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VOLUME LX-3 (2019)

COLUMN SYMMETRIC POLYNOMIALS

Eduardo DUBUC and Anders KOCK

Résumé. Nous étudions l'algèbre des polynômes en une m x n matrice de variables sur un anneau contenant les rationnels, sujette à la condition que le produit de deux variables appartenant à une même colonne est nul. Nous montrons que la sous-algèbre des polynômes invariants sous l'action des n! permutations des colonnes est un quotient de l'algèbre des polynômes en m variables; l'application quotient envoie la i-ème variable en la somme des entrées de la i-ème ligne de la matrice. Une application en géométrie différentielle synthétique est esquissée.

Abstract. We study the polynomial algebra (over a ring containing the rationals) in an m by n matrix of variables, and subject to the relation that says that the product of any two variables in the same column is zero. We show that the sub-algebra of polynomials, which are invariant under the n! permutations of the columns, is a quotient of the polynomial algebra in m variables; the quotient map sends the i'th variable to the sum of the entries in the i'th row of the matrix. An application in synthetic differential geometry is sketched.

Keywords. Symmetric polynomials, synthetic differential geometry.

Mathematics Subject Classification (2010). 13A50, 51K10

Introduction

Let A be a commutative ring. It is classical how symmetric polynomials in $A[x_1, \ldots, x_n]$ are uniquely expressible as polynomials in the n elementary symmetric polynomials, cf. e.g. [4] §29. For instance for $n = 2$, the two elementary polynomials are $\sigma_1 := x_1 + x_2$ and $\sigma_2 := x_1 x_2$; and the symmetric polynomial $x_1^2 + x_2^2$ may be expressed as $\sigma_1^2 - 2\sigma_2$:

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1 x_2.$$

Modulo the ideal I generated by x_1^2 and x_2^2, we therefore also have

$$x_1 x_2 = \frac{1}{2}(x_1 + x_2)^2,$$

provided $\frac{1}{2}$ exists in the base ring A.

In fact, we have more generally that if A contains the ring $\mathbb{Q}$ of rationals, then, modulo I, any symmetric polynomial in $A[x_1, \ldots, x_n]$ may be uniquely expressed as a polynomium in the single symmetric polynomium $x_1 + \ldots + x_n$, where I is the ideal generated by the x_i^2s. This is a well known and important fact, called "the symmetric functions property" in [2] Exercise I.3.3.

It is a result in this direction we intend to generalize from dimension 1 to dimension m. We are considering the polynomial ring in $m \times n$ variables $x_{i,j}$; the kind of symmetry we consider is not with respect to all the mn variables; we consider these variables organized in an $m \times n$ matrix, and we only consider invariance under the $n!$ permutations of the n columns. The result refers to what we can assert, modulo the ideal I generated by the degree 2 monomials $\{x_{ij} x_{i'j}\}_{j=1,\ldots,n,\ i=1,\ldots,m,\ i'=1,\ldots,m}$.

The result asserts that any polynomial, invariant under the $n!$ permutations of the columns can, modulo I, be expressed uniquely as a polymonial in the m "row-sums", $\{s_i = x_{i,1} + x_{i,2} + \ldots + x_{i,n}\}_{i=1,\ldots,m}$. The classical "symmetric functions property" is the special case where $m = 1$.

An application of this Theorem concerns formal exactness of closed differential 1-forms is sketched in Section 3 below.

Throughout A will be a commutative ring. It is assumed to contain $\mathbb{Q}$. All the A-modules which we consider are free. Therefore, we use terminology from linear algebra, as if A were a field.

1. Polynomials in a matrix of variables

1.1 The free commutative monoid

The free commutative monoid $M(X)$ on a set X is in a natural way a graded monoid. We call its elements *monomials* in X, we call X the set of *variables*; we write the monoid structure multiplicatively. We shall give an explicit presentation of $M(X)$.[1]

Let k be a positive integer; we let $[k]$ denote the set $[k] = \{1, 2, \ldots, k\}$. Then a monomial ω of degree k may be explicitly presented by a function $f : [k] \to X$; we write the monomium thus presented $\omega_f := x_{f(1)}x_{f(2)} \ldots x_{f(k)}$. Since the variables commute, it follows that two functions f and $f' : [k] \to X$ present the same monomium iff they differ by a permutation $\varepsilon : [k] \to [k]$ of $[k]$, i.e. $f' = f \circ \varepsilon$.

Later on in the proof of Proposition 1.5, we shall need a finer notation: We denote by $\|f\|$ the set of all functions $f \circ \varepsilon$ for $\varepsilon \in S_k$ (where S_k is the group of permutations of $[k]$). Thus $\|f\|$ is the orbit of f under the right action (by precomposition) of S_k. The monomials are actually indexed by these orbits, we have a well defined monomium $\omega_{\|f\|}$, and $\omega_{\|f\|} = \omega_{\|f'\|} \iff \|f\| = \|f'\|$.

1.2 The polynomial ring in a matrix of variables

If A is any commutative ring, the polynomial ring $A[X]$ with coefficients in A in a set X of indeterminates is the free commutative A-algebra on the set X. It may be constructed by a two-stage process: first, construct the free commutative monoid $M(X)$ on X, and then construct the free A-module on the set $M(X)$. It inherits its multiplication from that of $M(X)$. It is a *graded* A-algebra, with the degree-k part being the linear submodule with basis the monomials of degree k.

We shall be interested in some further structure which the algebra $A[X]$ has, in the case where the set X is given as a product set $[m] \times [n]$. We think of this X as the set of $m \times n$ matrices (m rows, n columns) with entries $x_{i,j}$ ($i \in [m], j \in [n]$), and write $A[M^{m\times n}] := A[[m] \times [n]] = A[x_{1,1}, \ldots, x_{m,n}]$.

[1] An equivalent description is that $M(X)$ is the set of finite *multi-subsets* of X.

A function $[k] \longrightarrow [m]\times[n]$ is given by a pair (f, g), where $f : [k] \to [m]$ and $g : [k] \to [n]$. The monomium presented by such function we denote $\omega_{(f,g)}$, or just $\omega_{f,g}$. Thus

$$\omega_{f,g} = \prod_{l\in[k]} x_{f(l),g(l)} \;=\; x_{f(1)g(1)}x_{f(2)g(2)} \,\ldots\, x_{f(k)g(k)} \tag{1}$$

Clearly, when g is monic, then so is any other g', for any other presentation (f', g') of the same monomium. Therefore, the following notion is well defined.

Definition 1.1. *The monomial $\omega_{f,g}$ is* admissible *if $g : [k] \to [n]$ is monic. A polynomium $\in A[M^{m\times n}]$ is called* admissible *if it is a linear combination of admissible monomials.*

So a monomium in the $x_{i,j}$'s is admissible if it does not contain two factors from any of the columns, like $x_{i,j} \cdot x_{i',j}$. In particular, it does not contain any squared factor $x_{i,j}^2$. Clearly, admissible polynomials are of degree $\leq n$.

If ω is not admissible, it is called inadmissible. If ω is inadmissible, then so is $\omega \cdot \theta$ for any monomium θ. It follows that the linear subspace of $A[M^{m\times n}]$ generated by the inadmissible monomials is an ideal $I \subseteq A[M^{m\times n}]$. The quotient algebra $A[M^{m\times n}]/I$ may be identified with the linear subspace (not a subalgebra) $A_a[M^{m\times n}] \subseteq A[M^{m\times n}]$ generated by the admissible monomials, with the projection morphism $A[M^{m\times n}] \longrightarrow A_a[M^{m\times n}]$ being the map which discards all terms containing an inadmissible factor. The algebra structure of $A_a[M^{m\times n}]$ is thus given by the multiplication table $\{x_{i,j} \cdot x_{i',j} = 0\}_{i\in[m],\, i'\in[m],\, j\in[n]}$, and no other relations.[2] The algebra $A_a[M^{m\times n}]$ inherits a grading from that of $A[M^{m\times n}]$. Note that in $A_a[M^{m\times n}]$ all non-zero elements are of degree $\leq n$.

Among the polynomials in $A[M^{m\times n}]$ we have the m "row-sums" s_i for $i = 1, \ldots, m$ (the sum of the entries in the ith row); they are all admissible:

$$s_i := \sum_{j\in[n]} x_{i,j} \;=\; x_{i,1} + x_{i,2},\, \ldots\, x_{i,n}. \tag{2}$$

[2] $A_a[M^{m\times n}]$ is an example of what sometimes is called a *Weil-algebra* over A; in particular, it is finite-dimensional as an A-module. Likewise, the algebra $A_{\leq n}[y_1, \ldots, y_m]$ to be considered below, is a Weil-algebra.

Consider any map $f : [k] \to [m]$. By the distributive law, i.e. by multiplying out the product, we have the second equality sign in

$$\prod_{l\in[k]} s_{f(l)} = \prod_{l\in[k]} \sum_{j\in[n]} x_{f(l)j} = \sum_{[k]\xrightarrow{g}[n]} \prod_{l\in[k]} x_{f(l)g(l)},$$

where g ranges over the set of all maps $[k] \to [n]$. The admissible terms here are those where g is injective, so modulo I, equivalently, discarding inadmissible terms,

$$\prod_{l\in[k]} s_{f(l)} \;=\; \sum_{[k]\overset{g}{\hookrightarrow}[n]} \prod_{l\in[k]} x_{f(l),g(l)} \qquad \text{in the algebra } A_a[M^{m\times n}]. \tag{3}$$

where g now ranges over the set of monic maps $[k] \hookrightarrow [n]$.

1.3 Column symmetric polynomials

Let σ be a permutation $\sigma : [n] \to [n]$, i.e. $\sigma \in S_n$. One may permute the n columns of the matrix X of variables $x_{i,j}$ by σ. More explicitly, σ permutes the monomials by the recipe:

$$\sigma \cdot \omega_{f,g} := \omega_{f,\sigma\circ g}\,. \tag{4}$$

This is well defined with respect to different presentations of the same monomial. Thus, the set of monomials carry a left action by S_n. If $g : [k] \to [n]$ is injective, then so is $\sigma \circ g$, for any permutation $\sigma : [n] \to [n]$, hence the subset of admissible monomials is stable under the action. The action clearly extends to an action on the polynomial algebras $A[M^{m\times n}]$ and $A_a[M^{m\times n}]$. Note that the subspace inclusion as well as the quotient morphism preserve the action.

The polynomials which are invariant under the action of S_n, we call *column symmetric*. These elements form subalgebras of $A[M^{m\times n}]$ and of $A_a[M^{m\times n}]$; they deserve the notation $\mathrm{sym}(A[M^{m\times n}])$ and $\mathrm{sym}(A_a[M^{m\times n}])$, respectively.

In the sequel we study the structure of the elements of the algebra

$$\mathrm{sym}(A_a[M^{m\times n}]) \subseteq A_a[M^{m\times n}].$$

This is where we need that the ring A contains $\mathbb{Q}$.

If a finite group S acts on an algebra C over a commutative ring A, the elements in C invariant under the action of S form a subalgebra $\mathrm{sym}_S(C)$ of S-*symmetric* or S-*invariant* elements. If A contains the field of rational numbers $\mathbb{Q}$ as a subring, we further have that the subalgebra $\mathrm{sym}_S(C) \subseteq C$, seen just as a linear subspace, is a retract, with retraction the symmetrization operator sym given, for $a \in C$, by

$$\mathrm{sym}(a) := p^{-1} \cdot \sum_{\sigma \in S} \sigma \cdot a, \tag{5}$$

where p is the cardinality of S. And we have

$$a \text{ is invariant} \iff a = \mathrm{sym}(a).$$

Proposition 1.2. *Any two admissible monomials $\omega_{f,g}$, $\omega_{f,g'}$ with the same $f : [k] \to [m]$ are in the same orbit of the action by S_n. It follows that $\mathrm{sym}(\omega_{f,g}) = \mathrm{sym}(\omega_{f,g'})$, see (5).*

Proof. Recall that if g and $g' : [k] \to [n]$ are monic, then we may find a permutation $[n] \xrightarrow{\tau} [n]$ with $\tau \circ g = g'$. There are in fact $(n-k)!$ such permutations. With such τ, we have $\tau \cdot \omega_{f,g} = \omega_{f,g'}$. It follows that $\mathrm{sym}(\omega_{f,g})$ and $\mathrm{sym}(\omega_{f,g'})$ have the same terms but in different order. □

The row-sum polynomials s_i, see (2), are clearly column-symmetric, and the product $\prod_{l \in [k]} s_{f(l)}$, as a k-fold product of homogeneous degree 1 polynomials, is a homogeneous degree k polynomial, and likewise column symmetric.

Proposition 1.3. *For any admissible monomium $\omega_{f,g}$ of degree k, we have (discarding inadmissible terms)*

$$\mathrm{sym}(\omega_{f,g}) = \frac{(n-k)!}{n!} \prod_{l \in [k]} s_{f(l)}.$$

Proof. Any $\sigma \in S_n$ defines, by restriction to the subset $[k] \subseteq [n]$, a monic map $g : [k] \hookrightarrow [n]$. Conversely, any monic $[k] \hookrightarrow [n]$ extends to a permutation $\sigma : [n] \to [n]$ in $(n-k)!$ different ways, by simple combinatorics.

Let $C(g) \subseteq S_n$ be the set of such extensions of g. These subsets of S_n are clearly disjoint. So we have

$$S_n = \coprod_{[k] \overset{g}{\hookrightarrow} [n]} C(g).$$

Therefore, we may rewrite $\sum_{\sigma \in S_n} \sigma \cdot \omega_{f,g}$ as follows

$$\sum_{[k] \overset{g}{\hookrightarrow} [n]} \sum_{\sigma \in C(g)} \prod_{l \in [k]} x_{f(l)\sigma(l)} = \sum_{[k] \overset{g}{\hookrightarrow} [n]} \sum_{\sigma \in C(g)} \prod_{l \in [k]} x_{f(l)g(l)}$$

since for each g and each $\sigma \in C(g)$, $\sigma(l) = g(l)$, for $l \in [k] \subseteq [n]$. Therefore, for a given g, the terms in the summation over $C(g)$ are equal, and there are $(n-k)!$ of them, so the equation continues

$$= \sum_{[k] \overset{g}{\hookrightarrow} [n]} (n-k)! \prod_{l \in [k]} x_{f(l)g(l)} = (n-k)! \sum_{[k] \overset{g}{\hookrightarrow} [n]} \prod_{l \in [k]} x_{f(l)g(l)} \,,$$

and this expression equals $(n-k)! \prod_{l \in [k]} s_{f(l)}$ by equation (3). Dividing by $n!$ now gives the desired equation. □

From the Proposition, we may deduce (recall that $\mathbb{Q}$ is a subring of A)

Proposition 1.4. *Every column symmetric admissible polynomial can be expressed in $A_a[M^{m \times n}]$ as a polynomial in the s_i's. (This expression can be interpreted as an expression, modulo the ideal I of inadmissibles, in the polynomial ring $A[M^{m \times n}]$.)*

Proof. Any admissible polynomial $h \in A_a[M^{m \times n}]$ is a linear combination of admissible monomials, and sym is linear; by Proposition 1.3 sym of an admissible monomium is a polynomial in the s_i's. Therefore also $\mathrm{sym}(h)$ is so. If h is furthermore column symmetric, $h = \mathrm{sym}(h)$, then h itself is expressed as a polynomial of the s_i's, $h = G(s_1, \ldots, s_m)$ for some polynomium $G \in A([m]) = A[y_1, \ldots, y_m]$. □

We shall formulate the results so far and some of its consequences in the category $\mathcal{A}$ of commutative A-algebras.

Consider the algebra $A[y_1, \ldots, y_m]$. Since it is the free algebra in the generators y_i, and $s_i \in \mathrm{sym}(A[M^{m\times n}])$, there is a unique algebra map (preserving degree)

$$A[y_1, \ldots, y_m] \xrightarrow{S} \mathrm{sym}(A[M^{m\times n}]) \subseteq A[M^{m\times n}]), \tag{6}$$

namely the one which sends $y_i \in A[y_1, \ldots, y_m]$ to s_i.

Let J be the ideal in $A[y_1, \ldots, y_m]$ generated by the monomials of degree $n+1$. The quotient algebra $A[y_1, \ldots, y_m]/J$ may be identified with the linear subspace (not a subalgebra) $A_{\leq n}[y_1, \ldots, y_m] \subseteq A[y_1, \ldots, y_m]$ of polynomials of degree less or equal to n , the algebra structure given by the multiplication table $\{y_{f(1)}\, y_{f(2)}\ \ldots\ y_{f(n+1)} = 0\}_{f:[n+1]\to[m]}$, and no other relations.

It follows immediately from the respective multiplication tables (alternatively since S sends the ideal J into the ideal I) that we have an algebra map:

$$A_{\leq n}[y_1, \ldots, y_m] \xrightarrow{s} \mathrm{sym}\, A_a[M^{m\times n}] \tag{7}$$

making the diagram below commutative:

$$\begin{array}{ccc} A[y_1, \ldots, y_m] & \xrightarrow{S} & \mathrm{sym}(A[M^{m\times n}]) \\ \downarrow & & \downarrow \\ A_{\leq n}[y_1, \ldots, y_m] & \xrightarrow{s} & \mathrm{sym}(A_a[M^{m\times n}]) \end{array} \tag{8}$$

The vertical maps are quotient maps which discard terms of degree $> n$, respectively inadmissible terms. Thus the map s discards the inadmissible terms from the values of S.

Proposition 1.5. *The algebra map s in (7) is injective.*

Proof. (We refer to the last paragraph in Subsection 1.1 for the notation $\|f\|$ for the orbit of f under precomposition with permutations.) Clearly the monomials $\omega_{\|f\|}$ of degree $\leq n$ make up a vector basis of $A_{\leq n}[y_1, \ldots, y_m]$. We may define an equivalence relation $\sim$ on the set of monomials of degree k in $A_a[M^{m\times n}]$, namely $\omega_{f,g} \sim \omega_{f',g'}$ iff $\|f\| = \|f'\|$. We let $B_{\|f\|}$ be the equivalence class defined by $\|f\|$. It follows that $A_{\leq n}[M^{m\times n}]$ is a direct sum of the subspaces $V_{\|f\|}$ spanned by the $B_{\|f\|}$. We show that $s(\omega_{\|f\|})$ lies in $V_{\|f\|}$; recall equation (3) and note that for any g, $\omega_{f,g} \in B_{\|f\|}$:

$$s(\omega_{\|f\|}) = \prod_{l \in [k]} s_{f(l)} = \sum_{[k] \overset{g}{\hookrightarrow} [n]} \prod_{l \in [k]} x_{f(l)g(l)} = \sum_{[k] \overset{g}{\hookrightarrow} [n]} \omega_{f,g}.$$

Thus the map s sends a linear base into a set of lineary independent vectors, and its injectivity follows. □

Remark. A similar argument proves that also the map S in (6) is injective.

The surjectivity of the map s is a reformulation of Proposition 1.4. Thus, combining Propositions 1.4 and 1.5, we have our main result:

Theorem 1.6. *The algebra map s in (7) is an isomorphism.*

We shall paraphrase this in geometric terms:

2. Geometric interpretation

2.1 The category of A-algebras and its dual

The following Section only is a reminder, to fix notation etc. As above, $\mathcal{A}$ denotes the category of commutative A-algebras (here just called algebras.)

The dual category $\mathcal{A}^{op}$ is essentially the category of affine schemes over A. The objects, viewed in this category, we here just call *spaces*, and the maps in it, we call *functions*. If $A \in \mathcal{A}$, we denote $\overline{A} \in \mathcal{A}^{op}$ the correponding space, and similarly for maps.

A main object in $\mathcal{A}$ is the polynomial ring $A[x]$ in one variable; as a space it is denoted R,

$$R := \overline{A[x]}.$$

Because $A[x]$ is the free algebra in one generator x, there is, for any algebra B, a 1-1 correspondence between the set of elements of B and the set of algebra maps $A[x] \to B$, with dual notation, with the set of functions $\overline{B} \to \overline{A[x]} = R$. Thus, we have the basic fact:

elements of an algebra B correspond to R-valued functions on the space $\overline{B}$.

Since $\overline{A[x_1,\ldots,x_n]}$ is a coproduct in $\mathcal{A}$ of n copies of $A[x]$, it follows that $\overline{A[x_1,\ldots,x_n]} = R^n$, the "$n$-dimensional vector space over R", product of n copies of R. Therefore, the elements of $A[x_1,\ldots,x_n]$ correspond to functions $R^n \to R$, explaining in tautological terms the relationship between *polynomials* in n variables and *functions* $R^n \to R$; all functions $R^n \to R$ in $\mathcal{A}^{op}$ are polynomial.

Any ideal I in an an algebra B gives quotient map $B \twoheadrightarrow B/I$, and hence in $\mathcal{A}^{op}$ defines a monic function $\overline{B/I} \hookrightarrow \overline{B}$.

It is convenient to give names to some standard spaces thus defined. The space corresponding to $A[y_1,\ldots,y_m]/J$, where $J \subset A[y_1,\ldots,y_m]$ is the ideal generated by monomials of degree $n+1$, is denoted $D_n(m) \subset R^m$,

$$D_n(m) = \overline{A_{\leq n}[y_1,\ldots,y_m]},$$

and deserves the name "the nth infinitesimal neighbourhood of $0 \in R^m$". In the standard description of finite limits with internal variables we have:

$$D_n(m) = \{(x_1,\ \ldots . x_m) \in R^m \mid \forall f : [n{+}1] \to [m]\ \ x_{f(1)}\ \ldots\ x_{f(n+1)} = 0\}.$$

Likewise with the ideal $I \subseteq A[M^{m\times n}]$ described in Section 1.2. In this case we have

$$D_1(m)^n = \overline{A_a[M^{m\times n}]};$$

this follows since $A[M^{m\times n}]/I$ is the coproduct in $\mathcal{A}$ of n copies of $A[x_1,\ldots,x_m]/J$, where J now is the ideal generated by monomials of degree 2.

With internal variables we have the description:

$$D_1(m)^n = \{(x_{1,1},\ \ldots\ x_{m,n}) \in R^{m\times n} \mid x_{i,j}\, x_{i',j} = 0\}$$

(where i and i' range over $[m]$ and j over $[n]$), which is easily understood by the isomorphism $R^{m\times n} = (R^m)^n$.

2.2 Orbit space

Let B be an algebra, and let S be a finite group acting on B. The subalgebra $\mathrm{sym}_S(B) \subseteq B$ of *invariant* or *symmetric* elements may be described in categorical terms, in the category $\mathcal{A}$, as the joint equalizer of the automorphisms of the form $B \xrightarrow{\sigma} B$ over all the $\sigma, \sigma' \ldots \in S$,

$$\mathrm{sym}_S(B) \hookrightarrow B \underset{\sigma}{\overset{\sigma'}{\underset{\vdots}{\rightrightarrows}}} B\ .$$

In the category $\mathcal{A}^{op}$, this becomes a joint coequalizer, thus the orbit object of the action of S,

$$\overline{B}/S \twoheadleftarrow \overline{B} \underset{\overline{\sigma}}{\overset{\overline{\sigma'}}{\underset{\vdots}{\leftleftarrows}}} \overline{B}\ .$$

The isomorphism s in the Theorem 1.6, see diagram (8), is displayed in the following commutative diagram:

$$\begin{array}{ccccccc}
A[y_1,\ldots,y_m] & \xrightarrow{S} & \mathrm{sym}(A[M^{m\times n}]) & \hookrightarrow & A[M^{m\times n}] & \underset{\sigma}{\overset{\sigma'}{\underset{\vdots}{\rightrightarrows}}} & A[M^{m\times n}] \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
A_{\leq n}[y_1,\ldots,y_m] & \underset{\cong}{\xrightarrow{s}} & \mathrm{sym}(A_a[M^{m\times n}]) & \hookrightarrow & A_a[M^{m\times n}] & \underset{\sigma}{\overset{\sigma'}{\underset{\vdots}{\rightrightarrows}}} & A_a[M^{m\times n}]
\end{array}$$

By a tautological rewriting, this diagram becomes

$$\begin{array}{ccccccc}
R^m & \xleftarrow{\overline{S}} & (R^m)^n/S_n & \twoheadleftarrow & (R^m)^n & \underset{\overline{\sigma}}{\overset{\overline{\sigma'}}{\underset{\vdots}{\leftleftarrows}}} & (R^m)^n \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
D_n(m) & \underset{\cong}{\xleftarrow{\overline{s}}} & D_1(m)^n/S_n & \twoheadleftarrow & D_1(m)^n & \underset{\overline{\sigma}}{\overset{\overline{\sigma'}}{\underset{\vdots}{\leftleftarrows}}} & D_1(m)^n
\end{array} \qquad (9)$$

The composite map $(R^m)^n \to R^m$ in the diagram is, in synthetic terms: "take an n tuple of vectors in R^m, and add them up". It is symmetric in the n arguments; and it restricts to a map

$$\mathrm{sum} : D_1(m)^n \to D_n(m).$$

Theorem 1.6 then can be expressed as follows:

Theorem 2.1. *The addition map* $\mathrm{sum} : D_1(m)^n \to D_n(m)$ *is the quotient map of* $D(m)^n$ *under permutations of the* n *factors, i.e. is universal among* S_n*-symmetric maps out of* $D_1(m)^n$.

The special case where $m = 1$ was called "the symmetric functions property" in the early days of synthetic differential geometry (see e.g. Exercise I.4.4 in [2]); in this form, it was used (see e.g. [2] Exercise I.8.3 and I.8.4, or [1] Proposition 3.4) to establish the Formal Integration for vector fields: extending a vector field $D \times M \longrightarrow M$ into a "formal flow" $D_\infty \times M \longrightarrow M$.

Remark. It is not hard to prove that the constructions and results so far can be presented in a coordinate-free way, i.e. referring to an abstract m-dimensional vector space V over R, rather than to R^m, thus replacing e.g. the subspace $D_n(m) \subseteq R^m$ by a subspace $D_n(V)$; see e.g. [3] 1.2 for the definition of this subobject.

3. Primitives for closed differential 1-forms

The following Section is sketchy, and is included to give an indication of the kind of motivation from synthetic differential geometry that lead to the algebraic result stated in Theorem 1.6 or Theorem 2.1. Therefore, we do not attempt to give the reasoning fully explained, or in its full generality (e.g. replacing the space R^m by an abstract vector space $V \cong R^m$, or even by an arbitrary manifold). Also, some of the structure involved, like the ring structure on R (= the co-ring structure on $A[x]$), we shall assume known. Details may be found in [3], and the references therein.

Two points x and y in R^m are called *first order neighbours* if $y - x \in D_1(m)$. In this case, we write $x \sim y$. The relation $\sim$ is symmetric and reflexive, but not transitive. A differential 1-form ω on R^m may synthetically be described as an R-valued function ω defined on pairs of 1st order neighbour points x, y in R^m, with $\omega(x, x) = 0$ for all x. It is *closed* if for any three points x, y, z with $x \sim y$, $y \sim z$ and $x \sim z$, we have $\omega(x, y) + \omega(y, z) = \omega(x, z)$. Now, in R^m, the data of a 1-form ω may be encoded by giving a function $\Omega(-;-) : R^m \times R^m \to R$, linear in the argument after the semicolon, and such that

$$\omega(x, y) = \Omega\,(x; y - x), \quad \text{for } \ x \sim y.$$

Closedness of ω implies that the bilinear $d\Omega(x; -, -) : R^m \times R^m \to R$ is symmetric (see Proposition 2.2.7 in [3]). Hence, by the symmetric functions property (for the given m, and for $n = 2$), or by simple polarization, we get that the bilinear form $d\Omega(x; -, -)$ only depends on the sum of the two arguments. From this, it is easy to conclude (essentially by the Taylor expansion in the proof of the quoted Proposition) that $\omega(x, y) + \omega(y, z)$ is independent of y, even without assuming that $x \sim z$.

If $f : R^m \to R$ is a function, we get a closed 1-form df on M by $df(x, y) := f(y) - f(x)$. If $\omega = df$, we say that f is a *primitive* of ω. We may attempt to find a primitive f of a given closed 1-form ω, in a neighbourhood of the form $x_0 + D_n(m)$, where $x_0 \in R^m$. For a chain $x_0 \sim x_1 \sim \ldots \sim x_n$ (with each $x_i \sim x_{i+1}$), we want to define $f(x_n)$ by the sum

$$\omega(x_0, x_1) + \omega(x_1, x_2) + \ldots + \omega(x_{n-1}, x_n); \tag{10}$$

is this "definition" of $f(x_n)$ independent of the "interpolating points" $x_1, \ldots, \ldots, x_{n-1}$? We may write $x_{i+1} = x_i + d_{i+1}$ with $d_i \in D(V)$ $(i = 0, \ldots, n-1)$. In this case, the first question is whether the proposed value of $f(x_0 + d_1 + \ldots + d_n)$ is independent the individual d_is $(i < n)$ and only depends on their sum. By the symmetric functions property, this will follow if the sum is independent of the order in which we take the increments d_i. But this independence follows because closedness of ω implies $\omega(x, x + d) + \omega(x + d, x + d + d') = \omega(x, x + d') + \omega(x + d', x + d + d')$, thus two consecutive summands in the proposed chain of d_is may be interchanged; and such transpositions generate the whole of S_n. So Theorem 2.1 allows us to define $f : x_0 + D_n(m) \to R$ by the formula (10).

It is then easy to conclude that $f(y) - f(x) = \omega(x, y)$ for any $y \sim x$, for any x in the "formal neighbourhood of x_0" (meaning the set of points which can be reached by a chain $x_0 \sim x_1 \sim \ldots \sim x$, starting in x_0. So f is a primitive of ω on this formal neighbourhood.

References

[1] M. Bunge and E. Dubuc, Local Concepts in Synthetic Differential Geometry and Germ Representability, in *Mathematical Logic and Theoretical Computer Science*, ed. Kueker, Lopez-Escobar and Smith, Marcel Dekker 1987.

[2] A. Kock, *Synthetic Differential Geometry*, London Math. Soc. Lecture Notes Series no. 51, Cambridge Univ. Press 1981 (2nd ed. Lon-

don Math. Soc. Lecture Notes Series no. 333, Cambridge Univ. Press 2006).

[3] A. Kock, *Synthetic Geometry of Manifolds*, Cambridge Tracts in Mathematics no. 180, Cambridge Univ. Press 2010.

[4] B.L. van der Waerden, *Algebra* Vol. I, Grundlehren der Math. Wissenschaften no. 33, 5th ed. Springer Verlag 1960.

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PERSISTENT DOUBLE LIMITS

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Résumé. En poursuivant notre étude des limites doubles dans les catégories doubles faibles, nous étudions le 'cas invariant' des limites doubles persistentes, lié aux limites pondérées flexibles dans les 2-catégories.
Abstract. Continuing our study of double limits in weak double categories, we investigate the 'invariant case' of persistent double limits, which is related to flexible weighted limits in 2- categories.
Keywords. double category, 2-category, double limit, persistent limit, flexible weighted limit.
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0. Introduction

Strict double limits were presented in a talk at the International Category Theory Meeting of Bangor, in 1989 [Pa], showing that any W-weighted limit in a 2-category can be obtained as a double limit, parametrised over a suitable *double* category $\mathbb{E}\mathrm{l}(W)$ of elements of the 2-functor W.

A crucial point of this talk was the introduction of *persistent (double) limits* of functors $\mathbb{I} \to \mathbb{A}$, defined by a property of invariance up to equivalence. They were characterised by a condition on the double category $\mathbb{I}$, namely the existence of a natural weak initial object, and shown to be related to flexible weighted limits in 2-categories, by examining various relevant cases.

The second part of Verity's thesis [Ve], in 1992, took on the study of persistent double limits (based on strict double categories) and proved that the so called (PIES)*-class of flexible weighted colimits in the 2-category

Cat (defined in [BKPS]) is the same as the class of persistent colimits in **Cat** (viewed as a double category with trivial vertical arrows).

Weak double categories and their double limits were introduced and studied in 1999, in our joint paper [GP1], the first of a series on weak double categories. In particular it is proved that, in a 2-category, the existence of all weighted limits (characterised in [St]) is equivalent to the existence of all double limits, based on weak double categories. We want to complete this topic.

Here, after reviewing our terminology for weak double categories, various notions of 'horizontal equivalence' are studied in Sections 2–4; in particular we introduce the crucial notions of 'equivalence cell' (in 2.2) and 'pointwise equivalence of lax functors' (in 3.3). Then Section 5 deals with limits and pseudo limits in double categories (the latter are introduced here).

Persistent limits are defined in 6.1, as the double limits invariant up to pointwise equivalence. Then they are characterised by two Persistence Theorems (in 6.4 and 6.6): essentially, a weak double category $\mathbb{I}$ parametrises persistent limits if and only if it has a component-wise natural weak initial object (as claimed in [Pa]), if and only if $\mathbb{I}$-based limits and pseudo limits coincide up to equivalence. Section 7 concludes the proof of these results.

The links between persistent and flexible limits in 2-categories are deferred to a sequel.

Categories and 2-categories are generally written as $\mathbf{A}, \mathbf{B}, ...$; double categories as $\mathbb{A}, \mathbb{B}, ...$

We thank the Referee for very accurate suggestions and remarks.

1. Terminology for double categories

Strict double categories were introduced and studied by C. Ehresmann [Eh1, Eh2], the weak notion in our series [GP1]–[GP5]. The strict case extends the more usual (if historically subsequent) notion of 2-category, while the weak one extends bicategories, introduced by Bénabou [Be]. The extension is made clear in Subsection 1.4.

We review our terminology for strict and weak double categories [GP1]. We end by recalling one of the main examples, the weak double category $\mathbb{C}\mathrm{at}$ of categories, functors and profunctors, which will play an important role in the proof of the Persistence Theorem.

1.1 Strict double categories

A (strict) *double category* $\mathbb{A}$ consists of the following structure.

(a) A set $\mathrm{Ob}\mathbb{A}$ of *objects* of $\mathbb{A}$.

(b) *Horizontal morphisms* $f\colon A \to A'$ between those objects; they form the category $\mathrm{Hor}_0\mathbb{A}$ *of the objects and horizontal maps* of $\mathbb{A}$, with composition written as gf (or $g.f$) and identities $1_A\colon A \to A$.

(c) *Vertical morphisms* $u\colon A \nrightarrow B$ (often denoted by a dot-marked arrow) between the same objects; they form the category $\mathrm{Ver}_0\mathbb{A}$ *of the objects and vertical maps* of $\mathbb{A}$, with composition generally written as $u \otimes v$, in diagrammatic order, and identities written as $e_A\colon A \nrightarrow A$.

(d) (*Double*) *cells* $\alpha\colon (u\ {}^{f}_{g}\ v)$ with a *boundary* formed of two vertical arrows u, v and two horizontal arrows f, g

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & A' \\ {\scriptstyle u}\big\downarrow & \alpha & \big\downarrow{\scriptstyle v} \\ B & \xrightarrow[\ g\ ]{} & B' \end{array} \qquad\qquad \alpha\colon (u\ {}^{f}_{g}\ v)\colon ({}^{A}_{B}\ {}^{A'}_{B'}). \qquad (1)$$

Writing $\alpha\colon (A\ {}^{A}_{g}\ v)$ or $\alpha\colon (e\ {}^{1}_{g}\ v)$ we mean that $f = 1_A$ and $u = e_A$. The cell α is also written as $\alpha\colon u \to v$ (with respect to its *horizontal* domain and codomain, which are vertical arrows) or as $\alpha\colon f \nrightarrow g$ (with respect to its *vertical* domain and codomain, which are horizontal arrows).

We refer now to the following diagrams of cells, where the first is called a *consistent matrix* $({}^{\alpha}_{\gamma}\ {}^{\beta}_{\delta})$ of cells

$$\begin{array}{ccccc} A & \xrightarrow{\ f\ } & A' & \xrightarrow{\ f'\ } & A'' \\ {\scriptstyle u}\big\downarrow & \alpha & \big\downarrow{\scriptstyle v} & \beta & \big\downarrow{\scriptstyle w} \\ B & \xrightarrow{\ g\ } & B' & \xrightarrow{\ g'\ } & B'' \\ {\scriptstyle u'}\big\downarrow & \gamma & \big\downarrow{\scriptstyle v'} & \delta & \big\downarrow{\scriptstyle w'} \\ C & \xrightarrow[\ h\ ]{} & C' & \xrightarrow[\ h'\ ]{} & C'' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\ 1\ } & A \\ {\scriptstyle u}\big\downarrow & 1_u & \big\downarrow{\scriptstyle u} \\ B & \xrightarrow[\ 1\ ]{} & B \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\ f\ } & A' \\ {\scriptstyle e}\big\downarrow & e_f & \big\downarrow{\scriptstyle e} \\ A & \xrightarrow[\ f\ ]{} & A' \end{array} \qquad (2)$$

(e) Cells have a *horizontal composition*, consistent with the horizontal composition of arrows and written as $(\alpha \,|\, \beta)\colon (u\ {}^{f'f}_{g'g}\ w)$, or $\alpha \,|\, \beta$; this composition gives the category $\mathrm{Hor}_1\mathbb{A}$ *of vertical arrows and cells* $\alpha\colon u \to v$ of $\mathbb{A}$, with identities $1_u\colon (u \, {}^{1}_{1} \, u)$.

(f) Cells have also a *vertical composition*, consistent with the vertical composition of arrows, and written in the following forms:

$$\left(\frac{\alpha}{\gamma}\right)\colon (u \otimes u' \; {}^{f}_{h} \; v \otimes v'), \qquad \frac{\alpha}{\gamma}, \qquad \alpha \otimes \gamma.$$

This composition gives the category $\mathrm{Ver}_1\mathbb{A}$ *of horizontal arrows and cells* $a\colon f \to g$ of $\mathbb{A}$, with identities $e_f\colon (e \, {}^{f}_{f} \, e)$.

(g) The two compositions satisfy the *interchange laws* (for binary and zeroary compositions), which means that we have, in diagram (2):

$$\begin{array}{cc} \left(\dfrac{\alpha \,|\, \beta}{\gamma \,|\, \delta}\right) = \left(\dfrac{\alpha}{\gamma} \,\Big|\, \dfrac{\beta}{\delta}\right), & \left(\dfrac{1_u}{1_{u'}}\right) = 1_{u\otimes u'}, \\ (e_f \,|\, e_{f'}) = e_{f'f}, & 1_{e_A} = e_{1_A}. \end{array} \qquad (3)$$

The first condition says that a consistent matrix has a precise *pasting*; the last says that an object A has an *identity cell* $\square_A = 1_{e_A} = e_{1_A}$. The expressions $(\alpha \,|\, f')$ and $(f \,|\, \beta)$ will stand for $(\alpha \,|\, e_{f'})$ and $(e_f \,|\, \beta)$, when this makes sense (i.e. when $A' = B'$ and $v = e_{A'}$ in diagram (2)).

1.2 Weak double categories

More generally, in a *weak double category* $\mathbb{A}$ the horizontal composition behaves categorically (and we still have ordinary categories $\mathrm{Hor}_0\mathbb{A}$ and $\mathrm{Hor}_1\mathbb{A}$), while the composition of vertical arrows is categorical up to *comparison cells*.

Namely we have a *left unitor* λu, a *right unitor* ρu (for a vertical arrow $u\colon A \to B$) and an *associator* $\kappa(u,v,w)$ (for three consecutive vertical arrows)

$$\begin{array}{c} \lambda u\colon e_A \otimes u \to u, \qquad \rho u\colon u \otimes e_B \to u, \\ \kappa(u,v,w)\colon u \otimes (v {\otimes} w) \to (u {\otimes} v) \otimes w, \end{array} \qquad (4)$$

$$\begin{array}{ccc} A \;=\!=\!=\; A & A \;=\!=\!=\; A & A \;=\!=\!=\; A \\ {\scriptstyle e\otimes u}\downarrow \quad \lambda u \quad \downarrow{\scriptstyle u} & {\scriptstyle u\otimes e}\downarrow \quad \rho u \quad \downarrow{\scriptstyle u} & {\scriptstyle u\otimes(v\otimes w)}\downarrow \quad \kappa u \quad \downarrow{\scriptstyle (u\otimes v)\otimes w} \\ B \;=\!=\!=\; B & B \;=\!=\!=\; B & D \;=\!=\!=\; D \end{array}$$

Interchange holds strictly, as above. The comparison cells are *special* (which means that their horizontal arrows are identities) and horizontally invertible, also called *special isocells*. Moreover they are assumed to be *natural* and *coherent*, in a sense made precise in [GP1], Section 7: after stating naturality with respect to double cells, the coherence axioms are similar to those of bicategories. The terminology of the strict case is extended to the present one, as far as possible.

