t = f(x) - f(0)$$

However, as one may not necessarily have additive inverses in a co-calculus category, the $f(0)$ term is placed on the left-hand side. The Second Fundamental Theorem of Calculus generalizes nicely to the multivariable case in the form of the general Stokes' Theorem, which includes a special case known as the Fundamental Theorem of Line Integration that we will discuss in Section 6. On the other hand, the First Fundamental Theorem of Calculus states that the derivative of the integral of a continuous function is equal to the original function:

$$\frac{\mathsf{d}}{\mathsf{d}x}\int_a^x f(t)\ \mathsf{d}t = f(x)$$

The First Fundamental Theorem of Calculus generalizes to the multivariable case only with the addition of a hypothesis that is vacuous in the single variable case. Indeed, the Poincaré Lemma (or rather, its usual proof by way of an explicit chain homotopy) provides necessary and sufficient conditions for when a multivariable integrable map satisfies the First Fundamental Theorem of Calculus. Thus, the second axiom **[c.2]** generalizes the Poincaré Lemma. The axioms of a co-calculus category will be studied in greater detail for real smooth functions in Section 6.

Example 2.11. $\mathbb{R}$-Vec, with the 'polynomial' codifferential and co-integral structure carried by the symmetric algebra monad Sym (2.6, 2.8), is a co-calculus category.

In fact, $\mathbb{R}$-Vec is even stronger: it is a (co)differential category with antiderivatives. Before defining this notion, we first need to recall certain natural transformations associated with algebra modalities and deriving transformations.

Definition 2.12. *The **coderiving transformation** [10] for an algebra modality* $(S, \mu, \eta, \mathsf{m}, \mathsf{u})$ *is the natural transformation* $\mathsf{d}^\circ_A : SA \otimes A \to SA$ *defined as follows:*

$$\mathsf{d}^\circ := (1 \otimes \eta)\mathsf{m}$$

As discussed in [10], while the coderiving transformation is of the same type as an integral transformation, in most cases it is **NOT** an integral transformation, as clearly seen in the example below.

Example 2.13. In $\mathbb{R}$-Vec, the coderiving transformation $\mathsf{d}^\circ_V : \mathsf{Sym}(V) \otimes V \to \mathsf{Sym}(V)$ is defined on generators by

$$\mathsf{d}^\circ_V((v_1 \otimes \ldots \otimes v_n) \otimes w) := v_1 \otimes \ldots \otimes v_n \otimes w \qquad (v_1, ..., v_n, w \in V).$$

If V has a basis X, then with respect to the isomorphism $\mathsf{Sym}(V) \cong \mathbb{R}[X]$, the coderiving transformation $\mathsf{d}^\circ_V : \mathbb{R}[X] \otimes V \to \mathbb{R}[X]$ is given by

$$\mathsf{d}^\circ_V((x_1^{n_1} \ldots x_k^{n_k}) \otimes x_i) = x_1^{n_1} \ldots x_i^{n_i+1} \ldots x^{n_k}.$$

For example, if $V = \mathbb{R}^3$, for the polynomial $p(x, y, z) = x^2y^3 + xz^5 + 1$ we compute that

$$\mathsf{d}^\circ_{\mathbb{R}^3}\left((x^2y^3 + xz^5 + 1) \otimes y\right) = x^2y^4 + xyz^5 + y\ .$$

The coderiving transformation is used in the construction of the integral transformation for a codifferential category with antiderivatives.

Definition 2.14. *For a codifferential category with algebra modality $(\mathsf{S}, \mu, \eta, \mathsf{m}, \mathsf{u})$ and deriving transformation d, define the following natural transformations [10], all of type $S \Rightarrow S$:*

(i) $\mathsf{L} := \mathsf{d}\mathsf{d}^\circ$

(ii) $\mathsf{K} := \mathsf{L} + S(0)$

(iii) $\mathsf{J} := \mathsf{L} + 1.$

A codifferential category is said to have ***antiderivatives*** *if K is a natural isomorphism.*

In [12] Ehrhard uses a slightly different definition of having antiderivatives, instead of asking that J be invertible. However, as shown in [10, Proposition 6.1], the invertibility of K implies that of J. Moreover, if K or J is invertible, then one can construct a co-integral category with integral transformation constructed using either K^{-1} or J^{-1}, and the two constructions give the same result when both are invertible. The reason to use K over J is that K being invertible immediately implies one has a co-calculus category. On the other hand, while J being invertible gives a co-integral category, one needs an added condition (known as the Taylor Property [10, Definition 5.3]) to also obtain a co-calculus category.

Theorem 2.15. [10] *A codifferential category with antiderivatives is a co-calculus category whose integral transformation is defined by* $\mathsf{s} := \mathsf{d}^\circ \mathsf{K}^{-1} = (\mathsf{J}^{-1} \otimes 1)\mathsf{d}^\circ$.

Example 2.16. With the structure of polynomial differentiation given above, $\mathbb{R}$-Vec is a codifferential category with antiderivatives, and its integral transformation is of the form given in the theorem above [10]. Indeed, in this case one finds that K_V is the identity on scalars and scalar multiplies a pure tensor $v_1 \otimes \ldots \otimes v_n$ by n:

$$\mathsf{K}_V(v_1 \otimes \ldots \otimes v_n) = n \cdot (v_1 \otimes \ldots \otimes v_n)$$

while on the other hand J_V is also the identity on scalars but instead scalar multiplies $v_1 \otimes \ldots \otimes v_n$ by $n + 1$:

$$\mathsf{J}_V(v_1 \otimes \ldots \otimes v_n) = (n + 1) \cdot (v_1 \otimes \ldots \otimes v_n).$$

For example, if $V = \mathbb{R}^3$, for the polynomial $p(x, y, z) = x^2y^3 + xz^5 + 1$ we compute that

$$\mathsf{K}_{\mathbb{R}^3}(x^2y^3 + xz^5 + 1) = 5x^2y^3 + 6xz^5 + 1$$
$$\mathsf{J}_{\mathbb{R}^3}(x^2y^3 + xz^5 + 1) = 6x^2y^3 + 7xz^5 + 1\ .$$

K is clearly invertible, and therefore so is J, and one can calculate that the resulting integral transformation $\mathsf{s} := \mathsf{d}^\circ \mathsf{K}^{-1} = (\mathsf{J}^{-1} \otimes 1)\mathsf{d}^\circ$ is precisely the one given above in Example 2.8.

Many more examples of (co)differential and (co-)integral categories can be found in [10, §7]. Our main focus in this paper is the differential and integral structure of arbitrary smooth functions.

3. Some fundamentals of finitary algebra

In Section 4, we shall give a construction of a particular algebra modality S^∞ on the category $\mathbb{R}$-Vec of real vector spaces, such that the category of S^∞-algebras is the category of C^∞-rings. The monad S^∞ is finitary, and so in the present section, we shall first review and discuss some basics on finitary monads and Lawvere theories, which will provide the basis of our approach to defining S^∞ and equipping it with further structure. While much of this material is standard, we also discuss certain results that are known among practitioners but whose statements we have not found to appear in the literature, such as Propositions 3.5 and 3.6.

3.1 Finitary monads on locally finitely presentable categories

Let us recall that an object C of a locally small category $\mathcal{C}$ is **finitely presentable** if the functor $\mathcal{C}(C, -) : \mathcal{C} \to \mathrm{Set}$ preserves filtered colimits. Here, following [15], we use the term *filtered colimit* to mean the colimit of a functor whose domain is not only filtered but also small[3]. We denote by $\mathcal{C}_\mathrm{f}$ the full subcategory of $\mathcal{C}$ consisting of the finitely presentable objects. Recall that $\mathcal{C}$ is **locally finitely presentable (l.f.p.)** iff $\mathcal{C}$ is cocomplete and the full subcategory $\mathcal{C}_\mathrm{f}$ is small and dense (in $\mathcal{C}$) [15, Corollary 7.3].

Example 3.1. $\mathbb{R}$-Vec is l.f.p., and a vector space is finitely presentable if and only if it is finite-dimensional. Therefore $\mathbb{R}\text{-Vec}_\mathrm{f}$ is equivalent to the category $\mathrm{Lin}_\mathbb{R}$ whose objects are the cartesian spaces $\mathbb{R}^n$ and whose morphisms are arbitrary $\mathbb{R}$-linear maps between these spaces.

A functor between l.f.p. categories is **finitary** if it preserves filtered colimits. Letting $\mathcal{C}$ be an l.f.p. category, a **finitary monad** on $\mathcal{C}$ is a monad on $\mathcal{C}$ whose underlying endofunctor is finitary. By [15, Proposition 7.6], we have the following well-known result, which will be of central importance to us:

Proposition 3.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be l.f.p. categories, and let $\iota : \mathcal{C}_\mathrm{f} \hookrightarrow \mathcal{C}$ denote the inclusion. Then there is an equivalence of categories*

$$[\mathcal{C}_\mathrm{f}, \mathcal{D}] \underset{\mathsf{Lan}_\iota}{\overset{\iota^*}{\underset{\sim}{\leftrightarrows}}} \mathrm{Fin}(\mathcal{C}, \mathcal{D}) \tag{2}$$

between the category $[\mathcal{C}_\mathrm{f}, \mathcal{D}]$ of functors from $\mathcal{C}_\mathrm{f}$ to $\mathcal{D}$ and the category $\mathrm{Fin}(\mathcal{C}, \mathcal{D})$ of finitary functors from $\mathcal{C}$ to $\mathcal{D}$. The functor ι^* is given by restriction along ι, and its pseudo-inverse Lan_ι is given by left Kan-extension along ι. Furthermore, a functor $F : \mathcal{C} \to \mathcal{D}$ is finitary if and only if it is a left Kan extension along ι, if and only if it is a left Kan extension of $F\iota$ along ι.

[3] Again following [15], we call a category *small* if it has but a (small) set of isomorphism classes.

In this paper, we shall be concerned with the case of Proposition 3.2 where $\mathcal{D} = \mathcal{C}$ for an l.f.p. category $\mathcal{C}$, in which case we have an equivalence $[\mathcal{C}_\mathrm{f}, \mathcal{C}] \simeq \mathrm{Fin}(\mathcal{C}, \mathcal{C})$. As described in [16, §4], the category $[\mathcal{C}_\mathrm{f}, \mathcal{C}]$ carries a monoidal product for which the equivalence

$$[\mathcal{C}_\mathrm{f}, \mathcal{C}] \simeq \mathrm{Fin}(\mathcal{C}, \mathcal{C})$$

is monoidal, so that finitary monads on $\mathcal{C}$ may be described equivalently as monoids in $[\mathcal{C}_\mathrm{f}, \mathcal{C}]$.

Example 3.3. Recalling that $\mathbb{R}$-Vec is l.f.p. and $\mathbb{R}\text{-Vec}_\mathrm{f} \simeq \mathrm{Lin}_\mathbb{R}$ (Example 3.1), we have an equivalence

$$[\mathrm{Lin}_\mathbb{R}, \mathbb{R}\text{-Vec}] \underset{\mathsf{Lan}_\iota}{\overset{\iota^*}{\underset{\sim}{\leftrightarrows}}} \mathrm{Fin}(\mathbb{R}\text{-Vec}, \mathbb{R}\text{-Vec}) \tag{3}$$

given by restriction and left Kan extension along the inclusion $\iota : \mathrm{Lin}_\mathbb{R} \hookrightarrow \mathbb{R}$-Vec. In §4, we will define a finitary monad on $\mathbb{R}$-Vec whose corresponding functor $\mathrm{Lin}_\mathbb{R} \to \mathbb{R}$-Vec sends $\mathbb{R}^n$ to the space $C^\infty(\mathbb{R}^n)$ of smooth, real-valued functions on $\mathbb{R}^n$.

Proposition 3.4. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be finitary functors between l.f.p. categories $\mathcal{C}$ and $\mathcal{D}$, and suppose that $\mathcal{D}$ is equipped with a functor $\otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ that preserves filtered colimits in each variable separately. Then the point-wise tensor product $F \otimes G = F(-) \otimes G(-) : \mathcal{C} \to \mathcal{D}$ is finitary.*

Proof. $F \otimes G$ is the composite $\mathcal{C} \xrightarrow{\langle F,G \rangle} \mathcal{D} \times \mathcal{D} \xrightarrow{\otimes} \mathcal{D}$, and since F and G are finitary and colimits in $\mathcal{D} \times D$ are point-wise, it follows that $\langle F, G\rangle$ preserves filtered colimits. Hence it suffices to show that $\otimes$ preserves filtered colimits. Every filtered colimit in $\mathcal{D}\times\mathcal{D}$ is of the form $\varinjlim \langle D, E\rangle = (\varinjlim D, \varinjlim E)$ for functors $D, E : \mathcal{J} \to \mathcal{D}$ on a small, filtered category $\mathcal{J}$, and our assumption on $\otimes$ entails that

$$(\varinjlim D) \otimes (\varinjlim E) \cong \varinjlim_{J \in \mathcal{J}} \varinjlim_{K \in \mathcal{J}} DJ \otimes EK \cong \varinjlim_{J \in \mathcal{J}} DJ \otimes EJ\ ,$$

since the diagonal functor $\Delta : \mathcal{J} \to \mathcal{J} \times \mathcal{J}$ is final as $\mathcal{J}$ is filtered [2, 2.19]. □

Given categories $\mathcal{C}$ and $\mathcal{D}$ and a functor $G : \mathcal{D} \to \mathcal{C}$, we shall say that G is **strictly monadic** if G has a left adjoint such that the comparison functor $\mathcal{D} \to \mathcal{C}^{\mathsf{T}}$ is an isomorphism, where $\mathcal{C}^{\mathsf{T}}$ denotes the category of algebras of the induced monad T on $\mathcal{C}$. Supposing that $\mathcal{C}$ is l.f.p., let us say that G is **strictly finitary monadic** if G is strictly monadic and the induced monad on $\mathcal{C}$ is finitary. In the latter case, since $\mathcal{C}^{\mathsf{T}}$ is necessarily l.f.p [1, Ch. 3], it then follows that $\mathcal{D}$ is l.f.p. also.

We shall require the following characterizations of categories of algebras of finitary monads on a given l.f.p. category. Given a functor $G : \mathcal{D} \to \mathcal{C}$, we shall say that a parallel pair of morphisms f, g in $\mathcal{D}$ is a G**-absolute pair** if the pair Gf, Gg has an absolute coequalizer in $\mathcal{C}$.

Proposition 3.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be l.f.p. categories, and let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Then the following are equivalent:*

1. *G is strictly finitary monadic;*
2. *G creates small limits, filtered colimits, and coequalizers of G-absolute pairs;*
3. *G preserves small limits and filtered colimits, and G creates coequalizers of G-absolute pairs.*

Proof. Suppose (1). Then G creates limits [8, Proposition 4.3.1], and since the induced monad T preserves filtered colimits it follows that G creates filtered colimits [8, Proposition 4.3.2]. Hence (2) holds, by Beck's Monadicity Theorem [24, III.7, Thm. 1].

Since $\mathcal{C}$ is not only cocomplete but also complete [1, 1.28], the creation of small limits and filtered colimits by G entails their preservation, so (2) implies (3).

Lastly, suppose (3). Then we deduce by [1, 1.66] that G has a left adjoint, and we deduce by Beck's Monadicity Theorem [24, III.7, Thm. 1] that G is strictly monadic. But since G preserves filtered colimits and its left adjoint F preserves arbitrary colimits, it follows that the induced monad $T = GF$ preserves filtered colimits, so (1) holds. $\square$

Proposition 3.6. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be l.f.p. categories, and suppose that we are given a commutative diagram of functors*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{U} & \mathcal{D} \\ & {\scriptstyle H}\searrow \quad \swarrow{\scriptstyle G} & \\ & \mathcal{C} & \end{array}$$

in which H and G are strictly finitary monadic. Then U is strictly finitary monadic.

Proof. Let us say that a functor F *preserves* (resp. *creates*) if F preserves (resp. creates) small limits, filtered colimits, and coequalizers of F-absolute pairs. By Proposition 3.5, both G and $H = GU$ preserve and create, so it follows by a straightforward argument that U creates. The result now follows from 3.5. □

3.2 Some basics on Lawvere theories

By definition, a **Lawvere theory** [18] is a small category $\mathcal{T}$ with a denumerable set of distinct objects $T^0, T^1, T^2, \ldots$ in which each object T^n $(n \in \mathbb{N})$ is equipped with a family of morphisms $(\pi_i : T^n \to T)_{i=1}^n$ that present T^n as an n-th power of the object $T = T^1$. We can and will assume that the given morphism $\pi_1 : T^1 \to T$ is the identity morphism.

Example 3.7. There is a Lawvere theory $\mathrm{Poly}_{\mathbb{R}}$ whose objects are the cartesian spaces $\mathbb{R}^n$ $(n \in \mathbb{N})$ and whose morphisms $p : \mathbb{R}^n \to \mathbb{R}^m$ are *algebraic maps*, i.e. maps $p = (p_1, \ldots, p_m)$ whose coordinate functions $p_j : \mathbb{R}^n \to \mathbb{R}$ $(j = 1, \ldots, m)$ are polynomial functions; equivalently, we may describe the morphisms of $\mathrm{Poly}_{\mathbb{R}}$ as m-tuples of formal polynomials in n variables.

Example 3.8. There is a Lawvere theory $\mathrm{Lin}_{\mathbb{R}}$ whose objects are the same as those of $\mathrm{Poly}_{\mathbb{R}}$ (3.7), but whose morphisms $M : \mathbb{R}^n \to \mathbb{R}^m$ are $\mathbb{R}$-*linear maps*, which we shall identify with their corresponding $m \times n$ matrices.

Given a Lawvere theory $\mathcal{T}$, a $\mathcal{T}$**-algebra** is a functor $A : \mathcal{T} \to \mathrm{Set}$ that preserves finite powers (or, equivalently, preserves finite products). Every $\mathcal{T}$-algebra A has an underlying set $|A| = A(T)$, and for each n the set $A(T^n)$ is an n-th power of the set $|A|$. Writing$|A|^n$ to denote the usual choice of

n-th power of $|A|$, i.e. the set of n-tuples of elements of $|A|$; we say that a $\mathcal{T}$-algebra A is *normal* if A sends each of the given power cones $(\pi_i : T^n \to T)_{i=1}^n$ to the usual n-th power cone $(\pi_i : |A|^n \to |A|)_{i=1}^n$ ([22, Definition 5.10], [23, 2.4]).

$\mathcal{T}$-algebras are the objects of a category in which the morphisms are natural transformations, and this category has an equivalent full subcategory consisting of the normal $\mathcal{T}$-algebras ([22, Theorem 5.14], [23, 2.5]).

The category of normal $\mathcal{T}$-algebras is equipped with a 'forgetful' functor to Set, given by evaluating at T, and this functor is strictly finitary monadic, so the category of normal $\mathcal{T}$-algebras is *isomorphic*[4] to the category of T-algebras for an associated finitary monad T on Set; e.g. see [23, 2.6].

Example 3.9. The category of normal $\mathrm{Poly}_{\mathbb{R}}$-algebras for the Lawvere theory $\mathrm{Poly}_{\mathbb{R}}$ in Example 3.7 is isomorphic to the category $\mathbb{R}$-Alg of commutative $\mathbb{R}$-algebras (e.g. by[5] [23, 2.9]).

Example 3.10. The category of normal $\mathrm{Lin}_{\mathbb{R}}$-algebras for the Lawvere theory $\mathrm{Lin}_{\mathbb{R}}$ in 3.8 is isomorphic to the category $\mathbb{R}$-Vec of $\mathbb{R}$-vector spaces (e.g. by[6] [23, 2.8]).

4. The free C^∞-ring modality on vector spaces

There is a Lawvere theory Smooth whose objects are the cartesian spaces $\mathbb{R}^n$ $(n \in \mathbb{N})$ and whose morphisms are arbitrary smooth maps between them. By a **C^∞-ring** we shall mean a normal Smooth-algebra[7]. Hence C^∞-rings are the objects of a category C^∞-Ring, the category of normal Smooth-algebras (§3.2).

With this definition, a C^∞-ring A : Smooth $\to$ Set is uniquely determined by its *underlying set* $X = A(\mathbb{R})$ and the mappings $\Phi_f = A(f)$:

[4]The category of *all* $\mathcal{T}$-algebras is merely *equivalent* to the category of T-algebras.

[5]It is well known that the category of *all* $\mathrm{Poly}_{\mathbb{R}}$-algebras is (merely) *equivalent* to the category of commutative $\mathbb{R}$-algebras.

[6]It is well known that the category of *all* $\mathrm{Lin}_{\mathbb{R}}$-algebras is (merely) *equivalent* to $\mathbb{R}$-Vec.

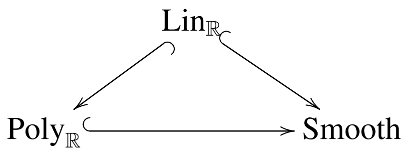
[7]More often, a C^∞-ring is defined as an arbitrary Smooth-algebra, but with the above definition we obtain an equivalent category, and one that is *strictly finitary monadic* over Set (§3.1, 3.2) and so *isomorphic* (rather than just equivalent) to a variety of algebras in Birkhoff's sense [24, III.8].

$X^m = A(\mathbb{R}^m) \to A(\mathbb{R}) = X$ associated to smooth, real-valued functions $f \in C^\infty(\mathbb{R}^m)$ ($m \in \mathbb{N}$). Hence A may be described equivalently as a pair (X, Φ) consisting of a set X and a suitable family of mappings Φ_f of the above form, called *operations*, satisfying certain conditions; this notation is as in [14], where the resulting conditions on Φ are also stated explicitly. A morphism of C^∞-rings $\phi : (X, \Phi) \to (Y, \Psi)$ is given by a mapping $\phi : X \to Y$ that preserves all of the operations Φ_f, Ψ_f, in the evident sense.

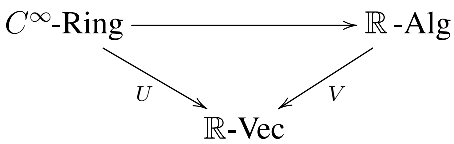
Note that there is a faithful inclusion

$$\mathrm{Poly}_{\mathbb{R}} \hookrightarrow \mathrm{Smooth},$$

where $\mathrm{Poly}_{\mathbb{R}}$ is the Lawvere theory considered in Example 3.7. This inclusion functor induces a functor from the category of normal Smooth-algebras to the category of normal $\mathrm{Poly}_{\mathbb{R}}$-algebras, given by pre-composition. In other words, we obtain a functor $C^\infty\text{-Ring} \to \mathbb{R}\text{-Alg}$, so that every C^∞-ring carries the structure of a commutative $\mathbb{R}$-algebra. Moreover, since every linear map is algebraic, and every algebraic map is smooth, we have a commutative diagram of faithful inclusions



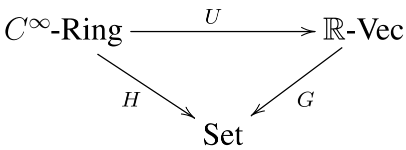
where $\mathrm{Lin}_{\mathbb{R}}$ is the Lawvere theory considered in Example 3.8. These inclusions induce a commutative diagram of functors



(4)

between the categories of normal algebras of these Lawvere theories, where we identify $\mathbb{R}$-Vec and $\mathbb{R}$-Alg with the categories of normal $\mathrm{Lin}_{\mathbb{R}}$-algebras and $\mathrm{Poly}_{\mathbb{R}}$-algebras, respectively (Example 3.10, Example 3.9).

The functor U in (4) participates in a commutative diagram



in which the forgetful functors H and G are strictly finitary monadic (by §3.2). Hence by Theorem 3.6 we deduce the following result:

Proposition 4.1. *There is a strictly finitary monadic functor* $U : C^\infty\text{-Ring} \to \mathbb{R}\text{-Vec}$ *that sends each* C^∞*-ring* A *to its underlying* $\mathbb{R}$*-vector space (which we denote also by* A*).*

Definition 4.2. *We denote by* $\mathsf{S}^\infty = (S^\infty, \mu, \eta)$ *the finitary monad on* $\mathbb{R}$-Vec *induced by the strictly finitary monadic functor* $U : C^\infty\text{-Ring} \to \mathbb{R}\text{-Vec}$*. We call* S^∞ *the* ***free*** C^∞***-ring monad on the category of real vector spaces****.*

Corollary 4.3. *The category* C^∞-Ring *of* C^∞*-rings is isomorphic to the category* $\mathbb{R}\text{-Vec}^{\mathsf{S}^\infty}$ *of* S^∞*-algebras for the finitary monad* S^∞ *on* $\mathbb{R}$-Vec.

We may of course apply similar reasoning to the functor $V : \mathbb{R}\text{-Alg} \to \mathbb{R}\text{-Vec}$ in (4), thus deducing also that V is strictly finitary monadic. The induced monad Sym on $\mathbb{R}$-Vec is described in Example 2.4. Hence we may make the following identifications:

$$\mathbb{R}\text{-Alg} = \mathbb{R}\text{-Vec}^{\mathsf{Sym}}, \qquad C^\infty\text{-Ring} = \mathbb{R}\text{-Vec}^{\mathsf{S}^\infty} . \tag{5}$$

Example 4.4. Letting $n \in \mathbb{N}$, it is well known that the set $C^\infty(\mathbb{R}^n)$ of all smooth, real-valued functions on $\mathbb{R}^n$ underlies the free C^∞-ring on n generators, i.e., the free C^∞-ring on the set $\{1, 2, ..., n\}$ [25]. The operations

$$\Phi_g \ : \ (C^\infty(\mathbb{R}^n))^m \to C^\infty(\mathbb{R}^n) \qquad (g \in C^\infty(\mathbb{R}^m))$$

carried by this C^∞-ring are given by

$$\Phi_g(f_1, ..., f_m) = g \circ \langle f_1, ..., f_m \rangle$$

where $\circ$ denotes right-to-left, non-diagrammatic composition. The projections $\pi_i \in C^\infty(\mathbb{R}^n)$ $(i = 1, ..., n)$ serve as generators, in the sense that the mapping $\pi_{(-)} : \{1, 2, ..., n\} \to C^\infty(\mathbb{R}^n)$ given by $i \mapsto \pi_i$ presents this C^∞-ring as free on the set $\{1, 2, ..., n\}$. Given a mapping $a : \{1, 2, ..., n\} \to A$ valued in a C^∞-ring (A, Ψ), the unique morphism of C^∞-rings $a' : C^\infty(\mathbb{R}^n) \to A$ such that $\pi_{(-)}a' = a$ is given by $a'(g) = \Psi_g(a(1), ..., a(n))$. From this we obtain the following:

Proposition 4.5. *The free C^∞-ring on the vector space $\mathbb{R}^n$ $(n \in \mathbb{N})$ is $C^\infty(\mathbb{R}^n)$, with operations as described above. The unit morphism $\eta_{\mathbb{R}^n}$: $\mathbb{R}^n \to C^\infty(\mathbb{R}^n)$ sends the standard basis vectors $e_1, ..., e_n \in \mathbb{R}^n$ to the projection functions $\pi_1, ..., \pi_n$. Given any linear map $\phi : \mathbb{R}^m \to A$ valued in a C^∞-ring (A, Ψ), there is a unique morphism of C^∞-rings $\phi^\sharp : C^\infty(\mathbb{R}^n) \to A$ such that $\eta_{\mathbb{R}^n}\phi^\sharp = \phi$, given by*

$$\phi^{\#}(g) = \Psi_g(\phi(e_1), ..., \phi(e_n)) \qquad (g \in C^\infty(\mathbb{R}^n)) \ .$$

Proof. The vector space $\mathbb{R}^n$ is free on the set $\{1, 2, ..., n\}$, so this follows from Example 4.4. □

Remark 4.6. By applying Proposition 4.5 and choosing the left adjoint to U suitably, we can and will assume that

$$S^\infty(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \ .$$

Accordingly, we will denote the restriction of S^∞ along the inclusion ι : $\mathrm{Lin}_{\mathbb{R}} \hookrightarrow \mathbb{R}$-Vec by

$$C^\infty = S^\infty\iota \ : \ \mathrm{Lin}_{\mathbb{R}} \longrightarrow \mathbb{R}\text{-Vec} \ .$$

Hence, since S^∞ is finitary, we deduce by Proposition 3.2 and Example 3.3 that S^∞ is a left Kan extension of $C^\infty : \mathrm{Lin}_{\mathbb{R}} \to \mathbb{R}$-Vec along ι. Symbolically,

$$S^\infty = \mathsf{Lan}_\iota C^\infty \ .$$

Hence

$$S^\infty(V) \quad \cong \quad \varinjlim_{(\mathbb{R}^n,\phi)\, \in\, \mathrm{Lin}_{\mathbb{R}}/V} \quad C^\infty(\mathbb{R}^n)$$

naturally in $V \in \mathbb{R}$-Vec, where $\mathrm{Lin}_{\mathbb{R}}/V$ denotes the comma category whose objects are pairs $(\mathbb{R}^n, \phi)$ consisting of an object $\mathbb{R}^n$ of $\mathrm{Lin}_{\mathbb{R}}$ and a morphism $\phi : \mathbb{R}^n \to V$ in $\mathbb{R}$-Vec. Equivalently, the maps

$$S^\infty(\phi) \ : \ S^\infty(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \longrightarrow S^\infty(V) \ , \qquad (\mathbb{R}^n, \phi) \in \mathrm{Lin}_{\mathbb{R}}/V \ ,$$

present $S^\infty(V)$ as a colimit of the composite functor $\mathrm{Lin}_{\mathbb{R}}/V \xrightarrow{\pi} \mathrm{Lin}_{\mathbb{R}} \xrightarrow{C^\infty}$ $\mathbb{R}$-Vec (where π is the forgetful functor).

Proposition 4.7. *The functor C^∞ sends each $\mathbb{R}$-linear map $h : \mathbb{R}^n \to \mathbb{R}^m$ to the map $C^\infty(h) : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^m)$ that sends each $g \in C^\infty(\mathbb{R}^n)$ to the composite*

$$\mathbb{R}^m \xrightarrow{h^*} \mathbb{R}^n \xrightarrow{g} \mathbb{R}$$

where h^* denotes the transpose (or adjoint) of h.

Proof. By definition C^∞ sends h to the unique C^∞-ring morphism $C^\infty(h) : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^m)$ such that $\eta_{\mathbb{R}^n} C^\infty(h) = h\eta_{\mathbb{R}^m}$. Hence, in view of Proposition 4.5 and Example 4.4 we deduce that $C^\infty(h) = (h\eta_{\mathbb{R}^m})^\#$ sends each $g \in C^\infty(\mathbb{R}^n)$ to

$$\begin{aligned} C^\infty(h)(g) &= (h\eta_{\mathbb{R}^m})^\#(g) \\ &= \Phi_g(\eta(h(e_1)), ..., \eta(h(e_n))) \\ &= g \circ \langle \eta(h(e_1)), ..., \eta(h(e_n)) \rangle \ . \end{aligned}$$

Letting (h_{ij}) be the matrix representation of h, we know that for each $j = 1, ..., n$,

$$h(e_j) = \sum_{i=1}^{m} h_{ij} e'_i$$

where $e'_1, ..., e'_m$ are the standard basis vectors for $\mathbb{R}^m$, so by linearity

$$\eta(h(e_j)) = \sum_{i=1}^{m} h_{ij}\pi_i = \pi_j \circ h^*.$$

Hence $C^\infty(h)(g) = g \circ \langle \pi_1 \circ h^*, ..., \pi_n \circ h^* \rangle = g \circ h^*$. □

We now employ a characterization of algebra modalities in [7] to show that S^∞ carries the structure of an algebra modality (Definition 2.3). Given a symmetric monoidal category $\mathcal{C}$, we shall denote by CMon($\mathcal{C}$) the category of commutative monoids in $\mathcal{C}$. If the forgetful functor CMon($\mathcal{C}$) $\to \mathcal{C}$ has a left adjoint, then we denote the induced monad on $\mathcal{C}$ by by Sym and call it the **symmetric algebra monad**, generalizing Example 2.4, and we say that *the symmetric algebra monad exists*.

Proposition 4.8. *Let $\mathcal{C}$ be a symmetric monoidal category $\mathcal{C}$ with reflexive coequalizers that are preserved by $\otimes$ in each variable, and assume that the symmetric algebra monad* Sym *on $\mathcal{C}$ exists. The following are in bijective correspondence:*

(1) algebra modalities $(\mathsf{S}, \mathsf{m}, \mathsf{u})$ *on* $\mathcal{C}$*;*

(2) pairs (S, λ) *consisting of a monad* S *on* $\mathcal{C}$ *and a monad morphism* $\lambda : \mathsf{Sym} \to \mathsf{S}$*;*

(3) pairs (S, M) *consisting of a monad* S *on* $\mathcal{C}$ *and a functor* $M : \mathcal{C}^{\mathsf{S}} \to$ CMon$(\mathcal{C})$ *that commutes with the forgetful functors valued in* $\mathcal{C}$.

Proof. We briefly sketch the correspondences; the verifications are straightforward, and the existence of a bijection between (1) and (2) is asserted in [7, Proposition 4.2], although with unnecessary blanket assumptions of additivity and finite biproducts.

Given $(\mathsf{S}, \mathsf{m}, \mathsf{u})$ as in (1), with $\mathsf{S} = (S, \mu, \eta)$, the associated monad morphism λ is obtained by defining $\lambda_C : \mathsf{Sym}(C) \to SC$ as the unique *monoid* morphism such that $\eta_C^{\mathsf{Sym}} \lambda_C = \eta_C$, where $\eta^{\mathsf{Sym}} : 1 \Rightarrow \mathsf{Sym}$ is the unit.

Given a monad S on $\mathcal{C}$, [2, Proposition A.26] yields a bijection between monad morphisms $\lambda : \mathsf{Sym} \to \mathsf{S}$ and functors $M : \mathcal{C}^{\mathsf{S}} \to \mathcal{C}^{\mathsf{Sym}}$ that commute with the forgetful functors to $\mathcal{C}$. But the above hypotheses entail that the forgetful functor $V :$ CMon$(\mathcal{C}) \to \mathcal{C}$ is a right adjoint and creates reflexive coequalizers, so by the well-known Crude Monadicity Theorem (in the form given in [21, Theorem 2.3.3.8]) we deduce that V is strictly monadic. Hence CMon$(\mathcal{C}) \cong \mathcal{C}^{\mathsf{Sym}}$ and the bijection between (2) and (3) is obtained.

