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INTERACTING OPEN DYNAMICS

Stéphane DUGOWSON

Résumé. Cet article présente les concepts de base d'une théorie systémique de l'interaction entre des dynamiques ouvertes non déterministes à temporalités variées. Elle comporte trois niveaux : la définition de ces dynamiques en tant que lax-foncteurs, la notion d'interaction — qui fait appel à des notions de requêtes, de synchronisations et de modes sociaux — et enfin l'engendrement de dynamiques globales ouvertes. L'aspect connectif des interactions est abordé, mais les autres aspects connectifs sont renvoyés à des travaux ultérieurs.
Abstract. This paper presents the basic concepts of a systemic theory of interaction between non-deterministic open dynamics with varying temporalities, which includes three stages: the definition of these dynamics as lax-functors, the notion of interaction — which uses some notions of requests, synchronizations and social modes (privacy) — and finally the generation of open global dynamics. Some connectivity structures of an interaction are defined, but the other aspects of dynamical connectivity are left to further work.
Keywords. Open Dynamics. Systemic. Interactivity. Lax functors. Categories. Complex Systems. Connectivity.
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Introduction

This article presents in English the fundamental concepts of our theory of interactivity between some open dynamics defined as kind of lax-functors to some 2-categories of sets with families of non-deterministic transitions as 1-cells. The origin of this work is linked to our research on connectivity structures [6], since connectivity has proven to be essentially dynamic in nature. In 2009, we began to study from a connectivity point of view some dynamics that were not necessarily deterministic, with durations taken in an arbitrary monoid. During a lecture in 2010 on these issues, an oral remark by Mme Andrée Ehresmann suggested that any small categories should be taken as duration systems. On this occasion, she mentioned her 1965 paper [1], where under the name of *guidable systems* she considers kinds of deterministic influenceable dynamical systems based on temporalities defined by small topological categories. At the end of the first section of the present paper, we precise some relations between this notion of "guidable systems" due to Mme Ehresmann and the one we developed on our side after the question of the interaction between our own non-deterministic dynamics based on various temporalities arose. In the course of our research on interactivity, we thus first considered open dynamics as defined by some functors said to be *disjunctive* and we sought to construct the global dynamics generated by the interaction of families of such open dynamics. The problem was to recognize that functors were insufficient because of a kind of instability: global dynamics were not always functors. We then had to extend our definitions to what we first called sub-functors ([11, 10, 13]), before Mathieu Anel and then Mme Ehresmann invited us to reformulate our definitions in terms of lax-functors [14]. Thanks to the lax-functorial stability theorem, presented at the beginning of the section 3, "Global Dynamics", we obtain a systemic theory where the dynamics generated by interactive families can in turn interact.

After this introduction and details on our notations and the 2-categories used in the paper, there are three sections:

- in the first section, we define what we call *open dynamics* thanks to notions of *multi-dynamics*, *mono-dynamics*, *clocks*, and morphisms between them. We also define some *parametric quotients* that are used in the third section. We then give a number of examples and, finally, we briefly describe some of the relations between our dynamics and Mme Ehresmann's guidable systems,

- in the second section we define precisely what we mean by an *interaction*. This is the only part of the article where we discuss connectivity structures, leaving the other connectivity aspects of dynamical interactivity to further work,

- finally, in the third section, the lax-functional stability theorem makes it possible to associate a number of global dynamics with a given interactive family, and we conclude the paper with two examples.

Notations and 2-categories at stake

Functions

The canonical inclusion $\varnothing \hookrightarrow \mathbf{R}$, that is the only real function defined on the empty set $\varnothing$, is denoted by $\underline{\varnothing}$. The restriction of a function f on a subset $D \subseteq \mathbf{R}$ is denoted $f_{|D}$. For any integer $k \in \mathbf{N}$, we set $\mathcal{C}^k = \bigcup_{D \in \mathcal{I}_\mathbf{R}} \mathcal{C}^k(D)$, where $\mathcal{I}_\mathbf{R}$ is the set of open real intervals and, for each interval D, $\mathcal{C}^k(D)$ is the set of real functions of class $\mathcal{C}^k$ defined on it[1]. The set $\mathcal{C}^0$ of all continuous real functions defined on open real intervals is also denoted by $\mathcal{C}$. Note that $\mathcal{C}^k(\varnothing) = \{\underline{\varnothing}\} \neq \varnothing$. For each interval $D \subseteq \mathbf{R}$, the set of metric maps $D \to \mathbf{R}$, that is the set of real Lipschitz maps with Lipschitz constant 1, is denoted $Lip_1(D)$.

Categories

As usual in our papers, for any category $\mathbf{D}$, we denote by $\dot{\mathbf{D}}$ the class of its objects, and $\overrightarrow{\mathbf{D}}$ the class of its arrows. For every arrow h, we denote by $\mathrm{dom}(h)$ its source object (or *domain*) and by $\mathrm{cod}(h)$ its target object (or *codomain*). $\mathbf{1} = (\bullet)$ is the terminal category, which has only one arrow $Id_\bullet$ that is also denoted by $\overrightarrow{0}$. The category of sets is denoted by $\mathbf{Sets}$. The discrete 2-category associated with any category $\mathbf{D}$ is again denoted as $\mathbf{D}$.

Transitions

For any sets U and V, we define a *transition from U to V* as a map $U \to \mathcal{P}(V)$ or, equivalently, as a binary relation $U \to V$. We often write $\varphi : U \rightsquigarrow V$ to indicate that φ is such a transition with $U = \mathrm{dom}(\varphi)$ and $V = \mathrm{cod}(\varphi)$. The *domain of definition* of φ is defined by $\mathrm{Def}_\varphi := \{u \in U, \varphi(u) \neq \varnothing\}$. Denoted

[1] If D is a singleton, we consider $\mathcal{C}^k(D)$ as the set of constant functions, *i.e.* $\mathcal{C}^k(D) \simeq \mathbf{R}$.

by $\psi \odot \varphi$, the composition of transitions $\varphi : U \rightsquigarrow V$ and $\psi : V \rightsquigarrow W$ is defined for all $u \in U$ by $\psi \odot \varphi(u) = \bigcup_{v \in \varphi(u)} \psi(v) \subseteq W$. A transition $f : U \rightsquigarrow V$ is said to be *hyper-deterministic*[2] if $card(f(u)) \leq 1$ for all $u \in U$. In this case, it is often considered as a partial function, and we denote it by writing $f : U \dashrightarrow V$. In particular, if $card(f(u)) = 1$ for all $u \in U$, it is said to be *deterministic* and it is considered and denoted as a total function $f : U \to V$.

The 2-categories $\mathbf{Tran}$ and $\mathbf{ParF}$

We denote by $\mathbf{Tran}$ the 2-category that has sets as objects, transitions as arrows with the composition defined above, and such that for each couple of sets (U, V) the category $\mathbf{Tran}(U, V)$ is given by ordering the set of transitions $U \rightsquigarrow V$ by *the constraint order* defined for all $\varphi, \psi \in \mathbf{Tran}(U, V)$ by

$$\varphi \leq \psi \Leftrightarrow \varphi \supseteq \psi$$

where $\varphi \supseteq \psi$ means that for all $u \in U$, $\varphi(u) \supseteq \psi(u)$. If $\varphi \leq \psi$, we say that ψ is more *constraining* than φ, or that φ is *laxer* than ψ. Thus, there exists a 2-cell $\varphi \Rightarrow \psi$ if and only if ψ is more constraining than φ.

We'll denote by $\mathbf{ParF}$ the sub-2-category of $\mathbf{Tran}$ obtained by keeping all sets as objects and, as 1-cells, only the hyper-deterministic transitions between them, that is partial functions. Thus we have these inclusions of 2-categories:

$$\mathbf{Sets} \subseteq \mathbf{ParF} \subseteq \mathbf{Tran}.$$

Given any small category $\mathbf{D}$, we write $\alpha : \mathbf{D} \rightharpoonup \mathbf{Tran}$ to indicate that α is a lax-functor from the discrete 2-category $\mathbf{D}$ to $\mathbf{Tran}$. Instead of $\alpha(S) \overset{\alpha(d)}{\rightsquigarrow} \alpha(T)$, the image of a $\mathbf{D}$-arrow $S \overset{d}{\to} T$ by α is denoted by $S^\alpha \overset{d^\alpha}{\rightsquigarrow} T^\alpha$.

Remark 0.1. *Of course, as a category,* $\mathbf{Tran}$ *coincides with* $\mathbf{Rel}$*, the category of sets with binary relations as arrows, but we prefer to emphasize the* transition *point of view with this notation. In [7], [8], [10] and [13], it was denoted* $\mathbf{P}$ *(for "possible").*

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[2]In our previous texts, these transitions were called "quasi-deterministic", but the expression "hyper-deterministic" is more coherent with the constraint order defined below.

Remark 0.2. *Given α and β two lax-functors from $\mathbf{D}$ to $\mathbf{Tran}$, we have to distinguish between the set — denoted* $\mathrm{nat}_{\mathbf{D}}(\alpha,\beta)$ *or* $\mathrm{nat}(\alpha,\beta)$ *— of all families of transitions* $(S^\alpha \overset{\delta_S}{\rightsquigarrow} S^\beta)_{S\in\dot{\mathbf{D}}}$ *such that*

$$\forall (S \xrightarrow{d} T) \in \overrightarrow{\mathbf{D}}, \delta_T \odot d^\alpha \subseteq d^\beta \odot \delta_S,$$

and the set $\mathrm{Nat}(\alpha,\beta)$ *of lax-natural transformations from α to β. Indeed, such a lax-natural transformation — denoted by $\delta : \alpha \looparrowright \beta$ — is defined not only by the data of the associated family* $(\delta_S)_{S\in\dot{\mathbf{D}}} \in \mathrm{nat}(\alpha,\beta)$*, but also by its domain α, and its codomain β. To underline this nuance, we sometimes write*

$$\mathrm{Nat}(\alpha,\beta) = \{(\alpha,\delta,\beta),\ \delta \in \mathrm{nat}(\alpha,\beta)\}.$$

For example, note that, if α_1, β_1, α_2 and β_2 are some lax-functors $\mathbf{D} \rightharpoonup \mathbf{Tran}$ such that $(\alpha_1,\beta_1) \neq (\alpha_2,\beta_2)$ but that, for all $S \in \dot{\mathbf{D}}$, $S^{\alpha_1} = S^{\alpha_2}$ and $S^{\beta_1} = S^{\beta_2}$ then, because domains or codomains differ, $\mathrm{Nat}(\alpha_1,\beta_1) \cap \mathrm{Nat}(\alpha_2,\beta_2) = \varnothing$*, while we can have, and we will often have,* $\mathrm{nat}(\alpha_1,\beta_1) \cap \mathrm{nat}(\alpha_2,\beta_2) \neq \varnothing$.

Some 2-categories of sets with L-families of transitions as arrows

For any non-empty set L we define a 2-category denoted by $\mathbf{Tran}^{\xrightarrow{L}}$ taking sets as 0-cells and, for each couple of sets (U,V), the category $\mathbf{Tran}^{\xrightarrow{L}}(U,V)$ being defined by

$$\mathbf{Tran}^{\xrightarrow{L}}(U,V) = (\mathbf{Tran}(U,V))^L.$$

In other words, for a given domain U and a given codomain V, a 1-cell φ in $\mathbf{Tran}^{\xrightarrow{L}}$ is an L-family $(\varphi_\lambda)_{\lambda\in L}$ of transitions $\varphi_\lambda : U \rightsquigarrow V$. We sometimes write $\varphi : U \rightsquigarrow_L V$ or $U \overset{\varphi}{\rightsquigarrow}_L V$ to indicate that φ is such a family.

The composition of 1-cells is naturally defined by

$$\varphi \odot \psi = (\varphi_\lambda)_{\lambda\in L} \odot (\psi_\lambda)_{\lambda\in L} = (\varphi_\lambda \odot \psi_\lambda)_{\lambda\in L},$$

and there is a 2-cell $\varphi \Rightarrow \psi$ if and only if $\varphi \leq \psi$, that is $\varphi_\lambda \supseteq \psi_\lambda$ for all $\lambda \in L$.

Similarly, we denote by $\mathbf{ParF}^{\xrightarrow{L}}$ the sub-2-category of $\mathbf{Tran}^{\xrightarrow{L}}$ obtained by keeping sets as objects and, as 1-cells, only the L-families of hyper-deterministic transitions between them, that is L-families of partial functions, and by $\mathbf{Sets}^{\xrightarrow{L}}$ the category of sets and, as arrows, L-families of total functions, so we have

$$\mathbf{Sets}^{\xrightarrow{L}} \subseteq \mathbf{ParF}^{\xrightarrow{L}} \subseteq \mathbf{Tran}^{\xrightarrow{L}}.$$

As in the case when L is a singleton, we write $\alpha : \mathbf{D} \rightharpoondown \mathbf{Tran}^{\xrightarrow{L}}$ to indicate that α is a lax-functor from the discrete 2-category $\mathbf{D}$ to $\mathbf{Tran}^{\xrightarrow{L}}$ and the image of a $\mathbf{D}$-arrow $S \overset{d}{\to} T$ by such an α is denoted by $S^\alpha \overset{d^\alpha}{\rightsquigarrow}_L T^\alpha$ instead of $\alpha(S) \overset{\alpha(d)}{\rightsquigarrow}_L \alpha(T)$.

1. Open dynamics

1.1 Multi-dynamics

1.1.1 L-dynamics on a category D

Let L be a non-empty set, and $\mathbf{D}$ a small category. A lax-functor $\alpha : \mathbf{D} \rightharpoondown \mathbf{Tran}^{\xrightarrow{L}}$ is said to be *disjunctive* if for all objects $S \neq T$ in $\mathbf{D}$, we have $S^\alpha \cap T^\alpha = \emptyset$.

Definition 1.1 (L-dynamics on $\mathbf{D}$). *A* multi-dynamic α on $\mathbf{D}$ with L as set of parameter values*, or simply an* L-dynamic on $\mathbf{D}$*, is a disjunctive lax-functor* $\alpha : \mathbf{D} \rightharpoondown \mathbf{Tran}^{\xrightarrow{L}}$.

For each $S \in \dot{\mathbf{D}}$, the elements of the set S^α are called the *states of* α *of type* S, and we denote by $st(\alpha)$ the set $\bigsqcup_{S \in \dot{\mathbf{D}}} S^\alpha$ of all states of α. The category $\mathbf{D}$ is called the *engine* of α, its arrows $(S \overset{d}{\to} T) \in \vec{\mathbf{D}}$ are called *durations*.

By definition of lax-functors between bicategories, an L-multi-dynamic α associates with each duration $(S \overset{d}{\to} T) \in \vec{\mathbf{D}}$ an L-family of transitions $d^\alpha = (d^\alpha_\lambda)_{\lambda \in L} : S^\alpha \rightsquigarrow_L T^\alpha$ such that, for each $\lambda \in L$ and any composable arrows $R \xrightarrow{d} S \xrightarrow{e} T$, we have

- (disjunctivity) $S \neq T \Rightarrow S^\alpha \cap T^\alpha = \emptyset$,
- (lax identity) $(Id_S)^\alpha_\lambda \subseteq Id_{S^\alpha}$,
- (lax composition) $(e \circ d)^\alpha_\lambda \subseteq e^\alpha_\lambda \odot d^\alpha_\lambda$.

A state $u \in S^\alpha$ such that $(Id_S)^\alpha_\lambda(u) = \varnothing$ is said to be *offside for the parameter value* $\lambda \in L$, and it is simply said to be *offside* if it is offside for all parameter values. A state that is not offside is said to be *onside*. If the lax-functor α is in fact a functor $\mathbf{D} \to \mathbf{Tran}^{\xrightarrow{L}}$, we say that the multi-dynamic α is *functorial* or *strict*. An L-dynamic on $\mathbf{D}$ is said to be *deterministic* (resp. *hyper-deterministic*) if for each duration $d \in \overrightarrow{\mathbf{D}}$ and each parameter value $\lambda \in L$, the transition d^α_λ is deterministic (resp. hyper-deterministic). In other words, a deterministic L-dynamic on $\mathbf{D}$ is a disjunctive functor[3] $\mathbf{D} \to \mathbf{Sets}^{\xrightarrow{L}}$, and a hyper-deterministic L-dynamic on $\mathbf{D}$ is a disjunctive lax-functor $\mathbf{D} \rightharpoonup \mathbf{ParF}^{\xrightarrow{L}}$.

Remark 1.2. *In [10], multi-dynamics were called* multi-dynamiques sous-catégoriques *— and* multi-dynamiques catégoriques *in the functorial case — whereas they were called* multi-dynamiques sous-fonctorielles *in [13].*

1.1.2 The category $\mathrm{MonoDyn}_\mathrm{D}$ of mono-dynamics on D

In the particular case where L is a singleton $\{*\}$, an L-dynamic is called a *mono-dynamic* (or simply a *dynamic*) on $\mathbf{D}$. Taking lax-natural transformations between mono-dynamics on $\mathbf{D}$ as morphisms, we obtain the category[4] $\mathbf{MonoDyn_D}$ of mono-dynamics on $\mathbf{D}$. These morphisms are called *dynamorphisms*, and we write $\delta : \alpha \looparrowright \beta$ to indicate that δ is a dynamorphism from α to β. Such a dynamorphism $\delta \in \mathbf{MonoDyn_D}(\alpha, \beta)$ is said to be *deterministic* (resp. *hyper-deterministic*) iff all transitions δ_S are deterministic (resp. hyper-deterministic).

Remark 1.3. *Following the remark 0.2, we have to distinguish between* $\mathrm{nat}(\alpha, \beta)$ *and* $\mathbf{MonoDyn_D}(\alpha, \beta) = \mathrm{Nat}(\alpha, \beta)$. *Nevertheless, as long as there is no ambiguity, we shall denote as usual by a same letter a lax-natural transformation* δ *and the corresponding family* $\delta = (\delta_S)_{S \in \dot{\mathbf{D}}}$ *of transitions. Also note that disjunctivity of* α *implies that a family of transitions* $(\delta_S : S^\alpha \rightsquigarrow S^\beta)_{S \in \dot{\mathbf{D}}}$ *can be seen as a single transition* $\delta : st(\alpha) \rightsquigarrow st(\beta)$

[3] Obviously, an L-dynamic that is deterministic is necessarily functorial.

[4] In [10], mono-dynamics were called *dynamiques sous-catégoriques* — and *dynamiques catégoriques* in the functorial case — and the category $\mathbf{MonoDyn_D}$ was denoted by $\mathbf{DySC_{(D)}}$. In [13], they were called *mono-dynamiques sous-fonctorielles*.

with, for each $S \in \dot{\mathbf{D}}$ *and each* $u \in S^\alpha$, $\delta(u) = \delta_S(u) \subseteq S^\beta \subseteq st(\beta)$. *Then, a dynamorphism* $\delta : \alpha \nrightarrow \beta$ *is often seen as such a transition* $st(\alpha) \rightsquigarrow st(\beta)$.

1.1.3 Clocks on $\mathbf{D}$

Definition 1.4. *A monodynamic* $\mathbf{h}$ *on* $\mathbf{D}$ *that is deterministic is called a* clock *on* $\mathbf{D}$*. Its states are called* $\mathbf{h}$−instants *(or simply* instants*).*

Thus, a clock on $\mathbf{D}$ is nothing but a disjunctive functor $\mathbf{D} \to \mathbf{Sets}$. A pre-order relation, called *anteriority* and denoted by $\leq_{\mathbf{h}}$, is defined on $st(\mathbf{h})$ by

$$(s \leq_{\mathbf{h}} t) \Leftrightarrow (\exists e \in \vec{\mathbf{D}}, e^{\mathbf{h}}(s) = t)$$

for all instants s and t. We define the category $\mathbf{Clocks_D}$ of clocks on $\mathbf{D}$ taking deterministic dynamorphisms as morphisms between them. It is equivalent to the topos of presheaves on $\mathbf{D}^{op}$.

1.1.4 The category $L - \mathbf{Dyn_D}$ of L-dynamics on $\mathbf{D}$

We denote by $L - \mathbf{Dyn_D}$ the category whose objects are L-dynamics on $\mathbf{D}$, and with arrows $\delta : \alpha \nrightarrow \beta$ — called $(\mathbf{D}, L)$*-dynamorphisms* — given by the families of transitions $(S^\alpha \overset{\delta_S}{\rightsquigarrow} S^\beta)_{S \in \dot{\mathbf{D}}}$ that are lax-natural from the mono-dynamic α_λ to the mono-dynamic β_λ for all $\lambda \in L$, that is such that

$$\forall \lambda \in L, \forall (S \overset{d}{\to} T) \in \vec{\mathbf{D}}, \delta_T \odot d_\lambda^\alpha \subseteq d_\lambda^\beta \odot \delta_S.$$

Following the remark 0.2, we can formulate this by writing

$$L - \mathbf{Dyn_D}(\alpha, \beta) = \{(\alpha, \delta, \beta), \delta \in \bigcap_{\lambda \in L} \mathrm{nat}(\alpha_\lambda, \beta_\lambda)\}$$

or even, with the usual omission of domain and codomain when there is no ambiguity, by $L - \mathbf{Dyn_D}(\alpha, \beta) = \bigcap_{\lambda \in L} \mathrm{nat}(\alpha_\lambda, \beta_\lambda)$.

1.1.5 The category $\mathrm{MultiDyn}_\mathbf{D}$ of multi-dynamics on $\mathbf{D}$

Let L and M be some non-empty sets, and $\alpha : \mathbf{D} \rightharpoondown \mathbf{Tran}^{\overset{L}{\to}}$ and $\beta : \mathbf{D} \rightharpoondown \mathbf{Tran}^{\overset{M}{\to}}$ be multi-dynamics on $\mathbf{D}$.

Definition 1.5. *A* $\mathbf{D}$-dynamorphism $\alpha \looparrowright \beta$ *is a couple* (θ, δ) *with* $\theta : L \to M$ *a map, and*[5] $\delta \in \bigcap_{\lambda \in L} \mathrm{nat}_{\mathbf{D}}(\alpha_\lambda, \beta_{\theta(\lambda)})$.

Thus, to be a dynamorphism, (θ, δ) must satisfy the lax-naturality condition

$$\forall \lambda \in L, \forall (S \xrightarrow{d} T) \in \overrightarrow{\mathbf{D}}, \delta_T \odot d^\alpha_\lambda \subseteq d^\beta_{\theta(\lambda)} \odot \delta_S.$$

We then obtain the category $\mathbf{MultiDyn_D}$ taking as objects all multi-dynamics on $\mathbf{D}$, and as arrows all $\mathbf{D}$-dynamorphisms between them. Naturally, such a dynamorphism (θ, δ) is said to be *(hyper-)deterministic* if for every object $S \in \dot{\mathbf{D}}$, δ_S is (hyper-)deterministic.

Remark 1.6. *For any set* L *with* $card(L) \geq 2$, $L - \mathbf{Dyn_D}$ *is a non-full subcategory of* $\mathbf{MultiDyn_D}$, *whereas* $\mathbf{MonoDyn_D}$ *is a full one. We can in particular consider dynamorphisms between mono-dynamics on* $\mathbf{D}$ *and multi-dynamics on* $\mathbf{D}$. *For example, if* L *is a non-empty set,* α *an* L*-dynamic on* $\mathbf{D}$, *and* $\mathbf{h}$ *a clock on the same engine, then a dynamorphism* $\mathfrak{s} : \mathbf{h} \looparrowright \alpha$ *is a couple* $\mathfrak{s} = (\lambda, \sigma)$ *with* $\lambda \in L$ *and* $\sigma = (S^{\mathbf{h}} \overset{\sigma_S}{\rightsquigarrow} S^\alpha)_{S \in \dot{\mathbf{D}}}$ *such that*

$$\forall (S \xrightarrow{d} T) \in \overrightarrow{\mathbf{D}}, \sigma_T \odot d^{\mathbf{h}} \subseteq d^\alpha_\lambda \odot \sigma_S,$$

whereas a dynamorphism $\tau : \alpha \looparrowright \mathbf{h}$ *is a family of transitions* $\tau = (S^\alpha \overset{\tau_S}{\rightsquigarrow} S^{\mathbf{h}})_{S \in \dot{\mathbf{D}}}$ *such that*

$$\forall (S \xrightarrow{d} T) \in \overrightarrow{\mathbf{D}}, \forall \lambda \in L, \tau_T \odot d^\alpha_\lambda \subseteq d^{\mathbf{h}} \odot \tau_S.$$

1.1.6 The category $\mathrm{MultiDyn}$ of multi-dynamics

Let $\alpha : \mathbf{D} \rightharpoonup \mathbf{Tran}^{\overset{L}{\to}}$ and $\beta : \mathbf{E} \rightharpoonup \mathbf{Tran}^{\overset{M}{\to}}$ be multi-dynamics with possibly different sets of parameter values and different engines.

Definition 1.7. *A* dynamorphism $\alpha \looparrowright \beta$ *consists, in addition to the data of* α *and* β, *of that of a triple* (θ, Δ, δ) *with*

- $\theta : L \to M$ *a map,*
- $\Delta : \mathbf{D} \to \mathbf{E}$ *a functor,*

[5] Using the notation explained in the remark 0.2.

- $\delta \in \bigcap_{\lambda \in L} \mathrm{nat}_{\mathbf{D}}(\alpha_\lambda, \beta_{\theta(\lambda)} \circ \Delta)$.

The last condition means that the lax-naturality condition

$$\forall \lambda \in L, \forall (S \overset{d}{\to} T) \in \overrightarrow{\mathbf{D}}, \delta_T \odot d^\alpha_\lambda \subseteq (\Delta d)^\beta_{\theta(\lambda)} \odot \delta_S$$

has to be satisfied. The category $\mathbf{MultiDyn}$ of multi-dynamics is then defined taking as objects all multi-dynamics, and as arrows all dynamorphisms between them. The full subcategory of $\mathbf{MultiDyn}$ obtained taking mono-dynamics (resp. clocks) as objects is denoted by $\mathbf{MonoDyn}$ (resp. $\mathbf{Clocks}$). In general, $\mathbf{MonoDyn_D}$ (resp. $\mathbf{Clocks_D}$) is a non-full subcategory of $\mathbf{MonoDyn}$ (resp. $\mathbf{Clocks}$).

1.2 Open dynamics: definition, realizations, quotients

1.2.1 Definition of open dynamics

Definition 1.8. *An* open dynamic A with engine $\mathbf{D}$ *is the data*

$$A = \Big((\alpha : \mathbf{D} \to \mathbf{Tran}^{\overset{L}{\to}}) \overset{\rho}{\nrightarrow} \mathbf{h} \Big)$$

of

- *a non-empty set* L *of parameter values,*
- *an* L*-dynamic* $\alpha \in L - \mathbf{Dyn_D}$*,*
- *a clock* $(\mathbf{h} : \mathbf{D} \to \mathbf{Sets}) \in \mathbf{Clocks_D}$*,*
- *a* deterministic *dynamorphism* $\rho \in \mathbf{MultiDyn_D}(\alpha, \mathbf{h})$ *called* datation.

An open dynamic with engine $\mathbf{D}$ is also called an open dynamic *on* $\mathbf{D}$. An open dynamic is said to be *intemporal* if its engine is the terminal category $\mathbf{1}$. The states of α are also called the states of A, thus we set: $st(A) = st(\alpha)$. If the parametric set L is a singleton, A is said to be an *open mono-dynamic* or, sometimes, an *opaque dynamic*.

Remark 1.9. *For each $\lambda \in L$, we naturally denote by A_λ the open mono-dynamic obtained by restricting parametric values to λ, that is*

$$A_\lambda = \Big((\alpha_\lambda : \mathbf{D} \to \mathbf{Tran}) \overset{\rho}{\looparrowright} (\mathbf{h} : \mathbf{D} \to \mathbf{Sets}) \Big).$$

According to the definitions given in **§ 2.4.2** of [10] and **§ 1.2.2** of [13], a *dynamorphism* from an open dynamic

$$A = \Big((\alpha : \mathbf{D} \to \mathbf{Tran}^{\overset{L}{\to}}) \overset{\rho}{\looparrowright} (\mathbf{h} : \mathbf{D} \to \mathbf{Sets}) \Big)$$

to an open dynamic

$$B = \Big((\beta : \mathbf{E} \to \mathbf{Tran}^{\overset{M}{\to}}) \overset{\tau}{\looparrowright} (\mathbf{k} : \mathbf{E} \to \mathbf{Sets}) \Big)$$

is a quadruplet $(\theta, \Delta, \delta, \varepsilon)$ with

- $(\theta, \Delta, \delta) \in \mathbf{MultiDyn}(\alpha, \beta)$,
- $(\Delta, \varepsilon) \in \mathbf{MonoDyn}(\mathbf{h}, \mathbf{k})$,
- this lax synchronization condition satisfied:

$$\forall S \in \dot{\mathbf{D}}, \tau_{\Delta_S} \odot \delta_S \subseteq \varepsilon_S \odot \rho_S.$$

We denote by $\mathbf{ODyn}$ the category of all open dynamics, with dynamorphisms as arrows.

1.2.2 Realizations of an open dynamic

Let $A = \Big((\alpha : \mathbf{D} \to \mathbf{Tran}^{\overset{L}{\to}}) \overset{\rho}{\looparrowright} (\mathbf{h} : \mathbf{D} \to \mathbf{Sets}) \Big)$ be an open dynamic.

Definition 1.10. *A* realization *(or a* solution*) of A is a hyper-deterministic dynamorphism $(\mathfrak{s} : \mathbf{h} \looparrowright \alpha) \in \mathbf{MultiDyn}_{\mathbf{D}}(\mathbf{h}, \alpha)$ such that the lax condition*

$$\rho \odot \mathfrak{s} \subseteq Id_{\mathbf{h}}$$

be satisfied.

In other words[6], a realization of A is a couple $\mathfrak{s} = (\lambda, \sigma)$ with $\lambda \in L$ and $\sigma : st(\mathbf{h}) \dashrightarrow st(\alpha)$ a partial function defined on a subset $\mathrm{Def}_\sigma \subseteq st(\mathbf{h})$ such that:

1. $\forall t \in \mathrm{Def}_\sigma, \rho(\sigma(t)) = t$,
2. $\forall S \in \dot{\mathbf{D}}, \forall t \in S^{\mathbf{h}} \cap \mathrm{Def}_\sigma, \sigma(t) \in S^{\alpha}$,
3. $\forall (S \overset{d}{\to} T) \in \overrightarrow{\mathbf{D}}, \forall t \in S^{\mathbf{h}}$,

$$d^{\mathbf{h}}(t) \in \mathrm{Def}_\sigma \Rightarrow \left[t \in \mathrm{Def}_\sigma \text{ and } \sigma(d^{\mathbf{h}}(t)) \in d^{\alpha}_{\lambda}(\sigma(t)) \right].$$

The set of realizations of A is denoted by $\mathfrak{S}_A$. Given $\mathfrak{s} = (\lambda, \sigma) \in \mathfrak{S}_A$, we call λ the *parametric part* or the *incoming part* of this realization, σ its *outgoing part*, and we set

$$\mathrm{In}(\mathfrak{s}) := \lambda \quad \text{and} \quad \mathrm{Out}(\mathfrak{s}) := \sigma.$$

Outgoing parts of realizations of A is often called *outgoing realizations of A* — or even simply *realizations*, if there is no ambiguity — and their set is denoted by Z_A. Thus, we have[7]

$$Z_A = \bigcup_{\lambda \in L} Z_{A_\lambda}.$$

A realization of A is said to be empty if its outgoing part is the empty function $st(\mathbf{h}) \supset \varnothing \hookrightarrow st(\alpha)$. This empty function is denoted by $\underline{\varnothing}_A$, or simply $\underline{\varnothing}$, if there is no ambiguity. We always have $Z_A \ni \underline{\varnothing}_A$, and we denote by Z^*_A the set of non-empty outgoing realizations of A:

$$Z^*_A = Z_A \setminus \{\underline{\varnothing}_A\}.$$

An open dynamic A is said to be *efficient* if the set Z^*_A is non empty.

[6] See [13], **§ 1.3.1.**

[7] For any $\lambda \in L$, the set of realizations of the open (mono) dynamic A_λ is simply given by $\mathfrak{S}_{A_\lambda} = \{\lambda\} \times Z_{A_\lambda}$, so this latter set Z_{A_λ} of outgoing parts of realizations of A_λ is often simply called the set of its realizations.

Realizations passing through a state.

Definition 1.11. *Given an open dynamic A, we say that a realization $\mathfrak{s} = (\lambda, \sigma)$ of A* passes through *a state $a \in st(A)$ — or equivalently that the outgoing part σ of $\mathfrak{s}$ passes through a — and we write*

$$\mathfrak{s} \rhd a \quad (\mathrm{or, equivalently} : \sigma \rhd \mathrm{a})$$

if $\sigma(\rho(a)) = a$.

More generally, if E is a set of states of A, we write

$$\mathfrak{s} \rhd E \quad (\mathrm{or, equivalently} : \sigma \rhd E)$$

to say that σ passes through every $a \in E$. If E is a finite set $E = \{a_1, ..., a_n\}$, we can also write

$$\sigma \rhd a_1, ..., a_n.$$

1.2.3 Parametric quotients

Let $\alpha : \mathbf{D} \rightarrow \mathbf{Tran}^{\overset{L}{\rightarrow}}$ be a multi-dynamic with engine $\mathbf{D}$ and parametric set L.

Proposition 1.12. *If $\sim$ is an equivalence relation on L, and $M = L/\sim$ is the quotient set of L by $\sim$, then the relation*

$$\forall \mu \in M, \beta_\mu = \bigcup_{\lambda \in \mu} \alpha_\lambda,$$

that is

- $\forall S \in \dot{\mathbf{D}}$, $S^\beta = S^\alpha$,
- $\forall (e : S \to T) \in \overrightarrow{\mathbf{D}}, \forall a \in S^\beta, e^\beta_\mu(a) = \bigcup_{\lambda \in \mu} e^\alpha_\lambda(a)$,

defines a multi-dynamic β on $\mathbf{D}$ with parametric set M.

Proof. For every $\mu \in L/\sim$, and each $S \in \dot{\mathbf{D}}$, we have[8]

$$(Id_S)^\beta_\mu = \bigcup_{\lambda \in \mu} (Id_S)^\alpha_\lambda \subseteq \bigcup_{\lambda \in \mu} Id_{S^\alpha} = Id_{S^\beta},$$

[8]Where the order relation $\leq$ is of course the constraint order $\varphi \leq \psi \Leftrightarrow \varphi \supseteq \psi$.

that is $(Id_S)^{\beta}_{\mu} \geq Id_{S^{\beta}}$.

