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PARTIAL ALGEBRAS AND EMBEDDING THEOREMS FOR (WEAKLY) MAL'TSEV CATEGORIES AND MATRIX CONDITIONS

Pierre-Alain JACQMIN

Résumé. Il est montré que la catégorie des ensembles munis d'une opération partielle de Mal'tsev est faiblement de Mal'tsev. De plus, pour toute petite catégorie faiblement de Mal'tsev et finiment complète, le foncteur de Yoneda la plonge pleinement dans une puissance de cette catégorie des algèbres partielles de Mal'tsev. Ces résultats sont en fait prouvés en utilisant le langage des 'conditions matricielles' de Z. Janelidze afin d'obtenir des théorèmes de plongement pour les catégories faiblement de Mal'tsev, de Mal'tsev, faiblement unitaires, unitaires, fortement unitaires et soustractives.

Abstract. We prove that the category of sets equipped with a partial Mal'tsev operation is a weakly Mal'tsev category. Moreover, for each small finitely complete weakly Mal'tsev category, the Yoneda embedding fully embeds it into a power of this category of partial Mal'tsev algebras. We actually prove these results using the language of 'matrix conditions' from Z. Janelidze, getting in this way embedding theorems for weakly Mal'tsev, Mal'tsev, weakly unital, unital, strongly unital and subtractive categories.

Keywords. embedding theorem, (weakly) Mal'tsev category, (weakly) unital category, partial algebra, closed homomorphism.

Mathematics Subject Classification (2010). 18B15, 08A55, 08B05.

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1. Introduction

Mal'tsev categories have been defined in [4] as finitely complete categories in which every binary relation is difunctional. This generalises the notion of a regular Mal'tsev category from [3]. A variety of universal algebras is a Mal'tsev category if and only if its corresponding theory contains a ternary term $p(x, y, z)$ satisfying the identities $p(x, x, y) = y = p(y, x, x)$ [13]. There are many characterisations of Mal'tsev categories in the literature. For instance, a finitely complete category is a Mal'tsev category if and only if, for any pullback of split epimorphisms,

$$
\begin{tikzcd}
P \arrow[r, shift left, two heads] \arrow[d, shift left, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[l, shift left, "r_Y"'] \arrow[d, shift left, two heads] \\
X \arrow[u, shift left, "l_X"] \arrow[r, shift left, two heads] & Z \arrow[l, shift left] \arrow[u, shift left]
\end{tikzcd}
$$

the induced morphisms l_X and r_Y are jointly strongly epimorphic [2]. In [15], N. Martins-Ferreira generalises this notion defining a weakly Mal'tsev category as a category in which the pullbacks as above exist and the morphisms l_X and r_Y are jointly epimorphic.

For a small category $\mathbb{C}$, the full Yoneda embedding $\mathbb{C} \to \mathrm{Set}^{\mathbb{C}^{\mathrm{op}}}$ preserves limits. This allows one to reduce the proofs of some statements about limits in any category to the particular case of Set, the category of sets. The aim of this paper is to construct a weakly Mal'tsev category $\mathbb{M}$ for which, if $\mathbb{C}$ is a small weakly Mal'tsev finitely complete category, the Yoneda embedding factors through $\mathbb{M}^{\mathbb{C}^{\mathrm{op}}}$.

$$
\begin{tikzcd}
 & \mathbb{M}^{\mathbb{C}^{\mathrm{op}}} \arrow[d] \\
\mathbb{C} \arrow[ur, "\phi"] \arrow[r, "Y"'] & \mathrm{Set}^{\mathbb{C}^{\mathrm{op}}}
\end{tikzcd}
$$

This functor ϕ is a full and faithful embedding which preserves and reflects finite limits. Up to a change of universe, it is then enough to prove some statements about finite limits in $\mathbb{M}$ in order to prove them in all weakly Mal'tsev finitely complete categories.

An object in this category $\mathbb{M}$ is a set A equipped with a partial operation $p\colon A^3 \to A$ which is defined (at least) for all triples of the form (x, x, y) and

(y, x, x) and which satisfies the axioms $p(x, x, y) = y = p(y, x, x)$. A homomorphism between such partial Mal'tsev algebras is a function $f\colon A \to B$ such that, if $p(x, y, z) \in A$ is defined, then $p(f(x), f(y), f(z)) \in B$ is also defined and equal to $f(p(x, y, z))$. In general, they fail to satisfy the converse property: if $p(f(x), f(y), f(z))$ is defined in B, then $p(x, y, z)$ is defined in A. Homomorphisms satisfying this additional property are said to be closed [7] (also called strong homomorphisms in [6]). We prove that the monomorphisms in $\mathbb{M}$ are exactly the injective homomorphisms and strong monomorphisms are exactly the injective closed homomorphisms. With this notion of a closed monomorphism, we get a similar embedding theorem for Mal'tsev categories: any small Mal'tsev category $\mathbb{C}$ admits a full and faithful embedding $\phi\colon \mathbb{C} \to \mathbb{M}^{\mathbb{C}^{\mathrm{op}}}$ which preserves and reflects finite limits and such that for each monomorphism f and each object $X \in \mathbb{C}^{\mathrm{op}}$, $\phi(f)_X$ is a closed monomorphism.

In [8], an embedding theorem for the smaller collection of regular Mal'tsev categories has been proved. More precisely, a regular Mal'tsev category $\mathbb{M}'$ has been constructed such that each small regular Mal'tsev category has a regular conservative embedding into a power of $\mathbb{M}'$. That category $\mathbb{M}'$ is also constructed using a partial ternary operation p satisfying the Mal'tsev identities. But one of the main differences between the embedding theorem of [8] and the ones of this paper is the fact that, in $\mathbb{M}'$, the domain of definition of p is determined as the solution set of some totally defined equation. Therefore, all monomorphisms in $\mathbb{M}'$ are closed, which is not the case in $\mathbb{M}$.

In order to establish at the same time embedding theorems for weakly Mal'tsev, Mal'tsev, weakly unital [14], unital [2], strongly unital [2] and subtractive [9] categories, we use the 'matrix conditions' introduced in [10]. For each extended matrix M of terms in a commutative algebraic theory, we construct the category of partial M-algebras Part_M (being $\mathbb{M}$ when M is the Mal'tsev matrix). This category Part_M has M-closed strong relations, its monomorphisms are exactly the injective homomorphisms and its strong monomorphisms are closed. Moreover, for some particular M's, closed epimorphisms are surjective and closed monomorphisms actually coincide with strong monomorphisms (see Propositions 3.8, 3.9 and Corollary 3.11). For a general M, we then prove an embedding theorem for small categories with M-closed relations and their 'weakly version', the categories with M-closed strong relations.

The paper is divided as follows. In Section 2, we recall the notions of a category with M-closed relations and with M-closed strong relations. In Section 3, we construct the category Part_M and study its closed monomorphisms. Section 4 is devoted to the proof of our embedding theorems, while in Section 5 we give some examples how to use these embedding theorems to make proofs using elements in the above contexts.

2. Categories with M-closed (strong) relations

In order to recall the general treatment of unital, strongly unital, Mal'tsev and subtractive categories introduced in [10], we first need to recall the notion of a $\mathcal{T}$-enrichment.

2.1 $\mathcal{T}$-enrichments

Let $\mathcal{T}$ be an algebraic theory (by that we will always mean a finitary one-sorted algebraic theory). An *internal $\mathcal{T}$-algebra* in a category $\mathbb{C}$ is an object A of $\mathbb{C}$ equipped with a structure of (ordinary) $\mathcal{T}$-algebra on $Y(A)$, where $Y: \mathbb{C} \to \mathrm{Set}^{\mathbb{C}^{\mathrm{op}}}$ is the Yoneda embedding. An *internal homomorphism of internal $\mathcal{T}$-algebras* is a morphism $f: A \to B$ in $\mathbb{C}$ such that $Y(f)$ is an ordinary homomorphism of algebras. This forms the category $\mathrm{Alg}_{\mathcal{T}}\,\mathbb{C}$ of internal $\mathcal{T}$-algebras.

A *$\mathcal{T}$-enrichment* on $\mathbb{C}$ is a section of the forgetful functor $\mathrm{Alg}_{\mathcal{T}}\,\mathbb{C} \to \mathbb{C}$. In order words, it is the assignment of an internal $\mathcal{T}$-algebra structure for each object A of $\mathbb{C}$ in such a way that every morphism is an internal $\mathcal{T}$-algebra homomorphism. A *$\mathcal{T}$-enriched category* is a category $\mathbb{C}$ with a fixed $\mathcal{T}$-enrichment. Thus, a $\mathcal{T}$-enriched category is a category $\mathbb{C}$ equipped with a factorisation $\mathrm{Hom}_{\mathbb{C}}$ of the functor $\hom_{\mathbb{C}}$ through $\mathrm{Alg}_{\mathcal{T}}$, the category of $\mathcal{T}$-algebras.

$$
\begin{tikzcd}
 & \mathrm{Alg}_{\mathcal{T}} \arrow[d, "U"] \\
\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \arrow[ur, "\mathrm{Hom}_{\mathbb{C}}"] \arrow[r, "\hom_{\mathbb{C}}"'] & \mathrm{Set}
\end{tikzcd}
$$

A *$\mathcal{T}$-enriched functor* between the $\mathcal{T}$-enriched categories $\mathbb{C}$ and $\mathbb{D}$ is a functor $F\colon \mathbb{C} \to \mathbb{D}$ such that for all $A, B \in \mathbb{C}$,

$$F\colon\ \mathrm{Hom}_{\mathbb{C}}(A,B) \to \mathrm{Hom}_{\mathbb{D}}(F(A),F(B))$$

is a homomorphism of $\mathcal{T}$-algebras.

If $\mathcal{K}$ is another algebraic theory, $\mathcal{T}$-enrichments of $\mathrm{Alg}_{\mathcal{K}}$ are in one-to-one correspondence with central morphisms $\mathcal{T} \to \mathcal{K}$ of algebraic theories. These are morphisms such that for every term t from $\mathcal{T}$, its interpretation t^ι as a term of $\mathcal{K}$ commutes with every term q of $\mathcal{K}$ in the sense that

$$\begin{aligned} &t^\iota(q(x_{11},\dots,x_{1m}),\dots,q(x_{n1},\dots,x_{nm})) \\ &\quad = q(t^\iota(x_{11},\dots,x_{n1}),\dots,t^\iota(x_{1m},\dots,x_{nm})) \end{aligned}$$

is a theorem in $\mathcal{K}$ (where n and m are the arities of t and q respectively) (see [5]). The theory $\mathcal{T}$ is said to be *commutative* [12] if the identity $\mathcal{T} \to \mathcal{T}$ is a central morphism, i.e., if every two operations in $\mathcal{T}$ commute with each other.

Notice that if $\mathbb{C}$ is a $\mathcal{T}$-enriched category and $\mathbb{P}$ a small category, then the equalities

$$t(\alpha_1,\dots,\alpha_n)_P = t(\alpha_{1,P},\dots,\alpha_{n,P})$$

for all n-ary terms t of $\mathcal{T}$, $P \in \mathbb{P}$ and natural transformations $\alpha_1,\dots,\alpha_n : F \Rightarrow G$ define a $\mathcal{T}$-enrichment on the functor category $\mathbb{C}^{\mathbb{P}}$. If $\mathcal{T}$ is commutative and $\mathbb{C}$ small, the Yoneda embedding factors through $\mathrm{Alg}_{\mathcal{T}}^{\mathbb{C}^{\mathrm{op}}}$ as a $\mathcal{T}$-enriched functor $Y_{\mathcal{T}}\colon \mathbb{C} \to \mathrm{Alg}_{\mathcal{T}}^{\mathbb{C}^{\mathrm{op}}}$.

$$\begin{tikzcd} & \mathrm{Alg}_{\mathcal{T}}^{\mathbb{C}^{\mathrm{op}}} \arrow[d, "U^{\mathbb{C}^{\mathrm{op}}}"] \\ \mathbb{C} \arrow[ur, "Y_{\mathcal{T}}"] \arrow[r, "Y"'] & \mathrm{Set}^{\mathbb{C}^{\mathrm{op}}} \end{tikzcd}$$

2.2 Categories with M-closed relations

Let again $\mathcal{T}$ be an algebraic theory. An *extended matrix of terms in $\mathcal{T}$* [10] is a matrix

$$M = \left(\begin{array}{ccc|c} t_{11} & \cdots & t_{1m} & u_1 \\ \vdots & & \vdots & \vdots \\ t_{n1} & \cdots & t_{nm} & u_n \end{array} \right) \tag{1}$$

where the t_{ij}'s and the u_i's are terms of $\mathcal{T}$ in the variables $x_1, \ldots, x_k$ with $n \geqslant 1$, $m \geqslant 0$ and $k \geqslant 0$.

Let $r = (r_i \colon R \to A)_{i \in \{1,\ldots,n\}}$ be an n-ary relation in a $\mathcal{T}$-enriched category $\mathbb{C}$. We say that r is *M-closed* when, for all object X in $\mathbb{C}$ and morphisms $x_1, \ldots, x_k \colon X \to A$, if for each $j \in \{1, \ldots, m\}$, the span $(t_{ij}(x_1, \ldots, x_k) \colon X \to A)_{i \in \{1,\ldots,n\}}$ factors through r then so does the span $(u_i(x_1, \ldots, x_k) \colon X \to A)_{i \in \{1,\ldots,n\}}$.

Now, if $r = (r_i \colon R \to A_i)_{i \in \{1,\ldots,n\}}$ is an n-ary relation in $\mathbb{C}$, we say that this relation r is *strictly M-closed* when, for all object X in $\mathbb{C}$ and families of morphisms $(x_{ii'} \colon X \to A_i)_{i \in \{1,\ldots,n\}, i' \in \{1,\ldots,k\}}$, if for each $j \in \{1, \ldots, m\}$, the span $(t_{ij}(x_{i1}, \ldots, x_{ik}) \colon X \to A_i)_{i \in \{1,\ldots,n\}}$ factors through r then so does the span $(u_i(x_{i1}, \ldots, x_{ik}) \colon X \to A_i)_{i \in \{1,\ldots,n\}}$.

Here is the link between M-closedness and strict M-closedness.

Theorem 2.1. *(Theorem 5.5 in [10]) Let $\mathcal{T}$ be an algebraic theory, M an extended matrix of terms in $\mathcal{T}$ as in (1) and $\mathbb{C}$ a finitely complete $\mathcal{T}$-enriched category. Then, the following conditions are equivalent:*

1. *Every relation $r \colon R \rightarrowtail A^n$ in $\mathbb{C}$ is M-closed.*

2. *Every relation $r \colon R \rightarrowtail A_1 \times \cdots \times A_n$ in $\mathbb{C}$ is strictly M-closed.*

If the above conditions are satisfied, we say that $\mathbb{C}$ *has M-closed relations*. This matrix notation allows an easy characterisation in the varietal context.

Theorem 2.2. *(Theorem 3.2 in [10]) Let $\mathcal{T} \to \mathcal{K}$ be a central morphism of algebraic theories. Let also M be an extended matrix of terms in $\mathcal{T}$ as in (1). Then, the $\mathcal{T}$-enriched category $\mathrm{Alg}_{\mathcal{K}}$ has M-closed relations if and only if there exists an m-ary term p in $\mathcal{K}$ such that*

$$p(t^{\iota}_{i1}(x_1, \ldots, x_k), \ldots, t^{\iota}_{im}(x_1, \ldots, x_k)) = u^{\iota}_i(x_1, \ldots, x_k)$$

is a theorem of $\mathcal{K}$ for each $i \in \{1, \ldots, n\}$ (where t^{ι} is the interpretation in $\mathcal{K}$ of the term t in $\mathcal{T}$ induced by the morphism $\mathcal{T} \to \mathcal{K}$).

Example 2.3. Let $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$ be the theory of sets, $\mathbb{C}$ a finitely complete category and M_{Mal} the extended matrix

$$M_{\mathrm{Mal}} = \left(\begin{array}{ccc|c} x & y & y & x \\ x & x & y & y \end{array} \right)$$

of terms in $\mathrm{Th}[\mathrm{Set}]$. Then $\mathbb{C}$ has M_{Mal}-closed relations if and only if $\mathbb{C}$ is a Mal'tsev category [4, 10].

If $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$ is the theory of pointed sets and $\mathbb{C}$ a finitely complete pointed category, then the following equivalences hold:

- $\mathbb{C}$ has M_{Uni}-closed relations if and only if $\mathbb{C}$ is unital [2, 10], where M_{Uni} is the extended matrix
$$M_{\mathrm{Uni}} = \left(\begin{array}{cc|c} x & 0 & x \\ 0 & x & x \end{array} \right).$$

- $\mathbb{C}$ has M_{StrUni}-closed relations if and only if $\mathbb{C}$ is strongly unital [2, 10], where M_{StrUni} is the extended matrix
$$M_{\mathrm{StrUni}} = \left(\begin{array}{ccc|c} x & 0 & 0 & x \\ y & y & x & x \end{array} \right).$$

- $\mathbb{C}$ has M_{Subt}-closed relations if and only if $\mathbb{C}$ is subtractive [9, 10], where M_{Subt} is the extended matrix
$$M_{\mathrm{Subt}} = \left(\begin{array}{cc|c} x & 0 & x \\ x & x & 0 \end{array} \right).$$

2.3 Categories with M-closed strong relations

We now weaken this notion of a category with M-closed relations, considering only strong relations. We recall that in a finitely complete category $\mathbb{C}$, a morphism m is said to be a *strong monomorphism* if it is orthogonal to any epimorphism e. This means that for any commutative square

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d] & B \arrow[d] \arrow[dl, dotted, "d"'] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

with e an epimorphism, there exists a (unique) diagonal d making the two triangles commutative. It is easy to see that since $\mathbb{C}$ has pullbacks, it implies that m is a monomorphism and even an extremal monomorphism. Strong monomorphisms are closed under composition and stable under pullbacks. Regular monomorphisms (i.e., equalisers) are strong monomorphisms. We say that a span $(r_i\colon R \to A_i)_{i\in\{1,\dots,n\}}$ is a *strong relation* if the induced morphism $r = (r_1, \dots, r_n)\colon R \to A_1 \times \cdots \times A_n$ is a strong monomorphism.

Theorem 2.4. *Let $\mathcal{T}$ be an algebraic theory, M an extended matrix of terms in $\mathcal{T}$ as in (1) and $\mathbb{C}$ a finitely complete $\mathcal{T}$-enriched category. Then, the following conditions are equivalent:*

1. *Every strong relation $r\colon R \rightarrowtail A^n$ in $\mathbb{C}$ is M-closed.*
2. *Every strong relation $r\colon R \rightarrowtail A_1 \times \cdots \times A_n$ in $\mathbb{C}$ is strictly M-closed.*

Proof. $2 \Rightarrow 1$ being trivial, let us prove $1 \Rightarrow 2$. So, let us consider a strong relation $r\colon R \rightarrowtail A_1 \times \cdots \times A_n$ in $\mathbb{C}$. Since r is strong, its pullback along $\pi_1 \times \cdots \times \pi_n$ is also strong, where $\pi_i\colon A_1 \times \cdots \times A_n \to A_i$ is the i-th projection.

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail, "s"'] \arrow[dr, phantom, very near start, "\lrcorner"] & R \arrow[d, tail, "r"] \\ (A_1 \times \cdots \times A_n)^n \arrow[r, "\pi_1\times\cdots\times\pi_n"'] & A_1 \times \cdots \times A_n \end{tikzcd}$$

We conclude the proof by Proposition 1.9 in [10] which says that r is strictly M-closed if and only if s is M-closed. $\square$

If the above conditions are satisfied, we say that $\mathbb{C}$ *has M-closed strong relations*. In view of the following examples, we could also have written that $\mathbb{C}$ is 'weakly with M-closed relations'.

Example 2.5. If $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$ and $\mathbb{C}$ is a finitely complete category, $\mathbb{C}$ has M_{Mal}-closed strong relations if and only if $\mathbb{C}$ is a weakly Mal'tsev category. Let us recall that $\mathbb{C}$ is *weakly Mal'tsev* [15] if for every pullback of split epimorphisms,

$$\begin{tikzcd}
X \times_Z Y \arrow[r, two heads, "p_Y"] \arrow[d, two heads, "p_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[l, bend left, "r_Y"] \arrow[d, two heads, "g"'] \\
X \arrow[r, two heads, "f"] \arrow[u, bend right, "l_X"'] & Z \arrow[l, bend left, "s"] \arrow[u, bend right, "t"']
\end{tikzcd}$$

the induced morphisms $l_X = (1_X, tf)$ and $r_Y = (sg, 1_Y)$ are jointly epimorphic. Such a characterisation holds because a binary relation is strictly M_{Mal}-closed precisely when it is difunctional [10] and by Corollary 5.1 in [11], $\mathbb{C}$ is weakly Mal'tsev if and only if every binary strong relation in $\mathbb{C}$ is difunctional.

Example 2.6. If $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$ and $\mathbb{C}$ is a finitely complete pointed category, $\mathbb{C}$ has M_{Uni}-closed strong relations if and only if $\mathbb{C}$ is weakly unital. We recall that $\mathbb{C}$ is *weakly unital* [14] if for all objects X and Y in $\mathbb{C}$, the product injections

$$X \xrightarrow{(1_X,0)} X \times Y \xleftarrow{(0,1_Y)} Y$$

are jointly epimorphic. In that case, if $r\colon R \rightarrowtail A^2$ is a strong relation and $x\colon X \to A$ a morphism such that $(x, 0)\colon X \to A^2$ and $(0, x)\colon X \to A^2$ factor through r, we consider the pullback s of r along x^2.

$$\begin{tikzcd}
S \arrow[r] \arrow[d, tail, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, tail, "r"] \\
X^2 \arrow[r, "x^2"'] & A^2
\end{tikzcd}$$

The relation s is strong, $(1_X, 0)$ and $(0, 1_X)\colon X \to X^2$ factor through it and we only have to prove that $(1_X, 1_X)$ also factors through s. But since $(1_X, 0)$ and $(0, 1_X)$ are jointly epimorphic, s is an epimorphism. Together with the fact that it is also a strong monomorphism, s is an isomorphism and so $(1_X, 1_X)$ factors through it.

Conversely, suppose that $\mathbb{C}$ has M_{Uni}-closed strong relations and let f, g : $X \times Y \to Z$ be morphisms such that $f(1_X, 0) = g(1_X, 0)$ and $f(0, 1_Y) = g(0, 1_Y)$. Their equaliser $e\colon E \rightarrowtail X \times Y$ is a strong relation through which $(p_X, 0)$ and $(0, p_Y): X \times Y \to X \times Y$ factor. Thus, by assumption, $1_{X\times Y} = (p_X, p_Y): X \times Y \to X \times Y$ also factors through it, so that e is an isomorphism and $f = g$.

3. The category of partial M-algebras

3.1 Part_M and its limits

We suppose from now on that $\mathcal{T}$ is a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$ as in (1). A *partial M-algebra* is a $\mathcal{T}$-algebra A equipped with a partial operation $p: A^m \to A$ such that

- for each $i \in \{1, \dots, n\}$ and all $a_1, \dots, a_k \in A$,

  $$p(t_{i1}(a_1, \dots, a_k), \dots, t_{im}(a_1, \dots, a_k))$$

  is defined and

  $$p(t_{i1}(a_1, \dots, a_k), \dots, t_{im}(a_1, \dots, a_k)) = u_i(a_1, \dots, a_k);$$

- for each r-ary operation symbol σ of $\mathcal{T}$ and all families of elements $(a_j^{j'} \in A)_{j\in\{1,\dots,m\}, j'\in\{1,\dots,r\}}$ such that $p(a_1^{j'}, \dots, a_m^{j'})$ is defined for each $j' \in \{1, \dots, r\}$, $p(\sigma(a_1^1, \dots, a_1^r), \dots, \sigma(a_m^1, \dots, a_m^r))$ is defined and the equality

  $$\begin{aligned} &p(\sigma(a_1^1, \dots, a_1^r), \dots, \sigma(a_m^1, \dots, a_m^r)) \\ &= \sigma(p(a_1^1, \dots, a_m^1), \dots, p(a_1^r, \dots, a_m^r)) \end{aligned}$$

  holds.

A *homomorphism $f: A \to B$ of partial M-algebras* is a homomorphism between the corresponding $\mathcal{T}$-algebras such that, for all $a_1, \dots, a_m \in A$ for which $p(a_1, \dots, a_m)$ is defined, $p(f(a_1), \dots, f(a_m))$ is also defined and

$$p(f(a_1), \dots, f(a_m)) = f(p(a_1, \dots, a_m)).$$

We denote by Part_M the corresponding category. We have a $\mathcal{T}$-enrichment on Part_M: if σ is an r-ary operation symbol of $\mathcal{T}$ and $f_1, \dots, f_r: A \to B$ are homomorphisms of partial M-algebras, we define $\sigma(f_1, \dots, f_r): A \to B$ by

$$\sigma(f_1, \dots, f_r)(a) = \sigma(f_1(a), \dots, f_r(a))$$

for all $a \in A$. Since $\mathcal{T}$ is commutative, $\mathrm{Alg}_{\mathcal{T}}$ has a $\mathcal{T}$-enrichment computed as above and so $\sigma(f_1, \dots, f_r)$ is a homomorphism of $\mathcal{T}$-algebras. Moreover, if $a_1, \dots, a_m \in A$ are such that $p(a_1, \dots, a_m)$ is defined, for each $j' \in \{1, \dots, r\}$, $p(f_{j'}(a_1), \dots, f_{j'}(a_m))$ is also defined. This implies

$$\begin{aligned}
&p(\sigma(f_1, \dots, f_r)(a_1), \dots, \sigma(f_1, \dots, f_r)(a_m)) \\
&= p(\sigma(f_1(a_1), \dots, f_r(a_1)), \dots, \sigma(f_1(a_m), \dots, f_r(a_m)))
\end{aligned}$$

is defined as well and equal to

$$\begin{aligned}
&\sigma(p(f_1(a_1), \dots, f_1(a_m)), \dots, p(f_r(a_1), \dots, f_r(a_m))) \\
&= \sigma(f_1(p(a_1, \dots, a_m)), \dots, f_r(p(a_1, \dots, a_m))) \\
&= \sigma(f_1, \dots, f_r)(p(a_1, \dots, a_m))
\end{aligned}$$

in view of the second condition in the definition of partial M-algebras. This proves $\sigma(f_1, \dots, f_r)$ is indeed a homomorphism of partial M-algebras.

Let us now describe small limits in Part_M. In order to do so, we consider a small diagram $D\colon \mathbb{J} \to \mathrm{Part}_M$. Let $(\lambda_j\colon L \to U_{\mathcal{T}}D(j))_{j\in\mathbb{J}}$ be the limit of $U_{\mathcal{T}}D$ in $\mathrm{Alg}_{\mathcal{T}}$, where $U_{\mathcal{T}}\colon\ \mathrm{Part}_M \to \mathrm{Alg}_{\mathcal{T}}$ is the forgetful functor. So L is given by

$$L = \{(a_j)_{j\in\mathbb{J}} \in \prod_{j\in\mathbb{J}} D(j) \mid D(d)(a_j) = a_{j'}\ \forall d\colon j \to j' \in \mathbb{J}\}$$

with

$$\sigma((a_j^1)_{j\in\mathbb{J}}, \dots, (a_j^r)_{j\in\mathbb{J}}) = (\sigma(a_j^1, \dots, a_j^r))_{j\in\mathbb{J}}$$

for each r-ary operation symbol σ of $\mathcal{T}$. Now, if $(a_j^1)_{j\in\mathbb{J}}, \dots, (a_j^m)_{j\in\mathbb{J}} \in L$, we define $p((a_j^1)_{j\in\mathbb{J}}, \dots, (a_j^m)_{j\in\mathbb{J}})$ if and only if $p(a_j^1, \dots, a_j^m)$ is defined for all $j \in \mathbb{J}$. In this case, we set

$$p((a_j^1)_{j\in\mathbb{J}}, \dots, (a_j^m)_{j\in\mathbb{J}}) = (p(a_j^1, \dots, a_j^m))_{j\in\mathbb{J}}.$$

This makes L a partial M-algebra. Indeed, for each $i \in \{1, \dots, n\}$ and each $(a_j^1)_{j\in\mathbb{J}}, \dots, (a_j^k)_{j\in\mathbb{J}} \in L$,

$$\begin{aligned}
&p(t_{i1}((a_j^1)_{j\in\mathbb{J}}, \dots, (a_j^k)_{j\in\mathbb{J}}), \dots, t_{im}((a_j^1)_{j\in\mathbb{J}}, \dots, (a_j^k)_{j\in\mathbb{J}})) \\
&= p((t_{i1}(a_j^1, \dots, a_j^k))_{j\in\mathbb{J}}, \dots, (t_{im}(a_j^1, \dots, a_j^k))_{j\in\mathbb{J}})
\end{aligned}$$

is defined since $p(t_{i1}(a_j^1,\dots,a_j^k),\dots,t_{im}(a_j^1,\dots,a_j^k))$ is for each $j\in\mathbb{J}$ and it is equal to

$$\begin{aligned}(p(t_{i1}(a_j^1,\dots,a_j^k),\dots,t_{im}(a_j^1,\dots,a_j^k)))_{j\in\mathbb{J}} &= (u_i(a_j^1,\dots,a_j^k))_{j\in\mathbb{J}}\\ &= u_i((a_j^1)_{j\in\mathbb{J}},\dots,(a_j^k)_{j\in\mathbb{J}}).\end{aligned}$$

We check the second condition analogously: Let σ be an r-ary operation symbol of $\mathcal{T}$ and for each $j'\in\{1,\dots,r\}$, $(a_j^{1,j'})_{j\in\mathbb{J}},\dots,(a_j^{m,j'})_{j\in\mathbb{J}}$ elements of L such that $p((a_j^{1,j'})_{j\in\mathbb{J}},\dots,(a_j^{m,j'})_{j\in\mathbb{J}})$ is defined (i.e., $p(a_j^{1,j'},\dots,a_j^{m,j'})$ is defined for each $j\in\mathbb{J}$). This implies

$$p(\sigma(a_j^{1,1},\dots,a_j^{1,r}),\dots,\sigma(a_j^{m,1},\dots,a_j^{m,r}))$$

is defined and equal to

$$\sigma(p(a_j^{1,1},\dots,a_j^{m,1}),\dots,p(a_j^{1,r},\dots,a_j^{m,r}))$$

for each $j\in\mathbb{J}$. Thus

$$\begin{aligned}&p(\sigma((a_j^{1,1})_{j\in\mathbb{J}},\dots,(a_j^{1,r})_{j\in\mathbb{J}}),\dots,\sigma((a_j^{m,1})_{j\in\mathbb{J}},\dots,(a_j^{m,r})_{j\in\mathbb{J}}))\\ &\quad= p((\sigma(a_j^{1,1},\dots,a_j^{1,r}))_{j\in\mathbb{J}},\dots,(\sigma(a_j^{m,1},\dots,a_j^{m,r}))_{j\in\mathbb{J}})\end{aligned}$$

is also defined in L and equal to

$$\begin{aligned}&(\sigma(p(a_j^{1,1},\dots,a_j^{m,1}),\dots,p(a_j^{1,r},\dots,a_j^{m,r})))_{j\in\mathbb{J}}\\ &\quad= \sigma(p((a_j^{1,1})_{j\in\mathbb{J}},\dots,(a_j^{m,1})_{j\in\mathbb{J}}),\dots,p((a_j^{1,r})_{j\in\mathbb{J}},\dots,(a_j^{m,r})_{j\in\mathbb{J}})),\end{aligned}$$

which shows that L is a partial M-algebra. Moreover, given a cone $(\mu_j\colon A\to D(j))_{j\in\mathbb{J}}$ over D, let f be the unique homomorphism of $\mathcal{T}$-algebras

$$\begin{aligned}f\colon A &\longrightarrow L\\ a &\longmapsto (\mu_j(a))_{j\in\mathbb{J}}\end{aligned}$$

such that $\lambda_j f=\mu_j$ for each $j\in\mathbb{J}$. If $a_1,\dots,a_m\in A$ are such that $p(a_1,\dots,a_m)$ is defined in A, $p(\mu_j(a_1),\dots,\mu_j(a_m))$ is defined in $D(j)$ for each $j\in\mathbb{J}$. Thus, $p(f(a_1),\dots,f(a_m))$ is also defined and equal to

$$\begin{aligned}p((\mu_j(a_1))_{j\in\mathbb{J}},\dots,(\mu_j(a_m))_{j\in\mathbb{J}}) &= (p(\mu_j(a_1),\dots,\mu_j(a_m)))_{j\in\mathbb{J}}\\ &= (\mu_j(p(a_1,\dots,a_m)))_{j\in\mathbb{J}}\\ &= f(p(a_1,\dots,a_m)),\end{aligned}$$

which proves that f is a homomorphism of partial M-algebras and the cone $(\lambda_j\colon L \to D(j))_{j\in\mathbb{J}}$ the limit of D. Therefore, Part_M is complete and $U_{\mathcal{T}}\colon \mathrm{Part}_M \to \mathrm{Alg}_{\mathcal{T}}$ preserves small limits, but it does not reflect them in general. Indeed, one could have defined p on a smaller subset of L^m in order to make L a partial M-algebra, but this would not have made it a limit in Part_M. This means $U_{\mathcal{T}}$ is not conservative in general. Here is a simple counterexample.

Counterexample 3.1. Let $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$ and $M = M_{\mathrm{Uni}}$ from Example 2.3. Let A be the pointed set $\{0, x\}$ endowed with the structure of a partial M_{Uni}-algebra given by $p(0,0) = 0$, $p(x,0) = x = p(0,x)$ and $p(x,x)$ undefined. Let also B be the partial M_{Uni}-algebra on $\{0,x\}$ given by $p(0,0) = 0$ and $p(x,0) = x = p(0,x) = p(x,x)$. Then, the identity map $A \to B$ is a bijective homomorphism but not an isomorphism in $\mathrm{Part}_{M_{\mathrm{Uni}}}$.

3.2 Strong monomorphisms in Part_M

In order to understand strong monomorphisms in Part_M, we need to construct a left adjoint to the forgetful functor $U\colon \mathrm{Part}_M \to \mathrm{Set}$. As an intermediate step, we consider the category m-Part where objects are sets X equipped with a partial operation $p\colon X^m \to X$ and morphisms are functions $f\colon X \to Y$ such that if $p(x_1,\dots,x_m)$ is defined for some $x_1,\dots,x_m \in X$, then $p(f(x_1),\dots,f(x_m))$ is also defined and equal to $f(p(x_1,\dots,x_m))$. The forgetful functor $U\colon \mathrm{Part}_M \to \mathrm{Set}$ thus factors as $\mathrm{Part}_M \to m$-$\mathrm{Part} \to \mathrm{Set}$.

Proposition 3.2. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$ as in (1). The forgetful functor*

$$U'\colon \mathrm{Part}_M \to m\text{-}\mathrm{Part}$$

has a left adjoint.