Any functor M as in (3) endows each free S-algebra SC with the structure of a commutative monoid in $\mathcal{C}$, which we may write as $(SC, \mathsf{m}_C, \mathsf{u}_C)$, and we thus obtain an algebra modality $(\mathsf{S}, \mathsf{m}, \mathsf{u})$. □

Corollary 4.9. *The free* C^∞*-ring monad* S^∞ *on* $\mathbb{R}$-Vec *carries the structure of an algebra modality* $(\mathsf{S}^\infty, \mathsf{m}, \mathsf{u})$.

Proof. CMon($\mathbb{R}$-Vec) $=$ $\mathbb{R}$-Alg, so this follows from Proposition 4.8 in view of (4). □

Remark 4.10. We call the algebra modality $(\mathsf{S}^\infty, \mathsf{m}, \mathsf{u})$ the **free C^∞-ring modality**. For each real vector space V, $(S^\infty(V), \mathsf{m}_V, \mathsf{u}_V)$ is the $\mathbb{R}$-algebra underlying the free C^∞-ring on V. In view of the proof of Proposition 4.8, the corresponding monad morphism $\lambda : \mathsf{Sym} \to \mathsf{S}^\infty$ consists of mappings

$$\lambda_V : \mathsf{Sym}(V) \longrightarrow S^\infty(V) \qquad (V \in \mathbb{R}\text{-Vec})$$

each characterized as the unique $\mathbb{R}$-algebra homomorphism with $\eta^{\mathsf{Sym}}_C \lambda_C = \eta_C$, where $\eta^{\mathsf{Sym}} : 1 \Rightarrow \mathsf{Sym}$ and $\eta : 1 \Rightarrow S^\infty$ denote the units. In the case where $V = \mathbb{R}^n$, we may identify $\mathsf{Sym}(\mathbb{R}^n)$ with the polynomial $\mathbb{R}$-algebra $\mathbb{R}[x_1, \ldots, x_n]$, and $\lambda_{\mathbb{R}^n}$ is simply the inclusion

$$\lambda_{\mathbb{R}^n} : \mathsf{Sym}(\mathbb{R}^n) = \mathbb{R}[x_1, \ldots, x_n] \hookrightarrow C^\infty(\mathbb{R}^n) \; .$$

Indeed, the latter is an $\mathbb{R}$-algebra homomorphism that sends the generators x_i to the generators π_i $(i = 1, ..., n)$.

5. Differential structure

Our goal in this section is to give codifferential structure for the free C^∞-ring modality S^∞ (Corollary 4.9). Note that this was also done in the original differential categories paper [5, §3], but for reasons explained in Section 1 we will instead employ a different approach: we will exploit the fact that S^∞ is a finitary monad, in order to obtain its differential structure by left Kan-extension from structure on the finite-dimensional spaces, which we will describe explicitly. This new approach will later enable us to also endow S^∞ with integral structure in Section 6. Moreover, we believe that it is helpful to have multiple viewpoints on this key example.

To demonstrate codifferential structure for S^∞, we will use the following theorem from [7]:

Theorem 5.1. [7, 6.1] *Suppose that $\mathcal{C}$ is an additive symmetric monoidal category with reflexive coequalizers that are preserved by the tensor product in each variable, and suppose that the symmetric algebra monad* Sym *on $\mathcal{C}$ exists. Then to equip $\mathcal{C}$ with the structure of a codifferential category (in the sense of [5]) is, equivalently, to equip $\mathcal{C}$ with the following three structures:*

- *a monad $\mathsf{S} = (S, \eta, \mu)$,*
- *a monad morphism $\lambda : \mathsf{Sym} \to \mathsf{S}$, and*
- *a natural transformation $\mathsf{d} : SC \to SC \otimes C \;\; (C \in \mathcal{C})$*

such that

(a) for each object C of $\mathcal{C}$,

$$\begin{array}{ccc} \mathsf{Sym}(C) & \xrightarrow{\lambda_C} & S(C) \\ {\scriptstyle \mathsf{d}_C^{\mathsf{Sym}}}\downarrow & & \downarrow{\scriptstyle \mathsf{d}_C} \\ \mathsf{Sym}(C) \otimes V & \xrightarrow[\lambda_C \otimes 1]{} & S(C) \otimes C \end{array}$$

commutes, where $\mathsf{d}^{\mathsf{Sym}}$ is the canonical deriving transformation on Sym;

(b) the chain rule axiom of Definition 2.5 holds for d.

It is important to note that this theorem gives codifferential structure in the original sense [5], not in the sense used in [10]. In particular, the above theorem gives codifferential structure satisfying the first four axioms of Definition 2.5, but not necessarily the last axiom (interchange). Hence, we will use the following corollary of this result:

Corollary 5.2. *To give a codifferential structure in the sense used in [10] is equivalently to give structure as in Theorem 5.1 such that the transformation d also satisfies the interchange axiom* **[d.5]***: $\mathsf{d}(\mathsf{d} \otimes 1) = \mathsf{d}(\mathsf{d} \otimes 1)(1 \otimes \sigma)$.*

In 4.10 we have already equipped S^∞ with a monad morphism $\lambda : \mathsf{Sym} \to \mathsf{S}^\infty$. We will define the deriving transformation first for the finitely presentable objects, i.e., the finite-dimensional vector spaces $\mathbb{R}^n$, and then we will use Proposition 3.2 and Example 3.3 both to extend this definition to arbitrary vector spaces and to facilitate the checking of the required axioms for a deriving transformation.

Definition 5.3. *For each $n \in \mathbb{N}$, define $\mathsf{d}^\flat_{\mathbb{R}^n} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n$ by*

$$\mathsf{d}^\flat_{\mathbb{R}^n}(f) := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \otimes e_i$$

where e_i denotes the i-th standard basis vector for $\mathbb{R}^n$.

Example 5.4. Consider the smooth function $f(x, y, z) = \sin(x)y^2 + x^2z^4 + 2$, so that $f \in C^\infty(\mathbb{R}^3)$. We compute that

$$\mathsf{d}^\flat_{\mathbb{R}^n}(f) = \big(\cos(x)y^2 + 2xz^4\big) \otimes e_1 + 2\sin(x)y \otimes e_2 + 4x^2z^3 \otimes e_3$$

Remark 5.5. Note that $C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n \cong (C^\infty(\mathbb{R}^n))^n$ is a free, finitely-generated $C^\infty(\mathbb{R}^n)$-module of rank n and hence may be identified with the $C^\infty(\mathbb{R}^n)$-module of smooth 1-forms on $\mathbb{R}^n$, whereupon the basis elements $1 \otimes e_i$ of this free module $C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n$ are identified with the basic 1-forms $\mathsf{d}x_i$ $(i = 1, ..., n)$ on $\mathbb{R}^n$ (noting that then $\mathsf{d}^\flat_{\mathbb{R}^n}(x_i) = \mathsf{d}x_i$ if one writes $x_i : \mathbb{R}^n \to \mathbb{R}$ for the i-th projection).

In particular, each element $\omega \in C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n$ can be expressed uniquely as

$$\omega = \sum_{i=1}^{n} f_i \otimes e_i = \sum_{i=1}^{n} f_i \mathsf{d}x_i$$

for smooth functions $f_i : \mathbb{R}^n \to \mathbb{R}$. Since $(C^\infty(\mathbb{R}^n))^n \cong C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, each such 1-form $\omega \in C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n$ corresponds to a smooth vector field $F = \langle f_1, ..., f_n \rangle : \mathbb{R}^n \to \mathbb{R}^n$ on $\mathbb{R}^n$.

Given $f \in C^\infty(\mathbb{R}^n)$, the 1-form $\mathsf{d}^\flat_{\mathbb{R}^n}(f)$ defined in Definition 5.3 is the usual differential of f (also known as the exterior derivative of the 0-form f), whose corresponding vector field is the **gradient** of f

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \right\rangle \ : \ \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

Lemma 5.6. *The maps* $\mathsf{d}^\flat_{\mathbb{R}^n}$ *in Definition 5.3 constitute a natural transformation*

$$\mathsf{d}^\flat \ : \ C^\infty(-) \Longrightarrow C^\infty(-) \otimes (-) \ : \ \mathrm{Lin}_{\mathbb{R}} \longrightarrow \mathbb{R}\text{-Vec} \ .$$

Proof. For this, we need to show that for any linear map $h : \mathbb{R}^n \to \mathbb{R}^m$,

$$\begin{array}{ccc} C^\infty(\mathbb{R}^n) & \xrightarrow{\mathsf{d}^\flat_{\mathbb{R}^n}} & C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n \\ {\scriptstyle C^\infty(h)}\downarrow & & \downarrow{\scriptstyle C^\infty(h)\otimes h} \\ C^\infty(\mathbb{R}^m) & \xrightarrow[\mathsf{d}^\flat_{\mathbb{R}^m}]{} & C^\infty(\mathbb{R}^m) \otimes \mathbb{R}^m \end{array}$$

commutes. Let (h_{ij}) be the matrix representation of h, let h^* denote the adjoint (or transpose) of h, and let $(e_i)_{i=1}^m$ and $(e_j)_{j=1}^n$ denote the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Then for $f \in C^\infty(\mathbb{R}^n)$,

$$\mathsf{d}^\flat_{\mathbb{R}^m}(C^\infty(h)(f))$$

$$\begin{aligned}
&= \mathsf{d}^\flat_{\mathbb{R}^m}(f \circ h^*) \qquad \text{(by Proposition 4.7)} \\
&= \sum_{i=1}^m \frac{\partial (f \circ h^*)}{\partial x_i} \otimes e_i \\
&= \sum_{i=1}^m \left[\sum_{j=1}^n \left(\frac{\partial f}{\partial x_j} \circ h^* \right) \frac{\partial h_j^*}{\partial x_i} \right] \otimes e_i \text{ (by the chain rule)} \\
&= \sum_{i=1}^m \left[\sum_{j=1}^n \left(\frac{\partial f}{\partial x_j} \circ h^* \right) h_{ij} \right] \otimes e_i \text{ (by the matrix representation of } h^*) \\
&= \sum_{j=1}^n \left(\frac{\partial f}{\partial x_j} \circ h^* \right) \otimes \left(\sum_{i=1}^m h_{ij} e_i \right) \text{ (by bilinearity of } \otimes) \\
&= \sum_{j=1}^n \left(\frac{\partial f}{\partial x_j} \circ h^* \right) \otimes h(e_j) \text{ (by the matrix representation of } h) \\
&= (C^\infty(h) \otimes h) \left(\sum_{j=1}^n \frac{\partial f}{\partial x_j} \otimes e_j \right) \\
&= (C^\infty(h) \otimes h) \left(\mathsf{d}^\flat_{\mathbb{R}^n}(f) \right)
\end{aligned}$$

as required. □

Lemma 5.7.

1. *The functor* $S^\infty(-) \otimes (-) : \mathbb{R}\text{-Vec} \to \mathbb{R}\text{-Vec}$ *is finitary.*
2. *The restriction of* $S^\infty(-) \otimes (-)$ *to* $\text{Lin}_\mathbb{R}$ *is precisely* $C^\infty(-) \otimes (-)$.
3. $S^\infty(-) \otimes (-)$ *is a left Kan extension of* $C^\infty(-) \otimes (-) : \text{Lin}_\mathbb{R} \to \mathbb{R}\text{-Vec}$ *along the inclusion* $\iota : \text{Lin}_\mathbb{R} \hookrightarrow \mathbb{R}\text{-Vec}$.

Proof. (2) follows from the fact that the restriction $S^\infty \iota$ of S^∞ to $\text{Lin}_\mathbb{R}$ is precisely C^∞ (by Remark 4.6). Since S^∞ and $1_{\mathbb{R}\text{-Vec}}$ are finitary, we deduce by Proposition 3.4 that (1) holds, and (3) then follows, by Example 3.3 and Proposition 3.2. □

Definition 5.8. *Using Lemma 5.7, we define*

$$\mathsf{d} \; : \; S^\infty(-) \Longrightarrow S^\infty(-) \otimes (-) \; : \; \mathbb{R}\text{-Vec} \to \mathbb{R}\text{-Vec}$$

to be the natural transformation

$$\mathsf{d} = \mathsf{Lan}_\iota(\mathsf{d}^\flat) \ : \ \mathsf{Lan}_\iota(C^\infty) \Longrightarrow \mathsf{Lan}_\iota(C^\infty(-) \otimes (-))$$

corresponding to $\mathsf{d}^\flat$ *under the equivalence*

$$\mathsf{Lan}_\iota : [\mathrm{Lin}_\mathbb{R}, \mathbb{R}\text{-Vec}] \to \mathrm{Fin}(\mathbb{R}\text{-Vec}, \mathbb{R}\text{-Vec})$$

of Example 3.3 and Proposition 3.2. In view of Lemma 5.7.(2), we note that $\iota^*(\mathsf{d}) = \mathsf{d}^\flat$ *in the notation of Example 3.3, i.e.,*

$$\mathsf{d}_{\mathbb{R}^n} = \mathsf{d}^\flat_{\mathbb{R}^n} \qquad (n \in \mathbb{N}).$$

Lemma 5.9. d *satisfies (a) of Theorem 5.1 for the objects* $C = \mathbb{R}^n$ $(n \in \mathbb{N})$*; that is,*

$$\begin{array}{ccc} \mathbb{R}[x_1 \dots x_n] & \xrightarrow{\lambda_{\mathbb{R}^n}} & C^\infty(\mathbb{R}^n) \\ {\scriptstyle \mathsf{d}^{\mathsf{Sym}}_{\mathbb{R}^n}}\downarrow & & \downarrow{\scriptstyle \mathsf{d}_{\mathbb{R}^n}} \\ \mathbb{R}[x_1 \dots x_n] \otimes \mathbb{R}^n & \xrightarrow[\lambda_{\mathbb{R}^n} \otimes 1]{} & C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n \end{array} \tag{6}$$

commutes.

Proof. This is immediate since by 4.10, $\lambda_{\mathbb{R}^n}$ is the inclusion, and when the formula for $\mathsf{d}_{\mathbb{R}^n} = \mathsf{d}^\flat_{\mathbb{R}^n}$ is applied to a polynomial, we recover the formula for $\mathsf{d}^{\mathsf{Sym}}_{\mathbb{R}^n}$ (see Example 2.6). □

Corollary 5.10. d *satisfies (a) of Theorem 5.1.*

Proof. 5.1.(a) requires that

$$\lambda\mathsf{d} = \mathsf{d}^{\mathsf{Sym}}(\lambda \otimes 1) \ : \ \mathsf{Sym} \Longrightarrow S^\infty(-) \otimes (-) \ : \ \mathbb{R}\text{-Vec} \longrightarrow \mathbb{R}\text{-Vec} \ .$$

The components at $\mathbb{R}^n$ of these two natural transformations $\lambda\mathsf{d}$ and $\mathsf{d}^{\mathsf{Sym}}(\lambda \otimes 1)$ are precisely the two composites in (6), so since Sym and $S^\infty(-)\otimes(-)$ are finitary functors and $\iota^* : \mathrm{Fin}(\mathbb{R}\text{-Vec}, \mathbb{R}\text{-Vec}) \to [\mathrm{Lin}_\mathbb{R}, \mathbb{R}\text{-Vec}]$ is an equivalence (Example 3.3), the result follows. □

Lemma 5.11. d *satisfies the chain rule for the objects* $\mathbb{R}^n$*; that is, the following diagram commutes:*

$$\begin{array}{ccccc} S^\infty(C^\infty(\mathbb{R}^n)) & & \xrightarrow{\mu} & & C^\infty(\mathbb{R}^n) \\ {\scriptstyle \mathsf{d}}\downarrow & & & & \downarrow{\scriptstyle \mathsf{d}} \\ S^\infty(C^\infty(\mathbb{R}^n)) \otimes C^\infty(\mathbb{R}^n) & \xrightarrow[\mu \otimes \mathsf{d}]{} & C^\infty(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n & \xrightarrow[m \otimes 1]{} & C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n \end{array}$$

Proof. By 4.6, we know that the maps

$$S^\infty(\phi) \;:\; S^\infty(\mathbb{R}^m) = C^\infty(\mathbb{R}^m) \longrightarrow S^\infty(C^\infty(\mathbb{R}^n))$$

for $(\mathbb{R}^m, \phi) \in \mathrm{Lin}_\mathbb{R} / C^\infty(\mathbb{R}^n)$, present $S^\infty(C^\infty(\mathbb{R}^n))$ as a colimit in $\mathbb{R}$-Vec. Hence, to check the commutativity of the diagram above, it suffices to let $\phi : \mathbb{R}^m \to C^\infty(\mathbb{R}^n)$ be a linear map and check that the diagram commutes when pre-composed by the map $S^\infty(\phi)$. So, we will first consider the upper-right composite:

$$C^\infty(\mathbb{R}^m) \xrightarrow{S^\infty(\phi)} S^\infty(C^\infty(\mathbb{R}^n)) \xrightarrow{\mu} C^\infty(\mathbb{R}^n) \xrightarrow{\mathsf{d}} C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n \qquad (7)$$

Since $C^\infty(\mathbb{R}^m) = S^\infty(\mathbb{R}^m)$ is the free S^∞-algebra on the vector space $\mathbb{R}^m$, we deduce by generalities on Eilenberg-Moore categories that the composite $S^\infty(\phi)\mu$ of the first two morphisms in (7) is the unique S^∞-algebra homomorphism $\phi^\# : C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^n)$ such that $\eta\phi^\# = \phi$. Hence, by Proposition 4.5 and Example 4.4 we deduce that $\phi^\# = S^\infty(\phi)\mu$ is given by

$$\phi^\#(g) = \Phi_g(\phi(e_1), ..., \phi(e_m)) = g \circ \langle \phi(e_1), ..., \phi(e_m) \rangle \qquad (g \in C^\infty(\mathbb{R}^m)) \;.$$

Hence, letting $\alpha_i = \phi(e_i)$ for each $i = 1, ..., m$ and letting $\alpha = \langle \alpha_1, ..., \alpha_m \rangle : \mathbb{R}^n \to \mathbb{R}^m$, we know that $\phi^\#(g) = g \circ \alpha$. Therefore

$$(S^\infty(\phi)\mu\mathsf{d})(g) = (\phi^\#\mathsf{d})(g) = \mathsf{d}(g \circ \alpha) = \sum_{i=1}^{n} \frac{\partial(g \circ \alpha)}{\partial x_i} \otimes e_i \;(\dagger).$$

We now calculate the lower-left composite when pre-composed by the map $S^\infty(\phi)$. By the naturality of d, $S^\infty(\phi)\mathsf{d} = \mathsf{d}(S^\infty(\phi) \otimes \phi)$. Also, $\phi^\# = S^\infty(\phi)\mu$, so we are considering the composite

$$(\mathsf{d})(\phi^\# \otimes \phi)(1 \otimes \mathsf{d})(m \otimes 1) : C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n.$$

We now calculate the result of applying this composite to each $g \in C^\infty(\mathbb{R}^m)$.

- Applying d to g gives
$$\sum_{j=1}^{m} \frac{\partial g}{\partial x_j} \otimes e_j.$$

- As above, $\phi^{\#}(g) = g \circ \alpha$, so applying $\phi^{\#} \otimes \phi$ to this gives

$$\sum_{j=1}^{m} \left(\frac{\partial g}{\partial x_j} \circ \alpha \right) \otimes \alpha_j .$$

- Applying $1 \otimes \mathsf{d}$ to this gives

$$\sum_{j=1}^{m} \left(\frac{\partial g}{\partial x_j} \circ \alpha \right) \otimes \sum_{i=1}^{n} \frac{\partial \alpha_j}{\partial x_i} \otimes e_i = \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_j} \circ \alpha \right) \otimes \frac{\partial \alpha_j}{\partial x_i} \otimes e_i$$

- And then applying $\mathsf{m} \otimes 1$ gives

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_j} \circ \alpha \right) \frac{\partial \alpha_j}{\partial x_i} \otimes e_i = \sum_{i=1}^{n} \left[\sum_{j=1}^{m} \left(\frac{\partial g}{\partial x_j} \circ \alpha \right) \frac{\partial \alpha_j}{\partial x_i} \right] \otimes e_i$$

However, by the ordinary chain rule for smooth functions, this last expression equals †, as required. □

Corollary 5.12. d *satisfies the chain rule ([d.4] in Definition 2.5).*

Proof. This follows from Lemma 5.11 by an argument as in Corollary 5.10, using Example 3.3, since the chain rule is the equality of a parallel pair of natural transformations $S^\infty(S^\infty(-)) \Rightarrow S^\infty(-) \otimes (-)$ between finitary endofunctors on $\mathbb{R}$-Vec. □

Lemma 5.13. d *satisfies the interchange rule for the objects* $\mathbb{R}^n$*; that is, the following diagram commutes:*

$$\begin{array}{ccccc} C^\infty(\mathbb{R}^n) & & \xrightarrow{\quad \mathsf{d} \quad} & & C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n \\ \downarrow{\scriptstyle \mathsf{d}} & & & & \downarrow{\scriptstyle \mathsf{d}\otimes 1} \\ C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n & \xrightarrow[\mathsf{d}\otimes 1]{} & C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n \otimes \mathbb{R}^n & \xrightarrow[1\otimes\sigma]{} & C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n \otimes \mathbb{R}^n \end{array}$$

Proof. Given some $f \in C^\infty(\mathbb{R}^n)$, by definition,

$$\mathsf{d}_{\mathbb{R}^n}(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \otimes e_i \ .$$

Then applying $\mathsf{d} \otimes 1$ to this, we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} \otimes e_j \otimes e_i, (\dagger)$$

and applying $1 \otimes \sigma$ to this gives

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} \otimes e_i \otimes e_j = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} \otimes e_j \otimes e_i.$$

But this is equal to †, by the symmetry of mixed partial derivatives. □

Corollary 5.14. d *satisfies the interchange rule ([d.5] in Definition 2.5).*

Proof. By an argument as in Lemma 5.7 , $S^\infty(-) \otimes (-) \otimes (-)$ is a finitary endofunctor on $\mathbb{R}$-Vec whose restriction to $\mathrm{Lin}_\mathbb{R}$ is precisely $C^\infty(-) \otimes (-) \otimes (-)$. The result now follows from Lemma 5.13 by an argument as in Corollaries 5.10 and 5.12. □

Theorem 5.15. $\mathbb{R}$-Vec *has the structure of a codifferential category when equipped with the free C^∞-ring monad S^∞.*

Proof. In view of Corollary 5.2, this follows from Remark 4.10, Definition 5.8, and Corollaries 5.10, 5.12, and 5.14. □

Remark 5.16. Recall that an algebra modality is said to have the *storage* or *Seely* isomorphisms [6, §3] if certain canonical morphisms $S(X) \otimes S(Y) \to S(X \times Y)$ and $k \to S(1)$ are isomorphisms; see [4, Definition 7.1], where the dual notion is defined. The algebra modality considered here does not have this property, as even for $X = Y = \mathbb{R}$ the canonical map $C^\infty(\mathbb{R}) \otimes C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R} \times \mathbb{R}) = C^\infty(\mathbb{R}^2)$ is not surjective, as its image does not contain[8] the function e^{xy}. As noted in the introduction, this is then a crucial example of a (co)differential category that does not have the Seely isomorphisms.

The fact that the canonical map $C^\infty(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ is not an isomorphism can be remedied by equipping each space $C^\infty(\mathbb{R}^k)$ with the topology of uniform convergence of all higher partial derivatives on

[8]Example from math.stackexchange.com page #2244402.

compact sets [28, §1] and considering a completed topological tensor product. Equipping $C^\infty(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}^m)$ with a suitable topology [28, §2], it can be shown that the completion $C^\infty(\mathbb{R}^n) \mathbin{\widehat{\otimes}} C^\infty(\mathbb{R}^m)$ of the latter topological tensor product is isomorphic to $C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ [28, Proposition 12].

5.1 S^∞-Derivations

The algebraic notion of a *derivation* provides a useful way to bring techniques of differential calculus to abstract algebra. If k is a commutative ring and A is a commutative k-algebra, then a *k-derivation* on A is a k-linear map $\partial : A \to M$ from A into a (left) A-module M, such that the Leibniz rule

$$\partial(fg) = f\partial(g) + g\partial(f) \tag{8}$$

holds for all $f, g \in A$. One can straightforwardly generalize the concept of a derivation to the setting of additive symmetric monoidal categories, as was done in [3]. However, it was subsequently realized by Blute, Lucyshyn-Wright, and O'Neill that a different generalization of derivations was required in the setting of codifferential categories, namely the notion of S-*derivation* [7], which is defined relative to the given differential structure. On the other hand, a different kind of generalization of derivations based on Lawvere theories had been introduced earlier by Dubuc and Kock [11], namely the notion of $\mathbb{T}$-*derivation* relative to a given *Fermat theory* $\mathbb{T}$. Previously, it was not clear how these two notions are related.

Our goal in this section is to show a precise sense in which they are related: we prove (Theorem 5.25) that the S^∞-derivations for the codifferential category given by the free C^∞-ring modality correspond precisely to derivations relative to the Fermat theory of smooth functions. This thus demonstrates the importance of the general notion of S-derivation defined in [7], and provides a key link between differential categories and previous work in categorical differential geometry.

We begin by recalling the notion of S-derivation for a codifferential category. For an algebra modality $(\mathsf{S}, \mathsf{m}, \mathsf{u})$ on a symmetric monoidal category, every S-algebra comes equipped with a commutative monoid structure [7, Theorem 2.12]. Indeed, if (A, ν) is an S-algebra for the monad $\mathsf{S} = (S, \mu, \eta)$ (where we recall that $\nu : S(A) \to A$ is a morphism satisfying certain equations involving η and μ), we define a commutative monoid structure on A

with multiplication $\mathsf{m}^{\nu} : A \otimes A \to A$ and unit $\mathsf{u}^{\nu} : k \to A$ defined respectively as follows:

$$\mathsf{m}^{\nu} := (\eta_A \otimes \eta_A)\mathsf{m}_A \nu \qquad \mathsf{u}^{\nu} := \mathsf{u}_A \nu$$

Notice that for free S-algebras $(S(C), \mu_C)$, $\mathsf{m}^{\mu_C} = \mathsf{m}_C$ and $\mathsf{u}^{\mu_C} = \mathsf{u}_C$. We may now also consider modules over an S-algebra (A, ν), or rather modules over the commutative monoid $(A, \mathsf{m}^{\nu}, \mathsf{u}^{\nu})$, which we recall are pairs (M, α) consisting of an object M and a morphism $\alpha : A \otimes M \to M$ satisfying the standard coherences.

Example 5.17. In $\mathbb{R}$-Vec, a Sym-algebra is precisely a commutative $\mathbb{R}$-algebra by (5). Given a commutative $\mathbb{R}$-algebra A, its associated Sym-algebra structure $\nu : \mathsf{Sym}(A) \to A$ is defined on generators by multiplication in A:

$$\nu(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = a_1 a_2 \ldots a_n$$

Conversely, given a Sym-algebra $\nu : \mathsf{Sym}(A) \to A$, the above construction equips A with the structure of a commutative monoid in the symmetric monoidal category $\mathbb{R}$-Vec, i.e., a commutative $\mathbb{R}$-algebra.

Example 5.18. In $\mathbb{R}$-Vec, as explained in Section 4, an S^{∞}-algebra is precisely a C^{∞}-ring. The above construction equips each C^{∞}-ring with its underlying commutative $\mathbb{R}$-algebra structure, mentioned in Section 4.

Definition 5.19. *Let $\mathcal{C}$ be a codifferential category with algebra modality* $(\mathsf{S}, \mathsf{m}, \mathsf{u})$ *and deriving transformation* d. *Given an* S*-algebra* (A, ν) *and an* $(A, \mathsf{m}^{\nu}, \mathsf{u}^{\nu})$*-module* (M, α)*, an* **S*-derivation*** *[7, Definition 4.12] is a morphism* $\partial : A \to M$ *such that the following diagram commutes:*

$$\begin{array}{ccccc} S(A) & & \xrightarrow{\quad\nu\quad} & & A \\ \downarrow{\scriptstyle \mathsf{d}_A} & & & & \downarrow{\scriptstyle \partial} \\ S(A) \otimes A & \xrightarrow[\nu \otimes \partial]{} & A \otimes M & \xrightarrow[\alpha]{} & M \end{array}$$

The canonical example of an S-derivation is the deriving transformation [7, Theorem 4.13]. Indeed for each object C, d_C is an S-derivation on the S-algebra $(S(C), \mu_C)$ valued in the module $(S(C) \otimes C, \mathsf{m}_C \otimes 1_C)$. S-derivations are the appropriate generalization of the classical notion of derivation, as every S-derivation is a derivation in the classical sense. The key difference is that classical derivations are axiomatized by the Leibniz rule, while S-derivations are axiomatized by the chain rule.

Example 5.20. By [7, Remark 5.8], Sym-derivations in $\mathbb{R}$-Vec correspond precisely to $\mathbb{R}$-derivations in the classical sense recalled above at (8). As explained above, for every vector space V, the deriving transformation $\mathsf{d}_V : \mathsf{Sym}(V) \to \mathsf{Sym}(V) \otimes V$ is a Sym-derivation, and therefore also an $\mathbb{R}$-derivation.

But what do S^∞-derivations correspond to? For this, we turn to Dubuc and Kock's generalized notion of derivation for Fermat theories. While we will not review Fermat theories in general (we invite the curious reader to learn about them in [11]), we will instead consider Dubuc and Kock's generalized derivations for the Fermat theory of smooth functions, which are explicitly described by Joyce in [14].

Definition 5.21. *Given a C^∞-ring (A, Φ) and an A-module M (that is, M is a module over the underlying ring structure of A), a C^∞**-derivation** [11, 14] is a map $\mathsf{D} : A \to M$ such that for each smooth function $f : \mathbb{R}^n \to \mathbb{R}$, the following equality holds:*

$$\mathsf{D}\left(\Phi_f(a_1, \dots, a_n)\right) = \sum_{i=1}^{n} \Phi_{\frac{\partial f}{\partial x_i}}(a_1, \dots, a_n) \cdot \mathsf{D}(a_i)$$

for all $a_1, \dots, a_n \in A$ (and where $\cdot$ is the A-module action).

Example 5.22. Consider the C^∞-ring $C^\infty(\mathbb{R})$. The polynomial ring $\mathbb{R}[x]$ can be made into a $C^\infty(\mathbb{R})$-module with respect to the action

$$f \cdot p = f(0)p$$

for $f \in C^\infty(\mathbb{R})$ and $p \in \mathbb{R}[x]$. Then the map $\mathsf{D} : C^\infty(\mathbb{R}) \to \mathbb{R}[x]$ defined as

$$\mathsf{D}[f] = f'(0)$$

is a C^∞-derivation since for all smooth functions $g : \mathbb{R}^n \to \mathbb{R}$ and $f_1, ..., f_n \in C^\infty(\mathbb{R})$ we can use Example 4.4 to compute that

$$\mathsf{D}(\Phi_g(f_1, ..., f_n)) = \mathsf{D}(g \circ \langle f_1, \dots, f_n \rangle) = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(f_1(0), \dots, f_n(0)) f_i'(0) \ .$$

To see why C^∞-derivations are precisely the same thing as S^∞-derivations, we will first take a look at equivalent definitions for each of these generalized derivations. In the presence of biproducts, arbitrary S-derivations in a codifferential category can equivalently be described as certain S-algebra morphisms [7, Definition 4.7]. This generalizes a well-known result on derivations in commutative algebra.

Theorem 5.23. [7, Theorem 4.1, Proposition 4.11] *Let $\mathcal{C}$ be a codifferential category with algebra modality $(\mathsf{S}, \mathsf{m}, \mathsf{u})$ and deriving transformation d. Let (A, ν) be an S-algebra and (M, α) an $(A, \mathsf{m}^\nu, \mathsf{u}^\nu)$-module, and suppose that $\mathcal{C}$ has finite biproducts $\oplus$. Then the pair $(A \oplus M, \beta)$ is an S-algebra where $\beta : S(A \oplus M) \to A \oplus M$ is defined as*

$$\beta := \langle S(\pi_1)\nu, \mathsf{d}_{A\oplus M}(S(\pi_1) \otimes \pi_2)(\nu \otimes 1)\alpha \rangle$$

where $\pi_1 : A \oplus M \to A$ and $\pi_2 : A \oplus M \to M$ are the projections. Furthermore, a morphism $\partial : A \to M$ is an S-derivation if and only if $\langle 1_A, \partial \rangle : (A, \nu) \to (A \oplus M, \beta)$ is an S-algebra morphism.

More general statements regarding the equivalence between S-derivations and S-algebra morphisms can be found in [7]. In the case of the free C^∞-ring monad, S^∞-algebra morphisms correspond precisely to C^∞-ring morphisms. Therefore to give an S^∞-derivation $\partial : A \to M$ amounts to giving a C^∞-ring morphism $\langle 1_A, \partial \rangle : A \to A \oplus M$, where $A \oplus M$ carries the C^∞-ring structure corresponding to the S^∞-algebra structure β in Theorem 5.23. This is similar to a result for algebras of Fermat theories that had been given earlier by Kock and Dubuc, stated here for C^∞-rings and C^∞-derivations:

Theorem 5.24. [11, Proposition 2.2] *Let (A, Φ) be a C^∞-ring and M an A-module. Then $(A \oplus M, \tilde{\Phi})$ is a C^∞-ring where for each smooth function $f : \mathbb{R}^n \to \mathbb{R}$, $\tilde{\Phi}_f : (A \oplus M)^n \to A \oplus M$ is defined as follows:*

$$\tilde{\Phi}_f\left((a_1, m_1), \ldots, (a_n, m_n)\right) = \left(\Phi_f(a_1, \ldots, a_n), \sum_{i=1}^{n} \Phi_{\frac{\partial f}{\partial x_i}}(a_1, \ldots, a_n) \cdot m_i\right)$$

for all $a_1, \ldots, a_n \in A$ (and where $\cdot$ is the A-module action). Furthermore, a map $\mathsf{D} : A \to M$ is a C^∞-derivation if and only if $\langle 1_A, \mathsf{D} \rangle : (A, \Phi) \to (A \oplus M, \tilde{\Phi})$ is a C^∞-ring morphism.