A *lax (double) functor* $F\colon \mathbb{X} \to \mathbb{A}$ between weak double categories amounts to assigning:

(a) two functors Hor_0F and Hor_1F, consistent with domain and codomain

$$\begin{array}{ccc} \mathrm{Hor}_1\mathbb{X} & \xrightarrow{\mathrm{Hor}_1F} & \mathrm{Hor}_1\mathbb{A} \\ {\scriptstyle \mathrm{Dom}}\downarrow & & \downarrow{\scriptstyle \mathrm{Dom}} \\ \mathrm{Hor}_0\mathbb{X} & \xrightarrow[\mathrm{Hor}_0F]{} & \mathrm{Hor}_0\mathbb{A} \end{array} \qquad \begin{array}{ccc} \mathrm{Hor}_1\mathbb{X} & \xrightarrow{\mathrm{Hor}_1F} & \mathrm{Hor}_1\mathbb{A} \\ {\scriptstyle \mathrm{Cod}}\downarrow & & \downarrow{\scriptstyle \mathrm{Cod}} \\ \mathrm{Hor}_0\mathbb{X} & \xrightarrow[\mathrm{Hor}_0F]{} & \mathrm{Hor}_0\mathbb{A} \end{array} \qquad (5)$$

(b) for any object X in $\mathbb{X}$ a special cell, the *unit comparison* of F

$$\underline{F}(X)\colon e_{FX} \to Fe_X\colon FX \to FX,$$

(c) for any vertical composite $u \otimes v\colon X \to\!\!\!\!\!\!\bullet\; Y \to\!\!\!\!\!\!\bullet\; Z$ in $\mathbb{X}$ a special cell, the *composition comparison*

$$\underline{F}(u, v)\colon Fu \otimes Fv \to F(u \otimes v)\colon FX \to FZ.$$

Again, these comparisons must satisfy axioms of naturality and coherence with the comparisons of $\mathbb{X}$ and $\mathbb{A}$ [GP2].

In a *pseudo* (resp. *strict*) functor these special cells are horizontally invertible (resp. identities).

1.3 Unitarity

From now on a weak double category $\mathbb{A}$ will be assumed to be *unitary*, which means that the unitors are identities and (therefore) the vertical identities behave as strict units.

As remarked in [GP2], by this assumption the composite of three vertical arrows is well-defined (without any brackets) *whenever one of them is an* *identity*: in this case, the associator $\kappa(u, v, w)$ is an identity (because of its coherence with the relevant unitors, which are assumed to be identities).

As a consequence, the *vertical composite of three cells is well-defined whenever each of their two triples of vertical arrows falls in the previous situation*, so that both associators are identities. We shall refer to this situation as a *normal ternary composition*, of arrows or cells.

In our main examples the composition of the vertical arrows (typically spans, cospans, profunctors...) is settled by choice, and can be made unitary by obvious constraints on the latter.

In the general case one can obtain unitarity by modifying the vertical composition of arrows

$$e_A \mathbin{\bar{\otimes}} u = u, \qquad u \mathbin{\bar{\otimes}} e_B = u \qquad (\text{for } u\colon A \to B). \tag{6}$$

The vertical composition of cells $\alpha \mathbin{\bar{\otimes}} \beta$ is also modified whenever vertical identities are present, as in the following two examples (with $\lambda' u = (\lambda u)^{-1}$, the horizontal inverse)

$$\begin{array}{ccccc} A & \xrightarrow{f} & A' & = & A' \\ {\scriptstyle u}\downarrow & \alpha & \downarrow{\scriptstyle v} & & \\ B & \xrightarrow{g} & B' & \rho v & \downarrow{\scriptstyle v} \\ {\scriptstyle u'}\downarrow & \beta & \downarrow{\scriptstyle e} & & \\ C & \xrightarrow[h]{} & B' & = & B' \end{array} \qquad \begin{array}{ccccccc} A & = & A & \xrightarrow{f} & A' & = & A' \\ & & {\scriptstyle e}\downarrow & \alpha & \downarrow{\scriptstyle v} & & \\ {\scriptstyle u}\downarrow & \lambda' u & A & \xrightarrow{g} & B' & \rho v & \downarrow{\scriptstyle v} \\ & & {\scriptstyle u}\downarrow & \beta & \downarrow{\scriptstyle e} & & \\ C & = & C & \xrightarrow[h]{} & B' & = & B' \end{array} \tag{7}$$

Finally, the new unitors are identities and the associators are also modified, letting $\bar{\kappa}(u, v, w)$ be an identity whenever at least one of u, v, w is a vertical identity.

1.4 Special cells and globular cells

In the *special* cells $\alpha\colon u \to v$ considered above the horizontal arrows are identities; similarly we speak of a *globular* cell $\varphi\colon f \to g$ when the vertical arrows are identities

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ {\scriptstyle u}\downarrow & \alpha & \downarrow{\scriptstyle v} \\ B & \xrightarrow[1]{} & B \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle e}\downarrow & \varphi & \downarrow{\scriptstyle e} \\ A & \xrightarrow[g]{} & A' \end{array} \tag{8}$$

By restricting the cells of the weak double category $\mathbb{A}$ in these two ways we get two important substructures.

(a) The *vertical bicategory* $\mathbf{Ver}\mathbb{A}$ is the graph $\mathrm{Ver}_0\mathbb{A}$ enriched with 2-cells $\alpha\colon u \to v$ given by the special cells of $\mathbb{A}$

$$\begin{array}{c} A \\ u \left(\begin{array}{c} \alpha \\ \to \end{array} \right) v \\ B \end{array} \tag{9}$$

with the original operations and comparisons. The horizontal inverse of a special isocell $\alpha\colon u \to v$ is written as $\alpha^{-1}\colon v \to u$.

A weak double category whose horizontal arrows are identities will be called a *vertical weak double category*. It is the same as a *bicategory written vertically*, with arrows and weak composition in the vertical direction and strict composition in the horizontal one.

(b) Because of unitarity, $\mathbb{A}$ contains also a *horizontal 2-category* $\mathbf{Hor}\mathbb{A}$, formed by the category $\mathrm{Hor}_0\mathbb{A}$ enriched with 2-cells $\varphi\colon f \to\!\!\!\!\!\to g\colon A \to A'$ provided by the globular cells of $\mathbb{A}$

$$A \underset{g}{\overset{f}{\rightrightarrows}} A' \quad \downarrow\varphi \tag{10}$$

(Without the unitarity assumption we should here modify the vertical composition of these cells, as in 1.3. This would leave us with two vertical compositions, which might result in misunderstandings.)

This cell φ is *vertically invertible*, or a *globular isocell*, if it has a vertical inverse in the 2-category $\mathbf{Hor}\mathbb{A}$, written as $\varphi^{\smile}\colon g \to\!\!\!\!\!\to f$

$$\varphi \otimes \varphi^{\smile} = e_f, \qquad\qquad \varphi^{\smile} \otimes \varphi = e_g. \tag{11}$$

Then the arrows f, g are said to be *vertically isomorphic*. A vertically invertible cell will always be a globular one. (A general notion of 'vertically invertible cells' in a weak double category would be ill-founded, in the same way as to speak of 'invertible spans' or 'invertible profunctors', in a strict sense.)

A double category whose vertical arrows are identities will be called a *horizontal double category*. It is the same as a 2-category (horizontally written, as usual.)

1.5 Dualities

A weak double category has a *horizontal opposite* $\mathbb{A}^{\mathrm{h}}$ (reversing the horizontal direction) and a *vertical opposite* $\mathbb{A}^{\mathrm{v}}$ (reversing the vertical direction). A strict structure also has a *transpose* $\mathbb{A}^{\mathrm{t}}$ (interchanging the horizontal and vertical issues).

The prefix 'co', as in *colimit*, *coequaliser* or *colax double functor*, always refers to horizontal duality, the main one. The prefix 'op' is not used here.

1.6 The weak double category of profunctors

The weak double category $\mathbb{C}\mathrm{at}$ *of categories, functors and profunctors* is a prime example, studied in [GP1] and to be used below.

An object is a small category, a horizontal arrow is a functor and a vertical arrow is a profunctor $u\colon X \nrightarrow Y$, defined as a functor $u\colon X^{\mathrm{op}}\times Y \to \mathbf{Set}$. A cell $\alpha\colon (u\,{}^{f}_{g}\,v)$ is a natural transformation

$$\alpha\colon u \to v.(f^{\mathrm{op}} \times g)\colon X^{\mathrm{op}} \times Y \to \mathbf{Set}.$$

Compositions and comparisons are known or easily defined.

A profunctor $u\colon X \nrightarrow Y$ determines a category $\perp u = X +_u Y$, called the *gluing*, or *collage*, of X and Y along u. It consists of the categorical sum $X + Y$ with new maps $\lambda\colon x \to y$ in the set $u(x, y)$, for $x \in \mathrm{Ob}X$, $y \in \mathrm{Ob}Y$, and no maps backwards; the composition of the new maps with the old ones is defined by the action of u.

This category, equipped with the inclusions $i\colon X \to \perp u$, $j\colon Y \to \perp u$ and the obvious cell ι

$$\iota\colon (u\,{}^{i}_{j}\,e), \qquad \iota\colon u \to e_{\perp u}(i^{\mathrm{op}} \times j)\colon X^{\mathrm{op}} \times Y \to \mathbf{Set}, \qquad \iota(x, y)\colon u(x, y) = \perp u(x, y), \tag{12}$$

gives the *cotabulator* of u, i.e. its colimit [GP1]. (Note that u can be recovered from these data.)

The Grothendieck construction is also well known: the profunctor u has a category $\top u = \mathbf{El}(u)$ *of elements* of u. It has objects (x, y, λ) with $x \in \mathrm{Ob}X$, $y \in \mathrm{Ob}Y$, $\lambda \in u(x, y)$ and maps (f, g) of $X \times Y$ which form a commutative square in the collage $X +_u Y$

$$\begin{gathered} (f, g) \colon (x, y, \lambda) \to (x', y', \lambda'), \\ f \colon x \to x', \qquad\qquad g \colon y \to y', \qquad\qquad g\lambda = \lambda' f. \end{gathered} \tag{13}$$

(The last condition means that $u(1_x, g)(\lambda) = u(f, 1_y)(\lambda')$ in $u(x, y')$.) This category is the *tabulator* $\top u$ of u, i.e. its limit, when equipped with the projections $p \colon \top u \to X$, $q \colon \top u \to Y$ and the following cell τ

$$\begin{gathered} \tau \colon (e \, {}^{p}_{q} \, u), \qquad \tau \colon e_{\top u} \to u(p^{\mathrm{op}} \times q) \colon (\top u)^{\mathrm{op}} \times \top u \to \mathbf{Set}, \\ \tau(x, y, \lambda; x', y', \lambda') \colon \top u(x, y, \lambda; x', y', \lambda') \to u(x, y'), \\ (f, g) \mapsto g\lambda = \lambda' f. \end{gathered} \tag{14}$$

The reader will note that a globular cell $\varphi \colon (X \, {}^{f}_{g} \, X')$ is the same as a natural transformation $\varphi \colon f \to g \colon X \to X'$, so that the horizontal 2-category $\mathbf{Hor}\mathbb{C}\mathrm{at}$ is the usual 2-category $\mathbf{Cat}$ of categories, functor and natural transformations.

2. Equivalence arrows and equivalence cells

$\mathbb{A}$ is always a (unitary) weak double category. We introduce two crucial notions of 'horizontal equivalence' that will be used to define the persistence property. (Vertical equivalences are defined in [GP1], Subsection 2.2.)

2.1 Definition (Horizontal equivalence)

A horizontal arrow $f \colon A \to A'$ in the weak double category $\mathbb{A}$ is a *horizontal equivalence* if it is an equivalence in the 2-category $\mathbf{Hor}\mathbb{A}$ introduced above (in 1.4).

This means that there exists a horizontal arrow $f' \colon A' \to A$ with vertically invertible cells $\eta\colon 1_A \to f'f$ and $\varepsilon\colon ff' \to 1_{A'}$

$$\begin{array}{ccccc} A & \xrightarrow{\quad 1 \quad} & & & A \\ {\scriptstyle e}\downarrow & & \eta & & \downarrow{\scriptstyle e} \\ A & \xrightarrow[f]{} & A' & \xrightarrow[f']{} & A \end{array} \qquad \begin{array}{ccccc} A' & \xrightarrow{f'} & A & \xrightarrow{f} & A' \\ {\scriptstyle e}\downarrow & & \varepsilon & & \downarrow{\scriptstyle e} \\ A' & & \xrightarrow[1]{\quad\quad} & & A' \end{array} \tag{15}$$

Then f' is called a (horizontal) pseudo inverse of f. As well known, one can always modify the cell ε (or the cell η) so that the four-tuple $(f, f', \eta, \varepsilon)$ forms an *adjoint horizontal equivalence*, satisfying the triangular equations

$$(\eta \,|\, e_f) \otimes (e_f \,|\, \varepsilon) = e_f, \qquad (e_{f'} \,|\, \eta) \otimes (\varepsilon \,|\, e_{f'}) = e_{f'}, \tag{16}$$

which means that the cells $(\eta \,|\, e_f)$, $(e_f \,|\, \varepsilon)$ are vertically inverse in $\mathbf{Hor}\mathbb{A}$, as well as $(e_{f'} \,|\, \eta)$ and $(\varepsilon \,|\, e_{f'})$. We also express this fact saying that the pair (η, ε) is *coherent*.

2.2 Definition (Equivalence cell)

A cell $\alpha\colon (u \,{}^f_g\, v)$ is a (horizontal) *equivalence cell* in $\mathbb{A}$ if there exists a cell α' which is made its pseudo inverse by globular isocells $\eta, \varepsilon, \eta', \varepsilon'$ (in $\mathbf{Hor}\mathbb{A}$), in the following sense:

$$\text{(eqc1)}\quad \begin{array}{ccccc} A & & \xrightarrow{\quad 1 \quad} & & A \\ {\scriptstyle e}\downarrow & & \eta & & \downarrow{\scriptstyle e} \\ A & \xrightarrow{f} & A' & \xrightarrow{f'} & A \\ {\scriptstyle u}\downarrow & \alpha & \downarrow{\scriptstyle v} & \alpha' & \downarrow{\scriptstyle u} \\ B & \xrightarrow[g]{} & B' & \xrightarrow[g']{} & B \end{array} \quad = \quad \begin{array}{ccccc} A & & \xrightarrow{\quad 1 \quad} & & A \\ {\scriptstyle u}\downarrow & & 1_u & & \downarrow{\scriptstyle u} \\ B & & \xrightarrow{\quad 1 \quad} & & B \\ {\scriptstyle e}\downarrow & & \eta' & & \downarrow{\scriptstyle e} \\ B & \xrightarrow[g]{} & B' & \xrightarrow[g']{} & B \end{array}$$

$$\text{(eqc2)}\quad \begin{array}{ccccc} A' & \xrightarrow{f'} & A & \xrightarrow{f} & A' \\ {\scriptstyle v}\downarrow & \alpha' & \downarrow{\scriptstyle u} & \alpha & \downarrow{\scriptstyle v} \\ B' & \xrightarrow{g'} & B & \xrightarrow{g} & B' \\ {\scriptstyle e}\downarrow & & \varepsilon' & & \downarrow{\scriptstyle e} \\ B' & & \xrightarrow[1]{\quad\quad} & & B' \end{array} \quad = \quad \begin{array}{ccccc} A' & \xrightarrow{f'} & A & \xrightarrow{f} & A' \\ {\scriptstyle e}\downarrow & & \varepsilon & & \downarrow{\scriptstyle e} \\ A' & & \xrightarrow{\quad 1 \quad} & & A' \\ {\scriptstyle v}\downarrow & & 1_v & & \downarrow{\scriptstyle v} \\ B' & & \xrightarrow[1]{\quad\quad} & & B' \end{array}$$

Note that f and g are horizontal equivalences, with pseudo inverses f' and g', respectively.

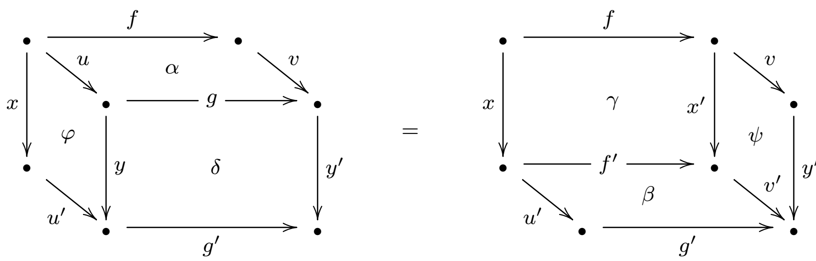
Again, one can modify (for instance) $\varepsilon, \varepsilon'$ so that the pairs (η, ε) and (η', ε') are coherent, and still satisfy the conditions above. This also follows from the remark below.

2.3 Remark

There is a weak double category $\mathbb{A}^{\downarrow}$, whose objects are the vertical arrows of $\mathbb{A}$. Its horizontal arrows $\alpha\colon u \to v$ are the cells of $\mathbb{A}$, its vertical arrows $u \to\!\!\!\!\!\cdot\;\, u'$ are squares $\varphi\colon x \otimes u' \to u \otimes y$ of vertical arrows inhabited by a special isocell, as in the left parallelogram below. A vertical identity E_u is inhabited by the horizontal identity $1_u\colon e \otimes u \to u \otimes e$.

Finally, a cell $\Gamma = (\gamma, \delta)\colon (\varphi \; {}^{\alpha}_{\beta} \; \psi)$ forms a 'commutative cube' of cells of $\mathbb{A}$

$$(\varphi \,|\, \alpha \otimes \delta) = (\gamma \otimes \beta \,|\, \psi), \tag{17}$$



An equivalence cell of $\mathbb{A}$ is the same as a horizontal equivalence $\alpha\colon u \to v$ of $\mathbb{A}^{\downarrow}$, i.e. an equivalence in the 2-category $\mathbf{Hor}\mathbb{A}^{\downarrow}$. The coherent cases also coincide.

$\mathbb{A}^{\downarrow}$ can be viewed as a substructure of $\mathbb{L}\mathrm{ax}(\mathbf{2}^{\mathrm{t}}, \mathbb{A})$, a weak double category introduced in Section 4, where $\mathbf{2}^{\mathrm{t}}$ is the strict double category containing a vertical arrow $0 \to\!\!\!\!\!\cdot\;\, 1$ and otherwise trivial.

2.4 Proposition

If in a cell $\alpha\colon (u \; {}^{f}_{g} \; v)$ of the weak double category $\mathbb{A}$ the arrows f, g are horizontal equivalences with pseudo inverses g, g' and vertically invertible *cells $\eta, \varepsilon, \eta', \varepsilon'$ (satisfying the coherence conditions* (16)*), there is at most one pseudo inverse cell $\alpha' \colon (v \, {}^{f'}_{g'} \, u)$ satisfying the equations* (eqc1, 2).

Proof. Supposing we have two candidates α' and α'', we get (using normal ternary compositions, see 1.3)

$$\begin{aligned} e_{f'} \otimes \alpha' \otimes e_{g'} &= 1_v \,|\, (e_{f'} \otimes \alpha' \otimes e_{g'}) = \\ &= (\breve{\varepsilon} \otimes (\alpha'' \,|\, \alpha) \otimes \varepsilon') \,|\, (e_{f'} \otimes \alpha' \otimes e_{g'}) \\ &= (\breve{\varepsilon} \,|\, e_{f'}) \otimes (\alpha'' \,|\, \alpha \,|\, \alpha') \otimes (\varepsilon' \,|\, e_{g'}), \end{aligned}$$

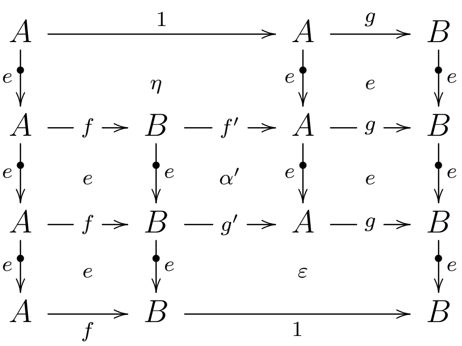
$$\begin{aligned} e_{f'} \otimes \alpha'' \otimes e_{g'} &= (e_{f'} \otimes \alpha'' \otimes e_{g'}) \,|\, 1_u \\ &= (e_{f'} \otimes \alpha'' \otimes e_{g'}) \,|\, (\eta \otimes (\alpha \,|\, \alpha') \otimes \breve{\eta'}) \\ &= (e_{f'} \,|\, \eta) \otimes (\alpha'' \,|\, \alpha \,|\, \alpha') \otimes (e_{g'} \,|\, \breve{\eta'}). \end{aligned}$$

But $(\varepsilon|e_{f'})$ and $(e_{f'}|\eta)$ are vertically inverse; the same holds for $(\varepsilon'|e_{g'})$ and $(e_{g'}|\eta')$. Therefore $\alpha' = \alpha''$. □

2.5 Proposition

An equivalence cell $\alpha \colon (A \, {}^{f}_{g} \, B)$ whose vertical arrows are identities is a globular isocell. Conversely, if α is a globular isocell and f, g are horizontal equivalences, then α is an equivalence cell.

Proof. Given a pseudo horizontal inverse $\alpha' \colon (B \, {}^{f'}_{g'} \, A)$, one gets a vertical inverse $\beta \colon (A \, {}^{g}_{f} \, B)$ in $\mathbf{Hor}\mathbb{A}$ by the following pasting



(18)

Conversely, given β, η, ε as above, we obtain α' by a similar pasting with the vertically inverses $\varepsilon^{\smallsmile}, \eta^{\smallsmile}$

$$\alpha' = (\varepsilon^{\smallsmile} \,|\, e_{f'}) \otimes (e_{g'} \,|\, \beta \,|\, e_{f'}) \otimes (e_{g'} \,|\, \eta^{\smallsmile}). \tag{19}$$

□

3. Pointwise equivalences of lax functors

We begin studying diagrams in $\mathbb{A}$, as lax (double) functors $F\colon \mathbb{I} \to \mathbb{A}$ where $\mathbb{I}$ is a small weak double category. (One could also consider the colax functors.)

3.1 Horizontal transformations of lax functors

As already recalled, a lax functor $F\colon \mathbb{I} \to \mathbb{A}$ has comparisons

$$\underline{F}I\colon e(FI) \to F(eI), \qquad \underline{F}(u,v)\colon Fu \otimes Fv \to F(u \otimes v). \tag{20}$$

A *horizontal transformation* $h\colon F \to G\colon \mathbb{I} \to \mathbb{A}$ of lax functors has the following components:

(a) for every object I in $\mathbb{I}$, a horizontal arrow $hI\colon FI \to GI$ in $\mathbb{A}$,

(b) for every vertical arrow $u\colon I \to\!\!\!\!\!\bullet\; J$ in $\mathbb{I}$, a cell $hu\colon Fu \to Gu$ in $\mathbb{A}$

$$\begin{array}{ccc} FI & \xrightarrow{hI} & GI \\ {\scriptstyle Fu}\downarrow & hu & \downarrow{\scriptstyle Gu} \\ FJ & \xrightarrow[hJ]{} & GJ \end{array} \tag{21}$$

satisfying a condition of naturality and two conditions of vertical coherence

$$\begin{aligned} &hu \,|\, G\alpha = F\alpha \,|\, hv && \text{for a cell } \alpha\colon u \to v \text{ in } \mathbb{I}, \\ &\underline{F}I \,|\, h(e_I) = e_{hI} \,|\, \underline{G}I && \text{for an object } I, \\ &\underline{F}(u,v) \,|\, hw = (hu \otimes hv) \,|\, \underline{G}(u,v) && \text{for } w = u \otimes v. \end{aligned} \tag{22}$$

These equations will be represented diagrammatically in the next subsection, in a more general situation.

The naturality condition above implies its 1-dimensional form: for every horizontal arrow $i\colon I \to I'$ in $\mathbb{I}$

$$Gi.hI = hI'.Fi. \qquad (23)$$

We want to say that h is a pointwise equivalence if all its components hI are horizontal equivalences and all its components hu are equivalence cells, as defined in Section 2. But then, choosing pseudo inverses $kI\colon GI \to FI$ and $ku\colon Gu \to Fu$, we only get a 'transformation' $k\colon G \to F$ that satisfies the naturality condition (23) up to vertically invertible comparison cells. We define now this notion of *pseudo horizontal transformation*, to be understood as a vertically-pseudo form of the previous notion, and will later define a pointwise equivalence in this more general frame.

3.2 Definition (Pseudo horizontal transformations)

A *pseudo horizontal transformation* $h\colon F \to G\colon \mathbb{I} \to \mathbb{A}$ of lax functors has components hI and hu as above, in 3.1(a), (b), and:

(c) for every horizontal arrow $i\colon I \to I'$ in $\mathbb{I}$, a vertically invertible comparison cell hi in $\mathbb{A}$

$$\begin{array}{ccccc} FI & \xrightarrow{hI} & GI & \xrightarrow{Gi} & GI' \\ {\scriptstyle e}\downarrow & & hi & & \downarrow{\scriptstyle e} \\ FI & \xrightarrow[Fi]{} & FI' & \xrightarrow[hI']{} & GI' \end{array} \qquad (24)$$

The following five axioms must be satisfied:

(pht1) (*naturality on a cell*) for a cell $\alpha\colon (u\ {}^{i}_{j}\ v)$ in $\mathbb{I}$

$$\begin{array}{ccccc} FI & \xrightarrow{hI} & GI & \xrightarrow{Gi} & GI' \\ {\scriptstyle Fu}\downarrow & hu & \downarrow{\scriptstyle Gu} & G\alpha & \downarrow{\scriptstyle Gv} \\ FJ & \xrightarrow{hJ} & GJ & \xrightarrow{Gj} & GJ' \\ {\scriptstyle e}\downarrow & & hj & & \downarrow{\scriptstyle e} \\ FJ & \xrightarrow[Fj]{} & FJ' & \xrightarrow[hJ']{} & GJ' \end{array} \quad = \quad \begin{array}{ccccc} FI & \xrightarrow{hI} & GI & \xrightarrow{Gi} & GI' \\ {\scriptstyle e}\downarrow & & hi & & \downarrow{\scriptstyle e} \\ FI & \xrightarrow{Fi} & FI' & \xrightarrow{hI'} & GJ' \\ {\scriptstyle Fu}\downarrow & F\alpha & \downarrow{\scriptstyle Fv} & hv & \downarrow{\scriptstyle Gv} \\ FJ & \xrightarrow[Fj]{} & FJ' & \xrightarrow[hJ']{} & GJ' \end{array}$$

(pht2) (*coherence with vertical identities*) for I in $\mathbb{I}$

$$\begin{array}{ccccc} FI & \xrightarrow{1} & FI & \xrightarrow{hI} & GI \\ {\scriptstyle e}\downarrow & \underline{F}I & \downarrow{\scriptstyle Fe} & he & \downarrow{\scriptstyle Ge} \\ FI & \xrightarrow[1]{} & FI & \xrightarrow[hI]{} & GI \end{array} \quad = \quad \begin{array}{ccccc} FI & \xrightarrow{hI} & GI & \xrightarrow{1} & GI \\ {\scriptstyle e}\downarrow & e_{hI} & \downarrow{\scriptstyle e} & \underline{G}I & \downarrow{\scriptstyle Ge} \\ FI & \xrightarrow[hI]{} & GI & \xrightarrow[1]{} & GI \end{array}$$

(pht3) (*coherence with vertical composition*) for a vertical composite $w = u \otimes v \colon I \to J \to K$ in $\mathbb{I}$

$$\begin{array}{ccccc} FI & \xrightarrow{1} & FI & \xrightarrow{hI} & GI \\ {\scriptstyle Fu}\downarrow & & & & \\ FJ & \underline{F}(u,v) & \downarrow{\scriptstyle Fw} & hw & \downarrow{\scriptstyle Gw} \\ {\scriptstyle Fv}\downarrow & & & & \\ FK & \xrightarrow[1]{} & FK & \xrightarrow[hK]{} & GK \end{array} \quad = \quad \begin{array}{ccccc} FI & \xrightarrow{hI} & GI & \xrightarrow{1} & GI \\ {\scriptstyle Fu}\downarrow & hu & \downarrow{\scriptstyle Gu} & & \\ FJ & \xrightarrow{hJ} & GJ & \underline{G}(u,v) & \downarrow{\scriptstyle Gw} \\ {\scriptstyle Fv}\downarrow & hv & \downarrow{\scriptstyle Gv} & & \\ FK & \xrightarrow[hK]{} & GK & \xrightarrow[1]{} & GK \end{array}$$

(pht4) (*coherence with horizontal identities*) for I in $\mathbb{I}$

$$\begin{array}{ccccc} FI & \xrightarrow{hI} & GI & \xrightarrow{1} & GI \\ {\scriptstyle e}\downarrow & & h1_I & & \downarrow{\scriptstyle e} \\ FI & \xrightarrow[1]{} & FI & \xrightarrow[hI]{} & GI \end{array} \quad = \quad e_{hI},$$

(pht5) (*coherence with horizontal composition*) for a horizontal composite $ji \colon I \to I' \to I''$

$$\begin{array}{ccccccc} FI & \xrightarrow{hI} & GI & \xrightarrow{Gi} & GI' & \xrightarrow{Gj} & GI'' \\ {\scriptstyle e}\downarrow & & hi & & {\scriptstyle e}\downarrow & e & \downarrow{\scriptstyle e} \\ FI & \xrightarrow{Fi} & FI' & \xrightarrow{hI'} & GI' & \xrightarrow{Gj} & GI'' \\ {\scriptstyle e}\downarrow & e & \downarrow{\scriptstyle e} & & hj & & \downarrow{\scriptstyle e} \\ FI & \xrightarrow[Fi]{} & FI' & \xrightarrow[Fj]{} & FI'' & \xrightarrow[hI'']{} & GI'' \end{array} \quad = \quad h(ji).$$

A horizontal transformation h is a pseudo horizontal transformation where all the comparison cells hi are identities. Then these cells become the (redundant) naturality condition (23), while (pht1–3) give the axioms of (22) and (pht4, 5) are vacuous.

3.3 Definition (Pointwise equivalences)

A pseudo horizontal transformation $h\colon F \to G$ is a *pointwise equivalence* (of lax functors) if:

(i) for every object I in $\mathbb{I}$, the horizontal arrow $hI\colon FI \to GI$ is a horizontal equivalence in $\mathbb{A}$,

(ii) for every vertical arrow $u\colon I \to\!\!\!\!\bullet\; J$ in $\mathbb{I}$, the cell $hu\colon Fu \to Gu$ is an equivalence cell in $\mathbb{A}$.

This is the same as an equivalence in a 2-category $\mathbf{Lax}(\mathbb{I}, \mathbb{A})$ of lax functors, as we shall see in the next section. Pointwise equivalence is thus an equivalence relation in the set of lax functors $\mathbb{I} \to \mathbb{A}$.

4. A weak double category of lax functors

We introduce a weak double category $\mathbb{L}\mathrm{ax}(\mathbb{I}, \mathbb{A})$ of lax functors $\mathbb{I} \to \mathbb{A}$, pseudo horizontal transformations, pseudo vertical transformations and modifications. It turns out that a pointwise equivalence $h\colon F \to G$ of lax functors is the same as a horizontal equivalence in $\mathbb{L}\mathrm{ax}(\mathbb{I}, \mathbb{A})$, i.e. an equivalence in the associated 2-category $\mathbf{Lax}(\mathbb{I}, \mathbb{A}) = \mathbf{Hor}\mathbb{L}\mathrm{ax}(\mathbb{I}, \mathbb{A})$.

In the present section we only use this 2-category; non-globular modifications will be used later, for the vertical arrows of the cones of a diagram.

4.1 Definition (Pseudo vertical transformations)

A *pseudo vertical transformation* $r\colon F \to\!\!\!\!\bullet\; G\colon \mathbb{I} \to \mathbb{A}$ of lax double functors consists of the following components:

(a) for every object I in $\mathbb{I}$, a vertical arrow $rI\colon FI \to\!\!\!\!\bullet\; GI$ in $\mathbb{A}$,

(b) for every $i\colon I \to I'$ in $\mathbb{I}$, a cell $ri\colon (rI \,{}^{Fi}_{Gi}\, rI')$ in $\mathbb{A}$,

(c) for every $u\colon I \to\!\!\!\!\bullet\; J$ in $\mathbb{I}$, a special isocell $ru\colon Fu \otimes rJ \to rI \otimes Gu$, the *naturality comparison*.

The following axioms must be satisfied:

(pvt1) (*naturality on a cell*) for a cell $\alpha\colon (u\ {}^{i}_{j}\ v)$ in $\mathbb{I}$

$$
\begin{array}{ccccc}
FI & \xrightarrow{Fi} & FI' & \xrightarrow{1} & FI' \\
{\scriptstyle Fu}\downarrow & F\alpha & \downarrow{\scriptstyle Fv} & & \downarrow{\scriptstyle rI'} \\
FJ & \xrightarrow{Fj} & FJ' & rv & GJ' \\
{\scriptstyle rJ}\downarrow & rj & \downarrow{\scriptstyle rJ'} & & \downarrow{\scriptstyle Gv} \\
GJ & \xrightarrow[Gj]{} & GJ' & \xrightarrow[1]{} & GJ'
\end{array}
\quad = \quad
\begin{array}{ccccc}
FI & \xrightarrow{1} & FI & \xrightarrow{Fi} & FI' \\
{\scriptstyle Fu}\downarrow & & \downarrow{\scriptstyle rI} & ri & \downarrow{\scriptstyle rI'} \\
FJ & ru & GI & \xrightarrow{Gi} & GJ' \\
{\scriptstyle rJ}\downarrow & & \downarrow{\scriptstyle Gu} & G\alpha & \downarrow{\scriptstyle Gv} \\
GJ & \xrightarrow[1]{} & GJ & \xrightarrow[Gj]{} & GJ'
\end{array}
$$

(pvt2) (*coherence with horizontal identities*) for I in $\mathbb{I}$, $r(1_I) = 1_{rI}$,

(pvt3) (*coherence with horizontal composition*) for $k = ji$ in $\mathbb{I}$, $r(k) = (ri\,|\,rj)$,

(pvt4) (*coherence with vertical identities*) for I in $\mathbb{I}$

$$
\begin{array}{ccc}
FI & \xrightarrow{1} & FI \\
{\scriptstyle rI}\downarrow & 1 & \downarrow{\scriptstyle rI} \\
GI & \xrightarrow{1} & GI \\
{\scriptstyle e}\downarrow & \underline{G}I & \downarrow{\scriptstyle Ge} \\
GI & \xrightarrow[1]{} & GI
\end{array}
\quad = \quad
\begin{array}{ccccc}
FI & \xrightarrow{1} & FI & \xrightarrow{1} & FI \\
{\scriptstyle e}\downarrow & \underline{F}I & \downarrow{\scriptstyle Fe} & & \downarrow{\scriptstyle rI} \\
FI & \xrightarrow{1} & FI & re_I & GI \\
{\scriptstyle rI}\downarrow & 1 & \downarrow{\scriptstyle rI} & & \downarrow{\scriptstyle Ge} \\
GI & \xrightarrow[1]{} & GI & \xrightarrow[1]{} & GI
\end{array}
$$

(pvt5) (*coherence with vertical composition*) for $w = u \otimes v\colon I \to J \to K$ in $\mathbb{I}$, the following pasting, where $\kappa' = \kappa^{-1}$

$$
\begin{array}{ccccccccccccc}
FI & = & \bullet & = & \bullet & = & \bullet & = & \bullet & = & \bullet & = & FI \\
 & & {\scriptstyle Fu}\downarrow & 1 & \downarrow{\scriptstyle Fu} & & \downarrow{\scriptstyle Fu} & & \downarrow{\scriptstyle rI} & & \downarrow{\scriptstyle rI} & 1 & \downarrow{\scriptstyle rI} \\
{\scriptstyle Fu\otimes Fv}\downarrow & \kappa & \bullet & = & \bullet & \kappa' & \bullet & ru & \bullet & \kappa & \bullet & = & GI \\
 & & {\scriptstyle Fv}\downarrow & & \downarrow{\scriptstyle rJ} & & \downarrow{\scriptstyle rJ} & & \downarrow{\scriptstyle Gu} & & & \underline{G} & \\
FK & & \bullet & rv & \bullet & & \bullet & = & \bullet & & \downarrow{\scriptstyle Gu\otimes Gv} & & \downarrow{\scriptstyle Gw} \\
{\scriptstyle rK}\downarrow & & {\scriptstyle rK}\downarrow & & \downarrow{\scriptstyle Gv} & & \downarrow{\scriptstyle Gv} & 1 & \downarrow{\scriptstyle Gv} & & & & \\
GK & = & \bullet & = & \bullet & = & \bullet & = & \bullet & = & \bullet & = & GK
\end{array}
$$

coincides with $(\underline{F}(u, v) \otimes 1_{rK}\,|\,rw)$.