Furthermore, for each couple of composable arrows $R \xrightarrow{f} S \xrightarrow{g} T$ in $\mathbf{D}$, we have

$$(g \circ f)^{\beta}_{\mu} = \bigcup_{\lambda \in \mu} (g \circ f)^{\alpha}_{\lambda} \geq \bigcup_{\lambda \in \mu} (g^{\alpha}_{\lambda} \odot f^{\alpha}_{\lambda}),$$

but for each $\lambda \in \mu$, we have $f^{\alpha}_{\lambda} \subseteq f^{\beta}_{\mu}$, and the same for g, and then

$$g^{\alpha}_{\lambda} \odot f^{\alpha}_{\lambda} \subseteq g^{\beta}_{\mu} \odot f^{\beta}_{\mu},$$

so

$$(g \circ f)^{\beta}_{\mu} \subseteq g^{\beta}_{\mu} \odot f^{\beta}_{\mu},$$

that is

$$(g \circ f)^{\beta}_{\mu} \geq g^{\beta}_{\mu} \odot f^{\beta}_{\mu}.$$

□

Definition 1.13 (Parametric quotient of a dynamic)**.** *The multi-dynamic* $\beta : \mathbf{D} \rightharpoonup \mathbf{Tran}^{\xrightarrow{M}}$ *defined in proposition 1.12 by*

$$\forall \mu \in M, \beta_{\mu} = \bigcup_{\lambda \in \mu} \alpha_{\lambda},$$

is called the parametric quotient of α by $\sim$ *and is denoted by* $\beta = \alpha/\sim$*.*
For any open dynamic

$$A = \Big((\alpha : \mathbf{D} \rightharpoonup \mathbf{Tran}^{\xrightarrow{L}}) \overset{\rho}{\looparrowright} (\mathbf{h} : \mathbf{D} \to \mathbf{Sets}) \Big)$$

and any equivalence relation $\sim$ *on* L*, we define in the same way the* quotient open dynamic $B = A/\sim$ *setting*

$$B = \Big(((\alpha/\sim) : \mathbf{D} \rightharpoonup \mathbf{Tran}^{\xrightarrow{(L/\sim)}}) \overset{\tilde{\rho}}{\looparrowright} (\mathbf{h} : \mathbf{D} \to \mathbf{Sets}) \Big)$$

where, for every $b \in S^{\alpha/\sim} = S^{\alpha}$*,* $\tilde{\rho}(b) = \rho(b)$*.*

1.3 Examples of open dynamics

Example 1.14 (Bushaw's dynamics)**.** In her 1965 article [1], Andrée Bastiani (-Ehresmann) cited Donald W. Bushaw's 1963 article [3] in which this one introduced some continuous *dynamical polysystems* that correspond — leaving aside topological aspects — to our deterministic open dynamics with the group $(\mathbf{R}, +)$ as engine and with clock the real *existential clock*[9] $\xi = \xi_{(\mathbf{R},+)}$ (defined by $st(\xi) = \mathbf{R}$ and $d^{\xi}(t) = t + d$ for all reals t and d):

$$(\alpha : (\mathbf{R}, +) \to \mathbf{Sets}^{\xrightarrow{L}}) \overset{\rho}{\looparrowright} (\xi : (\mathbf{R}, +) \to \mathbf{Sets})$$

such that the following additional condition ("non-anticipation") be satisfied: for all $\lambda_1, \lambda_2 \in L$ and $t_0 \in \mathbf{R}$ there exists a unique $\lambda \in L$ such that, for all states $s \in st(\alpha)$ with $\tau(s) = t_0$, we have

- $\forall d \in \mathbf{R}_-, d^{\alpha}_{\lambda}(s) = d^{\alpha}_{\lambda_1}(s)$,
- $\forall d \in \mathbf{R}_+, d^{\alpha}_{\lambda}(s) = d^{\alpha}_{\lambda_2}(s)$.

Thus, with each Bushaw's dynamical polysystem is canonically associated a deterministic open multi-dynamic on $\mathbf{R}$. Reciprocally, by choosing convenient topological structures on the set of states and on the set of parameter values, some Bushaw's dynamical polysystem(s) can be associated with each deterministic open multi-dynamic on $\mathbf{R}$ endowed with the existential clock ξ and satisfying the "non-anticipation" property.

Realizations of Bushaw's dynamics. $(\mathbf{R}, +)$ being a group, every non-empty outgoing realization of the considered deterministic open dynamic is defined on the whole real line and, with the topological assumptions of Bushaw's paper, it is necessarily continuous. For its part, Bushaw doesn't explicitly define the realizations (solutions) of his systems. Nevertheless, for each $\varphi \in L$, Bushaw denotes again by φ the map $E \times \mathbf{R} \to E$, where $E = st(\alpha)$, defined with our notations by $\varphi(e, d) = d^{\alpha}_{\varphi}(e)$. Then, for each given state $e \in E$, the map $\sigma : \mathbf{R} \ni t \mapsto \varphi(e, t - \tau(e)) \in E$ constitutes the single realization of the considered deterministic dynamic such that $\sigma(\tau(e)) = e$. It is defined on all $\mathbf{R}$, and it is continuous. Thus, for each λ, there is an

[9] About the existential clock of a category, see [6].

implicitly notion of realization that coincides with ours, even if some notion of partial solution could perhaps be closer to the spirit of his work (because of the local aspect of the parameters λ).

Example 1.15 (Φ, a deterministic intemporal mono-dynamic)**.** An intemporal dynamic is functorial if and only if it is deterministic, and in this case its behavior cannot depend on any parameter, since the image by the dynamic of the only duration $\overrightarrow{0}$ is necessarily the identity of the set of states. For example, we can consider the deterministic intemporal monodynamic Φ for which the set of states is $\{0,1\}$, that is

$$\Phi = \Big((\phi : \mathbf{1} \to \mathbf{Sets}) \overset{!}{\looparrowright} (\xi_{\mathbf{1}} : \mathbf{1} \to \mathbf{Sets}) \Big)$$

where

- $st(\phi) = \bullet^{\phi} = \{0,1\}$,
- $\overrightarrow{0}^{\phi} = Id_{\{0,1\}}$,
- $\xi_{\mathbf{1}}$ is the canonical clock[10] of $\mathbf{1}$, which has only one instant 0,

and $\phi \overset{!}{\looparrowright} \xi_{\mathbf{1}}$ is the necessarily constant dynamorphism.

Realizations of Φ. We immediately see that

$$\mathfrak{S}_{\Phi} = Z_{\Phi} = \{\underline{\varnothing}_{\Phi}, 0, 1\}.$$

Example 1.16 (Υ, a one-step deterministic cell)**.** We set

$$\Upsilon = \Big((\upsilon : \mathbf{D}_{\Upsilon} \to \mathbf{Tran}^{L_{\Upsilon}}_{\rightarrow}) \overset{!}{\looparrowright} (\zeta_{\mathbf{D}_{\Upsilon}} : \mathbf{D}_{\Upsilon} \to \mathbf{Sets}) \Big)$$

where

[10]That is both the existential clock and the essential clock of $\mathbf{1}$. About the existential clock and the essential clock of a category, see [6].

- $\mathbf{D}_\Upsilon = (T_0 \overset{d}{\to} T_1) \simeq (\bullet \to \bullet)$, the category with two objects and a single non-trivial arrow between them, which we call the *one-step category*,
- $\zeta_{\mathbf{D}_\Upsilon}$ is the *essential clock*[11] of $\mathbf{D}_\Upsilon$, for which the set of instants associated with each T_k is a singleton, say ${T_k}^{\zeta_{\mathbf{D}_\Upsilon}} = \{t_k\}$,
- $\forall k \in \{0,1\}$, $(T_k)^\upsilon = \{t_k\} \times \{0,1\}$,
- $L_\Upsilon = \{0,1\}^{\{0,1\}}$,
- $\forall \lambda \in L_\Upsilon$, $\forall k \in \{0,1\}$, $(Id_{(T_k)})^\upsilon_\lambda = Id_{((T_k)^\upsilon)}$ (since Υ is functorial),
- $\forall \lambda \in L_\Upsilon$, $\forall s \in \{0,1\}$, $d^\upsilon_\lambda(t_0, s) = (t_1, \lambda(s))$,
- $\upsilon \overset{!}{\looparrowright} \zeta_{\mathbf{D}_\Upsilon}$ is the unique possible deterministic dynamorphism here (since there is a unique instant for each temporal type $T_k \in \dot{\mathbf{D}}_\Upsilon$).

Realizations of Υ. An outgoing realization of Υ can be identified with some partial function $\sigma : \{t_0, t_1\} \dashrightarrow \{0,1\}$ such that $\mathrm{Def}_\sigma \in \{\varnothing, \{t_0\}, \{t_0, t_1\}\}$. With this identification, we can write $\mathfrak{S}_\Upsilon$ as the set of all couples (λ, σ) with $\lambda \in L_\Upsilon$ and $\sigma = \underline{\varnothing}_\Upsilon$, or $\sigma \in \{0,1\}^{\{t_0\}}$, or $\sigma \in \{0,1\}^{\{t_0,t_1\}}$ with $\sigma(t_1) = \lambda(\sigma(t_0))$. Then,

$$Z_\Upsilon = \{\underline{\varnothing}_\Upsilon\} \cup \{0,1\}^{\{t_0\}} \cup \{0,1\}^{\{t_0,t_1\}}.$$

Example 1.17 (Υ_*, a one-step hyper-deterministic cell)**.** This is a functorial hyper-deterministic variant of the example 1.16, keeping the same engine $\mathbf{D}_{\Upsilon_*} = \mathbf{D}_\Upsilon = (T_0 \overset{d}{\to} T_1)$, the same states and the same clock $\zeta = \zeta_{\mathbf{D}_\Upsilon}$ but including new parameter values to permit a state to ask to "*exit the game*". More precisely,

$$\Upsilon_* = \left((\upsilon_* : \mathbf{D}_{\Upsilon_*} \to \mathbf{Tran}^{\overset{L_{\Upsilon_*}}{\longrightarrow}}) \overset{!}{\looparrowright} \zeta \right)$$

where

[11] See [6].

- $\forall k \in \{0,1\}$, $(T_k)^{\upsilon_*} = (T_k)^{\upsilon} = \{t_k\} \times \{0,1\}$,
- $L_{\Upsilon_*} = \{*, 0, 1\}^{\{0,1\}}$,
- $\forall \lambda \in L_{\Upsilon_*}$, $\forall k \in \{0,1\}$, $(Id_{(T_k)})^{\upsilon_*}_{\lambda} = Id_{((T_k)^{\upsilon_*})}$ (because Υ_* is functorial),
- $\forall \lambda \in L_{\Upsilon_*}$, $\forall s \in \{0,1\}$,
  if $\lambda(s) = *$ then $d^{\upsilon_*}_{\lambda}(t_0, s) = \varnothing$,
  if $\lambda(s) \in \{0,1\}$ then, like with Υ, $d^{\upsilon_*}_{\lambda}(t_0, s) = \{(t_1, \lambda(s))\}$.

In other words, viewing $d^{\upsilon_*}_{\lambda}$ as a partial function, it is defined for $s \in \{0,1\}$ by :

- if $\lambda(s) = *$ then $(t_0, s) \notin \mathrm{Def}_{d^{\upsilon_*}_{\lambda}}$,
- if $\lambda(s) \in \{0,1\}$ then $d^{\upsilon_*}_{\lambda}(t_0, s) = (t_1, \lambda(s))$.

Realizations of Υ_*. As in the case of Υ, we can write $\mathfrak{S}_{\Upsilon_*}$ as the set of all couples (λ, σ) with $\lambda \in L_{\Upsilon_*}$ and $\sigma = \underline{\varnothing}_{\Upsilon_*}$, or $\sigma \in \{0,1\}^{\{t_0\}}$, or $\sigma \in \{0,1\}^{\{t_0,t_1\}}$ with $\sigma(t_1) = \lambda(\sigma(t_0))$ (which implies that $\lambda(\sigma(t_0)) \neq *$). And we have $Z_{\Upsilon_*} = Z_{\Upsilon}$.

Example 1.18 (Γ, a hyper-deterministic intemporal lax-dynamic)**.** This is a hyper-deterministic variant of the example 1.15, with the same set of states and the same clock, but depending on parameter values. Precisely, we set

$$\Gamma = \left((\gamma : \mathbf{1} \to \mathbf{Tran} \overset{L_\Gamma}{\to}) \overset{!}{\looparrowright} (\xi_1 : \mathbf{1} \to \mathbf{Sets}) \right)$$

where

- $st(\gamma) = \bullet^{\gamma} = \{0,1\}$,
- $L_\Gamma = \{a, b\}$, a set with two elements,
- $\overrightarrow{0}^{\gamma}_{a} = Id_{\{0,1\}}$,

- $\overrightarrow{0}^{\gamma}_{b}$ is defined as a transition[12] by $\overrightarrow{0}^{\gamma}_{b}(0) = \varnothing$ and $\overrightarrow{0}^{\gamma}_{b}(1) = \{1\}$,

- $\xi_{\mathbf{1}}$ is the canonical clock[13] of $\mathbf{1}$ and $\gamma \overset{!}{\looparrowright} \xi_{\mathbf{1}}$ is the constant dynamorphism.

Realizations of Γ. The set $Z^*_\Gamma = \{0,1\}$ of nonempty outgoing realizations of Γ is the same as for Φ, but now we have

$$\mathfrak{S}_\Gamma = \{(a, \underline{\varnothing}_\Gamma), (a,0), (a,1), (b, \underline{\varnothing}_\Gamma), (b,1)\}.$$

Example 1.19 ($\mathbb{W}$ = ו, an intemporal open dynamic with functions as states)**.** The open dynamic $\mathbb{W}$ — also denoted by the Hebrew letter ו (vav) — described in this example 1.19 has been given in [13] and [18] together with a dynamic denoted by $\mathbb{H}$ or by the Hebrew letter ה (hey) — see *infra*, example 1.20 — and a third one denoted by $\mathbb{Y}$ or י (yod) (example 1.21) to produce the interactive family that we will describe in the example 2.14, section 2.6. The choice of the Hebrew letter ו comes from the fact that this dynamic is intended to (approximately and partially) model the philosophical concept that P. M. Klein [19] named in the same letter. The dynamic $\mathbb{W}$ = ו is defined by

$$\mathbb{W} = \Big((\alpha_{\mathbb{W}} : \mathbf{1} \rightharpoondown \mathbf{Tran}^{\overset{L_{\mathbb{W}}}{\rightarrow}}) \overset{!}{\looparrowright} (\xi_{\mathbf{1}} : \mathbf{1} \to \mathbf{Sets}) \Big),$$

where[14]

- $st(\mathbb{W}) = \bullet^{\alpha_{\mathbb{W}}} = \mathcal{C}$,

- $L_{\mathbb{W}} = \mathcal{C}$,

- for all $\lambda \in L_{\mathbb{W}}$, the transition $\overrightarrow{0}^{\alpha_{\mathbb{W}}}_{\lambda}$ is defined for all $f \in st(\mathbb{W})$ by

$$\overrightarrow{0}^{\alpha_{\mathbb{W}}}_{\lambda}(f) = \begin{cases} \{f\} \text{ if } f \diamond \lambda, \\ \varnothing \text{ in other cases,} \end{cases}$$

[12]Equivalently, $\overrightarrow{0}^{\gamma}_{b}$ can be defined as a partial function by $0 \notin \mathrm{Def}_{\overrightarrow{0}^{\gamma}_{b}}$ and $\overrightarrow{0}^{\gamma}_{b}(1) = 1$.

[13]See the example 1.15.

[14]For the meaning of $\mathcal{C}$, see notations in the begining of the paper.

- $\xi_{\mathbf{1}}$ is the canonical clock of $\mathbf{1}$, and $\alpha_{\mathbb{W}} \overset{!}{\looparrowright} \xi_{\mathbf{1}}$ is the constant dynamorphism,

where $f \diamond \lambda$ stands for $f_{|\mathrm{Def}_f \cap \mathrm{Def}_\lambda} = \lambda_{|\mathrm{Def}_f \cap \mathrm{Def}_\lambda}$.

Realizations of $\mathbb{W}$. For each $\lambda \in L_{\mathbb{W}} = \mathcal{C}$, the empty realization $\underline{\varnothing}_{\mathbb{W}_\lambda} = \underline{\varnothing}_{\mathbb{W}}$ is the partial function $st(\xi_{\mathbf{1}}) = \{0\} \dashrightarrow st(\mathbb{W}) = \mathcal{C}$ with an empty domain (or, as a transition, the map $0 \mapsto \varnothing \subset \mathcal{C}$) whereas a nonempty realization of $\mathbb{W}_\lambda$ can be identified with its value on the only instant $0 \in st(\xi_{\mathbf{1}})$, this value being itself a real function $f \in \mathcal{C}$, possibly the empty real function $\underline{\varnothing}_{\mathbf{R}}$. Then, with this identification, we have

$$\mathfrak{S}_{\mathbb{W}} = \bigcup_{\lambda \in \mathcal{C}} \left(\{(\lambda, f), f \in \mathcal{C}, f \diamond \lambda\} \cup \{(\lambda, \underline{\varnothing}_{\mathbb{W}_\lambda})\} \right).$$

The set of nonempty outgoing realizations of $\mathbb{W}$ is then

$$Z^*_{\mathbb{W}} = Z_{\mathbb{W}} \smallsetminus \{\underline{\varnothing}_{\mathbb{W}}\} = \mathcal{C}.$$

Note that the empty real function $\underline{\varnothing}_{\mathbf{R}}$ belongs to $Z^*_{\mathbb{W}}$.

Example 1.20 ($\mathbb{H}$ = ה, a hyper-deterministic dynamic on $\mathbf{R}_+$)**.** The dynamic $\mathbb{H}$ — also referred to as ה, “hey” in the Hebrew alphabet — has been introduced in [13] and [18] under the name “history” to constitute an interactive family together with $\mathbb{Y}$ = י (*cf. infra*, example 1.21) and $\mathbb{W}$ = ו (*cf. supra*, example 1.19). It is a hyper-deterministic functorial open dynamic with engine $(\mathbf{R}_+, +)$ and with a clock $\mathbf{h}_{]T_0,+\infty[}$ having instants $t \in]T_0, +\infty[$ where T_0, called the *origin of times*, is taken to be $\{-\infty\} \cup \mathbf{R}$. We distinguish the origin of times T_0 with the origin of histories which here will be taken to be $-\infty$. More precisely, such a $T_0 \in \{-\infty\} \cup \mathbf{R}$ having been chosen, we set

$$\mathbb{H} = ה = \left(((\mathbf{R}_+, +) \overset{\alpha_{\mathbb{H}}}{\to} \mathbf{Tran} \overset{L_{\mathbb{H}}}{\to}) \overset{\tau_{\mathbb{H}}}{\looparrowright} \mathbf{h}_{]T_0,+\infty[} \right),$$

with

- $st(\alpha_{\mathbb{H}}) = \bigcup_{t \in]T_0,+\infty[} (\{t\} \times \mathcal{C}^1(]-\infty, t[))$,

- $st(\mathbf{h}_{]T_0,+\infty[}) =]T_0, +\infty[$,
- $L_{\mathbb{H}} = \mathcal{C}^*_{]T_0,\to[} := \bigcup_{u\in]T_0,+\infty]} \mathcal{C}(]T_0, u[)$,
- $\forall(t, f) \in st(\alpha_{\mathbb{H}})$, $\tau_{\mathbb{H}}(t, f) = t$,
- $\forall(t, f) \in st(\alpha_{\mathbb{H}})$, $\forall d \in \mathbf{R}^*_+$, $\forall u \in]T_0, +\infty]$, $\forall \lambda \in \mathcal{C}(]T_0, u[)$,
  - if $t + d \leq u$ and if there exists a (necessarily unique) $g \in \mathcal{C}^1(]-\infty, t+d[)$ such that $g_{|]-\infty,t[} = f$ and $g_{|]t,t+d[} = \lambda_{|]t,t+d[}$, then we set $d^{\alpha_{\mathbb{H}}}_{\lambda}(t, f) = (t + d, g)$,
  - in all other cases, we set $d^{\alpha_{\mathbb{H}}}_{\lambda}((t, f)) = \varnothing$ that is, viewing $d^{\alpha_{\mathbb{H}}}_{\lambda}$ as a partial function: $(t, f) \notin \mathrm{Def}_{d^{\alpha_{\mathbb{H}}}_{\lambda}}$.

Realizations of $\mathbb{H}$. It is easy to see that the outgoing part σ of a nonempty realization $(\lambda, \sigma) \in \mathfrak{S}_{\mathbb{H}}$ can be uniquely represented by a real function of class $\mathcal{C}^1$ defined on an interval of the form $]-\infty, a[$ or $]-\infty, a]$, with $a > T_0$, that coincides with λ on $]T_0, a[$. More precisely, with these representations, we verify that we can write

$$\mathfrak{S}_{\mathbb{H}} = \{(\underline{\varnothing}, \underline{\varnothing})\} \cup \left[\bigcup_{u\in]T_0,+\infty]} \left(\bigcup_{\lambda\in\mathcal{C}^1(]T_0,u[)} (\{\lambda\} \times \bigcup_{a\in]T_0,u]} E_{\lambda,a}) \right) \right]$$

with $E_{\lambda,+\infty} = \{\sigma \in \mathcal{C}^1(\mathbf{R}), \sigma_{|]T_0,+\infty[} = \lambda\}$ whereas

$$E_{\lambda,a} = \{\sigma \in \mathcal{C}^1(]-\infty, a[) \cup \mathcal{C}^1(]-\infty, a]), \sigma_{|]T_0,a[} = \lambda_{|]T_0,a[}\}$$

when $a < +\infty$. Thus, the set of outgoing realizations of $\mathbb{H}$ is

$$Z_{\mathbb{H}} = \{\underline{\varnothing}\} \cup \left(\bigcup_{a\in]T_0,+\infty]} \mathcal{C}^1(]-\infty, a[) \right) \cup \left(\bigcup_{a\in]T_0,+\infty[} \mathcal{C}^1(]-\infty, a]) \right).$$

For any nonempty realization $\sigma \in Z^*_{\mathbb{H}}$ of $\mathbb{H}$, we call the restriction $\sigma_{|]-\infty,T_0]}$ the *mythical part* of σ.

Example 1.21 ($\mathbb{Y}$ = י, a non-deterministic functorial mono-dynamic)**.** Introduced in [13] and [18] as a “future” dynamic together with $\mathbb{W}$ (*cf. supra*, example 1.19) and $\mathbb{H}$ (example 1.20), the dynamic that we designate by $\mathbb{Y}$ or י (yod) and call a “lipschitzian source”, is defined by

$$\mathbb{Y} = \left((\alpha_{\mathbb{Y}} : (\mathbf{R}_+, +) \to \mathbf{Tran}) \overset{\tau_{\mathbb{Y}}}{\looparrowright} \xi_{\mathbf{R}_+} \right),$$

where

- $\xi_{\mathbf{R}_+}$ is the existential clock associated with the monoïd $(\mathbf{R}_+, +)$, that is such that $st(\xi_{\mathbf{R}_+}) = \mathbf{R}_+$ and $d^{\xi_{\mathbf{R}_+}}(t) = t + d$ for all instants $t \in \mathbf{R}_+$ and all durations $d \in \mathbf{R}_+$,
- the set of states is $st(\alpha_{\mathbb{Y}}) = \mathbf{R}_+ \times \mathbf{R}$,
- for all states $(t, a) \in st(\alpha_{\mathbb{Y}})$,

  $\tau_{\mathbb{Y}}(t, a) = t$,

  and for all $d \in \mathbf{R}_+$, $d^{\alpha_{\mathbb{Y}}}(t, a) = \{t + d\} \times [a - d, a + d]$.

Realizations of $\mathbb{Y}$. It is immediate to see that a realization $\sigma \in Z_{\mathbb{Y}} = \mathfrak{S}_{\mathbb{Y}}$ is a partial function $\mathbf{R}_+ \dashrightarrow \mathbf{R}_+ \times \mathbf{R}$ defined on an interval D of the form $[0, a]$ or $[0, a[$ that can be identified with a metric map[15] $\sigma : D \to \mathbf{R}$:

$$Z_{\mathbb{Y}} \simeq \bigcup_{a \in \mathbf{R}_+ \cup \{+\infty\}} Lip_1([0, a[) \cup \bigcup_{a \in \mathbf{R}_+} Lip_1([0, a])$$

where $Lip_1(D) = \{\sigma : D \to \mathbf{R}, \sigma \,\mathrm{is\,a\,metric\,map}\}$.

1.4 Some relations with Bastiani (-Ehresmann)’s control systems

In her article [1], published in 1967, Andrée Bastiani (-Ehresmann) considered some *control systems* — called *systèmes guidables* in French — which, leaving aside topological aspects, seem to be quite close to some of our open systems which we have developed, as indicated in the introduction to this paper, with a view to proposing a theory of interactivity, from a first categorical generalization of some “closed” dynamical systems — namely mono-dynamics on monoids (see section 1.1.2) — a generalization itself prompted

[15] That is a Lipschitz function with Lipschitz constant 1.

by an oral remark by Mme Ehresmann. To lay the foundations for a further exploration of the possible connections between these two notions, we reformulated in our own language Bastiani (-Ehresmann)'s definitions, which was originally given in the language and notations introduced by Charles Ehresmann in his book *Catégories et Structures* [15] and which have been more recently rapidly mentioned again by Mme Ehresmann in two lectures [16, 17], with more current notations. Leaving aside, as announced, topological aspects, it then turns out that the definition of a control system given by Mme Ehresmann is equivalent to considering the data (F, q) of a disjunctive functor $\mathbf{G} \overset{F}{\to} \mathbf{ParF}$ and of a functor $\mathbf{G} \overset{q}{\to} \mathbf{H}$, with $\mathbf{G}$ and $\mathbf{H}$ some categories which we will assume to be small. Intuitively, the objects of $\mathbf{H}$ can be seen as *instants*, and its arrows as *durations* whereas the objects of $\mathbf{G}$ can be seen as "parameterized instants" and its arrows as "parameterized durations". With the functor F is then defined a set $E := \bigsqcup_{g\in\dot{\mathbf{G}}} F(g)$ whose elements we shall see as "parameterized states", and a partial action of $\mathbf{G}$ on E given for all $\gamma \in \overrightarrow{\mathbf{G}}$ and all $e \in \mathrm{Def}_{F(\gamma)} \subseteq F(\mathrm{dom}(\gamma)) \subseteq E$ by $\gamma.e := F(\gamma)(e)$. A *solution on a subcategory* $\mathbf{S} \subseteq \mathbf{H}$ of the control system (F, q) then consists in a couple $(\dot{\mathbf{S}} \overset{\varphi}{\to} E, \mathbf{S} \overset{\psi}{\to} \mathbf{G})$ where $\psi : \mathbf{S} \to \mathbf{G}$ is a functor and φ is a map that associates with each instant $t \in \dot{\mathbf{S}}$ a parameterized state $\varphi(t) \in E$, such that we have $\dot{q} \circ p \circ \varphi = Id_{\dot{\mathbf{S}}}$, $q \circ \psi = Id_{\overrightarrow{\mathbf{S}}}$ and, for all $h \in \overrightarrow{\mathbf{S}}$, $\psi(h).\varphi(t_1) = \varphi(t_2)$ where $t_1 = \mathrm{dom}(h)$ and $t_2 = \mathrm{cod}(h)$.

An interpretation of these definitions in relation with ours is given by the following association with each functorial hyper-deterministic open dynamic

$$A = \Big((\alpha : \mathbf{D} \to \mathbf{ParF}^{\overset{L}{\to}}) \overset{\rho}{\nrightarrow} (\mathbf{h} : \mathbf{D} \to \mathbf{Sets}) \Big)$$

of a Bastiani (-Ehresmann)'s control system

$$GS(A) = (\mathbf{G} \overset{F}{\to} \mathbf{ParF}, \mathbf{G} \overset{q}{\to} \mathbf{H}),$$

namely the one given by

- $\dot{\mathbf{H}} := st(\mathbf{h})$,
- $\overrightarrow{\mathbf{H}} := \{(t_1, d, t_2) \in \dot{\mathbf{H}} \times \overrightarrow{\mathbf{D}} \times \dot{\mathbf{H}}, d^{\mathbf{h}}(t_1) = t_2\}$, with obvious source, target and composition,

- $\mathbf{G} := L \times \mathbf{H}$ (where L is seen as a discrete category),
- $q : \mathbf{G} \to \mathbf{H}$ is the projection on $\mathbf{H}$, that is the forgetting of the parameter:

$$q\Big((\lambda, t_1) \overset{(\lambda,t_1,d,t_2)}{\longrightarrow} (\lambda, t_2)\Big) := \Big(t_1 \overset{(t_1,d,t_2)}{\longrightarrow} t_2\Big),$$

- for all $(\lambda, t) \in \dot{\mathbf{G}}$, $F(\lambda, t) := \{\lambda\} \times \rho^{-1}(t) \subset L \times st(\alpha)$,
- for all $((\lambda, t_1) \overset{(\lambda,t_1,d,t_2)}{\longrightarrow} (\lambda, t_2)) \in \overrightarrow{\mathbf{G}}$ and all $s \in \rho^{-1}(t_1)$,
  - if $s \in \mathrm{Def}_{d^\alpha_\lambda}$ then $F(\lambda, t_1, d, t_2)(\lambda, s) := (\lambda, d^\alpha_\lambda(s))$,
  - else $(\lambda, s) \notin \mathrm{Def}_{F(\lambda,t_1,d,t_2)}$.

It is then straightforward to verify that every realization $(\lambda, \sigma) \in \mathfrak{S}_A$ gives a solution (φ, ψ) of the control system $GS(A)$ over the full subcategory $\mathbf{S} \subseteq \mathbf{H}$ defined by $\dot{\mathbf{S}} = \mathrm{Def}_\sigma$, namely the couple (φ, ψ) given by $\varphi(t) = (\lambda, \sigma(t)) \in E = \bigsqcup_{g \in \dot{\mathbf{G}}} F(g)$ for every $t \in \dot{\mathbf{S}}$ and $\psi(t_1, d, t_2) = (\lambda, t_1, d, t_2) \in \overrightarrow{\mathbf{G}}$ for every $(t_1, d, t_2) \in \overrightarrow{\mathbf{S}}$.

The association $A \mapsto GS(A)$ gives us a first idea of the possible relationships between our open systems and Mme Ehresmann's control systems, each with their own limitations. Let us make a few comments on this. First, not that GS is not injective (up to isomorphism) since, for example

- the open dynamic $A = \Big((\xi : (\mathbf{R}_+, +) \to \mathbf{Sets}) \overset{Id}{\looparrowright} (\xi : (\mathbf{R}_+, +) \to \mathbf{Sets})\Big)$, where $st(\xi) = \mathbf{R}_+$ and, for every $d \in \mathbf{R}_+$ and every $t \in \mathbf{R}_+$, $d^\xi(t) = t + d$,
- and the open dynamic $B = \Big((\zeta : (\mathbf{R}_+, \leq) \to \mathbf{Sets}) \overset{Id}{\looparrowright} (\zeta : (\mathbf{R}_+, \leq) \to \mathbf{Sets})\Big)$, where for every $t \in \mathbf{R}_+$ we have $t^\xi = \{t\}$ and, for every $d = (t_1 \leq t_2) \in \overrightarrow{(\mathbf{R}_+, \leq)}$, we have $d^\zeta(t_1) = t_2$,

are not isomorphic, but $GS(A)$ and $GS(B)$ are essentially the same control systems, the important difference between the categories $(\mathbf{R}_+, +)$ and $(\mathbf{R}_+, \leq)$ being lost in translation.

In addition, while our open dynamics are not necessarily deterministic whereas Mme Ehresmann's control systems could be said to be hyperdeterministic, the formulation we obtained of a control system as a couple

$(\mathbf{G} \overset{F}{\to} \mathbf{ParF}, \mathbf{G} \overset{q}{\to} \mathbf{H})$ suggests a non-deterministic generalization, given by couples of the form $(\mathbf{G} \overset{F}{\to} \mathbf{Trans}, \mathbf{G} \overset{q}{\to} \mathbf{H})$. As we said in our introduction, the necessity to use lax-functors instead of functors in our own theory came from our treatment of interactivity (cf. theorem 3.1). If a theory of interacting "control systems" would be developed, it could lead as well to consider lax-functorial non-deterministic systems given by couples of the form $(\mathbf{G} \overset{F}{\rightharpoonup} \mathbf{Trans}, \mathbf{G} \overset{q}{\to} \mathbf{H})$ which could be the subject of further research.

On the other side, GS isn't surjective either. In particular, note that the design of control systems gives to their "parametrical" aspects — which are implied both in the category $\mathbf{G}$ of "parameterized instants" and in the set E of "parameterized states" — a local nature, as opposed to the parameters of our open dynamics, which are on the contrary global in nature, and this can be viewed as an advantage of control systems.

Finally, according to our definition of the realizations of an open dynamic, note that even in the case when a control system G is of the form $GS(A)$ with A an open dynamic in the sense of our theory, a solution of G that is defined over a subcategory $\mathbf{S} \subseteq \mathbf{H}$ that does not satisfy the property

$$\forall (t_1, d, t_2) \in \overrightarrow{\mathbf{H}},\ t_2 \in \dot{\mathbf{S}} \Rightarrow (t_1, d, t_2) \in \overrightarrow{\mathbf{S}}$$

cannot be obtained from a realization of A : thanks to a greater partiality, Bastiani (-Ehresmann)'s control systems have more solutions[16] than our open dynamics, and this can be seen as another advantage of control systems. Of course, it would be easy to broaden in turn our definition of realizations of open dynamics to include more partiality, but the real difficulties will then arise in interacting with other dynamics: how can a complex system work when some of its components are removed or added ? This type of question, linked to the philosophical problem known as the "Ship of Theseus", seems to us to be at the core of Andrée Ehresmann's research work, but in its current state our own theory does not yet allow us to address it correctly since our collective *global dynamics* need all their components to be "simultaneously"[17] active to obtain a realization defined at the corresponding instant.

[16] At least as they are defined in [1] since this notion of partial solutions does not appear at all in [16] and [17].

[17] For some *synchronization*.

2. Interactive families

The main purpose of this section is to give the definition of interactive families, namely interacting families of open dynamics. For this, we firstly give some reminders about *binary relations*, *multiple relations* and *multiple binary relations* (**§ 2.1**), then we give the definitions of an *interaction request* and of an *interaction relation* between some open dynamics (**§ 2.2**) and the definition of a *synchronization* between these dynamics (**§ 2.3**). An interaction request (or an interaction relation) and a synchronization then define an *interaction* in the family of open dynamics under consideration, and such an interaction — together with a third element, called *privacy* or *social mode* — leads in turn to the definition of an *interactive family* (**§ 2.4**). In **§ 2.5**, we associate four connectivity structures with any given interactive family, in particular the *realization connectivity structure of the interaction relation*, which is the most important and which we simply call *the connectivity structure* of the considered interactive family. Finally, in **§ 2.6**, we give some examples of interactive families.