We note that [11, Proposition 2.2] is in fact a more general statement than what is stated in Theorem 5.24. It is not difficult to see that the C^∞-ring $(A \oplus M, \tilde{\Phi})$ from Theorem 5.24 corresponds precisely to the S^∞-algebra $(A \oplus M, \beta)$ from Theorem 5.23. Therefore since S^∞-algebra morphisms are equivalently described as C^∞-ring morphisms, it follows that S^∞-derivations are equivalently described as C^∞-derivations:

Theorem 5.25. *For the codifferential category structure on* $\mathbb{R}$-Vec *induced by the free C^∞-ring monad S^∞, the following are in bijective correspondence:*

(i) S^∞*-derivations (Definition 5.19);*

(ii) C^∞*-derivations (Definition 5.21).*

An immediate consequence of this theorem is that universal S^∞-derivations correspond to universal C^∞-derivations. For arbitrary codifferential categories, universal S-derivations $\partial : A \to \Omega_A$ [7, Definition 4.14] are generalizations of Kähler differentials, where in particular, Ω_A is the generalization of the classical module of Kähler differentials of a commutative algebra. Similarly, universal derivations of Fermat theories [11, Theorem 2.3] provide a simultaneous generalization of both Kähler differentials and smooth 1-forms. Indeed, it is well known that for a smooth manifold M, the module of Kähler differentials (in the classical sense) of $C^\infty(M)$ is not, in general, the module of smooth 1-forms of M. We can explain this phenomenon in the following way: the module of Kähler differentials is the universal Sym-derivation, and not the universal C^∞-derivation. For $C^\infty(M)$, the universal C^∞-derivation (equivalently, the universal S^∞-derivation) is in fact the module of smooth 1-forms of M [11]. Looking back at arbitrary codifferential categories, this justifies the use of the more general S-derivations to study de Rham cohomology of S-algebras [26].

5.2 A quasi-codereliction

In a codifferential category where $S(C)$ admits a natural bialgebra structure, the differential structure can equivalently be axiomatized by a **codereliction** [4, 5], which is in particular an S-derivation. We will see that although S^∞ does not have this structure, it is still possible to construct a sort of 'quasi-codereliction' that satisfies identities that are similar to the axioms of a codereliction.

Definition 5.26. *An algebra modality* $(S, \mu, \eta, \mathsf{m}, \mathsf{u})$ *on an additive symmetric monoidal category* $(\mathcal{C}, \otimes, k, \sigma)$ *is said to be an* ***additive bialgebra modality*** *[4] if it comes equipped with a natural transformation* Δ *with components* $\Delta_C : S(C) \to S(C) \otimes S(C)$ *and a natural transformation* e *with components* $\mathsf{e}_C : S(C) \to k$*, such that*

- *for each* C *in* $\mathcal{C}$*,* $(SC, \mathsf{m}_C, \mathsf{u}_C, \Delta_C, \mathsf{e}_C)$ *is a commutative and cocommutative bimonoid (in the symmetric monoidal category* $\mathcal{C}$*);*
- *the following equations are satisfied:*

$$\eta\Delta = (\mathsf{u} \otimes \eta) + (\eta \otimes \mathsf{u}), \qquad \eta\mathsf{e} = 0;$$

- *for each pair of morphisms* $f : A \to B$ *and* $g : A \to B$*, the following equality holds:*

$$S(f + g) = \Delta_A \left(S(f) \otimes S(g)\right) \mathsf{m}_B;$$

- *for each zero morphism* $0 : A \to B$*, the following equality holds:*

$$S(0) = \mathsf{e}_A \mathsf{u}_B.$$

Definition 5.27. *A* ***codereliction*** *[4, 5] for an additive bialgebra modality is a natural transformation* $\varepsilon : S \to 1_{\mathcal{C}}$ *such that:*

[dc.1] ***Constant rule****:* $\mathsf{u}\varepsilon = 0$*;*

[dc.2] ***Leibniz/product rule****:* $\mathsf{m}\varepsilon = (\mathsf{e} \otimes \varepsilon) + (\varepsilon \otimes \mathsf{e})$*;*

[dc.3] ***Derivative of a linear function****:* $\eta\varepsilon = 1$*;*

[dc.4] ***Chain rule****:* $\mu\Delta(1 \otimes \epsilon) = \Delta(\mu \otimes \epsilon)(1 \otimes \Delta)(\mathsf{m} \otimes \epsilon)$.

The intuition for coderelictions is best understood as evaluating derivatives at zero. For additive bialgebra modalities, there is a bijective correspondence between deriving transformations and coderelictions [4]. Indeed, every codereliction induces a deriving transformation, defined by

$$\mathsf{d} := \Delta(1 \otimes \varepsilon) \ ,$$

and, conversely, every deriving transformation induces a codereliction:

$$\varepsilon := \mathsf{d}(\mathsf{e} \otimes 1) \ . \tag{9}$$

Therefore, note that the codereliction chain rule **[dc.4]** is then precisely the deriving transformation chain rule **[d.4]**. Post-composing both sides of the chain rule with $(\mathsf{e} \otimes 1)$, one obtains the following identity (called the alternative chain rule in [4])

$$\mu\varepsilon = \Delta(\mu \otimes \varepsilon)(\mathsf{e} \otimes \varepsilon) \ ,$$

which we can rewrite equivalently as

$$\mu\varepsilon = \mathsf{d}(\mu \otimes \varepsilon)(\mathsf{e} \otimes 1) \ . \tag{10}$$

We illustrate this equation (10) in terms of smooth functions in Example 5.31 below. Now since $S(C)$ is a bialgebra, the morphism $\mathsf{e}_C \otimes 1_A : S(C) \otimes A \to A$ is a $(S(C), \mathsf{m}, \mathsf{u})$-module action for every object A. Therefore, we obtain the following observation:

Lemma 5.28. *For every object C, $\varepsilon_C : S(C) \to C$ is an S-derivation on the free S-algebra $(S(C), \mu_C)$ valued in the $(S(C), \mathsf{m}_C, \mathsf{u}_C)$-module $(C, \mathsf{e}_C \otimes 1_C)$.*

In the presence of biproducts, additive bialgebra modalities are equivalently described as algebra modalities that have the Seely isomorphisms. Indeed, one can construct the Seely isomorphism from the bialgebra structure and vice-versa; see [4, §7] for these constructions. Therefore, the algebra modality S^∞ is not an additive bialgebra modality since it does not have the Seely isomorphisms, as mentioned in Remark 5.16. Specifically, what is missing from the additive bialgebra modality structure is the natural comultiplication $\Delta : \mathsf{S}^\infty(-) \Longrightarrow \mathsf{S}^\infty(-) \otimes \mathsf{S}^\infty(-)$. Indeed, if S^∞ was an additive bialgebra modality, then by [4, §7] the vector spaces $C^\infty(\mathbb{R}^n)$ would be bialgebras and the following composite

$$C^\infty(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}^m) \xrightarrow{C^\infty(\iota_1) \otimes C^\infty(\iota_2)}$$
$$C^\infty(\mathbb{R}^n \oplus \mathbb{R}^m) \otimes C^\infty(\mathbb{R}^n \oplus \mathbb{R}^m) \xrightarrow{\mathsf{m}} C^\infty(\mathbb{R}^n \oplus \mathbb{R}^m)$$

is an inverse of the following composite:

$$C^\infty(\mathbb{R}^n \oplus \mathbb{R}^m) \xrightarrow{\Delta} C^\infty(\mathbb{R}^n \oplus \mathbb{R}^m) \otimes C^\infty(\mathbb{R}^n \oplus \mathbb{R}^m) \xrightarrow{C^\infty(\pi_1)\otimes C^\infty(\pi_2)} C^\infty(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}^m)$$

where π_j and ι_j are the projection and injection maps of the biproduct respectively. Then the canonical map $C^\infty(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^n \oplus \mathbb{R}^m)$ would be an isomorphism, which is the not the case, as discussed in Remark 5.16.

On the other hand to construct a codereliction from a deriving transformation as in (9), one only needs to have a counit, which S^∞ does have. Indeed, define the natural transformation $\mathsf{e}^\flat : C^\infty \Rightarrow \mathbb{R}$ by declaring that for each finite-dimensional vector space $\mathbb{R}^n$, the map $\mathsf{e}^\flat_{\mathbb{R}^n} : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is given by evaluation at zero:

$$\mathsf{e}^\flat_{\mathbb{R}^n}(f) = f(\vec{0}) \ .$$

One can check that $\mathsf{e}^\flat_{\mathbb{R}^n}$ is also a C^∞-ring morphism.

Define $\mathsf{e} : S^\infty \Rightarrow \mathbb{R}$ as the image of $\mathsf{e}^\flat$ under the equivalence of Example 3.3, i.e., $\mathsf{e} = \mathsf{Lan}_\iota(\mathsf{e}^\flat)$ (recalling that $S^\infty = \mathsf{Lan}_\iota(C^\infty)$ and noting that the constant functor $\mathbb{R} : \mathbb{R}\text{-Vec} \to \mathbb{R}\text{-Vec}$ is finitary and is a left Kan extension, along ι, of the constant functor $\mathbb{R} : \mathrm{Lin}_{\mathbb{R}} \to \mathbb{R}\text{-Vec}$). We then define the natural transformation $\varepsilon : S^\infty \Rightarrow 1_{\mathbb{R}\text{-Vec}}$ in the same manner that a codereliction was defined in (9). Explicitly, ε is defined component-wise as follows:

$$\varepsilon_V := \ S^\infty(V) \xrightarrow{\mathsf{d}_V} S^\infty(V) \otimes V \xrightarrow{\mathsf{e}_V \otimes 1_V} V$$

In particular, $\varepsilon_{\mathbb{R}^n} : C^\infty(\mathbb{R}^n) \to \mathbb{R}^n$ is the linear map that evaluates the derivative at zero:

$$\varepsilon_{\mathbb{R}^n}(f) = \sum_{i=1}^n \mathsf{e}^\flat_{\mathbb{R}^n}\left(\frac{\partial f}{\partial x_i}\right) \otimes e_i = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\vec{0}) e_i = \left(\frac{\partial f}{\partial x_1}(\vec{0}), \dots, \frac{\partial f}{\partial x_n}(\vec{0})\right),$$

where the pure tensors in the second expression are taken in $\mathbb{R} \otimes \mathbb{R}^n = \mathbb{R}^n$, and $e_i \in \mathbb{R}^n$ denotes the i-th standard basis vector.

Example 5.29. For the smooth function $f(x, y, z) = 3\sin(x) + y^2 z^2 + 4z + 1$, we compute that

$$\varepsilon_{\mathbb{R}^3}(3\sin(x) + y^2 z^2 + 4z + 1) = (3, 0, 4) \ .$$

Now note that the first three codereliction axioms **[dc.1]**, **[dc.2]**, and **[dc.3]** involve the algebra modality structure and the counit e, but not the comultiplication. As a consequence of the deriving transformation axioms, it follows that ε satisfies **[dc.1]**, **[dc.2]**, and **[dc.3]**. The remaining axiom, the codereliction chain rule **[dc.4]**, involves the comultiplication, which we cannot express with S^{∞}. However by construction, ε satisfies the alternative chain rule (10), which in a sense replaces **[dc.4]**, and requires precisely that ε be an S^{∞}-derivation. This makes ε a sort of *quasi*-codereliction for S^{∞}. We summarize this result as follows:

Proposition 5.30. *The natural transformation $\varepsilon : \mathsf{S}^{\infty} \Rightarrow 1_{\mathbb{R}\text{-Vec}}$ satisfies the following equalities:*

[dqc.1] $\mathsf{u}\varepsilon = 0$*;*

[dqc.2] $\mathsf{m}\varepsilon = (\mathsf{e} \otimes \varepsilon) + (\varepsilon \otimes \mathsf{e})$*;*

[dqc.3] $\eta\varepsilon = 1$*;*

[dqc.4] $\mu\varepsilon = \mathsf{d}(\mu \otimes \varepsilon)(\mathsf{e} \otimes 1)$.

In particular, for every $\mathbb{R}$-vector space V, $\varepsilon_V : S^{\infty}(V) \to V$ is an S^{∞}-derivation (or equivalently a C^{∞}-derivation) on the free S^{∞}-algebra $(S^{\infty}(V), \mu)$, valued in the $(S^{\infty}(V), \mathsf{m}_V, \mathsf{u}_V)$-module $(V, \mathsf{e}_V \otimes 1_V)$.

Example 5.31. It may be useful for the reader to work out the identities **[dqc.1]** to **[dqc.4]** in detail in the case of $V = \mathbb{R}^n$. The first identity **[dqc.1]** says that for a constant function $c : \mathbb{R}^n \to \mathbb{R}$, $\varepsilon_{\mathbb{R}^n}(c(\vec{x})) = 0$. The second identity **[dqc.2]** states that for a pair of smooth functions $f, g : \mathbb{R}^n \to \mathbb{R}$, we have that:

$$\varepsilon_{\mathbb{R}^n}\left(f(\vec{x})g(\vec{x})\right) = \left(f(\vec{0})\frac{\partial g}{\partial x_1}(\vec{0}) + g(\vec{0})\frac{\partial f}{\partial x_1}(\vec{0}), \ldots, f(\vec{0})\frac{\partial g}{\partial x_n}(\vec{0}) + g(\vec{0})\frac{\partial f}{\partial x_n}(\vec{0})\right)$$

The third identity **[dqc.3]** amounts to the statement that $\varepsilon_{\mathbb{R}^n}(\pi_i) = e_i$ for each of the projections $\pi_i : \mathbb{R}^n \to \mathbb{R}$. The last identity **[dqc.4]** says that for a smooth function $g : \mathbb{R}^m \to \mathbb{R}$ and a tuple of smooth functions $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$, we have that:

$$\varepsilon_{\mathbb{R}^n}\left(g \circ \langle f_1, \ldots, f_m \rangle\right) =$$

$$\left(\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial g}{\partial x_i}\left(f_1(\vec{0}),\ldots,f_m(\vec{0})\right)\frac{\partial f_j}{\partial x_1}(\vec{0}),\ldots,\right.$$
$$\left.\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial g}{\partial x_i}\left(f_1(\vec{0}),\ldots,f_m(\vec{0})\right)\frac{\partial f_j}{\partial x_n}(\vec{0})\right)$$

6. Antiderivatives and integral structure

The goal of this section is to show that the codifferential category structure on $\mathbb{R}$-Vec induced by the free C^∞-ring monad has antiderivatives, and that therefore we obtain a calculus category (and hence also an integral category). Explicitly, we wish to show that the natural transformation $\mathsf{K} : S^\infty \Rightarrow S^\infty$ (Definition 2.14) is a natural isomorphism. However, the finitary functor $S^\infty : \mathbb{R}\text{-Vec} \to \mathbb{R}\text{-Vec}$ is a left Kan extension of its own restriction $C^\infty : \text{Lin}_{\mathbb{R}} \to \mathbb{R}\text{-Vec}$ (4.6). Hence, in keeping with the strategy of Section 5, it suffices to show that the restriction $\mathsf{K}^\flat : C^\infty \Rightarrow C^\infty$ of K is an isomorphism, as $\mathsf{K}^\flat$ is the image of K under the equivalence $\text{Fin}(\mathbb{R}\text{-Vec}, \mathbb{R}\text{-Vec}) \simeq [\text{Lin}_{\mathbb{R}}, \mathbb{R}\text{-Vec}]$ of Example 3.3.

Extending this notation, we shall write

$$\mathsf{L}^\flat,\ \mathsf{K}^\flat,\ \mathsf{J}^\flat\ :\ C^\infty \Rightarrow C^\infty$$

to denote the restrictions of the transformations $\mathsf{L}, \mathsf{K}, \mathsf{J} : S^\infty \Rightarrow S^\infty$ defined in Definition 2.14.

In order to show that $\mathsf{K}^\flat$ is an isomorphism, we begin by first taking a look at the coderiving transformation d° (Definition 2.12) and its components $\mathsf{d}^\circ_{\mathbb{R}^n}$ for the finite-dimensional spaces $\mathbb{R}^n$. Recall that $\mathsf{d}^\circ_{\mathbb{R}^n} : C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n \to C^\infty(\mathbb{R}^n)$ is defined as follows:

$$\mathsf{d}^\circ_{\mathbb{R}^n} = (1_{C^\infty(\mathbb{R}^n)} \otimes \eta_{\mathbb{R}^n})\mathsf{m}_{\mathbb{R}^n}$$

where $\mathsf{m}_{\mathbb{R}^n}$ is the standard multiplication of $C^\infty(\mathbb{R}^n)$ and $\eta_{\mathbb{R}^n} : \mathbb{R}^n \to C^\infty(\mathbb{R}^n)$ is the linear map that sends the standard basis vectors $e_i \in \mathbb{R}^n$ to the projection maps $\pi_i : \mathbb{R}^n \to \mathbb{R}$ (4.5). Recalling from 5.5 that each element $\omega \in C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n$ can be expressed uniquely as a sum

$$\omega = \sum_{i=1}^{n} f_i \otimes e_i\ ,$$

with $f_1, ..., f_n \in C^\infty(\mathbb{R}^n)$, we compute that the resulting smooth function $\mathsf{d}^\circ_{\mathbb{R}^n}(\omega) : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\begin{aligned}\mathsf{d}^\circ_{\mathbb{R}^n}(\omega)(\vec{v}) &= \mathsf{d}^\circ_{\mathbb{R}^n}\left(\sum_{i=0}^{n} f_i \otimes e_i\right)(\vec{v}) \\ &= \sum_{i=0}^{n} f_i(\vec{v})\pi_i(\vec{v}) \\ &= \sum_{i=0}^{n} f_i(\vec{v})v_i \\ &= (f_1(\vec{v}), \ldots, f_n(\vec{v})) \cdot \vec{v}\end{aligned}$$

where the symbol $\cdot$ on the right-hand side denotes the usual dot product. Equivalently,

$$\mathsf{d}^\circ_{\mathbb{R}^n}(\omega)(\vec{v}) = F(\vec{v}) \cdot \vec{v}$$

where $F = \langle f_1, ..., f_n \rangle : \mathbb{R}^n \to \mathbb{R}^n$ is the vector field corresponding to $\omega = \sum_{i=1}^{n} f_i \otimes e_i$ as discussed in 5.5.

Now that we have computed the coderiving transformation d° for the spaces $\mathbb{R}^n$, we can now explicitly describe the transformations $\mathsf{L}^\flat, \mathsf{K}^\flat, \mathsf{J}^\flat : C^\infty \Rightarrow C^\infty$. Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, we recall from 5.5 that the vector field corresponding to $\mathsf{d}_{\mathbb{R}^n}(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \otimes e_i$ is precisely the gradient $\nabla f = \left\langle \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \right\rangle : \mathbb{R}^n \to \mathbb{R}^n$ of f. Using this, we obtain a simple expression for $\mathsf{L}^\flat_{\mathbb{R}^n}(f) = \mathsf{L}_{\mathbb{R}^n}(f) : \mathbb{R}^n \to \mathbb{R}$:

$$\mathsf{L}^\flat_{\mathbb{R}^n}(f)(\vec{v}) = \mathsf{d}^\circ_{\mathbb{R}^n}(\mathsf{d}_{\mathbb{R}^n}(f))(\vec{v}) = \nabla(f)(\vec{v}) \cdot \vec{v}$$

By definition $\mathsf{K}^\flat_{\mathbb{R}_n} = \mathsf{K}_{\mathbb{R}_n} = \mathsf{L}_{\mathbb{R}_n} + S^\infty(0)$ where $S^\infty(0) = C^\infty(0) : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, and by Proposition 4.7 we deduce that $C^\infty(0)$ is given by simply evaluating at zero:

$$C^\infty(0)(f)(\vec{v}) = f(\vec{0}) \ .$$

It is interesting to note that $C^\infty(0) = \mathsf{e}_{\mathbb{R}^n}\mathsf{u}_{\mathbb{R}^n}$, where $\mathsf{e}_{\mathbb{R}^n}$ is the counit map defined in Section 5.2. Therefore, $\mathsf{K}^\flat_{\mathbb{R}^n}(f) = \mathsf{K}_{\mathbb{R}^n}(f) : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\mathsf{K}^\flat_{\mathbb{R}^n}(f)(\vec{v}) = \nabla(f)(\vec{v}) \cdot \vec{v} + f(\vec{0}) \ .$$

Lastly, we find that $\mathsf{J}^\flat_{\mathbb{R}^n}(f) = \mathsf{J}_{\mathbb{R}^n}(f) : \mathbb{R}^n \to \mathbb{R}$ is given as follows:

$$\mathsf{J}^\flat_{\mathbb{R}^n}(f)(\vec{v}) = \nabla(f)(\vec{v}) \cdot \vec{v} + f(\vec{v}) \ .$$

We wish to show that $\mathsf{K}^\flat$ and $\mathsf{J}^\flat$ are natural isomorphisms. To do so, we need to make use of the *Fundamental Theorem of Line Integration*, which relates the gradient and line integration. Recall that for any continuous map $F : \mathbb{R}^n \to \mathbb{R}^n$ and a curve C parametrized by a given smooth path $r : [a,b] \to \mathbb{R}^n$, the **line integral of F along C** is defined as

$$\int_C F \cdot \mathsf{d}r := \int_a^b F(r(t)) \cdot r'(t) \ \mathsf{d}t \ .$$

The **Fundamental Theorem of Line Integration** states that for every C^1 function $f : \mathbb{R}^n \to \mathbb{R}$ and any smooth path $r : [a,b] \to \mathbb{R}^n$, we have the following equality:

$$\int_C \nabla f \cdot \mathsf{d}r = f(r(t))\Big|_a^b = f(r(b)) - f(r(a))$$

Note that the Fundamental Theorem of Line Integration is a higher-dimensional generalization of the Second Fundamental Theorem of Calculus.

Another basic tool that we will require is a compatibility relation between integration and differentiation called the **Leibniz integral rule**, to the effect that partial differentiation and integration commute when they act on independent variables:

$$\frac{\partial}{\partial x} \int_a^b f(x,t) \ \mathsf{d}t = \int_a^b \frac{\partial f(x,t)}{\partial x} \ \mathsf{d}t$$

for any C^1 function $f : \mathbb{R}^2 \to \mathbb{R}$ and any constants $a, b \in \mathbb{R}$, noting that this equation also holds under more general hypotheses, such as those in [27, §8.1, Thm. 1]. As a consequence, if $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a C^1 function, which we shall write as a function $f(\vec{x}, t)$ of a vector variable $\vec{x}$ and a scalar variable t, then any pair of constants $a, b \in \mathbb{R}$ determines a C^1 function

$g(\vec{x}) = \int_a^b f(\vec{x},t)\ \mathsf{d}t$ of $\vec{x}$ whose gradient $\nabla g : \mathbb{R}^n \to \mathbb{R}^n$ is

$$\nabla\left(\int\limits_a^b f(\vec{x},t)\ \mathsf{d}t\right) = \int\limits_a^b \nabla(f(\vec{x},t))\ \mathsf{d}t\ ,$$

where the right-hand side is an $\mathbb{R}^n$-valued integral and is regarded as a function of $\vec{x} \in \mathbb{R}^n$. As will be our convention throughout the sequel, the gradient in each case is taken with respect to the variable $\vec{x}$.

Proposition 6.1. $\mathsf{K}^\flat : C^\infty \Rightarrow C^\infty$ *is a natural isomorphism.*

Proof. For each n, define $\mathsf{K}^*_{\mathbb{R}^n} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ as follows:

$$\mathsf{K}^*_{\mathbb{R}^n}(f)(\vec{v}) = \int\limits_0^1 \int\limits_0^1 \nabla(f)(st\vec{v}) \cdot \vec{v}\ \mathsf{d}s\ \mathsf{d}t + f(\vec{0})\ ,$$

for each $f \in C^\infty(\mathbb{R}^n)$ and $\vec{v} \in \mathbb{R}^n$, noting that the resulting function $\mathsf{K}^*_{\mathbb{R}^n}(f) : \mathbb{R}^n \to \mathbb{R}$ is smooth, as a consequence of the Leibniz integral rule. We shall use the Fundamental Theorem of Line Integration to show that $\mathsf{K}^*_{\mathbb{R}^n}$ is an inverse of $\mathsf{K}^\flat_{\mathbb{R}^n} = \mathsf{K}_{\mathbb{R}^n}$.

For each $\vec{v} \in \mathbb{R}^n$, define the smooth path $r_{\vec{v}} : [0,1] \to \mathbb{R}^n$ by

$$r_{\vec{v}}(t) = t\vec{v}\ .$$

Note that $r_{\vec{v}}$ is a parametrization of the straight line between $\vec{0}$ and $\vec{v}$, which we will denote as $C_{\vec{v}}$. The derivative of $r_{\vec{v}}$ is simply the constant function that maps everything to $\vec{v}$: $r'_{\vec{v}}(t) = \vec{v}$. Now the Fundamental Theorem of Line Integration implies that for every smooth function $f : \mathbb{R}^n \to \mathbb{R}$, the following equality holds:

$$\begin{aligned} \int\limits_0^1 \nabla(f)(t\vec{v}) \cdot \vec{v}\ \mathsf{d}t &= \int\limits_0^1 \nabla(f)(r_{\vec{v}}(t)) \cdot r'_{\vec{v}}(t)\ \mathsf{d}t \\ &= \int\limits_{C_{\vec{v}}} \nabla f \cdot dr \qquad (11) \\ &= f(r_{\vec{v}}(1)) - f(r_{\vec{v}}(0)) \\ &= f(\vec{v}) - f(\vec{0})\ . \end{aligned}$$

This is the key identity to the proof that $\mathsf{K}^\flat$ is an isomorphism. In fact, later we will see that the Second Fundamental Theorem of Calculus rule **[c.1]** from the definition of a co-calculus category (Definition 2.9) is precisely this instance of the Fundamental Theorem of Line Integration.

Now observe that for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$, the following equality holds:

$$\mathsf{K}_{\mathbb{R}^n}(f)(\vec{0}) = \underbrace{\nabla(f)(\vec{0}) \cdot \vec{0}}_{=0} + f(\vec{0}) = f(\vec{0})$$

Using the Fundamental Theorem of Line Integration twice and playing with the bounds of the integral using limits, we show that $\mathsf{K}_{\mathbb{R}^n}\mathsf{K}^*_{\mathbb{R}^n} = 1$:

$$\begin{aligned}
\mathsf{K}^*_{\mathbb{R}^n}(\mathsf{K}_{\mathbb{R}^n}(f))(\vec{v}) &= \int_0^1 \int_0^1 \nabla(\mathsf{K}_{\mathbb{R}^n}(f))(st\vec{v}) \cdot v \; \mathsf{d}s \; \mathsf{d}t + \mathsf{K}_{\mathbb{R}^n}(f)(\vec{0}) \\
&= \lim_{u \to 0^+} \int_u^1 \int_0^1 \nabla(\mathsf{K}_{\mathbb{R}^n}(f))(st\vec{v}) \cdot v \; \mathsf{d}s \; \mathsf{d}t + f(\vec{0}) \\
&= \lim_{u \to 0^+} \int_u^1 \frac{t}{t} \int_0^1 \nabla(\mathsf{K}_{\mathbb{R}^n}(f))(st\vec{v}) \cdot \vec{v} \; \mathsf{d}s \; \mathsf{d}t + f(\vec{0}) \\
&= \lim_{u \to 0^+} \int_u^1 \frac{1}{t} \int_0^1 \nabla(\mathsf{K}_{\mathbb{R}^n}(f))(st\vec{v}) \cdot t\vec{v} \; \mathsf{d}s \; \mathsf{d}t + f(\vec{0}) \\
&= \lim_{u \to 0^+} \int_u^1 \frac{1}{t} \left(\mathsf{K}_{\mathbb{R}^n}(f)(t\vec{v}) - \mathsf{K}_{\mathbb{R}^n}(f)(\vec{0}) \right) \; \mathsf{d}t + f(\vec{0}) \\
&= \lim_{u \to 0^+} \int_u^1 \frac{1}{t} \left(\nabla(f)(t\vec{v}) \cdot t\vec{v} + f(\vec{0}) - f(\vec{0}) \right) \; \mathsf{d}t + f(\vec{0}) \\
&= \lim_{u \to 0^+} \int_u^1 \frac{1}{t} \; \nabla(f)(t\vec{v}) \cdot t\vec{v} \; \mathsf{d}t + f(\vec{0})
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{u\to 0^+} \int_u^1 \frac{t}{t}\, \nabla(f)(t\vec{v})\cdot \vec{v}\, \mathsf{d}t + f(\vec{0}) \\
&= \lim_{u\to 0^+} \int_u^1 \nabla(f)(t\vec{v})\cdot \vec{v}\, \mathsf{d}t + f(\vec{0}) \\
&= \int_0^1 \nabla(f)(t\vec{v})\cdot \vec{v}\, \mathsf{d}t + f(\vec{0}) \\
&= f(\vec{v}) - f(\vec{0}) + f(\vec{0}) \\
&= f(\vec{v}).
\end{aligned}
$$

To prove that $\mathsf{K}^*_{\mathbb{R}^n}\mathsf{K}_{\mathbb{R}^n} = 1$, we will begin with a few preliminary observations. First, the following identities also hold for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$:

$$
\mathsf{K}^*_{\mathbb{R}^n}(f)(\vec{0}) = \underbrace{\int_0^1\int_0^1 \nabla(f)(st\vec{0})\cdot\vec{0}\, \mathsf{d}s\, \mathsf{d}t}_{=0} + f(\vec{0}) = f(\vec{0}).
$$

Next, as a consequence of the gradient version of the chain rule, for any scalar $t \in \mathbb{R}$ we have that

$$
\nabla(f(t\vec{x}))(\vec{v})\cdot\vec{w} = \nabla(f)(t\vec{v})\cdot t\vec{w},
$$

where the gradient on the left-hand side is taken with respect to the variable $\vec{x}$ and then explicitly evaluated at $\vec{v}$; we will use similar notation in the sequel.

The last observation we need is that the gradient interacts nicely with our line integral, as a consequence of the Leibniz integral rule and the chain rule:

$$
\nabla\left(\int_0^1 f(t\vec{x})\, \mathsf{d}t\right)(\vec{v})\cdot\vec{w} = \int_0^1 \nabla(f(t\vec{x}))(\vec{v})\cdot\vec{w}\, \mathsf{d}t = \int_0^1 \nabla(f)(t\vec{v})\cdot t\vec{w}\, \mathsf{d}t \tag{12}
$$

With all these observations and using similar techniques from before, we can prove that $\mathsf{K}^*_{\mathbb{R}^n}\mathsf{K}_{\mathbb{R}^n} = 1$:

$$
\mathsf{K}_{\mathbb{R}^n}(\mathsf{K}^*_{\mathbb{R}^n}(f))(\vec{v}) = \nabla(\mathsf{K}^*_{\mathbb{R}^n}(f))(\vec{v})\cdot\vec{v} + \mathsf{K}^*_{\mathbb{R}^n}(f)(\vec{0})
$$

$$= \nabla\left(\int_0^1\int_0^1 \nabla(f)(st\vec{x})\cdot\vec{x}\,\mathsf{d}s\,\mathsf{d}t + f(\vec{0})\right)(\vec{v})\cdot\vec{v} + f(\vec{0})$$

$$= \nabla\left(\int_0^1\int_0^1 \nabla(f)(st\vec{x})\cdot\vec{x}\,\mathsf{d}s\,\mathsf{d}t\right)(\vec{v})\cdot\vec{v} + \underbrace{\nabla(f(\vec{0}))(\vec{v})\cdot\vec{v}}_{=0} + f(\vec{0})$$

$$= \nabla\left(\lim_{u\to 0^+}\int_u^1\int_0^1 \nabla(f)(st\vec{x})\cdot\vec{x}\,\mathsf{d}s\,\mathsf{d}t\right)(\vec{v})\cdot\vec{v} + f(\vec{0})$$

$$= \nabla\left(\lim_{u\to 0^+}\int_u^1 \frac{t}{t}\int_x^1 \nabla(f)(st\vec{x})\cdot\vec{x}\,\mathsf{d}s\,\mathsf{d}t\right)(\vec{v})\cdot\vec{v} + f(\vec{0})$$

$$= \nabla\left(\lim_{u\to 0^+}\int_u^1 \frac{1}{t}\int_x^1 \nabla(f)(st\vec{x})\cdot t\vec{x}\,\mathsf{d}s\,\mathsf{d}t\right)(\vec{v})\cdot\vec{v} + f(\vec{0})$$

$$= \nabla\left(\lim_{u\to 0^+}\int_u^1 \frac{1}{t}\left(f(t\vec{x}) - f(\vec{0})\right)\,\mathsf{d}t\right)(\vec{v})\cdot\vec{v} + f(\vec{0})$$

$$= \nabla\left(\lim_{u\to 0^+}\int_u^1 \frac{1}{t}f(t\vec{x})\,\mathsf{d}t\right)(\vec{v})\cdot\vec{v} - \nabla\left(\lim_{u\to 0^+}\int_u^1 \frac{1}{t}f(\vec{0})\,\mathsf{d}t\right)(\vec{v})\cdot\vec{v}$$
$$+ f(\vec{0})$$

$$= \lim_{u\to 0^+}\int_u^1 \frac{1}{t}\nabla(f(t\vec{x}))(\vec{v})\cdot\vec{v}\,\mathsf{d}t - \lim_{u\to 0^+}\int_u^1 \frac{1}{t}\underbrace{\nabla(f(\vec{0}))(\vec{v})\cdot\vec{v}}_{=0}\,\mathsf{d}t + f(\vec{0})$$

$$= \lim_{u\to 0^+}\int_u^1 \frac{1}{t}\nabla(f)(t\vec{v})\cdot t\vec{v}\,\mathsf{d}t + f(\vec{0})$$

$$= \lim_{u\to 0^+}\int_u^1 \frac{t}{t}\nabla(f)(t\vec{v})\cdot\vec{v}\,\mathsf{d}t + f(\vec{0})$$

$$= \lim_{u\to 0^+} \int_u^1 \nabla(f)(t\vec{v}) \cdot \vec{v} \, \mathsf{d}t + f(\vec{0})$$
$$= \int_0^1 \nabla(f)(t\vec{v}) \cdot \vec{v} \, \mathsf{d}t + f(\vec{0})$$
$$= f(\vec{v}) - f(\vec{0}) + f(\vec{0})$$
$$= f(\vec{v})$$

Therefore we conclude that $\mathsf{K}^\flat : C^\infty \Rightarrow C^\infty$ is a natural isomorphism. $\square$

Corollary 6.2. $\mathsf{K} : \mathsf{S}^\infty \Rightarrow \mathsf{S}^\infty$ *is a natural isomorphism.*

Therefore by Theorem 2.15, we obtain the following result:

Theorem 6.3. *The monad* S^∞ *on* $\mathbb{R}$-Vec *has the structure of a codifferential category with antiderivatives and therefore also the structure of a co-calculus category (and thus a co-integral category).*

Before giving an explicit description of the induced integral transformation s, we take a look at the inverse of J. Using J^{-1} will simplify calculating s. Recall that K being a natural isomorphism implies that J is a natural isomorphism. One can then construct J^{-1} from K^{-1} with the aid of μ [10]. However, in the present case, for the finite-dimensional spaces $\mathbb{R}^n$, we will see that $\mathsf{J}^{-1}_{\mathbb{R}^n}$ can be described by a considerably simpler formula that our integral formula for $\mathsf{K}^{-1}_{\mathbb{R}^n} = \mathsf{K}^*_{\mathbb{R}^n}$ in Proposition 6.1. For this reason, and for the sake of completeness, we will give a stand-alone proof that J is invertible, by directly defining an inverse of $\mathsf{J}_{\mathbb{R}^n}$ by means of an integral formula.