For a consecutive pseudo vertical transformation $s\colon G \dashrightarrow H$, the vertical composite $t = r \otimes s$ has obvious components tI and tf, with special isocells tu obtained by pasting the ones of r and s by means of the associativity isocells κ of $\mathbb{A}$, as shown in the diagram below:

$$\begin{gathered} tI = rI \otimes sI\colon FI \dashrightarrow HI, \qquad tf = rf \otimes sf\colon (tI\ {}^{Ff}_{Hf}\ tI') \\ tu\colon Fu \otimes tJ \to tI \otimes Hu\colon FI \dashrightarrow HJ, \end{gathered} \tag{25}$$

$$tu \quad = \quad \begin{array}{ccccccccccc}
FI & = & \bullet & = & \bullet & = & \bullet & = & \bullet & = & FI \\
{\scriptstyle Fu}\downarrow & & {\scriptstyle Fu}\downarrow & & {\scriptstyle rI}\downarrow & & \downarrow{\scriptstyle rI} & 1 & \downarrow{\scriptstyle rI} & & \\
FJ & \kappa' & \bullet & ru & \bullet & \kappa & \bullet & = & \bullet & \kappa' & \downarrow{\scriptstyle tI} \\
 & & {\scriptstyle rJ}\downarrow & & {\scriptstyle Gu}\downarrow & & \downarrow{\scriptstyle Gu} & & \downarrow{\scriptstyle sI} & & \\
{\scriptstyle tJ}\downarrow & & \bullet & = & \bullet & & \bullet & su & \bullet & & HI \\
 & & {\scriptstyle sJ}\downarrow & 1 & {\scriptstyle sJ}\downarrow & & \downarrow{\scriptstyle Hu} & & \downarrow{\scriptstyle Hu} & & \downarrow{\scriptstyle Hu} \\
HJ & = & \bullet & = & \bullet & = & \bullet & = & \bullet & = & HJ
\end{array}$$

4.2 Definition (Modifications)

A *modification* $\mu\colon (r\ {}^{h}_{k}\ s)\colon ({}^{F}_{G}\ {}^{F'}_{G'})$, where h, k are pseudo horizontal transformations and r, s are pseudo vertical transformations (of lax functors $\mathbb{I} \to \mathbb{A}$), has components μI in $\mathbb{A}$, for every I in $\mathbb{I}$

$$\begin{array}{ccc} FI & \xrightarrow{hI} & F'I \\ {\scriptstyle rI}\downarrow & \mu I & \downarrow{\scriptstyle sI} \\ GI & \xrightarrow[kI]{} & G'I \end{array} \tag{26}$$

They must satisfy two conditions:

(mod1) (*horizontal coherence*) for every $i\colon I \to I'$ in $\mathbb{I}$ we have

$$\begin{array}{ccccc} FI & \xrightarrow{hI} & F'I & \xrightarrow{F'i} & F'I' \\ {\scriptstyle e}\downarrow & & hi & & \downarrow{\scriptstyle e} \\ FI & \xrightarrow{Fi} & FI' & \xrightarrow{hI'} & F'I' \\ {\scriptstyle rI}\downarrow & ri & \downarrow{\scriptstyle rI'} & \mu I' & \downarrow{\scriptstyle sI'} \\ GI & \xrightarrow[Gi]{} & GI' & \xrightarrow[kI']{} & G'I' \end{array} \quad = \quad \begin{array}{ccccc} FI & \xrightarrow{hI} & F'I & \xrightarrow{F'i} & F'I' \\ {\scriptstyle rI}\downarrow & \mu I & \downarrow{\scriptstyle sI} & si & \downarrow{\scriptstyle sI'} \\ GI & \xrightarrow{kI} & G'I & \xrightarrow{G'i} & G'I' \\ {\scriptstyle e}\downarrow & & ki & & \downarrow{\scriptstyle e} \\ GI & \xrightarrow[Gi]{} & GI' & \xrightarrow[kI']{} & G'I' \end{array}$$

(mod2) (*vertical coherence*) for every $u\colon I \nrightarrow J$ in $\mathbb{I}$ we have

$$
\begin{array}{ccccc}
FI & \xrightarrow{1} & FI & \xrightarrow{hi} & F'I \\
{\scriptstyle Fu}\downarrow & & {\scriptstyle rI}\downarrow & \mu I & \downarrow{\scriptstyle sI} \\
FJ & ru & GI & \xrightarrow{kI} & G'I \\
{\scriptstyle rJ}\downarrow & & {\scriptstyle Gu}\downarrow & ku & \downarrow{\scriptstyle G'u} \\
GJ & \xrightarrow[1]{} & GJ & \xrightarrow[kJ]{} & G'J
\end{array}
\quad = \quad
\begin{array}{ccccc}
FI & \xrightarrow{hI} & F'I & \xrightarrow{1} & F'I \\
{\scriptstyle Fu}\downarrow & hu & \downarrow{\scriptstyle F'u} & & \downarrow{\scriptstyle sI} \\
FJ & \xrightarrow{hJ} & F'J & su & G'I \\
{\scriptstyle rJ}\downarrow & \mu J & \downarrow{\scriptstyle sJ} & & \downarrow{\scriptstyle G'u} \\
GJ & \xrightarrow[kJ]{} & G'J & \xrightarrow[1]{} & G'J
\end{array}
$$

4.3 Theorem

For any weak double categories $\mathbb{I}$ and $\mathbb{A}$ we have a weak double category $\mathbb{L}\mathrm{ax}(\mathbb{I}, \mathbb{A})$ whose objects are the lax functors $\mathbb{I} \to \mathbb{A}$; the arrows are pseudo horizontal and pseudo vertical transformations, while the double cells are modifications.

The associated 2-category $\mathbf{Lax}(\mathbb{I}, \mathbb{A}) = \mathbf{Hor}\mathbb{L}\mathrm{ax}(\mathbb{I}, \mathbb{A})$ has trivial vertical arrows and globular modifications $\varphi\colon (e_F \, {}^{h}_{k} \, e_G)$.

Proof. By straightforward computation. □

4.4 Theorem (Characterisation of pointwise equivalences)

A pseudo horizontal transformation $h\colon F \to G\colon \mathbb{I} \to \mathbb{A}$ is a pointwise equivalence (of lax functors) if and only if it is an equivalence in the 2-category $\mathbf{Lax}(\mathbb{I}, \mathbb{A}) = \mathbf{Hor}\mathbb{L}\mathrm{ax}(\mathbb{I}, \mathbb{A})$.

Proof. We have to prove that a pointwise equivalence $h\colon F \to G$ (see 3.3) has a pseudo inverse $k\colon G \to F$ in $\mathbf{Lax}(\mathbb{I}, \mathbb{A})$, the other implication being obvious.

For every I in $\mathbb{I}$, the horizontal arrow $hI\colon FI \to GI$ is a horizontal equivalence in $\mathbb{A}$. We choose a horizontal arrow $kI\colon GI \to FI$ pseudo inverse to hI, with coherent vertically invertible cells $\eta I, \varepsilon I$

$$
\begin{array}{ccccc}
FI & & \xrightarrow{\quad 1 \quad} & & FI \\
{\scriptstyle e}\downarrow & & \eta I & & \downarrow{\scriptstyle e} \\
FI & \xrightarrow[hI]{} & GI & \xrightarrow[kI]{} & FI
\end{array}
\qquad
\begin{array}{ccccc}
GI & \xrightarrow{kI} & FI & \xrightarrow{hI} & GI \\
{\scriptstyle e}\downarrow & & \varepsilon I & & \downarrow{\scriptstyle e} \\
GI & & \xrightarrow[\quad 1 \quad]{} & & GI
\end{array}
\qquad (27)
$$

For every vertical arrow $u\colon I \to J$ in $\mathbb{I}$, the cell $hu\colon Fu \to Gu$ is an equivalence cell in $\mathbb{A}$. We can choose a pseudo inverse $ku\colon Gu \to Fu$ consistent with the previous choice of kI, kJ.

From conditions (eqc1, 2) we deduce that

$$
\begin{array}{ccccc}
FI & \xrightarrow{\quad 1 \quad} & & & FI \\
{\scriptstyle e}\downarrow & & \eta I & & \downarrow{\scriptstyle e} \\
FI & \xrightarrow{hI} & GI & \xrightarrow{kI} & FI \\
{\scriptstyle Fu}\downarrow & hu & \downarrow{\scriptstyle Gu} & ku & \downarrow{\scriptstyle Fu} \\
FJ & \xrightarrow[hJ]{} & GJ & \xrightarrow[kJ]{} & FJ
\end{array}
\quad = \quad
\begin{array}{ccccc}
FI & \xrightarrow{\quad 1 \quad} & & & FI \\
{\scriptstyle Fu}\downarrow & & 1_{Fu} & & \downarrow{\scriptstyle Fu} \\
FJ & \xrightarrow{\quad 1 \quad} & & & FJ \\
{\scriptstyle e}\downarrow & & \eta J & & \downarrow{\scriptstyle e} \\
FJ & \xrightarrow[hJ]{} & GJ & \xrightarrow[kJ]{} & FJ
\end{array}
\qquad (28)
$$

$$
\begin{array}{ccccc}
GI & \xrightarrow{kI} & FI & \xrightarrow{hI} & GI \\
{\scriptstyle Gu}\downarrow & ku & \downarrow{\scriptstyle Fu} & hu & \downarrow{\scriptstyle Gu} \\
GJ & \xrightarrow{kJ} & FJ & \xrightarrow{hJ} & GJ \\
{\scriptstyle e}\downarrow & & \varepsilon J & & \downarrow{\scriptstyle e} \\
GJ & \xrightarrow[\quad 1 \quad]{} & & & GJ
\end{array}
\quad = \quad
\begin{array}{ccccc}
GI & \xrightarrow{kI} & FI & \xrightarrow{hI} & GI \\
{\scriptstyle e}\downarrow & & \varepsilon I & & \downarrow{\scriptstyle e} \\
GI & \xrightarrow{\quad 1 \quad} & & & GI \\
{\scriptstyle Gu}\downarrow & & 1_{Gu} & & \downarrow{\scriptstyle Gu} \\
GJ & \xrightarrow[\quad 1 \quad]{} & & & GJ
\end{array}
\qquad (29)
$$

For a horizontal arrow $i\colon I \to I'$, the naturality comparison

$$ki\colon Fi.kI \to kI'.Gi$$

is obtained from the following pasting of globular isocells, where the central one is the vertical inverse $(hi)^{\smile}$ of the naturality comparison cell hi

$$
\begin{array}{ccccccccc}
GI & \xrightarrow{kI} & FI & \xrightarrow{Fi} & FI' & \xrightarrow{\quad 1 \quad} & & & FI' \\
{\scriptstyle e}\downarrow & e & \downarrow{\scriptstyle e} & e & \downarrow{\scriptstyle e} & & \eta I' & & \downarrow{\scriptstyle e} \\
GI & \xrightarrow{kI} & FI & \xrightarrow{Fi} & FI' & \xrightarrow{hI'} & GI' & \xrightarrow{kI'} & FI' \\
{\scriptstyle e}\downarrow & e & \downarrow{\scriptstyle e} & & (hi)^{\smile} & & \downarrow{\scriptstyle e} & e & \downarrow{\scriptstyle e} \\
GI & \xrightarrow{kI} & FI & \xrightarrow{hI} & GI & \xrightarrow{Gi} & GI' & \xrightarrow{kI'} & FI' \\
{\scriptstyle e}\downarrow & & \varepsilon I & & \downarrow{\scriptstyle e} & e & \downarrow{\scriptstyle e} & e & \downarrow{\scriptstyle e} \\
GI & \xrightarrow[\quad 1 \quad]{} & & & GI & \xrightarrow[Gi]{} & GI' & \xrightarrow[kI']{} & FI'
\end{array}
\qquad (30)
$$

We have now a pseudo horizontal transformation $k\colon G \to F$, which is made pseudo inverse to h by the globular modifications $\eta\colon 1 \to kh$ and $\varepsilon\colon hk \to 1$. □

4.5 Theorem (Vertical transport)

Let $h\colon F \to G\colon \mathbb{I} \to \mathbb{A}$ be a pseudo horizontal transformation of lax functors. Suppose we are given for each I a horizontal arrow $kI\colon FI \to GI$ and a globular isocell $\varphi I\colon kI \to\!\!\!\!\bullet\; hI$, without further conditions

$$\begin{array}{ccc} FI & \xrightarrow{kI} & GI \\ {\scriptstyle e}\downarrow & \varphi I & \downarrow{\scriptstyle e} \\ FI & \xrightarrow[hI]{} & GI \end{array} \tag{31}$$

Then there is precisely one way of making k into a pseudo horizontal transformation $F \to G$, so that φ is a vertically invertible globular modification $\varphi\colon (e_F \, {}^{k}_{h} \, e_G)$.

Proof. The components ku (for $u\colon I \to\!\!\!\!\bullet\; J$) and ki (for $i\colon I \to I'$) are (necessarily) defined as the following normal ternary composites, in order that the family (φI) satisfy the axioms (mod1, 2) of Definition 4.2

$$\begin{array}{ccc} FI & \xrightarrow{kI} & GI \\ {\scriptstyle e}\downarrow & \varphi I & \downarrow{\scriptstyle e} \\ FI & \xrightarrow{hI} & GI \\ {\scriptstyle Fu}\downarrow & hu & \downarrow{\scriptstyle Gu} \\ FJ & \xrightarrow{hJ} & GJ \\ {\scriptstyle e}\downarrow & (\varphi J)^{\smallsmile} & \downarrow{\scriptstyle e} \\ FJ & \xrightarrow[kJ]{} & GJ \end{array} \qquad \begin{array}{ccccc} FI & \xrightarrow{kI} & GI & \xrightarrow{Gi} & GI' \\ {\scriptstyle e}\downarrow & \varphi I & \downarrow{\scriptstyle e} & e & \downarrow{\scriptstyle e} \\ FI & \xrightarrow{hI} & GI & \xrightarrow{Gi} & GI' \\ {\scriptstyle e}\downarrow & & hi & & \downarrow{\scriptstyle e} \\ FI & \xrightarrow{Fi} & FI' & \xrightarrow{hI'} & GI' \\ {\scriptstyle e}\downarrow & e & \downarrow{\scriptstyle e} & (\varphi I')^{\smallsmile} & \downarrow{\scriptstyle e} \\ FI & \xrightarrow[Fi]{} & FI' & \xrightarrow[kI']{} & GI' \end{array} \tag{32}$$

Long but straightforward verifications show that, with these additional components, k is indeed a pseudo horizontal transformation $F \to G$. $\square$

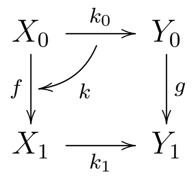
4.6 Remark

Even if, in the previous theorem, the transformation $h\colon F \to G$ is strict, the transformation k need not be, as we can see from the formulas (32), where the cells φI and φJ are unrelated, as well as φI and $\varphi I'$. Nor is every pseudo horizontal transformation isomorphic to a strict one, as the following example shows.

However, a crucial part of the proof of our 'Persistence Theorem' will be a strictification result of this kind, under strong conditions on $\mathbb{I}$ (in Theorem 6.3).

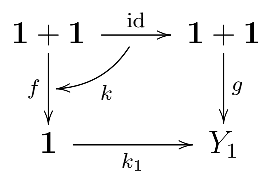
4.7 An example

Let $\mathbb{I}$ be the horizontal double category associated to the ordinal $\mathbf{2}$: we have a horizontal arrow $0 \to 1$ and the required identities. Let $\mathbb{A}$ be the horizontal double category $\mathbf{Cat}$, with trivial vertical arrows. A double functor $F\colon \mathbf{2} \to \mathbf{Cat}$ amounts to a functor $f\colon X_0 \to X_1$ in $\mathbf{Cat}$. A pseudo horizontal transformation $k\colon F \to G\colon \mathbf{2} \to \mathbf{Cat}$ 'is' a square in $\mathbf{Cat}$ inhabited by a functorial isomorphism, its component $k = k(0 \to 1)$



(33)

and it is strict when $k\colon gk_0 \cong k_1 f$ is the identity. Consider the following instance



(34)

where Y_1 is the indiscrete groupoid on the objects $0, 1$, the functor g is the inclusion and $k_1(0) = 0$, so that the square is inhabited by a (unique) functorial isomorphism k.

Here $k\colon F \to G$ is not isomorphic to any strict horizontal transformation $F \to G$. Indeed, in the diagram above, the functors f, g are given by F and G, and the identity on top cannot be changed, since all isomorphisms in its codomain are trivial; no choice of the bottom functor can make the diagram commute.

5. Limits and pseudo limits

Limits of lax double functors $F\colon \mathbb{I} \to \mathbb{A}$ were studied in [GP1]. Our definition of persistence will be based on these limits, but in its study we also need 'pseudo limits', which we introduce here; they are again actual limits rather then bilimits, and determined up to horizontal isomorphism by a strict universal property.

After recalling our definition of a terminal object in a weak double category $\mathbb{D}$ (from [GP1], Subsection 1.8) we define the weak double category $\mathrm{Ps}\mathbb{C}\mathrm{one}(F)$ of pseudo cones of F, and its full substructure $\mathbb{C}\mathrm{one}(F)$ of cones. Their terminal objects, if they exist, give $\mathrm{psLim}F$ and $\mathrm{Lim}\, F$.

For weighted limits in 2-categories we refer to [St, K1, K2, AK]; for the flexible ones to [BKP, BKPS].

$F\colon \mathbb{I} \to \mathbb{A}$ is always a lax double functor between weak double categories; $\mathbb{I}$ is small.

5.1 Terminal object

A (horizontal double) *terminal object* of the weak double category $\mathbb{D}$ is an object T such that:

(t1) for every object A there is precisely one map $t\colon A \to T$ (also written as tA),

(t2) for every vertical map $u\colon A \to\!\!\!\!\bullet\; B$ there is precisely one cell $\tau\colon u \to e_T$ (also written as τu)

$$\begin{array}{ccc} A & \xrightarrow{tA} & T \\ {\scriptstyle u}\downarrow & \tau & \downarrow{\scriptstyle e} \\ B & \xrightarrow[tB]{} & T \end{array} \qquad (35)$$

Actually the 2-dimensional property (t2) implies (t1), by applying it to the vertical identity e_A.

5.2 Comments

As discussed in [GP2], Subsection 6.4, this notion amounts to a *unitary* lax right adjoint to the double functor $D\colon \mathbb{D} \to \mathbb{1}$ (taking values in the singleton double category).

A general lax right adjoint to D would be a pair (T, t) of $\mathbb{D}$, formed of an object T equipped with a vertical arrow $t\colon T \to\!\!\!\!\!\!\bullet\; T$, that takes the place of e_T in diagram (35). The universal property yields special cells $\eta\colon e \to t$ and $\mu\colon t \otimes t \to t$ completing the lax functor $\mathbb{1} \to \mathbb{A}$, and giving a monad in the vertical bicategory $\mathbf{Ver}\mathbb{A}$. There are few instances of weak double categories lacking a terminal object, but having a 'terminal pair' of this kind (see [GP2]).

5.3 Diagonalisation

An object A of the weak double category $\mathbb{A}$ determines a strict double functor constant at A, and denoted by the same letter (or by DA when useful)

$$A\colon \mathbb{I} \to \mathbb{A}. \tag{36}$$

Let us note that the strictness of this functor comes from the unitarity of $\mathbb{A}$, otherwise we would have a unitary pseudo double functor with comparison

$$\underline{DA}(u, v) = \lambda(e_A) = \rho(e_A)\colon e_A \otimes e_A \to e_A,$$

for all consecutive vertical arrows u, v in $\mathbb{I}$.

It is also useful to note that a *unitary* lax functor $S\colon \mathbb{A} \to \mathbb{B}$ *preserves diagonalisation*, in the sense that $S.DA = D(SA)$; for a general lax functor S one should proceed in a more complex way.

We have thus a diagonal double functor $D\colon A \to \mathbb{L}\mathrm{ax}(\mathbb{I}, \mathbb{A})$, and we define $\mathrm{Ps}\mathbb{C}\mathrm{one}(F)$ as the double comma $D \downarrow\!\downarrow F$ (see [GP2], Subsection 2.5)

$$\begin{array}{ccc} \mathrm{Ps}\mathbb{C}\mathrm{one}(F) & \xrightarrow{\;P\;} & \mathbb{A} \\ \big\downarrow & \swarrow_{\pi} & \big\downarrow{\scriptstyle D} \\ \mathbb{1} & \xrightarrow[F]{} & \mathbb{L}\mathrm{ax}(\mathbb{I}, \mathbb{A}) \end{array} \tag{37}$$

5.4 Pseudo cones and cones

A *(horizontal) pseudo cone* of the lax functor $F\colon \mathbb{I} \to \mathbb{A}$ will be a pair $(A, h\colon A \to F)$ comprising an object A of $\mathbb{A}$ (the *vertex* of the cone) and a pseudo horizontal transformation of lax functors $h\colon A \to F\colon \mathbb{I} \to \mathbb{A}$.

As defined in 3.2, h amounts to assigning the following data in $\mathbb{A}$:

(a) a horizontal map $hI\colon A \to FI$ for every object I in $\mathbb{I}$,

(b) a cell $hu\colon (A\ {}^{hI}_{hJ}\ Fu)$ for every vertical arrow $u\colon I \to\!\!\!\!\bullet\ J$ in $\mathbb{I}$,

(c) a globular isocell hi for every horizontal arrow $i\colon I \to I'$ in $\mathbb{I}$,

$$\begin{array}{ccccc} A & \xrightarrow{hI} & FI & \xrightarrow{Fi} & FI' \\ {\scriptstyle e}\downarrow & & hi & & \downarrow{\scriptstyle e} \\ A & & \xrightarrow[hI']{\qquad\qquad} & & FI' \end{array} \tag{38}$$

under the axioms (pht1–5) of naturality and coherence.

It is a *cone* when all the comparison cells hi are vertical identities.

5.5 Morphisms

(a) A *horizontal morphism* $f\colon (A, h) \to (A', k)$ of pseudo cones of $F\colon \mathbb{I} \to \mathbb{A}$ is a horizontal arrow $f\colon A \to A'$ in $\mathbb{A}$ that commutes with the cone elements, as follows

(i)	$hI = kI.f\colon A \to A' \to FI,$	for I in $\mathbb{I}$,
(ii)	$hu = (e_f \mid ku)\colon e_A \to e_{A'} \to Fu,$	for $u\colon I \to\!\!\!\!\bullet\ J$,
(iii)	$hi = (e_f \mid ki)\colon eA \to e_{A'} \to eFI',$	for $i\colon I \to I'$,

$$\begin{array}{ccccccc} A & \xrightarrow{f} & A' & \xrightarrow{kI} & FI & \xrightarrow{Fi} & FI' \\ {\scriptstyle e}\downarrow & e_f & {\scriptstyle e}\downarrow & & ki & & \downarrow{\scriptstyle e} \\ A & \xrightarrow[f]{} & A' & & \xrightarrow[kI']{\qquad\qquad} & & FI' \end{array} \quad = \quad \begin{array}{ccccc} A & \xrightarrow{hI} & FI & \xrightarrow{Fi} & FI' \\ {\scriptstyle e}\downarrow & & hi & & \downarrow{\scriptstyle e} \\ A & & \xrightarrow[hI']{\qquad\qquad} & & FI' \end{array}$$

Horizontal morphisms compose, forming a category. For strict cones, condition (iii) is automatically satisfied.

(b) A *vertical morphism* $(u, \xi)\colon (A, h) \to\!\!\!\!\bullet\ (B, k)$ of pseudo cones of $F\colon \mathbb{I} \to \mathbb{A}$ is given by a vertical arrow $u\colon A \to\!\!\!\!\bullet\ B$ (viewed as a vertical transformation $Du\colon DA \to\!\!\!\!\bullet\ DB$, constant at u) and a modification $\xi\colon (u\ {}^{h}_{k}\ e_F)$.

We have thus for every I in $\mathbb{I}$ a cell $\xi I\colon (u\ {}^{hI}_{kI}\ FI)$ in $\mathbb{A}$ that satisfies the conditions (mod1, 2) of Definition 4.2.

5.6 Double cells

A double cell of pseudo cones

$$\begin{array}{ccc} (A,h) & \xrightarrow{\ f\ } & (A',h') \\ {\scriptstyle (u,\xi)}\big\downarrow\!\!\bullet & \alpha & \bullet\!\!\big\downarrow{\scriptstyle (v,\zeta)} \\ (B,k) & \xrightarrow[\ g\ ]{} & (B',k') \end{array} \tag{39}$$

is a cell $\alpha\colon (u\,{}^{f}_{g}\,v)$ in $\mathbb{A}$ such that, for every I in $\mathbb{I}$

$$\begin{array}{ccccc} A & \xrightarrow{\ f\ } & A' & \xrightarrow{\ h'I\ } & FI \\ {\scriptstyle u}\big\downarrow & \alpha & \big\downarrow{\scriptstyle v} & \zeta I & \big\downarrow{\scriptstyle e} \\ B & \xrightarrow[\ g\ ]{} & B' & \xrightarrow[\ k'I\ ]{} & FI \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{\ hI\ } & FI \\ {\scriptstyle u}\big\downarrow & \xi I & \big\downarrow{\scriptstyle e} \\ B & \xrightarrow[\ kI\ ]{} & FI \end{array}$$

They compose 'as in $\mathbb{A}$'. We have now defined the weak double category $\mathrm{Ps}\mathbb{C}\mathrm{one}(F)$ of pseudo cones of F.

The weak double category $\mathbb{C}\mathrm{one}(F)$ of cones of F is the full (on arrows and cells) substructure of $\mathrm{Ps}\mathbb{C}\mathrm{one}(F)$ determined by the strict cones (A,h), where $h\colon A \to F$ is a (strict) horizontal transformation.

5.7 Definition (Limits)

A (horizontal double) *limit* of a lax functor $F\colon \mathbb{I} \to \mathbb{A}$ is a terminal object of the weak double category $\mathbb{C}\mathrm{one}(F)$. A *pseudo limit* of F is a terminal object of $\mathrm{Ps}\mathbb{C}\mathrm{one}(F)$.

Explicitly, we have a strict (resp. pseudo) cone $(L, p\colon L \to F)$ such that:

(lim1) for every strict (resp. pseudo) cone $(A, h\colon A \to F)$ there is a unique horizontal morphism $f\colon A \to L$ in $\mathbb{A}$ such that $h = pf\colon A \to L \to F$, in the 2-category $\mathbf{Lax}(\mathbb{I},\mathbb{A})$,

(lim2) for every vertical arrow $(u,\xi)\colon (A,h) \to\!\!\!\bullet\ (A,k)$ of strict (resp. pseudo) cones there is a unique cell $\alpha\colon (u\,{}^{f}_{g}\,L)$ in $\mathbb{A}$ such that, for every I in $\mathbb{I}$

$$\begin{array}{ccccc} A & \xrightarrow{\ f\ } & L & \xrightarrow{\ pI\ } & FI \\ {\scriptstyle u}\big\downarrow & \alpha & \big\downarrow{\scriptstyle e} & e & \big\downarrow{\scriptstyle e} \\ B & \xrightarrow[\ g\ ]{} & L & \xrightarrow[\ pI\ ]{} & FI \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{\ hI\ } & FI \\ {\scriptstyle u}\big\downarrow & \xi I & \big\downarrow{\scriptstyle e} \\ B & \xrightarrow[\ kI\ ]{} & FI \end{array} \tag{40}$$

These conditions will be called, respectively, the 1-*dimensional* and the 2-*dimensional* universal property of the limit. We speak of a 1-*dimensional limit* when only the first is assumed.

When u is a vertical identity, the usual argument shows that the globular cell ξ is vertically invertible if and only if α is.

In [GP1] we proved that all limits in a weak double category can be constructed from products, equalisers (of pairs of horizontal arrows) and tabulators (all of them being limits of double functors defined on strict double categories). In particular, this was used to show that $\mathbb{C}\mathrm{at}$ has all limits; below we give a direct proof of this fact, extended to pseudo limits.

In a horizontal double category $\mathbb{A}$ the tabulator of a vertical identity e_X is the same as the cotensor product $\mathbf{2} \pitchfork X$ in the corresponding 2-category $\mathbf{A}$. Since all weighted limits in $\mathbf{A}$ can be constructed from products, equalisers and cotensors by $\mathbf{2}$ [St], it follows that the existence of all weighted limits in $\mathbf{A}$ amounts to that of all double limits in $\mathbb{A}$. This point will be further analysed in a sequel.

5.8 Proposition

The weak double category $\mathbb{C}\mathrm{at}$ *and the horizontal double category* $\mathbf{Cat}$ *have all limits and pseudo limits.*

Proof. The proof is standard. Let $F\colon \mathbb{I} \to \mathbb{C}\mathrm{at}$ be a lax functor. We begin by constructing its pseudo limit $(L, p\colon L \to F)$

An object of the category L is a pseudo cone $x\colon \mathbf{1} \to F$ with vertex at the singleton category. This is a pseudo horizontal transformation with components:

- for every I in $\mathbb{I}$, an object xI in FI,

- for every $u\colon I \nrightarrow J$, an element $xu \in Fu(xI, xJ)$, i.e. an arrow $xu\colon xI \to xJ$ in the gluing $FI +_{Fu} FJ$,

- for every $i\colon I \to I'$ an isomorphism $xi\colon Fi(xI) \to xI'$,

under the axioms of naturality and coherence coming from (pht1–5) in Definition 3.2.

A morphism $m\colon x \to y$ of L is a double cell $m\colon (\mathbf{1}\ {}^{x}_{y}\ F)$ of $\mathrm{Ps}\mathbb{C}\mathrm{one}(F)$, i.e. a globular modification with components mI in $\mathbb{C}\mathrm{at}$

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{x} & F \\ {\scriptstyle e}\downarrow\!\!\bullet & m & \bullet\!\!\downarrow{\scriptstyle e} \\ \mathbf{1} & \xrightarrow[y]{} & F \end{array} \qquad\qquad \begin{array}{ccc} \mathbf{1} & \xrightarrow{xI} & FI \\ {\scriptstyle e}\downarrow\!\!\bullet & mI & \bullet\!\!\downarrow{\scriptstyle e} \\ \mathbf{1} & \xrightarrow[yI]{} & FI \end{array} \qquad (41)$$

which just amounts to a morphism $mI\colon xI \to yI$ in the category FI.

For every $u\colon I \nrightarrow J$ and $i\colon I \to I'$, the following squares must commute in $FI +_{Fu} FJ$ and FI', respectively

$$\begin{array}{ccc} xI & \xrightarrow{mI} & yI \\ {\scriptstyle xu}\downarrow & = & \downarrow{\scriptstyle yu} \\ xJ & \xrightarrow[mJ]{} & yJ \end{array} \qquad\qquad \begin{array}{ccc} Fi(xI) & \xrightarrow{Fi(mI)} & Fi(yI) \\ {\scriptstyle xi}\downarrow & = & \downarrow{\scriptstyle yi} \\ xI' & \xrightarrow[mI']{} & yI' \end{array} \qquad (42)$$

We shall write $mu = yu.mI = mJ.xu\colon xI \to yJ$ the diagonal of the left square above, in $FI +_{Fu} FJ$.

The pseudo cone $p\colon L \to F$ has the following components

- for every I in $\mathbb{I}$, the functor $pI\colon L \to FI$ sends the object x to $pI(x) = xI \in FI$,

- for every $u\colon I \nrightarrow J$, the double cell $pu\colon (L\ {}^{pI}_{pJ}\ Fu)$ in $\mathbb{C}\mathrm{at}$ sends the morphism $m\colon x \to y$ of L to the morphism $mu\colon xI \to yJ$ of $FI +_{Fu} FJ$,

- for every $i\colon I \to I'$, the isomorphism pi sends $(xi\colon Fi(pI(x)) \to pI'(x))$ to $(xi\colon Fi(xI) \to xI')$.

Every pseudo cone $(A, h\colon A \to F)$ factorises uniquely through the cone $(L, p\colon L \to F)$ via the functor

$$\begin{gathered} f\colon A \to L, \\ f(a)I = hI(a), \qquad f(a)u = hu(a), \qquad f(a)i = hi(a). \end{gathered} \qquad (43)$$

The limit $(L_0, p\colon L_0 \to F)$ is obtained by restricting L to the full subcategory of strict cones $x\colon \mathbf{1} \to F$, where all the isomorphisms xi are identities.

The same construction holds in $\mathbf{Cat}$, noting that a lax functor $F\colon \mathbb{I} \to \mathbf{Cat}$ sends a vertical arrow $u\colon I \nrightarrow J$ to the vertical identity e_X of the category $FI = X = FJ$, whose cotabulator is the cartesian product $\mathbf{2} \times X$. $\square$

6. Persistence and strictification

This section contains the main results of this article. We define 'persistent limits', i.e. the weak double categories $\mathbb{I}$ which parametrise the latter. The Persistence Theorem 6.4 characterises them, by a sort of 'initiality property' of $\mathbb{I}$.

We also prove that all *pseudo* limits automatically have a 'persistence property', which leads to a second characterisation of persistent limits, in Theorem 6.6, as those limits which are 'equivalent to pseudo limits'.

The section ends with elementary examples of persistent double limits, which in a 2-category amount to cotensors by $\mathbf{2}$, comma objects, inserters and equifiers.

6.1 Definition (Persistence)

(a) Let $\mathbb{I}$ be a weak double category. We say that $\mathbb{I}$-*limits are persistent*, or that $\mathbb{I}$ *parametrises persistent limits*, if given the following items

- a weak double category $\mathbb{A}$,

- two lax functors $F, G\colon \mathbb{I} \to \mathbb{A}$ having limits $(A, h\colon A \to F)$ and $(B, k\colon B \to G)$,

- a pointwise equivalence $m\colon F \to G$,

there exists a horizontal equivalence $f\colon A \to B$ linked to $m\colon F \to G$ by a vertically invertible cell ϑ of $\mathbf{Lax}(\mathbb{I}, \mathbb{A})$

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{k} & G \\ {\scriptstyle e}\downarrow & & \vartheta & & \downarrow{\scriptstyle e} \\ A & \xrightarrow[h]{} & F & \xrightarrow[m]{} & G \end{array} \tag{44}$$

(b) We say that $\mathbb{I}$-*limits are persistent in* $\mathbb{A}$ when all this holds for a given weak double category $\mathbb{A}$. The rest will show that persistence in $\mathbb{C}\mathrm{at}$ (or even in the horizontal double category $\mathbf{Cat}$) implies persistence in every weak double category. This is even more interesting since $\mathbb{C}\mathrm{at}$ and $\mathbf{Cat}$ have all (small) limits and pseudo limits.

6.2 Definition

The following property of 'weak local horizontal initiality' will be used to characterise the previous ones.

We say that the weak double category $\mathbb{I}$ is *grounded* if every connected component of the ordinary category $\mathrm{Hor}_0\mathbb{I}$ has a *natural weak initial object*.

By this we mean an endofunctor $\Phi\colon \mathrm{Hor}_0\mathbb{I} \to \mathrm{Hor}_0\mathbb{I}$ which is trivial on every arrow $i\colon I \to I'$ of $\mathbb{I}$, and is equipped with a natural transformation $\varphi\colon \Phi \to 1$.

In elementary terms, for every object I there is an object ΦI and a morphism $\varphi I\colon \Phi I \to I$ such that, for every $i\colon I \to I'$ in $\mathbb{I}$ we have:

$$\Phi I = \Phi I', \qquad \varphi I' = i.\varphi I. \tag{45}$$

Note that the first condition in (45) is redundant. Note also that each ΦI comes equipped with an idempotent endomorphism $pI = \varphi(\Phi I)\colon \Phi I \to \Phi I$. (This is the identity if and only if ΦI is the initial object of its connected component in $\mathrm{Hor}_0\mathbb{I}$, if and only if such an object exists.)

6.3 Theorem (Strictification)

Let $F\colon \mathbb{I} \to \mathbb{A}$ be a lax (or colax) functor, where $\mathbb{I}$ is grounded. Then every pseudo cone $(A, h\colon A \to F)$ is vertically isomorphic to a strict cone $(A, h'\colon A \to F)$, by a globular isocell $\vartheta\colon (A\,{}^{h'}_{h}\,F)$.

Proof. We will use Theorem 4.5 on vertical transport.

We modify the pseudo cone $(A, h\colon A \to F)$ as follows: for every I we define $h'I$ and ϑI by means of the natural family $(\varphi I\colon \Phi I \to I)_I$ introduced above

$$h'I = F\varphi I.h\Phi I\colon A \to F\Phi I \to FI, \tag{46}$$

$$\vartheta I \;=\; \begin{array}{ccccc} A & \xrightarrow{h\Phi I} & F\Phi I & \xrightarrow{F\varphi I} & FI \\ {\scriptstyle e}\downarrow & & h\varphi I & & \downarrow{\scriptstyle e} \\ A & & \xrightarrow[hI]{\qquad\qquad} & & FI \end{array} \tag{47}$$

The cells ϑI are vertically invertible. By Theorem 4.5 these data can be uniquely completed to a pseudo cone (A, h') and a vertical isomorphism $\vartheta\colon (A\,{}^{h'}_{h}\,F)$ of pseudo cones of F.

Proving that (A, h') is actually a strict cone will conclude the proof. For $i \colon I \to I'$, the comparison $h'i$ is defined in the right-hand diagram (32), as the following pasting

$$\begin{array}{ccccccc}
A & \xrightarrow{h\Phi I} & F\Phi I & \xrightarrow{F\varphi I} & FI & \xrightarrow{Fi} & FI' \\
{\scriptstyle e}\downarrow & & h\varphi I & & {\scriptstyle e}\downarrow & e & \downarrow{\scriptstyle e} \\
A & & \xrightarrow{\qquad hI \qquad} & & FI & \xrightarrow{Fi} & FI' \\
{\scriptstyle e}\downarrow & & & hI & & & \downarrow{\scriptstyle e} \\
A & \xrightarrow{1} & A & & \xrightarrow{\qquad hI' \qquad} & & FI' \\
{\scriptstyle e}\downarrow & e & \downarrow{\scriptstyle e} & & (h\varphi I')\breve{} & & \downarrow{\scriptstyle e} \\
A & \xrightarrow[1]{} & A & \xrightarrow[h\Phi I']{} & F\Phi I' & \xrightarrow[F\varphi I']{} & FI'
\end{array} \qquad (48)$$

By coherence of the pseudo horizontal transformation $h \colon A \to F$ on the horizontal composite $i.\varphi I \colon \Phi I \to I \to I'$ (axiom (pht5) of Definition 3.2), the two upper rows of cells compose (vertically) to $h(i.\varphi I) = h(\varphi I')$, and then the vertical composition with the lowest row gives the identity. □

6.4 Persistence Theorem, I

The following conditions on the weak double category $\mathbb{I}$ *are equivalent:*

(i) $\mathbb{I}$*-limits are persistent,*

(ii) $\mathbb{I}$*-limits are persistent in the weak double category* $\mathbb{C}\mathrm{at}$ *of categories, functors and profunctors,*

(iii) $\mathbb{I}$ *is grounded,*

(iv) for every double functor $F \colon \mathbb{I} \to \mathbb{A}$*, every pseudo cone* $(A, h \colon A \to F)$ *is vertically isomorphic to a strict cone, in the weak double category* $\mathrm{Ps}\mathbb{C}\mathrm{one}(F)$.

Proof. (i) $\Rightarrow$ (ii) is obvious and (iii) $\Rightarrow$ (iv) is the Strictification Theorem 6.3. Here we prove that (iv) $\Rightarrow$ (i); the last point (ii) $\Rightarrow$ (iii) will be proved in the next section.

We are given the data of Definition 6.1, and in particular a pseudo horizontal transformation $m \colon F \to G \colon \mathbb{I} \to \mathbb{A}$ which is a pointwise equivalence.

We assume that all the pseudo cones of F and G can be strictified, up to vertical isomorphism, and we construct a horizontal equivalence $f\colon A \to B$ and a vertically invertible (globular) cell $\vartheta\colon (A \,{}^{kf}_{mh}\, G)$.

The limit cone $(A, h\colon A \to F)$ gives a pseudo cone $(A, mh\colon A \to G)$. By hypothesis there exists a strict cone $(A, h'\colon A \to G)$ with a vertically invertible cell $\vartheta\colon (A \,{}^{h'}_{mh}\, G)$.

By the universal property of the limit $(B, k\colon B \to G)$ there is a unique $f\colon A \to B$ such that $h' = kf\colon A \to B \to G$. We have thus obtained the required $\vartheta\colon (A \,{}^{kf}_{mh}\, G)$, and we are left with proving that f is a horizontal equivalence.

The pointwise equivalence $m\colon F \to G$ has a pseudo inverse $n\colon G \to F$ with globular isocells

$$\begin{array}{ccccc} F & \xrightarrow{\quad 1 \quad} & & & F \\ {\scriptstyle e}\downarrow & & \eta & & \downarrow{\scriptstyle e} \\ F & \xrightarrow[m]{} & G & \xrightarrow[n]{} & F \end{array} \qquad\qquad \begin{array}{ccccc} G & \xrightarrow{n} & F & \xrightarrow{m} & G \\ {\scriptstyle e}\downarrow & & \varepsilon & & \downarrow{\scriptstyle e} \\ G & & \xrightarrow[1]{\quad\quad} & & G \end{array} \tag{49}$$

and n is also a pointwise equivalence. By the argument above we get a morphism $g\colon B \to A$ and a vertically invertible cell $\varphi\colon (B \,{}^{hg}_{nk}\, F)$.

We have now a vertically invertible cell $\sigma\colon (A \,{}^{hgf}_{h}\, F)$

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & A & \xrightarrow{h} & F \\ {\scriptstyle e}\downarrow & e & \downarrow{\scriptstyle e} & & \varphi & & \downarrow{\scriptstyle e} \\ A & \xrightarrow{f} & B & \xrightarrow{k} & G & \xrightarrow{n} & F \\ {\scriptstyle e}\downarrow & & \vartheta & & {\scriptstyle e}\downarrow & e & \downarrow{\scriptstyle e} \\ A & \xrightarrow{h} & F & \xrightarrow{m} & G & \xrightarrow{n} & F \\ {\scriptstyle e}\downarrow & e & \downarrow{\scriptstyle e} & & \eta^{\smile} & & \downarrow{\scriptstyle e} \\ A & \xrightarrow[h]{} & F & & \xrightarrow[1]{\quad\quad} & & F \end{array} \tag{50}$$

By the 2-dimensional universal property of the limit (A, h) there exists a unique cell $\alpha\colon (A \,{}^{gf}_{1}\, A)$ such that $\sigma = (\alpha \,|\, h)$. By the usual argument, already recalled in 5.7, the cell α is also vertically invertible.