Proof. Let X be an object of m-Part. Let us add the constant operation symbols c_x for all $x \in X$ in $\mathcal{T}$ to form the theory $\mathcal{T}'$. We denote by I the set

$$\begin{aligned} I &= \{1,\dots,n\} \sqcup \{(x_1,\dots,x_m) \in X^m \,|\, p(x_1,\dots,x_m) \text{ is defined}\} \\ &= \{1,\dots,n\} \sqcup \mathrm{dom}(p) \end{aligned}$$

and, for each $i = (x_1, \dots, x_m) \in \mathrm{dom}(p)$, $t_{ij}(y_1, \dots, y_k)$ is the k-ary term c_{x_j} of $\mathcal{T}'$ for each $j \in \{1, \dots, m\}$ and $u_i(y_1, \dots, y_k)$ the k-ary term $c_{p(i)}$ of $\mathcal{T}'$. Let $\mathcal{Q}$ be the quasivariety of $\mathcal{T}'$-algebras satisfying, for all r-ary (respectively r'-ary) terms τ and τ' of $\mathcal{T}$ and all indices $i_1, \dots, i_r, i'_1, \dots, i'_{r'}$ in I, the following implication: if

$$\begin{aligned}&\tau(t_{i_1 j}(y_{11}, \dots, y_{1k}), \dots, t_{i_r j}(y_{r1}, \dots, y_{rk}))\\&= \tau'(t_{i'_1 j}(y'_{11}, \dots, y'_{1k}), \dots, t_{i'_{r'} j}(y'_{r'1}, \dots, y'_{r'k}))\end{aligned}$$

for each $j \in \{1, \dots, m\}$, then

$$\begin{aligned}&\tau(u_{i_1}(y_{11}, \dots, y_{1k}), \dots, u_{i_r}(y_{r1}, \dots, y_{rk}))\\&= \tau'(u_{i'_1}(y'_{11}, \dots, y'_{1k}), \dots, u_{i'_{r'}}(y'_{r'1}, \dots, y'_{r'k})).\end{aligned}$$

For an object A of the quasivariety $\mathcal{Q}$, we define p in A via the equalities

$$\begin{aligned}&p(\tau(t_{i_1 1}(a_{11}, \dots, a_{1k}), \dots, t_{i_r 1}(a_{r1}, \dots, a_{rk})), \dots\\&\qquad\qquad \dots, \tau(t_{i_1 m}(a_{11}, \dots, a_{1k}), \dots, t_{i_r m}(a_{r1}, \dots, a_{rk})))\\&= \tau(u_{i_1}(a_{11}, \dots, a_{1k}), \dots, u_{i_r}(a_{r1}, \dots, a_{rk}))\end{aligned}$$

for all r-ary terms τ of $\mathcal{T}$, all indices $i_1, \dots, i_r \in I$ and all families of elements $(a_{j'i'} \in A)_{j' \in \{1,\dots,r\}, i' \in \{1,\dots,k\}}$. We do not define p for any other elements of A^m. In view of the implications defining $\mathcal{Q}$, this partial operation p is well-defined. We see that the first condition defining partial M-algebras is satisfied by choosing τ to be the identity term $\tau(y) = y$. The second condition is also satisfied: Let σ be an r-ary term of $\mathcal{T}$, $\tau^{j'}$ an $r^{j'}$-ary term of $\mathcal{T}$ for each $j' \in \{1, \dots, r\}$, $i^{j'}_{j''} \in I$ for all $j' \in \{1, \dots, r\}$ and $j'' \in \{1, \dots, r^{j'}\}$, and $a^{j'}_{j''i'} \in A$ for all $j' \in \{1, \dots, r\}$, $j'' \in \{1, \dots, r^{j'}\}$ and $i' \in \{1, \dots, k\}$. Then,

$$p((\sigma((\tau^{j'}((t_{i^{j'}_{j''} j}(a^{j'}_{j''1}, \dots, a^{j'}_{j''k}))^{r^{j'}}_{j''=1}))^{r}_{j'=1}))^{m}_{j=1})$$

is defined in view of the $(r^1 + \cdots + r^r)$-ary term

$$\sigma(\tau^1(y_{11}, \dots, y_{1r^1}), \dots, \tau^r(y_{r1}, \dots, y_{rr^r}))$$

of $\mathcal{T}$. Moreover, it is equal to

$$\begin{aligned}&\sigma((\tau^{j'}((u_{i^{j'}_{j''}}(a^{j'}_{j''1},\dots,a^{j'}_{j''k}))^{r^{j'}}_{j''=1}))^{r}_{j'=1})\\ &= \sigma((p((\tau^{j'}((t_{i^{j'}_{j''}j}(a^{j'}_{j''1},\dots,a^{j'}_{j''k}))^{r^{j'}}_{j''=1}))^{m}_{j=1}))^{r}_{j'=1})\end{aligned}$$

as required. So A has been endowed with a structure of partial M-algebra. We consider the function $f\colon X \to U'(A)\colon x \mapsto c_x$. It is a morphism in m-Part. Indeed, if $i = (x_1,\dots,x_m) \in \mathrm{dom}(p)$, choosing τ to be the identity term $\tau(y) = y$ and $i_1 = i$, we have

$$\begin{aligned}p(f(x_1),\dots,f(x_m)) &= p(c_{x_1},\dots,c_{x_m})\\ &= p(t_{i1}(a_1,\dots,a_k),\dots,t_{im}(a_1,\dots,a_k))\\ &= u_i(a_1,\dots,a_k)\\ &= c_{p(i)}\\ &= f(p(x_1,\dots,x_m)).\end{aligned}$$

If $g\colon A \to A'$ is a morphism in $\mathcal{Q}$, it can be considered as a homomorphism of partial M-algebras making the triangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f'"'] & U'(A) \arrow[d, "g"] \\ & U'(A') \end{tikzcd}$$

commutative. Indeed, the above triangle commutes since g is a $\mathcal{T}'$-homomorphism and when

$$\begin{aligned}&p(\tau(t_{i_1 1}(a_{11},\dots,a_{1k}),\dots,t_{i_r 1}(a_{r1},\dots,a_{rk})),\dots\\ &\qquad\dots,\tau(t_{i_1 m}(a_{11},\dots,a_{1k}),\dots,t_{i_r m}(a_{r1},\dots,a_{rk})))\end{aligned}$$

is defined in A,

$$\begin{aligned}&p(g(\tau(t_{i_1 1}(a_{11},\dots,a_{1k}),\dots,t_{i_r 1}(a_{r1},\dots,a_{rk}))),\dots\\ &\qquad\dots,g(\tau(t_{i_1 m}(a_{11},\dots,a_{1k}),\dots,t_{i_r m}(a_{r1},\dots,a_{rk}))))\\ &= p(\tau(t_{i_1 1}(g(a_{11}),\dots,g(a_{1k})),\dots,t_{i_r 1}(g(a_{r1}),\dots,g(a_{rk}))),\dots\\ &\qquad\dots,\tau(t_{i_1 m}(g(a_{11}),\dots,g(a_{1k})),\dots,t_{i_r m}(g(a_{r1}),\dots,g(a_{rk}))))\end{aligned}$$

is defined in A' and equal to

$$\begin{aligned}&\tau(u_{i_1}(g(a_{11}),\dots,g(a_{1k})),\dots,u_{i_r}(g(a_{r1}),\dots,g(a_{rk})))\\&= g(\tau(u_{i_1}(a_{11},\dots,a_{1k}),\dots,u_{i_r}(a_{r1},\dots,a_{rk}))).\end{aligned}$$

We have thus defined a functor $F'\colon \mathcal{Q} \to (X \downarrow U')$ where $(X \downarrow U')$ is the comma category of morphisms $X \to U'(A)$ in *m*-Part.

On the other hand, if $f\colon X \to U'(A)$ is an object of $(X \downarrow U')$, A admits a $\mathcal{T}'$-algebra structure considering $c_x = f(x)$ for each $x \in X$. Moreover, for each $i \in I$ and $a_1,\dots,a_k \in A$,

$$p(t_{i1}(a_1,\dots,a_k),\dots,t_{im}(a_1,\dots,a_k)) = u_i(a_1,\dots,a_k).$$

So, if τ is an r-ary term of $\mathcal{T}$, $i_1,\dots,i_r \in I$ and $(a_{j'i'} \in A)_{j'\in\{1,\dots,r\},i'\in\{1,\dots,k\}}$,

$$\begin{aligned}&p(\tau(t_{i_11}(a_{11},\dots,a_{1k}),\dots,t_{i_r1}(a_{r1},\dots,a_{rk})),\dots\\&\qquad\qquad\dots,\tau(t_{i_1m}(a_{11},\dots,a_{1k}),\dots,t_{i_rm}(a_{r1},\dots,a_{rk})))\\&= \tau(u_{i_1}(a_{11},\dots,a_{1k}),\dots,u_{i_r}(a_{r1},\dots,a_{rk}))\end{aligned}$$

since A is a partial M-algebra. Hence, A satisfies the implications defining $\mathcal{Q}$ and this makes $G'\colon (X \downarrow U') \to \mathcal{Q}$ a functor.

Since the equality above holds in A for any object $f\colon X \to U'(A)$ of $(X \downarrow U')$, the identity map on A defines a morphism $\varepsilon_f\colon F'G'(f) \to f$ in $(X \downarrow U')$. This gives a natural transformation $\varepsilon\colon F'G' \Rightarrow 1_{(X\downarrow U')}$. Moreover, $G'F' = 1_{\mathcal{Q}}$ and we have constructed an adjunction $F' \dashv G'$. But $\mathcal{Q}$ is a quasivariety, so it has an initial object. Therefore $(X \downarrow U')$ has also an initial object which is the reflection of X along U'. $\square$

To construct the reflection of the set X along the forgetful functor *m*-Part $\to$ Set is much easier. It suffices to consider the identity map $1_X\colon X \to X$ where the partial operation p on X is nowhere defined. This gives a left adjoint Set $\to$ *m*-Part. Composed with the left adjoint *m*-Part $\to$ Part_M given by the above proposition, we have constructed the left adjoint $F\colon \mathrm{Set} \to \mathrm{Part}_M$ to the forgetful functor $U\colon \mathrm{Part}_M \to \mathrm{Set}$. We remark that in the particular case $X = \varnothing$, the quasivariety $\mathcal{Q}$ described above is the quasivariety $\mathcal{Q}_M$ of $\mathcal{T}$-algebras satisfying, for all r-ary (respectively r'-ary) terms τ and τ' of $\mathcal{T}$ and all indices $i_1, \dots, i_r, i'_1, \dots, i'_{r'}$ in $\{1, \dots, n\}$, the following implication: if

$$\begin{aligned}&\tau(t_{i_1 j}(a_{11}, \dots, a_{1k}), \dots, t_{i_r j}(a_{r1}, \dots, a_{rk}))\\&\quad = \tau'(t_{i'_1 j}(a'_{11}, \dots, a'_{1k}), \dots, t_{i'_{r'} j}(a'_{r'1}, \dots, a'_{r'k}))\end{aligned}$$

for each $j \in \{1, \dots, m\}$, then

$$\begin{aligned}&\tau(u_{i_1}(a_{11}, \dots, a_{1k}), \dots, u_{i_r}(a_{r1}, \dots, a_{rk}))\\&\quad = \tau'(u_{i'_1}(a'_{11}, \dots, a'_{1k}), \dots, u_{i'_{r'}}(a'_{r'1}, \dots, a'_{r'k})).\end{aligned}$$

The functor $F' \colon \mathcal{Q} \to (X \downarrow U')$ is then nothing but the left adjoint $\mathcal{Q}_M \to \mathrm{Part}_M$ to the forgetful functor $\mathrm{Part}_M \to \mathcal{Q}_M$. The left adjoint $F \colon \mathrm{Set} \to \mathrm{Part}_M$ can thus be also obtained by composing $F' \colon \mathcal{Q}_M \to \mathrm{Part}_M$ with the usual free functor $\mathrm{Set} \to \mathcal{Q}_M$.

We now consider the case $X = \{1, \dots, m+1\}$ with p defined only by $p(1, \dots, m) = m + 1$. We denote by $X \to U'(F_M)$ its reflection along $U' \colon \mathrm{Part}_M \to$ *m*-Part and g its restriction $g \colon \{1, \dots, m\} \hookrightarrow X \to U(F_M)$. The function g is such that $p(g(1), \dots, g(m))$ is defined in F_M and universal with that property, i.e., if $h \colon \{1, \dots, m\} \to U(A)$ is a function to a partial M-algebra A where $p(h(1), \dots, h(m))$ is defined, there exists a unique homomorphism of partial M-algebras $\overline{h} \colon F_M \to A$ such that $U(\overline{h}) \circ g = h$.

$$\begin{tikzcd}
\{1, \dots, m\} \arrow[rr, "g"] \arrow[dr, "\forall h"'] & & U(F_M) \arrow[dl, dotted, "\exists!\, U(\overline{h})"] \\
& U(A) &
\end{tikzcd}$$

Since $U \colon \mathrm{Part}_M \to \mathrm{Set}$ preserves kernel pairs, monomorphisms in Part_M are exactly the injective homomorphisms. We can now study strong monomorphisms: Let $f \colon A \rightarrowtail B$ be such a monomorphism. Consider also the homomorphism $e \colon F(\{1, \dots, m\}) \to F_M$ given by the universal property of $F(\{1, \dots, m\})$ and the function $g \colon \{1, \dots, m\} \to U(F_M)$. If $\overline{h}, \overline{k} \colon F_M \to C$ are homomorphisms of partial M-algebras such that $\overline{h}e = \overline{k}e$, then $\overline{h}g = \overline{k}g$ and $\overline{h} = \overline{k}$. Thus e is actually an epimorphism in Part_M. If $a_1, \dots, a_m \in A$ are such that $p(f(a_1), \dots, f(a_m))$ is defined, we can construct a commutative square as below with $k(j) = a_j$ and

$\overline{h}(g(j)) = f(a_j)$ for each $j \in \{1, \dots, m\}$.

$$\begin{tikzcd} F(\{1,\dots,m\}) \arrow[r, two heads, "e"] \arrow[d, "k"'] & F_M \arrow[d, "\overline{h}"] \arrow[dl, dotted] \\ A \arrow[r, tail, "f"'] & B \end{tikzcd}$$

Since f is supposed to be a strong monomorphism, $\overline{h}$ factors through f. Hence, $p(a_1, \dots, a_m)$ is defined as well. Therefore, strong monomorphisms in Part_M reflect the m-uples where p is defined, i.e., if $p(f(a_1), \dots, f(a_m))$ is defined, then so is $p(a_1, \dots, a_m)$. Following the terminology of [7] in universal partial algebra, homomorphisms in Part_M which reflect the m-uples where p is defined are said to be *closed*. This leads us to the following proposition.

Proposition 3.3. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$. Strong monomorphisms in Part_M are closed.*

The homomorphism from Counterexample 3.1 is an example of a bijective homomorphism of partial M-algebras which is not closed. Note that isomorphisms in Part_M are exactly the bijective closed homomorphisms. Indeed, in view of the next lemma, closedness of a bijective homomorphism $f\colon B \to C$ is exactly what we need to prove the inverse map $f^{-1}\colon C \to B$ is a homomorphism of partial M-algebras.

Lemma 3.4. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$. Let also $g\colon A \to B$ be a function between partial M-algebras and $f\colon B \rightarrowtail C$ a closed monomorphism in Part_M. If fg is a homomorphism of partial M-algebras, then so is g.*

Proof. Let σ be an r-ary operation symbol of $\mathcal{T}$ and $a_1, \dots, a_r \in A$. Since

$$\begin{aligned} f(g(\sigma(a_1, \dots, a_r))) &= \sigma(fg(a_1), \dots, fg(a_r)) \\ &= f(\sigma(g(a_1), \dots, g(a_r))) \end{aligned}$$

and f is injective, g is a homomorphism of $\mathcal{T}$-algebras.

Besides, if $a_1, \dots, a_m \in A$ are such that $p(a_1, \dots, a_m)$ are defined in A, $p(fg(a_1), \dots, fg(a_m))$ is defined in C and $p(g(a_1), \dots, g(a_m))$ is defined in

B since f is closed. We can also compute

$$\begin{aligned} f(p(g(a_1),\ldots,g(a_m))) &= p(fg(a_1),\ldots,fg(a_m)) \\ &= fg(p(a_1,\ldots,a_m)), \end{aligned}$$

which implies

$$p(g(a_1),\ldots,g(a_m)) = g(p(a_1,\ldots,a_m))$$

since f is injective. $\square$

We now want to prove that, in some cases, closed monomorphisms in Part_M are exactly the strong monomorphisms. To achieve this, we need to study the properties of closed monomorphisms.

Proposition 3.5. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$. Closed monomorphisms in Part_M are stable under pullbacks.*

Proof. We consider a closed monomorphism $f\colon A \rightarrowtail B$ in Part_M and its pullback along $g\colon C \to B$.

$$\begin{tikzcd} P \arrow[r, tail, "f'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ A \arrow[r, tail, "f"'] & B \end{tikzcd}$$

If $(a_1,c_1),\ldots,(a_m,c_m) \in P$, $p((a_1,c_1),\ldots,(a_m,c_m))$ is defined if and only if $p(a_1,\ldots,a_m)$ and $p(c_1,\ldots,c_m)$ are defined. But if $p(c_1,\ldots,c_m)$ is defined, $p(g(c_1),\ldots,g(c_m)) = p(f(a_1),\ldots,f(a_m))$ is also defined. Since f is closed, this further implies $p(a_1,\ldots,a_m)$ and so $p((a_1,c_1),\ldots,(a_m,c_m))$ are defined. Thus f' is a closed monomorphism. $\square$

Let us recall the following well-known proposition, which will be used in the particular case $\mathbb{C} = \mathrm{Part}_M$ and $\mathcal{R}$ the class of closed monomorphisms.

Proposition 3.6. *Let $\mathcal{R}$ be a class of monomorphisms in the finitely complete category $\mathbb{C}$ which is stable under pullbacks and contains regular monomorphisms. A morphism e in $\mathbb{C}$ is orthogonal to all elements of $\mathcal{R}$ if and only if, when e factors as fg with $f \in \mathcal{R}$, then f is an isomorphism. In this case, e is an epimorphism.*

Proposition 3.7. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$. If $\mathcal{R}$ denotes the class of closed monomorphisms in Part_M and $\mathcal{R}^\perp$ its orthogonal class, $(\mathcal{R}^\perp, \mathcal{R})$ is a factorisation system.*

Proof. Since $\mathcal{R}$ contains regular monomorphisms, is stable under pullbacks and closed under composition, we only have to prove that each homomorphism $f\colon A \to B$ of partial M-algebras factors as an element of $\mathcal{R}^\perp$ followed by a closed monomorphism. Let B' be the smallest sub-$\mathcal{T}$-algebra of B satisfying the conditions:

- $f(a) \in B'$ for each $a \in A$,
- if $b_1, \ldots, b_m \in B'$ are such that $p(b_1, \ldots, b_m)$ is defined in B, then $p(b_1, \ldots, b_m) \in B'$.

We consider the unique structure of partial M-algebra on B' making the inclusion $i\colon B' \hookrightarrow B$ a closed monomorphism. Then, f factors as if' with $f'\colon A \to B'$ a homomorphism of partial M-algebras by Lemma 3.4. Moreover, if $f' = f''g$ with f'' a closed monomorphism, the image of f'' contains B' by definition of B'. Thus f'' is surjective and so an isomorphism. By Proposition 3.6, $f' \in \mathcal{R}^\perp$. $\square$

Epimorphisms in Part_M thus factor as an epimorphism orthogonal to closed monomorphisms followed by a closed monomorphism (which is also an epimorphism). Therefore, to prove that closed monomorphisms are orthogonal to epimorphisms (i.e., are strong monomorphisms), it suffices to prove that closed epimorphisms are surjective. Indeed, in that case, this would imply that the only epimorphisms which are closed monomorphisms are the isomorphisms. This will be true for some particular M's.

Proposition 3.8. *Let M be an extended matrix of terms in* $\mathrm{Th}[\mathrm{Set}_*]$. *Closed epimorphisms in* Part_M *are surjective.*

Proof. Firstly, we notice that all partial M-algebras with one element are isomorphic (since $p(0, \ldots, 0)$ has to be defined). If this partial M-algebra is the unique one, the result is trivial. Hence, we suppose that there exists a partial M-algebra C with a non-zero element $c \in C$. Now, we also suppose we have a closed epimorphism $f\colon A \twoheadrightarrow B$ in Part_M which is not surjective.

Let $\mathrm{Im}(f)$ be the set-theoretical image of f and $D = D' = B \setminus \mathrm{Im}(f) \neq \varnothing$. Notice that $0 \in \mathrm{Im}(f)$. We define a partial m-ary operation p on

$$\mathrm{Im}(f) \sqcup D \sqcup D'$$

in the following way:

1. $p(t_{i1}(x_1, \dots, x_k), \dots, t_{im}(x_1, \dots, x_k))$ is defined as $u_i(x_1, \dots, x_k)$ for all $i \in \{1, \dots, n\}$ and all $x_1, \dots, x_k \in \mathrm{Im}(f) \sqcup D \sqcup D'$;

2. p restricted on $(\mathrm{Im}(f) \sqcup D)^m$ is defined as in B via the isomorphism of pointed sets $\mathrm{Im}(f) \sqcup D \cong B$;

3. p restricted on $(\mathrm{Im}(f) \sqcup D')^m$ is defined as in B via the isomorphism of pointed sets $\mathrm{Im}(f) \sqcup D' \cong B$;

4. p is defined nowhere else than required by one of the above conditions.

Let us prove this p is well-defined. There is no problem with condition 1 alone. Indeed, let us suppose by contradiction there exist $i, i' \in \{1, \dots, n\}$ and $x_1, \dots, x_k, x'_1, \dots, x'_k \in \mathrm{Im}(f) \sqcup D \sqcup D'$ satisfying $t_{ij}(x_1, \dots, x_k) = t_{i'j}(x'_1, \dots, x'_k)$ for all $j \in \{1, \dots, m\}$, but $u_i(x_1, \dots, x_k) \neq u_{i'}(x'_1, \dots, x'_k)$. Without loss of generality, we can suppose $u_i(x_1, \dots, x_k) \neq 0$. We consider any homomorphism of pointed sets $g\colon \mathrm{Im}(f) \sqcup D \sqcup D' \to C$ which sends $u_i(x_1, \dots, x_k)$ to c and $u_{i'}(x'_1, \dots, x'_k)$ to 0. Then

$$t_{ij}(g(x_1), \dots, g(x_k)) = t_{i'j}(g(x'_1), \dots, g(x'_k))$$

for each $j \in \{1, \dots, m\}$ and therefore

$$\begin{aligned}
c &= g(u_i(x_1, \dots, x_k)) \\
&= u_i(g(x_1), \dots, g(x_k)) \\
&= p(t_{i1}(g(x_1), \dots, g(x_k)), \dots, t_{im}(g(x_1), \dots, g(x_k))) \\
&= p(t_{i'1}(g(x'_1), \dots, g(x'_k)), \dots, t_{i'm}(g(x'_1), \dots, g(x'_k))) \\
&= u_{i'}(g(x'_1), \dots, g(x'_k)) \\
&= g(u_{i'}(x'_1, \dots, x'_k)) \\
&= 0,
\end{aligned}$$

which is a contradiction.

Since B is a (well-defined) partial M-algebra, there is no problem with condition 2 alone nor with condition 3 alone. The cohabitation of conditions 2 and 3 does not cause any problem neither. Indeed, the only way it could, is to have $b_1, \dots, b_m \in \mathrm{Im}(f)$ such that $p(b_1, \dots, b_m)$ is defined but does not belong to $\mathrm{Im}(f)$. If we write $b_i = f(a_i)$ for some $a_i \in A$, this means $p(f(a_1), \dots, f(a_m))$ is defined. But since f is closed, it implies $p(a_1, \dots, a_m)$ is defined and

$$p(b_1, \dots, b_m) = p(f(a_1), \dots, f(a_m)) = f(p(a_1, \dots, a_m)) \in \mathrm{Im}(f).$$

By symmetry, it remains to check there is no problem with the cohabitation of conditions 1 and 2. If there is one, it means there exist $x_1, \dots, x_k \in \mathrm{Im}(f) \sqcup D \sqcup D'$ and $i \in \{1, \dots, n\}$ such that $t_{ij}(x_1, \dots, x_k) \in \mathrm{Im}(f) \sqcup D$ for each $j \in \{1, \dots, m\}$ but $p(t_{i1}(x_1, \dots, x_k), \dots, t_{im}(x_1, \dots, x_k))$ defined as in B (via $\mathrm{Im}(f) \sqcup D \cong B$) is not $u_i(x_1, \dots, x_k)$. We denote by

$$q \colon \mathrm{Im}(f) \sqcup D \sqcup D' \to \mathrm{Im}(f) \sqcup D$$

the homomorphism of pointed sets which coequalises the two copies of D. This implies

$$t_{ij}(x_1, \dots, x_k) = q(t_{ij}(x_1, \dots, x_k)) = t_{ij}(q(x_1), \dots, q(x_k))$$

for each $j \in \{1, \dots, m\}$. Since we have already shown there is no problem with condition 1 alone, we can write using this condition

$$\begin{aligned} u_i(x_1, \dots, x_k) &= p(t_{i1}(x_1, \dots, x_k), \dots, t_{im}(x_1, \dots, x_k)) \\ &= p(t_{i1}(q(x_1), \dots, q(x_k)), \dots, t_{im}(q(x_1), \dots, q(x_k))) \\ &= u_i(q(x_1), \dots, q(x_k)). \end{aligned}$$

But since B is a partial M-algebra, if we compute using condition 2, we also get

$$\begin{aligned} &p(t_{i1}(x_1, \dots, x_k), \dots, t_{im}(x_1, \dots, x_k)) \\ &= p(t_{i1}(q(x_1), \dots, q(x_k)), \dots, t_{im}(q(x_1), \dots, q(x_k))) \\ &= u_i(q(x_1), \dots, q(x_k)). \end{aligned}$$

This discussion proves p is well defined.

The first condition to be a partial M-algebra is satisfied by $\mathrm{Im}(f)\sqcup D\sqcup D'$ in view of condition 1. In the case $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$, the second one resumes to $p(0, \ldots, 0) = 0$ which is true since it holds in B. Thus $\mathrm{Im}(f) \sqcup D \sqcup D'$ is a partial M-algebra. Now we consider the two obvious homomorphisms of partial M-algebras $g_1, g_2 \colon B \to \mathrm{Im}(f)\sqcup D\sqcup D'$. They satisfy $g_1 f = g_2 f$ but $g_1 \neq g_2$ since $D = D' \neq \varnothing$. This is a contradiction since f was supposed to be an epimorphism. □

If $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$, there are two partial M-algebras with at most one element, i.e., the empty partial M-algebra and the singleton one $\{\star\}$ (in which $p(\star, \ldots, \star)$ has to be defined since $n \geqslant 1$). Therefore, the first argument in the previous proof does not hold if we replace $\mathrm{Th}[\mathrm{Set}_*]$ by $\mathrm{Th}[\mathrm{Set}]$. For instance, if $M = \left(\begin{array}{c|c} x & y \end{array}\right)$, the category Part_M is equivalent to the arrow category $0 \to 1$. With this M, the unique homomorphism of partial M-algebras $\varnothing \to \{\star\}$ is an injective closed epimorphism, but not an isomorphism. However, if M is such that there exists a partial M-algebra with at least two elements, the same proof can be repeated to get the following proposition.

Proposition 3.9. *Let M be an extended matrix of terms in* $\mathrm{Th}[\mathrm{Set}]$ *such that there exists a partial M-algebra with at least two elements. Closed epimorphisms in* Part_M *are surjective.*

Counterexample 3.10. If $\mathcal{T}$ is the theory of commutative monoids and M is the trivial matrix $\left(\begin{array}{c|c} x & x \end{array}\right)$, Part_M is isomorphic to the category of commutative monoids. There, the inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ is an injective closed epimorphism but not an isomorphism.

As explained above, Propositions 3.8 and 3.9 admit the following corollary.

Corollary 3.11. *In* $\mathrm{Part}_{M_{\mathrm{Mal}}}$, $\mathrm{Part}_{M_{\mathrm{Uni}}}$, $\mathrm{Part}_{M_{\mathrm{StrUni}}}$ *and* $\mathrm{Part}_{M_{\mathrm{Subt}}}$, *closed monomorphisms coincide with strong monomorphisms.*

We now prove that Part_M has M-closed strong relations.

Proposition 3.12. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$ as in (1). Every relation $r \colon R \rightarrowtail A_1 \times \cdots \times A_n$ which is a closed monomorphism in* Part_M *is strictly M-closed. In particular,* Part_M *has M-closed strong relations.*

Proof. Consider a family of morphisms $(x_{ii'} \colon X \to A_i)_{i\in\{1,\dots,n\},i'\in\{1,\dots,k\}}$ in Part_M for which the morphism

$$(t_{1j}(x_{11},\dots,x_{1k}),\dots,t_{nj}(x_{n1},\dots,x_{nk}))\colon X \to A_1\times\cdots\times A_n$$

factors as rw_j for each $j\in\{1,\dots,m\}$.

$$\begin{tikzcd} & & R \arrow[d, tail, "r"] \\ X \arrow[rru, "w_j"] \arrow[rr, "(t_{1j}(x_{11},\dots,x_{1k}),\dots,t_{nj}(x_{n1},\dots,x_{nk}))"'] & & A_1\times\cdots\times A_n \end{tikzcd}$$

We know that for all $x\in X$ and each $i\in\{1,\dots,n\}$,

$$p(t_{i1}(x_{i1}(x),\dots,x_{ik}(x)),\dots,t_{im}(x_{i1}(x),\dots,x_{ik}(x)))$$

is defined and equal to $u_i(x_{i1}(x),\dots,x_{ik}(x))$. Using the description of small products in Part_M, we can say that, for all $x\in X$, $p(rw_1(x),\dots,rw_m(x))$ is defined and equal to

$$(u_1(x_{11}(x),\dots,x_{1k}(x)),\dots,u_n(x_{n1}(x),\dots,x_{nk}(x))).$$

Since r is closed, $p(w_1(x),\dots,w_m(x))$ is defined in R and we can consider the function $w\colon X\to R\colon x\mapsto p(w_1(x),\dots,w_m(x))$ which satisfies

$$rw = (u_1(x_{11},\dots,x_{1k}),\dots,u_n(x_{n1},\dots,x_{nk})).$$

Finally, Lemma 3.4 tells us w is a homomorphism of partial M-algebras since rw is and r is a closed monomorphism, which concludes the proof. $\square$

4. The embedding theorems

Now that the preliminary work on Part_M has been done, we can prove our embedding theorems for small categories with M-closed relations and for small categories with M-closed strong relations. In order to prove both at the same time, we are going to use a set of monomorphisms, closed under composition, stable under pullbacks and which contains regular monomorphisms.

Theorem 4.1. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$ as in (1). Let also $\mathcal{R}$ be a set of monomorphisms in the small finitely complete $\mathcal{T}$-enriched category $\mathbb{C}$ such that $\mathcal{R}$ is closed under composition, stable under pullbacks and contains regular monomorphisms. Suppose also that all n-ary relations $R \rightarrowtail A^n$ in $\mathcal{R}$ are M-closed in $\mathbb{C}$. Then, there exists a full and faithful $\mathcal{T}$-enriched embedding $\phi\colon \mathbb{C} \hookrightarrow \mathrm{Part}_M^{\mathbb{C}^{\mathrm{op}}}$ which preserves and reflects finite limits. Moreover, for each monomorphism $f\colon A \rightarrowtail B$ in $\mathcal{R}$ and each $X \in \mathbb{C}^{\mathrm{op}}$, $\phi(f)_X$ is a closed monomorphism in Part_M.*

Proof. We would like to factorise the $\mathcal{T}$-enriched Yoneda embedding $Y_{\mathcal{T}} : \mathbb{C} \to \mathrm{Alg}_{\mathcal{T}}^{\mathbb{C}^{\mathrm{op}}}$ through $\mathrm{Part}_M^{\mathbb{C}^{\mathrm{op}}}$.

$$\begin{tikzcd} & \mathrm{Part}_M^{\mathbb{C}^{\mathrm{op}}} \arrow[d, "U_{\mathcal{T}}^{\mathbb{C}^{\mathrm{op}}}"] \\ \mathbb{C} \arrow[ur, dotted, "\phi"] \arrow[r, "Y_{\mathcal{T}}"'] & \mathrm{Alg}_{\mathcal{T}}^{\mathbb{C}^{\mathrm{op}}} \end{tikzcd}$$

In order to do so, let us provide $\mathbb{C}(X,Y)$ with a structure of partial M-algebra, for all objects $X, Y \in \mathbb{C}$. Thus, let $f_1, \ldots, f_m : X \to Y$ be morphisms in $\mathbb{C}$. We define $p(f_1, \ldots, f_m)$ if and only if there exist morphisms $x_1, \ldots, x_k\colon X \to W$, a relation $r\colon Z \rightarrowtail W^n$ in $\mathcal{R}$, and morphisms $g_1, \ldots, g_m\colon X \to Z$ and $f\colon Z \to Y$ such that, for all $j \in \{1, \ldots, m\}$, $fg_j = f_j$ and $rg_j = (t_{1j}(x_1, \ldots, x_k), \ldots, t_{nj}(x_1, \ldots, x_k))$.

$$\begin{tikzcd} & X \arrow[dl, bend right, "{(t_{1j}(x_1,\ldots,x_k),\ldots,t_{nj}(x_1,\ldots,x_k))}"'] \arrow[d, "g_j"] \arrow[dr, bend left, "f_j"] & \\ W^n & Z \arrow[l, tail, "r"] \arrow[r, "f"'] & Y \end{tikzcd}$$

In this case, since r is M-closed, there exists a unique $h\colon X \to Z$ such that $rh = (u_1(x_1, \ldots, x_k), \ldots, u_n(x_1, \ldots, x_k))$ and we define $p(f_1, \ldots, f_m) = fh$.

$$\begin{tikzcd} & Z \arrow[d, tail, "r"] \\ X \arrow[ur, dotted, "\exists!\, h"] \arrow[r, "{(u_1(x_1,\ldots,x_k),\ldots,u_n(x_1,\ldots,x_k))}"'] & W^n \end{tikzcd}$$

Let us first prove the independence of the choices. So, suppose $x'_1, \ldots, x'_k : X \to W'$, $r' \colon Z' \rightarrowtail W'^n$, $g'_1, \ldots, g'_m \colon X \to Z'$, $f' \colon Z' \to Y$ and $h' \colon X \to Z'$ also satisfy the above conditions and let us prove $fh = f'h'$. We consider the following pullback

$$\begin{tikzcd}
Z_1 \arrow[r, "q_1"] \arrow[d, tail, "r_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, tail, "r"] \\
(W \times W')^n \arrow[r, "\pi_1^n"'] & W^n
\end{tikzcd}$$

where $\pi_1 \colon W \times W' \to W$ is the first projection. We also consider the unique morphisms $l_1^1, \ldots, l_1^m \colon X \to Z_1$ such that $q_1 l_1^j = g_j$ and

$$r_1 l_1^j = (t_{1j}((x_1, x'_1), \ldots, (x_k, x'_k)), \ldots, t_{nj}((x_1, x'_1), \ldots, (x_k, x'_k)))$$

for each $j \in \{1, \ldots, m\}$. Let also $h_1 \colon X \to Z_1$ be the unique morphism such that $q_1 h_1 = h$ and

$$r_1 h_1 = (u_1((x_1, x'_1), \ldots, (x_k, x'_k)), \ldots, u_n((x_1, x'_1), \ldots, (x_k, x'_k))).$$

Similarly, we define Z_2, r_2, q_2, $l_2^1, \ldots, l_2^m$ and h_2 using the pullback of r' along π_2^n where $\pi_2 \colon W \times W' \to W'$ is the second projection. Since $\mathcal{R}$ is stable under pullbacks, $r_1, r_2 \in \mathcal{R}$. We also construct their intersection,

$$\begin{tikzcd}
P \arrow[r, tail, "r_3"] \arrow[d, tail, "r_4"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z_2 \arrow[d, tail, "r_2"] \\
Z_1 \arrow[r, tail, "r_1"'] & (W \times W')^n
\end{tikzcd}$$

the unique morphism $h_3 \colon X \to P$ such that $r_3 h_3 = h_2$ and $r_4 h_3 = h_1$ and, for each $j \in \{1, \ldots, m\}$, the unique morphism $l_3^j \colon X \to P$ such that $r_3 l_3^j = l_2^j$ and $r_4 l_3^j = l_1^j$. Again, $r_3, r_4 \in \mathcal{R}$. Finally, we consider the following equaliser diagram.

$$\begin{tikzcd}
E \arrow[r, tail, "e"] & P \arrow[r, shift left, "f q_1 r_4"] \arrow[r, shift right, "f' q_2 r_3"'] & Y
\end{tikzcd}$$

For each $j \in \{1, \ldots, m\}$, l_3^j factors as $e l_4^j = l_3^j$ since

$$f q_1 r_4 l_3^j = f q_1 l_1^j = f g_j = f_j = f' g'_j = f' q_2 l_2^j = f' q_2 r_3 l_3^j.$$

Hence, for all such j, the morphism

$$(t_{1j}((x_1, x'_1), \dots, (x_k, x'_k)), \dots, t_{nj}((x_1, x'_1), \dots, (x_k, x'_k)))$$

factors as $r_1 r_4 e l^j_4$. But since the relation $r_1 r_4 e \colon E \rightarrowtail (W \times W')^n$ is in $\mathcal{R}$, it is M-closed and so there exists a unique morphism $l_5 \colon X \to E$ such that

$$r_1 r_4 e l_5 = (u_1((x_1, x'_1), \dots, (x_k, x'_k)), \dots, u_n((x_1, x'_1), \dots, (x_k, x'_k))).$$

The equalities $r_1 r_4 h_3 = r_1 h_1 = r_1 r_4 e l_5$ imply that $h_3 = e l_5$ and it remains to compute

$$\begin{aligned} fh &= f q_1 h_1 = f q_1 r_4 h_3 = f q_1 r_4 e l_5 \\ &= f' q_2 r_3 e l_5 = f' q_2 r_3 h_3 = f' q_2 h_2 \\ &= f' h'. \end{aligned}$$

Now that we have shown p is well-defined, let us prove it makes $\mathbb{C}(X, Y)$ a partial M-algebra. If $i \in \{1, \dots, n\}$ and $x_1, \dots, x_k \in \mathbb{C}(X, Y)$, we can set $W = Y$, $r = 1_{Y^n}$,

$$g_j = (t_{1j}(x_1, \dots, x_k), \dots, t_{nj}(x_1, \dots, x_k)),$$

$f = \pi_i \colon Y^n \to Y$ the i-th projection and

$$h = (u_1(x_1, \dots, x_k), \dots, u_n(x_1, \dots, x_k)).$$

This shows that $p(t_{i1}(x_1, \dots, x_k), \dots, t_{im}(x_1, \dots, x_k))$ is defined and equal to $fh = u_i(x_1, \dots, x_k)$.