Proposition 6.4. $\mathsf{J}^\flat : C^\infty \Rightarrow C^\infty$ *is a natural isomorphism.*

Proof. For each n, define $\mathsf{J}^*_{\mathbb{R}^n} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ as follows:

$$\mathsf{J}^*_{\mathbb{R}^n}(f)(\vec{v}) = \int_0^1 f(t\vec{v}) \, \mathsf{d}t \ ,$$

noting that the Leibniz integral rule entails that $\mathsf{J}^*_{\mathbb{R}^n}(f)$ is indeed smooth. Again, we wish to use the Fundamental Theorem of Line Integration to show that this is indeed the inverse of $\mathsf{J}^\flat_{\mathbb{R}^n} = \mathsf{J}_{\mathbb{R}^n}$.

Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, define the smooth function $\tilde{f} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ simply as multiplying f by a scalar: $\tilde{f}(\vec{v}, t) = tf(\vec{v})$. Its gradient $\nabla(\tilde{f}) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$ is given by

$$\begin{aligned}\nabla(\tilde{f})(\vec{v},t) = & \left(\frac{\partial \tilde{f}}{\partial x_1}(\vec{v},t), \frac{\partial \tilde{f}}{\partial x_1}(\vec{v},t), \ldots, \frac{\partial \tilde{f}}{\partial x_n}(\vec{v},t), \frac{\partial \tilde{f}}{\partial t}(\vec{v},t)\right) \\ = & \left(t\frac{\partial f}{\partial x_1}(\vec{v}), \ldots, t\frac{\partial f}{\partial x_n}(\vec{v}), f(\vec{v})\right) \\ = & \left(t\nabla(f)(\vec{v}), f(\vec{v})\right) .\end{aligned}$$

As a consequence, we obtain the following identities:

$$\nabla(\tilde{f})(\vec{v},t)\cdot(\vec{w},1) = t\nabla(f)(\vec{v})\cdot\vec{w}+f(\vec{v}) = \nabla(f)(\vec{v})\cdot t\vec{w}+f(\vec{v}) = \nabla(\tilde{f})(\vec{v},1)\cdot(t\vec{w},1) .$$

Now using this above identity and the Fundamental Theorem of Line Integration, we show that $\mathsf{J}_{\mathbb{R}^n}\mathsf{J}^*_{\mathbb{R}^n} = 1$:

$$\begin{aligned}\mathsf{J}^*_{\mathbb{R}^n}\left(\mathsf{J}_{\mathbb{R}^n}(f)\right)(\vec{v}) &= \int_0^1 \mathsf{J}^{\flat}_{\mathbb{R}^n}(f)(t\vec{v})\ \mathsf{d}t \\ &= \int_0^1 \left(\nabla(f)(t\vec{v})\cdot t\vec{v} + f(t\vec{v})\right)\ \mathsf{d}t \\ &= \int_0^1 \left(\nabla(\tilde{f})(t\vec{v},t)\cdot(\vec{v},1)\right)\ \mathsf{d}t \\ &= \tilde{f}(\vec{v},1) - \tilde{f}(\vec{0},0) \\ &= f(\vec{v})\end{aligned}$$

Having thus shown that $\mathsf{J}^*_{\mathbb{R}^n}$ is a retraction of $\mathsf{J}_{\mathbb{R}^n}$, and having already noted above that J is invertible since K is invertible, we may at this point deduce that $\mathsf{J}^*_{\mathbb{R}^n} = \mathsf{J}^{-1}_{\mathbb{R}^n}$. However, in order to construct a standalone proof that J is invertible, we now show directly that $\mathsf{J}^*_{\mathbb{R}^n}\mathsf{J}_{\mathbb{R}^n} = 1$, by using the interchange identity between the gradient and the line integral (12):

$$\mathsf{J}_{\mathbb{R}^n}\left(\mathsf{J}^*_{\mathbb{R}^n}(f)\right)(\vec{v}) = \nabla\left(\mathsf{J}^*_{\mathbb{R}^n}(f)\right)(\vec{v})\cdot\vec{v} + \mathsf{J}^*_{\mathbb{R}^n}(f)(\vec{v})$$

$$
\begin{aligned}
&= \nabla \left(\int_0^1 f(t\vec{x})\ \mathsf{d}t \right) (\vec{v}) \cdot \vec{v} + \int_0^1 f(t\vec{v})\ \mathsf{d}t \\
&= \int_0^1 \nabla(f)(t\vec{v}) \cdot t\vec{v}\ \mathsf{d}t + \int_0^1 f(t\vec{v})\ \mathsf{d}t \\
&= \int_0^1 \left(\nabla(f)(t\vec{v}) \cdot t\vec{v} + f(t\vec{v})\right)\ \mathsf{d}t \\
&= \int_0^1 \left(\nabla(\tilde{f})(t\vec{v}, t) \cdot (\vec{v}, 1)\right)\ \mathsf{d}t \\
&= \tilde{f}(\vec{v}, 1) - \tilde{f}(\vec{0}, 0) \\
&= f(\vec{v})
\end{aligned}
$$

□

Corollary 6.5. $\mathsf{J} : \mathsf{S}^\infty \Rightarrow \mathsf{S}^\infty$ *is a natural isomorphism.*

We now compute the induced integral transformation s for the finite-dimensional vector spaces $\mathbb{R}^n$, that is, we compute a formula for the map

$$
\mathsf{s}_{\mathbb{R}^n} : C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n \longrightarrow C^\infty(\mathbb{R}^n) ,
$$

which we recall is defined as $\mathsf{s}_{\mathbb{R}^n} = \mathsf{d}^\circ_{\mathbb{R}^n}\mathsf{K}^{-1}_{\mathbb{R}^n} = (\mathsf{J}^{-1}_{\mathbb{R}^n} \otimes 1_{\mathbb{R}^n})\mathsf{d}^\circ_{\mathbb{R}^n}$ (Theorem 2.15). Given any element $\omega = \sum_{i=1}^n f_i \otimes e_i$ of $C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n$, expressed as in 5.5, we compute that

$$
\begin{aligned}
\mathsf{s}_{\mathbb{R}^n}(\omega)(\vec{v}) &= \mathsf{d}^\circ \left(\left(\mathsf{J}^{-1}_{\mathbb{R}^n} \otimes 1_{\mathbb{R}^n}\right) \left(\sum_{i=1}^n f_i \otimes e_i \right) \right) (\vec{v}) \\
&= \mathsf{d}^\circ \left(\sum_{i=1}^n \left(\int_0^1 f_i(t\vec{x})\ \mathsf{d}t \right) \otimes e_i \right) (\vec{v}) \\
&= \left(\int_0^1 f_1(t\vec{v})\ \mathsf{d}t, \ldots, \int_0^1 f_n(t\vec{v})\ \mathsf{d}t \right) \cdot \vec{v}
\end{aligned}
$$

$$= \int_0^1 (f_1(t\vec{v}), \ldots, f_n(t\vec{v})) \cdot \vec{v}\, \mathsf{d}t$$

$$= \int_0^1 F(t\vec{v}) \cdot \vec{v}\, \mathsf{d}t$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is the vector field $F = \langle f_1, \ldots, f_n \rangle$ corresponding to ω as in 5.5. Equivalently, $\mathsf{s}_{\mathbb{R}}(\omega)(\vec{v})$ can be described as the line integral

$$\mathsf{s}_{\mathbb{R}^n}(\omega)(\vec{v}) = \int_{C_{\vec{v}}} F \cdot \mathsf{d}r$$

of the vector field F along the directed line segment $C_{\vec{v}}$ from the origin to the point $\vec{v}$ (for which one parametrization is $r = r_{\vec{v}}$, as discussed in the proof of Proposition 6.1). Recalling that ω is a 1-form on $\mathbb{R}^n$ (5.5), this line integral is more succinctly described as follows:

Theorem 6.6. *The integral transformation* s *carried by the free* C^∞*-ring modality* S^∞ *sends each 1-form* $\omega \in C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n$ *to the function* $\mathsf{s}_{\mathbb{R}^n}(\omega) \in C^\infty(\mathbb{R}^n)$ *whose value at each* $\vec{v} \in \mathbb{R}^n$ *is the integral of* ω *along the directed line segment* $C_{\vec{v}}$ *from* $\vec{0}$ *to* $\vec{v}$*:*

$$\mathsf{s}_{\mathbb{R}^n}(\omega)(\vec{v}) = \int_{C_{\vec{v}}} \omega \ .$$

Remark 6.7. For brevity, we will write the integral $\int_{C_{\vec{v}}} \omega$ in Theorem 6.6 as $\int_{\vec{v}} \omega$, as it can be thought of as an integral *over* $\vec{v}$*, considered as a position vector*. Correspondingly, we will denote the function $\mathsf{s}_{\mathbb{R}^n}(\omega) : \mathbb{R}^n \to \mathbb{R}$ by $\int_{(\text{-})} \omega$.

Example 6.8. It is illustrative to consider what the above formulae produce when the input is a 1-form ω with polynomial coefficients. For example, writing $\vec{x} = (x_1, x_2)$ for a general point in $\mathbb{R}^2$, let ω be the 1-form $\omega = x_1^2 x_2^5\, \mathsf{d}x_1 + x_1^3\, \mathsf{d}x_2$ on $\mathbb{R}^2$ (with the notation of 5.5), whose corresponding vector field F is given by $F(x_1, x_2) = (x_1^2 x_2^5, x_1^3)$. Then $F(tx_1, tx_2) =$

$((tx_1)^2(tx_2)^5, (tx_1)^3) = (t^7x_1^2x_2^5, t^3x_1^3)$ so that $\mathsf{s}_{\mathbb{R}^n}(\omega)(\vec{x})$ is the integral

$$\int_{\vec{x}} \omega = \int_0^1 F(t\vec{x}) \cdot \vec{x} \, \mathsf{d}t = \int_0^1 (t^7x_1^2x_2^5x_1 + t^3x_1^3x_2) \, \mathsf{d}t = \frac{1}{8}x_1^3x_2^5 + \frac{1}{4}x_1^3x_2.$$

More generally, one can readily show that when applied to any 1-form

$$\omega = \sum_{i=1}^{n} p_i \, \mathsf{d}x_i = \sum_{i=1}^{n} p_i \otimes e_i$$

on $\mathbb{R}^n$ with polynomial coefficients p_i, the above formulae for s reproduce the integral transformation for polynomials as described in Example 2.8. The formula for arbitrary smooth functions thus explains the seemingly odd choice of summing all the coefficients when integrating a particular term.

Let us now examine what the identities of a co-calculus category (Definition 2.9) amount to in the specific co-calculus category that we have developed here. The Second Fundamental Theorem of Calculus rule **[c.1]** is precisely the special case of the Fundamental Theorem of Line Integration that we used extensively in the proofs of Propositions 6.1 and 6.4, namely (11). Indeed, given a smooth function $f \in C^\infty(\mathbb{R}^n)$, one has that

$$\begin{aligned}\mathsf{s}_{\mathbb{R}^n}\left(\mathsf{d}_{\mathbb{R}^n}(f)\right)(\vec{v}) + S^\infty(0)(f)(\vec{v}) &= \mathsf{s}_{\mathbb{R}^n}\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \otimes e_i\right)(\vec{v}) + f(\vec{0}) \\ &= \int_0^1 \nabla(f)(t\vec{v}) \cdot \vec{v} \, \mathsf{d}t + f(\vec{0}) = f(\vec{v}) \, .\end{aligned}$$

On the other hand the Poincaré condition **[c.2]** is essentially the statement of its namesake, the Poincaré Lemma, for 1-forms on Euclidean spaces. Explicitly, **[c.2]** says that closed 1-forms are exact and that the integral transformation s provides a canonical choice of 0-form to serve as 'antiderivative' for each closed 1-form. So if ω is a closed 1-form over $\mathbb{R}^n$, then ω is exact by being the exterior derivative of the 0-form $\mathsf{s}_{\mathbb{R}^n}(\omega)$, that is, $\mathsf{d}_{\mathbb{R}^n}(\mathsf{s}_{\mathbb{R}^n}(\omega)) = \omega$.

We now take a look at the Rota-Baxter rule **[s.2]** for the integral transformation s (Definition 2.7). Continuing to identify $C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n$ with the

$C^\infty(\mathbb{R}^n)$-module of smooth 1-forms on $\mathbb{R}^n$ as in Remark 5.5, we will employ the usual notation $f\omega$ for the product of a function $f \in C^\infty(\mathbb{R}^n)$ and a 1-form ω. Given two 1-forms $\omega, \nu \in C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n$, the Rota-Baxter rule **[s.2]** gives the following equality, with the notation of Remark 6.7:

$$\left(\int_{\vec{v}} \omega\right)\left(\int_{\vec{v}} \nu\right) = \int_{\vec{v}} \left(\int_{(\text{-})} \nu\right)\omega + \int_{\vec{v}} \left(\int_{(\text{-})} \omega\right)\nu$$

The Rota-Baxter identity also admits a nice (and possibly more explicit) expression in terms of vector fields. Indeed, given two vector fields $F : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^n$, then the Rota-Baxter rule **[s.2]** implies that the following equality holds:

$$\begin{aligned}\left(\int_0^1 F(t\vec{v})\cdot\vec{v}\,\mathsf{d}t\right)\left(\int_0^1 G(t\vec{v})\cdot\vec{v}\,\mathsf{d}t\right) &= \\ \int_0^1 (F(t\vec{v})\cdot\vec{v})\left(\int_0^t G(u\vec{v})\cdot\vec{v}\,\mathsf{d}u\right)\,\mathsf{d}t& \\ +\int_0^1 \left(\int_0^t F(u\vec{v})\cdot\vec{v}\,\mathsf{d}u\right)(G(t\vec{v})\cdot\vec{v})\,\mathsf{d}t&\end{aligned}$$

A further consequence of the Rota-Baxter rule, for arbitraryvector spaces V, is that the integral transformation $\mathsf{s}_V : S^\infty(V) \otimes V \to S^\infty(V)$ induces a *Rota-Baxter operator* on the free C^∞-ring $S^\infty(V)$, as we will show in Proposition 6.10.

Definition 6.9. *Let R be a commutative ring. A (commutative)* ***Rota-Baxter algebra*** *[13] (of weight 0) over R is a pair (A, P) consisting of a (commutative) R-algebra A and an R-linear map $\mathsf{P} : A \to A$ such that P satisfies the Rota-Baxter identity; that is, for each $a, b \in A$, the following equality holds:*

$$\mathsf{P}(a)\mathsf{P}(b) = \mathsf{P}(a\mathsf{P}(b)) + \mathsf{P}(\mathsf{P}(a)b). \tag{13}$$

The map P is called a ***Rota-Baxter operator***.

As discussed in [10], the latter Rota-Baxter identity (13) corresponds to a formulation of the integration by parts rule that involves only integrals and no derivatives—as we will soon illustrate in Example 6.11. We refer the reader to [13] for more details on Rota-Baxter algebras.

Now for an arbitrary $\mathbb{R}$-vector space V and any element $\vec{v} \in V$, it readily follows from the Rota-Baxter rule **[s.2]** in Definition 2.7 that the corresponding linear map $v : \mathbb{R} \to V$ induces a Rota-Baxter operator $\mathsf{P}_v : S^\infty(V) \to S^\infty(V)$ defined as the following composite

$$\mathsf{P}_v := \; S^\infty(V) \xrightarrow{1\otimes v} S^\infty(V) \otimes V \xrightarrow{\mathsf{s}_V} S^\infty(V) \;, \tag{14}$$

making the pair $(S^\infty(V), \mathsf{P}_v)$ a Rota-Baxter algebra over $\mathbb{R}$. Summarizing, we obtain the following new observation:

Proposition 6.10. *Free C^∞-rings are commutative Rota-Baxter algebras over $\mathbb{R}$, with Rota-Baxter operators defined as in (14).*

Example 6.11. A particularly important example arises when we let $V = \mathbb{R}$ and we take $\vec{v}$ to be the element $1 \in \mathbb{R}$ (whose corresponding linear map is the identity on $\mathbb{R}$). In this case, the corresponding Rota-Baxter operator P_1 on $S^\infty(\mathbb{R}) = C^\infty(\mathbb{R})$ is essentially the integral transformation:

$$\mathsf{P}_1 := \; C^\infty(\mathbb{R}) \xrightarrow{\cong} C^\infty(\mathbb{R}) \otimes \mathbb{R} \xrightarrow{\mathsf{s}_V} C^\infty(\mathbb{R}) \;.$$

Letting $f \in C^\infty(\mathbb{R})$, we can use the substitution rule to compute that the function $\mathsf{P}_1(f) \in C^\infty(\mathbb{R})$ is given by

$$\mathsf{P}_1(f)(x) = \mathsf{s}_{\mathbb{R}}(f \otimes 1)(x) = \int_0^1 f(tx)x \,\mathsf{d}t = \int_0^x f(u)\,\mathsf{d}u \;.$$

Expressed in this form, the Rota-Baxter algebra $(C^\infty(\mathbb{R}), \mathsf{P}_1)$ is often considered the canonical example of a Rota-Baxter algebra (of weight 0). For a pair of smooth functions $f, g \in C^\infty(\mathbb{R})$, the Rota-Baxter identity is

$$\mathsf{P}_1(f)(x)\mathsf{P}_1(g)(x) = \left(\int_0^x f(u)\,\mathsf{d}u\right)\left(\int_0^x g(u)\,\mathsf{d}u\right)$$

$$= \int_0^x f(u) \left(\int_0^u g(t)\, \mathsf{d}t \right) \mathsf{d}u + \int_0^x \left(\int_0^u f(t)\, \mathsf{d}t \right) g(u)\, \mathsf{d}u$$
$$= \mathsf{P}_1 \left(f \mathsf{P}_1(g) \right)(x) + \mathsf{P}_1 \left(\mathsf{P}_1(f) g \right) .$$

One interesting consequence of Rota-Baxter algebra structure is that the Rota-Baxter operator induces a new *non-unital* Rota-Baxter algebra structure. If (A, P) is a Rota-Baxter algebra over R, then define a new associative binary operation $*_{\mathsf{P}}$ by

$$a *_{\mathsf{P}} b = a\mathsf{P}(b) + \mathsf{P}(a)b .$$

This new multiplication $*_{\mathsf{P}}$ is called the double product and endows A with a non-unital R-algebra structure, for which P is again a Rota-Baxter operator. If A is commutative, then the double product is also commutative. Also note that by R-linearity of P, the Rota-Baxter identity can then be re-expressed as:

$$\mathsf{P}(a *_{\mathsf{P}} b) = \mathsf{P}(a)\mathsf{P}(b)$$

which implies that P is a non-unital Rota-Baxter algebra homomorphism.

Corollary 6.12. *In addition to its underlying unital $\mathbb{R}$-algebra structure, each free C^∞-ring carries a further non-unital, commutative $\mathbb{R}$-algebra structure, with the same addition operation but with multiplication given by the double product induced by the Rota-Baxter operator defined in* (14).

Example 6.13. Consider the Rota-Baxter algebra $(C^\infty(\mathbb{R}), \mathsf{P}_1)$ from Example 6.11. In this case, the induced double product $*_{\mathsf{P}_1}$ is given by

$$(f *_{\mathsf{P}_1} g)(x) = f(x) \left(\int_0^x g(t)\, \mathsf{d}t \right) + \left(\int_0^x f(t)\, \mathsf{d}t \right) g(x)$$
$$= f(x)\mathsf{P}_1(g)(x) + \mathsf{P}_1(f)(x)g(x) .$$

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

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DIFFEOLOGICAL MORITA EQUIVALENCE

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Résumé. Nous introduisons une nouvelle notion d'équivalence de Morita pour les groupoïdes difféologiques, généralisant la notion originale pour les groupoÏdes de Lie. Pour cela, nous développons une théorie des actions de groupoïdes difféologiques, fibrés et bi-fibrés. Nous définissons une notion de fibré principal qui utilise la notion de subduction, généralisant la notion de fibré principal pour un group(oïde) de Lie. Nous disons que deux groupoïdes difféologiques sont Morita équivalents si, et seulement si, il existe un fibré bi-principal entre eux. Utilisant le produit tensoriel de Hilsum-Skandalis, nous définissons en outre une composition des bi-fibrés difféologiques, et obtenons une bi-catégorie **DiffeolBiBund**. Notre principal résultat est le suivant: un bi-fibré est bi-principal si, et seulement si, il est faiblement inversible dans cette bi-catégorie. Ceci généralise un théorème bien connu de la théorie des groupoïdes de Lie. Comme application, nous prouvons que les espaces d'orbites de deux groupoïdes difféologiques Morita équivalents sont difféomorphes. Nous montrons également que les propriétés d'un groupoïde difféologique d'être fibrant, et sa catégorie d'actions, sont des invariants de Morita.

Abstract. We introduce a new notion of *Morita equivalence* for *diffeological groupoids*, generalising the original notion for Lie groupoids. For this we develop a theory of *diffeological groupoid actions*, *-bundles* and *-bibundles*. We define a notion of *principality* for these bundles, which uses the notion of a *subduction*, generalising the notion of a Lie group(oid) principal bundle.

We say two diffeological groupoids are Morita equivalent if and only if there exists a *biprincipal* bibundle between them. Using a Hilsum-Skandalis tensor product, we further define a composition of diffeological bibundles, and obtain a bicategory **DiffeolBiBund**. Our main result is the following: a bibundle is biprincipal if and only if it is *weakly invertible* in this bicategory. This generalises a well known theorem from the Lie groupoid theory. As an application of the framework, we prove that the *orbit spaces* of two Morita equivalent diffeological groupoids are diffeomorphic. We also show that the property of a diffeological groupoid to be *fibrating*, and its *category of actions*, are Morita invariants.
Keywords. Diffeology, Lie groupoids, diffeological groupoids, bibundles, Hilsum-Skandalis products, Morita equivalence, orbit spaces.
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1. Introduction

Diffeology originates from the work of J.-M. Souriau [Sou80; Sou84] and his students [DI83; Don84; Igl85] in the 1980s. The main objects of this theory are *diffeological spaces*, a type of generalised smooth space that extends the traditional notion of a smooth manifold. They make for a convenient framework that deals well with (singular) quotients, function spaces (or otherwise infinite-dimensional objects), fibred products (or otherwise singular subspaces), and other constructions that lie beyond the realm of classical differential topology. As many of these constructions naturally occur in differential topology and -geometry, and since they cannot be studied with their standard tools, diffeology has become a useful addition to the geometer's toolbox.

Diffeological groupoids have recently garnered attention in the mathematical physics of general relativity [BFW13; Gł19], foliation theory [ASZ19; GZ19; Mac20], the theory of algebroids [AZ], the theory of (differentiable) stacks [RV18; WW19], and even in relation to noncommutative geometry [IZL18; IZP20]. In all but one of these fields (general relativity), the notion of *Morita equivalence* is an important one. Yet, as the authors of [GZ19, p.3] point out: "The theory of Morita equivalence for diffeological groupoids has not been developed yet." In the current paper we present one possible development of such a notion, based on the results of the author's Master thesis [vdS20]. This development is a generalisation of the theory of Hilsum-Skandalis bibundles and the Morita equivalence of Lie groupoids, where many definitions and proofs, and certainly the general idea, extend quite straightforwardly to the diffeological case. The main exception is that we need to replace surjective submersions with so-called *subductions*. This special type of smooth map is, even on smooth manifolds, slightly weaker than the notion of a surjective submersion, but it turns out that they still share enough of their properties so that the entire theory can be developed[1]. This development proceeds roughly as follows: based on the notions of *actions* and *bundles* defined in Section 4, we define a diffeological version of a *bibundle* between groupoids (Definition 5.1). These stand in analogy to *bimodules* for rings, and can be treated as a generalised type of morphism between groupoids. This gives a *bicategory* **DiffeolBiBund** of diffeological groupoids, bibundles, and *biequivariant maps* (Theorem 5.17). Using the aforementioned notion of a *subduction* (Definition 2.16), we define *biprincipality* of bibundles, and with this, we obtain a notion of *Morita equivalence* for diffeological groupoids (Definition 5.3). In the bicategory we also get a notion of equivalence, by way of the *weak isomorphisms*. A morphism in a bicategory is called *weakly invertible* if it is invertible *up to 2-isomorphism*. Two objects in a bicategory are called *weakly isomorphic* if there exists a weakly invertible morphism between them. The main point of this paper is to prove a *Morita theorem* for diffeological groupoids, characterising the weakly invertible bibundles, and hence realising Morita equivalence as a particular instance of weak isomorphism:

Theorem 5.31 (Morita theorem)**.** *A diffeological bibundle is weakly invertible if and only if it is biprincipal. In other words, two diffeological groupoids are Morita equivalent if and only if they are weakly isomorphic in the bicategory* **DiffeolBiBund**.

A Morita theorem for Lie groupoids has been known in the literature for some time, see e.g. [Lan01b, Proposition 4.21]. Throughout the paper, we shall point out some differences between the diffeological- and Lie theories. The main difference is that, due to technical constraints, a Morita theorem for Lie groupoids only holds in the restricted setting of *left principal* bibundles.

[1]This is essentially due to the fact that the subductions are the *strong epimorphisms* in the category of diffeological spaces [BH11, Proposition 5.10].

The main improvement of Theorem 5.31 over the classical Lie Morita theorem, besides the generalisation to diffeology, is therefore that it considers also a more general class of bibundles. Besides this improvement, with this paper we hope to contribute a complete account of the basic theory of bibundles and Morita equivalence of groupoids, providing detailed proofs and constructions of most necessary technical results, and culminating in a proof of the main Theorem 5.31. A brief outline of the contents of the paper is as follows.

We briefly recall the definition of a diffeology in Section 2. In particular, we describe the diffeologies of fibred products (pullbacks) and quotients, since they will be important to describe the smooth structure of the orbit space and space of composable arrows of a groupoid. We also define and study the behaviour of *subductions*, especially in relation to fibred products.

In Section 3 we define *diffeological groupoids*, and highlight some examples from the literature.

Sections 4 and 5 contain the main contents of this paper. In them, we define the notions of smooth groupoid *actions* and *-bundles*. For the latter we give a new notion of *principality*, generalising the notion of a principal Lie group(oid) bundle. This leads naturally to the definition of a *biprincipal bibundle*, and hence to our definition of *Morita equivalence*. The remainder of Section 5 is dedicated to a proof of Theorem 5.31.

In Section 6, we describe some *Morita invariants*, by generalising some well-known theorems from the Lie theory. We prove: the property of a diffeological groupoid to be *fibrating* is preserved under our notion of Morita equivalence; the *orbit spaces* of two Morita equivalent diffeological groupoids are diffeomorphic; and the categories of representations of two Morita equivalent diffeological groupoids are categorically equivalent.

Lastly, in Section 7, we discuss the question of diffeological Morita equivalence between Lie groupoids. We end the paper with the open Question 7.6, and some suggestions for future research.

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2. Diffeology

One of the main conveniences of *diffeology*[2] is that the category **Diffeol** of diffeological spaces and smooth maps (Definition 2.2) is complete, cocomplete, (locally) Cartesian closed, and in fact a quasitopos [BH11, Theorem 3.2]. This means that we can perform many categorical constructions that are unavailable in the category **Mnfd** of smooth manifolds. From these, the ones that are important for us are pullbacks and quotients. We discuss both of these explicitly below. The approach of diffeology has been compared to other theories of generalised smooth spaces in [Sta11; BIKW17]. For some historical remarks we refer to [IZ13b; IZ17] and [vdS20, Chapter I]. The main reference for this section is the textbook [IZ13a] by Iglesias-Zemmour, in which nearly all of the theory below is already developed.

Definition 2.1. *A* parametrisation *on a set X is a function $U \to X$ defined on an open subset $U \subseteq \mathbb{R}^m$ of Euclidean space, for arbitrary $m \in \mathbb{N}_{\geq 0}$. We denote by* $\mathrm{Param}(X)$ *the set of all parametrisations on X.*

The basic idea behind diffeology is that it determines which parametrisations are *'smooth'*, in such a way that it captures the properties of ordinary smooth functions on smooth manifolds. The precise definition is as follows:

Definition 2.2 (Axioms of Diffeology)**.** *Let X be a set. A* diffeology *on X is a collection of parametrisations $\mathcal{D}_X \subseteq \mathrm{Param}(X)$, containing what we call* plots*, satisfying the following three axioms:*

- (Covering) *Every constant parametrisation $U \to X$ is a plot.*
- (Smooth Compatibility) *For every plot $\alpha : U_\alpha \to X$ in $\mathcal{D}_X$ and every smooth function $h : V \to U_\alpha$ between open subsets of Euclidean space, we have that $\alpha \circ h \in \mathcal{D}_X$.*
- (Locality) *If $\alpha : U_\alpha \to X$ is a parametrisation, and $(U_i)_{i\in I}$ an open cover of U_α such that each restriction $\alpha|_{U_i}$ is a plot of X, then $\alpha \in \mathcal{D}_X$.*

[2] The etymology of the word is explained in the afterword to [IZ13a]. Souriau first used the term *"différentiel"*, as in 'differential' (from the Latin *differentia*, "difference"). Through a suggestion by Van Est, the name was later changed to *"difféologie,"* as in *"topologie"* ('topology', from the Ancient Greek *tópos*, "place," and *-(o)logy*, "study of"). Hence the term: diffeology.

A set X, paired with a diffeology: $(X, \mathcal{D}_X)$, is called a diffeological space*. Although, usually we shall just write X.*

A function $f : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ between diffeological spaces is called smooth *if for every plot $\alpha \in \mathcal{D}_X$ of X, the composition $f \circ \alpha \in \mathcal{D}_Y$ is a plot of Y. The set of all smooth functions between such diffeological spaces is denoted $C^\infty(X, Y)$, and smoothness is preserved by composition. The category of diffeological spaces and smooth maps is denoted by* $\mathbf{Diffeol}$*, and the isomorphisms in this category are called* diffeomorphisms*.*

Example 2.3. Any open subset $U \subseteq \mathbb{R}^m$ of Euclidean space, for $m \in \mathbb{N}_{\geq 0}$, gets a canonical diffeology $\mathcal{D}_U$, called the *Euclidean diffeology*. Its plots are the parametrisations that are smooth in the ordinary sense of the word. Similarly, we get a canonical diffeology $\mathcal{D}_M$ for any smooth manifold M, called the *manifold diffeology*. With respect to these diffeologies, the notion of smoothness defined in Definition 2.2 agrees with the ordinary one. Hence the inclusion functor $\mathbf{Mnfd} \hookrightarrow \mathbf{Diffeol}$ is fully faithful, and we can adopt the previous definition without causing any confusion.

Example 2.4. Any set X carries two canonical diffeologies. First, the largest diffeology, $\mathcal{D}_X^\bullet := \mathrm{Param}(X)$, called the *coarse diffeology*, containing all possible parametrisations. Letting $X^\bullet$ denote the diffeological space with the coarse diffeology, it is easy to see that every function $Z \to X^\bullet$ is smooth. On the other hand, the smallest diffeology on X is $\mathcal{D}_X^\circ$, containing all locally constant parametrisations. This is called the *discrete diffeology*. Similar to the above, we find that every function $X^\circ \to Y$ is smooth.

Example 2.5. For any two diffeological spaces X and Y, there is a natural diffeology on the space of smooth functions $C^\infty(X, Y)$ called the *standard functional diffeology* [IZ13a, Article 1.57]. It is the smallest diffeology that makes the evaluation map $(f, x) \mapsto f(x)$ smooth. With these diffeologies, $\mathbf{Diffeol}$ becomes Cartesian closed.

2.1 Generating families

The Axiom of Locality in Definition 2.2 ensures that the smoothness of a parametrisation, or of a function between diffeological spaces, can be checked locally. This allows us to introduce the following notions, which will help us study interesting constructions, and will often simplify proofs.

Definition 2.6. *Consider a family $\mathcal{F} \subseteq \mathrm{Param}(X)$ of parametrisations on X. There exists a smallest diffeology on X that contains $\mathcal{F}$. We denote this diffeology by $\langle \mathcal{F} \rangle$, and call it the* diffeology generated by $\mathcal{F}$. *If $\mathcal{D}_X = \langle \mathcal{F} \rangle$, we say $\mathcal{F}$ is a* generating family *for $\mathcal{D}_X$. The elements of $\mathcal{F}$ are called* generating plots.

The plots of the diffeology generated by $\mathcal{F}$ are characterised as follows: a parametrisation $\alpha : U_\alpha \to X$ is a plot in $\langle \mathcal{F} \rangle$ if and only if α is locally either constant, or factors through elements of $\mathcal{F}$. Concretely, this means that for all $t \in U_\alpha$ there exists an open neighbourhood $t \in V \subseteq U_\alpha$ such that $\alpha|_V$ is either constant, or of the form $\alpha|_V = F \circ h$, where $F : W \to X$ is an element in $\mathcal{F}$, and $h : V \to W$ is a smooth function between open subsets of Euclidean space. When the family $\mathcal{F}$ is *covering*, in the sense that $\bigcup_{F \in \mathcal{F}} \mathrm{im}(F) = X$, then the condition for $\alpha|_V$ to be constant becomes redundant, and the plots in $\langle \mathcal{F} \rangle$ are locally just of the form $\alpha|_V = F \circ h$.

