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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

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POLYNOMIALS AS SPANS

Ross STREET *

Résumé. L'article définit les polynômes dans une bicatégorie $\mathscr{M}$. Les polynômes dans les bicatégories $\mathrm{Spn}\mathscr{C}$ des spans dans une catégorie finiment complète $\mathscr{C}$ coïncident avec les polynômes dans $\mathscr{C}$ comme définis par Nicola Gambino et Joachim Kock, et par Mark Weber. Lorsque $\mathscr{M}$ est *calibré*, nous obtenons une autre bicatégorie $\mathrm{Poly}\mathscr{M}$. Nous démontrons que les polynômes dans $\mathscr{M}$ ont des représentations comme pseudofoncteurs $\mathscr{M}^{\mathrm{op}} \to \mathrm{Cat}$. En utilisant des *tabulations*, nous produisons des calibrations pour la bicatégorie des relations dans une catégorie régulière et pour la bicatégorie des distributeurs entre catégories, en fournissant ainsi de nouveaux exemples de bicatégories de "polynômes".

Abstract. The paper defines polynomials in a bicategory $\mathscr{M}$. Polynomials in bicategories $\mathrm{Spn}\mathscr{C}$ of spans in a finitely complete category $\mathscr{C}$ agree with polynomials in $\mathscr{C}$ as defined by Nicola Gambino and Joachim Kock, and by Mark Weber. When $\mathscr{M}$ is *calibrated*, we obtain another bicategory $\mathrm{Poly}\mathscr{M}$. We see that polynomials in $\mathscr{M}$ have representations as pseudofunctors $\mathscr{M}^{\mathrm{op}} \to \mathrm{Cat}$. Using *tabulations*, we produce calibrations for the bicategory of relations in a regular category and for the bicategory of two-sided modules (distributors) between categories thereby providing new examples of bicategories of "polynomials".

Keywords. span; partial map; powerful morphism; polynomial functor; exponentiable morphism; calibrated bicategory; right lifting.

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1. Introduction

Polynomials in an internally complete (= "locally cartesian closed") category $\mathscr{E}$ were shown by Gambino-Kock [9] to be the morphisms of a bicategory. Weber [30] defined polynomials in any category $\mathscr{C}$ with pullbacks and proved they formed a bicategory. While both these papers are quite beautiful and accomplish further advances, I felt the need to better understand the composition of polynomials. Perhaps what I have produced is merely a treatment of polynomials for bicategory theorists.

The starting point was to view polynomials as spans of spans so that composition could be viewed as the more familiar composition of spans using pullbacks; see Bénabou [3]. A polynomial from X to Y in a category $\mathscr{C}$ is a diagram of the shape $X \xleftarrow{m_2} E \xrightarrow{m_1} S \xrightarrow{p} Y$ with m_2 a powerful (= exponentiable) morphisms in $\mathscr{C}$. Such diagrams can be thought of as generalizing spans: a span $X \xrightarrow{(m_2,S,p)} Y$ amounts to the case where $E = S$ and m_1 is the identity. Our simple idea was to make the diagram more complicated by including an identity thus:

$$X \xleftarrow{m_2} E \xrightarrow{m_1} S \xleftarrow{1_S} S \xrightarrow{p} Y \ ,$$

resulting in a span

$$X \xleftarrow{(m_1,E,m_2)} S \xrightarrow{(1_S,S,p)} Y$$

of spans from X to Y.

Of course, the bicategory of spans does not have all bicategorical pullbacks. Fortunately, polynomials are not general spans and sufficient pullbacks can be constructed. Indeed, that is what Weber's distributivity pullbacks around a pair of composable morphisms in $\mathscr{C}$ construct. That construction requires the use of powerful morphisms in $\mathscr{C}$. Here we define a morphism in a bicategory to be a *right lifter* when every morphism into its codomain has a right lifting through it. For spans in $\mathscr{C}$ to be right lifters, one leg must be powerful.

We introduce the term *calibration* for a class of morphisms, called *neat*, in a bicategory; the technical use of this word comes from Bénabou [4] who used it for categories. A bicategory with a distinguished calibration is called *calibrated*. Polynomials in a calibrated bicategory $\mathscr{M}$ are spans with one leg a right lifter and the other leg neat. This suffices for the construction of a tricategory [10] of polynomials in $\mathscr{M}$ in which all the 3-morphisms are invertible. However, for two reasons, we decided to centre attention here on the bicategory $\mathrm{Poly}\mathscr{M}$ obtained by taking isomorphism classes of 2-morphisms. One reason is that it covers our present examples, the other is the possibility of iterating the construction without moving to higher level categories.

A *polynomic bicategory* $\mathscr{M}$ is one in which the neat morphisms are all the groupoid fibrations (see Section 3) in $\mathscr{M}$. We prove that $\mathrm{Spn}\mathscr{C}$ is polynomic for any finitely complete $\mathscr{C}$. In this case the polynomials are the polynomials in $\mathscr{C}$ in the sense of Weber [30].

The bicategory $\mathrm{Rel}\mathscr{E}$ of relations in a regular category $\mathscr{E}$ is calibrated by morphisms which are isomorphic to graphs of monomorphisms in $\mathscr{E}$. In Example 10.3 for $\mathscr{E}$ a topos, we give a reinterpretation of the bicategory of polynomials in $\mathrm{Rel}\mathscr{E}$ as a Kleisli construction.

By providing a calibration for the bicategory Mod of two-sided modules between categories, we obtain another example. Again, in Example 10.6, we give a reinterpretation of the bicategory of polynomials in Mod as a Kleisli construction.

It must be pointed out that the meaning of polynomial in a bicategory is different from the meaning in Section 4 of Weber [30] which is about polynomials in 2-categories. Weber is dealing with the 2-category as a Cat-enriched category, taking the polynomials to be diagrams of the same shape as in the case of ordinary categories, and accommodating the presence of 2-cells. In particular, if a category is regarded as a 2-category with only identity 2-cells, then his polynomials in the 2-category are just polynomials in the category. To define a polynomial, in the sense of this paper, in such a 2-category would require the specification of a calibration on the category and then a polynomial would reduce to a single morphism (called "neat") in that calibration.

I am grateful to the Australian Category Seminar, especially Yuki Maehara, Richard Garner, Michael Batanin and Charles Walker, for comments during and following my talks on this topic. I am also particularly grateful to the diligent and insightful referee for suggesting important improvements, mainly that I should add the detail to the previously vaguely expressed Examples 10.3 and 10.6; there are some facts involved that may be unfamiliar.

2. Bipullbacks and cotensors

Recall that the pseudopullback (also called iso-comma category) of two functors $\mathscr{C} \xrightarrow{F} \mathscr{E} \xleftarrow{P} \mathscr{D}$ is the category $F/_{\mathrm{ps}}P$ whose objects (C, α, D) consist of objects $C \in \mathscr{C}$ and $D \in \mathscr{D}$ with $\alpha : FC \xrightarrow{\cong} PD$, and whose morphisms $(u, v) : (C, \alpha, D) \to (C', \alpha', D')$ consist of morphisms $u : C \to C'$ in $\mathscr{C}$ and $v : D \to D'$ in $\mathscr{D}$ such that $(Pv)\alpha = \alpha'(Fu)$. We have a universal square of functors

$$\begin{array}{ccc} F/_{\mathrm{ps}}P & \xrightarrow{\mathrm{dom}} & \mathscr{D} \\ {\scriptstyle \mathrm{cod}}\downarrow & \overset{\xi}{\underset{\cong}{\Longleftarrow}} & \downarrow{\scriptstyle F} \\ \mathscr{C} & \xrightarrow[P]{} & \mathscr{E} \end{array} \tag{2.1}$$

containing an invertible natural transformation ξ.

A square

$$\begin{array}{ccc} P & \xrightarrow{d} & A \\ {\scriptstyle c}\downarrow & \overset{\theta}{\underset{\cong}{\Longleftarrow}} & \downarrow{\scriptstyle n} \\ B & \xrightarrow[p]{} & C \end{array} \tag{2.2}$$

in a bicategory $\mathscr{A}$ is a *bipullback* of the cospan $A \xrightarrow{n} C \xleftarrow{p} B$ when, for all objects K of $\mathscr{A}$, the induced functor

$$\mathscr{A}(K, P) \longrightarrow \mathscr{A}(K, n)/_{\mathrm{ps}}\mathscr{A}(K, p)\ ,\ u \mapsto (du, \theta u, cu)\ ,$$

is an equivalence of categories.

In a bicategory $\mathscr{A}$, we write $A^{\mathbf{2}}$ for the (bicategorical) cotensor (or power) of A with the ordinal **2**; this means that the category $\mathscr{A}(K, A^{\mathbf{2}})$ is equivalent to the arrow category of $\mathscr{A}(K, A)$, pseudonaturally in $K \in \mathscr{A}$. The identity morphism in $\mathscr{A}(A^{\mathbf{2}}, A^{\mathbf{2}})$ corresponds to a morphism (arrow)

$$A^{\mathbf{2}} \underset{\mathrm{c}}{\overset{\mathrm{d}}{\rightrightarrows}} A \quad (\Downarrow \lambda)$$

in $\mathscr{A}(A^{\mathbf{2}}, A)$.

Example 2.1. For $\mathscr{A} = \mathscr{V}\text{-}\mathrm{Cat}$ in the sense of [14], the $\mathscr{V}$-category $A^{\mathbf{2}}$ is the usual arrow $\mathscr{V}$-category.

Example 2.2. Recall (from [5] for instance) the definition of the bicategory $\mathscr{V}\text{-Mod}$ of $\mathscr{V}$-categories and their modules for a nice symmetric closed monoidal base category $\mathscr{V}$. The objects are $\mathscr{V}$-categories. The homcategories are defined to be the $\mathscr{V}$-functor categories

$$\mathscr{V}\text{-Mod}(A, B) = [B^{\mathrm{op}} \otimes A, \mathscr{V}]$$

whose objects $m : B^{\mathrm{op}} \otimes A \to \mathscr{V}$ are called modules from A to B. Composition is defined by the coends $(n \circ m)(c, a) = \int^b m(b, a) \otimes n(c, b)$. Let $I_*\mathbf{2}$ denote the free $\mathscr{V}$-category on the category $\mathbf{2}$. The cotensor of the $\mathscr{V}$-category A with the ordinal $\mathbf{2}$ in the bicategory $\mathscr{V}\text{-Mod}$ is the $\mathscr{V}$-category $(I_*\mathbf{2})^{\mathrm{op}} \otimes A$. This is because of the calculation

$$\begin{aligned}\mathscr{V}\text{-Mod}(K, (I_*\mathbf{2})^{\mathrm{op}} \otimes A) &= [I_*\mathbf{2} \otimes A^{\mathrm{op}} \otimes K, \mathscr{V}] \\ &\cong [A^{\mathrm{op}} \otimes K, \mathscr{V}^{\mathbf{2}}] \\ &\cong [A^{\mathrm{op}} \otimes K, \mathscr{V}]^{\mathbf{2}} \\ &\cong \mathscr{V}\text{-Mod}(K, A)^{\mathbf{2}}\ .\end{aligned}$$

Let $\partial_0 : \mathbf{1} \to \mathbf{2}$ be the functor $0 \mapsto 1$; it is right adjoint to $! : \mathbf{2} \to \mathbf{1}$. It follows that $\mathrm{c} : A^{\mathbf{2}} \to A$ in $\mathscr{V}\text{-Mod}$ is $((I_*!)^{\mathrm{op}} \otimes A)_* : (I_*\mathbf{2})^{\mathrm{op}} \otimes A \to A$. In particular, when $\mathscr{V} = \mathrm{Set}$, c is the module $\mathrm{pr}_{2\,*}$ induced by the second projection functor $\mathbf{2}^{\mathrm{op}} \times A \to A$.

Remark 2.3. The phenomenon described in Example 2.2 has to do with the fact that the pseudofunctor $(-)_* : \mathscr{V}\text{-Cat} \to \mathscr{V}\text{-Mod}$, taking each $\mathscr{V}$-functor $f : A \to B$ to the module $f_* : A \to B$ with $f_*(b, a) = B(b, fa)$, preserves bicolimits and $\mathscr{V}\text{-Mod}$ is self dual.

3. Groupoid fibrations

Let $p : E \to B$ be a functor. A morphism $\chi : e' \to e$ in E is called *cartesian*[1] for p when the square (3.3) is a pullback for all $k \in E$.

$$\begin{array}{ccc} E(k, e') & \xrightarrow{E(k,\chi)} & E(k, e) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B(pk, pe') & \xrightarrow[B(pk,p\chi)]{} & B(pk, pe) \end{array} \tag{3.3}$$

[1]Classically called "strong cartesian"

Note that all invertible morphisms in E are cartesian. If p is fully faithful then all morphisms of E are cartesian.

We call the functor $p : E \to B$ a *groupoid fibration* when

(i) for all objects $e \in E$ and morphisms $\beta : b \to pe$ in B, there exist a morphism $\chi : e' \to e$ in E and isomorphism $b \cong pe'$ whose composite with $p\chi$ is β, and

(ii) every morphism of E is cartesian for p.

From the pullback (3.3), it follows that groupoid fibrations are conservative (that is, reflect invertibility).

We call the functor $p : E \to B$ an *equivalence relation fibration* or *er-fibration* when it is a groupoid fibration and the only endomorphisms $\xi : x \to x$ in E which map to identities under p are identities. It follows (using condition (ii)) that p is faithful. Note that if p is an equivalence then it is an er-fibration.

Write $\mathrm{GFib}B$ for the 2-category whose objects are groupoid fibrations $p : E \to B$, and whose hom categories are given by the following pseudopullbacks.

$$\begin{array}{ccc} \mathrm{GFib}B(p,q) & \longrightarrow & [E,F] \\ \big\downarrow & \overset{\cong}{\Longleftarrow} & \big\downarrow {\scriptstyle [E,q]} \\ \mathbf{1} & \underset{\lceil p \rceil}{\longrightarrow} & [E,B] \end{array} \tag{3.4}$$

So objects of $\mathrm{GFib}B(p,q)$ are pairs (f,ϕ) where $f : E \to F$ is a functor and $\phi : qf \Rightarrow p$ is an invertible natural transformation. If ϕ is an identity then (f,ϕ) is called *strict*.

Let Gpd be the 2-category of groupoids, functors and natural transformations. Write $\mathrm{Hom}(B^{\mathrm{op}},\mathrm{Gpd})$ for the 2-category of pseudofunctors (= homomorphisms of bicategories [3]) $T : B^{\mathrm{op}} \to \mathrm{Gpd}$, pseudo-natural transformations, and modifications [16].

Recall that the Grothendieck construction $\mathrm{pr} : \wr T \to B$ on a pseudofunctor $T : B^{\mathrm{op}} \to \mathrm{Gpd}$ is the projection functor from the category $\wr T$ whose objects are pairs (t,b) with $b \in B$ and $t \in Tb$, and whose morphisms $(\tau,\beta) : (t,b) \to (t',b')$ consist of morphisms $\beta : b \to b'$ in B and $\tau : t \to (T\beta)t'$ in Tb. This construction is the object assignment for a

2-functor

$$\wr : \mathrm{Hom}(B^{\mathrm{op}}, \mathrm{Gpd}) \longrightarrow \mathrm{GFib}B \tag{3.5}$$

which actually lands in the sub-2-category of strict morphisms. Note that the pullback

$$\begin{array}{ccc} Tb & \longrightarrow & \wr T \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ \mathbf{1} & \xrightarrow[\ulcorner b \urcorner]{} & B \end{array}$$

is also a bipullback (see [12]); this suggests that we can reconstruct a pseudofunctor T from a groupoid fibration $p : E \to B$ by defining Tb to be the pseudopullback of $\ulcorner b \urcorner : \mathbf{1} \to B$ and p.

Proposition 3.1. *The 2-functor* (3.5) *is a biequivalence.*

A category which is both a groupoid and a preorder is the same as an equivalence relation; that is, a set of objects equipped with an equivalence relation thereon. Let ER be the 2-category of equivalence relations, functors and natural transformations. Note that the 2-functor $\mathrm{Set} \to \mathrm{ER}$ taking each set to the identity relation is a biequivalence. Write $\mathrm{ERFib}B$ for the full sub-2-category of $\mathrm{GFib}B$ with objects the er-fibrations.

Proposition 3.2. *The biequivalence* (3.5) *restricts to a biequivalence*

$$\wr : \mathrm{Hom}(B^{\mathrm{op}}, \mathrm{ER}) \xrightarrow{\sim} \mathrm{ERFib}B \ ,$$

and so further restricts to a biequivalence

$$[B^{\mathrm{op}}, \mathrm{Set}] \xrightarrow{\sim} \mathrm{ERFib}B \ .$$

Let $\mathscr{E}$ and $\mathscr{B}$ be bicategories. Baklović [2] and Buckley [6] say that a morphism $x : Z \to X$ in $\mathscr{E}$ is *cartesian* for a pseudofunctor $P : \mathscr{E} \to \mathscr{B}$ when the following square is a bipullback in Cat for all objects K of $\mathscr{E}$.

$$\begin{array}{ccc} \mathscr{E}(K,Z) & \xrightarrow{\mathscr{E}(K,x)} & \mathscr{E}(K,X) \\ {\scriptstyle P}\downarrow & \overset{\cong}{\Longleftarrow} & \downarrow{\scriptstyle P} \\ \mathscr{B}(PK,PZ) & \xrightarrow[\mathscr{B}(PK,Px)]{} & \mathscr{B}(PK,PX) \end{array} \tag{3.6}$$

A 2-cell $\sigma : x' \Rightarrow x : Z \to X$ in $\mathscr{E}$ is called *cartesian* for P when it is cartesian (as a morphism of $\mathscr{E}(Z,X)$) for the functor $P : \mathscr{E}(Z,X) \to \mathscr{B}(PZ,PX)$. Note that all equivalences are cartesian morphisms and all invertible 2-cells are cartesian.

Definition 3.3. *A pseudofunctor $P : \mathscr{E} \to \mathscr{B}$ is a* groupoid fibration *when*

(i) for all $X \in \mathscr{E}$ and $f : B \to PX$ in $\mathscr{B}$, there exist a morphism $x : Z \to X$ in $\mathscr{E}$ and an equivalence $B \simeq PZ$ whose composite with Px is isomorphic to f,

(ii) every morphism of $\mathscr{E}$ is cartesian for P, and

(iii) every 2-cell of $\mathscr{E}$ is cartesian for P.

A morphism $p : E \to B$ in a tricategory $\mathscr{T}$ is called a groupoid fibration *when, for all objects K of $\mathscr{T}$, the pseudofunctor $\mathscr{T}(K,p) : \mathscr{T}(K,E) \to \mathscr{T}(K,B)$ is a groupoid fibration between bicategories.*

Definition 3.4. *A pseudofunctor $F : \mathscr{A} \to \mathscr{B}$ is called* conservative *when*

(a) if Ff is an equivalence in $\mathscr{B}$ for a morphism f in $\mathscr{A}$ then f is an equivalence;

(b) if $F\alpha$ is an isomorphism in $\mathscr{B}$ for a 2-cell α in $\mathscr{A}$ then α is an isomorphism.

A morphism $f : A \to B$ in a tricategory $\mathscr{T}$ is conservative *when, for all objects K of $\mathscr{T}$, the pseudofunctor $\mathscr{T}(K,f) : \mathscr{T}(K,A) \to \mathscr{T}(K,B)$ is conservative.*

Proposition 3.5. *Groupoid fibrations are conservative.*

Proof. If Px is an equivalence, we see from the bipullback (3.6) that each functor $\mathscr{E}(K,x)$ is too. Since these equivalences can be chosen to be adjoint equivalences, they become pseudonatural in K and so, by the bicategorical Yoneda Lemma [23], are represented by an inverse equivalence for x. This proves (a) in the Definition of conservative. Similarly, for (b), look at the pullback (3.3) for the functor $p = (\mathscr{E}(Z,X) \xrightarrow{P} \mathscr{B}(PZ,PX))$. □

For a pseudofunctor $P : \mathscr{E} \to \mathscr{B}$ between bicategories, write $\mathscr{B}/P$ for the bicategory whose objects are pairs $(B \xrightarrow{f} PE, E)$, where E is an object of $\mathscr{E}$ and $f : B \to PE$ is a morphism of $\mathscr{B}$, and whose homcategories are defined by pseudopullbacks

$$\begin{array}{ccc} \mathscr{B}/P((f,E),(f',E')) & \xrightarrow{\mathrm{c}} & \mathscr{E}(E,E') \\ & & \downarrow{\scriptstyle P} \\ {\scriptstyle \mathrm{d}}\downarrow & \Downarrow\cong & \mathscr{B}(PE,PE') \\ & & \downarrow{\scriptstyle \mathscr{B}(f,1)} \\ \mathscr{B}(B,B') & \xrightarrow[\mathscr{B}(1,f')]{} & \mathscr{B}(B,PE') \end{array} \qquad (3.7)$$

Write $\mathscr{E}/\mathscr{E}$ for $\mathscr{B}/P$ in the case P is the identity pseudofunctor of $\mathscr{E}$. There is a canonical pseudofunctor $J_P : \mathscr{E}/\mathscr{E} \to \mathscr{B}/P$ taking the object $(X \xrightarrow{u} E, E)$ to $(PX \xrightarrow{Pu} PE, E)$.

Proposition 3.6. *The pseudofunctor $P : \mathscr{E} \to \mathscr{B}$ between bicategories satisfies condition (i) in the Definition 3.3 of groupoid fibration if and only if $J_P : \mathscr{E}/\mathscr{E} \to \mathscr{B}/P$ is surjective on objects up to equivalence. Also, $P : \mathscr{E} \to \mathscr{B}$ satisfies condition (ii) if and only if the effect of $J_P : \mathscr{E}/\mathscr{E} \to \mathscr{B}/P$ on homcategories is an equivalence. Condition (iii) is automatic if all 2-cells in $\mathscr{E}$ are invertible.*

Example 3.7. Each biequivalence of bicategories is a groupoid fibration.

Example 3.8. Let H be an object of the bicategory $\mathscr{B}$. Write $\mathscr{B}/H$ for the bicategory $\mathscr{B}/P$ where P is the constant pseudofunctor $\mathbf{1} \to \mathscr{B}$ at H. The "take the domain" pseudofunctor

$$\mathrm{dom} : \mathscr{B}/H \to \mathscr{B} \qquad (3.8)$$

is a groupoid fibration. For, it is straightforward to see that the canonical pseudo-functor $(\mathscr{B}/H)/(\mathscr{B}/H) \to \mathscr{B}/\mathrm{dom}$ is a biequivalence, so it remains to prove each 2-cell

$$\sigma : (g,\psi) \Rightarrow (f,\phi) : (A \xrightarrow{a} H) \to (B \xrightarrow{b} H)$$

in $\mathscr{B}/H$ is cartesian for (3.9). The condition for a 2-cell is $(b\sigma)\psi = \phi$. Take another 2-cell $\tau : (h,\theta) \Rightarrow (f,\phi)$ in $\mathscr{B}/H$ (so that $(b\tau)\theta = \phi$) and a 2-cell $\upsilon : h \Rightarrow g$ in $\mathscr{B}$ such that $\sigma\upsilon = \tau$. Then we have $(b\sigma)(b\upsilon)\theta = (b\sigma\upsilon)\theta = (b\tau)\theta = \phi = (b\sigma)\psi$ with $b\sigma$ invertible. So $(b\upsilon)\theta = \psi$. We conclude that $\upsilon : (h,\theta) \Rightarrow (g,\psi)$ is a 2-cell in $\mathscr{B}/H$, as required.

Note that (3.8) is not a local groupoid fibration in general; that is, the functor $\mathrm{dom}_{a,b} : \mathscr{B}/H(A \xrightarrow{a} H, B \xrightarrow{b} H) \to \mathscr{B}(A,B)$ is generally not a groupoid fibration.

Example 3.9. Apparently more generally, let $f : H \to K$ be a morphism in the bicategory $\mathscr{B}$. Write

$$f_* : \mathscr{B}/H \to \mathscr{B}/K \tag{3.9}$$

for the pseudofunctor which composes with f. On applying Example 3.8 with $\mathscr{B}$ and H replaced by $\mathscr{B}/K$ and $H \xrightarrow{f} K$, up to biequivalence we obtain this example.

Proposition 3.10. *Up to biequivalence, pseudofunctors of the form* (3.8) *are precisely the groupoid fibrations $P : \mathscr{E} \to \mathscr{B}$ for which the domain bicategory has a terminal object. Moreover, if such a P has a left biadjoint, it is a biequivalence.*

Proof. Let T be a terminal object of $\mathscr{E}$. Make a choice of morphism $!_E : E \to T$ for each object E of $\mathscr{E}$. Then the pseudofunctor

$$\hat{J}_P : \mathscr{E} \longrightarrow \mathscr{B}/PT \ , \ E \mapsto (PE \xrightarrow{P!_E} PT)$$

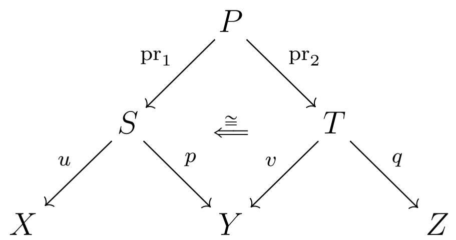
is a biequivalence over $\mathscr{B}$; it is a tripullback of the biequivalence J_P along the canonical $\mathscr{B}/PT \to \mathscr{B}/P$. So P is biequivalent to (3.8) with $H = PT$.

For the second sentence, if we suppose the dom of (3.8) has a left biadjoint then it preserves terminal objects. The bicategory $\mathscr{B}/H$ has the terminal object $1_H : H \to H$. So $H = \mathrm{dom}(H \xrightarrow{1_H} H)$ is terminal in $\mathscr{B}$. So dom is a biequivalence. □

4. Spans in a bicategory

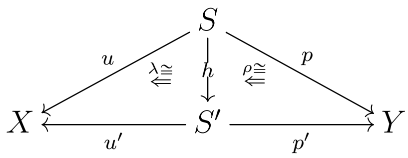
Spans in a bicategory $\mathscr{A}$ with bipullbacks (= iso-comma objects) will be recalled; compare [23] Section 3.

A *span* from X to Y in the bicategory $\mathscr{A}$ is a diagram $X \xleftarrow{u} S \xrightarrow{p} Y$; we write $(u, S, p) : X \to Y$. The composite of $X \xleftarrow{u} S \xrightarrow{p} Y$ and $Y \xleftarrow{v} T \xrightarrow{q} Z$ is obtained from the diagram



(4.10)

(where P is the bipullback of p and v) as the span $X \xleftarrow{u\mathrm{pr}_1} P \xrightarrow{q\mathrm{pr}_2} Z$. A *morphism* $(\lambda, h, \rho) : (u, S, p) \to (u', S', p')$ *of spans* is a morphism $h : S \to S'$ in $\mathscr{M}$ equipped with invertible 2-cells as shown in the two triangles below.



(4.11)

A 2-cell $\sigma : h \Rightarrow k : (u, S, p) \to (u', S', p')$ between such morphisms is a 2-cell $\sigma : h \Rightarrow k : S \to S'$ in $\mathscr{M}$ which is compatible with the 2-cells in the triangles in the sense that $\lambda = \lambda'.u'\sigma$ and $\rho' = p'\sigma.\rho$. We write $\mathrm{Spn}\mathscr{A}(X, Y)$ for the bicategory of spans from X to Y.

Composition pseudofunctors

$$\mathrm{Spn}\mathscr{A}(Y, Z) \times \mathrm{Spn}\mathscr{A}(X, Y) \longrightarrow \mathrm{Spn}\mathscr{A}(X, Z)$$

are defined on objects by composition of spans (4.10) and on morphisms by using the universal properties of bipullback.

In this way, we obtain a tricategory $\mathrm{Spn}\mathscr{A}$. The associator equivalences are obtained using the horizontal and vertical stacking properties of pseudopullbacks. The identity span on X has the form $(1_X, X, 1_X)$ and the unitor equivalences are obtained using the fact that a pseudopullback of the cospan $X \xrightarrow{f} Y \xleftarrow{1_Y} Y$ is given by the span $X \xleftarrow{1_X} X \xrightarrow{f} Y$ equipped with the canonical isomorphism $1_Y f \cong f \cong f 1_X$ in $\mathscr{A}$.

For $e : X \to Y$ in $\mathscr{A}$, let $e_* = (1_X, X, f) : X \to Y$. Notice that $e^* = (e, X, 1_X) : Y \to X$ is a right biadjoint for e_* in the tricategory $\mathrm{Spn}\mathscr{A}$.

Proposition 4.1. *Let $\mathscr{A}$ be a finitely complete bicategory. The following conditions on a span (u, S, p) from X to Y in $\mathscr{A}$ are equivalent:*

(i) the morphism $(u, S, p) : X \to Y$ has a right biadjoint in the tricategory $\mathrm{Spn}\mathscr{A}$*;*

(ii) the morphism $u : S \to X$ is an equivalence in $\mathscr{A}$;

(iii) the morphism $(u, S, p) : X \to Y$ is equivalent in $\mathrm{Spn}\mathscr{A}$ *to f_* for some morphism f in $\mathscr{A}$;*

(iv) the morphism $(u, S, p) : X \to Y$ is a groupoid fibration in the tricategory $\mathrm{Spn}\mathscr{A}$*.*

Proof. The equivalence of (i), (ii) and (iii) is essentially as in the case where $\mathscr{A}$ is a category; see [7]. We will prove the equivalence of (ii) and (iv). Put $\mathfrak{S} = \mathrm{Spn}\mathscr{A}$ to save space. Using Proposition 3.10 and the fact that the span $K \xleftarrow{\mathrm{pr}_1} K \times X \xrightarrow{\mathrm{pr}_2} X$ is a terminal object in the bicategory $\mathfrak{S}(K, X)$, we see that the pseudofunctor $P_K := \mathfrak{S}(K, X) \xrightarrow{\mathfrak{S}(K, p_* u^*)} \mathfrak{S}(K, Y)$ is a groupoid fibration if and only if the canonical pseudofunctor J_{P_K} in the diagram (4.12) is a biequivalence.

$$
\begin{array}{ccc}
 & \mathfrak{S}(K,X) & \\
 \swarrow^{\mathfrak{S}(K,u^*)} & \overset{\simeq}{\Longrightarrow} & \searrow^{J_{P_K}} \\
\mathfrak{S}(K,S) & \xrightarrow[J_{\mathfrak{S}(K,p_*)}]{} & \mathfrak{S}(K,Y)/(\mathrm{pr}_1, K \times S, p\mathrm{pr}_2)
\end{array}
\tag{4.12}
$$

However, we see easily that J_{P_K} does factor up to equivalence as shown in (4.12) where $J_{\mathfrak{S}(K,p_*)}$ is a biequivalence. So $p_* u^* : X \to Y$ is a groupoid fibration if and only if $\mathfrak{S}(K, u^*)$ is a biequivalence for all K; that is, if and only u is an equivalence in $\mathscr{A}$. $\square$

Remark 4.2. (a) In fact (ii) implies (iv) in Proposition 4.1 requires no assumption on the bicategory $\mathscr{A}$. For, it is straightforward to check (compare Example 3.9) that $p_* : \mathrm{Spn}\mathscr{A}(K, S) \to \mathrm{Spn}\mathscr{A}(K, Y)$ is a groupoid fibration for all K; this does not even require bipullback in $\mathscr{A}$ since we only need the hom bicategories of $\mathrm{Spn}\mathscr{A}$.

(b) Given that p_* is a groupoid fibration, we can prove the converse (iv) implies (ii) by noting that $p_* u^*$ is a groupoid fibration if and only if u^* is (compare (i) of Proposition 5.2). So, provided $\mathrm{Spn}\mathscr{A}(K, Y)$ has a terminal object (as guaranteed by the finite bicategorical limits in $\mathscr{A}$), we deduce that u^* is a biequivalence using Proposition 3.10 and $u_* \dashv u^*$.