2.1 Binary, multiple and multiple binary relations

2.1.1 Binary relations

Given E and E' two sets, a binary relation B from E to E' is defined by its *domain* $E = \mathrm{dom}(B)$, its *codomain* $E' = \mathrm{cod}(B)$ and its *graph* $|B| \subset E \times E'$. According to the introduction of the paper, we also consider such a binary relation as a (not necessarily deterministic) transition $E \rightsquigarrow E'$, that is a map $E \to \mathcal{P}(E')$, setting for any $e \in E$

$$B(e) = \{e' \in E', (e, e') \in |B|\}.$$

Then the *image* of B is defined by

$$\mathrm{Im}(B) = \bigcup_{e \in E} B(e) \subset E',$$

the *converse binary relation*, denoted as B^{-1} or $B^{\top}$, is defined by its graph

$$|B^{\top}| := |B|^{\top} = \{(e, e')^{\top}, (e, e') \in |B|\} \quad \text{where}\, (e, e')^{\top} := (e', e)$$

or, equivalently, by

$$\forall e' \in E', B^{\top}(e') = \{e \in E, B(e) \ni e'\},$$

and the *domain of definition* of B is given by

$$\mathrm{Def}_B = \{e \in E, B(e) \neq \varnothing\} = \mathrm{Im}(B^{\top}).$$

The set of binary relations from E to E' is denoted by $\mathbf{BR}_{(E,E')}$, and the class of all binary relations is denoted by $\mathbf{BR}$.

2.1.2 Multiple relations

In this section and the next, we recall the definitions we gave in [12] and [11] about *multiple relations* and *multiple binary relations*[18]. Given $\mathcal{E} = (E_i)_{i\in I}$ a family of sets indexed by a set I, the product $\prod_{i\in I} E_i$ is also denoted as $\Pi_I\mathcal{E}$ or $\Pi\mathcal{E}$.

Definition 2.1. *A multiple relation R is the data $R = (I, \mathcal{E}, |R|)$ of*

- *a set $I = \mathrm{ar}(R)$, called the* index set *or the* arity *of R,*
- *an I-family of sets $\mathcal{E} = (E_i)_{i\in I}$ called the* context of *R,*
- *a subset $|R| \subseteq \Pi_I\mathcal{E}$, called the* graph of *$R$.*

The class of all multiple relations with a given index set I — which are also called *I-relations* — is denoted by $\mathbf{MR}_I$. Given a context $\mathcal{E} = (E_i)_{i\in I}$ on I, the set of multiple relations with context $\mathcal{E}$ is denoted as $\mathbf{MR}_{\mathcal{E}}$. For example, if 2 denotes the set $\{0,1\}$, the class $\mathbf{MR}_2$ can be seen as the class $\mathbf{BR}$ of all binary relations between sets and, given (E_0, E_1) a couple of sets, we have $\mathbf{MR}_{(E_0,E_1)} = \mathbf{BR}_{(E_0,E_1)}$.

If R and S are multiple relations in a context $\mathcal{E}$, we'll denote by $R \cap S$ their intersection, that is the multiple relation in the same context such that $|R \cap S| = |R| \cap |S|$, and we define an order $(\mathbf{MR}_{\mathcal{E}}, \subseteq)$ by putting $R \subseteq S$ when $R = R \cap S$. If $J \subseteq I$, we put $\mathcal{E}_{|J} = (E_j)_{j\in J}$, $\Pi_J\mathcal{E} = \Pi(\mathcal{E}_{|J}) = \prod_{j\in J} E_j$ and we designate by 0_J the minimum element of $(\mathbf{MR}_{\mathcal{E}_{|J}}, \subseteq)$, that is the empty J-relation $0_J = (J, \mathcal{E}_{|J}, \varnothing)$, and by 1_J its maximum element, that is the plain

[18] Or, as well: *binary multiple relations*.

J-relation $1_J = (J, \mathcal{E}_{|J}, \Pi_J\mathcal{E})$. Note that in the case where $J = \varnothing$, we have $0_\varnothing \neq 1_\varnothing$, since the graph of $1_\varnothing$ is a singleton $\Pi_\varnothing\mathcal{E} = \{\bullet\}$, whereas $|0_J| = \varnothing$. If $R \in \mathbf{MR}_{\mathcal{E}_{|J}}$, we also denote by $R_{|K}$ the restriction of R on $K \subseteq J$, that is the K-relation defined by $R_{|K} := (K, \mathcal{E}_{|K}, |R|_{|K})$, where $|R|_{|K} = \{y_{|K}, y \in |R|\}$ or, equivalently, $|R|_{|K} = \{x \in \Pi_K\mathcal{E}, \exists y \in |R|, \forall k \in K, x_k = y_k\}$. Finally, we denote by $\mathbf{MR}_{\subseteq\mathcal{E}}$ the set of multiple relations *inside* the context $\mathcal{E}$, that is the set of all multiple relations $R = (J, \mathcal{E}_{|J}, |R|)$ with $J \subseteq I$ and $|R| \subseteq \Pi_J\mathcal{E}$. In other words $\mathbf{MR}_{\subseteq\mathcal{E}} = \bigcup_{J\subseteq I} \mathbf{MR}_{\mathcal{E}_{|J}} \subset \bigcup_{J\subseteq I} \mathbf{MR}_J$. The set $\mathbf{MR}_{\subseteq\mathcal{E}}$ can be endowed with a "gluing operator" $\otimes$ defined[19] for a J_1-relation R_1 and a J_2-relation R_2 as the $(J_1 \cup J_2)$-relation $R_1 \otimes R_2$ containing all "glued" families $x_1 + x_2$ with some *compatible* $x_n \in |R_n|$, that is such that x_1 and x_2 have the same restrictions on $J_1 \cap J_2$. In other words, for every $x \in \Pi_{J_1\cup J_2}\mathcal{E}$, we have $x \in |R_1 \otimes R_2|$ if and only if $x_{|J_1} \in |R_1|$ and $x_{|J_2} \in |R_2|$. Note also that the relation $\mathbf{1} = 1_\varnothing$ is neutral for this operator, giving $(\mathbf{MR}_{\subseteq\mathcal{E}}, \otimes, \mathbf{1})$ a structure of a commutative monoïd, whereas 0_I is an annihilating element.

Remark 2.2. *The intersection of two multiple relations in a given context is nothing but a peculiar case of the gluing operator $\otimes$ applied to relations inside a same context and having a same arity.*

2.1.3 Multiple binary relations

Definition 2.3. *A* multiple binary relation *Q is the data $(I, \mathcal{W}, \mathcal{M}, |Q|)$ of*

- *a set $I = \mathrm{ar}(Q)$, called the* index set *or the* arity *of Q,*
- *an I-family of sets $\mathcal{W} = (W_i)_{i\in I}$ called the* incoming context of *Q,*
- *an I-family of sets $\mathcal{M} = (M_i)_{i\in I}$ called the* outgoing context of *Q,*
- *a subset $|Q| \subseteq \Pi_I\mathcal{E}$ called the* graph of *Q, where $\mathcal{E} = (E_i)_{i\in I}$ is given by $E_i = W_i \times M_i$ for all $i \in I$ and is called the* product context of *Q.*

The class of all multiple binary relations with a given index set I — which are also called *I-multiple binary relations* or *I-binary relations* — is denoted by $\mathbf{MBR}_I$. The set of all multiple binary relations with given

[19] See [12], section § **1.5.1**, where it was denoted by $\bowtie$.

incoming context $\mathcal{W} = (W_i)_{i\in I}$ and outgoing context $\mathcal{M} = (M_i)_{i\in I}$ is denoted by $\mathbf{MBR}_{(\mathcal{W},\mathcal{M})}$ and, as in the case of multiple relations, we'll denote $\mathbf{MBR}_{\subseteq(\mathcal{W},\mathcal{M})}$ the set of multiple binary relations *inside* the context $(\mathcal{W},\mathcal{M})$, that is the set of all multiple relations $R = (J, \mathcal{W}_{|J}, \mathcal{M}_{|J}, |R|)$ with $J \subseteq I$, and $|R| \subseteq \Pi_J \mathcal{E} = \prod_{j\in J}(W_j \times M_j)$. A gluing operator $\otimes$ is defined on $\mathbf{MBR}_{\subseteq(\mathcal{W},\mathcal{M})}$ exactly in the same way that for $\mathbf{MR}_{\subseteq\mathcal{E}}$: if, for $n \in \{1,2\}$, we have $R_n = (J_n, \mathcal{W}_{|J_n}, \mathcal{M}_{|J_n}, |R_n|)$, then $R_1 \otimes R_2$ designates the multiple binary relation R with arity $J = J_1 \cup J_2$, with context $(\mathcal{W}_{|J}, \mathcal{M}_{|J})$ and with graph $|R| = \{y \in \Pi_J \mathcal{E}, \forall n \in \{1,2\}, y_{|J_n} \in |R_n|\}$. When R_1 and R_2 have the same arity, $R_1 \otimes R_2$ can be simply denoted by $R_1 \cap R_2$, and we obtain an order on $\mathbf{MBR}_{(\mathcal{W}_{|J},\mathcal{M}_{|J})}$ by putting $R_1 \subseteq R_2$ iff $R_1 \cap R_2 = R_1$.

2.1.4 Type conversions between $\mathbf{MBR}_I$, $\mathbf{MR}_{2I}$, $\mathbf{MR}_I$ and $\mathbf{BR}$

With any $Q = (I, \mathcal{W}, \mathcal{M}, |Q|) \in \mathbf{MBR}_I$, we associate the I-multiple relation $mr(Q) := (I, \mathcal{E}, |Q|)|$ where $\mathcal{E} = (W_i \times M_i)_{i\in I}$. Note that the gluing operator $\otimes$ defined on $\mathbf{MBR}_{\subseteq(\mathcal{W},\mathcal{M})}$ can then be defined from the operator $\otimes$ defined on $\mathbf{MR}_{\subseteq\mathcal{E}}$ by the fact that, for R_1 and R_2 belonging to $\mathbf{MBR}_{\subseteq(\mathcal{W},\mathcal{M})}$, we have $mr(R_1 \otimes R_2) = mr(R_1) \otimes mr(R_2)$. Of course, if R_1 and R_2 have the same arity, we also have $mr(R_1 \cap R_2) = mr(R_1) \cap mr(R_2)$. With each $Q \in \mathbf{MBR}_I$, we also associate the binary relation[20] $br(Q) : \Pi_I \mathcal{W} \rightsquigarrow \Pi_I \mathcal{M}$ that has graph $|br(Q)|$ given by $|Q|$ after an obvious re-indexing. Note that the applications $mr : \mathbf{MBR}_I \to \mathbf{MR}_I$ and $br : \mathbf{MBR}_I \to \mathbf{BR}$ so defined are injective on non-empty relations[21]. In particular, we will often define the graph $|Q|$ of a multiple binary relation $Q \in \mathbf{MBR}_{(\mathcal{W},\mathcal{M})}$ by giving, for all $w \in \Pi_I \mathcal{W}$, the set $br(Q)(w) \subset \Pi_I \mathcal{M}$.

Note also that, applying notations for binary relations, we have:

$$\mathrm{Im}(br(Q)) = \bigcup_{w\in\Pi_I(\mathcal{W})} br(Q)(w) \subset \Pi_I \mathcal{M},$$

$$\forall \mu \in \Pi_I \mathcal{M}, br(Q)^{\intercal}(\mu) = \{w \in \Pi_I(\mathcal{W}), br(Q)(w) \ni \mu\},$$

and

$$\mathrm{Def}_{br(Q)} = \{w \in \Pi_I(\mathcal{W}), br(Q)(w) \neq \varnothing\} = \mathrm{Im}(br(Q)^{\intercal}).$$

[20]Recall that we often see binary relations as (not necessarily deterministic) transitions.

[21]Because if $|Q| \neq \varnothing$, then $\Pi_I \mathcal{W} \times \Pi_I \mathcal{M} \neq \varnothing$ and, in this case, this product characterizes all sets W_i and M_i.

Moreover, by putting $2I = I_0 \cup I_1$ where, for $k \in \{0,1\}$, $I_k = I \times \{k\}$, we define canonical reciprocal bijections

$$mr_2 : \mathbf{MBR}_I \leftrightarrow \mathbf{MR}_{2I} : mbr$$

in a trivial way: for any $Q = (I, \mathcal{W}, \mathcal{M}, |Q|) \in \mathbf{MBR}_I$, where $\mathcal{W} = (W_i)_{i\in I}$ and $\mathcal{M} = (M_i)_{i\in I}$, we set $mr_2(Q) = (2I, \mathcal{D}, \widetilde{|Q|})$ where $\mathcal{D} = (D_j)_{j\in I_0\cup I_1}$ with, for each $i \in I$, $D_{(i,0)} = W_i$ and $D_{(i,1)} = M_i$, and $\widetilde{|Q|}$ is the image of $|Q|$ given by the canonical bijection $\Pi_I(W_i \times M_i) \to \Pi_{2I}\mathcal{D}$.

2.2 Interaction relations in a family of open dynamics

From now on, I denotes a non-empty set and $\mathcal{A} = (A_i)_{i\in I}$ an I-family of open dynamics $A_i = \Big((\alpha_i : \mathbf{D}_i \rightharpoonup \mathbf{Tran}^{\overset{L_i}{\to}}) \overset{\rho_i}{\looparrowright} (\mathbf{h}_i : \mathbf{D}_i \to \mathbf{Sets})\Big)$. For each $i \in I$, the set Z_{A_i} of outgoing realizations of A_i is simply denoted by Z_i — thus Z_i^* denotes the set of nonempty realizations of A_i — and, for any $\lambda \in L_i$, the set of (outgoing) realizations of the open mono-dynamic $(A_i)_\lambda$ is denoted by $Z_{i,\lambda}$ instead of $Z_{(A_i)_\lambda}$. We also put $\mathcal{Z} := (Z_i)_{i\in I}$, $\mathcal{Z}^* := (Z_i^*)_{i\in I}$, $\mathcal{L} := (L_i)_{i\in I}$ and $\mathcal{E} := (E_i)_{i\in I}$ where, for each $i \in I$, $E_i := Z_i \times L_i$. The elements $\mathfrak{q}$ of $\Pi_I\mathcal{E} \simeq \Pi_I\mathcal{Z} \times \Pi_I\mathcal{L}$ are often denoted as in the following form:

$$\mathfrak{q} = \begin{pmatrix} \lambda_i \\ \sigma_i \end{pmatrix}_{i\in I} \tag{1}$$

with, for all $i \in I$, $\sigma_i \in Z_i$ and $\lambda_i \in L_i$. The coefficients of $\mathfrak{q}$ is sometimes designated for all $i \in I$ by $\mathfrak{q}_i := \sigma_i$ and $\mathfrak{q}^i := \lambda_i$. With these notations, such a $\mathfrak{q} \in \Pi_I\mathcal{E}$ is said to be *coherent (for the family $\mathcal{A}$)* if, for all $i \in I$, $\mathfrak{q}_i \in Z_{i,\mathfrak{q}^i}$. More generally, a set $C \subset \Pi_I\mathcal{E}$ is said to be *coherent* if all its elements are coherent, and a multiple binary relation $Q \in \mathbf{MBR}_{(\mathcal{Z},\mathcal{L})}$ is said to be *coherent* if its graph $|Q|$ is also coherent. The multiple binary relation whose graph is the maximal coherent one is denoted $\Omega_\mathcal{A}$. Then a multiple binary relation $Q \in \mathbf{MBR}_{(\mathcal{Z},\mathcal{L})}$ is coherent if $Q \subseteq \Omega_\mathcal{A}$, that is if $|Q| \subseteq |\Omega_\mathcal{A}|$. Remark that

$$|\Omega_\mathcal{A}| = (\Pi_I\mathfrak{S}_i)^\top := \{\mathfrak{q} \in \Pi_I\mathcal{E}, \mathfrak{q}^\top \in \Pi_I\mathfrak{S}_i\},$$

where $\mathfrak{q}^\top := \begin{pmatrix} \sigma_i \\ \lambda_i \end{pmatrix}_{i\in I} \in \prod_{i\in I}(L_i \times Z_i)$. With every multiple binary relation $Q \in \mathbf{MBR}_{(\mathcal{Z},\mathcal{L})}$ we associate its *coherent part* $\check{Q} := Q \cap \Omega_\mathcal{A}$, that is the multiple binary relation such that $|\check{Q}| = |Q| \cap |\Omega_{\mathcal{A}}|$.

In the following, multiple binary relations $Q \in \mathbf{MBR}_{(\mathcal{Z},\mathcal{L})}$ are also called *interaction requests* for the family $\mathcal{A} = (A_i)_{i\in I}$. Such a request Q is said to be

- *normal* if $\mathrm{Def}_{br(Q)} \supseteq \Pi_I \mathcal{Z}^*$,
- *admissible* if $\check{Q} \neq \emptyset$,
- *functional* if $br(Q)$ is a (partial) function $\Pi_I \mathcal{Z} \dashrightarrow \Pi_I \mathcal{L}$, that is if for every $(\sigma_i)_{i\in I} \in \Pi_I \mathcal{Z}$ one has

$$card(br(Q)((\sigma_i)_{i\in I})) \leq 1,$$

- *strongly functional* if for every $i \in I$ and for every $(\sigma_j)_{j\in I\setminus\{i\}} \in \Pi_{j\neq i} Z_j$, one has

$$card(\{\lambda_i \in L_i, \exists \mathfrak{q} \in |Q|, (\forall j \neq i, \mathfrak{q}_j = \sigma_j) \text{ and } \mathfrak{q}^i = \lambda_i\}) \leq 1.$$

We'll denote by $\mathbf{MBR}^{\#}_{(\mathcal{Z},\mathcal{L})}$ the set of admissible interaction requests.

Definition 2.4. *[Interaction relations] An* interaction relation for $\mathcal{A}$ *is a coherent interaction request for* $\mathcal{A}$*.*

The set of interaction relations for the family $\mathcal{A}$ of open dynamics is denoted by $\mathbf{IR}_{\mathcal{A}}$. Note that

$$mr((\mathbf{IR}_{\mathcal{A}})^{\top}) = \mathbf{MR}_{(\mathfrak{S}_i)_{i\in I}}.$$

Given $R \in \mathbf{IR}_{\mathcal{A}}$ an interaction relation for $\mathcal{A}$, we say that

- R is *normal* if there exists a *normal* interaction request $Q \in \mathbf{MBR}_{(\mathcal{Z},\mathcal{L})}$ such that $\check{Q} = R$,
- R is *efficient* if $\mathrm{Def}_{br(R)} \subsetneq \Pi_I \mathcal{Z}$,
- R is *functional* (resp. *strongly functional*) if it is so as a request.

Example 2.5. For the inclusion order, $\Omega_{\mathcal{A}}$ is the greatest interaction relations for $\mathcal{A}$. It is normal but not efficient. Indeed, we have $\Omega_{\mathcal{A}} = \widetilde{Q_M}$, where Q_M designates the greatest interaction request in the context given by $\mathcal{A}$, that is such that $|Q_M| = \Pi_I \mathcal{E}$, and Q_M is a normal interaction request since $\mathrm{Def}_{br(Q)} = \Pi_I \mathcal{Z}$. $\Omega_{\mathcal{A}}$ is not efficient since $\mathrm{Def}_{br(\Omega_{\mathcal{A}})} = \Pi_I \mathcal{Z}$ (because for all $\sigma_i \in Z_i$, there exists $\lambda_i \in L_i$ such that $\sigma_i \in Z_{i,\lambda_i}$). Note also that, in general, $\Omega_{\mathcal{A}}$ is not functional. We could say that the graph $|\Omega_{\mathcal{A}}|$ is too large to define an efficient interaction relation: interacting is restricting possibilities so, roughly speaking, the smaller is the graph of an interaction relation, the stronger is this interaction.

Example 2.6. Let $I = \{1, 2\}$ and $A_1 = A_2 = A$ with A the open functorial non-deterministic mono-dynamic defined by $A = \left((\alpha : (\mathbf{N}, +) \to \mathbf{Tran}) \overset{\rho}{\looparrowright} \mathbf{h}\right)$ with $st(\alpha) = \bullet^\alpha := \mathbf{N} \times \mathbf{R}$, $st(\mathbf{h}) = \mathbf{N}$ and for all $(n, r) \in st(\alpha)$,

- $\rho(n, r) = n$,
- $\forall d \in \mathbf{N}^*, d^\alpha(n, r) = \{n + d\} \times \mathbf{R}$.

The set Z_A of (outgoing) realizations of A can be seen as the set of finite or infinite sequences $\sigma = (s_n)_{n \in \mathbf{N}_\sigma}$ of reals, with $\mathbf{N}_\sigma$ an initial segment of $\mathbf{N}$, and we have $Z_1 = Z_2 = Z_A$. The set of parameter values of the mono-dynamic A is a singleton, thus we can write $L_1 = L_2 = \{*\}$. Let's now consider the interaction relation R given by the graph

$$|R| = \left\{ \begin{pmatrix} * & * \\ \sigma & \sigma \end{pmatrix}, \sigma \in Z \right\}.$$

Then R is obviously a non-normal, functional efficient interaction relation. The lack of normality means that relations between outgoing realizations of the two dynamics Z_1 and Z_2 are not founded on parameter values, but are directly established. Seeing parameter values as data that dynamics can receive from others, we could say that such a non-normal interaction relation is a "paranormal" relation.

2.3 Synchronizations

Recall that I denotes a non-empty set and $\mathcal{A} = (A_i)_{i \in I}$ an I-family of open dynamics $A_i = \left((\alpha_i : \mathbf{D}_i \rightharpoondown \mathbf{Tran} \overset{L_i}{\to}) \overset{\rho_i}{\looparrowright} (\mathbf{h}_i : \mathbf{D}_i \to \mathbf{Sets})\right)$.

Let's begin with the notion of a synchronization of an open dynamic by another, denoting 1 and 0 their index in the family $\mathcal{A}$.

Definition 2.7. *A* synchronization of A_1 by A_0 *is the data* (Δ, δ) *of*

- *a map* $\Delta : \dot{\mathbf{D}}_0 \to \dot{\mathbf{D}}_1$ *defined on the objects of* $\mathbf{D}_0$*,*
- *a map* $\delta : st(\mathbf{h}_0) \to st(\mathbf{h}_1)$ compatible *with* Δ *in the meaning that*

$$\forall S \in \dot{\mathbf{D}}_0, \forall s \in S^{\mathbf{h}_0}, \delta(s) \in (\Delta S)^{\mathbf{h}_1},$$

and such that δ *is monotonic, which means that* δ *is*

- *either increasing:* $\forall (s_0, t_0) \in st(\mathbf{h}_0)^2, s_0 \leq_{\mathbf{h}_0} t_0 \Rightarrow \delta(s_0) \leq_{\mathbf{h}_1} \delta(t_0)$,
- *or decreasing:* $\forall (s_0, t_0) \in st(\mathbf{h}_0)^2, s_0 \leq_{\mathbf{h}_0} t_0 \Rightarrow \delta(t_0) \leq_{\mathbf{h}_1} \delta(s_0)$,

where $\leq_{\mathbf{h}_i}$ *denotes the pre-order on* $\mathbf{h}_i$*-instants*[22].

We write $(\Delta, \delta) : \mathbf{h}_0 \Rsh \mathbf{h}_1$ to indicate that (Δ, δ) is a synchronization of $\mathbf{h}_1$ by $\mathbf{h}_0$. Such a synchronization is said to be *rigid* if (Δ, δ) is a (necessarily deterministic) dynamorphism $\mathbf{h}_0 \looparrowright \mathbf{h}_1$. Otherwise, it is called *flexible*[23].

Definition 2.8. *A* synchronization of the family $\mathcal{A}$ with conductor $i_0 \in I$ *is a family of synchronizations* $((\Delta_i, \delta_i) : \mathbf{h}_{i_0} \Rsh \mathbf{h}_i)_{i \in I}$*, with* $(\Delta_{i_0}, \delta_{i_0}) = Id_{\mathbf{h}_{i_0}}$.

Remark 2.9. *More complex synchronization systems could be usefully considered, which we will not do in this paper.*

2.4 Interactive families

We can now define an *interactive family* [24] as a family of open dynamics endowed with an interaction request (for example an interaction relation), a family of synchronizations between some of these dynamics and a third element, called *privacy* or social mode, which is an equivalence relation on the families of parametric values. More precisely:

[22] See section 1.1.3.

[23] In [11] and [10], only rigid synchronizations had been considered, while the much more general idea of flexible synchronizations appeared in [13].

[24] In [10], we used the expression "dynamical families", but this one presents a risk of confusion with the notion of "families of dynamics", and we finally prefer to use the expression "interactive families".

Definition 2.10. *We call* interactive family *the data* $(I, \mathcal{A}, R, i_0, (\Delta_i, \delta_i)_{i\in I}, \sim)$ *of*

- *a non-empty set* I*,*
- *an* I*-family* $\mathcal{A} = (A_i)_{i\in I}$ *of open dynamics, say*

$$A_i = (\rho_i : (\alpha_i : \mathbf{D}_i \rightharpoondown \mathbf{Tran}^{\overset{L_i}{\to}}) \looparrowright \mathbf{h}_i),$$

- *an* interaction $(R, i_0, (\Delta_i, \delta_i)_{i\in I})$ for $\mathcal{A}$*, that is*
  an admissible interaction request $R \in \mathbf{MBR}^{\#}_{(\mathcal{Z},\mathcal{L})}$ *for* $\mathcal{A}$*,*
  an element $i_0 \in I$*,*
  a synchronization $((\Delta_i, \delta_i) : \mathbf{h}_{i_0} \looparrowright \mathbf{h}_i)_{i\in I}$ *of* $\mathcal{A}$ *with conductor* i_0*,*
- *an equivalence relation* $\sim$ *on the set* $\Pi_I\mathcal{L} = \prod_{i\in I} L_i$*, called the* intimacy *or the* social mode *of the interactive family.*

Remark 2.11. *We'll see in section 3.7 the role of the intimacy of an interactive family.*

An interactive family with components $\mathcal{A}$ and its interaction $(R, i_0, (\Delta_i, \delta_i)_{i\in I})$ are said to be *normal*, *efficient* or *functional* if it is the case for the interaction relation $\breve{R}$, respectively.

Let $\mathcal{F} = (I, \mathcal{A}, R, i_0, (\Delta_i, \delta_i)_{i\in I}, \sim)$ and $\mathcal{G} = (I, \mathcal{A}, Q, i_0, (\Delta_i, \delta_i)_{i\in I}, \backsim)$ be two interactive families defined on a same family $\mathcal{A} = (A_i)_{i\in I}$ of open dynamics and sharing a same synchronization $((\Delta_i, \delta_i)_{i\in I}$. If $\breve{Q} = \breve{R}$, the two interactions Q and R are said to be *strongly equivalent*. If, in addition, the restriction of the equivalence relation $\sim_{|M}$ on the set $M = \mathrm{Im}(br(\breve{R})) \subset \Pi_I\mathcal{L}$ is equal to the restriction $\backsim_{|M}$, then the two interactive families $\mathcal{F}$ and $\mathcal{G}$ are said to be *strongly equivalent*.

2.5 Connectivity structures of an interactive family

Even if we do not discuss in this article the notion of *connective dynamics*[25], it should be noted that we have developed the theory of open dynamics and

[25] See [7] and [8].

their interactions as an extension of our research on connectivity spaces[26]. This perspective also explains that, regarding topological aspects, we emphasize the connectivity point of view[27]. In the present paper we limit ourselves regarding these types of matters to defining a main connectivity structure of an interactive family and three other connectivity structures, based on its interaction request. At this stage, we do not include in these definitions any considerations about synchronizations.

We begin with some very brief reminders about connectivity spaces and structures and about the connectivity structure of a multiple relation. A *connectivity space*[28] X is a pair $(|X|, \kappa(X))$ where $|X|$ is a set called the *carrier* of X and $\mathcal{K} = \kappa(X) \subseteq \mathcal{P}(|X|)$ is called the *connectivity structure* of X and is such that for every $\mathcal{I} \in \mathcal{P}(\mathcal{K})$ we have the implication $\bigcap_{K\in\mathcal{I}} K \neq \varnothing \Rightarrow \bigcup_{K\in\mathcal{I}} K \in \mathcal{K}$. Every element $K \in \mathcal{K}$ is said to be a *connected subset of* $|X|$, or is simply said to be *connected* (to itself). When $|X|$ is non-empty, the empty subset is always connected, because it is the union of the empty family, whose intersection is then non-empty. A connectivity space is said to be *finite* when its carrier is a finite set and it said to be *integral* if every singleton subset is connected. The morphisms between two connectivity spaces are the functions which transform connected subsets into connected subsets.

Given some context $\mathcal{E} = (E_i)_{i\in I}$, *the connectivity space of a multiple relation* $R = (J, \mathcal{E}_{|J}, |R|) \in \mathbf{MR}_{\subseteq\mathcal{E}}$ has been defined in [12] as the space having J as carrier and having as connectivity structure the set $\mathcal{K}_R \subseteq \mathcal{P}(J)$ of subsets K of J that are *non-splittable* for R, that is such that there does not exist a partition $K = K_1 \sqcup K_2$ with $R_{|K} = R_{|K_1} \otimes R_{|K_2}$.

The notion of connectivity structure of a multiple relation naturally extends to the case of a multiple binary relation: given some context $(\mathcal{W}, \mathcal{M})$ for a given index set I, *the connectivity space of a multiple binary relation* $R = (J, \mathcal{W}_{|J}, \mathcal{M}_{|J}, |R|) \in \mathbf{MBR}_{\subseteq(\mathcal{W},\mathcal{M})}$ is the connectivity space having J as carrier and having as connected subsets $K \subseteq J$ the ones that are *non-splittable* for R, that is such that there does not exist a partition $K = K_1 \sqcup K_2$ with $R_{|K} = R_{|K_1} \otimes R_{|K_2}$. In other words, the connectivity structure of a multiple binary relation $R \in \mathbf{MBR}_{\subseteq(\mathcal{W},\mathcal{M})}$ is the one of the multiple relation

[26]See [9].

[27]For the relation between connectivity and topology, see in particular [4].

[28]See [2], [5] and [6].

$mr(R) \in \mathbf{MR}_{\subseteq \mathcal{E}}$.

For example, given a family $\mathcal{A} = (A_i)_{i \in I}$ of open dynamics, the connectivity structure of the interaction $\Omega = \Omega_{\mathcal{A}}$ is the discrete integral one[29], because the coherence property is local, that could be written $\Omega = \bigotimes_{i \in I} \Omega_{|\{i\}}$.

Given $R \in \mathbf{MBR}_{(\mathcal{Z},\mathcal{L})}$ an interaction request for a family $\mathcal{A} = (A_i)_{i \in I}$ of open dynamics — that is[30] : $|R| \subseteq \Pi_I \mathcal{E} = \Pi_I (Z_i \times L_i)$ with Z_i the set of outgoing realizations of A_i and L_i the set of its parameter values — we denote by $\underline{R}$ the I-multiple relation with context $\mathcal{Z}$ obtained by projection (*i.e.* restriction) of R on $\Pi_I \mathcal{Z}$, that is

$$|\underline{R}| = \{(\sigma_i)_{i \in I} \in \Pi \mathcal{Z}, \exists (\lambda_i)_{i \in I} \in \Pi \mathcal{L}, \begin{pmatrix} \lambda_i \\ \sigma_i \end{pmatrix}_{i \in I} \in |R|\}.$$

In the same way, $\underline{\breve{R}}$ denotes the projection of $\breve{R}$ on $\Pi_I \mathcal{Z}$, where we remind that $\breve{R} = R \cap \Omega_{\mathcal{A}}$ denotes the coherent part of R. Then, we obtain four connectivity structures on I naturally associated with the interaction R, that is :

- $\mathcal{K}_R$, the connectivity structure of $R \in \mathbf{MBR}_{(\mathcal{Z},\mathcal{L})}$,
- $\mathcal{K}_{\breve{R}}$, the connectivity structure of $\breve{R} \in \mathbf{MBR}_{(\mathcal{Z},\mathcal{L})}$,
- $\mathcal{K}_{\underline{R}}$, the connectivity structure of $\underline{R} \in \mathbf{MR}_{\mathcal{Z}}$,
- $\mathcal{K}_{\underline{\breve{R}}}$ the connectivity structure of $\underline{\breve{R}} \in \mathbf{MR}_{\mathcal{Z}}$.

Proposition 2.12. *For any interaction request $R \in \mathbf{MBR}_{(\mathcal{Z},\mathcal{L})}$ for a given family $\mathcal{A} = (A_i)_{i \in I}$ of open dynamics, we have $\mathcal{K}_{\underline{R}} \subseteq \mathcal{K}_R$ and $\mathcal{K}_{\underline{\breve{R}}} \subseteq \mathcal{K}_{\breve{R}} \subseteq \mathcal{K}_R$. Moreover, if R is a normal request, $\mathcal{K}_{\underline{R}}$ is the discrete integral connectivity structure*[31]*, so in this case, we have*

$$\mathcal{K}_{\underline{R}} \subseteq \mathcal{K}_{\underline{\breve{R}}} \subseteq \mathcal{K}_{\breve{R}} \subseteq \mathcal{K}_R.$$

[29] That is the structure for which the only connected parts of I are the singletons $\{i\}$ and the empty set, see [6].

[30] Using notations of section 2.2.

[31] See footnote 29.

Proof. Let $K \in \mathcal{K}_{\underline{R}}$. Suppose $K \notin \mathcal{K}_R$: then there is a partition $K = K_1 \sqcup K_2$ such that $R_{|K} = R_{|K_1} \otimes R_{|K_2}$. Then

$$|\underline{R}_{|K}| = \left\{ (\sigma_k)_{k\in K} \in \Pi_K \mathcal{Z}, \exists (\lambda_k)_{k\in K} \in \Pi_K \mathcal{L}, \forall n \in \{1,2\}, \begin{pmatrix} \lambda_k \\ \sigma_k \end{pmatrix}_{k\in K_n} \in |R_{|K_n}| \right\},$$

so $|\underline{R}_{|K}| = |\underline{R}_{|K_1} \otimes \underline{R}_{|K_2}|$ that is absurd. Thus $K \in \mathcal{K}_R$, so $\mathcal{K}_{\underline{R}} \subseteq \mathcal{K}_R$. The same reasoning applied to $\breve{R}$ proves that $\mathcal{K}_{\underline{\breve{R}}} \subseteq \mathcal{K}_{\breve{R}}$.