Now, let σ be an r-ary operation symbol of $\mathcal{T}$ with $r > 0$ and

$$(f^{j'}_j \in \mathbb{C}(X, Y))_{j \in \{1,\dots,m\}, j' \in \{1,\dots,r\}}$$

be families of morphisms such that $p(f^{j'}_1, \dots, f^{j'}_m)$ is defined for each $j' \in \{1, \dots, r\}$ using the diagrams below.

$$\begin{tikzcd} & X \arrow[dl, bend right, "(t_{1j}(x^{j'}_1,\dots,x^{j'}_k),\dots,t_{nj}(x^{j'}_1,\dots,x^{j'}_k))"'] \arrow[d, "g^{j'}_j"] \arrow[dr, bend left, "f^{j'}_j"] & \\ (W_{j'})^n & Z_{j'} \arrow[l, tail, "r^{j'}"] \arrow[r, "f^{j'}"'] & Y \end{tikzcd}$$

$$
\begin{tikzcd}
 & Z_{j'} \arrow[d, tail, "r^{j'}"] \\
X \arrow[ur, "h^{j'}"] \arrow[r, "{(u_1(x_1^{j'},\dots,x_k^{j'}),\dots,u_n(x_1^{j'},\dots,x_k^{j'}))}"'] & (W_{j'})^n
\end{tikzcd}
$$

We consider the pullbacks

$$
\begin{tikzcd}
S_{j'} \arrow[r, "q^{j'}"] \arrow[d, tail, "s^{j'}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z_{j'} \arrow[d, tail, "r^{j'}"] \\
(W_1 \times \cdots \times W_r)^n \arrow[r, "\pi_{j'}^n"'] & (W_{j'})^n
\end{tikzcd}
$$

where $\pi_{j'} \colon W_1 \times \cdots \times W_r \to W_{j'}$ is the j'-th projection as usual. We denote by $l_j^{j'}$ the unique morphism $X \to S_{j'}$ such that $q^{j'} l_j^{j'} = g_j^{j'}$ and

$$s^{j'} l_j^{j'} = (t_{1j}(x_1, \dots, x_k), \dots, t_{nj}(x_1, \dots, x_k))$$

where $x_{i'}$ is the factorisation $(x_{i'}^1, \dots, x_{i'}^r) \colon X \to W_1 \times \cdots \times W_r$. Let also $h_1^{j'} \colon X \to S_{j'}$ be the unique morphism satisfying $q^{j'} h_1^{j'} = h^{j'}$ and

$$s^{j'} h_1^{j'} = (u_1(x_1, \dots, x_k), \dots, u_n(x_1, \dots, x_k)).$$

We now consider the intersection of the $s^{j'}$'s

$$
\begin{tikzcd}
 & Z \arrow[dl, tail, "t^1"'] \arrow[dr, tail, "t^r"] & \\
S^1 \arrow[dr, tail, "s^1"'] & \cdots & S^r \arrow[dl, tail, "s^r"] \\
 & (W^1 \times \cdots \times W^r)^n &
\end{tikzcd}
$$

and the unique morphisms $l_j, h \colon X \to Z$ such that $t^{j'} l_j = l_j^{j'}$ and $t^{j'} h = h_1^{j'}$. Since this intersection can be built using pullbacks and compositions, $s^1 t^1 = \cdots = s^r t^r \in \mathcal{R}$. Thus, we end up with the commutative diagrams

$$
\begin{tikzcd}
 & X \arrow[dl, bend right, "{(t_{1j}(x_1,\dots,x_k),\dots,t_{nj}(x_1,\dots,x_k))}"'] \arrow[d, "l_j"] \arrow[dr, bend left, "{\sigma(f_j^1,\dots,f_j^r)}"] & \\
(W^1 \times \cdots \times W^r)^n & Z \arrow[l, tail, "s^1 t^1"] \arrow[r, "{\sigma(f^1 q^1 t^1,\dots,f^r q^r t^r)}"'] & Y
\end{tikzcd}
$$

and

$$\begin{tikzcd} & & Z \arrow[d, tail, "s^1t^1"] \\ X \arrow[urr, "h"] \arrow[rr, "{(u_1(x_1,\dots,x_k),\dots,u_n(x_1,\dots,x_k))}"'] & & (W^1 \times \cdots \times W^r)^n \end{tikzcd}$$

proving that $p(\sigma(f_1^1, \dots, f_1^r), \dots, \sigma(f_m^1, \dots, f_m^r))$ is defined and equal to

$$\sigma(f^1q^1t^1h, \dots, f^rq^rt^rh) = \sigma(p(f_1^1, \dots, f_m^1), \dots, p(f_1^r, \dots, f_m^r)).$$

If $r = 0$, we also have $p(\sigma, \dots, \sigma) = \sigma$. To see it, we can use for instance the commutative diagram below.

$$\begin{tikzcd} & X \arrow[dl, bend right, "{(t_{1j}(1_X,\dots,1_X),\dots,t_{nj}(1_X,\dots,1_X))}"'] \arrow[d] \arrow[dr, bend left, "\sigma"] & \\ X^n & X^n \arrow[l, tail, "1_{X^n}"] \arrow[r, "\sigma"'] & Y \end{tikzcd}$$

We have therefore provided $\mathbb{C}(X, Y)$ with a structure of partial M-algebra.

In view of the definition of a $\mathcal{T}$-enrichment, if $x\colon X' \to X$ and $y\colon Y \to Y'$ are morphisms in $\mathbb{C}$,

$$- \circ x\colon \mathbb{C}(X, Y) \to \mathbb{C}(X', Y)$$

and

$$y \circ -\colon \mathbb{C}(X, Y) \to \mathbb{C}(X, Y')$$

are homomorphisms of $\mathcal{T}$-algebras. Let us prove they are actually homomorphisms of partial M-algebras. So let $f_1, \dots, f_m\colon X \to Y$ be morphisms of $\mathbb{C}$ such that $p(f_1, \dots, f_m)$ is defined via the following diagrams.

$$\begin{tikzcd} & X \arrow[dl, bend right, "{(t_{1j}(x_1,\dots,x_k),\dots,t_{nj}(x_1,\dots,x_k))}"'] \arrow[d, "g_j"] \arrow[dr, bend left, "f_j"] & \\ W^n & Z \arrow[l, tail, "r"] \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & Z \arrow[d, tail, "r"] \\ X \arrow[ur, "h"] \arrow[r, "{(u_1(x_1,\dots,x_k),\dots,u_n(x_1,\dots,x_k))}"'] & W^n \end{tikzcd}$$

Thus, in view of the commutative diagrams

$$\begin{tikzcd} & X' \arrow[dl, bend right, "{(t_{1j}(x_1x,\dots,x_kx),\dots,t_{nj}(x_1x,\dots,x_kx))}"'] \arrow[d, "g_jx"] \arrow[dr, bend left, "f_jx"] & \\ W^n & Z \arrow[l, tail, "r"] \arrow[r, "f"'] & Y \end{tikzcd}$$

and

$$\begin{tikzcd}
& & Z \arrow[d, tail, "r"] \\
X' \arrow[urr, "hx"] \arrow[rr, "{(u_1(x_1x,\dots,x_kx),\dots,u_n(x_1x,\dots,x_kx))}"'] & & W^n
\end{tikzcd}$$

$p(f_1x,\dots,f_mx)$ is defined and equal to $fhx = p(f_1,\dots,f_m)x$, which shows that $- \circ x$ is a homomorphism of partial M-algebras. Besides, since the diagram

$$\begin{tikzcd}
& X \arrow[dl, bend right, "{(t_{1j}(x_1,\dots,x_k),\dots,t_{nj}(x_1,\dots,x_k))}"'] \arrow[d, "g_j"] \arrow[dr, bend left, "yf_j"] & \\
W^n & Z \arrow[l, tail, "r"] \arrow[r, "yf"'] & Y'
\end{tikzcd}$$

commutes, $p(yf_1,\dots,yf_m)$ is defined and equal to $yfh = yp(f_1,\dots,f_m)$, which proves that $y \circ -$ is a homomorphism of partial M-algebras. We have thus constructed a functor $\phi\colon \mathbb{C} \to \mathrm{Part}_M^{\mathbb{C}^{\mathrm{op}}}$ as announced.

$$\begin{tikzcd}
& \mathrm{Part}_M^{\mathbb{C}^{\mathrm{op}}} \arrow[d, "U_{\mathcal{T}}^{\mathbb{C}^{\mathrm{op}}}"] \\
\mathbb{C} \arrow[ur, "\phi"] \arrow[r, "Y_{\mathcal{T}}"'] & \mathrm{Alg}_{\mathcal{T}}^{\mathbb{C}^{\mathrm{op}}}
\end{tikzcd}$$

This ϕ preserves $\mathcal{T}$-enrichment since $Y_{\mathcal{T}}$ and $U_{\mathcal{T}}$ do and $U_{\mathcal{T}}$ is faithful. It is full and faithful since $Y_{\mathcal{T}}$ is full and faithful and $U_{\mathcal{T}}$ is faithful.

Since ϕ is full and faithful, it reflects isomorphisms. Thus, it will reflect finite limits if it preserves them. So, let $(\lambda_j\colon L \to D(j))_{j\in\mathbb{J}}$ be the limit of $D\colon \mathbb{J} \to \mathbb{C}$ with $\mathbb{J}$ a finite category. We would like to prove that for all $X \in \mathbb{C}$,

$$(\phi(\lambda_j)_X\colon \mathbb{C}(X,L) \to \mathbb{C}(X,D(j)))_{j\in\mathbb{J}}$$

is a limit in Part_M. But since $Y_{\mathcal{T}}$ preserves limits, and in view of the description of small limits in Part_M (Section 3.1), we only have to prove that, if $f_1,\dots,f_m\colon X \to L$ are such that $p(\lambda_j f_1,\dots,\lambda_j f_m)$ is defined for all $j \in \mathbb{J}$, then $p(f_1,\dots,f_m)$ is also defined.

Thus, to prove that ϕ preserves the terminal object, we have to show that $p(!,\dots,!)$ is defined where $!$ is the unique morphism $X \to 1$. This is obvious in view of the diagram below.

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "{(t_{1j}(!,\dots,!),\dots,t_{nj}(!,\dots,!))}"'] \arrow[d, "!"] \arrow[dr, bend left, "!"] & \\
1^n = 1 & 1 \arrow[l, tail, "1_1"] \arrow[r, "1_1"'] & 1
\end{tikzcd}$$

Moreover, ϕ preserves the binary product $Y \times Y'$.

$$\begin{tikzcd}
 & Y \times Y' \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
Y & & Y'
\end{tikzcd}$$

Indeed, suppose $f_1, \dots, f_m \colon X \to Y$ and $f'_1, \dots, f'_m \colon X \to Y'$ are such that $p(f_1, \dots, f_m)$ and $p(f'_1, \dots, f'_m)$ are defined using the following diagrams.

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "{(t_{1j}(x_1,\dots,x_k),\dots,t_{nj}(x_1,\dots,x_k))}"'] \arrow[d, "g_j"] \arrow[dr, bend left, "f_j"] & \\
W^n & Z \arrow[l, tail, "r"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "{(t_{1j}(x'_1,\dots,x'_k),\dots,t_{nj}(x'_1,\dots,x'_k))}"'] \arrow[d, "g'_j"] \arrow[dr, bend left, "f'_j"] & \\
W'^n & Z' \arrow[l, tail, "r'"] \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

We consider again the pullback

$$\begin{tikzcd}
Z_1 \arrow[r, "q_1"] \arrow[d, tail, "r_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, tail, "r"] \\
(W \times W')^n \arrow[r, "\pi_1^n"'] & W^n
\end{tikzcd}$$

and the unique morphisms $l_1^1, \dots, l_1^m \colon X \to Z_1$ such that $q_1 l_1^j = g_j$ and

$$r_1 l_1^j = (t_{1j}((x_1, x'_1), \dots, (x_k, x'_k)), \dots, t_{nj}((x_1, x'_1), \dots, (x_k, x'_k)))$$

for all $j \in \{1, \dots, m\}$. We define similarly Z_2, r_2, q_2 and $l_2^1, \dots, l_2^m$. We also consider the intersection

$$\begin{tikzcd} P \arrow[r, tail, "r_3"] \arrow[d, tail, "r_4"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Z_2 \arrow[d, tail, "r_2"] \\ Z_1 \arrow[r, tail, "r_1"'] & (W \times W')^n \end{tikzcd}$$

and the unique morphisms $l_3^1, \dots, l_3^m \colon X \to P$ such that $r_4 l_3^j = l_1^j$ and $r_3 l_3^j = l_2^j$ for all $j \in \{1, \dots, m\}$. Then, since the diagram below is commutative,

$$\begin{tikzcd} & X \arrow[dl, bend right, "(t_{1j}((x_1,x_1'),\dots,(x_k,x_k')),\dots,t_{nj}((x_1,x_1'),\dots,(x_k,x_k')))"'] \arrow[d, "l_3^j"] \arrow[dr, bend left, "(f_j,f_j')"] & \\ (W \times W')^n & P \arrow[l, tail, "r_1 r_4"] \arrow[r, "(fq_1r_4, f'q_2r_3)"'] & Y \times Y' \end{tikzcd}$$

$p((f_1, f_1'), \dots, (f_m, f_m'))$ is also defined and ϕ preserves finite products.

Finally, to prove that ϕ preserves equalisers, it is enough to show that $\phi(e)_X = e \circ - \colon \mathbb{C}(X, Y) \to \mathbb{C}(X, Y')$ is a closed homomorphism for each $X \in \mathbb{C}^{\mathrm{op}}$ and each regular monomorphism $e \colon Y \rightarrowtail Y'$. To conclude the proof, we are going to prove the more general fact that $\phi(e)_X$ is a closed homomorphism for each $e \colon Y \rightarrowtail Y'$ in $\mathcal{R}$ and each $X \in \mathbb{C}^{\mathrm{op}}$. So let $f_1, \dots, f_m \colon X \to Y$ be such that $p(ef_1, \dots, ef_m)$ is defined using the diagram below.

$$\begin{tikzcd} & X \arrow[dl, bend right, "(t_{1j}(x_1,\dots,x_k),\dots,t_{nj}(x_1,\dots,x_k))"'] \arrow[d, "g_j"] \arrow[dr, bend left, "ef_j"] & \\ W^n & Z \arrow[l, tail, "r"] \arrow[r, "f"'] & Y' \end{tikzcd}$$

We consider the pullback of e along f

$$\begin{tikzcd} Z' \arrow[r, "f'"] \arrow[d, tail, "r'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, tail, "e"] \\ Z \arrow[r, "f"'] & Y' \end{tikzcd}$$

and the unique morphisms $g'_1, \dots, g'_m \colon X \to Z'$ satisfying $f'g'_j = f_j$ and $r'g'_j = g_j$ for each $j \in \{1, \dots, m\}$. Then, considering the diagram

$$\begin{tikzcd}
& X \arrow[dl, bend right, "{(t_{1j}(x_1,\dots,x_k),\dots,t_{nj}(x_1,\dots,x_k))}"'] \arrow[d, "g'_j"] \arrow[dr, bend left, "f_j"] & \\
W^n & Z' \arrow[l, tail, "rr'"] \arrow[r, "f'"'] & Y
\end{tikzcd}$$

we see that $p(f_1, \dots, f_m)$ is defined, which concludes the proof. $\square$

Taking $\mathcal{R}$ to be the whole set of monomorphisms in $\mathbb{C}$, we immediately get the following corollary.

Corollary 4.2. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$. Let also $\mathbb{C}$ be a small finitely complete $\mathcal{T}$-enriched category with M-closed relations. There exists a full and faithful $\mathcal{T}$-enriched embedding $\phi\colon \mathbb{C} \hookrightarrow \mathrm{Part}_M^{\mathbb{C}^{\mathrm{op}}}$ which preserves and reflects finite limits. Moreover, for each monomorphism $f\colon A \rightarrowtail B$ and each $X \in \mathbb{C}^{\mathrm{op}}$, $\phi(f)_X$ is a closed monomorphism in Part_M.*

And now with $\mathcal{R}$ the set of strong monomorphisms.

Corollary 4.3. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$. Let also $\mathbb{C}$ be a small finitely complete $\mathcal{T}$-enriched category with M-closed strong relations. There exists a full and faithful $\mathcal{T}$-enriched embedding $\phi\colon \mathbb{C} \hookrightarrow \mathrm{Part}_M^{\mathbb{C}^{\mathrm{op}}}$ which preserves and reflects finite limits. Moreover, for each strong monomorphism $f\colon A \rightarrowtail B$ and each $X \in \mathbb{C}^{\mathrm{op}}$, $\phi(f)_X$ is a closed monomorphism in Part_M.*

Remark 4.4. Notice that Corollaries 4.2 and 4.3 characterise categories with M-closed relations (resp. with M-closed strong relations) among small finitely complete $\mathcal{T}$-enriched categories. Indeed, if we have such an embedding, to prove that a (strong) relation $r\colon R \rightarrowtail A^n$ in $\mathbb{C}$ is M-closed, it is enough to prove that $\phi(r)_X$ is M-closed in Part_M for all $X \in \mathbb{C}^{\mathrm{op}}$, which is true by Proposition 3.12.

5. Applications

Our embedding theorems give a way to reduce the proofs of some statements in finitely complete $\mathcal{T}$-enriched categories with M-closed strong relations to the particular case of Part_M as follows: Consider a statement P of the form $\psi \Rightarrow \omega$, where ψ and ω can be expressed as conjunctions of the conditions that some finite diagrams are commutative, some finite cones are limit cones and some equalities as $t(f_1, \dots, f_r) = g$ hold where t is an r-ary term of $\mathcal{T}$ and $f_1, \dots, f_r, g$ are parallel morphisms. Then, P is valid in all finitely complete $\mathcal{T}$-enriched $\mathcal{V}$-categories with M-closed strong relations (for all universes $\mathcal{V}$) if and only if it is valid in Part_M (for all universes). Indeed, in view of Proposition 3.12, the 'only if part' is obvious. Conversely, if $\mathbb{C}$ is a finitely complete $\mathcal{T}$-enriched category with M-closed strong relations, we can consider it is small up to a change of universe. Therefore, by Corollary 4.3, it suffices to prove P in $\mathrm{Part}_M^{\mathbb{C}^{\mathrm{op}}}$. Since every part of the statement P is 'componentwise', it is enough to prove it in Part_M. Note that the conditions that some morphisms are monomorphisms, isomorphisms, or factor through some given monomorphisms can also be expressed using finite limits.

Similarly, to prove this statement P in all finitely complete $\mathcal{T}$-enriched categories with M-closed relations, it is enough to prove it in Part_M (for all universes) supposing that each monomorphism considered in the statement P is closed.

Let us now give two concrete examples, the first one taking place in the 'weakly strongly unital context', i.e., for pointed finitely complete categories with M_{StrUni}-closed strong relations (see Example 2.3). This lemma has been proved in [1] as Lemma 1.8.18 in the strongly unital case, we now slightly improve it.

Lemma 5.1. *Consider the following diagram in a pointed finitely complete* *category with M_{StrUni}-closed strong relations*

$$\begin{tikzcd}
 & X \times R \arrow[d, shift right, "1_X\times r_1"'] \arrow[d, shift left, "1_X\times r_2"] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\
X \arrow[r, "(1_X{,}0)"] \arrow[dr, shift left, "h"] & X \times Y \arrow[d, "\psi"] & Y \arrow[l, "(0{,}1_Y)"'] \arrow[dl, "f"] \\
 & Z \arrow[ul, shift left, "g"] &
\end{tikzcd}$$

where $\psi(1_X, 0) = h$, $\psi(0, 1_Y) = f$, $gh = 1_X$, $gf = 0$ and (r_1, r_2) is the kernel pair of f. Then $(1_X \times r_1, 1_X \times r_2)$ is the kernel pair of ψ.

Proof. By Corollary 4.3, it is enough to prove it in $\mathrm{Part}_{M_{\mathrm{StrUni}}}$. First of all, let us compute, for all $x \in X$ and $y \in Y$

$$\begin{aligned}
\psi(x, y) &= \psi(p(x, 0, 0), p(0, 0, y)) \\
&= \psi(p((x, 0), (0, 0), (0, y))) \\
&= p(\psi(x, 0), \psi(0, 0), \psi(0, y)) \\
&= p(h(x), 0, f(y))
\end{aligned}$$

which is always defined. Next, let $x, x' \in X$ and $y, y' \in Y$ be such that $\psi(x, y) = \psi(x', y')$. We have

$$\begin{aligned}
x &= p(x, 0, 0) = p(gh(x), 0, gf(y)) = g(\psi(x, y)) \\
&= g(\psi(x', y')) = p(gh(x'), 0, gf(y')) = p(x', 0, 0) \\
&= x'
\end{aligned}$$

and

$$\begin{aligned}
f(y) &= \psi(0, y) = \psi(p(x, x, 0), p(y, 0, 0)) \\
&= \psi(p((x, y), (x, 0), (0, 0))) = p(\psi(x, y), \psi(x, 0), \psi(0, 0)) \\
&= p(\psi(x', y'), \psi(x', 0), \psi(0, 0)) = \psi(p(x', x', 0), p(y', 0, 0)) \\
&= \psi(0, y') = f(y').
\end{aligned}$$

Then,

$$X \times R = \{(x, y_1, y_2) \in X \times Y \times Y \mid f(y_1) = f(y_2)\}$$

in which p is defined componentwise. If $(x, y_1, y_2) \in X \times R$, we have

$$\begin{aligned}\psi(x, y_1) &= p(h(x), 0, f(y_1)) \\ &= p(h(x), 0, f(y_2)) \\ &= \psi(x, y_2).\end{aligned}$$

The kernel pair of ψ is given by

$$\{(x, y, x', y') \in X \times Y \times X \times Y \mid \psi(x, y) = \psi(x', y')\}$$

in which p is also defined componentwise. It is thus isomorphic to $X \times R$ via the mutually inverse homomorphisms $(x, y_1, y_2) \mapsto (x, y_1, x, y_2)$ and $(x, y, x', y') \mapsto (x, y, y')$. □

To conclude, we prove a well-known fact in Mal'tsev categories.

Proposition 5.2. *(Theorem 2.2 in [4]) In a Mal'tsev category, every internal category is a groupoid.*

Proof. If

$$\begin{tikzcd} A = (A_1 \times_{c,d} A_1 \arrow[r, "m"] & A_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & A_0 \arrow[l, bend left=60, "e"]) \end{tikzcd}$$

is an internal category, we have to prove that $\mathrm{Iso}(A) \rightarrowtail A_1$ is an isomorphism where $\mathrm{Iso}(A)$ is the object of isomorphisms of A, constructed via a limit of a finite diagram involving e, d, c and m. Thus, by Corollary 4.2, it is enough to prove this statement in $\mathrm{Part}_{M_{\mathrm{Mal}}}$.

We write π_1 and π_2 for the projections of the pullback of c along d.

$$\begin{tikzcd} A_1 \times_{c,d} A_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_1 \arrow[d, "d"] \\ A_1 \arrow[r, "c"'] & A_0 \end{tikzcd}$$

Let us first prove that

$$A_1 \times_{c,d} A_1 \xrightarrow{(\pi_2, m)} A_1 \times A_1$$

is a monomorphism. So let

$$X \underset{f'}{\overset{f}{\rightrightarrows}} Y \xrightarrow{g} Z$$

be morphisms in A such that $m(f,g) = m(f',g)$. Then

$$\begin{aligned} f &= m(f, 1_Y) \\ &= m(p(f, 1_Y, 1_Y), p(g, g, 1_Y)) \\ &= p(m(f,g), m(1_Y, g), m(1_Y, 1_Y)) \\ &= p(m(f',g), m(1_Y, g), m(1_Y, 1_Y)) \\ &= f' \end{aligned}$$

and (π_2, m) is a monomorphism. We can therefore suppose it is closed (using the last part of Corollary 4.2). Let us now prove that every map $f\colon X \to Y$ in A is invertible (i.e., that $\mathrm{Iso}(A) \rightarrowtail A_1$ is surjective). We know that

$$p((1_Y, 1_Y), (f, 1_Y), (1_X, f)) \in A_1 \times_{c,d} A_1$$

is defined since $p(1_Y, 1_Y, f)$ and $p(1_Y, f, f)$ are and (π_2, m) is a closed monomorphism. Thus, applying π_1, we deduce that $p(1_Y, f, 1_X)$ is defined. It remains to compute

$$d(p(1_Y, f, 1_X)) = p(Y, X, X) = Y,$$

$$c(p(1_Y, f, 1_X)) = p(Y, Y, X) = X,$$

$$\begin{aligned} m(f, p(1_Y, f, 1_X)) &= m(p(f, 1_X, 1_X), p(1_Y, f, 1_X)) \\ &= p(m(f, 1_Y), m(1_X, f), m(1_X, 1_X)) \\ &= p(f, f, 1_X) \\ &= 1_X \end{aligned}$$

and similarly for $m(p(1_Y, f, 1_X), f) = 1_Y$. Therefore, $\mathrm{Iso}(A) \rightarrowtail A_1$ is bijective and can also be supposed to be closed. This means it is an isomorphism. $\square$

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INTERNAL GROUPOIDS AND EXPONENTIABILITY

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Résumé. Nous étudions les objets et les morphismes exponentiables dans la 2-catégorie $\mathbf{Gpd}(\mathcal{C})$ des groupoides internes à une catégorie $\mathcal{C}$ avec sommes finies lorsque $\mathcal{C}$ est : (1) finiment complète, (2) cartésienne fermée et (3) localement cartésienne fermée. Parmi les exemples auxquels on s'intéresse on trouve, en particulier, (1) les espaces topologiques, (2) les espaces compactement engendrés, (3) les ensembles, respectivement. Nous considérons aussi les morphismes pseudo-exponentiables dans les catégories "pseudo-slice" $\mathbf{Gpd}(\mathcal{C})//B$. Comme ces dernières sont les catégories de Kleisli d'une monade T sur la catégorie "slice" stricte sur B, nous pouvons appliquer un théorème général de Niefield [17] qui dit que si TY est exponentiable dans une 2-catégorie $\mathcal{K}$, alors Y est pseudo-exponentiable dans la catégorie de Kleisli $\mathcal{K}_T$. Par conséquent, nous verrons que $\mathbf{Gpd}(\mathcal{C})//B$ est pseudo-cartésienne fermée, lorsque $\mathcal{C}$ est la catégorie des espaces compactement engendrés et chaque B_i est faiblement de Hausdorff, et $\mathbf{Gpd}(\mathcal{C})$ est localement pseudo-cartésienne fermée quand $\mathcal{C}$ est la catégorie des ensembles ou une catégorie localement cartésienne fermée quelconque.

Abstract. We study exponentiable objects and morphisms in the 2-category $\mathbf{Gpd}(\mathcal{C})$ of internal groupoids in a category $\mathcal{C}$ with finite coproducts when $\mathcal{C}$ is: (1) finitely complete, (2) cartesian closed, and (3) locally cartesian closed. The examples of interest include (1) topological spaces, (2) compactly generated spaces, and (3) sets, respectively. We also consider pseudo-exponentiable morphisms in the pseudo-slice categories $\mathbf{Gpd}(\mathcal{C})//B$. Since the latter is the Kleisli category of a monad T on the strict slice over B, we can apply a general theorem from Niefield [17] which states that if TY

is exponentiable in a 2-category $\mathcal{K}$, then Y is pseudo-exponentiable in the Kleisli category $\mathcal{K}_T$. Consequently, we will see that $\mathbf{Gpd}(\mathcal{C})//B$ is pseudo-cartesian closed, when $\mathcal{C}$ is the category of compactly generated spaces and each B_i is weak Hausdorff, and $\mathbf{Gpd}(\mathcal{C})$ is locally pseudo-cartesian closed when $\mathcal{C}$ is the category of sets or any locally cartesian closed category.
Keywords. exponentiability, internal groupoids, topological groupoids
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1. Introduction

Suppose $\mathcal{C}$ is a category with finite limits. An object Y of $\mathcal{C}$ is *exponentiable* if the functor $- \times Y : \mathcal{C} \to \mathcal{C}$ has a right adjoint, usually denoted by $(\ )^Y$, and $\mathcal{C}$ is called *cartesian closed* if every object is exponentiable. A morphism $q\colon Y \to B$ is *exponentiable* if q is exponentiable in the slice category $\mathcal{C}/B$, and $\mathcal{C}$ is called *locally cartesian closed* if every morphism is exponentiable. Note that if $q\colon Y \to B$ is exponentiable and $r\colon Z \to B$, we follow the abuse of notation and write the exponential as $r^q\colon Z^Y \to B$.

It is well known that the class of exponentiable morphisms is closed under composition and pullback along arbitrary morphisms. For proofs of these and other properties of exponentiability, we refer the reader to Niefield [16].

An *internal groupoid* G in $\mathcal{C}$ is a diagram of the form

$$\begin{tikzcd} G_2 \arrow[r, "c"] & G_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & G_0 \arrow[l, "u"' description] \end{tikzcd}$$

where $G_2 = G_1 \times_{G_0} G_1$, making G an internal category in $\mathcal{C}$ in which every morphism is invertible. Unless otherwise stated, the morphism in the pullback is $t\colon G_1 \to G_0$ when G_1 appears on the left in $G_1 \times_{G_0} G_1$ and s when it is on the right. When $\mathcal{C}$ is the category of topological spaces, we say G is a topological groupoid.

Let $\mathbf{Gpd}(\mathcal{C})$ denote the 2-category whose objects are groupoids in $\mathcal{C}$, morphisms $\sigma\colon G \to H$ are "internal homomorphisms," i.e., morphisms $\sigma_0\colon G_0 \to H_0$ and $\sigma_1\colon G_1 \to H_1$ of $\mathcal{C}$ compatible with the groupoid structure, and 2-cells $\sigma \Rightarrow \sigma'\colon G \to H$ are "internal natural transformations," i.e, morphisms $\alpha\colon G_0 \to H_1$ of $\mathcal{C}$ such that the following diagram is defined and commutes

$$\begin{tikzcd} G_1 \arrow[r, "\langle \alpha s, \sigma_1' \rangle"] \arrow[d, "\langle \sigma_1, \alpha t \rangle"'] & H_2 \arrow[d, "c"] \\ H_2 \arrow[r, "c"'] & H_1 \end{tikzcd} \tag{1}$$

Note that for an object of a 2-category $\mathbb{C}$ to be 2-exponentiable, one requires that the 2-functor $- \times Y : \mathbb{C} \longrightarrow \mathbb{C}$ has a right 2-adjoint, i.e., there is an isomorphism of categories $\mathbb{C}(X \times Y, Z) \cong \mathbb{C}(X, Z^Y)$ natural in X and Z. One can similarly define 2-exponentiable morphisms of $\mathbb{C}$.

It is well known that the 2-category $\mathbf{Cat}(\mathcal{C})$ of internal categories in $\mathcal{C}$ is cartesian closed whenever $\mathcal{C}$ is, and the construction of exponentials restricts to $\mathbf{Gpd}(\mathcal{C})$ (see Bastiani/Ehresmann [1], Johnstone [10]). Since the construction of the exponentials H^G depends only on the exponentiability of G_0, G_1, and G_2 in $\mathcal{C}$, we will see that G is exponentiable in $\mathbf{Gpd}(\mathcal{C})$ whenever G_0, G_1, and G_2 are exponentiable in $\mathcal{C}$, for any merely finitely complete category $\mathcal{C}$. However, $\mathbf{Cat}(\mathcal{C})$ and $\mathbf{Gpd}(\mathcal{C})$ are not locally cartesian closed even when $\mathcal{C}$ is. In fact, $q \colon Y \longrightarrow B$ is exponentiable in $\mathbf{Cat}$ if and only if it satisfies a factorization lifting property (FLP) known as the Conduché-Giraud condition (see Conduché [3], Giraud [7]). In the groupoid case, q satisfies FLP if and only if it is a fibration in the sense of Grothendieck [8].