Remark 4.3. If $\mathscr{C}$ is a finitely complete category (regarded as a bicategory with only identity 2-cells) then the tricategory $\mathrm{Spn}\mathscr{C}$ has only identity 3-cells; it is a bicategory. We are interested in spans in such a bicategory $\mathscr{A} = \mathrm{Spn}\mathscr{C}$. The problem is that bipullbacks do not exist in this kind of $\mathscr{A}$ in general. Hence we must hone our concepts to restricted kinds of spans in $\mathscr{A}$.

5. More on bipullbacks and groupoid fibrations

In Section 4, we defined groupoid fibrations in a tricategory. This applies in a bicategory $\mathscr{A}$ regarded as a tricategory by taking only identity 3-cells. Then the 2-cells in each $\mathscr{A}(A, B)$ are invertible (identities in fact) so condition (iii) of Definition 3 is automatic.

Proposition 5.1. *Suppose $p : E \to B$ is a morphism in a bicategory $\mathscr{A}$ for which $E^{\mathbf{2}}$ and $B^{\mathbf{2}}$ exist. Then $p : E \to B$ is a groupoid fibration in $\mathscr{A}$ if and only if the square*

$$\begin{array}{ccc} E^{\mathbf{2}} & \xrightarrow{\ \mathrm{c}\ } & E \\ {\scriptstyle p^{\mathbf{2}}}\downarrow & \overset{\cong}{\Longleftarrow} & \downarrow{\scriptstyle p} \\ B^{\mathbf{2}} & \xrightarrow[\ \mathrm{c}\ ]{} & B \end{array}$$

is a bipullback.

Proof. Since all concepts are defined representably, it suffices to check this for the bicategory $\mathscr{A} = \mathrm{Cat}$ where the bipullback of c and p is the comma category B/p. So the square in the Proposition is a bipullback if and only if the canonical functor $j_p : E^{\mathbf{2}} \to B/p$ is an equivalence. We have the result by looking at Proposition 3.6 in the bicategory case. □

Proposition 5.2. *Suppose $\mathscr{A}$ is a bicategory.*

(i) Suppose $r \cong pq$ with p a groupoid fibration in $\mathscr{A}$. Then r is a groupoid fibration if and only if q is.

(ii) In the bipullback (2.2) *in $\mathscr{A}$, if p is a groupoid fibration then so is d.*

(iii) Suppose (2.2) *is a bipullback in $\mathscr{A}$ and p is a groupoid fibration. For any square*

$$\begin{array}{ccc} K & \xrightarrow{\;u\;} & A \\ {\scriptstyle v}\downarrow & \overset{\psi}{\Longleftarrow} & \downarrow{\scriptstyle n} \\ B & \xrightarrow[\;p\;]{} & C \end{array} \tag{5.13}$$

in $\mathscr{A}$ with ψ not necessarily invertible, there exists a diagram

$$\begin{array}{ccccc} & & K & & \\ & {\scriptstyle v}\swarrow \;\overset{\lambda}{\Leftarrow} & \downarrow{\scriptstyle h} & \overset{\rho\cong}{\Leftarrow}\; \searrow{\scriptstyle u} & \\ B & \xleftarrow[\;c\;]{} & P & \xrightarrow[\;d\;]{} & A \end{array}\;, \tag{5.14}$$

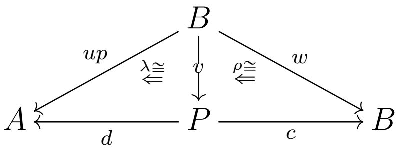
(with λ invertible if and only if ψ is) which pastes onto (2.2) *to yield ψ. This defines on objects an inverse equivalence of the functor from the category of such (λ, h, ρ) to the category of diagrams* (5.13) *obtained by pasting onto* (2.2)*.*

Proof. These are essentially standard facts about groupoid fibrations, especially (i) and (ii). For (iii) we use the groupoid fibration property of p to lift the 2-cell $\psi : nu \Rightarrow pv$ to a 2-cell $\chi : w \Rightarrow v$ with an invertible 2-cell $\nu : nu \Rightarrow pw$ such that $\psi = (p\chi)\nu$. Now use the bipullback property of (2.2) to factor the square $\nu : nu \Rightarrow pw$ as a span morphism $(\sigma, h, \rho) : (w, K, u) \to (c, P, d)$ pasted onto (2.2). Then λ is the composite of σ and χ. $\square$

The next result is related to Proposition 5 of [22].

Proposition 5.3. *In the bipullback square* (2.2) *in the bicategory $\mathscr{A}$, if p is a groupoid fibration and n has a right adjoint $n \dashv u$ then c has a right adjoint $c \dashv v$ such that the mate $\hat{\theta} : dv \Rightarrow up$ of $\theta : nd \Rightarrow pc$ is invertible.*

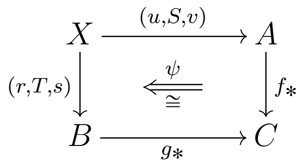
Proof. Let $\varepsilon : nu \Rightarrow 1_C$ be the counit of $n \dashv u$. By the groupoid fibration property of p, there exists $\chi : w \Rightarrow 1_B$ and an invertible $\tau : nup \Rightarrow pw$ such that $(p\chi)\tau = \varepsilon p$. By the bipullback property of (2.2), there exists a span morphism



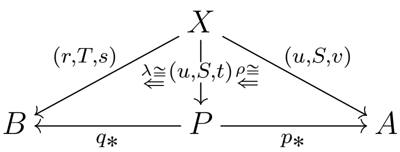
whose pasting onto θ is τ. Then $c \dashv v$ with counit $cv \overset{\rho^{-1}}{\Longrightarrow} w \overset{\chi}{\Rightarrow} 1_B$ and we see that $\hat{\theta} = \rho$ is invertible. □

Proposition 5.4. *Suppose $\mathscr{C}$ is a category with pullbacks. Then the pseudo-functor $(-)_* : \mathscr{C} \to \mathrm{Spn}\mathscr{C}$ takes pullbacks to bipullbacks.*

Proof. Let the span $(p, P, q) : A \to B$ be the pullback of the cospan (f, C, g) in $\mathscr{C}$. Consider a square



in $\mathrm{Spn}\mathscr{C}$. The isomorphism ψ amounts to an isomorphism $h : (u, S, fv) \to (r, T, gs)$ of spans. In particular, $fv = gsh$, so, by the pullback property, there exists a unique $t : S \to P$ such that $pt = v$ and $qt = sh$. Then we have a morphism of spans

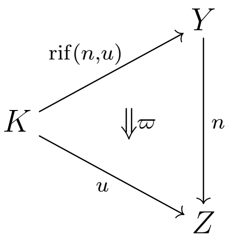


in which λ is $h : (u, S, qt) \to (r, T, s)$ and ρ is an identity. □

6. Lifters

Let $\mathscr{M}$ be a bicategory.

We use the notation



(6.15)

to depict a right lifting $\mathrm{rif}(n, u)$ (see [27]) of u through n. The defining property is that pasting a 2-cell $v \Longrightarrow \mathrm{rif}(n, u)$ onto the triangle to give a 2-cell $nv \Longrightarrow u$ defines a bijection.

A morphism $n : Y \to Z$ is called a *right lifter* when $\mathrm{rif}(n, u)$ exists for all $u : K \to Z$.

Example 6.1. Left adjoint morphisms in any $\mathscr{M}$ are right lifters (since the lifting is obtained by composing with the right adjoint).

Example 6.2. Composites of right lifters are right lifters.

Example 6.3. Suppose $\mathscr{M} = \mathrm{Spn}\mathscr{C}$ with $\mathscr{C}$ a finitely complete category. If $f : A \to B$ is powerful (in the sense of [25], elsewhere called exponentiable, and meaning that the functor $\mathscr{C}/B \to \mathscr{C}/A$, which pulls back along f, has a right adjoint Π_f) in $\mathscr{C}$ then $f^* : B \to A$ is a right lifter. The formula is $\mathrm{rif}(f^*, (v, T, q)) = (w, U, r)$ where

$$(U \xrightarrow{(w,r)} K \times B) = \Pi_{1_K \times f}(T \xrightarrow{(v,q)} K \times A)\ .$$

Example 6.4. Suppose $m = (m_1, E, m_2)$ is a morphism in $\mathscr{M} = \mathrm{Spn}\mathscr{C}$ with $\mathscr{C}$ a finitely complete category. Then m is a right lifter if and only if m_1 is powerful. The previous Examples imply "if". Conversely, we can apply the Dubuc Adjoint Triangle Theorem (see Lemma 2.1 of [25] for example) to see that $\mathscr{M}(K, {m_1}^*)$ has a right adjoint for all K because $\mathscr{M}(K, m) \cong \mathscr{M}(K, m_{2*})\mathscr{M}(K, {m_1}^*)$ and the unit of $m_{2*} \dashv {m_2}^*$ is an equalizer. Taking K to be the terminal object, we conclude that m_1 is powerful.

Example 6.5. Let $\mathscr{E}$ be a regular category and let $\mathrm{Rel}\mathscr{E}$ be the locally ordered bicategory of relations in $\mathscr{E}$ as characterized in [7]. The objects are the same as for $\mathscr{E}$ and the morphisms $(r_1, R, r_2) : X \to Y$ are jointly monomorphic spans $X \xleftarrow{r_1} R \xrightarrow{r_2} Y$ in $\mathscr{E}$. Let $\mathrm{Sub}X = \mathrm{Rel}\mathscr{E}(1, X)$ be the partially ordered set of subobjects of $X \in \mathscr{E}$. If $f : Y \to X$ is a morphism of $\mathscr{E}$ then pulling back subobjects of X along f defines an order-preserving function $f^{-1} : \mathrm{Sub}X \to \mathrm{Sub}Y$ whose right adjoint, if it exists, is denoted by $\forall_f : \mathrm{Sub}Y \to \mathrm{Sub}X$. A similar analysis as in the span case yields that $(r_1, R, r_2) : X \to Y$ is a right lifter in $\mathrm{Rel}\mathscr{E}$ if and only if $\forall_{r_1}$ exists.

Proposition 6.6. *Suppose* (2.2) *is a bipullback in* $\mathscr{M}$ *with* p *a groupoid fibration. If* n *is a lifter then so is* c *and, for all morphisms* $b : K \to B$*, the canonical 2-cell*

$$d \circ \mathrm{rif}(c, b) \Longrightarrow \mathrm{rif}(n, p \circ b)$$

is invertible.

Proof. For all $K \in \mathscr{M}$, we have a bipullback square

$$\begin{array}{ccc} \mathscr{M}(K,P) & \xrightarrow{\mathscr{M}(K,d)} & \mathscr{M}(K,A) \\ {\scriptstyle \mathscr{M}(K,c)}\downarrow & \overset{\cong}{\Longleftarrow} & \downarrow{\scriptstyle \mathscr{M}(K,n)} \\ \mathscr{M}(K,B) & \xrightarrow[\mathscr{M}(K,p)]{} & \mathscr{M}(K,C) \end{array}$$

in Cat with $\mathscr{M}(K, p)$ a groupoid fibration and $\mathscr{M}(K, n) \dashv \mathrm{rif}(n, -)$. By Proposition 5.3, $\mathscr{M}(K, c)$ has a right adjoint, so that c is a lifter, and $\mathscr{M}(K, d)\mathrm{rif}(c, -) \cong \mathrm{rif}(n, -)\mathscr{M}(K, p)$. Evaluating this last isomorphism at b we obtain the isomorphism displayed in the present Proposition. □

7. Distributivity pullbacks

We now recall Definitions 2.2.1 and 2.2.2 from Weber [30] of *pullback* and *distributivity pullback* around a composable pair (f, g) of morphisms in a category $\mathscr{C}$.

$$\begin{array}{ccccc} X & \xrightarrow{p} & Z & \xrightarrow{g} & A \\ {\scriptstyle q}\downarrow & & & & \downarrow{\scriptstyle f} \\ Y & & \xrightarrow[r]{} & & B \end{array} \qquad (7.16)$$

A pullback (p, q, r) around $Z \xrightarrow{g} A \xrightarrow{f} B$ is a commutative diagram (7.16) in which the span (q, X, gp) is a pullback of the cospan (r, B, f) in $\mathscr{C}$.

A morphism $t : (p', q', r') \to (p, q, r)$ of pullbacks around (f, g) is a morphism $t : Y' \to Y$ in $\mathscr{C}$ such that $rt = r'$. For such a morphism, using the pullback properties, it follows that there is a unique morphism $s : X' \to X$ in $\mathscr{C}$ such that $ps = p'$ and $qs = tq'$.[2] This gives a category $\mathrm{PB}(f, g)$. It is worth noting, also using the pullback properties, that the commuting square $qs = tq'$ exhibits the span (s, X', q') as a pullback of the cospan (q, Y, t).

The diagram (7.16) is called a distributivity pullback around (f, g) when it is a terminal object of the category $\mathrm{PB}(f, g)$.

$$\begin{array}{ccc} Y & \xrightarrow{r_*} & B \\ {\scriptstyle p_*q^*}\downarrow & \overset{\Longleftarrow}{\cong} & \downarrow{\scriptstyle f^*} \\ Z & \xrightarrow[g_*]{} & A \end{array} \tag{7.17}$$

Proposition 7.1. *Let $Z \xrightarrow{g} A \xrightarrow{f} B$ be a composable pair of morphisms in a category $\mathscr{C}$ with pullbacks. The diagram* (7.16) *is a pullback around (f, g) in the category $\mathscr{C}$ if and only if there is a square of the form* (7.17) *in the bicategory $\mathrm{Spn}\mathscr{C}$. The diagram* (7.16) *is a distributivity pullback around (f, g) in $\mathscr{C}$ if and only the diagram* (7.17) *is a bipullback in $\mathrm{Spn}\mathscr{C}$.*

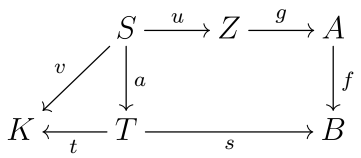
Proof. Passage around the top and right sides of (7.17) produces the pullback span of the cospan $Y \xrightarrow{r} B \xleftarrow{f} A$. Passage around the left and bottom sides produces the left and top sides of (7.16). That these passages be isomorphic says (7.16) is a pullback.

Suppose (7.16) is a distributivity pullback. We will show that (7.17) is a bipullback. Take a square of the form

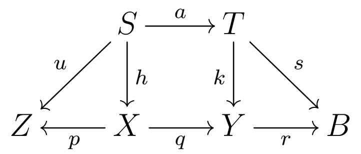
$$\begin{array}{ccc} K & \xrightarrow{s_*t^*} & B \\ {\scriptstyle u_*v^*}\downarrow & \overset{\Longleftarrow}{\cong} & \downarrow{\scriptstyle f^*} \\ Z & \xrightarrow[g_*]{} & A \end{array} \tag{7.18}$$

[2]Rather than the single t, Weber's definition takes the pair (s, t) as the morphism of pullbacks around (f, g) although he has a typographical error in the condition $rt = r'$.

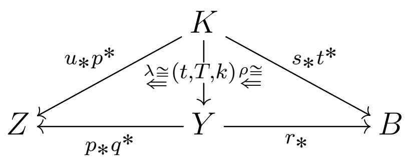
in $\mathrm{Spn}\mathscr{C}$. This square amounts to a diagram



in $\mathscr{C}$ in which the right-hand region is a pullback around (f, g). By the distributivity property, there exists a unique pair (h, k) such that the diagram

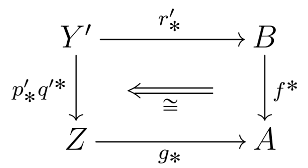


commutes; moreover, the square is a pullback. Thus we have a span morphism



which pastes onto (7.17) to yield (7.18); in fact ρ is an identity. To prove the bipullback 2-cell property, suppose we have span morphisms $e : (v', S', u') \to (v, S, u)$ and $j : (t', T', s') \to (t, T, s)$ such that composing the first with g_* is the composite of the second with f^*. Then, in obvious notation, $j : (u', a', s') \to (u, a, s)$ is a morphism in $\mathrm{PB}(f, g)$. By the terminal property of (p, q, r), we have $k' = kj$. This gives the span morphism $j : (t', T', k') \to (t, T, k)$ which is unique as required.

Conversely, suppose (7.17) is a bipullback. We must see that (p, q, r) is terminal in $\mathrm{PB}(f, g)$. Take another object (p', q', r') of $\mathrm{PB}(f, g)$. We have the square



which allows us to use the bipullback property to obtain a span morphism

$$(p'_* q'^*, Y', r'_*) \to (p_* q^*, Y, r_*)$$

in $\mathrm{Spn}\mathscr{C}$ which is compatible with the squares. Since the span $Y' \to Y$ in this morphism composes with r_* to be isomorphic to r'_*, we see that it has the form k_* for some $k : Y' \to Y$ in $\mathscr{C}$. Thus we have our unique $k : (p', q', r') \to (p, q, r)$ in $\mathrm{PB}(f, g)$. □

8. Polynomials in calibrated bicategories

Recall from [3] Section 7 that the *Poincaré category* $\Pi\mathscr{K}$ of a bicategory $\mathscr{K}$ has the same objects as $\mathscr{K}$, however, the homset $\Pi\mathscr{K}(H, K)$ is the set $\pi_0(\mathscr{K}(H, K))$ of undirected path components of the homcategory $\mathscr{K}(H, K)$. Composition is induced by composition of morphisms in $\mathscr{K}$. The *classifying category* $\mathrm{Cl}\mathscr{K}$ of $\mathscr{K}$ is obtained by taking isomorphism classes of morphisms in each category $\mathscr{K}(H, K)$. If $\mathscr{K}$ is locally groupoidal then $\Pi\mathscr{K}$ is equivalent to $\mathrm{Cl}\mathscr{K}$.

We adapt Bénabou's notion of "catégorie calibrée" [4] to our present purpose.

Definition 8.1. *A class $\mathscr{P}$ of morphisms, whose members are called* neat *("propres" in French), in a bicategory $\mathscr{M}$ is called a* calibration of $\mathscr{M}$ *when it satisfies the following conditions*

P0. all equivalences are neat and, if p is neat and there exists an invertible 2-cell $p \cong q$, then q is neat;

P1. for all neat p, the composite $p \circ q$ is neat if and only if q is neat;

P2. every neat morphism is a groupoid fibration;

P3. every cospan of the form

$$S \xrightarrow{p} Y \xleftarrow{n} T \ ,$$

with n a right lifter and p neat, has a bipullback (8.19) *in $\mathscr{M}$ with $\tilde{p}$ neat.*

$$\begin{array}{ccc} P & \xrightarrow{\tilde{p}} & T \\ {\scriptstyle \tilde{n}}\downarrow & \overset{\theta}{\underset{\cong}{\Leftarrow}} & \downarrow{\scriptstyle n} \\ S & \xrightarrow[p]{} & Y \end{array} \tag{8.19}$$

A bicategory equipped with a calibration is called calibrated.

Notice that the class GF *of all groupoid fibrations in any bicategory $\mathscr{M}$ satisfies all the conditions for a calibration except perhaps the bipullback existence part of P3 (automatically $\tilde{p}$ will be a groupoid fibration by (ii) of Proposition 5.2).*

A bicategory $\mathscr{M}$ is called polynomic *when* GF *is a calibration of $\mathscr{M}$.*

Definition 8.2. *Let $\mathscr{M} = (\mathscr{M}, \mathscr{P})$ be a calibrated bicategory. A* polynomial *(m, S, p) from X to Y in $\mathscr{M}$ is a span*

$$X \xleftarrow{m} S \xrightarrow{p} Y$$

in $\mathscr{M}$ with m a right lifter and p neat. A polynomial morphism (λ, h, ρ) : $(m, S, p) \to (m', S', p')$ *is a diagram*

$$\begin{array}{ccccc} & {\scriptstyle m}\swarrow & S & \searrow{\scriptstyle p} & \\ & \overset{\lambda}{\Leftarrow} & \downarrow{\scriptstyle h} & \overset{\rho\cong}{\Leftarrow} & \\ X & \xleftarrow[m']{} & S' & \xrightarrow[p']{} & Y \end{array} \tag{8.20}$$

in which ρ (but not necessarily λ) is invertible. By part (i) of Proposition 5.2 we know that h must be a groupoid fibration. (Indeed, by condition P1, h is neat; this is not really needed and is the only use made herein of the "only if" in P1.) We call (λ, h, ρ) strong *when λ is invertible. A 2-cell $\sigma : h \Rightarrow k : (m, S, p) \to (m', S', p')$ is a 2-cell $\sigma : h \Rightarrow k : S \to S'$ in $\mathscr{M}$ compatible with λ and ρ. By Proposition 3.5, we know that σ must be invertible. Write* $\mathrm{Poly}\mathscr{M}(X, Y)$ *for the Poincaré category of the bicategory of polynomials from X to Y so obtained.*

We write $\mathbf{h} = [\lambda, h, \rho] : (m, S, p) \to (m', S', p')$ for the isomorphism class of the polynomial morphism $(\lambda, h, \rho) : (m, S, p) \to (m', S', p')$. We also write λ_h and ρ_h when several morphisms are involved.

Proposition 8.3. *If $\mathscr{C}$ is a finitely complete category then the bicategory* $\mathrm{Spn}\mathscr{C}$ *is polynomic.*

Proof. Take a cospan $S \xrightarrow{p} Y \xleftarrow{n} T$ in $\mathrm{Spn}\mathscr{C}$ with p a groupoid fibration and $n = (n_1, F, n_2)$ a lifter. By Proposition 4.1, we can suppose p is actually p_* for some $p : S \to Y$ in $\mathscr{C}$. From Example 6.4, we know that n_1 is powerful. Form the pullback span $S \xleftarrow{f} P \xrightarrow{g} F$ of the cospan $S \xrightarrow{p} Y \xleftarrow{n_2} F$ in $\mathscr{C}$. By Proposition 5.4, we have a bipullback

$$\begin{array}{ccc} P & \xrightarrow{\quad g_* \quad} & F \\ {\scriptstyle f_*}\big\downarrow & \overset{\cong}{\Longleftarrow} & \big\downarrow{\scriptstyle (n_2)_*} \\ S & \xrightarrow[\quad p_* \quad]{} & Y\ . \end{array}$$

Since n_1 is powerful, Proposition 2.2.3 of Weber [30] implies we have a distributivity pullback

$$\begin{array}{ccc} V \xrightarrow{\ a\ } P & \xrightarrow{\ g\ } & F \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle n_1} \\ W & \xrightarrow[\quad r \quad]{} & T \end{array}$$

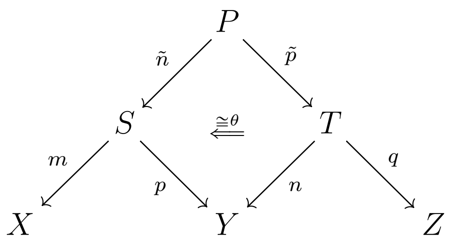
around (n_1, g). By Proposition 7.1, we have the bipullback

$$\begin{array}{ccc} W & \xrightarrow{\quad r_* \quad} & T \\ {\scriptstyle a_* q^*}\big\downarrow & \underset{\cong}{\Longleftarrow} & \big\downarrow{\scriptstyle (n_1)^*} \\ P & \xrightarrow[\quad g_* \quad]{} & F \end{array}$$

in $\mathrm{Spn}\mathscr{C}$. Paste this second bipullback on top of the first to obtain a bipullback of the cospan $S \xrightarrow{p_*} Y \xleftarrow{(n_2)_*(n_1)^*} T$ as required. □

The class of equivalences in any bicategory is a calibration. In Section 10, we will provide an example of a calibration strictly between equivalences and GP.

In a calibrated bicategory $\mathscr{M}$, polynomials can be composed as in the diagram (8.21); this is made possible by Example 6.2, Proposition 6.6, condition P3 and the "if" part of condition P1. Identity spans are also identity polynomials.

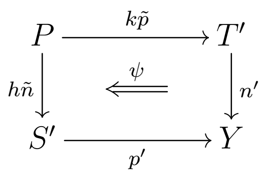


(8.21)

Indeed, this composition of polynomials is the effect on objects of functors

$$\circ : \mathrm{Poly}\mathscr{M}(Y,Z) \times \mathrm{Poly}\mathscr{M}(X,Y) \longrightarrow \mathrm{Poly}\mathscr{M}(X,Z)\ . \tag{8.22}$$

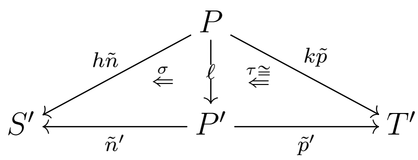
The effect on morphisms is defined using part (iii) of Proposition 5.2 as follows. Take morphisms $\mathbf{h} : (m, S, p) \to (m', S', p')$ and $\mathbf{k} : (n, T, q) \to (n', T', q')$. We have a square



in which

$$\psi = (n'k\tilde{p} \overset{\lambda_k\tilde{p}}{\Longrightarrow} n\tilde{p} \overset{\theta\cong}{\Longrightarrow} p\tilde{n} \overset{\rho_h\tilde{n}\cong}{\Longrightarrow} p'h\tilde{n})\ .$$

Now we use Proposition 5.2 to obtain, in obvious primed notation, a diagram



which leads to the polynomial morphism

$$((\lambda_h\tilde{n})(m'\sigma)), \ell, (q'\tau)(\rho_k\tilde{p}) : (m\tilde{n}, P, q\tilde{p}) \longrightarrow (m'\tilde{n}', P', q'\tilde{p}')$$

whose isomorphism class is the desired

$$\mathbf{k} \circ \mathbf{h} : (n, T, q) \circ (m, S, p) \to (n', T', q') \circ (m', S', p')\ .$$

Proposition 8.4. *There is a bicategory* $\mathrm{Poly}\mathscr{M}$ *of polynomials in a calibrated bicategory* $\mathscr{M}$*. The objects are those of* $\mathscr{M}$*, the homcategories are the* $\mathrm{Poly}\mathscr{M}(X,Y)$*. Composition is given by the functors* (8.22)*. The vertical and horizontal stacking properties of bipullbacks provide the associativity isomorphisms.*

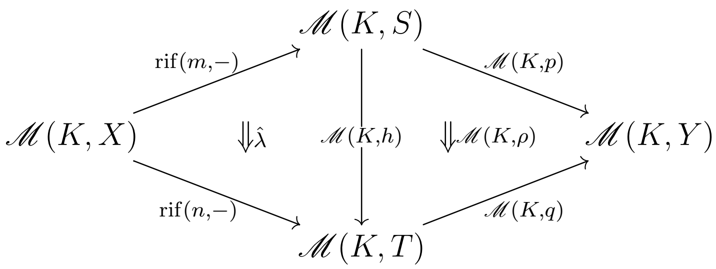
We write $\mathrm{Poly}_{\mathrm{s}}\mathscr{M}$ for the sub-bicategory of $\mathrm{Poly}\mathscr{M}$ obtained by restricting to the strong polynomial morphisms.

Example 8.5. If $\mathscr{C}$ is a finitely complete category then the bicategory $\mathrm{PolySpn}\mathscr{C}$ is biequivalent to the bicategory denoted by $\mathrm{Poly}_{\mathscr{C}}$ in Gambino-Kock [9] and by $\mathrm{Poly}(\mathscr{C})$ in Walker [28]. Moreover, $\mathrm{Poly}_{\mathrm{s}}\mathrm{Spn}\mathscr{C}$ is biequivalent to Walker's bicategory $\mathrm{Poly}_{\mathrm{c}}(\mathscr{C})$. Note that the isomorphism classes $\mathbf{h}$ of polynomial morphisms have canonical representatives of the form f_* (since each span $(u,S,v) : U \to V$ with u invertible is isomorphic to $(1_U, U, v\,u^{-1})$).

Proposition 8.6. *If the bicategory* $\mathscr{M}$ *is calibrated then, for each* $K \in \mathscr{M}$*, there is a pseudofunctor* $\mathbb{H}_K : \mathrm{Poly}\mathscr{M} \longrightarrow \mathrm{Cat}$ *taking the polynomial* $X \xleftarrow{m} S \xrightarrow{p} Y$ *to the composite functor*

$$\mathscr{M}(K,X) \xrightarrow{\mathrm{rif}(m,-)} \mathscr{M}(K,S) \xrightarrow{\mathscr{M}(K,p)} \mathscr{M}(K,Y)\ .$$

The 2-cell $\mathbf{h} : (m,S,p) \to (n,T,q)$ *in* $\mathrm{Poly}\mathscr{M}$ *is taken to the natural transformation obtained by the pasting*



where $\hat{\lambda}$ *is the mate under the adjunctions of the natural transformation* $\mathscr{M}(K,\lambda) : \mathscr{M}(K,n)\mathscr{M}(K,h) \Rightarrow \mathscr{M}(K,m)$.

Proof. We will show that polynomial 2-cells

$$\alpha : (\lambda_h, h, \rho_h) \Rightarrow (\lambda', h', \rho') : (m,S,p) \to (n,T,g)$$

are taken to identities. Since

$$\mathscr{M}(K,\lambda) = \Big(\mathscr{M}(K,n)\mathscr{M}(K,h) \xRightarrow{\mathscr{M}(K,n)\mathscr{M}(K,\alpha)} \mathscr{M}(K,n)\mathscr{M}(K,h') \xRightarrow{\mathscr{M}(K,\lambda')} \mathscr{M}(K,m)\Big)\ ,$$

it follows that

$$\hat{\lambda} = \Big(\mathscr{M}(K,h)\mathrm{rif}(m,-) \xRightarrow{\mathscr{M}(K,\alpha)\mathrm{rif}(m,-)} \mathscr{M}(K,h')\mathrm{rif}(m,-) \overset{\hat{\lambda}'}{\Rightarrow} \mathrm{rif}(n,-)\Big)\ .$$

Using this and that $\rho' = (g\alpha)\rho$, we have the identity

$$(g\hat{\lambda})(\rho\,\mathrm{rif}(m,u) = (g\hat{\lambda}')(\rho'\,\mathrm{rif}(m,u) : f\,\mathrm{rif}(m,u) \Longrightarrow g\,\mathrm{rif}(m,u)$$

induced by α as claimed.

That $\mathbb{H}_K$ is a pseudofunctor follows from Proposition 6.6. □

We can put somewhat more structure on the image of the pseudofunctor $\mathbb{H}_K$. Recall the definition (for example, in [21] Section 3) of the 2-category $\mathscr{V}$-Act of $\mathscr{V}$-actegories for a monoidal category $\mathscr{V}$.

Composition in $\mathscr{M}$ yields a monoidal structure on the category $\mathscr{V}_K = \mathscr{M}(K,K)$ and a right $\mathscr{V}_K$-actegory structure on each category $\mathbb{H}_K X = \mathscr{M}(K,X)$:

$$-\circ- : \mathscr{M}(K,X)\times\mathscr{M}(K,K) \longrightarrow \mathscr{K}(K,X)\ .$$

We can replace the codomain Cat of $\mathbb{H}_K$ in Proposition 8.6 by $\mathscr{V}_K$-Act.