Now, let's prove that $\mathcal{K}_{\breve{R}} \subseteq \mathcal{K}_R$: let $K \in \mathcal{K}_{\breve{R}}$, and suppose $K \notin \mathcal{K}_R$. Then, as previously, there is a partition $K = K_1 \sqcup K_2$ such that $R_{|K} = R_{|K_1} \otimes R_{|K_2}$. By putting $\Omega = \Omega_{\mathcal{A}}$, we thus have $\breve{R}_{|K} = R_{|K} \cap \Omega_{|K} = (R_{|K_1} \otimes R_{|K_2}) \cap (\Omega_{|K_1} \otimes \Omega_{|K_2})$. But $\cap$ is nothing but $\otimes$ in the case of a same arity so, by associativity and commutativity, we have $\breve{R}_{|K} = (R_{|K_1} \cap \Omega_{|K_1}) \otimes (R_{|K_2} \cap \Omega_{|K_2}) = \breve{R}_{|K_1} \otimes \breve{R}_{|K_2}$, which is absurd, because we assumed that $K \in \mathcal{K}_{\breve{R}}$.

Finally, if R is a normal request, then $|\underline{R}| = \mathrm{Def}_{br(R)} = \Pi_I \mathcal{Z}$, so its connectivity structure is the discrete integral one, that is finer than the others. □

Definition 2.13. *Given an interactive family $\mathcal{F} = (I, \mathcal{A}, R, i_0, (\Delta_i, \delta_i)_{i\in I}, \sim)$ we call $\mathcal{K}_{\mathcal{F}} := \mathcal{K}_{\underline{\breve{R}}}$ the* manifest connectivity structure of $\mathcal{F}$ *(or simply the* connectivity structure of $\mathcal{F}$*), and $\mathcal{K}_{\breve{R}}$ the* plain connectivity structure of $\mathcal{F}$.

2.6 Examples of interactive families

Example 2.14 (The $\mathbb{WHY}$ = יהו family)**.** As a first example of an interactive family, let us recall the $\mathbb{WHY}$ family, also denoted by יהו, that we have introduced in [13] and that we have also described in [18], on the occasion of our work with philosopher Pierre Michel Klein concerning his philosophical theory of time, *Metachronology* [19]. As its name suggests, this family involves the open dynamics $\mathbb{Y}$ = י, $\mathbb{H}$ = ה and $\mathbb{W}$ = ו (cf. *supra* examples 1.21, 1.20 and 1.19). More precisely, it is defined by $\mathbb{WHY} = (I, \mathcal{A}, Q, i_0, (\Delta_i, \delta_i)_{i\in I}, \sim)$ with

- $I = \{1, 2, 3\}$,
- $\mathcal{A} = (A_i)_{i\in I}$ where $A_1 = \mathbb{Y}$, $A_2 = \mathbb{H}$ with the origin of times being taken equal to $T_0 = 0$ for simplicity, and $A_3 = \mathbb{W}$,

- the graph of the interaction request Q for $\mathcal{A}$ contains all the families

$$\begin{pmatrix} \lambda_1 = * & \lambda_2 \in L_{\mathbb{H}} & \lambda_3 \in L_{\mathbb{W}} = \mathcal{C} \\ \sigma_1 \in Z^*_{\mathbb{Y}} & \sigma_2 \in Z^*_{\mathbb{H}} & \sigma_3 \in Z^*_{\mathbb{W}} = \mathcal{C} \end{pmatrix}$$

  such that $\lambda_3 = \sigma_2$ and λ_2 is the restriction of σ_1 to the interior of its domain of definition Def_{σ_1},

- the conductor is given by $i_0 = 2$,

- $\Delta_1 = Id_{\mathbf{R}_+}$ and $\delta_1 : st(\mathbf{h}_{\mathbb{H}}) =]0, +\infty[\hookrightarrow [0, +\infty[= st(\mathbf{h}_{\mathbb{Y}})$, the inclusion map,

- $\Delta_3 = (\mathbf{R}_+ \xrightarrow{!} \mathbf{1})$ and $\delta_3 : st(\mathbf{h}_{\mathbb{H}}) \xrightarrow{!} \{\bullet\} = st(\mathbf{h}_{\mathbb{W}})$, which is necessarily constant,

- the social mode $\sim$ (that was not included in our previous definitions of an interactive family) is taken equal to the maximal equivalence relation on $\Pi\mathcal{L} = \{*\} \times L_{\mathbb{H}} \times L_{\mathbb{W}}$, *i.e* $\mu \sim \nu$ for all μ and ν in $\Pi\mathcal{L}$.

Note that the interaction request Q is normal, and that the manifest connectivity structure $\mathcal{K}_{\mathbb{WHY}} = \mathcal{K}_{\underline{\check{Q}}}$ is the indiscrete one, that is $\mathcal{K}_{\mathbb{WHY}} = \mathcal{P}(I)$.

Example 2.15 (A borromean family)**.** Our second example of an interactive family is $\mathcal{F} = (I, \mathcal{A}, Q, i_0, (\Delta_i, \delta_i)_{i\in I}, \sim)$ with

- $I = \{1, 2, 3\}$,

- $\mathcal{A} = (A_i)_{i\in I}$ with, for each $i \in I$, $A_i = \Upsilon$, the open dynamic given in the example 1.16,

- the graph of the interaction request Q for $\mathcal{A}$ contains all the families

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} \in (Z_\Upsilon \times L_\Upsilon)^3$$

  that satisfy $\{i \in I, \lambda_i(0) = 1\} \neq \varnothing$,

- the conductor is given by $i_0 = 1$,

- $\delta_i = Id_{\{t_0,t_1\}}$ (and $\Delta_i = Id_{\{T_0,T_1\}}$) for every $i \in I$,

- the intimacy (social mode) $\sim$ is defined on L^3_Υ by

$$(\lambda_1, \lambda_2, \lambda_3) \sim (\mu_1, \mu_2, \mu_3) \Leftrightarrow \lambda_1(0) = \mu_1(0).$$

Note that we can write

$$|Q| = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} \in (Z_\Upsilon \times L_\Upsilon)^3, \{\lambda_1, \lambda_2, \lambda_3\} \cap \{\varphi_{10}, \varphi_{11}\} \neq \varnothing \right\},$$

where φ_{kl} denotes the map $\{0,1\} \to \{0,1\}$ such that $\varphi_{kl}(0) = k$ and $\varphi_{kl}(1) = l$, so that we have $L_\Upsilon = \{\varphi_{00}, \varphi_{01}, \varphi_{10}, \varphi_{11}\}$.

The interaction request Q is obviously normal, and it is easy to see that the manifest connectivity structure $\mathcal{K}_\mathcal{F} = \mathcal{K}_{\underline{\breve{Q}}}$ and the plain connectivity structure $\mathcal{K}_{\breve{Q}}$ are both the integral borromean one[32], that is $\mathcal{K}_\mathcal{F} = \mathcal{P}(I) \smallsetminus \{\{1,2\}, \{2,3\}, \{1,3\}\}$.

3. Global dynamics

In this section, we associate with any interactive family some global dynamics, *i.e.* some open dynamics produced by the family in question in order to incorporate in a certain way the different dynamics composing the family. The difference between these global dynamics lies in the choice of the social mode applied : if it is the social mode belonging to the family itself, we obtain the one we'll call the global dynamic *demanded* by the considered interactive family. But other choices of a social mode can be made, starting with the trivial equivalence relation (equality) which leads to what we call the *transparent global dynamic*, on which other global dynamics are modelled. Among other possibilities, we also define the *responsible global dynamic* and the J*-global dynamic* — which results from the choice of the set J of indices for which the parameter values can be determined from the outside — and we finally introduce the most "closed" global dynamic (*i. e.* a mono-dynamic that cannot be influenced (normally) by some other dynamics), which we call the *opaque global dynamic* generated by the family.

[32]Cf. [6].

3.1 The lax-functorial stability theorem

Theorem 3.1 (Lax-functorial Stability Theorem)**.** *Let $\mathcal{F} = (I, \mathcal{A}, R, i_0, (\Delta_i, \delta_i)_{i \in I})$ be an interactive family, with $\mathcal{A} = (A_i)_{i \in I}$ and, for each $i \in I$*

$$A_i = (\rho_i : (\alpha_i : \mathbf{D}_i \rightharpoondown \mathbf{Tran}^{\overset{L_i}{\to}}) \nrightarrow \mathbf{h}_i),$$

and let $\mathbf{E} = \mathbf{D}_{i_0}$ and $M = \mathrm{Im}(br(\breve{R}))$. Then we obtain an M-dynamic $\beta : \mathbf{E} \rightharpoondown \mathbf{Tran}^{\overset{M}{\to}}$ by putting for every $S \in \dot{\mathbf{E}}$

$$S^\beta = \{(a_i)_{i \in I} \in \prod_{i \in I} (\Delta_i S)^{\alpha_i}, \forall i \in I, \rho_i(a_i) = \delta_i(\rho_{i_0}(a_{i_0}))\},$$

and, for every $(d : S \to T) \in \overrightarrow{\mathbf{E}}$, $a = (a_i)_{i \in I} \in S^\beta$ and $\mu \in M$, by defining $d^\beta_\mu(a)$ as the set of the $b = (b_i)_{i \in I} \in T^\beta$ such that

$$\exists (\sigma_i)_{i \in I} \in br(\breve{R})^{-1}(\mu), \forall i \in I, \sigma_i \rhd a_i, b_i \tag{2}$$

and

$$\rho_{i_0}(b_{i_0}) = d^{\mathbf{h}_{i_0}}(\rho_{i_0}(a_{i_0})). \tag{3}$$

Proof. First, we have $M \neq \varnothing$, because R is an admissible request so $P = \breve{R}$ and $M = \mathrm{Im}(br(P))$ are not empty. Then, following the section § 1.1.1, we have to check that β is a disjunctive lax-functor, *i.e.* that these three conditions are satisfied:

1. (Disjunctivity) $\forall (S, T) \in \dot{\mathbf{E}}^2$, $S \neq T \Rightarrow S^\beta \cap T^\beta = \varnothing$,
2. (Lax identity) $\forall S \in \dot{\mathbf{E}}, \forall \mu \in M, (Id_S)^\beta_\mu \subseteq Id_{S^\beta}$,
3. (Lax composition) for every $(S \overset{d}{\to} T \overset{e}{\to} U)$ in $\mathbf{E}$ and every $\mu \in M$,

$$(e \circ d)^\beta_\mu \subseteq e^\beta_\mu \odot d^\beta_\mu.$$

1. Disjunctivity. Suppose $S \neq T$ but $S^\beta \cap T^\beta \neq \varnothing$, then we would have an element $(a_i)_{i \in I} \in S^\beta \cap T^\beta$ and — as $\Delta_{i_0} = Id_{\mathbf{D}_{i_0}}$ and α_{i_0} is disjunctive — we would have $a_{i_0} \in S^{\alpha_{i_0}} \cap T^{\alpha_{i_0}} = \varnothing$, that is absurd.

2. Lax identity. Let $S \in \dot{\mathbf{E}}$ and $\mu \in M$. We want to check that $(Id_S)^\beta_\mu \subseteq Id_{S^\beta}$. In other words, we want to check that if $S^\beta \neq \varnothing$ and $a = (a_i)_{i\in I} \in S^\beta$ then $(Id_S)^\beta_\mu(a) \subseteq \{a\}$, that is $(Id_S)^\beta_\mu(a) = \varnothing$ or $(Id_S)^\beta_\mu(a) = \{a\}$. But if $(Id_S)^\beta_\mu(a)$ is not empty and $a' = (a'_i)_{i\in I}$ is an element of it, then for every $i \in I$, there is an outgoing realization $\sigma_i \in Z_i$ such that $\sigma_i \triangleright a_i, a'_i$ and then — using $\sigma_i \triangleright a'_i$, the definition of S^β, the condition (3) and $\sigma_i \triangleright a_i$ — we have

$$a'_i = \sigma_i(\rho_i(a'_i)) = \sigma_i(\delta_i(\rho_{i_0}(a'_{i_0}))) = \sigma_i(\delta_i(\rho_{i_0}(a_{i_0}))) = \sigma_i(\rho_i(a_i)) = a_i.$$

Thus $a' = a$, and we have proved that $(Id_S)^\beta_\mu \subseteq Id_{S^\beta}$.

3. Lax composition. We have to check that given any $(S \overset{d}{\to} T \overset{e}{\to} U)$ in $\mathbf{E}$, any $\mu \in M$ and any state $a = (a_i)_{i\in I} \in S^\beta$, we have $(e \circ d)^\beta_\mu(a) \subseteq (e^\beta_\mu \odot d^\beta_\mu)(a)$. In other words, supposing $(e \circ d)^\beta_\mu(a)$ not empty and taking any $c = (c_i)_{i\in I} \in (e \circ d)^\beta_\mu(a) \subseteq U^\beta$, we have to prove the existence of a state $b \in d^\beta_\mu(a) \subseteq T^\beta$ such that $c \in e^\beta_\mu(b)$.

To express such a state $b = (b_i)_{i\in I}$, let us set $t_0 := \rho_{i_0}(a_{i_0}) \in S^{\mathbf{h}_{i_0}}$, $t_1 := d^{\mathbf{h}_{i_0}}(t_0) \in T^{\mathbf{h}_{i_0}}$, and $t_2 := e^{\mathbf{h}_{i_0}}(t_1) \in U^{\mathbf{h}_{i_0}}$. Note that by the definition of $(e \circ d)^\beta_\mu(a)$ we also have $t_2 = \rho_{i_0}(c_{i_0})$ and that there exists a family $(\sigma_i)_{i\in I} \in br(\breve{R})^{-1}(\mu)$ of outgoing realizations such that $\sigma_i \triangleright a_i, c_i$ for every $i \in I$. Given $(\sigma_i)_{i\in I}$ such a family, it suffices now to prove that each σ_i is defined for the instant $\delta_i(t_1)$, and that $b_i := \sigma_i(\delta_i(t_1))$ is a suitable choice. Note first that since $\sigma_{i_0} \triangleright c_{i_0}$, we have $\rho_{i_0}(c_{i_0}) \in \mathrm{Def}_{\sigma_{i_0}}$, that is $t_2 \in \mathrm{Def}_{\sigma_{i_0}}$. But $t_1 \leq_{\mathbf{h}_{i_0}} t_2$, so, according to the properties of realizations (see section § 1.2.2), $\delta_{i_0}(t_1) = t_1 \in \mathrm{Def}_{\sigma_{i_0}}$.

Let us now consider the case of an $i \neq i_0$. By definition of a synchronization, the map δ_i is either increasing or decreasing. If it is increasing, then $\delta_i(t_1) \leq_{\mathbf{h}_i} \delta_i(t_2)$, but $c_i = \sigma_i(\rho_i(c_i)) = \sigma_i(\delta_i(t_2))$, so $\delta_i(t_1) \in \mathrm{Def}_{\sigma_i}$. If δ_i is decreasing, then $\delta_i(t_1) \leq_{\mathbf{h}_i} \delta_i(t_0)$, but $a_i = \sigma_i(\rho_i(a_i)) = \sigma_i(\delta_i(t_0))$, so $\delta_i(t_0) \in \mathrm{Def}_{\sigma_i}$ and thus we have again $\delta_i(t_1) \in \mathrm{Def}_{\sigma_i}$. Now, let's put $b_i = \sigma_i(\delta_i(t_1))$ for every $i \in I$. Then $b \in d^\beta_\mu(a)$, since

- for every $i \in I$, $t_1 \in T^{\mathbf{h}_{i_0}} \Rightarrow \delta_i(t_1) \in (\Delta_i T)^{\mathbf{h}_i}$, and then $b_i = \sigma_i(\delta_i(t_1)) \in (\Delta_i T)^{\alpha_i}$,
- by definition of a realization $\rho_{i_0}(b_{i_0}) = \rho_{i_0}(\sigma_{i_0}(t_1)) = t_1$ and, for every $i \in I$, $\rho_i(b_i) = \rho_i(\sigma_i(\delta_i(t_1))) = \delta_i(t_1) = \delta_i(\rho_{i_0}(b_{i_0}))$,

- by construction, we have $\sigma_i \rhd a_i, b_i$ for every $i \in I$.

But we also have $c \in e^{\beta}_{\mu}(b)$, since

- $c \in U^{\beta}$,
- $\rho_{i_0}(c_{i_0}) = t_2 = (e \circ d)^{\mathbf{h}_{i_0}}(t_0) = e^{\mathbf{h}_{i_0}}(t_1) = e^{\mathbf{h}_{i_0}}(\rho_{i_0}(b_{i_0}))$,
- and for all $i \in I$, $\sigma_i \rhd b_i, c_i$,

and this concludes the proof. □

3.2 The transparent global dynamic

Thanks to the theorem 3.1, it is immediate to check that the definition below is consistent.

Definition 3.2. *Using the same notations than above, the* transparent global dynamic associated with an interactive family $\mathcal{F}$ *is the open dynamics denoted* $[\mathcal{F}]_1$ *defined by*

$$[\mathcal{F}]_1 = \Big((\beta : \mathbf{E} \rightharpoonup \mathbf{Tran}^{\overset{M}{\to}}) \overset{\tau}{\looparrowright} (\mathbf{k} : \mathbf{E} \to \mathbf{Sets}) \Big)$$

where β *and thus, in particular,* M *and* $\mathbf{E}$*, are the one associated with* $\mathcal{F}$ *by the theorem 3.1, the clock* $\mathbf{k}$ *is given by* $\mathbf{k} = \mathbf{h}_{i_0}$ *and the datation* $\tau : st(\beta) \to st(\mathbf{k})$ *is defined by*

$$\forall S \in \dot{\mathbf{E}}, \forall a = (a_i)_{i \in I} \in S^{\beta}, \tau(a) = \rho_{i_0}(a_{i_0}).$$

3.3 The demanded global dynamic

The parametric set M of the transparent global dynamic $[\mathcal{F}]_1$ associated with an interactive family $\mathcal{F}$ is generally "too big" in the sense that very often some parts of the parametric values are not intended to be externally controlled and should instead be determined by the realizations of the dynamics that compose the interactive family itself. The social mode (or intimacy) $\sim$ of $\mathcal{F}$ — which does not play a role in the definition of the transparent global dynamic — is precisely used to "reduce" the parametric set, thanks to the notion of the parametric quotient of an open dynamic by an equivalence relation on the set of parametric values (see definition 1.13). The response of the new global dynamic thus obtained is the same for two distinct parametric values, as long as they are equivalent: it is up to it to take into account, or not, the requests made to it from outside, by constructing its response on all the possibilities given to it by the different equivalent parametric values of a same equivalence class.

Definition 3.3. *Using the same notations than above, and denoting again* $\sim$ *the restriction of the intimacy* $\sim$ *of* $\mathcal{F}$ *to the subset* $M = \mathrm{Im}(br(\breve{R})) \subseteq \Pi_I\mathcal{L}$*, the global dynamic* demanded *by* $\mathcal{F}$ *— also called the* demanded global dynamic of $\mathcal{F}$ *— is defined as the open dynamic denoted* $[\mathcal{F}]_\sim$ *given by* $[\mathcal{F}]_\sim = [\mathcal{F}]_1/\sim$.

3.4 The responsible global dynamic

In this section, we associate with any interactive request an intimacy called "responsible intimacy" that intuitively allows the interactive family to choose the parametric values of each dynamic at stake when these values are susceptible to be determined by the realizations of the other dynamics of the family. More precisely, using the same notations as previously, if Q designates a (not necessarily coherent) admissible interaction request for a family $\mathcal{A} = (A_i)_{i\in I}$ of open dynamics, we define the responsible intimacy $\asymp_Q$ for Q as the equivalence relation on $\Pi_I\mathcal{L}$ setting, for any $((\mu_i)_{i\in I}, (\lambda_i)_{i\in I}) \in (\Pi_I\mathcal{L})^2$, $(\mu_i)_{i\in I} \asymp_Q (\lambda_i)_{i\in I}$ iff we have, for all $i \in I$: ($\mu_i = \lambda_i$ or $\mu_i \in N_i \ni \lambda_i$), where $N_i \subseteq L_i$ is defined as the set

$$N_i := \left\{ l \in L_i, \forall(\mathfrak{p},\mathfrak{q}) \in |Q|^2, \left(\mathfrak{p}^i = l \,\text{and}\, \forall k \neq i, \mathfrak{p}_k = \mathfrak{q}_k\right) \Rightarrow \mathfrak{q}^i = l \right\}.$$

The *responsible global dynamic* $[\mathcal{F}]_{\asymp_Q}$ generated by an interactive family $\mathcal{F} = (I, \mathcal{A}, Q, i_0, (\Delta_i, \delta_i)_{i\in I}, \sim)$ is then defined as the demanded global dynamic of the interactive family $(I, \mathcal{A}, Q, i_0, (\Delta_i, \delta_i)_{i\in I}, \asymp_Q)$. Note that $[\mathcal{F}]_{\asymp_Q}$ does not depend on the social mode $\sim$ demanded by $\mathcal{F}$ itself, and that two different interaction requests Q and R can result in two different social modes $\asymp_Q$ and $\asymp_R$ even if $\breve{Q} = \breve{R}$.

3.5 The J-global dynamic

Let $\mathcal{F} = (I, \mathcal{A}, Q, i_0, (\Delta_i, \delta_i)_{i \in I}, \sim)$ be an interactive family as previously, and let $J \subseteq I$ a subset whose elements $j \in J$ intuitively represent the indices such that the corresponding dynamics A_j could be influenced from outside the global dynamic we want to define. To achieve this, we can use a similar construction to the one we described for the responsible global dynamic, but taking for each $i \in I$ the set N_i given by: $N_i = \emptyset$ if $i \in J$ and $N_i = L_i$ if $i \notin J$. In this way, we get a global dynamics $[\mathcal{F}]_{\sim_J}$ that we'll call the J-global dynamic associated with $\mathcal{F}$. When $J = I$, we obtain the transparent global dynamic $[\mathcal{F}]_1 = [\mathcal{F}]_{\sim_I}$ associated with $\mathcal{F}$.

3.6 The opaque global dynamic

The transparent global dynamic $[\mathcal{F}]_1$ is the "most open" of the global dynamics associated with an interactive family (so much so that it is generally "too open"). At the other end, the *opaque global dynamic* is the most "closed" of them, since its parametric set is reduced to a singleton. It is obtained by making the quotient of $[\mathcal{F}]_1$ by the maximum equivalence relationship on M, for which M is the only equivalence class. Denoting again M this equivalence relation, we thus have:

Definition 3.4. *The* opaque global dynamic associated with $\mathcal{F}$ *is the open mono-dynamic denoted* $[\mathcal{F}]_0$ *defined by* $[\mathcal{F}]_0 = [\mathcal{F}]_1/M$.

In other words, the opaque global dynamic generated by $\mathcal{F}$ is its $\emptyset$-global dynamic: $[\mathcal{F}]_0 = [\mathcal{F}]_{\sim_\emptyset}$. The following proposition immediately follows from the definitions:

Proposition 3.5. *If the interaction request of an interactive family is strongly functional, then its responsible global dynamic is the opaque one.*

Remark 3.6. *With a strongly functional interaction request, the responsible global dynamic (which in this case is the opaque one) generated by a family can be non-deterministic even if all the dynamics at stake are deterministic (for example if the request is that each dynamic has a similar behavior than another, then each one can be entirely determined by the others without the responsible global dynamic being deterministic). This shows a certain instability of determinism.*

3.7 Examples of global dynamics

The examples in this section are the global dynamics associated with the interactive families proposed as examples in the section § 2.6.

Example 3.7 (The global dynamic of the $\mathbb{WHY}$ family)**.** Like in the example 1.19, given two (partial) functions a and b on $\mathbf{R}$, we use the notation $a \diamond b$ to say that $a_{|\mathrm{Def}_a \cap \mathrm{Def}_b} = b_{|\mathrm{Def}_a \cap \mathrm{Def}_b}$. Then one checks that the interactive family $\mathbb{WHY}$ of the example 2.14 generates as global dynamic the opaque global dynamic $\mathbb{S} = [\mathbb{WHY}]_\sim$ given by

$$\mathbb{S} = \left((\alpha_\mathbb{S} : \mathbf{D}_\mathbb{S} \to \mathbf{Tran}) \overset{\tau}{\looparrowright} \mathbf{h}_\mathbb{S}\right)$$

where

- $\mathbf{D}_\mathbb{S} = \mathbf{D}_\mathbb{H} = (\mathbf{R}_+, +)$,
- $\mathbf{h}_\mathbb{S} = \mathbf{h}_\mathbb{H}$, so $st(\mathbf{h}_\mathbb{S}) =]0, +\infty[$ and for all $t \in]0, +\infty[$ and all $d \in \mathbf{R}_+$, we have $d^{\mathbf{h}_\mathbb{S}}(t) = t + d$,
- $st(\mathbb{S}) = \{(t, r, f, w) \in \mathbf{R}_+^* \times \mathbf{R} \times \mathcal{C}^1 \times \mathcal{C}, \mathrm{Def}_f =]-\infty, t[\}$,
- $\forall (t, r, f, w) \in st(\mathbb{S}), \tau(t, r, f, w) = t$,

and such that a state $(t, r, f, w) \in st(\mathbb{S})$ is onside iff $r = f(t^-) := \lim_{s \to t^-} f(s)$, $f_{|[0,t[} \in Lip_1([0, t[)$ and $f \diamond w$, and that in this case the set $d^{\alpha_\mathbb{S}}(t, r, f, w)$ is given for any duration $d \in \mathbf{R}_+$ as the set of all states $(t + d, q, g, w)$ such that

- $g_{|]-\infty,t[} = f$,
- $q = g((t + d)^-)$,
- $g_{|[0,t+d[} \in Lip_1([0, t + d[)$,
- $g \diamond w$.

Example 3.8 (The global dynamic of a borromean family)**.** One checks that the borromean interactive family considered in the example 2.15 results in a functorial global dynamics isomorphic (in $\mathbf{ODyn}$) to the one given by

$$\mathbb{U} = \left((\mathbf{u} : \mathbf{D}_\Upsilon \to \mathbf{Tran}^{M_\mathbb{U}}_{\to}) \overset{!}{\looparrowright} (\zeta_{\mathbf{D}_\Upsilon} : \mathbf{D}_\Upsilon \to \mathbf{Sets}) \right)$$

where we recall that $\mathbf{D}_\Upsilon$ is one-step category $(T_0 \overset{d}{\to} T_1)$ and $\zeta_{\mathbf{D}_\Upsilon}$ is its *essential clock* with instants ${T_k}^{\zeta_{\mathbf{D}_\Upsilon}} = \{t_k\}$ (cf. example 1.16), and with

- $\forall k \in \{0,1\}$, $(T_k)^\mathbf{u} = \{t_k\} \times \{0,1\}^3$,
- $M_\mathbb{U} = \{0,1\}$,
- $\forall \mu \in M_\mathbb{U}$, $\forall k \in \{0,1\}$, $(Id_{(T_k)})^\mathbf{u}_\mu = Id_{((T_k)^\mathbf{u})}$ (since $\mathbf{u}$ is functorial),
- for $\mu \in M_\mathbb{U}$ and $(a,b,c) \in \{0,1\}^3$, $d^\mathbf{u}_\mu(t_0; a,b,c)$ is the set of all states of the form $(t_1; a', b', c') \in (T_1)^\mathbf{u}$ such that

  if ($\mu = 0$ and $a = 0$ and $(b,c) \neq (0,0)$) then $a' = 0$,

  if ($\mu = 0$ and $(a,b,c) = (0,0,0)$) then ($a' = 0$ and ($b' = 1$ or $c' = 1$)),

  if ($\mu = 0$ and $(a,b,c) = (1,0,0)$) then ($b' = 1$ or $c' = 1$),

  if ($\mu = 1$ and $a = 0$) then $a' = 1$,

- $\mathbf{u} \overset{!}{\looparrowright} \zeta_{\mathbf{D}_\Upsilon}$ is the unique possible deterministic dynamorphism, for which the date of $(t_k; a,b,c)$ is t_k.

Conclusion

This article presents the basics of our theory of interacting open dynamics, but various important questions are not addressed at all. In particular, while we have presented the connectivity structures of interactions, a subsequent article will have to take up the theme of the connectivity structures of the dynamics themselves (theme that we had addressed in [7] and [8] in the case of mono-dynamics), in order to clarify the relationships between the connectivity structures of the dynamics of an interactive family, the connectivity structure of the interaction and the connectivity structure of the global dynamic generated by such a family.

In addition, the study of the relationships between our theory and Andrée Ehresmann's suggests addressing some ideas that are currently absent from our theory, in particular the question of how an interactive family can continue to produce a global dynamic when certain dynamics of this family cease to function, or when new dynamics enter the dance. More generally, we hope to address the question of self-organization, which is currently largely absent from our theory. On the other hand, as we have seen, the study of the relationships between the two theories suggests a non-deterministic extension of Andrée Ehresmann's guidable systems, which should be studied. Furthermore, it would be interesting to clarify the relation with other "compositional theories" such as David Spivak's dynamical theory.

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Contents

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PROJECTIVE COVERS OF 2-STAR-PERMUTABLE CATEGORIES

*Vasileios Aravantinos-Sotiropoulos**

Résumé. Nous introduisons la notion de star-symétrie pour les relations dans une catégorie multipointée et nous l'utilisons pour obtenir une caractérisation des revêtements projectifs des catégories 2-star-permutables. Cela généralise les résultats de Rosicky-Vitale pour les catégories régulières de Mal'tsev [19] et aussi ceux de Gran-Rodelo pour les catégories régulières soustractives [13]. Nous appliquons la caractérisation en termes de star-symétrie pour retrouver les conditions syntaxiques définissant les variétés E-soustractives au sens de Ursini [20]
Abstract. We introduce the notion of star-symmetry for relations in a multi-pointed category and use it to obtain a characterization of the projective covers of 2-star-permutable categories. This generalizes the results of Rosický-Vitale for regular Mal'tsev categories [19], as well as those of Gran-Rodelo for regular subtractive categories [13]. We apply the characterization in terms of star-symmetry to recover the syntactic conditions defining E-subtractive varieties in the sense of Ursini [20].
Keywords. Multi-pointed category, star relation, Mal'tsev category, subtractive category, projective cover, variety of algebras.
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1. Introduction

The notion of *multi-pointed category* has in recent years been introduced and studied as a setting where certain pointed and non-pointed contexts of interest in Categorical and Universal Algebra can be treated simultaneously. A multi-pointed category is simply a category $\mathcal{C}$ equipped with an *ideal* $\mathcal{N}$ of morphisms in the sense of Ehresmann [6], i.e. a collection of morphisms in $\mathcal{C}$ such that $fg \in \mathcal{N}$ whenever $f \in \mathcal{N}$ or $g \in \mathcal{N}$. The *pointed context* is captured by taking $\mathcal{N}$ to be the class of zero morphisms in a pointed category, while non-pointed settings, which are referred to as the *total context*, are captured by choosing $\mathcal{N}$ to be the class of all morphisms of a category. This has allowed the unification and extension of various results and characterizations known in pointed and non-pointed Categorical Algebra to the context of multi-pointed categories. First, in the article [12] the authors introduced the notion of a multi-pointed category with a *good theory of ideals* and unified results from the realm of *ideal determined* categories, on one hand, and *Barr-exact Goursat* categories, on the other. Next, in [11], notions of permutability of equivalence relations in multi-pointed categories were introduced and studied in connection with certain diagrammatic characterizations, known for regular *subtractive* categories and *Goursat* categories. Furthermore, in [10] the authors considered generalizations of homological lemmas, such as the *3 × 3 Lemma* and the *Short Five Lemma*. In non-pointed contexts the appropriate notion of exact sequence is that of *exact fork*, which is a sequence consisting of a kernel pair together with its coequalizer. Then, in a more general multi-pointed context, the pertinent notion becomes that of a *star-exact* sequence, which unifies the pointed and non-pointed versions, and allows for the aforementioned multi-pointed homological lemmas. Finally, in [14] the notion of 2-star-permutable category was studied as a common extension of both regular subtractive and regular Mal'tsev categories and characterizations of these categories via diagrams such as regular pushouts were generalized to a multi-pointed context. In the present note we want to add to this list a characterization of *projective covers* of regular 2-star-permutable multi-pointed categories.

There has been a lot of interest and work carried out in the literature on obtaining characterizations for the projective covers of various types of regular categories. The first result of this kind appears already in the work of Freyd in [7] in connection with his construction of the free abelian category on a given (pre-)additive one. About 3 decades later, Carboni and Vitale gave beautiful constructions for free *regular* and *exact* categories. Since abelian categories are in particular exact, these constructions can be used to recover in a nice conceptual manner the aforementioned one by Freyd, as well as other results on abelian categories (see [19]). One important feature of these *regular and exact completions* is that they apply to any category which is merely *weakly lex*, i.e. which is only required to have weak finite limits [5]. Then it turns out that any such category $\mathcal{C}$ appears as a projective cover inside both its regular completion and its exact completion and, furthermore, that a free exact category is the exact completion of any one of its projective covers. Such and other motivations have led various authors to establish characterizations for the projective covers of regular and exact categories that are extensive [15], Mal'tsev [19], protomodular, semi-abelian [8], unital, subtractive [13], Goursat [18] and others.

In this note we look at regular Mal'tsev and regular subtractive categories as special cases of the notion of 2-star-permutable category, following the line of research in [11], [14]. The aim here is to obtain a characterization of the projective covers of 2-star-permutable multi-pointed categories, thus unifying and subsuming the known characterizations in the Mal'tsev [19] and subtractive [13] settings. To accomplish this we first prove that 2-star-permutability is equivalent to a certain symmetry property of reflexive relations (3.2, 3.4), which specializes to known characterizations in both the total and pointed contexts. In the total context it becomes the well-known statement [4] that a regular category is Mal'tsev if and only if every reflexive relations in it is symmetric, while in the pointed context it says that a regular category is subtractive if and only if every reflexive relation in it is 0-symmetric [1, 16]. We then introduce the appropriate "weakening" of this symmetry property in the context of multi-pointed categories with only weak finite limits and weak kernels (3.8) and prove that this weakened property gives the desired characterization of projective covers (3.12). This result yields, in particular, a characterization of when the regular completion and the exact completion of a category with weak finite limits are 2-star-permutable. Finally, we apply the result to the case of varieties of universal algebras which have a non-empty set of constants, allowing us to recover the syntactic conditions defining *E-subtractive* varieties in the sense of Ursini [20].