Symmetrically we get a vertically invertible cell $\beta\colon (B \,{}^{fg}_{1}\, B)$, whence f is a horizontal equivalence and the proof is concluded. □

6.5 Proposition (The persistence property of pseudo limits)

Every weak double category $\mathbb{I}$ parametrises persistent *pseudo* limits, *in the following sense. Given*

- a weak double category $\mathbb{A}$*,*

- two lax functors $F, G\colon \mathbb{I} \to \mathbb{A}$ *having* pseudo *limits* $(A, h\colon A \to F)$ *and* $(B, k\colon B \to G)$*,*

- a pointwise equivalence $m\colon F \to G$*,*

there exists a horizontal equivalence $f\colon A \to B$ *such that* $mh = kf$*.*

Proof. By Theorem 4.4 there is a pseudo horizontal transformation $n\colon G \to F$ with vertically invertible globular modifications

$$\eta\colon (F\ {}^{\ 1}_{nm}\ F), \qquad\qquad \varepsilon\colon (G\ {}^{\ 1}_{mn}\ G).$$

The pseudo cone $(A, h\colon A \to F)$ of F gives a pseudo cone $(A, mh\colon A \to G)$ of G, whence there is precisely one $f\colon A \to B$ in $\mathbb{A}$ such that $mh = kf\colon A \to B \to G$ in the 2-category $\mathbf{Lax}(\mathbb{I}, \mathbb{A})$, and it is sufficient to prove that it is a horizontal equivalence.

Symmetrically, there is precisely one pseudo horizontal transformation $g\colon B \to A$ such that $nk = hg\colon B \to A \to F$.

Now $gf\colon A \to A$ gives $h.gf = nk.f = nm.h\colon A \to F$, where the pseudo horizontal transformation $nm\colon F \to F$ is vertically isomorphic to 1_F in $\mathbf{Lax}(\mathbb{I}, \mathbb{A})$.

By the 2-dimensional universal property of (A, h), the invertible vertical arrow $(e, \eta)\colon (A, h) \to (A, nmh) = (A, h.gf)$ of pseudo cones gives a vertically invertible cell $\alpha\colon (A\ {}^{\ 1}_{gf}\ A)$ in $\mathbb{A}$ such that $(\alpha \,|\, hI) = \eta I$, for all I

$$\begin{array}{ccccc} A & \xrightarrow{\ 1\ } & A & \xrightarrow{\ hI\ } & FI \\ {\scriptstyle e}\big\downarrow & \alpha & \big\downarrow{\scriptstyle e} & e & \big\downarrow{\scriptstyle e} \\ A & \xrightarrow[\ gf\ ]{} & A & \xrightarrow[\ hI\ ]{} & FI \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{\ hI\ } & FI \\ {\scriptstyle e}\big\downarrow & \eta I & \big\downarrow{\scriptstyle e} \\ A & \xrightarrow[\ (hgf)I\ ]{} & FI \end{array} \qquad (51)$$

Symmetrically there is a vertically invertible cell $\beta\colon (B\ {}^{\ 1}_{fg}\ B)$. $\square$

6.6 Persistence Theorem, II

$\mathbb{I}$-limits are persistent if and only if:

(v) for every double functor $F\colon \mathbb{I} \to \mathbb{A}$ having a limit $(A, h\colon A \to F)$ and a pseudo limit $(A', h'\colon A' \to F)$, the canonical morphism $f\colon A \to A'$ (determined by $h = h'f$) is a horizontal equivalence.

Proof. First we assume that $\mathbb{I}$-limits are persistent, whence condition 6.4(iv) holds. By hypothesis, the pseudo limit $(A', h'\colon A' \to F)$ is vertically isomorphic to a strict cone $(A', k'\colon A' \to F)$, by means of a vertically invertible cell $\vartheta\colon (A' \;{}^{h'}_{k'}\; F)$. There is thus a unique $g\colon A' \to A$ such that $k' = hg\colon A' \to A \to F$.

Now $h'fg = hg = k'$ and we have a vertically invertible cell

$$\vartheta\colon (A' \;{}^{h'}_{h'fg}\; F),$$

between pseudo cones. By the 2-dimensional universal property of (A', h') there exists a unique cell $\alpha\colon (A' \;{}^{1}_{fg}\; A')$, vertically invertible as well, such that $\vartheta = (\alpha \,|\, h')$.

Similarly, the vertically invertible cell

$$(f \,|\, \vartheta)\colon (A \;{}^{h'f}_{k'f}\; F) = (A \;{}^{h}_{hgf}\; F),$$

is a cell between cones of F and gives a unique (vertically invertible) cell $\beta\colon (A \;{}^{1}_{gf}\; A)$ such that $(f \,|\, \vartheta) = (\beta \,|\, h)$. Finally $f\colon A \to A'$ is a horizontal equivalence.

Second, we assume that (v) holds for a weak double category $\mathbb{A}$ with all pseudo limits over $\mathbb{I}$, and prove that $\mathbb{I}$-limits in $\mathbb{A}$ are persistent. In particular this will be true for $\mathbb{A} = \mathbb{C}\mathrm{at}$, so that Theorem 6.4 implies that $\mathbb{I}$-limits are persistent without restrictions.

We are given the data of Definition 6.1, with a pointwise equivalence $m\colon F \to G\colon \mathbb{I} \to \mathbb{A}$. The lax functors F, G are supposed to have a limit and a pseudo limit, which we write as

$$\begin{aligned} &(A, h\colon A \to F), \qquad && (A', h'\colon A' \to F), \\ &(B, k\colon B \to G), \qquad && (B', k'\colon B' \to G). \end{aligned} \tag{52}$$

By (v) and the persistence property of pseudo limits (Proposition 6.5) we have three horizontal equivalences f, g, c coherent with the limit cones or pseudo cones

$$
\begin{array}{ccc}
A & \xrightarrow{f} & A' \\
 & & \downarrow{\scriptstyle c} \\
B & \underset{g'}{\overset{g}{\rightleftarrows}} & B'
\end{array}
\qquad\qquad
\begin{array}{c}
h = h'f \colon A \to F, \\
mh' = k'c \colon A' \to G, \\
k = k'g \colon B \to G.
\end{array}
\tag{53}
$$

Taking a pseudo inverse $g' \colon B' \to B$ with $g'g \;\cong\; 1_B$ and $gg' \;\cong\; 1_{B'}$ (vertically isomorphic), we have a horizontal equivalence $d = g'cf \colon A \to B$ which satisfies the required coherence property

$$dq = (k'g)(g'cf) \;\cong\; k'cf = mh'f = mh. \tag{54}$$

□

6.7 Examples

We end this section by considering some elementary double limits, based on a grounded small double category $\mathbb{I}$ and therefore persistent in any weak double category. In a 2-category they give well-known flexible limits.

(a) When $\mathbb{I}$ is a category (viewed as a horizontal double category), two cases stand out: *products* and *idempotent-splittings*, based respectively on a discrete category or on the category formed of an object $*$ and a non-trivial idempotent arrow $* \to *$ (so that $*$ is a weak initial object).

(b) The *tabulator* $\top u$ is the limit of the vertical arrow $u \colon X_0 \to\!\!\!\!\bullet\; X_1$, as a diagram based on the formal vertical-arrow $\mathbf{2}^t$. In a 2-category it gives the *path-object* PX, or *cell-representer*, or *cotensor* $\mathbf{2} \pitchfork X$.

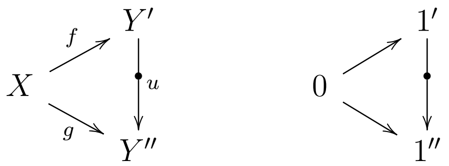
(c) The *extabulator* $\top(f, g, u)$ will be the limit of the left diagram, based on the double category represented on the right

$$
\begin{array}{ccc}
X' & \xrightarrow{f} & Y' \\
 & & \big\downarrow\!\bullet\, {\scriptstyle u} \\
X'' & \xrightarrow[g]{} & Y''
\end{array}
\qquad\qquad
\begin{array}{ccc}
0' & \longrightarrow & 1' \\
 & & \big\downarrow\!\bullet \\
0'' & \longrightarrow & 1''
\end{array}
\tag{55}
$$

A tabulator is a particular case, when f and g are identities; conversely, extabulators are generated by tabulators and finite limits of functors based on horizontal double categories (by the construction theorem of double limits in [GP1]).

In a 2-category we have two arrows f, g with the same codomain $Y' = Y''$ and we get the *comma-object* $(f \downarrow g)$, as a 2-limit.

(d) The *intabulator* $\top_0(f, g, u)$ will be the limit of the left diagram, based on the double category represented on the right

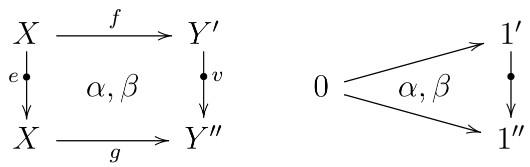


(56)

The tabulator of $u \colon Y' \rightarrow\!\!\!\!\!\bullet\; Y''$ can be obtained from the product $Y' \times Y''$ and the intabulator $\top_0(p_1, p_2, u)$; conversely, intabulators are generated as in (c).

In a 2-category we get the *inserter* of two parallel arrows $f, g \colon X \to Y$.

(e) The (double) *equifier* $\mathrm{Eq}(\alpha, \beta)$ will be the limit of the left diagram, based on the double category represented on the right



(57)

In a 2-category it gives the *ordinary equifier* of two 2-cells $\alpha, \beta \colon f \to g \colon X \to Y$.

7. Concluding the proof of the Persistence Theorem

We end by completing the proof of the first Persistence Theorem 6.4. (Let us note that the proof of the second Persistence Theorem, in 6.6, also depends on the former.)

7.1 Indiscrete profunctors

The remaining part of the proof will be based on 'indiscrete diagrams' in the weak double category $\mathbb{C}\mathrm{at}$ of categories, functors and profunctors.

We say that the profunctor $t\colon C \to\!\!\!\!\!\bullet\; D$ is *indiscrete* if the categories C, D are indiscrete (i.e. chaotic groupoids) and the functor $t\colon C^{\mathrm{op}} \times D \to \mathbf{Set}$ is constant at the singleton set 1 (or at any singleton set).

Indiscrete profunctors are closed under composition and identities of indiscrete categories.

An indiscrete profunctor $t\colon C \to\!\!\!\!\!\bullet\; D$ has an obvious 'terminal-filling' property: given any profunctor $u\colon X \to\!\!\!\!\!\bullet\; Y$ and any functors $f\colon X \to C$ and $g\colon Y \to D$, there always is a unique cell $\alpha\colon (u\; {}^{f}_{g}\; t)$ of $\mathbb{C}\mathrm{at}$ (by the terminal property of 1 in $\mathbf{Set}$)

$$\begin{array}{ccc} X & \xrightarrow{f} & C \\ {\scriptstyle u}\downarrow & \alpha & \downarrow{\scriptstyle t} \\ Y & \xrightarrow[g]{} & D \end{array} \qquad \alpha(x,y)\colon u(x,y) \to t(fx, gy) = 1. \tag{58}$$

Now, every category X has a unique functor $f\colon X \to \mathbf{1}$ to the terminal category (which is indiscrete). If X is a non-empty indiscrete category, f is a horizontal equivalence in $\mathbb{C}\mathrm{at}$, by choosing any functor $f'\colon \mathbf{1} \to X$ (i.e. any object of X) and the unique globular cell η, necessarily vertically invertible

$$\begin{array}{ccccc} X & & \xrightarrow{1} & & X \\ {\scriptstyle e}\downarrow & & \eta & & \downarrow{\scriptstyle e} \\ X & \xrightarrow[f]{} & \mathbf{1} & \xrightarrow[f']{} & X \end{array} \qquad (ff' = \mathrm{id}\mathbf{1}). \tag{59}$$

Any profunctor $u\colon X \to\!\!\!\!\!\bullet\; Y$ has a unique cell $\alpha\colon u \to e_{\mathbf{1}}$

$$\begin{array}{ccccc} X & \xrightarrow{f} & \mathbf{1} & \dashrightarrow^{f'} & X \\ {\scriptstyle u}\downarrow & \alpha & \downarrow{\scriptstyle e} & \alpha' & \downarrow{\scriptstyle u} \\ Y & \xrightarrow[g]{} & \mathbf{1} & \dashrightarrow_{g'} & Y \end{array} \tag{60}$$

If $u\colon X \to\!\!\!\!\!\bullet\; Y$ is an indiscrete profunctor between non-empty (indiscrete) categories, then α is an equivalence cell. In fact, after choosing two arbitrary pseudo inverses $f'\colon \mathbf{1} \to X$, $g'\colon \mathbf{1} \to Y$ there is a unique filling-cell α' as above. Then the vertically invertible cells $\eta, \varepsilon, \eta', \varepsilon'$ of Definition 2.2 are also uniquely determined and automatically satisfy the coherence conditions (eqc1, 2).

7.2 Indiscrete diagrams

By an *indiscrete diagram* $F\colon \mathbb{I} \to \mathbb{C}\mathrm{at}$ we mean a pseudo functor such that:

(i) for each I, FI is an indiscrete category,

(ii) for each $u\colon I \nrightarrow J$, the profunctor $Fu\colon FI \nrightarrow FJ$ is indiscrete.

By the previous point, any lax or colax functor $F\colon \mathbb{I} \to \mathbb{C}\mathrm{at}$ satisfying the conditions (i), (ii) is automatically pseudo.

7.3 Lemma

(a) An indiscrete diagram $F\colon \mathbb{I} \to \mathbb{C}\mathrm{at}$ *is determined by an ordinary functor* $F_0\colon \mathrm{Hor}_0\mathbb{I} \to \mathbf{Set}$, *obtained as the composite*

$$\mathrm{Ob}.\mathrm{Hor}_0F\colon \mathrm{Hor}_0\mathbb{I} \to \mathbf{Cat} \to \mathbf{Set}.$$

(b) Given the indiscrete diagram F, for any lax functor $G\colon \mathbb{I} \to \mathbb{C}\mathrm{at}$ *there is a bijection between horizontal transformations* $k\colon G \to F$ *and natural transformations* $k_0\colon G_0 \to F_0$*, where again*

$$G_0 = \mathrm{Ob}.\mathrm{Hor}_0G\colon \mathrm{Hor}_0\mathbb{I} \to \mathbf{Set}.$$

(c) Let $G\colon \mathbb{I} \to \mathbb{C}\mathrm{at}$ *be the constant diagram at the terminal category* $\mathbf{1}$*, which is also indiscrete. Then there is a unique horizontal transformation* $h\colon F \to G$*. If every category* FI *is non empty (for* I *in* $\mathbb{I}$*), then* h *is a pointwise equivalence.*

Note. One can also prove that *pseudo* horizontal transformations $k\colon G \to F$ amount to *transformations* $k_0\colon G_0 \to F_0$ which are not assumed to satisfy the naturality condition.

Proof. We begin by proving (b), which implies (a). A horizontal transformation $k\colon G \to F$ consists of the following components:

(i) a functor $kI\colon GI \to FI$ for each I,

(ii) a cell $ku\colon (Gu\ {}^{kI}_{kJ}\ Fu)$ for each $u\colon I \to\!\!\!\!\!\bullet\ J$.

Now, kI is determined by its function on objects

$$k_0 I\colon \mathrm{Ob}(GI) \to \mathrm{Ob}(FI),$$

while the cell ku always exists and is unique, because of the terminal-filling property of the profunctor Fu. Finally the naturality of k amounts to the naturality of k_0, while its vertical coherence is automatic, because of the same terminal property (taking into account that a composite of indiscrete profunctors is indiscrete).

(c) Follows immediately from 7.1. □

7.4 Proposition

The (horizontal double) limit of an indiscrete diagram $F\colon \mathbb{I} \to \mathbb{C}\mathrm{at}$ always exists, and is given by the indiscrete category $L = \mathrm{Indisc}(S)$ whose set of objects S is the ordinary limit of the functor $F_0\colon \mathrm{Hor}_0\mathbb{I} \to \mathbf{Set}$.

Proof. A cone (A, h) of F is a category A with a horizontal transformation $h\colon DA \to F$. By Lemma 7.3 this is the same as an ordinary cone

$$h_0\colon (DA)_0 \to F_0\colon \mathrm{Hor}_0\mathbb{I} \to \mathbf{Set},$$

which corresponds to a function $(DA)_0 \to S = \mathrm{Lim}\, F_0$ of sets, and therefore to a functor $A \to L$, with values in $L = \mathrm{Indisc}(S)$.

A horizontal morphism $f\colon (A, h) \to (B, k)$ of cones of F is an ordinary functor $f\colon A \to B$ that satisfies condition (i), (ii) of 5.5. But the first is the cone-condition for $h_0\colon A \to F_0$, while the second

$$hu = (e_f \,|\, ku)\colon eA \to Fu,$$

is automatically satisfied, because Fu is an indiscrete profunctor.

We have thus proved that the 1-dimensional limit of F is indeed L. It will be sufficient to prove that the 2-dimensional property (lim2) is automatically satisfied.

For any vertical morphism $(u, \xi)\colon (A, h) \nrightarrow (B, k)$ we want to find a unique cell $\alpha\colon (u \, {}^{f}_{g} \, L)$ in $\mathbb{C}\mathrm{at}$ such that condition (40) holds:

$$(\alpha \,|\, pI) = \xi I\colon u \to e_{FI} \qquad \text{(for } I \text{ in } \mathbb{I}).$$

But L is indiscrete, so that there is precisely one cell $\alpha\colon (u \, {}^{f}_{g} \, L)$, and the condition above is also satisfied, because FI is also indiscrete. □

7.5 Corollary

If $F, G\colon \mathbb{I} \to \mathbb{C}\mathrm{at}$ *are indiscrete diagrams, then*

$$\mathrm{Lim}\,(F \times G) \;\cong\; \mathrm{Lim}\,F \times \mathrm{Lim}\,G.$$

Proof. Obvious. □

7.6 Completing the Persistence Theorem

For the last point of the proof of Theorem 6.4, we suppose that $\mathbb{I}$-limits are persistent in $\mathbb{C}\mathrm{at}$ and want to prove that the weak double category $\mathbb{I}$ is grounded.

Let $F\colon \mathbb{I} \to \mathbb{C}\mathrm{at}$ be the indiscrete diagram corresponding to the following functor $F_0\colon \mathrm{Hor}_0\mathbb{I} \to \mathbf{Set}$

$$\begin{aligned} F_0(I) &= \{x\colon X \to I \,|\, x \text{ horizontal in } \mathbb{I}\}, \\ F_0(i\colon I \to I')&\colon x \mapsto ix. \end{aligned} \tag{61}$$

Let $G\colon \mathbb{I} \to \mathbb{C}\mathrm{at}$ be the constant diagram at the (indiscrete) terminal category $\mathbf{1}$. Since each category $F(I)$ is non empty, we know from Lemma 7.3(c) that there is a unique horizontal transformation $m\colon F \to G$, which is a pointwise equivalence.

The double functors F and G have a limit, by Proposition 7.4. By hypothesis, the induced morphism $\mathrm{Lim}\,F \to \mathrm{Lim}\,G$ is a horizontal equivalence. But $\mathrm{Lim}\,G \;\cong\; \mathbf{1}$, whence the category $\mathrm{Lim}\,F$ is non empty.

There exists an object of $\mathrm{Lim}\, F$. This is an element of the set $\mathrm{Lim}\, F_0$, i.e. a cone $\varphi\colon \mathbf{1} \to F_0$, or – in other words – a natural family of arrows $(\varphi I\colon \Phi I \to I)_{I \in Ob\mathbb{I}}$

$$\varphi I' = i.\varphi I\colon \Phi I \to I \to I' \qquad (\text{for } i\colon I \to I' \text{ in } \mathbb{I}), \tag{62}$$

(whence $\Phi I = \Phi I'$), which precisely means that $\mathbb{I}$ is grounded.

7.7 Remarks

In the Bangor talk [Pa] the Persistence Theorem was stated by considering lax functors $\mathbb{I} \to \mathbf{A}$ with values in a 2-category, viewed as a horizontal double category (with trivial vertical arrows).

In fact, one can deduce from the present Persistence Theorem I that the two approaches are equivalent: a restricted persistence hypothesis concerning these restricted diagrams would have the same consequence on the weak double category $\mathbb{I}$.

We start again from the indiscrete diagram $F\colon \mathbb{I} \to \mathbb{C}\mathrm{at}$ corresponding to the functor $F_0\colon \mathrm{Hor}_0\mathbb{I} \to \mathbf{Set}$ of (61)

$$F_0(I) = \{x\colon X \to I \mid x \text{ horizontal in } \mathbb{I}\}, \tag{63}$$

and we modify it into a double functor $\mathbb{I} \to \mathbf{Cat}$ with values in the horizontal double category of the 2-category $\mathbf{Cat}$.

First we let S be a set large enough so that $F_0 I \times S \;\cong\; S$ (for all I), which is possible because the set $F_0 I$ is not empty. For each object I we choose an isomorphism

$$\vartheta I\colon F_0 I \times S \to S.$$

Letting $H_0\colon \mathrm{Hor}_0\mathbb{I} \to \mathbf{Set}$ be constant at S, the functor

$$F_0 \times H_0\colon \mathrm{Hor}_0\mathbb{I} \to \mathbf{Set},$$

can be transported along the isomorphisms ϑI, producing a functor $K_0\colon \mathrm{Hor}_0\mathbb{I} \to \mathbf{Set}$ and a functorial isomorphism $\vartheta\colon F_0 \times H_0 \to K_0$. The new functor K_0 is not constant, but it is constant on objects: $K_0 I = S$, for all I.

Let $H, K\colon \mathbb{I} \to \mathbb{C}\mathrm{at}$ be the indiscrete diagrams corresponding to H_0, K_0; again, $F \times H \;\cong\; K$.

Now for every $u\colon I \nrightarrow J$ the profunctor Ku is constant at the singleton; but $KI = KJ = \mathrm{Indisc}(S)$, whence Ku is the vertical identity of KI, and K factorises through the 2-category $\mathbf{Cat}$.

As in 7.6, the (unique) horizontal transformation $K \to G$ (with values in the constant diagram at $\mathbf{1}$) is a pointwise equivalence. By the hypothesis of 'restricted persistence' on $\mathbb{I}$, we deduce that the category $\mathrm{Lim}\, K$ is not empty. But $\mathrm{Lim}\, K \;\cong\; \mathrm{Lim}\, F \times \mathrm{Lim}\, H$ (by Corollary 7.5), so $\mathrm{Lim}\, F$ is not empty and we conclude as at the end of 7.6 that $\mathbb{I}$ is grounded.

References

[AK] M.H. Albert, G.M. Kelly, The closure of a class of colimits, J. Pure Appl. Algebra 51 (1988), 1–17.

[Be] J. Bénabou, Introduction to bicategories, in: Reports of the Midwest Category Seminar, Lecture Notes in Math. Vol. 47, Springer, Berlin 1967, pp. 1–77.

[BKP] G.J. Bird, G.M. Kelly, A.J. Power, Two-dimensional monad theory, J. Pure Appl. Algebra 59 (1989), 1–41.

[BKPS] G.J. Bird, G.M. Kelly, A.J. Power, R.H. Street, Flexible limits for 2-categories, J. Pure Appl. Algebra 61 (1989), 1–27.

[Eh1] C. Ehresmann, Catégories structurées, Ann. Sci. Ecole Norm. Sup. 80 (1963), 349–425.

[Eh2] C. Ehresmann, Catégories et structures, Dunod, Paris 1965.

[GP1] M. Grandis, R. Paré, Limits in double categories, Cah. Topol. Géom. Différ. Catég. 40 (1999), 162–220.

[GP2] M. Grandis, R. Paré, Adjoint for double categories, Cah. Topol. Géom. Différ. Catég. 45 (2004), 193–240.

[GP3] M. Grandis, R. Paré, Kan extensions in double categories (On weak double categories, Part III), Theory Appl. Categ. 20 (2008), No. 8, 152–185.

[GP4] M. Grandis, R. Paré, Lax Kan extensions for double categories (On weak double categories, Part IV), Cah. Topol. Géom. Différ. Catég. 48 (2007), 163–199.

[GP5] M. Grandis, R. Paré, Span and cospan representations of weak double categories, Categ. Gen. Algebr. Struct. Appl. 6 (2017), 85–105. Online publication (2016): http://www.cgasa.ir/volume_3074.html

[K1] G.M. Kelly, Basic concepts of enriched category theory, Cambridge University Press, Cambridge 1982.

[K2] G.M. Kelly, Elementary observations on 2-categorical limits, Bull. Austral. Math. Soc. 39 (1989), 301–317.

[Pa] R. Paré, Double limits, International Category Theory Meeting, Bangor 1989, unpublished talk.

[St] R.H. Street, Limits indexed by category valued 2-functors, J. Pure Appl. Algebra 8 (1976), 149–181.

[Ve] D. Verity, Enriched categories, internal categories and change of base, Ph.D. Dissertation, University of Cambridge, 1992. Published in: Repr. Theory Appl. Categ. No. 20 (2011), 1–266.

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COMPATIBILITÉ ENTRE DEUX CONCEPTIONS D'ALGÈBRE SUR UNE OPÉRADE

Jacques PENON

Résumé. Opérades de May et opérade de Burroni ont chacune leurs algèbres. Nous donnons ici un contexte général qui va nous permettre de montrer, facilement, l'équivalence entre ces deux conceptions d'algèbres.
Abstract. May's operads and Burroni's operads have each one their algebras. Here, we give a general context that enable to prove, easily, the equivalence between two algebra's kinds.
Keywords. Operad. Monoidal category. Monoid. Cartesian monad. Enriched category.
Mathematics Subject Classification (2010). 18D50.

Introduction

Opérade de May (voir [4]) et opérade de Burroni (un cas particulier des $\mathbb{T}$-catégories) (voir [1]), possèdent chacune des algèbres a priori de conception différente et bien qu'une opérade de May puisse être vu comme un cas particulier de celle de Burroni, ses algèbres se généralisent mal dans le cadre de celle de Burroni. Qui plus est, lorsque c'est le cas (comme par exemple dans un topos muni d'une monade cartésienne) l'équivalence entre les deux types d'algèbres n'est pas immédiate à vérifier (voir [3]).
Dans cet article, après avoir rappelé brièvement les définitions des deux types d'opérades et de leurs algèbres (voir la section 1), nous montrons que derrière cette problématique se cache une structure relativement nouvelle (voir [5]) naturellement présente dans la catégorie de base (voir la section 2)

qui permet d'éclairer la question en la généralisant mais encore de trivialiser bon nombre de démonstrations (voir les sections 3 et 4).

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1. Rappel des deux conceptions

• En 1972 P.May définit le concept d'opérade dans [4]. Rappelons en ici la définition.

Définition 1.1. : Une *opérade* (de May) $\underline{\Omega}$ est la donnée :
- d'une collection (Ω, π) (c.a.d. $\Omega \in |\underline{\mathbb{E}ns}|$ et $\pi : \Omega \to \mathbb{N}$ une application),
- un élément particulier $e \in \Omega$,
- une application $m : \Omega^{(2)} \to \Omega$ où $\Omega^{(2)}$ est défini par :
$\Omega^{(2)} = \{(s,(s_o,...,s_{n-1})) \in \Omega \times Mo(\Omega)/\pi(s) = n\}$ ($Mo(\Omega)$ désignant l'ensemble des suites finies (ou listes) d'éléments de Ω).
Toutes ces données devant satisfaire les axiomes suivants :
(Pos) $\pi(e) = 1$ et $\forall(s,(s_o,...,s_{n-1})) \in \Omega^{(2)}$, $\pi.m(s,(s_o,...,s_{n-1})) = \sum_{j\in[n]} \pi(s_j)$ (où $[n] = \{0,...,n-1\}$),
(Ug) $m(e,(s)) = s$,
(Ud) $m(s,(e,...,e)) = s$, où $(e,...,e)$ est une liste de longueur $\pi(s)$,
(Ass) $\forall s \in \Omega, \forall(s_o,...,s_{n-1}) \in \Omega^n, \forall(\bar{s}_o,...,\bar{s}_{n-1}) \in Mo(\Omega)^n$ tels que $n = \pi(s)$,
$\forall j \in [n]$, $\pi(s_j) = \mathrm{L}(\bar{s_j})$ (où ici $\mathrm{L}(-)$ désigne la fonction longueur d'une liste), alors :

$$m(s,(m(s_0,\bar{s}_0),...,m(s_{n-1},\bar{s}_{n-1})) = m(m(s,(s_o,...,s_{n-1})),\bar{s}_0...\bar{s}_{n-1}).$$

(Ici $\bar{s}_0...\bar{s}_{n-1}$ désigne la concaténation des listes $\bar{s}_0, ..., \bar{s}_{n-1}$).

Exemple 1.2. : A chaque ensemble E on associe une opérade $\underline{\Omega}_E$ que P. May appelle *l'opérade des endomorphismes de* E. Elle est donnée par :
- L'ensemble $\Omega_E = \{(n, f)/n \in \mathbb{N}$ et $f : E^n \to E$ une application$\}$ (Dans la suite de la section les éléments de Ω_E seront souvent notés "$f : E^n \to E$" ou, abusivement, f au lieu de (n, f)),
- $\pi : \Omega_E \to \mathbb{N}$ est l'application $(n, f) \mapsto n$,
- $e = (1, Id_E)$, où on identifie E^1 et E,
- Pour tout $(f, (f_0, ..., f_{n-1})) \in \Omega_E^{(2)}$, c.a.d. $f : E^n \to E$ et $\forall j \in [n], f_j : E^{m_j} \to E$, alors :

$$m(f, (f_0, ..., f_{n-1})) =$$
$$(E^{m_0+...+m_{n-1}} \simeq E^{m_0} \times ... \times E^{m_{n-1}} \overset{f_0\times...\times f_{n-1}}{\longrightarrow} E^n \overset{f}{\longrightarrow} E).$$

Définition 1.3. : $\underline{\Omega}$ et $\underline{\Omega}'$ étant des opérades, un *morphisme* $\phi : \underline{\Omega} \to \underline{\Omega}'$ est une application $\Omega \to \Omega'$ telle que :
(MP) $\forall s \in \Omega,\ \pi'.\phi(s) = \pi(s)$,
(MU) $\phi(e) = e'$,
(MC) $\forall(s, (s_0, ..., s_{n-1})) \in \Omega^{(2)}, \phi.m(s, (s_0, ..., s_{n-1})) =$
$$m'(\phi(s), (\phi(s_0), ..., \phi(s_{n-1})))$$

• On en vient maintenant à la première conception d'algèbre sur une opérade.

Définition 1.4. : $\underline{\Omega}$ étant une opérade, une *algèbre* (au sens de P.May) sur $\underline{\Omega}$ est la donnée successivement :
- d'un ensemble A,
- d'un morphisme d'opérade $a : \underline{\Omega} \to \underline{\Omega}_A$.

Remarque 1.5. : Cette définition est assez intuitive car on associe à chaque "opération formelle" $s \in \Omega$ une "opération effective" $a(s) : A^n \to A$, (où $n = \pi(s)$).

Définition 1.6. : (A, a) et (A', a') étant deux algèbres sur une opérade $\underline{\Omega}$, un *morphisme* $(A, a) \to (A', a')$ est une application $f : A \to A'$ telle que, pour tout $s \in \Omega$, le carré suivant commute (où $n = \pi(s)$) :

$$\begin{array}{ccc} A^n & \xrightarrow{f^n} & A'^n \\ {\scriptstyle a(s)}\downarrow & & \downarrow{\scriptstyle a'(s)} \\ A & \xrightarrow[f]{} & A' \end{array}$$

• A peu près à la même époque (en 1971), A.Burroni définit le concept de $\mathbb{T}$-catégorie (voir [1]) qui, il ne le savait pas encore, généralise celui d'opérade.

Remarque 1.7. : Dans la définition qui va suivre, on se focalise uniquement sur les $\mathbb{T}$-catégories dont l'objet des objets est égal à 1 (l'objet final de la catégorie de base). Ce sont elles que nous baptiserons (en accord avec l'usage actuel (voir [3])) opérade sur $\mathbb{T}$.

• On se donne une catégorie $\underline{E}$ à limites à gauche finies et $\mathbb{M} = (M, \eta, \mu)$ une monade cartésienne sur $\underline{E}$ (c.a.d. M commute aux produits fibrés et les deux transformations naturelles η et μ sont cartésiennes ce qui signifie qu'elles envoient une flèche quelconque sur un produit fibré (voir [2])) . Dans $\underline{E}$ on choisit un objet final 1 et des produits fibrés. On munit la catégorie $\underline{E}/M(1)$ d'une structure monoïdale pour laquelle :
- l'unité est $I = (1, \eta_1 : 1 \to M(1))$,
- le produit tensoriel est $(C, \pi) \otimes (C', \pi') = (\hat{C}, \hat{\pi})$ où $\hat{C}$ est l'objet de $\underline{E}$ obtenu par le produit fibré suivant:

$$\begin{array}{ccc} \hat{C} & \xrightarrow{pr_1} & M(C') \\ {\scriptstyle pr_0}\downarrow & & \downarrow{\scriptstyle M(!)} \\ C & \xrightarrow[\pi]{} & M(1) \end{array}$$

et $\hat{\pi} = (\hat{C} \xrightarrow{pr_1} M(C') \xrightarrow{M(\pi')} M^2(1) \xrightarrow{\mu_1} M(1))$.
Grâce à la cartésianité de $\mathbb{M}$, les isomorphismes naturels $(C, \pi) \simeq (C, \pi) \otimes I$ et $(C, \pi) \otimes ((C', \pi') \otimes (C'', \pi'')) \simeq ((C, \pi) \otimes (C, \pi')) \otimes (C'', \pi'')$ se construisent facilement.
On note $\mathcal{C}oll(\mathbb{M})$ la catégorie monoïdale obtenue. Ces objets sont maintenant appelés des collections sur $\mathbb{M}$.

Définition 1.8. : On appelle *opérade* sur $\mathbb{M}$(selon A.Burroni) un monoïde dans la catégorie monoïdale $\mathcal{C}oll(\mathbb{M})$.

Remarque 1.9. : Cette définition généralise celle de P.May car, si on prend $\mathbb{M} = \mathbb{M}o = (Mo, \eta, \mu)$ la monade des monoïdes où $\underline{E} = \underline{\mathbb{E}ns}$, alors $\underline{\mathbb{E}ns}/Mo(1)$ s'identifie à la catégorie des collections de May. (Ω, π, e, m) étant maintenant une opérade de May, on lui fait correspondre le monoïde dans $\mathcal{C}oll(\mathbb{M}o)$ dont l'unité $I \to (\Omega, \pi)$ est l'application constante sur e et sa multiplication $(\Omega, \pi) \otimes (\Omega, \pi) \to (\Omega, \pi)$, en tant qu'application, s'identifie à $m : \Omega^{(2)} \to \Omega$.

• Le grand intérêt de telles opérades généralisées est qu'on leur associe canoniquement une monade sur $\underline{E}$. Lorsque $\underline{\Omega} = ((\Omega, \pi), e, m)$ est une opérade sur $\mathbb{M}$, la monade $\underline{\tilde{\Omega}} = (\tilde{\Omega}, \tilde{\eta}, \tilde{\mu})$ obtenue a son endofoncteur qui, sur un objet X, est donné par le produit fibré :

$$\begin{array}{ccc} \tilde{\Omega}(X) & \xrightarrow{\pi'_X} & \Omega \\ {\scriptstyle \tilde{\pi}_X}\downarrow & & \downarrow{\scriptstyle \pi} \\ M(X) & \xrightarrow[M(!)]{} & M(1) \end{array}$$

$\tilde{\eta}_X$ et $\tilde{\mu}_X$ s'obtiennent respectivement en utilisant les flèches e et m (ce calcul sera repris et généralisé à la section 3).
Dans cette manière de voir les opérades, une *algèbre* sur l'opérade $\underline{\Omega}$ est alors tout simplement une algèbre d'Eilenberg-Moore sur la monade $\underline{\tilde{\Omega}}$.
Dans le cas où $\mathbb{M} = \mathbb{M}o$ (pour $\underline{E} = \underline{\mathbb{E}ns}$) les deux conceptions d'algèbre sur une opérade sont équivalentes (c.a.d. qu'on a une équivalence entre la catégorie des algèbres de May sur $\underline{\Omega}$ et la catégorie des algèbres d'Eilenberg-Moore sur la monade $\underline{\tilde{\Omega}}$).
Dans [3] T.Leinster généralise cette équivalence, dans le cas où $\underline{E}$ est un topos et $\mathbb{M}$ est une monade cartésienne quelconque (Il faut pour cela construire, dans ce cadre, une opérade des endomorphismes associée à un objet quelconque X de $\underline{E}$).

Remarque 1.10. : 1) Nous reviendrons sur ces différentes généralisations dans les sections suivantes.
2) En essayant de reprendre la preuve du résultat de T.Leinster et ayant

buté pendant un moment sur la méthode à suivre, nous nous sommes rendu compte qu'il manquait un ingrédient décisif qui, contre toute attente, allait trivialiser la situation. Nous allons voir, dans le contexte qui va suivre, comment nous abordons la question.

2. Les catégories $\mathbb{V}$-tensorisées

Donnons nous au départ une catégorie monoïdale quelconque
$\mathbb{V} = (\underline{V}, \otimes, I, u_g, u_d, ass)$.