The main use of this construction is that we may encounter families of parametrisations that are not quite diffeologies, but that contain functions that we nevertheless want to be smooth. On the other hand, calculations may sometimes be simplified by finding a suitable generating family for a given diffeology. This simplification lies in the following result, saying that smoothness has only to be checked on generating plots:

Proposition 2.7. *Let $f : X \to Y$ be a function between diffeological spaces, such that $\mathcal{D}_X$ is generated by some family $\mathcal{F}$. Then f is smooth if and only if for all $F \in \mathcal{F}$ we have $f \circ F \in \mathcal{D}_Y$.*

Example 2.8. The *wire diffeology* (called the *spaghetti diffeology* by Souriau) is the diffeology $\mathcal{D}_{\mathrm{wire}}$ on $\mathbb{R}^2$ generated by $C^\infty(\mathbb{R}, \mathbb{R}^2)$. The resulting diffeological space is not diffeomorphic to the ordinary $\mathbb{R}^2$, since the identity map $\mathrm{id}_{\mathbb{R}^2} : (\mathbb{R}^2, \mathcal{D}_{\mathbb{R}^2}) \to (\mathbb{R}^2, \mathcal{D}_{\mathrm{wire}})$ is not smooth.

Example 2.9. The charts of a smooth atlas on a manifold define a generating family for the manifold diffeology from Example 2.3. Since a manifold may have many atlases, this shows that similarly any diffeology may have many generating families.

2.2 Quotients

We use the terminology from Section 2.1 to define a natural diffeology on a quotient $X/{\sim}$. This question relates to a more general one: given a function $f : X \to Y$, and a diffeology $\mathcal{D}_X$ on the domain, what is the smallest diffeology on Y such that f remains smooth? The following provides an answer:

Definition 2.10. *Let $f : X \to Y$ be a function between sets, and let $\mathcal{D}_X$ be a diffeology on X. The* pushforward diffeology *on Y is the diffeology $f_*(\mathcal{D}_X) := \langle f \circ \mathcal{D}_X \rangle$, where $f \circ \mathcal{D}_X$ is the family of parametrisations of the form $f \circ \alpha$, for $\alpha \in \mathcal{D}_X$. The pushforward diffeology is the smallest diffeology on Y that makes f smooth.*

We can now use this to define a natural diffeology on a quotient space:

Definition 2.11. *Let X be a diffeological space, and let $\sim$ be an equivalence relation on the set X. We denote the equivalence classes by $[x]$. The* quotient *$X/{\sim}$ is the collection of all equivalence classes, and comes with a* canonical projection map *$p : X \to X/{\sim}$, which sends $x \mapsto [x]$. The* quotient diffeology *on $X/{\sim}$ is defined as the pushforward diffeology $p_*(\mathcal{D}_X)$ of $\mathcal{D}_X$ along the canonical projection map. Naturally, with respect to this diffeology, the canonical projection map becomes smooth.*

The quotient diffeology will be used extensively, where the equivalence relation will often be defined by the orbits of a group(oid) action, or as the fibres of some smooth surjection. The existence of the quotient diffeology for arbitrary quotients should be contrasted to the situation for smooth manifolds, where quotients often carry no natural differentiable structure at all, but where instead one could appeal to the *Godement criterion* ([Ser65, Theorem 2, p. 92]). The following is an example of a quotient that does not exist as a smooth manifold, but whose diffeological structure is still quite rich:

Example 2.12. The *irrational torus* is the diffeological space defined by the quotient of $\mathbb{R}$ by an additive subgroup: $T_\theta := \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z})$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is an arbitrary irrational number. Equivalently, it can be described as the leaf space of the Kronecker foliation on the 2-torus with irrational slope. The topology of this quotient contains only the two trivial open sets, yet its quotient diffeology is non-trivial[3]. They were first classified in [DI83], whose result is (amazingly) directly analogous to the classification of the irrational rotation algebras [Rie81]. This example is treated in detail in [vdS20, Section 2.3].

2.3 Fibred products

The second construction we need is that of *fibred products*, which are the pullbacks in the category **Diffeol**. Recall that if $f : X \to Z$ and $g : Y \to Z$ are two functions between sets with a common codomain, then the fibred product of sets is (up to unique bijection)

$$X \times_Z^{f,g} Y := \{(x, y) \in X \times Y : f(x) = g(y)\}.$$

When each set is equipped with a diffeology, we shall construct a diffeology on the fibred product in two steps. First we describe a natural diffeology on the product $X \times Y$, and then show how this descends to a diffeology on the fibred product as a subset.

Definition 2.13. *Let X and Y be two diffeological spaces. The* product diffeology *on the Cartesian product $X \times Y$ is defined as*

$$\mathcal{D}_{X\times Y} := \langle \mathcal{D}_X \times \mathcal{D}_Y \rangle,$$

where $\mathcal{D}_X \times \mathcal{D}_Y$ is the family of parametrisations of the form $\alpha_1 \times \alpha_2$, for $\alpha_1 \in \mathcal{D}_X$ and $\alpha_2 \in \mathcal{D}_Y$. The plots in $\mathcal{D}_{X\times Y}$ are exactly the parametrisations $\alpha : U_\alpha \to X \times Y$ such that $\mathrm{pr}_1 \circ \alpha$ and $\mathrm{pr}_2 \circ \alpha$ are plots of X and Y, respectively. We assume that products are always furnished with their product diffeologies.

It is clear that both projection maps pr_1 and pr_2 are smooth with respect to the product diffeology. The smooth functions into $X \times Y$ behave exactly as one would expect, where $f : A \to X \times Y$ is smooth if and only if the components $f_1 = \mathrm{pr}_1 \circ f$ and $f_2 = \mathrm{pr}_2 \circ f$ are smooth.

Next we define how the diffeology on a set X transfers to any of its subsets:

[3] This shows that there are meaningful notions of smooth space that do not rely on the regnant philosophy of "smooth space = topological space + extra structure."

Definition 2.14. *Consider a diffeological space X, and an arbitrary subset $A \subseteq X$. Let $i_A : A \hookrightarrow X$ denote the natural inclusion map. The* subset diffeology *on A is defined as*

$$\mathcal{D}_{A \subseteq X} := \{\alpha \in \mathrm{Param}(A) : i_A \circ \alpha \in \mathcal{D}_X\}.$$

That is, α is a plot of A if and only if, when seen as a parametrisation of X, it is also a plot. We assume that a subset of a diffeological space is always endowed with its subset diffeology.

Since the fibred product $X \times_Z^{f,g} Y$ is a subset of the product $X \times Y$, the following definition is a natural combination of Definitions 2.13 and 2.14:

Definition 2.15. *Let $f : X \to Z$ and $g : Y \to Z$ be two smooth maps between diffeological spaces. The* fibred product diffeology $\mathcal{D}_{X \times_Z^{f,g} Y}$ *on the set $X \times_Z^{f,g} Y$ is the subset diffeology it gets from the product diffeology on $X \times Y$. Concretely:*

$$\mathcal{D}_{X \times_Z^{f,g} Y} = \{\alpha \in \mathcal{D}_{X \times Y} : f \circ \alpha_1 = g \circ \alpha_2\}.$$

That is, the plots of the fibred product are just plots of $X \times Y$, whose components satisfy an extra condition. We assume that all fibred products are equipped with their fibred product diffeologies.

2.4 Subductions

Subductions are a special class of smooth functions that generalise the notion of surjective submersion from the theory of smooth manifolds. Since there is no unambiguous notion of tangent space in diffeology (cf. [CW16]), the definition looks somewhat different. For (more) detailed proofs of the results in this section, we refer to [IZ13a, Article 1.46] and surrounding text, and [vdS20, Section 2.6].

Definition 2.16. *A surjective function $f : X \to Y$ between diffeological spaces is called a* subduction *if $f_*(\mathcal{D}_X) = \mathcal{D}_Y$. Note that subductions are automatically smooth.*

In the case that f is a subduction, since it is then particularly a surjection, the family of parametrisations $f \circ \mathcal{D}_X$ is covering, and hence the plots of $\mathcal{D}_Y$

are all locally of the form $f \circ \alpha$, where $\alpha \in \mathcal{D}_X$. In other words, f is a subduction if and only if f is smooth and the plots of Y can locally be *lifted* along f to plots of X:

Lemma 2.17. *Let $f : X \to Y$ be a function between diffeological spaces. Then f is a subduction if and only if the following two conditions are satisfied:*

1. *The function f is smooth.*
2. *For every plot $\alpha : U_\alpha \to Y$, and any point $t \in U_\alpha$, there exists an open neighbourhood $t \in V \subseteq U_\alpha$ and a plot $\beta : V \to X$, such that $\alpha|_V = f \circ \beta$.*

Since many of the functions we encounter will naturally be smooth already, the notion of subductiveness is effectively captured by condition *(2)* in this lemma. This can also be seen in the following simple example:

Example 2.18. Consider the product $X \times Y$ of two diffeological spaces X and Y. The projection maps pr_1 and pr_2 are both subductions.

Example 2.19. For a surjective function $\pi : X \to B$ we get an equivalence relation on X, where two points are identified if and only if they inhabit the same π-fibre. The equivalence classes are exactly the π-fibres themselves. We denote the quotient set of this equivalence relation by X/π, and equip it with the quotient diffeology whenever X is a diffeological space. If π is a subduction, then there is a diffeomorphism $B \cong X/\pi$ [IZ13a, Article 1.52].

For subsequent use, we state here some useful properties of subductions with respect to composition:

Lemma 2.20. *We have the following properties for subductions:*

1. *If f and g are two subductions, then the composition $f \circ g$ is a subduction as well.*
2. *Let $f : Y \to Z$ and $g : X \to Y$ be two smooth maps such that the composition $f \circ g$ is a subduction. Then so is f.*
3. *Let $\pi : X \to B$ be a subduction, and $f : B \to Y$ an arbitrary function. Then f is smooth if and only if $f \circ \pi$ is smooth. In fact, f is a subduction if and only if $f \circ \pi$ is a subduction.*

Proof. (1) This is [IZ13a, Article 1.47].

(2) Assume $f : Y \to Z$ and $g : X \to Y$ are smooth, such that $f \circ g$ is a subduction. Take a plot $\alpha : U_\alpha \to Z$. Since the composition is a subduction, for every $t \in U_\alpha$ we can find an open neighbourhood $t \in V \subseteq U_\alpha$ and a plot $\beta : V \to X$ such that $\alpha|_V = (f \circ g) \circ \beta$. Since g is smooth, we get a plot $g \circ \beta \in \mathcal{D}_Y$, which is a local lift of α along f. The result follows by Lemma 2.17.

(3) If f is smooth, it follows immediately that $f \circ \pi$ is smooth. Suppose now that $f \circ \pi$ is smooth. We need to show that f is smooth. For that, take a plot $\alpha : U_\alpha \to B$. Since π is a subduction, we can find an open cover $(V_t)_{t \in U_\alpha}$ of U_α together with a family of plots $\beta_t : V_t \to X$ such that $\alpha|_{V_t} = \pi \circ \beta_t$. It follows that each restriction $f \circ \alpha|_{V_t} = f \circ \pi \circ \beta_t$ is smooth, and by the Axiom of Locality it follows that $f \circ \alpha \in \mathcal{D}_Y$, and hence that f is smooth. The claim about when f is a subduction follows from *(2)*. □

We also collect the following noteworthy claim:

Proposition 2.21 ([IZ13a, Article 1.49])**.** *An injective subduction is a diffeomorphism.*

We recall now some elementary results on the interaction between subductions and fibred products, as obtained in [vdS20, Section 2.6]. We point out that if f is a subduction, an arbitrary restriction $f|_A$ may no longer be a subduction. We know from Example 2.18 that the second projection map pr_2 of a product $X \times Y$ is a subduction, but it is not always the case that the restriction of this projection to a fibred product $X \times_Z^{f,g} Y$ is a subduction as well. The following result shows that, to ensure this, it suffices to assume that f is a subduction:

Lemma 2.22. *Let $f : X \to Z$ be a subduction, and let $g : Y \to Z$ be a smooth map. Then the restricted projection map*

$$\mathrm{pr}_2|_{X \times_Z^{f,g} Y} : X \times_Z^{f,g} Y \longrightarrow Y$$

is also a subduction. In other words, in **Diffeol***, subductions are preserved under pullback.*

Proof. Consider a plot $\alpha : U_\alpha \to Y$. By composition, this gives another plot $g \circ \alpha \in \mathcal{D}_Z$. Now, since f is a subduction, for every $t \in U_\alpha$ we can find a plot $\beta : V \to X$ defined on an open neighbourhood $t \in V \subseteq U_\alpha$ such that $g \circ \alpha|_V = f \circ \beta$. This gives a plot $(\beta, \alpha|_V) : V \to X \times_Z Y$ that satisfies $\mathrm{pr}_2|_{X \times_Z Y} \circ (\beta, \alpha|_V) = \alpha|_V$. The result follows by Lemma 2.17. $\square$

The next result shows how two subductions interact with fibred products:

Lemma 2.23. *Consider the following two commuting triangles of diffeological spaces and smooth maps:*

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & Y_1 \\ & {\scriptstyle r}\searrow \quad \swarrow{\scriptstyle R} & \\ & A & \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} X_2 & \xrightarrow{g} & Y_2 \\ & {\scriptstyle l}\searrow \quad \swarrow{\scriptstyle L} & \\ & A, & \end{array}$$

where both f and g are subductions. Then the map

$$(f \times g)|_{X_1 \times_A X_2} : X_1 \times_A^{r,l} X_2 \longrightarrow Y_1 \times_A^{R,L} Y_2; \qquad (x_1, x_2) \longmapsto (f(x_1), g(x_2))$$

is also a subduction.

Proof. Clearly $f \times g$ is smooth, so we are left to show that the second condition in Lemma 2.17 is fulfilled. For that, take a plot

$$(\alpha_1, \alpha_2) : U \longrightarrow Y_1 \times_A^{R,L} Y_2,$$

i.e., we have two plots $\alpha_1 \in \mathcal{D}_{Y_1}$ and $\alpha_2 \in \mathcal{D}_{Y_2}$ such that $R \circ \alpha_1 = L \circ \alpha_2$. Now fix a point $t \in U$ in the domain. Then since both f and g are subductive, we can find two plots $\beta_1 : U_1 \to X_1$ and $\beta_2 : U_2 \to X_2$, defined on open neighbourhoods of $t \in U$, such that $\alpha_1|_{U_1} = f \circ \beta_1$ and $\alpha_2|_{U_2} = g \circ \beta_2$. Now the plot

$$(\beta_1|_{U_1 \cap U_2}, \beta_2|_{U_1 \cap U_2}) : U_1 \cap U_2 \longrightarrow X_1 \times X_2$$

takes values in the fibred product because

$$r \circ \beta_1|_{U_2} = R \circ f \circ \beta_1|_{U_2} = R \circ \alpha_1|_{U_1 \cap U_2} = L \circ \alpha_2|_{U_1 \cap U_2} = l \circ \beta_2|_{U_1},$$

and we see that it lifts $(\alpha_1, \alpha_2)|_{U_1 \cap U_2}$ along $f \times g$. $\square$

By setting $A = \{*\}$ to be the one-point space, this lemma gives in particular that the product $f \times g$ of two subductions is again a subduction.

To end this section, we should also mention the existence of the notion of a *local subduction* (or *strong subduction*):

Definition 2.24. *A smooth surjection* $f : X \to Y$ *is called a* local subduction *if for each* $y \in Y$*, each* $x \in f^{-1}(\{y\})$*, and any* pointed plot *of the form* $\alpha : (U_\alpha, 0) \to (Y, y)$*, there exists another pointed plot* $\beta : (V, 0) \to (X, x)$*, defined on an open neighbourhood* $0 \in V \subseteq U_\alpha$*, such that* $\alpha|_V = f \circ \beta$*.*

Compare this to a definition of a subduction, where in general the plot β does not have to hit the point x in the domain of f. Note also that *local* subduction does not mean *locally a subduction everywhere*.

Proposition 2.25 ([IZ13a, Article 2.16])**.** *The local subductions between smooth manifolds are exactly the surjective submersions.*

Due to the above proposition, the notion of a local subduction will be of interest when studying Lie groupoids in the framework of diffeological Morita equivalence we develop below. See Section 7.1.

3. Diffeological Groupoids

We assume that the reader is familiar with the definition of a (Lie) groupoid. A textbook reference for that theory is [Mac05]. To fix our notation, we give here an informal description of a set-theoretic groupoid. A *groupoid* consists of two sets: G_0 and G, together with five *structure maps*. A groupoid will be denoted $G \rightrightarrows G_0$, or just G. Here G_0 is the set of objects of the groupoid, and G is the set of arrows. The five structure maps are

1. The *source map* $\mathrm{src} : G \to G_0$,
2. The *target map* $\mathrm{trg} : G \to G_0$,
3. The *unit map* $\mathrm{u} : G_0 \to G$, mapping $x \mapsto \mathrm{id}_x$,
4. The *inversion map* $\mathrm{inv} : G \to G$, mapping $g \mapsto g^{-1}$,
5. And the *composition*:

$$\mathrm{comp} : G \times_{G_0}^{\mathrm{src},\mathrm{trg}} G \longrightarrow G; \qquad (g, h) \mapsto g \circ h.$$

The composition is associative, and the identities and inverses behave as such. We say $G \rightrightarrows G_0$ is a *Lie groupoid* if both G and G_0 are smooth manifolds such that the source and target maps are submersions, and each of the other structure maps are smooth. The definition of a diffeological groupoid is a straightforward generalisation of this:

Definition 3.1. *A* diffeological groupoid *is a groupoid internal to the category of diffeological spaces. Concretely, this means that it is a groupoid $G \rightrightarrows G_0$ such that the object space G_0 and arrow space G are endowed with diffeologies that make all of the structure maps smooth.*

As diffeology subsumes smooth manifolds, so do diffeological groupoids capture Lie groupoids. Note the main difference with the definition of a Lie groupoid is that we put no extra assumptions on the source and target maps, whereas to make sense of the composition in Lie groupoids we need $G \times_{G_0}^{\mathrm{src},\mathrm{trg}} G$ to be a smooth manifold, for which it suffices to assume the source and target maps are submersions. We do, however, have:

Proposition 3.2. *The source and target maps of a diffeological groupoid are subductions.*

Proof. The smooth structure map $\mathrm{u} : G_0 \to G$, sending each object to its identity arrow, is a global smooth section of the source map, and hence by Lemma 2.20*(2)* the source map must be a subduction. Since the inversion map is a diffeomorphism, it follows that the target map is a subduction as well. □

Definition 3.3. *Let $G \rightrightarrows G_0$ be a diffeological groupoid. The* isotropy group *at $x \in G_0$ is the collection G_x consisting of all arrows in G from and to x:*

$$G_x := \mathrm{Hom}_G(x, x) = \mathrm{src}^{-1}(\{x\}) \cap \mathrm{trg}^{-1}(\{x\}).$$

Definition 3.4. *Let $G \rightrightarrows G_0$ be a diffeological groupoid. The* orbit *of an object $x \in G_0$ is defined as*

$$\mathrm{Orb}_G(x) := \{y \in G_0 : \exists x \xrightarrow{g} y\} = \mathrm{trg}(\mathrm{src}^{-1}(\{x\})).$$

The orbit space *of the groupoid is the space G_0/G consisting of these orbits. We furnish the orbit space with the quotient diffeology from Definition 2.11, so that $\mathrm{Orb}_G : G_0 \to G_0/G$ is a subduction.*

The orbit space of a Lie groupoid is not necessarily (canonically) a smooth manifold. The flexibility of diffeology allows us to study the smooth structure of orbit spaces of all diffeological groupoids. Below we give some examples of diffeological groupoids.

Example 3.5. Let X be a diffeological space, and let R be an equivalence relation on X. We define the *relation groupoid* $X \times_R X \rightrightarrows X$ as follows. The space of arrows consists of exactly those pairs $(x, y) \in X \times X$ such that xRy. With the composition $(z, y) \circ (y, x) := (z, x)$, this becomes a diffeological groupoid. The orbit space $X/(X \times_R X)$ is just the quotient X/R. When X is a smooth manifold, the relation groupoid becomes a Lie groupoid (even when the quotient is not a smooth manifold).

Example 3.6. Let $G \rightrightarrows G_0$ be a diffeological groupoid. We can then consider the subgroupoid of G that only consists of elements in isotropy groups:

$$I_G := \bigcup_{x \in G_0} G_x \subseteq G.$$

This becomes a diffeological groupoid $I_G \rightrightarrows G_0$ called the *isotropy groupoid*. This has been studied in [Bos07, Example 2.1.9] in the context of Lie groupoids. Note that if $G \rightrightarrows G_0$ is a Lie groupoid, then generally I_G is not a submanifold of G, so the isotropy groupoid may no longer be a Lie groupoid.

Example 3.7. The *thin fundamental groupoid* (or *path groupoid*) $\Pi^{\text{thin}}(M)$ of any smooth manifold M is a diffeological groupoid [CLW16, Proposition A.25].

Example 3.8. The *groupoid of* Σ*-evolutions* of a Cauchy surface is a diffeological groupoid [Gł19, Section II.2.2].

Example 3.9. For any smooth surjection $\pi : X \to B$ between diffeological spaces, the fibres $X_b := \pi^{-1}(\{b\})$ get the subset diffeology from X. We then have a diffeological groupoid $\mathbf{G}(\pi) \rightrightarrows B$ called the *structure groupoid*, whose space of arrows is defined as

$$\mathbf{G}(\pi) := \bigcup_{a,b \in B} \operatorname{Diff}(X_a, X_b).$$

Structure groupoids play an important rôle in the theory of diffeological fibre bundles [IZ13a, Chapter 8]. In general, they are too big to be Lie groupoids. They also generalise the notion of a *frame groupoid* for a smooth vector bundle. Related to this, in [vdS20, Section 3.4] structure groupoids are used to define a notion of *smooth linear representations* for diffeological groupoids.

Example 3.10. If we are given a diffeological space X, the *germ groupoid* $\mathbf{Germ}(X) \rightrightarrows X$ consists of all *germs* of local diffeomorphisms on X. Even if X itself is a smooth manifold, this is generally not a Lie groupoid. Germ groupoids are used in [IZL18; IZP20]. A detailed construction of the diffeological structure of this groupoid appears in [vdS20, Section 6.1].

4. Diffeological Groupoid Actions and -Bundles

In the following two sections we generalise the theory of Lie groupoid bibundles to the diffeological setting. The development we present here (as in [vdS20, Chapter IV]) is analogous to the development of the Lie version, save that we need to find a suitable replacement for the notion of a surjective submersion. Some of the proofs from the Lie theory can be performed almost *verbatim* in our setting. These proofs already appear in the literature in various places: [Blo08; dHo12; Lan01a; MM05], and also in the different setting of [MZ15]. We adopt many definitions and proofs from those sources, and point out how the diffeological theory subtly differs from the Lie theory. This difference mainly stems from the existence of quotients and fibred products of diffeological spaces, whereas in the Lie theory more care has to be taken. Ultimately, this extra care is what leads to a restricted Morita theorem for Lie groupoids, whereas the diffeological theorem is more general. In this section specifically we introduce *diffeological groupoid actions* and *-bundles*, two notions that form the ingredients for the main theory on bibundles.

4.1 Diffeological groupoid actions

The most basic notion for the upcoming theory is that of a *groupoid action*. For diffeological groupoids, the definition is the same as for Lie groupoids:

Definition 4.1. *Take a diffeological groupoid $G \rightrightarrows G_0$, and a diffeological space X. A* smooth left groupoid action *of G on X* along *a smooth map $l_X : X \to G_0$ is a smooth function*

$$G \times_{G_0}^{\mathrm{src}, l_X} X \longrightarrow X; \qquad (g, x) \longmapsto g \cdot x,$$

satisfying the following three conditions:

1. *For* $g \in G$ *and* $x \in X$ *such that* $\mathrm{src}(g) = l_X(x)$ *we have* $l_X(g \cdot x) = \mathrm{trg}(g)$.
2. *For every* $x \in X$ *we have* $\mathrm{id}_{l_X(x)} \cdot x = x$.
3. *We have* $h \cdot (g \cdot x) = (h \circ g) \cdot x$ *whenever defined, i.e. when* $\mathrm{src}(g) = l_X(x)$ *and the arrows are composable.*

The smooth map $l_X : X \to G_0$ *is called the* left moment map. *In-line, we denote an action by* $G \curvearrowright^{l_X} X$. *To save space, we may write* $(g, x) \mapsto gx$ *instead.*

Right actions are defined similarly: a smooth right groupoid action *of* G *on* X along $r_X : X \to G_0$ *is a smooth map*

$$X \times_{G_0}^{r_X,\mathrm{trg}} G \longrightarrow X; \qquad (x, g) \longmapsto xg,$$

satisfying $r_X(xg) = \mathrm{src}(g)$, $x \cdot \mathrm{id}_{r_X(x)} = x$ *and* $(x \cdot g) \cdot h = x \cdot (g \circ h)$ *whenever defined. Note how the rôle of the source and target maps are switched with respect to the definition of a left action. Right actions will be denoted by* $X\, {}^{r_X}\!\curvearrowleft G$, *and* r_X *is called the* right moment map.

Example 4.2. Any diffeological groupoid $G \rightrightarrows G_0$ acts on its own arrow space from the left and right by composition, which gives actions $G \curvearrowright^{\mathrm{trg}} G$ and $G\, {}^{\mathrm{src}}\!\curvearrowleft G$ that are both defined by $(g, h) \mapsto g \circ h$.

Definition 4.3. *The* orbit *of a point* $x \in X$ *in the space of an action* $G \curvearrowright^{l_X} X$ *is defined as*

$$\mathrm{Orb}_G(x) := \{gx : g \in \mathrm{src}^{-1}(\{l_X(x)\})\}.$$

The quotient space *(or* orbit space*) of the action is defined as the collection of all orbits, and denoted* X/G. *With the quotient diffeology, the* orbit projection map $\mathrm{Orb}_G : X \to X/G$ *becomes a subduction.*

The following gives a notion of morphism between actions:

Definition 4.4. *Consider two smooth groupoid actions* $G \curvearrowright^{l_X} X$ *and* $G \curvearrowright^{l_Y} Y$. *A smooth map* $\varphi : X \to Y$ *is called* G-equivariant *if* $l_X = l_Y \circ \varphi$ *and it commutes with the actions whenever defined:* $\varphi(gx) = g\varphi(x)$.

Definition 4.5. *The* (smooth left) action category $\mathbf{Act}(G \rightrightarrows G_0)$ *of a diffeological groupoid* $G \rightrightarrows G_0$ *is the category consisting of smooth left actions* $G \curvearrowright^{l_X} X$ *as objects, and G-equivariant maps as morphisms. This forms the analogue of the category of (left) modules from ring theory. We show in Section 6.3 that the action category is in some sense a Morita invariant.*

4.1.1 The balanced tensor product

We now give an important construction that will later allow us to define the *composition* of bibundles.

Construction 4.6. Consider a diffeological groupoid $H \rightrightarrows H_0$, with a smooth left action $H \curvearrowright^{l_Y} Y$ and a smooth right action $X \,^{r_X}\!\curvearrowleft H$. On the fibred product $X \times_{H_0}^{r_X, l_Y} Y$ we define the following smooth right H-action. The moment map is $R := r_X \circ \mathrm{pr}_1|_{X \times_{H_0} Y} = l_Y \circ \mathrm{pr}_2|_{X \times_{H_0} Y}$, and the action is given by:

$$\left(X \times_{H_0}^{r_X, l_Y} Y\right) \times_{H_0}^{R, \mathrm{trg}} H \longrightarrow X \times_{H_0}^{r_X, l_Y} Y; \qquad ((x,y),h) \longmapsto (x \cdot h, h^{-1} \cdot y).$$

It is clear that this action is also smooth, and we call it the *diagonal H-action*. The *balanced tensor product* is the diffeological space defined as the orbit space of this smooth groupoid action:

$$X \otimes_H Y := \left(X \times_{H_0}^{r_X, l_Y} Y\right) / H.$$

The orbit of a pair of points (x, y) in the balanced tensor product will be denoted $x \otimes y$. Whenever we encounter a term of the form $x \otimes y \in X \otimes_H Y$, we assume that it is well defined, i.e. $r_X(x) = l_Y(y)$. The terminology is explained by the following useful identity:

$$xh \otimes y = x \otimes hy.$$

In the literature on Lie groupoids, this space is sometimes called the *Hilsum-Skandalis tensor product*, named after a construction appearing in [HS87].

We note that this marks the first difference with the development of the Lie theory of bibundles and Morita equivalence. There, the balanced tensor product can only be defined when both $X \times_{H_0}^{r_X, l_Y} Y$ and the quotient by the diagonal H-action are smooth manifolds. This is usually only done after (bi)bundles are defined, and some principality conditions are presupposed. The principality then exactly ensures the existence of canonical differentiable structures on the fibred product and quotient. Here, the flexibility of diffeology allows us to define the balanced tensor product in an earlier stage of the development, and we do so to demonstrate this conceptual difference.

4.2 Diffeological groupoid bundles

A groupoid bundle is a smooth map, whose domain carries a groupoid action, such that the fibres of the map are preserved by this action:

Definition 4.7. *A* smooth left diffeological groupoid bundle *is a smooth left groupoid action* $G \curvearrowright^{l_X} X$ *together with a* G-invariant *smooth map* $\pi : X \to B$. *We denote such bundles by* $G \curvearrowright^{l_X} X \xrightarrow{\pi} B$*, and also call them* (left) G-bundles. Right *bundles are defined similarly, and denoted* $B \xleftarrow{\pi} X \, {}^{r_X}\!\curvearrowleft G$.

The next definition gives a notion of morphism between bundles:

Definition 4.8. *Take two left* G*-bundles* $G \curvearrowright^{l_X} X \xrightarrow{\pi_X} B$ *and* $G \curvearrowright^{l_Y} Y \xrightarrow{\pi_Y} B$ *over the same base. A* G-bundle morphism *is a* G*-equivariant smooth map* $\varphi : X \to Y$ *such that* $\pi_X = \pi_Y \circ \varphi$. *We make a similar definition for right bundles.*

In order to define Morita equivalence, we need to define a notion of when a bundle is *principal*. For Lie groupoid bundles, these generalise the ordinary notion of smooth principal bundles of Lie groups and manifolds. That definition involves the notion of a surjective submersion. As we have mentioned, this notion needs to be generalised to diffeology. Proposition 2.25 suggests that we could take *local subductions*, since they directly generalise the surjective submersions. However, it turns out that *subductions* behave sufficiently like submersions for the theory to work. The following definition then generalises the fact that the underlying bundle of a principal Lie groupoid bundle has to be a submersion:

Definition 4.9. *A diffeological groupoid bundle* $G \curvearrowright^{l_X} X \xrightarrow{\pi} B$ *is called* subductive *if the underlying map* $\pi : X \to B$ *is a subduction.*

The following generalises the fact that the action of a principal Lie groupoid bundle has to be free and transitive on the fibres:

Definition 4.10. *A diffeological groupoid bundle* $G \curvearrowright^{l_X} X \xrightarrow{\pi} B$ *is called* pre-principal *if the* action map $A_G : G \times_{G_0}^{\mathrm{src}, l_X} X \to X \times_B^{\pi,\pi} X$ *mapping* $(g, x) \mapsto (gx, x)$ *is a diffeomorphism.*

Combining these two:

Definition 4.11. *A diffeological groupoid bundle is called* principal *if it is both subductive and pre-principal.*

This definition serves as our generalisation of principal Lie groupoid bundles, cf. [Blo08, Definition 2.10] and [dHo12, Section 3.6]. Clearly any principal Lie groupoid bundle in the sense described in those references is also a principal diffeological groupoid bundle. Note that in the Lie theory, most constructions (such as the balanced tensor product) depend on the submersiveness of the underlying bundle map, so it makes little sense to define pre-principality for Lie groupoids. However, as we have already seen, in the diffeological case these constructions can be carried out more generally, and this will allow us to see what parts of the development of the theory depend on either the subductiveness or pre-principality of the bundles, rather than on full principality. In our development of the theory, some proofs can therefore be performed separately, whereas in the Lie theory they have to be performed at once. We hope this makes for clearer exposition.

Note also that when a bundle $G \curvearrowright^{l_X} X \xrightarrow{\pi} B$ is pre-principal, the action map induces a diffeomorphism $X/\pi \cong X/G$, and when the bundle is subductive, Example 2.19 gives a diffeomorphism $B \cong X/\pi$. For a principal bundle we therefore have $B \cong X/G$.

Example 4.12. The action of any diffeological groupoid $G \rightrightarrows G_0$ on its own arrow space (Example 4.2) forms a bundle $G \curvearrowright^{\mathrm{trg}} G \xrightarrow{\mathrm{src}} G_0$. From Proposition 3.2 it follows that this bundle is principal.

4.2.1 The division map of a pre-principal bundle

The material in this section is similar to [Blo08, Section 3.1] for Lie groupoids. If a bundle $G \curvearrowright^{l_X} X \xrightarrow{\pi} B$ is pre-principal, the fact that the action map is bijective gives that the action $G \curvearrowright^{l_X} X$ has to be *free*, and *transitive* on the π-fibres. This means that for every two points $x, y \in X$ such that $\pi(x) = \pi(y)$, there exists a *unique* arrow $g \in G$ such that $gy = x$. We denote this arrow by $\langle x, y \rangle_G$, and the map $\langle \cdot, \cdot \rangle_G$ is called the *division map*[4]:

[4]The notational resemblance to an inner-product is not accidental. The division map plays a very similar rôle to the inner product of a Hilbert C*-module. For more on this analogy, see [Blo08, Section 3].

Definition 4.13. *Let $G \curvearrowright^{l_X} X \xrightarrow{\pi} B$ be a pre-principal G-bundle, and let A_G denote its action map. Then the* division map *associated to this bundle is the smooth map*

$$\langle \cdot, \cdot \rangle_G : X \times_B^{\pi,\pi} X \xrightarrow{A_G^{-1}} G \times_{G_0}^{\mathrm{src}, l_X} X \xrightarrow{\mathrm{pr}_1|_{G \times_{G_0} X}} G.$$

We summarise some algebraic properties of the division map that will be used in our proofs throughout later sections. The proofs are straightforward, and use the uniqueness property described above.