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2. Preliminaries

2.1 Regular categories and relations

A finitely complete category $\mathcal{E}$ is called *regular* when every kernel pair in $\mathcal{E}$ has a coequalizer and moreover regular epimorphisms in $\mathcal{E}$ are stable under pullbacks. Equivalently, $\mathcal{E}$ is regular if it admits (regular epi, mono) factorizations of morphisms and these are stable under pullback.

A *relation* R from X to Y in any finitely complete category is a subobject $\langle r_0, r_1 \rangle : R \rightarrowtail X \times Y$. When $Y = X$ we will say that R is a relation on X and also denote this by a parallel pair $R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X$. The *opposite* relation R° is the relation given by $\langle r_1, r_0 \rangle : R \rightarrowtail Y \times X$. Any morphism $f : X \to Y$ can be considered as a relation by identifying it with its graph $\langle 1_X, f \rangle : X \rightarrowtail X \times Y$. Then we will write f° to denote the opposite of the latter relation.

In the context of a regular category $\mathcal{E}$ [2] it is possible to define a composition of relations which, moreover, is associative. If R is a relation from X to Y and S is a relation from Y to Z, then we denote their composition by SR, which is a relation from X to Z. The diagonal relations $\Delta_X = \langle 1_X, 1_X \rangle : X \rightarrowtail X \times X$ act as identities for the composition of relations on either side. Furthermore, if a relation R is given by the subobject $\langle r_0, r_1 \rangle : R \rightarrowtail X \times Y$, then we can write it as

$R = r_1 r_0^\circ$ in the above notation.

If $\mathrm{Eq}(f)$ denotes the kernel pair of a morphism $f : X \to Y$, then as a relation on X we have $\mathrm{Eq}(f) = f^\circ f$.

Let $f : X \to Y$ be a morphism and S be a relation on Y. We denote by $f^{-1}(S)$ the *inverse image* of the relation S along f, which is the relation on X defined as the pullback of the subobject $S \rightarrowtail Y \times Y$ along the morphism $f \times f : X \times X \to Y \times Y$. Then in the calculus of relations we have that $f^{-1}(S) = f^\circ S f$.

2.2 Projective covers

Let $\mathcal{E}$ be a category with a full subcategory $\mathcal{C}$. We say that $\mathcal{C}$ is a *projective cover* of $\mathcal{E}$ if the following two conditions hold:

- Every object of $\mathcal{C}$ is a regular projective in $\mathcal{E}$.
- For every object $E \in \mathcal{E}$ there exists a regular epimorphism $P \twoheadrightarrow E$ with $P \in \mathcal{C}$.

A regular epimorphism $P \twoheadrightarrow E$ with $P \in \mathcal{C}$ is called a $\mathcal{C}$-*cover* of E.

Even if $\mathcal{E}$ has limits of some type, $\mathcal{C}$ will in general only have *weak* limits of that type. So if $\mathcal{E}$ has finite limits (e.g. if it is regular), then $\mathcal{C}$ will be *weakly lex*, i.e. will have all weak finite limits. To construct the weak limit of a diagram in $\mathcal{C}$ one first constructs the actual limit in the ambient category $\mathcal{E}$ and then one takes a $\mathcal{C}$-cover of the latter limit.

Finally, every weakly lex category $\mathcal{C}$ appears as a projective cover inside both its regular completion $\mathcal{C}_{reg}$ and its exact completion $\mathcal{C}_{ex}$ in the sense of [5].

2.3 Multi-pointed categories and stars

We first recall here some basic notions introduced in [12].

A *multi-pointed category* is a pair $(\mathcal{C}, \mathcal{N})$ consisting of a category $\mathcal{C}$ and a distinguished class $\mathcal{N}$ of morphisms in $\mathcal{C}$ which is an *ideal*. The latter, as mentioned in the introduction, means that for any pair of arrows $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, either $f \in \mathcal{N}$ or $g \in \mathcal{N}$ implies that $gf \in \mathcal{N}$. The elements of $\mathcal{N}$ are usually referred to as *null* morphisms.

We will often by abuse say that $\mathcal{C}$ is a multi-pointed category and suppress the ideal $\mathcal{N}$ if there is no possibility of confusion. Before moving on, let us recall here the main examples of multi-pointed categories that we shall consider.

- A simple first example of multi-pointed category is obtained by taking any category $\mathcal{C}$ and defining $\mathcal{N}$ to be the collection of all morphisms in $\mathcal{C}$. This class of examples is known as the *total context*.

- A second example of importance arises when $\mathcal{C}$ is pointed (i.e. has a zero object) and $\mathcal{N}$ is defined as the collection of zero morphisms, i.e. the morphisms that factor through the zero object. This general class of examples is referred to as the *pointed context*.

- The previous example can in fact be seen as a special case of a more general class of multi-pointed categories, the so called *proto-pointed context* introduced in [12]. This refers to a category $\mathcal{C}$ in which every object has a smallest subobject and where a morphism $f : X \to Y$ is defined to be a null morphism precisely when it factors through the smallest subobject of Y. In the case of a variety $\mathbb{V}$ of universal algebras these morphisms are exactly those whose image is the subalgebra E_Y of Y generated by the constants. This latter situation has been called the *algebraic proto-pointed context* in [11] and is actually the motivation for the term "multi-pointed". Indeed, a proto-pointed category is the category-theoretic notion that corresponds to varieties with potentially more than one constant, such as unital rings and Heyting algebras, just as that of pointed category corresponds to varieties possessing a unique constant.

An $\mathcal{N}$-*kernel* of a morphism $f : X \to Y$ is a morphism $k : K \to X$ such that $fk \in \mathcal{N}$ and which is universal with this property, i.e. whenever $fg \in \mathcal{N}$ there is a unique morphism u such that $ku = g$. Note that k is then necessarily a monomorphism. Observe also that in the total context the $\mathcal{N}$-kernels are just identities, while in the pointed context we obtain the usual notion of kernel.

In the more general proto-pointed setting the kernel of f is generally the inverse image of the smallest subobject of Y, which in the algebraic case becomes precisely the subalgebra of X consisting of those elements that map to the subalgebra generated by the constants in Y. So for example, if $f : X \to Y$ is a morphism in the proto-pointed category **Heyt** of Heyting algebras, then the $\mathcal{N}$-kernel of f is the subalgebra $\{x \in X | f(x) = 0 \vee f(x) = 1\}$. Similarly, if f lives in the category **Ring** of unitary rings, then its kernel in the above sense is $\{x \in X | (\exists n \in \mathbb{Z}) f(x) = n \cdot 1\}$. Note how the latter is indeed a subring of X and hence defines a subobject in the category **Ring**, whereas the ordinary kernel does not.

Since we shall have occasion to deal with categories that only have weak finite limits, we will also correspondingly require the notion of *weak $\mathcal{N}$-kernel* of a morphism $f : X \to Y$. This is defined as $\mathcal{N}$-kernels above, but by only requiring existence of the factorization, not necessarily uniqueness.

If $\mathcal{N}$-kernels exist for all morphisms in $\mathcal{C}$, then we shall say that $\mathcal{C}$ is a *multi-pointed category with kernels.* Similarly for weak $\mathcal{N}$-kernels.

We also record here for future use the following basic observation on the behavior of $\mathcal{N}$-kernels under pullback. For the sake of completeness, we also give the easy proof.

Lemma 2.1. *Consider the following pullback square in a multi-pointed category $(\mathcal{C}, \mathcal{N})$.*

$$\begin{array}{ccc} K' & \xrightarrow{k'} & X \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ K & \xrightarrow[k]{} & Y \end{array}$$

If k is the $\mathcal{N}$-kernel of some $f : Y \to Z$, then k' is the $\mathcal{N}$-kernel of $fg : X \to Z$.

Proof. Let $h : A \to X$ be such that $fgh \in \mathcal{N}$. Then, since k is the $\mathcal{N}$-kernel of f, there exists a unique $u : A \to K$ such that $ku = gh$. Now the universal property of the pullback gives a unique $v : A \to K'$ such that $g'v = u$ and $k'v = h$. Finally, note that k' is monomorphic because k is monomorphic. $\square$

A pair of morphisms $r = (r_0, r_1) : R \rightrightarrows X$ is called a *star* if $r_0 \in \mathcal{N}$. When it is moreover jointly monomorphic, we say that it is a *star relation*. In the total context this just defines a relation in the ordinary sense, whereas in the pointed case it is a relation whose first projection is zero. However, a more motivating example can be identified in the proto-pointed setting of the category **Ring** of unitary rings. Given any ideal $I \subseteq A$ inside the unitary ring A, we have an associated star relation R_I on A defined by $R_I := \bigcup_{n \in \mathbb{Z}} \{n\} \times (n + I)$. This star relation clearly uniquely determines the ideal I, but furthermore has the advantage that it is a subalgebra of $A \times A$ and hence lives in the category **Ring**, while the ideal I itself generally does not.

Given a relation R on an object X represented by the jointly monomorphic pair $r = (r_0, r_1) : R \rightrightarrows X$ and assuming $\mathcal{N}$-kernels exist, we define the *star* of R to be the relation R^* on X represented by the pair $(r_0 k_0, r_1 k_0)$ where $k_0 : K_0 \to R$ is the $\mathcal{N}$-kernel of r_0. Equivalently, one could say that R^* is the largest subrelation of R which is a star. In particular, when $R = \mathrm{Eq}(f)$ is the kernel pair of a morphism $f : X \to Y$, then $R^* = \mathrm{Eq}(f)^*$ is called the *star-kernel* of f.

In the context of a regular multi-pointed category it is possible to use the usual calculus of relations to develop a calculus of star relations, as is done in [11]. We shall not really need much of this though. We just record here the fact that, given relations R, S on an object X, we have that $(RS)^* = RS^*$.

Finally, we record an observation on how the star of a relation on an object X can be computed as a certain pullback involving the $\mathcal{N}$-kernel $\kappa_X : K_X \rightarrowtail X$ of the identity $1_X : X \to X$. Observe that this is just a generalization of the fact that the 0-class of a relation $R \rightarrowtail X \times X$ in a pointed category can be computed as the pullback of that relation along $\langle 0, 1 \rangle : X \to X \times X$. Since the more general statement does not appear in the literature, we also provide a proof.

Lemma 2.2. *Consider a relation* $R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X$ *in the multi-pointed category* $(\mathcal{E}, \mathcal{N})$ *with kernels. Then the* $\mathcal{N}$*-kernel* k_0 *of* r_0 *is obtained as the following pullback.*

$$\begin{array}{ccc} K_0 & \overset{k_0}{\rightarrowtail} & R \\ {\scriptstyle \langle \overline{r_0}, r_1 k_0 \rangle} \downarrow & & \downarrow {\scriptstyle \langle r_0, r_1 \rangle} \\ K_X \times X & \underset{\kappa_X \times 1_X}{\rightarrowtail} & X \times X \end{array}$$

where $\kappa_X : K_X \rightarrowtail X$ is the $\mathcal{N}$-kernel of the identity $1_X : X \to X$.

Proof. We consider the $\mathcal{N}$-kernel $k_0 : K_0 \rightarrowtail R$ of r_0 and we will show that there is a pullback square as indicated.

First, observe that $r_0 k_0 \in \mathcal{N}$ implies that there is a $\overline{r_0} : K_0 \to K_X$ such that $\kappa_X \overline{r_0} = r_0 k_0$, giving the indicated morphism $K_0 \to K_X \times X$ in the above commutative diagram.

Now assume that $f = \langle f_0, f_1 \rangle : Z \to K_X \times X$ and $g : Z \to R$ are such that $(\kappa_X \times 1_X) f = \langle r_0, r_1 \rangle g$. Then $r_0 g = \kappa_X f_0$ and $r_1 g = f_1$. Since $\kappa_X \in \mathcal{N}$, the first of these implies that $r_0 g \in \mathcal{N}$ and hence there exists a unique $h : Z \to K_0$ such that $k_0 h = g$. Then also $\langle \overline{r_0}, r_1 k_0 \rangle h = \langle \overline{r_0} h, r_1 k_0 h \rangle = \langle \overline{r_0} h, r_1 g \rangle = \langle f_0, f_1 \rangle = f$, where $\overline{r_0} h = f_0$ follows because $\kappa_X \overline{r_0} h = r_0 k_0 h = r_0 g = \kappa_X f_0$ and κ_X is monomorphic. □

3. 2-star-permutable categories

Let us recall the definition of 2-star-permutability from [11].

Definition 3.1. *Let $\mathcal{C}$ be a regular multi-pointed category with kernels. We say that $\mathcal{C}$ is* 2-star-permutable *if for any two effective equivalence relations R, S on an object $X \in \mathcal{C}$ we have $RS^* = SR^*$.*

In the total context, since the star of any relation is that relation itself, the definition says that effective equivalence relations are permutable, which yields precisely the regular Mal'tsev categories [4].

In the case of a pointed variety of universal algebras, the star of a relation R on X is the subrelation $R^* = \{(0, x) \in X \times X | (0, x) \in R\}$. More generally, in any pointed context, the star of the relation $\langle r_0, r_1 \rangle : R \rightarrowtail X \times X$ is the relation $\langle 0, c \rangle : C \rightarrowtail X \times X$ where $c : C \rightarrowtail X$ is the *0-class* of R, i.e. where the mono $c : C \rightarrowtail X$ is given by $c = r_1 \ker(r_0)$. Thus, the above definition says precisely that effective equivalence relations are *0-permutable* and this is known to characterize regular subtractive categories (see [16], [11] and [1] for the varietal case).

We first want to present an equivalent characterization of 2-star-permutability in terms of a symmetry property of reflexive relations. The symmetry property in question will be the following.

Definition 3.2. *Let $\mathcal{E}$ be a multi-pointed category with kernels and* $R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X$ *a relation in $\mathcal{E}$. We say that R is* left star-symmetric *if $R^* \leq (R^\circ)^*$. We say that it is* star-symmetric *if $R^* = (R^\circ)^*$, i.e. if both R and R° are left star-symmetric.*

Observe that in the pointed context left star-symmetry becomes the usual notion of *left 0-symmetry*, i.e. the statement that R satisfies the implication $(0, x) \in_A R \implies (x, 0) \in_A R$ for any generalized element $x : A \to X$ of X. In an algebraic proto-pointed setting it is the implication $(e, x) \in_A R \implies (x, e) \in_A R$ *for every* $e \in E_X$, where E_X is the subalgebra generated by the constants. In the total context on the other hand, R being left star-symmetric just means that $R \leq R^\circ$, which is to say that R is a symmetric relation in the ordinary sense. In particular, in this case left star-symmetry and star-symmetry become equivalent.

Indeed, note more generally that for any generalized elements $x, y : A \to X$ in $\mathcal{E}$ we have that $(x, y) \in_A R^*$ precisely if $(x, y) \in_A R$ and $x \in \mathcal{N}$. Thus, R being left-star symmetric is saying that whenever $(n, y) \in_A R$ with $n \in \mathcal{N}$, then also $(y, n) \in_A R$.

We will need the following lemma, from [11], for the proof of our next proposition.

Lemma 3.3. *For any morphism $f : X \to Y$ and every relation S on Y in a multi-pointed category we have $(f^{-1}(S))^* = (f^{-1}(S^*))^*$.*

We can now present new equivalent characterizations of 2-star-permutability using the notion of star-symmetry. In fact, this allows us to also deduce that 2-star-permutability is equivalent to having the equality $RS^* = SR^*$ for any two equivalence relations R, S on the same object, not just effective ones. This does not seem to have appeared in the literature before.

Proposition 3.4. *For a regular multi-pointed category $\mathcal{C}$ with kernels the following are equivalent:*

1. *$\mathcal{C}$ is 2-star-permutable.*

2. *For any two equivalence relations R, S on an object $X \in \mathcal{C}$ we have $RS^* = SR^*$.*

3. *Every reflexive relation E in $\mathcal{C}$ is left star-symmetric, i.e. $E^* \leq (E^\circ)^*$.*

4. *Every reflexive relation E in $\mathcal{C}$ is star-symmetric, i.e. $E^* = (E^\circ)^*$.*

Proof. 1. $\implies$ 4. Let $E \overset{e_0}{\underset{e_1}{\rightrightarrows}} X$ be a reflexive relation with diagonal $\delta : X \to E$. Set $R \coloneqq \mathrm{Eq}(e_0) = e_0^\circ e_0$ and $S \coloneqq \mathrm{Eq}(e_1) = e_1^\circ e_1$, so that both R and S are effective equivalence relations on E. Observe that $\delta^{-1}(SR) = \delta^\circ e_1^\circ e_1 e_0^\circ e_0 \delta = e_1 e_0^\circ = E$ and $\delta^{-1}(RS) = \delta^\circ e_0^\circ e_0 e_1^\circ e_1 \delta = e_0 e_1^\circ = E^\circ$. Now using the assumption (1) and 3.3 we have

$$\begin{aligned} RS^* = SR^* &\implies (RS)^* = (SR)^* \\ &\implies \delta^{-1}((RS)^*) = \delta^{-1}((SR)^*) \\ &\implies \delta^{-1}((RS)^*)^* = \delta^{-1}((SR)^*)^* \\ &\implies \delta^{-1}(RS)^* = \delta^{-1}(SR)^* \\ &\implies (E^\circ)^* = E^*. \end{aligned}$$

4. $\implies$ 2. Consider the reflexive relation $E \coloneqq SR$ on X. Then we have $E^* = (E^\circ)^* \implies (SR)^* = (RS)^* \implies SR^* = RS^*$.

2. $\implies$ 1. Clear.

3. $\iff$ 4. Clear by considering both reflexive relations E and E°. ☐

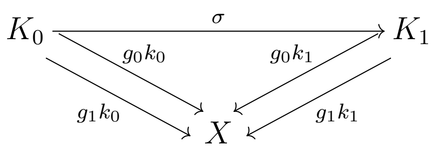
It should be observed that conditions (3) and (4) above can be formulated in any finitely complete multi-pointed category $(\mathcal{C}, \mathcal{N})$ with kernels, thus enlarging the class of categories for which the notion of 2-star-permutability can be considered to include non-regular ones. This generalizes the fact that the notion of Mal'tsev category can be formulated as a finitely complete category where every reflexive relation is symmetric [4], as well as the fact that subtractive categories can be defined as pointed finitely complete categories where every reflexive relation is 0-symmetric [16]. The following definition therefore appears pertinent.

Definition 3.5. *A multi-pointed category* $(\mathcal{C}, \mathcal{N})$ *is said to be* star-Mal'tsev *if every reflexive relation in* $\mathcal{C}$ *is left star-symmetric. Equivalently, if every reflexive relation in* $\mathcal{C}$ *is star-symmetric.*

With this terminology, 3.4 says that a regular multi-pointed category is 2-star-permutable if and only if it is star-Mal'tsev.

We now want to characterize the projective covers of 2-star-permutable regular multi-pointed categories, or, in other words, of regular star-Mal'tsev categories. In doing so, the following notion will play a key role. It is the appropriate adaptation of the notion of star-symmetry to the context of a multi-pointed category with only weak finite limits and weak kernels.

Definition 3.6. *Let* $(\mathcal{C}, \mathcal{N})$ *be a weakly lex multi-pointed category with weak* $\mathcal{N}$*-kernels. A graph* $G \underset{g_1}{\overset{g_0}{\rightrightarrows}} X$ *in* $\mathcal{C}$ *is said to be* left star-symmetric *if , given weak* $\mathcal{N}$*-kernels* $k_0 : K_0 \to G$ *and* $k_1 : K_1 \to G$ *of* g_0 *and* g_1 *respectively, there exists a* $\sigma : K_0 \to K_1$ *such that the following diagram serially commutes*



i.e. such that $g_1k_1\sigma = g_0k_0$ *and* $g_0k_1\sigma = g_1k_0$ *both hold. We say that it is* star-symmetric *if both* G *and its opposite graph* $G \underset{g_0}{\overset{g_1}{\rightrightarrows}} X$ *are left star-symmetric.*

In other words, a graph G is left star-symmetric if a "weak star" of G factors through a weak star of the opposite graph. Note also that the definition does not depend on the chosen weak $\mathcal{N}$-kernels because any two weak $\mathcal{N}$-kernels of the same morphism factor through each other. Furthermore, it is clear that when G is a relation and $\mathcal{N}$-kernels exist the definition says precisely that $G^* \leq (G^\circ)^*$, i.e. that G is a left star-symmetric relation.

Remark 3.7. It is easy to see that in the total context we get the usual definition of a symmetric graph, since both $\mathcal{N}$-kernels are identities in this case. In the pointed context one of the two commutativities required above becomes trivial because $g_0 k_0 = 0 = g_1 k_1$ and we obtain the notion of a *left 0-symmetric* graph.

We now introduce the categories that will appear in our characterization of the projective covers of 2-star-permutable categories. These are the multi-pointed categories with weak finite limits and weak kernels which satisfy the appropriate "weakening" of the star-Mal'tsev property. Our terminology is inspired by that of Rosický-Vitale in [19] for the total context.

Definition 3.8. *We will say that a weakly lex multi-pointed category $\mathcal{C}$ with weak kernels is* star-G-Mal'tsev *if every reflexive graph in $\mathcal{C}$ is left star-symmetric. Equivalently, if every reflexive graph is star-symmetric.*

In what follows, we will be considering regular categories $\mathcal{E}$ together with a projective cover $\mathcal{C}$ of $\mathcal{E}$. We are thus interested in how ideals of morphisms in the projective cover are related to ideals of morphisms in the ambient regular category. A thorough analysis of this situation is contained in [9], from which we now borrow and record below the main points that will be of use in the remainder of this paper.

First, if $\mathcal{N}$ is an ideal of morphisms in the regular category $\mathcal{E}$, then we denote by $\mathcal{N}_{\mathcal{C}}$ the restriction of $\mathcal{N}$ to $\mathcal{C}$. It is then clear that $\mathcal{N}_{\mathcal{C}}$ is an ideal in $\mathcal{C}$.

Second, if we are given an ideal $\mathcal{N}$ in the projective cover $\mathcal{C}$, then we define $\mathcal{N}^{\mathcal{E}}$ to be the collection of morphisms $f : X \to Y$ in $\mathcal{E}$ for which there exists a commutative square

$$\begin{array}{ccc} P & \xrightarrow{n} & Q \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where p and q are regular epimorphisms and $n \in \mathcal{N}$. It is again not hard to check that $\mathcal{N}^{\mathcal{E}}$ is an ideal in $\mathcal{E}$.

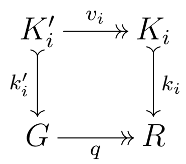
Lemma 3.9. *[9] Let $\mathcal{E}$ be a regular category having a projective cover $\mathcal{C}$.*

1. *For any ideal $\mathcal{N}$ in $\mathcal{E}$, if $\mathcal{E}$ has $\mathcal{N}$-kernels, then $\mathcal{C}$ has weak $\mathcal{N}_{\mathcal{C}}$-kernels, which can be computed by taking a projective cover of the domain of the $\mathcal{N}$-kernel in $\mathcal{E}$.*
2. *For any ideal $\mathcal{N}$ in $\mathcal{C}$, the category $\mathcal{C}$ has weak $\mathcal{N}$-kernels if and only if $\mathcal{E}$ has $\mathcal{N}^{\mathcal{E}}$-kernels.*
3. *For any ideal $\mathcal{N}$ in $\mathcal{C}$ we have $(\mathcal{N}^{\mathcal{E}})_{\mathcal{C}} = \mathcal{N}$.*
4. *For any ideal $\mathcal{N}$ in $\mathcal{C}$, regular epimorphisms are $\mathcal{N}^{\mathcal{E}}$-saturating in $\mathcal{E}$ (see 3.11).*

Before presenting our characterization, we find it useful to isolate the following fundamental observation.

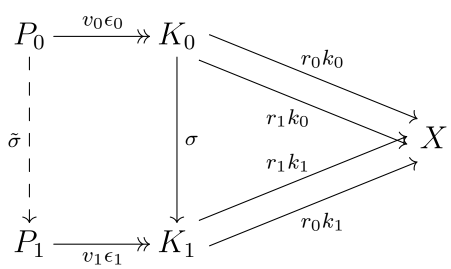
Lemma 3.10. *Let $\mathcal{C}$ be a projective cover of the regular multi-pointed category $(\mathcal{E}, \mathcal{N})$ with kernels. Consider a graph $G \underset{g_1}{\overset{g_0}{\rightrightarrows}} X$ in $\mathcal{C}$ with its image factorization $\langle g_0, g_1 \rangle = G \xrightarrow{q} R \xrightarrow{\langle r_0, r_1 \rangle} X \times X$ in $\mathcal{E}$. Then G is a left star-symmetric graph if and only if R is a left star-symmetric relation.*

Proof. Consider $\mathcal{N}$-kernels $k_i : K_i \to R$ of r_i, for $i = 0, 1$. Form the pullbacks below for $i = 0, 1$ and then take $\mathcal{C}$-covers $\epsilon_i : P_i \twoheadrightarrow K'_i$.



By 2.1 we know that $k_i' : K_i' \rightarrowtail G$ is the $\mathcal{N}$-kernel of $r_i q = g_i$. Thus, we have that $u_i := k_i'\epsilon_i : P_i \to G$ is a weak $\mathcal{N}$-kernel of g_i in $\mathcal{C}$ for $i = 0, 1$ by 3.9.

Assume first that R is left star-symmetric, so that there exists a morphism $\sigma : K_0 \to K_1$ such that $r_1 k_1 \sigma = r_0 k_0$ and $r_0 k_1 \sigma = r_1 k_0$. By projectivity of P_0 and the fact that $v_1\epsilon_1$ is a regular epimorphism, there exists a morphism $\tilde{\sigma} : P_0 \to P_1$ making the following diagram commute.



Now we have

$$\begin{aligned} g_1 u_1 \tilde{\sigma} &= r_1 q k_1' \epsilon_1 \tilde{\sigma} \\ &= r_1 k_1 v_1 \epsilon_1 \tilde{\sigma} \\ &= r_1 k_1 \sigma v_0 \epsilon_0 \\ &= r_0 k_0 v_0 \epsilon_0 \\ &= r_0 q k_0' \epsilon_0 \\ &= g_0 u_0 \end{aligned}$$

and similarly

$$\begin{aligned} g_0 u_1 \tilde{\sigma} &= r_0 q k_1' \epsilon_1 \tilde{\sigma} \\ &= r_0 k_1 v_1 \epsilon_1 \tilde{\sigma} \\ &= r_0 k_1 \sigma v_0 \epsilon_0 \\ &= r_1 k_0 v_0 \epsilon_0 \\ &= r_1 q k_0' \epsilon_0 \\ &= g_1 u_0 \end{aligned}$$

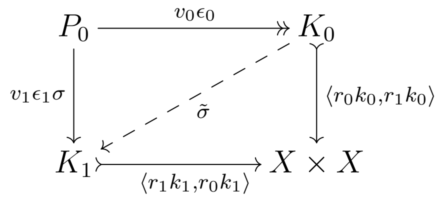
proving that G is left star-symmetric.

Conversely, assume that G is left star-symmetric. This means that there exists a $\sigma : P_0 \to P_1$ such that $g_1 u_1 \sigma = g_0 u_0$ and $g_0 u_1 \sigma = g_1 u_0$. We can then again calculate as follows:

$$\begin{aligned} r_1 k_1 v_1 \epsilon_1 \sigma &= r_1 q k_1' \epsilon_1 \sigma \\ &= g_1 k_1' \epsilon_1 \sigma \\ &= g_1 u_1 \sigma \\ &= g_0 u_0 \\ &= r_0 q k_0' \epsilon_0 \\ &= r_0 k_0 v_0 \epsilon_0 \end{aligned}$$

$$\begin{aligned} r_0 k_1 v_1 \epsilon_1 \sigma &= r_0 q k_1' \epsilon_1 \sigma \\ &= g_0 k_1' \epsilon_1 \sigma \\ &= g_0 u_1 \sigma \\ &= g_1 u_0 \\ &= r_1 q k_0' \epsilon_0 \\ &= r_1 k_0 v_0 \epsilon_0 \end{aligned}$$

This means that the square below commutes and so we obtain the indicated morphism $\tilde{\sigma}$ because $v_0 \epsilon_0$ is a regular epimorphism and $\langle r_1 k_1, r_0 k_1 \rangle$ is monomorphic, being the star of the relation R°.

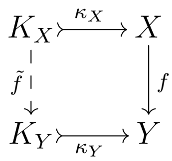


The commutation of the bottom triangle is precisely left star-symmetry of R. □

In order to prove our main result, we will need to impose an additional condition on the regular category $\mathcal{E}$ regarding the behavior of regular epimorphisms with respect to $\mathcal{N}$-kernels. This condition is familiar from the literature (see [9,11,14]) and is indeed mild enough that it includes all examples of interest. We now proceed to introduce the necessary notions.

Consider any object X in the multi-pointed category $(\mathcal{E},\mathcal{N})$. We will denote by $\kappa_X : K_X \rightarrowtail X$ the $\mathcal{N}$-kernel of the identity morphism 1_X. Observe that by definition the generalized elements of K_X correspond precisely to the generalized elements of X that are in $\mathcal{N}$. Hence, K_X should be thought of as consisting of the "trivial elements" of the object X. Indeed, in the algebraic proto-pointed setting K_X is exactly what we have earlier in the text denoted by E_X, namely the subalgebra of X generated by the constants of the theory.

Now given any morphism $f : X \to Y$ in $\mathcal{E}$, we have a uniquely induced morphism $\tilde{f} : K_X \to K_Y$ making the following square commute.



Then we can introduce the following definition.

Definition 3.11. *A morphism $f : X \to Y$ in a multi-pointed category $(\mathcal{E},\mathcal{N})$ is called* saturating *if the induced morphism $\tilde{f} : K_X \to K_Y$ is a regular epimorphism.*

Note that in the pointed context all morphisms are saturating, since $K_X = 0$ for any object X. The same holds in any algebraic proto-pointed setting, since every element $e \in X$ which is generated by constants is preserved under all homomorphisms $f : X \to Y$. Furthermore, it is not hard to see that regular epimorphisms are saturating in any proto-pointed context, not just the varietal one. In the total context, on the other hand, it is clear that the saturating morphisms are exactly the regular epimorphisms. In fact, that all regular epimorphisms are saturating is precisely what we shall require below.

Now we can present the main result of this note.

Theorem 3.12. *Let $\mathcal{C}$ be a projective cover of the regular multi-pointed category with kernels $(\mathcal{E}, \mathcal{N})$ and assume regular epimorphisms in $\mathcal{E}$ are saturating. Then $(\mathcal{E}, \mathcal{N})$ is 2-star-permutable if and only if $(\mathcal{C}, \mathcal{N}_{\mathcal{C}})$ is star-G-Mal'tsev.*

Proof. Assume first that $\mathcal{E}$ is 2-star-permutable and consider any reflexive graph $G \underset{g_1}{\overset{g_0}{\rightrightarrows}} X$ in $\mathcal{C}$ with splitting $\delta : X \to G$. Consider its image factorization $\langle g_0, g_1 \rangle = G \overset{q}{\twoheadrightarrow} R \overset{\langle r_0, r_1 \rangle}{\rightarrowtail} X \times X$ in the regular category $\mathcal{E}$. Then the relation R on X is reflexive as well, since $r_i q \delta = g_i \delta = 1_X$ for $i = 0, 1$. Since $\mathcal{E}$ is 2-star-permutable, we know by 3.4 that R must be star-symmetric. Now 3.10 implies that G is (left) star-symmetric.

Conversely, assume that $(\mathcal{C}, \mathcal{N}_{\mathcal{C}})$ is star-G-Mal'tsev. Consider any reflexive relation $E \underset{e_1}{\overset{e_0}{\rightrightarrows}} X$ in $\mathcal{E}$. We want to show that E is left star-symmetric.

Take a $\mathcal{C}$-cover $p : \tilde{X} \twoheadrightarrow X$ of X and consider the inverse image relation $E' := p^{-1}(E)$. i.e. form the following pullback

$$\begin{array}{ccc} E' & \xrightarrow{\langle e_0', e_1' \rangle} & \tilde{X} \times \tilde{X} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p \times p} \\ E & \xrightarrow{\langle e_0, e_1 \rangle} & X \times X \end{array}$$

Now again take a $\mathcal{C}$-cover $\epsilon : G \twoheadrightarrow E'$ and set $g_0 := e_0'\epsilon$ and $g_1 := e_1'\epsilon$, so that we have a graph $G \underset{g_1}{\overset{g_0}{\rightrightarrows}} \tilde{X}$ in $\mathcal{C}$. Observe that the relation E' is reflexive, being the inverse image of a reflexive relation. It follows that the graph G is also reflexive. Indeed, if $\delta' : \tilde{X} \to E'$ is the diagonal of E', then by projectivity of $\tilde{X}$ we can lift to a $\tilde{\delta} : \tilde{X} \to G$ such that $\epsilon\tilde{\delta} = \delta'$ and then $g_i\tilde{\delta} = e_i'\epsilon\tilde{\delta} = e_i'\delta' = 1_{\tilde{X}}$ for $i = 0, 1$.

Now consider $\mathcal{N}$-kernels $k_i : K_i \to E$ of e_i and $k_i' : K_i' \to E$ of e_i' in $\mathcal{E}$ for $i = 0, 1$. We then have induced morphisms $u_i : K_i' \to K_i$ such that $k_i u_i = q k_i'$. We claim that the u_i are regular epimorphisms.

$$\begin{array}{ccc} K_i' & \xrightarrow{k_i'} & E' \\ {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle q} \\ K_i & \xrightarrow{k_i} & E \end{array}$$

To see this for u_0 we consider the following two commutative diagrams. In the first one the right-hand square is a pullback by construction, while the left-hand square is a pullback by 2.2. In the second diagram we know only that the right-hand square is a pullback, again by 2.2.