In [11], Johnstone characterized pseudo-exponentiable morphisms in the pseudo-slice $\mathbf{Cat}/\!/B$, where the morphisms commute up to specified natural transformation, as those satisfying a certain pseudo-factorization lifting property, and Niefield [17] later obtained this result as a consequence of a general theorem about pseudo-exponentiable objects in the Kleisli bicategory of a pseudo-monad on a bicategory. In a related note, Palmgren [18] showed that every groupoid homomorphism is pseudo-exponentiable, so that $\mathbf{Gpd}/\!/B$ is locally pseudo-cartesian closed. Although Palmgren includes a complete proof, we will see that his result follows from the characterization in [17].

The goal of this paper is to generalize these results so that we can eventually apply them to categories of topological groupoids arising in the study of orbifolds. We begin, in Section 2, by recalling a general construction from Niefield [15] of cartesian closed coreflective subcategories of the category $\mathbf{Top}$ of all topological spaces (see also Bunge/Niefield [2]), which includes compactly generated spaces as a special case, and leads to cartesian closed coreflective subcategories of $\mathbf{Top}$. In the next two sections, we consider exponentiable objects of $\mathbf{Gpd}(\mathcal{C})$ and its slices when $\mathcal{C}$ is not locally cartesian closed, and apply this to $\mathbf{Top}$ and its subcategories. In this process we will need the internal version of the notion of fibration. This has been developed in full detail for arbitrary 2-categories in [20]. However, for internal groupoids, the descriptions given in the literature for $q\colon G \longrightarrow B$ to be an internal cloven fibration are equivalent to the existence of a right inverse for the arrow $\langle s, q_1 \rangle\colon G_1 \longrightarrow G_0 \times_{B_0} B_1$. The reason this naive internalization of the Grothendieck condition works is the fact that in groupoids all arrows of the domain of a fibration are cartesian. We conclude, in Section 5, with the construction of a pseudo-monad on $\mathbf{Gpd}(\mathcal{C})/B$, in the case where $\mathcal{C}$ also has finite coproducts, and thus obtain pseudo-cartesian closed slices of $\mathbf{Gpd}(\mathcal{C})$ when $\mathcal{C}/B$ is cartesian closed. This includes the case where $\mathcal{C} = \mathbf{Sets}$, giving another proof of Palmgren's result, as well as certain slices of $\mathbf{Top}$ considered in Section 2.

2. Exponentiability in Categories of Spaces

In this section, we recall some general results about cartesian closed coreflective subcategories of $\mathbf{Top}$ and their slices. It is well known that the exponentiable topological spaces Y are those for which the collection $\mathcal{O}(Y)$ is a continuous lattice, in the sense of Scott [19]. This is equivalent to local compactness for Hausdorff (or more generally sober [9]) spaces Y. The sufficiency of this condition goes back to R.H. Fox [6] and the necessity appeared in Day/Kelly [5]. A characterization of exponentiable morphisms of $\mathbf{Top}$ was established by Niefield in [15] and published in [16], where it was shown that the inclusion of a subspace Y of B is exponentiable if and only if it is locally closed, i.e., of the form $U \cap F$, with U open and F closed in B.

There are several general expositions of cartesian closed coreflective subcategories of $\mathbf{Top}$. One, we recall here, follows from a general construction presented in [15] and later included in Bunge/Niefield [2].

Let $\mathcal{M}$ be a class of topological spaces. Given a space X, let $\hat{X}$ denote the set X with the topology generated by the collection

$$\{f\colon M \longrightarrow X \mid M \in \mathcal{M}\}$$

of continuous maps. We say X is $\mathcal{M}$*-generated* if $X = \hat{X}$, and let $\mathbf{Top}_{\mathcal{M}}$ denote the full subcategory of $\mathbf{Top}$ consisting of $\mathcal{M}$-generated spaces. Then one can show that $\mathbf{Top}_{\mathcal{M}}$ is a coreflective subcategory of $\mathbf{Top}$ with coreflection $\widehat{\ }\colon \mathbf{Top} \longrightarrow \mathbf{Top}_{\mathcal{M}}$.

In particular, $\mathbf{Top}_{\mathcal{K}}$ and $\mathbf{Top}_{\mathcal{E}}$ are the categories of compactly generated and exponentiably generated spaces, when $\mathcal{K}$ and $\mathcal{E}$ are the classes of compact Hausdorff spaces and all exponentiable spaces, respectively. Moreover, it is not difficult to show that if $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathbf{Top}_{\mathcal{M}}$, then $\mathbf{Top}_{\mathcal{M}} = \mathbf{Top}_{\mathcal{N}}$. Thus, since every locally compact Hausdorff space is known to be compactly generated, adding all such spaces to $\mathcal{K}$ does not increase $\mathbf{Top}_{\mathcal{K}}$.

The following theorem is a special case of the one in [15] and later included in [2]. We include a proof here for completeness.

Theorem 2.1. *If $\mathcal{M}$ is a class of exponentiable objects of $\mathbf{Top}$ such that $M \times N \in \mathbf{Top}_{\mathcal{M}}$, for all $M, N \in \mathcal{M}$, then $\mathbf{Top}_{\mathcal{M}}$ is cartesian closed.*

Proof. The product in $\mathbf{Top}_{\mathcal{M}}$ is given by

$$X \hat{\times} Y = \varinjlim_{L \to X \times Y} L = \varinjlim_{\substack{M \to X \\ N \to Y}} M \times N = \varinjlim_{N \to Y} ((\varinjlim_{M \to X} M) \times N) = \varinjlim_{N \to Y} X \times N$$

where the second equality holds since each $M \times N \in \mathbf{Top}_{\mathcal{M}}$ and the third since $- \times N$ preserves colimits as N is exponentiable. Thus,

$$\begin{aligned} \mathbf{Top}_{\mathcal{M}}(X \hat{\times} Y, Z) &= \mathbf{Top}(\varinjlim_{N \to Y} X \times N, Z) = \varprojlim_{N \to Y} \mathbf{Top}(X \times N, Z) \\ &= \varprojlim_{N \to Y} \mathbf{Top}(X, Z^N) = \mathbf{Top}_{\mathcal{M}}(X, \widehat{\varprojlim_{N \to Y} Z^N}) \end{aligned}$$

□

Although $\mathbf{Top}_{\mathcal{M}}$ is generally not locally cartesian closed, there are many cases of cartesian closed slices. In fact, we know of no nontrivial case (i.e., $\mathbf{Top}_{\mathcal{M}} \neq \mathbf{Sets}$) for which $\mathbf{Top}_{\mathcal{M}}$ is locally cartesian closed. The following general proposition leads to examples of such slices.

Proposition 2.2. *If Y is exponentiable in $\mathcal{C}$ and B is any object for which the diagonal $\Delta\colon B \longrightarrow B \times B$ is exponentiable, then every morphism $q\colon Y \longrightarrow B$ is exponentiable.*

Proof. Since the horizontal morphisms in the pullbacks

$$\begin{tikzcd} Y \arrow[r, "\langle id,q\rangle"] \arrow[d, "q"'] & Y \times B \arrow[d, "q\times id"] \\ B \arrow[r, "\Delta"'] & B \times B \end{tikzcd} \qquad \begin{tikzcd} Y \times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d] \\ Y \arrow[r] & 1 \end{tikzcd}$$

are exponentiable, factoring $q = \pi_2\langle id, q\rangle$, yields the desired result. □

Corollary 2.3. *If the diagonal $B \longrightarrow B \hat{\times} B$ is exponentiable in $\mathbf{Top}_{\mathcal{M}}$, then $\mathbf{Top}_{\mathcal{M}}/B$ is cartesian closed.*

For examples of spaces satisfying the hypotheses of Corollary 2.3, we use:

Proposition 2.4. *If $\mathbf{Top}_{\mathcal{M}}$ is closed under locally closed subspaces of all M in $\mathcal{M}$, then inclusions of locally closed subspaces are exponentiable in $\mathbf{Top}_{\mathcal{M}}$.*

Proof. Suppose B is $\mathcal{M}$-generated and $q\colon Y \longrightarrow B$ is the inclusion of a locally closed subspace. Then for all $p\colon X \longrightarrow B$ in $\mathbf{Top}_{\mathcal{M}}$, since $p^{-1}(Y)$ is locally closed, one can show that $X \hat{\times}_B Y = p^{-1}(Y) = X \times_B Y$ is the product in $\mathbf{Top}_{\mathcal{M}}/B$. Then $\mathbf{Top}_{\mathcal{M}}/B(X \hat{\times}_B Y, Z) = \mathbf{Top}/B(X \times_B Y, Z) = \mathbf{Top}/B(X, Z^Y) = \mathbf{Top}_{\mathcal{M}}/B(X, \widehat{Z^Y})$, since locally closed inclusions are exponentiable in $\mathbf{Top}$. □

Corollary 2.5. *Locally closed inclusions are exponentiable in the categories $\mathbf{Top}_{\mathcal{K}}$ of compactly generated spaces and $\mathbf{Top}_{\mathcal{E}}$ of exponentially generated spaces.*

Proof. Locally closed subspaces of compact Hausforff spaces are compactly generated and locally closed subspaces of exponentiable space are exponentiable. □

An $\mathcal{M}$-generated space X is called *$\mathcal{M}$-Hausdorff* (respectively, *locally $\mathcal{M}$-Hausdorff*) if the diagonal $B \longrightarrow B \hat{\times} B$ is closed (respectively, locally closed). A $\mathcal{K}$-Hausdorff space is also known as a weak Hausdorff compactly generated space or a k-space in the literature Lewis [12]. Note that weak Hausdorff compactly generated spaces also form a cartesian closed category but the only exponentiable morphisms there are the open maps [12].

Corollary 2.6. *If $\mathbf{Top}_{\mathcal{M}}$ is closed under locally closed subspaces of all M in $\mathcal{M}$, and B is $\mathcal{M}$-Hausdorff (more generally, locally $\mathcal{M}$-Hausdorff), then $\mathbf{Top}_{\mathcal{M}}/B$ is cartesian closed.*

Proof. Apply Corollary 2.3 and Proposition 2.4. □

In particular, we get:

Corollary 2.7. *If B is a weak Hausdorff space, then $\mathbf{Top}_{\mathcal{K}}/B$ is cartesian closed.*

3. Exponentiable Topological Groupoids

In this section, we consider exponentiable topological groupoids, but first some general results in $\mathbf{Gpd}(\mathcal{C})$, where $\mathcal{C}$ is a finitely complete category with finite coproducts. As noted in the introduction, G is exponentiable in $\mathbf{Gpd}(\mathcal{C})$, whenever G_0, G_1, and G_2 are exponentiable in $\mathcal{C}$. It is not true that $q\colon G \longrightarrow B$ is exponentiable in $\mathbf{Gpd}(\mathcal{C})/B$ whenever the $q_i\colon G_i \longrightarrow B_i$ are exponentable for i=0,1,2, since even when $\mathcal{C} = \mathbf{Sets}$, for $G \longrightarrow B$ to be exponentiable it is necessary that it is a fibration. Moreover, one cannot use Proposition 2.2 to obtain exponentiable morphisms of $\mathbf{Gpd}(\mathcal{C})$, since the diagonal $\Delta\colon B \longrightarrow B \times B$ is rarely exponentiable. In fact, when $\mathcal{C} = \mathbf{Sets}$, this is the case if and only if B is discrete.

When $\mathcal{C}$ is cartesian closed, the exponential H^G in $\mathbf{Gpd}(\mathcal{C})$ can be constructed as follows. The object of objects $(H^G)_0$ needs to encode triples of arrows $\langle \sigma_0\colon G_0 \longrightarrow H_0, \sigma_1\colon G_1 \longrightarrow H_1, \sigma_2\colon G_2 \longrightarrow H_2 \rangle$ that fit in the appropriate commutative diagrams to form an internal functor $G \longrightarrow H$; i.e.,

$$\begin{tikzcd} G_1 \arrow[r, "\sigma_1"] \arrow[d, "s"'] & H_1 \arrow[d, "s"] \\ G_0 \arrow[r, "\sigma_0"'] & H_0 \end{tikzcd} \qquad \begin{tikzcd} G_1 \arrow[r, "\sigma_1"] \arrow[d, "t"'] & H_1 \arrow[d, "t"] \\ G_0 \arrow[r, "\sigma_0"'] & H_0 \end{tikzcd} \qquad \begin{tikzcd} G_1 \arrow[r, "\sigma_1"] & H_1 \\ G_0 \arrow[r, "\sigma_0"'] \arrow[u, "u"] & H_0 \arrow[u, "u"'] \end{tikzcd}$$

$$\begin{tikzcd} G_2 \arrow[r, "\sigma_2"] \arrow[d, "c"'] & H_2 \arrow[d, "c"] \\ G_1 \arrow[r, "\sigma_1"'] & H_1 \end{tikzcd} \qquad \begin{tikzcd} G_2 \arrow[r, "\sigma_2"] \arrow[d, "\pi_1"'] & H_2 \arrow[d, "\pi_1"] \\ G_1 \arrow[r, "\sigma_1"'] & H_1 \end{tikzcd} \qquad \begin{tikzcd} G_2 \arrow[r, "\sigma_2"] \arrow[d, "\pi_2"'] & H_2 \arrow[d, "\pi_2"] \\ G_1 \arrow[r, "\sigma_1"'] & H_1 \end{tikzcd}$$

Hence, it is obtained as the equalizer

$$(H^G)_0 \rightarrowtail H_0^{G_0} \times H_1^{G_1} \times H_2^{G_2} \underset{g_0}{\overset{f_0}{\rightrightarrows}} H_0^{G_1} \times H_0^{G_1} \times H_1^{G_0} \times H_1^{G_2} \times H_1^{G_2} \times H_1^{G_2}$$

where

$$f_0 = \langle H_0^s \pi_1, H_0^t \pi_1, u^{G_0}\pi_1, H_1^c \pi_2, H_1^{\pi_1}\pi_2, H_1^{\pi_2}\pi_2 \rangle$$

and

$$g_0 = \langle s^{G_1}\pi_2, t^{G_1}\pi_2, H_1^u \pi_2, c^{G_2}\pi_3, \pi_1^{G_2}\pi_3, \pi_2^{G_2}\pi_3 \rangle$$

The object of arrows $(H^G)_1$ needs to encode internal natural tranformations $\alpha\colon \sigma \Rightarrow \sigma'$ between internal functors $\sigma, \sigma'\colon G \rightrightarrows H$. These are given by quintuples $\langle \sigma, \sigma', \alpha, \beta_1, \beta_2 \rangle$, where $\alpha\colon G_0 \to H_1$ and $\beta_1, \beta_2\colon G_1 \rightrightarrows H_2$, that make the following diagrams commute,

$$\begin{tikzcd} G_0 \arrow[r, "\alpha"] \arrow[dr, "\sigma_0"'] & H_1 \arrow[d, "s"] \\ & H_0 \end{tikzcd} \quad \begin{tikzcd} G_0 \arrow[r, "\alpha"] \arrow[dr, "\sigma'_0"'] & H_1 \arrow[d, "t"] \\ & H_0 \end{tikzcd} \quad \begin{tikzcd} G_1 \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] & H_2 \arrow[d, "c"] \\ H_2 \arrow[r, "c"'] & H_1 \end{tikzcd}$$

$$\begin{tikzcd} G_1 \arrow[r, "\beta_1"] \arrow[dr, "\sigma_1"'] & H_2 \arrow[d, "\pi_1"] \\ & H_1 \end{tikzcd} \quad \begin{tikzcd} G_1 \arrow[r, "\beta_1"] \arrow[d, "t"'] & H_2 \arrow[d, "\pi_2"] \\ G_0 \arrow[r, "\alpha"'] & H_1 \end{tikzcd} \quad \begin{tikzcd} G_1 \arrow[r, "\beta_2"] \arrow[dr, "\sigma'_1"'] & H_2 \arrow[d, "\pi_2"] \\ & H_1 \end{tikzcd} \quad \begin{tikzcd} G_1 \arrow[r, "\beta_2"] \arrow[d, "s"'] & H_2 \arrow[d, "\pi_1"] \\ G_0 \arrow[r, "\alpha"'] & H_1 \end{tikzcd}$$

(Note that the last five encode commutativity of the naturality square (1).) Hence, it is obtained as the equalizer $(H^G)_1$ of the parallel pair,

$$(H^G)_0 \times (H^G)_0 \times H_1^{G_0} \times H_2^{G_1} \times H_2^{G_1} \underset{g_1}{\overset{f_1}{\rightrightarrows}} H_0^{G_0} \times H_0^{G_0} \times H_1^{G_1} \times H_1^{G_1} \times H_1^{G_1} \times H_1^{G_1} \times H_1^{G_1}$$

where

$$f_1 = \langle \pi_1\pi_1,\ \pi_1\pi_2,\ c^{G_1}\pi_4,\ \pi_2\pi_1,\ H_1^t\pi_3,\ \pi_2\pi_2,\ H_1^s\pi_3 \rangle$$

and

$$g_1 = \langle s^{G_0}\pi_3,\ t^{G_0}\pi_3,\ c^{G_1}\pi_5,\ \pi_1^{G_1}\pi_4, \pi_2^{G_1}\pi_4, \pi_2^{G_1}\pi_5, \pi_1^{G_1}\pi_5 \rangle$$

The source and target maps $(H^G)_1 \to (H^G)_0$ are given by first and second projection. The unit map $(H^G)_0 \to (H^G)_1$ has the identity map in the first and second coordinate and $u^{G_0}\pi_1$ in the third coordinate. We describe the last two coordinates using the transpose. Note that $(H^G)_0$ is a subobject of $H_0^{G_0} \times H_1^{G_1} \times H_2^{G_2}$. So consider

$$H_0^{G_0} \times H_1^{G_1} \times H_2^{G_2} \times G_1 \xrightarrow{\langle \pi_1, \pi_2, \Delta_{G_1}\pi_4 \rangle} H_0^{G_0} \times H_1^{G_1} \times G_1 \times G_1$$

$$\xrightarrow{id_{H_0^{G_0}} \times id_{H_1^{G_1}} \times s \times id_{G_1}} H_0^{G_0} \times H_1^{G_1} \times G_0 \times G_1$$

$$\xrightarrow{\langle \mathrm{ev}\langle \pi_1, \pi_3 \rangle, \mathrm{ev}\langle \pi_2, \pi_4 \rangle \rangle} H_0 \times H_1$$

$$\xrightarrow{u \times id_{H_1}} H_1 \times H_1$$

When we take the subobject $(H^G)_0 \rightarrowtail H_0^{G_0} \times H_1^{G_1} \times H_2^{G_2}$, this restricts to a map

$$\tau \colon (H^G)_0 \times G_1 \longrightarrow H_1 \times_{H_0} H_1 \cong H_2$$

Its transpose $\hat{\tau} \colon (H^G)_0 \longrightarrow H_2^{G_1}$ is the projection of the fourth coordinate of the unit. The fifth coordinate is obtained in a similar fashion, but starting with the mapping

$$H_0^{G_0} \times H_1^{G_1} \times H_2^{G_2} \times G_1 \xrightarrow{\langle \pi_1, \pi_2, \Delta_{G_1}\pi_4 \rangle} H_0^{G_0} \times H_1^{G_1} \times G_1 \times G_1$$

$$\xrightarrow{id_{H_0^{G_0}} \times id_{H_1^{G_1}} \times id_{G_1} \times t} H_0^{G_0} \times H_1^{G_1} \times G_1 \times G_0$$

$$\xrightarrow{\langle \mathrm{ev}\langle \pi_2, \pi_3 \rangle, \mathrm{ev}\langle \pi_1, \pi_4 \rangle \rangle} H_1 \times H_0$$

$$\xrightarrow{id_{H_1} \times u} H_1 \times H_1$$

Composition in $(H^G)_1$ can be expressed using projections in the first two coordinates and the appropriate composites in H_1 in the last three coordinates of the map. This makes H^G the "groupoid of homomorphisms"

$G \longrightarrow H$ and the adjunction can be established using only the exponentiability of G_0, G_1, and G_2. Thus:

Proposition 3.1. *If G_0, G_1, and G_2 are exponentiable in $\mathcal{C}$, then G is exponentiable in $\mathbf{Gpd}(\mathcal{C})$.*

To obtain a partial converse to Proposition 3.1, we use the left and right adjoints to $(\ )_0\colon \mathbf{Gpd}(\mathcal{C}) \longrightarrow \mathcal{C}$ which we recall are given by

$$\begin{tikzcd} L_0(X)\colon X \arrow[r, "id"] & X \arrow[loop above, "id"] \arrow[r, shift left=2, "id"] \arrow[r, shift right=2, "id"'] & X \arrow[l, "id" description] \end{tikzcd} \qquad \text{and}$$

$$\begin{tikzcd} R_0(X)\colon X \times X \times X \arrow[r, "\pi_{13}"] & X \times X \arrow[loop above, "\langle \pi_2, \pi_1 \rangle"] \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] & X \arrow[l, "\Delta" description] \end{tikzcd}$$

respectively.

Proposition 3.2. *If G is exponentiable in $\mathbf{Gpd}(\mathcal{C})$, then G_0 is exponentiable in $\mathcal{C}$. The converse holds if s (or equivalently, t) is exponentiable.*

Proof. Suppose G is exponentiable in $\mathbf{Gpd}(\mathcal{C})$. Then G_0 is exponentiable in $\mathcal{C}$, since

$$\begin{aligned} \mathcal{C}(X \times G_0, Y) &\cong \mathcal{C}((L_0X \times G)_0, Y) \cong \mathbf{Gpd}(\mathcal{C})(L_0X \times G, R_0Y) \\ &\cong \mathbf{Gpd}(\mathcal{C})(L_0X, (R_0Y)^G) \cong \mathcal{C}(X, (R_0Y)_0^G) \end{aligned}$$

For the converse, suppose G_0 and $s\colon G_1 \longrightarrow G_0$ are exponentiable in $\mathcal{C}$. Then G_1 is exponentiable since composition preserves exponentiability. To see that G_2 is exponentiable, consider the pullback

$$\begin{tikzcd} G_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & G_1 \arrow[d, "s"] \\ G_1 \arrow[r, "t"'] & G_0 \end{tikzcd}$$

where π_1 is exponentiable since s is and pullback preserves exponentiability, and so G_2 is exponentiable since G_1 is. Thus, G is exponentiable in $\mathbf{Gpd}(\mathcal{C})$ by Proposition 3.1. $\square$

Note that if G is an étale groupoid, in the sense of Moerdijk/Pronk [14], then s and t are local homeomorphisms in $\mathbf{Top}$, and we conjecture that H^G is étale when H is also étale and G_1/G_0 is compact. Thus, G is exponentiable in $\mathbf{Gpd}(\mathbf{Top})$ if and only if G_0 is exponentiable in $\mathbf{Top}$. Of course, all étale groupoids are exponentiable in $\mathbf{Gpd}(\mathbf{Top}_{\mathcal{K}})$, since $\mathbf{Top}_{\mathcal{K}}$ is cartesian closed.

An exponentiable internal groupoid of interest is the groupoid $\mathbb{I}$ with two objects and one nontrivial isomorphism. It is well known that $\mathbb{I}$ makes sense in $\mathbf{Gpd}(\mathcal{C})$, for any finitely complete $\mathcal{C}$ with finite coproducts, where $\mathbb{I}_0 = 1 + 1$ and $\mathbb{I}_1 = 1 + 1 + 1 + 1$. In particular, the exponentials $B^{\mathbb{I}}$ will play a role when we consider the pseudo-slices $\mathbf{Cat}//B$ in Section 5. We know that $B^{\mathbb{I}}$ is exponentiable whenever $B_0^{\mathbb{I}}$, $B_1^{\mathbb{I}}$, and $B_2^{\mathbb{I}}$ are.

Using our construction of exponentials, one can see that $B^{\mathbb{I}}$ can be described as follows. Since $B_0^{\mathbb{I}}$ can be thought of as the "object of homomorphisms $\mathbb{I} \longrightarrow B$," i.e., morphisms $b_s \longrightarrow b_t$ in B, we can take $B_0^{\mathbb{I}} = B_1$. Then $B_1^{\mathbb{I}}$ becomes $(B^{\mathbb{I}})_1 = B_2 \times_{B_1} B_2$ via the pullback

$$\begin{tikzcd} B_2 \times_{B_1} B_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B_2 \arrow[d, "c"] \\ B_2 \arrow[r, "c"'] & B_1 \end{tikzcd}$$

i.e., the "object of squares"

$$\begin{tikzcd} b_s \arrow[r, "\alpha"] \arrow[d, "\beta_s"'] & b_t \arrow[d, "\beta_t"] \\ \bar{b}_s \arrow[r, "\bar{\alpha}"'] & \bar{b}_t \end{tikzcd}$$

and $B_1^{\mathbb{I}} \overset{s}{\underset{t}{\rightrightarrows}} B_0^{\mathbb{I}}$ is given by $B_2 \times_{B_1} B_2 \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} B_2 \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} B_1$, i.e., $s(\beta_s \overset{\alpha}{\underset{\bar{\alpha}}{\rightrightarrows}} \beta_t) = \beta_s$ and $t(\beta_s \overset{\alpha}{\underset{\bar{\alpha}}{\rightrightarrows}} \beta_t) = \beta_t$. Finally, $B_2^{\mathbb{I}} = (B_2 \times_{B_1} B_2) \times_{B_1} (B_2 \times_{B_1} B_2)$ is the "object of commutative diagrams" with composition

$$\begin{tikzcd} b_s \arrow[r, "\alpha"] \arrow[d, "\beta_s"'] & b_t \arrow[r, "\alpha'"] \arrow[d, "\beta_t"] & b_{t'} \arrow[d, "\beta_{t'}"] \\ \bar{b}_s \arrow[r, "\bar{\alpha}"'] & \bar{b}_t \arrow[r, "\bar{\alpha}'"'] & \bar{b}_{t'} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} b_s \arrow[r, "\alpha'\alpha"] \arrow[d, "\beta_s"'] & b_{t'} \arrow[d, "\beta_{t'}"] \\ \bar{b}_s \arrow[r, "\bar{\alpha}'\bar{\alpha}"'] & \bar{b}_{t'} \end{tikzcd}$$

Thus, we get the following corollary of Proposition 3.2.

Corollary 3.3. *If B^{II} is exponentiable in $\mathbf{Gpd}(\mathcal{C})$, then B_1 is exponentiable in $\mathcal{C}$. The converse holds if the arrows $B_2 \underset{\pi_1}{\overset{c}{\rightrightarrows}} B_1$ are exponentiable in $\mathcal{C}$.*

Proof. The first part holds by Proposition 3.2, since $B_0^{\mathrm{II}} = B_1$. So, assume that B_1 and $B_2 \underset{\pi_1}{\overset{c}{\rightrightarrows}} B_1$ are exponentiable in $\mathcal{C}$. Then $B_1^{\mathrm{II}} = B_2 \times_{B_1} B_2 \xrightarrow{\pi_1} B_2$ is exponentiable being a pullback of $c\colon B_2 \longrightarrow B_1$, and so B^{II} is exponentiable in $\mathbf{Gpd}(\mathcal{C})$ by Proposition 3.2, since $s\colon B_1^{\mathrm{II}} \longrightarrow B_0^{\mathrm{II}}$ is given by $B_2 \times_{B_1} B_2 \xrightarrow{\pi_1} B_2 \xrightarrow{\pi_1} B_1$. □

Recall [4] that a topological groupoid G is called an *orbigroupoid* if s and t are étale and $\langle s, t\rangle\colon G_1 \longrightarrow G_0 \times G_0$ is a proper map.

Proposition 3.4. *If B is an orbigroupoid, then so is B^{II}.*

Proof. Suppose B is an orbigroupoid. Since s is étale and

$$\begin{tikzcd} B_2 \arrow[r, "\pi_1"] \arrow[d, "c"'] & B_1 \arrow[d, "s"] \\ B_1 \arrow[r, "s"'] & B_0 \end{tikzcd}$$

is a pullback (as B is a groupoid), it follows that $c\colon B_2 \longrightarrow B_1$ and hence the projections $B_2 \times_{B_1} B_2 \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} B_2$ are étale. Thus, $B_1^{\mathrm{II}} \underset{t}{\overset{s}{\rightrightarrows}} B_0^{\mathrm{II}}$ are étale, as desired. To see that $\langle s,t\rangle\colon B_1^{\mathbb{I}} \longrightarrow B_0^{\mathbb{I}} \times B_0^{\mathbb{I}}$ is proper, consider the diagram

$$\begin{tikzcd}
B_1^{\mathbb{I}} \arrow[r, "\langle s,t\rangle"] \arrow[d, equal] & B_0^{\mathbb{I}} \times B_0^{\mathbb{I}} \arrow[d, equal] \\
B_2 \times_{B_1} B_2 \arrow[r] \arrow[d, "c\pi_1"'] & B_1 \times B_1 \arrow[d, "s\times t"] \\
B_1 \arrow[r, "\langle s,t\rangle"'] & B_0 \times B_0
\end{tikzcd}$$

which is a pullback as B is a groupoid. Since the bottom row is proper it follows that the top one is, and so $B^{\mathbb{I}}$ is an orbigroupoid. $\square$

4. Exponentiable Morphisms of Groupoids

In this section, we consider exponentiable morphisms in $\mathbf{Gpd}(\mathcal{C})$. When $\mathcal{C} = \mathbf{Sets}$, or any topos, we know that these are precisely the fibrations. Though the categories $\mathcal{C}$ of spaces of interest are not even locally cartesian closed, we will see that if $q\colon G \longrightarrow B$ is a fibration (in the sense defined below) and each $q_i\colon G_i \longrightarrow B_i$ is exponentiable in $\mathcal{C}$, then q is exponentiable in $\mathbf{Gpd}(\mathcal{C})$.

Suppose $q\colon G \longrightarrow B$ is exponentiable in $\mathbf{Gpd}(\mathcal{C})/B$. Then, as in Proposition 3.2, we know $q_0\colon G_0 \longrightarrow B_0$ is exponentiable in $\mathcal{C}$, since

$$(\ )_0\colon \mathbf{Gpd}(\mathcal{C})/B \longrightarrow \mathcal{C}/B_0$$

has left and right adjoints, given by $(X \xrightarrow{p} B_0) \mapsto (L_0X \xrightarrow{L_0p} L_0B_0 \xrightarrow{\varepsilon} B)$, where ε is the counit of the adjunction $L_0 \dashv (\ )_0$, and $(X \xrightarrow{p} B_0) \mapsto (B \times_{R_0B_0} R_0X \xrightarrow{\pi_1} B)$, where $B \longrightarrow R_0B_0$ is the unit of the adjuncion $(\ )_0 \dashv R_0$.

Definition 4.1. *A morphism $q\colon G \longrightarrow B$ is a* fibration *in $\mathbf{Gpd}(\mathcal{C})$ if*

$$\langle s, q_1\rangle\colon G_1 \longrightarrow G_0 \times_{B_0} B_1$$

has a right inverse, or equivalently, $\langle q_1, t\rangle\colon G_1 \longrightarrow B_1 \times_{B_0} G_0$ has a right inverse in $\mathcal{C}$.

Remark 4.2. When $\mathcal{C}$ is the category of all topological spaces (or any concrete category), this says q is a fibration, in the sense of Grothendieck [8]; i.e., given a and $\beta\colon q_0 a \longrightarrow \bar{b}$, there exists $\alpha\colon a \longrightarrow \bar{a}$ such that $q_1\alpha = \beta$, but our condition is stronger since $(a, \beta) \mapsto \alpha$ must be a morphism of $\mathcal{C}$.

Our notion is equivalent to the notion of a cloven strict internal fibration as given in [20] for the 2-category **Gpd**$(\mathcal{C})$. Note that the description for **Gpd**$(\mathcal{C})$ can be simplified this way because we do not need to worry about cartesian arrows: for a fibration between groupoids all arrows in the domain are cartesian.

Lemma 4.3. *An arrow $q\colon G \longrightarrow B$ in* **Gpd**$(\mathcal{C})$ *is a fibration in our sense precisely when it is representably a cloven strict internal fibration.*

Proof. Let $q\colon G \longrightarrow B$ be a fibration in **Gpd**$(\mathcal{C})$ with $\theta\colon G_0 \times_{B_0} B_1 \longrightarrow G_1$ a right inverse to $\langle s, q_1\rangle$. Let H be any groupoid in $\mathcal{C}$. We need to show that the induced functor

$$q_* = \mathbf{Gpd}(\mathcal{C})(H, q)\colon \mathbf{Gpd}(\mathcal{C})(H, G) \longrightarrow \mathbf{Gpd}(\mathcal{C})(H, B)$$

is a cloven strict fibration in **Cat**. So let $\varphi\colon H \longrightarrow G$ be an internal functor, viewed as object in **Gpd**$(\mathcal{C})(H, G)$ and let $\alpha\colon q\varphi \Rightarrow \psi$ be an internal natural transformation, viewed as an arrow in **Gpd**$(\mathcal{C})(H, B)$, Then α gives rise to a morphism $\alpha\colon H_0 \longrightarrow B_1$ in $\mathcal{C}$, with $s\alpha = q_0\varphi_0$. Hence this gives us $\langle \varphi_0, \alpha\rangle\colon H_0 \longrightarrow G_0 \times_{B_0} B_1$. It follows that the composition $\theta\langle\varphi_0, \alpha\rangle\colon H_0 \longrightarrow G_1$ is the required lifting. This defines a cleavage, because the internal categories here are groupoids. The fact that for any $f\colon H \longrightarrow H'$, the induced square is a morphism of fibrations follows immediately from the fact that we are working with groupoids.

Conversely, suppose that $q\colon G \longrightarrow B$ is representably a cloven internal fibration in **Gpd**$(\mathcal{C})$. This implies that

$$q_* = \mathbf{Gpd}(\mathcal{C})(H, q)\colon \mathbf{Gpd}(\mathcal{C})(H, G) \longrightarrow \mathbf{Gpd}(\mathcal{C})(H, B)$$

is a cloven strict fibration in **Cat** for each H in **Gpd**$(\mathcal{C})$. Now take H to be the strict comma square,

$$\begin{tikzcd} H \arrow[r, "r"] \arrow[d, "p"'] \arrow[dr, phantom, "\stackrel{\alpha}{\Rightarrow}"] & B \arrow[d, "id_B"] \\ G \arrow[r, "q"'] & B \end{tikzcd}$$

Then we may take H_0 to be the pullback

$$\begin{tikzcd} H_0 \arrow[r] \arrow[d] & B_1 \arrow[d, "s"] \\ G_0 \arrow[r, "q_0"'] & B_0 \end{tikzcd}$$

and $p_0 = \pi_1\colon H_0 \longrightarrow G_0$ and $\alpha = \pi_2\colon H_0 \longrightarrow B_1$.

Note that we have $\alpha\colon qp \Rightarrow r$, an arrow in $\mathbf{Gpd}(\mathcal{C})(H, B)$, and p is such that $q_*(p) = qp$. Hence the cleavage gives us a lifting $\tilde{\alpha}\colon q \Rightarrow \tilde{r}$ in $\mathbf{Gpd}(H, G)$ represented by $\tilde{\alpha}\colon H_0 \longrightarrow G_1$ such that $s\tilde{\alpha} = p_0$ and $q_1\tilde{\alpha} = \alpha = \pi_2$. So we get that $\langle s, q_1\rangle\tilde{\alpha} = id_{G_0\times_{B_0}B_1}$ as required. $\square$

Note that $B^{\mathbb{I}}$ becomes a groupoid over B via $B^{\mathbb{I}} \underset{t}{\overset{s}{\rightrightarrows}} B$ defined by $s_0 = s$, $t_0 = t$, $s_1\colon B_2 \times_{B_1} B_2 \xrightarrow{\pi_2} B_2 \xrightarrow{\pi_1} B_1$, and $t_1\colon B_2 \times_{B_1} B_2 \xrightarrow{\pi_1} B_2 \xrightarrow{\pi_2} B_1$, i.e., $s_1(\beta_s \underset{\bar{\alpha}}{\overset{\alpha}{\rightrightarrows}} \beta_t) = \alpha$, and $t_1(\beta_s \underset{\bar{\alpha}}{\overset{\alpha}{\rightrightarrows}} \beta_t) = \bar{\alpha}$.