Proposition 4.14. *Let $G \curvearrowright^{l_X} X \xrightarrow{\pi} B$ be a pre-principal G-bundle. Its division map $\langle \cdot, \cdot \rangle_G$ satisfies the following properties:*

1. *The source and targets are* $\mathrm{src}(\langle x_1, x_2 \rangle_G) = l_X(x_2)$ *and* $\mathrm{trg}(\langle x_1, x_2 \rangle_G) = l_X(x_1)$.
2. *The inverses are given by $\langle x_1, x_2 \rangle_G^{-1} = \langle x_2, x_1 \rangle_G$.*
3. *For every $x \in X$ we have $\langle x, x \rangle_G = \mathrm{id}_{l_X(x)}$.*
4. *Whenever well-defined, we have $\langle g x_1, x_2 \rangle_G = g \circ \langle x_1, x_2 \rangle_G$.*

Proposition 4.15. *Let $\varphi : X \to Y$ be a bundle morphism between two pre-principal G-bundles $G \curvearrowright^{l_X} X \xrightarrow{\pi_X} B$ and $G \curvearrowright^{l_Y} Y \xrightarrow{\pi_Y} B$. Denoting the division maps of these bundles respectively by $\langle \cdot, \cdot \rangle_G^X$ and $\langle \cdot, \cdot \rangle_G^Y$, we have for all $x_1, x_2 \in X$ in the same π_X-fibre that:*

$$\langle x_1, x_2 \rangle_G^X = \langle \varphi(x_1), \varphi(x_2) \rangle_G^Y.$$

Proof. Observe that $\langle \varphi(x_1), \varphi(x_2) \rangle_G^Y$ is the unique arrow that satisfies $\langle \varphi(x_1), \varphi(x_2) \rangle_G^Y \varphi(x_2) = \varphi(x_1)$. However, by G-equivariance we get $\varphi(x_1) = \varphi\left(\langle x_1, x_2 \rangle_G^X x_2\right) = \langle x_1, x_2 \rangle_G^X \varphi(x_2)$, from which the claim immediately follows. □

4.2.2 Invertibility of G-bundle morphisms

We now prove a result that generalises the fact that morphisms between principal Lie group bundles are always diffeomorphisms. In our case we shall do the proof in two separate lemmas.

Lemma 4.16. *Consider a G-bundle morphism $\varphi : X \to Y$ between a pre-principal bundle $G \curvearrowright^{l_X} X \xrightarrow{\pi_X} B$ and a bundle $G \curvearrowright^{l_Y} Y \xrightarrow{\pi_Y} B$ whose underlying action $G \curvearrowright^{l_Y} Y$ is free. Then φ is injective.*

Proof. Since $G \curvearrowright^{l_X} X \xrightarrow{\pi_X} B$ is pre-principal, we get a smooth division map $\langle \cdot, \cdot \rangle_G^X$. To start the proof, suppose that we have two points $x_1, x_2 \in X$ satisfying $\varphi(x_1) = \varphi(x_2)$. Since φ preserves the fibres, we get that

$$\pi_X(x_1) = \pi_Y \circ \varphi(x_1) = \pi_Y \circ \varphi(x_2) = \pi_X(x_2).$$

Hence the pair (x_1, x_2) defines an element in $X \times_B X$, so we get an arrow $\langle x_1, x_2 \rangle_G^X \in G$, satisfying $\langle x_1, x_2 \rangle_G^X x_2 = x_1$. If we apply φ to this equation and use its G-equivariance, we get $\varphi(x_1) = \langle x_1, x_2 \rangle_G^X \varphi(x_2)$. However, by assumption, $\varphi(x_1) = \varphi(x_2)$ and the action $G \curvearrowright^{l_Y} Y$ is free, so we must have that $\langle x_1, x_2 \rangle_G^X$ is the identity arrow at $l_Y \circ \varphi(x_2) = l_X(x_2)$. Hence we get the desired result:

$$x_1 = \langle x_1, x_2 \rangle_G^X x_2 = \mathrm{id}_{l_X(x_2)} x_2 = x_2.$$

□

Lemma 4.17. *Consider a G-bundle morphism $\varphi : X \to Y$ from a subductive bundle $G \curvearrowright^{l_X} X \xrightarrow{\pi_X} B$ to a pre-principal bundle $G \curvearrowright^{l_Y} Y \xrightarrow{\pi_Y} B$. Then φ is a subduction.*

Proof. Denote the smooth division map of $G \curvearrowright^{l_Y} Y \xrightarrow{\pi_Y} B$ by $\langle \cdot, \cdot \rangle_G^Y$. Then φ and $\langle \cdot, \cdot \rangle_G^Y$ combine into a smooth map

$$\psi : X \times_B^{\pi_X, \pi_Y} Y \longrightarrow X; \qquad (x, y) \longmapsto \langle y, \varphi(x) \rangle_G^Y x.$$

Note that this is well defined because if $\pi_X(x) = \pi_Y(y)$, then $\pi_Y \circ \varphi(x) = \pi_Y(y)$ as well, and moreover $l_Y \circ \varphi(x) = l_X(x)$, showing that the action on the right hand side is allowed. The G-equivariance of φ then gives

$$\varphi \circ \psi = \mathrm{pr}_2|_{X \times_B Y}.$$

Since π_X is a subduction, so is $\mathrm{pr}_2|_{X \times_B Y}$ by Lemma 2.22, and by Lemma 2.20*(2)* it follows φ is a subduction. □

Proposition 4.18. *Any bundle morphism from a principal groupoid bundle to a pre-principal groupoid bundle is a diffeomorphism. In particular, both must then be principal.*

Proof. By Lemma 4.17 any such bundle morphism is a subduction, and since in particular the underlying action of a pre-principal bundle is free, it must also be injective by Lemma 4.16. The result follows by Proposition 2.21. That the second bundle is principal too follows from the fact that a bundle map preserves the fibres, so the projection of the second bundle can be written as the composition of a diffeomorphism and a subduction. ☐

5. Diffeological Bibundles and Morita Equivalence

This section contains the main definition of this paper: the notion of a *biprincipal bibundle*, which immediately gives our definition of *Morita equivalence*. The definition of groupoid bibundles for diffeology are a straightforward adaptation of the definition in the Lie case:

Definition 5.1. *Let $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ be two diffeological groupoids. A* diffeological (G, H)-bibundle *consists of a smooth left action $G \curvearrowright^{l_X} X$ and a smooth right action $X\, {}^{r_X}\!\curvearrowleft H$ such that the left moment map l_X is H-invariant and the right moment map r_X is G-invariant, and moreover such that the actions commute: $(g \cdot x) \cdot h = g \cdot (x \cdot h)$, whenever defined. We draw:*

$$\begin{array}{ccccc} G & \curvearrowright & X & \curvearrowleft & H \\ \downarrow\downarrow & \swarrow l_X & & r_X \searrow & \downarrow\downarrow \\ G_0 & & & & H_0, \end{array}$$

and denote them by $G \curvearrowright^{l_X} X\, {}^{r_X}\!\curvearrowleft H$ in-line. Underlying each bibundle are two groupoid bundles: the left underlying G-bundle $G \curvearrowright^{l_X} X \xrightarrow{r_X} H_0$ *and the* right underlying H-bundle $G_0 \xleftarrow{l_X} X\, {}^{r_X}\!\curvearrowleft H$. *It is the properties of these underlying bundles that will determine the behaviour of the bundle itself.*

Definition 5.2. *Consider a diffeological bibundle $G \curvearrowright^{l_X} X\, {}^{r_X}\!\curvearrowleft H$. We say this bibundle is* left pre-principal *if the left underlying bundle $G \curvearrowright^{l_X} X \xrightarrow{r_X} H_0$ is pre-principal. We say it is* right pre-principal *if the right underlying bundle $G_0 \xleftarrow{l_X} X\, {}^{r_X}\!\curvearrowleft H$ is pre-principal. We make similar definitions for subductiveness and principality. Notice that, in this convention, if a bibundle*

$G \curvearrowright^{l_X} X {}^{r_X}\curvearrowleft H$ *is* left *subductive, then its* right *moment map* r_X *is a subduction (and vice versa)*[5].

We now have the main definition of this theory:

Definition 5.3. *A diffeological bibundle is called:*

1. pre-biprincipal *if it is both left- and right pre-principal*[6]*;*
2. bisubductive *if it is both left- and right subductive;*
3. biprincipal *if it is both left- and right principal.*

Two diffeological groupoids G and H are called Morita equivalent *if there exists a biprincipal bibundle between them, and in that case we write $G \simeq_{\mathrm{ME}} H$.*

Compare this to the original definition [MRW87, Definition 2.1] of equivalence for locally compact Hausdorff groupoids. We will prove in Corollary 5.23 that Morita equivalence forms a genuine equivalence relation.

Example 5.4. Since submersions between manifolds are subductions with respect to the manifold diffeologies, we see that if two *Lie* groupoids $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are Morita equivalent in the *Lie* sense (e.g. [CM18, Definition 2.15]), then they are Morita equivalent in the *diffeological* sense. We remark on the converse question in Section 7.1.

In fact, many elementary examples of Morita equivalences between Lie groupoids generalise straightforwardly to analogously defined diffeological groupoids. We refer to [vdS20, Section 4.3] for some of these examples. For us, the most important one is:

Example 5.5. Consider a diffeological groupoid $G \rightrightarrows G_0$. There exists a canonical (G, G)-bibundle structure on the space of arrows G, which is called the *identity bibundle*. The actions are just the composition in G itself,

[5] Note: [dHo12, Section 4.6] defines this differently, where "[a] bundle is left (resp. right) principal if only the right (resp. left) underlying bundle is so." We suspect this may be a typo, since it apparently conflicts with their use of terminology in the proof of [dHo12, Theorem 4.6.3]. We stick to the terminology defined above, where *left* principality pertains to the *left* underlying bundle.

[6] The prefixes *bi-* and *pre-* commute: "bi-(pre-principal) = pre-(biprincipal)".

as in Example 4.2. Note that the identity bibundle is always biprincipal, because the action map has a smooth inverse $(g_1, g_2) \mapsto (g_1 \circ g_2^{-1}, g_2)$. This proves that any diffeological groupoid is Morita equivalent to itself, through the identity bibundle $G \curvearrowright^{\mathrm{trg}} G \,^{\mathrm{src}}\!\curvearrowleft G$.

Construction 5.6. Consider a diffeological bibundle $G \curvearrowright^{l_X} X \,^{r_X}\!\curvearrowleft H$. The *opposite bibundle* $H \curvearrowright^{l_{\overline{X}}} \overline{X} \,^{r_{\overline{X}}}\!\curvearrowleft G$ is defined as follows. The underlying diffeological space does not change, $\overline{X} := X$, but the moment maps switch, meaning that $l_{\overline{X}} := r_X$ and $r_{\overline{X}} := l_X$, and the actions are defined as follows:

$$\begin{aligned} &H \curvearrowright^{r_X} \overline{X}; \qquad h \cdot x := x h^{-1}, \\ &\overline{X} \,^{l_X}\!\curvearrowleft G; \qquad x \cdot g := g^{-1} x. \end{aligned}$$

Here the actions on the right-hand sides are the original actions of the bibundle. It is easy to see that performing this operation twice gives the original bibundle back. It is also important to note that for all properties defined in Definition 5.2, taking the opposite merely switches the words 'left' and 'right'.

The following extends Proposition 4.14*(4)*:

Lemma 5.7. *Consider a left pre-principal bibundle $G \curvearrowright^{l_X} X \,^{r_X}\!\curvearrowleft H$, and also the opposite G-action $\overline{X} \,^{l_X}\!\curvearrowleft G$. Then, whenever defined, we have:*

$$\langle x_1, x_2 g \rangle_G = \langle x_1, x_2 \rangle_G \circ g.$$

Proof. This follows directly from Proposition 4.14 and the definition of the opposite action:

$$\langle x_1, x_2 g \rangle_G = \langle x_1, g^{-1} x_2 \rangle_G = \left(g^{-1} \circ \langle x_2, x_1 \rangle_G\right)^{-1} = \langle x_1, x_2 \rangle_G \circ g. \quad \square$$

5.1 Induced actions

A bibundle $G \curvearrowright X \curvearrowleft H$ allows us to transfer a groupoid action $H \curvearrowright Y$ to a groupoid action $G \curvearrowright X \otimes_H Y$. This is called the *induced action*, and, together with the balanced tensor product, will be crucial to define the composition of bibundles. The idea is that G acts on the first component of $X \otimes_H Y$.

Construction 5.8. Consider a diffeological bibundle $G \curvearrowright^{l_X} X \,{}^{r_X}\!\curvearrowleft H$, and a smooth action $H \curvearrowright^{l_Y} Y$. We construct a smooth left G-action on the balanced tensor product $X \otimes_H Y$. The left moment map is defined as

$$L_X : X \otimes_H Y \longrightarrow G_0; \qquad x \otimes y \longmapsto l_X(x).$$

This is well defined because l_X is H-invariant, and smooth by Lemma 2.20*(3)*. For an arrow $g \in G$ with $\mathrm{src}(g) = L_X(x \otimes y) = l_X(x)$, define the action as:

$$G \curvearrowright^{L_X} X \otimes_H Y; \qquad g \cdot (x \otimes y) := (gx) \otimes y.$$

Note that the right hand side is well defined because r_X is G-invariant and the G- and H-actions commute, so $r_X(gx) = l_Y(y)$ and the expression does not change if we replace $x \otimes y$ by $xh \otimes h^{-1}y$ for arbitrary $h \in H$. Since there can be no confusion, we will drop all parentheses and write $gx \otimes y$ instead. That the action is smooth follows because $(g, (x, y)) \mapsto (gx, y)$ is smooth (on the appropriate domains) and by another application of Lemma 2.20*(3)*. Hence we obtain the *induced action* $G \curvearrowright^{L_X} X \otimes_H Y$.

Now suppose that we are given a smooth H-equivariant map $\varphi : Y_1 \to Y_2$ between two smooth actions $H \curvearrowright^{l_1} Y_1$ and $H \curvearrowright^{l_2} Y_2$. We define a map

$$\mathrm{id}_X \otimes \varphi : X \otimes_H Y_1 \longrightarrow X \otimes_H Y_2; \qquad x \otimes y \longmapsto x \otimes \varphi(y).$$

The underlying map $X \times_{H_0} Y_1 \to X \times_{H_0} Y_2 : (x, y) \mapsto (x, \varphi(y))$ is clearly smooth. Then by composition of the projection onto $X \otimes_H Y_2$ and Lemma 2.20*(3)*, we find $\mathrm{id}_X \otimes \varphi$ is smooth. Moreover, it is G-equivariant:

$$\mathrm{id}_X \otimes \varphi(gx \otimes y) = gx \otimes \varphi(y) = g\left(\mathrm{id}_X \otimes \varphi(x \otimes y)\right).$$

Definition 5.9. *A diffeological bibundle $G \curvearrowright^{l_X} X \,{}^{r_X}\!\curvearrowleft H$ defines an* induced action functor*:*

$$\begin{aligned} X \otimes_H - : \mathbf{Act}(H \rightrightarrows H_0) &\longrightarrow \mathbf{Act}(G \rightrightarrows G_0), \\ \left(H \curvearrowright^{l_Y} Y\right) &\longmapsto \left(G \curvearrowright^{L_X} X \otimes_H Y\right), \\ \varphi &\longmapsto \mathrm{id}_X \otimes \varphi. \end{aligned}$$

sending each smooth left H-action $\left(H \curvearrowright^{l_Y} Y\right) \mapsto \left(G \curvearrowright^{L_X} X \otimes_H Y\right)$ and each H-invariant map $\varphi \mapsto \mathrm{id}_X \otimes \varphi$. We will use this functor in Section 6.3.

5.2 The bicategory of diffeological groupoids and -bibundles

Combining the balanced tensor product (Construction 4.6) and the induced action of a bibundle (Construction 5.8), we can define a notion of composition for diffeological bibundles, and thereby obtain a new sort of category of diffeological groupoids[7]. Specifically, in Theorem 5.17 we will see that we obtain a *bicategory* $\mathbf{DiffeolBiBund}$. A bicategory is like a category where the axioms of composition hold merely up to *canonical 2-isomorphism*. For the precise definition we refer to e.g. [Mac71; Lac10]. The point of this section is to give precise definitions for this bicategorical structure, with the first ingredient being the following:

Definition 5.10. *Let $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ and $G \overset{l_Y}{\curvearrowright} Y \overset{r_Y}{\curvearrowleft} H$ be two bibundles between the same two diffeological groupoids. A smooth map $\varphi : X \to Y$ is called a* bibundle morphism *if it is a bundle morphism between both underlying bundles. We also say that φ is* biequivariant. *Concretely, this means that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{r_X} & H_0 \\ {\scriptstyle l_X}\big\downarrow & \searrow^{\varphi} & \big\uparrow{\scriptstyle r_Y} \\ G_0 & \xleftarrow[l_Y]{} & Y, \end{array} \qquad \textit{that is:} \qquad \begin{aligned} l_X &= l_Y \circ \varphi, \\ r_X &= r_Y \circ \varphi, \end{aligned}$$

and that φ is equivariant with respect to both actions. These will be the 2-morphisms in $\mathbf{DiffeolBiBund}$. The isomorphisms of bibundles are exactly the diffeomorphic biequivariant maps. These will be the 2-isomorphisms in $\mathbf{DiffeolBiBund}$.

The composition of bibundles is defined as follows:

Construction 5.11. Consider two diffeological bibundles $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ and $H \overset{l_Y}{\curvearrowright} Y \overset{r_Y}{\curvearrowleft} K$. We shall define on $X \otimes_H Y$ a (G, K)-bibundle structure using the induced actions from Construction 5.8. On the left we take the induced G-action along $L_X : X \otimes_H Y \to G_0$, which we recall maps $x \otimes y \mapsto l_X(x)$, defined by

$$G \overset{L_X}{\curvearrowright} X \otimes_H Y; \qquad g(x \otimes y) := (gx) \otimes y.$$

[7]The most straightforward way to obtain a (2-)category of diffeological groupoids is to consider the *smooth functors* and *smooth natural transformations*. We will not be studying this category in the current paper.

Analogous to Construction 5.8, swapping left actions for right actions, we get an induced K-action on the right along $R_Y : X \otimes_H Y \to K_0$, which maps $x \otimes y \mapsto r_Y(y)$, given by

$$X \otimes_H Y \,{}^{R_Y}\!\!\curvearrowleft K; \qquad (x \otimes y)k := x \otimes (yk).$$

It is easy to see that these two actions form a bibundle $G \curvearrowright^{L_X} X \otimes_H Y \,{}^{R_Y}\!\!\curvearrowleft K$, which we also call the *balanced tensor product*.

From this construction we can see that the composition of bibundles will not be strictly associative, and this is where the bicategorical structure becomes important. The following two propositions characterise the compositional structure of the balanced tensor product *up to biequivariant diffeomorphism*. The first of these shows that the identity bibundle (Example 5.5) is a *weak identity:*

Proposition 5.12. *Let $G \curvearrowright^{l_X} X \,{}^{r_X}\!\!\curvearrowleft H$ be a diffeological bibundle. Then there are biequivariant diffeomorphisms*

$$\begin{array}{ccc} G \curvearrowright^{L_G} G \otimes_G X \,{}^{R_X}\!\!\curvearrowleft H & & G \curvearrowright^{L_X} X \otimes_H H \,{}^{R_H}\!\!\curvearrowleft H \\ \Downarrow \varphi & \textit{and} & \Downarrow \\ G \curvearrowright^{l_X} X \,{}^{r_X}\!\!\curvearrowleft H & & G \curvearrowright^{l_X} X \,{}^{r_X}\!\!\curvearrowleft H. \end{array}$$

Proof. The idea of the proof is briefly sketched on [Blo08, p.8]. The map $\varphi : G \otimes_G X \to X$ is defined by the action: $g \otimes x \mapsto gx$. This map is clearly well defined, and by an easy application of Lemma 2.20*(3)* also smooth. Further note that φ intertwines the left moment maps:

$$l_X \circ \varphi(g \otimes x) = l_X(gx) = \mathrm{trg}(g) = L_G(g \otimes x),$$

and similarly we find it intertwines the right moment maps. Associativity of the G-action and the fact that it commutes with the H-action directly ensure that φ is biequivariant. Moreover, we claim that the smooth map $\psi : X \to G \otimes_G X$ defined by $x \mapsto \mathrm{id}_{l_X(x)} \otimes x$ is the inverse of φ. It follows easily that $\varphi \circ \psi = \mathrm{id}_X$, and the other side follows from the defining property of the balanced tensor product:

$$\psi \circ \varphi(g \otimes x) = \psi(gx) = \mathrm{id}_{l_X(gx)} \otimes gx = (\mathrm{id}_{\mathrm{trg}(g)} \circ g) \otimes x = g \otimes x.$$

It follows from an analogous argument that the identity bibundle of H acts like a weak right inverse. □

The second proposition shows that the balanced tensor product is associative *up to canonical biequivariant diffeomorphism:*

Proposition 5.13. *Let* $G \curvearrowright^{l_X} X \,^{r_X}\!\curvearrowleft H$, $H \curvearrowright^{l_Y} Y \,^{r_Y}\!\curvearrowleft H'$, *and* $H' \curvearrowright^{l_Z} Z \,^{r_Z}\!\curvearrowleft K$ *be diffeological bibundles. Then there exists a biequivariant diffeomorphism*

$$\begin{array}{c} G \curvearrowright^{L_{X \otimes_H Y}} (X \otimes_H Y) \otimes_{H'} Z \,^{R_Z}\!\curvearrowleft K \\ \Big\Downarrow A \\ G \curvearrowright^{L_X} X \otimes_H (Y \otimes_{H'} Z) \,^{R_{Y \otimes_{H'} Z}}\!\curvearrowleft K, \end{array} \qquad A : (x \otimes y) \otimes z \longmapsto x \otimes (y \otimes z).$$

Proof. That the map A is smooth follows by Lemma 2.20*(3)*, because the corresponding underlying map $((x, y), z) \mapsto (x, (y, z))$ is a diffeomorphism. The inverse of this diffeomorphism on the underlying fibred product induces exactly the smooth inverse of A, showing that A is a diffeomorphism. Furthermore, it is easy to check that A is biequivariant. □

Combining Propositions 5.12 and 5.13 gives that the balanced tensor product of bibundles does indeed behave like the composition in a bicategory. Next to the composition of arrows in a bicategory, we also need to describe the compositional structure of the 2-arrows. The following elementary result says that the ordinary *vertical composition* of biequivariant maps is again biequivariant:

Proposition 5.14. *Consider two biequivariant smooth maps:*

$$\begin{array}{c} G \curvearrowright^{l_X} X \,^{r_Y}\!\curvearrowleft H \\ \Big\Downarrow \varphi \\ G \curvearrowright^{l_Y} Y \,^{r_Y}\!\curvearrowleft H \\ \Big\Downarrow \psi \\ G \curvearrowright^{l_Z} Z \,^{r_Z}\!\curvearrowleft H. \end{array}$$

Then the composition $\psi \circ \varphi : X \to Z$ *is also biequivariant.*

Next to vertical composition, a bicategory should also allow for *horizontal composition* of 2-arrows. Again, the construction of this composition follows the Lie groupoid theory:

Construction 5.15. Consider the following situation of four bibundles and two biequivariant maps:

$$\begin{array}{ccc} G \curvearrowright^{l_X} X \,^{r_X}\!\curvearrowleft H & & H \curvearrowright^{l_P} P \,^{r_P}\!\curvearrowleft K \\ \Downarrow \varphi & \text{and} & \Downarrow \psi \\ G \curvearrowright^{l_Y} Y \,^{r_Y}\!\curvearrowleft H & & H \curvearrowright^{l_Q} Q \,^{r_Q}\!\curvearrowleft K. \end{array}$$

The goal will be to construct a biequivariant map

$$\begin{array}{c} G \curvearrowright^{L_X} X \otimes_H P \,^{R_P}\!\curvearrowleft K \\ \Downarrow \varphi\otimes\psi \\ G \curvearrowright^{L_Y} Y \otimes_H Q \,^{R_Q}\!\curvearrowleft K, \end{array}$$

called the *horizontal composition* of φ and ψ. The most obvious choice for the underlying function is the following:

$$\varphi \times \psi|_{X \times_H P} : X \times_H^{r_X, l_P} P \longrightarrow Y \times_H^{r_Y, l_Q} Q.$$

The biequivariance of φ and ψ ensures that the image of this function indeed lands in the fibred product $Y \times_H^{r_Y, l_Q} Q$, showing it is well defined and smooth. Projecting down to the balanced tensor products, we define:

$$\varphi \otimes \psi : X \otimes_H P \longrightarrow Y \otimes_H Q; \qquad x \otimes p \longmapsto \varphi(x) \otimes \psi(p).$$

To show that this will again form a 2-arrow in $\mathbf{DiffeolBiBund}$, we have the following counterpart to Proposition 5.14:

Proposition 5.16. *The map $\varphi \otimes \psi : X \otimes_H P \to Y \otimes_H Q$ from Construction 5.15 is a well-defined smooth biequivariant map of diffeological bibundles.*

Proof. We start by showing that $\varphi \otimes \psi$ is a well-defined function on the balanced tensor products. For that, take an element $x \otimes p \in X \otimes_H P$, and take an arbitrary arrow $h \in H$ with $\mathrm{trg}(h) = r_X(x) = l_P(p)$, such that $x \otimes p = xh \otimes h^{-1}p$. Using the fact that both φ and ψ are biequivariant, and the defining relation of the balanced tensor product $Y \otimes_H Q$, we calculate:

$$\begin{aligned} \varphi \otimes \psi(xh \otimes h^{-1}p) &:= \varphi(xh) \otimes \psi(h^{-1}p) \\ &= \varphi(x)h \otimes h^{-1}\psi(p) \\ &= \varphi(x) \otimes \psi(p) \\ &=: \varphi \otimes \psi(x \otimes p), \end{aligned}$$

showing that $\varphi \otimes \psi$ is indeed well defined. Next, observe that we have the following commutative diagram of functions:

$$\begin{array}{ccc} X \times_H^{r_X, l_P} P & \xrightarrow{\varphi \times \psi|_{X \times_H P}} & Y \times_H^{r_Y, l_Q} Q \\ \downarrow & & \downarrow \\ X \otimes_H P & \xrightarrow[\varphi \otimes \psi]{} & Y \otimes_H Q, \end{array}$$

where the vertical arrows are the canonical projections. It follows immediately from Lemma 2.20*(3)* that $\varphi \otimes \psi$ is also smooth.

Lastly, we show that $\varphi \otimes \psi$ is biequivariant with respect to the G- and K-actions. An easy calculation using the biequivariance of φ shows that

$$\begin{aligned} L_Y \circ (\varphi \otimes \psi)(x \otimes p) &= L_Y(\varphi(x) \otimes \psi(p)) \\ &= l_Y \circ \varphi(x) \\ &= l_X(x) \\ &= L_X(x \otimes p), \end{aligned}$$

and similarly we find $R_Q \circ (\varphi \otimes \psi) = R_P$. Moreover, $\varphi \otimes \psi$ commutes with the left G-actions:

$$\begin{aligned} \varphi \otimes \psi\,(g \cdot (x \otimes p)) &= \varphi \otimes \psi\,((gx) \otimes p) \\ &= \varphi(gx) \otimes \psi(p) \\ &= (g\varphi(x)) \otimes \psi(p) \\ &= g \cdot (\varphi(x) \otimes \psi(p)) \\ &= g \cdot (\varphi \otimes \psi(x \otimes p)), \end{aligned}$$

and we similarly find that it commutes with the right K-actions. We have thus proved that $\varphi \otimes \psi$ defines a smooth biequivariant map, as desired. □

We have now described all bicategorical ingredients for $\mathbf{DiffeolBiBund}$, and so it remains to check that they do indeed satisfy the axioms of a bicategory. The proof of this is directly analogous to the one for the Lie theory, which is explained in [Blo08, Proposition 2.12], and we therefore leave out the details.

Theorem 5.17. *There is a bicategory* $\mathbf{DiffeolBiBund}$ *consisting of diffeological groupoids as objects, diffeological bibundles as morphisms with balanced tensor product as composition, and biequivariant smooth maps as 2-morphisms.*

Proof idea. Note that the way in which we have the defined the bicategorical structure of $\mathbf{DiffeolBiBund}$ is a direct generalisation of the Lie groupoid theory (in the sense that, when restricted to Lie groupoids, it is the exact same). Furthermore, that the axioms of a bicategory hold for Lie groupoids ([Blo08, Proposition 2.12]) is not dependent on the (left or right) principality of the bibundles (save for the fact that this is needed to ensure the existence of the balanced tensor product), but is rather a property of the underlying functions. Given the results in this section, it is therefore clear that those proofs generalise directly to the diffeological setting. □

As we remarked in Section 4.1, the balanced tensor product for Lie groupoids can only be constructed for *left* (or *right*) *principal* bibundles. This means that in the Lie theory, the category of bibundles only consists of the left (or right) principal bibundles, since otherwise the composition cannot be defined. For diffeology we obtain a bicategory of *all* bibundles.

5.3 Properties of bibundles under composition and isomorphism

We study how the properties of diffeological bibundles defined in Definition 5.2 are preserved under the balanced tensor product and biequivariant diffeomorphism. These results will be crucial in characterising the weakly invertible bibundles. First we show that left subductive and left pre-principal bibundles are closed under composition.

Proposition 5.18. *The balanced tensor product preserves left subductiveness.*

Proof. Consider the balanced tensor product $G\curvearrowright^{L_X} X \otimes_H Y \,^{R_Y}\!\curvearrowleft K$ of two left subductive bibundles $G\curvearrowright^{l_X} X \,^{r_X}\!\curvearrowleft H$ and $H\curvearrowright^{l_Y} Y \,^{r_Y}\!\curvearrowleft K$. We need to show that the right moment map $R_Y : X \otimes_H Y \to K_0$ is a subduction. But, note that it fits into the following commutative diagram:

$$\begin{array}{ccc} X \times_{H_0}^{r_X, l_Y} Y & \xrightarrow{\pi} & X \otimes_H Y \\ {\scriptstyle \mathrm{pr}_2|_{X \times_{H_0} Y}}\big\downarrow & & \big\downarrow{\scriptstyle R_Y} \\ Y & \xrightarrow[r_Y]{} & K_0. \end{array}$$

Here π is the canonical quotient projection. The restricted projection $\mathrm{pr}_2|_{X \times_{H_0} Y}$ is a subduction by Lemma 2.22, since r_X is a subduction. Moreover, r_Y is a subduction, so the bottom part of the diagram is a subduction. It follows by Lemma 2.20*(3)* that R_Y is a subduction. □

Note that, even though R_Y only explicitly depends on the moment map r_Y, the proof still depends on the subductiveness of r_X as well.

To prove that the balanced tensor product of two left pre-principal bibundles is again left pre-principal, we need the following lemma, describing how the division map interacts with the bibundle structure, extending the list in Proposition 4.14 on the algebraic properties of the division map.

Lemma 5.19. *Let $G\curvearrowright^{l_X} X \,^{r_X}\!\curvearrowleft H$ be a left pre-principal bibundle, and denote its division map by $\langle\cdot,\cdot\rangle_G$. Then, whenever defined:*

$$\langle x_1, x_2 h\rangle_G = \langle x_1 h^{-1}, x_2\rangle_G, \quad \textit{or equivalently:} \quad \langle x_1 h, x_2 h\rangle_G = \langle x_1, x_2\rangle_G.$$

Proof. The arrow $\langle x_1 h, x_2 h\rangle_G \in G$ is the unique one that sends $x_2 h$ to $x_1 h$. Now, since the actions commute, we can multiply both of these terms from the right by h^{-1}, which gives the equation $\langle x_1 h, x_2 h\rangle_X x_2 = x_1$, and this immediately gives our result. □

Proposition 5.20. *The balanced tensor product preserves left pre-principality.*

Proof. To start the proof, take two left pre-principal bibundles, with our usual notation: $G\curvearrowright^{l_X} X \,^{r_X}\!\curvearrowleft H$ and $H\curvearrowright^{l_Y} Y \,^{r_Y}\!\curvearrowleft K$. Denote their division maps by $\langle\cdot,\cdot\rangle_G^X$ and $\langle\cdot,\cdot\rangle_H^Y$, respectively. Using these, we will construct a smooth inverse of the action map of the balanced tensor product. Let us denote the action map of the balanced tensor product by

$$\Phi : G \times_{G_0}^{\mathrm{src}, L_X} (X \otimes_H Y) \longrightarrow (X \otimes_H Y) \times_{K_0}^{R_Y, R_Y} (X \otimes_H Y),$$

mapping $(g, x \otimes y) \mapsto (gx \otimes y, x \otimes y)$. After some calculations (which we describe below), we propose the following map as an inverse for Φ:

$$\begin{aligned} \Psi : (X \otimes_H Y) \times_{K_0}^{R_Y, R_Y} (X \otimes_H Y) &\longrightarrow G \times_{G_0}^{\mathrm{src}, L_X} (X \otimes_H Y); \\ (x_1 \otimes y_1, x_2 \otimes y_2) &\longmapsto \left(\left\langle x_1 \langle y_1, y_2 \rangle_H^Y, x_2 \right\rangle_G^X, x_2 \otimes y_2 \right). \end{aligned}$$

It is straightforward to check that every action and division occurring in this expression is well defined. We need to check that Ψ is independent on the representations of $x_1 \otimes y_1$ and $x_2 \otimes y_2$. Only the first component Ψ_1 of Ψ could be dependent on the representations, so we focus there. Suppose we have two arrows $h_1, h_2 \in H$ satisfying $\mathrm{trg}(h_i) = r_X(x_i) = l_Y(y_i)$, so that $x_i h_i \otimes h_i^{-1} y_i = x_i \otimes y_i$. For the division of y_2 and y_1 we then use Proposition 4.14 to get:

$$\begin{aligned} \langle h_1^{-1} y_1, h_2^{-1} y_2 \rangle_H^Y &= h_1^{-1} \circ \langle y_1, h_2^{-1} y_2 \rangle_H^Y \\ &= h_1^{-1} \circ \left(h_2^{-1} \circ \langle y_2, y_1 \rangle_H^Y \right)^{-1} \\ &= h_1^{-1} \circ \langle y_1, y_2 \rangle_H^Y \circ h_2. \end{aligned}$$

Then, using this and Lemma 5.19, we get:

$$\begin{aligned} \Psi_1(x_1 h_1 \otimes h_1^{-1} y_1, x_2 h_2 \otimes h_2^{-1} y_2) &= \left\langle x_1 h_1 \langle h_1^{-1} y_1, h_2^{-1} y_2 \rangle_H^Y, x_2 h_2 \right\rangle_G^X \\ &= \left\langle (x_1 h_1) \left(h_1^{-1} \circ \langle y_1, y_2 \rangle_H^Y \circ h_2 \right), x_2 h_2 \right\rangle_G^X \\ &= \langle (x_1 \langle y_1, y_2 \rangle) h_2, x_2 h_2 \rangle_G^X \\ &= \left\langle x_1 \langle y_1, y_2 \rangle_H^Y, x_2 \right\rangle_G^X. \end{aligned}$$

Since the second component of Ψ is, by construction, independent on the representation, it follows that Ψ is a well-defined function. We now need to show that Ψ is smooth. The second component is clearly smooth, because it is just the projection onto the second component of the fibred product. That the other component is smooth follows from Lemmas 2.20 and 2.23. Writing

$$\psi : ((x_1, y_1), (x_2, y_2)) \longmapsto \langle x_1 \langle y_1, y_2 \rangle_H^Y, x_2 \rangle_G^X$$

and $\pi : X \times_{H_0}^{r_X, l_Y} Y \to X \otimes_H Y$ for the canonical projection, we get a commutative diagram

$$\begin{array}{ccc} \left(X \times_{H_0}^{r_X, l_Y} Y\right) \times_{K_0}^{\overline{r_Y}, \overline{r_Y}} \left(X \times_{H_0}^{r_X, l_Y} Y\right) & \xrightarrow{\psi} & \\ \Big\downarrow {\scriptstyle (\pi \times \pi)|_{\mathrm{dom}(\psi)}} & & G. \\ (X \otimes_H Y) \times_{K_0}^{R_Y, R_Y} (X \otimes_H Y) & \xrightarrow{\Psi_1} & \end{array}$$

Here we temporarily use the notation $\overline{r_Y} := r_Y \circ \mathrm{pr}_2|_{X \times_{H_0} Y}$, which satisfies $R_Y \circ \pi = \overline{r_Y}$. Therefore by Lemma 2.23 the left arrow in this diagram is a subduction. Since the map ψ is evidently smooth, it follows by Lemma 2.20*(3)* that the first component Ψ_1, and hence Ψ itself, must be smooth.