To see this, we need a $\mathscr{V}_K$-actegory morphism structure on each functor

$$\mathbb{H}_K(m,S,p) = p\,\mathrm{rif}(m,-) : \mathscr{M}(K,X) \to \mathscr{M}(K,Y)\ .$$

However, for each $a \in \mathscr{V}_K$ and $u \in \mathscr{M}(K,X)$, we have the canonical $m\mathrm{rif}(m,u)a \overset{\varpi a}{\Longrightarrow} ua$ which induces a 2-cell $\mathrm{rif}(m,u)a \Rightarrow \mathrm{rif}(m,ua)$. Whiskering this with $p : S \to Y$, we obtain the component at (u,a) of a natural transformation:

$$\begin{array}{ccc}
\mathscr{M}(K,X)\times\mathscr{V}_K & \xrightarrow{\ -\circ-\ } & \mathscr{M}(K,X) \\
{\scriptstyle p\ \mathrm{rif}(m,-)\times 1_{\mathscr{V}_K}}\downarrow & \Longrightarrow & \downarrow{\scriptstyle p\ \mathrm{rif}(m,-)} \\
\mathscr{M}(K,Y)\times\mathscr{V}_K & \xrightarrow[\ -\circ-\ ]{} & \mathscr{M}(K,Y)\ .
\end{array}$$

The axioms for an actegory morphism are satisfied and each $\mathbb{H}_K(m,S,p)$ is a 2-cell in $\mathscr{V}_K$-Act.

In fact, we have a pseudofunctor

$$\mathbb{H} : \mathrm{Poly}\mathscr{M} \longrightarrow \mathrm{Hom}(\mathscr{M}^{\mathrm{op}}, \mathrm{Act})$$

where Act is the 2-category of pairs $(\mathscr{V},\mathscr{C})$ consisting of a monoidal category $\mathscr{V}$ and a category $\mathscr{C}$ on which it acts.

9. Bipullbacks from tabulations

Tabulations in a bicategory, in the sense intended here, appeared in [7] to characterize bicategories of spans.

For any bicategory $\mathscr{M}$, we write $\mathscr{M}_*$ for the sub-bicategory obtained by restricting to left adjoint morphisms. For each left adjoint morphism $f : X \to Y$ in $\mathscr{M}$, we write $f^* : Y \to X$ for a right adjoint.

Definition 9.1. *The bicategory $\mathscr{M}$ is said to* have tabulations from the terminal *when the following conditions hold:*

(i) the bicategory $\mathscr{M}_*$ has a terminal object 1 with the property that, for all objects U, the unique-up-to-isomorphism left-adjoint morphism $!_U : U \to 1$ is terminal in the category $\mathscr{M}(U,1)$;

(ii) for each morphism $u : 1 \to X$ in $\mathscr{M}$, there is a diagram (9.24)*, called a* tabulation *of u, in which $p : U \to X$ is a left adjoint morphism and such that the diagram*

$$\begin{array}{ccc} \mathscr{M}(K,U) & \longrightarrow & \mathbf{1} \\ {\scriptstyle \mathscr{M}(K,p)}\downarrow & \stackrel{\lambda}{\Longrightarrow} & \downarrow{\scriptstyle \lceil u!_K \rceil} \\ \mathscr{M}(K,X) & \xrightarrow[1_{\mathscr{M}(K,X)}]{} & \mathscr{M}(K,X) \end{array} \qquad (9.23)$$

where the natural transformation λ has component $pw \xrightarrow{\rho_u w} u!_U w \xrightarrow{u!} u!_K$ at $w \in \mathscr{M}(K,U)$, exhibits $\mathscr{M}(K,U)$ as a bicategorical comma object in Cat.

$$\begin{array}{ccc} & U & \\ {}^{!_U}\swarrow & \overset{\rho_u}{\Longleftarrow} & \searrow^{p} \\ 1 & \xrightarrow[u]{} & X \end{array} \tag{9.24}$$

Remark 9.2. (a) The bicategorical comma property of the diagram (9.23) implies p is an er-fibration in $\mathscr{M}$.

(b) Notice that condition (ii) in this Definition does agree with combined conditions T1 and T2 in the definition of tabulation in [7] for morphisms with domain 1. This is because all left adjoints $K \to 1$ are isomorphic to $!_K$ using condition (i) of our Definition.

(c) Using (b) and Proposition 1(d) of [7], we see that the mate $p!^*_U \Rightarrow u$ of $\rho_u : p \Rightarrow u!_U$ is invertible. Let us denote the unit of the adjunction $!_U \dashv !^*_U$ by $\eta_U : 1_U \Longrightarrow !^*_U\, !_U$. So we can replace u up to isomorphism by $p!^*_U$ and ρ_u by $p\eta_U$.

(d) If $\mathscr{M}$ has tabulations from the terminal and we have a morphism $p : U \to X$ such that (9.23) has the bicategorical comma property with $u = p!^*_U : 1 \to X$ and $\rho_u = p\eta_U$ then p is a left adjoint. This is because a tabulation of $u : 1 \to X$ does exist in which the right leg is a left adjoint and the comma property implies the right leg is isomorphic to pe for some equivalence e.

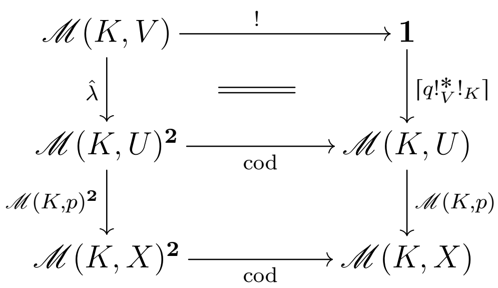
(e) Another way to express the comma object condition (9.23) is to say (9.25) is a bipullback for $\hat{\lambda}w = \lambda_w = (pw \xrightarrow{\rho_u w} u!_U w \xrightarrow{u!} u!_K)$.

$$\begin{array}{ccc} \mathscr{M}(K,U) & \xrightarrow{\quad ! \quad} & \mathbf{1} \\ {\scriptstyle\hat{\lambda}}\downarrow & = & \downarrow{\scriptstyle\ulcorner u!_K \urcorner} \\ \mathscr{M}(K,X)^{\mathbf{2}} & \xrightarrow[\mathrm{cod}]{} & \mathscr{M}(K,X) \end{array} \tag{9.25}$$

Let Tab be the class of morphisms $p : U \to X$ in $\mathscr{M}$ which occur in a tabulation (9.24).

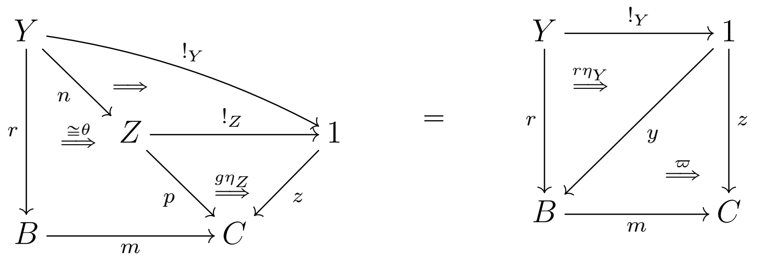
Theorem 9.3. *The class* Tab *is a calibration for any bicategory* $\mathscr{M}$ *which has tabulations from the terminal.*

Proof. We must prove properties P0–P3 for a calibration. Property P0 is obvious. For P1, take $V \xrightarrow{q} U \xrightarrow{p} X$ with $p \in \mathrm{Tab}$. By (c) and (d) of Remark 9.2, p can be assumed to come from the tabulation of $u = p!_U^*$. If q is to come from a tabulation it must be of $v = q!_V^*$. If pq is to come from a tabulation it must be of $w = pq!_V^* = pv$. Contemplate the following diagram in which the $\hat{\lambda}$ comes from v.



By Remark 9.2(a), p is a groupoid fibration; incidentally, this gives P2. So the bottom square is a bipullback (see Proposition 5.1). Therefore, the top square is a bipullback if and only if the pasted square is a bipullback. By Remark 9.2(d) and (e), this says $q \in \mathrm{Tab}$ if and only if $pq \in \mathrm{Tab}$. This proves P1.

It remains to prove P3. We start with a cospan $Z \xrightarrow{p} C \xleftarrow{m} B$ with m a right lifter and $p \in \mathrm{Tab}$. Put $z = p!_Z^* : 1 \to C$ and tabulate $y = \mathrm{rif}(m, z) : 1 \to B$ as $y = r\, !_Y^*$ for $r : Y \to B$ in Tab. Using the tabulation property of Z, we induce n and invertible θ as in the diagram (9.26) in which the triangle containing ϖ exhibits the right lifting $\mathrm{rif}(m, z)$.



(9.26)

It is the region containing θ in (9.26) that we will show is a bipullback. For all $K \in \mathscr{M}$, we must show that the left-hand square in the diagram

$$\begin{array}{ccccc} \mathscr{M}(K,Y) & \xrightarrow{\mathscr{M}(K,n)} & \mathscr{M}(K,Z) & \longrightarrow & \mathbf{1} \\ {\scriptstyle \mathscr{M}(K,r)}\downarrow & \overset{\cong}{\Longrightarrow} & {\scriptstyle \mathscr{M}(K,p)}\downarrow & \overset{\lambda}{\Longrightarrow} & \downarrow{\scriptstyle \lceil z!_K \rceil} \\ \mathscr{M}(K,B) & \xrightarrow[\mathscr{M}(K,m)]{} & \mathscr{M}(K,C) & \xrightarrow[1_{\mathscr{M}(K,C)}]{} & \mathscr{M}(K,C) \end{array}$$

is a bipullback. However, the right-hand square has the comma property. So the bipullback property of the left-hand square is equivalent to the comma property of the pasted diagram. However, using (9.26), we see that the pasted composite is equal to the pasted composite

$$\begin{array}{ccccc} \mathscr{M}(K,Y) & \longrightarrow & \mathbf{1} & \longrightarrow & \mathbf{1} \\ {\scriptstyle \mathscr{M}(K,r)}\downarrow & \overset{\lambda}{\Longrightarrow} & {\scriptstyle \lceil y!_K \rceil}\downarrow & \overset{\lceil \varpi!_K \rceil}{\Longrightarrow} & \downarrow{\scriptstyle \lceil z!_K \rceil} \\ \mathscr{M}(K,B) & \xrightarrow[1_{\mathscr{M}(K,B)}]{} & \mathscr{M}(K,B) & \xrightarrow[\mathscr{M}(K,m)]{} & \mathscr{M}(K,C) \end{array}$$

Here, the left-hand square has the comma property and $y!_K$ is the value of the right adjoint $\mathrm{rif}(m,-)$ to $\mathscr{M}(K,m)$ at $z!_K$. So the pasted composite does have the comma property, as required. □

10. Calibrations of $\mathrm{Spn}\mathscr{C}$, of $\mathrm{Rel}\mathscr{E}$ and of Mod

If $\mathscr{C}$ is a category with finite limits, its terminal object $\mathbf{1}$ clearly has the property (i) in the definition of tabulations from the terminal. Then, from Remark 9.2 and [7], we know that $\mathrm{Spn}\mathscr{C}$ has tabulations from the terminal.

A span from $\mathbf{1}$ to X has the form $(!, U, p) : \mathbf{1} \to X$ for some $p : U \to X$ in $\mathscr{C}$. A tabulation of the span is provided by the diagram

$$\begin{array}{ccc} & U & \\ {\scriptstyle !_{U*}}\swarrow & \overset{\rho_u}{\Longleftarrow} & \searrow{\scriptstyle p_*} \\ 1 & \xrightarrow[(!,U,p)]{} & X \end{array} .$$

It follows that Tab consists of spans of the form p_* for some morphism p in $\mathscr{C}$. Using Proposition 4.1, we deduce:

Proposition 10.1. *For the bicategory* $\mathrm{Spn}\mathscr{C}$ *of spans in a finitely complete category* $\mathscr{C}$*,* $\mathrm{Tab} = \mathrm{GF}$.

With this, Theorem 9.3 provides another proof of Proposition 8.3.

Here is the result in the case of bicategories of relations.

Proposition 10.2. *The bicategory* $\mathrm{Rel}\mathscr{E}$ *of relations in a regular category* $\mathscr{E}$ *has tabulations from the terminal. Moreover the calibration* Tab *of* $\mathrm{Rel}\mathscr{E}$ *consists of those relations isomorphic to* p_* *for some monomorphism* p *in* $\mathscr{E}$*.*

Proof. The existence of tabulations was shown in [7]. The right leg of a tabulation of $(!, R, p) : 1 \to X$ is of course $p : R \to X$ which must be a monomorphism for the span $(!, R, p)$ to be a relation. □

With this and Example 6.5, we obtain a different notion of "polynomials" in a regular category; again they are the morphisms of a bicategory $\mathrm{PolyRel}\mathscr{E}$.

Example 10.3. An elementary topos $\mathscr{E}$ admits two basic constructions, the power object (or relation classifier) $\mathcal{P}X$ and the partial map classifier $\widetilde{X}$; see [15, 11]. Both define object assignments for monads on $\mathscr{E}$. There is a distributive law $d_X : \mathcal{P}\widetilde{X} \to \widetilde{\mathcal{P}X}$ between the two monads. We claim that, for a topos $\mathscr{E}$, the classifying category of $\mathrm{PolyRel}\mathscr{E}$ is equivalent to the opposite of the Kleisli category $\mathscr{E}_{\widetilde{\mathcal{P}(-)}}$ for the composite monad $X \mapsto \widetilde{\mathcal{P}X}$. To see this, we need some detail on the monads involved.

The (covariant) power endofunctor $\mathcal{P}$ on $\mathscr{E}$ is defined on morphisms $u : X \to Y$ by direct image $\exists_u : \mathcal{P}X \to \mathcal{P}Y$. The partial map classifier takes u to $\widetilde{u} : \widetilde{X} \to \widetilde{Y}$ corresponding to the partial map $u : \widetilde{X} \to Y$ which is u with X as domain of definition. The unit $\sigma : 1_{\mathscr{E}} \Longrightarrow \mathcal{P}$ for the monad $\mathcal{P}$ has components $\sigma_X : X \Longrightarrow \mathcal{P}X$ corresponding to the identity relation on X. Similarly, the unit $\eta : 1_{\mathscr{E}} \Longrightarrow \widetilde{(-)}$ for the monad $\widetilde{(-)}$ has components $\eta_X : X \Longrightarrow \widetilde{X}$ corresponding to the identity partial map on X.

Rather than examine the multiplications for these monads, we take the "no iteration" or "mw-" point of view (see [19, 1, 20, 17]) from which the Kleisli bicategory is easily obtained. For $\mathcal{P}$, the extra data needed are functions

$$\mathscr{E}(X, \mathcal{P}Y) \longrightarrow \mathscr{E}(\mathcal{P}X, \mathcal{P}Y) \, ;$$

they take $X \xrightarrow{f} \mathcal{P}Y$ to the supremum-preserving extension $\mathcal{P}X \xrightarrow{f'} \mathcal{P}Y$ of f along σ_X. The Kleisli category for $\mathcal{P}$ is the classifying category $\mathrm{ClRel}\mathscr{E}$ of the bicategory of relations in $\mathscr{E}$. For $\widetilde{(-)}$, the extra data needed are functions

$$\mathscr{E}(X, \widetilde{Y}) \longrightarrow \mathscr{E}(\widetilde{X}, \widetilde{Y}) \;;$$

they take $X \xrightarrow{f} \widetilde{Y}$ to the bottom-preserving extension $\widetilde{X} \xrightarrow{f_1} \widetilde{Y}$ of f along η_X. The Kleisli category for $\widetilde{(-)}$ is the classifying category $\mathrm{ClPar}\mathscr{E}$ of the bicategory $\mathrm{Par}\mathscr{E}$ of partial maps in $\mathscr{E}$: it is the subbicategory of $\mathrm{Spn}\mathscr{E}$ whose morphisms are restricted to those spans $X \xleftarrow{i} U \xrightarrow{f} Y$ for which the left leg i is a monomorphism.

To give a distributive law $d_X : \mathcal{P}\widetilde{X} \to \widetilde{\mathcal{P}X}$ is equally to give an extension $\widehat{\mathcal{P}}$ of the monad $\mathcal{P}$ to a monad on the Kleisli category $\mathrm{ClPar}\mathscr{E}$ of $\widetilde{(-)}$. Indeed, we can extend $\mathcal{P}$ to a pseudomonad $\widehat{\mathcal{P}}$ on $\mathrm{Par}\mathscr{E}$. We use the facts that $\mathcal{P}$ preserves pullbacks of monomorphisms along arbitrary morphisms and that the square

$$\begin{array}{ccc} U & \xrightarrow{\quad i \quad} & X \\ {\scriptstyle \sigma_U}\downarrow & & \downarrow{\scriptstyle \sigma_X} \\ \mathcal{P}U & \xrightarrow[\exists_i]{\quad\quad} & \mathcal{P}X \end{array}$$

is a pullback when i is a monomorphism. These imply that we can define $\widehat{\mathcal{P}}$ on objects to be $\mathcal{P}$ and on partial maps by

$$\widehat{\mathcal{P}}(X \xleftarrow{i} U \xrightarrow{f} Y) = (\mathcal{P}X \xleftarrow{\exists_i} \mathcal{P}U \xrightarrow{\exists_f} \mathcal{P}Y)$$

to obtain a pseudofunctor, and that $X \xleftarrow{1_X} X \xrightarrow{\sigma_X} \mathcal{P}X$ provides a pseudonatural unit. Again, rather than a multiplication for $\widehat{\mathcal{P}}$, we supply the functor

$$\mathrm{Par}\mathscr{E}(X, \mathcal{P}Y) \longrightarrow \mathrm{Par}\mathscr{E}(\mathcal{P}X, \mathcal{P}Y)\,,\ (X \xleftarrow{i} U \xrightarrow{f} \mathcal{P}Y) \mapsto (\mathcal{P}X \xleftarrow{\exists_i} \mathcal{P}U \xrightarrow{f'} \mathcal{P}Y)\,.$$

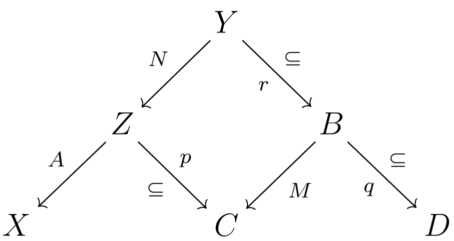
The Kleisli category $\mathscr{E}_{\widetilde{\mathcal{P}(-)}}$ of the composite monad $\widetilde{\mathcal{P}(-)}$ on $\mathscr{E}$ is the classifying category for the Kleisli bicategory $(\mathrm{Par}\mathscr{E})_{\widehat{\mathcal{P}}}$ of the pseudomonad $\widehat{\mathcal{P}}$ on $\mathrm{Par}\mathscr{E}$.

The claim at the beginning of this example will follow after we see that $\mathrm{PolyRel}\mathscr{E}$ is biequivalent to the opposite of $(\mathrm{Par}\mathscr{E})_{\widehat{\mathcal{P}}}$. To see this, notice that the objects of the two bicategories are the same: they are the objects of $\mathscr{E}$. Also, we have the pseudonatural equivalence

$$\mathrm{PolyRel}\mathscr{E}(X, C) \simeq \mathrm{Par}\mathscr{E}(C, \mathcal{P}X)$$

of hom categories under which the polynomial $X \xleftarrow{(a_1,A,a_2)} Z \xrightarrow{(1_Z,Z,p)} C$ corresponds to the partial map $C \overset{p}{\leftarrow} Z \overset{a}{\rightarrow} \mathcal{P}X$ where a classifies the relation (a_1, A, a_2).

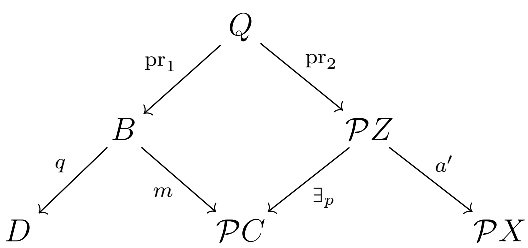
What remains is to see that span composition of polynomials transports to Kleisli composition. We shall write for the case $\mathscr{E} = \mathrm{Set}$ and appeal to topos internal logic to justify the argument in general. First we look at composition in $\mathrm{PolyRel}\mathscr{E}$. So that we can make use of the notation in the construction of pseudopullback in (9.26), we look at the following span composite.



The object Y and relation N are obtained from the subobjects $Z \subseteq C$ and $B \subseteq D$, and relations A and M. Referring to (9.26), we see that

$$Y = \{b \in B : bMc \text{ implies } c \in Z\}$$

and N is the restriction of the relation M. Now we look at composition of the corresponding morphisms in the Kleisli bicategory; this is given by the diagram



in which the diamond is a pullback while m and a classify the relations M and A. We therefore have an isomorphism

$$Q = \{b \in B : m(b) \subseteq Z\} \cong Y$$

under which $q \circ \mathrm{pr}_1$ and $a' \circ \mathrm{pr}_2$ transport to $q \circ r$ and the classifier of $A \circ N$.

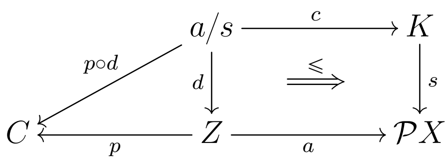
Incidentally, using this biequivalence, we can view the pseudofunctor $\mathbb{H}_K$ of Proposition 8.6 as a pseudofunctor

$$(\mathrm{Par}\mathscr{E})^{\mathrm{op}}_{\hat{\mathcal{P}}} \longrightarrow \mathrm{Ord}$$

into ordered sets taking $C \overset{p}{\leftarrow} Z \overset{a}{\rightarrow} \mathcal{P}X$ to the order-preserving function

$$\mathrm{Rel}\mathscr{E}(K, X) \xrightarrow{\mathrm{rif}(a,-)} \mathrm{Rel}\mathscr{E}(K, Z) \xrightarrow{p\circ -} \mathrm{Rel}\mathscr{E}(K, C)$$

whose value at a relation $(s_1, S, s_2) : K \to X$ is the relation $(c, a/s, p \circ d) : K \to C$ as in the diagram



in which the square has the comma property and s classifies the relation (s_1, S, s_2).

Next we look at the bicategory $\mathrm{Mod} = \mathscr{V}\text{-}\mathrm{Mod}$, where $\mathscr{V} = \mathrm{Set}$; see Example 2.2.

Proposition 10.4. *The bicategory* Mod *has tabulations from the terminal.*

Proof. We need to check the validity of conditions (i) and (ii) defining the having of tabulations from the terminal. Since the terminal object $\mathbf{1}$ of Cat is Cauchy complete (idempotents split) [18], every left-adjoint module $K \to \mathbf{1}$ is isomorphic to $!_{K*} : K \to \mathbf{1}$ where $!_K : K \to \mathbf{1}$ is the unique functor. The module $!_{K*}$, as a functor $\mathbf{1}^{\mathrm{op}} \times K \to \mathrm{Set}$, is constant at a one-point set. So condition (i) holds.

For condition (ii), take a module $u : \mathbf{1} \to X$ regarded as a functor $u : X^{\mathrm{op}} \to \mathrm{Set}$. Form the comma category U of u as in the square

$$\begin{array}{ccc} U & \longrightarrow & \mathbf{1} \\ {\scriptstyle p}\downarrow & \overset{\rho_u}{\Longrightarrow} & \downarrow{\scriptstyle \lceil u \rceil} \\ X & \underset{\mathrm{yon}_X}{\longrightarrow} & [X^{\mathrm{op}}, \mathrm{Set}]\ . \end{array} \tag{10.27}$$

The natural transformation in the square has components $\rho_{ux} : X(x, p-) \to ux$ which reinterprets as a 2-cell

$$\begin{array}{ccc} & U & \\ {\scriptstyle !_{U*}}\swarrow & \overset{\rho_u}{\Longleftarrow} & \searrow{\scriptstyle p_*} \\ 1 & \underset{u}{\longrightarrow} & X \end{array}$$

in Mod. In fact, we see that $U \xrightarrow{p} X$ is $\wr u$ in the sense of Proposition 3.2. So $\wr u!_{K*}$ is $K^{\mathrm{op}} \times U \xrightarrow{1_{K^{\mathrm{op}}} \times p} K^{\mathrm{op}} \times X$. Using Proposition 3.2, we see that the comma construction $\mathrm{Mod}(K, X)/\lceil u!_{K*} \rceil$ is biequivalent to

$$\mathrm{ERFib}(K^{\mathrm{op}} \times X)/(K^{\mathrm{op}} \times U \xrightarrow{1_{K^{\mathrm{op}}} \times p} K^{\mathrm{op}} \times X) \sim \mathrm{ERFib}(K^{\mathrm{op}} \times U)$$

and, again by Proposition 3.2, this is biequivalent to $\mathrm{Mod}(K, U)$, as required for the comma property of diagram (10.27). □

Corollary 10.5. *The bicategory* Mod *is calibrated by* Tab*. All morphisms are right lifters and, up to equivalence, the neat morphisms are those of the form* $p_* : E \to B$ *where* p *is a discrete fibration.*

A polynomial from X to Y in Mod is therefore a span $X \overset{m}{\longleftarrow} E \xrightarrow{p} Y$ where m is a module from E to X and p is a discrete fibration. The module m is equivalent to a functor $E \longrightarrow \mathrm{Psh}X$, so, as we see from page 312 of [24], the polynomial is equivalent to a parametric right adjoint functor $\mathrm{Psh}X \longrightarrow \mathrm{Psh}Y$. As pointed out by Weber, in Remark 2.12 of [29], this is equivalent to a polynomial $X \overset{d}{\leftarrow} D \xrightarrow{c} E \xrightarrow{p} Y$ in the category Cat where $X \overset{d}{\leftarrow} D \xrightarrow{c} E$ is a two-sided discrete fibration and p is a discrete fibration.

Example 10.6. The bicategory $\mathrm{PolyMod}$ is biequivalent to the opposite of the Kleisli bicategory for the composite $X \mapsto \mathrm{Fam}^{\mathrm{op}}[X^{\mathrm{op}}, \mathrm{Set}]$ of the colimit-completion pseudomonad and the product-completion pseudomonad (modulo obvious size issues).

To see this, note that the coproduct completion $\mathrm{Fam}X$ of a category X can be efficiently described, in the terminology of Section 4 of [13], as the lax comma object

$$\begin{array}{ccc} \mathrm{Fam}X & \xrightarrow{\;!\;} & \mathbf{1} \\ {\scriptstyle \mathrm{forget}}\downarrow & \overset{\lambda}{\rightsquigarrow} & \downarrow{\scriptstyle X} \\ \mathrm{Set} & \xrightarrow[\subset]{} & \mathrm{Cat} \end{array}$$

so that functors $f : Y \to \mathrm{Fam}X$ are in 2-natural bijection with pairs $(\tilde{f}, \phi)$ where $\tilde{f} : Y \to \mathrm{Set}$ is a functor and $\phi : \tilde{f} \rightsquigarrow Z!_Y$ is a lax natural transformation. The Grothendieck fibration construction transforms such $(\tilde{f}, \phi)$ into a commutative triangle

$$\begin{array}{ccc} E & \xrightarrow{(\hat{f}, q)} & X \times Y \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\ & Y & \end{array}$$

for which the data are a discrete opfibration $q : E \to Y$ and an arbitrary functor $\hat{f} : E \to X$; as Lawvere pointed out early in the decade of the 1970s, we might think of this as a *2-dimensional partial map* $(q, \hat{f}) : Y \to X$ between categories. This gives a pseudonatural equivalence of categories

$$[Y, \mathrm{Fam}X] \simeq 2\mathrm{Par}(Y, X)$$

The product completion of X is $\mathrm{Fam}^{\mathrm{op}}X := \mathrm{Fam}(X^{\mathrm{op}})^{\mathrm{op}}$. Objects (I, x) of $\mathrm{Fam}^{\mathrm{op}}X$ are functors $x : I \to X$ from a small discrete category (set) I to X, Morphisms $(u, \xi) : (I, x) \to (J, y)$ are diagrams in Cat of the form

$$\begin{array}{ccc} I & \xleftarrow{\;u\;} & J \\ & {\scriptstyle x}\searrow \overset{\theta}{\Longrightarrow} \swarrow{\scriptstyle y} & \\ & X & \end{array} \quad .$$

Functors $f : Y \to \mathrm{Fam}^{\mathrm{op}}X$ correspond, up to equivalence, to spans $Y \xleftarrow{p} E \xrightarrow{g} X$ where p is a discrete fibration; we shall call such a span a *2-dimensional partial opmap* from Y to X. This gives a pseudonatural equivalence of categories

$$[Y, \mathrm{Fam}^{\mathrm{op}}X] \simeq 2\mathrm{Parop}_{\mathrm{op}}(Y, X)\ . \tag{10.28}$$

While there is a size problem with $\mathrm{Fam}^{\mathrm{op}}$ as a monad on Cat, we do have what would be its Kleisli bicategory, namely, $2\mathrm{Par}_{\mathrm{op}}$ whose objects are small categories, whose homs are the categories $2\mathrm{Par}_{\mathrm{op}}(Y, X)$, and whose composition is that of spans. There is also a size problem with Psh as a monad

$$X \xrightarrow{k} Y \ \mapsto\ [X^{\mathrm{op}}, \mathrm{Set}] \xrightarrow{\exists_k = \mathrm{lan}(k^{\mathrm{op}}, -)} [Y^{\mathrm{op}}, \mathrm{Set}]$$

on Cat but we do have its Kleisli bicategory Mod whose objects are small categories, whose homs are given by $\mathrm{Mod}(Y, X) = [X^{\mathrm{op}} \times Y, \mathrm{Set}]$, and composition is that of modules (see Example 2.2). Modulo the size problem, the monad Psh extends to a monad $\widehat{\mathrm{Psh}}$ on $2\mathrm{Par}_{\mathrm{op}}$: this is one way of seeing that we have a distributive law $\partial : \mathrm{Psh}\mathrm{Fam}^{\mathrm{op}} \Longrightarrow \mathrm{Fam}^{\mathrm{op}}\mathrm{Psh}$. The value of $\widehat{\mathrm{Psh}}$ at a 2-partial opmap $Y \xleftarrow{p} E \xrightarrow{g} X$ is

$$[Y^{\mathrm{op}}, \mathrm{Set}] \xleftarrow{\exists_p} [E^{\mathrm{op}}, \mathrm{Set}] \xrightarrow{\exists_g} [X^{\mathrm{op}}, \mathrm{Set}]\ . \tag{10.29}$$

There are several things to be said about this most of which are better understood by looking at the equivalent span where presheaves are replaced by discrete fibrations:

$$\mathrm{DFib}Y \xleftarrow{p_*} \mathrm{DFib}E \xrightarrow{g_*} \mathrm{DFib}X\ .$$

Here $g_* : \mathrm{DFib}E \to \mathrm{DFib}X$ is defined on the discrete fibration $r : F \to E$ by factoring the composite $g \circ r : F \to X$ as $g \circ r = s \circ j$ where $j : F \to F'$ is final and $s : F' \to X$ is a discrete fibration; this uses the comprehensive factorization of functors described in [26, 25]. In particular, $p_*(r) = p \circ r$ since the composite is already a discrete fibration. It follows that, if $p : E \to X$ is a discrete fibration then so is $p_* : \mathrm{DFib}E \to \mathrm{DFib}X$. Also, if further, the left square

$$\begin{array}{ccc} F & \xrightarrow{\ g\ } & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[\ f\ ]{} & X \end{array} \qquad \begin{array}{ccc} \mathrm{DFib}F & \xrightarrow{\ g_*\ } & \mathrm{DFib}E \\ {\scriptstyle q_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ \mathrm{DFib}Y & \xrightarrow[\ f_*\ ]{} & \mathrm{DFib}X \end{array}$$

is a pullback, then so is the right square. Using this, we conclude that (10.29) is again a 2-partial opmap and that $\widehat{\mathrm{Psh}}$ is a pseudofunctor. To see that the unit for the monad Psh, which is given by Yoneda embedding $\mathrm{y}_X : X \to \mathrm{Psh}X$, lifts to $2\mathrm{Par}^{\mathrm{op}}$, we must see that y_X seen as a 2-partial opmap, is pseudonatural in $X \in 2\mathrm{Par}_{\mathrm{op}}$; this follows from the fact that, for all discrete fibrations $p : E \to X$, the square

$$\begin{array}{ccc} E & \xrightarrow{E/-} & \mathrm{DFib}E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_*} \\ X & \xrightarrow[X/-]{} & \mathrm{DFib}X \end{array}$$

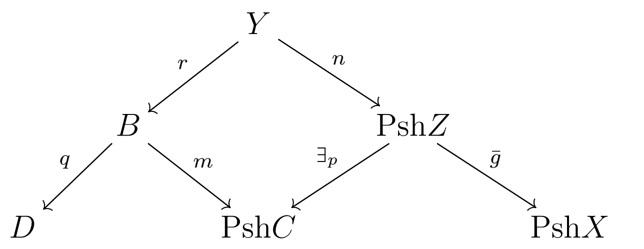
is a pullback, which is another form of the Yoneda Lemma. Rather than examine the multiplication for $\widehat{\mathrm{Psh}}$, as in Example 10.3, we take the "no iteration" or "mw-" point of view. We need to supply functors

$$\mathrm{P} : 2\mathrm{Par}_{\mathrm{op}}(C, \mathrm{Psh}X) \longrightarrow 2\mathrm{Par}_{\mathrm{op}}(\mathrm{Psh}C, \mathrm{Psh}X) \ . \qquad (10.30)$$

An object of the domain is a span $C \xleftarrow{p} Z \xrightarrow{g} \mathrm{Psh}X$ where p is a discrete fibration. Define

$$\mathrm{P}(C \xleftarrow{p} Z \xrightarrow{g} \mathrm{Psh}X) = (\mathrm{Psh}C \xleftarrow{p_*} \mathrm{Psh}Z \xrightarrow{\bar{g}} \mathrm{Psh}X)$$

where $\bar{g} = \mathrm{lan}(\mathrm{y}_Z, g)$ is the colimit-preserving extension of g. Thus the composite of $D \xleftarrow{q} B \xrightarrow{m} \mathrm{Psh}C$ and $C \xleftarrow{p} Z \xrightarrow{g} \mathrm{Psh}X$ in the Kleisli bicategory $(2\mathrm{Par}_{\mathrm{op}})_{\widehat{\mathrm{Psh}}}$ of the pseudomonad $\widehat{\mathrm{Psh}}$ is the following composite of spans in Cat.