Proposition 4.4. *The morphisms $B^{\mathbb{I}} \times_B G \xrightarrow{s\pi_1} B$ and $G \times_B B^{\mathbb{I}} \xrightarrow{t\pi_2} B$ are fibrations, for all $q\colon G \longrightarrow B$. In particular, $s\colon B^{\mathbb{I}} \longrightarrow B$ and $t\colon B^{\mathbb{I}} \longrightarrow B$ are fibrations, for all B.*

Proof. This result follows from the general theory on fibrations as spelled out in Theorem 14 [20] for instance, where it is shown that any span which is the comma object of some opspan is a split bifibration. However, in this particular case, there is also a short straightforward argument: For $s\pi_1$, the morphism $\langle s, (s\pi_1)_1\rangle\colon (B^{\mathbb{I}} \times_B G)_1 \longrightarrow (B^{\mathbb{I}} \times_B G)_0 \times_{B_0} B_1$ is given by

$$\begin{tikzcd} b_s \arrow[r, "\alpha"] \arrow[d, "\beta_s"'] & b_t \arrow[d, "\beta_t"] & & b_s \arrow[r, "\alpha"] \arrow[d, "\beta_s"'] & b_t \\ qa_s \arrow[r, "q\gamma"'] & qa_t \arrow[r, phantom, "\mapsto"] & & qa_s & \end{tikzcd}$$

and so

$$\begin{tikzcd} b_s \arrow[r, "\alpha"] \arrow[d, "\beta_s"'] & b_t & & b_s \arrow[r, "\alpha"] \arrow[d, "\beta_s"'] & b_t \arrow[d, "\beta_s\alpha^{-1}"] \\ qa_s & \arrow[r, phantom, "\mapsto"] & & qa_s \arrow[r, "qid"'] & qa_s \end{tikzcd}$$

is a right inverse to $\langle s, (s\pi_1)_1\rangle$. The proof for t is similar. $\square$

Now, for "discrete" groupoids L_0B, we know

$$\mathbf{Gpd}(\mathcal{C})/L_0B \cong \mathbf{Gpd}(\mathcal{C}/B)$$

and so $q\colon G \longrightarrow L_0B$ is exponentiable in $\mathbf{Gpd}(\mathcal{C})$ if each $q_i\colon G_i \longrightarrow B$ is exponentiable in $\mathcal{C}$. Thus, if $\mathcal{C}$ is cartesian closed, then $\mathbf{Gpd}(\mathcal{C})/L_0B$ is cartesian closed whenever the diagonal on B is exponentiable in $\mathcal{C}$. In particular, $\mathbf{Gpd}(\mathbf{Top}_{\mathcal{M}})/L_0B$ is cartesian closed whenever B is $\mathcal{M}$-Hausdorff, e.g., weak Hausdorff in the case where $\mathcal{M} = \mathcal{K}$.

For the non-discrete case, given $q\colon G \longrightarrow B$ and $r\colon H \longrightarrow B$, to see how to define the exponentials $r^q\colon H^G \longrightarrow B$ when q is exponentiable in $\mathbf{Gpd}(\mathcal{C})/B$, consider the case where $\mathcal{C} = \mathbf{Sets}$. Recall that the fiber of $(H^G)_0$ over b in B is the set of homomorphisms $\sigma\colon G_b \longrightarrow H_b$ between the fibers of G and H over b. A morphism $\Sigma\colon \sigma \longrightarrow \sigma'$ over $\beta\colon b \longrightarrow b'$ in B is a family of morphisms $\Sigma_\alpha\colon \sigma a \longrightarrow \sigma' a'$ of H over β indexed by the morphisms $\alpha\colon a \longrightarrow a'$ of G over β such that the diagram

$$\begin{tikzcd}
\sigma\bar{a} \arrow[r, "\sigma\bar{\alpha}"] \arrow[d, "\Sigma_{\alpha\bar{\alpha}}"'] & \sigma a \arrow[d, "\Sigma_{\bar{\alpha}'\alpha}"] \arrow[dl, "\Sigma_\alpha"'] \\
\sigma' a' \arrow[r, "\sigma'\bar{\alpha}'"'] & \sigma'\bar{a}'
\end{tikzcd} \qquad (2)$$

commutes, for all $\bar{a} \xrightarrow{\bar{\alpha}} a \xrightarrow{\alpha} a' \xrightarrow{\bar{\alpha}'} \bar{a}'$ such that $q(\bar{\alpha}) = id_b$ and $q(\bar{\alpha}') = id_{b'}$. Defining the morphisms s, t, u and i is straightforward, but for composition, one must assume q is a fibration. Then, let $r\colon G_0 \times_{B_0} B_1 \longrightarrow G_1$ be a right inverse of $\langle s, q_1\rangle$. Suppose $\sigma \xrightarrow{\Sigma} \sigma' \xrightarrow{\Sigma'} \sigma''$ is a composable pair over $b \xrightarrow{\beta} b' \xrightarrow{\beta'} b''$, and define $\sigma \xrightarrow{\Sigma'\Sigma} \sigma''$ as follows. Given $a \xrightarrow{\alpha''} a''$ over $b \xrightarrow{\beta'\beta} b''$, consider

$$\begin{tikzcd}
a \arrow[rr, "\alpha''"] \arrow[dr, "\alpha"'] & & a'' \\
& a' \arrow[ur, "\alpha'"'] &
\end{tikzcd}$$

where $\alpha = r(a, \beta)$ and $\alpha' = \alpha''\alpha^{-1}$, and define $(\Sigma'\Sigma)_{\alpha''} = \Sigma'_{\alpha'}\Sigma_\alpha$. Then it is not difficult to show that H^G is a groupoid over B and that this provides a right adjoint to the functor $- \times_B G\colon \mathbf{Gpd}/B \longrightarrow \mathbf{Gpd}/B$.

Theorem 4.5. *If $q\colon G \longrightarrow B$ is a fibration and $q_i\colon G_i \longrightarrow B_i$ is exponentiable in $\mathcal{C}$, for $i = 0, 1, 2$, then q is exponentiable in $\mathbf{Gpd}(\mathcal{C})/B$.*

Proof. Given $H \longrightarrow B$, define $(H^G)_0 \longrightarrow B_0$ by the equalizer

$$(H^G)_0 \rightarrowtail H_0^{G_0} \times_{B_0} (B_0 \times_{B_1} H_1^{G_1}) \times_{B_0} (B_0 \times_{B_2} H_2^{G_2}) \overset{f_0}{\underset{g_0}{\rightrightarrows}} X_0$$

in $\mathcal{C}/B_0$, capturing the fact that

$$(\sigma_0\colon G_0 \longrightarrow H_0, \sigma_1\colon G_1 \longrightarrow H_1, \sigma_2\colon G_2 \longrightarrow H_2)$$

is a "homomorphism of groupoids", where $H_0^{G_0} \longrightarrow B_0$, $H_1^{G_1} \longrightarrow B_1$ and $H_2^{G_2} \longrightarrow B_2$ are the exponentials,

$$\begin{tikzcd} B_0 \times_{B_1} H_1^{G_1} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & H_1^{G_1} \arrow[d] \\ B_0 \arrow[r, "u"'] & B_1 \end{tikzcd}$$

and

$$\begin{tikzcd} B_0 \times_{B_2} H_2^{G_2} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & H_2^{G_2} \arrow[d] \\ B_0 \arrow[r, "{(u,u)}"'] & B_2 \end{tikzcd}$$

are pullbacks in $\mathcal{C}$, and the morphisms f_0 and g_0 ensure that σ_0, σ_1 and σ_2 are compatible with s, t, u, c and the projections. In detail, for s, X_0 has a factor of the form $H_0^{B_0 \times_{B_1} G_1}$ whose projections of f_0 and g_0 are given by

$$\begin{tikzcd}[column sep=small] & H_0^{G_0} \arrow[dr, "H_0^s"] & \\ H_0^{G_0} \times_{B_0} (B_0 \times_{B_1} H_1^{G_1}) \times_{B_0} (B_0 \times_{B_2} H_2^{G_2}) \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & & H_0^{B_0 \times_{B_1} G_1} \\ & B_0 \times_{B_1} H_1^{G_1} \cong (B_0 \times_{B_1} H_1)^{B_0 \times_{B_1} G_1} \arrow[ur, "(s\pi_2)^{B_0 \times_{B_1} G_1}"'] & \end{tikzcd}$$

The factor of X_0 for t is defined similarly: just replace both occurrences of s by t in this diagram.

The factor of X_0 for u is of the form $(B_0 \times_{B_1} H_1)^{G_0}$ and the projections of f_0 and g_0 for this factor are given by

$$\begin{tikzcd}
& H_0^{G_0} \arrow[dr, "(q_0,u)^{G_0}"] & \\
H_0^{G_0} \times_{B_0} (B_0 \times_{B_1} H_1^{G_1}) \times_{B_0} (B_0 \times_{B_2} H_2^{G_2}) \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & & (B_0 \times_{B_1} H_1)^{G_0} \\
& B_0 \times_{B_1} H_1^{G_1} \cong (B_0 \times_{B_1} H_1)^{B_0 \times_{B_1} G_1} \arrow[ur, "(B_0 \times_{B_1} H_1)^{(q_0,u)}"'] &
\end{tikzcd}$$

The factor of X_0 for c is of the form $(B_0 \times_{B_1} H_1)^{B_0 \times_{B_2} G_2}$ and the projections of f_0 and g_0 for this factor are given by

$$\begin{tikzcd}
& B_0 \times_{B_2} H_2^{G_2} \cong (B_0 \times_{B_2} H_2)^{B_0 \times_{B_2} G_2} \arrow[dr, "(B_0 \times_{B_2} c)^{B_0 \times_{B_2} G_2}"] & \\
H_0^{G_0} \times_{B_0} (B_0 \times_{B_1} H_1^{G_1}) \times_{B_0} (B_0 \times_{B_2} H_2^{G_2}) \arrow[ur, "\pi_3"] \arrow[dr, "\pi_2"'] & & (B_0 \times_{B_1} H_1)^{B_0 \times_{B_2} G_2} \\
& B_0 \times_{B_1} H_1^{G_1} \cong (B_0 \times_{B_1} H_1)^{B_0 \times_{B_1} G_1} \arrow[ur, "(B_0 \times_{B_1} H_1)^{B_0 \times_{B_1} c}"'] &
\end{tikzcd}$$

The factors of X_0 for the commutativity with the two projections from the objects of composable pairs to the objects of arrows are given by two additional copies of $(B_0 \times_{B_1} H_1)^{B_0 \times_{B_2} G_2}$ and the projections of f_0 and g_0 are obtained by replacing c in this diagram by π_1 and π_2 respectively.

We conclude that

$$\begin{aligned} X_0 = H_0^{B_0 \times_{B_1} G_1} \times_{B_0} H_0^{B_0 \times_{B_1} G_1} \times_{B_0} (B_0 \times_{B_1} H_1)^{G_0} \times_{B_0} (B_0 \times_{B_1} H_1)^{B_0 \times_{B_2} G_2} \\ \times_{B_0} (B_0 \times_{B_1} H_1)^{B_0 \times_{B_2} G_2} \times_{B_0} (B_0 \times_{B_1} H_1)^{B_0 \times_{B_2} G_2} \end{aligned}$$

and the maps f_0 and g_0 are given by

$$\begin{aligned} f_0 = (H_0^s \pi_1,\, H_0^t \pi_1,\, (q_0, u)^{G_0} \pi_1,\, (B_0 \times_{B_2} c)^{B_0 \times_{B_2} G_2} \pi_3, \\ (B_0 \times_{B_2} \pi_1)^{B_0 \times_{B_2} G_2} \pi_3,\, (B_0 \times_{B_2} \pi_2)^{B_0 \times_{B_2} G_2} \pi_3) \end{aligned}$$

and

$$g_0 = ((s\pi_2)^{B_0\times_{B_1}G_1}\pi_2,\ (t\pi_2)^{B_0\times_{B_1}G_1}\pi_2,\ (B_0\times_{B_1}H_1)^{(q_0,u)}\pi_2,$$
$$(B_0\times_{B_1}H_1)^{B_0\times_{B_1}c}\pi_2,\ (B_0\times_{B_1}H_1)^{B_0\times_{B_1}\pi_1}\pi_2,\ (B_0\times_{B_1}H_1)^{B_0\times_{B_1}\pi_2}\pi_2)$$

To define $(H^G)_1 \longrightarrow B_1$ we use an equalizer over B_1 of the form

$$(H^G)_1 \rightarrowtail X_2 \underset{g_1}{\overset{f_1}{\rightrightarrows}} X_1$$

where X_2 is given by

$$((H^G)_0\times_{B_0}H_1^{G_1}\times_{B_0}(H^G)_0)\times_{B_1}H_2^{(B_0\times_{B_1}G_1)\times_{G_0}G_1}\times_{B_1}H_2^{(B_0\times_{B_1}G_1)\times_{G_0}G_1}$$

and $H_1^{G_1}\longrightarrow B_1 \underset{t}{\overset{s}{\rightrightarrows}} B_0$ appear in the product over B_0 via the usual convention. The morphisms f_1 and g_1 are defined to encode the commutativity of the diagram (2) defining Σ in $\mathbf{Gpd}(\mathbf{Sets})$. The $H_2^{(B_0\times_{B_1}G_1)\times_{G_0}G_1}$ and $H_2^{(B_0\times_{B_1}G_1)\times_{G_0}G_1}$ components in X_2 have been added to be able to express commutativity of the top left triangle and bottom right triangle (respectively) in (2). To make our diagrams a bit more managable we will write G_1' for $B_0\times_{B_1}G_1$. Commutativity of the top left triangle is then expressed by commutativity of the following three diagrams:

$$\begin{tikzcd}
H_2^{G_1'\times_{G_0}G_1} \arrow[drr, "\pi_1^{G_1'\times_{G_0}G_1}"] & & \\
((H^G)_0\times_{B_0}H_1^{G_1}\times_{B_0}(H^G)_0)\times_{B_1}H_2^{G_1'\times_{G_0}G_1}\times_{B_1}H_2^{G_1'\times_{G_0}G_1} \arrow[u, "\pi_2"] \arrow[d, "\pi_2\pi_1\pi_1"'] & & H_1^{G_1'\times_{G_0}G_1} \\
H_1^{G_1'} \arrow[urr, "H_1^{\pi_1}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
H_2^{G_1'\times_{G_0}G_1} \arrow[drr, "\pi_2^{G_1'\times_{G_0}G_1}"] & & \\
((H^G)_0\times_{B_0}H_1^{G_1}\times_{B_0}(H^G)_0)\times_{B_1}H_2^{G_1'\times_{G_0}G_1}\times_{B_1}H_2^{G_1'\times_{G_0}G_1} \arrow[u, "\pi_2"] \arrow[d, "\pi_2\pi_1"'] & & H_1^{G_1'\times_{G_0}G_1} \\
H_1^{G_1} \arrow[urr, "H_1^{\pi_2}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
& H_2^{G_1'\times_{G_0}G_1} \arrow[dr, "c^{G_1'\times_{G_0}G_1}"] & \\
((H^G)_0 \times_{B_0} H_1^{G_1} \times_{B_0} (H^G)_0) \times_{B_1} H_2^{G_1'\times_{G_0}G_1} \times_{B_1} H_2^{G_1'\times_{G_0}G_1} \arrow[ur, "\pi_2"] \arrow[dr, "\pi_2\pi_1"'] & & H_1^{G_1'\times_{G_0}G_1} \\
& H_1^{G_1} \arrow[ur, "H_1^c"'] &
\end{tikzcd}$$

The diagrams for the commutativity of the bottom right triangle are constructed similarly.

So we need that

$$X_1 = H_1^{G_1'\times_{G_0}G_1} \times_{B_1} H_1^{G_1'\times_{G_0}G_1} \times_{B_1} H_1^{G_1'\times_{G_0}G_1} \times_{B_1} H_1^{G_1\times_{G_0}G_1'} \times_{B_1} H_1^{G_1\times_{G_0}G_1'} \times_{B_1} H_1^{G_1\times_{G_0}G_1'},$$

and

$$f_1 = (\pi_1^{G_1'\times_{G_0}G_1}\pi_2, \pi_2^{G_1'\times_{G_0}G_1}\pi_2, c^{G_1'\times_{G_0}G_1}\pi_2, \pi_1^{G_1\times_{G_0}G_1'}\pi_3, \pi_2^{G_1\times_{G_0}G_1'}\pi_3, c^{G_1\times_{G_0}G_1'}\pi_3)$$
$$g_1 = (H_1^{\pi_1}\pi_2\pi_1\pi_1, H_1^{\pi_2}\pi_2\pi_1, H_1^c\pi_2\pi_1, H_1^{\pi_1}\pi_2\pi_1, H_1^{\pi_2}\pi_2\pi_3\pi_1, H_1^c\pi_2\pi_1)$$

Note that $s, t\colon (H^G)_1 \longrightarrow (H^G)_0$ are given by the projections. The morphisms $i\colon (H^G)_1 \longrightarrow (H^G)_1$ and $u\colon (H^G)_0 \longrightarrow (H^G)_1$ are induced by $i^{G_1}\colon H_1^{G_1} \longrightarrow H_1^{G_1}$ and

$$\langle id, \varphi, id\rangle\colon (H^G)_0 \longrightarrow (H^G)_0 \times_{B_0} \times H_1^{G_1} \times_{B_0} (H^G)_0$$

respectively, where φ is the composition

$$(H^G)_0 \rightarrowtail H_0^{G_0} \times_{B_0} (B_0 \times_{B_1} H_1^{G_1}) \times_{B_0} (B_0 \times_{B_2} H_2^{G_2}) \xrightarrow{\pi_2\pi_2} H_1^{G_1}$$

To define composition, let $\theta\colon G_0 \times_{B_0} B_1 \longrightarrow G_1$ denote the right inverse of $\langle s, q_1\rangle$, which exists since q is a fibration, and consider the diagram

$$\begin{tikzcd}
(H^G)_1 \times_{(H^G)_0} (H^G)_1 \arrow[r] \arrow[d] & (H^G)_1 \arrow[d] \\
H_1^{G_1} \times_{B_1} B_2 \times_{B_1} H_1^{G_1} \arrow[r] \arrow[d] & H_1^{G_1} \arrow[d] \\
B_2 \arrow[r, "c"'] & B_1
\end{tikzcd}$$

where the vertical compositions are the "projections" and the unnamed horizontal morphisms are to be determined. It suffices to define a morphism $H_1^{G_1} \times_{B_1} B_2 \times_{B_1} H_1^{G_1} \longrightarrow H_1^{G_1}$ so that the bottom square commutes, since all other components can be derived from this map. Now, θ induces a morphism

$$\theta' \colon G_1 \times_{B_1} B_2 \xrightarrow{\langle \pi_1, s\pi_1, \pi_1\pi_2 \rangle} G_1 \times_{B_1} (G_0 \times_{B_0} B_1) \xrightarrow{\langle \theta\pi_2, c(i\theta\pi_2, \pi_1) \rangle} G_1 \times_{G_0} G_1$$

and hence, $(H_1^{G_1} \times_{B_1} B_2 \times_{B_1} H_1^{G_1}) \times_{B_1} G_1 \longrightarrow (H_1^{G_1} \times_{B_1} G_1) \times_{B_0} (H_1^{G_1} \times_{B_1} G_1) \longrightarrow H_1 \times_{H_0} H_1 \xrightarrow{c} H_1$, whose transpose gives the desired morphism. As in the case of $\mathcal{C} = \mathbf{Sets}$, this defines the exponential $H^G \longrightarrow B$. $\square$

Remark 4.6. Since each one of our fibrations in $\mathbf{Gpd}(\mathcal{C})$ is a fibration in $\mathbf{Cat}(\mathcal{C})$ as used in [21], Theorem 4.5 describes a special case of Theorem 2.17 in that paper. We include the proof given here, because it gives an explicit construction of the exponential groupoid in the slice category and shows where each assumption is used.

By Theorem 4.5, a fibration $q \colon G \longrightarrow B$ is exponentiable in $\mathbf{Gpd}(\mathbf{Top})/B$, if each $q_i \colon G_i \longrightarrow B_i$ is exponentiable in $\mathbf{Top}$, for $i = 0, 1, 2$. Now, if $\mathcal{C}/B_i$ is cartesian closed, for $i = 0, 1, 2$, then every fibration is exponentiable in $\mathbf{Gpd}(\mathcal{C})/B$. This is the case when $\mathcal{C}$ is cartesian closed and each diagonal $\Delta \colon B_i \longrightarrow B_i \times B_i$ is exponentiable in $\mathcal{C}$, e.g., $\mathcal{C} = \mathbf{Top}_{\mathcal{M}}$ and the B_i are locally $\mathcal{M}$-Hausdorff. By the following lemma, we need not assume the $i = 2$ case.

Lemma 4.7. *Suppose $\mathcal{C}$ is a finitely complete category.*

(a) If X and Y have exponentiable diagonals, then so does $X \times Y$.

(b) If B is a groupoid in $\mathcal{C}$ and B_1 has an exponentiable diagonal, then so does B_2.

Proof. For (a), suppose X and Y have exponentiable diagonals. Then the diagonal on $X \times Y$ is exponentiable, since it can be factored

$$X \times Y \xrightarrow{id_X \times \Delta} X \times (Y \times Y) \xrightarrow{\Delta \times id_{Y \times Y}} (X \times X) \times (Y \times Y) \xrightarrow{\varphi} (X \times Y) \times (X \times Y)$$

where the first two morphisms are exponentiable being pullbacks of exponentiables and φ is an isomorphism.

For (b), suppose B_1 has an exponentiable diagonal. Then $B_1 \times B_1$ does, by (a). Since there is a monomorphism $\psi\colon B_2 \longrightarrow B_1 \times B_1$, we see that the diagram

$$\begin{tikzcd} B_2 \arrow[r, "\Delta"] \arrow[d, "\psi"'] & B_2 \times B_2 \arrow[d, "\psi\times\psi"] \\ B_1 \times B_1 \arrow[r, "\Delta"'] & (B_1 \times B_1) \times (B_1 \times B_1) \end{tikzcd}$$

is a pullback, and it follows that B_2 has an exponentiable diagonal. □

Thus, we get the following corollaries to Theorem 4.5:

Corollary 4.8. *If G_0, G_1, and G_2 are exponentiable spaces, and B_0 and B_1 are locally Hausdorff, then every fibration $q\colon G \longrightarrow B$ is exponentiable in* $\mathbf{Gpd}(\mathbf{Top})$.

Corollary 4.9. *If B_0 and B_1 have exponentiable diagonals in a cartesian closed category $\mathcal{C}$, then every fibration $q\colon G \longrightarrow B$ is exponentiable in* $\mathbf{Gpd}(\mathcal{C})$.

Corollary 4.10. *Every fibration is exponentiable in $\mathbf{Gpd}(\mathbf{Top}_{\mathcal{M}})/B$, if B_0 and B_1 are locally $\mathcal{M}$-Hausdorff.*

Corollary 4.11. *The following are equivalent.*

(a) $s\colon B^{\mathbb{I}} \longrightarrow B$ is exponentiable in $\mathbf{Gpd}(\mathcal{C})$.

(b) $s\colon B_1 \longrightarrow B_0$ is exponentiable in $\mathcal{C}$.

(c) $t\colon B^{\mathbb{I}} \longrightarrow B$ is exponentiable in $\mathbf{Gpd}(\mathcal{C})$.

(d) $t\colon B_1 \longrightarrow B_0$ is exponentiable in $\mathcal{C}$.

Proof. Since $si = t$ and i is an isomorphism, we know (b) and (d) are equivalent. We will establish the equivalence of (a) and (b). The proof for (c) and (d) is similar.

First, (a) implies (b) follows from the remark at the beginning of this section. For the converse, it suffices to show that $s_1\colon B_1^{\mathbb{I}} \longrightarrow B_1$ and $s_2\colon B_2^{\mathbb{I}} \longrightarrow B_2$

are exponentiable in $\mathcal{C}$, since $s\colon B^{\mathbb{I}} \longrightarrow B$ is a fibration by Proposition 4.4. We know the first one is exponentiable, as it is given by

$$s_1\colon B_2 \times_{B_1} B_2 \xrightarrow{\pi_2} B_2 \xrightarrow{\pi_1} B_1$$

which is a composition of exponentiables when $s\colon B_1 \longrightarrow B_0$ is exponentiable, since the diagrams

$$\begin{tikzcd} B_2 \times_{B_1} B_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1\pi_1"'] & B_2 \arrow[d, "s\pi_1"] \\ B_1 \arrow[r, "s"'] & B_0 \end{tikzcd} \qquad \begin{tikzcd} B_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B_1 \arrow[d, "t"] \\ B_1 \arrow[r, "s"'] & B_0 \end{tikzcd}$$

are pullbacks in $\mathcal{C}$. To see that $s_2\colon B_2^{\mathbb{I}} \longrightarrow B_2$ is exponentiable, note that $s_2 = \pi_1\pi_2 \times \pi_2\pi_1$ and the square

$$\begin{tikzcd} B_2^{\mathbb{I}} \arrow[r, "\pi_1\times\pi_2"] \arrow[d, "s_2"'] & B_2 \times_{B_0} B_2 \arrow[d, "c(c\times c)"] \\ B_2 \arrow[r, "c"'] & B_1 \end{tikzcd}$$

is a pullback. Thus, it suffices to show that

$$B_2 \times_{B_0} B_2 = (B_1 \times_{B_0} B_1) \times_{B_0} (B_1 \times_{B_0} B_1) \xrightarrow{c\times c} B_1 \times_{B_0} B_1 \xrightarrow{c} B_1$$

is exponentiable. Since s is exponentiable and

$$\begin{tikzcd} B_1 \times_{B_0} B_1 \arrow[r, "\pi_1"] \arrow[d, "c"'] & B_1 \arrow[d, "s"] \\ B_1 \arrow[r, "s"'] & B_0 \end{tikzcd}$$

is a pullback, we know c is exponentiable. Since

$$\begin{tikzcd} (B_1 \times_{B_0} B_1) \times_{B_0} (B_1 \times_{B_0} B_1) \arrow[r, "\pi_2\pi_1\times\pi_1\pi_2"] \arrow[d, "c\times c"'] & B_1 \times_{B_0} B_1 \arrow[d, "s\pi_2"] \\ B_1 \times_{B_0} B_1 \arrow[r, "s\pi_2"'] & B_0 \end{tikzcd}$$

is a pullback and $s\pi_2$ is a composition of exponentiable morphisms, it follows that $c \times c$ is exponentiable. $\square$

Corollary 4.12. *If $s\colon B_1 \longrightarrow B_0$ (respectively, $t\colon B_1 \longrightarrow B_0$) and $q_i\colon G_i \longrightarrow B_i$ are exponentiable in $\mathcal{C}$, for $i = 0, 1, 2$, then $s\pi_1\colon B^{\mathbb{I}} \times_B G \longrightarrow B$ (respectively, $t\pi_2\colon G \times_B B^{\mathbb{I}} \longrightarrow B$) is exponentiable in $\mathbf{Gpd}(\mathcal{C})/B$.*

Proof. Since pullback and composition preserve exponentiability, the result follows from Proposition 4.4, Theorem 4.5, and Corollary 4.11. □

Corollary 4.13. *If B_0 and B_1 have exponentiable diagonals in a cartesian closed category $\mathcal{C}$, then $s\pi_1\colon B^{\mathbb{I}} \times_B G \longrightarrow B$ and $t\pi_2\colon G \times_B B^{\mathbb{I}} \longrightarrow B$ are exponentiable in $\mathbf{Gpd}(\mathcal{C})$, for all $q\colon G \longrightarrow B$.*

Proof. By Lemma 4.7(b), since B_1 has an exponentiable diagonal, so does B_2. Thus, applying Proposition 2.2, we see that every morphism $X \longrightarrow B_i$ is exponentiable in $\mathcal{C}$, for $i = 0, 1, 2$, and so the desired result follows from Corollary 4.12. □

Corollary 4.14. *If G_0, G_1, G_2, and B_1 are exponentiable spaces and B_0 and B_1 are locally Hausdorff, then $s\pi_1\colon B^{\mathbb{I}} \times_B G \longrightarrow B$ and $t\pi_2\colon G \times_B B^{\mathbb{I}} \longrightarrow B$ are exponentiable in $\mathbf{Gpd}(\mathbf{Top})$, for all $q\colon G \longrightarrow B$.*

Corollary 4.15. *If B_0 and B_1 are locally $\mathcal{M}$-Hausdorff, then $s\pi_1\colon B^{\mathbb{I}} \times_B G \longrightarrow B$ and $t\pi_2\colon G \times_B B^{\mathbb{I}} \longrightarrow B$ are exponentiable in $\mathbf{Gpd}(\mathbf{Top}_{\mathcal{M}})$, for all $q\colon G \longrightarrow B$.*

5. Pseudo-Exponentiability of Morphisms of Groupoids

In this section, we use a general theorem from Niefield [17] for monads and their Kleisli categories to show that $G \longrightarrow B$ is pseudo-exponentiable in $\mathbf{Gpd}(\mathcal{C})//B$ if $s\pi_1\colon B^{\mathbb{I}} \times_B G \longrightarrow B$ is exponentiable in $\mathbf{Gpd}(\mathcal{C})/B$, e.g., $s\colon B_1 \longrightarrow B_0$ and $G_i \longrightarrow B_i$ are exponentiable in $\mathcal{C}$, for $i = 0, 1, 2$. Consequently, $\mathbf{Gpd}(\mathcal{C})//B$ is pseudo-cartesian closed whenever B_0 and B_1 have exponentiable diagonals in a cartesian closed category $\mathcal{C}$. In particular, $\mathbf{Gpd}(\mathcal{C})$ is locally pseudo-cartesian closed when $\mathcal{C}$ is locally cartesian closed, e.g., $\mathcal{C} = \mathbf{Sets}$.

The general result in [17], i.e.,Theorem 3.4, was proved for pseudo-monads on a bicategory since one of the examples there was not a 2-category. Restricting to the strict case we get:

Theorem 5.1. *Suppose $\mathcal{K}$ is a 2-category with finite 2-products and T, μ, η is a 2-monad on $\mathcal{K}$ such that $\eta T \cong T\eta$ and the induced morphism*

$$\rho\colon T(X \times TY) \longrightarrow TX \times TY$$

is an isomorphism, for all X, Y in $\mathcal{K}$. If TY is 2-exponentiable in $\mathcal{K}$, then Y is pseudo-exponentiable in the Kleisli 2-category $\mathcal{K}_T$.

Before applying this theorem to $\mathcal{K} = \mathbf{Gpd}(\mathcal{C})/B$, we recall the definition of pseudo-exponentiability. First, a diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

is a *pseudo-product* in a 2-category $\mathcal{K}$ if the induced functor

$$\mathcal{K}(Z, X \times Y) \xrightarrow{\varphi_Z} \mathcal{K}(Z, X) \times \mathcal{K}(Z, Y)$$

is an equivalence of categories, for all Z. Since the definition of 2-product requires that φ_Z is an isomorphism, for all Z, it follows that every 2-product is necessarily a pseudo-product in $\mathcal{K}$. An object Y is *pseudo-exponentiable* if the pseudo-functor $- \times Y\colon \mathcal{K} \longrightarrow \mathcal{K}$ has a right pseudo-adjoint, i.e., for every object Z, there is an object Z^Y together with an equivalence

$$\mathcal{K}(X \times Y, Z) \xrightarrow{\theta_{X,Z}} \mathcal{K}(X, Z^Y)$$

which are pseudo-natural in X and Z.

As before, we are assuming that $\mathcal{C}$ is a finitely complete category with finite coproducts. Then there is an internal groupoid

$$\begin{tikzcd} B^{\mathbb{I}} \times_B B^{\mathbb{I}} \arrow[r, "c"] & B^{\mathbb{I}} \arrow[loop above, "i"] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & B \arrow[l, "u" description] \end{tikzcd}$$

in $\mathbf{Gpd}(\mathcal{C})$, where as usual, we write $B^{\mathbb{I}}$ on the left of $\times_B$, when $t\colon B^{\mathbb{I}} \longrightarrow B$ and on the right when $s\colon B^{\mathbb{I}} \longrightarrow B$. Note that s and t are as in Section 4 and

c, i and u are defined analogously. Thus, as in [17] (see also Street [20]), we get a monad on $\mathbf{Gpd}(\mathcal{C})/B$ defined by

$$T(G \xrightarrow{q} B) = B^{\mathbb{I}} \times_B G \xrightarrow{s\pi_1} B \qquad \eta\colon G \xrightarrow{\langle uq, id\rangle} B^{\mathbb{I}} \times_B G$$

$$\mu\colon B^{\mathbb{I}} \times_B B^{\mathbb{I}} \times_B G \xrightarrow{c \times id} B^{\mathbb{I}} \times_B G$$

and it is not difficult to show that the 2-Kleisli category is (isomorphic to) the pseudo-slice $\mathbf{Gpd}(\mathcal{C})//B$ whose objects are homomorphism $q\colon G \longrightarrow B$, morphisms are triangles

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "q"'] & \arrow[d, phantom, "\xrightarrow{\varphi}"] & H \arrow[dl, "r"] \\ & B & \end{tikzcd} \quad \text{or equivalently} \quad \begin{tikzcd} G \arrow[rr, "\langle \hat{\varphi}, f\rangle"] \arrow[dr, "q"'] & & B^{\mathbb{I}} \times_B H \arrow[dl, "s\pi_1"] \\ & B & \end{tikzcd}$$

and 2-cells $\theta\colon (f, \varphi) \longrightarrow (g, \psi)$ are 2-cells $\theta\colon f \longrightarrow g$ such that

$$\begin{tikzcd} & q \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ rf \arrow[rr, "r\theta"'] & & rg \end{tikzcd}$$

To show that $\rho\colon B^{\mathbb{I}} \times_B (G \times_B B^{\mathbb{I}} \times_B H) \longrightarrow (B^{\mathbb{I}} \times_B G) \times_B (B^{\mathbb{I}} \times_B H)$ is an isomorphism, note that $\pi_i \rho = \pi_i$, for $i = 1, 2, 4$, and

$$\begin{tikzcd} B^{\mathbb{I}} \times_B (G \times_B B^{\mathbb{I}} \times_B H) \arrow[r, "\rho"] \arrow[d, "\langle \pi_1, \pi_3\rangle"'] & (B^{\mathbb{I}} \times_B G) \times_B (B^{\mathbb{I}} \times_B H) \arrow[d, "\pi_3"] \\ B^{\mathbb{I}} \times_B B^{\mathbb{I}} \arrow[r, "c"'] & B^{\mathbb{I}} \end{tikzcd}$$

Then one can show that ρ is invertible with $\pi_i \rho^{-1} = \pi_i$, for $i = 1, 2, 4$, and

$$\begin{tikzcd} (B^{\mathbb{I}} \times_B G) \times_B (B^{\mathbb{I}} \times_B H) \arrow[r, "\rho^{-1}"] \arrow[d, "\langle i\pi_1, \pi_3\rangle"'] & B^{\mathbb{I}} \times_B (G \times_B B^{\mathbb{I}} \times_B H) \arrow[d, "\pi_3"] \\ B^{\mathbb{I}} \times_B B^{\mathbb{I}} \arrow[r, "c"'] & B^{\mathbb{I}} \end{tikzcd}$$

To show $\eta T \cong T\eta$, it suffices to show $(\eta T)_{B^{\mathbb{I}}} \cong T\eta_{B^{\mathbb{I}}}$, where $t\colon B^{\mathbb{I}} \longrightarrow B$, since $\eta_G = \eta_{B^{\mathbb{I}}} \times_B G$. Now, $(\eta T)_{B^{\mathbb{I}}}$ and $T\eta_{B^{\mathbb{I}}}$ are given by

$$B^{\mathbb{I}} \xrightarrow{\langle s, id\rangle} B \times_B B^{\mathbb{I}} \xrightarrow{\langle u, id\rangle} B^{\mathbb{I}} \times_B B^{\mathbb{I}} \quad \text{and} \quad B^{\mathbb{I}} \xrightarrow{\langle id, t\rangle} B^{\mathbb{I}} \times_B B \xrightarrow{\langle id, u\rangle} B^{\mathbb{I}} \times_B B^{\mathbb{I}}$$

Then one can show that the desired isomorphism is induced by the following morphism $\theta\colon (B^{\mathrm{I\!I}})_0 \longrightarrow (B^{\mathrm{I\!I}})_1 \times_{B_1} (B^{\mathrm{I\!I}})_1$. First, recall that

$$(B^{\mathrm{I\!I}})_0 \cong B_1 \quad \text{and} \quad (B^{\mathrm{I\!I}})_1 \times_{B_1} (B^{\mathrm{I\!I}})_1 \cong (B_2 \times_{B_1} B_2) \times_{B_1} (B_2 \times_{B_1} B_2)$$

Then $\pi_1\theta$ and $\pi_2\theta$ are given by

$$B_1 \xrightarrow{\langle us,id,us,id\rangle} (B_1 \times_{B_0} B_1) \times_{B_1} (B_1 \times_{B_0} B_1) \cong B_2 \times_{B_1} B_2$$

and

$$B_1 \xrightarrow{\langle id,ut,id,ut\rangle} (B_1 \times_{B_0} B_1) \times_{B_1} (B_1 \times_{B_0} B_1) \cong B_2 \times_{B_1} B_2$$

respectively.