Thus, we are left to show that Ψ is an inverse for Φ. That Ψ is a right inverse for Φ now follows by simple calculation using Proposition 4.14 and Lemma 5.19:

$$\begin{aligned} \Psi \circ \Phi(g, x \otimes y) &= \Psi(gx \otimes y, x \otimes y) \\ &= \left(\langle gx \langle y, y \rangle_H^Y, x \rangle_G^X, x \otimes y\right) \\ &= \left(g \circ \langle x, x \rangle_G^X, x \otimes y\right) \\ &= (g, x \otimes y). \end{aligned}$$

For the other direction, we calculate:

$$\begin{aligned} \Phi \circ \Psi(x_1 \otimes y_1, x_2 \otimes y_2) &= \Phi\left(\left\langle x_1 \langle y_1, y_2 \rangle_H^Y, x_2 \right\rangle_G^X, x_2 \otimes y_2\right) \\ &= \left(\left\langle x_1 \langle y_1, y_2 \rangle_H^Y, x_2 \right\rangle_G^X x_2 \otimes y_2, x_2 \otimes y_2\right) \\ &= \left(x_1 \langle y_1, y_2 \rangle_H^Y \otimes y_2, x_2 \otimes y_2\right) \\ &= \left(x_1 \otimes \langle y_1, y_2 \rangle_H^Y y_2, x_2 \otimes y_2\right) \\ &= (x_1 \otimes y_1, x_2 \otimes y_2). \end{aligned}$$

Here in the second to last step we use the properties of the balanced tensor product to move the arrow $\langle y_1, y_2 \rangle_H^Y$ over the tensor symbol. Hence we conclude that Φ is a diffeomorphism, which proves that $G \curvearrowright^{L_X} X \otimes_H Y \,^{R_Y}\!\curvearrowleft K$ is a left pre-principal bibundle. □

Next we show that left subductiveness and left pre-principality are also preserved under biequivariant diffeomorphism.

Proposition 5.21. *Left pre-principality is preserved by biequivariant diffeomorphism.*

Proof. Suppose that $\varphi : X \to Y$ is a biequivariant diffeomorphism from a left pre-principal bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ to another diffeological bibundle $G \overset{l_Y}{\curvearrowright} Y \overset{r_Y}{\curvearrowleft} H$. Denote their left action maps by A_X and A_Y, respectively. The following square commutes because of biequivariance:

$$\begin{array}{ccc} G \times^{\mathrm{src}, l_X}_{G_0} X & \xrightarrow{A_X} & X \times^{r_X, r_X}_{H_0} X \\ {\scriptstyle (\mathrm{id}_G \times \varphi)|_{G \times_{G_0} X}} \big\downarrow & & \big\downarrow {\scriptstyle (\varphi \times \varphi)|_{X \times_{H_0} X}} \\ G \times^{\mathrm{src}, l_Y}_{G_0} Y & \xrightarrow[A_Y]{} & Y \times^{r_Y, r_Y}_{H_0} Y. \end{array}$$

It is easy to see that both vertical maps are diffeomorphisms. Hence it follows A_Y must be a diffeomorphism as well. □

Proposition 5.22. *Left subductiveness is preserved by biequivariant diffeomorphism.*

Proof. Suppose that $\varphi : X \to Y$ is a biequivariant diffeomorphism from a left subductive bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ to $G \overset{l_Y}{\curvearrowright} Y \overset{r_Y}{\curvearrowleft} H$. That the first bundle is left subductive means that r_X is a subduction, but since φ intertwines the moment maps, it follows immediately that $r_Y = r_X \circ \varphi^{-1}$ is a subduction as well. □

Of course, these four propositions all hold for their respective 'right' versions as well. This can be proved formally, without repeating the work, by using opposite bibundles.

Corollary 5.23. *Morita equivalence defines an equivalence relation between diffeological groupoids.*

Proof. Morita equivalence is reflexive by the existence of identity bibundles, which are always biprincipal (Example 5.5). It is also easy to check that the opposite bibundle (Construction 5.6) of a biprincipal bibundle is again biprincipal, showing that Morita equivalence is symmetric. Transitivity follows directly from Propositions 5.18 and 5.20 and their opposite versions. □

5.4 Weak invertibility of diffeological bibundles

In this section we prove the main Morita Theorem 5.31. As we explained in the Introduction, in the bicategory of diffeological groupoids we get a notion of *weak isomorphism*. Let us describe these explicitly: a bibundle $G \curvearrowright X \curvearrowleft H$ is weakly invertible if and only if there exists a second bibundle $H \curvearrowright Y \curvearrowleft G$, such that $X \otimes_H Y$ is biequivariantly diffeomorphic to G and $Y \otimes_G X$ is biequivariantly diffeomorphic to H. The Morita theorem says that such a weak inverse exists if and only if the bibundle is biprincipal. Let us recall the corresponding statement in the Lie theory: a (say) left principal bibundle has a left principal weak inverse if and only if it is biprincipal [Lan01b, Proposition 4.21]. Here both the original bibundle and its weak inverse have to be left principal, since everything takes place in a bicategory of Lie groupoids and left principal bibundles. According to Theorem 5.17 we get a bicategory of arbitrary bibundles, and the question of weak invertibility becomes a slightly more general one, since we do not start out with a bibundle that is already left principal. Instead we have to infer left principality from bare weak invertibility, where neither the weak inverse may be assumed to be left principal.

One direction of the claim in the main theorem is relatively straightforward, and is the same as for Lie groupoids:

Proposition 5.24. *Let* $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ *be a biprincipal bibundle. Then its opposite bundle* $H \overset{r_X}{\curvearrowright} \overline{X} \overset{l_X}{\curvearrowleft} G$ *is a weak inverse.*

Proof. We construct biequivariant diffeomorphisms

$$G \overset{L_X}{\curvearrowright} X \otimes_H \overline{X} \overset{R_{\overline{X}}}{\curvearrowleft} G \;\overset{\varphi_G}{\Longrightarrow}\; G \overset{\mathrm{trg}}{\curvearrowright} G \overset{\mathrm{src}}{\curvearrowleft} G, \qquad \text{and} \qquad H \overset{L_{\overline{X}}}{\curvearrowright} \overline{X} \otimes_G X \overset{R_X}{\curvearrowleft} H \;\overset{\varphi_H}{\Longrightarrow}\; H \overset{\mathrm{trg}}{\curvearrowright} H \overset{\mathrm{src}}{\curvearrowleft} H.$$

Since the original bundle is pre-biprincipal, we have a division map $\langle \cdot, \cdot \rangle_G : X \times_{H_0}^{r_X, r_X} \overline{X} \to G$. We define a new function

$$\varphi_G : X \otimes_H \overline{X} \longrightarrow G; \qquad x_1 \otimes x_2 \longmapsto \langle x_1, x_2 \rangle_G.$$

This is independent on the representation of the tensor product by Lemma 5.19, and smooth by Lemma 2.20*(3)* since $\varphi_G \circ \pi = \langle \cdot, \cdot \rangle_G$, where π is the canonical projection onto the balanced tensor product. We check that φ_G is biequivariant. It is easy to check that φ_G intertwines the moment maps, for example:

$$\mathrm{src} \circ \varphi_G(x_1 \otimes x_2) = \mathrm{src}\left(\langle x_1, x_2 \rangle_G\right) = l_X(x_2) = R_{\overline{X}}(x_1 \otimes x_2).$$

The left G-equivariance of φ_G follows directly out of Proposition 4.14, and the right G-equivariance follows from Lemma 5.7. Hence φ_G is a genuine bibundle morphism.

Since the original bundle is biprincipal, so is its opposite, and therefore by Propositions 5.18 and 5.20 it follows that both balanced tensor products are also biprincipal. Therefore φ_G is in particular a left G-equivariant bundle morphism from a principal bundle $G \curvearrowright^{L_X} X \otimes_H \overline{X} \xrightarrow{R_{\overline{X}}} G_0$ to a pre-principal bundle $G \curvearrowright^{\mathrm{trg}} G \xrightarrow{\mathrm{src}} G_0$, and hence a diffeomorphism by Proposition 4.18. This proves that the opposite bibundle is a weak right inverse. Note that we already need full biprincipality of the original bibundle for this. To prove that it is also a weak left inverse we make an analogous construction for φ_H, which we leave to the reader. □

The rest of this section will be dedicated to proving the converse of this claim, i.e., that a weakly invertible bibundle is biprincipal. First let us remark that by imitating a result from the Lie theory, we can obtain a partial result in this direction. Let us denote by $\mathbf{DiffeolBiBund}_{\mathrm{LP}}$ the bicategory of diffeological groupoids and left principal bibundles. Note that by Section 5.3 left principality is preserved by the balanced tensor product, so this indeed forms a subcategory.

Theorem 5.25. *A left principal diffeological bibundle has a left principal weak inverse if and only if it is biprincipal. That is, the weakly invertible bibundles in* $\mathbf{DiffeolBiBund}_{\mathrm{LP}}$ *are exactly the biprincipal ones.*

Proof. This follows by combining Proposition 5.24 with an adaptation of an argument from the Lie groupoid theory as in [MM05, Proposition 2.9]. A more detailed proof (for diffeological groupoids) is in [vdS20, Proposition 4.61]. □

This theorem is the most direct analogue of [Lan01b, Proposition 4.21] in the setting of diffeology. Our main theorem will be a further generalisation of this, which says that the same claim holds in the larger bicategory

DiffeolBiBund of *all* bibundles. We break the proof down in several steps, starting with the implication of bisubductiveness:

Proposition 5.26. *A weakly invertible diffeological bibundle is bisubductive.*

Proof. Suppose we have a bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ that admits a weak inverse $H \overset{l_Y}{\curvearrowright} Y \overset{r_Y}{\curvearrowleft} G$. Let us denote the included biequivariant diffeomorphisms by $\varphi_G : X \otimes_H Y \to G$ and $\varphi_H : Y \otimes_G X \to H$, as usual. Since the identity bibundles of G and H are both biprincipal, it follows by Proposition 5.22 that the moment maps L_X, R_X, L_Y and R_Y are all subductions. Together with the original moment maps, we get four commutative squares, each of the form:

$$\begin{array}{ccc} X \times_{H_0}^{r_X, l_Y} Y & \xrightarrow{\pi} & X \otimes_H Y \\ {\scriptstyle \mathrm{pr}_1|_{X \times_{H_0} Y}} \big\downarrow & & \big\downarrow {\scriptstyle L_X} \\ X & \xrightarrow[l_X]{} & G_0. \end{array}$$

Here $\pi : X \times_{H_0}^{r_X, l_Y} Y \to X \otimes_H Y$ is the quotient map of the diagonal H-action. By Lemma 2.20*(3)* it follows that, since L_X is a subduction, so is the composition $l_X \circ \mathrm{pr}_1|_{X \times_{H_0} Y}$, and in turn by Lemma 2.20*(2)* it follows l_X is a subduction. In a similar fashion we find that r_X, l_Y and r_Y are all subductions as well. □

This proposition gets us halfway to proving that weakly invertible bibundles are biprincipal. To prove that they are pre-biprincipal, it is enough to construct smooth division maps. We will give this construction below (Construction 5.29), which follows from a careful reverse engineering of the division map of a pre-principal bundle. Recall from Proposition 5.20 that the smooth inverse of the action map contains the information of both the G-division map and the H-division map. Specifically, the first component of the inverse is of the form $\langle x_1 \langle y_1, y_2 \rangle_H^Y, x_2 \rangle_G^X$, in which if we set $y_1 = y_2$, we simply reobtain the G-division map $\langle x_1, x_2 \rangle_G^X$. The question is if this "reobtaining" can be done in a smooth way. This is not so obvious at first. Namely, if we vary (x_1, x_2) smoothly within $X \times_{H_0}^{r_X, r_X} X$, can we guarantee that y_1 and y_2 vary smoothly with it, while still retaining the equalities $r_X(x_i) = l_Y(y_i)$ and $y_1 = y_2$? The elaborate Construction 5.29 proves that this can indeed be done. An essential part of our argument will be supplied by the following two lemmas.

Lemma 5.27. *When $G\overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ is a weakly invertible bibundle, admitting a weak inverse $H\overset{l_Y}{\curvearrowright} Y \overset{r_Y}{\curvearrowleft} G$, then all four actions are free.*

Proof. This follows from an argument that is used in the proof of [Blo08, Proposition 3.23]. Suppose we have an arrow $h \in H$ and a point $y \in Y$ such that $hy = y$. By Proposition 5.26 it follows that in particular l_X is surjective, so we can find $x \in X$ such that $y \otimes x \in Y \otimes_G X$. Then

$$h(y \otimes x) = (hy) \otimes x = y \otimes x.$$

But by Proposition 5.21 the bundle $H\overset{L_Y}{\curvearrowright} Y \otimes_G X \xrightarrow{R_X} G_0$, which is equivariantly diffeomorphic to the identity bundle on H, is pre-principal. So, the left action $H \curvearrowright Y \otimes_G X$ is free, and hence $h = \mathrm{id}_{L_Y(y\otimes x)} = \mathrm{id}_{l_Y(y)}$, proving that $H \curvearrowright Y$ is also free. That the three other actions are free follows analogously. □

Lemma 5.28. *Let $X \overset{r_X}{\curvearrowleft} H$ and $H\overset{l_Y}{\curvearrowright} Y$ be smooth actions, so that we can form the balanced tensor product $X \otimes_H Y$. Suppose that $H \curvearrowright Y$ is free. Then $x_1 \otimes y = x_2 \otimes y$ if and only if $x_1 = x_2$. Similarly, if $X \curvearrowleft H$ is free, then $x \otimes y_1 = x \otimes y_2$ if and only if $y_1 = y_2$.*

Proof. If $x_1 = x_2$ to begin with, the implication is trivial. Suppose therefore that $x_1 \otimes y = x_2 \otimes y$, which means that there exists an arrow $h \in H$ such that $(x_1h^{-1}, hy) = (x_2, y)$. In particular $hy = y$, which, because the action on Y is free, implies $h = \mathrm{id}_{l_Y(y)}$, and it follows that $x_1 = x_1\mathrm{id}_{l_Y(y)}^{-1} = x_2$. □

We shall now describe how the division map arises from local data:

Construction 5.29. For this construction to work, we start with a diffeological bibundle $G\overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$, admitting a weak inverse $H\overset{l_Y}{\curvearrowright} Y \overset{r_Y}{\curvearrowleft} G$. Then consider a pointed plot $\alpha : (U_\alpha, 0) \to (X \times_{H_0}^{r_X,r_X} X, (x_1, x_2))$. We split α into the components (α_1, α_2), which in turn give two pointed plots $\alpha_i : (U_\alpha, 0) \to (X, x_i)$ satisfying $r_X \circ \alpha_1 = r_X \circ \alpha_2 : U_\alpha \to H_0$. This equation gives a plot of H_0, and since by Proposition 5.26 the moment map $l_Y : Y \to H_0$ is a subduction, for every $t \in U_\alpha$ we can find a plot $\beta : V \to Y$, defined on an open neighbourhood $t \in V \subseteq U_\alpha$, such that $r_X \circ \alpha_i|_V = l_Y \circ \beta$. From this equation it follows that the smooth maps $(\alpha_i|_V, \beta) : V \to X \times_{H_0}^{r_X, l_Y} Y$ define two plots of the underlying space of the balanced tensor product. Applying the canonical quotient projection map $\pi : X \times_{H_0}^{r_X, l_Y} Y \to X \otimes_H Y$, we thus get two full-fledged plots $s \mapsto \alpha_i|_V(s) \otimes \beta(s)$ of the balanced tensor product. We combine these two plots to define yet another smooth map:

$$\Omega^\alpha|_V := (\pi \circ (\alpha_1|_V, \beta), \pi \circ (\alpha_2|_V, \beta)) : V \longrightarrow (X \otimes_H Y) \times_{G_0}^{R_Y, R_Y} (X \otimes_H Y).$$

Note that $\Omega^\alpha|_V$ lands in the right codomain because

$$R_Y \circ \pi \circ (\alpha_i|_V, \beta) = r_Y \circ \beta,$$

irrespective of $i \in \{1, 2\}$. We also note that the codomain of $\Omega^\alpha|_V$ is exactly the *do*main of the inverse $\Psi = (\Psi_1, \Psi_2)$ of the action map of the balanced tensor product $G \curvearrowright^{L_X} X \otimes_H Y \xrightarrow{R_Y} H_0$ (given explicitly in Proposition 5.20). In particular we then get a smooth map

$$\Psi_1 \circ \Omega^\alpha|_V : V \xrightarrow{\Omega^\alpha|_V} (X \otimes_H Y) \times_{G_0}^{R_Y, R_Y} (X \otimes_H Y) \xrightarrow{\Psi_1} G.$$

We now extend this map to the entire domain U_α, and show that it is independent on the choice of plot β. For that, pick two points $t, \bar{t} \in U_\alpha$, so that by subductiveness of the left moment map l_Y we can find two plots, $\beta : V \to Y$ and $\overline{\beta} : \overline{V} \to Y$, defined on open neighbourhoods of t and $\bar{t}$, respectively, such that $r_X \circ \alpha_i|_V = l_Y \circ \beta$ and $r_X \circ \alpha_i|_{\overline{V}} = l_Y \circ \overline{\beta}$. Following the above construction, we get two smooth maps:

$$\begin{aligned} \Omega^\alpha|_V &: s \longmapsto (\alpha_1|_V(s) \otimes \beta(s), \alpha_2|_V(s) \otimes \beta(s)), \\ \overline{\Omega}^\alpha|_{\overline{V}} &: s \longmapsto \left(\alpha_1|_{\overline{V}}(s) \otimes \overline{\beta}(s), \alpha_2|_{\overline{V}}(s) \otimes \overline{\beta}(s)\right). \end{aligned}$$

We now remark an important characterisation of Ψ, as a consequence of it being a diffeomorphism and inverse to the action map. Namely, when evaluated, $\Psi_1(x_1 \otimes y_1, x_2 \otimes y_2)$ is the *unique* arrow $g \in G$ satisfying the equation $gx_2 \otimes y_2 = x_1 \otimes y_1$. Therefore, $\Psi_1 \circ \Omega^\alpha|_V(s) \in G$ is the unique arrow such that

$$[\Psi_1 \circ \Omega^\alpha|_V(s)] \cdot (\alpha_2|_V(s) \otimes \beta(s)) = \alpha_1|_V(s) \otimes \beta(s).$$

By Lemma 5.27 all of the four actions of the original bibundles are free. Consequently, applying Lemma 5.28, since the second component in each term is just $\beta(s)$, this means that $\Psi_1 \circ \Omega^\alpha|_V(s)$ is the unique arrow in G such that

$$\Psi_1 \circ \Omega^\alpha|_V(s) \cdot \alpha_2|_V(s) = \alpha_1|_V(s),$$

where the tensor with $\beta(s)$ can be removed. But, for exactly the same reasons, if we take $s \in V \cap \overline{V}$, then $\Psi_1 \circ \overline{\Omega}^\alpha|_{\overline{V}}(s) \in G$ is *also* the unique arrow such that

$$\Psi_1 \circ \overline{\Omega}^\alpha|_{V\cap\overline{V}}(s) \cdot \alpha_2|_{V\cap\overline{V}}(s) = \alpha_1|_{V\cap\overline{V}}(s),$$

proving that

$$\Psi_1 \circ \Omega^\alpha|_{V\cap\overline{V}} = \Psi_1 \circ \overline{\Omega}^\alpha|_{V\cap\overline{V}}.$$

This shows that on the overlaps $V \cap \overline{V}$ the map $\Psi_1 \circ \Omega^\alpha|_{V\cap\overline{V}}$ does *not* depend on the plots β and $\overline{\beta}$. This allows us to extend $\Psi_1 \circ \Omega^\alpha|_V$, in a well-defined way, to the entire domain of U_α. We do this as follows. For every $t \in U_\alpha$ there exists a plot $\beta_t : V_t \to Y$, defined on an open neighbourhood $V_t \ni t$, such that $r_X \circ \alpha_i|_{V_t} = l_Y \circ \beta_t$. Clearly, this gives an open cover $(V_t)_{t\in U_\alpha}$ of U_α. For $t \in U_\alpha$ we then set $\Psi_1 \circ \Omega^\alpha(t) := \Psi_1 \circ \Omega^\alpha|_{V_t}(t)$. Hence we get a well-defined function $\Psi_1 \circ \Omega^\alpha : U_\alpha \to G$, which is smooth by the Axiom of Locality.

The main observation now is that, as the plot α is centred at (x_1, x_2), we get that $\Psi_1 \circ \Omega^\alpha(0)$ is the unique arrow in G such that $\Psi_1 \circ \Omega^\alpha(0) \cdot x_2 = x_1$. This is exactly the property that characterises the division $\langle x_1, x_2\rangle_G$!

Proposition 5.30. *A weakly invertible diffeological bibundle is pre-biprincipal.*

Proof. The bulk of the work has been done in Construction 5.29. Start with a diffeological bibundle $G \curvearrowright^{l_X} X {}^{r_X}\curvearrowleft H$ and a weak inverse $H \curvearrowright^{l_Y} Y {}^{r_Y}\curvearrowleft G$. We shall define a smooth division map $\langle \cdot, \cdot \rangle_G$ for the left G-action. For $(x_1, x_2) \in X \times^{r_X, r_X}_{H_0} X$, we know by the Axiom of Covering that the constant map $\mathrm{const}_{(x_1,x_2)} : \mathbb{R} \to X \times^{r_X, r_X}_{H_0} X$ is a plot centred at (x_1, x_2). We use the shorthand $\Psi_1 \circ \Omega^{(x_1,x_2)}$ to denote the map $\Psi_1 \circ \Omega^\alpha$ defined by the plot $\alpha = \mathrm{const}_{(x_1,x_2)}$, and then write:

$$\langle x_1, x_2\rangle_G := \Psi_1 \circ \Omega^{(x_1,x_2)}(0).$$

That just leaves us to show that this map is smooth. For that, take an arbitrary plot $\alpha : U_\alpha \to X \times_{H_0}^{r_X,r_X} X$ of the fibred product. We need to show that $\langle\cdot,\cdot\rangle_G \circ \alpha$ is a plot of G. For any $t \in U_\alpha$, we have that

$$\langle \alpha_1(t), \alpha_2(t)\rangle_G = \Psi_1 \circ \Omega^{\alpha(t)}(0)$$

is the unique arrow in G such that

$$\Psi_1 \circ \Omega^{\alpha(t)}(0) \cdot \mathrm{const}^2_{\alpha(t)}(0) = \mathrm{const}^1_{\alpha(t)}(0),$$

where const^i denotes the ith component of the constant plot. But then $\mathrm{const}^i_{\alpha(t)}(0) = \alpha_i(t)$, and we already know that $\Psi_1 \circ \Omega^\alpha(t) \in G$ is the unique arrow that sends $\alpha_2(t)$ to $\alpha_1(t)$, so we have:

$$\Psi_1 \circ \Omega^{\alpha(t)}(0) = \Psi_1 \circ \Omega^\alpha(t), \qquad \text{which means} \qquad \langle\cdot,\cdot\rangle_G \circ \alpha = \Psi_1 \circ \Omega^\alpha.$$

But the right hand side $\Psi_1 \circ \Omega^\alpha : U_\alpha \to G$ is a plot of G as per Construction 5.29, proving that the map $\langle\cdot,\cdot\rangle_G$ is smooth. It is quite evident from its construction that it satisfies exactly the properties of a division map, and it is now easy to verify that

$$\left(\langle\cdot,\cdot\rangle_G, \mathrm{pr}_2|_{X\times_{H_0}X}\right) : X \times_{H_0}^{r_X,r_X} X \longrightarrow G \times_{G_0}^{\mathrm{src},l_X} X$$

is a smooth inverse of the action map (see Section 4.2.1). The fact that it lands in the right codomain, i.e., $\mathrm{src}(\langle x_1, x_2\rangle_G) = l_X(x_2)$, follows from the properties of Ψ as the inverse of the action map of the balanced tensor product. Therefore $G \curvearrowright^{l_X} X \xrightarrow{r_X} H_0$ is a pre-principal bundle. An analogous argument will show that $G_0 \xleftarrow{l_Y} X\, {}^{r_X}\!\curvearrowleft H$ is also pre-principal, and hence we have proved the claim. □

We can now prove our main theorem:

Theorem 5.31. *A bibundle is weakly invertible in* $\mathbf{DiffeolBiBund}$ *if and only if it is biprincipal. That means: two diffeological groupoids are Morita equivalent if and only if they are equivalent in* $\mathbf{DiffeolBiBund}$.

Proof. One of the implications is just Proposition 5.24. The other now follows from a combination of Propositions 5.26 and 5.30. □

This significantly generalises [Lan01b, Proposition 4.21], not only in that we have a generalisation to a diffeological setting, but also in that it considers a more general type of bibundle. It justifies the bicategory $\mathbf{DiffeolBiBund}$ as being the appropriate setting for Morita equivalence of diffeological groupoids. It also shows that the assumptions of left principality of the Lie groupoid bibundles appear to be more like technical necessities for getting a well defined bicategory of Lie groupoids and bibundles, rather than being meaningful assumptions on the underlying smooth structure of the bibundles. In Section 7.1 we discuss other aspects of diffeological Morita equivalence between Lie groupoids. A possible *category of fractions* approach to Morita equivalence of diffeological groupoids is discussed in [vdS20, Chapter V].

6. Some Morita Invariants

In theories of Morita equivalence, there are often interesting properties that are naturally Morita invariant. In this section we discuss some results that generalise several well known Morita invariants of Lie groupoids to the diffeological setting. These include: invariance of the orbit spaces (Definition 3.4), of being *fibrating* (Definition 6.2), and of the action categories (Definition 4.5). The proofs are taken from [vdS20, Chapter IV].

6.1 Invariance of orbit spaces

It is a well known result that if two Lie groupoids $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are Morita equivalent (in the Lie groupoid sense), then there is a *homeo*morphism between their orbit spaces G_0/G and H_0/H, see e.g. [CM18, Lemma 1]. In fact, it turns out that the orbit spaces are even diffeologically diffeomorphic [Wat20, Theorem 3.8][8]. The following theorem extends this result further from Lie groupoids to arbitrary diffeological groupoids. The construction of the underlying function is the same as for the Lie groupoid case, which is sketched in the proof of [CM18, Lemma 1], and which we describe below in detail.

[8]The author thanks the anonymous referee for bringing this result to his attention. We should also like to note that several other variants of this statement hold, namely in the settings of *differentiable-* and *subcartesian spaces*, as proved in [CM18].

Theorem 6.1. *If $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are two Morita equivalent diffeological groupoids, then there is a diffeomorphism $G_0/G \cong H_0/H$ between their orbit spaces.*

Proof. Let $G \curvearrowright^{l_X} X \,^{r_X}\!\curvearrowleft H$ be the bibundle instantiating the Morita equivalence. Our first task will be to construct a function $\Phi : G_0/G \to H_0/H$ between the orbit spaces. The idea is to lift a point $a \in G_0$ of the base of the groupoid to its l_X-fibre, which by right principality is just an H-orbit in X, and then to project this orbit down to the other base H_0 along the right moment map r_X. The fact that the bundle is biprincipal ensures that this can be done in a consistent fashion.

We are dealing with *four* actions here, so we need to slightly modify our notation to avoid confusion. If $a \in G_0$ is an object in the groupoid G, we shall denote its orbit by $\mathrm{Orb}_{G_0}(a)$, which, as usual, is just the set of all points $a' \in G_0$ such that there exists an arrow $g : a \to a'$ in G. Similarly, for $b \in H_0$ we write $\mathrm{Orb}_{H_0}(b)$. On the other hand, we have two actions on X, for whose orbits we use the standard notations $\mathrm{Orb}_G(x)$ and $\mathrm{Orb}_H(x)$, where $x \in X$.

Now, start with a point $a \in G_0$, and consider its fibre $l_X^{-1}(a)$ in X. Since the bibundle is right subductive, the map l_X is surjective, so this fibre is non-empty and we can find a point $x_a \in l_X^{-1}(a)$. We claim that the expression $\mathrm{Orb}_{H_0} \circ r_X(x_a)$ is independent on the choice of the point x_a in the fibre. For that, take another point $x'_a \in l_X^{-1}(a)$. This gives the equation $l_X(x_a) = l_X(x'_a)$, and since the bibundle is right pre-principal, we get a unique arrow $h \in H$ such that $x'_a = x_a h$. From the definition of a right groupoid action, this in turn gives the equations $r_X(x'_a) = \mathrm{src}(h)$ and $r_X(x_a) = \mathrm{trg}(h)$, which proves the claim. To summarise, whenever $x_a, x'_a \in l_X^{-1}(a)$ are two points in the same l_X-fibre, then we have:

$$\mathrm{Orb}_{H_0} \circ r_X(x_a) = \mathrm{Orb}_{H_0} \circ r_X(x'_a). \tag{1}$$

Next we want to show that neither is this expression dependent on the point $a \in G_0$, but rather on its orbit $\mathrm{Orb}_{G_0}(a)$. For this, take another point $b \in \mathrm{Orb}_{G_0}(a)$, so there exists some arrow $g : a \to b$ in G. Pick then $x \in l_X^{-1}(a)$ and $y \in l_X^{-1}(b)$. This means that $\mathrm{src}(g) = l_X(x)$ and $\mathrm{trg}(g) = l_X(y)$, which means that if we let g act on the point x we get a point $gx \in l_X^{-1}(b)$, in the same l_X-fibre as y. Then using equation (1) applied to gx and y, and the G-invariance of the right moment map r_X, we immediately get:

$$\mathrm{Orb}_{H_0} \circ r_X(x) = \mathrm{Orb}_{H_0} \circ r_X(gx) = \mathrm{Orb}_{H_0} \circ r_X(y).$$

Using this, we can now conclude that there is a well-defined function

$$\Phi : G_0/G \longrightarrow H_0/H; \qquad \mathrm{Orb}_{G_0}(a) \longmapsto \mathrm{Orb}_{H_0} \circ r_X(x_a),$$

that is neither dependent on the point a in the orbit $\mathrm{Orb}_{G_0}(a)$, nor on the choice of the point $x_a \in l_X^{-1}(a)$ in the fibre. Note that this function exists by virtue of right subductivity (and the Axiom of Choice), which ensures that the left moment map l_X is a surjection (and for each a there exists an x_a).

Either by replacing $G \curvearrowright^{l_X} X \,{}^{r_X}\!\curvearrowleft H$ by its opposite bibundle, or by switching the words 'left' and 'right', the above argument analogously gives a function going the other way:

$$\Psi : H_0/H \longrightarrow G_0/G; \qquad \mathrm{Orb}_{H_0}(b) \longmapsto \mathrm{Orb}_{G_0} \circ l_X(y_b),$$

where now $y_b \in r_X^{-1}(b)$ is some point in the fibre of the right moment map r_X. We claim that Φ and Ψ are mutual inverses. To see this, pick a point $a \in G_0$, a point $x_a \in l_X^{-1}(a)$, a point $y_{r_X(x_a)} \in r_X^{-1}(r_X(x_a))$. Then we can write

$$\Psi \circ \Phi\left(\mathrm{Orb}_{G_0}(a)\right) = \Psi\left(\mathrm{Orb}_{H_0}(r_X(x_a))\right) = \mathrm{Orb}_{G_0}\left(l_X(y_{r_X(x_a)})\right).$$

We also have, by choice, the equation $r_X(x_a) = r_X(y_{r_X(x_a)})$, so by left pre-principality there exists an arrow $g \in G$ such that $gx_a = y_{r_X(x_a)}$. By definition of a left groupoid action, this then further gives

$$\mathrm{src}(g) = l_X(x_a) = a \qquad \text{and} \qquad \mathrm{trg}(g) = l_X(y_{r_X(x_a)}).$$

This proves that the right-hand side of the previous equation is equal to

$$\mathrm{Orb}_{G_0}\left(l_X(y_{r_X(x_a)})\right) = \mathrm{Orb}_{G_0}(a),$$

which gives $\Psi \circ \Phi = \mathrm{id}_{G_0/G}$. Through a similar argument, using right pre-principality, we obtain that $\Phi \circ \Psi = \mathrm{id}_{H_0/H}$.