Suppose the left square in diagram (10.31) is in Mod and the right is in CAT.

$$\begin{array}{ccc} K & \xrightarrow{t_*} & B \\ {\scriptstyle h}\downarrow & \overset{\theta}{\underset{\cong}{\Rightarrow}} & \downarrow{\scriptstyle m} \\ Z & \xrightarrow[p_*]{} & C \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{t} & B \\ {\scriptstyle h}\downarrow & \overset{\phi}{\underset{\cong}{\Rightarrow}} & \downarrow{\scriptstyle m} \\ \mathrm{Psh}Z & \xrightarrow[\exists_p]{} & \mathrm{Psh}C \end{array} \qquad (10.31)$$

An easy evaluation shows that isomorphisms θ are in bijection with isomorphisms ϕ. It follows that Y, r and n agree with the construction in (9.26) and we have the biequivalence

$$\mathrm{PolyMod}^{\mathrm{op}} \simeq (2\mathrm{Par}_{\mathrm{op}})_{\widehat{\mathrm{Psh}}}$$

from which we obtain the claim of this example's first paragraph.

Incidentally, using this biequivalence, we can view the pseudofunctor $\mathbb{H}_K$ of Proposition 8.6 as the pseudofunctor

$$(2\mathrm{Par}_{\mathrm{op}})^{\mathrm{op}}_{\widehat{\mathrm{Psh}}} \longrightarrow \mathrm{Cat}$$

taking the morphism $Y \xleftarrow{p} S \xrightarrow{m} \mathrm{Psh}$ to the functor

$$[K, \mathrm{Psh}X] \longrightarrow [K, \mathrm{Psh}Y]\ ,\ \ell \mapsto \bar{\ell} \tag{10.32}$$

where

$$(\bar{\ell}k)y = \sum_{s \in S_y} \mathrm{Psh}X(ms, \ell k)$$

for $k \in K$, for $y \in Y$ and for S_y the fibre of $p : S \to Y$ over y.

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ON TRUNCATED QUASI-CATEGORIES

Alexander CAMPBELL and Edoardo LANARI

Résumé. Pour chaque $n \geq -1$, une quasi-catégorie est dite n-tronquée si ses espaces de morphismes sont des $(n-1)$-types d'homotopie. Dans ce travail, nous étudions la structure de catégorie de modèles pour les quasi-catégories n-tronquées. Nous montrons que cette structure peut être construire comme une localisation de Bousfield de la structure de catégorie de modèles de Joyal pour les quasi-catégories par rapport à l'inclusion du bord du $(n+2)$-simplexe. En outre, nous établissons l'équivalence de Quillen attendue entre les catégories et les quasi-catégories 1-tronquées, ainsi qu'entre les quasi-catégories n-tronquées et les $(n,1)$-Θ-espaces de Rezk.

Abstract. For each $n \geq -1$, a quasi-category is said to be n-truncated if its hom-spaces are $(n-1)$-types. In this paper we study the model structure for n-truncated quasi-categories, which we prove can be constructed as the Bousfield localisation of Joyal's model structure for quasi-categories with respect to the boundary inclusion of the $(n+2)$-simplex. Furthermore, we prove the expected Quillen equivalences between categories and 1-truncated quasi-categories and between n-truncated quasi-categories and Rezk's $(n,1)$-Θ-spaces.

Keywords. Quasi-category, truncated quasi-category, homotopy n-type, Bousfield localisation, Quillen model category, complete Segal space.

Mathematics Subject Classification (2020). 18N40, 18N50, 18N55, 18N60.

Contents

1. Introduction

Quasi-categories were introduced by Boardman and Vogt [5, §IV.2], and were developed by Joyal [14, 15] and Lurie [20] among others as a model for $(\infty, 1)$-categories: (weak) infinite-dimensional categories in which every morphism above dimension 1 is (weakly) invertible. Among the $(\infty, 1)$-categories are the $(n, 1)$-categories, which have no non-identity morphisms above dimension n.[1] In [20, §2.3.4], Lurie identified the quasi-categories that model $(n, 1)$-categories (for $n \geq 1$) as those in which every inner horn above dimension n has a unique filler. Moreover, he proved that a quasi-category is equivalent to such a quasi-category precisely when its hom-spaces are homotopy $(n-1)$-types (i.e. Kan complexes whose homotopy groups are trivial above dimension $n-1$); in [16, §26], Joyal called quasi-categories with this latter property *n-truncated*, and stated without proof a collection of assertions on n-truncated quasi-categories.

In this paper, we prove (Theorem 3.28) that, for each $n \geq -1$, the n-truncated quasi-categories are the fibrant objects of the Bousfield localisation of Joyal's model structure for quasi-categories with respect to

[1]This description is accurate only for $n \geq 1$; it is natural to identify $(0, 1)$-categories with posets and $(-1, 1)$-categories with truth values (i.e. 0 and 1). See [3] for a discussion of this point.

the boundary inclusion $\partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$. (Note that the existence of the model structure for n-truncated quasi-categories was stated without proof in Joyal's notes [16, §26.5]. However, our construction and identification of this model structure as the Bousfield localisation of Joyal's model structure for quasi-categories with respect to the boundary inclusion $\partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$ is new to this paper; see Remark 3.29.) Moreover, we prove (Theorem 4.14) Joyal's assertion (stated without proof in [16, §26.6]) that, if $n \geq 1$, a morphism of quasi-categories is a weak equivalence in this model structure if and only if it is essentially surjective on objects and an $(n-1)$-equivalence on hom-spaces.

Furthermore, we prove (Theorem 5.9) that the two Quillen equivalences

$$[\Delta^{\mathrm{op}}, \mathbf{Set}] \underset{t^!}{\overset{t_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} [(\Delta \times \Delta)^{\mathrm{op}}, \mathbf{Set}]$$

$$[(\Delta \times \Delta)^{\mathrm{op}}, \mathbf{Set}] \underset{i_1^*}{\overset{p_1^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} [\Delta^{\mathrm{op}}, \mathbf{Set}]$$

established by Joyal and Tierney [17] between the model structures for quasi-categories and complete Segal spaces remain Quillen equivalences between the model structures for n-truncated quasi-categories and Rezk's $(n,1)$-Θ-spaces [22], which are another model for $(n,1)$-categories. We also prove (Theorem 5.1) that the nerve functor $N \colon \mathbf{Cat} \longrightarrow \mathbf{sSet}$ is the right adjoint of a Quillen equivalence between the folk model structure for categories and the model structure for 1-truncated quasi-categories, and hence (Theorem 5.11) that the composite adjunction

$$\mathbf{Cat} \underset{N}{\overset{\tau_1}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} [\Delta^{\mathrm{op}}, \mathbf{Set}] \underset{t^!}{\overset{t_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} [(\Delta \times \Delta)^{\mathrm{op}}, \mathbf{Set}],$$

whose right adjoint is Rezk's "classifying diagram" functor [21], is a Quillen equivalence between the model structures for categories and Rezk's $(1,1)$-Θ-spaces.

The need for the $n = 1$ case of these results arose during the first-named author's work on the paper [6], wherein they serve as part of the proofs that certain adjunctions

$$\mathbf{Bicat}_{\mathrm{s}} \underset{N}{\overset{\tau_b}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} [\Theta_2^{\mathrm{op}}, \mathbf{Set}] \underset{t^!}{\overset{t_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} [(\Theta_2 \times \Delta)^{\mathrm{op}}, \mathbf{Set}]$$

are Quillen equivalences between Lack's model structure for bicategories [19], the Bousfield localisation of Ara's model structure for 2-quasi-categories [1] with respect to the boundary inclusion $\partial\Theta_2[1;3] \longrightarrow \Theta_2[1;3]$, and Rezk's model structure for $(2,2)$-Θ-spaces [22].

We begin this paper in §2 with a collection of some preliminary notions and results pertaining to simplicial sets and n-types. Our study of n-truncated quasi-categories begins in §3, where we construct the model structure for n-truncated quasi-categories, and continues in §4, where we characterise the weak equivalences of this model structure. Finally, in §5 we prove the aforementioned Quillen equivalences between the model categories of categories and 1-truncated quasi-categories and between the model categories of n-truncated quasi-categories and Rezk's $(n,1)$-Θ-spaces. In an appendix §A, we recall some of the basic theory of Bousfield localisations of model categories, including two criteria for detecting Quillen equivalences between Bousfield localisations.

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2. Simplicial preliminaries

In this section, we collect some preliminary notions and results pertaining to simplicial sets and homotopy n-types (as modelled by Kan complexes) that we will use in the following sections on truncated quasi-categories. For further background on simplicial sets, see for example [10], [11], and [7, Chapitre 2].

We begin with the definition of (homotopy) n-types, which we will use in the definition of truncated quasi-categories in §3.

Definition 2.1. Let $n \geq 0$ be an integer. A Kan complex X is said to be an *n-type* if, for each object (i.e. 0-simplex) $x \in X_0$ and each integer $m > n$, the homotopy group $\pi_m(X, x)$ is trivial (i.e. $\pi_m(X, x) \cong 1$).

Example 2.2. Every discrete (i.e. constant) simplicial set is a 0-type. Furthermore, a Kan complex X is a 0-type if and only if the unit morphism $X \longrightarrow \mathrm{disc}(\pi_0 X)$ of the adjunction

$$\mathbf{Set} \underset{\mathrm{disc}}{\overset{\pi_0}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{sSet} \tag{2.3}$$

is a homotopy equivalence.

It is natural to extend the notion of n-type to lower values of n as follows. Recall that a Kan complex X is said to be *contractible* if the unique morphism $X \longrightarrow \Delta^0$ is a homotopy equivalence.

Definition 2.4. A Kan complex is said to be a *(-1)-type* if it is either empty or contractible, and is said to be a *(-2)-type* if it is contractible.

In our study of truncated quasi-categories, we will use the following well-known alternative characterisation of n-types in terms of a lifting property (whose proof is a standard exercise).

Proposition 2.5. *Let $n \geq -2$ be an integer. A Kan complex is an n-type if and only if it has the right lifting property with respect to the boundary inclusion $\partial\Delta^m \longrightarrow \Delta^m$ for every $m \geq n + 2$.*

We will see in Proposition 3.12 that n-truncated quasi-categories can be characterised by the same lifting property. For this reason, we now record this lifting property in the following definition and explore some of its consequences.

Definition 2.6. Let $n \geq -1$ be an integer. A simplicial set X is said to be *n-acyclic* if it has the right lifting property with respect to the boundary inclusion $\partial\Delta^m \longrightarrow \Delta^m$ for every $m > n$.

In these terms, Proposition 2.5 states that, for every $n \geq -2$, a Kan complex is an n-type if and only if it is $(n + 1)$-acyclic. Similarly, we will prove in Proposition 3.12 that, for every $n \geq -1$, a quasi-category is n-truncated if and only if it is $(n+1)$-acyclic. This lifting property will be very useful, as it yields a large class of morphisms with respect to which n-truncated quasi-categories have the right lifting property.

Definition 2.7. Let $n \geq -1$ be an integer. A morphism of simplicial sets $f\colon X \longrightarrow Y$ is said to be *n-bijective* if the function $f_k\colon X_k \longrightarrow Y_k$ is bijective for each $0 \leq k \leq n$.

Lemma 2.8. *Let $n \geq -1$ be an integer. A simplicial set is n-acyclic if and only if it has the right lifting property with respect to every n-bijective monomorphism of simplicial sets.*

Proof. Since the boundary inclusion $\partial\Delta^m \longrightarrow \Delta^m$ is an n-bijective monomorphism for every $m > n$, any simplicial set with the stated lifting property is n-acyclic. Note that any class of morphisms defined by a left lifting property is stable under pushout and closed under coproducts and countable composition. The converse then follows from the fact that any n-bijective monomorphism can be decomposed into a countable composite of pushouts of coproducts of the boundary inclusions $\partial\Delta^m \longrightarrow \Delta^m$ for $m > n$, as in [10, §II.3.8]. □

Remark 2.9. If $n = -1$, the condition in Definition 2.7 is vacuous, and so every morphism of simplicial sets is (-1)-bijective. Hence the $n = -1$ case of Lemma 2.8 states that a simplicial set is (-1)-acyclic if and only if it is an injective object in the category of simplicial sets, i.e. a contractible Kan complex, i.e. a (-2)-type. Furthermore, a simplicial set is 0-acyclic if and only if it is a (-1)-type.

In §3, we will use the following consequence of Lemma 2.8 to prove that the model structures for n-truncated quasi-categories are cartesian.

Lemma 2.10. *Let $n \geq -1$ be an integer. For every simplicial set A and n-acyclic simplicial set X, the internal hom simplicial set X^A is n-acyclic.*

Proof. It is required to prove that X^A has the right lifting property with respect to the boundary inclusion $b_m\colon \partial\Delta^m \longrightarrow \Delta^m$ for every $m > n$. By adjunction, this is so if and only if X has the right lifting property with respect to the morphism $b_m \times A\colon \partial\Delta^m \times A \longrightarrow \Delta^m \times A$ for every $m > n$. But the n-acyclic simplicial set X has this lifting property by Lemma 2.8, since the morphism $b_m \times A$ is an n-bijective monomorphism for every $m > n$. □

The following lemma shows that the property of n-acyclicity can be understood as a weakening of the property of n-coskeletality. (Recall that a simplicial set X is said to be *n-coskeletal* if the unit morphism $X \longrightarrow \mathrm{cosk}_n X$ to its n-coskeleton is an isomorphism; dually, X is said to be *n-skeletal* if the counit morphism $\mathrm{sk}_n X \longrightarrow X$ from its n-skeleton is an isomorphism.)

Lemma 2.11. *Let $n \geq -1$ be an integer. A simplicial set X is n-acyclic if and only if the unit morphism $X \longrightarrow \mathrm{cosk}_n X$ is a trivial fibration.*

Proof. Let X be a simplicial set. By definition, the unit morphism $X \longrightarrow \mathrm{cosk}_n X$ is a trivial fibration if and only if it has the right lifting property with respect to the boundary inclusion $\partial\Delta^m \longrightarrow \Delta^m$ for each $m \geq 0$. By adjointness, this is so if and only if X has the right lifting property with respect to the inclusion $\mathrm{sk}_n\Delta^m \cup \partial\Delta^m \longrightarrow \Delta^m$ for each $m \geq 0$. If $m \leq n$, this inclusion is an identity, and so the lifting property is satisfied trivially. If $m > n$, this inclusion is the boundary inclusion $\partial\Delta^m \longrightarrow \Delta^m$. Thus the two properties in the statement are seen to be equivalent. □

Remark 2.12. By an argument similar to the proof of Lemma 2.8, one can show that a simplicial set is n-coskeletal if and only if it has the *unique* right lifting property with respect to the boundary inclusion $\partial\Delta^m \longrightarrow \Delta^m$ for each $m > n$. This gives another sense in which the property of n-acyclicity is a weakening of the property of n-coskeletality.

Now, recall that (the simplicial analogue of) Whitehead's theorem states that a morphism of Kan complexes $f\colon X \longrightarrow Y$ is a homotopy equivalence if and only if (i) the induced function $\pi_0(f)\colon \pi_0 X \longrightarrow \pi_0 Y$ is a bijection and (ii) for every integer $n \geq 1$ and every object x of X, the induced function $\pi_n(f)\colon \pi_n(X, x) \longrightarrow \pi_n(Y, fx)$ is a bijection (and hence an isomorphism of groups). We will use the following weakenings of these properties in our characterisation of the weak equivalences in the model structures for n-truncated quasi-categories in §4.

Definition 2.13. Let $n \geq 0$ be an integer. A morphism of Kan complexes $f\colon X \longrightarrow Y$ is said to be a *homotopy n-equivalence* if

(i) the induced function $\pi_0(f)\colon \pi_0 X \longrightarrow \pi_0 Y$ is a bijection, and

(ii) the induced function $\pi_k(f)\colon \pi_k(X, x) \longrightarrow \pi_k(Y, fx)$ is a bijection (and hence an isomorphism of groups) for every integer $1 \leq k \leq n$ and every object $x \in X$.

Thus a morphism of Kan complexes is a homotopy 0-equivalence if and only if it is inverted by the functor $\pi_0\colon \mathbf{sSet} \longrightarrow \mathbf{Set}$. We similarly define a morphism of Kan complexes to be a *homotopy* (-1)*-equivalence* if it is inverted by the functor $\pi_{-1}\colon \mathbf{sSet} \longrightarrow \{0 < 1\}$ that sends the empty simplicial set to 0 and every nonempty simplicial set to 1. Thus a morphism of Kan complexes is a homotopy (-1)-equivalence if either (i) its domain and codomain are both empty, or (ii) its domain and codomain are both nonempty. Furthermore, we define a morphism of Kan complexes to be a *homotopy* (-2)*-equivalence* if it is inverted by the unique functor $\pi_{-2}\colon \mathbf{sSet} \longrightarrow \mathbf{1}$ to the terminal category; thus every morphism of Kan complexes is a homotopy (-2)-equivalence.

For each $n \geq -2$, a morphism of n-types is a homotopy equivalence if and only if it is a homotopy n-equivalence: if $n \geq 0$, this follows from Whitehead's theorem; if $n = -2, -1$, this follows from the fact that any morphism between contractible Kan complexes is a homotopy equivalence.

Remark 2.14. It is a standard result (cf. [12, §1.5] and [7, §9.2]) that, for each integer $n \geq -2$, the n-types are the fibrant objects of the Bousfield localisation of the model structure for Kan complexes with respect to the boundary inclusion $\partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$, and that a morphism of Kan complexes is a weak equivalence in this Bousfield localisation if and only if it is a homotopy n-equivalence in the sense of the above definitions. In §§3–4, we will generalise both of these statements to n-truncated quasi-categories.

We will use the following two properties of the class of homotopy n-equivalences in §4.

Lemma 2.15. *Let $n \geq -2$ be an integer and let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be morphisms of Kan complexes. If two of the morphisms f, g, gf are homotopy n-equivalences, then so is the third.*

Proof. This is proved by any of the standard arguments proving that the class of morphisms of Kan complexes described in the statement of Whitehead's theorem enjoys the same property. □

Lemma 2.16. *Let $n \geq -2$ be an integer. An $(n+1)$-bijective morphism of Kan complexes is a homotopy n-equivalence.*

Proof. The cases $n = -2, -1$ are immediate. Suppose $n \geq 0$. The result follows from the facts that the set of connected components of a simplicial set depends only on its 1-skeleton, and that, for each integer $k \geq 1$, the kth homotopy groups of a Kan complex depend only on its $(k+1)$-skeleton (since their elements are pointed homotopy classes of morphisms to X from the (simplicial) k-sphere, whose homotopy type can be modelled by a k-skeletal simplicial set, e.g. $\Delta^k/\partial\Delta^k$ or $\partial\Delta^{k+1}$). □

3. Truncated quasi-categories

Throughout this section, let $n \geq -1$ be an integer.

Remark 3.1. As mentioned in §1, some of the results of §§3–5 are stated without proof in Joyal's notes [16, §26]. These results will be indicated below by references to the numbered paragraphs of those notes in which they are stated. (Given that one of the purposes of this paper is to provide proofs for these statements, we beg the reader's patience if we spell out the occasional "obvious" argument.)

As recalled in Remark 2.14, the (homotopy) n-types are the fibrant objects of the Bousfield localisation of the model structure for Kan complexes with respect to the boundary inclusion $\partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$, and a morphism of Kan complexes is a weak equivalence in this Bousfield localisation if and only if it is a homotopy n-equivalence. The goal of this section and the next is to prove the analogous results for quasi-categories. In this section, we prove that the n-truncated quasi-categories are the fibrant objects of the Bousfield localisation of Joyal's model structure for quasi-categories with respect to the same boundary inclusion (Theorem 3.28). In §4, we will prove that a morphism of quasi-categories is a weak equivalence in this Bousfield localisation if and only if it is a categorical n-equivalence (Theorem 4.14). (Note that the first of these two results is new to this paper, whereas the second was stated without proof in [16, §26.6].)

We refer the reader to Appendix A for the necessary background on Bousfield localisations, and to [14], [17, §1], and [20, Chapter 1] for a more than sufficient background in the theory of quasi-categories. In particular, recall that there is a (left proper and combinatorial) cartesian model structure due to Joyal on the category of simplicial sets whose cofibrations are the monomorphisms and whose fibrant objects are the quasi-categories [15, Theorem 6.12]. We call this model structure the *model structure for quasi-categories*; the weak equivalences and fibrations between fibrant objects of this model structure will be called *weak categorical equivalences* and *isofibrations* respectively. (Note that, following [9], we will sometimes denote the category of simplicial sets equipped with the model structures for Kan complexes and quasi-categories by $\mathbf{sSet}_{\mathrm{K}}$ and $\mathbf{sSet}_{\mathrm{J}}$ respectively.)

To begin, let us recall the definition of the hom-spaces of a quasi-category. For each pair of objects (i.e. 0-simplices) x, y of a quasi-category X, their *hom-space* $\mathrm{Hom}_X(x,y)$ is the Kan complex defined by the pullback

$$\begin{array}{ccc} \mathrm{Hom}_X(x,y) & \longrightarrow & X^{\Delta^1} \\ \downarrow \quad \lrcorner & & \downarrow {\scriptstyle (X^{\delta^1}, X^{\delta^0})} \\ \Delta^0 & \xrightarrow[(x,y)]{} & X \times X \end{array} \tag{3.2}$$

in the category of simplicial sets. By [9, Proposition 4.5], this hom-space construction defines the right adjoint of a Quillen adjunction

$$\partial\Delta^1 \backslash \mathbf{sSet}_{\mathrm{J}} \underset{\mathrm{Hom}}{\overset{\Sigma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{sSet}_{\mathrm{K}} \tag{3.3}$$

between the category of bipointed simplicial sets (note that $\partial\Delta^1 \cong \Delta^0 + \Delta^0$) equipped with the model structure induced by the model structure for quasi-categories and the category of simplicial sets equipped with the model structure for Kan complexes, whose left adjoint sends a simplicial set U to its *(two-point) suspension* ΣU, defined by the pushout

$$\begin{array}{ccc} U \times \partial\Delta^1 & \xrightarrow{\mathrm{pr}_2} & \partial\Delta^1 \\ \downarrow & & \downarrow{\scriptstyle (\perp,\top)} \\ U \times \Delta^1 & \longrightarrow & \Sigma U \end{array} \tag{3.4}$$

in the category of simplicial sets; note that the simplicial set ΣU has precisely two 0-simplices, which we denote by $\perp$ and $\top$, as in the diagram above.

Next, recall that one can assign to each category A a quasi-category NA via the nerve functor $N\colon \mathbf{Cat} \longrightarrow \mathbf{sSet}$, which defines the fully faithful right adjoint of an adjunction

$$\mathbf{Cat} \underset{N}{\overset{\tau_1}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{sSet} \tag{3.5}$$

whose left adjoint sends a simplicial set X to its fundamental category $\tau_1 X$ (see [10, §II.4]). If X is a quasi-category, then its fundamental category $\tau_1 X$ is isomorphic to its *homotopy category* $\mathrm{ho}\, X$, which was first constructed by Boardman and Vogt [5, §IV.2] (for a detailed proof, see [15, Chapter 1]). The homotopy category $\mathrm{ho}\, X$ of a quasi-category X has the same set of objects as X, and its hom-sets $(\mathrm{ho}\, X)(x, y) \cong \pi_0(\mathrm{Hom}_X(x, y))$ are isomorphic to the sets of connected components of the hom-spaces of X; thus the unit morphism $X \longrightarrow N(\mathrm{ho}\, X)$ of the adjunction (3.5) is a bijection on objects, and is given on hom-spaces by the unit morphism $\mathrm{Hom}_X(x, y) \longrightarrow \mathrm{disc}(\pi_0(\mathrm{Hom}_X(x, y)))$ of the adjunction $\pi_0 \dashv \mathrm{disc}$ (2.3). A morphism (i.e. a 1-simplex) in a quasi-category X is said to be an *isomorphism* if it is sent by the unit morphism $X \longrightarrow N(\mathrm{ho}\, X)$ to an isomorphism in $\mathrm{ho}\, X$.

A morphism of quasi-categories $f\colon X \longrightarrow Y$ is said to be *essentially surjective on objects* if the induced functor between homotopy categories $\mathrm{ho}(f)\colon \mathrm{ho}\, X \longrightarrow \mathrm{ho}\, Y$ is essentially surjective on objects. A fundamental theorem of quasi-category theory states that a morphism of quasi-categories $f\colon X \longrightarrow Y$ is an equivalence of quasi-categories (i.e.

a weak categorical equivalence between quasi-categories) if and only if it is essentially surjective on objects and a homotopy equivalence on hom-spaces, that is, for each pair of objects $x, y \in X$, the induced morphism of hom-spaces $f\colon \mathrm{Hom}_X(x,y) \longrightarrow \mathrm{Hom}_Y(fx,fy)$ is a homotopy equivalence of Kan complexes.

We now recall the definition of n-truncated quasi-categories from [16, §26].

Definition 3.6. A quasi-category X is said to be *n-truncated* if, for each pair of objects $x, y \in X$, the hom-space $\mathrm{Hom}_X(x,y)$ is an $(n-1)$-type.

Remark 3.7. In [20, Proposition 2.3.4.18], Lurie proved that a quasi-category is n-truncated if and only if it is equivalent to an *n-category* in the sense of [20, Definition 2.3.4.1]. We will not use this result in the present paper.

Before proceeding with the study of the homotopy theory of n-truncated quasi-categories, let us examine the low dimensional cases of this definition. By definition, a quasi-category is 1-truncated if and only if its hom-spaces are 0-types. For example, the nerve NA of a category A is a 1-truncated quasi-category, since its hom-spaces are the discrete simplicial sets $\mathrm{Hom}_{NA}(a,b) \cong \mathrm{disc}(A(a,b))$ given by the hom-sets of A, and since every discrete simplicial set is a 0-type.

Proposition 3.8 ([16, §26.1])**.** *A quasi-category X is 1-truncated if and only if the unit morphism $X \longrightarrow N(\mathrm{ho}\,X)$ is an equivalence of quasi-categories. In particular, the nerve of a category is a 1-truncated quasi-category.*

Proof. Let X be a quasi-category. By construction, the unit morphism $X \longrightarrow N(\mathrm{ho}\,X)$ is bijective on objects, and therefore is an equivalence if and only if it is a homotopy equivalence on hom-spaces, that is, if and only if the unit morphism $\mathrm{Hom}_X(x,y) \longrightarrow \mathrm{disc}(\pi_0(\mathrm{Hom}_X(x,y)))$ is a homotopy equivalence for each pair of objects $x, y \in X$. But this is so precisely when each hom-space $\mathrm{Hom}_X(x,y)$ is a 0-type (see Example 2.2), that is, precisely when X is 1-truncated. □

Remark 3.9. For any quasi-category X, the unit morphism $X \longrightarrow N(\mathrm{ho}\,X)$ is an isofibration. Hence a quasi-category X is 1-truncated if and only if the unit morphism $X \longrightarrow N(\mathrm{ho}\,X)$ is a trivial fibration.

Recall that a category is a *preorder* if each of its hom-sets has at most one element. A category is a preorder if and only if it is equivalent to a poset (partially ordered set): the quotient of a preorder by the congruence $x \sim y \iff x \leq y \;\&\; y \leq x$ defines an equivalent poset, which we call its *poset quotient*; conversely, any category equivalent to a preorder is evidently a preorder, and a poset is in particular a preorder.

Proposition 3.10 ([16, §26.2])**.** *A quasi-category is* 0*-truncated if and only if it is* 1*-truncated and its homotopy category is equivalent to a poset. In particular, the nerve of a preorder is a* 0*-truncated quasi-category.*

Proof. A Kan complex is a (-1)-type if and only if it is a 0-type and its set of connected components has at most one element. Hence a quasi-category X is 0-truncated if and only if it is 1-truncated and its homotopy category is a preorder, that is, equivalent to a poset. □

A quasi-category is (-1)-truncated if and only if it is empty or a contractible Kan complex, that is, if and only if it is a (-1)-type: if X is a nonempty (-1)-truncated quasi-category, then its hom-spaces are contractible, and so the unique morphism $X \longrightarrow \Delta^0$ is surjective on objects and a homotopy equivalence on hom-spaces, and is thus an equivalence of quasi-categories, and hence a trivial fibration. Similarly, one could define a quasi-category to be (-2)-truncated if it is a (-2)-type, i.e. a contractible Kan complex.

We now proceed towards the main goal of this section, which is to prove that the n-truncated quasi-categories are the fibrant objects of the Bousfield localisation of the model structure for quasi-categories with respect to the boundary inclusion $\partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$. Our first step will be to show that n-truncated quasi-categories can be characterised in terms of a lifting property. To this end, it will be convenient to use an alternative model for the hom-spaces of a quasi-category.

Recall that a morphism of simplicial sets $f\colon X \longrightarrow Y$ is said to be a *right fibration* if it has the right lifting property with respect to the horn inclusion $\Lambda^m_k \longrightarrow \Delta^m$ for every $m \geq 1$ and $0 < k \leq m$ (see [14, §2] or [20, Chapter 2]). For each object x of a quasi-category X, one obtains by the join and slice constructions of [14, §3] a right fibration $X/x \longrightarrow X$ whose domain is the *slice quasi-category* X/x (see [20, §1.2.9]). The slice quasi-category construction defines the right adjoint of an adjunction

$$\Delta^0\backslash\mathbf{sSet} \underset{\text{slice}}{\overset{-\star\Delta^0}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{sSet}$$

whose left adjoint sends a simplicial set U to the *right cone* of U, i.e. the join $U \star \Delta^0$ with base point $\Delta^0 \cong \emptyset \star \Delta^0 \longrightarrow U \star \Delta^0$. Thus, for each $k \geq 0$, a k-simplex of the slice quasi-category X/x is given by a $(k+1)$-simplex of X whose final vertex is x; the right fibration $X/x \longrightarrow X$ sends a k-simplex of X/x to the face opposite the last vertex of the corresponding $(k+1)$-simplex of X. (See [14, §3] and [20, §§1.2.8–9] for further details.)