Theorem 5.2. *If $s\colon B_1 \longrightarrow B_0$ and $q_i\colon G_i \longrightarrow B_i$ are exponentiable in $\mathcal{C}$, for $i = 0, 1, 2$, then $q\colon G \longrightarrow B$ is pseudo-exponentiable in $\mathbf{Gpd}(\mathcal{C})/\!\!/B$.*

Proof. Apply Corollary 4.12 and Theorem 5.1. □

In particular, we get the following corollaries:

Corollary 5.3. *If G_0, G_1, G_2, and B_1 are exponentiable (e.g., locally compact) and B_0 and B_1 are locally Hausdorff spaces, then every morphism $q\colon G \longrightarrow B$ is pseudo-exponentiable in $\mathbf{Gpd}(\mathbf{Top})$.*

Corollary 5.4. *If B_0 and B_1 have exponentiable diagonals in a cartesian closed category $\mathcal{C}$, then $\mathbf{Gpd}(\mathcal{C})/\!\!/B$ is pseudo-cartesian closed.*

Corollary 5.5. *If B_0 and B_1 are locally $\mathcal{M}$-Hausdorff, then $\mathbf{Gpd}(\mathbf{Top}_{\mathcal{M}})/\!\!/B$ is pseudo-cartesian closed.*

Corollary 5.6. *If B_0 and B_1 are compactly generated weak Hausdorff spaces, then $\mathbf{Gpd}(\mathbf{Top}_{\mathcal{K}})/\!\!/B$ is pseudo-cartesian closed.*

Corollary 5.7. *If $\mathcal{C}$ is locally cartesian closed, then $\mathbf{Gpd}(\mathcal{C})$ is locally pseudo-cartesian closed.*

Corollary 5.8. $\mathbf{Gpd}(\mathbf{Sets})$ *is locally pseudo-cartesian closed.*

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GOURSAT COMPLETIONS

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Résumé. Nous caractérisons les catégories avec limites finies faibles dont les complètions régulières sont des catégories de Goursat.

Abstract. We characterize categories with weak finite limits whose regular completions give rise to Goursat categories.

Keywords. regular category, projective cover, Goursat category, 3-permuta– ble variety.

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1. Introduction

The construction of the free exact category over a category with finite limits was introduced in [3]. It was later improved to the construction of the free exact category over a category with finite weak limits (*weakly lex*) in [4]. This was possible because the uniqueness of the finite limits of the original category was never used in the construction; only the existence. In [4], the authors also considered the free regular category over a weakly lex one.

An important property of the free exact (or regular) construction is that such categories always have enough (regular) projectives. In fact, an exact category $\mathbb{A}$ may be seen as the exact completion of a weakly lex category if and only if it has enough projectives. If so, then $\mathbb{A}$ is the exact completion of any of its *projective covers*. Such a phenomenon is captured by varieties of universal algebras: they are the exact completions of their full subcategory of free algebras.

Having this link in mind, our main interest in studying this subject is to characterize projective covers of certain algebraic categories through simple properties involving projectives and to relate those properties to the known varietal characterizations in terms of the existence of operations of their varietal theories. Such kind of studies have been done for the projective covers of categories which are: Mal'tsev [11], protomodular and semi-abelian [5], (strongly) unital and subtractive [6].

The aim of this work is to obtain characterizations of the weakly lex categories whose regular completion is a Goursat (=3-permutable) category (Propositions 4.5 and 4.7). We then relate them to the existence of the quaternary operations which characterize the varieties of universal algebras which are 3-permutable (Remark 4.8).

2. Preliminaries

In this section, we briefly recall some elementary categorical notions needed in the following.

A category with finite limits is **regular** if regular epimorphisms are stable under pullback, and if kernel pairs have coequalizers. Equivalently, any arrow $f : A \longrightarrow B$ has a unique factorization $f = ir$ (up to isomorphism), where r is a regular epimorphism and i is a monomorphism and this factorization is pullback stable.

A **relation** R from X to Y is a subobject $\langle r_1, r_2 \rangle : R \rightarrowtail X \times Y$. The opposite relation of R, denoted R^o, is the relation from Y to X given by the subobject $\langle r_2, r_1 \rangle : R \rightarrowtail Y \times X$. A relation R from X to X is called a relation on X. We shall identify a morphism $f : X \longrightarrow Y$ with the relation $\langle 1_X, f \rangle : X \rightarrowtail X \times Y$ and write f^o for its opposite relation. Given two relations $R \rightarrowtail X \times Y$ and $S \rightarrowtail Y \times Z$ in a regular category, we write $SR \rightarrowtail X \times Z$ for their relational composite. With the above notations, any relation $\langle r_1, r_2 \rangle : R \rightarrowtail X \times Y$ can be seen as the relational composite $r_2 r_1^o$. The properties collected in the following lemma are well known and easy to prove (see for instance [1]):

Lemma 2.1. *Let $f : X \longrightarrow Y$ be an arrow in a regular category $\mathbb{C}$, and let $f = ir$ be its (regular epimorphism, monomorphism) factorization. Then:*

1. *$f^o f$ is the kernel pair of f, thus $1_X \leqslant f^o f$; moreover, $1_X = f^o f$ if and only if f is a monomorphism;*

2. *ff^o is (i,i), thus $ff^o \leqslant 1_Y$; moreover, $ff^o = 1_Y$ if and only if f is a regular epimorphism;*

3. *$ff^of = f$ and $f^off^o = f^o$.*

A relation R on X is **reflexive** if $1_X \leqslant R$, **symmetric** if $R^o \leqslant R$, and **transitive** if $RR \leqslant R$. As usual, a relation R on X is an **equivalence relation** when it is reflexive, symmetric and transitive. In particular, a kernel pair $\langle f_1, f_2\rangle : \mathrm{Eq}(f) \rightarrowtail X \times X$ of a morphism $f : X \longrightarrow Y$ is an equivalence relation.

By dropping the assumption of uniqueness of the factorization in the definition of a limit, one obtains the definition of a weak limit. We call **weakly lex** any category with weak finite limits.

An object P in a category is (regular) **projective** if, for any arrow $f : P \longrightarrow X$ and for any regular epimorphism $g : Y \twoheadrightarrow X$ there exists an arrow $h : P \longrightarrow Y$ such that $gh = f$. We say that a full subcategory $\mathbb{C}$ of $\mathbb{A}$ is a **projective cover** of $\mathbb{A}$ if two conditions are satisfied:

- any object of $\mathbb{C}$ is regular projective in $\mathbb{A}$;
- for any object X in $\mathbb{A}$, there exists a ($\mathbb{C}$-)cover of X, that is an object C in $\mathbb{C}$ and a regular epimorphism $C \twoheadrightarrow X$.

When $\mathbb{A}$ admits a projective cover, one says that $\mathbb{A}$ has *enough projectives*.

Remark 2.2. If $\mathbb{C}$ is a projective cover of a weakly lex category $\mathbb{A}$, then $\mathbb{C}$ is also weakly lex [4]. For example, let X and Y be objects in $\mathbb{C}$ and $X \longleftarrow W \longrightarrow Y$ a weak product of X and Y in $\mathbb{A}$. Then, for any cover $\bar{W} \twoheadrightarrow W$ of W, $X \longleftarrow \bar{W} \longrightarrow Y$ is a weak product of X and Y in $\mathbb{C}$. Furthermore, if $\mathbb{A}$ is a regular category and $X \longleftarrow P \longrightarrow Y$ a weak product of X and Y in $\mathbb{C}$, then the induced morphism $P \twoheadrightarrow X \times Y$ is a regular epimorphism. Similar remarks apply to all weak finite limits.

3. Goursat categories

In this section we review the notion of Goursat category and the characterizations of Goursat categories through regular images of equivalence relations and through Goursat pushouts.

Definition 3.1. [2, 1] *A regular category $\mathbb{C}$ is called a* ***Goursat category*** *when the equivalence relations in $\mathbb{C}$ are 3-permutable, i.e. $RSR = SRS$ for any pair of equivalence relations R and S on the same object.*

When $\mathbb{C}$ is a regular category, (R, r_1, r_2) is a relation on X and $f : X \twoheadrightarrow Y$ is a regular epimorphism, we define the **regular image of** R **along** f to be the relation $f(R)$ on Y induced by the (regular epimorphism, monomorphism) factorization $\langle s_1, s_2\rangle\psi$ of the composite $(f \times f)\langle r_1, r_2\rangle$:

$$\begin{tikzcd}
R \arrow[r, two heads, dotted, "\psi"] \arrow[d, tail, "\langle r_1, r_2\rangle"'] & f(R) \arrow[d, tail, dotted, "\langle s_1, s_2\rangle"] \\
X \times X \arrow[r, two heads, "f\times f"'] & Y \times Y.
\end{tikzcd}$$

Note that the regular image $f(R)$ can be obtained as the relational composite $f(R) = fRf^o = fr_2r_1^of^o$. When R is an equivalence relation, $f(R)$ is also reflexive and symmetric. In a general regular category $f(R)$ is not necessarily an equivalence relation. This is the case in a *Goursat category* according to the following theorem.

Theorem 3.2. [1] *A regular category $\mathbb{C}$ is a Goursat category if and only if for any regular epimorphism $f : X \twoheadrightarrow Y$ and any equivalence relation R on X, the regular image $f(R) = fRf^o$ of R along f is an equivalence relation.*

If $\langle e_1, e_2\rangle : E \rightarrowtail X \times X$ is a reflexive relation, then the regular image of e_2 along the kernel pair of e_1 is given by $e_2(\mathrm{Eq}(e_1)) = e_2e_1^oe_1e_2^o = EE^o$. Goursat categories may also be characterized by such regular images:

Theorem 3.3. [1] *A regular category $\mathbb{C}$ is a Goursat category if and only if for any reflexive relation E, EE^o is an equivalence relation.*

Goursat categories are well known in Universal Algebra. In fact, by a classical theorem in [10], a variety of universal algebras is a Goursat category precisely when its theory has two quaternary operations p and q such that the identities $p(x, y, y, z) = x$, $q(x, y, y, z) = z$ and $p(x, x, y, y) = q(x, x, y, y)$ hold. Accordingly, the varieties of groups, Heyting algebras and implication algebras are Goursat categories. The category of topological groups, Hausdorff groups, right complemented semi-groups are also Goursat categories.

There are many known characterizations of Goursat categories (see [1, 7, 8, 9] for instance). In particular the following characterization, through Goursat pushouts, will be useful:

Theorem 3.4. [7] *Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:*

(i) *$\mathbb{C}$ is a Goursat category;*

(ii) *any commutative diagram of type* (I) *in $\mathbb{C}$, where α and β are regular epimorphisms and f and g are split epimorphisms*

$$\begin{tikzcd} X \arrow[r, two heads, "\alpha"] \arrow[d, shift left, "f"] \arrow[dr, phantom, "\text{(I)}"] & U \arrow[d, shift left, "g"] \\ Y \arrow[u, shift left, "s"] \arrow[r, two heads, "\beta"'] & W, \arrow[u, shift left, "t"] \end{tikzcd} \qquad \begin{array}{l} g\alpha=\beta f \\ \alpha s=t\beta \end{array}$$

*(which is necessarily a pushout) is a **Goursat pushout**: the morphism $\lambda : \mathrm{Eq}(f) \longrightarrow \mathrm{Eq}(g)$, induced by the universal property of kernel pair $\mathrm{Eq}(g)$ of g, is a regular epimorphism.*

Remark 3.5. Diagram (I) is a Goursat pushout precisely when the regular image of $\mathrm{Eq}(f)$ along α is (isomorphic to) $\mathrm{Eq}(g)$. From Theorem 3.4, it then follows that a regular category $\mathbb{C}$ is a Goursat category if and only if for any commutative diagram of type (I) one has $\alpha(\mathrm{Eq}(f)) = \mathrm{Eq}(g)$.

Note that Theorem 3.2 characterizes Goursat categories through the property that regular images of equivalence relations are equivalence relations, while Theorem 3.4 characterizes them through the property that regular images of certain kernel pairs are kernel pairs.

4. Projective covers of Goursat categories

In this section, we characterize the categories with weak finite limits whose regular completion are Goursat categories.

Definition 4.1. *Let $\mathbb{C}$ be a weakly lex category:*

1. *a* ***pseudo-relation*** *on an object X of $\mathbb{C}$ is a pair of parallel arrows* $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$; *a pseudo-relation is a relation if r_1 and r_2 are jointly monomorphic;*

2. *a pseudo-relation $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ on X is said to be:*

   - ***reflexive*** *when there is an arrow $r : X \longrightarrow R$ such that $r_1 r = 1_X = r_2 r$;*
   - ***symmetric*** *when there is an arrow $\sigma : R \longrightarrow R$ such that $r_2 = r_1\sigma$ and $r_1 = r_2\sigma$;*
   - ***transitive*** *if by considering a weak pullback*

$$\begin{tikzcd} W \arrow[r, "p_2"] \arrow[d, "p_1"'] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_2"'] & X, \end{tikzcd}$$

   there is an arrow $t : W \longrightarrow R$ such that $r_1 t = r_1 p_1$ and $r_2 t = r_2 p_2$.
   - *a* ***pseudo-equivalence relation*** *if it is reflexive, symmetric and transitive.*

Remark that the transitivity of a pseudo-relation $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ does not depend on the choice of the weak pullback of r_1 and r_2; in fact, if

$$\begin{tikzcd} \bar{W} \arrow[r, "\bar{p_2}"] \arrow[d, "\bar{p_1}"'] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_2"'] & X, \end{tikzcd}$$

is another weak pullback, the factorization $\bar{W} \longrightarrow W$ composed with the transitivity arrow $t : W \longrightarrow R$ ensures that the pseudo-relation is transitive also with respect to the second weak pullback.

The following property from [12] (Proposition 1.1.9) will be useful in the sequel:

Proposition 4.2. [12] *Let $\mathbb{C}$ be a projective cover of a regular category $\mathbb{A}$. Let $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X$ be a pseudo-relation in $\mathbb{C}$ and consider its (regular epimorphism, monomorphism) factorization in $\mathbb{A}$*

$$\begin{tikzcd} R \arrow[rr, "{(r_1,r_2)}"] \arrow[dr, twoheadrightarrow, "p"'] & & X \times X. \\ & E \arrow[ur, tail, "{(e_1,e_2)}"'] & \end{tikzcd}$$

Then, R is a pseudo-equivalence relation in $\mathbb{C}$ if and only if E is an equivalence relation in $\mathbb{A}$.

In order to characterize the projective covers $\mathbb{C}$ of Goursat categories $\mathbb{A}$, we should consider good properties characterizing Goursat categories which easily translate to the weakly lex context. A possible translation of the property in Theorem 3.2 should replace equivalence relations in $\mathbb{A}$ with pseudo-equivalence relations in $\mathbb{C}$ and regular epimorphisms in $\mathbb{A}$ with split epimorphisms in $\mathbb{C}$ (a regular epimorphism in $\mathbb{A}$ with a projective codomain is necessarily a split epimorphism). Thus, we introduce:

Definition 4.3. *Let $\mathbb{C}$ be a weakly lex category. We call $\mathbb{C}$ a **weak Goursat category** if, for any pseudo-equivalence relation $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X$ and any split epimorphism $X \overset{f}{\underset{}{\rightleftarrows}} Y$, the composite $R \overset{fr_1}{\underset{fr_2}{\rightrightarrows}} Y$ is also a pseudo-equivalence relation.*

Lemma 4.4. *If $\mathbb{C}$ is a regular weak Goursat category, then $\mathbb{C}$ is a Goursat category.*

Proof. We shall prove that for any reflexive relation $\langle e_1, e_2 \rangle : E \rightarrowtail X \times X$, EE^o is an equivalence relation (Theorem 3.3).

Consider the (pseudo-)equivalence relation $\mathrm{Eq}(e_1) \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} E$ and the split epimorphism e_2 (which is split by the reflexivity arrow). By assumption $\mathrm{Eq}(e_1) \overset{e_2\pi_1}{\underset{e_2\pi_2}{\rightrightarrows}} X$ is a pseudo-equivalence relation. Its (regular epimorphism, monomorphism) factorization defines the regular image $e_2(\mathrm{Eq}(e_1)) = EE^o$

$$\begin{tikzcd}
\mathrm{Eq}(e_1) \arrow[r, two heads] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & e_2(\mathrm{Eq}(e_1)) = EE^o \arrow[d, shift right] \arrow[d, shift left] \\
E & X; \arrow[l, dashed, shift right] \arrow[l, leftarrow, shift left, "e_2"']
\end{tikzcd}$$

thus EE^o is an equivalence relation. $\square$

We use Remark 2.2 repeatedly in the next results.

Proposition 4.5. *Let $\mathbb{C}$ be a projective cover of a regular category $\mathbb{A}$. Then $\mathbb{A}$ is a Goursat category if and only if $\mathbb{C}$ is a weak Goursat category.*

Proof. Since $\mathbb{C}$ is a projective cover of a regular category $\mathbb{A}$, $\mathbb{C}$ is weakly lex.

Suppose that $\mathbb{A}$ is a Goursat category. Let $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X$ be a pseudo-equivalence relation in $\mathbb{C}$ and let $X \underset{\leftarrow - -}{\overset{f}{\longrightarrow}} Y$ be a split epimorphism in $\mathbb{C}$. For the (regular epimorphism, monomorphism) factorizations of $\langle r_1, r_2 \rangle$ and $\langle fr_1, fr_2 \rangle$ we get the following diagram

$$\begin{tikzcd}
R \arrow[rr, "\langle r_1, r_2 \rangle"] \arrow[dr, two heads, "p"] \arrow[ddd, equal] & & X \times X \arrow[ddd, two heads, "f \times f"] \\
& E \arrow[ur, tail, "\langle e_1, e_2 \rangle"'] \arrow[d, dotted, "w"] & \\
& S \arrow[dr, tail, "\langle s_1, s_2 \rangle"] & \\
R \arrow[ur, two heads, "q"] \arrow[rr, "\langle fr_1, fr_2 \rangle"'] & & Y \times Y,
\end{tikzcd} \tag{1}$$

where $w : E \longrightarrow S$ is induced by the strong epimorphism p

$$\begin{tikzcd}
R \arrow[r, two heads, "p"] \arrow[d, two heads, "q"'] & E \arrow[d, "(f \times f)\langle e_1, e_2 \rangle"] \arrow[dl, dotted, "w"'] \\
S \arrow[r, tail, "\langle s_1, s_2 \rangle"'] & Y \times Y.
\end{tikzcd}$$

Then w is a regular epimorphism and by the commutativity of the right side of (1), one has $S = f(E)$. By Proposition 4.2, we know that E is an equivalence relation in $\mathbb{A}$. Since $\mathbb{A}$ is a Goursat category and f is a regular epimorphism (being a split one), then $S = f(E)$ is also an equivalence relation in $\mathbb{A}$ (Theorem 3.2) and by Proposition 4.2, we can conclude that $R \underset{fr_2}{\overset{fr_1}{\rightrightarrows}} X$ is a pseudo-equivalence relation in $\mathbb{C}$.

Conversely, suppose that $\mathbb{C}$ is a weak Goursat category. Let $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ be an equivalence relation in $\mathbb{A}$ and $f : X \twoheadrightarrow Y$ a regular epimorphism. We are going to show that $f(R) = S$

$$\begin{tikzcd}
R \arrow[r, two heads, "h"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & f(R) = S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
X \arrow[r, two heads, "f"'] & Y
\end{tikzcd}$$

is an equivalence relation; it is obviously reflexive and symmetric. In order to conclude that $\mathbb{A}$ is a Goursat category, we must prove that S is transitive.

We begin by covering the regular epimorphism f in $\mathbb{A}$ with a split epimorphism $\bar{f}$ in $\mathbb{C}$. For that we take the cover $y : \bar{Y} \twoheadrightarrow Y$, consider the pullback of y and f in $\mathbb{A}$ and take its cover $\alpha : \bar{X} \twoheadrightarrow X \times_Y \bar{Y}$

$$\begin{tikzcd}
\bar{X} \arrow[dr, two heads, "\alpha"] \arrow[drr, bend left, two heads, "\bar{f}=f'\alpha"] \arrow[ddr, bend right, two heads, "x=y'\alpha"'] & & \\
& X \times_Y \bar{Y} \arrow[r, two heads, "f'"] \arrow[d, two heads, "y'"'] & \bar{Y} \arrow[d, two heads, "y"] \arrow[ull, dashed, bend right] \\
& X \arrow[r, two heads, "f"'] & Y.
\end{tikzcd}$$

Since $\bar{f} = f'\alpha$ is a regular epimorphism in $\mathbb{A}$ with a projective codomain, it is a split epimorphism. Note that the above outer diagram is a *regular pushout*, so that

$$f^o y = x\bar{f}^o \quad \text{and} \quad y^o f = \bar{f} x^o \tag{2}$$

(Proposition 2.1 in [1]).

Next, we take the inverse image $x^{-1}(R)$ in $\mathbb{A}$, which is an equivalence relation since R is, and cover it to obtain a pseudo-equivalence $W \rightrightarrows \bar{X}$ in $\mathbb{C}$. By assumption $W \rightrightarrows \bar{X} \xrightarrow{\bar{f}} \bar{Y}$ is a pseudo-equivalence relation in $\mathbb{C}$ so it factors through an equivalence relation, say $V \underset{v_2}{\overset{v_1}{\rightrightarrows}} \bar{Y}$, in $\mathbb{A}$. We have

$$
\begin{tikzcd}
W \arrow[rrr, equal] \arrow[d, two heads, "w"'] & & & W \arrow[dr, two heads, "v"] \arrow[ddd] & & \\
x^{-1}(R) \arrow[rrrr, dotted, two heads, "\gamma"] \arrow[dr, two heads, "\pi_R"] \arrow[dd, tail, "\langle \rho_1,\rho_2 \rangle"'] & & & & V \arrow[dr, dotted, two heads, "\lambda"] \arrow[ddl, tail, "\langle v_1,v_2 \rangle"'] & \\
& R \arrow[rrrr, two heads, "h"] \arrow[dd, tail, "\langle r_1,r_2 \rangle"'] & & & & S \arrow[dd, tail, "\langle s_1,s_2 \rangle"] \\
\bar{X} \times \bar{X} \arrow[rrr, two heads, "\bar{f}\times\bar{f}"] \arrow[dr, two heads, "x\times x"'] & & & \bar{Y} \times \bar{Y} \arrow[drr, two heads, "y\times y"] & & \\
& X \times X \arrow[rrrr, two heads, "f\times f"'] & & & & Y \times Y,
\end{tikzcd}
$$

where γ and λ are induced by the strong epimorphisms w and v, respectively

$$
\begin{tikzcd}
W \arrow[r, two heads, "w"] \arrow[dd, two heads, "v"'] & x^{-1}(R) \arrow[d, tail, "\langle \rho_1,\rho_2 \rangle"] \arrow[ddl, dotted, "\gamma"'] \\
& \bar{X} \times \bar{X} \arrow[d, two heads, "\bar{f}\times\bar{f}"] \\
V \arrow[r, tail, "\langle v_1,v_2 \rangle"'] & \bar{Y} \times \bar{Y}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
W \arrow[r, two heads, "v"] \arrow[dd, two heads, "h\pi_R w"'] & V \arrow[d, tail, "\langle v_1,v_2 \rangle"] \arrow[ddl, dotted, "\lambda"'] \\
& \bar{Y} \times \bar{Y} \arrow[d, two heads, "y\times y"] \\
S \arrow[r, tail, "\langle s_1,s_2 \rangle"'] & Y \times Y.
\end{tikzcd}
$$

Since γ is a regular epimorphism, we have $V = \bar{f}(x^{-1}(R))$. Since λ is a regular epimorphism, we have $S = y(V)$. One also has $V = y^{-1}(S)$ because

$$
\begin{array}{rcll}
y^{-1}(S) & = & y^o S y & \\
& = & y^o f(R) y & \\
& = & y^o f R f^o y & \\
& = & \bar{f} x^o R x \bar{f}^o & \text{(by (2))} \\
& = & \bar{f}(x^{-1}(R)) & \\
& = & V. &
\end{array}
$$

Finally, S is transitive since

$$\begin{array}{rcll} SS & = & yy^oSyy^oSyy^o & \text{(Lemma 2.1(2))} \\ & = & yy^{-1}(S)y^{-1}(S)y^o & \\ & = & yVVy^o & \\ & = & yVy^o & \text{(since V is an equivalence relation)} \\ & = & y(V) & \\ & = & S. & \end{array}$$

□

We may also consider weak Goursat categories through a property which is more similar to the one mentioned in Theorem 3.2:

Lemma 4.6. *Let $\mathbb{C}$ be a projective cover of a regular category $\mathbb{A}$. Then $\mathbb{C}$ is a weak Goursat category if and only if for any commutative diagram in $\mathbb{C}$*

$$\begin{tikzcd} R \arrow[r, shift left, "\varphi"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[l, shift left, dashed] \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, shift right, "f"'] & Y \arrow[l, shift right, dashed] \end{tikzcd} \tag{3}$$

such that f and φ are split epimorphism and R is a pseudo-equivalence relation, S is a pseudo-equivalence relation.

Proof. $(i) \Rightarrow (ii)$ Since $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X$ is a pseudo-equivalence relation, by assumption $R \overset{fr_1}{\underset{fr_2}{\rightrightarrows}} X$ is also a pseudo-equivalence relation and then its (regular epimorphism, monomorphism) factorization gives an equivalence relation $E \overset{e_1}{\underset{e_2}{\rightrightarrows}} Y$ in $\mathbb{A}$ (Proposition 4.2). We have the following commutative diagram

$$\begin{tikzcd} R \arrow[r, equal] \arrow[dd, shift right, "r_1"'] \arrow[dd, shift left, "r_2"] & R \arrow[rr, shift left, "\varphi"] \arrow[dd, shift right, "fr_1"'] \arrow[dd, shift left, "fr_2"] \arrow[dr, "\rho"] & & S \arrow[ll, shift left, dashed] \arrow[dl, dotted, "\sigma"'] \arrow[ddll, bend left, shift right, "s_1"'] \arrow[ddll, bend left, shift left, "s_2"] \\ & & E \arrow[dl, shift right, "e_1"'] \arrow[dl, shift left, "e_2"] & \\ X \arrow[r, shift right, "f"'] & Y \arrow[l, shift right, dashed] & & \end{tikzcd}$$

where $\sigma : S \longrightarrow E$ is induced by the strong (split) epimorphism φ

$$\begin{tikzcd}
R \arrow[r, shift left, "\varphi"] \arrow[d, two heads, "\rho"'] & S \arrow[l, dashed, shift left] \arrow[d, "\langle s_1,s_2 \rangle"] \arrow[dl, dotted, "\sigma"'] \\
E \arrow[r, tail, "\langle e_1,e_2 \rangle"'] & Y \times Y.
\end{tikzcd}$$

Then σ is a regular epimorphism and $S \overset{s_1}{\underset{s_2}{\rightrightarrows}} Y$ is a pseudo-equivalence relation (Proposition 4.2).

$(ii) \Rightarrow (i)$ Let $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X$ be a pseudo-equivalence relation in $\mathbb{C}$ and $X \underset{\longleftarrow}{\overset{f}{\longrightarrow}} Y$ a split epimorphism. The following diagram is of the type (3)

$$\begin{tikzcd}
R \arrow[r, equal] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & R \arrow[d, shift right, "fr_1"'] \arrow[d, shift left, "fr_2"] \\
X \arrow[r, shift right, "f"'] & Y. \arrow[l, dashed, shift right]
\end{tikzcd}$$

Since $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X$ is a pseudo-equivalence relation, then by assumption $R \overset{fr_1}{\underset{fr_2}{\rightrightarrows}} Y$ is also a pseudo-equivalence relation. $\square$

Alternatively, weak Goursat categories may be characterized through a property more similar to the one mentioned in Remark 3.5. A diagram of type (I) in a weakly lex context should have the regular epimorphisms α and β replaced by split epimorphisms; we call it of type (II). Note that such a diagram does not necessarily commute with the splittings of α and β.

Proposition 4.7. *Let $\mathbb{C}$ be a projective cover of a regular category $\mathbb{A}$. The following conditions are equivalent:*

(i) $\mathbb{A}$ is a Goursat category;

(ii) $\mathbb{C}$ is a weak Goursat category;

(iii) For any commutative diagram of type (II) *in* $\mathbb{C}$

$$\begin{tikzcd}
F \arrow[r, shift left, "\lambda"] \arrow[d, shift right, "\beta_1"'] \arrow[d, shift left, "\beta_2"] & G \arrow[l, dashed, shift left] \arrow[d, shift right, "\rho_1"'] \arrow[d, shift left, "\rho_2"] \\
X \arrow[r, shift left, "\alpha"] \arrow[d, shift left, "f"] \arrow[dr, phantom, "\text{(II)}"] & U \arrow[l, dashed, shift left] \arrow[d, shift left, "g"] \\
Y \arrow[u, shift left, "s"] \arrow[r, shift right, "\beta"'] & W \arrow[l, dashed, shift right] \arrow[u, shift left, "t"]
\end{tikzcd}$$

where F is a weak kernel pair of f and λ is a split epimorphism, G is a weak kernel pair of g.

Proof. $(i) \Leftrightarrow (ii)$ By Proposition 4.5.

$(i) \Rightarrow (iii)$ If we take the kernel pairs of f and g, then the induced morphism $\bar{\alpha} : \mathrm{Eq}(f) \longrightarrow \mathrm{Eq}(g)$ is a regular epimorphism by Theorem 3.4. Moreover, the induced morphism $\varphi : F \longrightarrow \mathrm{Eq}(f)$ is also a regular epimorphism. We get

$$\begin{tikzcd}
F \arrow[r, shift left, "\lambda"] \arrow[d, two heads, "\varphi"] & G \arrow[l, dashed, shift left] \arrow[d, dotted, "\omega"] \arrow[dd, bend left=60, shift right, "\rho_1"'] \arrow[dd, bend left=60, shift left, "\rho_2"] \\
\mathrm{Eq}(f) \arrow[r, two heads, "\bar{\alpha}"] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "f_2"] & \mathrm{Eq}(g) \arrow[d, shift right, "g_1"'] \arrow[d, shift left, "g_2"] \\
X \arrow[r, shift left, "\alpha"] \arrow[d, shift left, "f"] & U \arrow[l, dashed, shift left] \arrow[d, shift left, "g"] \\
Y \arrow[u, shift left, "s"] \arrow[r, shift right, "\beta"'] & W, \arrow[l, dashed, shift right] \arrow[u, shift left, "t"]
\end{tikzcd}$$

where $w : G \longrightarrow \mathrm{Eq}(g)$ is induced by the strong (split) epimorphism λ

$$\begin{tikzcd}
F \arrow[r, shift left, "\lambda"] \arrow[d, two heads, "\bar{\alpha}.\varphi"'] & G \arrow[l, dashed, shift left] \arrow[dl, dotted, "w"'] \arrow[d, "\langle \rho_1, \rho_2 \rangle"] \\
\mathrm{Eq}(g) \arrow[r, tail, "\langle g_1, g_2 \rangle"'] & U \times U.
\end{tikzcd}$$

This implies that ω is a regular epimorphism and then $G \underset{\rho_2}{\overset{\rho_1}{\rightrightarrows}} U$ is a weak kernel pair of g.

$(iii) \Rightarrow (ii)$ Consider diagram (3) in $\mathbb{C}$ where $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ is a pseudo-equivalence relation. We want to prove that $S \underset{s_2}{\overset{s_1}{\rightrightarrows}} Y$ is also a pseudo-equivalence. Take the (regular epimorphism, monomorphism) factorization of R and S in $\mathbb{A}$ and the induced morphism μ making the following diagram commutative

$$
\begin{tikzcd}
R \arrow[rr, shift left, "\varphi"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \arrow[dr, two heads, "\rho"] & & S \arrow[ll, shift left, dashed] \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \arrow[dr, two heads, "\sigma"] & \\
& U \arrow[rr, dotted, two heads, "\mu"] \arrow[dl, shift right, "u_1"'] \arrow[dl, shift left, "u_2"] & & V \arrow[dl, shift right, "v_1"'] \arrow[dl, shift left, "v_2"] \\
X \arrow[rr, shift right, "f"'] & & Y. \arrow[ll, shift right, dashed] &
\end{tikzcd}
$$

Since μ is a regular epimorphism, $V = f(U)$ and consequently, V is reflexive and symmetric, as the regular image of the equivalence relation U (Theorem 3.2).