Définition 2.1. On appelle *catégorie* $\mathbb{V}$-*tensorisée* (à gauche) (encore appelée $\mathbb{V}$-module dans [5]) la donnée :
- d'une catégorie $\underline{E}$,
- d'un foncteur $\wedge : \underline{V} \times \underline{E} \to \underline{E}$ (appelé produit tensoriel extérieur),
- d'une première famille, dans $\underline{E}$, d'isomorphismes

$$s_X : I \wedge X \to X$$

qui est naturelle en $X \in |\underline{E}|$(dans la suite on omettra l'indice X de s_X),
- d'une seconde famille, dans $\underline{E}$, d'isomorphismes

$$am_{(A,B,X)} : (A \otimes B) \wedge X \to A \wedge (B \wedge X)$$

qui est naturelle en $(A, B, X) \in |\mathbb{V}| \times |\mathbb{V}| \times |\underline{E}|$ (de même, dorénavant, on omettra A, B, X dans $am_{A,B,X}$), vérifiant les deux axiomes de cohérence suivants, où $A, B, C \in |\mathbb{V}|$ et $X \in |\underline{E}|$:
(UD)

$$\begin{array}{ccc} (A \otimes I) \wedge X & \xrightarrow{am} & A \wedge (I \wedge X) \\ & {\scriptstyle u_d \wedge Id} \searrow \quad \swarrow {\scriptstyle Id \wedge s} & \\ & A \wedge X & \end{array}$$

(AM)

$$\begin{array}{ccc} ((A \otimes B) \otimes C) \wedge X & \xrightarrow{ass \wedge Id} & (A \otimes (B \otimes C)) \wedge X \\ \downarrow {\scriptstyle am} & & \downarrow {\scriptstyle am} \\ (A \otimes B) \wedge (C \wedge X) & & A \wedge ((B \otimes C)) \wedge X) \\ & {\scriptstyle am} \searrow \quad \swarrow {\scriptstyle Id \wedge am} & \\ & A \wedge (B \wedge (C \wedge X)) & \end{array}$$

Remarque 2.2. (voir [5]): Comme dans le cas des catégories monoïdales, des deux axiomes (UD) et (AM) on en déduit la commutativité (UG) suivante:

$$\begin{array}{ccc} (I \otimes A) \wedge X & \xrightarrow{am} & I \wedge (A \wedge X) \\ & {\scriptstyle u_g \wedge Id} \searrow \quad \swarrow {\scriptstyle s} & \\ & A \wedge X & \end{array}$$

Exemples 2.3. : 1) Une catégorie monoïdale $\mathbb{V}$ est elle-même $\mathbb{V}$-tensorisée en prenant $\wedge = \otimes,\ s = u_g,\ am = ass.$
2) $\underline{C}$ étant une catégorie, soit $\mathbb{V}$ la catégorie monoïdale stricte $[\underline{C}, \underline{C}]$ des endofoncteurs de $\underline{C}$(où $\otimes$ est la composisition des foncteurs (pour les objets) et la composition horizontale des transformations naturelles (pour les flèches)). $\underline{C}$ devient alors une catégorie $\mathbb{V}$-tensorisée "stricte",où $\wedge$ est le foncteur $[\underline{C}, \underline{C}] \times \underline{C} \to \underline{C},\ (F, X) \mapsto F(X),\ s = Id,\ am = Id.$
3) Soit $\underline{E}$ une catégorie à limites à gauche finies et $\mathbb{M} = (M, \eta, \mu)$ une monade cartésienne sur $\underline{E}$. On considère la catégorie monoïdale $\mathbb{V} = \mathcal{C}oll(\mathbb{M})$ des collections sur $\mathbb{M}$ (voir la section 1). On construit ensuite un foncteur $\wedge : \underline{V} \times \underline{E} \to \underline{E}$ défini sur les objets par $(C, \pi) \wedge X = P$ où P est donné par le produit fibré suivant :

$$\begin{array}{ccc} P & \xrightarrow{pr_1} & M(X) \\ {\scriptstyle pr_0} \downarrow & & \downarrow {\scriptstyle M(!)} \\ C & \xrightarrow[\pi]{} & M(1) \end{array}$$

La construction de $s_X : I \wedge X \to X$ provient de la cartésianité du carré suivant :

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & M(X) \\ {\scriptstyle !} \downarrow & & \downarrow {\scriptstyle M(!)} \\ 1 & \xrightarrow[\eta_1]{} & M(1) \end{array}$$

et celle de $am : (\bar{A} \otimes \bar{B}) \wedge X \to \bar{A} \wedge (\bar{B} \wedge X)$ (où $\bar{A} = (A, \pi),\ \bar{B} = (B, \pi) \in \mathcal{C}oll(\mathbb{M})$ et $X \in |\underline{E}|$), de la cartésianité du composé des carrés cartésiens suivants :

$$\begin{array}{ccccccc}
\bar{A}\wedge(\bar{B}\wedge X) & \xrightarrow{pr_1} & M(\bar{B}\wedge X) & \xrightarrow{M(pr_1)} & M^2(X) & \xrightarrow{\mu_X} & M(X) \\
\downarrow{\scriptstyle Id\wedge pr_0} & & \downarrow{\scriptstyle M(pr_0)} & & \downarrow{\scriptstyle M^2(!)} & & \downarrow{\scriptstyle M(!)} \\
\bar{A}\wedge B & \xrightarrow[pr_1]{} & M(B) & \xrightarrow[M(\pi)]{} & M^2(1) & \xrightarrow[\mu_1]{} & M(1)
\end{array}$$

Définition 2.4. : Soit $\mathbb{E}=(\underline{E},\wedge,s,am)$ une catégorie $\mathbb{V}$-tensorisée. On dit qu'elle est *enrichissable* si, pour tout $X\in|\underline{E}|$, le foncteur $(-)\wedge X:\underline{V}\to\underline{E}$ admet un adjoint à droite. Dans ce cas, on note $(-)^X:\underline{E}\to\underline{V}$ le choix d'un adjoint à droite et $Ev^X:(-)^X\wedge X\to(-)$, où simplement Ev, la co-unité de l'adjonction.

Proposition 2.5. : Si $\mathbb{E}$ est enrichissable il existe une structure de catégorie $\mathbb{V}$-enrichie canonique sur $\underline{E}$ (notée $\mathcal{E}$).

Preuve : - Tout d'abord on pose $|\mathcal{E}|=|\underline{E}|$.
- Pour $X,Y\in|\underline{E}|$, on pose $\mathcal{E}(X,Y)=(-)^X(Y)$.
- Pour $X\in|\underline{E}|$, on considère $id_X:I\to\mathcal{E}(X,X)$ l'unique flèche de $\underline{V}$ telle que le triangle suivant commute :

$$\begin{array}{ccc}
I\wedge X & \xrightarrow{id_X\wedge Id} & \mathcal{E}(X,X)\wedge X \\
{\scriptstyle s}\searrow & & \swarrow{\scriptstyle Ev} \\
 & X &
\end{array}$$

- Pour $X,Y,Z\in|\underline{E}|$, la flèche $c_{X,Y,Z}:\mathcal{E}(Y,Z)\otimes\mathcal{E}(X,Y)\to\mathcal{E}(X,Z)$ (la composition) est l'unique flèche de $\underline{V}$ qui fait commuter le diagramme suivant :

$$\begin{array}{ccc}
(\mathcal{E}(Y,Z)\otimes\mathcal{E}(X,Y))\wedge X & \xrightarrow{c_{X,Y,Z}\wedge Id} & \mathcal{E}(X,Z)\wedge X \\
\downarrow{\scriptstyle am} & & \\
\mathcal{E}(Y,Z)\wedge(\mathcal{E}(X,Y))\wedge X) & & \downarrow{\scriptstyle Ev} \\
\searrow{\scriptstyle Id\wedge Ev} & & \\
\mathcal{E}(Y,Z)\wedge Y & \xrightarrow{Ev} & Z
\end{array}$$

• Reprenons l'exemple 3 dans 2.3 où $\mathbb{V}=\mathcal{C}oll(\mathbb{M})$.

Proposition 2.6. : Si $\underline{V} = \underline{E}/M(1)$ est cartésienne fermée alors, en tant que catégorie $\mathbb{V}$-tensorisée, $\mathbb{E} = (\underline{E}, \wedge, s, am)$ est enrichissable.

Preuve : Pour chaque $X \in |\underline{E}|$, le foncteur $(-) \wedge X : \underline{V} \to \underline{E}$ est le composé suivant :

$$\underline{E}/M(1) \xrightarrow{(-)\times X_*} \underline{E}/M(1) \xrightarrow{U} \underline{E}$$

où U est le foncteur d'oubli canonique, $(-) \times (-)$ est le produit cartésien dans $\underline{V}$, et $X_* = (M(X), M(!)) \in |\underline{V}|$. Or $(-) \times X_*$ et U admettent des adjoints à droite; d'où la conclusion voulue.

Remarque 2.7. : 1) En particulier lorsque $\underline{E}$ est un topos, $\mathbb{E}$ est enrichissable, en tant que catégorie $\mathbb{V}$-tensorisée, où $\mathbb{V} = \mathcal{C}oll(\mathbb{M})$.
2) Si $\mathcal{E}$ désigne la catégorie enrichie dans $\mathbb{V}$ obtenue, on voit que pour chaque $X \in |\underline{E}|$, $\mathcal{E}(X, X)$ a une structure de monoïde dans $\mathbb{V}$. C'est donc une opérade sur $\mathbb{M}$ qui, dans le cas ensembliste (c.a.d. $\underline{E} = \underline{\mathbb{E}ns}$ et $\mathbb{M} = \mathbb{M}o$) n'est autre que l'opérade des endomorphismes de X (voir sa définition en 1.2).

3. Catégorie tensorisée et monade

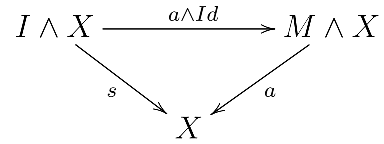
• Comme à la section précédente, donnons nous une catégorie monoïdale $\mathbb{V} = (\underline{V}, \otimes, ...)$ mais aussi une catégorie $\mathbb{V}$-tensorisée $\mathbb{E} = (\underline{E}, \wedge, s, am)$. Considérons maintenant un monoïde $\mathcal{M} = (M, e, m)$ dans $\mathbb{V}$. On lui associe canoniquement une monade, notée $\mathcal{M}^\wedge$, de la façon suivante :
- L'endofoncteur de $\mathcal{M}^\wedge$ est donné par $M \wedge (-)$.
- La transformation naturelle $\eta : Id \to M \wedge (-)$ est donnée, pour chaque $X \in |\underline{E}|$, par :

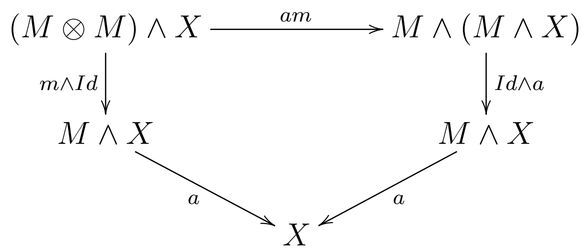
$$\eta_X = (X \xrightarrow{s^{-1}} I \wedge X \xrightarrow{e \wedge Id} M \wedge X)$$

- La transformation naturelle $\mu : M \wedge (M \wedge (-)) \to M \wedge (-)$ est donnée, pour chaque $X \in |\underline{E}|$, par :

$$\mu_X = (M \wedge (M \wedge X) \xrightarrow{am^{-1}} (M \otimes M) \wedge X \xrightarrow{m \wedge Id} M \wedge X)$$

Remarques 3.1. : 1) Une algèbre sur la monade $\mathcal{M}^\wedge$ est un couple (X, a) où $X \in |\underline{E}|$ et $a : M \wedge X \to X$ est une flèche de $\underline{E}$ faisant commuter les diagrammes suivants :





2) Si on reprend l'exemple 3 de 2.3, on retrouve la monade associée à une opérade signalée à la section 1 (on a $\underline{\Omega}^\wedge \simeq \tilde{\underline{\Omega}}$).

Si maintenant $h : \mathcal{M} \to \mathcal{M}'$ est un morphisme de monoïde dans $\mathbb{V}$, on lui associe un morphisme de monade $\mathcal{M}^\wedge \to \mathcal{M}'^\wedge$ sur $\underline{E}$, noté $h^\wedge$. Il est défini sur un objet X par $h^\wedge_X = h \wedge Id_X : M \wedge X \to M' \wedge X$.
Supposons maintenant que $\underline{V}$ admette un objet final 1. Alors 1 a une unique structure de monoïde dans $\mathbb{V}$ et, pour tout monoïde $\mathcal{M}$ de $\mathbb{V}$, la flèche unique $! : M \to 1$ est un morphisme de monoïde. Elle induit donc un morphisme de monade $!^\wedge : \mathcal{M}^\wedge \to 1^\wedge$.

Remarque 3.2. : Toujours en reprenant l'exemple 3 de 2.3, on voit que si $\mathbb{V} = \mathcal{C}oll(\mathbb{M})$ où $\mathbb{M}$ est une monade cartésienne sur $\underline{E}$, alors $\underline{V} = \underline{E}/M(1)$ a un objet final $\mathbb{I} = (M(1), Id_{M1})$. Dans ce cas particulier, on a un isomorphisme canonique de monade $\mathbb{I}^\wedge \simeq \mathbb{M}$.

• Le théorème qui suit répond à la problèmatique énoncée dans l'introduction de cet article. Il donne en fait une généralisation du théorème de P. Leinster (voir [3]).

Théorème 3.3. : Soit $\mathbb{V} = (\underline{V}, \otimes, ...)$ est une catégorie monoïdale, soit $\mathbb{E} = (\underline{E}, \wedge, ...)$ une catégorie $\mathbb{V}$-tensorisée enrichissable et $\mathcal{M} = (M, e, m)$ un monoïde dans $\mathbb{V}$. Alors on a un isomorphisme canonique :

$$Alg(\mathcal{M}^\wedge) \simeq \mathbb{V}\text{-}Cat(\mathcal{M}, \mathcal{E})$$

où $\mathcal{E}$ désigne la catégorie enrichie sur $\mathbb{V}$ canonique associée à $\mathbb{E}$.

Preuve : Notons γ l'isomorphisme à construire.
- Pour $(X, a) \in |Alg(\mathcal{M}^\wedge)|$, $\gamma(X, a)$ est donné par $\bar{a} : M \to \mathcal{E}(X, X)$, qui est l'unique flèche de $\underline{V}$ rendant commutatif le triangle suivant :

$$\begin{array}{ccc} M \wedge X & \xrightarrow{\bar{a} \wedge Id} & \mathcal{E}(X, X) \wedge X, \\ & {\scriptstyle a} \searrow \quad \swarrow {\scriptstyle Ev} & \\ & X & \end{array}$$

- Pour une flèche $f : (X, a) \to (X', a')$ de $Alg(\mathcal{M}^\wedge)$, la flèche $\gamma(f) : \gamma(X, a) \to \gamma(X', a')$ est la $\mathbb{V}$-transformation naturelle définie par $\gamma(f) = \bar{f} : I \to \mathcal{E}(X, X')$, qui est l'unique flèche de $\mathbb{V}$ faisant commuter le carré suivant :

$$\begin{array}{ccc} I \wedge X & \xrightarrow{\bar{f} \wedge Id} & \mathcal{E}(X, X') \wedge X \\ {\scriptstyle s} \downarrow & & \downarrow {\scriptstyle Ev} \\ X & \xrightarrow[f]{} & X' \end{array}$$

Remarques 3.4. : 1) Appliquons le théorème à l'exemple 3 de 2.3, au cas ensembliste (où $\mathbb{M} = \mathbb{M}o$). Un monoïde $\underline{\Omega}$ de $\mathcal{C}oll(\mathbb{M}o)$ peut être vu comme une opérade au sens de May (voir 1.9) et $\underline{\Omega}^\wedge \simeq \tilde{\underline{\Omega}}$ (voir 3.1(2)). Alors $Alg(\underline{\Omega}^\wedge) \simeq Alg(\tilde{\underline{\Omega}})$. Le terme de gauche, dans l'isomorphisme du théorème, correspond donc à la deuxième conception d'algèbre sur une opérade.
2) D'autre part, pour chaque $X \in |\underline{\mathbb{E}ns}|$, puisque $\mathcal{E}(X, X)$ correspond à l'opérade des endomorphismes de X (remarque 2.7), à $(X, a) \in Alg(\tilde{\underline{\Omega}})$ correspond bien, par le théorème précédent et la remarque 1, à une algèbre, au sens de May, sur l'opérade $\underline{\Omega}$.

4. Cartésianité

Définition 4.1. : 1) Soit $F : \underline{A} \times \underline{B} \to \underline{C}$ un foncteur quelconque. On dit que F est *cartésien* si, pour tout couple de flèches $(A \xrightarrow{a} A', B \xrightarrow{b} B')$ de

$\underline{A} \times \underline{B}$ le carré suivant est cartésien :

$$\begin{array}{ccc} F(A,B) & \xrightarrow{F(a,Id)} & F(A',B) \\ {\scriptstyle F(Id,b)}\downarrow & & \downarrow{\scriptstyle F(Id,b)} \\ F(A,B') & \xrightarrow[F(a,Id)]{} & F(A',B') \end{array}$$

2) On dit qu'une catégorie $\mathbb{V}$-tensorisée $\mathbb{E} = (\underline{E}, \wedge, ...)$ est *cartésienne* si son produit tensoriel extérieur $\wedge : \underline{V} \times \underline{E} \to \underline{E}$ est cartésien.
3) Lorsque $\mathbb{V}$ possède un objet final 1, on dit qu'une catégorie $\mathbb{V}$-tensorisée $\mathbb{E} = (\underline{E}, \wedge, ...)$ est *fortement cartésienne*, si elle est cartésienne et si $1^\wedge$ est une monade cartésienne.

Exemple 4.2. : Lorsque $\mathbb{V} = \mathcal{C}oll(\mathbb{M})$ où $\mathbb{M}$ est une monade cartésienne sur une catégorie $\underline{E}$ à limites à gauche finies, alors $\mathbb{E} = (\underline{E}, \wedge, ...)$, en tant que catégorie $\mathbb{V}$-tensorisée, est fortement cartésienne.

Proposition 4.3. : Soit $\mathbb{E} = (\underline{E}, \wedge, ...)$ une catégorie $\mathbb{V}$-tensorisée cartésienne et soit $h : \mathcal{M} \to \mathcal{M}'$ un morphisme de monoïdes dans $\mathbb{V}$. Alors à h correspond $h^\wedge : \mathcal{M}^\wedge \to \mathcal{M}'^\wedge$ qui est un morphisme cartésien entre ces deux monades (i.e. La transformation naturelle $h^\wedge : M \wedge (-) \to M' \wedge (-)$ est cartésienne).

Corollaire 4.4. : Si $\mathbb{E}$ est une catégorie $\mathbb{V}$-tensorisée fortement cartésienne, alors pour tout monoïde $\mathcal{M}$ de $\mathbb{V}$, $\mathcal{M}^\wedge$ est une monade cartésienne.

References

[1] A.BURRONI, $\mathbb{T}$-*catégories*, Cahiers de Topologie et Géométrie Différentielle, (1971),Vol.XII, pages 215-321.

[2] A.CARBONI AND P.JOHNSTONE, *Connected limits, familial representability and Artin glueing*, Mathematical Structures in Computer Science (1995) pages 441-459.

[3] T.LEINSTER, *General operads and Multicategories*, arXiv: math /9810053 [math CT],(1998).

[4] J.P.MAY, *The Geometry of Iterated Loop Spaces*, Lect. Notes. in Math. Springer-Verlag,(1972),Vol.271.

[5] X.ROCHARD, *Théorie tannakienne non additive*, Thèse (1998).

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

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LAX DISTRIBUTIVE LAWS FOR TOPOLOGY, I

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Résumé. Pour un quantaloïde $\mathcal{Q}$, considéré comme une bicatégorie, Walters a introduit des catégories enrichies dans $\mathcal{Q}$. Nous étendons ici l'étude des deux dernières décennies des catégories enrichies dans un quantale avec une monade en introduisant des catégories enrichies dans $\mathcal{Q}$ avec une monade et élargissant ainsi son éventail de catégories pour inclure, entre autres, des structures métriques dites partielles. Nous faisons cela en établissant des lois distributives relâchées d'une monade $\mathbb{T}$ sur la monade des préfaisceaux discrets du petit quantaloïde $\mathcal{Q}$, les données primaires de la théorie, plutôt que les extensions relâchées de $\mathbb{T}$ dans la catégorie des relations avex valeurs dans $\mathcal{Q}$ qu'elles décrivent de manière équivalente. La partie centrale du travail établit une correspondance de Galois entre des lois distributives relâchées et des structures algébriques d'Eilenberg-Moore sur l'ensemble des préfaisceaux discrets sur l'ensemble d'objets de $\mathcal{Q}$. Nous faisons une comparaison précise de ces structures avec la notion introduite par Hofmann dans le cas d'un quantale commutatif, appelée ici théories topologiques naturelles, et décrivons les extensions de monade relâchées introduites par Hofmann comme minimales. Tout au long de cet article, divers exemples anciens et nouveaux de structures ordonnées, métriques et topologiques illustrent la théorie développée, qui inclut la prise en compte des foncteurs algébriques et des foncteurs de changement de base en toute généralité.

Abstract. For a quantaloid $\mathcal{Q}$, considered as a bicategory, Walters introduced categories enriched in $\mathcal{Q}$. Here we extend the study of monad-quantale-enriched categories of the past two decades by introducing monad-quantaloid-enriched categories and thereby enlarging its range of example categories to

include, among others, so-called partial metric structures. We do so by making lax distributive laws of a monad $\mathbb{T}$ over the discrete presheaf monad of the small quantaloid $\mathcal{Q}$ the primary data of the theory, rather than the lax monad extensions of $\mathbb{T}$ to the category of $\mathcal{Q}$-relations that they equivalently describe. The central piece of the paper establishes a Galois correspondence between such lax distributive laws and lax Eilenberg-Moore $\mathbb{T}$-algebra structures on the set of discrete presheaves over the object set of $\mathcal{Q}$. We give a precise comparison of these structures with the considerably more restrictive notion introduced by Hofmann in the case of a commutative quantale, called natural topological theories here, and describe the lax monad extensions introduced by him as minimal. Throughout the paper, a variety of old and new examples of ordered, metric and topological structures illustrate the theory developed, which includes the consideration of algebraic functors and change-of-base functors in full generality.
Keywords. Quantaloid, quantale, monad, discrete presheaf monad, lax distributive law, lax λ-algebra, lax monad extension, monad-quantaloid-enriched category, topological theory, natural topological theory, algebraic functor, change-of-base functor.
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1. Introduction

For monads $\mathbb{S}$ and $\mathbb{T}$ on a category $\mathcal{C}$, liftings of $\mathbb{S}$ along the forgetful functor $\mathcal{C}^{\mathbb{T}} \longrightarrow \mathcal{C}$ of the Eilenberg-Moore category of $\mathbb{T}$, or extensions of $\mathbb{T}$ along the insertion functor $\mathcal{C} \longrightarrow \mathcal{C}_{\mathbb{S}}$ to the Kleisli category of $\mathbb{S}$, correspond precisely to Beck's [4] *distributive laws* $\lambda : TS \longrightarrow ST$ of $\mathbb{T}$ over $\mathbb{S}$; see [3] and II.3 of [26] for a compact account of these correspondences. For $\mathcal{C} = \mathbf{Set}, \mathbb{T} = \mathbb{L}$ the free monoid (or *list*) monad, and $\mathbb{S}$ the free Abelian group monad, their algebraic prototype interpretes the left-hand terms of the equations

$$x(y+z) = xy + xz \quad \text{and} \quad (x+y)z = xz + yz$$

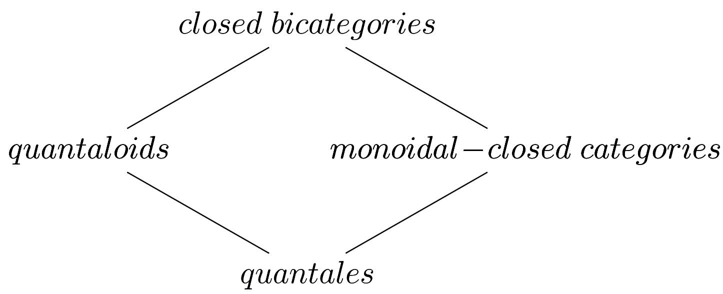
as elements of the free monoid LSX over (the underlying set of) the free Abelian group SX over some alphabet X and assigns to them the right-hand terms in SLX, to then obtain the category of (unital) rings as the Eilenberg-Moore algebras of a composite monad $\mathbb{SL}$ as facilitated by λ. Similarly, keeping $\mathbb{T} = \mathbb{L}$ but letting now $\mathbb{S} = \mathbb{P}$ be the power set monad, the distributive law

$$\lambda_X : L\mathsf{P}X \longrightarrow \mathsf{P}LX, \quad (A_1, ..., A_n) \mapsto A_1 \times ... \times A_n,$$

produces a composite monad whose Eilenberg-Moore category is the category of *quantales*, *i.e.*, of the monoid objects in the monoidal-closed category **Sup** of sup-lattices (see [31, 46]), characterized as the complete lattices with a monoid structure whose multiplication distributes over arbitrary suprema in each variable. Ever since the appearance of Beck's original work, distributive laws have been, and continue to be, studied from a predominantly algebraic perspective, at many levels of generality; see, for example, [54, 33, 21, 6]. But what is their role in topology, if any?

As a unification of the settings used by Lawvere [37] and by Manes [40] and Barr [2] for their respective descriptions of metric spaces and topological spaces, the viewpoint of *monoidal topology* [12, 15, 14, 49, 24, 26] has been that some key categories of analysis and topology are described as categories of lax $(\mathbb{T}, \mathsf{V})$-algebras, also called $(\mathbb{T}, \mathsf{V})$-*categories*, where V is a quantale and $\mathbb{T}$ a $\mathbf{Set}$-monad with a lax extension to the category V-$\mathbf{Rel}$ of sets and V-valued relations (or *matrices* [5]) as morphisms. For example, for $\mathsf{V} = 2$ the two-element chain and for $\mathbb{T} = \mathbb{U}$ the ultrafilter monad with its lax Barr extension to relations, one obtains the Manes-Barr presentation of topological spaces in terms of ultrafilter convergence (with just two axioms that generalize reflexivity and transitivity of ordered sets). With the same monad, but now with $\mathsf{V} = [0, \infty]$ being Lawvere's extended real half-line and addition playing the role of the tensor product, one obtains Lowen's [38] category of *approach spaces*, which incorporates both Barr's **Top** and Lawvere's **Met** in a satisfactory manner. Perhaps one of the best successes of the subject so far has been the strictly equational characterization of exponential objects in the lax setting of $(\mathbb{T}, \mathsf{V})$-categories. For the extensive literature on the subject, we must refer the reader to the literature list in [26], in particular the Notes to Chapters III and IV of [26], which also list many important related approaches, such as that of Burroni [10] (which drew Lambek's [36] multicategories into the setting) and the thesis of Möbus [44] (which, beyond compactness and Hausdorff separation, explored a wide range of topological concepts in the relational monadic setting).

In the $(\mathbb{T}, \mathsf{V})$-setting, it had been realized early on that V-$\mathbf{Rel}$ is precisely the Kleisli category of the V-power set monad $\mathbb{P}_{\mathsf{V}}$ (with $\mathsf{P}_{\mathsf{V}}X = \mathsf{V}^X$), and it was therefore plausible that *lax* extensions $\hat{T}$ of $\mathbb{T}$ to V-$\mathbf{Rel}$ correspond to *monotone lax* distributive laws of $\mathbb{T}$ over $\mathbb{P}_{\mathsf{V}}$ (see [48] and Exercise III.1.I of [26]). In this paper we present the lax distributive laws and their equivalent lax monad extensions, together with their isomorphic model categories (*i.e.*, lax λ-algebras *vs.* $(\mathbb{T}, \mathsf{V})$-categories) at a considerably generalized level, by replacing the quantale V by a small *quantaloid* $\mathcal{Q}$, *i.e.*, by a small category (rather than a monoid) enriched in the category **Sup** of complete lattices and their suprema preserving maps (see [47, 56, 57, 23]). For this to work, $\mathbb{T}$ must now be a monad on the comma category $\mathbf{Set}/\mathcal{Q}_0$, with $\mathcal{Q}_0$ the set of objects of $\mathcal{Q}$, rather than just on $\mathbf{Set}$ as in the quantale case when $\mathcal{Q}_0 \cong 1$ is a singleton set. However, noting that every $\mathbf{Set}$-monad $\mathbb{T}$ lifts to a $\mathbf{Set}/\mathcal{Q}_0$-monad when $\mathcal{Q}_0$ carries a Eilenberg-Moore $\mathbb{T}$-algebra structure, one realizes immediately that the range of applications is not at all reduced by moving to the comma category. The opposite is true, even when $\mathbb{T}$ is the identity monad and λ the identity transformation of the discrete presheaf monad $\mathsf{P}_{\mathcal{Q}}$, where lax λ-algebras are simply $\mathcal{Q}$-categories, as first considered in Walters' pioneering note [62]. More generally then, in the hierarchy



we add a monad to the enrichment through quantaloids, thus complementing the corresponding past efforts for quantales and monoidal-closed categories, and leaving the field open for future work on closed bicategories. In doing so, our focus is not on a generalization *per se*, but rather on the expansion of the range of meaningful examples. In fact, through the consideration of quantaloids that arise from quantales via the well-studied Freyd-Grandis "diagonal construction", originating with [18], presented in [20], and used by many authors (see, for example, [29, 45, 58]), we demonstrate that the quantaloidic context allows for the incorporation of many "partially defined" structures, which typically relax the reflexivity condition of the total context in a meaningful way.

In carrying out this work, we underline the role of lax distributive laws as the *primary* data in the study of topological categories, rather than as some secondary data derived from lax monad extensions, the establishment of which can be tedious (see [12, 49]). In fact, in analyzing step by step the correspondence between the two entities (as we do in Section 6 of this paper), we see that lax distributive laws minimize the number of variables in, and often the computational effort for, checking the required inequalities. It is therefore consequential that here we express $(\mathbb{T}, \mathsf{V})$-categories directly as *lax λ-algebras*, without prior reference to the lax monad extension which the ambient lax distributive law λ corresponds to. Thus, their axioms are entirely expressed in terms of maps, rather than V-relations, and of the two $\mathbf{Set}$-monads at play, $\mathbb{T}$ and $\mathbb{P}_{\mathsf{V}}$. We note that, to date, the strict counterpart of the notion of lax λ-algebra as introduced in Section 4 does not seem to have been explored to a great extent (beyond one example discussed in [61])– and may indeed be of much lesser importance than the lax version –, but must in any case not be confused with a different notion appearing in IV.3 of Manes' book [41].

In [24], Hofmann gave the notion of a (lax) *topological theory* which, in the presence of the $\mathbf{Set}$-monad $\mathbb{T}$ and the commutative quantale V, concentrates all needed information about the specific Barr-type lax extension of T to V-$\mathbf{Rel}$ into a (lax) $\mathbb{T}$-algebra structure $\xi : T\mathsf{V} \longrightarrow \mathsf{V}$ on the set V, such that ξ makes the monoid operations $\otimes : \mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V}$ and $\mathsf{k} : 1 \longrightarrow \mathsf{V}$ (lax) $\mathbb{T}$-homomorphisms and satisfies a monotonicity and naturality condition. While in [16] we characterized the Barr-Hofmann lax extensions of $\mathbb{T}$ arising from such theories among all lax extensions, the two main results of this paper clarify the role of Hofmann's notion in the quantale setting and extend it considerably to the more general context of a quantaloid $\mathcal{Q}$. First, in Section 5 we establish a Galois correspondence between monotone lax distributive laws of a given monad $\mathbb{T}$ on $\mathbf{Set}/\mathcal{Q}_0$ and certain lax $\mathbb{T}$-algebra structures ξ on $\mathsf{P}_{\mathcal{Q}}\mathcal{Q}_0$. The lax distributive laws closed under this correspondence, called *maximal*, give rise to new types of lax monad extension that don't seem to have been explored earlier. Secondly, in Theorem 8.2, we give a precise comparison of our notion of topological theory (as given in Definition 5.4) with Hofmann's more restrictive notion. We also give a context in which the Hofmann-type extensions are characterized as *minimal* (see Theorem 8.5). Let us emphasize that the conditions on the cartesian binary and nullary monoid operations used by Hofmann don't compare easily with the conditions on the multiplication and unit of the discrete presheaf monad as used in our setting, and they don't seem to be amenable to direct extension from the context of a commutative quantale to that of a quantaloid. For an overview chart on the relationships between lax distributive laws, lax monad extensions, and topological theories, we refer to Section 8.

A comparison of this paper with its successor [35] seems to be in order, where we present the non-discrete counterpart of the theory presented here, thus considering monads on the category $\mathcal{Q}$-$\mathbf{Cat}$ of small $\mathcal{Q}$-categories and their lax distributive laws over the (full) presheaf monad. While it is clear from the outset that such setting will make for a more satisfactory theory, simply because the full presheaf monad, unlike its discrete counterpart, is lax idempotent (or *of Kock-Zöberlein type*), we should emphasize that the *prior* consideration of the discrete case in this paper seems to be a necessary step in order for [35] to be able to resort to a viable array of monads on $\mathcal{Q}$-$\mathbf{Cat}$. Indeed, only with a lax extension of a monad on $\mathbf{Set}/\mathcal{Q}_0$ at hand is it easy to "lift" monads on $\mathbf{Set}/\mathcal{Q}_0$ to $\mathcal{Q}$-$\mathbf{Cat}$, as first demonstrated in [60] in the case of a quantale.

For general categorical background, we refer the reader to [39, 1, 7, 32].

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2. Quantaloid-enriched categories

A *quantaloid* is a category $\mathcal{Q}$ enriched in the monoidal-closed category **Sup** [31] of complete lattices with suprema-preserving maps; hence, the homsets of $\mathcal{Q}$ are complete lattices, and composition of morphisms from either side preserves arbitrary suprema and has therefore right adjoints. As a consequence, one has binary operations $\searrow$ and $\swarrow$ representing the "internal homs”, that is: for $u : r \longrightarrow s,\ v : s \longrightarrow t,\ w : r \longrightarrow t$ in $\mathcal{Q}$ one has the morphisms $(v \searrow w) : r \longrightarrow s,\ (w \swarrow u) : s \longrightarrow t$ given by the equivalences

$$u \leq v \searrow w \iff v \circ u \leq w \iff v \leq w \swarrow u.$$

A *lax homomorphism* $\varphi : \mathcal{Q} \longrightarrow \mathcal{R}$ of quantaloids is a lax functor (thus satisfying the rules $1_{\varphi t} \leq \varphi 1_t$ and $\varphi v \circ \varphi u \leq \varphi(v \circ u)$) which maps hom-sets monotonely; φ is a (strict) *homomorphism* if φ is a functor preserving suprema taken in the hom-sets. We denote the category of small quantaloids and their (lax) homomorphisms by **Qnd** (**LaxQnd**). The set-of-objects functor

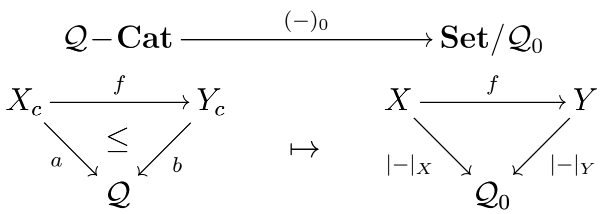
$$(-)_0 : \mathbf{LaxQnd} \longrightarrow \mathbf{Set}, \mathcal{Q} \mapsto \mathrm{ob}\mathcal{Q} =: \mathcal{Q}_0$$

has a right adjoint $(-)_c$ which provides each set X with the chaotic order and considers it as a category X_c with $(X_c)_0 = X$, so that for all $x, y \in X$ there is exactly one morphism $x \longrightarrow y$, denoted by (x, y); having singleton hom-sets only, X_c is trivially a quantaloid, and every **Set**-map becomes a homomorphism.

Throughout the paper, let $\mathcal{Q}$ be a small quantaloid. A small $\mathcal{Q}$-*category* is a set X provided with a lax homomorphism $a : X_c \longrightarrow \mathcal{Q}$. Its object part $a : X \longrightarrow \mathcal{Q}_0$ assigns to every $x \in X$ its *array* (also called *type* or *extent*) $ax \in \mathcal{Q}_0$, often denoted by $|x| = |x|_X = ax$, and its morphism part gives for all $x, y \in X$ $\mathcal{Q}$-morphisms $a(x, y) : |x| \longrightarrow |y|$, subject to the rules

$$1_{|x|} \leq a(x, x), \quad a(y, z) \circ a(x, y) \leq a(x, z).$$

A $\mathcal{Q}$-*functor* $f : (X, a) \longrightarrow (Y, b)$ is an array-preserving map $f : X \longrightarrow Y$ with $a(x, y) \leq b(fx, fy)$ for all $x, y \in X$. In other words then, the resulting category $\mathcal{Q}$-**Cat** of small $\mathcal{Q}$-categories and their $\mathcal{Q}$-functors is the lax comma category of small chaotic quantaloids over $\mathcal{Q}$, and one has the set-of-objects functor



to the comma category of sets over $\mathcal{Q}_0$. In what follows, we will often write X instead of X_c or (X, a).

An easily proved (see [51]), but useful, fact is:

Proposition 2.1. *The functor* $(-)_0$ *is topological (in the sense of* [22]*) and, as a consequence,* $\mathcal{Q}$-**Cat** *is totally complete and totally cocomplete (in the sense of* [55]*).*

Proof. The $(-)_0$-initial structure a on X with respect to a family of $(\mathbf{Set}/\mathcal{Q}_\mathbf{0})$-morphisms $f_i : X \longrightarrow Y_i$ with each Y_i carrying the $\mathcal{Q}$-category structure $b_i (i \in I)$ is given by

$$a(x, y) = \bigwedge_{i \in I} b_i(f_i x, f_i y),$$

with $x, y \in X$. □

Incidentally, it seems fitting to note here that topologicity of a faithful functor is characterized as total cocompleteness when the concrete category in question is considered as a category enriched over a certain quantaloid: see [19, 52].

Next, one easily sees that every lax homomorphism $\varphi : \mathcal{Q} \longrightarrow \mathcal{R}$ of quantaloids induces the *change-of-base functor*

$$B_\varphi : \mathcal{Q} - \mathbf{Cat} \longrightarrow \mathcal{R} - \mathbf{Cat},\ (\mathbf{X}, \mathbf{a}) \mapsto (\mathbf{X}, \varphi\mathbf{a}),$$

which commutes with the underlying **Set**-functors. More precisely, with B_{φ_0} denoting the effect of B_φ on the underlying sets over $\mathcal{Q}_0$, one has the commutative diagram of functors which exhibits $(B_\varphi, B_{\varphi_0})$ as a morphism of topological functors:

$$\begin{array}{ccc} \mathcal{Q}-\mathbf{Cat} & \xrightarrow{B_\varphi} & \mathcal{R} - \mathbf{Cat} \\ {\scriptstyle (-)_0}\downarrow & & \downarrow{\scriptstyle (-)_0} \\ \mathbf{Set}/\mathcal{Q}_0 & \xrightarrow[B_{\varphi_0}]{} & \mathbf{Set}/\mathcal{R}_0 \end{array}$$

Obviously, B_φ preserves $(-)_0$-initiality when φ preserves infima. Let us also mention that, if we order the hom-sets of **LaxQnd** by

$$\varphi \leq \psi \iff \forall u : r \longrightarrow s \text{ in } \mathcal{Q} : \varphi r = \psi r, \varphi s = \psi s \text{ and } \varphi u \leq \psi u,$$

then $\varphi \leq \psi$ gives a natural transformation $B_\varphi \longrightarrow B_\psi$ whose components at the **Set**-level are identity maps; thus a 2-functor $B_{(-)} : \mathbf{LaxQnd} \longrightarrow \mathbf{CAT}$ emerges.

The one-object quantaloids are the (unital) *quantales*, *i.e.*, the complete lattices V that come with a monoid structure whose binary operation $\otimes$ preserves suprema in each variable. We generally denote the $\otimes$-neutral element by k; so, in quantaloidic terms, $\mathsf{k} = 1_*$, when we denote by $*$ the only object of V as a category. Let us record here a well-known list of relevant quantales V with their induced categories $\mathsf{V}\!-\!\mathbf{Cat}$.

Example 2.2. (1) The terminal quantaloid $\mathsf{1}$ is a quantale, and $1\!-\!\mathbf{Cat} = \mathbf{Set}$. The initial quantale is (as a lattice) the two-element chain $2 = \{\bot < \top\}$, with $\otimes = \wedge, \mathsf{k} = \top$, and $2\!-\!\mathbf{Cat}$ is the category **Ord** of preordered sets and monotone maps. (*In what follows, we suppress the prefix "pre" in "preorder(ed)", adding "separated" whenever antisymmetry is required.*)

(2) $[0, \infty]$ denotes the extended real line, ordered by the natural $\geq$ (so that 0 becomes the largest and ∞ the least element) and considered as a quantale with the binary operation +, naturally extended to ∞. (This is the monoidal-closed category first considered by Lawvere [37].) A $[0, \infty]$-category is a generalized metric space, *i.e.,* a set X provided with a function $a : X \times X \longrightarrow [0, \infty]$ with $a(x, x) = 0$ and $a(x, z) \leq a(x, y) + a(y, z)$ for all $x, y, z \in X$; $[0, \infty]$-functors are non-expanding maps. We write $\mathbf{Met} = [0, \infty]$-**Cat** for the resulting category and allow ourselves to *call its objects just metric spaces*. The only homomorphism $2 \longrightarrow [0, \infty]$ of quantales has both a left and a right adjoint, hence there is an embedding $\mathbf{Ord} \longrightarrow \mathbf{Met}$ that is both reflective and coreflective.

(3) The quantale $[0, \infty]$ is of course isomorphic to the unit interval $[0, 1]$, ordered by the natural $\leq$ and provided with the multiplication. Interpreting $a(x, y) \in [0, 1]$ as the probability that $x, y \in X$ be related under a given random order $\tilde{a}$ on X, we call $(X, a) \in [0, 1]\!-\!\mathbf{Cat}$ a *probabilistic ordered set* and denote the resulting cateory by **ProbOrd**, which, of course, is just an isomorphic guise of **Met**.