For each pair of objects x, y of a quasi-category X, the *right hom-space* $\mathrm{Hom}^R_X(x, y)$ is defined by the pullback

$$\begin{array}{ccc} \mathrm{Hom}^R_X(x,y) & \longrightarrow & X/y \\ \big\downarrow & \lrcorner & \big\downarrow \\ \Delta^0 & \xrightarrow[x]{} & X \end{array}$$

in the category of simplicial sets [20, §1.2.2]. Since the projection $X/y \longrightarrow X$ is a right fibration, it follows that the right hom-space $\mathrm{Hom}^R_X(x, y)$ is a Kan complex (see [20, Proposition 1.2.2.3]). A k-simplex of $\mathrm{Hom}^R_X(x, y)$ is given by a $(k+1)$-simplex of X whose last vertex is y and whose face opposite the last vertex is the degenerate k-simplex on x.

Importantly, for each pair of objects x, y of a quasi-category X, there is a homotopy equivalence $\mathrm{Hom}^R_X(x, y) \simeq \mathrm{Hom}_X(x, y)$ between the right hom-space and the hom-space (see [20, Corollary 4.2.1.8]). Hence a quasi-category is n-truncated if and only if each of its right hom-spaces is an $(n-1)$-type.

The characterisation of n-truncated quasi-categories in terms of a lifting property depends on the following lemma. Recall from Definition 2.6 that a simplicial set is said to be n-acyclic if it has the right lifting property with respect to the boundary inclusion $\partial\Delta^m \longrightarrow \Delta^m$ for every $m > n$.

Lemma 3.11. *Let $f\colon X \longrightarrow Y$ be a right fibration of simplicial sets. Then the following properties are equivalent.*

(i) *f has the right lifting property with respect to the boundary inclusion $\partial\Delta^m \longrightarrow \Delta^m$ for every $m > n$.*

(ii) *For every 0-simplex $y \in Y_0$, the fibre $f^{-1}(y)$ is an $(n-1)$-type.*

(iii) *For every 0-simplex $y \in Y_0$, the fibre $f^{-1}(y)$ is n-acyclic.*

Proof. Since the fibres of a right fibration are Kan complexes, the equivalence (ii) $\iff$ (iii) follows from Proposition 2.5. Furthermore, since any pullback of a morphism satisfying the lifting property (i) inherits this lifting property, we have the implication (i) $\implies$ (iii).

It remains to prove the implication (iii) $\implies$ (i). If $n = -1$, this implication is precisely [20, Lemma 2.1.3.4], which states that a right fibration whose fibres are contractible is a trivial fibration. In fact, the proof of the cited result proves moreover that, for each $k \geq 0$, if the fibres of a right fibration each have the right lifting property with respect to the boundary inclusion $\partial\Delta^k \longrightarrow \Delta^k$, then the right fibration also has the right lifting property with respect to that boundary inclusion. This proves the implication (iii) $\implies$ (i) for an arbitrary $n \geq -1$. □

By applying Lemma 3.11 to the right fibrations of the form $X/x \longrightarrow X$, we can characterise the n-truncated quasi-categories by the following lifting property.

Proposition 3.12 ([16, §§26.1–3])**.** *A quasi-category is n-truncated if and only if it has the right lifting property with respect to the boundary inclusion $\partial\Delta^m \longrightarrow \Delta^m$ for every $m \geq n+2$.*

Proof. By the homotopy equivalences between the hom-spaces and the right hom-spaces of a quasi-category [20, Corollary 4.2.1.8], a quasi-category X is n-truncated if and only if the right hom-space $\mathrm{Hom}^R_X(x, y)$ is an $(n-1)$-type for each pair of objects $x, y \in X$. We thus have that a quasi-category X is n-truncated if and only if every fibre of the right fibration $X/y \longrightarrow X$ is an $(n-1)$-type for every object $y \in X$. By Lemma 3.11, this is so if and only if the right fibration $X/y \longrightarrow X$ has the right lifting property with respect to the boundary inclusion $\partial\Delta^m \longrightarrow \Delta^m$ for every $m > n$ and every $y \in X$. By adjointness (see [14, Lemma 3.6]), this lifting property is satisfied if and only if X has the right lifting property with respect to the pushout-join $(\partial\Delta^m \star \Delta^0) \cup (\Delta^m \star \emptyset) \longrightarrow \Delta^m \star \Delta^0$ for every $m > n$. But this pushout-join is none other than the boundary inclusion $\partial\Delta^{m+1} \longrightarrow \Delta^{m+1}$ [14, Lemma 3.3]. Hence we have shown that a quasi-category X is n-truncated if and only if it has the right lifting property with respect to the boundary inclusion $\partial\Delta^{m+1} \longrightarrow \Delta^{m+1}$ for every $m > n$, as required. □

In the terminology of Definition 2.6, Proposition 3.12 states that a quasi-category is n-truncated if and only if it is $(n+1)$-acyclic. Thus we may deduce that the class of n-truncated quasi-categories inherits the following properties from the class of $(n+1)$-acyclic simplicial sets.

Corollary 3.13. *A quasi-category is n-truncated if and only if it has the right lifting property with respect to every $(n+1)$-bijective monomorphism of simplicial sets.*

Proof. This is a consequence of Proposition 3.12 and Lemma 2.8. □

Corollary 3.14. *For every simplicial set A and n-truncated quasi-category X, the internal hom simplicial set X^A is an n-truncated quasi-category.*

Proof. We have by [15, Corollary 2.19] that X^A is a quasi-category. Hence by Proposition 3.12, X^A is an n-truncated quasi-category if and only if it is $(n+1)$-acyclic. The result then follows from Corollary 2.10. □

Remark 3.15. The result of Corollary 3.14 was proved by Lurie as [20, Corollary 2.3.4.20]. Our proof of this result is more direct and elementary than Lurie's proof, which uses the corresponding result for n-categories (in his sense) [20, Proposition 2.3.4.8] and the fact that any quasi-category is equivalent to a minimal quasi-category [20, Proposition 2.3.3.8].

Corollary 3.16. *A quasi-category X is n-truncated if and only if the unit morphism $X \longrightarrow \mathrm{cosk}_{n+1}X$ is a trivial fibration.*

Proof. This is a consequence of Proposition 3.12 and Lemma 2.11. □

In Propositions 2.5 and 3.12, n-types and n-truncated quasi-categories were both characterised by the same lifting property. Hence we may deduce the following corollary.

Corollary 3.17 ([16, §§26.1–3])**.** *A Kan complex is an n-truncated quasi-category if and only if it is an n-type.*

Proof. By definition, every Kan complex is a quasi-category. Hence by Proposition 3.12, a Kan complex is an n-truncated quasi-category if and only if it is $(n+1)$-acyclic, which is so, by Proposition 2.5, precisely when it is an n-type. □

Next, we deduce from Proposition 3.12 a further characterisation of n-truncated quasi-categories as the quasi-categories that are local (see (A.8)) with respect to the boundary inclusion $\partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$ in the model structure $\mathbf{sSet}_{\mathrm{J}}$ for quasi-categories. As explained in Appendix A, this will require a model for the derived hom-spaces of the model category $\mathbf{sSet}_{\mathrm{J}}$, which we will obtain by Lemma A.12 from the Quillen adjunction (3.18) below.

Let **qCat** and **Kan** denote the full subcategories of **sSet** consisting of the quasi-categories and the Kan complexes respectively. By [15, Theorem 4.19], the full inclusion **Kan** $\longrightarrow$ **qCat** has a right adjoint $J\colon \mathbf{qCat} \longrightarrow \mathbf{Kan}$, which sends a quasi-category X to its maximal sub Kan complex $J(X)$. By [15, Lemma 4.18], a simplex of X belongs to the simplicial subset $J(X)$ if and only if each of its 1-simplices is an isomorphism in X. Note that, by [15, Proposition 4.27], the functor J sends isofibrations to Kan fibrations.

Let X be a quasi-category. By [15, Corollary 5.11], there is an adjunction

$$\mathbf{sSet}_{\mathrm{J}}^{\mathrm{op}} \underset{J(X^{-})}{\overset{X^{(-)}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{sSet}_{\mathrm{K}} \tag{3.18}$$

whose right adjoint sends a simplicial set A to the Kan complex $J(X^A)$, and whose left adjoint sends a simplicial set U to the full sub-quasi-category $X^{(U)}$ of X^U consisting of the morphisms of simplicial sets $U \longrightarrow X$ which factor through $J(X)$, i.e. which send each 1-simplex of U to an isomorphism in X. Moreover, by [15, Theorems 5.7, 5.10], this adjunction is a Quillen adjunction between (the opposite of) the model structure $\mathbf{sSet}_{\mathrm{J}}$ for quasi-categories and the model structure $\mathbf{sSet}_{\mathrm{K}}$ for Kan complexes as indicated.

Hence, for each simplicial set A and quasi-category X, Lemma A.12 applied to the Quillen adjunction (3.18) implies that the Kan complex $J(X^A)$ is a model for the derived hom-space $\underline{\mathrm{Ho}\,\mathbf{sSet}_{\mathrm{J}}}(A,X)$ from A to X in the model structure for quasi-categories. We may therefore deduce the following lemma.

Lemma 3.19. *A quasi-category X is local with respect to a morphism $f\colon A \longrightarrow B$ in the model structure for quasi-categories if and only if the morphism $J(X^f)\colon J(X^B) \longrightarrow J(X^A)$ is a homotopy equivalence of Kan complexes.*

Proof. By definition (see Appendix A), a quasi-category X is local with respect to a morphism $f\colon A \longrightarrow B$ in the model category $\mathbf{sSet}_{\mathrm{J}}$ if and only if this morphism is sent to an isomorphism by the functor

$$\underline{\mathrm{Ho}\,\mathbf{sSet}_{\mathrm{J}}}(-,X) : \mathrm{Ho}\,\mathbf{sSet}_{\mathrm{J}}^{\mathrm{op}} \longrightarrow \mathscr{H}.$$

Since $X^{(\Delta^0)} \cong X$, Lemma A.12 implies that this functor is naturally isomorphic to the derived right adjoint of the Quillen adjunction (3.18). Therefore, since every object of $\mathbf{sSet}_{\mathrm{J}}$ is cofibrant, a morphism of simplicial sets is sent to an isomorphism by the functor $\underline{\mathrm{Ho}\,\mathbf{sSet}_{\mathrm{J}}}(-,X)$ if and only if it is sent to a homotopy equivalence of Kan complexes by the right Quillen functor $J(X^-)\colon \mathbf{sSet}_{\mathrm{J}}^{\mathrm{op}} \longrightarrow \mathbf{sSet}_{\mathrm{K}}$, as required. □

Remark 3.20. A Kan complex is local with respect to a given morphism in the model structure for Kan complexes if and only if it is local with respect to that morphism in the model structure for quasi-categories. This can be seen as a consequence either of the fact that the model structure for Kan complexes is a Bousfield localisation of the model structure for quasi-categories (cf. [1, Lemma A.4]), or of the standard result that for any simplicial set A and Kan complex X, the Kan complex X^A is a model for the derived hom-space from A to X in the model category $\mathbf{sSet}_{\mathrm{K}}$ (see [12, Example 17.1.4]), which coincides with our model for the derived hom-space from A to X in the model category $\mathbf{sSet}_{\mathrm{J}}$.

Remark 3.21. An alternative model for the derived hom-spaces of the model category $\mathbf{sSet}_{\mathrm{J}}$ involves the following adjunction (which we will meet again in §5). Let $k\colon \Delta \longrightarrow \mathbf{sSet}$ denote the functor that sends the ordered set $[m]$ to the nerve of its groupoid reflection, i.e. the nerve of the contractible groupoid with the set of objects $\{0,\ldots,m\}$. This functor induces an adjunction

$$\mathbf{sSet}_{\mathrm{J}} \underset{k^!}{\overset{k_!}{\rightleftarrows}} \mathbf{sSet}_{\mathrm{K}} \quad (\perp) \tag{3.22}$$

whose left adjoint is the left Kan extension of $k\colon \Delta \longrightarrow \mathbf{sSet}$ along the Yoneda embedding $\Delta \longrightarrow \mathbf{sSet}$. By [15, Theorem 6.22], this adjunction is a Quillen adjunction between the model structures for quasi-categories and Kan complexes as indicated. Note that, since $k([0]) = \Delta^0$, the right adjoint functor $k^!$ sends a quasi-category to a Kan complex with the same set of objects.

One can show by another application of Lemma A.12 that for each simplicial set A and quasi-category X, the Kan complex $k^!(X^A)$ is a model for the derived hom-space from A to X in the model category $\mathbf{sSet}_{\mathrm{J}}$, which is homotopy equivalent to the Kan complex $J(X^A)$ by [15, Proposition 6.26]. For our purposes, either of these models $J(X^A)$ or $k^!(X^A)$ for the derived hom-space would suffice; but one must be chosen, and we have chosen the former.

Using Lemma 3.19, we are now able to prove the following proposition.

Proposition 3.23. *A quasi-category is n-truncated if and only if it is local with respect to the boundary inclusion $\partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$ in the model structure for quasi-categories.*

Proof. By Lemma 3.19, it is required to prove that a quasi-category X is n-truncated if and only if the Kan fibration

$$J(X^{b_{n+2}})\colon J(X^{\Delta^{n+2}}) \longrightarrow J(X^{\partial\Delta^{n+2}}) \tag{3.24}$$

induced by the boundary inclusion $b_{n+2}\colon \partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$ is a homotopy equivalence of Kan complexes, or equivalently a trivial fibration.

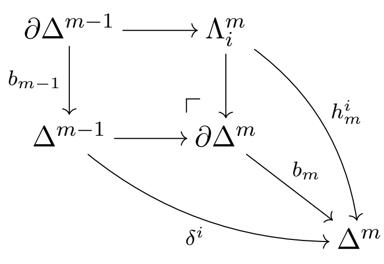
Let X be an n-truncated quasi-category. We will prove that the morphism (3.24) is a trivial fibration. Since $n \geq -1$, the boundary inclusion $\partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$ is 0-bijective, and so by [15, Lemma 5.9] (see also [15, Corollary 5.11]) the following square is a pullback square.

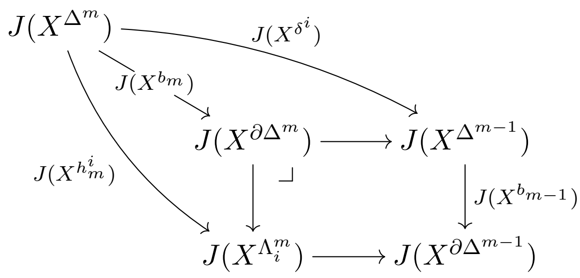
$$\begin{array}{ccc} J(X^{\Delta^{n+2}}) & \longrightarrow & X^{\Delta^{n+2}} \\ \downarrow & \lrcorner & \downarrow \\ J(X^{\partial\Delta^{n+2}}) & \longrightarrow & X^{\partial\Delta^{n+2}} \end{array}$$

Hence it suffices to prove that the morphism $X^{\Delta^{n+2}} \longrightarrow X^{\partial\Delta^{n+2}}$ is a trivial fibration. By adjointness, this is so if and only if X has the right lifting property with respect to the pushout-product of the boundary inclusion $\partial\Delta^m \longrightarrow \Delta^m$ with the $(n+1)$-bijective boundary inclusion $\partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$ for every $m \geq 0$. But every such pushout-product is an $(n+1)$-bijective monomorphism, and so X has the desired lifting property by Corollary 3.13. Therefore the morphism (3.24) is a trivial fibration.

Conversely, let X be a quasi-category and suppose that the morphism (3.24) is a trivial fibration. By Proposition 3.12, it remains to prove that X has the right lifting property with respect to the boundary inclusion $\partial\Delta^m \longrightarrow \Delta^m$ for every $m \geq n+2$. Since trivial fibrations are surjective on 0-simplices, it suffices to prove that the morphism $J(X^{b_m})\colon J(X^{\Delta^m}) \longrightarrow J(X^{\partial\Delta^m})$ is a trivial fibration for every $m \geq n+2$.

We prove by induction that the morphism $J(X^{b_m})$ is a trivial fibration for every $m \geq n+2$. The base case $m = n+2$ of the induction is precisely the assumption that the morphism (3.24) is a trivial fibration. Now suppose $m > n+2$, and let $0 < i < m$ be an integer (which exists since $n \geq -1$). We then have a diagram of monomorphisms as on the left below,





and hence a diagram of Kan fibrations as on the right above. In this latter diagram, the morphism $J(X^{b_{m-1}})$ is a trivial fibration by the induction hypothesis, and hence so is its pullback. Since the morphism h^i_m is an inner horn inclusion, the morphism $X^{h^i_m}$ is a trivial fibration, and hence so is the morphism $J(X^{h^i_m})$. It then follows from the two-of-three property that the morphism $J(X^{b_m})$ is a trivial fibration. This completes the proof by induction. □

As a special case of this result, we recover the following well-known characterisation of n-types (cf. [12, Proposition 1.5.1]).

Corollary 3.25. *A Kan complex X is an n-type if and only if it is local with respect to the boundary inclusion $\partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$ in the model structure for Kan complexes.*

Proof. By Remark 3.20, a Kan complex is local with respect to the boundary inclusion $\partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$ in the model structure for Kan complexes if and only if it is local with respect to it in the model structure for quasi-categories. Hence the result follows from Proposition 3.23 and Corollary 3.17. □

Remark 3.26. A Kan complex X is local with respect to the boundary inclusion $\partial\Delta^0 = \emptyset \longrightarrow \Delta^0$ if and only if the unique morphism $X \cong X^{\Delta^0} \longrightarrow X^{\emptyset} = \Delta^0$ is a homotopy equivalence, that is, if and only if X is contractible. Hence Corollary 3.25 holds for all $n \geq -2$.

We may now apply Smith's existence theorem (Theorem A.11) to deduce the existence of the Bousfield localisation of the model structure for quasi-categories whose fibrant objects are precisely the n-truncated quasi-categories. We break the statement of the following result into two parts: the first part was stated without proof in [16, §26.5], and the second part is new to this paper.

Theorem 3.27 ([16, §26.5])**.** *There exists a model structure on the category of simplicial sets whose cofibrations are the monomorphisms and whose fibrant objects are the n-truncated quasi-categories. This model structure is cartesian and left proper.*

Theorem 3.28. *The model structure of* Theorem 3.27 *is the Bousfield localisation of Joyal's model structure for quasi-categories with respect to the boundary inclusion $\partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$, and is combinatorial.*

Proof. Since the model category $\mathbf{sSet}_\mathrm{J}$ is left proper and combinatorial, there exists by Theorem A.11 a Bousfield localisation of $\mathbf{sSet}_\mathrm{J}$ whose fibrant objects are precisely the quasi-categories that are local with respect to the single morphism $\partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$. By Proposition 3.23, these fibrant objects are precisely the n-truncated quasi-categories. Theorem A.11 further implies that this model structure is left proper and combinatorial. The model structure is cartesian by Proposition A.7 and Corollary 3.14, since $\mathbf{sSet}_\mathrm{J}$ is a cartesian model category in which every object is cofibrant. □

Remark 3.29. In [16, §26.5], the model structure of Theorem 3.27 is defined as the Bousfield localisation of the model structure $\mathbf{sSet}_\mathrm{J}$ for quasi-categories with respect to the (large) class of "weak categorical n-equivalences" (defined therein as the morphisms of simplicial sets satisfying the property stated in Lemma 4.1 below). However, our identification of this model structure with the Bousfield localisation of $\mathbf{sSet}_\mathrm{J}$ with respect to the boundary inclusion $\partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$, or indeed with respect to any small set of morphisms, is not contained in [16].

We will call the model structure of Theorem 3.27 the *model structure for n-truncated quasi-categories*. Similarly, one can prove by Corollary 3.25 and Theorem A.11 that the n-types are the fibrant objects of the Bousfield localisation of the model structure for Kan complexes with respect to the boundary inclusion $\partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$, as recalled in Remark

2.14. Since every n-type is an n-truncated quasi-category by Corollary 3.17, this model structure for n-types is also a Bousfield localisation of the model structure for n-truncated quasi-categories; indeed, the following proposition implies that it is the Bousfield localisation of this model structure with respect to the unique morphism $\Delta^1 \longrightarrow \Delta^0$ (cf. Examples A.5 and A.9).

Proposition 3.30. *A quasi-category is a Kan complex if and only if it is local with respect to the unique morphism* $\Delta^1 \longrightarrow \Delta^0$ *in the model structure for quasi-categories.*

Proof. Let X be a quasi-category. By Lemma 3.19, it suffices to prove that X is a Kan complex if and only if the induced morphism of Kan complexes $J(X) \longrightarrow J(X^{\Delta^1})$ is a homotopy equivalence. To prove this, consider the following commutative diagram of Kan complexes.

$$\begin{array}{ccc} & J(X) & \\ \swarrow & & \searrow \\ J(X)^{\Delta^1} & \longrightarrow & J(X^{\Delta^1}) \end{array}$$

In this diagram, the left-diagonal morphism is a homotopy equivalence, since $\Delta^1 \longrightarrow \Delta^0$ is a homotopy equivalence. Hence, by the two-of-three property, it remains to show that X is a Kan complex if and only if the bottom morphism in this diagram is a homotopy equivalence. But this bottom morphism is both a monomorphism and a Kan fibration, since, by [15, Proposition 5.3], it is the image under the functor J of the inclusion $X^{(\Delta^1)} \longrightarrow X^{\Delta^1}$ of the replete full sub-quasi-category of X^{Δ^1} consisting of the isomorphisms in X, which is both a monomorphism and an isofibration. Hence the bottom morphism is a homotopy equivalence if and only if it is surjective on objects, which is so precisely when every morphism in the quasi-category X is an isomorphism, that is, precisely when X is a Kan complex. □

We have constructed the model structure for n-truncated quasi-categories as the Bousfield localisation of the model structure for quasi-categories with respect to the boundary inclusion $\partial\Delta^{n+2} \longrightarrow \Delta^{n+2}$. However, as in Remark A.10, this model structure can also be described as the Bousfield localisation of the model structure for quasi-categories with respect to any of a variety of alternative morphisms. To conclude this section, we give one such alternative morphism. This will be derived as an instance of a more general proposition, which we will prove by an application of the following standard result.

Consider a commutative diagram of simplicial sets as displayed below,

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & A & \end{array}$$

in which the morphisms p and q are Kan fibrations. A standard result states that the morphism f is a weak homotopy equivalence if and only if, for each 0-simplex $a \in A_0$, the induced morphism between fibres $f_a \colon p^{-1}(a) \longrightarrow q^{-1}(a)$ is a homotopy equivalence of Kan complexes.

Let $\Sigma\colon \mathbf{sSet} \longrightarrow \mathbf{sSet}$ denote the (two-point) suspension functor, that is, the composite of the left adjoint of the adjunction (3.3) with the functor $\partial\Delta^1\backslash\mathbf{sSet} \longrightarrow \mathbf{sSet}$ that forgets the base points. Since the adjunction (3.3) is a Quillen adjunction, the suspension functor preserves monomorphisms and sends weak homotopy equivalences to weak categorical equivalences.

Proposition 3.31. *Let $f\colon A \longrightarrow B$ be a morphism of simplicial sets. A quasi-category X is local with respect to the morphism $\Sigma(f)\colon \Sigma A \longrightarrow \Sigma B$ in the model structure for quasi-categories if and only if, for each pair of objects $x, y \in X$, the hom-space $\mathrm{Hom}_X(x,y)$ is local with respect to the morphism $f\colon A \longrightarrow B$ in the model structure for Kan complexes.*

Proof. Let $f\colon A \longrightarrow B$ be a morphism of simplicial sets and let X be a quasi-category. By Lemma 3.19, X is local with respect to the morphism $\Sigma(f)$ in $\mathbf{sSet}_{\mathrm{J}}$ if and only if the morphism of Kan complexes

$$J(X^{\Sigma(f)})\colon J(X^{\Sigma B}) \longrightarrow J(X^{\Sigma A}) \tag{3.32}$$

is a homotopy equivalence. By Lemma 3.19 and Remark 3.20, for each pair of objects x, y of X, the hom-space $\mathrm{Hom}_X(x,y)$ is local with respect to the morphism f in $\mathbf{sSet}_{\mathrm{K}}$ if and only if the morphism of Kan complexes

$$\mathrm{Hom}_X(x,y)^f\colon \mathrm{Hom}_X(x,y)^B \longrightarrow \mathrm{Hom}_X(x,y)^A \tag{3.33}$$

is a homotopy equivalence. Hence it is required to prove that the morphism (3.32) is a homotopy equivalence if and only if the morphism (3.33) is a homotopy equivalence for each pair of objects x, y of X.

From the commutative diagram of simplicial sets on the left below

$$\begin{array}{ccc} & \partial\Delta^1 & \\ {\scriptstyle (\bot,\top)}\swarrow & & \searrow{\scriptstyle (\bot,\top)} \\ \Sigma A & \xrightarrow[\Sigma(f)]{} & \Sigma B \end{array} \qquad \begin{array}{ccc} J(X^{\Sigma B}) & \xrightarrow{J(X^{\Sigma(f)})} & J(X^{\Sigma A}) \\ & \searrow \quad \swarrow & \\ & J(X \times X) & \end{array}$$

we obtain the commutative diagram on the right above, in which the diagonal morphisms are Kan fibrations. By the standard result recalled above, the morphism $J(X^{\Sigma(f)})$ is a homotopy equivalence if and only if, for each pair of objects x, y of X, the induced morphism between the fibres over (x, y) is a homotopy equivalence. Therefore, the result follows from the observation that, for each pair of objects x, y of X, this induced morphism between the fibres is none other than the morphism (3.33). This can be seen as follows.

For each simplicial set U, since the functor $V \mapsto X^V$ sends pushouts to pullbacks, the quasi-category $X^{\Sigma U}$ is given by the pullback on the right below.

$$\begin{array}{ccccc} \mathrm{Hom}_X(x,y)^U & \longrightarrow & X^{\Sigma U} & \longrightarrow & (X^{\Delta^1})^U \\ \big\downarrow & \lrcorner & \big\downarrow & \lrcorner & \big\downarrow \\ \Delta^0 & \xrightarrow[(x,y)]{} & X \times X & \longrightarrow & (X \times X)^U \end{array}$$

Since the functor $(-)^U$ preserves limits, we see by the pasting lemma for pullbacks that the fibre of the isofibration $X^{\Sigma U} \longrightarrow X \times X$ over a pair of objects (x, y) is the Kan complex $\mathrm{Hom}_X(x, y)^U$, and hence, upon application of the limit preserving functor J, that this Kan complex is also the fibre of the Kan fibration $J(X^{\Sigma U}) \longrightarrow J(X \times X)$ over (x, y). A further application of the pasting lemma to the diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_X(x,y)^B & \xrightarrow{\mathrm{Hom}_X(x,y)^f} & \mathrm{Hom}_X(x,y)^A & \longrightarrow & \Delta^0 \\
\big\downarrow \lrcorner & & \big\downarrow \lrcorner & & \big\downarrow{\scriptstyle (x,y)} \\
J(X^{\Sigma B}) & \xrightarrow[J(X^{\Sigma(f)})]{} & J(X^{\Sigma A}) & \longrightarrow & J(X \times X)
\end{array}$$

shows that the morphism (3.33) is the pullback of the morphism (3.32), seen as a morphism of simplicial sets over $J(X \times X)$, along the morphism $(x, y) \colon \Delta^0 \longrightarrow J(X \times X)$, as required. □

By applying this proposition to the morphism $\partial\Delta^{n+1} \longrightarrow \Delta^{n+1}$, we obtain an alternative characterisation of n-truncated quasi-categories as local objects, and thus an alternative description of the model structure for n-truncated quasi-categories as a Bousfield localisation of the model structure for quasi-categories.

Corollary 3.34. *A quasi-category is n-truncated if and only if it is local with respect to the morphism $\Sigma(\partial\Delta^{n+1} \longrightarrow \Delta^{n+1})$ in the model structure for quasi-categories. Hence the model structure for n-truncated quasi-categories is the Bousfield localisation of the model structure for quasi-categories with respect to the morphism $\Sigma(\partial\Delta^{n+1} \longrightarrow \Delta^{n+1})$.*

Proof. By Corollary 3.25 (or Remark 3.26, if $n = -1$), a Kan complex is an $(n-1)$-type if and only if it is local with respect to the boundary inclusion $\partial\Delta^{n+1} \longrightarrow \Delta^{n+1}$ in the model structure for Kan complexes. Hence the result follows from Proposition 3.31. □

4. Categorical n-equivalences

Throughout this section, let $n \geq 0$ be an integer.

A morphism of simplicial sets is said to be a *weak categorical n-equivalence* if it is a weak equivalence in the model structure for n-truncated quasi-categories established in Theorems 3.27 and 3.28. Since this model structure is a Bousfield localisation of the model structure for quasi-categories, the class of weak categorical n-equivalences enjoys the following characterisation.

Lemma 4.1 ([16, §26.5])**.** *A morphism of simplicial sets $f\colon A \longrightarrow B$ is a weak categorical n-equivalence if and only if the function*

$$(\mathrm{Ho}\,\mathbf{sSet}_{\mathrm{J}})(f, X) : (\mathrm{Ho}\,\mathbf{sSet}_{\mathrm{J}})(B, X) \longrightarrow (\mathrm{Ho}\,\mathbf{sSet}_{\mathrm{J}})(A, X)$$

is a bijection for each n-truncated quasi-category X.

Proof. Since the weak categorical n-equivalences are the weak equivalences in the model structure for n-truncated quasi-categories, which is a Bousfield localisation of the model structure for quasi-categories, this is an instance of Lemma A.2. □

The main goal of this section is to prove that a morphism of quasi-categories is a weak categorical n-equivalence if and only if it is a categorical n-equivalence, in the sense of the following definitions. (We reiterate that this result was stated without proof in [16, §26.6].)

Definition 4.2. If $n \geq 1$, a morphism of quasi-categories $f\colon X \longrightarrow Y$ is said to be a *categorical n-equivalence* if it is essentially surjective on objects, and if for each pair of objects $x, y \in X$, the induced morphism of hom-spaces $f = f_{x,y}\colon \mathrm{Hom}_X(x, y) \longrightarrow \mathrm{Hom}_Y(fx, fy)$ is a homotopy $(n-1)$-equivalence.

Let us first examine the lowest dimensional case of this definition.

Proposition 4.3. *A morphism of quasi-categories is a categorical* 1-*equivalence if and only if it is sent by the fundamental category functor $\tau_1\colon$ **sSet** $\longrightarrow$ **Cat** to an equivalence of categories.*

Proof. Recall that the restriction of the fundamental category functor to the full subcategory of quasi-categories is naturally isomorphic to the homotopy category functor. Let $f\colon X \longrightarrow Y$ be a morphism of quasi-categories. By definition, f is essentially surjective on objects if and only if the induced functor between homotopy categories $\mathrm{ho}(f)\colon\ \mathrm{ho}\,X \longrightarrow \mathrm{ho}\,Y$ is essentially surjective on objects. By construction, the functor $\mathrm{ho}(f)$ is fully faithful if and only if f is a homotopy 0-equivalence on hom-spaces. Therefore the morphism of quasi-categories f is a categorical 0-equivalence if and only if the functor $\mathrm{ho}(f)$ is an equivalence of categories. □

Similarly, let us make the following definition (cf. Proposition 3.10). Recall that the category **Pos** of posets is a reflective subcategory of **Cat**; the poset reflection of a category A is the poset quotient of its preorder reflection, where the latter is the preorder whose objects are those of A and in which one has $a \leq b$ if and only if the hom-set $A(a, b)$ is nonempty. Thus one obtains a composite adjunction

$$\mathbf{Pos} \underset{N}{\overset{\tau_p}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{sSet} \tag{4.4}$$

whose fully faithful right adjoint sends a poset to its nerve, and whose left adjoint sends a simplicial set to the poset reflection of its fundamental category.