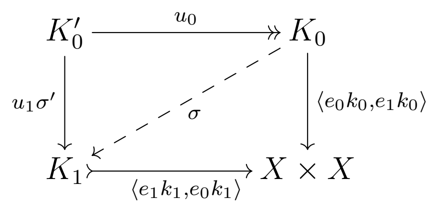
$$\begin{array}{ccccc} K_0' & \xrightarrow{k_0'} & E' & \xrightarrow{q} & E \\ {\scriptstyle \langle \overline{e_0'}, e_1' k_0' \rangle}\downarrow & & \downarrow{\scriptstyle \langle e_0', e_1' \rangle} & & \downarrow{\scriptstyle \langle e_0, e_1 \rangle} \\ K_{\tilde{X}} \times \tilde{X} & \xrightarrow{\kappa_{\tilde{X}} \times 1_{\tilde{X}}} & \tilde{X} \times \tilde{X} & \xrightarrow{p \times p} & X \times X \end{array}$$

$$\begin{array}{ccccc} K_0' & \xrightarrow{u_0} & K_0 & \xrightarrow{k_0} & E \\ {\scriptstyle \langle \overline{e_0'}, e_1' k_0' \rangle}\downarrow & & \downarrow{\scriptstyle \langle \overline{e_0}, e_1 k_0 \rangle} & & \downarrow{\scriptstyle \langle e_0, e_1 \rangle} \\ K_{\tilde{X}} \times \tilde{X} & \xrightarrow{\tilde{p} \times p} & K_X \times X & \xrightarrow{\kappa_X \times 1_X} & X \times X \end{array}$$

Since $(p \times p)(\kappa_{\tilde{X}} \times 1_{\tilde{X}}) = (\kappa_X \times 1_X)(\tilde{p} \times p)$, we deduce that the outer rectangle in the second diagram is a pullback. Then by the usual pullback-cancellation property we have that the left-hand square is a pullback as well. But since both p and $\tilde{p}$ are regular epimorphisms, so is $\tilde{p} \times p$, since $\mathcal{E}$ is regular, and hence we deduce that the pullback u_0 is a regular epimorphism.

By the assumption that $\mathcal{C}$ is star-G-Mal'tsev, the reflexive graph G is left star-symmetric and so by 3.10 its image relation E' is also left star-symmetric. Thus, there exists a $\sigma' : K_0' \to K_1'$ such that $e_1' k_1' \sigma' = e_0' k_0'$ and $e_0' k_1' \sigma' = e_1' k_0'$.

Finally, consider the commutative square below.



Since u_0 is a regular epimorphism and $\langle e_1 k_1, e_0 k_1 \rangle$ is monomorphic (being the star of the relation E°), we get the indicated factorization $\sigma : K_0 \to K_1$, which shows that E is left star-symmetric. Thus, by 3.4 it follows that $\mathcal{E}$ is 2-star-permutable. $\square$

The above result yields a characterization of when the regular and exact completion (in the sense of [5]) of a weakly lex multi-pointed category are 2-star-permutable.

Corollary 3.13. *Let $(\mathcal{C}, \mathcal{N})$ be a weakly lex multi-pointed category with weak kernels. Then $(\mathcal{C}_{reg}, \mathcal{N}^{\mathcal{C}_{reg}})$ is 2-star-permutable if and only if $(\mathcal{C}, \mathcal{N})$ is star-G-Mal'tsev.*

Proof. $\mathcal{C}$ appears as a projective cover inside $\mathcal{C}_{reg}$. Then 3.12 indeed applies to give the result because by 3.9 we know that $\mathcal{C}_{reg}$ has $\mathcal{N}^{\mathcal{C}_{reg}}$-kernels, regular epimorphisms in $\mathcal{C}_{reg}$ are $\mathcal{N}^{\mathcal{C}_{reg}}$-saturating and $(\mathcal{N}^{\mathcal{C}_{reg}})_{\mathcal{C}} = \mathcal{N}$. $\square$

In the exact same way we get the corresponding result about the exact completion $\mathcal{C}_{ex}$.

Corollary 3.14. *Let* $(\mathcal{C}, \mathcal{N})$ *be a weakly lex multi-pointed category with weak kernels. Then* $(\mathcal{C}_{ex}, \mathcal{N}^{\mathcal{C}_{ex}})$ *is 2-star-permutable if and only if* $(\mathcal{C}, \mathcal{N})$ *is star-G-Mal'tsev.*

Remark 3.15. We should comment here on how 3.12 extends the characterizations of projective covers for regular Mal'tsev categories, due to Rosicky-Vitale [19], and for regular subtractive categories, due to Gran-Rodelo [13].

For the Mal'tsev case, it is immediately clear from the definitions that we obtain exactly the same characterization as in [19], i.e. our star-G-Mal'tsev categories are exactly the G-Mal'tsev ones introduced therein. Note that in that paper G-Mal'tsev is initially defined by requiring that every reflexive graph be both symmetric and transitive, but this is equivalent to just requiring symmetry and that is in fact implicitly proved in [19].

In the pointed context, it is not immediate from the definitions that our star-G-Mal'tsev, which we should probably call *0-G-Mal'tsev* in this case, yields the *w-subtractive* categories of Gran-Rodelo [13]. Of course, since both characterize projective covers of the same class of regular categories, they turn out to be equivalent, since any weakly lex category can always be considered a projective cover of its regular completion. On the other hand, a direct proof of the equivalence of the two notions is also not too hard to construct.

Now suppose we are in an algebraic proto-pointed context and that the set of constants of the variety is nonempty. It was proved in [11] that in this case 2-star-permutability is equivalent to a priori more general properties such as *3-star-permutability* and the *symmetric saturation property*, but also to the syntactic condition defining *E-subtractive varieties* in the sense of [20]. We would like to conclude this note by showing how the equivalence with the latter notion can also directly be obtained from our characterization in terms of star-symmetry.

Corollary 3.16. *Let* $\mathbb{V}$ *be a variety of universal algebras and let* $E_{\mathbb{V}} \neq \emptyset$ *be its algebra of constants (i.e. the free* $\mathbb{V}$*-algebra on the empty set).*

Then $\mathbb{V}$ *is 2-star-permutable if and only if the following syntactic condition holds:*

For every $e \in E_{\mathbb{V}}$ *there exists a binary term* $s_e(x, y)$ *such that* $s_e(x, x) = e$ *and* $s_e(x, e) = x$.

Proof. Suppose 2-star-permutability holds and fix any $e \in E_{\mathbb{V}}$. We then consider a graph $F(x, y) \underset{g_1}{\overset{g_0}{\rightrightarrows}} F(x)$ between free algebras on 2 and 1 generator respectively, where g_0, g_1 are defined by setting $g_0(x) = x$, $g_0(y) = e$ and $g_1(x) = g_1(y) = x$. This graph is reflexive, since it is clearly split by the map $\delta : F(x) \to F(x, y)$ defined by $\delta(x) = x$. Since free algebras are projective, we can apply 3.12 (i.e. $\mathcal{C}$ here is the full subcategory of free algebras) to deduce that this graph must be star-symmetric.

Now we have $(e, x) = (g_0(y), g_1(y))$, so by the star-symmetry we must also have $(x, e) = (g_0(s_e), g_1(s_e))$ for some $s_e(x, y) \in F(x, y)$. Thus, $x = g_0(s_e(x, y)) = s_e(x, e)$ and $e = g_1(s_e(x, y)) = s_e(x, x)$.

Conversely, suppose we have binary terms $s_e(x, y)$ for all $e \in E_{\mathbb{V}}$ with the indicated properties. We will show that any reflexive relation $R \rightarrowtail X \times X$ in the variety $\mathbb{V}$ is left star-symmetric.

Indeed, assume that $(e, x) \in R$ for some $e \in E_X$ and $x \in X$. Since R is reflexive, we also have $(x, x) \in R$. By compatibility with the operations we then must have $(s_e(x, e), s_e(x, x)) \in R$, i.e. that $(x, e) \in R$. This concludes the proof. □

As particular examples of E-subtractive varieties one has the categories **Ring** of unitary rings, as well as the categories **Heyt**, **Bool** of Heyting and Boolean algebras respectively. These are all in fact already Mal'tsev, but of course one also has examples of subtractive varieties which are not, such as that of *implication algebras* [17].

Remark 3.17. Since our main result is stated for any regular category, without requiring exactness, it can equally well be applied to *quasi-varieties* of universal algebras, since these are still regular categories. This encompasses further interesting examples, such as the category **RedRng** of *reduced rings*, i.e. unitary rings R satisfying $(\forall x \in R)(\forall n \geq$

$1)(x^n = 0 \implies x = 0)$. As a non-Mal'tsev example here one has the quasi-variety of *BCK algebras* (see [3], for example).

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THE FULLNESS AXIOM AND EXACT COMPLETION OF HOMOTOPY CATEGORIES

Jacopo Emmenegger

Résumé. Nous adoptons une formulation catégorique du “Fullness Axiom” de Aczel de la Théorie Constructive des Ensembles, dans le but de dériver la propriété que la complétion exacte est localement cartésienne fermée. Nous montrons, en tant qu'application, que cette formulation est vérifiée dans la catégorie homotopique de toute catégorie de modèles satisfaisant des faibles conditions additionnelles, en obtenant ainsi en particulier que la complétion exacte de la catégorie des espaces topologiques et classes homotopiques des applications continues est localement cartésienne fermée. Dans la perspective de la théorie des types, ces résultats donnent une motivation générale pour la fermeture cartésienne locale de la catégorie des setoïdes. Pourtant, les résultats et les démonstrations sont formulés seulement dans le langage des catégories, et les lecteurs n'ont besoin d'aucune connaissance préalable de la théorie des types ou de la théorie constructive des ensembles.

Abstract. We use a category-theoretic formulation of Aczel's Fullness Axiom from Constructive Set Theory to derive the local cartesian closure of an exact completion. As an application, we prove that such a formulation is valid in the homotopy category of any model category satisfying mild requirements, thus obtaining in particular the local cartesian closure of the exact completion of topological spaces and homotopy classes of maps. Under a type-theoretic reading, these results provide a general motivation for the local cartesian closure of the category

of setoids. However, results and proofs are formulated solely in the language of categories, and no knowledge of type theory or constructive set theory is required on the reader's part.
Keywords. Exact completion, homotopy category, fullness axiom, local cartesian closure, weak limits.
Mathematics Subject Classification (2020). 18D15; 18A35; 18E08; 55U35; 18B15; 18A15; 18F60.

Introduction

In the paper that generalises the exact completion construction to an arbitrary category with *weak* finite limits, where a universal arrow is not required to be unique, Carboni and Vitale advocated a deeper study of that construction applied to homotopy categories [9]. These categories, indeed, form a large class of natural examples of categories with weak finite limits, in the sense that they do not arise as projective covers of finitely complete categories. A first step in this direction was made by Gran and Vitale in [13], where they provide a complete characterisation of those exact completions of categories with weak finite limits (henceforth ex/wlex completions) that produce a pretopos, and apply this result to show that the exact completion of the category of topological spaces and homotopy classes of maps is indeed a pretopos. However, the problem of determining whether it is also locally cartesian closed is explicitly left open.

The author has given a complete characterisation of locally cartesian closed ex/wlex completions in [10]. That characterisation is however not much suited to the study of the ex/wlex completion of a homotopy category $\mathrm{Ho}\,\mathbb{M}$, when instead a formulation in terms of the original Quillen model category $\mathbb{M}$ would be preferable. The present paper provides a condition ensuring the local cartesian closure of the ex/wlex completion $(\mathrm{Ho}\,\mathbb{M})_{\mathrm{ex}}$ for a large class of model categories. Somewhat surprisingly, this condition turns out to be what Carboni and Rosolini named weak local cartesian closure in [8], that is, simply existence of weak dependent products.

As we shall prove in the last section, the homotopy quotient of a weak dependent product in $\mathbb{M}$ is a *dependent full diagram* in $\mathrm{Ho}\,\mathbb{M}$. The latter is a generalisation to arbitrary categories with weak finite limits of a concept introduced in [11] to prove the local cartesian closure of the exact completion of a well-pointed category with finite products and weak equalisers. Anther precursor is the axiom F for a class of small maps in [5], which is proved to be stable under ex/reg completions. We shall comment on the (tight) relation between this axiom and dependent full diagrams in Remark 3.2. Indeed, both the universal property of dependent full diagrams and axiom F are inspired by Aczel's Fullness Axiom from the constructive set theory CZF [1, 2]. The Fullness Axiom is a collection principle asserting the existence of what Aczel calls *full sets*, that is, sets containing enough total relations (a.k.a. multi-valued functions). This axiom is strictly weaker than the Power Set Axiom (and it is regarded as a predicative principle) but strong enough, in particular, to entail the Exponentiation Axiom, which asserts that functions between two sets form a set.

We prove that also in the general case of an ex/wlex completion, existence of dependent full diagrams in $\mathbb{C}$ is enough to derive the local cartesian closure of $\mathbb{C}_{\mathrm{ex}}$. That this should be possible follows from the observation, due to Erik Palmgren, that arrows $V \to Y$ out of a weak product $Z \leftarrow V \to X$ may be understood as families indexed on Z of multi-valued functions from X to Y. A more robust formulation of this observation is in Remark 2.1. In addition, dependent full diagrams endow the internal logic with universal quantification and implication which, in turn, can be used to extract from multi-valued functions only the functional ones (cf. Lemma 2.6). At this point it is enough to construct a suitable equivalence relation to obtain an exponential in $\mathbb{C}_{\mathrm{ex}}$.

In order to prove that dependent full diagrams are homotopy quotients of weak dependent products, we exploit the concepts of *path category* and *weak homotopy* Π*-type*, recently introduced by van den Berg and Moerdijk in [6]. A path category is a slight strengthening of Brown's fibration category. In particular, the subcategory $\mathbb{M}_{\mathrm{f}}$ on the fibrant objects of a model category $\mathbb{M}$ is a path category as soon as all the objects of $\mathbb{M}$ are cofibrant. Weak homotopy Π-types in a path category $\mathbb{C}$ are what van den Berg and Moerdijk use to derive the local cartesian closure of $(\mathrm{Ho}\,\mathbb{C})_{\mathrm{ex}}$, the homotopy exact completion of $\mathbb{C}$. We show that if $\mathbb{M}$ is right proper, then weak homotopy Π-types arise as fibrant replacements of weak dependent products, so that a weak dependent product in $\mathbb{M}$ gives rise to a weak homotopy Π-type in $\mathbb{M}_\mathrm{f}$. Furthermore, in the same way as pullbacks along fibrations enjoy the additional universal property of homotopy pullbacks, also weak homotopy Π-types enjoy an additional universal property up to homotopy, which shows that the homotopy quotient maps weak homotopy Π-types to dependent full diagrams.

Under a type-theoretic reading, the results in the present paper provide a general motivation for the local cartesian closure of the category of setoids in Martin-Löf type theory. Indeed, the category of contexts of Martin-Löf type theory is a path category [4], see also [12], and ∏-types endow it with weak homotopy Π-types. More generally, we obtain a more elementary proof of the local cartesian closure of a homotopy exact completion. It should be noted that, under the reading of arrows out of a weak product as multi-valued functions, single-valued functions in a homotopy category $\mathrm{Ho}\,\mathbb{C}$ appear as "homotopy-irrelevant" arrows. Indeed, an arrow k out of a homotopy limit, say a homotopy pullback of $f\colon X \to Y$ and $g\colon Z \to Y$, induce an arrow in $(\mathrm{Ho}\,\mathbb{C})_\mathrm{ex}$ out of the actual pullback of f and g in if and only if values of k only depend on pairs (x, z) and not on the homotopy witnessing $f(x) \simeq g(z)$. The analogy with the role of homotopy-irrelevant fibrations in the argument for the local cartesian closure of $(\mathrm{Ho}\,\mathbb{C})_\mathrm{ex}$ from van den Berg and Moerdijk [6] may be worth further investigation. Indeed, in type-theoretic terminology, these are the proof-irrelevant setoid families whose importance has been stressed by Palmgren [16].

Furthermore, the observation that dependent full diagrams naturally arise as homotopy quotients of weak dependent products shows that existence of the former is not just a particular feature of the category of types in Martin-Löf type theory, the only example in [11]. On the contrary, it provides a large class of examples of categories with weak finite limits and dependent full diagrams. In particular, we obtain the local cartesian closure of the exact completion of the category of spaces and homotopy classes of maps, thus answering a question left open in [13].

The first half of the paper is devoted to the proof that existence of dependent full diagrams imply the local cartesian closure of the exact completion. In order to simplify the presentation, we split the argument in two steps. In Section 2, after a brief recap on ex/wlex completions, we define a non-indexed version of a full diagram in $\mathbb{C}$ and, assuming that $\mathbb{C}_{ex}$ (or, equivalently, $\mathbb{C}$) has the needed structure for implication and universal quantification, we construct from it an exponential in $\mathbb{C}_{ex}$. Section 3 contains the definition of the more general dependent full diagrams and the proof that their existence gives rise both to right adjoints to inverse images and to non-indexed full diagrams. Finally, Section 4 covers the case of homotopy categories.

Full diagrams

We briefly recall some basic facts about weak limits and the exact completion which are essential to our treatment. For additional background notions and notations we refer to [9, 8] and to section 1 of [10]. In this section and the next one $\mathbb{C}$ denotes a category with weak finite limits and $\mathbb{C}_{ex}$ denotes its ex/wlex completion, that we shall refer to as the exact completion of $\mathbb{C}$. Regular epis are denoted with a triangle head, like in $A \twoheadrightarrow B$, while hook arrows $A \hookrightarrow B$ denote monos.

Recall that weak limits are defined as usual limits but without requiring uniqueness of a universal arrow. An arrow $f\colon V \to Y$ in $\mathbb{C}$ from a weak product $Z \leftarrow V \to X$ of Z and X is *determined by projections* [10] if it coequalises every pair of arrows jointly coequalised by the two weak product projections. A *weak exponential* [8, Definition 2.1] in $\mathbb{C}$ of two objects Y and X consists of an object W, a weak product $W \leftarrow V \to X$ and an arrow $V \to Y$ which is determined by projections, such that for any object W', weak product $W' \leftarrow V' \to X$ and arrow $V' \to Y$ determined by projections, there are (not necessarily unique) arrows $h\colon W' \to W$ and $k\colon V' \to V$ making the obvious diagram commute.

An object X in a category $\mathbb{E}$ is called *(regular) projective* if, for every regular epi $g\colon A \twoheadrightarrow B$ and arrow $f\colon X \to B$, there is a *lift* of f against g, *i.e.* an arrow $f'\colon X \to A$ such that $gf' = f$. A *projective cover* of an exact category $\mathbb{E}$ is a full subcategory $\mathbb{P}$ such that *(i)* every object in $\mathbb{P}$ is projective in $\mathbb{E}$, and *(ii)* every object in $\mathbb{E}$ is covered by an object in $\mathbb{P}$, *i.e.* for every A in $\mathbb{E}$ there are X in $\mathbb{P}$ and a regular epi $X \twoheadrightarrow A$. $\mathbb{E}$ has *enough projectives* if it has a projective cover. Whenever we are given a projective cover $\mathbb{P}$ of an exact category $\mathbb{E}$, we adopt the convention of using letters from P to Z for objects in $\mathbb{P}$. The reader should however keep in mind that $\mathbb{P}$ is not in general closed under limits that exist in $\mathbb{E}$.

Recall from [9] that, for a category $\mathbb{C}$ with weak finite limits, the exact completion $\mathbb{C}_{\text{ex}}$ can be described as the category whose objects are pseudo equivalence relations $R \rightrightarrows X$ in $\mathbb{C}$ and whose arrows from $R \rightrightarrows X$ to $S \rightrightarrows Y$ are equivalence classes of those arrows $X \to Y$ of $\mathbb{C}$ that map related elements to related elements, where $f, f' \colon X \to Y$ are equivalent if they are related in $S \rightrightarrows Y$. The full subcategory of $\mathbb{C}_{\text{ex}}$ on the free pseudo equivalence relations, *i.e.* those with equal legs, is a projective cover of $\mathbb{C}_{\text{ex}}$. It is equivalent to $\mathbb{C}$ via the embedding of $\mathbb{C}$ into $\mathbb{C}_{\text{ex}}$, that maps an object X to the pair of identities $\text{id}_X, \text{id}_X \colon X \rightrightarrows X$. Conversely, a projective cover $\mathbb{P}$ of an exact category $\mathbb{E}$ has weak finite limits and its exact competion $\mathbb{P}_{\text{ex}}$ is equivalent to $\mathbb{E}$ [9, Thm. 16]. A weak limit in $\mathbb{P}$ is obtained covering with a projective the corresponding limit in $\mathbb{E}$. More generally, a cone in $\mathbb{P}$ over a diagram $\mathcal{D}$ in $\mathbb{P}$ is a weak limit if and only if the unique arrow from the cone into the limit in $\mathbb{E}$ of $\mathcal{D}$ is a regular epi. see [10, Lemma 1.7].

Given a projective cover $\mathbb{P}$ of $\mathbb{E}$ exact and a weak product $Z \xleftarrow{p_1} V \xrightarrow{p_2} X$ in $\mathbb{P}$, an arrow $f \colon V \to Y$ is determined by projections in $\mathbb{P}$ if and only if it factors in $\mathbb{E}$ through the regular epi $\langle p_1, p_2 \rangle \colon V \twoheadrightarrow Z \times X$. It follows that $\mathbb{P}$ has weak exponentials if $\mathbb{E}$ is cartesian closed: given $Y, X \in \mathbb{P}$, one just need cover Y^X with a projective W and to do the same with $W \times X$. However, as argued in the last section of [10], the universal property of weak exponentials does not seem to be suited to prove the cartesian closure of $\mathbb{C}_{\text{ex}}$ when $\mathbb{C}$ only has weak finite limits. A complete characterisation in terms of what we called *extensional simple products* is presented in [10], but here we look at yet another (weakly) universal property.

Remark 2.1. Let $\mathbb{E}$ be exact with a projective cover $\mathbb{P}$ and let Z, X, Y be three objects in $\mathbb{P}$. There is an isomorphism between the poset of subobjects $\text{Sub}_{\mathbb{E}}(Z \times X \times Y)$ and the poset reflection $(\mathbb{P}/(Z, X, Y))_{\text{po}}$ of the category of spans

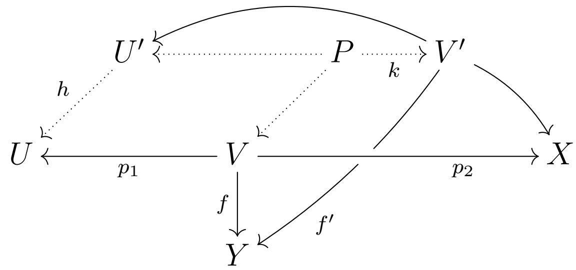
$$\begin{array}{ccc} & V & \\ \swarrow & \downarrow & \searrow \\ Z & X & Y \end{array}$$

in $\mathbb{P}$ over Z, X and Y [9, Lemma 35]. This isomorphism restricts between those subobjects $R \hookrightarrow Z \times X \times Y$ such that $R \to Z \times X$ is regular epic, and those (equivalence classes of) spans such that $Z \leftarrow V \to X$ is a weak product. This restricts further between those subobjects such that $R \to Z \times X$ is iso, *i.e.* essentially graphs of arrows $Z \times X \to Y$, and those spans such that $Z \leftarrow V \to X$ is a weak product and $V \to Y$ is determined by projections.

The previous remark allows us to understand arrows $f\colon V \to Y$ in a category $\mathbb{C}$ with weak finite limits, where $Z \leftarrow V \to X$ is a weak product, as families, indexed by Z, of total relations (*i.e.* multi-valued functions) from X to Y. Such a total relation is functional (*i.e.* a single-valued function) precisely when f is determined by projections. This reading suggests that, in order to have a suitable universal property with respect to arbitrary arrows $V \to Y$ out of a weak product, we should look for some property of closure with respect to (families of) total relations from X to Y. A promising notion, that indeed proves to be useful, is that of a *full set* from the constructive set theory CZF.

A set f is full for two sets a and b if it consists of multi-valued functions from a to b and, for every multi-valued function r from a to b, there is $s \in f$ such that $s \subseteq r$. The *Fullness Axiom* states that, for any two sets a and b, there is a set which is full for a and b. This axiom was introduced in the context of Constructive Zermelo-Fraenkel set theory (CZF) by Peter Aczel in [1] in order to provide a simpler formulation of the axiom schema of Subset Collection. This axiom implies in particular the so-called Exponentiation Axiom, that the class of functions between two sets is a set. The next definition is inspired by the notion of full set.

Definition 2.2. *Let X and Y be two objects in a category $\mathbb{C}$ with weak finite limits. A* full diagram from X to Y *consists of a weak product $U \xleftarrow{p_1} V \xrightarrow{p_2} X$ and an arrow $f\colon V \to Y$ such that, for every object U', weak product $U' \leftarrow V' \to X$ and arrow $f\colon V' \to Y$, there are an arrow $h\colon U' \to U$, a weak pullback $U' \leftarrow P \to V$ of $U' \to U \leftarrow V$ and an arrow $k\colon P \to V'$ such that the diagram below commutes.*

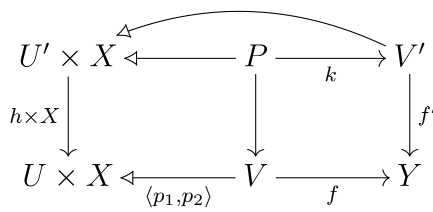


(1)

Remark 2.3.

1. The notion of full diagram is independent of the specific weak product in the following sense. If the pair $U \leftarrow V \rightarrow X$, $V \rightarrow Y$ is a full diagram from X to Y, then any other weak product $U \leftarrow W \rightarrow X$ together with the composite $W \rightarrow V \rightarrow Y$ is a full diagram from X to Y.

2. In the case of a projective cover $\mathbb{P}$ of an exact category $\mathbb{E}$, diagram (1) in Definition 2.2 can be written in $\mathbb{E}$ as



where the left-hand square is covering, *i.e.* the induced arrow from P to the pullback of $\langle p_1, p_2 \rangle$ and $h \times X$ is a regular epi.

Lemma 2.4. *Suppose that $\mathbb{C}$ has binary products. Then weak exponentials are full diagrams and any full diagram from X to Y gives rise to a weak exponential of Y and X.*

Proof. Since $\mathbb{C}$ has binary products, every weak product retracts onto the product of the same objects. Also, an arrow from a weak product is determined by projections if and only if it factors through the retraction onto the product. It follows that we may assume that the domain of a weak evaluation is a product, rather than just a weak product.

Let then W be a weak exponential of Y and X with weak evaluation $e\colon W \times X \rightarrow Y$. Given $f\colon V \rightarrow Y$ from a weak product $U' \leftarrow V \rightarrow X$

we can take $U \times X$ as P in (1): the arrow $k\colon U \times X \to V$ is a section of the retraction $V \to U \times X$ and the arrow $h\colon U \to W$ is obtained by the universal property of the weak exponential applied to the composite $fk\colon U \times X \to Y$.

For the converse, let $U \leftarrow V \to X$, $f\colon V \to Y$ form a full diagram and let $s\colon U \times X \hookrightarrow V$ be a section of the retraction $V \to U \times X$. We shall prove that $fs\colon U \times X \to Y$ exhibits U as a weak exponential of Y and X. Given $f'\colon U' \times X \to Y$, the universal property of full diagrams yields an arrow $h\colon U' \to U$ and a commutative diagram

$$\begin{array}{ccccc} U' \times X & \longleftarrow & P & \longrightarrow & U' \times X \\ {\scriptstyle h\times X}\downarrow & & \downarrow & & \downarrow{\scriptstyle f'} \\ U \times X & \longleftarrow & V & \xrightarrow{\ f\ } & Y \end{array}$$

where the left-hand square is a weak pullback. It follows that $P \to U' \times X$ has a section s' such that the diagram

$$\begin{array}{ccc} U' \times X & \overset{s'}{\hookrightarrow} & P \\ {\scriptstyle h\times X}\downarrow & & \downarrow \\ U \times X & \overset{s}{\hookrightarrow} & V \end{array}$$

commutes. The equation $fs(h \times X) = f'$ then follows immediately. □

In particular, a category with finite limits has full diagrams if and only if it has weak exponentials. Lemma 2.6 shows that, whenever the internal logic of $\mathbb{C}_{\mathrm{ex}}$ (equivalently, of $\mathbb{C}$) supports implication and universal quantification, the left-to-right implication also holds when $\mathbb{C}$ only has weak finite limits.

First, recall that descent in exact categories allows us to prove the following, where $\mathbb{X}_{\mathrm{po}}$ denotes the poset reflection of the category $\mathbb{X}$. See also [10, Remark 1.9].

Lemma 2.5. $\mathbb{C}_{\mathrm{ex}}$ *has right adjoints to inverse images if and only if* $\mathbb{C}$ *has right adjoints to weak pullback functors, i.e. the functors* $(\mathbb{C}/X)_{\mathrm{po}} \to (\mathbb{C}/Y)_{\mathrm{po}}$ *induced by weak pullback along arrows* $f\colon Y \to X$.

Proof. One direction follows by the existence of natural isomorphisms $\mathrm{Sub}_{\mathbb{C}_{\mathrm{ex}}}(\Gamma_{\mathrm{ex}}X) \cong (\mathbb{C}/X)_{\mathrm{po}}$. For the other direction apply Theorem 2 in Section 3.7 of [3]. □

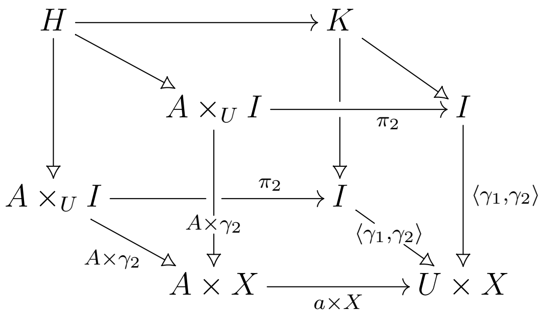
We now need a lemma which, for convenience, we formulate using an exact category $\mathbb{E}$ with a fixed projective cover $\mathbb{P}$.

Lemma 2.6. *Let $\mathbb{E}$ be an exact category with a projective cover $\mathbb{P}$ and suppose that $\mathbb{E}$ has right adjoints to inverse images along any arrow. If $\mathbb{P}$ has full diagrams, then for every X in $\mathbb{P}$ and B in $\mathbb{E}$ there are an object W in $\mathbb{P}$ and an arrow $W \times X \to B$ in $\mathbb{E}$ which are weakly terminal with respect to objects Z in $\mathbb{P}$ and arrows $Z \times X \to B$ in $\mathbb{E}$.*

Proof. Let $b\colon Y \twoheadrightarrow B$ be a cover of B with Y in $\mathbb{P}$, and take a full diagram $U \xleftarrow{p_1} V \xrightarrow{p_2} X$, $f\colon V \to Y$. The idea is to extract from U (codes of) functional relations. Let $\gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle \colon I \hookrightarrow U \times X \times B$ be the image factorisation of $\langle p_1, p_2, bf \rangle \colon V \to U \times X \times B$ and denote with $\phi\colon F \hookrightarrow U$ the subobject defined by the formula in context

$$u : U \mid (\forall x : X)(\forall y, y' : B)\, \gamma(u, x, y) \wedge \gamma(u, x, y') \Rightarrow y = y'.$$

In other words, given an arrow $a\colon A \to U$ in $\mathbb{E}$, consider the diagram

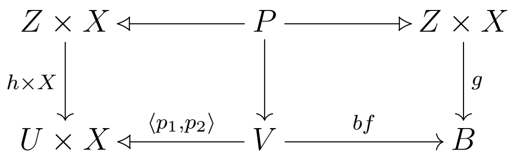


where all the squares are pullback. Then

$$a \leq \phi \iff A \times_U I \xrightarrow{\pi_2} I \xrightarrow{\gamma_3} B \text{ coequalises } H \rightrightarrows A \times_U I. \qquad (2)$$

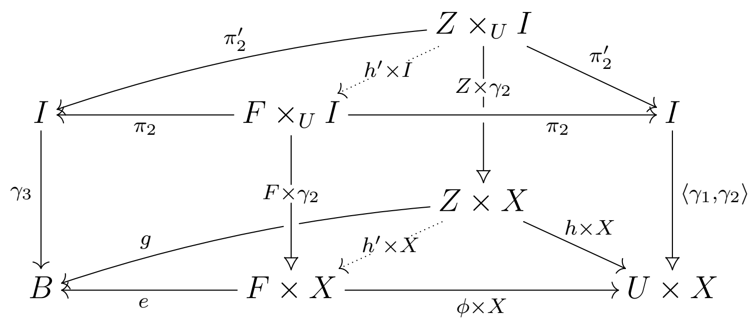
The existence of an arrow $e\colon F \times X \to B$ follows from (2) taking $a = \phi$. We shall show that e satisfies the required universal property. It then follows easily that $W \times X \to B$ satisfies it as well for any cover $W \twoheadrightarrow F$. In particular, $\mathbb{P}$ will have weak exponentials.

Given $g\colon Z \times X \to B$ with $Z \in \mathbb{P}$, let $g'\colon V' \to Y$ be a cover of g, *i.e.* be such that the right-hand square in diagram (3) is covering. By the universal property of a full diagram, we have, in particular, an arrow $h\colon Z \to U$ and a commutative diagram



(3)

where the left-hand square is covering. It follows that the induced arrow $q\colon P \to Z \times_U I$ is a regular epi. Consider now the solid arrows in diagram (4) below. The two right-hand squares are pullback and the front left-hand square commutes by definition of e.



(4)

In order to obtain an arrow $h'\colon Z \to F$ such that $e(h' \times X) = g$, it is enough to show that the square with side g commutes. Indeed, in this case, there is $h'\colon Z \to F$ such that $\phi h' = h$ by (2). It follows that the upper triangle(s) and the square with dotted sides in diagram (4) commute. Since $Z \times \gamma_2$ is (regular) epic, the lower left-hand triangle commutes as well.

To see that the square with side g in diagram (4) commutes note that, precomposing its two sides with $q\colon P \twoheadrightarrow Z \times_U I$ yields the right-hand square in diagram (3). The claim follows from the fact that q is (regular) epic. □

Theorem 2.7. *Suppose that $\mathbb{C}$ has right adjoints to weak pullback functors. If $\mathbb{C}$ has full diagrams, then $\mathbb{C}_{\text{ex}}$ is cartesian closed.*

Proof. In the terminology of [10], Lemmas 2.5 and 2.6 prove that if a category has weak finite limits, right adjoints to weak pullback functors and full diagrams, then it has extensional exponentials too. The statement follows from Lemma 2.13 in [10]. □

Below we collect together the results in this section.

Corollary 2.8. *Suppose that $\mathbb{C}$ has right adjoints to weak pullback functors, and consider the following.*

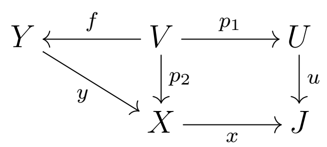
1. *$\mathbb{C}$ has full diagrams.*
2. *$\mathbb{C}_{\mathrm{ex}}$ is cartesian closed.*
3. *$\mathbb{C}$ has weak exponentials.*

We have $1 \Rightarrow 2 \Rightarrow 3$. If $\mathbb{C}$ has binary products, then $3 \Rightarrow 1$.