Both, $[0,\infty]$ and $[0,1]$ are embeddable into the quantale $\mathbf{\Delta}$ of all *distance distribution functions* $\varphi : [0,\infty] \longrightarrow [0,1]$, required to satisfy the left-continuity condition $\varphi(\beta) = \sup_{\alpha<\beta}\varphi(\alpha)$, for all $\beta \in [0,\infty]$. Its order is inherited from $[0,1]$, and its monoid structure is given by the commutative *convolution* product $(\varphi \otimes \psi)(\gamma) = \sup_{\alpha+\beta\leq\gamma}\varphi(\alpha)\psi(\beta)$; the $\otimes$-neutral function κ satisfies $\kappa(0) = 0$ and $\kappa(\alpha) = 1$ for all $\alpha > 0$. Interpreting $a(x,y)(\alpha)$ as the probability that a given randomized metric $\tilde{a} : X \times X \longrightarrow [0,\infty]$ satsisfies $\tilde{a}(x,y) < \alpha$, one calls the objects (X,a) in $\mathbf{\Delta}\text{-}\mathbf{Cat}$ *probabilistic metric spaces* [25, 30], and we denote their category by **ProbMet**.

The quantale homomorphisms $\sigma : [0,\infty] \longrightarrow \mathbf{\Delta}$ and $\tau : [0,1] \longrightarrow \mathbf{\Delta}$, defined by $\sigma(\alpha)(\gamma) = 0$ if $\gamma \leq \alpha$, and 1 otherwise, and $\tau(u)(\gamma) = u$ if $\gamma > 0$, and 0 otherwise, induce full embeddings of **Met** and **ProbOrd** into **ProbMet**, respectively. Their significance lies in the fact that they present $\mathbf{\Delta}$ as a coproduct of $[0,\infty]$ and $[0,1]$ in the category of commutative quantales and their homomorphisms, since every $\varphi \in \mathbf{\Delta}$ has a presentation $\varphi = \sup_{\gamma\in[0,\infty]}\sigma(\gamma) \otimes \tau(\varphi(\gamma))$.

(4) The powerset 2^M of a (multiplicative) monoid M (with neutral element e_M) becomes a quantale when ordered by inclusion and provided with the composition $B \circ A = \{\beta\alpha \mid \alpha, \beta \in M\}$ for $A, B \subseteq M$; in fact, it is the free quantale over the monoid M. The objects of 2^M-**Cat** are sets X equipped with a family $(\leq_\alpha)_{\alpha\in M}$ of relations on them satisfying the rules $x \leq_{e_M} x$ and $(x \leq_\alpha y,\ y \leq_\beta z \Rightarrow x \leq_{\beta\alpha} z)$; morphisms must preserve each relation of the family; see [26] V.1.4. Every homomorphism $\varphi : M \longrightarrow N$ of monoids may be considered a homomorphism $\varphi : 2^M \longrightarrow 2^N$ of quantales via direct image, while its right adjoint given by inverse image is in general only a lax homomorphism $\varphi^{-1} : 2^N \longrightarrow 2^M$. Still, 2-functoriality of $(-)\text{-}\mathbf{Cat}$ produces adjunctions $\varphi(-) \dashv \varphi^{-1}(-) : 2^N\text{-}\mathbf{Cat} \longrightarrow 2^M\text{-}\mathbf{Cat}$. In particular, when considering $1 \longrightarrow M \longrightarrow 1$ with 1 trivial, one sees that there is a coreflective embedding of **Ord** into $2^M\text{-}\mathbf{Cat}$, as well as a reflective one.

(5) Every *frame, i.e.*, every complete lattice in which binary infima distribute over arbitrary suprema, may be considered a quantale; in fact,

these are precisely the commutative quantales in which every element is idempotent. For example, in addition to 2 of (1), $([0,\infty],\geq)$ may be considered a quantale $[0,\infty]_{\max}$ when, instead of $\alpha+\beta$ as in (2), the binary operation is given by $\max\{\alpha,\beta\}$. The resulting category $[0,\infty]_{\max}-\mathbf{Cat}$ is the category **UMet** of generalized *ultrametric* spaces (X,a) whose distance function must satisfy $a(x,z)\leq \max\{a(x,y),a(y,z)\}$ instead of the weaker triangle inequality.

A quantale V is called *divisible* [28] if for all $u\leq v$ in V there are $a,b\in\mathsf{V}$ with $a\otimes v=u=v\otimes b$; it is easy to see that then one may choose $a=u\swarrow v$ and $b=v\searrow u$. Applying the defining property to $u=\mathsf{k}$ and $v=\top$ the top element, so that $\top=\top\otimes\mathsf{k}=\top\otimes\top\otimes b\leq\top\otimes b=\mathsf{k}$, one sees that such a quantale must be *integral*, *i.e.*, $\mathsf{k}=\top$. Of the quantales of Example 2.2, all but $\mathbf{\Delta}$ (3) and 2^M(4) are divisible; 2^M is not even integral, unless the monoid M is trivial.

We refer to [18, 20] for the the Freyd-Grandis construction of freely adjoining a proper orthogonal factorization system to a category. In the case of a quantaloid $\mathcal{Q}$ it produces the quantaloid $\mathsf{D}\mathcal{Q}$ of "diagonals" of $\mathcal{Q}$ (so named in [58], after the prior treatments in [29, 45]), which has a particulary easy description when the quantaloid is a divisible quantale V: the objects of the quantaloid DV are the elements of V, and there is a morphism $(u,d,v):u\longrightarrow v$ in DV if $d\in\mathsf{V}$ satisfies $d\leq u\wedge v$; for ease of notation, we write $d:u\rightsquigarrow v$, keeping in mind that it is essential to keep track of the domain u and the codomain v. The composite $e\circ d$ of d with $e:v\rightsquigarrow w$ in DV is defined by $e\otimes(v\searrow d)=(e\swarrow v)\otimes d$ in V, and $v:v\rightsquigarrow v$ serves as the identity morphism on v in DV. The order of the hom-sets of DV is inherited from V.

The quantale V is fully embedded into DV by the homomorphism $\iota=\iota_{\mathsf{V}}:\mathsf{V}\longrightarrow\mathsf{DV},\ v\mapsto(v:\mathsf{k}\rightsquigarrow\mathsf{k})$, of quantaloids. There are lax homomorphisms, known as the *backward* and *forward globalization* functors (see [17, 45, 59]),

$$\begin{array}{ll}\delta:\mathsf{DV}\longrightarrow\mathsf{V}, & \gamma:\mathsf{DV}\longrightarrow\mathsf{V},\\ (d:u\rightsquigarrow v)\mapsto\ v\searrow d & (d:u\rightsquigarrow v)\mapsto d\swarrow u\end{array}$$

which, from a factorization perspective, play the role of the *domain* and *codomain* functors. They satisfy $\delta\iota_{\mathsf{V}}=1_{\mathsf{V}}=\gamma\iota_{\mathsf{V}}$ and therefore make V a retract of DV. Consequently, the full embedding $\mathsf{V}-\mathbf{Cat}\longrightarrow\mathsf{DV}-\mathbf{Cat}$ induced by ι has retractions, facilitated by δ and γ (see Example 7.5).

More importantly, when one considers V as a V-category $(\mathsf{V},\ \mathsf{h})$ with $\mathsf{h}(u,v) = v \swarrow u$, there is a full reflective embedding

$$E_{\mathsf{V}} : \mathsf{DV}\!-\!\mathbf{Cat} \longrightarrow \mathsf{V}\!-\!\mathbf{Cat}/\mathsf{V}$$

which provides a DV-category (X,a) with the V-category structure d defined by $d(x,y) = a(x,y) \swarrow a(x,x)$ and considers it as a V-category over V via $tx = a(x,x)$. Conversely, the reflector provides a V-category (X,d) that comes equipped with a V-functor $t : X \longrightarrow \mathsf{V}$, with the DV-category structure a, defined by $a(x,y) = d(x,y) \otimes tx$; see [35].

The quantaloids DV induced by the divisible quantales V of Example 2.2 are of interest in what follows. Here we mention only a couple of easy cases.

Example 2.3. (1) The quantaloid $\mathsf{D2}$ has objects $\bot, \top$, and there are exactly two morphisms $\bot, \top : \top \rightsquigarrow \top$ while all other hom-sets are trivial, each of them containing only $\bot$. The object part of a $\mathsf{D2}$-category structure on a set X is given by its fibre over $\top$, *i.e.*, by a subset $A \subseteq X$ and an order on A; in other words, by a truly *partial* (!) order on X. A $\mathsf{D2}$-functor $f : (X,A) \longrightarrow (Y,B)$ is a map $f : X \longrightarrow Y$ with $f^{-1}B = A$ whose restriction to A is monotone. We write **ParOrd** for $\mathsf{D2}$-**Cat**.

(2) For a $\mathsf{D}([0,\infty])$-category (X,a) one must have (in the natural order $\leq$ of $[0,\infty]$) $|x| \leq a(x,x) \leq |x|$ for all $x \in X$, so that the object part of the structure $a : X \times X \longrightarrow [0,\infty]$ is determined by its morphism part. Since $\alpha \circ \beta = (\alpha \swarrow \nu) + \beta = \alpha - \nu + \beta$ for $\nu \leq \alpha, \beta \in [0,\infty]$, the defining conditions on a may now be stated as

$$a(x,x) \leq a(x,y),\ a(x,z) \leq a(x,y) - a(y,y) + a(y,z) \quad (x,y,z \in X).$$

With $\mathsf{D}([0,\infty])$-functors $f : (X,a) \longrightarrow (Y,b)$ required to satisfy

$$b(f(x),f(y)) \leq a(x,y), \quad b(f(x),f(x)) = a(x,x) \quad (x,y \in X)$$

one obtains the category **ParMet** of *partial metric spaces*, as originally considered in [43]; see also [9]. (For example, when one thinks of $a(x,y)$ as of the cost of transporting goods from location x to location y, which will entail some fixed overhead costs $a(x,x)$ and $a(y,y)$ at these locations, the term $-a(y,y)$ in the "partial triangle inequality" justifies itsself since the operator should not pay the overhead twice at the intermediate location y.) For $\mathsf{V} = [0, \infty]$, the full embedding E_{V} in fact gives an isomorphism

$$\mathbf{ParMet} \cong \mathbf{Met}/[0, \infty]$$

of categories; *i.e.*, partial metric spaces and their non-expanding maps may equivalently be considered as metric spaces (X, d) that come with a "norm" $t : X \longrightarrow [0, \infty]$ satisfying $ty - tx \leq d(x, y)$ for all $x, y \in X$, the morphisms of which are norm-preserving and non-expanding. The presentation of **ParMet** as a comma category makes it easy to relate it properly to $\mathbf{Met}$, as we may look at the forgetful functor $\Sigma : \mathbf{Met}/[0, \infty] \longrightarrow \mathbf{Met}$ and its right adjoint $X \mapsto (\pi_2 : X \times [0, \infty] \longrightarrow [0, \infty])$ (with the direct product taken in **Met**). When expressed in terms of partial metrics, Σ is equivalently described by

$$B_\gamma : \mathbf{ParMet} \longrightarrow \mathbf{Met},\ (X, a) \mapsto (X, \tilde{a}),\ \tilde{a}(x, y) = a(x, y) - a(x, x),$$

and its right adjoint assigns to $(X, d) \in \mathbf{Met}$ the set $X \times [0, \infty]$ provided with the partial metric d^+, defined by

$$d^+((x, \alpha), (y, \beta)) = d(x, y) + \max\{\alpha, \beta\}$$

for all $x, y \in X, \alpha, \beta \in [0, \infty]$. For a recent discussion of partial metrics we refer to [27].

3. Encoding a quantaloid by its discrete presheaf monad

For a quantaloid $\mathcal{Q}$ one forms the category $\mathcal{Q}\text{-}\mathbf{Rel}$ of *$\mathcal{Q}$-relations*, as follows: its objects are those of $\mathbf{Set}/\mathcal{Q}_0$, *i.e.*, sets X that come with an *array* (or *type*) map $a = a_X : X \longrightarrow \mathcal{Q}_0$, also denoted by $|-| = |-|_X$, and a morphism $\varphi : X \longrightarrow\!\!\!\!\!\!\mapsto Y$ in $\mathcal{Q}\text{-}\mathbf{Rel}$ is given by a family of morphisms $\varphi(x, y) : |x|_X \longrightarrow |y|_Y$ $(x \in X, y \in Y)$ in $\mathcal{Q}$; its composite with $\psi : Y \longrightarrow\!\!\!\!\!\!\mapsto Z$ is defined by

$$(\psi \circ \varphi)(x, z) = \bigvee_{y \in Y} \psi(y, z) \circ \varphi(x, y).$$

A map $f : X \longrightarrow Y$ over $\mathcal{Q}_0$ may be seen as a $\mathcal{Q}$-relation via its $\mathcal{Q}$-*graph* or its $\mathcal{Q}$-*cograph*, as facilitated by the functors

$$\mathbf{Set}/\mathcal{Q}_0 \xrightarrow{(-)_\circ} \mathcal{Q}\text{–}\mathbf{Rel} \xleftarrow{(-)^\circ} (\mathbf{Set}/\mathcal{Q}_0)^{\mathrm{op}}$$
$$f_\circ(x,y) = \left\{ \begin{array}{ll} 1_{|x|} & \text{if } f(x)=y \\ \bot & \text{else} \end{array} \right\} = f^\circ(y,x).$$

For X in $\mathbf{Set}/\mathcal{Q}_0$ and $s \in \mathcal{Q}_0$, a $\mathcal{Q}$-*presheaf* σ on X with array $|\sigma| = s$ is a $\mathcal{Q}$-relation $\sigma : X \longrightarrow\!\!\!\!\!\mapsto \{s\}$ (where $\{s\}$ is considered as a set over $\mathcal{Q}_0$ via the inclusion map); hence, σ is a family $(\sigma_x : |x| \longrightarrow s)_{x\in X}$ of $\mathcal{Q}$-morphisms with specified common codomain. By definition then, the hom-functor $\mathcal{Q}$–$\mathbf{Rel}(-,\{s\}) : \mathcal{Q}\text{–}\mathbf{Rel}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ assigns to X the set of $\mathcal{Q}$-presheaves with array s; its left adjoint provides every element of a given set with the constant array s. Universal quantification over $s \in \mathcal{Q}_0$ produces the functor

$$(\mathcal{Q}\text{–}\mathbf{Rel}(-,\{s\}))_{s\in\mathcal{Q}_0} : \mathcal{Q}\text{–}\mathbf{Rel}^{\mathrm{op}} \longrightarrow \mathbf{Set}^{\mathcal{Q}_0}$$

whose left adjoint is the coproduct of the left adjoints of its components, By composition with the category equivalence $\mathbf{Set}/\mathcal{Q}_0 \simeq \mathbf{Set}^{\mathcal{Q}_0}$ we obtain the $\mathcal{Q}$-presheaf functor P, assigning to X the set $\mathsf{P}X = \mathsf{P}_\mathcal{Q}X$ of $\mathcal{Q}$-presheaves on X; its left adjoint turns out to be (the opposite of) the $\mathcal{Q}$-cograph functor. (The $\mathcal{Q}$-graph functor is produced similarly.) We may describe the morphism part of P and the correspondence under the adjunction by

$$\frac{X \overset{\varphi}{\longrightarrow\!\!\!\!\!\mapsto} Y}{Y \xrightarrow{\overleftarrow{\varphi}} \mathsf{P}X} \quad (\overleftarrow{\varphi}(y))_x = \varphi(x,y) \qquad \begin{array}{c} \mathbf{Set}/\mathcal{Q}_0 \underset{\mathsf{P}}{\overset{(-)^\circ}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{Q}\text{–}\mathbf{Rel}^{\mathrm{op}} \\ (\mathsf{P}Y \xrightarrow{\varphi^\odot} \mathsf{P}X) \longleftarrow\!\!\!\!\dashv (X \overset{\varphi}{\longrightarrow\!\!\!\!\!\mapsto} Y). \\ \tau \mapsto \tau\circ\varphi \end{array}$$

The unit y and counit ε of the adjunction are given by

$$\mathsf{y}_X = \overleftarrow{1_X^\circ} : X \longrightarrow \mathsf{P}X,\ (\mathsf{y}y)_x = 1_{|y|} \Longrightarrow x = y;$$

$$\varepsilon_X : X \longrightarrow\!\!\!\!\!\mapsto \mathsf{P}X,\ \varepsilon_X(x,\sigma) = \sigma_x : |x| \longrightarrow |\sigma|.$$

The adjunction induces the monad $\mathbb{P} = \mathbb{P}_\mathcal{Q} = (\mathsf{P}, \mathsf{s}, \mathsf{y})$ on $\mathbf{Set}/\mathcal{Q}_0$; for future reference, we record here explicitly its functor $\mathsf{P} : \mathbf{Set}/\mathcal{Q}_0 \longrightarrow \mathbf{Set}/\mathcal{Q}_0$ and multiplication s as well:

$$(X \xrightarrow{f} Y) \mapsto f_! := (f^\circ)^\odot : \mathsf{P}X \longrightarrow \mathsf{P}Y, \quad (f_!\sigma)_y = \bigvee_{x \in f^{-1}y} \sigma_x;$$
$$\sigma \mapsto \sigma \circ f^\circ$$

$$\mathsf{s}_X = \varepsilon_X^\odot : \mathsf{PP}X \longrightarrow \mathsf{P}X, \quad (\mathsf{s}_X\Sigma)_x = \bigvee_{\sigma \in \mathsf{P}X} \Sigma_\sigma \circ \sigma_x.$$
$$\Sigma \mapsto \Sigma \circ \varepsilon_X$$

One notes that $\mathcal{Q}{-}\mathbf{Rel}$ is a (large) quantaloid that inherits the pointwise order of its hom-sets from $\mathcal{Q}$. The full embedding $\mathcal{Q} \longrightarrow \mathcal{Q}{-}\mathbf{Rel}$, which interprets every $s \in \mathcal{Q}_0$ as the set $\{s\}$ over $\mathcal{Q}_0$, is therefore a homomorphism of quantaloids. Its image serves as a generating set in $\mathcal{Q}{-}\mathbf{Rel}$. As outlined earlier, under the category equivalence $\mathbf{Set}/\mathcal{Q}_0 \simeq \mathbf{Set}^{\mathcal{Q}_0}$ the set $\mathsf{P}X$ over $\mathcal{Q}_0$ corresponds to $(\mathcal{Q}{-}\mathbf{Rel}(X, \{s\}))_{s \in \mathcal{Q}_0}$, which lives in $\mathbf{Sup}^{\mathcal{Q}_0}$. The corresponding order on $\mathsf{P}X$ is described by

$$\sigma \leq \sigma' \iff |\sigma| = |\sigma'| \text{ and } \forall x \in X\ (\sigma_x \leq \sigma'_x).$$

For $f : X \longrightarrow Y$ in $\mathbf{Set}/\mathcal{Q}_0$, the map $f_! : \mathsf{P}X \longrightarrow \mathsf{P}Y$, considered as a morphism in $\mathbf{Sup}^{\mathcal{Q}_0}$, preserves suprema and, therefore, has a right adjoint $f^! : \mathsf{P}Y \longrightarrow \mathsf{P}X$ which actually preserves suprema as well and is easily described in $\mathbf{Set}/\mathcal{Q}_0$ by

$$\forall \tau \in \mathsf{P}Y,\ x \in X\ ((f^!\tau)_x = \tau_{fx});$$

since
$$f^! = (f_\circ)^\odot,$$
the adjunction $f_! \dashv f^!$ follows from $f_\circ \dashv f^\circ$ in $\mathcal{Q}{-}\mathbf{Rel}$ and the monotonicity of $(-)^\odot$ on hom-sets, which we explain next.

The sets $\mathbf{Set}(Y, \mathsf{P}X)$ with their pointwise order inherited from $\mathsf{P}X$ make the bijections

$$\mathcal{Q}{-}\mathbf{Rel}(X, Y) \longrightarrow \mathbf{Set}/\mathcal{Q}_0(Y, \mathsf{P}X),\ \varphi \mapsto \overleftarrow{\varphi},$$

order isomorphisms. Since

$$\overleftarrow{\psi \circ \varphi} = \varphi^\odot \cdot \overleftarrow{\psi}$$

(for $\psi : Y \longrightarrow\!\!\!\!\!\!\mapsto Z$), monotonicity of $(\psi \mapsto \psi \circ \varphi)$ in ψ makes the maps

$$\mathbf{Set}/\mathcal{Q}_0(Z, \mathsf{P}Y) \longrightarrow \mathbf{Set}/\mathcal{Q}_0(Z, \mathsf{P}X),\ g \mapsto \varphi^{\odot} \cdot g,$$

monotone. This proves item (1) of the following Lemma.

Lemma 3.1. *For $\varphi, \varphi' : X \longrightarrow\!\!\!\!\!\!\mapsto Y$ in $\mathcal{Q}-\mathbf{Rel}$ and $f : X \longrightarrow Y,\ g, g' : Z \longrightarrow \mathsf{P}Y, h : W \longrightarrow Z$ in $\mathbf{Set}/\mathcal{Q}_0$ one has:*

(1) $\varphi \leq \varphi',\ g \leq g' \Rightarrow \varphi^{\odot} \cdot g \cdot h \leq \varphi'^{\odot} \cdot g' \cdot h$;

(2) $\mathsf{y}_X \leq f^! \cdot \mathsf{y}_Y \cdot f,\ \ f^! \cdot \mathsf{s}_Y = \mathsf{s}_X \cdot (f^!)_!$.

Proof. The inequality of (2) follows from the naturality of y and the adjunction $f_! \dashv f^!$. For the stated equality, using

$$\varphi^{\odot} = \mathsf{s}_X \cdot (\overleftarrow{\varphi})_!$$

we can show more generally

$$\varphi^{\odot} \cdot \mathsf{s}_Y = \mathsf{s}_X \cdot (\overleftarrow{\varphi})_! \cdot \mathsf{s}_Y = \mathsf{s}_X \cdot \mathsf{s}_{\mathsf{P}X} \cdot (\overleftarrow{\varphi})_{!!} = \mathsf{s}_X \cdot (\mathsf{s}_X)_! \cdot (\overleftarrow{\varphi})_{!!} = \mathsf{s}_X \cdot (\varphi^{\odot})_!.$$

□

Let us finally mention that, of course, there is a functorial dependency of $\mathsf{P}_{\mathcal{Q}}$ on the quantaloid $\mathcal{Q}$, which we may describe briefly, as follows. Let $\vartheta : \mathcal{Q} \longrightarrow \mathcal{R}$ be a lax homomorphism of quantaloids, and let $B_{\vartheta_0} : \mathbf{Set}/\mathcal{Q}_0 \longrightarrow \mathbf{Set}/\mathcal{R}_0$ be the induced "discrete change-of-base functor" (as in Section 2). We can then regard ϑ as a lax natural transformation

$$\vartheta : B_{\vartheta_0}\mathsf{P}_{\mathcal{Q}} \longrightarrow \mathsf{P}_{\mathcal{R}}B_{\vartheta_0},$$

so that

$$(B_{\vartheta_0}f)_! \cdot \vartheta_X \leq \vartheta_Y \cdot B_{\vartheta_0}(f_!)$$

for all $f : X \longrightarrow Y$ in $\mathbf{Set}/\mathcal{Q}_0$; indeed, for $X \in \mathbf{Set}/\mathcal{Q}_0$, one defines $\vartheta_X : B_{\vartheta_0}\mathsf{P}_{\mathcal{Q}}X \longrightarrow \mathsf{P}_{\mathcal{R}}B_{\vartheta_0}X$ by

$$\sigma = (\sigma_x)_{x \in X} \mapsto \vartheta\sigma = (\vartheta(\sigma_x))_{x \in X}.$$

In fact, ϑ is now a lax monad morphism, as described by the following two diagrams:

$$\begin{array}{ccc} & B_{\vartheta_0} & \\ {}^{B_{\vartheta_0}\mathsf{y}^{\mathcal{Q}}}\swarrow & \geq & \searrow^{\mathsf{y}^{\mathcal{R}}B_{\vartheta_0}} \\ B_{\vartheta_0}\mathsf{P}_{\mathcal{Q}} & \xrightarrow[\vartheta]{} & \mathsf{P}_{\mathcal{R}}B_{\vartheta_0} \end{array} \qquad \begin{array}{ccccc} B_{\vartheta_0}\mathsf{P}_{\mathcal{Q}}\mathsf{P}_{\mathcal{Q}} & \xrightarrow{\vartheta\mathsf{P}_{\mathcal{Q}}} & \mathsf{P}_{\mathcal{R}}B_{\vartheta_0}\mathsf{P}_{\mathcal{Q}} & \xrightarrow{\mathsf{P}_{\mathcal{R}}\vartheta} & \mathsf{P}_{\mathcal{R}}\mathsf{P}_{\mathcal{R}}B_{\vartheta_0} \\ {\scriptstyle B_{\vartheta_0}\mathsf{s}^{\mathcal{Q}}}\downarrow & & \geq & & \downarrow{\scriptstyle \mathsf{s}^{\mathcal{R}}B_{\vartheta_0}} \\ B_{\vartheta_0}\mathsf{P}_{\mathcal{Q}} & & \xrightarrow[\vartheta]{\qquad\qquad} & & \mathsf{P}_{\mathcal{R}}B_{\vartheta_0} \end{array}$$

Note that, if $\mathcal{Q}, \mathcal{R}$ are quantales, these properties simplify considerably, since then B_{ϑ_0} may be treated as being the identity functor of $\mathbf{Set}$. Furthermore, if $\vartheta : \mathcal{Q} \longrightarrow \mathcal{R}$ is a homomorphism of quantaloids, the lax natural transformation ϑ becomes strict and makes the two diagrams commute strictly. Consequently, in the strict case one obtains a morphism $\mathbb{P}_{\mathcal{Q}} \longrightarrow \mathbb{P}_{\mathcal{R}}$ of monads.

We will return to ϑ as a lax monad morphism in Section 7 where we discuss change-of-base functors in greater generality.

4. Monads laxly distributing over the presheaf monad, and their lax algebras

Let $\mathbb{T} = (T, m, e)$ be a monad on $\mathbf{Set}/\mathcal{Q}_0$. We wish to generate certain lax extensions of $\mathbb{T}$ to $\mathcal{Q}-\mathbf{Rel}$, *i.e.*, to the (dual of the) Kleisli category of the presheaf monad $\mathbb{P}_{\mathcal{Q}}$. Since, as it is well known, strict extensions are provided by distributive laws $T\mathsf{P} \longrightarrow \mathsf{P}T$ (see [26]), we should consider a *lax distributive law* $\lambda : T\mathsf{P} \longrightarrow \mathsf{P}T$ instead, that is: a family $\lambda_X : T\mathsf{P}X \longrightarrow \mathsf{P}TX$ $(X \in \mathbf{Set}/\mathcal{Q}_0)$ of morphisms in $\mathbf{Set}/\mathcal{Q}_0$ satisfying the following inequalities for all maps $f : X \longrightarrow Y$ over $\mathcal{Q}_0$:

(a)
$$\begin{array}{ccc} T\mathsf{P}X & \xrightarrow{\lambda_X} & \mathsf{P}TX \\ {\scriptstyle T(f_!)}\downarrow & \leq & \downarrow{\scriptstyle (Tf)_!} \\ T\mathsf{P}Y & \xrightarrow[\lambda_Y]{} & \mathsf{P}TY \end{array}$$

$(Tf)_! \cdot \lambda_X \leq \lambda_Y \cdot T(f_!)$
(lax naturality of λ);

(b)
$$\begin{array}{ccc} & TX & \\ {}^{T\mathsf{y}_X}\swarrow & \geq & \searrow^{\mathsf{y}_{TX}} \\ T\mathsf{P}X & \xrightarrow[\lambda_X]{} & \mathsf{P}TX \end{array}$$

$\mathsf{y}_{TX} \leq \lambda_X \cdot T\mathsf{y}_X$
(lax $\mathbb{P}_{\mathcal{Q}}$-unit law);

(c) $$\begin{array}{ccccc} T\mathsf{PP}X & \xrightarrow{\lambda_{\mathsf{P}X}} & \mathsf{P}T\mathsf{P}X & \xrightarrow{(\lambda_X)_!} & \mathsf{PP}TX \\ {\scriptstyle T\mathsf{s}_X}\downarrow & & \geq & & \downarrow{\scriptstyle \mathsf{s}_{TX}} \\ T\mathsf{P}X & & \xrightarrow[\lambda_X]{\hspace{6em}} & & \mathsf{P}TX \end{array}$$

$\mathsf{s}_{TX} \cdot (\lambda_X)_! \cdot \lambda_{\mathsf{P}X} \ \leq\ \lambda_X \cdot T\mathsf{s}_X$ (lax $\mathbb{P}_{\mathcal{Q}}$-multiplication law);

(d) $$\begin{array}{ccc} & \mathsf{P}X & \\ {\scriptstyle e_{\mathsf{P}X}}\swarrow & \geq & \searrow{\scriptstyle (e_X)_!} \\ T\mathsf{P}X & \xrightarrow[\lambda_X]{} & \mathsf{P}TX \end{array}$$

$(e_X)_! \ \leq\ \lambda_X \cdot e_{\mathsf{P}X}$ (lax $\mathbb{T}$-unit law);

(e) $$\begin{array}{ccccc} TT\mathsf{P}X & \xrightarrow{T\lambda_X} & T\mathsf{P}TX & \xrightarrow{\lambda_{TX}} & \mathsf{P}TTX \\ {\scriptstyle m_{\mathsf{P}X}}\downarrow & & \geq & & \downarrow{\scriptstyle (m_X)_!} \\ T\mathsf{P}X & & \xrightarrow[\lambda_X]{\hspace{6em}} & & \mathsf{P}TX \end{array}$$

$(m_X)_! \cdot \lambda_{TX} \cdot T\lambda_X \ \leq\ \lambda_X \cdot m_{\mathsf{P}X}$ (lax $\mathbb{T}$-multiplication law).

Each of these laws is said to hold *strictly* (at f or X) if the respective inequality sign may be replaced by an equality sign; for a *strict distributive law*, all lax laws must hold strictly everywhere.

The lax distributive law λ is called *monotone* if

$$f \leq g \Rightarrow \lambda_X \cdot Tf \leq \lambda_X \cdot Tg$$

for all $f, g : Y \longrightarrow \mathsf{P}X$ in $\mathbf{Set}/\mathcal{Q}_0$. For simplicity, in what follows, we refer to a monotone lax distributive law $\lambda : T\mathsf{P} \longrightarrow T\mathsf{P}$ just as a *monotone distributive law*, which indirectly emphasizes the fact that the ambient 2-cell structure is given by order; we also say that $\mathbb{T}$ *distributes monotonely* over $\mathbb{P}_{\mathcal{Q}}$ by λ in this case, adding *strictly* when λ is strict.

Example 4.1. (1) For every quantaloid $\mathcal{Q}$, the identity monad on $\mathbf{Set}/\mathcal{Q}_0$ distributes strictly and monotonely over $\mathsf{P}_{\mathcal{Q}}$, via the identity transformation 1_{P}.

(2) For every quantale V, the *list*-monad $\mathbb{L}$ on **Set**, *i.e.*, the free-monoid monad with underlying **Set**-functor $LX = \bigcup_{n\geq 0} X^n$, distributes strictly and monotonely over P_{V}, via $\otimes_X : L\mathsf{P}_{\mathsf{V}}X \longrightarrow \mathsf{P}_{\mathsf{V}}LX$ defined by

$$(\sigma^1, ..., \sigma^n) \mapsto \sigma,\ \sigma_{(x_1,...,x_m)} = \left\{ \begin{array}{ll} \sigma^1_{x_1} \otimes ... \otimes \sigma^n_{x_n} & \text{if } m = n, \\ \bot & \text{else.} \end{array} \right\}$$

For $\mathsf{V} = 2$, so that $\mathsf{P}_2 \cong \mathsf{P}$ is the (covariant) power set functor, we in particular obtain the strict monotone distributive law

$$\times_X : LPX \longrightarrow PLX,\ (A_1, ..., A_n) \mapsto A_1 \times ... \times A_n,$$

that was mentioned in the Introduction.

(3) For every quantale V, the **Set**-monad $\mathbb{L}$ may be extended to $\mathbf{Set}/\mathsf{V}$: using the monoid structure of V, one maps every $(X, a) \in \mathbf{Set}/\mathsf{V}$ to $(LX, \zeta \cdot La)$, with $\zeta : L\mathsf{V} \longrightarrow \mathsf{V}$ the monoid homomorphism with $\zeta(v) = v$, *i.e.*, $\zeta : (v_1, ..., v_n) \mapsto v_1 \otimes ... \otimes v_n$. For the quantaloid $\mathcal{Q} = \mathsf{DV}$ (as described in Section 2 when V is divisible) and $\mathbb{L}$ considered as a $\mathbf{Set}/\mathsf{V}$-monad, one now obtains a strict monotone distributive law $\otimes : L\mathsf{P}_{\mathcal{Q}} \longrightarrow \mathsf{P}_{\mathcal{Q}}L$ defined just as in (2), with the understanding that $\sigma = \otimes_X(\sigma^1, ..., \sigma^n)$ is now given by $\mathcal{Q}$-arrows

$$\sigma_{(x_1,...,x_m)} : |x_1| \otimes ... \otimes |x_m| \rightsquigarrow |\sigma| = |\sigma^1| \otimes ... \otimes |\sigma^n|.$$

(4) (See [34].) For every quantale $\mathsf{V} = (\mathsf{V}, \otimes, \mathsf{k})$, the power set monad $\mathbb{P} = \mathbb{P}_2$ of **Set** distributes monotonely over $\mathbb{P}_\mathsf{V}$ by the law $\delta : \mathsf{P}\mathsf{P}_\mathsf{V} \longrightarrow \mathsf{P}_\mathsf{V}\mathsf{P}$ which, when we write $\mathsf{P}_\mathsf{V}X = \mathsf{V}^X$ as the set of maps $X \longrightarrow \mathsf{V}$, is defined by

$$\delta_X : \mathsf{P}(\mathsf{V}^X) \longrightarrow \mathsf{V}^{\mathsf{P}X}, \quad (\delta_X\mathcal{F})(A) = \bigwedge_{x\in A} \bigvee_{\sigma\in\mathcal{F}} \sigma(x),$$

for all $\mathcal{F} \subseteq \mathsf{V}^X, A \subseteq X$.

(5) Let $\mathbb{U} = (U, \Sigma, \dot{(-)})$ denote the ultrafilter monad on $\mathbf{Set}$; so, U assigns to a set X the set of ultrafilters on X, the unit assigns to a point in X its principal ultrafilter on X, and the monad multiplication is given by the so-called Kowalsky sum; see [40, 2, 26]. For every *completely distributive* quantale V (see [63, 26]), one defines a monotone distributive law $\beta : U\mathsf{P}_\mathsf{V} \longrightarrow \mathsf{P}_\mathsf{V}U$ by

$$\beta_X : U(\mathsf{V}^X) \longrightarrow \mathsf{V}^{UX}, \quad (\beta_X\mathfrak{z})(\mathfrak{x}) = \bigwedge_{A\in\mathfrak{x}, C\in\mathfrak{z}} \bigvee_{x\in A, \sigma\in C} \sigma(x),$$

for all ultrafilters $\mathfrak{z}$ on V^X, $\mathfrak{x}$ on X; compare with Corollary IV.2.4.5 of [26] and see [34].

Returning to the general context of a quantaloid $\mathcal{Q}$ and a monad $\mathbb{T}$ on $\mathbf{Set}/\mathcal{Q}_0$, we define:

Definition 4.2. *For a monotone distributive law* $\lambda : T\mathsf{P} \longrightarrow \mathsf{P}T$*, a* lax λ-algebra (X, p) *over* $\mathcal{Q}$ *is a set* X *over* $\mathcal{Q}_0$ *with a map* $p : TX \longrightarrow \mathsf{P}X$ *over* $\mathcal{Q}_0$ *satisfying*

(f)
$$\begin{array}{ccc} & X & \\ {\scriptstyle e_X}\swarrow & \geq & \searrow{\scriptstyle \mathsf{y}_X} \\ TX & \xrightarrow[p]{} & \mathsf{P}X \end{array}$$

$\mathsf{y}_X \leq p \cdot e_X$ (lax unit law);

(g)
$$\begin{array}{ccc} TTX \xrightarrow{Tp} T\mathsf{P}X \xrightarrow{\lambda_X} \mathsf{P}TX \xrightarrow{p_!} \mathsf{PP}X \\ {\scriptstyle m_X}\downarrow \qquad\qquad \geq \qquad\qquad \downarrow{\scriptstyle \mathsf{s}_X} \\ TX \xrightarrow[p]{\qquad\qquad\qquad\qquad} \mathsf{P}X \end{array}$$

$\mathsf{s}_X \cdot p_! \cdot \lambda_X \cdot Tp \leq p \cdot m_X$ (lax multiplication law).

A lax λ-homomorphism $f : (X, p) \longrightarrow (Y, q)$ *of lax* λ*-algebras must satisfy*

(h)
$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle p}\downarrow & \leq & \downarrow{\scriptstyle q} \\ \mathsf{P}X & \xrightarrow[f_!]{} & \mathsf{P}Y \end{array}$$

$f_! \cdot p \leq q \cdot Tf$ (lax homomorph. law).

The resulting category is denoted by $(\lambda, \mathcal{Q})-\mathbf{Alg}$.

Example 4.3. (1) For $\mathbb{T}$ the identity monad on $\mathbf{Set}/\mathcal{Q}_0$ and $\lambda = 1_{\mathsf{P}_\mathcal{Q}}$, there is an isomorphism $(\lambda, \mathcal{Q}) - \mathbf{Alg} \cong \mathcal{Q} - \mathbf{Cat}$ that commutes with the forgetful functors to $\mathbf{Set}/\mathcal{Q}_0$. Indeed, a lax homomorphism $a : X_c \longrightarrow \mathcal{Q}$ of quantaloids constitutes a $\mathcal{Q}$-relation $a : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$, such that $p = \overleftarrow{a} : X \longrightarrow \mathsf{P}X$ satisfies the lax unit- and multiplication laws (f) and (g), and conversely; similarly for the morphisms of the two categories.

(2) In Section 6 we will elaborate on the correspondence between monotone distributive laws λ of $\mathbb{T}$ over $\mathsf{P}_\mathcal{Q}$ and lax extensions $\hat{\mathbb{T}}$ of the monad $\mathbb{T}$ to $\mathcal{Q}-\mathbf{Rel}$. The λ-algebra axioms for $p : TX \longrightarrow \mathsf{P}_\mathcal{Q}X$ may then be expressed in terms of a $\mathcal{Q}$-relation $X \relbar\joinrel\mapstochar\joinrel\rightarrow TX$. In the case of $\mathcal{Q}$ being a commutative quantale V,

$$(\lambda, \mathsf{V})-\mathbf{Alg} \cong (\mathbb{T}, \mathsf{V})-\mathbf{Cat}$$

becomes the familiar category of $(\mathbb{T}, \mathsf{V})$-*categories* $(X, a : TX \relbar\joinrel\mapstochar\joinrel\rightarrow X)$ (as defined in [26], but see Remark 6.7), satisfying the lax-algebra conditions conditions

$$\mathsf{k} \leq a(e_X(x), x),\ a(\mathfrak{y}, z) \otimes \hat{T}a(\mathfrak{X}, \mathfrak{y}) \leq a(m_X(\mathfrak{X}), z)$$

for all $z \in X, \mathfrak{y} \in TX, \mathfrak{X} \in TTX$; morphisms, *i.e.*, $(\mathbb{T}, \mathsf{V})$-*functors* $f : (X, a) \longrightarrow (Y, b)$, satisfy $a(x, x') \leq b(fx, fx')$ for all $x, x' \in X$. For example, in the case of Example 4.1(2), with $\mathbb{T} = \mathbb{L}$ and $\mathsf{V} = 2$, one obtains the category **MulOrd** of *multiordered sets* X (carrying a reflexive and transitive relation $LX \longrightarrow\!\!\!\!\!\!\mapsto X$). For $\mathsf{V} = [0, \infty]$ one obtains the category **MulMet** of *multimetric spaces* $(X, a : LX \times X \longrightarrow [0, \infty])$, defined to satisfy the conditions $a((x), x) = 0$ and

$$\begin{aligned} &a((\underbrace{x_{1,1}, \ldots, x_{1,n_1}}_{\mathfrak{x}_1}, \ldots, \underbrace{x_{m,1}, \ldots, x_{m,n_m}}_{\mathfrak{x}_m}), z) \\ &\leq a(\mathfrak{x}_1, y_1) + \ldots + a(\mathfrak{x}_m, y_m) + a((y_1, \ldots, y_m), z); \end{aligned}$$

morphisms $f : (X, a) \longrightarrow (Y, b)$ are non-expanding maps:

$$b((fx_1, ..., fx_n), fy) \leq a((x_1, ..., x_n), y)$$

.