Definition 4.5. A morphism of quasi-categories is said to be a *categorical* 0*-equivalence* if it is sent by the functor $\tau_p \colon \mathbf{sSet} \longrightarrow \mathbf{Pos}$ to an isomorphism of posets.

Remark 4.6. Unpacking this definition, one finds that a morphism of quasi-categories $f \colon X \longrightarrow Y$ is a categorical 0-equivalence if and only if it satisfies the following two properties:

(i) for each object $z \in Y$, there exists an object $x \in X$ and a pair of morphisms $Fx \longrightarrow z$ and $z \longrightarrow Fx$ in Y, and

(ii) for each pair of objects $x, y \in X$, the induced morphism

$$f \colon \mathrm{Hom}_X(x, y) \longrightarrow \mathrm{Hom}_Y(fx, fy)$$

is a homotopy (-1)-equivalence.

If Y is a 0-truncated quasi-category, then any endomorphism in Y is necessarily an isomorphism, and so a morphism of quasi-categories $f \colon X \longrightarrow Y$ satisfies property (i) if and only if it is essentially surjective on objects.

Remark 4.7. To prevent a proliferation of cases, we have made the global assumption $n \geq 0$ in this section. The $n = -1$ case of the problem of this section is easily dispensed with: since the model structure for

(-1)-truncated quasi-categories coincides with the model structure for (-1)-types, a morphism of simplicial sets is a weak categorical (-1)-equivalence if and only if it is inverted by the functor $\pi_{-1}\colon \mathbf{sSet} \longrightarrow \{0 < 1\}$ that sends the empty simplicial set to 0 and every nonempty simplicial set to 1.

Next, we establish a few useful properties of the class of categorical n-equivalences.

Lemma 4.8. *Let $f\colon X \longrightarrow Y$ be a morphism of n-truncated quasi-categories. Then the following properties are equivalent.*

(i) *f is an equivalence of quasi-categories.*

(ii) *f is a weak categorical n-equivalence.*

(iii) *f is a categorical n-equivalence.*

Proof. The equivalence (i) $\iff$ (ii) is a consequence of the fact that the model structure for n-truncated quasi-categories is a Bousfield localisation of the model structure for quasi-categories.

To prove the equivalence (i) $\iff$ (iii), recall that a morphism of quasi-categories is an equivalence if and only if it is essentially surjective on objects and a homotopy equivalence on hom-spaces, and that a morphism between $(n-1)$-types is a homotopy equivalence if and only if it is a homotopy $(n-1)$-equivalence. Since the hom-spaces of n-truncated quasi-categories are $(n-1)$-types, we see that a morphism of n-truncated quasi-categories is an equivalence if and only if it is a categorical n-equivalence (by Remark 4.6 if $n = 0$). □

Lemma 4.9. *Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be morphisms of quasi-categories. If two of the morphisms f, g, gf are categorical n-equivalences, then so is the third.*

Proof. The class of categorical 0-equivalences was defined as the class of morphisms of quasi-categories inverted by a functor, and therefore satisfies the stated property.

Note that by the functoriality of the hom-space construction, the composite morphism $gf\colon X \longrightarrow Z$ is given on hom-spaces by the composite morphism

$$\mathrm{Hom}_X(x,x') \xrightarrow{\ f\ } \mathrm{Hom}_Y(fx,fx') \xrightarrow{\ g\ } \mathrm{Hom}_Z(gfx,gfx'). \tag{4.10}$$

Suppose $n \geq 1$. We must consider three cases. In the first, suppose f and g are categorical n-equivalences. Since the class of essentially surjective on objects morphisms of quasi-categories and the class of homotopy $(n-1)$-equivalences of Kan complexes are both closed under composition (by Lemma 2.15), we have that the composite morphism $gf\colon X \longrightarrow Z$ is a categorical n-equivalence.

In the second case, suppose that g and gf are categorical n-equivalences. To show that f is essentially surjective on objects, it suffices to show that the functor $\mathrm{ho}(f)\colon \mathrm{ho}\,X \longrightarrow \mathrm{ho}\,Y$ is essentially surjective on objects. This follows from the assumptions (which hold since $n \geq 1$) that the functor $\mathrm{ho}(gf)$ is essentially surjective on objects and that the functor $\mathrm{ho}(g)$ is fully faithful. Since gf is given on hom-spaces by the composite (4.10), we have that f is a homotopy $(n-1)$-equivalence on hom-spaces by Lemma 2.15.

In the third case, suppose that f and gf are categorical n-equivalences. Since gf is essentially surjective on objects, it follows that g is essentially surjective on objects. To show that g is a homotopy $(n-1)$-equivalence on hom-spaces, let y, y' be a pair of objects of Y. Since f is essentially surjective on objects, there exist objects $x, x' \in X$ and isomorphisms $u\colon fx \cong y$ and $v\colon fx' \cong y'$ in Y. Thus we have a commutative diagram of quasi-categories as on the left below in which the vertical morphisms are equivalences of quasi-categories (by the Quillen adjunction (3.18)),

$$\begin{array}{ccccc}
 & & Y & \xrightarrow{\ g\ } & Z \\
 & {\scriptstyle (y,y')}\nearrow & \uparrow{\scriptstyle Y^{(\delta^0)}} & & \uparrow{\scriptstyle Z^{(\delta^0)}} \\
\partial\Delta^1 & \xrightarrow{(u,v)} & Y^{(\Delta^1)} & \xrightarrow{g^{(\Delta^1)}} & Z^{(\Delta^1)} \\
 & {\scriptstyle (fx,fx')}\searrow & \downarrow{\scriptstyle Y^{(\delta^1)}} & & \downarrow{\scriptstyle Z^{(\delta^1)}} \\
 & & Y & \xrightarrow[\ g\ ]{} & Z
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Hom}_Y(y,y') & \xrightarrow{\ g\ } & \mathrm{Hom}_Z(gy,gy') \\
\uparrow{\scriptstyle Y^{(\delta^0)}} & & \uparrow{\scriptstyle Z^{(\delta^0)}} \\
\mathrm{Hom}_{Y^{(\Delta^1)}}(u,v) & \xrightarrow{g^{(\Delta^1)}} & \mathrm{Hom}_{Z^{(\Delta^1)}}(gu,gv) \\
\downarrow{\scriptstyle Y^{(\delta^1)}} & & \downarrow{\scriptstyle Z^{(\delta^1)}} \\
\mathrm{Hom}_Y(fx,fx') & \xrightarrow[\ g\ ]{} & \mathrm{Hom}_Z(gfx,gfx')
\end{array}$$

and which therefore induces a commutative diagram of Kan complexes as on the right above in which the vertical morphisms are homotopy equivalences, and hence also homotopy $(n-1)$-equivalences. Hence, by Lemma 2.15, the morphism $g\colon \mathrm{Hom}_Y(y,y') \longrightarrow \mathrm{Hom}_Z(gy,gy')$ is a homotopy $(n-1)$-equivalence if and only if the morphism $g\colon \mathrm{Hom}_Y(fx,fx') \longrightarrow \mathrm{Hom}_Z(gfx,gfx')$ is a homotopy $(n-1)$-equivalence. But the latter morphism is a homotopy $(n-1)$-equivalence by Lemma 2.15, since the composite morphism (4.10) and its first factor are homotopy $(n-1)$-equivalences by assumption. □

By construction (3.4), the suspension ΣU of an n-skeletal simplicial set U is $(n+1)$-skeletal (since it is a colimit of $(n+1)$-skeletal simplicial sets). Hence the n-skeleta of the hom-spaces $\mathrm{Hom}_X(x,y)$ of a quasi-category X depend only on the $(n+1)$-skeleton of X. This implies that an $(n+1)$-bijective morphism of quasi-categories $f\colon X \longrightarrow Y$ induces n-bijective morphisms on hom-spaces $f\colon \mathrm{Hom}_X(x,y) \longrightarrow \mathrm{Hom}_Y(fx,fy)$. We may therefore deduce the following lemma from Lemma 2.16.

Lemma 4.11. *An $(n+1)$-bijective morphism of quasi-categories is a categorical n-equivalence.*

Proof. Let $f\colon X \longrightarrow Y$ be an $(n+1)$-bijective morphism of quasi-categories. Then f is a 0-bijection, and hence in particular (essentially) surjective on objects (if $n=0$, note that this implies property (i) of Remark 4.6). Furthermore, for each pair of objects x,y of X, the induced morphism on hom-spaces $\mathrm{Hom}_X(x,y) \longrightarrow \mathrm{Hom}_Y(fx,fy)$ is n-bijective as above, and hence is a homotopy $(n-1)$-equivalence by Lemma 2.16. Therefore f is a categorical n-equivalence. □

Following [16, §26.7], define a *categorical n-truncation* of a simplicial set A to be a fibrant replacement of A in the model structure for n-truncated quasi-categories, that is, an n-truncated quasi-category X together with a weak categorical n-equivalence $A \longrightarrow X$. In the next two propositions, we will prove that the $(n+1)$-coskeleton of a quasi-category is a model for its categorical n-truncation (cf. [2, §1] or [7, §9.1], where the $(n+1)$-coskeleton of a Kan complex is given as a model for its nth Postnikov truncation). We will then use these results to prove the main theorem of this section.

Proposition 4.12. *Let X be a quasi-category. Then its $(n+1)$-coskeleton $\mathrm{cosk}_{n+1}X$ is an n-truncated quasi-category, and the unit morphism $X \longrightarrow \mathrm{cosk}_{n+1}X$ is a categorical n-equivalence.*

Proof. First, to prove that $\mathrm{cosk}_{n+1}X$ is a quasi-category, it is required to prove that it has the right lifting property with respect to the inner horn inclusion $h^k_m \colon \Lambda^m_k \longrightarrow \Delta^m$ for every $m \geq 2$ and $0 < k < m$. By adjointness, this is so if and only if X has the right lifting property with respect to the morphism $\mathrm{sk}_{n+1}(h^k_m) \colon \mathrm{sk}_{n+1}\Lambda^m_k \longrightarrow \mathrm{sk}_{n+1}\Delta^m$. Consider the following three cases. If $m \leq n+1$, then the morphism $\mathrm{sk}_{n+1}(h^k_m)$ is the inner horn inclusion h^k_m, with respect to which X has the right lifting property since it is a quasi-category. If $m = n+2$, then the morphism $\mathrm{sk}_{n+1}(h^k_m)$ is isomorphic to the inclusion $\Lambda^m_k \longrightarrow \partial\Delta^m$, with respect to which X has the right lifting property, since it has this property with respect to the composite $\Lambda^m_k \longrightarrow \partial\Delta^m \longrightarrow \Delta^m$, since it is a quasi-category. If $m > n+2$, then the morphism $\mathrm{sk}_{n+1}(h^m_k)$ is an isomorphism, with respect to which therefore X has the unique right lifting property.

Next, to show that the quasi-category $\mathrm{cosk}_{n+1}X$ is n-truncated, it suffices to observe that the unit morphism $\mathrm{cosk}_{n+1}X \longrightarrow \mathrm{cosk}_{n+1}\mathrm{cosk}_{n+1}X$ is an isomorphism (since cosk_{n+1} is an idempotent monad), for then cosk_{n+1} is n-truncated by Corollary 3.16.

Finally, since the unit morphism $X \longrightarrow \mathrm{cosk}_{n+1}X$ is an $(n+1)$-bijective morphism of quasi-categories, it is a categorical n-equivalence by Lemma 4.11. □

Let $J = k([1])$ denote the nerve of the "free-living isomorphism", i.e. the nerve of the groupoid reflection of the ordered set $\{0 < 1\}$. By [15, Proposition 6.18], for any simplicial set A and quasi-category X, the hom-set $(\mathrm{Ho}\,\mathbf{sSet}_{\mathrm{J}})(A, X)$ is in bijection with the set of J-homotopy classes of morphisms $A \longrightarrow X$, where two such morphisms f, g belong to the same J-homotopy class if and only if there exists a morphism $h \colon J \times A \longrightarrow X$ such that $h \circ (\{0\} \times \mathrm{id}) = f$ and $h \circ (\{1\} \times \mathrm{id}) = g$.

Proposition 4.13. *Let A be a simplicial set. Then the unit morphism $A \longrightarrow \mathrm{cosk}_{n+1}A$ is a weak categorical n-equivalence.*

Proof. Let $\eta_A \colon A \longrightarrow \mathrm{cosk}_{n+1}A$ denote the unit morphism in question.

By Lemma 4.1, it is required to prove that the function

$$(\mathrm{Ho}\,\mathbf{sSet}_{\mathrm{J}})(\eta_A, X) : (\mathrm{Ho}\,\mathbf{sSet}_{\mathrm{J}})(\mathrm{cosk}_{n+1}A, X) \longrightarrow (\mathrm{Ho}\,\mathbf{sSet}_{\mathrm{J}})(A, X)$$

is a bijection for each n-truncated quasi-category X, which, without loss of generality, we may assume to be $(n+1)$-coskeletal by Lemma 3.16.

Let X be an $(n+1)$-coskeletal quasi-category. To show that the function displayed above is injective, let $f, g \colon \mathrm{cosk}_{n+1}A \longrightarrow X$ be a pair of morphisms of simplicial sets, and let $h \colon J \times A \longrightarrow X$ be a J-homotopy from $f\eta_A$ to $g\eta_A$. Then the morphism

$$J \times \mathrm{cosk}_{n+1}A \cong \mathrm{cosk}_{n+1}(J \times A) \xrightarrow{\mathrm{cosk}_{n+1}(h)} \mathrm{cosk}_{n+1}X \cong X$$

defines a J-homotopy from f to g (where we have used that the functor cosk_{n+1} preserves products and that J is 0-coskeletal). Hence the function is injective. To show that it is surjective, let $f \colon X \longrightarrow Y$ be a morphism of simplicial sets. Then the morphism

$$\mathrm{cosk}_{n+1}X \xrightarrow{\mathrm{cosk}_{n+1}(f)} \mathrm{cosk}_{n+1}Y \cong Y$$

defines an extension of f along the unit morphism η_A. Hence the function is surjective, and is therefore a bijection. □

We are now ready to prove the main theorem of this section.

Theorem 4.14 ([16, §26.6])**.** *A morphism of quasi-categories is a weak categorical n-equivalence if and only if it is a categorical n-equivalence.*

Proof. This statement is true of morphisms of n-truncated quasi-categories by Lemma 4.8. Let $f \colon X \longrightarrow Y$ be a morphism of quasi-categories. In the commutative diagram displayed below,

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \downarrow & & \downarrow \\ \mathrm{cosk}_{n+1}X & \xrightarrow[\mathrm{cosk}_{n+1}(f)]{} & \mathrm{cosk}_{n+1}Y \end{array}$$

the vertical morphisms are weak categorical n-equivalences by Proposition 4.13 and categorical n-equivalences by Proposition 4.12, and the bottom morphism is a morphism of n-truncated quasi-categories by Proposition 4.12. Since the class of weak categorical n-equivalences and the class of categorical n-equivalences both satisfy the two-of-three property (the one since it is the class of weak equivalences of a model category by definition, the other by Lemma 4.9), it follows that f inherits from $\mathrm{cosk}_{n+1}(f)$ the property that it is a weak categorical n-equivalence if and only if it is a categorical n-equivalence. □

Remark 4.15. In [16, §26.6], it is incorrectly stated that a morphism of quasi-categories is a (weak) categorical 0-equivalence if and only if it is essentially surjective on objects and a homotopy (-1)-equivalence on hom-spaces. This statement can be corrected by replacing the property "essentially surjective on objects" by the weaker property (i) in Remark 4.6. For a counterexample, let C be the category freely generated by the graph displayed below,

$$\bullet \rightleftarrows \bullet$$

and let $1 \longrightarrow C$ be the functor corresponding to either of the two objects of C. This functor is not essentially surjective on objects, but its poset reflection is an isomorphism. Hence the nerve of this functor is an example of a categorical 0-equivalence that is not essentially surjective on objects.

5. Some Quillen equivalences

In this final section, we use the criteria proved at the end of Appendix A to prove Quillen equivalences between the model categories of categories and 1-truncated quasi-categories and between the model categories of n-truncated quasi-categories and Rezk's $(n,1)$-Θ-spaces.

To begin, recall that the adjunction $\tau_1 \dashv N\colon \mathbf{Cat} \longrightarrow \mathbf{sSet}$ (3.5), whose right adjoint sends a category A to its nerve NA and whose left adjoint sends a simplicial set X to its fundamental category $\tau_1 X$, is a Quillen adjunction, and moreover a homotopy reflection (i.e. its derived right adjoint is fully faithful), between the folk model structure for categories (whose weak equivalences are the equivalences of categories) and Joyal's model structure for quasi-categories [15, Proposition 6.14]. Using Theorem A.14 and the results of §3, we can show that this adjunction is moreover a Quillen equivalence between the folk model structure for categories and the model structure for 1-truncated quasi-categories.

Theorem 5.1. *The adjunction*

$$\mathbf{Cat} \underset{N}{\overset{\tau_1}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{sSet}$$

is a Quillen equivalence between the folk model structure for categories and the model structure for 1*-truncated quasi-categories.*

Proof. By Theorem A.14, we must prove that the nerve of a category is a 1-truncated quasi-category, and that, for any 1-truncated quasi-category X, the unit morphism $X \longrightarrow N(\tau_1 X)$ is an equivalence of quasi-categories. These both follow from Proposition 3.8. □

Corollary 5.2 ([16, §26.6])**.** *A morphism of simplicial sets is a weak categorical* 1*-equivalence if and only if it sent by the functor* $\tau_1 \colon \mathbf{sSet} \longrightarrow \mathbf{Cat}$ *to an equivalence of categories.*

Proof. Since the functor τ_1 is the left adjoint of a Quillen equivalence by Theorem 5.1, and since every simplicial set is cofibrant in the model structure for 1-truncated quasi-categories, this is an instance of the fact that the left adjoint of a Quillen equivalence preserves and reflects weak equivalences between cofibrant objects. □

Recall the adjunction $\tau_p \dashv N \colon \mathbf{Pos} \longrightarrow \mathbf{sSet}$ (4.4), whose fully faithful right adjoint sends a poset to its nerve, and whose left adjoint sends a simplicial set to the poset reflection of its fundamental category. We now show that this adjunction is a Quillen equivalence between the trivial model structure (i.e. the unique model structure whose weak equivalences are the isomorphisms) on the category of posets and the model structure for 0-truncated quasi-categories.

Theorem 5.3. *The adjunction*

$$\mathbf{Pos} \underset{N}{\overset{\tau_p}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{sSet}$$

is a Quillen equivalence between the trivial model structure for posets and the model structure for 0*-truncated quasi-categories.*

Proof. To see that this adjunction is a Quillen adjunction between the trivial model structure for posets and the model structure for quasi-categories, it suffices to observe that each weak categorical equivalence is sent by the functor τ_p to an isomorphism of posets. But this functor is the composite of the functor $\tau_1 \colon \mathbf{sSet} \longrightarrow \mathbf{Cat}$, which sends each weak categorical equivalence to an equivalence of categories, and the poset reflection functor $\tau_p \colon \mathbf{Cat} \longrightarrow \mathbf{Pos}$, which is easily shown to invert equivalences of categories.

It remains to verify conditions (i) and (ii) of Theorem A.14. Firstly, by Proposition 3.10, the nerve NA of a poset A is a 0-truncated quasi-category, which verifies condition (i). Secondly, a 0-truncated quasi-category X is in particular 1-truncated, and so by Proposition 3.8 the unit morphism $X \longrightarrow N(\operatorname{ho} X)$ is an equivalence of quasi-categories. But by Proposition 3.10, $\operatorname{ho} X$ is a preorder and hence $N(\operatorname{ho} X)$ is a 0-truncated quasi-category. This verifies condition (ii). □

Corollary 5.4 ([16, §26.6])**.** *A morphism of simplicial sets is a weak categorical* 0*-equivalence if and only if it is sent by the functor* $\tau_p \colon \mathbf{sSet} \longrightarrow \mathbf{Pos}$ *to an isomorphism of posets.*

Proof. Since the functor τ_p is the left adjoint of a Quillen equivalence by Theorem 5.3, and since every simplicial set is cofibrant in the model structure for 0-truncated quasi-categories, this is another instance of the fact that the left adjoint of a Quillen equivalence preserves and reflects weak equivalences between cofibrant objects. □

Recall that a categorical n-truncation of a simplicial set A is an n-truncated quasi-category X together with a weak categorical n-equivalence $A \longrightarrow X$.

Corollary 5.5 ([16, §26.7])**.** *For each simplicial set A, the unit morphism $A \longrightarrow N(\tau_1 A)$ is a categorical* 1*-truncation of A, and the unit morphism $A \longrightarrow N(\tau_p A)$ is a categorical* 0*-truncation of A.*

Proof. By Corollaries 5.2 and 5.4, it suffices to show that these unit morphisms are sent to isomorphisms by the functors τ_1 and τ_p respectively.

In each case, this is an instance of the fact that each component of the unit of an adjunction whose right adjoint is fully faithful is sent by the left adjoint to an isomorphism. □

Next, we prove, for each $n \geq -1$, two Quillen equivalences between the model categories of n-truncated quasi-categories and Rezk's $(n,1)$-Θ-spaces. In [17], Joyal and Tierney established two Quillen equivalences

$$[\Delta^{\mathrm{op}}, \mathbf{Set}] \underset{t^!}{\overset{t_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} [(\Delta \times \Delta)^{\mathrm{op}}, \mathbf{Set}] \tag{5.6}$$

$$[(\Delta \times \Delta)^{\mathrm{op}}, \mathbf{Set}] \underset{i_1^*}{\overset{p_1^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} [\Delta^{\mathrm{op}}, \mathbf{Set}]$$

between Joyal's model structure for quasi-categories and Rezk's model structure for complete Segal spaces on the category of bisimplicial sets (defined in [21]). Suffice it to recall that the functor $t^!$ sends a simplicial set A to the bisimplicial set $t^!(A)$ whose nth column $t^!(A)_n$ is the simplicial set $k^!(A^{\Delta^n})$ (where $k^!$ denotes the right adjoint of the Quillen adjunction (3.22)), that the functor i_1^* sends a bisimplicial set X to its zeroth row X_{*0}, and that there are natural isomorphisms $t_! p_1^* \cong \mathrm{id}$ and $i_1^* t^! \cong \mathrm{id}$. For each complete Segal space X, we refer to the elements of the set X_{00} as the objects of X; there is an evident bijection between the objects of a quasi-category A and the objects of its associated complete Segal space $t^!(A)$.

For each $n \geq -1$, Rezk constructed in [22] a Bousfield localisation of the model structure for complete Segal spaces, whose fibrant objects are the complete Segal spaces X each of whose hom-spaces $\mathrm{Hom}_X(x,y)$ is an $(n-1)$-type [22, Proposition 11.20]. Rezk calls complete Segal spaces with this property $(n,1)$-Θ-spaces, but for convenience we will call them *n-truncated complete Segal spaces*, and we will call this model structure the *model structure for n-truncated complete Segal spaces*. Recall from [21, §5.1] that for each pair of objects x, y of a complete Segal space X, the hom-space $\mathrm{Hom}_X(x,y)$ is defined to be the pullback

$$\begin{array}{ccc} \mathrm{Hom}_X(x,y) & \longrightarrow & X_1 \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle (d_1,d_0)} \\ \Delta^0 & \xrightarrow[(x,y)]{} & X_0 \times X_0 \end{array}$$

in the category of simplicial sets (where X_n denotes the nth column of the bisimplicial set X). By comparison with the definition of the hom-spaces of a quasi-category (3.2), one sees that there is a canonical isomorphism

$$\mathrm{Hom}_{t^!(A)}(x,y) \cong k^!(\mathrm{Hom}_A(x,y)) \tag{5.7}$$

for each pair of objects x,y in a quasi-category A, since the right adjoint functor $k^!$ preserves limits.

We now apply Theorem A.15 to prove that the two Quillen equivalences (5.6) remain Quillen equivalences between the Bousfield localisations for n-truncated quasi-categories and n-truncated complete Segal spaces. The following proposition shows that these Quillen equivalences satisfy the hypotheses of that theorem.

Proposition 5.8. *Let $n \geq -1$ be an integer.*

1. *A quasi-category A is n-truncated if and only if the complete Segal space $t^!(A)$ is n-truncated.*

2. *A complete Segal space X is n-truncated if and only if its underlying quasi-category $i_1^*(X)$ is n-truncated.*

Proof. (1) Let A be a quasi-category. For each pair of objects $x,y \in A$, there is a homotopy equivalence $\mathrm{Hom}_{t^!(A)}(x,y) \simeq \mathrm{Hom}_A(x,y)$ by the isomorphism (5.7) and [15, Proposition 6.26]. Hence the hom-spaces of A are $(n-1)$-types if and only if the hom-spaces of $t^!(A)$ are $(n-1)$-types, that is, A is an n-truncated quasi-category if and only if $t^!(A)$ is an n-truncated complete Segal space.

(2) Let X be a complete Segal space. There is a span of weak equivalences in the model structure for complete Segal spaces

$$X \longleftarrow p_1^*(i_1^*X) \longrightarrow t^!(i_1^*X),$$

where the left-pointing arrow is the counit of the Quillen equivalence $p_1^* \dashv i_1^*$ and the right-pointing arrow is the transpose of the canonical isomorphism $t_!(p_1^*(i_1^*X)) \cong i_1^*X$ under the Quillen equivalence $t_! \dashv t^!$, both of which are weak equivalences since X is fibrant. Hence X is weakly equivalent to the complete Segal space $t^!(i_1^*X)$, and so X is n-truncated if and only if $t^!(i_1^*X)$ is n-truncated, which by (1) is so if and only if the quasi-category i_1^*X is n-truncated. □

Hence the adjunctions (5.6) satisfy the hypotheses of Theorem A.15, and we may deduce the following theorem.

Theorem 5.9. *For each integer $n \geq -1$, the adjunctions*

$$[\Delta^{\mathrm{op}}, \mathbf{Set}] \underset{t^!}{\overset{t_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} [(\Delta \times \Delta)^{\mathrm{op}}, \mathbf{Set}]$$

$$[(\Delta \times \Delta)^{\mathrm{op}}, \mathbf{Set}] \underset{i_1^*}{\overset{p_1^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} [\Delta^{\mathrm{op}}, \mathbf{Set}]$$

are Quillen equivalences between the model structure for n-truncated quasi-categories on the category of simplicial sets and the model structure for n-truncated complete Segal spaces on the category of bisimplicial sets.

Proof. By [17], these adjunctions are Quillen equivalences between the model structures for quasi-categories and complete Segal spaces. For each $n \geq -1$, the model structures for n-truncated quasi-categories and n-truncated complete Segal spaces are Bousfield localisations of the former model structures, and so it remains to show that these adjunctions satisfy the conditions of Theorem A.15. But this is precisely what was shown in Proposition 5.8. □

Remark 5.10. In [22, §11], Rezk defines the model structure for n-truncated complete Segal spaces as the Bousfield localisation of the model structure for complete Segal spaces with respect to the morphism denoted therein by $V[1](\partial\Delta^{n+1} \longrightarrow \Delta^{n+1})$. One can show that the left adjoint functor $t_!$ sends this morphism to the morphism of simplicial sets $\Sigma(k_!(\partial\Delta^{n+1} \longrightarrow \Delta^{n+1}))$. Since there is a natural weak homotopy equivalence id $\longrightarrow k_!$ [15, Theorem 6.22], and since the suspension functor Σ sends weak homotopy equivalences to weak categorical equivalences, it follows from Corollary 3.34 that the model structure for n-truncated quasi-categories is the Bousfield localisation of the model structure for quasi-categories with respect to the morphism $\Sigma(k_!(\partial\Delta^{n+1} \longrightarrow \Delta^{n+1}))$. Hence one could alternatively prove Proposition 5.8(1) and that half of Theorem 5.9 concerning the adjunction $t_! \dashv t^!$ by [12, Proposition 3.1.12] and [12, Theorem 3.3.20] respectively.

To conclude, we combine two previous theorems to deduce a Quillen equivalence between the folk model structure for categories and the model structure for 1-truncated complete Segal spaces. The right adjoint of this Quillen equivalence is the composite functor

$$t^!N\colon \mathbf{Cat} \longrightarrow [(\Delta \times \Delta)^{\mathrm{op}}, \mathbf{Set}],$$

which sends a category A to its *classifying diagram* [21, §3.5], which is the bisimplicial set whose nth column is the nerve of the maximal subgroupoid of the category $A^{[n]}$, and which was shown directly by Rezk to be a complete Segal space [21, Proposition 6.1].

Theorem 5.11. *The composite adjunction*

$$\mathbf{Cat} \underset{N}{\overset{\tau_1}{\rightleftarrows}} [\Delta^{\mathrm{op}}, \mathbf{Set}] \underset{t^!}{\overset{t_!}{\rightleftarrows}} [(\Delta \times \Delta)^{\mathrm{op}}, \mathbf{Set}],$$

whose right adjoint is Rezk's classifying diagram functor, is a Quillen equivalence between the folk model structure for categories and the model structure for 1*-truncated complete Segal spaces.*

Proof. This adjunction is the composite of the Quillen equivalence of Theorem 5.1 and the $n = 1$ case of one of the Quillen equivalences of Theorem 5.9, and is therefore a Quillen equivalence. □

A. Bousfield localisations

In this appendix, we recall some of the basic theory of Bousfield localisations of model categories (mostly those results that we use which are difficult to find explicitly stated in the literature in the form we use them), including two criteria for detecting Quillen equivalences between Bousfield localisations. We assume familiarity with the basic theory of model categories, such as is contained in [13, Chapter 1]; our approach is particularly influenced by the insightful appendices [17, §7] and [15, Appendix E].

We begin with the notion of a Bousfield localisation of a model category (after [17, Definition 7.20], in contrast to [12, Definition 3.3.1], where a Bousfield localisation is defined with respect to a given class of morphisms). Recall that a *model category* is a locally small complete and cocomplete category equipped with a *model structure*, which is determined by its classes $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of *cofibrations*, *weak equivalences*, and *fibrations*.

Definition A.1. A *Bousfield localisation* of a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a category $\mathcal{M}$ is a model structure $(\mathcal{C}_{\text{loc}}, \mathcal{W}_{\text{loc}}, \mathcal{F}_{\text{loc}})$ on the same category $\mathcal{M}$ such that $\mathcal{C}_{\text{loc}} = \mathcal{C}$ and $\mathcal{W} \subseteq \mathcal{W}_{\text{loc}}$.

We will often denote the model category determined by a Bousfield localisation of (the model structure of) a model category $\mathcal{M}$ by $\mathcal{M}_{\text{loc}}$, and call the morphisms belonging to the classes $\mathcal{W}_{\text{loc}}$ and $\mathcal{F}_{\text{loc}}$ *local weak equivalences* and *local fibrations* respectively; the fibrant objects of the model category $\mathcal{M}_{\text{loc}}$ we will call *local fibrant objects.* It is immediate from the definition that the adjunction

$$\mathcal{M}_{\text{loc}} \underset{1_{\mathcal{M}}}{\overset{1_{\mathcal{M}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{M},$$

whose left and right adjoints both are the identity functor on (the underlying category of) $\mathcal{M}$, is a Quillen adjunction. Hence every local fibration and local fibrant object is in particular a fibration and a fibrant object (in the model category $\mathcal{M}$) respectively. Moreover, the derived right adjoint of this Quillen adjunction is fully faithful, which is to say that the Quillen adjunction is a *homotopy reflection* in the sense of [15, Definition E.2.15] (the term *homotopy localisation* is used in [17]); it follows that a morphism between local fibrant objects is a weak equivalence if and only if it is a local weak equivalence [17, Proposition

7.18]. Furthermore, it follows by a factorisation and retract argument that a morphism between local fibrant objects is a fibration if and only if it is a local fibration [17, Proposition 7.21].