Dependent full diagrams

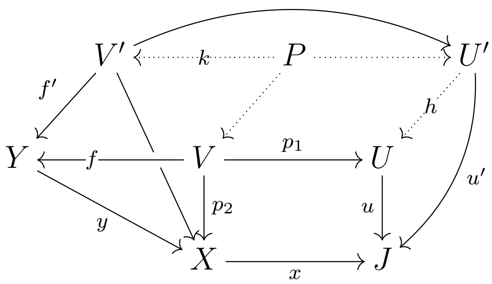
Recall that $\mathbb{C}$ denotes a category with weak finite limits and $\mathbb{C}_{\mathrm{ex}}$ its exact completion. In this section we define an indexed version of full diagrams, whose existence will endow the internal logic of $\mathbb{C}$ (hence of $\mathbb{C}_{\mathrm{ex}}$) with implication and universal quantification.

Definition 3.1. *Let $y\colon Y \to X$ and $x\colon X \to J$ be two arrows in a category with weak finite limits. A* dependent full diagram over x, y *is a commutative diagram*



(5)

such that the square is a weak pullback and, for every such diagram u', f' over y, x as below, there are an arrow $h\colon U' \to U$, a weak pullback $V \leftarrow P \to U'$ of $V \xrightarrow{p_1} U \xleftarrow{h} U'$ and an arrow $k\colon P \to V'$ making the diagram below commute.



The same observations as in Remark 2.3 apply, mutatis mutandis, to dependent full diagrams. Moreover, it is not difficult to see that dependent full diagrams generalise full families of pseudo-relations from [11], in the sense that the two notions coincide in well-pointed categories with finite products and weak equalisers.

Remark 3.2. Another category-theoretic version of Aczel's Fullness Axiom was introduced in [5] in the context of Algebraic Set Theory [15]. There one deals with classes of arrows, called *small maps*, in categories which are at least regular. According to the properties satisfied by the small maps, various set theories may be interpreted in this structure. In particular, in order to interpret the Fullness Axiom in a suitable category $\mathbb{E}$ equipped with a class of small maps $\mathcal{S}$, van den Berg and Moerdijk introduce a condition $\mathrm{F}(\mathcal{S})$, called **(F)** in [5, Section 3.7]. By taking the class $\mathcal{S}$ to consist of all arrows of $\mathbb{E}$, condition $\mathrm{F} = \mathrm{F}(\mathrm{Ar}\,\mathbb{E})$ makes sense for any regular category. If a regular category $\mathbb{E}$ has a projective cover $\mathbb{P}$, then a straightforward but lengthy computation shows that F holds in $\mathbb{E}$ if and only if $\mathbb{P}$ has dependent full diagrams. Moreover, Proposition 6.2.5 in [5] entails that condition F is stable under ex/reg completion. As a reg/wlex completion is in particular a regular category with enough projectives [9], it follows immediately that $\mathbb{C}_{\mathrm{ex}} \equiv (\mathbb{C}_{\mathrm{reg/wlex}})_{\mathrm{ex/reg}}$ satisfies F whenever $\mathbb{C}$ has dependent full diagrams.

Recall that a *weak dependent product* of two composable arrows $Y \xrightarrow{y} X \xrightarrow{x} J$ in a category with weak finite limits is a commutative diagram as (5) such that the square is a weak pullback, $f\colon V \to Y$ is determined by projections and, for every such diagram $u'\colon U' \to J, f'\colon V' \to Y$ over y and x, there are arrow $U' \to U$ and $V' \to V$ making the obvious diagram commute.

As for the non-indexed case, as soon as $\mathbb{C}$ has pullbacks, we may regard weak dependent products in $\mathbb{C}$ as those dependent full diagrams whose weak pullback is a pullback.

Lemma 3.3. *If $\mathbb{C}$ has pullbacks, weak dependent products are dependent full diagrams and any dependent full diagram over y, x gives rise to a weak dependent product of y, x.*

Proof. The lemma is proven similarly to Lemma 2.4. □

Lemma 3.4. *If $\mathbb{C}$ has dependent full diagrams, then it has full diagrams.*

Proof. A full diagram for two objects X and Y can be obtained as a dependent full diagram over $V \to U \to T$, where T is weakly terminal, U is a weak product of X and T, V is a weak product of U and Y and the arrows are the obvious projections. □

Lemma 3.5. *Let $\mathbb{C}$ be a category with weak finite limits. $\mathbb{C}$ has dependent full diagrams if and only if every slice of $\mathbb{C}$ has them.*

Proof. It follows from the fact that the forgetful functor $\mathbb{C}/J \to \mathbb{C}$ preserves and reflects weak pullbacks and dependent full diagrams. □

Lemma 3.6. *If $\mathbb{C}$ has dependent full diagrams, then $\mathbb{C}_{\mathrm{ex}}$ has right adjoints to inverse images.*

Proof. Using a choice of dependent full diagrams in $\mathbb{C}$ it is possible to define, for every $f\colon Y \to X$ in $\mathbb{C}$, functors $\forall_f^{\mathrm{w}}\colon (\mathbb{C}/Y)_{\mathrm{po}} \to (\mathbb{C}/X)_{\mathrm{po}}$. These functors are right adjoint to weak pullback functors by the universal property of dependent full diagrams. The statement follows from Lemma 2.5. □

Theorem 3.7. *If $\mathbb{C}$ has dependent full diagrams, then $\mathbb{C}_{\mathrm{ex}}$ is locally cartesian closed.*

Proof. It only remains to put together the previous results. Lemma 3.6 ensures that $\mathbb{C}_{\mathrm{ex}}$ has right adjoints to inverse images, whereas Lemmas 3.4 and 3.5 imply that $\mathbb{C}/X$ has full diagrams for every X in $\mathbb{C}$. Hence Theorem 2.7 yields the cartesian closure of $\mathbb{C}_{\mathrm{ex}}/(\Gamma_{\mathrm{ex}}X)$. The general statement follows now "descending" along a cover $\Gamma_{\mathrm{ex}}X \twoheadrightarrow A$ as in the proof of Theorem 3.6 in [10]. □

We again collect together the results of this section.

Corollary 3.8. *Consider the following.*

1. $\mathbb{C}$ *has dependent full diagrams.*
2. $\mathbb{C}_{\mathrm{ex}}$ *is locally cartesian closed.*
3. $\mathbb{C}$ *has weak dependent products.*

We have 1 $\Rightarrow$ *2* $\Rightarrow$ *3. If* $\mathbb{C}$ *has pullbacks, then 3* $\Rightarrow$ *1.*

Full diagrams in homotopy categories

In this section we show that, under mild assumptions on a model category $\mathbb{M}$, the homotopy category $\mathrm{Ho}\,\mathbb{M}$ has dependent full diagrams if $\mathbb{M}$ has weak dependent products. Using well-known results, this implies that the exact completion of the homotopy categories on spaces and CW-complexes yields locally cartesian closed pretoposes.

$\xrightarrow{\sim}$ For basic notions on model categories and homotopical algebra we refer to [14]. Fibrations, cofibrations and weak equivalences are denoted as $\twoheadrightarrow$, $\rightarrowtail$ and $\xrightarrow{\sim}$, respectively. A path object factorisation for an object A is denoted as $A \overset{\sim}{\rightarrowtail} PA \twoheadrightarrow A \times A$, and a fibrewise path object factorisation for a fibration $p\colon A \twoheadrightarrow B$ as $A \overset{\sim}{\rightarrowtail} P_pA \twoheadrightarrow A \times_B A$. Since we shall not be concerned here with cylinder objects, we say that two arrows $f, g\colon C \to A$ in $\mathbb{M}$ are homotopic, written $f \simeq g$, if they are right homotopic (*i.e.* homotopic with respect to the path object PA). Similarly, we shall write $f \simeq_p g$ to mean that they are fibrewise right homotopic over the fibration p. Note that, since every fibration $p\colon A \twoheadrightarrow B$ is fibrant in the model category structure on $\mathbb{M}/B$ induced by that one on $\mathbb{M}$, fibrewise (right) homotopy is an equivalence relation on arrows $f, f'\colon X \to A$ such that $pf = pf'$. If every object in $\mathbb{M}$ is cofibrant, then it is also a congruence.

When every object in a model category $\mathbb{M}$ is cofibrant, the homotopy category $\mathrm{Ho}\,\mathbb{M}$ is equivalent to the category obtained quotienting the full subcategory $\mathbb{M}_f$ of $\mathbb{M}$ on fibrant objects by the homotopy relation [14]. Moreover, $\mathbb{M}_{\mathrm{f}}$ is a category of fibrant objects in the sense of Brown [7] where, in addition, every acyclic fibration has a section and where weak equivalences and homotopy equivalences coincide. A category of fibrant objects satisfying these additional properties is called a *path category* by van den Berg and Moerdijk [6]. More explicitly, a path category may be axiomatised as follows (cf. [6]). It has a terminal object and two classes of distinguished arrows closed under isomorphism and composition, called weak equivalences and fibrations, such that: *(i)* weak equivalences are closed under 2-out-of-6, *(ii)* terminal arrows are fibrations, *(iii)* pullbacks along fibrations exist and (acyclic) fibrations are stable under pullback, and *(iv)* every acyclic fibration has a section.

Definition 4.1 ([6], Definition 5.2)**.** *Let $g\colon B \twoheadrightarrow A$ and $f\colon A \twoheadrightarrow I$ be two fibrations in a path category $\mathbb{C}$. A commuting diagram*

$$\begin{array}{ccccc} B & \xleftarrow{\;e\;} & U \times_I A & \longrightarrow\!\!\!\!\!\rightarrow & U \\ & \overset{g}{\searrow} & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow{\scriptstyle u} \\ & & A & \xrightarrow[f]{} \!\!\!\!\!\rightarrow & I \end{array}$$

is a homotopy weak dependent product of f and g *if for every such commutative diagram $u'\colon U' \twoheadrightarrow I$, $e'\colon U' \times_I A \to B$, there is $k\colon u' \to u$ over I such that $e(k \times A) \simeq_g e'$.*

Homotopy weak dependent products are called weak homotopy Π-types in [6]. As observed in [6], the (weak) universal property also holds when the arrow u' is not a fibration.

When the path category is $\mathbb{M}_{\mathrm{f}}$, a weak homotopy dependent product arises as fibrant replacement of a weak dependent product in $\mathbb{M}$. To prove this fact we need the following result, which is a reformulation for a model category of Theorem 2.38 from [6].

Theorem 4.2. *Let $\mathbb{M}$ be a model category and let A and B be cofibrant objects. Then every commutative square*

$$\begin{array}{ccc} A & \xrightarrow{k} & C \\ {\scriptstyle f}\downarrow\wr & & \downarrow\!\!\downarrow{\scriptstyle g} \\ B & \xrightarrow[l]{} & D \end{array} \tag{6}$$

has a homotopy diagonal filler, i.e. an arrow $d\colon B \to C$ such that $gd = l$ and $df \simeq_g k$. Moreover, such a filler is unique up to fibrewise homotopy over g.

Proof. We shall first show that every commuting square (6) has a lower filler, *i.e.* an arrow d such that $gd = l$. This fact, in turn, allows us to obtain a homotopy witnessing the fact that the previously constructed lower filler is in fact a homotopy diagonal filler, and another homotopy witnessing its uniqueness.

Consider a factorisation of $\langle f, k\rangle\colon A \to B \times_D C$ into an acyclic cofibration $c\colon A \overset{\sim}{\rightarrowtail} E$ followed by a fibration $p\colon E \twoheadrightarrow B \times_D C$. In particular, E is cofibrant. From 2-out-of-3 we obtain that $\pi_1 p\colon E \twoheadrightarrow B$ is an acyclic fibration and, since B is cofibrant, it has a section $s\colon B \overset{\sim}{\to} E$. But then $d \coloneqq \pi_2 ps\colon B \to C$ is the required lower filler, as $gd = l\pi_1 ps = l$.

Therefore every commuting square (6) has a lower filler. In particular, we obtain a homotopy $s\pi_1 p \simeq_{(\pi_1 p)} \mathrm{id}_X$ as a lower filler in

$$\begin{array}{ccc} B & \longrightarrow & P_{\pi_1 p}E \\ {\scriptstyle s}\downarrow{\scriptstyle \wr} & & \downarrow\!\!\downarrow \\ E & \xrightarrow[\langle s\pi_1 p, \mathrm{id}_X\rangle]{} & E \times_B E \end{array}$$

where the top horizontal arrow is s followed by reflexivity of $P_{\pi_1 p}E$. Since E is cofibrant, it is $df = \pi_2 p(s\pi_1 p)c \simeq_g \pi_2 pc = k$.

Finally, given another homotopy diagonal filler d', the homotopy witnessing $d \simeq_g d'$ is obtained as a lower filler in

$$\begin{array}{ccc} A & \longrightarrow & P_g C \\ {\scriptstyle f}\downarrow{\scriptstyle \wr} & & \downarrow\!\!\downarrow \\ B & \xrightarrow[\langle d, d'\rangle]{} & C \times_D C \end{array}$$

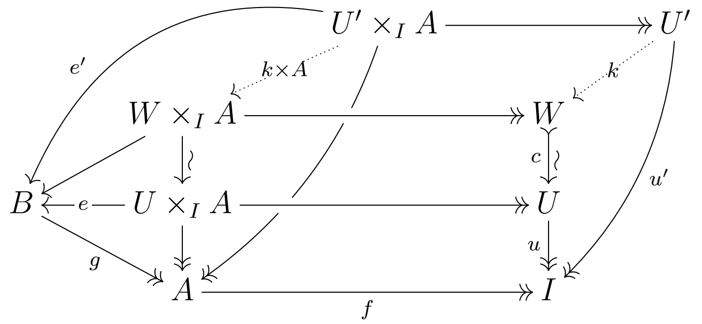
where the top horizontal arrow is the concatenation $df \simeq_g k \simeq_g d'f$. $\square$

Remark 4.3. The argument used in the previous proof can be adapted to work in a path category, so to provide an alternative proof of Theorem 2.38 in [6]. To this aim it is enough to observe that, in a path category, the existence of lower fillers for commutative squares as in (6) is enough to derive that the homotopy relation is a congruence, as in the proof of Theorem 2.14 in [6].

Corollary 4.4. *Let $\mathbb{M}$ be a right proper model category where every object is cofibrant. If $\mathbb{M}$ has weak dependent products, then $\mathbb{M}_{\mathrm{f}}$ has homotopy weak dependent products for every pair of composable fibrations.*

Proof. Let $f\colon A \twoheadrightarrow I$ and $g\colon B \twoheadrightarrow A$ be two fibrations in $\mathbb{M}_{\mathrm{f}}$ and let $w\colon W \to I$, $d\colon W \times_I A \to B$ be a weak dependent product of them. Factor w as an acyclic cofibration $c\colon W \overset{\sim}{\rightarrowtail} U$ followed by a fibration $u\colon U \twoheadrightarrow I$. Since $\mathbb{M}$ is right proper, $W \times_I A \to U \times_I A$ is also a weak equivalence, hence we obtain $e\colon U \times_I A \to B$ as homotopy diagonal filler.

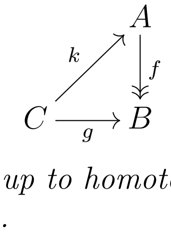
The required universal property is depicted in the diagram below



where $e(ck \times A) \simeq_g e'$ since e is just a homotopy diagonal filler. □

Homotopy weak dependent products also enjoy another universal property with respect to certain homotopy diagrams. This is proved below in Lemma 4.9 and it is a consequence of the following result.

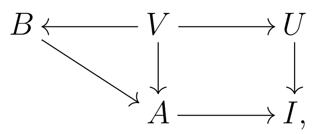
Proposition 4.5 ([6], Proposition 2.31)**.** *Let $\mathbb{C}$ be a path category and let*



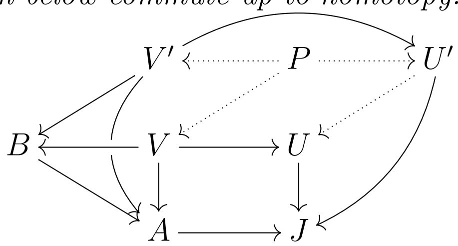
be a diagram that commutes up to homotopy. Then there is $k'\colon C \to A$ such that $k' \simeq k$ and $fk' = g$.

Remark 4.6. Proposition 4.5 has the important consequence that pullbacks in $\mathbb{M}_\mathfrak{f}$ along fibrations are homotopy pullbacks and so are mapped to weak pullbacks in $\operatorname{Ho}\mathbb{M}$.

Definition 4.7. *Let $f\colon A \to I$ and $g\colon B \to A$ be two arrows in a path category $\mathbb{C}$. A diagram*



that commutes up to homotopy and where the square is a homotopy pullback, is homotopy full over f, g *if, for every such diagram over $B \to A \to I$ commuting up to homotopy, there are an arrow $U' \to U$, a homotopy pullback $V \leftarrow P \to U'$ of $V \to U \leftarrow U'$ and an arrow $P \to V'$ making the diagram below commute up to homotopy.*

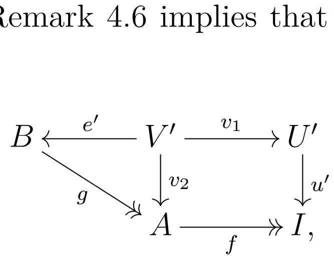


Remark 4.8. Let $\mathbb{C}$ be a path category. Since $\operatorname{Ho}\mathbb{C}$ is the quotient of $\mathbb{C}$ by the homotopy relation (cf. Theorem 2.16 in [6]), the image in $\operatorname{Ho}\mathbb{C}$ of a homotopy full diagram over f, g is a full diagram over $[f], [g]$.

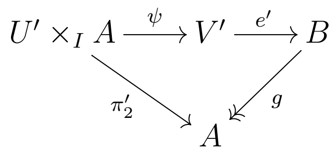
Lemma 4.9. *Let $\mathbb{C}$ be a path category and let $f\colon A \twoheadrightarrow I$ and $g\colon B \twoheadrightarrow A$ be two fibrations. A homotopy weak dependent product of f and g is a homotopy full diagram over f, g.*

Proof. Let $u\colon U \twoheadrightarrow I$, $e\colon U \times_I A \to B$ be a homotopy weak dependent product of f and g. Remark 4.6 implies that $U \times_I A$ is a homotopy pullback.

Let now



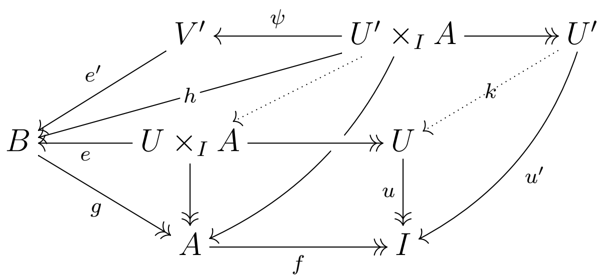
be commutative up to homotopy and such that the square is a homotopy pullback. Hence there is an arrow $\psi\colon U' \times_I A \to V'$ such that $v_1\psi \simeq \pi'_1$ and $v_2\psi \simeq \pi'_2$. In particular, the diagram below commutes up to homotopy



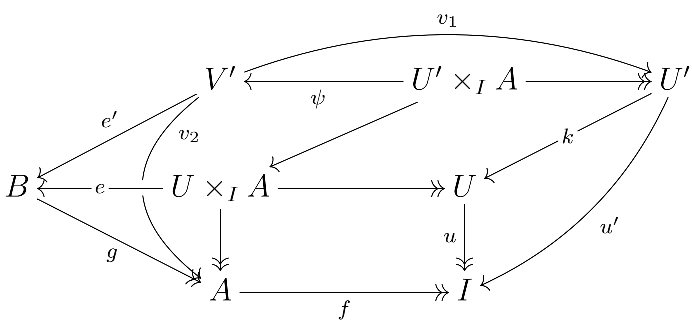
(7)

and Proposition 4.5 implies that there is $h\colon U' \times_I A \to B$ that makes the above triangle commute and which is homotopic to $e'\psi$.

The universal property of the homotopy weak dependent product then yields an arrow $k\colon U' \to U$ such that everything in the diagram below commutes strictly except for the two top-left triangles with common edge h, which only commute up to homotopy.



Hence, as required, the square with two dotted sides above is a homotopy pullback and the diagram below commutes up to homotopy.



□

Theorem 4.10. *Let* $\mathbb{M}$ *be a right proper model category where every object is cofibrant. If* $\mathbb{M}$ *has weak dependent products, then* $\operatorname{Ho}\mathbb{M}$ *has dependent full diagrams and, in turn,* $(\operatorname{Ho}\mathbb{M})_{\mathrm{ex}}$ *is locally cartesian closed.*

Proof. Lemma 4.9 and Remark 4.8 yield a full diagram over $[f], [g]$ whenever f and g are both fibrations. Since arrows in $\mathbb{M}$ factor as weak equivalences and fibrations and the former are isomorphisms in $\mathrm{Ho}\,\mathbb{M}$, this is enough to conclude that $\mathrm{Ho}\,\mathbb{M}$ has dependent full diagrams. The last statement is an application of Theorem 3.7. □

As an application of Theorem 4.10, consider the two standard model structures on the category of topological spaces by Quillen [17] and by Strøm [18], which we denote by Top_Q and Top_S, respectively. The latter is right proper and every space is cofibrant. Furthermore, Carboni and Rosolini showed that Top has weak dependent products [8]. Therefore $(\mathrm{Ho}\,\mathsf{Top}_S)_{\mathrm{ex}}$ not only is a pretopos, as proved in [13], but it is also locally cartesian closed. This answers a question left open by Gran and Vitale in [13]. In addition, although in Top_Q the cofibrant objects are just the CW-complexes, Top_Q is Quillen equivalent to simplicial sets with the Quillen model structure. This latter category does satisfy the hypothesis of our theorem, therefore $(\mathrm{Ho}\,\mathsf{Top}_Q)_{\mathrm{ex}}$ is locally cartesian closed too.

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A simplified categorical approach to several Galois theories

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Résumé. Nous étudions le concept de structure de Galois et epimorphisme de Galois dans un contexte général. Notamment, une structure de Galois pour un épimorphisme $\pi\colon M \to B$ dans une catégorie $\mathcal{C}$ est l'action d'un groupe objet qui munit M d'une structure d'espace homogène dans la catégorie relative $\mathcal{C}_B$.

Abstract. We discuss the concept of Galois structure and Galois epimorphism in a general setting. Namely, a Galois structure for an epimorphism $\pi\colon M \to B$ in some category $\mathcal{C}$ is the action of a group object that gives to M the structure of principal homogeneous space in the relative category $\mathcal{C}_B$.

Keywords. Galois theory, Differential algebra, Foliation, Groupoid, Principal bundle.

Mathematics Subject Classification (2010). 18B40, 57M10, 12F10, 12H05, 53C12.

1. Introduction

From its very starting point in the theory of polynomial equations with one variable [12], Galois theory proposes a systematic use of the principal homogeneous structure of the space of solutions of an equation. This idea was systematically applied by E. Vessiot [29] in his general approach to differential Galois theory. Today there are several Galois theories, with different domains of application.

It is clear that there is some common mathematical core within all these theories. This is usually explained through analogy. Most texts dedicated to several Galois theories develop them separately, establish some bridges, and point out these analogies between them, as in the book of R. and D. Douady [9].

There is a categorical approach to Galois theory initiated by Grothendieck ([14], see [10] for a more accessible exposition) and continued in [1] (see also [16]). This theory is further developed by Dubuc [11] and culminated by Joyal-Tierney [17]. A different approach to Galois theory is considered by G. Janelidze and F. Bourceux ([15], see also [2], chapter 5). This categorical Galois theory does not cover some natural incarnations of Galois theory, as differential Galois theory [25]. The main difference between Grothedieck approach and ours is the following: we do not see the Galois group as a set-theoretical group acting on an object but as a group object of the category. This line of thinking is inspired by some facts of differential Galois theory. For instance, the Galois group of a strongly normal extension [20] is an algebraic group defined over the constants, which can be seen as a particular kind a group object in the category of differential algebraic varieties. Some years ago A. Pillay generalized E. Kolchin's theory of strongly normal extensions [26]. A generalized strongly normal extension is a differential field extension whose group of automorphisms admits a natural structure of differential algebraic group, that is, a group object in the category of differential algebraic varieties.

Our framework also explains how some Galois theories are naturally extended. Most of them allow Galois structures (Definition 2.7) with Galois groups in some specific class of group objects in a category. By modifying the category, or by extending the class of possible Galois groups we obtain different extensions of Galois theory. For instance, classical Galois theory extends to Hopf-Galois theory by allowing a broader class of group objects.

We give some examples of how the proposed general definitions apply to the cases of classical Galois theory (algebraic and topological), and differential Galois theory. Then we explore the category of foliated smooth manifolds. Epimorphims in such category are partial Ehresmann conections. When examining Galois structures there naturally appear G-invariant connections. This is not surprising, G-invariant connections were in fact introduced in the context of Galois theory by E. Vessiot in the beggining to 20th century: they are the so-called automorphic systems appearing in [29].

We prove uniqueness of the Galois group for the irreducible case, Theorem 4.3. Finally we compare the real smooth and the complex algebraic cases.

2. General definitions

2.1 Split of groupoid actions

Let us consider $\mathcal{C}$ a category with binary products, kernels of pairs of morphisms, and a final object $\{\star\}$. Thus, there are also fibered products (pullbacks) as well as finite limits. We may define group objects and groupoid objects in $\mathcal{C}$.

Let G be a group object in $\mathcal{C}$. For each object X, the set $G(X) = \mathrm{Hom}(X, G)$ of X-elements of G is a group. An action of G in an object M is a morphism,

$$\alpha\colon G \times M \to M,$$

satisfying $\alpha \circ (\mu \times \mathrm{Id}_M) = \mu \circ (\mathrm{Id}_G \times \alpha)$ and $\alpha \circ ((e_G \circ \pi_M) \times \mathrm{Id}_M) = \mathrm{Id}_M$.[1] The action α induces a group morphism $\alpha\colon G(\{\star\}) \to \mathrm{Aut}(M)$, $g \mapsto \alpha \circ \langle g \circ \pi_M, \mathrm{Id}_M \rangle$.

From the action α we can form the *action groupoid* $G \ltimes M \rightrightarrows M$, with objects of objects M and object of arrows $G \ltimes M$. The source map is the projection π_2 onto the second factor M, and the target map is α. In terms of sets and elements, we have:

$$s(g, x) = x, \quad t(g, x) = \alpha(g, x), \quad (h, gx) \circ (g, x) = (hg, x).$$

Definition 2.1. *We say that a groupoid object $\mathcal{G} \rightrightarrows M$ splits in $\mathcal{C}$ if there is an action $\alpha\colon G \times M \to M$ an action of a group object and a groupoid isomorphism $\varphi\colon G \ltimes M \xrightarrow{\sim} \mathcal{G}$. In such a case, we say that G is a splitting group, α is a splitting action and φ is a splitting morphism for $\mathcal{G}$ in $\mathcal{C}$.*

Example 2.2. Let us remark that it is not in general possible to recover the group G from the action groupoid $G \ltimes M$. For instance, in the category of sets, let us consider two free and transitive actions of $\mathbb{Z}_4$ and $\mathbb{K}_4$ in a set $X = \{p_1, p_2, p_3, p_4\}$ of four elements. Since the actions are free and transitive we

[1] Where e_G represent the identity $e_G\colon \{\star\} \to G$ and π_M represents the unique morphism $\pi_M\colon M \to \{\star\}$.

have that the action groupoid is, in both cases, the total equivalence realtion $X \times X$. Therefore we have groupoid isomorphism:

$$\mathbb{K}_4 \ltimes X \xrightarrow{\sim} X \times X \xleftarrow{\sim} \mathbb{Z}_4 \ltimes X$$

$$(\sigma, p) \longrightarrow (p, \sigma \cdot p), (p, \tau \cdot p) \longleftarrow (\tau, p)$$

Thus, a split groupoid object may have different realizations as an action groupoid.

2.2 Normal epimorphisms

Let us recall that an action of a group (set) G in an object X is a group morphism $\phi\colon G \to \mathrm{Aut}(X)$. We say that $q\colon X \to Y$ is a *categorical quotient* of the action of G in X if:

1. For all $g \in G$, $q \circ \phi(g) = q$. In other words, q is G-invariant.
2. For all morphisms $f\colon X \to Z$ such that for all $g \in G$ $f \circ \phi(g) = f$ (i.e. f is G-invariant) there exists a unique $\bar{f}\colon Y \to Z$ such that $\bar{f} \circ q = f$.

Categorical quotients are epimorphisms and are unique up to isomorphims. Let us consider $\pi\colon M \to B$ an epimorphism in $\mathcal{C}$. The group group $\mathrm{Aut}_B(M)$ acts on M.

Definition 2.3. *We say that π is normal if it is the categorical quotient of M by the action of the group (set) $\mathrm{Aut}_B(M)$.*

Some categorical approaches to Galois theory rely in the notion of strict epimorphism ([1, I.10.2] see also [18, Def. 5.1.6]).

Definition 2.4. *Let $\pi\colon M \to B$ be an epimorphism.*

(a) A morphism $f\colon M \to Z$ is π-compatible if for any pair of morphisms $x, y\colon X \rightrightarrows M$ such that $\pi \circ x = \pi \circ y$ also $f \circ x = f \circ z$.

(b) π is a strict epimorphism if for any π-compatible f there is a unique $\bar{f}\colon B \to Z$ such that $f = \bar{f} \circ \pi$.

Proposition 2.5. *Let $\pi\colon M \to B$ be an epimorphism in a category $\mathcal{C}$.*

(a) *If π is normal then it is strict.*

(b) *Assume that any arrow with codomain M is invertible. Then, if π is strict π is normal.*

Proof. Let us consider an object Z an the composition map

$$\pi^*\colon \mathrm{Hom}(B,Z) \to \mathrm{Hom}(M,Z).$$

The image of π^* consists of π-compatible morphisms. Moreover, let us assume that $f\colon M \to Z$ is π-compatible. Then, for any $\sigma \in \mathrm{Aut}_B(M)$ we have $\pi \circ \sigma = \pi \circ \mathrm{Id}_M$ and therefore $f \circ \sigma = f$. It means that π-compatible morphisms are invariant under the action of $\mathrm{Aut}_B(M)$. In general we have a chain,[2]

$$\pi^*(\mathrm{Hom}(B,Z)) \subseteq \{\pi\text{-compatible morphisms}\} \subseteq \mathrm{Hom}(M,Z)^{\mathrm{Aut}_B(M)}.$$

Let us note the following:

(i) The epimorphism π is normal if and only if for any Z we have the equality between the first and third members of the chain.

(ii) The epimorphism π is strict if and only if for any Z we have the equality between the first and second members of the chain.

(a) Assume π normal. Then the three members of the above chain coincide. In particular, any π-compatible morphism factorizes.
(b) Assume that π is strict. We need to prove that any $\mathrm{Aut}_B(M)$ invariant morphism $f\colon M \to Z$ is π-compatible. Let $a, b\colon X \rightrightarrows M$ be a pair of morphisms such that $\pi \circ a = \pi \circ b$. Since f is $\mathrm{Aut}_B(M)$ invariant we have $f = f \circ (b \circ a^{-1})$ and from this $f \circ a = f \circ b$. Hence f is π-compatible. □

Remark 2.6. Let us recall that the notions of regular and effective epimorphism.

(a) An epimorphism $q\colon Y \to X$ is said to be regular if it is the coequalizer of a pair of morphisms $Z \rightrightarrows Y \to Z$.

[2] Here $\mathrm{Hom}(M,Z)^{\mathrm{Aut}_B(M)}$ stands for the set of $\mathrm{Aut}_B(M)$-invariant morphisms in $\mathrm{Hom}(M,Z)^{\mathrm{Aut}_B(M)}$.

(b) An epimorphism $q\colon Y \to X$ is said to be effective if it has a kernel pair and it is the coequalizer of a congruence of its kernel pair $\mathrm{KP}_q \rightrightarrows Y \to X$.

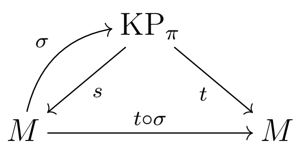
In a general category we have:

$$\text{effective} \implies \text{regular} \implies \text{strict}\,.$$

Moreover, in a category with pullbacks it is known that strict epimorphisms are effective. Therefore a *normal* epimorphism in a category with pullbacks is effective. If additionally, as stated in Proposition 2.5 (b), the epimorphism $\pi\colon M \to B$ satisfies that any arrow with codomain M is invertible, then π is normal if and only if it is effective. This equivalence between effectiveness and normality seems to be a key aspect in classical Galois theory.

2.3 Galois structures

The kernel pair of π, $\mathrm{KP}_\pi = M \times_B M \rightrightarrows M$, is a congruence (equivalence relation) in M, and therefore a grupoid object in $\mathcal{C}$. We set the source (s) and target (t) maps to be the first and second projection respectively. It represents the endomorphisms of M over B in the following sense: let $\mathrm{KP}_\pi(M)$ be the set of sections of the source map (s); the composition with the target map yields a bijection.



$$\begin{array}{ccc} \mathrm{KP}_\pi(M) & \xrightarrow{\ \sim\ } & \mathrm{End}_B(M) \\ \sigma & \xrightarrow{\ \sim\ } & t \circ \sigma \end{array}$$

Let us consider an splitting action $\alpha\colon G \times M \to M$ of KP_π. The splitting isomorphism is necessarily

$$\langle \pi_2, \alpha\rangle\colon G \ltimes M \xrightarrow{\sim} \mathrm{KP}_\pi \quad (g, x) \mapsto (x, \alpha(g, x)),$$

which is completely determined by α. In other words, an splitting action of KP_π is an action that gives $\pi\colon M \to B$ the structure of *principal homogeneous space* modeled over $G \times B \to B$ in the relative category $\mathcal{C}_B$ of arrows over B.

Let us note that the splitting action α induces a bijection between $G(M)$ and $\mathrm{KP}_\pi(M)$ and therefore a bijection,

$$G(M) \xrightarrow{\sim} \mathrm{End}_B(M), \quad g \mapsto \alpha_g = \alpha \circ \langle g, \mathrm{Id}_M \rangle.$$

However, such bijection is not compatible with the composition. We have

$$\alpha_g \circ \alpha_h = \alpha \circ \langle g, \mathrm{Id}_M \rangle \circ \alpha \circ \langle h, \mathrm{Id}_M \rangle = \alpha \circ \langle g \circ \alpha_h, \alpha_h \rangle$$

on the other hand,

$$\alpha_{gh} = \alpha \circ \langle g, \alpha_h \rangle.$$

It follows that, if $g = g \circ \alpha_h$ then, $\alpha_{gh} = \alpha_g \circ \alpha_h$. We see that this is satisfied if $g \in G(B)$, given that $\alpha_h \in \mathrm{End}_B(M)$ induces the identity in B. For *normal* epimorphisms this condition is optimal, as $G(M)^{\mathrm{Aut}_B(M)} = G(B)$. We have thus,

$$\begin{array}{ccccc} & & G(M) & \xrightarrow{\sim} & \mathrm{End}_B(M) \\ & & \uparrow & & \uparrow \\ G(\{\star\}) & \hookrightarrow & G(B) & \hookrightarrow & \mathrm{Aut}_B(M) \end{array}$$

where the maps in the lower row are injective group morphisms.