(3) (See [34].) For any quantale V and the monotone distributive law δ of Example 4.1(4) that makes the powerset monad $\mathbb{P} = \mathbb{P}_2$ distribute over $\mathbb{P}_{\mathsf{V}}$,

$$(\delta, \mathsf{V})\text{-}\mathbf{Alg} = \mathsf{V}\text{-}\mathbf{Cls}$$

is the category of V-*valued closure spaces* $(X, c : \mathsf{P}X \longrightarrow \mathsf{V}^X)$ (see [50]). When V is integral, at every "level" $u \in \mathsf{V}$ they give rise to the "c-closure" $A^{(u)} = \{x \in X \mid c(A)(x) \geq u\}$ of $A \subseteq X$. Considering now the full reflective subcategory of $\mathsf{V}\text{-}\mathbf{Cls}$ of those spaces (X, c) for which c is a homomorphism of join-semilattices, so that the finite additivity conditions

$$c(\emptyset) = \bot \text{ and } c(A \cup B) = c(A) \vee c(B)$$

for all $A, B \subseteq X$ are satisfied, one obtains for $\mathsf{V} = 2,\ [0, \infty]$, or $\mathbf{\Delta}$, respectively topological spaces (as described by a closure operation), approach spaces (as described by a point-set distance function [38]), or probabilistic approach spaces [8]; in general, we call them V-*valued topological spaces*. Since lax δ-homomorphisms provide the "right" morphisms in each of the three cases, we denote the resulting category by $\mathsf{V}-\mathbf{Top}$ and obtain in the special cases the categories

$$2-\mathbf{Top}=\mathbf{Top},\quad [0,\infty]-\mathbf{Top}=\mathbf{App},\quad \mathbf{\Delta}-\mathbf{Top}=\mathbf{ProbApp}.$$

(4) As shown in [34], for a completely distributive quantale V and the monotone distributive law β of Example 4.1(5) that makes $\mathbb{U}$ distribute over $\mathbb{P}_{\mathsf{V}}$,

$$(\beta,\mathsf{V})-\mathbf{Alg}\cong\mathsf{V}-\mathbf{Top}$$

is the category of V-valued topological spaces; see also Example 7.3. Considering for V the quantales $2, [0,\infty]$, and $\mathbf{\Delta}$, in this way one obtains respectively the ultrafilter characterization of the objects of the categories **Top** of topological spaces ([2, 26]), **App** of approach spaces ([38, 12, 26]), and **ProbApp** of probabilistic approach spaces ([8, 64, 25, 30]).

In generalization of Proposition 2.1 one easily proves:

Proposition 4.4. $(\lambda,\mathcal{Q})$–$\mathbf{Alg}$ *is topological over* $\mathbf{Set}/\mathcal{Q}_0$ *and, hence, totally complete and totally cocomplete.*

Proof. For any family of λ-algebras (Y_i,q_i) and $\mathbf{Set}/\mathcal{Q}_0$-maps $f_i:X\longrightarrow Y_i$ $(i\in I)$, the fixed set X obtains its initial structure p with respect to the forgetful functor $(\lambda,\mathcal{Q})-\mathbf{Alg}\longrightarrow\mathbf{Set}/\mathcal{Q}_0$ as

$$p:=\bigwedge_{i\in I}(f_i)^!\cdot q_i\cdot Tf_i$$

which, in pointwise terms, reads as $(p\mathfrak{x})_x=\bigwedge_{i\in I}(q_i(Tf_i(\mathfrak{x})))_{f_ix}$, for all $x\in X,\mathfrak{x}\in TX$. $\square$

5. Topological theories and maximal lax distributive laws

In addition to the given small quantaloid $\mathcal{Q}$, in this section we restrict ourselves to considering monads $\mathbb{T}$ on $\mathbf{Set}/\mathcal{Q}_0$ that are liftings of $\mathbf{Set}$-monads along the forgetful functor $\Sigma : \mathbf{Set}/\mathcal{Q}_0 \longrightarrow \mathbf{Set}$. The following proposition (which remains valid when $\mathbf{Set}$ is replaced by an arbitrary category) states that these are completely described by Eilenberg-Moore algebra structures on $\mathcal{Q}_0$, just as we have encountered them in the special case of the list monad in Example 4.1(3).

Proposition 5.1. *Let $\mathbb{T} = (T, m, e)$ be a monad on $\mathbf{Set}$. Then there is a bijective correspondence between $\mathbb{T}$-algebra structures $\zeta : T\mathcal{Q}_0 \longrightarrow \mathcal{Q}_0$ and monads $\mathbb{T}' = (T', m', e')$ on $\mathbf{Set}/\mathcal{Q}_0$ with*

$$\Sigma T' = T\Sigma,\ \Sigma e' = e\Sigma,\ \Sigma m' = m\Sigma.$$

Proof. For a "Σ-lifting" $\mathbb{T}'$ of $\mathbb{T}$, let ζ be the array function of the $\mathbf{Set}/\mathcal{Q}_0$-object $T'(\mathcal{Q}_0, 1_{\mathcal{Q}_0})$, whose domain must necessarily be $T\mathcal{Q}_0$. For any $\mathbf{Set}/\mathcal{Q}_0$-object (X, a), the unique $\mathbf{Set}/\mathcal{Q}_0$-morphism $a : (X, a) \longrightarrow (\mathcal{Q}_0, 1_{\mathcal{Q}_0})$ to the terminal object is being mapped by T' to

$$(TX, a_{TX}) \xrightarrow{Ta} (T\mathcal{Q}_0, \zeta), \text{ so that } a_{TX} = \zeta \cdot Ta \quad (*).$$

The object assignment by T' is therefore uniquely determined by ζ, and so is its morphism assignment, by faithfulness of Σ. Furthermore, since necessarily

$$e'_{(X,a)} = e_X : (X, a) \longrightarrow (TX, \zeta \cdot Ta),$$
$$m'_{(X,a)} = m_X : (TTX, \zeta \cdot T\zeta \cdot TTa) \longrightarrow (TX, \zeta \cdot Ta),$$

in $\mathbf{Set}/\mathcal{Q}_0$, one has $\zeta \cdot Ta \cdot e_X = a_X$ and $\zeta \cdot Ta \cdot m_X = \zeta \cdot T\zeta \cdot TTa$ which, for $X = \mathcal{Q}_0$ and $a = 1_{\mathcal{Q}_0}$, amount to the $\mathbb{T}$-algebra laws $\zeta \cdot e_{\mathcal{Q}_0} = 1_{\mathcal{Q}_0}$ and $\zeta \cdot m_{\mathcal{Q}_0} = \zeta \cdot T\zeta$.

Conversely, with T' defined by $(*)$, these laws similarly give the lifting $\mathbb{T}'$ of $\mathbb{T}$ along Σ. □

In what follows, we will not distinguish notationally between $\mathbb{T}'$ and $\mathbb{T}$. So, we are working with a $\mathbf{Set}$-monad $\mathbb{T} = (T, m, e)$ and a fixed $\mathbb{T}$-algebra structure $\zeta : T\mathcal{Q}_0 \longrightarrow \mathcal{Q}_0$ on $\mathcal{Q}_0$ that allows us to treat $\mathbb{T}$ as a monad on $\mathbf{Set}/\mathcal{Q}_0$. For such $\mathbb{T}$ and a monotone distributive law $\lambda : T\mathsf{P} \longrightarrow \mathsf{P}T$ we consider the $\mathbf{Set}/\mathcal{Q}_0$-maps

$$\xi := (T\mathsf{P}\mathcal{Q}_0 \xrightarrow{\lambda_{\mathcal{Q}_0}} \mathsf{P}T\mathcal{Q}_0 \xrightarrow{\zeta_!} \mathsf{P}\mathcal{Q}_0),$$

$$\theta := (T\mathsf{P}\mathsf{P}\mathcal{Q}_0 \xrightarrow{\lambda_{\mathsf{P}\mathcal{Q}_0}} \mathsf{P}T\mathsf{P}\mathcal{Q}_0 \xrightarrow{\xi_!} \mathsf{P}\mathsf{P}\mathcal{Q}_0).$$

Proposition 5.2. *ξ and θ are lax $\mathbb{T}$-algebra structures on $\mathsf{P}\mathcal{Q}_0$ and $\mathsf{PP}\mathcal{Q}_0$, respectively, making $\mathsf{y}_{\mathcal{Q}_0} : (\mathcal{Q}_0, \zeta) \longrightarrow (\mathsf{P}\mathcal{Q}_0, \xi)$ and $\nu_{\mathcal{Q}_0} : (\mathsf{PP}\mathcal{Q}, \theta) \longrightarrow (\mathsf{P}\mathcal{Q}_0, \xi)$ lax $\mathbb{T}$-homomorphisms, that is, producing the following laxly commuting diagrams:*

$$
\begin{array}{ccc}
\mathsf{P}\mathcal{Q}_0 & \xrightarrow{e_{\mathsf{P}\mathcal{Q}_0}} & T\mathsf{P}\mathcal{Q}_0 \\
 & {\scriptstyle 1_{\mathsf{P}\mathcal{Q}_0}} \searrow \;\; \leq & \downarrow {\scriptstyle \xi} \\
 & & \mathsf{P}\mathcal{Q}_0
\end{array}
\qquad
\begin{array}{ccc}
TT\mathsf{P}\mathcal{Q}_0 & \xrightarrow{T\xi} & T\mathsf{P}\mathcal{Q}_0 \\
{\scriptstyle m_{\mathsf{P}\mathcal{Q}_0}} \downarrow & \geq & \downarrow {\scriptstyle \xi} \\
T\mathsf{P}\mathcal{Q}_0 & \xrightarrow[\xi]{} & \mathsf{P}\mathcal{Q}_0
\end{array}
\qquad
\begin{array}{ccc}
T\mathcal{Q}_0 & \xrightarrow{T\mathsf{y}_{\mathcal{Q}_0}} & T\mathsf{P}\mathcal{Q}_0 \\
{\scriptstyle \zeta} \downarrow & \leq & \downarrow {\scriptstyle \xi} \\
\mathcal{Q}_0 & \xrightarrow[\mathsf{y}_{\mathcal{Q}_0}]{} & \mathsf{P}\mathcal{Q}_0
\end{array}
$$

$$
\begin{array}{ccc}
\mathsf{PP}\mathcal{Q}_0 & \xrightarrow{e_{\mathsf{PP}\mathcal{Q}_0}} & T\mathsf{PP}\mathcal{Q}_0 \\
 & {\scriptstyle 1_{\mathsf{PP}\mathcal{Q}_0}} \searrow \;\; \leq & \downarrow {\scriptstyle \theta} \\
 & & \mathsf{PP}\mathcal{Q}_0
\end{array}
\qquad
\begin{array}{ccc}
TT\mathsf{PP}\mathcal{Q}_0 & \xrightarrow{T\theta} & T\mathsf{PP}\mathcal{Q}_0 \\
{\scriptstyle m_{\mathsf{PP}\mathcal{Q}_0}} \downarrow & \geq & \downarrow {\scriptstyle \theta} \\
T\mathsf{PP}\mathcal{Q}_0 & \xrightarrow[\theta]{} & \mathsf{PP}\mathcal{Q}_0
\end{array}
\qquad
\begin{array}{ccc}
T\mathsf{PP}\mathcal{Q}_0 & \xrightarrow{T\mathsf{s}_{\mathcal{Q}_0}} & T\mathsf{P}\mathcal{Q}_0 \\
{\scriptstyle \theta} \downarrow & \leq & \downarrow {\scriptstyle \xi} \\
\mathsf{PP}\mathcal{Q}_0 & \xrightarrow[\mathsf{s}_{\mathcal{Q}_0}]{} & \mathsf{P}\mathcal{Q}_0
\end{array}
$$

Moreover, ξ (θ) is a strict $\mathbb{T}$-algebra structure on $\mathsf{P}\mathcal{Q}_0$ ($\mathsf{PP}\mathcal{Q}_0$) if λ satisfies the lax $\mathbb{T}$-unit and -multiplication laws (d) *and* (e) *strictly at $\mathcal{Q}_0$ (at $\mathsf{P}\mathcal{Q}_0$, respectively); and $\mathsf{y}_{\mathcal{Q}_0}$ ($\mathsf{s}_{\mathcal{Q}_0}$) is a strict $\mathbb{T}$-homomorphism if λ satisfies the lax $\mathbb{P}_{\mathcal{Q}}$-unit law* (b) *(the lax $\mathbb{P}_{\mathcal{Q}}$-multiplication law* (c), *respectively) strictly at $\mathcal{Q}_0$.*

Proof. Lax unit law for ξ: By (d), $\xi \cdot e_{\mathsf{P}\mathcal{Q}_0} \geq \zeta_! \cdot (e_{\mathcal{Q}_0})_! = (\zeta \cdot e_{\mathcal{Q}_0})_! = 1_{\mathsf{P}\mathcal{Q}_0}$, with equality holding when λ satisfies (d) strictly at $\mathcal{Q}_0$.

Lax multiplication law for ξ: By (e),

$$
\begin{aligned}
\xi \cdot m_{\mathsf{P}\mathcal{Q}_0} &\geq \zeta_! \cdot (m_{\mathcal{Q}_0})_! \cdot \lambda_{T\mathcal{Q}_0} \cdot T\lambda_{\mathcal{Q}_0} \\
&= \zeta_! \cdot (T\zeta)_! \cdot \lambda_{T\mathcal{Q}_0} \cdot T\lambda_{\mathcal{Q}_0} = \zeta_! \cdot \lambda_{\mathcal{Q}_0} \cdot T(\zeta_!) \cdot T\lambda_{\mathcal{Q}_0} = \xi \cdot T\xi,
\end{aligned}
$$

with equality holding when λ satisfies (e) strictly at $\mathcal{Q}_0$.

One proceeds similarly for the (lax) unit and multiplication laws for θ.

Lax homomorphism law for $\mathsf{y}_{\mathcal{Q}_0}$: By (b), $\xi \cdot T\mathsf{y}_{\mathcal{Q}_0} \geq \zeta_! \cdot \mathsf{y}_{T\mathcal{Q}_0} = \mathsf{y}_{\mathcal{Q}_0} \cdot \zeta$, with equality holding when λ satisfies (b) strictly at $\mathcal{Q}_0$.

Lax homomorphism law for $\mathsf{s}_{\mathcal{Q}_0}$: By (c),

$$\begin{aligned}\xi\cdot T\mathsf{s}_{\mathcal{Q}_0} &\geq \zeta_!\cdot \mathsf{s}_{T\mathcal{Q}_0}\cdot(\lambda_{\mathcal{Q}_0})_!\cdot\lambda_{\mathsf{P}\mathcal{Q}_0}\\ &= \mathsf{s}_{\mathcal{Q}_0}\cdot\zeta_{!!}\cdot(\lambda_{\mathcal{Q}_0})_!\cdot\lambda_{\mathsf{P}\mathcal{Q}_0} = \mathsf{s}_{\mathcal{Q}_0}\cdot\xi_!\cdot\lambda_{\mathsf{P}\mathcal{Q}_0} = \mathsf{s}_{\mathcal{Q}_0}\cdot T\mathsf{y}_{\mathcal{Q}_0},\end{aligned}$$

with equality holding when λ satisfies (c) strictly at $\mathcal{Q}_0$. □

Remark 5.3. (1) Let $t := |-|_{\mathsf{P}\mathcal{Q}_0}$ denote the array map of $\mathsf{P}\mathcal{Q}_0$ (that assigns to a $\mathcal{Q}_0$-indexed family of $\mathcal{Q}$-morphisms in $\mathsf{P}\mathcal{Q}_0$ their common codomain). Then $|-|_{T\mathsf{P}\mathcal{Q}_0} = \zeta\cdot Tt$ (see $(*)$ of Proposition 5.1), and since ξ is a map over $\mathcal{Q}_0$, we must have $t\cdot\xi = \zeta\cdot Tt$. In other words, $t : (\mathsf{P}\mathcal{Q}_0,\xi)\longrightarrow(\mathcal{Q}_0,\zeta)$ is a strict $\mathbb{T}$-homomorphism.

(2) From $\xi = \zeta_!\cdot\lambda_{\mathcal{Q}_0}$ one obtains $\lambda_{\mathcal{Q}_0}\leq\zeta^!\cdot\xi$ by adjunction, and the lax naturality (a) of λ at t then gives

$$\lambda_{\mathsf{P}\mathcal{Q}_0}\leq (Tt)^!\cdot\lambda_{\mathcal{Q}_0}\cdot T(t_!)\leq (Tt)^!\cdot\zeta^!\cdot\xi\cdot T(t_!).$$

Consequently, one obtains an upper bound for θ:

$$\theta = \xi_!\cdot\lambda_{\mathsf{P}\mathcal{Q}_0}\leq\xi_!\cdot(Tt)^!\cdot\zeta^!\cdot\xi\cdot T(t_!).$$

We now embark on a converse path, by establishing a monotone distributive law from a given map ξ, in addition to ζ, and by choosing θ maximally.

Definition 5.4. *Let $\mathbb{T}$ be a $\mathbf{Set}$-monad that comes with a $\mathbb{T}$-algebra structure ζ on the object set $\mathcal{Q}_0$ of the small quantaloid $\mathcal{Q}$. A* topological theory *for $\mathbb{T}$ and $\mathcal{Q}$ is a $\mathbf{Set}$-map $\xi : T\mathsf{P}\mathcal{Q}_0\longrightarrow\mathsf{P}\mathcal{Q}_0$ which is array compatible, satisfies the lax $\mathbb{T}$-algebra and homomorphism laws, and is monotone, as follows:*

0. $t\cdot\xi = \zeta\cdot Tt$ *(with t as in Remark 5.3(1));*
1. $1_{\mathsf{P}\mathcal{Q}_0}\leq\xi\cdot e_{\mathsf{P}\mathcal{Q}_0},\ \xi\cdot T\xi\leq\xi\cdot m_{\mathsf{P}\mathcal{Q}_0}$*;*
2. $\mathsf{y}_{\mathcal{Q}_0}\cdot\zeta\leq\xi\cdot T\mathsf{y}_{\mathcal{Q}_0},\ \mathsf{s}_{\mathcal{Q}_0}\cdot\theta\leq\xi\cdot T\mathsf{s}_{\mathcal{Q}_0}$ *($\theta := \xi_!\cdot(\zeta\cdot Tt)^!\cdot\xi\cdot T(t_!)$);*
3. $\forall f,g : Y\longrightarrow\mathsf{P}\mathcal{Q}_0$ *in* $\mathbf{Set}/\mathcal{Q}_0$ *($f\leq g\Rightarrow\xi\cdot Tf\leq\xi\cdot Tg$).*

The theory is strict *if the inequality signs in conditions 1 and 2 may be replaced by equality signs.*

Proposition 5.2 produces for every (strict) monotone distributive law a (strict) topological theory. We will call this theory *induced* by the given law.

Theorem 5.5. *For* $\mathbb{T}, \mathcal{Q}, \zeta$ *as in* Definition 5.4 *and a topological theory* ξ*,*

$$\lambda_X^\xi := (\zeta \cdot Ta)^! \cdot \xi \cdot T(a_!)$$

for all $X = (X, a) \in \mathbf{Set}/\mathcal{Q}_0$ *defines a monotone distributive law* λ^ξ *for* $\mathbb{T}$ *and* $\mathcal{Q}$*. This law is largest amongst all laws that induce the given theory* ξ*.*

Proof. We check monotonicity of $\lambda = \lambda^\xi$ and each of the conditions (a)-(e), considering $f : (X, a) \longrightarrow (Y, b)$ in $\mathbf{Set}/\mathcal{Q}_0$. Note that $c := \zeta \cdot Ta$ is the array function of TX. With t the array function of $\mathsf{P}\mathcal{Q}_0$ (see Remark (1)), an easy inspection shows that $s := t \cdot a_!$ is the array function of $\mathsf{P}X$.

Monotonicity: For $g, h : Y \longrightarrow\!\!\!\!\mapsto \mathsf{P}X$ in $\mathbf{Set}/\mathcal{Q}_0$, monotonicity of ξ gives

$$\lambda_X \cdot Tg = c^! \cdot \xi \cdot T(a_! \cdot g) \leq c^! \cdot \xi \cdot T(a_! \cdot h) = \lambda_X \cdot Th.$$

(a) With the adjunction $(Tf)_! \dashv (Tf)^!$, from $b \cdot f = a$ one obtains $(Tf)_! \cdot (Ta)^! \leq (Tb)^!$. Hence,

$$(Tf)_! \cdot \lambda_X = (Tf)^! \cdot (Ta)^! \cdot \zeta^! \cdot \xi \cdot T(a_!) \leq (Tb)^! \cdot \zeta^! \cdot \xi \cdot T(b_!) \cdot T(f_!) = \lambda_Y \cdot T(f_!).$$

(b) Condition 2 for a lax topological theory and Lemma 3.1(2) give

$$\lambda_X \cdot T\mathsf{y}_X = c^! \cdot \xi \cdot T(a_!) \cdot T\mathsf{y}_X = c^! \cdot \xi \cdot T\mathsf{y}_{\mathcal{Q}_0} \cdot Ta \geq c^! \cdot \mathsf{y}_{\mathcal{Q}_0} \cdot c \geq \mathsf{y}_{TX}.$$

(c) The adjunction $(T(a_!))_! \dashv (T(a_!))^!$ gives $(T(a_!))_! \cdot (Ts)^! \leq (Tt)^!$. Hence, with Condition 2 for a lax topological theory and Lemma 3.1(2) one obtains

$$\begin{aligned} \lambda_X \cdot T\mathsf{s}_X &= c^! \cdot \xi \cdot T(a_!) \cdot T\mathsf{s}_X = c^! \cdot \xi \cdot T\mathsf{s}_{\mathcal{Q}_0} \cdot T(a_{!!}) \\ &\geq c^! \cdot \mathsf{s}_{\mathcal{Q}_0} \cdot \theta \cdot T(a_{!!}) = \mathsf{s}_{TX} \cdot (c^!)_! \cdot \theta \cdot T(a_{!!}) \\ &= \mathsf{s}_{TX} \cdot (c^!)_! \cdot \xi_! \cdot (Tt)^! \cdot \zeta^! \cdot \xi \cdot T(t_!) \cdot T(a_{!!}) \\ &\geq \mathsf{s}_{TX} \cdot (c^!)_! \cdot \xi_! \cdot (T(a_!))_! \cdot (Ts)^! \cdot \zeta^! \cdot \xi \cdot T(s_!) \\ &= \mathsf{s}_{TX} \cdot (c^!)_! \cdot \xi_! \cdot (T(a_!))_! \cdot (\zeta \cdot Ts)^! \cdot \xi \cdot T(s_!) \\ &= \mathsf{s}_{TX} \cdot (\lambda_X)_! \cdot \lambda_{\mathsf{P}X}. \end{aligned}$$

(d) From $\zeta \cdot e_{\mathcal{Q}_0} = 1_{\mathcal{Q}_0}$ one obtains $(e_{\mathcal{Q}_0})_! \leq \zeta^!$ by adjunction. Together with Condition 3 for a lax topological theory, this gives

$$\begin{aligned} \lambda_X \cdot e_{\mathsf{P}X} &= c^! \cdot \xi \cdot T(a_!) \cdot e_{\mathsf{P}X} = c^! \cdot \xi \cdot e_{\mathsf{P}\mathcal{Q}_0} \cdot a_! \geq (\zeta \cdot Ta)^! \cdot a_! \\ &\geq (Ta)^! \cdot (e_{\mathcal{Q}_0})_! \cdot a_! = (Ta)^! \cdot (Ta)_! \cdot (e_X)_! \quad \geq (e_X)_!. \end{aligned}$$

(e) With $d := \zeta \cdot Tc$ the array function of TTX, from $c \cdot m_X = d$ one obtains $(m_X)_! \cdot d^! \leq c^!$ by adjunction, so that condition 3 for a topological theory gives

$$\begin{aligned}\lambda_X\cdot m_{\mathsf{P}X} &= c^! \cdot \xi \cdot T(a_!) \cdot m_{\mathsf{P}X} = c^! \cdot \xi \cdot m_{\mathsf{P}\mathcal{Q}_0} \cdot TT(a_!)\\ &\geq c^! \cdot \xi \cdot T\xi \cdot TT(a_!) \geq (m_X)_! \cdot d^! \cdot \xi \cdot T\xi \cdot TT(a_!)\\ &\geq (m_X)_! \cdot d^! \cdot \xi \cdot T(c_!) \cdot T(c^!) \cdot T(\xi) \cdot TT(a_!)\\ &= (m_X)_! \cdot \lambda_{TX} \cdot T\lambda_X.\end{aligned}$$

Next we show that the topological theory ξ' induced by $\lambda = \lambda^\xi$ equals ξ. Indeed, since ζ is surjective, one has $\zeta_\circ \circ \zeta^\circ = 1_{T\mathcal{Q}_0}$ and therefore

$$\xi' = \zeta_! \cdot \lambda_{\mathcal{Q}_0} = \zeta_! \cdot \zeta^! \cdot \xi = (\zeta_\circ \circ \zeta^\circ)^\odot \cdot \xi = \xi.$$

Finally, let $\kappa : T\mathsf{P} \longrightarrow \mathsf{P}T$ be any monotone distributive law inducing ξ, so that $\zeta_! \cdot \kappa_{\mathcal{Q}_0} = \xi$. Then

$$\begin{aligned}\lambda_X &= c^! \cdot \xi \cdot T(a_!) = (Ta)^! \cdot \zeta^! \cdot \zeta_! \cdot \kappa_{\mathcal{Q}_0} \cdot T(a_!)\\ &\geq (Ta)^! \cdot \kappa_{\mathcal{Q}_0} \cdot T(a_!) = (Ta)^! \cdot T(a_!) \cdot \kappa_X \geq \kappa_X.\end{aligned}$$ □

Remark 5.6. (1) When stated in pointwise terms, the definition of $\lambda = \lambda^\xi$ reads as

$$(\lambda_X \mathfrak{z})_{\mathfrak{x}} = (\xi \cdot T(a_!)(\mathfrak{z}))_{\zeta\cdot Ta(\mathfrak{x})},$$

for all $X = (X, a) \in \mathbf{Set}/\mathcal{Q}_0, \mathfrak{x} \in TX, \mathfrak{z} \in T\mathsf{P}X$.

(2) *For a topological theory ξ, the structure θ as in* Definition 5.4 *always satisfies the lax $\mathbb{T}$-unit and -multiplication laws of* Proposition 5.2, since ξ is induced by the monotone distributive law λ^ξ.

Corollary 5.7. *For a quantaloid $\mathcal{Q}$ and a $\mathbf{Set}$-monad $\mathbb{T}$ that comes equipped with a $\mathbb{T}$-algebra structure ζ on the set of objects of $\mathcal{Q}$, the assignments*

$$(\xi \mapsto \lambda^\xi), \quad (\lambda \mapsto \xi^\lambda := \zeta_! \cdot \lambda_{\mathcal{Q}_0})$$

define an adjunction between the ordered set of topological theories for $\mathbb{T}$ and $\mathcal{Q}$ and the conglomerate of monotone distributive laws $T\mathsf{P}_\mathcal{Q} \longrightarrow \mathsf{P}_\mathcal{Q}T$, ordered componentwise.

Definition 5.8. *A monotone distributive law λ is* maximal *if it is closed under the correspondence of Corollary 5.7, that is, if it is induced by some topological theory or, equivalently, by ξ^λ. More explicitly then, λ is maximal if, and only if, for all $X = (X, a) \in \mathbf{Set}/\mathcal{Q}_0$,*

$$\lambda_X = (Ta)^! \cdot \zeta^! \cdot \zeta_! \cdot \lambda_{\mathcal{Q}_0} \cdot T(a_!).$$

Note that this condition simplifies to $\lambda_X = (Ta)^! \cdot \lambda_{\mathcal{Q}_0} \cdot T(a_!)$ when ζ is bijective.

Corollary 5.9. *Maximal monotone distributive laws correspond bijectively to topological theories.*

Example 5.10. (1) For $\mathbb{T}$ and λ identical (as in Example 4.1(1)), with $\zeta = 1_{\mathcal{Q}_0}$ also the induced map $\xi = 1_{\mathsf{P}\mathcal{Q}_0}$ is identical, but the maximal law λ^ξ associated with it (by Theorem 5.5) is not; for a set X with array function $|-| : X \longrightarrow \mathcal{Q}_0$ one has

$$\lambda^\xi_X : \mathsf{P}_\mathcal{Q}X \longrightarrow \mathsf{P}_\mathcal{Q}X,\ (\lambda^\xi_X\sigma)_y = \bigvee\{\sigma_x \mid x \in X, |x| = |y|\},$$

for all $\sigma \in \mathsf{P}X, y \in X$.

(2) For $\mathbb{T} = \mathbb{L}$ and the strict distributive law $\otimes$ of Example 4.1(2), the induced map $\xi : L\mathsf{V} \longrightarrow \mathsf{V}$ with $(v_1, ..., v_n) \mapsto v_1 \otimes ... \otimes v_n$ is in fact the Eilenberg-Moore structure of the monoid $(\mathsf{V}, \otimes, \mathsf{k})$. The maximal law $\lambda^\xi_X : L(\mathsf{V}^X) \longrightarrow \mathsf{V}^{LX}$ maps $(\sigma_1, ..., \sigma_n)$ to the map $LX \longrightarrow \mathsf{V}$ with constant value $\bigvee\{\sigma_1(z_1) \otimes ... \otimes \sigma_n(z_n) \mid z_1, ..., z_n \in X\}$, for every set X.

(3) For $\mathcal{Q} = \mathsf{DV}$ with V divisible, $\mathbb{T} = \mathbb{L}$, and the distributive law and the map $\zeta : L\mathsf{V} \longrightarrow \mathsf{V}$ as in Example 4.1(3) (which coincides with the map ξ of (2) above), the now induced map $\xi : L(\mathsf{PV}) \longrightarrow \mathsf{PV} = \mathsf{P}_\mathcal{Q}\mathsf{V}$ is given by

$$(\xi(\sigma^1, ..., \sigma^n))_u = \bigvee_{v_1 \otimes ... \otimes v_n = u} \sigma^1_{v_1} \otimes ... \otimes \sigma^n_{v_n} : \quad u \rightsquigarrow |\sigma^1| \otimes ... \otimes |\sigma^n|,$$

for all $\sigma^1, ..., \sigma^n \in \mathsf{PV}, u \in \mathsf{V}$.

(4) The map $\xi : \mathsf{PV} \longrightarrow \mathsf{V}$ induced by the law δ of Example 4.1(4) has constant value $\top$.

(5) The map $\xi : U\mathsf{V} \longrightarrow \mathsf{V}$ induced by the ultrafilter monad and the law β as in Example 4.1(5) is given by

$$\xi(\mathfrak{z}) = \bigwedge_{C \in \mathfrak{z}} \bigvee C$$

(which may be written as $\xi(\mathfrak{z}) = \bigvee_{C \in \mathfrak{z}} \bigwedge C$ if V is completely distributive), for every ulltrafilter $\mathfrak{z}$ on V; it plays a central role in [24].

While typically maximal monotone distributive laws are rather special and often allow only for trivial λ-algebras, especially when $\mathcal{Q}$ is a quantale (see Remark 8.4(2)), they do lead to interesting categories $(\lambda, \mathcal{Q})-\mathbf{Alg}$ when $\mathcal{Q}$ is a multi-object quantaloid, including the case when $\mathcal{Q} = \mathsf{DV}$ for a quantale V. We can mention here only the easiest case.

Example 5.11. Consider the maximal law $\lambda = \lambda^{\xi}$ induced by the identity map $\xi = 1_{\mathsf{P}\mathcal{Q}_0}$ of Example 5.10(1), for any quantaloid $\mathcal{Q}$ and $\mathbb{T}$ the identity monad on $\mathbf{Set}/\mathcal{Q}_0$. Writing $a(x, y) := (py)_x$ for $x, y \in X$ and a lax λ-algebra structure $p : X \longrightarrow \mathsf{P}X$ on a set X with array map $|-| : X \longrightarrow \mathcal{Q}_0$, conditions (f), (g) of Definition 4.2 translate to

$$1_{|x|} \leq a(x, x), \quad (|y| = |y'| \Longrightarrow a(y', z) \circ a(x, y) \leq a(x, z))$$

for all $x, y, y', z \in X$. Since in particular $a(y, y) \circ a(x, x) \leq a(x, y)$ whenever $|x| = |y|$, these conditions are equivalent to

$$(|x| = |y| \Longrightarrow 1_{|x|} \leq a(x, y)), \quad a(y, z) \circ a(x, y) \leq a(x, z)$$

for all $x, y, z \in X$. Consequently then, $(\lambda, \mathcal{Q}) - \mathbf{Alg}$ can be seen as the full subcategory of $\mathcal{Q}-\mathbf{Cat}$ containing those $\mathcal{Q}$-categories (X, a) satisfying $1_{|x|} \leq a(x, y))$ for all x, y with the same array. In the case of $\mathcal{Q} = \mathsf{D}[0, \infty]$ (see Example 2.3(2)), this is the full subcategory of **ParMet** of those partial metric spaces (X, a) satisfying the array-invariance condition

$$a(x, x) = a(y, y) \Longrightarrow a(x, y) = a(x, x)$$

for all $x, y \in X$.

6. Lax distributive laws of $\mathbb{T}$ over $\mathbb{P}_{\mathcal{Q}}$ versus lax extensions of $\mathbb{T}$ to $\mathcal{Q}-\mathbf{Rel}$

In this section we give a precise account of the bijective correspondence between monotone distributive laws of $\mathbb{T}$ over $\mathbb{P}_{\mathcal{Q}}$ and so-called lax extensions of $\mathbb{T}$ to $\mathcal{Q}-\mathbf{Rel}$, *i.e.*, to the Kleisli category of $\mathbb{P}_{\mathcal{Q}}$, where $\mathbb{T}$ is now again an arbitrary monad of $\mathbf{Set}/\mathcal{Q}_0$, *i.e.*, not necessarily a lifting of a $\mathbf{Set}$-monad as in Section 5.

Remark 6.1. For future reference, we give a list of identities that will be used frequently in what follows. In part they have already been used in Section 3, and they all follow from the discrete presheaf adjunction that induces $\mathbb{P}_{\mathcal{Q}}$. For morphisms $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y,\ \psi : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ in $\mathcal{Q}-\mathbf{Rel}$ and $f : X \longrightarrow Y,\ g : X \longrightarrow Z,\ h : Z \longrightarrow Y$ in $\mathbf{Set}/\mathcal{Q}_0$ one has:

(1) $\overleftarrow{\varphi} = \varphi^{\odot} \cdot \mathsf{y}_Y, \quad \varphi = \overleftarrow{\varphi}^{\circ} \circ \varepsilon_X, \quad \varphi^{\odot} = \mathsf{s}_X \cdot \overleftarrow{\varphi}_!, \quad (\varphi^{\odot})^{\circ} = \varepsilon_Y \circ \varphi;$

(2) $\overleftarrow{\psi \circ \varphi} = \varphi^{\odot} \cdot \overleftarrow{\psi}, \quad g_! \cdot \overleftarrow{\varphi} = \overleftarrow{\varphi \cdot g^{\circ}}, \quad \overleftarrow{h^{\circ} \circ \varphi} = \overleftarrow{\varphi} \cdot h;$

(3) $\overleftarrow{f^{\circ}} = \mathsf{y}_Y \cdot f = f_! \cdot \mathsf{y}_X, \quad \overleftarrow{1_X^{\circ}} = \mathsf{y}_X, \quad 1_X^{\circ} = \mathsf{y}_X^{\circ} \circ \varepsilon_X, \quad \overleftarrow{\varepsilon_X} = 1_{\mathsf{P}X}.$

In what follows, we analyze which of the inequalities required for lax extensions and distributive laws correspond to each other, starting with the most general scenario. Hence, initially we consider mere families $\lambda_X : T\mathsf{P}X \longrightarrow \mathsf{P}TX$ $(X \in \mathbf{Set}/\mathcal{Q}_0)$ of maps in $\mathbf{Set}/\mathcal{Q}_0$, which we will call $(\mathbb{T}, \mathcal{Q})$-*distribution families*, and contrast them with families

$$\hat{T}\varphi : TX \relbar\joinrel\mapstochar\joinrel\rightarrow TY \ (\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y \text{ in } \mathcal{Q}-\mathbf{Rel}),$$

which we refer to as $(\mathbb{T}, \mathcal{Q})$-*extension families*. Certainly, a distribution family $\lambda = (\lambda_X)_X$ determines an extension family

$$\Phi(\lambda) = \hat{T} = (\hat{T}\varphi)_{\varphi} \text{ with } \overleftarrow{\hat{T}\varphi} := \lambda_X \cdot T\overleftarrow{\varphi},$$

also visualized by

$$\begin{array}{ccc} (\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y) & \mapsto & (\hat{T}\varphi : TX \relbar\joinrel\mapstochar\joinrel\rightarrow TY) \\ (\overleftarrow{\varphi} : Y \longrightarrow \mathsf{P}X) & \mapsto & \begin{array}{ccc} TY & \xrightarrow{\ \overleftarrow{\hat{T}\varphi}\ } & \mathsf{P}TX \\ & {\scriptstyle T\overleftarrow{\varphi}} \searrow \quad \nearrow {\scriptstyle \lambda_X} & \\ & T\mathsf{P}X. & \end{array} \end{array}$$

We see immediately that we may retrieve $(\lambda_X)_X$ from $(\hat{T}\varphi)_{\varphi}$, by choosing φ such that $\overleftarrow{\varphi} = 1_{\mathsf{P}X}$, which is the case precisely when $\varphi = \varepsilon_X : X \relbar\joinrel\mapstochar\joinrel\rightarrow \mathsf{P}X$ (the co-unit of the adjunction presented in Section 3). Hence, when assigning to any extension family $\hat{T} = (\hat{T}\varphi)_{\varphi}$ the distribution family

$$\Psi(\hat{T}) = \lambda = (\lambda_X)_X \; rmwith \; \lambda_X := \overleftarrow{\hat{T}\varepsilon_X},$$

we certainly have $\Psi\Phi(\lambda) = \lambda$ for all distribution families λ. The following Proposition clarifies which extension families correspond bijectively to distribution families. We call an extension family $\hat{T}$ *monotone* if it satisfies

$$\forall \varphi, \varphi' : X \relbar\joinrel\mapsto Y \quad (\varphi \leq \varphi' \Longrightarrow \hat{T}\varphi \leq \hat{T}\varphi'),$$

and monotonicity of a lax distribution family is defined as monotonicity for a lax distributive law in Section 4.