The model category axioms imply that a morphism is a local fibration if and only if it has the right lifting property with respect to the class of morphisms $\mathcal{C} \cap \mathcal{W}_{\text{loc}}$, whose members we call *local trivial cofibrations*. Hence a Bousfield localisation of a model category is determined by its class $\mathcal{W}_{\text{loc}}$ of local weak equivalences. Alternatively, a Bousfield localisation of a model category is determined by its class of local fibrant objects, since this class determines the local weak equivalences by the following argument (cf. [15, Proposition E.1.10] and [12, §3.5]). We denote the homotopy category of a model category $\mathcal{M}$ by $\operatorname{Ho}\mathcal{M}$; we will typically not distinguish an object or morphism of $\mathcal{M}$ from its image under the localisation functor $\mathcal{M} \longrightarrow \operatorname{Ho}\mathcal{M}$.

Lemma A.2. *Let $\mathcal{M}_{\text{loc}}$ be a Bousfield localisation of a model category $\mathcal{M}$. A morphism $f\colon A \longrightarrow B$ in $\mathcal{M}$ is a local weak equivalence if and only if the function*

$$(\operatorname{Ho}\mathcal{M})(f, X) : (\operatorname{Ho}\mathcal{M})(B, X) \longrightarrow (\operatorname{Ho}\mathcal{M})(A, X) \tag{A.3}$$

is a bijection for each local fibrant object X.

Proof. A morphism $f\colon A \longrightarrow B$ in $\mathcal{M}$ is a local weak equivalence if and only if it is (sent to) an isomorphism in the homotopy category $\operatorname{Ho}\mathcal{M}_{\text{loc}}$, which is so, by the Yoneda lemma, if and only if the function

$$(\operatorname{Ho}\mathcal{M}_{\text{loc}})(f, X) : (\operatorname{Ho}\mathcal{M}_{\text{loc}})(B, X) \longrightarrow (\operatorname{Ho}\mathcal{M}_{\text{loc}})(A, X) \tag{A.4}$$

is a bijection for each local fibrant object X (since every object of $\operatorname{Ho}\mathcal{M}_{\text{loc}}$ is isomorphic to a local fibrant object). By taking cofibrant replacements in $\mathcal{M}$, we may suppose $f\colon A \longrightarrow B$ to be a morphism between cofibrant objects. For each cofibrant object C and local fibrant object X, the sets $(\operatorname{Ho}\mathcal{M}_{\text{loc}})(C, X)$ and $(\operatorname{Ho}\mathcal{M})(C, X)$ are in bijection with the sets of homotopy classes of morphisms $C \longrightarrow X$ in the model categories $\mathcal{M}_{\text{loc}}$ and $\mathcal{M}$ respectively. But these latter sets coincide, since any cylinder object for C in the model category $\mathcal{M}$ is also a cylinder object for C in the model category $\mathcal{M}_{\text{loc}}$; hence there is a bijection $(\operatorname{Ho}\mathcal{M}_{\mathrm{loc}})(C,X) \cong (\operatorname{Ho}\mathcal{M})(C,X)$. The functions (A.3) and (A.4) correspond under these bijections, and so one is a bijection if and only if the other is. □

Hence a Bousfield localisation of a model category $\mathcal{M}$ can equivalently be defined as a model structure with the same underlying category and the same class of cofibrations as $\mathcal{M}$, but with fewer fibrant objects. This alternative definition makes it easy to recognise Bousfield localisations, as in the following example.

Example A.5. On the category **sSet** of simplicial sets, the model structure for Kan complexes is a Bousfield localisation of the model structure for quasi-categories, since the cofibrations are the monomorphisms in both model structures, and since every Kan complex is a quasi-category (see [17, §1] for details). Hence a morphism of Kan complexes is a homotopy equivalence if and only if it is an equivalence of quasi-categories.

We may thus regard a Bousfield localisation of a given model category as determined by its local fibrant objects. Given this perspective, the following lemma will be found useful (cf. [15, Proposition E.2.23] and [12, Proposition 3.3.15]). We say that two objects in a model category $\mathcal{M}$ are *weakly equivalent* if they are isomorphic in the homotopy category $\operatorname{Ho}\mathcal{M}$, that is, if they are connected by a zig-zag of weak equivalences in $\mathcal{M}$.

Lemma A.6. *Let $\mathcal{M}_{\mathrm{loc}}$ be a Bousfield localisation of a model category $\mathcal{M}$. A fibrant object of $\mathcal{M}$ is local fibrant if and only if it is weakly equivalent in $\mathcal{M}$ to a local fibrant object.*

Proof. The condition being obviously necessary, we prove its sufficiency. Suppose X is a fibrant object of $\mathcal{M}$ that is weakly equivalent to a local fibrant object Y. Since both objects are fibrant in $\mathcal{M}$, they are connected by a span of weak equivalences $X \longleftarrow Z \longrightarrow Y$ in which the object Z is fibrant. Hence it suffices to consider the two cases in which (i) there exists a weak equivalence $X \longrightarrow Y$, or (ii) there exists a weak equivalence $Y \longrightarrow X$.

In case (i), take a factorisation of the weak equivalence $X \longrightarrow Y$ into a trivial cofibration $X \longrightarrow W$ followed by a trivial fibration $W \longrightarrow Y$.

Since a trivial fibration is in particular a local fibration, W is a local fibrant object. Since X is fibrant, the trivial cofibration $X \longrightarrow W$ has a retraction, whence X is a retract of the local fibrant object W, and is therefore local fibrant.

In case (ii), let $X \longrightarrow X'$ be a local fibrant replacement of X. The composite $Y \longrightarrow X \longrightarrow X'$ is then a local weak equivalence between local fibrant objects, and hence is a weak equivalence. It then follows from the two-of-three property that $X \longrightarrow X'$ is a weak equivalence, and so X is local fibrant by case (i). □

One can use the following criterion involving local fibrant objects to determine when a Bousfield localisation of a cartesian model category is cartesian, at least when every object is cofibrant. Recall that a model category $\mathcal{M}$ is said to be *cartesian* (or *cartesian closed* [17, Definition 7.29]) if its underlying category is cartesian closed, its terminal object is cofibrant, and the product functor $- \times - \colon \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ is a *left Quillen bifunctor* [13, Definition 4.2.1].

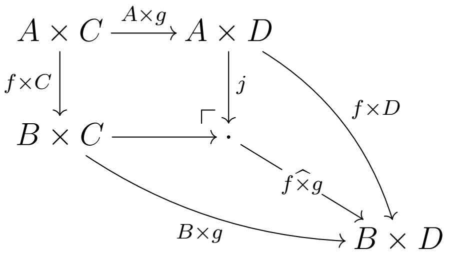
Proposition A.7. *Let $\mathcal{M}$ be a cartesian model category in which every object is cofibrant. A Bousfield localisation of $\mathcal{M}$ is cartesian if and only if the internal hom object X^A is local fibrant for every object A and every local fibrant object X of $\mathcal{M}$.*

Proof. The condition is necessary since every object is cofibrant and the internal hom functor of a cartesian model category is a right Quillen bifunctor.

To prove sufficiency, note that any Bousfield localisation of $\mathcal{M}$ inherits the properties that the terminal object is cofibrant and that the pushout-product of any two cofibrations is a cofibration; hence it remains to show that the pushout-product of a local trivial cofibration with a cofibration is a local trivial cofibration, or equivalently a local weak equivalence.

First, observe that for any local weak equivalence $f \colon A \longrightarrow B$ and any object C, the morphism $f \times C \colon A \times C \longrightarrow B \times C$ is a local weak equivalence. This follows from Lemma A.2, since for any local fibrant object X, the function $(\mathrm{Ho}\,\mathcal{M})(B \times C, X) \longrightarrow (\mathrm{Ho}\,\mathcal{M})(A \times C, X)$ is isomorphic to the function $(\mathrm{Ho}\,\mathcal{M})(B, X^C) \longrightarrow (\mathrm{Ho}\,\mathcal{M})(A, X^C)$ by [13, Theorem 4.3.2], and the latter function is a bijection by Lemma A.2 since X^C is local fibrant by assumption.

Now, let $f\colon A \longrightarrow B$ be a local trivial cofibration and let $g\colon C \longrightarrow D$ be a cofibration. Then in the diagram



we have that the morphisms $f \times C$, its pushout j, and $f \times D$ are local trivial cofibrations, and hence by the two-of-three property for local weak equivalences that the pushout-product $f \widehat{\times} g$ is a local trivial cofibration. $\square$

Let us now recall an existence theorem for Bousfield localisations due to Smith, which will enable us to recognise when a class of fibrant objects in a model category is the class of local fibrant objects for a Bousfield localisation of that model category. To state this theorem, it will be helpful to first recall some results from [13, Chapter 5] concerning the canonical enrichments of homotopy categories and derived adjunctions over the *classical homotopy category*, that is, the homotopy category $\mathrm{Ho}\,\mathbf{sSet}_{\mathrm{K}}$ of the category of simplicial sets equipped with the model structure for Kan complexes, which we denote by $\mathscr{H}$. By [10, §IV.3] (see also [13, Theorem 4.3.2]), $\mathscr{H}$ is a cartesian closed category, with terminal object Δ^0, and as such may be considered as a base for enriched category theory (for which, see [18, Chapter 1]). We use underlines to indicate $\mathscr{H}$-enriched categories; in particular, we denote the self-enrichment of $\mathscr{H}$ by $\underline{\mathscr{H}}$.

By [13, Theorem 5.5.3], for any model category $\mathcal{M}$, its homotopy category $\mathrm{Ho}\,\mathcal{M}$ admits a canonical enrichment over the cartesian closed category $\mathscr{H}$; we denote this $\mathscr{H}$-enriched category by $\underline{\mathrm{Ho}\,\mathcal{M}}$ and refer to its hom-objects as the *derived hom-spaces* of the model category $\mathcal{M}$. (Note that the $\mathscr{H}$-enrichment of the homotopy category $\mathrm{Ho}\,\mathcal{M}^{\mathrm{op}}$ defines an $\mathscr{H}$-enriched category isomorphic to the opposite of $\underline{\mathrm{Ho}\,\mathcal{M}}$.) Furthermore, by [13, Theorem 5.6.2], for any Quillen adjunction as on the left below,

$$\mathcal{M} \underset{G}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{N} \qquad\qquad \underline{\mathrm{Ho}\,\mathcal{M}} \underset{\mathbf{R}G}{\overset{\mathbf{L}F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \underline{\mathrm{Ho}\,\mathcal{N}}$$

its derived adjunction underlies an $\mathscr{H}$-enriched adjunction between $\mathscr{H}$-enriched homotopy categories as on the right above. We refer to the right adjoint of this $\mathscr{H}$-enriched adjunction as the *$\mathscr{H}$-enriched right derived functor* of G.

Now, let S be a set of morphisms in a model category $\mathcal{M}$. We say that an object X of $\mathcal{M}$ is *S-local*, or *local with respect to S*, if the induced morphism between derived hom-spaces

$$\underline{\mathrm{Ho}\,\mathcal{M}}(f, X) : \underline{\mathrm{Ho}\,\mathcal{M}}(B, X) \longrightarrow \underline{\mathrm{Ho}\,\mathcal{M}}(A, X) \tag{A.8}$$

is an isomorphism in $\mathscr{H}$ for each morphism $f\colon A \longrightarrow B$ belonging to S (cf. [12, Definition 3.1.4]). The following theorem gives sufficient conditions for the existence of the (necessarily unique) Bousfield localisation of $\mathcal{M}$ whose local fibrant objects are the S-local fibrant objects of $\mathcal{M}$; if it exists, we call this Bousfield localisation the *Bousfield localisation of $\mathcal{M}$ with respect to S*, and denote it by $\mathrm{L}_S\mathcal{M}$ (cf. [12, Definition 3.3.1]).

Example A.9. It follows from Proposition 3.30 that the model structure for Kan complexes on the category of simplicial sets is the Bousfield localisation of the model structure for quasi-categories with respect to the single morphism $\Delta^1 \longrightarrow \Delta^0$.

Remark A.10. The set of morphisms S may be thought of as a "presentation" of the Bousfield localisation $\mathrm{L}_S\mathcal{M}$ of $\mathcal{M}$ (if it exists). In general, a Bousfield localisation $\mathcal{M}_{\mathrm{loc}}$ of a left proper (see below) model category $\mathcal{M}$ admits many such presentations: in particular, one can always take S to be the (large) set of local weak equivalences; if the model category $\mathcal{M}_{\mathrm{loc}}$ is cofibrantly generated, one can take S to be a small set of generating trivial cofibrations for $\mathcal{M}_{\mathrm{loc}}$.

To state the existence theorem, we require the following technical conditions. A model category is said to be *left proper* if any pushout of a weak equivalence along a cofibration is a weak equivalence (see

[12, §13.1]; any model category in which every object is cofibrant is left proper), and is said to be *combinatorial* if it is cofibrantly generated and its underlying category is locally presentable (see [8, §2]). Every model category considered in this paper is both left proper and combinatorial.

Theorem A.11 (Smith). *Let $\mathcal{M}$ be a left proper combinatorial model category and let S be a small set of morphisms in $\mathcal{M}$. Then there exists a Bousfield localisation $\mathrm{L}_S\mathcal{M}$ of $\mathcal{M}$ whose local fibrant objects are precisely the S-local fibrant objects of $\mathcal{M}$. The model category $\mathrm{L}_S\mathcal{M}$ is left proper and combinatorial.*

Proof. See [4, Theorem 4.7]. Note that any Bousfield localisation of a left proper model category is left proper: any local weak equivalence admits a factorisation into a local trivial cofibration followed by a weak equivalence, and hence, if $\mathcal{M}$ is left proper, so too does any pushout of a local weak equivalence along a cofibration. □

To determine whether an object X of a model category $\mathcal{M}$ is local with respect to some set of morphisms, one needs a model for the functor $\underline{\mathrm{Ho}\,\mathcal{M}}(-, X) : \mathrm{Ho}\,\mathcal{M}^{\mathrm{op}} \longrightarrow \mathscr{H}$, which appeared in (A.8). In practice, such models can be easily recognised with the help of (the dual of) the following lemma, which implies that the derived right adjoint of a Quillen adjunction $F \dashv G \colon \mathcal{M}^{\mathrm{op}} \longrightarrow \mathbf{sSet}_{\mathrm{K}}$ is naturally isomorphic to the functor $\underline{\mathrm{Ho}\,\mathcal{M}}(-, X) : \mathrm{Ho}\,\mathcal{M}^{\mathrm{op}} \longrightarrow \mathscr{H}$ if $F(\Delta^0)$ is weakly equivalent to X in $\mathcal{M}$.

Lemma A.12. *Let*

$$\mathcal{M} \underset{G}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{sSet}_{\mathrm{K}}$$

be a Quillen adjunction between a model category $\mathcal{M}$ and the category of simplicial sets equipped with the model structure for Kan complexes. For each object A of $\mathcal{M}$, the following are equivalent.

(i) *The $\mathscr{H}$-enriched right derived functor of G is $\mathscr{H}$-naturally isomorphic to the $\mathscr{H}$-enriched representable functor $\underline{\mathrm{Ho}\,\mathcal{M}}(A, -) : \underline{\mathrm{Ho}\,\mathcal{M}} \longrightarrow \underline{\mathscr{H}}$.*

(ii) *The objects* $F(\Delta^0)$ *and* A *are weakly equivalent in the model category* $\mathcal{M}$.

Proof. By the $\mathscr{H}$-enriched derived adjunction $\mathbf{L}F \dashv \mathbf{R}G\colon \underline{\mathrm{Ho}\,\mathcal{M}} \longrightarrow \underline{\mathscr{H}}$ of the Quillen adjunction $F \dashv G$, and since Δ^0 is the terminal object of the cartesian closed category $\mathscr{H}$, there exist isomorphisms in $\mathscr{H}$

$$(\mathbf{R}G)X \cong \underline{\mathscr{H}}(\Delta^0, (\mathbf{R}G)X) \cong \underline{\mathrm{Ho}\,\mathcal{M}}((\mathbf{L}F)\Delta^0, X)$$

$\mathscr{H}$-natural in $X \in \underline{\mathrm{Ho}\,\mathcal{M}}$. Since Δ^0 is a cofibrant object of $\mathbf{sSet}_{\mathrm{K}}$, there exists an isomorphism $(\mathbf{L}F)\Delta^0 \cong F(\Delta^0)$ in $\mathrm{Ho}\,\mathcal{M}$. Hence the $\mathscr{H}$-enriched functor $\mathbf{R}G\colon \underline{\mathrm{Ho}\,\mathcal{M}} \longrightarrow \underline{\mathscr{H}}$ is represented by the object $F(\Delta^0)$. The result then follows from the $\mathscr{H}$-enriched Yoneda lemma. □

We conclude this section with two criteria for detecting Quillen equivalences between Bousfield localisations, which are stated in terms of local fibrant objects. These criteria distill presumably standard arguments; we apply them in §5 to prove the Quillen equivalences mentioned in §1. First, we give a necessary and sufficient condition for an adjunction to remain a Quillen adjunction after Bousfield localisation.

Proposition A.13. *Let* $F \dashv G\colon \mathcal{M} \longrightarrow \mathcal{N}$ *be a Quillen adjunction between model categories, and let* $\mathcal{N}_{\mathrm{loc}}$ *be a Bousfield localisation of* $\mathcal{N}$*. The adjunction* $F \dashv G\colon \mathcal{M} \longrightarrow \mathcal{N}_{\mathrm{loc}}$ *is a Quillen adjunction if and only if the functor* G *sends each fibrant object of* $\mathcal{M}$ *to a fibrant object of* $\mathcal{N}_{\mathrm{loc}}$.

Proof. The condition is necessary since right Quillen functors preserve fibrant objects. To prove the converse, it suffices by [17, Proposition 7.15] to prove that $F\colon \mathcal{N}_{\mathrm{loc}} \longrightarrow \mathcal{M}$ preserves cofibrations and that $G\colon \mathcal{M} \longrightarrow \mathcal{N}_{\mathrm{loc}}$ preserves fibrations between fibrant objects. The first holds since $\mathcal{N}$ and $\mathcal{N}_{\mathrm{loc}}$ share the same class of cofibrations and since $F\colon \mathcal{N} \longrightarrow \mathcal{M}$ preserves cofibrations. The second holds since the hypothesis implies that G sends each fibration between fibrant objects in $\mathcal{M}$ to a fibration between local fibrant objects in $\mathcal{N}$, which by [17, Proposition 7.21] is a fibration in $\mathcal{N}_{\mathrm{loc}}$. □

Recall that a Quillen adjunction $F \dashv G\colon \mathcal{M} \longrightarrow \mathcal{N}$ is said to be a homotopy reflection if its derived right adjoint $\mathbf{R}G\colon \mathrm{Ho}\,\mathcal{M} \longrightarrow \mathrm{Ho}\,\mathcal{N}$ is fully faithful [15, Definition E.2.15].

Theorem A.14. *Let $F \dashv G\colon \mathcal{M} \longrightarrow \mathcal{N}$ be a homotopy reflection, and let $\mathcal{N}_{\mathrm{loc}}$ be a Bousfield localisation of $\mathcal{N}$. The adjunction $F \dashv G\colon \mathcal{M} \longrightarrow \mathcal{N}_{\mathrm{loc}}$ is a Quillen equivalence if and only if the following conditions are satisfied:*

(i) *G sends each fibrant object of $\mathcal{M}$ to a fibrant object of $\mathcal{N}_{\mathrm{loc}}$, and*

(ii) *for every cofibrant fibrant object X of $\mathcal{N}_{\mathrm{loc}}$, there exists a fibrant object A of $\mathcal{M}$ and a weak equivalence $X \longrightarrow GA$ in $\mathcal{N}$.*

Proof. Suppose that the adjunction $F \dashv G\colon \mathcal{M} \longrightarrow \mathcal{N}_{\mathrm{loc}}$ is a Quillen equivalence. Condition (i) holds by Proposition A.13. To prove condition (ii), let X be a cofibrant fibrant object of $\mathcal{N}_{\mathrm{loc}}$. Then the composite morphism

$$X \xrightarrow{\eta_X} GFX \xrightarrow{Gr} G(FX)^f,$$

where η is the unit of the adjunction $F \dashv G$ and $r\colon FX \longrightarrow (FX)^f$ is a fibrant replacement of FX in $\mathcal{M}$, is the component of the derived unit of this Quillen equivalence at the cofibrant object X, and is therefore a local weak equivalence between local fibrant objects, and hence also a weak equivalence in $\mathcal{N}$.

Conversely, suppose that the conditions (i) and (ii) hold. Condition (i) implies that the adjunction $F \dashv G\colon \mathcal{M} \longrightarrow \mathcal{N}_{\mathrm{loc}}$ is a Quillen adjunction by Proposition A.13. To show that this Quillen adjunction is a Quillen equivalence, it suffices to show that its derived right adjoint $\mathbf{R}G\colon \mathrm{Ho}\,\mathcal{M} \longrightarrow \mathrm{Ho}\,\mathcal{N}_{\mathrm{loc}}$ is an equivalence of categories. Since the original homotopy reflection is equal to the composite Quillen adjunction

$$\mathcal{M} \underset{G}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{N}_{\mathrm{loc}} \underset{1_{\mathcal{N}}}{\overset{1_{\mathcal{N}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{N},$$

we have that the composite of the functor $\mathbf{R}G\colon \mathrm{Ho}\,\mathcal{M} \longrightarrow \mathrm{Ho}\,\mathcal{N}_{\mathrm{loc}}$ with the fully faithful functor $\mathbf{R}1_{\mathcal{N}}\colon \mathrm{Ho}\,\mathcal{N}_{\mathrm{loc}} \longrightarrow \mathrm{Ho}\,\mathcal{N}$ is fully faithful, and hence that the functor $\mathbf{R}G\colon \mathrm{Ho}\,\mathcal{M} \longrightarrow \mathrm{Ho}\,\mathcal{N}_{\mathrm{loc}}$ is fully faithful. Since every object of $\mathrm{Ho}\,\mathcal{N}_{\mathrm{loc}}$ is isomorphic to a cofibrant fibrant object of $\mathcal{N}_{\mathrm{loc}}$, condition (ii) implies that the functor $\mathbf{R}G\colon \mathrm{Ho}\,\mathcal{M} \longrightarrow \mathrm{Ho}\,\mathcal{N}_{\mathrm{loc}}$ is essentially surjective on objects, and therefore an equivalence of categories. $\square$

Finally, we give a necessary and sufficient condition for a Quillen equivalence to remain a Quillen equivalence after Bousfield localisation.

Theorem A.15. *Let $F \dashv G\colon \mathcal{M} \longrightarrow \mathcal{N}$ be a Quillen equivalence between model categories, and let $\mathcal{M}_{\mathrm{loc}}$ and $\mathcal{N}_{\mathrm{loc}}$ be Bousfield localisations of $\mathcal{M}$ and $\mathcal{N}$ respectively. The adjunction $F \dashv G\colon \mathcal{M}_{\mathrm{loc}} \longrightarrow \mathcal{N}_{\mathrm{loc}}$ is a Quillen equivalence if and only if a fibrant object A of $\mathcal{M}$ is fibrant in $\mathcal{M}_{\mathrm{loc}}$ precisely when GA is fibrant in $\mathcal{N}_{\mathrm{loc}}$.*

Proof. The composite Quillen adjunction

$$\mathcal{M}_{\mathrm{loc}} \underset{1_{\mathcal{M}}}{\overset{1_{\mathcal{M}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{M} \underset{G}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{N} \tag{A.16}$$

is a homotopy reflection, since it is a composite of homotopy reflections. Hence the adjunction $F \dashv G\colon \mathcal{M}_{\mathrm{loc}} \longrightarrow \mathcal{N}_{\mathrm{loc}}$ is a Quillen equivalence if and only if the homotopy reflection (A.16) satisfies the conditions of Theorem A.14. It will therefore suffice to show that these conditions are equivalent to that of the present theorem.

Suppose that the homotopy reflection (A.16) satisfies the conditions (i) and (ii) of Theorem A.14, and let A be a fibrant object of $\mathcal{M}$. If A is fibrant in $\mathcal{M}_{\mathrm{loc}}$, then by condition (i), GA is a fibrant object of $\mathcal{N}_{\mathrm{loc}}$. Conversely, suppose GA is a fibrant object of $\mathcal{N}_{\mathrm{loc}}$. Let $(GA)^c \longrightarrow GA$ be a cofibrant replacement of GA in $\mathcal{N}$, chosen to be a trivial fibration. Then $(GA)^c$ is a cofibrant fibrant object of $\mathcal{N}_{\mathrm{loc}}$, and so by condition (ii), there exists a fibrant object B of $\mathcal{M}_{\mathrm{loc}}$ and a weak equivalence $(GA)^c \longrightarrow GB$ in $\mathcal{N}$. Hence the objects GA and GB are weakly equivalent in $\mathcal{N}$, and since the derived right adjoint of the Quillen equivalence $F \dashv G\colon \mathcal{M} \longrightarrow \mathcal{N}$ is fully faithful, it follows that A and B are weakly equivalent in $\mathcal{M}$. Hence A is a fibrant object in $\mathcal{M}_{\mathrm{loc}}$ by Lemma A.6.

To prove the converse, suppose that a fibrant object A of $\mathcal{M}$ is fibrant in $\mathcal{M}_{\mathrm{loc}}$ precisely when GA is fibrant in $\mathcal{N}_{\mathrm{loc}}$. One half of this assumption is precisely condition (i) of Theorem A.14. To verify condition (ii) of that theorem, let X be a cofibrant fibrant object of $\mathcal{N}_{\mathrm{loc}}$. For any fibrant replacement $r\colon FX \longrightarrow (FX)^f$ of FX in $\mathcal{M}$, the composite morphism

$$X \xrightarrow{\eta_X} GFX \xrightarrow{Gr} G(FX)^f$$

gives the component at X of the derived unit of the original Quillen equivalence, and is thus a weak equivalence in $\mathcal{N}$. Hence $G(FX)^f$ is weakly equivalent in $\mathcal{N}$ to the fibrant object X of $\mathcal{N}_{\mathrm{loc}}$, and is therefore itself fibrant in $\mathcal{N}_{\mathrm{loc}}$ by Lemma A.6. The other half of our assumption now implies that $(FX)^f$ is a fibrant object of $\mathcal{M}_{\mathrm{loc}}$, thus verifying condition (ii). □

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FLOWS REVISITED: THE MODEL CATEGORY STRUCTURE AND ITS LEFT DETERMINEDNESS

Philippe GAUCHER

Résumé. Les flots sont un modèle topologique de la concurrence qui permet d'encoder la notion de raffinement de l'observation et de comprendre les propriétés homologiques des branchements et des confluences des chemins d'exécution. Intuitivement, ce sont des d-espaces au sens de Grandis sans espace topologique sous-jacent. Ils ont seulement un type d'homotopie sous-jacent. Cette note a deux objectifs. Premièrement de donner une nouvelle construction de la catégorie de modèles des flots plus conceptuelle grâce au travail d'Isaev. Cela permet d'éviter des arguments topologiques difficiles. Deuxièmement nous prouvons que cette catégorie de modèles est déterminée gauche en adaptant un argument de Olschok. L'introduction contient quelques spéculations sur ce qu'on s'attend à trouver en localisant cette catégorie de modèles minimale.

Abstract. Flows are a topological model of concurrency which enables to encode the notion of refinement of observation and to understand the homological properties of branchings and mergings of execution paths. Roughly speaking, they are Grandis' d-spaces without an underlying topological space. They just have an underlying homotopy type. This note is twofold. First, we give a new construction of the model category structure of flows which is more conceptual thanks to Isaev's results. It avoids the use of difficult topological arguments. Secondly, we prove that this model category is left determined by adapting an argument due to Olschok. The introduction contains some speculations about what we expect to find out by localizing this minimal model

category structure.
Keywords. Left determined model category, combinatorial model cate
causal structure, bisimulation.
Mathematics Subject Classification (2010). 18C35, 55U35, 18G55, 68

Contents

1. Introduction

1.1 Topological models of concurrency

There is a multitude of topological models of concurrency: flows which are the subject of this paper and which are introduced in [6], but also d-spaces [18], streams [28], inequilogical spaces [19], spaces with distinguished cubes [21] [20], multipointed d-spaces [14] etc... All these mathematical devices contain the same basic examples coming from concurrency (e.g. the geometric realizations of precubical sets), a local ordering modeling the direction of time and its irreversibility, execution paths, a set or a topological space of states and a notion of homotopy between execution paths to model concurrency. Grandis' d-spaces give rise to a vast literature studying the directed fundamental category and the directed components of directed spaces whatever the definition we give to this notion of *directed space*.

This paper belongs to the sequence of papers [6] [9] [15] [12] [10] [11] [7] [8]. The main feature of the model category of flows is to enable the formalization and the study of the notion of refinement of observation (the cofibrant replacement functor plays a crucial role in the formalization indeed).

The model category of flows also enables the study of the homological properties of the branching areas and merging areas of execution paths in concurrent systems, in particular a long exact sequence, and their interaction with the refinement of observation, actually their invariance with respect to them. Since flows have also a labeled version (see [13, Section 6]), they can be used for modeling the path spaces of process algebras for any synchronization algebra [13] [16].

1.2 Speculative digression

In our line of research, the objects are multipointed, i.e. they are equipped with a distinguished set of states. The set of states provided by the description of a concurrent process is not forgotten. It is exactly the same phenomenon as in the formalism of simplicial set. A simplicial set is equipped with a set of vertices which comes from the description of the space. This family of topological models of concurrency differs from other topological models of concurrency like Grandis' d-spaces or streams which are not multipointed.

The interest of our approach is that it is already possible to build model category structures such that the weak equivalences preserve the causal structure of a process. Indeed, all flows are fibrant. By Ken Brown's lemma, two flows are therefore weakly equivalent if and only if they can be related by a zig-zag of trivial fibrations. Every trivial fibration satisfies the right lifting property with respect to any cofibration, and therefore with respect to any reasonable notion of extension of paths. Thus two weakly equivalent flows are bisimilar in Joyal-Nielsen-Winskel's sense [27] for any reasonable notion of extension of paths.

The main drawback of a lot of, and to the best of our knowledge, actually all other model categories introduced in directed homotopy is that their weak equivalences destroy the causal structure for example by identifying the directed segment up to weak equivalence with a point. Indeed, the directed segment should not be contractible (in a directed sense) in directed homotopy. To understand the reason, consider the well-known example of the Swiss Flag example (cf. Figure 1). It consists of two processes concurrently executing the instructions $PA.PB.VA.VB$ and $PB.PA.VB.VA$ where Px means taking the control of a shared ressource x and Vx means releasing it. We suppose that at most one process can take the control of a given shared ressource.