Definition 2.7. *Let $\pi\colon M \to B$ be an epimorphism in $\mathcal{C}$. A Galois structure for π is an splitting action $\alpha\colon G \times M \to M$ for KP_π such that the induced group morphism*

$$G(\{\star\}) \xrightarrow{\sim} \mathrm{Aut}_B(M), \quad g \mapsto \alpha_g$$

is an isomorphism.

Definition 2.8. *We say that an epimorphism $\pi\colon M \to B$ of $\mathcal{C}$ is Galois if satisfies the following conditions:*

(i) it is normal;

(ii) it admits a unique (up to isomorphism) Galois structure.

We call Galois group of π the group object Gal_π appearing in the unique Galois structure.

Note that if α is a Galois structure for π then we have isomorphisms,

$$G(\{\star\}) \xrightarrow{\sim} G(B) \xrightarrow{\sim} \mathrm{Aut}_B(M).$$

Given an splitting action α for KP_π as a groupoid object in $\mathcal{C}$ we may define an splitting action

$$\tilde{\alpha}\colon (G \times B) \times_B M \to M, \quad ((g,b),x) \mapsto \alpha(g,x),$$

for KP_π as a groupoid object in $\mathcal{C}_B$. In some cases, a splitting action may fail to be a Galois structure in the category $\mathcal{C}$ but be so in the relative category $\mathcal{C}_B$ of arrows over B.

Example 2.9. Let $\mathbf{Set}$ be the category of sets and $\pi\colon M \to B$ be a surjective map. In any case KP_π splits in $\mathbf{Set}_B$, and any splittig action is a Galois structure. The group object acting is a family of groups indexed by B and acting freely and transitively on the fibers of π. It is Galois if and only if the fibers have 1, 2 or 3 points.

However, KP_π splits in $\mathbf{Set}$ if and only if all fibers of π have exactly the same cardinal. Finally, π is Galois in $\mathbf{Set}$ if and only if it is a bijection, otherwise we may have the uniqueness for the Galois structure, but $G \subsetneq \mathrm{Aut}_B(M)$.

Example 2.10. Let $\mathbf{Mnf}$ be the category of smooth manifolds with smooth maps. By direct examination of the definition we have that a an splitting action for a submersion $\pi\colon M \to B$ is an structure of a principal bundle for some structure Lie group G. The splitting actions is far from being unique, moreover, G represents a very small part of $\mathrm{Aut}_B(M)$.

2.4 Galois correspondence

Let us recall that a congruence (internal equivalence relation) in M is a subobject of $M \times M$ having the reflexive, symmetric and transitive property. We say that a congruence $R \subseteq M \times M$ is *effective* if it is the kernel pair of an effective epimorphism. The class of an effective epimorphism up to isomorphisms of the codomain is called an effective quotient. We have then a diagram:

$$R \rightrightarrows M \to M/R.$$

The class $\mathrm{Rel}(M)$ of effective of congruences in M is partially ordered. For two congruences represented by monomorphisms $i\colon R \hookrightarrow M \times M$ and $i'\colon R' \hookrightarrow M \times M$ we say that $R \leq R'$ if there is $j\colon R \hookrightarrow R'$ such that $i' \circ j = i$. Analogously the class $\mathrm{Quot}(M)$ of effective quotients of M is ordered. For two effective quotients represented by effective epimorphisms $q\colon M \to Z$ and $q'\colon M \to Z'$ we say $q \geq q'$ if q' is q-compatible, so that there is $p\colon Z \to Z'$ such that $p \circ q = q'$. There is a natural bijective Galois connection between $\mathrm{Rel}(M)$ and $\mathrm{Quot}(M)$ of effective quotients of M given by the adjunctions:

$$\mathrm{KP}\colon \mathrm{Quot}(M) \to \mathrm{Rel}(M), \quad (q\colon M \to Z) \mapsto \mathrm{KP}_q = M \times_Z M,$$

$$\mathrm{coeq}\colon \mathrm{Rel}(M) \to \mathrm{Quot}(M), \quad R \mapsto (q\colon M \to M/R).$$

The quotient by a group action $\alpha\colon G \times M \to M$ is also understood in the above terms. We have $M/G = \mathrm{coeq}(\alpha, \pi_2)$ if such coequalizer exists in $\mathcal{C}$. Under suitable assumptions on the existence and nature of quotients by group actions, the general Galois connection gives rise to the classical Galois correspondence.

Theorem 2.11. *Let $\pi\colon M \to B$ be a Galois epimorphism. Let us assume the following:*

(a) any subgrupoid object of the action groupoid $\mathrm{Gal}_\pi \ltimes M$ is of the form $H \ltimes M$ where H is a subgroup object of Gal_π;

(b) for any subgroup object $H \subseteq \mathrm{Gal}_\pi$ it does exists the effective quotient M/H.

Then the following sentences hold:

(i) The assignation:

$$H \subseteq G \;\rightsquigarrow\; q_H\colon M \to M/H,$$

establishes an order reversing bijective correspondence between the partially ordered class $\mathrm{Sub}(G)$ of subgroup objects of G and the partially ordered class $\mathrm{Quot}_{\geq\pi}(M)$ of intermediate effective quotients of M.

(ii) Let us consider any effective intermediate quotient $q\colon M \to Z$ with corresponding subgroup $H \subseteq \mathrm{Gal}_\pi$. The restriction of the Galois structure α to $H \times M$ is a Galois structure for q.

Proof. (i) It is clear that the assignation reverses order, for $H \subseteq H'$ we have $q_H \geq q'_H$. In order to see that it is bijective, let us construct its inverse correspondence. Let $q\colon M \to Z$ be a representative of an effective quotient with $q \geq \pi$. The kernel pair KP_q is an effective congruence in M and $\mathrm{KP}_q \leq \mathrm{KP}_\pi$. The splitting isomorphim establishes an isomorphism of KP_q with a subgroupoid object of $\mathrm{Gal}_\pi \ltimes M$ which, by condition (a), is of the form $H_q \ltimes M$ for a subgroup object H_q depending on q. We have that the effective epimorphism $q\colon M \to Z$ is equivalent to $q_{H_q}\colon M \to M/H_q$. Then we have:

$$H \rightsquigarrow q_H \rightsquigarrow H, \quad q \rightsquigarrow H_q \rightsquigarrow q.$$

(ii) It is enough to note that the splitting isomorphism $\langle \pi_2, \alpha \rangle$ maps $H \ltimes M$ onto KP_q. □

3. Classical Galois theory

3.1 Covering spaces

Let **Top** be the category of topological spaces. A covering map $\pi\colon M \to B$, with M and B connected, is a *Galois cover* if $\pi \times \mathrm{Id}_M\colon M \times_B M \to M$ is a trivial covering space. There is a Galois theory for covering spaces, analogous to classical Galois theory (see, for instance [19]).

Theorem 3.1. *Let $\pi\colon M \to B$ be a surjective local homeomorphism with M and B connected. The following are equivalent:*

(a) π is a Galois cover.

(b) π is a Galois in **Top**.

(c) KP_π splits in **Top**.

In any case, the Galois group object is $\mathrm{Gal}_\pi = \mathrm{Aut}_B(M)$ with the discrete topology.

Proof. $(c) \Rightarrow (a)$. Let us assume that there is an splitting isomorphism $\varphi\colon G \times M \xrightarrow{\sim} M \times_B M$. Then we have that the projection on the second factor $G \times M \to M$ is a local homeomorphism. Thus, G is discrete and $M \times_B M \to M$ is a trivial cover. It follows that π is a Galois cover. We also have $(b) \Rightarrow (c)$.

Let us see $(a) \Rightarrow (b)$. We assume that $M \times_B M \to M$ is a trivial cover, thus there is a trivialization,

$$\begin{array}{ccc} G \times M & \xrightarrow[\sim]{\varphi} & M \times_B M \\ & {\scriptstyle \bar{\pi}} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & M & \end{array}$$

with G a discrete topological space. Let us check that there is a group structure on G such that it is isomorphic to $\mathrm{Aut}_B(M)$ and φ is the action of $\mathrm{Aut}_B(M)$ in M.

For each $g \in G$ let us consider the map $\sigma(g)\colon M \to M$ defined by the formula $\sigma(g)(x) = \pi_2(\varphi(g, x))$. It is a continuous map that induces the identity on B and thus, an automorphism of M over B. On the other hand, let σ be an automorphism of M over B. Then, the map $x \mapsto \varphi^{-1}(x, \sigma(x))$ is a section of $\bar{\pi}$. Since $\bar{\pi}$ is trivial, then there is a unique g in G such that $\varphi^{-1}(x, \sigma(x)) = (g, x)$. We define this g to be $g(\sigma)$. It is easy to check that those bijections inverse of each other. With the group operation in G induced by $\sigma_{gh} = \sigma_g \circ \sigma_h$ then we have that φ is a splitting morphism and thus π admits a Galois structure, where the action of G in M is isomorphic to that of $\mathrm{Aut}_B(M)$ endowed with the discrete topology, and thus unique.

Let us discuss the normality of π. In this context, it means that the action of $\mathrm{Aut}_B(M)$ is transitive on the fibers. Let m_1, m_2 be two points of M in the same fiber. Let g be the element of G such that $\varphi(g, m_1) = (m_1, m_2)$. Then, it is clear that $\sigma(g)(m_1) = m_2$. $\square$

Note that Galois covers are under the hypothesis of Theorem 2.11. The subgroupoids of $G \ltimes M$ are of the form $H \ltimes M$ with H a subgroup of G and the quotient M/H exists in **Top**. We obtain the well known correspondence between intermediate coverings and subgroups of G.

3.2 Algebraic Galois extensions

Let $\mathbf{Cmm}$ be the category of commutative rings with unit. The dual category $\mathbf{Cmm}^{\rm op}$ is the category of affine schemes.

Let us consider an extension of rings $i\colon K \hookrightarrow L$. The dual map $i^*\colon \operatorname{Spec}(L) \to \operatorname{Spec}(K)$ is an epimorphism in $\mathbf{Cmm}^{op}$. In this case the kernel pair is $\operatorname{Spec}(L\otimes_K L) \rightrightarrows \operatorname{Spec}(L)$ where the source and target maps are the dual of the canonical embeddings $a\mapsto a\otimes 1$ and $a \mapsto 1\otimes a$ respectively.

Group objects is $\mathbf{Cmm}^{\rm op}$ are commutative Hopf algebras. Thus, splitting actions in $\mathbf{Cmm}^{\rm op}$ are the already known Hopf-Galois structures, in the sense of Chase and Sweedler [7]. It is well known that Hopf-Galois structures are not unique in general.

Let us revisit classical Galois theory. Let us consider i to be a finite extension of fields. Classically, it is called a Galois extension if it satisfies one of the following equivalent conditions (see [27] pp. 140-141):

(a) L is separable and normal[3] over K.

(b) $|\operatorname{Aut}_K(L)| = \dim_K L$.

(c) $L\otimes_K L$ (with L-algebra structure given by the embedding $a\mapsto a\otimes 1$) is a finite trivial[4] L-algebra.

Let us consider $i\colon K\hookrightarrow L$ a Galois extension, and let G be $\operatorname{Aut}_K(L)$. Then, it is well known that the trivialization of $L\otimes_K L$ can be realized as a split. We have the trivial finite L-algebra $\operatorname{Maps}(G,L)$ and an isomorphism:

$$\varphi\colon L\otimes_K L \xrightarrow{\sim} \operatorname{Maps}(G,L) = \prod_{g\in G} L, \quad a\otimes b\mapsto f_{a\otimes b},$$

where $f_{a\otimes b}(g) = g(a)b$. Now we have that $\operatorname{Maps}(G,L) = \operatorname{Maps}(G,K)\otimes_K L$. Thus, in the dual category we have that the map,

$$\varphi^*\colon \operatorname{Spec}(\operatorname{Maps}(G,K))\times_K \operatorname{Spec}(L) \xrightarrow{\sim} \mathrm{KP}_{i^*},$$

[3]It is clear that our categorical definition of normality coincides, in this context, with the classical definition $L^{\operatorname{Aut}_K(L)} = K$.

[4]A finite trivial L-algebra is an L-algebra isomorphic to a direct product of a finite number of copies of L, $\prod_{i\in I} L$.

is a splitting isomorphism of the groupoid KP_{i^*}. Noting that $\mathrm{Maps}(G,K) = \mathrm{Maps}(G,\mathbb{Z}) \otimes_{\mathbb{Z}} K$ we see that the splitting isomorphism can be defined in the category $\mathbf{Cmm}^{op}$ and not only in the relative category $\mathbf{Cmm}_K^{op}$. We may state the following result.

Proposition 3.2. *Let us consider $i\colon K \hookrightarrow L$ a finite separable field extension, and $i^*\colon \mathrm{Spec}(L) \to \mathrm{Spec}(\mathrm{K})$ its dual morphism. The following are equivalent:*

(a) $i\colon K \hookrightarrow L$ is a Galois extension.

(b) i^* is Galois in $\mathbf{Cmm}^{\mathrm{op}}$.

(c) i^* is Galois in $\mathbf{Cmm}_K^{\mathrm{op}}$.

In such a case, if $G = \mathrm{Aut}_K(L)$, there is a natural action of G in $L \otimes_K L$ such that $(L \otimes_K L)^G$ is a Hopf K-algebra canonically isomorphic to $\mathrm{Maps}(G,K)$.

Let us fix a Galois extension $i\colon K \hookrightarrow L$ with group G. Let H be a subgroup of G. Then, we realize the field of invariants L^H as the equalizer, $L^H \to L \rightrightarrows L \otimes_{L^H} L$. Therefore, in the dual category $\mathrm{Spec}(L^H)$ appears as the effective quotient of $\mathrm{Spec}(L)$ by the action of the group object H. Moreover, since $G \ltimes \mathrm{Spec}(L)$ is the spectrum of a L-trivial algebra, we have that any subgroupoid is of the form $H \ltimes \mathrm{Spec}(L)$. We are under the hypothesis of Theorem 2.11, which in this particular case gives the classical Galois correspondence between intermediate field extensions and subgroups.

4. Foliated manifolds

4.1 Smooth foliated manifolds

Let $\mathbf{FMn}$ be the category of smooth manifolds endowed with regular foliations. Objects are pairs $(M,\mathcal{D})$ where M is a smooth manifold and $\mathcal{D}$ is an involutive linear subbundle of TM. Morphisms $f\colon (M,\mathcal{D}) \to (M',\mathcal{D}')$ are smooth maps $f\colon M \to M'$ such that for all $p \in M$ the differential d_pf induces a linear epimorphism from $\mathcal{D}_p$ to $\mathcal{D}'_p$. This implies that f maps leaves of $\mathcal{D}$ onto leaves of $\mathcal{D}'$ by local submersions. A manifold B admits two trivial structures of foliated manifold (B,TB), with only a leaf B and $(B,0_B)$ with point leaves.

Let $(G, \mathcal{D}_G)$ be a group object in **FMn**. It is clear that G is a Lie group. The existence of the identity element implies that the map,

$$(\{\star\}, 0_\star) \to (G, \mathcal{D}_G), \quad \star \mapsto e,$$

is a morphism of foliated manifolds, so that $\operatorname{rank}(\mathcal{D}_G) \leq \operatorname{rank}(0_\star) = 0$. It follows $\mathcal{D}_G = 0_G$. By abuse of notation we write G instead of $(G, 0_G)$. If is also clear that an action of G in $(M, \mathcal{D})$ to the category of foliated manifolds is an action of G in M by symmetries of $\mathcal{D}$. That is, for any $p \in M$ and $g \in G$ $d_pL_g(\mathcal{D}_p) = \mathcal{D}_{gp}$.

A *flat Ehresmann connection* in a submersion $\pi\colon M \to B$ is an involutive subbundle $\mathcal{F} \subset TM$ such that for each $p \in M$ the differential $d_p\pi$ is an isomorphism of $\mathcal{F}_p$ with $T_{\pi(p)}B$. We say that a foliated manifold $(M, \mathcal{F})$ is *irreducible* if it contains a dense leaf. Let us first analyze the case in which the basis M has a trivial structure of foliated manifold.

Proposition 4.1. *Let* $\pi\colon (M, \mathcal{F}) \to (B, TB)$ *be an epimorphism of foliated manifolds with* $\operatorname{rank}(\mathcal{F}) = \dim(B)$*. Then* π *is a submersion and* $\mathcal{F}$ *is a flat Ehresmann connection.*

Proof. For all $p \in M$ we have that $d_p\pi$ maps $\mathcal{F}_p$ onto T_pB. Therefore $d_p\pi$ is surjective for all $p \in M$ and π is a submersion. It is clear that $\mathcal{F}$ is a flat Ehresmann connection. □

Proposition 4.2. *Let* $\pi\colon (M, \mathcal{F}) \to (B, TB)$ *be a epimorphism of irreducible foliated manifolds with* $\operatorname{rank}\mathcal{F} = \dim B$*. The following are equivalent.*

(a) KP_π *splits in* **FMn**.

(b) π *is Galois in* **FMn**.

(c) There is a Lie group G *acting on* M *such that* π *is a principal* G*-bundle and* $\mathcal{L}$ *is a* G*-invariant connection.*

(d) The above, with a unique G*.*

In such a case G *is* $\operatorname{Aut}_{(B,TB)}(M, \mathcal{F})$.

Proof. Cases (a) and (c) are equivalent from the very definition of splitting action. It is also clear that (b) and (d) are equivalent. It remains to prove that (c) implies (d). Let us consider two principal structures $\beta\colon M \times H \to M$ and $\alpha\colon M \times G \to M$ such that $\mathcal{F}$ is simultaneously G and H-invariant. Let us see that these actions are conjugated by a Lie group isomorphism.

Let $\mathcal{L}$ be a dense leaf in M. We consider in $\mathcal{L}$ its intrinsic structure as smooth manifold, so that the projection $\mathcal{L} \to B$ is an étale map with arc-connected Hausdorff domain. Let us note that M and B are necessarily connected. Let x be any point of $\mathcal{L}$; there is a unique $h \in H$ such that $\alpha(x, g) = \beta(x, h)$. Let $\mathcal{L}'$ be the leaf of $\mathcal{F}$ passing through $\alpha(x, g) = \beta(x, h)$. Let us denote R^α_g and R^β_h the right translations by g and h respectively. Then, $R^\alpha_g|_{\mathcal{L}}$ and $R^\beta_h|_{\mathcal{L}}$ are homeomorphisms of $\mathcal{L}$ into $\mathcal{L}'$ that project onto the identity on B. They coincide on the point x, and thus they are the same, $R^\alpha_g|_{\mathcal{L}} = R^\beta_h|_{\mathcal{L}}$. Maps R^α_g and R^β_h are smooth and they coincide along the dense subset $\mathcal{L}$, thus they are equal. Finally, the map $G \to H$ that assigns to each g the only element h such that $\alpha(x, g) = \beta(x, h)$ is a group isomorphism. It is defined by composing and inverting smooth maps, so that, it is a Lie group isomorphism conjugating the actions α and β.

Moreover, the same argument proves that any automorphism $\varphi \in \mathrm{Aut}_{(B,TB)}(M, \mathcal{F})$ must be a translation by an element of G. □

The same idea can be generalized to the case in which the foliated structure of the basis is not trivial, but irreducible. Let $\pi\colon M \to B$ be a manifold submersion, and $\mathcal{D}$ a foliation in M. Let us recall that a flat $\mathcal{D}$*-connection* (or a flat partial connection in the direction of $\mathcal{D}$) is a foliation $\mathcal{F}$ in M that for all $p \in M$ the differential $d_p\pi$ maps $\mathcal{F}_p$ isomorphically onto $\mathcal{D}_p$. Note that a flat Ehresmann connection is the same that a flat TB-connection.

As in Proposition 4.1 if $\pi\colon (M, \mathcal{F}) \to (B, \mathcal{D})$ is an submersion of foliated manifolds with $\operatorname{rank}\mathcal{F} = \operatorname{rank}\mathcal{D}$ then $\mathcal{F}$ is a flat $\mathcal{D}$-connection.

Theorem 4.3. *Let $\pi\colon (M, \mathcal{F}) \to (B, \mathcal{D})$ be a epimorphism of irreducible foliated manifolds with* $\operatorname{rank}\mathcal{F} = \operatorname{rank}\mathcal{D}$*. The following are equivalent.*

(a) KP_π *splits in* **FMn**.

(b) π *is Galois in* **FMn**.

(c) There is a Lie group G acting on M such that π is a principal G-bundle and $\mathcal{D}$ is a $\mathcal{D}$-partial G-invariant connection.

(d) The above, with a unique G.

In such a case G is $\operatorname{Aut}_{(B,\mathcal{D})}(M,\mathcal{F})$.

Proof. Let us consider $\mathcal{L}$ a dense leaf of $\mathcal{F}$. Then $\pi(\mathcal{L})$ is a dense leaf of $\mathcal{D}$. We may proceed as in the proof of Proposition 4.2 replacing the role of B by $\pi(\mathcal{F})$. □

4.2 Galois correspondence

From now on let $\pi\colon (M,\mathcal{F}) \to (B,\mathcal{D})$ be a Galois submersion of irreducible foliated manifolds with $\operatorname{rank}\mathcal{F} = \operatorname{rank}\mathcal{D}$ with Galois group G. Let us check that we are under the hypothesis of Theorem 2.11.

Proposition 4.4. *Any object subgroupoid of the action groupoid $G \ltimes (M,\mathcal{F})$ is of the form $H \ltimes (M,\mathcal{F})$ with H a Lie subgroup of G.*

Proof. Let $(\mathcal{G},\mathcal{D}') \rightrightarrows (M,\mathcal{D})$ be a subgroupoid object of the action groupoid. Then $\mathcal{G}$ is a Lie subgroupoid of $G \ltimes M$ and if $(g,p) \in \mathcal{G}$ implies that the $\{g\} \times \mathcal{L}_p \subset \mathcal{G}$ where $\mathcal{L}_p$ is the leaf of $\mathcal{F}$ through p.

Let $\mathcal{L}$ be a dense leaf of $\mathcal{F}$. Note that for any $g \in G$ and $p \in M$ the poinf (g,p) is an accumulation point of $\{g\} \times \mathcal{L}$. Therefore if $(g,p) \in \mathcal{G}$ implies $\{g\} \times \mathcal{L} \subseteq \mathcal{G}$ and therefore $\{g\} \times M \subseteq \mathcal{G}$. It follows that $\mathcal{G} = S \times M$ for some submanifold $S \subseteq G$. From the groupoid composition and inversion it follows that $S = H$ a Lie subgroup of G. □

By a G-manifold we mean a manifold X endowed with a left action of G. To any G-manifold X it corresponds an associated bundle with fiber X,

$$M \times_G X \to B$$

defined as the quotient of the direct product $M \times X$ by the equivalence relation $(pg,x) \sim (p,gx)$ for all $p \in M$, $g \in G$, $x \in X$. The G-invariant $\mathcal{D}$-connection induces an *associated $\mathcal{D}$-connection* $\mathcal{F} \times_G 0_X$ which is the projection on $M \times_G X$ of the direct product $\mathcal{F} \times_G 0_X$. We have that,

$$(M \times_G X, \mathcal{F} \times_G 0_X) \to (B,\mathcal{D})$$

is an epimorphism of foliated manifolds and $\mathcal{F} \times 0_X$ is a flat Ehresmann $\mathcal{D}$-connection. In particular if H is a Lie subgroup of G and $X = G/H$ is an homogeneous G-space we have,

$$M \times_G (G/H) = M/H$$

and the induced associated $\mathcal{D}$-connection is just the projection of $\mathcal{F}$ onto M/H. Therefore, in this case, Theorem 2.11 gives us a Galois correspondence between Lie subgroups of G and associated $\mathcal{D}$-connections in associated bundles whose fibers are homogeneous G-spaces.

4.3 Galois structures over $(B, \mathcal{D})$

Let us discuss Galois structures in the relative category $\mathbf{FMn}_{(B,\mathcal{D})}$ whose objects are smooth maps of foliated manifolds $(Z, \mathcal{D}_Z) \to (B, \mathcal{D})$. A *group bundle* $G \to B$ is a smooth bundle by Lie groups, where composition, inversion and identity depends smoothly on the base point. A *group $\mathcal{D}$-connection* in $G \to B$ is a $\mathcal{D}$-connection $\mathcal{D}$ in G such that leaves are compatible with composition. Linear bundles and linear $\mathcal{D}$-connections are the most usual examples of group bundles and group connections. Group bundles over B endowed with group $\mathcal{D}$-connections are group objects in $\mathbf{FMn}_{(B,\mathcal{D})}$. They are the smooth geometric counterpart of differential algebraic groups of finite dimension discussed by Buium in [5].

In the case of trivial foliated structure in the basis, group objects are locally Lie groups after change of basis, as the following result explains.

Proposition 4.5. *Let B be simply connected, and $q\colon (G, \mathcal{L}) \to B$ a group bundle with group connection (and therefore a group object in $\mathbf{FMn}_{(B,TB)}$). Let x be a point in M and G_x the fiber of G over x, then $(G, \mathcal{L}) \simeq (G_x, \{0\}) \times (B, TB)$.*

Proof. The argument is local, so we have to see that for each $x \in B$ there is a neighborhood U of x such that $(G|_U, \mathcal{L}|_U) \simeq (G_x, \{0\}) \times (U, TU)$. If this is the case, for each homotopy class of a path γ connecting x and y in B we have a group isomorphism $\gamma_*\colon G_x \to G_y$. If B is simply connected, those homotopy classes are unique for each y and the isomorphisms γ_* give us the trivialization of the group connection.

In fact, there are neighborhoods U of x in B, V_x of e_x (the identity element) in G_x, and V of e_x in G, and a decomposition $V \simeq U \times V_x$, such that the horizontal leaves of $\mathcal{L}$ in V have the form $\{g_x\} \times U$ for fixed $g \in V_x$.

Let us see that, for each $h_x \in G_x$ the leaf $\mathcal{F}$ of $\mathcal{L}$ that passes through h_x projects onto U. We may also assume that we take U small enough so that each connected component of $G|_U$ contains exactly one connected component of G_x. Let y be an accumulation point of $q(\mathcal{F})$ inside U. Let us consider h_y an element in G_y in the same connected component of $G|_U$ than h_x. Then there is a leaf $\mathcal{F}'$ of $\mathcal{L}|_U$ passing through h_x. Let U' be $q(\mathcal{F}')$ which is an open subset that intersects $q(\mathcal{F})$. By successive composition of $\mathcal{F}'$ with the leafs of $\mathcal{L}$ in $V|_{U'}$ we have that the connected component of $G|_U'$ containing h_y decomposes in leaves of $\mathcal{L}$. In particular, $\mathcal{F} \cap G|_{U'}$ is part of a leaf of such a decomposition. Finally, $y \in q(\mathcal{F})$. We have seen that $q(\mathcal{F})$ is an open subset that contains all its accumulation points inside U, so that $q(\mathcal{F}) = U$. Thus, $G|_U$ decomposes in leaves of $\mathcal{L}$. □

For the non-simply connected case, the classification of group connections may follow a similar path to the classification of linear connections. Classes of group connections may be given by classes of representations of the fundamental group $\Pi_1(x, B)$ into the group $\mathrm{Aut}(G_x)$ of automorphisms of the fiber. In the case of simply connected B there is no distinction between Galois structures in $\mathbf{FMn}$ or in $\mathbf{FMn}_{(B,TB)}$.

Corollary 4.6. *Let B be simply connected and let $\pi\colon (M, \mathcal{L}) \to (B, TB)$ be a submersion of foliated manifolds with $\mathrm{rank}\,\mathcal{L} = \dim\, B$. Then KP_π splits in $\mathbf{FMn}$ if and only if it splits in $\mathbf{FMn}_{(B,TB)}$.*

In the non-simply connected case, non trivial irreducible linear connections give us examples of splitting actions in the relative category. For instance, we may take, $B = S^1 \times S^1$. We take $G = \mathbb{R} \times B$ and $\mathcal{D} = \langle \partial_\theta + u\partial_u, \partial_\phi + \alpha u \partial u \rangle$ where u is the coordinate in $\mathbb{R}$ and α is an irrational number. Then, we have $(G, \mathcal{D}) \to (B, TB)$ is a group bundle with an irreducible group connection, locally isomorphic to the trivial additive bundle. The action of G on itself is an splitting action in $\mathbf{FMn}_{(B,TB)}$.

4.4 Foliated complex algebraic varieties

Let $\mathbf{FVar}$ be the category of complex regular foliated varieties. Objects are $(M, \mathcal{D})$ where M is a complex variety and $\mathcal{D}$ is an involutive Zariski closed linear subbundle of TM. A foliated variety is called *irreducible* if it has a Zariski dense leaf, or equivalently, it does not have rational first integrals (except locally constant functions). Group objects in $\mathbf{FVar}$ are complex algebraic groups.

In this category, we can state Galois theory exactly in a way totally analogous to what has been done in $\mathbf{FMn}$.

Theorem 4.7. *Let $\pi\colon (M, \mathcal{F}) \to (B, \mathcal{D})$ be a submersion of irreducible foliated varieties with $\operatorname{rank}\mathcal{F} = \operatorname{rank}\mathcal{D}$. The following are equivalent.*

(a) KP_π *splits* $\mathbf{FVar}$.

(b) π *is Galois in* $\mathbf{FVar}$.

(c) There is an algebraic group G acting on M such that π is a principal G-bundle and $\mathcal{D}$ is a $\mathcal{D}$-partial G-invariant connection.

(d) The above, with a unique G.

In such a case G is $\operatorname{Aut}_{(B,\mathcal{D})}(M, \mathcal{L})$.

Proof. Totally analogous to the proofs given in Proposition 4.2 and Theorem 4.3. □

It is interesting to make the connection of this Galois theory with differential algebra. Let us fix $\pi\colon (M, \mathcal{L}) \to (B, \mathcal{D})$ a Galois submersion of irreducible foliated varieries with Galois group G and $\operatorname{rank}\mathcal{F} = \operatorname{rank}\mathcal{D} = r$. Let us note that, by elimination, it is always possible to find a system of commuting rational vector fields $\vec{D}_1, \ldots, \vec{D}_r$ that span $\mathcal{D}$ on the generic point of B. Let us fix $\Delta_B = (\vec{D}_1, \ldots, \vec{D}_r)$. We have that the field of rational functions $(\mathbb{C}(B), \Delta_B)$ is a differential field whose field of constants is $\mathbb{C}$.

The $\mathcal{D}$-connection $\mathcal{F}$ induce lifts of the rational vector fields $\vec{D}_j$ to $\mathcal{F}$-horizontal rational vector fields $\vec{F}_i$ in M that span $\mathcal{F}$ on the generic point of M. We set $\Delta_M = (\vec{F}_1, \ldots, \vec{F}_m)$ so that $(\mathbb{C}(M), \Delta_M)$ is also a differential field whose field of constants is $\mathbb{C}$. Since the projection of $\vec{F}_j$ is $\vec{D}_j$ we have that $\pi^*\colon (\mathbb{C}(B), \Delta_B) \hookrightarrow (\mathbb{C}(M), \Delta_M)$ is an differential field extension. We have the following geometric characterization of strongly normal extensions due to Bialynicki-Virula.

Proposition 4.8 ([3], in [4] p. 18)**.** *Let $(K, \Delta) \hookrightarrow (F, \Delta')$ be a differential field extension with K relatively algebraically closed in F and algebraically closed field of constants $C = K^{\Delta} = F^{\Delta'}$. The following are equivalent:*

1. *It is strongly normal in the sense of Kolchin.*
2. *There are a connected algebraic group G over C and a K-variety W such that:*
    (a) *W is a principal homogeneous space modeled over $G_K = G \times_C \operatorname{Spec}(K)$.*
    (b) *The field of rational functions in W is F.*
    (c) *The group G acts faithfully on F by differential automorphisms fixing K.*

Moreover the pair (G, W) is uniquely determined up to isomorphism and we have $G(C) = \operatorname{Aut}_{\Delta}(F/K)$.

This geometric characterization immediately yields the following.

Proposition 4.9. *Let $\pi\colon (M, \mathcal{L}) \to (B, \mathcal{D})$ a Galois submersion of irreducible foliated varieries with Galois group G, and Δ_B, Δ_M as above. Assume any of the following equivalent hypothesis:*

1. *$\mathbb{C}(B)$ is relatively algebraically closed in $\mathbb{C}(M)$;*
2. *$\pi\colon M \to B$ has connected fibers;*
3. *G is connected.*

The differential field extension:

$$\pi^*\colon (\mathbb{C}(B), \Delta_B) \hookrightarrow (\mathbb{C}(M), \Delta_M)$$

is a strongly normal extension in the sense of Kolchin with Galois group G.

Proof. Let us consider: $M_B = M \times_B \operatorname{Spec}\mathbb{C}(B)$ and $G_B = M \times_C \operatorname{Spec}\mathbb{C}(B)$ as $\mathbb{C}(B)$-varieties. The splitting isomorhism:

$$M \times_{\mathbb{C}} G \xrightarrow{\sim} M \times_B M$$

changes of basis to an isomorphism of $\mathbb{C}(B)$-varieties,

$$M_B \times_{\mathbb{C}(B)} G_B \xrightarrow{\sim} M_B \times_{\mathbb{C}(B)} M_B$$

And therefore M_B is a principal homogenous space over G_B. The field of rational functions in M_B is also $\mathbb{C}(M)$. For any $g \in G$ we have a field automorphism,

$$R_g^*\colon \mathbb{C}(M) \to \mathbb{C}(M)$$

that fixes $\mathbb{C}(B)$ and the derivations $\vec{F}_j$ in Δ_M. This gives an inclusion,

$$G \to \operatorname{Aut}_\Delta(\mathbb{C}(M)/\mathbb{C}(B)), \quad g \mapsto R_g^*$$

and we conclude by Bialynicki-Virula's Proposition 4.8. □

Remark 4.10. The applications to differential algebra seem to go further. There have been several generalizations of differential Galois theory theory [26, 6] and a geometric characterization of strongly normal extensions [21, 22] which is very much in the flavour of Definition 2.7. We expect upcoming research clarifying how all those theories relate with the framework proposed here.

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CAHIERS DE TOPOLOGIE ET
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