To finish the proof, it suffices to prove that both Φ and Ψ are smooth. Again, due to the symmetry of the situation, we shall only prove that Φ is smooth. The proof for Ψ will follow analogously. Since Orb_{G_0} is a subduction, to prove that Φ is smooth it suffices by Lemma 2.20*(3)* to prove that $\Phi \circ \mathrm{Orb}_{G_0}$ is smooth. Since the left moment map l_X is a surjection, using the Axiom of Choice we pick a section $\sigma : G_0 \to X$, which replaces our earlier notation of $\sigma(a) =: x_a$. From the way Φ is defined, we see that we get a commutative diagram:

$$\begin{array}{ccccc} G_0 & \xrightarrow{\sigma} & X & \xrightarrow{r_X} & H_0 \\ {\scriptstyle \mathrm{Orb}_{G_0}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{Orb}_{H_0}} \\ G_0/G & & \xrightarrow[\Phi]{\hspace{6em}} & & H_0/H. \end{array}$$

We are therefore to show that $\mathrm{Orb}_{H_0} \circ r_X \circ \sigma$ is smooth. For this, pick a plot $\alpha : U_\alpha \to G_0$ of the base space. By right subductivity, the left moment map l_X is a subduction, so locally $\alpha|_V = l_X \circ \beta$, where β is some plot of X. Now, note that, for all $t \in V$, both the points $\beta(t)$ and $\sigma \circ l_X \circ \beta(t)$ are elements of the fibre $l_X^{-1}(l_X \circ \beta(t))$. Therefore, by equation (1) we get:

$$\mathrm{Orb}_{H_0} \circ r_X \circ \sigma \circ \alpha|_V = \mathrm{Orb}_{H_0} \circ r_X \circ \sigma \circ l_X \circ \beta = \mathrm{Orb}_{H_0} \circ r_X \circ \beta.$$

The right-hand side of this equation is clearly smooth (and no longer dependent on the choice of section σ). By the Axiom of Locality for G_0, it follows that $\mathrm{Orb}_{H_0} \circ r_X \circ \sigma \circ \alpha$ is globally smooth, and since the plot α was arbitrary, this proves that $\Phi \circ \mathrm{Orb}_{G_0}$ is smooth. Hence, Φ is smooth. After an analogous argument that shows Ψ is smooth, the desired diffeomorphism between the orbit spaces follows. □

Note that in the proof of [Wat20, Theorem 3.8], instead of a global (not necessarily smooth) section $\sigma : G_0 \to X$ of the left moment map $l_X : X \to G_0$, they use the fact that l_X is a surjective submersion to find a *local smooth* section. Our proof shows that it is not necessary for σ to be smooth, highlighting another difference between the rôle of surjective submersions and subductions[9].

6.2 Invariance of fibration

The theory of diffeological (principal) fibre bundles is shown in [IZ13a, Chapter 8] to be fully captured by the following notion:

[9] We thank the anonymous referee for pointing out this difference between the proofs.

Definition 6.2. *A diffeological groupoid $G \rightrightarrows G_0$ is called* fibrating *(or a* fibration groupoid*) if the* characteristic map $(\mathrm{trg}, \mathrm{src}) : G \to G_0 \times G_0$ *is a subduction.*

This leads to a theory of diffeological fibre bundles that is able to treat the standard smooth locally trivial (principal) fibre bundles of smooth manifolds, but also bundles that are not (and could not meaningfully be) locally trivial. It is then natural to ask if this property of diffeological groupoids is invariant under Morita equivalence. The following theorem proves that this is the case:

Theorem 6.3. *Let $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ be two Morita equivalent diffeological groupoids. Then $G \rightrightarrows G_0$ is fibrating if and only if $H \rightrightarrows H_0$ is fibrating.*

Proof. Because Morita equivalence is an equivalence relation, it suffices to prove that if $G \rightrightarrows G_0$ is fibrating, then so is $H \rightrightarrows H_0$. Denoting the characteristic maps of these groupoids by $\chi_G = (\mathrm{trg}_G, \mathrm{src}_G)$ and $\chi_H = (\mathrm{trg}_H, \mathrm{src}_H)$, assume that G is fibrating, so that χ_G is a subduction. Our goal is to show χ_H is also a subduction.

To begin with, take an arbitrary plot $\alpha = (\alpha_1, \alpha_2) : U_\alpha \to H_0 \times H_0$, and fix an element $t \in U_\alpha$. We thus need to find a plot $\Phi : W \to H$, defined on an open neighbourhood $t \in W \subseteq U_\alpha$, such that $\alpha|_W = \chi_H \circ \Phi$. Morita equivalence yields a biprincipal bibundle $G \curvearrowright^{l_X} X \, ^{r_X}\!\curvearrowleft H$. To construct the plot Φ, we use almost all of the structure of this bibundle.

The right moment map $r_X : X \to H_0$ is a subduction, so for each of the components α_i of α we get a plot $\beta_i : U_i \to X$, defined on an open neighbourhood $t \in U_i \subseteq U_\alpha$, such that $\alpha_i|_{U_i} = r_X \circ \beta_i$. Define $U := U_1 \cap U_2$, which is another open neighbourhood of $t \in U_\alpha$, and introduce the notation

$$\beta := (\beta_1|_U, \beta_2|_U) : U \longrightarrow X \times X.$$

Composing with the left moment map $l_X : X \to G_0$, we get $(l_X \times l_X) \circ \beta : U \to G_0 \times G_0$. It is here that we use that $G \rightrightarrows G_0$ is fibrating. Because of that, we can find an open neighbourhood $t \in V \subseteq U \subseteq U_\alpha$ and a plot $\Omega : V \to G$ such that

$$\chi_G \circ \Omega = (l_X \times l_X) \circ \beta|_V . \tag{2}$$

This means that $\mathrm{trg}_G \circ \Omega = l_X \circ \beta_1|_V$ and $\mathrm{src}_G \circ \Omega = l_X \circ \beta_2|_V$. Let $\varphi_G : X \otimes_H \overline{X} \to G$ be the biequivariant diffeomorphism from Proposition 5.24. Using the plot Ω we just obtained, we get another plot $\varphi_G^{-1} \circ \Omega : V \to X \otimes_H \overline{X}$. Now, since the canonical projection $\pi_H : X \times_{H_0}^{r_X, r_X} \overline{X} \to X \otimes_H \overline{X}$ of the diagonal H-action is a subduction, we can find an open neighbourhood $t \in W \subseteq V$ and a plot $\omega : W \to X \times_{H_0}^{r_X, r_X} \overline{X}$ such that

$$\pi_H \circ \omega = \varphi_G^{-1} \circ \Omega|_W. \tag{3}$$

Note that the plot ω decomposes into its components $\omega_1, \omega_2 : W \to X$, which satisfy $r_X \circ \omega_1 = r_X \circ \omega_2$. Using the biequivariance of φ_G and the defining relation $L_X \circ \pi_H = l_X \circ \mathrm{pr}_1|_{X \times_{H_0} \overline{X}}$ we find:

$$\begin{aligned} l_X \circ \beta_1|_W &= \mathrm{trg}_G \circ \Omega|_W \\ &= L_X \circ \varphi_G^{-1} \circ \Omega|_W \\ &= L_X \circ \pi_H \circ \omega \\ &= l_X \circ \mathrm{pr}_1|_{X \times_{H_0} \overline{X}} \circ \omega \\ &= l_X \circ \omega_1, \end{aligned}$$

where the first equality follows from the equation (2), and the third one from (3). Similarly, we find $l_X \circ \beta_2|_W = l_X \circ \omega_2$. These two equalities give two well-defined plots, one for each $i \in \{1, 2\}$, given by

$$\beta_i|_W \otimes \omega_i := \pi_G \circ (\beta_i|_W, \omega_i) : W \xrightarrow{(\beta_i|_W, \omega_i)} \overline{X} \times_{G_0}^{l_X, l_X} X \xrightarrow{\pi_G} \overline{X} \otimes_G X,$$

where $\pi_G : \overline{X} \times_{G_0}^{l_X, l_X} X \to \overline{X} \otimes_G X$ is the canonical projection of the diagonal G-action. We can now apply the biequivariant diffeomorphism $\varphi_H : \overline{X} \otimes_G X \Rightarrow H$ from Proposition 5.24 to get two plots in H. It is from these two plots that we will create Φ. Here it is absolutely essential that we have constructed the plot ω such that $r_X \circ \omega_1 = r_X \circ \omega_2$, because that means that the sources of these two plots in H will be equal, and hence they can be composed if we first invert one of them component-wise. To see this, use the biequivariance of φ_H to calculate

$$\begin{aligned} \mathrm{src}_H \circ \varphi_H \circ (\beta_i|_W \otimes \omega_i) &= R_X \circ (\beta_i|_W \otimes \omega_i) \\ &= r_X \circ \mathrm{pr}_2|_{\overline{X} \times_{G_0} X} \circ (\beta_i|_W, \omega_i) \\ &= r_X \circ \omega_i, \end{aligned}$$

and similarly:

$$\begin{aligned}\mathrm{trg}_H \circ \varphi_H \circ (\beta_i|_W \otimes \omega_i) &= L_{\overline{X}} \circ (\beta_i|_W \otimes \omega_i)\\ &= r_X \circ \mathrm{pr}_1|_{\overline{X} \times_{G_0} X} \circ (\beta_i|_W, \omega_i)\\ &= r_X \circ \beta_i|_W\\ &= \alpha_i|_W.\end{aligned}$$

Of course, if we switch $\beta_i|_W \otimes \omega_i$ to $\omega_i \otimes \beta_i|_W$, which is defined in the obvious way, then the right-hand sides of the above two equations will switch. So, for every $s \in W$, the expression $\varphi_H(\omega_2(s) \otimes \beta_2(s))$ is an arrow in H from $r_X \circ \beta_2(s) = \alpha_2(s)$ to $r_X \circ \omega_2(s)$, and $\varphi_H(\beta_1(s) \otimes \omega_1(s))$ is an arrow from $r_X \circ \omega_1(s) = r_X \circ \omega_2(s)$ to $r_X \circ \beta_1(s) = \alpha_1(s)$, which can hence be composed to give an arrow from $\alpha_2(s)$ to $\alpha_1(s)$. This is exactly the kind of arrow we want. Therefore, for every $s \in W$, we get a commutative triangle in the groupoid H, which defines for us the plot $\Phi : W \to H$:

$$\begin{array}{ccc} \alpha_2(s) & \xrightarrow{\ \Phi(s)\ } & \alpha_1(s) \\ & {\scriptstyle \varphi_H(\omega_2(s)\otimes\beta_2(s))} \searrow \quad \nearrow {\scriptstyle \varphi_H(\beta_1(s)\otimes\omega_1(s))} & \\ & r_X \circ \omega_1(s). & \end{array}$$

The map Φ is clearly smooth, because inversion and multiplication in H are smooth. Hence we have defined the plot Φ, and by the above diagram it is clear that it satisfies

$$\chi_H \circ \Phi = (\mathrm{trg}_H \circ \Phi, \mathrm{src}_H \circ \Phi) = \alpha|_W.$$

Thus we may at last conclude that χ_H is a subduction, and hence that $H \rightrightarrows H_0$ is also fibrating. $\square$

6.3 Invariance of representations

In the Morita theory of rings, it holds that two rings are Morita equivalent if and only if their categories of modules are equivalent. For groupoids, even discrete ones, this is no longer an "if and only if" proposition, but merely an "only if". Nevertheless, it is known that the result transfers to Lie groupoids as well [Lan01a, Theorem 6.6], and here we shall prove that it transfers also to diffeology.

Theorem 6.4. *Suppose that $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are Morita equivalent diffeological groupoids. Then the action categories $\mathbf{Act}(G \rightrightarrows G_0)$ and $\mathbf{Act}(H \rightrightarrows H_0)$ are categorically equivalent.*

Proof. If $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are Morita equivalent, there exists a biprincipal bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$. Recall from Definition 4.5 the notion of action categories and from Definition 5.9 that of induced action functors. We claim that

$$\begin{aligned} X \otimes_H - &: \mathbf{Act}(H \rightrightarrows H_0) \longrightarrow \mathbf{Act}(G \rightrightarrows G_0), \\ \overline{X} \otimes_G - &: \mathbf{Act}(G \rightrightarrows G_0) \longrightarrow \mathbf{Act}(H \rightrightarrows H_0) \end{aligned}$$

are mutually inverse functors up to natural isomorphism. To see this, take a left H action $H \overset{l_Y}{\curvearrowright} Y$. Then

$$\begin{aligned} \left(\overline{X} \otimes_G -\right) \circ (X \otimes_H -) \, [H \overset{l_Y}{\curvearrowright} Y] &= \left(\overline{X} \otimes_G -\right) \left[G \overset{L_X}{\curvearrowright} X \otimes_H Y\right] \\ &= H \overset{L_{\overline{X}}}{\curvearrowright} \left(\overline{X} \otimes_G (X \otimes_H Y)\right). \end{aligned}$$

Therefore, we need to construct a natural biequivariant diffeomorphism

$$\mu_Y : \overline{X} \otimes_G (X \otimes_H Y) \longrightarrow Y.$$

For this, we collect the biequivariant diffeomorphisms from Propositions 5.12, 5.13 and 5.24. Let us denote them by

$$\begin{aligned} &A_Y : \overline{X} \otimes_G (X \otimes_H Y) \longrightarrow \left(\overline{X} \otimes_G X\right) \otimes_H Y, \\ &\varphi_H : \overline{X} \otimes_G X \longrightarrow H, \\ &M_Y : H \otimes_H Y \longrightarrow Y, \end{aligned}$$

describing the association up to isomorphism, the division map of the bibundle, and the left action $H \curvearrowright Y$, respectively. We then define

$$\mu_Y := M_Y \circ (\varphi_H \otimes \mathrm{id}_Y) \circ A_Y.$$

Note that $(\varphi_H \otimes \mathrm{id}_Y)$ is still a biequivariant diffeomorphism. The naturality square of the natural transformation $\mu : \left(\overline{X} \otimes_G -\right) \circ (X \otimes_H -) \Rightarrow \mathrm{id}_{\mathbf{Act}(H)}$ then becomes:

$$\begin{array}{ccc} \overline{X} \otimes_G (X \otimes_H Y) & \xrightarrow{\ \mu_Y\ } & Y \\ {\scriptstyle \mathrm{id}_{\overline{X}} \otimes (\mathrm{id}_X \circ \varphi)} \big\downarrow & & \big\downarrow {\scriptstyle \varphi} \\ \overline{X} \otimes_G (X \otimes_H Z) & \xrightarrow[\ \mu_Z\ ]{} & Z, \end{array}$$

where $\varphi : Y \to Z$ is an H-equivariant smooth map. It follows from the structure of these maps that the naturality square commutes. The top right corner of the diagram becomes:

$$\begin{aligned}\varphi \circ \mu_Y (x_1 \otimes (x_2 \otimes y)) &= \varphi \circ M_Y \circ (\varphi_H \otimes \mathrm{id}_Y) \circ A_Y (x_1 \otimes (x_2 \otimes y)) \\ &= \varphi \circ M_Y \circ (\varphi_H \otimes \mathrm{id}_Y) ((x_1 \otimes x_2) \otimes y) \\ &= \varphi \circ M_Y (\varphi_H(x_1 \otimes x_2) \otimes y) \\ &= \varphi (\varphi_H(x_1 \otimes x_2) y) \\ &= \varphi_H(x_1 \otimes x_2)\varphi(y),\end{aligned}$$

where the very last step follows from H-equivariance of φ. Following a similar calculation, the bottom left corner evaluates as

$$\begin{aligned}\mu_Z \circ (\mathrm{id}_{\overline{X}} \otimes (\mathrm{id}_X \otimes \varphi)) &= M_Z \circ (\varphi_H \otimes \mathrm{id}_Z) \circ A_Z \circ (\mathrm{id}_{\overline{X}} \otimes (\mathrm{id}_X \otimes \varphi)) \\ &= M_Z \circ (\varphi_H \otimes \mathrm{id}_Z) \circ ((\mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X) \otimes \varphi) \\ &= M_Z \circ (\varphi_H \otimes \varphi),\end{aligned}$$

which, when evaluated, gives exactly the same as the above expression for the top right corner. This proves that μ is natural, and since every of its components is an H-equivariant diffeomorphism, it follows that μ is a natural isomorphism. The fact that the composition $(X \otimes_H -) \circ \left(\overline{X} \otimes_G -\right)$ is naturally isomorphic to $\mathrm{id}_{\mathbf{Act}(G)}$ follows from an analogous argument. Hence the categories $\mathbf{Act}(G \rightrightarrows G_0)$ and $\mathbf{Act}(H \rightrightarrows H_0)$ are equivalent, as was to be shown. □

7. Discussion and Suggestions for Future Research

7.1 Diffeological bibundles between Lie groupoids

As we saw in Example 5.4, if two Lie groupoids are *Lie* Morita equivalent (i.e. Morita equivalent in the Lie groupoid sense [CM18, Definition 2.15]), then they are also *diffeologically* Morita equivalent. This is simply due to the fact that surjective submersions between smooth manifolds are in particular also subductions, and hence a Lie principal groupoid bundle is also diffeologically principal. But, what if $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are two *Lie* groupoids, such that there exists a *diffeological* biprincipal bibundle $G \curvearrowright^{l_X} X \, {}^{r_X}\curvearrowleft H$

between them. What does that say about the *Lie* Morita equivalence of G and H? This still remains an open question (Question 7.6). In this section we discuss some related results, which also pertain to our choice of subductions over *local* subductions for the development of the general theory. A slightly more detailed discussion is in [vdS20, Section 4.4.3]. In light of Proposition 2.25, the source and target maps of a Lie groupoid are local subductions (cf. Proposition 3.2), and we can therefore introduce the following class of diffeological groupoids:

Definition 7.1. *We say a diffeological groupoid $G \rightrightarrows G_0$ is* locally subductive *if its source and target maps are local subductions*[10]. *Clearly, every Lie groupoid is a locally subductive diffeological groupoid.*

Looking at the structure of the proofs in Sections 4 and 5, it appears as if they can be generalised to a setting where we replace all subductions by local subductions. In doing so, we would get a theory of locally subductive groupoids, locally subductive groupoid bundles, and the corresponding notions for bibundles and Morita equivalence, which, as it appears, would follow the same story as we have so far presented. An upside to that framework would be that it directly returns the original theory of Morita equivalence for Lie groupoids, once we restrict our diffeological spaces to smooth manifolds. In this section we shall prove that, even in the slightly more general setting of Section 5, the diffeological bibundle theory reduces to the Lie groupoid theory in the correct way. We do this by proving that the moment maps of a biprincipal bibundle between locally subductive groupoids have to be local subductions as well (Lemma 7.3). In hindsight, this provides more justification for our choice of starting with subductions instead of local subductions. One consequence of this choice is that it allows for groupoid bundles that are truly *pseudo*-bundles, in the sense of [Per16]. The notion of pseudo-bundles seems to be the correct notion in the setting of diffeology to generalise all bundle constructions on manifolds, at least if we want to treat (internal) tangent bundles as such (see [CW16]). There exists diffeological spaces whose internal tangent bundle is not a local subduction [CW16, Example 3.17]. If we had defined principality of a groupoid bundle to include

[10] It would be tempting to call such groupoids *"diffeological Lie groupoids,"* but this would conflict with earlier established terminology of so-called *diffeological Lie groups* in [IZ13a, Article 7.1] and [Les03; Mag18].

local subductiveness, these examples would not be treatable by our theory of Morita equivalence.

Lemma 7.2. *Let $G \curvearrowright^{l_X} X \,^{r_X}\!\curvearrowleft H$ be a diffeological bibundle, where $H \rightrightarrows H_0$ is a locally subductive groupoid. Then the canonical quotient projection map $\pi_H : X \times_{H_0}^{r_X, r_X} \overline{X} \to X \otimes_H \overline{X}$ is a local subduction.*

Proof. Let $\alpha : (U_\alpha, 0) \to (X \otimes_H \overline{X}, x_1 \otimes x_2)$ be a pointed plot of the balanced tensor product. Since π_H is already a subduction, we can find a plot $\beta : V \to X \times_{H_0} \overline{X}$, defined on an open neighbourhood $0 \in V \subseteq U_\alpha$ of the origin, such that $\alpha|_V = \pi_H \circ \beta$. This plot decomposes into two plots $\beta_1, \beta_2 \in \mathcal{D}_X$ on X, satisfying $r_X \circ \beta_1 = r_X \circ \beta_2$. We use the notation $\alpha|_V = \beta_1 \otimes \beta_2$. In particular, we get an equality $x_1 \otimes x_2 = \beta_1(0) \otimes \beta_2(0)$ inside the balanced tensor product, which means that we can find an arrow $h \in H$ such that $\beta_i(0) = x_i h$. The target must be $\operatorname{trg}(h) = r_X(x_1) = r_X(x_2)$. This arrow allows us to write a pointed plot $r_X \circ \beta_i : (V, 0) \to (H_0, \operatorname{trg}(h^{-1}))$, so that now we can use that $H \rightrightarrows H_0$ is locally subductive. Since the target map of H is a local subduction, we can find a pointed plot $\Omega : (W, 0) \to (H, h^{-1})$ such that $r_X \circ \beta_i|_W = \operatorname{trg}_H \circ \Omega$. This relation means that, for every $t \in W$, we have a well-defined action $\beta_i(t) \cdot \Omega(t) \in X$. Hence we get a pointed plot

$$\Psi : (W, 0) \longrightarrow (X \times_{H_0}^{r_X, r_X} \overline{X}, (x_1, x_2)); \qquad t \longmapsto (\beta_1(t)\Omega(t), \beta_2(t)\Omega(t)) .$$

It then follows by the definition of the balanced tensor product that

$$\pi_H \circ \Psi(t) = \beta_1|_W(t)\Omega(t) \otimes \beta_2|_W(t)\Omega(t) = \beta_1|_W(t) \otimes \beta_2|_W(t) = \alpha|_W(t),$$

proving that π_H is a local subduction. □

Lemma 7.3. *If $G \curvearrowright^{l_X} X \,^{r_X}\!\curvearrowleft H$ is a biprincipal bibundle between locally subductive groupoids, then the moment maps l_X and r_X are local subductions as well.*

Proof. If the bibundle $G \curvearrowright^{l_X} X \,^{r_X}\!\curvearrowleft H$ is biprincipal, we get two biequivariant diffeomorphisms $\varphi_G : X \otimes_H \overline{X} \to G$ and $\varphi_H : \overline{X} \otimes_G X \to H$ (Proposition 5.24). It follows that the local subductivity of the source and target maps of G and H transfer to the four moment maps of the balanced tensor products. For example, the left moment map $L_X : X \otimes_H \overline{X} \to G_0$ can be written as $L_X = \mathrm{trg}_G \circ \varphi_G$, where the right hand side is clearly a local subduction. We know as well that L_X fits into a commutative square with the original moment map l_X:

$$\begin{array}{ccc} X \times_{H_0}^{r_X, r_X} \overline{X} & \xrightarrow{\pi_H} & X \otimes_H \overline{X} \\ {\scriptstyle \mathrm{pr}_1|_{X \times_{H_0} \overline{X}}} \downarrow & & \downarrow {\scriptstyle L_X} \\ X & \xrightarrow[l_X]{} & G_0. \end{array}$$

Since local subductions compose, and since by Lemma 7.2 the projection π_H is a local subduction, we find that the upper right corner $L_X \circ \pi_H$ must be a local subduction. Hence the composition $l_X \circ \mathrm{pr}_1|_{X \times_{H_0} \overline{X}}$ is a local subduction, which by an argument that is analogous to the proof of Lemma 2.20*(2)* gives the local subductiveness of l_X. That the right moment map r_X is a local subduction follows from a similar argument. □

The lemma suggests that, if we refine our notion of principality to something we might call *pure-principality*, by passing from subductions to local subductions, then biprincipality between locally subductive groupoids means the same thing as this new notion of pure-principality. Let us make this precise.

Definition 7.4. *Two diffeological groupoids are called* purely Morita equivalent *if there exists a biprincipal bibundle between them, such that the two underlying moment maps are local subductions.*

Clearly, pure Morita equivalence implies ordinary Morita equivalence in the sense of Definition 5.3, since local subductions are, in particular, subductions. The question is if the converse implication holds as well. We have a partial answer, since Lemma 7.3 can now be restated as follows:

Proposition 7.5. *Two locally subductive groupoids are Morita equivalent if and only if they are purely Morita equivalent.*

Especially in light of the existence of subductions that are not local subductions (see e.g. [IZ13a, Exercise 61, p.60]), and the fact that the proof of Lemma 7.3 relies so heavily on the assumption that the groupoids are locally subductive, it seems that the ordinary diffeological Morita equivalence of Definition 5.3 is not equivalent to pure-Morita equivalence in general. We do not, however, know of an explicit counter-example. This discussion leaves us an open question:

Question 7.6. *Does diffeological Morita equivalence reduce to Lie Morita equivalence on Lie groupoids? That is to ask, if two Lie groupoids are diffeologically Morita equivalent, are they also Lie Morita equivalent?*

If two Lie groupoids G and H are diffeologically Morita equivalent, then there exists a diffeological biprincipal bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$, where X is a diffeological space. A positive answer to Question 7.6 could consist of a proof that X is in fact a smooth manifold. Since G and H are both manifolds, it follows that $X \otimes_H \overline{X}$ and $\overline{X} \otimes_G X$ are also manifolds. We do not know if this is sufficient to imply that X itself has to be a manifold. One suggestion is to use [IZ13a, Article 4.6], which gives a characterisation for when a quotient of a diffeological space by an equivalence relation is a smooth manifold. Since the balanced tensor products are quotients of diffeological spaces, one may try to use this result to obtain a special family of plots for their underlying fibred products. This could potentially be used to define an atlas on X.

7.2 Directions for future research

We list here some possible directions for future research. These are also proposed at the end of [vdS20, Section 1.2.3].

- Finding an answer to the open Question 7.6 about *diffeological* Morita equivalence between *Lie* groupoids.
- The construction of a theory of bibundles for a more general framework of generalised smooth spaces. One possibility is to look at the *generalised spaces* of [BH11, Definition 4.11] (subsuming diffeology), or even to look at arbitrary classes of sheaves. What is the relation between our theory of Morita equivalence and the discussion in [MZ15]? A theory of principal bibundles seems to exist in a general setting for groupoids in ∞-toposes: [nL18].
- What is the precise relation between differentiable stacks and diffeological groupoids (cf. [WW19])? Using our notion of Morita equi-

valence, what types of objects are *"diffeological stacks"* (i.e., Morita equivalence classes of diffeological groupoids)?

- Can the *Hausdorff Morita equivalence* for holonomy groupoids of singular foliations introduced in [GZ19] be understood as a Morita equivalence between diffeological groupoids?
- Can the bridge between diffeology and noncommutative geometry that is being built in [Ber16; IZL18; ASZ19; IZP20] be strengthened by our theory of Morita equivalence? Morita equivalence of Lie groupoids is already an important concept in relation to noncommutative geometry, especially for the theory of groupoid C*-algebras. Can this link be extended to the diffeological setting, possibly through a theory of groupoid C*-algebras for (a large class of) diffeological groupoids? If such a theory exists, what is the relation between Morita equivalence of diffeological groupoids and the Morita equivalence of their groupoid C*-algebras? Is Morita equivalence preserved just like in the Lie case?

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CORRECTIONS TO: A CONSTRUCTION OF 2-FILTERED BICOLIMITS OF CATEGORIES

Eduardo J. DUBUC and Ross STREET

Résumé. Martin Szyld a fait remarquer que le Lemme 1.14 ne tient pas pour une catégorie 2 pré-2-filtrée telle que définie dans notre article. Ici, nous montrons comment résoudre le problème.
Abstract. Martin Szyld pointed out that Lemma 1.14 does not hold for a pre-2-filtered 2-category as defined in our paper. Here we show how to resolve the problem.
Keywords. 2-filtered; bicolimit.
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We are grateful to Martin Szyld for pointing out that Lemma 1.14 does not hold for a pre-2-filtered 2-category (Definition 1.1). That lemma was used to prove Lemma 1.20 which is essential for Theorems 2.4 and 2.5. The problem can be resolved as follows.

1. *On page 94, delete Lemma 1.14.*
2. *On page 99, delete Lemma 1.20 and the line preceding it.*
3. *At the bottom of page 99 and on page 100, delete all the material from the beginning of Section 2 and Definition 2.1 down to (but excluding) Lemma 2.2, and replace the deleted material by the following:*

2. 2-Filtered 2-Categories

We refer here to Definition 1.1 of pre 2-filtered.

2.1a Definition. A 2-category $\mathcal{A}$ is defined to be *pseudo 2-filtered* when it satisfies the following three axioms:

FF1. A stronger form of the axiom F1 of pre 2-filtered.

$$Given\ \begin{array}{ccc} & & A \\ & {}^{f_1}\nearrow & \\ E_1 & & \\ & {}_{g_1}\searrow & \\ & & B \end{array} ,\ \begin{array}{ccc} & & A \\ & {}^{f_2}\nearrow & \\ E_2 & & \\ & {}_{g_2}\searrow & \\ & & B \end{array} \quad there\ exists\ \begin{array}{ccccc} & & A & & \\ & {}^{f_1}\nearrow & & \searrow^{u} & \\ E_1 & & \gamma_1\Downarrow & & C \\ & {}_{g_1}\searrow & & \nearrow_{v} & \\ & & B & & \end{array} ,\ \begin{array}{ccccc} & & A & & \\ & {}^{f_2}\nearrow & & \searrow^{u} & \\ E_2 & & \gamma_2\Downarrow & & C \\ & {}_{g_2}\searrow & & \nearrow_{v} & \\ & & B & & \end{array} ,$$

$with\ \gamma_1\ and\ \gamma_2\ invertible\ 2\text{-}cells.$

F2. Axiom F2 of pre 2-filtered.

F3. Given two 2-cells as in axiom F2, with $B = A$, and $u_1 = v_1$, $u_2 = v_2$, then there is a single 2-cell ε such that the LL-equation in F2 holds with ε in place of both α and β.

Remark Given two 2-cells as in axiom F2, with $B,\ C_1,\ C_2$ all equal to A, and $u_1,\ v_1,\ u_2,\ v_2$ all equal to id_A, then there exists $A \overset{w}{\longrightarrow} C$ such that $w\,\gamma_1 = w\,\gamma_2$. Note that this Kennison axiom BF2, see Definition 2.6.

Proof. It follows immediately from axioms F2 and F3 that there is an invertible 2-cell $\varepsilon : w_1 \Longrightarrow w_2$, such that $\varepsilon\,\gamma_1 = \varepsilon\,\gamma_2$. Cancelling εg, we deduce that $w\,\gamma_1 = w\,\gamma_2$ with $w = w_1$. □

2.1b Definition. A 2-category $\mathcal{A}$ is defined to be *2-filtered* when it is pseudo 2-filtered, non empty, and satisfies in addition the following axiom.

F0.

$$\textit{Given} \quad \begin{matrix} A \\ \\ B \end{matrix} \quad \textit{there exists} \quad \begin{matrix} A & & \\ & \searrow^{u} & \\ & & C \\ & \nearrow_{v} & \\ B & & \end{matrix} \; .$$

When $\mathcal{A}$ is a trivial 2-category (the only 2-cells are the identities), axiom F0 is the usual axiom in the definition of filtered category, while our axiom FF1 is equivalent to the conjunction of the two axioms PS1 and PS2 in the definition of pseudofiltered category (cf [1] Exposé I).

As was the case for axiom F1, in the presence of axiom WF3, axiom FF1 can be replaced by the weaker version in which we do not require the 2-cells γ_1 and γ_2 to be invertible.

The following properties of the construction LL follow for pseudo 2-filtered 2-categories and not for pre 2-filtered 2-categories.

4. *The replacements for the deleted lemmas, to appear just before Theorem 2.4, are:*

Lemma Given any pair of equivalent premorphisms

$$\begin{matrix} & A & \\ {}^{x}\nearrow & & \searrow^{u_1} \\ F & \xi_1\Downarrow & C_1 \\ {}_{y}\searrow & & \nearrow_{v_1} \\ & A & \end{matrix} \;\sim\; \begin{matrix} & A & \\ {}^{x}\nearrow & & \searrow^{u_2} \\ F & \xi_2\Downarrow & C_2 \\ {}_{y}\searrow & & \nearrow_{v_2} \\ & A & \end{matrix}$$

, if $u_1 = v_1$ and $u_2 = v_2$, then we can choose a homotopy defined by a single (invertible) 2-cell ε, $w_1 \stackrel{\varepsilon}{\Longrightarrow} w_2$, $(\varepsilon, \varepsilon) : \xi_1 \Rightarrow \xi_2$.

Proof. It follows immediately from axioms F2 and F3. □

Lemma Given two arrows $x \underset{\xi_2}{\overset{\xi_1}{\rightrightarrows}} y$ in FA, if $\lambda_A(\xi_1) = \lambda_A(\xi_2)$ in $\mathcal{L}(F)$, then there exists $A \stackrel{w}{\longrightarrow} C$ such that $w\,\xi_1 = w\,\xi_2$ in FC.

Proof. Recall the definition of λ_A in Theorem 1.19. By the previous Lemma with $C_1 = C_2 = C$, and u_1, v_1, u_2, v_2 all equal to id_C, it follows that there is a 2-cell $w_1 \stackrel{\varepsilon}{\Longrightarrow} w_2$, such that $\varepsilon\, \xi_1 = \varepsilon\, \xi_2$. Since ε is invertible, it follows that $w_1\, \xi_1 = w_1\, \xi_2$. Compare with the Remark after Definition 2.1a. □

7. *Minor corrections: on the second last line of page 82, delete the apostrophe in "ours"; on line 3 of page 84, replace "grateful to" by "grateful for".*

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