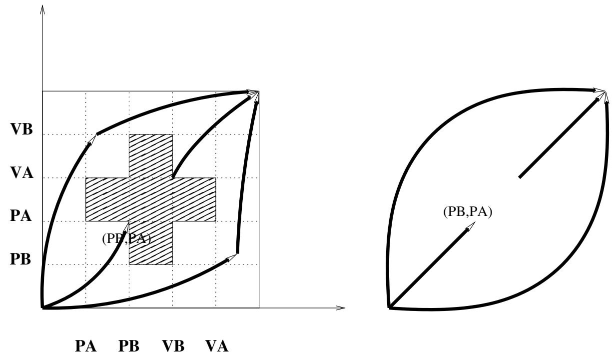

Figure 1: Swiss Flag example

There are in our example two shared ressources A and B which can be for example buffers to temporarily store data. The execution paths start from the bottom left corner, ends to the top right corner, and are supposed to be nondecreasing with respect to each axis of coordinates. Then the point of coordinates (PB, PA) is a deadlock because it is impossible from it to reach the desired final state. As soon as the directed segment becomes weakly equivalent to a point, the deadlock (PB, PA) could disappear up to weak equivalence (the picture on the right is an object having the same causal structure). It means that a weak equivalence could break the causal structure as soon as the directed segment is contractible in a directed sense and, in this case, if left properness is assumed.

The weak equivalences of the model category of flows do preserve the causal structure. However, they are too restrictive. It is not even possible up to weak equivalence to replace in a flow a directed segment by a more refined one (cf. Figure 2), by adding additional points in the middle of the segment in the distinguished set of states. This annoying behaviour can be overcome by adding weak equivalences by homotopical localization. One of the challenges of our line of research is precisely to understand the homotopical localization of the model category of (labeled) flows with respect to this kind of maps: they are called T-homotopy equivalences in [10].

The paper [7] proves not only that the class of weak equivalences of this homotopical localization contains more equivalences than the dihomotopy equivalences of flows as defined in [9], but also that there is no hope to obtain

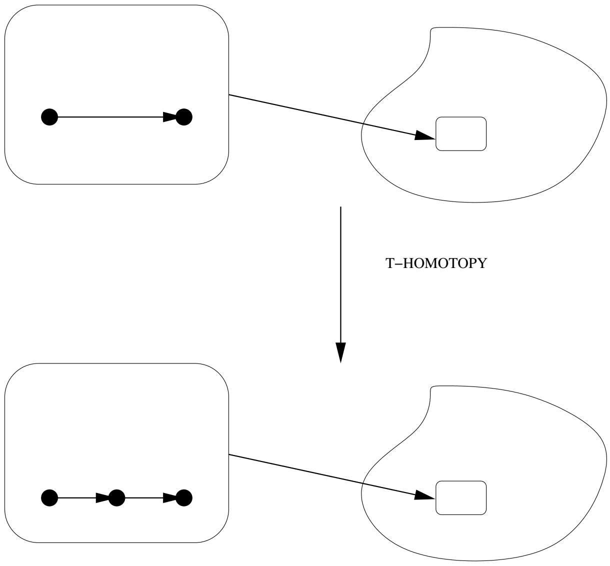

Figure 2: Replacement of a directed segment by a more refined one

a model category structure on flows such that the weak equivalences are exactly these dihomotopy equivalences, even if a notion of fibrant object (the homotopy continuous flows [9, Definition 4.3]) with the associated Whitehead theorem exists for dihomotopy equivalences [9, Theorem 4.6].

By now, we only know by studying examples that the weak equivalences of this homotopical localization seem to be, in the labeled case, dihomotopy equivalences in the sense of [9] up to a kind of bisimulation. In particular, it means that the weak equivalences of this homotopical localization likely preserve causality, but not the branching and merging homologies, and not the underlying homotopy type of a flow as defined in [11] which is, roughly speaking, the homotopy type of the space obtained after removing the directed structure of a flow.

It is actually possible to homotopically localize the model category of labeled flows by the whole class of bisimulations in Joyal-Nielsen-Winskel's sense since this class of maps is accessible. Then another strange phenomenon occurs. We would then have to deal in the localization with weak equivalences breaking the causal structure. The latter phenomenon is explained in [17, Theorem 12.4] within the combinatorial framework of Cattani-Sassone higher dimensional transition systems but it can be easily adapted and generalized to many other frameworks of directed homotopy, including the one of flows.

1.3 Purpose of this note

The construction of the model category of flows as carried out in [6] is quite long and tricky. It makes use of rather complicated topological lemmas, in particular because colimits of flows are difficult to understand. Indeed, flows are roughly speaking *small categories without identities (precategories ? pseudocategories ?) enriched over topological spaces* [1] Therefore new paths are created as soon as states are identified in a colimit, which may generate complicated modifications of the topology of the path space (e.g. see the proof of [6, Theorem 15.2] which is not only complicated but also contains a flaw which will be fixed in a subsequent paper). The first purpose of this note is to drastically simplify this construction using [25]. We then explain in a second part why the model category of flows is left determined in the sense of [30] by adapting an argument due to Marc Olschok for the model category of topological spaces. The latter fact is a new result (it was mentioned without proof in [17, Section 12]). In the (complicated) quest of finding out better model categories with as much weak equivalences as possible preserving the causal structure, this result means that it is not possible in the framework of flows to remove weak equivalences without changing the set of generating cofibrations.

1.4 Organization

Section 2 recalls what we need to use from Isaev's paper. Section 3 explains the new construction of the model category structure of flows (Theorem 3.11). Section 4 recalls the notion of left determined model category and proves that the category of flows is left determined (Theorem 4.3). Section 5 makes some final comments.

[1]There is no known relation between the model category of flows and the model category of topologically enriched small categories of [3]: the obvious adjunction is not even a Quillen adjunction; moreover the terminology of "flow" must not mislead the reader, it has nothing to do with a similar terminology in Morse theory.

1.5 Notations

All categories are locally small. The category of sets is denoted by $\mathbf{Set}$. The set of maps in a category $\mathcal{K}$ from X to Y is denoted by $\mathcal{K}(X,Y)$. The initial (final resp.) object, if it exists, is always denoted by $\varnothing$ ($\mathbf{1}$ resp.). The identity of an object X is denoted by Id_X. The composite of two maps $f : A \to B$ and $g : B \to C$ is denoted by $g.f$. A subcategory is always isomorphism-closed (replete). Let f and g be two maps of a category $\mathcal{K}$. Denote by $f \boxslash g$ when f satisfies the *left lifting property* (LLP) with respect to g, or equivalently g satisfies the *right lifting property* (RLP) with respect to f. Let $\mathcal{C}$ be a class of maps. Let us introduce the notations $\mathbf{inj}(\mathcal{C}) = \{g \in \mathcal{K}, \forall f \in \mathcal{C}, f \boxslash g\}$ and $\mathbf{cof}(\mathcal{C}) = \{f \mid \forall g \in \mathbf{inj}(\mathcal{C}), f \boxslash g\}$. The class of morphisms of $\mathcal{K}$ that are transfinite compositions of pushouts of elements of $\mathcal{C}$ is denoted by $\mathbf{cell}(\mathcal{C})$. We refer to [1] for locally presentable categories, to [29] for combinatorial model categories. We refer to [24] and to [23] for more general model categories.

2. Isaev approach for constructing model categories

Let $\mathcal{K}$ be a locally presentable category. A combinatorial model category structure is characterized by its set of generating cofibrations and by its class of fibrant objects by [26, Proposition E.1.10]. Therefore, for a given set of maps I, there exists at most one combinatorial model category structure on $\mathcal{K}$ such that the set of generating cofibrations is I and such that all objects are fibrant. In [25], several methods are expounded to obtain model category structures such that all objects are fibrant. We summarize in the next theorem what we are going to need in this note.

Theorem 2.1. *[25, Theorem 4.3, Proposition 4.4, Proposition 4.5 and Corollary 4.6] Let $\mathcal{K}$ be a locally presentable category. Let I be a set of maps of $\mathcal{K}$ such that the domains of the maps of I are I-cofibrant (i.e. belong to $\mathbf{cof}(I)$). Suppose that for every map $i : U \to V \in I$, the relative codiagonal map $V \sqcup_U V \to V$ factors as a composite $V \sqcup_U V \stackrel{\gamma_0 \sqcup \gamma_1}{\to} C_U(V) \to V$ such that the left-hand map belongs to $\mathbf{cof}(I)$. Let $J_I = \{\gamma_0 : V \to C_U(V) \mid U \to V \in I\}$. Suppose that there exists a path functor $\mathrm{Path} : \mathcal{K} \to \mathcal{K}$, i.e. an endofunctor of $\mathcal{K}$ equipped with two natural transformations $\tau : \mathrm{Id} \Rightarrow \mathrm{Path}$ and*

$\pi : \mathrm{Path} \Rightarrow \mathrm{Id} \times \mathrm{Id}$ *such that the composite* $\pi.\tau$ *is the diagonal. Moreover we suppose that the path functor satisfies the following hypotheses:*

1. *With* $\pi = (\pi_0, \pi_1)$*,* $\pi_0 : \mathrm{Path}(X) \to X$ *and* $\pi_1 : \mathrm{Path}(X) \to X$ *have the RLP with respect to* I*.*

2. *The map* $\pi : \mathrm{Path}(X) \to X \times X$ *has the RLP with respect to the maps of* J_I*.*

Then there exists a unique model category structure on $\mathcal{K}$ *such that the set of generating cofibrations is* I *and such that the set of generating trivial cofibrations is* J_I*. Moreover, all objects are fibrant.*

Unlike in [25], we can drop the hypothesis about the smallness of the domains and the codomains of the maps of I with respect to I by [2, Proposition 1.3] because the ambient category is supposed to be locally presentable . Note that every map of J_I is a split monomorphism since the composite $V \sqcup_U V \to C_U(V) \to V$ is the relative codiagonal. Therefore every object is fibrant indeed.

3. The model category of flows

Notation 3.1. *The category* $\mathbf{Top}$ *denotes a bicomplete locally presentable cartesian closed full subcategory of the category of general topological spaces containing all CW-complexes.*

The category of Δ-generated spaces, i.e. the colimits of simplices, or equivalently the colimits of the segment $[0,1]$ by [4, Proposition 3.17], satisfies these hypotheses [5]. It is also possible to add weak separability hypotheses like this one: for every continuous map $g : \Delta^n \to X$ where Δ^n is the topological n-simplex with $n \geqslant 0$, $g(\Delta^n)$ is closed in X. For a tutorial about these topological spaces, see for example [14, Section 2]. We take $\mathbf{Top}$ to be equipped with the standard Quillen model category structure.

Notation 3.2. *The internal hom functor is denoted by* $\mathbf{TOP}(-,-)$*.*

Definition 3.3. *[6] A* flow X *consists of a topological space* $\mathbb{P}X$ *of execution paths, a discrete space* X^0 *of states, two continuous maps* s *and* t *from* $\mathbb{P}X$

to X^0 *called the source and target map respectively, and a continuous and associative map*

$$* : \{(x,y) \in \mathbb{P}X \times \mathbb{P}X; t(x) = s(y)\} \longrightarrow \mathbb{P}X$$

such that $s(x*y) = s(x)$ *and* $t(x*y) = t(y)$. *A morphism of flows* $f : X \longrightarrow Y$ *consists of a set map* $f^0 : X^0 \longrightarrow Y^0$ *together with a continuous map* $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ *such that* $f(s(x)) = s(f(x))$, $f(t(x)) = t(f(x))$ *and* $f(x*y) = f(x)*f(y)$. *The corresponding category is denoted by* $\mathbf{Flow}$.

Notation 3.4. *For a topological space* X, *let* $\mathrm{Glob}(X)$ *be the flow defined by* $\mathrm{Glob}(X)^0 = \{0,1\}$ *and* $\mathbb{P}\mathrm{Glob}(X) = X$ *with* s *and* t *being the constant functions* $s = 0$ *and* $t = 1$. *The Glob mapping induces a functor from the category* $\mathbf{Top}$ *of topological spaces to the category* $\mathbf{Flow}$ *of flows.*

We need to recall the two following easy propositions:

Proposition 3.5. *[6, Proposition 13.2] A morphism of flows* $f : X \longrightarrow Y$ *satisfies the RLP with respect to* $\mathrm{Glob}(U) \longrightarrow \mathrm{Glob}(V)$ *if and only if for any* $\alpha, \beta \in X^0$, $\mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$ *satisfies the RLP with respect to* $U \longrightarrow V$.

Proposition 3.6. *[6, Proposition 16.2] Let* f *be a morphism of flows. Then the following conditions are equivalent:*

1. f *is bijective on states*
2. f *satisfies the RLP with respect to* $R : \{0,1\} \longrightarrow \{0\}$ *and* $C : \varnothing \subset \{0\}$.

We will also need this new proposition which does not seem to be proved in one of our previous papers about flows:

Proposition 3.7. *The globe functor* $\mathrm{Glob} : \mathbf{Top} \to \mathbf{Flow}$ *preserves connected colimits (i.e. colimits such that the underlying small category is connected).*

Note that the connectedness hypothesis is necessary. Indeed, V and W being two topological spaces, the flow $\mathrm{Glob}(V \sqcup W)$ has two states whereas the flow $\mathrm{Glob}(V) \sqcup \mathrm{Glob}(W)$ has four states.

Proof. Let V be a topological space. Giving a map from the flow $\mathrm{Glob}(V)$ to a flow X is equivalent to choosing two states α and β of X (the image of the states 0 and 1 of $\mathrm{Glob}(V)$) and a continuous map from V to $\mathbb{P}_{\alpha,\beta}X$. Thus the following natural bijection of sets holds

$$\mathbf{Flow}(\mathrm{Glob}(V), X) \cong \bigsqcup_{(\alpha,\beta)\in X^0\times X^0} \mathbf{Top}(V, \mathbb{P}_{\alpha,\beta}X). \tag{1}$$

We obtain the sequence of natural bijections ($\varinjlim V_i$ being a connected colimit of topological spaces)

$$\begin{aligned}
\mathbf{Flow}(\mathrm{Glob}(\varinjlim V_i), X) &\cong \bigsqcup_{(\alpha,\beta)\in X^0\times X^0} \mathbf{Top}(\varinjlim V_i, \mathbb{P}_{\alpha,\beta}X)\\
&\cong \bigsqcup_{(\alpha,\beta)\in X^0\times X^0} \varprojlim \mathbf{Top}(V_i, \mathbb{P}_{\alpha,\beta}X)\\
&\cong \varprojlim \bigsqcup_{(\alpha,\beta)\in X^0\times X^0} \mathbf{Top}(V_i, \mathbb{P}_{\alpha,\beta}X)\\
&\cong \varprojlim \mathbf{Flow}(\mathrm{Glob}(V_i), X)\\
&\cong \mathbf{Flow}(\varinjlim \mathrm{Glob}(V_i), X),
\end{aligned}$$

the first and the fourth isomorphisms by (1), the second and the fifth isomorphisms by definition of a (co)limit and the third isomorphism by the connectedness of the limit. The proof is complete using the Yoneda lemma. □

Notation 3.8. *[6, Notation 7.6] Let U be a topological space. Let X be a flow. The flow $\{U, X\}_S$ is defined as follows:*

1. *The set of states of $\{U, X\}_S$ is X^0.*
2. *For $\alpha, \beta \in X^0$, let $\mathbb{P}_{\alpha,\beta}\{U, X\}_S = \mathbf{TOP}(U, \mathbb{P}_{\alpha,\beta}X)$.*
3. *For $\alpha, \beta, \gamma \in X^0$, the composition law*

   $$* : \mathbb{P}_{\alpha,\beta}\{U, X\}_S \times \mathbb{P}_{\beta,\gamma}\{U, X\}_S \longrightarrow \mathbb{P}_{\alpha,\gamma}\{U, X\}_S$$

   is the composite

   $$\begin{aligned}\mathbb{P}_{\alpha,\beta}\{U, X\}_S \times \mathbb{P}_{\beta,\gamma}\{U, X\}_S \cong \mathbf{TOP}\left(U, \mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X\right)&\\ \longrightarrow \mathbf{TOP}\left(U, \mathbb{P}_{\alpha,\gamma}X\right)&\end{aligned}$$

   induced by the composition law of X.

The flow $\{U,X\}_S$ is functorial with respect to U and X (contravariant with respect to U and covariant with respect to X). The flow $\{\varnothing,Y\}_S$ is the flow having the same set of states as Y and exactly one non-constant execution path between two points of Y^0. The flow $\{\{0\},X\}_S$ is canonically isomorphic to X for all flows X. Maybe the latter assertion deserves a little explanation because it is precisely why cartesian closedness matters (the fact that $\{0\}$ is exponentiable is actually sufficient). For all topological spaces U, we have the natural bijections

$$\mathbf{Top}(U,\mathbb{P}_{\alpha,\beta}X)\cong\mathbf{Top}(U\times\{0\},\mathbb{P}_{\alpha,\beta}X)\cong\mathbf{Top}(U,\mathbf{TOP}(\{0\},\mathbb{P}_{\alpha,\beta}X)).$$

Thus by Yoneda, we obtain the homeomorphism

$$\mathbb{P}_{\alpha,\beta}X\cong\mathbf{TOP}(\{0\},\mathbb{P}_{\alpha,\beta}X)).$$

Notation 3.9. *Let $n\geqslant 1$. Denote by $\mathbf{D}^n=\{b\in\mathbb{R}^n,|b|\leqslant 1\}$ the n-dimensional disk, and by $\mathbf{S}^{n-1}$ the $(n-1)$-dimensional sphere. By convention, let $\mathbf{D}^0=\{0\}$ and $\mathbf{S}^{-1}=\varnothing$. Let $I^{gl}_+=\{\mathrm{Glob}(\mathbf{S}^{n-1})\subset\mathrm{Glob}(\mathbf{D}^n)\mid n\geqslant 0\}\cup\{C:\varnothing\to\{0\},R:\{0,1\}\to\{0\}\}$.*

We recall an elementary lemma about the model category **Top** which is a straightforward consequence of the fact that the Quillen model structure is Cartesian monoidal:

Lemma 3.10. *Let $i:U\to V$ be a cofibration of* **Top**. *Then for all topological spaces X, the map $i^*:\mathbf{TOP}(V,X)\to\mathbf{TOP}(U,X)$ is a fibration. If moreover i is a weak equivalence, then the map $i^*:\mathbf{TOP}(V,X)\to\mathbf{TOP}(U,X)$ is a trivial fibration.*

We can now easily carry out the construction of the model category structure.

Theorem 3.11. *There exists a unique model category structure such that I^{gl}_+ is the set of generating cofibrations and such that all objects are fibrant.*

Proof. We have to check the hypotheses of Theorem 2.1. The category **Flow** is locally presentable since **Top** is locally presentable (see for example the proof of [7, Proposition 6.11]). That all $\mathrm{Glob}(\mathbf{S}^{n-1})$ for all $n\geqslant 0$ are I^{gl}_+-cofibrant comes from the fact that the $(n-1)$-sphere is cofibrant in **Top**. We can factor the relative codiagonal map $\mathbf{D}^n \sqcup_{\mathbf{S}^{n-1}} \mathbf{D}^n \to \mathbf{D}^n$ as a composite $\mathbf{D}^n \sqcup_{\mathbf{S}^{n-1}} \mathbf{D}^n \subset \mathbf{D}^{n+1} \to \mathbf{D}^n$ for all $n \geqslant 0$. Thus for $U \to V$ being one of the maps $\mathrm{Glob}(\mathbf{S}^{n-1}) \subset \mathrm{Glob}(\mathbf{D}^n)$ for $n \geqslant 0$, we set $C_U(V) = \mathrm{Glob}(\mathbf{D}^{n+1})$. We have a pushout diagram of topological spaces

$$\begin{array}{ccc} \mathbf{S}^n & \longrightarrow & \mathbf{D}^n \sqcup_{\mathbf{S}^{n-1}} \mathbf{D}^n \\ \downarrow & & \downarrow \\ \mathbf{D}^{n+1} & \longrightarrow & \mathbf{D}^{n+1} \end{array}$$

which gives rise to the pushout diagram of flows

$$\begin{array}{ccc} \mathrm{Glob}(\mathbf{S}^n) & \longrightarrow & \mathrm{Glob}(\mathbf{D}^n) \sqcup_{\mathrm{Glob}(\mathbf{S}^{n-1})} \mathrm{Glob}(\mathbf{D}^n) \\ \downarrow & & \downarrow \\ \mathrm{Glob}(\mathbf{D}^{n+1}) & \longrightarrow & \mathrm{Glob}(\mathbf{D}^{n+1}) \end{array}$$

for all $n \geqslant 0$ by Proposition 3.7. This implies that for $U \to V$ being one of the maps $\mathrm{Glob}(\mathbf{S}^{n-1}) \subset \mathrm{Glob}(\mathbf{D}^n)$ for $n \geqslant 0$, the map $V \sqcup_U V \to C_U(V)$ belongs to $\mathbf{cell}(I^{gl}_+)$. The map $C : \varnothing \to \{0\}$ gives rise to the relative codiagonal map $\{0\} \sqcup \{0\} \to \{0\}$. Thus we set $C_\varnothing(\{0\}) = \{0\}$. In this case, the map $V \sqcup_U V \to C_U(V)$ is $R : \{0,1\} \to \{0\}$ which belongs to $\mathbf{cell}(I^{gl}_+)$. The map $R : \{0,1\} \to \{0\}$ gives rise to the relative codiagonal map $\mathrm{Id}_{\{0\}}$. Thus we set $C_{\{0,1\}}(\{0\}) = \{0\}$. In this case, the map $V \sqcup_U V \to C_U(V)$ is $\mathrm{Id}_{\{0\}}$ which belongs to $\mathbf{cell}(I^{gl}_+)$. The set of generating trivial cofibrations will be therefore the set of maps $\mathrm{Glob}(\mathbf{D}^n) \subset \mathrm{Glob}(\mathbf{D}^{n+1})$ for $n \geqslant 0$. Let $\mathrm{Path}(X) = \{[0,1], X\}_S$ for all flows X. The composite map $\{0,1\} \subset [0,1] \to \{0\}$ yields a natural composite map of flows $X \cong \{\{0\}, X\}_S \to \mathrm{Path}(X) \to \{\{0,1\}, X\}_S$ which is constant on states and which gives rise to the composite continuous map $\mathbb{P}_{\alpha,\beta}X \to \mathbf{TOP}([0,1], \mathbb{P}_{\alpha,\beta}X) \to \mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\alpha,\beta}X$ on the spaces of paths for all $(\alpha,\beta) \in X^0 \times X^0$. We obtain a natural composite map of flows $X \xrightarrow{\tau} \mathrm{Path}(X) \xrightarrow{\pi} X \times X$ since the set of states of $X \times X$ is $X^0 \times X^0$ and the space of paths from (α,α') to (β,β') is $\mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\alpha',\beta'}X$ by [6, Theorem 4.17]. We have obtained a path object in the sense of Theorem 2.1. Since the maps π_0 and π_1 are bijective on states, they satisfy the RLP with respect to $\{C : \varnothing \to \{0\}, R : \{0,1\} \to \{0\}\}$ by Proposition 3.6. By Proposition 3.5, the maps π_0 and π_1 satisfy the RLP with respect to $\mathrm{Glob}(\mathbf{S}^{n-1}) \subset \mathrm{Glob}(\mathbf{D}^n)$ for $n \geqslant 0$ if and only if the evaluation maps $\mathbf{TOP}([0,1], \mathbb{P}_{\alpha,\beta}X) \rightrightarrows \mathbb{P}_{\alpha,\beta}X$ on 0 and 1 satisfy the RLP with respect to the inclusion $\mathbf{S}^{n-1} \subset \mathbf{D}^n$ for $n \geqslant 0$ and for all $(\alpha, \beta) \in X^0 \times X^0$, i.e. if and only if the evaluation maps $\mathbf{TOP}([0,1], \mathbb{P}_{\alpha,\beta}X) \rightrightarrows \mathbb{P}_{\alpha,\beta}X$ are trivial fibrations for all $(\alpha, \beta) \in X^0 \times X^0$. The latter fact is a consequence of Lemma 3.10 and from the fact that the inclusions $\{0\} \subset [0,1]$ and $\{1\} \subset [0,1]$ are trivial cofibrations of $\mathbf{Top}$. Finally we have to check that the map $\pi : \mathrm{Path}(X) \to X \times X$ satisfies the RLP with respect to the maps $\mathrm{Glob}(\mathbf{D}^n) \subset \mathrm{Glob}(\mathbf{D}^{n+1})$ for $n \geqslant 0$. By Proposition 3.5 again, it suffices to prove that the map $\mathbf{TOP}([0,1], \mathbb{P}_{\alpha,\beta}X) \to \mathbf{TOP}(\{0,1\}, \mathbb{P}_{\alpha,\beta}X) = \mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\alpha,\beta}X$ is a fibration of topological spaces for all $(\alpha, \beta) \in X^0 \times X^0$. By Lemma 3.10 again, this comes from the fact that the inclusion $\{0,1\} \subset [0,1]$ is a cofibration of $\mathbf{Top}$. □

This model category structure coincides with the one of [6].

4. Left determinedness of the model category of flows

Let us now recall the definition of a left determined model category:

Definition 4.1. *Let I be a set of maps of a locally presentable category $\mathcal{K}$. A class of maps $\mathcal{W}$ is a* localizer (with respect to I) *or an I*-localizer *if $\mathcal{W}$ satisfies:*

- *Every map satisfying the RLP with respect to the maps of I belongs to $\mathcal{W}$.*
- *$\mathcal{W}$ is closed under retract and satisfies the 2-out-of-3 property.*
- *The class of maps $\mathbf{cof}(I) \cap \mathcal{W}$ is closed under pushout and transfinite composition.*

The class of all maps is an I-localizer. The class of I-localizers is closed under arbitrarily large intersection. Therefore there exists a smallest I-localizer denoted by $\mathcal{W}_I$.

Definition 4.2. *[30] A combinatorial model category $\mathcal{K}$ with the set of generating cofibrations I is* left determined *if the class of weak equivalences is $\mathcal{W}_I$.*

Consider a combinatorial model category $\mathcal{K}$ such that all objects are fibrant with a class of weak equivalences $\mathcal{W}$ and a set of generating cofibrations I. The localizer $\mathcal{W}_I$ could be strictly smaller than $\mathcal{W}$. If $\mathcal{W}_I$ is the class of weak equivalences of a model category structure on $\mathcal{K}$, then all objects of this model category structure are fibrant, and therefore $\mathcal{W} = \mathcal{W}_I$. To the best of our knowledge, we can only say, using [30, Theorem 2.2], that every combinatorial model category such that all objects are fibrant is left determined if we assume Vopěnka's principle. Since in any model category, two fibrant objects are weakly equivalent if and only if they are related by a span of trivial fibrations, and since all trivial fibrations belong to the smallest localizer, it is also true that there is an equivalence of categories $\mathcal{K}[\mathcal{W}_I^{-1}] \simeq \mathcal{K}[\mathcal{W}^{-1}]$ between the categorical localizations of $\mathcal{K}$ with respect to $\mathcal{W}_I$ and $\mathcal{W}$ if all objects of the combinatorial model category $\mathcal{K}$ are fibrant. Note that in [25], a localizer is just the class of weak equivalences of a model category structure. In the latter sense, a model category of fibrant objects has a minimal localizer indeed.

In our case, it is possible to conclude that the model category is left determined without assuming Vopěnka's principle by adapting a technique we learned from Marc Olschok for the model category of topological spaces.

Theorem 4.3. *The model category of flows is left determined.*

Proof. Let $f : X \to Y$ be a weak equivalence of flows. Then f factors as a composite $f = f_2.f_1$ where f_1 is a trivial cofibration, i.e. $f_1 \in \mathbf{cof}(\{\mathrm{Glob}(\mathbf{D}^n) \subset \mathrm{Glob}(\mathbf{D}^{n+1}) \mid n \geqslant 0\})$ and where f_2 is a trivial fibration. In particular f_2 satisfies the RLP with respect to $C : \varnothing \to \{0\}$ and $R : \{0, 1\} \to \{0\}$. Thus f_2 is bijective on states by Proposition 3.6. The functor $X \mapsto X^0$ from $\mathbf{Flow}$ to $\mathbf{Set}$ is colimit-preserving since it has a right adjoint (the functor taking a set S to the flow with the set of states S and exactly one path between each pair of states). Therefore f_1 is bijective on states since the maps $\mathrm{Glob}(\mathbf{D}^n) \subset \mathrm{Glob}(\mathbf{D}^{n+1})$ for all $n \geqslant 0$ are bijective on states. We deduce that f is bijective on states. Consider the commutative diagram of flows

$$
\begin{array}{ccccc}
X & \xrightarrow{\exists!\overline{f}} & N_f & \xrightarrow{p'} & X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
Y & \xrightarrow{\tau} & \{[0,1],Y\}_S & \xrightarrow{\pi_0} & Y \\
 & & \downarrow{\scriptstyle \pi_1} & & \\
 & & Y & &
\end{array}
$$

where the existence of $\overline{f}$ comes from the universal property of the pullback. All arrows are bijective on states. Using the fact that the functor $\mathbb{P} : \mathbf{Flow} \to \mathbf{Top}$ is limit-preserving by [6, Theorem 4.17], one obtains the commutative diagram of topological spaces

$$
\begin{array}{ccccc}
\mathbb{P}X & \xrightarrow{\mathbb{P}\overline{f}} & \mathbb{P}N_f & \xrightarrow{\mathbb{P}p'} & \mathbb{P}X \\
\downarrow{\scriptstyle \mathbb{P}f} & & \downarrow{\scriptstyle \mathbb{P}f'} & & \downarrow{\scriptstyle \mathbb{P}f} \\
\mathbb{P}Y & \xrightarrow{\mathbb{P}\tau} & \mathbf{TOP}([0,1],\mathbb{P}Y) & \xrightarrow{\mathbb{P}(\pi_0)} & \mathbb{P}Y \\
 & & \downarrow{\scriptstyle \mathbb{P}(\pi_1)} & & \\
 & & \mathbb{P}Y & &
\end{array}
$$

By [22, Proposition 4.64], the map $\mathbb{P}(\pi_1).\mathbb{P}(f')$ is a Hurewicz fibration, and therefore a fibration of the model category of $\mathbf{Top}$. By Proposition 3.5, $\pi_1.f'$ satisfies the RLP with respect to all trivial cofibrations of flows, i.e. $\pi_1.f'$ is a fibration of flows. The maps $\mathbb{P}(\pi_0)$ and $\mathbb{P}(\pi_1)$ are trivial fibrations of the model category of $\mathbf{Top}$ by Lemma 3.10. By Proposition 3.5 and Proposition 3.6, π_0 and π_1 satisfy the RLP with respect to all cofibrations of $\mathbf{Flow}$, i.e. π_0 and π_1 are trivial fibrations of flows. Thus p' is a trivial fibration of $\mathbf{Flow}$ since it is a pullback of a trivial fibration. Since f is a weak equivalence of flows by hypothesis, we deduce by the 2-out-of-3 property that f' is a weak equivalence of $\mathbf{Flow}$. Thus $\pi_1.f'$ is a trivial fibration of flows as well. We have $p'.\overline{f} = \mathrm{Id}_X$. Since p' is a trivial fibration, it belongs to the smallest localizer. Therefore by the 2-out-of-3 property, $\overline{f}$ belongs to the smallest localizer. Since $\pi_1.f'$ is a trivial fibration, it belongs to the smallest localizer as well. Since $\pi_1.f'.\overline{f} = \pi_1.\tau.f = f$, we deduce that f belongs to the smallest localizer. □

5. Concluding remarks

The hypothesis that $\mathbf{Top}$ is locally presentable can be removed. Theorem 3.11 and Theorem 4.3 hold by working in any bicomplete cartesian closed full subcategory of the general category of topological spaces containing all CW-complexes. But then, we have to check that all domains and all codomains of the maps of I^+_{gl} are small relative to $\mathbf{cell}(I^+_{gl})$. This is done in [6, Section 11] and there is no known way to avoid the use of some difficult topological arguments. However, the model category of flows is left proper but not cellular because of the presence of $R : \{0, 1\} \to \{0\}$ in the generating cofibrations. So outside the framework of locally presentable categories, we have no tools to prove the existence of any homotopical localization and to study the homotopical localization of $\mathbf{Flow}$ with respect to the refinement of observation.

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