To conclude that S is a pseudo-equivalence relation, we just need to prove that V is transitive. We apply our assumption to the diagram

$$
\begin{tikzcd}
F \arrow[rr, shift left, two heads, "\lambda=\chi'\beta"] \arrow[dr, two heads, "\beta"] \arrow[dd, two heads, "\delta=\alpha'\beta"'] & & G \arrow[ll, shift left, dashed] \arrow[dd, two heads, "\alpha"] \\
& \mathrm{Eq}(r_1) \times_{\varphi(\mathrm{Eq}(r_1))} G \arrow[ur, two heads, "\chi'"] \arrow[dl, two heads, "\alpha'"] & \\
\mathrm{Eq}(r_1) \arrow[rr, two heads, "\chi"'] \arrow[d, shift right] \arrow[d, shift left] & & \varphi(\mathrm{Eq}(r_1)) \arrow[d, shift right] \arrow[d, shift left] \\
R \arrow[rr, shift left, two heads, "\varphi"] \arrow[d, shift left, "r_1"] \arrow[rr, phantom, shift right=3ex, "\text{(II)}"] & & S \arrow[ll, shift left, dashed] \arrow[d, shift left, "s_1"] \\
X \arrow[u, shift left, "e_R"] \arrow[rr, shift right, two heads, "f"'] & & Y \arrow[u, shift left, "e_S"] \arrow[ll, shift right, dashed]
\end{tikzcd}
$$

where G is a cover of the regular image $\varphi(\mathrm{Eq}(r_1))$ and F is a cover of the pullback $\mathrm{Eq}(r_1) \times_{\varphi(Eq(r_1))} G$. Note that $\lambda = \chi'\beta$ is a regular epimorphism in $\mathbb{A}$ with a projective codomain, so it is a split epimorphism. Since δ is a regular epimorphism, then $F \rightrightarrows R$ is a weak kernel pair of r_1. By assumption $G \rightrightarrows S$ is a weak kernel pair of s_1, thus $\varphi(\mathrm{Eq}(r_1)) = \mathrm{Eq}(s_1)$. We then have

$$
\begin{array}{rclc}
VV & = & v_2 v_1^o v_1 v_2^o & \text{(since } V \text{ is symmetric)} \\
& = & v_2 \sigma\sigma^o v_1^o v_1 \sigma\sigma^o v_2^o & \text{(Lemma 2.1(2))} \\
& = & s_2 s_1^o s_1 s_2^o & (v_i\sigma = s_i) \\
& = & s_2 \varphi r_1^o r_1 \varphi^o s_2^o & (\varphi(\mathrm{Eq}(r_1)) = \mathrm{Eq}(s_1)) \\
& = & f r_2 r_1^o r_1 r_2^o f^o & (s_i\varphi = f r_i\,) \\
& = & f u_2 \rho\rho^o u_1^o u_1 \rho\rho^o u_2^o f^o & (u_i\rho = r_i) \\
& = & f u_2 u_1^o u_1 u_2^o f^o & \text{(Lemma 2.1(2))} \\
& = & fUUf^o & \text{(since } U \text{ is an equivalence relation)} \\
& = & fUf^o & \text{(since } U \text{ is an equivalence relation)} \\
& = & V. & (\, f(U) = V)
\end{array}
$$

$\square$

Remark 4.8. When $\mathbb{A}$ is a 3-permutable variety and $\mathbb{C}$ its subcategory of free algebras, then the property stated in Proposition 4.7 (iii) is precisely what is needed to obtain the existence of the quaternary operations p and q which characterize 3-permutable varieties. Let X denote the free algebra on one element. Diagram (II) below belongs to $\mathbb{C}$

$$
\begin{tikzcd}
F \arrow[rr, equal] \arrow[d, two heads, "\mu"] & & F \arrow[d, two heads, "\lambda\mu"] \\
\mathrm{Eq}(\nabla_2 + \nabla_2) \arrow[rr, dotted, "\lambda"] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & & \mathrm{Eq}(\nabla_3) \arrow[d, shift right] \arrow[d, shift left] \\
4X \arrow[rr, shift left, "1_X+\nabla_2+1_X"] \arrow[d, shift left, two heads, "\nabla_2+\nabla_2"] \arrow[drr, phantom, "\text{(II)}"] & & 3X \arrow[ll, shift left, dashed] \arrow[d, shift left, two heads, "\nabla_3"] \\
2X \arrow[u, shift left, "\iota_2+\iota_1"] \arrow[rr, shift right, "\nabla_2"'] & & X. \arrow[ll, shift right, dashed] \arrow[u, shift left, "\iota_2"]
\end{tikzcd}
$$

If F is a cover of $\mathrm{Eq}(\nabla_2 + \nabla_2))$, then $F \rightrightarrows 4X$ is a weak kernel pair of $\nabla_2 + \nabla_2$. By assumption $F \rightrightarrows 3X$ is a weak kernel pair of ∇_3, so that $\lambda\mu$ is surjective. We then conclude that λ is surjective and the existence of the quaternary operations p and q follows from Theorem 3 in [7].

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

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Level ϵ

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Résumé. Lawvere a observé que certains 'gros' topos en géométrie algébrique suggèrent l'existence d'un 'niveau infinitésimal', étroitement lié aux algèbres locales de dimension finie. Motivés par cette observation, nous proposons une définition élémentaire de *level* ϵ associée à un morphisme géométrique local, établissons quelques propriétés de base pertinentes suggérées par l'intuition géométrique et donnons une description concrète du niveau ϵ déterminé par plusieurs morphismes géométriques pré-cohésifs.

Abstract. Lawvere has observed that certain 'gros' toposes in algebraic geometry suggest the existence of an 'infinitesimal level', closely related to finite-dimensional local algebras. Motivated by this observation we propose an elementary definition of *level* ϵ associated to a local geometric morphism, establish some relevant basic properties suggested by geometric intuition, and give concrete descriptions of the level ϵ determined by several pre-cohesive geometric morphisms.

Keywords. Axiomatic Cohesion, graphic toposes, algebraic geometry, SDG.

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1. Introduction

The formulation of Axiomatic Cohesion [14] and its development in the last ten years naturally invites to revisit the ideas and concrete problems outlined in [12]. We consider here a specific question in the dimension theory proposed in Section II of the latter reference:

> The infinitesimal spaces, which contain the base topos in its non-Becoming aspect, are a crucial step toward determinate

> Becoming, but fall short of having among themselves enough connected objects, i.e. they do not in themselves constitute fully a 'category of cohesive unifying Being.' In examples the four adjoint functors relating their topos to the base topos coalesce into two (by the theorem that a finite-dimensional local algebra has a unique section of its residue field) and the infinitesimal spaces may well negate the largest essential subtopos of the ambient one which has that property. This level may be called 'dimension ϵ'; calling the levels (i.e. the subtoposes essential over the base) 'dimensions' does not imply that they are linearly ordered nor that the Aufhebung process touches each of them. The infinitesimal spaces provide (in many ways) a good example of a non trivial unity-and-identity-of-opposites inside the ambient topos of Being: explicitly recognizing the *two* inclusions, as spaces which could be called infinitesimal and formal spaces respectively, may help clarify the confusing but powerful interplay between these two classes which are opposite but in themselves identical. The calculation of the ϵ-skeleton and ϵ-coskeleton, of a space which is neither, needs to be carried out, and also the calculation of the Aufhebung of dimension ϵ.

Our purpose is to confirm the suggestion that, in many examples of cohesion, there exists a "largest essential subtopos of the ambient one which has" the property that "the four adjoint functors to the base coalesce into two". In fact, we turn the suggestion into a rigorous definition of the level ϵ determined, if it exists, by a local geometric morphism $\mathcal{E} \to \mathcal{S}$. When it exists, it is an essential subtopos $\mathcal{E}_\epsilon \to \mathcal{E}$ with special properties. We prove that level ϵ exists in many examples and we give an explicit description. In particular, if $\mathcal{E}$ is the Zariski topos determined by the field of complex numbers, the site for $\mathcal{E}_\epsilon$ will be shown to be closely related to local algebras, as suggested by the quotation above.

Although some of the theory is developed in more generality, the typical topos of Being that we have in mind is the domain of a pre-cohesive geometric morphism as defined, for example, in [16]. We recall most of the definitions but the reader is assumed to be familiar with the ideas therein.

In Section 2 we recall in more detail the basics of the dimension theory mentioned above. One way to start is to fix a local geometric morphisms

$p : \mathcal{E} \to \mathcal{S}$ (typically with extra properties), consider its centre $\mathcal{S} \to \mathcal{E}$ as 'level 0' and study the levels above it. In Section 3 we analyse the subtoposes above the centre of a local presheaf topos. The level ϵ determined by a local geometric morphism is defined in Section 4. In the remaining sections we calculate the level ϵ of several examples. In Section 5 we analyse the level ϵ of local presheaf toposes in general and in some simple cases. In Section 6 we show that the Weil topos (determined by the field $\mathbb{C}$ of complex numbers) underlies a subquality of the Gaeta topos determined by the same field. This is shown to be level ϵ in Section 7.

Remark 1.1. Since Gaeta toposes are perhaps not yet widely known, we include here a brief description. If $\mathcal{D}$ is a small extensive category then the finite families $(D_i \to D \mid i \in I)$ of maps in $\mathcal{D}$ such that the induced $\sum_{i\in I} D_i \to D$ is an isomorphism form the basis of a Grothendieck topology. The associated topos $G\mathcal{D}$ of sheaves is called the *Gaeta topos* (of $\mathcal{D}$) and it is equivalent to the category of finite-product preserving functors $\mathcal{D}^{\mathrm{op}} \to \mathbf{Set}$. The 'Gaeta topology' is subcanonical so that the Yoneda embedding of $\mathcal{D}$ into the topos of presheaves factors through the Gaeta topos but, moreover, the factorization $\mathcal{D} \to G\mathcal{D}$ preserves finite coproducts. See, for example, the end of page 3 in [12] or Section 2 in [15]. If T is an algebraic theory whose category $\mathcal{A}$ of finitely presented algebras is coextensive then we may naturally refer to $G(\mathcal{A}^{\mathrm{op}})$ as the Gaeta topos determined by T. For instance, we have the Gaeta toposes determined rigs, by distributive lattices or by k-algebras, where k is a ring. We may even push the terminology further and simply speak (as we have done above) of the Gaeta topos determined by $\mathbb{C}$, instead of the Gaeta topos determined by the theory of $\mathbb{C}$-algebras.

In Section 8 we show that the Zariski topos determined by $\mathbb{C}$ (which is not a presheaf topos) also has a level ϵ and that it coincides with that of the Gaeta topos.

2. Levels and dimensions

In this section we recall some of the material in Section II of [12] which proposes to consider essential subtoposes of a given topos of spaces (or a *category of Being*) as a refined notion of 'dimensions' in that topos. The quotations in this section are taken from that reference. First notice that reflective subcategories of a fixed category $\mathcal{X}$ may be partially ordered as follows.

Lemma 2.1. *If the adjunctions $F \dashv U : \mathcal{A} \to \mathcal{X}$ and $L \dashv R : \mathcal{C} \to \mathcal{X}$ are such that $U : \mathcal{A} \to \mathcal{X}$ and $R : \mathcal{C} \to \mathcal{X}$ are full and faithful then the following conditions are equivalent:*

(i) *There is an adjunction $H \dashv K : \mathcal{A} \to \mathcal{C}$ with K full and faithful such that $RK \simeq U$ (or, equivalently, $HL \simeq F$).*

(ii) *$U : \mathcal{A} \to \mathcal{X}$ factors (up to iso) through $R : \mathcal{C} \to \mathcal{X}$.*

(iii) *$F : \mathcal{X} \to \mathcal{A}$ factors (up to iso) through $L : \mathcal{X} \to \mathcal{C}$.*

(iv) *$\nu_U : U \to RLU$ is invertible, where $\nu : 1_{\mathcal{X}} \to RL$ is the unit of $L \dashv R$.*

(v) *$F\nu : F \to FRL$ is invertible.*

Proof. Clearly (i) implies (iii), and (iii) trivially implies (v) since R is full and faithful.

Assume (v) and let $\xi : LR \to 1_{\mathcal{C}}$ be the counit of $L \dashv R$. Define $H = FR$ and $K = LU$; we show that $H \dashv K$ by showing that the 2-cells

$$
\begin{tikzcd}
\mathcal{C} \arrow[rrrr, "1_{\mathcal{C}}"] \arrow[dr, "R"'] & & \arrow[d, phantom, "\xi^{-1}\Downarrow"] & & \mathcal{C} \\
& \mathcal{X} \arrow[rr, "1_{\mathcal{X}}"] \arrow[dr, "F"'] & \arrow[d, phantom, "\eta\Downarrow"] & \mathcal{X} \arrow[ur, "L"'] & \\
& & \mathcal{A} \arrow[ur, "U"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \mathcal{C} \arrow[dr, "R"] & & \\
& \mathcal{X} \arrow[ur, "L"] \arrow[rrrd, "F"] \arrow[r, phantom, "(F\nu)^{-1}\Downarrow"] \arrow[d, phantom, "\varepsilon\Downarrow"] & & \mathcal{X} \arrow[dr, "F"] & \\
\mathcal{A} \arrow[ur, "U"] \arrow[rrrr, "1_{\mathcal{A}}"'] & & & & \mathcal{A}.
\end{tikzcd}
$$

satisfy the triangular identities, where η and ε are the unit and counit of $F \dashv U$. One of these triangular identities is trivial. The other is

$$
\begin{tikzcd}
& & \mathcal{C} \arrow[rrrr, "1_{\mathcal{C}}"] \arrow[dr, "R"] & & \arrow[d, phantom, "\xi^{-1}\Downarrow"] & & \mathcal{C} \\
& \mathcal{X} \arrow[ur, "L"] \arrow[rrrd, "F"] \arrow[r, phantom, "(F\nu)^{-1}\Downarrow"] \arrow[d, phantom, "\varepsilon\Downarrow"] & & \mathcal{X} \arrow[rr, "1_{\mathcal{X}}"] \arrow[dr, "F"'] & \arrow[d, phantom, "\eta\Downarrow"] & \mathcal{X} \arrow[ur, "L"'] & \\
\mathcal{A} \arrow[ur, "U"] \arrow[rrrr, "1_{\mathcal{A}}"'] & & & & \mathcal{A} \arrow[ur, "U"'] & &
\end{tikzcd}
= 1_{LU}.
$$

This equation is equivalent to

$$\begin{tikzcd}[column sep=small]
 & & \mathcal{C} \arrow[rrrr, "1_{\mathcal{C}}"] \arrow[dr, "R"'] & & & & \mathcal{C} \\
 & \mathcal{X} \arrow[ur, "L"] & & \mathcal{X} \arrow[rr, "1_{\mathcal{X}}"] \arrow[dr, "F"'] \arrow[ur, phantom, "\xi^{-1}\Downarrow"] & & \mathcal{X} \arrow[ur, "L"'] & \\
 \mathcal{A} \arrow[ur, "U"] & & & & \mathcal{A} \arrow[ur, "U"'] \arrow[u, phantom, "\eta\Downarrow"] & &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 \mathcal{A} \arrow[rrrr, "1_{\mathcal{A}}"] \arrow[dr, "U"'] & & & & \mathcal{A} \arrow[dr, "U"] & \\
 & \mathcal{X} \arrow[rr, "1_{\mathcal{X}}"] \arrow[dr, "L"'] \arrow[urr, phantom, "\varepsilon^{-1}\Downarrow"] & & \mathcal{X} \arrow[ur, "F"'] & & \mathcal{X} \arrow[dr, "L"] & \\
 & & \mathcal{C} \arrow[ur, "R"'] \arrow[u, phantom, "\nu\Downarrow"] & & & & \mathcal{C}.
\end{tikzcd}$$

To see that this equation holds, replace on the right $U\varepsilon^{-1}$ by η_U, and then replace $L\nu$ by ξ_L^{-1}. Observe furthermore, that the counit of $H \dashv K$ is invertible, thus K is fully faithful. We conclude that (v) implies (i).

The proof that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i) is very similar. $\square$

If the equivalent conditions of Lemma 2.1 hold then we may say that the reflective subcategory $L \dashv R$ is *above* $F \dashv U$.

Remark 2.2. In the situation of Lemma 2.1, we may as well assume (as we do in what follows) that the adjunction $H \dashv K : \mathcal{A} \to \mathcal{C}$ is given by $FR \dashv LU$ with unit and counit given by

$$1_{\mathcal{C}} \xrightarrow{\xi^{-1}} LR \xrightarrow{\quad L\eta_R \quad} LUFR \quad \text{and} \quad FRLU \xrightarrow{\quad (F\nu_U)^{-1} \quad} FU \xrightarrow{\ \varepsilon\ } 1_{\mathcal{A}}$$

respectively. Observe as well that, if F preserves finite limits, then H also preserves them; and if F has a left adjoint, then H also has a left adjoint.

From now on we restrict attention to the case where the ambient category $\mathcal{X}$ is a fixed topos $\mathcal{E}$. In this case Lemma 2.1 and Remark 2.2 imply the following.

Corollary 2.3. *Given subtoposes $j : \mathcal{E}_j \to \mathcal{E}$ and $k : \mathcal{E}_k \to \mathcal{E}$ the following conditions are equivalent.*

(i) *$\mathcal{E}_k$ is above $\mathcal{E}_j$.*

(ii) *$j_* : \mathcal{E}_j \to \mathcal{E}$ factors through $k_* : \mathcal{E}_k \to \mathcal{E}$.*

(iii) *$j^* : \mathcal{E} \to \mathcal{E}_j$ factors through $k^* : \mathcal{E} \to \mathcal{E}_k$.*

(iv) *The natural transformation $\nu_{j_*} : j_* \to k_* k^* j_*$ is an isomorphism.*

(v) *The natural transformation $j^*\nu : j^* \to j^*k_*k^*$ is an isomorphism.*

When this is the case, we can take as witness of the fact that $\mathcal{E}_k$ is above $\mathcal{E}_j$ the geometric morphism $h : \mathcal{E}_j \to \mathcal{E}_k$ such that $h^* = j^*k_*$, $h_* = k^*j_*$ with unit and counit given by

$$1_k \xrightarrow{\xi'^{-1}} k^*k_* \xrightarrow{k^*\nu_{k_*}} k^*j_*j^*k_* \text{ and } j^*k_*k^*j_* \xrightarrow{(j^*\nu'_{j_*})^{-1}} j^*j_* \xrightarrow{\xi} 1_j$$

respectively, where ν, ξ are the unit and counit of $j : \mathcal{E}_j \to \mathcal{E}$ and ν', ξ' are the corresponding ones for $k : \mathcal{E}_k \to \mathcal{E}$.

A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is called *essential* if the inverse image f^* has a left adjoint $f_! : \mathcal{F} \to \mathcal{E}$. Following [12], essential subtoposes of $\mathcal{E}$ will be called *levels*. Notice that for any given level $l : \mathcal{E}_l \to \mathcal{E}$, the leftmost adjoint $l_!$ is full and faithful (because the direct image l_* is). Levels may be partially ordered according to their underlying subtoposes as in Corollary 2.3.

> The basic idea is simply to identify dimensions with levels and then try to determine what the general dimensions are in particular examples. More precisely, a space may be said to have (less than or equal to) the dimension grasped by a given level if it belongs to the negative (left adjoint inclusion) incarnation of that level.

So, for any level $l : \mathcal{E}_l \to \mathcal{E}$ and any X in $\mathcal{E}$, the counit $l_!(l^*X) \to X$ may be called the *l-skeleton* of X. The object X is said to be *l-skeletal* if its l-skeleton is an iso, so that $l_! : \mathcal{E}_l \to \mathcal{E}$ is the full subcategory of l-skeletal objects. On the other hand, and in accordance with standard terminology, the objects in the full subcategory $l_* : \mathcal{E}_l \to \mathcal{E}$ will be called *l-sheaves*.

A subtopos $\mathcal{E}_j \to \mathcal{E}$ is *way-above* a level $l : \mathcal{E}_l \to \mathcal{E}$ if j is above l and, moreover, $l_! : \mathcal{E}_l \to \mathcal{E}$ factors through $j_* : \mathcal{E}_j \to \mathcal{E}$. The *Aufhebung* of level l is (when it exists) the smallest level of $\mathcal{E}$ that is way-above l.

The Aufhebung of a level need not be easy to calculate. For an illustration of the complexity of the issue see [11], [7], [20], [13] and [8].

Recall that a geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is *local* if $p_* : \mathcal{E} \to \mathcal{S}$ has a fully faithful right adjoint (usually denoted by $p^!$). For such a p, the subtopos

$p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$ is a level called the *centre* of p and it is sometimes convenient to think of it as the smallest non-trivial level of $\mathcal{E}$ (or 'dimension 0'), especially, if p has further properties:

> Within the class of all levels over the base (of course it is a set in fact if the category of Being is a topos), the base itself is often further distinguished by having a still further left adjoint to its discrete inclusion, this extra functor therefore assigning to every space in Being its set of components.

So let us fix a local and essential geometric morphism $p : \mathcal{E} \to \mathcal{S}$. Recall that *essential* means that the fully faithful $p^* : \mathcal{S} \to \mathcal{E}$ has a further left adjoint $p_! : \mathcal{E} \to \mathcal{S}$. As quoted above, this left adjoint is thought of as assigning, to each space (i.e. an object in $\mathcal{E}$), its associated set (i.e. object in $\mathcal{S}$) of pieces or connected components. To aid the intuitive discussion, the centre of p will be called *level 0* (of p) and its Aufhebung will be called *level 1* (of p).

> Because of the special feature of dimension zero of having a components functor to it (usually there is no analogue of that functor in higher dimensions), the definition of dimension one is equivalent to the quite plausible condition: the smallest dimension such that the set of components of an arbitrary space is the same as the set of of components of the skeleton at that dimension of the space, or more pictorially: if two points of any space can be connected by anything, then they can be connected by a curve. Here of course by "curve" we mean any figure in (i.e. map to) the given space whose domain is one-dimensional.

See Proposition 17 in [11] and Proposition in p. 19 of [20]. We give a different proof:

Proposition 2.4. *For any level l above level 0, l is way-above 0 if and only if, for every X in $\mathcal{E}$, $p_!(l_!(l^*X)) \to p_!X$ is an isomorphism (where $l_!(l^*X) \to X$ is the l-skeleton of X).*

Proof. Apply Lemma 2.1 to $l^* \dashv l_* : \mathcal{E}_l \to \mathcal{E}$ and $p_! \dashv p^* : \mathcal{S} \to \mathcal{E}$. □

Notice that this result does not assume that level 1 exists. The levels way-above level 0 may be considered as the levels above level 1 even if the latter does no exist.

Corollary 2.5. *Assume that level 1 exists. If a level l is above level 0 then, l is above 1 if and only if $p_!(l_!(l^*X)) \to p_!X$ is an isomorphism for every X in $\mathcal{E}$.*

We also recall a general description of levels in presheaf toposes.

Definition 2.6. An *ideal* of a small category $\mathcal{C}$ is a class of maps $\mathcal{I}$ in $\mathcal{C}$ that satisfies the following two conditions:

1. (Right ideal) For every $g : D \to C$ in $\mathcal{I}$ and $h : E \to D$ in $\mathcal{C}$, $gh \in \mathcal{I}$.
2. (Left ideal) For every $f : C \to B$ in $\mathcal{C}$ and $g : D \to C$ in $\mathcal{I}$, $fg \in \mathcal{I}$.

An ideal is called *idempotent* if for every $f \in \mathcal{I}$ there are $g, h \in \mathcal{I}$ such that $f = gh$.

Theorem 4.4 in [7] shows that levels of the presheaf topos $\widehat{\mathcal{C}}$ are in bijective correspondence with idempotent ideals of $\mathcal{C}$. If $\mathcal{I}$ is such an idempotent ideal then the associated Grothendieck topology J is such that, for each C in $\mathcal{C}$, a sieve S on C is J-covering if and only if it contains all the maps in $\mathcal{I}$ with codomain C.

3. Subtoposes above the centre of a local map

Let $p : \mathcal{E} \to \mathcal{S}$ be a local geometric morphism.

Proposition 3.1. *Let $j : \mathcal{E}_j \to \mathcal{E}$ be a subtopos and assume that the following diagram*

$$\begin{tikzcd} \mathcal{E}_j \arrow[r, "j"] \arrow[dr, "f"'] & \mathcal{E} \arrow[d, "p"] \\ & \mathcal{S} \end{tikzcd}$$

commutes so that $p_*j_* = f_* : \mathcal{E}_j \to \mathcal{S}$ and $j^*p^* = f^* : \mathcal{S} \to \mathcal{E}_j$. Then the following are equivalent, where ν is the unit of $j : \mathcal{E}_j \to \mathcal{E}$:

1. *The subtopos $j : \mathcal{E}_j \to \mathcal{E}$ is above the centre of p.*
2. *$p^! : \mathcal{S} \to \mathcal{E}$ factors through $j_* : \mathcal{E}_j \to \mathcal{E}$.*
3. *$p_* : \mathcal{E} \to \mathcal{S}$ factors through $j^* : \mathcal{E} \to \mathcal{E}_j$.*
4. *The natural transformation $\nu_{p^!} : p^! \to j_* j^* p^!$ is an isomorphism.*
5. *The natural transformation $p_*\nu : p_* \to p_* j_* j^*$ is an isomorphism.*
6. *The geometric morphism f is local and $j_* f^! \simeq p^!$.*

In this case, we may assume that $f^! = j^* p^!$ and that the unit $\overline{\eta}$ and counit $\overline{\varepsilon}$ of $f_* \dashv f^!$ are given by

$$1_{\mathcal{E}_j} \xrightarrow{\xi^{-1}} j^* j_* \xrightarrow{j^*\eta_{j_*}} j^* p^! p_* j_* = f^! f_*$$

$$f_* f^! = p_* j_* j^* p^! \xrightarrow{(p_*\nu_{p^!})^{-1}} p_* p^! \xrightarrow{\varepsilon} 1_{\mathcal{S}}$$

respectively, where η and ε are the unit and counit of $p_* \dashv p^!$ respectively, and ξ is the (iso) counit of $j^* \dashv j_*$.

Proof. Corollary 2.3 tells us that items 1 to 5 are equivalent; furthermore, Remark 2.2 tells us that we can take unit and counit of $f_* = p_* j_* \dashv j^* p^!$ as given in the statement of the proposition. So that f is local. Observe that $j_* f^! = j_* j^* p^! \simeq p^!$ via the iso $\nu_{p^!}$. So any one of the first five conditions implies 6. Finally, almost immediately, 6 implies 2. □

So subtoposes above the centre determine local maps towards the base. Moreover, if p is essential then so is f. Indeed, $f_! = p_! j_*$ and the composites

$$1_{\mathcal{E}_j} \xrightarrow{\xi^{-1}} j^* j_* \xrightarrow{j^*\sigma_{j_*}} j^* p^* p_! j_* = f^* f_!$$

$$f_! f^* = p_! j_* j^* p^* \xrightarrow{(p_!\nu_{p^*})^{-1}} p_! p^* \xrightarrow{\tau} 1_{\mathcal{S}}$$

are the unit and counit of $f_! \dashv f^*$, where σ and τ are the unit and counit of $p_! \dashv p^*$.

For instance, consider a small category $\mathcal{C}$ with a terminal object so that the canonical geometric morphism $\widehat{\mathcal{C}} \to \mathbf{Set}$ is local. See C3.6.3(b) in [5].

Lemma 3.2. *Let $\mathcal{D} \to \mathcal{C}$ be a full subcategory such that idempotents split in $\mathcal{D}$. Then, the induced essential subtopos $\widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ is above the centre of p if and only if the subcategory $\mathcal{D} \to \mathcal{C}$ contains the terminal object.*

Proof. If the subcategory $\mathcal{D} \to \mathcal{C}$ contains the terminal object then $\widehat{\mathcal{D}} \to \mathbf{Set}$ is local by C3.6.3(b) in [5] and it is straightforward to check that there is a natural iso as in item 6 of Proposition 3.1, so $\widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ is above the centre of p.

Conversely, assume that $\widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ is above the centre of p. Then $\widehat{\mathcal{D}} \to \mathbf{Set}$ is local by item 6 of Proposition 3.1 so, as idempotents split by hypothesis, $\mathcal{D}$ must have a terminal object $1_{\mathcal{D}}$ by C3.6.3(b) in [5]. So it remains to show that the inclusion $\mathcal{D} \to \mathcal{C}$ preserves the terminal object. To do this let $1_{\mathcal{C}}$ be the terminal object of $\mathcal{C}$ and notice that, by item 3 of Proposition 3.1, it must be the case that, for every X in $\widehat{\mathcal{C}}$, $X1_{\mathcal{C}} \cong X1_{\mathcal{D}}$. Taking $X = \mathcal{C}(\_, 1_{\mathcal{D}})$ we may conclude that $1_{\mathcal{D}}$ has a point and, as $\mathcal{D} \to \mathcal{C}$ is fully faithful, it must be the case that the composite $1_{\mathcal{D}} \to 1_{\mathcal{C}} \to 1_{\mathcal{D}}$ is the identity on $1_{\mathcal{D}}$. So the point $1_{\mathcal{C}} \to 1_{\mathcal{D}}$ is an iso. $\square$

On the other hand, it is not the case that subtoposes that induce local maps are above the centre in general. For example, let $\mathcal{C} = \{0 < \frac{1}{2} < 1\}$ be the total order with three elements and consider the full subcategory $\mathcal{D} = \{0 < \frac{1}{2}\} \to \mathcal{C}$ then the following diagram

$$\begin{tikzcd} \widehat{\mathcal{D}} \arrow[r] \arrow[dr] & \widehat{\mathcal{C}} \arrow[d] \\ & \mathbf{Set} \end{tikzcd}$$

commutes and both morphisms to $\mathbf{Set}$ are local, but the subtopos $\widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ is not above the center of p; as one may show, for example, by checking that $p_* : \widehat{\mathcal{C}} \to \mathbf{Set}$ does not invert the unit of the subtopos.

Assume from now on that every object of $\mathcal{C}$ has a point so that the canonical $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive. Let J be a Grothendieck topology on $\mathcal{C}$ and let $\mathrm{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$ be the associated subtopos.

Lemma 3.3. *The following are equivalent:*

1. *The subtopos $\mathrm{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$ is above the centre of p.*

2. *For every C in $\mathcal{C}$ and $S \in JC$, S contains all points of C.*

3. *The maximal sieve is the only J-cover of* 1.

Proof. By Corollary 4.5 in [16], the centre of $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ coincides with the subtopos of sheaves for the double negation topology. In other words, $p^! : \mathbf{Set} \to \widehat{\mathcal{C}}$ coincides with the subtopos $\mathrm{Sh}(\mathcal{C}, K) \to \widehat{\mathcal{C}}$ where a sieve on C is K-covering if and only if it contains all points of C. So $\mathrm{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$ is above the centre of p if and only if, for every C in $\mathcal{C}$, $JC \subseteq KC$. In other words, the first two items are equivalent. The second item trivially implies the third. The third item easily implies the second. $\square$

4. Subqualities and level ϵ

Recall that if we denote the counit of $p_* \dashv p^!$ by ε, and the unit and counit of $p^* \dashv p_*$ by α and β then the following diagram commutes

$$\begin{tikzcd} p^* \arrow[r, "\eta_{p^*}"] \arrow[d, "p^*\varepsilon^{-1}"'] & p^! p_* p^* \arrow[d, "p^!\alpha^{-1}"] \\ p^* p_* p^! \arrow[r, "\beta_{p^!}"'] & p^! \end{tikzcd}$$

and the composite is denoted by $\phi : p^* \to p^!$. Following item (c) in Definition 2 of [14] we could say that the *Nullstellensatz* holds (for p) if $\phi : p^* \to p^!$ is monic. Recall that the Nullstellensatz holds if and only if p is hyperconnected. See [6] for explicit proofs of the equivalences between different formulations of the Nullstellensatz.

Lemma 4.1. *If the subtopos $j : \mathcal{E}_j \to \mathcal{E}$ is above the centre of $p : \mathcal{E} \to \mathcal{S}$ and $f = pj : \mathcal{E}_j \to \mathcal{S}$ denotes the composite local geometric morphism then, $\overline{\phi} : f^* \to f^!$ and $j^*\phi : j^*p^* \to j^*p^!$ are equal. Therefore, if p is hyperconnected then so is f.*

Proof. Observe that in the commutative diagram below

$$\begin{tikzcd}
j^*p^* \arrow[r, "j^*p^*\varepsilon^{-1}"] \arrow[dr, "j^*\phi"'] & j^*p^*p_*p^! \arrow[r, "j^*p^*p_*\nu_{p^!}"] \arrow[d, "j^*\beta_{p^!}"] & j^*p^*p_*j_*j^*p^! \arrow[d, "j^*\beta_{j_*j^*p^!}"] \\
 & j^*p^! \arrow[r, "j^*\nu_{p^!}"] \arrow[dr, "id"'] & j^*j_*j^*p^! \arrow[d, "\xi_{j^*p^!}"] \\
 & & j^*p^!
\end{tikzcd}$$

the top composite equals $f^*\overline{\varepsilon}^{-1}$ by Proposition 3.1, while the composite on the right, as in Section 6.1 of [18], is $\overline{\beta}_{f^!}$, so the top-right composite is $\overline{\phi} : f^* \to f^!$. □

It is convenient to slightly extend the terminology in [14] and say that a local geometric morphism $q : \mathcal{Q} \to \mathcal{S}$ is a *quality type* if the canonical transformation $\phi : q^* \to q^!$ is an isomorphism. Such special adjunctions are also called *quintessential localizations* in [4]. Notice that, trivially, quality types satisfy the Nullstellensatz, so they are hyperconnected.

Fix a local geometric morphism $p : \mathcal{E} \to \mathcal{S}$ and call its centre level 0 as in Section 2.

Definition 4.2. A *subquality* of $p : \mathcal{E} \to \mathcal{S}$ is a subtopos $j : \mathcal{E}_j \to \mathcal{E}$ above level 0 and such that the composite $pj : \mathcal{E}_j \to \mathcal{S}$ is a quality type.

Compare with the notion of *quality* introduced in [14]. Roughly speaking, while quality is a (special kind of) functor to a quality type, a subquality is a (special kind of) functor from a quality type.

Lemma 4.3. *Let $\mathcal{E}_j \to \mathcal{E}$ be a subtopos above level 0. Then, $\mathcal{E}_j \to \mathcal{E}$ is a subquality of p if and only if $j^*\phi : j^*p^* \to j^*p^!$ is an isomorphism. Hence, if p satisfies the Nullstellensatz then, $\mathcal{E}_j \to \mathcal{E}$ is a subquality of p if and only if $\phi_A : p^*A \to p^!A$ is j-dense for every A in $\mathcal{S}$.*

Proof. Follows from Lemma 4.1. □

A subquality of p is said to be *essential* if it is so as a subtopos of $\mathcal{E}$. In this case, the subquality is a level of $\mathcal{E}$ and it is above the centre of $\mathcal{E}$.

Definition 4.4. In case it exists, *level* ϵ (of p) is the largest essential subquality of p.

Intuitively, level ϵ is an 'infinitesimal' dimension so it should not be above level 1. More generally, essential subqualities should not be way-above level 0. In the context of Proposition 2.4 we can make this precise as follows. Let Ω be the subobject classifier of $\mathcal{E}$ and recall (Proposition 3 in [14]) that if p is a quality type and $p_!\Omega = 1$ then $\mathcal{S}$ is degenerate. Intuitively, the condition $p_!\Omega = 1$ is a positive way of saying that p is not a quality type.

Proposition 4.5. *Let the local $p : \mathcal{E} \to \mathcal{S}$ be essential and hyperconnected. Let $l : \mathcal{E}_l \to \mathcal{E}$ be an essential subquality of p. If $p_!\Omega = 1$ and l is way-above 0 then $\mathcal{S}$ is degenerate.*

Proof. Let $\rho : l_!(l^*\Omega) \to \Omega$ be the l-skeleton of Ω. As l is way-above level 0 by hypothesis, Proposition 2.4 implies that $p_!\rho : p_!(l_!(l^*\Omega)) \to p_!\Omega = 1$ is an isomorphism. As $pl : \mathcal{E}_l \to \mathcal{S}$ is a quality type, $p_!(l_!(l^*\Omega)) \cong p_*(l_*(l^*\Omega))$. So $p_*(l_*(l^*\Omega)) = 1$.

Let Ω_l be the subobject classifier in $\mathcal{E}_l$. It is well-known that $l_*\Omega_l$ is a retract of Ω in $\mathcal{E}$. So $\Omega_l \cong l^*(l_*\Omega_l)$ is a retract of $l^*\Omega$, and then $p_*(l_*\Omega_l)$ is a retract of $p_*(l_*(l^*\Omega)) = 1$. That is, $p_*(l_*\Omega_l) = 1$. As $pl : \mathcal{E}_l \to \mathcal{S}$ is hyperconnected, $p_*l_* : \mathcal{E}_l \to \mathcal{S}$ preserves the subobject classifier. Altogether, the subobject classifier of $\mathcal{S}$ is terminal. □

Corollary 4.6. *Let $p : \mathcal{E} \to \mathcal{S}$ be essential and hyperconnected. Assume that $p_!\Omega = 1$ and that level ϵ of p exists. If ϵ is way-above 0 then $\mathcal{E}$ is degenerate.*

5. Level ϵ in presheaf toposes

Consider a small category $\mathcal{C}$ with terminal object so that the canonical geometric morphism $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is local. Without loss of generality we may assume that idempotents split in $\mathcal{C}$.

Corollary 5.1. *If $\mathcal{D} \to \mathcal{C}$ is a full subcategory closed under splitting of idempotents then, the essential subtopos $\widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ is a subquality of p if and only if $\mathcal{D} \to \mathcal{C}$ contains the terminal object and every object of $\mathcal{D}$ has a unique point.*

Proof. It follows from Lemma 3.2 above and Proposition 4.5 in [19] which implies that the restriction $\widehat{\mathcal{D}} \to \mathbf{Set}$ is a quality type if and only if every object in $\mathcal{D}$ has a unique point. $\square$

Let $\mathcal{C}_! \to \mathcal{C}$ be the full subcategory of all objects in $\mathcal{C}$ that have exactly one point. For later reference we emphasize the following consequence of Corollary 5.1:

Lemma 5.2. *The subcategory $\mathcal{C}_! \to \mathcal{C}$ is the largest full subcategory $\mathcal{D} \to \mathcal{C}$ of $\mathcal{C}$ such that $\widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ is an essential subquality.*

We discuss below some related sufficient conditions for this subquality to be level ϵ. In order to do so recall (C2.2.18 in [5]) that an object B in a site $(\mathcal{B}, J)$ is *J-irreducible* if every J-covering sieve on B is the maximal sieve. The Grothendieck coverage J is said to be *rigid* if, for every object B in $\mathcal{B}$, the family of all morphisms from J-irreducible objects to B generates a J-covering sieve. If J is rigid and $\mathcal{I} \to \mathcal{B}$ is the full subcategory of J-irreducible objects then the Comparison Lemma implies that restriction along the inclusion $\mathcal{I} \to \mathcal{B}$ restricts to an equivalence $\mathrm{Sh}(\mathcal{B}, J) \cong \widehat{\mathcal{I}}$.

Proposition 5.3. *If every Grothendieck coverage on $\mathcal{C}$ is rigid then $\widehat{\mathcal{C}_!} \to \widehat{\mathcal{C}}$ is level ϵ of the local $\widehat{\mathcal{C}} \to \mathbf{Set}$.*

Proof. If every Grothendieck coverage on $\mathcal{C}$ is rigid then the levels of $\widehat{\mathcal{C}}$ are all induced by full subcategories of $\mathcal{C}$. So the result follows from Lemma 5.2. $\square$

At first glance, Proposition 5.3 may look difficult to apply so let us derive a simpler sufficient condition.

Corollary 5.4. *If $\mathcal{C}$ is finite then $\widehat{\mathcal{C}_!} \to \widehat{\mathcal{C}}$ is level ϵ of the local $\widehat{\mathcal{C}} \to \mathbf{Set}$.*

Proof. If $\mathcal{C}$ is finite then every coverage of $\mathcal{C}$ is rigid (see C2.2.21 in [5]). $\square$

In particular, graphic toposes [11] have a level ϵ of this simple kind. On the other hand, it is worth mentioning that Proposition 5.3 also applies to non-finite examples such as the sites studied in [8]. For instance, Δ or the category of non-empty finite sets. It follows that simplicial sets and the classifier of non-trivial Boolean algebras have a level ϵ. In this cases, though, level ϵ coincides with level 0 because, in the respective sites, the terminal object is the only object with exactly one point.

In order to discuss a simple example where ϵ does not coincide with 0 we borrow the 4-element graphic monoid discussed in p. 62 of [11]. Consider first, as an auxiliary step, the pre-cohesive topos $p : \widehat{\Delta_1} \to \mathbf{Set}$ of reflexive graphs. Let G be the graph with two nodes and a non-trivial loop displayed below

$$\circlearrowright \perp \qquad \top$$

and let M be the monoid of endomorphisms of G that are either constant or don't collapse the non-trivial loop. There are four such maps, two constants, the identity, and the unique map α that sends $\top$ to $\perp$ but does not collapse the loop. If we split the constants, we obtain the (non-full) subcategory of $\widehat{\Delta_1}$ pictured below

$$\begin{tikzcd} 1 \arrow[r, shift left, "\top"] \arrow[r, shift right, "\perp"'] & G \arrow[loop right, "\alpha"] \end{tikzcd}$$

where 1 is terminal, $\alpha\alpha = \alpha$ and $\alpha\perp = \perp = \alpha\top$. (Notice that, we are not drawing constant endos or the unique map to the terminal.) It is then clear that we may describe an object of $\widehat{M}$ as a reflexive graph equipped with an idempotent function on its edges that sends each edge x to a loop $x \cdot \alpha$ on the domain of x, preserving the identity loops. As suggested in [11] we call $x \cdot \alpha$ the *preparation to do* x. Alternatively, as a graph equipped with a distinguished subset of loops containing the trivial ones, and a domain-preserving retraction for the inclusion of distinguished loops into edges.

To calculate level ϵ of the pre-cohesive $\widehat{M}$ we split all idempotents. Let $s : D \to G$ be the split monic that results from splitting α in $\widehat{\Delta_1}$ and let $r : G \to D$ be its retraction. We may picture the idempotent-splitting N of M as the (non-full) subcategory of $\widehat{\Delta_1}$ suggested below

$$\begin{tikzcd} & D \arrow[d, shift right, "s"'] \\ 1 \arrow[ur, bend left, "\ddagger"] \arrow[r, shift left, "\top"] \arrow[r, shift right, "\perp"'] & G \arrow[u, shift right, "r"'] \arrow[loop right, "\alpha"] \end{tikzcd}$$

with $r\perp = \ddagger = r\top : 1 \to D$ and $s\ddagger = \perp : 1 \to G$.

It is then clear that the full subcategory $N_! \to N$ is that determined by D and 1 and, by Corollary 5.4, $\widehat{N_!} \to \widehat{N} \cong \widehat{M}$ is level ϵ of the pre-cohesive $\widehat{M} \to \mathbf{Set}$. It is then possible to check that the ϵ-skeletal objects in $\widehat{M}$ are those that consist only of distinguished loops. On the other hand, the sheaves for level ϵ are the objects such that for each distinguished loop d and each node n there exists a unique edge to n with preparation d.

The topos $\widehat{M}$ does not have many levels so it is easy to see that the Aufhebung of level ϵ coincides with the top level, that is, the whole of $\widehat{M}$. Similarly, level 1 must also be the top level in this case.

Consider again a small category $\mathcal{C}$ with a terminal object and such that every object has a point, so that the canonical $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive.

Definition 5.5. A morphism $f : D \to C$ in $\mathcal{C}$ is a *pseudo-constant* if for any two points $a, b : 1 \to D$ in $\mathcal{C}$, $fa = fb : 1 \to C$.

In other words, the pseudo-constants are those morphisms that are constant on points. We think of a pseudo-constant as a morphism that factors through an object that has exactly one point. Notice that if D has exactly one point then every map $D \to C$ is a pseudo-constant.

Proposition 5.6. *If J is a Grothendieck topology on $\mathcal{C}$ such that the subtopos $\mathrm{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$ is above the centre of p then the following are equivalent:*

1. *The subtopos $\mathrm{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$ is a subquality of p.*

2. *For every C in $\mathcal{C}$, the sieve of all the pseudo-constants with codomain C is J-covering.*

3. *For every C in $\mathcal{C}$, JC contains a sieve of pseudo-constants.*

Proof. First consider the canonical $\phi : p^* \to p^!$ in the present context. For A in $\mathbf{Set}$ and C in $\mathcal{C}$, the function $\phi_{A,C} : A = (p^*A)C \to (p^!A)C = A^{\mathcal{C}(1,C)}$ sends $a \in A$ to the constant function in $A^{\mathcal{C}(1,C)}$ that collapses everything to a.

If the first item holds then, for every A in $\mathbf{Set}$, $\phi_A : p^*A \to p^!A$ is J-dense by Lemma 4.3. So ϕ_A must be locally surjective (w.r.t. J) by Corollary III.7.6 in [17]. In particular, $\phi_{\mathcal{C}(1,C)}$ must be so. Take the identity id in codomain of $\phi_{\mathcal{C}(1,C),C} : \mathcal{C}(1, C) \to \mathcal{C}(1, C)^{\mathcal{C}(1,C)}$. Local surjectivity implies the existence of a J-cover $(f_i : C_i \to C \mid i \in I)$ such that for every

$i \in I$, $id \cdot f_i = f_i(\_) \in \mathcal{C}(1, C)^{\mathcal{C}(1,C_i)}$ is constant. In other words, each f_i is a pseudo-constant. So the third item holds. The third item trivially implies the second. If the second item holds then we can use the sieve mentioned there to prove that $\phi_A : p^* A \to p^! A$ is locally surjective. □

Notice that pseudo-constants in $\mathcal{C}$ form and ideal in the sense of Definition 2.6.

Proposition 5.7. *If pseudo-constants in $\mathcal{C}$ form an idempotent ideal then the pre-cohesive $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ has a level ϵ and it coincides with the largest subquality of p.*

Proof. The Grothendieck topology J on $\mathcal{C}$ determined by the idempotent ideal of pseudo-constants is such that a sieve on C is J-covering if and only if it contains all the pseudo-constants with codomain C. It follows that the terminal object is only covered by the identity so $\mathrm{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$ is above the centre of p by Lemma 3.3. Proposition 5.6 implies that the essential subtopos $\mathrm{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$ is a subquality of p and that every topology J' inducing a subquality of p must satisfy $J \subseteq J'$. □

In the case of reflexive graphs, simplicial sets, or the Gaeta topos determined by the theory of distributive lattices, the site satisfies that every pseudo-constant factors through a point so, in these cases, level ϵ exists and coincides with the centre.

Definition 5.8. We say that $\mathcal{C}$ has *enough little figures* if for every pseudo-constant $D \to C$ there is a commutative diagram

$$\begin{tikzcd} D \arrow[r] \arrow[dr] & B \arrow[d] \\ & C \end{tikzcd}$$

such that B has exactly one point.

The intuition behind the terminology is that a map $B \to C$ whose domain has exactly one point is to be thought of as a 'little figure' of C, or a figure of C with 'little' domain.

Corollary 5.9. *If $\mathcal{C}$ has enough little figures then $\widehat{\mathcal{C}_!} \to \widehat{\mathcal{C}}$ is level ϵ of the pre-cohesive $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ and it coincides with the largest subquality of p.*

Proof. An object C in $\mathcal{C}$ has exactly one point if and only if the identity on C is a pseudo-constant. So a little figure (which is of course a pseudo-constant) factors trivially as a composite of pseudo-constants. Therefore, if $\mathcal{C}$ has enough little figures then the ideal of pseudo-constants is idempotent. Proposition 5.7 implies that level ϵ exists and that it coincides with the largest subquality of p. It remains to show that level ϵ coincides with the indicated presheaf subtopos, but notice that the Grothendieck topology determined by the ideal of pseudo-constants is rigid because a sieve on C is covering if and only if it contains all the little figures of $\mathcal{C}$; that is all the morphisms whose domain has exactly one point. The irreducible objects w.r.t. to this topology are exactly those in $\mathcal{C}_!$ so level ϵ coincides with $\widehat{\mathcal{C}_!} \to \widehat{\mathcal{C}}$. $\square$

6. The Weil subquality of the Gaeta topos of $\mathbb{C}$

All algebras we consider are commutative and unital as in [1]. The following is a straightforward generalization of Definition 2.14 in [3] allowing an arbitrary base field instead of $\mathbb{R}$. Let k be a field.

Definition 6.1. A *Weil algebra (over k)* is a k-algebra A such that:

1. A is local, say, with unique maximal ideal $\mathfrak{m}$.
2. The composite $k \to A \to A/\mathfrak{m}$ is an isomorphism.
3. A is a finite k-algebra (i.e. it is finitely generated as a k-module).
4. $\mathfrak{m}^n = 0$ for some n.

It is known that there is some redundancy in this definition. Compare with the definition of *algèbre local* in [21], or the definition in I.16 of [9]. What we need to relate Weil algebras with the material in the present paper is the following, surely folk, result.

Lemma 6.2. *For any local $\mathbb{C}$-algebra A the following are equivalent:*

1. *A is a Weil algebra over $\mathbb{C}$.*
2. *A is finitely generated.*
3. *A is Artinian.*

Proof. The first item implies the second because, as A is a finite k-algebra by hypothesis, then it is finitely generated. Indeed, any basis for the finite dimensional vector space A generates A as a k-algebra.

If A is finitely generated then it is a Jacobson ring by Exercises 5.23 and 5.24 in [1], so every prime ideal is an intersection of maximal ideals. As A is local, it has a unique prime ideal (which must coincide with the maximal one). In this case, the algebra is Artinian by Exercise 8.2 op. cit.

Finally, if A is Artinian and $\mathfrak{m}$ is the unique maximal ideal of A then the composite $\mathbb{C} \to A \to A/\mathfrak{m}$ must be an iso by the 'weak' version of Hilbert's Nullstellensatz (Corollary 7.10 op. cit.). Also, A is a finite k-algebra by Exercise 8.3 op. cit. Moreover, $\mathfrak{m}$ must be the nilradical of A, so $\mathfrak{m}$ is nilpotent by Proposition 8.4 op. cit. $\square$

Let $\mathbf{Ring}$ be the category of rings and $\mathbb{C}/\mathbf{Ring}$ be the coslice category of $\mathbb{C}$-algebras. The full subcategory of finitely generated $\mathbb{C}$-algebras will be denoted by $(\mathbb{C}/\mathbf{Ring})_{f.g.} \to \mathbb{C}/\mathbf{Ring}$. The category $\mathcal{D} = ((\mathbb{C}/\mathbf{Ring})_{f.g.})^{\mathrm{op}}$ is essentially small and extensive. The associated Gaeta topos will be denoted by $\mathfrak{G} = \mathfrak{G}(\mathcal{D})$ and call it the *Gaeta topos of* $\mathbb{C}$.

The Gaeta topos of $\mathbb{C}$ is well-known to be a presheaf topos. To recall that description define a ring to be *(directly) indecomposable* if it has exactly two idempotents and let $(\mathbb{C}/\mathbf{Ring})_{f.g.i.} \to (\mathbb{C}/\mathbf{Ring})_{f.g.}$ be the full subcategory of those finitely generated algebras that are indecomposable. Let $\mathcal{C} = ((\mathbb{C}/\mathbf{Ring})_{f.g.i.})^{\mathrm{op}}$ so that the obvious inclusion $\mathcal{C} \to \mathcal{D}$ is the subcategory of those objects in $\mathcal{D}$ that are 'connected' in the sense that they have no non-trivial coproduct decompositions.

The Gaeta topos $\mathfrak{G}$ may be identified with the topos $\widehat{\mathcal{C}}$ of presheaves on $\mathcal{C}$. By Hilbert's Nullstellensatz, every object in $\mathcal{C}$ has a point so the canonical geometric morphism $p : \mathfrak{G} \to \mathbf{Set}$ is pre-cohesive. Moreover, there are certainly objects in $\mathcal{C}$ that have more than one point so p is Sufficiently Cohesive.

If we let $\mathcal{W} \to (\mathbb{C}/\mathbf{Ring})_{f.g.i.}$ be the full subcategory of Weil algebras then $\mathfrak{W} = \mathbf{Set}^{\mathcal{W}}$ is the *Weil topos* discussed in [3].

Proposition 6.3. *The Weil topos $\mathfrak{W}$ is an essential subquality of the Gaeta topos $\mathfrak{G}$.*

Proof. By Lemma 6.2, the full subcategory $\mathcal{C}_!{}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} = (\mathbb{C}/\mathbf{Ring})_{f.g.i.}$

coincides with $\mathcal{W} \to (\mathbb{C}/\mathbf{Ring})_{f.g.i.}$. So $\mathfrak{W} = \mathbf{Set}^{\mathcal{W}} = \widehat{\mathcal{C}_!} \to \widehat{\mathcal{C}} = \mathfrak{G}$ is an essential subquality by Lemma 5.2. □

7. The Weil subquality is level ϵ of the Gaeta topos

Let $\mathcal{D}$ be a category with terminal object and let $L : \mathcal{D}_\bullet \to \mathcal{D}$ be the full subcategory determined by the objects whose points are jointly epic.

Lemma 7.1. *If LA has a point and $e : LA \to V$ in $\mathcal{D}$ is an epic pseudo-constant then $V = 1$.*

Proof. As e is epic and the points of LA are jointly epic, the family of all composites

$$1 \longrightarrow LA \xrightarrow{e} V$$

is jointly epic but, as e is pseudo-constant, there is only one such map, so we have an epic $1 \to V$ which of course is also split monic, so V is terminal. □

Natural further hypotheses allow us to deal with more pseudo-constants.

Lemma 7.2. *Assume that $L : \mathcal{D}_\bullet \to \mathcal{D}$ has an epic-preserving right adjoint with monic counit. If X has a point then, for every epic pseudo-constant $f : X \to Y$, Y has exactly one point.*

Proof. Let R be the right adjoint to L and denote the counit by β. As $L(R1) = 1$, $L(RX)$ must have a point because X does by hypothesis. Also, the map Rf is epic by hypotheses and then so is $L(Rf) : L(RX) \to L(RY)$. Moreover, it is a pseudo-constant because, $\beta_Y : L(RY) \to Y$ is monic and $\beta_Y(L(Rf)) = f\beta_X$. Lemma 7.1 implies that $L(RY) = 1$. As every point of Y factors through $\beta : L(RY) \to Y$, Y has exactly one point. □

The next result supplies many little figures.

Proposition 7.3. *If every map in $\mathcal{D}$ factors as an epi followed by a mono and $L : \mathcal{D}_\bullet \to \mathcal{D}$ has an epi-preserving right adjoint with monic counit then every pseudo-constant whose domain has a point factors via an object with exactly one point.*

Proof. Let f be pseudo-constant whose domain has a point. By hypothesis, $f = me$ for some monic m and epic e. Then the codomain of e has exactly one point by Lemma 7.2. □

The proof of Proposition 7.3 is, in essence, that in [2]. This will become evident below where we discuss the context of Cornulier's Mathoverflow answer.

Let $\mathbf{Ring}$ be the category of (commutative unital) rings and consider the full subcategory $\mathbf{Red} \to \mathbf{Ring}$ of reduced rings (i.e. those whose only nilpotent element is 0).

Lemma 7.4. *This inclusion* $\mathbf{Red} \to \mathbf{Ring}$ *has a left adjoint that preserves monomorphisms. Moreover, the unit of the adjunction is regular epic.*

Proof. The left adjoint sends R in $\mathbf{Ring}$ to $R/\mathrm{Nil}(R)$ where $\mathrm{Nil}(R)$ is the nilradical of R. See Proposition 1.7 in [1]. The unit $R \to R/\mathrm{Nil}(R)$ is a regular epimorphism and the left adjoint $\mathbf{Ring} \to \mathbf{Red}$ preserves monos because if $m : R \to S$ is a monomorphism then $m^*\mathrm{Nil}(S) = \mathrm{Nil}(R)$ as subsets of R. □

Let $\mathbb{C}$ be the field of complex numbers and consider the coslice category $\mathbb{C}/\mathbf{Ring}$ of $\mathbb{C}$-*algebras*.

Lemma 7.5. *A finitely generated* $\mathbb{C}$*-algebra* R *is reduced (as a ring) if and only if the family of all maps* $R \to \mathbb{C}$ *is jointly monic.*

Proof. If the family of maps $R \to \mathbb{C}$ is jointly monic then R is, as a ring, a subobject of a power of $\mathbb{C}$ so it is reduced. Conversely, assume that R is reduced. That is, the nilradical $\mathrm{Nil}(R)$ is trivial. For finitely generated algebras over a field, the nilradical equals the Jacobson radical (see Exercise 5.24 in [1]), so the intersection of the maximal ideals in R is 0. In other words, the collection of all maps $R \to \mathbb{C}$ is jointly monic. □

Let $(\mathbb{C}/\mathbf{Ring})_{f.g.}$ be the category of finitely generated $\mathbb{C}$-algebras.

Lemma 7.6. *Every pseudo-constant in* $((\mathbb{C}/\mathbf{Ring})_{f.g.})^{\mathrm{op}}$ *whose domain has a point factors via an object with exactly one point.*

Proof. It is enough to check that $\mathcal{D} = ((\mathbb{C}/\mathbf{Ring})_{f.g.})^{\mathrm{op}}$ satisfies the hypotheses of Proposition 7.3. It is well-known that $\mathcal{D}$ has epi/regular-mono factorizations so it remains to show that the inclusion $\mathcal{D}_\bullet \to \mathcal{D}$ has an epi-preserving right adjoint with monic counit. We show that the inclusion ${\mathcal{D}_\bullet}^{\mathrm{op}} \to (\mathbb{C}/\mathbf{Ring})_{f.g.}$ satisfies the dual conditions. Bear in mind that, by Lemma 7.5, the full subcategory ${\mathcal{D}_\bullet}^{\mathrm{op}} \to (\mathbb{C}/\mathbf{Ring})_{f.g.}$ may be identified that of f.g. algebras that are reduced as rings.

The reflective subcategory $\mathbf{Red} \to \mathbf{Ring}$ of Lemma 7.4 induces another one such $\mathbb{C}/\mathbf{Red} \to \mathbb{C}/\mathbf{Ring}$. Also, the left adjoint $\mathbb{C}/\mathbf{Ring} \to \mathbb{C}/\mathbf{Red}$ is again obtained by quotienting by the nilradical so the unit is again regular epic. Moreover, it preserves monos because the canonical $\mathbb{C}/\mathbf{Red} \to \mathbf{Red}$ reflects monos.

By Noetherianity, the nilradical of a finitely generated algebra is finitely generated so the left adjoint $\mathbb{C}/\mathbf{Ring} \to \mathbb{C}/\mathbf{Red}$ restricts to finitely generated algebras. That is, we have the reflective ${\mathcal{D}_\bullet}^{\mathrm{op}} \to (\mathbb{C}/\mathbf{Ring})_{f.g.}$ satisfying the necessary conditions. $\square$

Recall from Proposition 6.3 that the Weil topos $\mathfrak{W}$ is a subtopos $\mathfrak{W} \to \mathfrak{G}$ of the Gaeta topos $\mathfrak{G}$ and that the subtopos is actually and essential subquality of the pre-cohesive $\mathfrak{G} \to \mathbf{Set}$.

Theorem 7.7 (The Weil subquality is level ϵ of the Gaeta topos)**.** *The essential subquality $\mathfrak{W} \to \mathfrak{G}$ is level ϵ of the pre-cohesive $p : \mathfrak{G} \to \mathbf{Set}$ and it coincides with the largest subquality of p.*

Proof. We identify the Gaeta topos $\mathfrak{G}$ with $\widehat{\mathcal{C}}$ where $\mathcal{C}$ is the opposite of the category of finitely generated complex algebras with exactly two idempotents. By Corollary 5.9 it is enough to prove that $\mathcal{C}$ has enough little figures so let $f : X \to Y$ be a pseudo-constant in $\mathcal{C}$. Then f is a pseudo-constant in $\mathcal{D}$ and X has a point because every object of $\mathcal{C}$ has a point. By Lemma 7.6, f factors (in $\mathcal{D}$) via a object with exactly one point. This object is necessarily in $\mathcal{C}$ so the factorization of f is inside $\mathcal{C}$. $\square$

We see Theorem 7.7 as a confirmation of Lawvere's suggestion (quoted in the beginning of the paper) that "the infinitesimal spaces may well negate the largest essential subtopos of the ambient one which" has the property that "the four adjoint functors relating their topos to the base topos coalesce into two".

8. The Weil subquality is level ϵ of the Zariski topos of $\mathbb{C}$

We show that the level ϵ of the Gaeta topos for $\mathbb{C}$ factors through the Zariski topos and, as a level of the latter, it is level ϵ. Some of the ideas involving restricted subqualities may be formulated at an elementary level. We deal with these first. Let $p : \mathcal{E} \to \mathcal{S}$ be a local geometric morphism.

Proposition 8.1. *Let $j : \mathcal{E}_j \to \mathcal{E}$ and $k : \mathcal{E}_k \to \mathcal{E}$ be subtoposes of $\mathcal{E}$ and assume that k is above j. If $\mathcal{E}_j \to \mathcal{E}$ is above the centre of p (so that $\mathcal{E}_k \to \mathcal{E}$ is also above the centre of p) then the following hold:*

1. *The subtopos $\mathcal{E}_j \to \mathcal{E}_k$ is above the centre of $pk : \mathcal{E}_k \to \mathcal{S}$.*
2. *If $\mathcal{E}_j \to \mathcal{E}$ is a subquality of p, then the subtopos $\mathcal{E}_j \to \mathcal{E}_k$ is a subquality of $pk : \mathcal{E}_k \to \mathcal{S}$.*
3. *If $\mathcal{E}_j \to \mathcal{E}$ is essential, then so is $\mathcal{E}_j \to \mathcal{E}_k$.*
4. *If a subtopos $\mathcal{E}_l \to \mathcal{E}_k$ is above the centre of $pk : \mathcal{E}_k \to \mathcal{S}$, then the subtopos $\mathcal{E}_l \to \mathcal{E}_k \to \mathcal{E}$ is above the centre of p.*

Proof. 1. According to Corollary 2.3 (and with the same notation introduced there) the unit of $\mathcal{E}_j \to \mathcal{E}_k$ is $(k^*\nu_{k_*}) \cdot \xi'^{-1}$. When we apply $p_*k_* : \mathcal{E}_k \to \mathcal{E}$ we observe that $p_*k_*k^*\nu_{k_*}$ is an iso since $p_*k_*k^* \simeq p_*$, given that k is above the centre of p, and $p_*\nu$ is an iso, given that j is above the centre of p.

2. This follows at once since $\mathcal{E}_j \to \mathcal{S}$ is a quality type regardless of whether we consider $\mathcal{E}_j$ as a subtopos of $\mathcal{E}$ or of $\mathcal{E}_k$.

3. This follows at once form Remark 2.2.

4. We must show that p_* inverts the unit of $\mathcal{E}_l \to \mathcal{E}_k \to \mathcal{E}$ assuming that p_*k_* inverts the unit of $\mathcal{E}_l \to \mathcal{E}_k$; but this follows at once since p_* inverts the unit of $k : \mathcal{E}_k \to \mathcal{E}$ because $\mathcal{E}_k$ is above the centre of p. □

Proposition 8.1 allows to show that, if level ϵ is not just that but is also the largest subquality then we can restrict it to subtoposes that contain it. More precisely:

Corollary 8.2. *Assume that $p : \mathcal{E} \to \mathcal{S}$ has a largest subquality $\mathcal{E}_j \to \mathcal{E}$ and that it is essential (so that $\mathcal{E}_j \to \mathcal{E}$ is level ϵ of p). If $k : \mathcal{E}_k \to \mathcal{E}$ is a subtopos above j then the subtopos $\mathcal{E}_j \to \mathcal{E}_k$ is the largest subquality of $pk : \mathcal{E}_k \to \mathcal{S}$ and it is essential (so that $\mathcal{E}_j \to \mathcal{E}_k$ is level ϵ of pk).*

Proof. By the second item of Proposition 8.1, the subtopos $\mathcal{E}_j \to \mathcal{E}_k$ is a subquality of $pk : \mathcal{E}_k \to \mathcal{S}$ and it is essential by the third item. Now assume that $\mathcal{E}_l \to \mathcal{E}_k$ is a subquality of $pk : \mathcal{E}_k \to \mathcal{S}$. Then $\mathcal{E}_l \to \mathcal{E}_k \to \mathcal{E}$ is a subquality of p by the fourth item. So it is above $\mathcal{E}_j \to \mathcal{E}$ by hypothesis and then, $\mathcal{E}_l \to \mathcal{E}_k$ is above $\mathcal{E}_j \to \mathcal{E}_k$. $\square$

We can now start to discuss the example. It is convenient to give first an alternative presentation of the Gaeta topos of $\mathbb{C}$ discussed in Section 6. As in that Section, let $\mathcal{D}$ be the opposite of the category of finitely generated $\mathbb{C}$-algebras. Let J_G be the Gaeta coverage on $\mathcal{D}$. The basic covering families are those of the form

$$(D_i \to D \mid i \in I)$$

such that I is finite and the induced $\sum_{i\in I} D_i \to D$ is an isomorphism. The intimate relation between products in $\mathcal{D}^{\mathrm{op}}$ and idempotents implies that the J_G-cocovering families in $\mathcal{D}^{\mathrm{op}}$ are those of the form

$$(A \to A[a_i^{-1}] \mid i \in I)$$

where I is a finite set, $\sum_{i\in I} a_i = 1$ and, for every $i, j \in I$, $i \neq j$ implies $a_i a_j = 0$.

Let $\mathcal{C} \to \mathcal{D}$ be the full subcategory determined by the (f.g.) algebras that have exactly two idempotents. As every object of $\mathcal{D}$ is a finite coproduct of objects in $\mathcal{C}$, the inclusion $\mathcal{C} \to \mathcal{D}$ is J_G-dense and so the Comparison Lemma (C2.2.3 in [5]) implies that restricting along the inclusion $\mathcal{C} \to \mathcal{D}$ underlies an equivalence $\mathrm{Sh}(\mathcal{D}, J_G) \to \widehat{\mathcal{C}} = \mathfrak{G}$ between the topos of sheaves $\mathrm{Sh}(\mathcal{D}, J_G)$ and the topos of presheaves $\widehat{\mathcal{C}}$ that we used to define the Gaeta topos in Section 6.

Let J_Z be the Zariski coverage on $\mathcal{D}$. It is well-known that J_Z-cocovering basic families in $\mathcal{D}^{\mathrm{op}}$ are those of the form

$$(A \to A[a_i^{-1}] \mid i \in I)$$

where I is a finite set and the ideal generated by $(a_i \mid i \in I)$ contains 1. The topos $\mathrm{Sh}(\mathcal{D}, J_Z)$ will be denoted by $\mathfrak{Z}$ and is called the *Zariski topos* (determined by the field $\mathbb{C}$) and $\mathfrak{Z} = \mathrm{Sh}(\mathcal{D}, J_Z) \to \mathbf{Set}$ is pre-cohesive (see [14] and also Example 1.5 in [6]).

The above description of J_Z and J_G implies that every J_G-cover is a J_Z cover. That is, the Zariski topos is a subtopos of the Gaeta topos. This presentation of the Zariski topos as a subtopos

$$\mathfrak{Z} = \mathrm{Sh}(\mathcal{D}, J_Z) \longrightarrow \mathrm{Sh}(\mathcal{D}, J_G) \xrightarrow[\cong]{} \widehat{\mathcal{C}} = \mathfrak{G}$$

(of the Gaeta topos) whose direct image is restriction along $\mathcal{C} \to \mathcal{D}$ is motivated by the discussion starting at the end of p. 109 in [10].

Lemma 8.3. *The subtopos $\mathfrak{Z} \to \mathfrak{G}$ is above the Weil subquality $\mathfrak{W} \to \mathfrak{G}$.*

Proof. Recall from Section 6 that we identified the Weil subquality with the geometric inclusion induced by the full subcategory $\mathcal{C}_! \to \mathcal{C}$. Consider now the full inclusion $\mathcal{C}_! \to \mathcal{C} \to \mathcal{D}$. By Lemma C2.3.9 in [5] there exists a smallest coverage K on $\mathcal{C}_!$ such that the inclusion into $\mathcal{D}$ is cover reflecting. In that result, K is defined as the Grothendieck coverage generated by the sieves of the form $R \cap \mathcal{C}_!$ where R is J_Z-covering. We show below that all these sieves contain an iso, which will allow us to conclude that K is trivial.

Consider an object A in $\mathcal{C}_!{}^{\mathrm{op}}$. Since it has a unique maximal ideal, Exercise 5.24 in [1] implies that the nilradical of A is a maximal ideal. Thus, by Exercise 1.10 loc. cit., every element of A is either nilpotent or invertible. As A is non-trivial, a Zariski cover cannot be generated by nilpotents, so every J_Z-cocover of A contains an isomorphism. In other words, for every C in $\mathcal{C}_!$, the only J_Z-covering sieve of C as an object of $\mathcal{D}$ is the maximal one. This implies that K is trivial.

The proof of Lemma C2.3.9 cited above shows that the outer square below

$$\begin{tikzcd} \mathrm{Sh}(\mathcal{C}_!, K) \arrow[rr] \arrow[d] & & \mathrm{Sh}(\mathcal{D}, J_Z) \arrow[d] \arrow[dl, dotted] \\ \widehat{\mathcal{C}_!} \arrow[r] & \widehat{\mathcal{C}} \arrow[r] & \widehat{\mathcal{D}} \end{tikzcd}$$

is a pullback. As the right vertical map factors through $\widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$, the inner polygon is also a pullback and, since the left vertical map is an isomorphism, $\mathrm{Sh}(\mathcal{D}, J_Z) \to \widehat{\mathcal{C}}$ is above $\widehat{\mathcal{C}_!} \to \widehat{\mathcal{C}}$. □

We may now identify level ϵ of the Zariski topos of $\mathbb{C}$.

Theorem 8.4 (The Weil subquality is level ϵ of the Zariski topos)**.** *The subtopos $\mathfrak{W} \to \mathfrak{Z}$ is level ϵ of the pre-cohesive $p : \mathfrak{Z} \to \mathbf{Set}$ and it coincides with the largest subquality of p.*

Proof. By Theorem 7.7, the subtopos $\mathfrak{W} \to \mathfrak{G}$ is level ϵ of the pre-cohesive $p : \mathfrak{G} \to \mathbf{Set}$ and it coincides with the largest subquality of p. Lemma 8.3 shows $\mathfrak{Z} \to \mathfrak{G}$ is above $\mathfrak{W} \to \mathfrak{G}$ so Corollary 8.2 applies. $\square$

Corollary 8.2 suggests the question: what is the largest subtopos of the Zariski topos that contains the Weil topos? In any case, the calculation of the Aufhebung of ϵ still needs to be carried out.

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