Proposition 6.2. Φ *and* Ψ *establish a bijective correspondence between all* $(\mathbb{T}, \mathcal{Q})$*-distribution families and those* $(\mathbb{T}, \mathcal{Q})$*-extension families* $\hat{T} = (\hat{T}\varphi)_\varphi$ *which satisfy the* left-op-whiskering *condition*

$$(0) \quad \hat{T}(h^\circ \circ \varphi) = (Th)^\circ \circ \hat{T}\varphi$$

for all $\varphi : X \relbar\joinrel\mapsto Y$ *in* $\mathcal{Q}$–$\mathbf{Rel}$, $h : Z \longrightarrow Y$ *in* $\mathbf{Set}/\mathcal{Q}_0$. *The correspondence restricts to a bijective correspondence between the conglomerate* $(\mathbb{T}, \mathcal{Q}) -$ DIS *of all monotone distribution families and the conglomerate* $(\mathbb{T}, \mathcal{Q}) -$ EXT *of all monotone extension families satisfying* (0).

Proof. For a distribution family λ and $\hat{T} := \Phi(\lambda)$, let us first verify the identity (0), using the definition of $\hat{T}$ and Remark 6.1(2):

$$\overleftarrow{(Th)^\circ \circ \hat{T}\varphi} = \overleftarrow{\hat{T}\varphi} \cdot Th = \lambda_X \cdot T\overleftarrow{\varphi} \cdot Th = \lambda_X \cdot T(\overleftarrow{h^\circ \circ \varphi}) = \overleftarrow{\hat{T}(h^\circ \circ \varphi)}.$$

Monotonicity of $\hat{T}$ follows trivially from the corresponding property of λ.

Next, for any extension family $\hat{T}$ satisfying (0), we must show $\Phi\Psi(\hat{T}) = \hat{T}$. Indeed, with $\lambda := \Psi(\hat{T})$, the definition of $\Phi(\lambda)$ and Remark 6.1(1) give

$$\begin{aligned} \overleftarrow{(\Phi\Psi(\hat{T}))\varphi} &= \lambda_X \cdot T\overleftarrow{\varphi} = \overleftarrow{\hat{T}\varepsilon_X} \cdot T\overleftarrow{\varphi} = (\hat{T}\varepsilon_X)^\odot \cdot \mathsf{y}_{T\mathsf{P}X} \cdot T\overleftarrow{\varphi} \\ &= (\hat{T}\varepsilon_X)^\odot \cdot ((T\overleftarrow{\varphi})^\circ)^\odot \cdot \mathsf{y}_{TY} = (\hat{T}(\overleftarrow{\varphi}^\circ \circ \varepsilon_X))^\odot \cdot \mathsf{y}_{TY} \\ &= (\hat{T}\varphi)^\odot \cdot \mathsf{y}_{TY} = \overleftarrow{\hat{T}\varphi}. \end{aligned}$$

That monotonicity of λ follows from the monotonicity of $\hat{T}$ and (0) is clear once one has observed that

$$\lambda_X \cdot Tf = \overleftarrow{\hat{T}\varepsilon_X} \cdot Tf = \overleftarrow{(Tf)^\circ \cdot \hat{T}\varepsilon_X} = \overleftarrow{\hat{T}(f^\circ \cdot \varepsilon_X)}$$

for all $f : Y \longrightarrow \mathsf{P}X$ in $\mathbf{Set}/\mathcal{Q}_0$. $\square$

Before pursuing the bijective correspondence further, let us contrast condition (0) with some other natural conditions for an extension family, as follows.

Proposition 6.3. *Let the monotone extension family $\hat{T}$ satisfy $\hat{T}\psi \circ \hat{T}\varphi \leq \hat{T}(\psi \circ \varphi)$ for all $\varphi, \psi \in \mathcal{Q}$–$\mathbf{Rel}$. Then the following conditions are equivalent when universally quantified over the variables occurring in them (with maps $f : X \longrightarrow Y,\ h : Z \longrightarrow Y$ over $\mathcal{Q}_0$):*

(i) $1^\circ_{TX} \leq \hat{T}(1^\circ_X), \quad \hat{T}(h^\circ \circ \varphi) = (Th)^\circ \circ \hat{T}\varphi$;

(ii) $1^\circ_{TX} \leq \hat{T}(1^\circ_X), \quad \hat{T}(\psi \circ f_\circ) = \hat{T}\psi \circ (Tf)_\circ$;

(iii) $(Tf)^\circ \leq \hat{T}(f^\circ), \quad (Tf)_\circ \leq \hat{T}(f_\circ)$.

Proof. (i) $\Rightarrow$ (iii) The hypotheses, the adjunction $f_\circ \dashv f^\circ$, and the monotonicity give

$$1^\circ_{TX} \leq \hat{T}(1^\circ_X) \leq \hat{T}(f^\circ \circ f_\circ) = (Tf)^\circ \circ \hat{T}(f_\circ),$$

so that $(Tf)_\circ \leq \hat{T}(f_\circ)$ follows with the adjunction $(Tf)_\circ \dashv (Tf)^\circ$. Furthermore,

$$(Tf)^\circ = (Tf)^\circ \circ 1_{TY} \leq (Tf)^\circ \circ \hat{T}(1^\circ_Y) = \hat{T}(f^\circ \circ 1^\circ_Y) = \hat{T}(f^\circ).$$

(iii) $\Rightarrow$ (i) One uses (iii) and the general hypotheses on $\hat{T}$ to obtain:

$$\begin{aligned} (Th)^\circ \circ \hat{T}\varphi &\leq \hat{T}(h^\circ) \circ \hat{T}\varphi \leq \hat{T}(h^\circ \circ \varphi) \\ &\leq (Th)^\circ \circ (Th)_\circ \circ \hat{T}(h^\circ \circ \varphi) \leq (Th)^\circ \circ \hat{T}(h_\circ) \circ \hat{T}(h^\circ \circ \varphi) \\ &\leq (Th)^\circ \circ \hat{T}(h_\circ \circ h^\circ \circ \varphi) \leq (Th)^\circ \circ \hat{T}\varphi. \end{aligned}$$

(i) $\Leftrightarrow$ (ii): One proceeds analogously to (i) $\Leftrightarrow$ (iii). $\square$

In what follows we compare the conditions on $\lambda \in (\mathbb{T}, \mathcal{Q}) - \mathrm{DIS}$ encountered in Section 4 with some relevant conditions on the related family $\hat{T} \in (\mathbb{T}, \mathcal{Q}) - \mathrm{EXT}$ under the correspondence of Proposition 6.2, so that $\hat{T} = \Phi(\lambda), \lambda = \Psi(\hat{T})$, all to be read as universally quantified over all new variables ($\varphi : X \longrightarrow\!\!\!\!\!\!\!\!\;\mapsto Y, \psi : Y \longrightarrow\!\!\!\!\!\!\!\!\;\mapsto Z, f : X \longrightarrow Y, g : Y \longrightarrow X$) occurring in them.

(a)
$$\begin{array}{ccc} T\mathsf{P}X & \xrightarrow{\lambda_X} & \mathsf{P}TX \\ {\scriptstyle T(f_!)}\downarrow & \leq & \downarrow{\scriptstyle (Tf)_!} \\ T\mathsf{P}Y & \xrightarrow[\lambda_Y]{} & \mathsf{P}TY \end{array}$$
(1) $\hat{T}\psi \circ (Tg)^\circ \leq \hat{T}(\psi \circ g^\circ)$

(b)
$$\begin{array}{ccc} & TX & \\ {\scriptstyle T\mathsf{y}_X}\swarrow & \geq & \searrow{\scriptstyle \mathsf{y}_{TX}} \\ T\mathsf{P}X & \xrightarrow[\lambda_X]{} & \mathsf{P}TX \end{array}$$
(2) $1_{TX}^\circ \leq \hat{T}(1_X^\circ)$

(2') $(Tf)^\circ \leq \hat{T}(f^\circ)$

(c)
$$\begin{array}{ccccc} T\mathsf{PP}X & \xrightarrow{\lambda_{\mathsf{P}X}} & \mathsf{P}T\mathsf{P}X & \xrightarrow{(\lambda_X)_!} & \mathsf{PP}TX \\ {\scriptstyle T\mathsf{s}_X}\downarrow & & \geq & & \downarrow{\scriptstyle \mathsf{s}_{TX}} \\ T\mathsf{P}X & & \xrightarrow[\lambda_X]{} & & \mathsf{P}TX \end{array}$$
(3) $\hat{T}\psi \circ \hat{T}\varphi \leq \hat{T}(\psi \circ \varphi)$

(3') $(\hat{T}\varphi)^\odot \cdot \overleftarrow{\hat{T}\varepsilon_Y} \leq \overleftarrow{\hat{T}\varepsilon_X} \cdot T\varphi^\odot$

(d)
$$\begin{array}{ccc} & \mathsf{P}X & \\ {\scriptstyle e_{\mathsf{P}X}}\swarrow & \geq & \searrow{\scriptstyle (e_X)_!} \\ T\mathsf{P}X & \xrightarrow[\lambda_X]{} & \mathsf{P}TX \end{array}$$
(4) $\varphi \circ e_X^\circ \leq e_Y^\circ \circ \hat{T}\varphi$

(e)
$$\begin{array}{ccccc} TT\mathsf{P}X & \xrightarrow{T\lambda_X} & T\mathsf{P}TX & \xrightarrow{\lambda_{TX}} & \mathsf{P}TTX \\ {\scriptstyle m_{\mathsf{P}X}}\downarrow & & \geq & & \downarrow{\scriptstyle (m_X)_!} \\ T\mathsf{P}X & & \xrightarrow[\lambda_X]{} & & \mathsf{P}TX \end{array}$$
(5) $\hat{T}\hat{T}\varphi \circ m_X^\circ \leq m_Y^\circ \circ \hat{T}\varphi$

Proposition 6.4. *Let* $\lambda \in (\mathbb{T}, \mathcal{Q})-\mathrm{DIS}$ *and* $\hat{T} \in (\mathbb{T}, \mathcal{Q})-\mathrm{EXT}$ *be related under the correspondence of Proposition* 6.2*, so that* $\hat{T} = \Phi(\lambda), \lambda = \Psi(\hat{T})$. *Then:* (a) $\Leftrightarrow$ (1), (b) $\Leftrightarrow$ (2) $\Leftrightarrow$ (2′), (a)&(c) $\Rightarrow$ (3) $\Leftrightarrow$ (3′) $\Rightarrow$ (c), (2′)&(3) $\Rightarrow$ (a), (d) $\Leftrightarrow$ (4), (e) $\Leftrightarrow$ (5),
and in each of these implications or equivalences one may replace the inequality sign by an equality sign on both sides of the implication or equivalence sign.

Proof. (a)$\Rightarrow$(1): The hypothesis (a) and Remark 6.1 give
$$\begin{aligned} \overleftarrow{\hat{T}\psi \circ (Tg)^\circ} &= (Tg)_! \cdot \overleftarrow{\hat{T}\psi} = (Tg)_! \cdot \lambda_X \cdot T\overleftarrow{\psi} \leq \lambda_Z \cdot T(g_! \cdot \overleftarrow{\psi}) \\ &= \lambda_Z \cdot T((g^\circ)^\odot \cdot \overleftarrow{\psi}) = \overleftarrow{\hat{T}(\psi \circ g^\circ)}, \end{aligned}$$

with equality holding when equality holds in (a).

(1)⇒(a): The hypotheses (0), (1), the naturality of ε and the repeated application of Remark 6.1 give the inequality (a), with equality holding when equality holds in (1):

$$\begin{aligned}(Tf)_! \cdot \lambda_X &= ((Tf)^\circ)^\odot \cdot \overleftarrow{\hat{T}\varepsilon_X} = \overleftarrow{\hat{T}\varepsilon_X \cdot (Tf)^\circ} \\ &\leq \overleftarrow{\hat{T}(\varepsilon_X \circ f^\circ)} = \overleftarrow{\hat{T}((f_!)^\circ \circ \varepsilon_Y} = \overleftarrow{(T(f_!))^\circ \circ \hat{T}\varepsilon_Y} \\ &= \overleftarrow{\hat{T}\varepsilon_Y} \cdot T(f_!) = \lambda_Y \cdot T(f_!).\end{aligned}$$

(b)⇒ $(2')$: $\overleftarrow{\hat{T}(f^\circ)} = \lambda_Y \cdot T\overleftarrow{f^\circ} = \lambda_Y \cdot T\mathsf{y}_Y \cdot Tf \geq \mathsf{y}_{TY} \cdot Tf = \overleftarrow{(Tf)^\circ}$.

$(2')$ ⇒(2)⇒(b): Consider $f = 1_X$ and use the same steps as in (b)⇒ $(2')$. Trivially then, equality holds in (b) if, and only if, equality holds in (2), or $(2')$.

(a)&(c)⇒ $(3')$: With $\lambda := \Psi(\hat{T})$, inequality $(3')$ follows from (a) and (c) and Remark 6.1, with equality holding if it holds in both (a) and (c), as follows:

$$\begin{aligned}\lambda_X \cdot T(\varphi^\odot) &= \lambda_X \cdot T\mathsf{s}_X \cdot T(\overleftarrow{\varphi}_!) \geq \mathsf{s}_{TX} \cdot (\lambda_X)_! \cdot \lambda_{\mathsf{P}Y} \cdot T(\overleftarrow{\varphi}_!) \\ &\geq \mathsf{s}_{TX} \cdot (\lambda_X)_! \cdot (T\overleftarrow{\varphi})_! \cdot \lambda_Y = \mathsf{s}_{TX} \cdot (\overleftarrow{\hat{T}\varphi})_! \cdot \lambda_Y \\ &= (\hat{T}\varphi)^\odot \cdot \lambda_Y.\end{aligned}$$

$(3')$ ⇒(c): Inequality (c) follows when one puts $\varphi = \varepsilon_X$ in $(3')$, with equality holding when it holds in $(3')$:

$$\begin{aligned}\lambda_X \cdot T\mathsf{s}_X &= \lambda_X \cdot T(\varepsilon_X^\odot) \\ &\geq (\hat{T}\varepsilon_X)^\odot \cdot \lambda_{\mathsf{P}X} = \mathsf{s}_{TX} \cdot (\overleftarrow{\hat{T}\varepsilon_X})_! \cdot \lambda_{\mathsf{P}X} = \mathsf{s}_{TX} \cdot (\lambda_X)_! \cdot \lambda_{\mathsf{P}X}.\end{aligned}$$

$(3')$ ⇒(3): With $\lambda_X = \overleftarrow{\hat{T}\varepsilon_X}$ one obtains (3) from (3') and Remark 6.1, as follows:

$$\begin{aligned}\overleftarrow{\hat{T}\psi \circ \hat{T}\varphi} &= (\hat{T}\varphi)^\odot \cdot \overleftarrow{\hat{T}\psi} = (\hat{T}\varphi)^\odot \cdot \lambda_Y \cdot T\overleftarrow{\psi} \\ &\leq \lambda_X \cdot T\varphi^\odot \cdot T\overleftarrow{\psi} = \lambda_X \cdot T(\overleftarrow{\psi \circ \varphi}) = \overleftarrow{\hat{T}(\psi \circ \varphi)}.\end{aligned}$$

(3)⇒ $(3')$: One exploits the naturality of ε and (3) (putting $\psi = \varepsilon_Y$) to obtain:

$$\begin{aligned}(\hat{T}\varphi)^\odot \cdot \overleftarrow{\hat{T}\varepsilon_Y} &= \overleftarrow{\hat{T}\varepsilon_Y \circ \hat{T}\varphi} \leq \overleftarrow{\hat{T}(\varepsilon_Y \circ \varphi)} \\ &= \overleftarrow{\hat{T}((\varphi^\odot)^\circ \circ \varepsilon_X)} = \overleftarrow{(T\varphi^\odot)^\circ \circ \hat{T}\varepsilon_X} = \overleftarrow{\hat{T}\varepsilon_X} \cdot T\varphi^\odot,\end{aligned}$$

with equality holding precisely when equality holds in (3).

$(2')$&(3)⇒(1): $\hat{T}\varphi \circ (Tg)^\circ \leq \hat{T}\varphi \circ \hat{T}(g^\circ) \leq \hat{T}(\varphi \circ g^\circ)$.

(d) $\Longleftrightarrow$ (4): We show " $\Rightarrow$ "; the implication " $\Leftarrow$ " follows similarly, with $\varphi = \varepsilon_X$:

$$\begin{aligned} \overleftarrow{\varphi \circ e_X^\circ} &= (e_X)_! \cdot \overleftarrow{\varphi} \leq \lambda_X \cdot e_{\mathsf{P}X} \cdot \overleftarrow{\varphi} \\ &= \lambda_X \cdot T\overleftarrow{\varphi} \cdot e_Y = \overleftarrow{\hat{T}\varphi} \cdot e_Y = (\hat{T}\varphi)^\odot \cdot \overleftarrow{e_Y^\circ} = \overleftarrow{e_Y^\circ \circ \hat{T}\varphi}. \end{aligned}$$

(e) $\Longleftrightarrow$ (5): Since again " $\Leftarrow$ " follows by putting $\varphi = \varepsilon_X$, we show only " $\Rightarrow$ ":

$$\begin{aligned} \overleftarrow{\hat{T}\hat{T}\varphi \circ m_X^\circ} &= (m_X)_! \cdot \overleftarrow{\hat{T}\hat{T}\varphi} = (m_X)_! \cdot \lambda_{TX} \cdot T(\overleftarrow{\hat{T}\varphi}) \\ &= (m_X)_! \cdot \lambda_{TX} \cdot T\lambda_X \cdot TT\overleftarrow{\varphi} \leq \lambda_X \cdot m_{\mathsf{P}X} \cdot TT\overleftarrow{\varphi} \\ &= \lambda_X \cdot T\overleftarrow{\varphi} \cdot m_Y = \overleftarrow{\hat{T}\varphi} \cdot m_Y = (\hat{T}\varphi)^\odot \cdot \overleftarrow{m_Y^\circ} \\ &= \overleftarrow{m_Y^\circ \circ \hat{T}\varphi}. \end{aligned}$$

□

A *lax extension* $\hat{T}$ of the monad $\mathbb{T}$ to $\mathcal{Q}-\mathbf{Rel}$ is a monotone $(\mathbb{T}, \mathcal{Q})$-extension family satisfying conditions (0), (2)-(5) for all $\varphi : X \longrightarrow\!\!\!\!\!\mapsto Y, \psi : Y \longrightarrow\!\!\!\!\!\mapsto Z$ in $\mathcal{Q}-\mathbf{Rel}$ and $h : Z \longrightarrow Y$ in $\mathbf{Set}/\mathcal{Q}_0$, *i.e.*, a left-whiskering lax functor $\hat{T} : \mathcal{Q}-\mathbf{Rel} \longrightarrow \mathcal{Q}-\mathbf{Rel}$ that coincides with T on objects and makes $e^\circ : \hat{T} \longrightarrow\!\!\!\!\!\mapsto 1_{\mathbf{Set}/\mathcal{Q}_0}$ and $m^\circ : \hat{T} \longrightarrow\!\!\!\!\!\mapsto \hat{T}\hat{T}$ lax natural transformations. We have proved in Propositions 6.2, 6.3 and 6.4 the following theorem (which corrects and considerably generalizes Exercise III.1.I in [26]):

Theorem 6.5. *There is a bijective correspondence between the monotone distributive laws of the monad $\mathbb{T}$ over $\mathbb{P}_\mathcal{Q}$ and the lax extensions $\hat{T}$ of $\mathbb{T}$ to $\mathcal{Q}$–*$\mathbf{Rel}$*. These lax extensions are equivalently described as monotone $(\mathbb{T}, \mathcal{Q})$-extension families $\hat{T}$ satisfying the following inequalities (for all f, φ, ψ as above):*

1. $(Tf)_\circ \leq \hat{T}(f_\circ)$,
2. $(Tf)^\circ \leq \hat{T}(f^\circ)$,
3. $(= (3))$ $\hat{\mathrm{T}}\psi \circ \hat{\mathrm{T}}\varphi \leq \hat{\mathrm{T}}(\psi \circ \varphi)$,
4. $(e_Y)_\circ \circ \varphi \leq \hat{T}\varphi \circ (e_X)_\circ$,
5. $(m_Y)_\circ \circ \hat{T}\hat{T}\varphi \leq \hat{T}\varphi \circ (m_X)_\circ$.

For a lax extension $\hat{T}$ of the monad $\mathbb{T}$ to $\mathcal{Q}$–!$\mathbf{Rel}$ we can now define:

Definition 6.6. *A* $(\mathbb{T},\mathcal{Q})$-category (X,α) *is a set* X *over* $\mathcal{Q}_0$ *equipped with a* $\mathcal{Q}$*-relation* $\alpha : X \longrightarrow\!\!\!\!\!\mapsto TX$ *satisfying the lax unit and multiplication laws*

$$1_X^\circ \leq e_X^\circ \circ \alpha, \quad \hat{T}\alpha \circ \alpha \leq m_X^\circ \circ \alpha.$$

A $(\mathbb{T},\mathcal{Q})$-functor $f : (X,\alpha) \longrightarrow (Y,\beta)$ *must satisfy*

$$\alpha \circ f^\circ \leq (Tf)^\circ \circ \beta.$$

Hence, the structure of a $(\mathbb{T},\mathcal{Q})$*-category* (X,α) *consists of a family of* $\mathcal{Q}$*-morphisms* $\alpha(x,\mathfrak{x}) : |x|_X \longrightarrow |\mathfrak{x}|_{TX}$ $(x \in X, \mathfrak{x} \in TX)$*, subject to the conditions*

$$1_{|x|} \leq \alpha(x, e_X x), \quad \hat{T}\alpha(\mathfrak{y},\mathfrak{Z}) \circ \alpha(x,\mathfrak{y}) \leq \alpha(x, m_X\mathfrak{Z}),$$

for all $x \in X, \mathfrak{y} \in TX, \mathfrak{Z} \in TTX$*. The* $(\mathbb{T},\mathcal{Q})$*-functoriality condition for* f *reads in pointwise form as*

$$\alpha(x,\mathfrak{y}) \leq \beta(fx, Tf(\mathfrak{y}))$$

for all $x \in X, \mathfrak{y} \in TX$*. The emerging category is denoted by*

$$(\mathbb{T},\mathcal{Q})\!-\!\mathbf{Cat};$$

only if there is the danger of ambiguity will we write $(\mathbb{T},\hat{T},\mathcal{Q})$–$\mathbf{Cat}$ *to stress the dependency on the chosen extension* $\hat{T}$*.*

Remark 6.7. When $\mathcal{Q}$ is a commutative quantale V, then the structure of a $(\mathbb{T},\mathsf{V})$-category (X,α) may be given equivalently by a V-relation $TX \longrightarrow\!\!\!\!\!\mapsto X$, and the notion takes on the familiar meaning (as presented in [26]). However, it is important to note that, because of the switch in direction of the V-relation $\alpha : X \longrightarrow\!\!\!\!\!\mapsto TX$ (as a lax coalgebra structure) to a lax algebra structure $TX \longrightarrow\!\!\!\!\!\mapsto X$ as in [26], $(\mathbb{T},\hat{T},\mathsf{V})-\mathbf{Cat}$ defined here actually becomes $(\mathbb{T},\mathsf{V},\check{T})-\mathbf{Cat}$ as defined in [26], III.1, with $\check{T}\varphi := (\hat{T}(\varphi^\circ))^\circ$ and $\varphi^\circ : Y \longrightarrow\!\!\!\!\!\mapsto X,\ \varphi^\circ(y,x) = \varphi(x,y)$, for all V-relations $\varphi : X \longrightarrow\!\!\!\!\!\mapsto Y,\ x \in X,\ y \in Y$ (see Exercise III.1.J in [26]).

Before presenting further examples, let us point out that $(\mathbb{T},\mathcal{Q})$-categories and $(\mathbb{T},\mathcal{Q})$-functors are just disguised lax λ-algebras with their lax homomorphisms, since $\mathcal{Q}$-relations $\alpha : X \longrightarrow\!\!\!\!\!\mapsto TX$ are in bijective correspondence with $\mathbf{Set}/\mathcal{Q}_0$-morphisms $p : TX \longrightarrow \mathsf{P}X$ under the adjunction of Section 3.

Proposition 6.8. *When λ and $\mathbb{T}, \hat{T}$ are related by the correspondence of* Theorem 6.5*, then there is a (natural) isomorphism*

$$(\lambda, \mathcal{Q})\text{-}\mathbf{Alg} \cong (\mathbb{T}, \hat{T}, \mathcal{Q})\text{-}\mathbf{Cat}$$

of categories which commutes with the underlying $\mathbf{Set}/\mathcal{Q}_0$*-functor.*

Proof. Given a $(\mathbb{T}, \mathcal{Q})$-category structure α on X, repeated applications of the rules of Remark 6.1 confirm that $\overleftarrow{\alpha}$ makes X a lax λ-algebra:

$$\overleftarrow{\alpha} \cdot e_X = \overleftarrow{e_X^\circ \circ \alpha} \geq \overleftarrow{1_X^\circ} = \mathsf{y}_X,$$

$$\overleftarrow{\alpha} \cdot m_X = \overleftarrow{m_X^\circ \circ \alpha} \geq \overleftarrow{\hat{T}\alpha \circ \alpha} = \alpha^{\odot} \cdot \overleftarrow{\hat{T}\alpha} = \mathsf{s}_X \cdot \overleftarrow{\alpha}_! \cdot \lambda_X \cdot T\overleftarrow{\alpha}.$$

Conversely, given a lax λ-algebra structure p on X, putting $\alpha := p^\circ \cdot \varepsilon_X$ one has $\overleftarrow{\alpha} = p$, and the same computational steps as above show $\overleftarrow{e_X^\circ \cdot \alpha} \geq \overleftarrow{1_X^\circ}$ and $\overleftarrow{m_x^\circ \cdot \alpha} \geq \overleftarrow{\hat{T}\alpha \cdot \alpha}$, so that α is a $(\mathbb{T}, \mathcal{Q})$-category structure on X.

A $(\mathbb{T}, \mathcal{Q})$-functor $f : (X, \alpha) \longrightarrow (Y, \beta)$ gives a lax λ-homomorphism $f : (X, \overleftarrow{\alpha}) \longrightarrow (Y, \overleftarrow{\beta})$, since

$$f_! \cdot \overleftarrow{\alpha} = \overleftarrow{\alpha \circ f^\circ} \leq \overleftarrow{(Tf)^\circ \circ \beta} = \overleftarrow{\beta} \cdot Tf,$$

and conversely. □

Example 6.9. (1) Let $\mathbb{T}$ be a $\mathbf{Set}$-monad with a lax extension $\tilde{T}$ to $\mathbf{Rel} = 2\text{-}\mathbf{Rel}$ that we now wish to extend further to $\mathsf{D}2\text{-}\mathbf{Rel}$. As in Proposition 5.1, we first consider a $\mathbb{T}$-algebra structure $\zeta : T2 \longrightarrow 2$, which then allows us to consider $\mathbb{T}$ as a monad on $\mathbf{Set}/2$, the category of sets X with a given subset A (see Example 2.3(2)). Of course, one now wishes to compute $T(X, A)$ as the pair (TX, TA). Since the array function of X is the characteristic function c_A of A, this is possible precisely when the $\mathbf{Set}$-functor T satisfies the pullback transformation condition

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \top \\ X & \xrightarrow[c_A]{} & 2 \end{array} \quad \Longrightarrow \quad \begin{array}{ccccc} TA & \longrightarrow & T1 & \longrightarrow & 1 \\ \downarrow & \lrcorner & & & \downarrow \top \\ TX & \xrightarrow[Tc_A]{} & T2 & \xrightarrow[\zeta]{} & 2, \end{array}$$

and this condition certainly holds when T is taut (*i.e.,* preserves pullbacks of monomorphisms) and $\zeta^{-1}1 = T1$. Since a morphism $\varphi : (X,A) \longrightarrow\!\!\!\!\!\!\mapsto (Y,B)$ (where $A \subseteq X, B \subseteq Y$) in $\mathsf{D2}-\mathbf{Rel}$ is completely determined by the restricted relation $\varphi_{\mathrm{rest}} : A \longrightarrow\!\!\!\!\!\!\mapsto B$, one may now declare $\mathfrak{x}$ to be $\hat{T}\varphi$-related to $\mathfrak{y}$ if, and only if, $\mathfrak{x} \in TA, \mathfrak{y} \in TB$ and $\mathfrak{x}$ is $\tilde{T}\varphi$-related to $\mathfrak{y}$, to obtain a lax extension of $\mathbb{T}$ to $\mathsf{D2}-\mathbf{Rel}$.

With $\tilde{T}$ and the $\mathbb{T}$-algebra structure ζ on 2 given such that $\zeta^{-1}1 = T1$, the objects (X, A, α) of the category $(\mathbb{T}, \hat{T}, \mathsf{D2}) - \mathbf{Cat}$ may be described as sets X with a subset A such that $(A, \alpha) \in (\mathbb{T}, \tilde{T}, 2)$–$\mathbf{Cat}$; morphisms $f : (X, A, \alpha) \longrightarrow (Y, B, \beta)$ are maps $f : X \longrightarrow Y$ with $f^{-1}B = A$ whose restrictions $A \longrightarrow B$ are $(\mathbb{T}, 2)$-functors. The list monad $\mathbb{L}$ (with $\zeta : L2 \longrightarrow 2$ given by $\wedge$) and the ultrafilter monad $\mathbb{U}$ both satisfy our hypotheses, and $(\mathbb{T}, \hat{T}, \mathsf{D2}) - \mathbf{Cat}$ then describes the categories of **ParMulOrd** and **ParTop** of *partial multi-ordered sets* and *partial topological spaces*, respectively.

(2) Expanding on Examples 4.1(2),(3) and Example 4.3(2), with $\mathbb{L}$ laxly extended to $\mathsf{D}[0,\infty]-\mathbf{Rel}$, one obtains as $(\mathbb{L}, \mathsf{D}[0,\infty])-\mathbf{Cat}$ the category **ParMulMet** of *partial multi-metric spaces*; its objects X may be described as sets carrying a distance function $a : LX \times X \longrightarrow [0,\infty]$ (see Remark 6.7), subject to the conditions

$$
\begin{aligned}
&\max(\sum_{i=1}^{n} a(x_i, x_i),\ a(y,y)) \leq a((x_1, ..., x_n)y),\\
&a((\underbrace{x_{1,1}, \ldots, x_{1,n_1}}_{\mathfrak{x}_1}, \ldots, \underbrace{x_{m,1}, \ldots, x_{m,n_m}}_{\mathfrak{x}_m}), z)\\
\leq\ &\Big(\sum_{i=1}^{m} a(\mathfrak{x}_i, y_i) - a(y_i, y_i)\Big) + a((y_1, \ldots, y_m), z);
\end{aligned}
$$

their morphisms $f : (X,a) \longrightarrow (Y,b)$ must satisfy

$$
\begin{aligned}
&b(f(x), f(x)) = a(x,x),\\
&b((f(x_1), \ldots, f(x_n)), f(y)) \leq a((x_1, \ldots, x_n), y)
\end{aligned}
$$

for all $x, x_1, ..., x_n, y \in X$.

7. Algebraic functors, change-of-base functors

Here we consider the standard types of functors arising from a variation in the two parameters defining the categories $(\lambda, \mathcal{Q})-\mathbf{Alg} \cong (\mathbb{T}, \hat{T}, \mathcal{Q})-\mathbf{Cat}$, which have been discussed earlier, in the quantale-monad-enriched case (see [14, 26]) as well as for $\mathbb{T}$ in more general settings (see [11]), but not in the current monad-quantaloid-enriched context, which does require some extra pecautions.

Let us first consider two monads $\mathbb{T} = (T, m, e)$, $\mathbb{S} = (S, n, d)$ on $\mathbf{Set}/\mathcal{Q}_0$, both monotonely distributing over $\mathbb{P}_\mathcal{Q}$, via the monotone distributive laws λ, κ, respectively; equivalently, both coming equipped with lax extensions $\hat{T}$ and $\hat{S}$ to $\mathcal{Q}-\mathbf{Rel}$, respectively. An *algebraic morphism*

$$h : (\mathbb{T}, \hat{T}) \longrightarrow\!\!\!\!\!\!\!+\;\; (\mathbb{S}, \hat{S})$$

of lax extensions is a family of $\mathcal{Q}$-relations $h_X : TX \longrightarrow\!\!\!\!\!\!\!+\;\; SX$ $(X \in \mathbf{Set}/\mathcal{Q}_0)$, satisfying the following conditions for all $f : X \longrightarrow Y$ in $\mathbf{Set}/\mathcal{Q}_0$, $\varphi : X \longrightarrow\!\!\!\!\!\!\!+\;\; Y$, $\alpha : X \longrightarrow\!\!\!\!\!\!\!+\;\; TX$ in $\mathcal{Q}-\mathbf{Rel}$:

a. $h_X \circ (Tf)^\circ \leq (Sf)^\circ \circ h_Y$, (lax naturality)
b. $e_X^\circ \leq d_X^\circ \circ h_X$, (lax unit law)
c. $\hat{S}h_X \circ h_{TX} \circ m_X^\circ \leq n_X^\circ \circ h_X$, (lax multiplication law)
d. $\hat{S}\varphi \circ h_X \leq h_Y \circ \hat{T}\varphi$, (lax compatability)
e. $\hat{S}(h_X \circ \alpha) \leq \hat{S}h_X \circ \hat{S}\alpha$. (strictness at h)

Note that, because of the lax functoriality of $\hat{S}$, "$\leq$" in condition e actually amounts to "=". Putting now $\tau_X := \overleftarrow{h_X}$ and exploiting Remark 6.1, we may equivalently call a family of $\mathbf{Set}/\mathcal{Q}_0$-morphisms $\tau_X : SX \longrightarrow \mathsf{P}_\mathcal{Q}TX$ (with X running through $\mathbf{Set}/\mathcal{Q}_0$) an *algebraic morphism* $\tau : \kappa \longrightarrow \lambda$ of monotone distributive laws if the following conditions hold for all $f : X \longrightarrow Y$, $g : Y \longrightarrow \mathsf{P}_\mathcal{Q}X$, $p : TX \longrightarrow \mathsf{P}_\mathcal{Q}X$ in $\mathbf{Set}/\mathcal{Q}_0$:

a′. $(Tf)_! \cdot \tau_X \le \tau_Y \cdot Sf$; (lax naturality)
b′. $\mathsf{y}_{TX} \cdot e_X \le \tau_X \cdot d_X$; (lax unit law)
c′. $(m_X)_! \cdot \mathsf{s}_{TTX} \cdot (\tau_{TX})_! \cdot \kappa_{TX} \cdot S\tau_X \le \tau_X \cdot n_X$; (lax mult. law)
d′. $\mathsf{s}_{TX} \cdot (\tau_X)_! \cdot \kappa_X \cdot Sg \le \mathsf{s}_{TX} \cdot (\lambda_X)_! \cdot (Tg)_! \cdot \tau_Y$; (lax compatibility)
e′. $\kappa_X \cdot S\mathsf{s}_X \cdot S(p_!) \cdot S\tau_X \le \mathsf{s}_{SX} \cdot (\kappa_X)_! \cdot (Sp)_! \cdot \kappa_{TX} \cdot S\tau_X$. (strictness at p)

A routine calculation shows:

Proposition 7.1. *Every algebraic morphism $h : (\mathbb{T}, \hat{T}) \longrightarrow (\mathbb{S}, \hat{S})$ of lax extensions induces the* algebraic functor

$$A_h : (\mathbb{T}, \hat{T}, \mathcal{Q})-\mathbf{Cat} \longrightarrow (\mathbb{S}, \hat{S}, \mathcal{Q})-\mathbf{Cat},\ (X, \alpha) \mapsto (X, h_X \circ \alpha).$$

When h is equivalently described as an algebraic morphism $\tau : \kappa \longrightarrow \lambda$, then A_h is equivalently described as the algebraic functor

$$A_\tau : (\lambda, \mathcal{Q})-\mathbf{Alg} \longrightarrow (\kappa, \mathcal{Q})-\mathbf{Alg},\ (X, p) \mapsto (X, \nu_X \cdot p_! \cdot \tau_X).$$

Considering $\mathbb{S}$ and $\hat{S}$ identical or, equivalently, $\kappa = 1_{\mathsf{P}}$, with the algebraic morphism $h_X = e_X^\circ$ or, equivalently, $\tau_X = \mathsf{y}_{TX} \cdot e_X$, one obtains:

Corollary 7.2. *For every monad $\mathbb{T}$ on $\mathbf{Set}/\mathcal{Q}_0$ with lax extension $\hat{T}$ and corresponding monotone distributive law λ, there is an algebraic functor*

$$A : (\mathbb{T}, \mathcal{Q})-\mathbf{Cat} \longrightarrow \mathcal{Q}-\mathbf{Cat}, (X, \alpha) \mapsto (X, e_X^\circ \circ \alpha)$$

that is equivalently described by

$$A : (\lambda, \mathcal{Q})-\mathbf{Alg} \longrightarrow \mathcal{Q}-\mathbf{Cat}, (X, p) \mapsto (X, p \cdot e_X).$$

Example 7.3. (See [34].) For the powerset monad $\mathbb{P} = \mathbb{P}_2$ and the ultrafilter monad $\mathbb{U}$ with their monotone distributive laws δ and β over $\mathbb{P}_{\mathsf{V}}$ of Example 4.1(4),(5) and their corresponding lax extensions $\hat{\mathsf{P}}$ and $\overline{U}$ to $\mathsf{V}-\mathbf{Rel}$, where $\mathsf{V} = (\mathsf{V}, \otimes, \mathsf{k})$ is a commutative and completely distributive quantale, the algebraic morphism h with $h_X : UX \relbar\joinrel\mapsto \mathsf{P}X,\quad h_X(\mathfrak{x}, A) = \mathsf{k}$ if $A \in \mathfrak{x} \in UX$, and $h_X(\mathfrak{x}, A) = \bot$ else, induces the algebraic functor

$$(\mathbb{U}, \mathsf{V})-\mathbf{Cat} \longrightarrow (\mathbb{P}, \mathsf{V})-\mathbf{Cat} \cong \mathsf{V}-\mathbf{Cls},$$

which actually takes values in $\mathsf{V}-\mathbf{Top}$ and facilitates the isomorphism of categories $(\mathbb{U}, \mathsf{V})-\mathbf{Cat} \cong \mathsf{V}-\mathbf{Top}$ already mentioned in equivalent form in Example 4.3(4).

In order to describe change-of-base functors in the general setting of this paper, let us now consider a lax homomorphism $\vartheta : \mathcal{Q} \longrightarrow \mathcal{R}$ of quantaloids, so that we have a lax natural transformation $\vartheta : B_{\vartheta_0}\mathsf{P}_{\mathcal{Q}} \longrightarrow \mathsf{P}_{\mathcal{R}}B_{\vartheta_0}$ (see the end of Section 3), and a $\mathbf{Set}$-monad $\mathbb{T} = (T, m, e)$ which, according to Proposition 5.1, has been lifted to $\mathbf{Set}/\mathcal{Q}_0$ and $\mathbf{Set}/\mathcal{R}_0$ via $\mathbb{T}$-algebra structures $\zeta : T\mathcal{Q}_0 \longrightarrow \mathcal{Q}_0$ and $\eta : T\mathcal{R}_0 \longrightarrow \mathcal{R}_0$, respectively, such that $\vartheta_0 : \mathcal{Q}_0 \longrightarrow \mathcal{R}_0$ is a $\mathbb{T}$-homomorphism. The liftings of $\mathbb{T}$ to $\mathbf{Set}/\mathcal{Q}_0$ and $\mathbf{Set}/\mathcal{R}_0$ commute with the "discrete change-of-base functor" B_{ϑ_0}, that is: $B_{\vartheta_0}T = TB_{\vartheta_0}, B_{\vartheta_0}e = eB_{\vartheta_0}, B_{\vartheta_0}m = mB_{\vartheta_0}$. (These provisions are, of course, trivially satisfied when $\mathcal{Q}$ and $\mathcal{R}$ are quantales.)

Extendinging now B_{ϑ_0} to a functor $\tilde{B_\vartheta} : \mathcal{Q}-\mathbf{Rel} \longrightarrow \mathcal{R}-\mathbf{Rel}$ by $(\tilde{B_\vartheta}\varphi)(x,y) = \vartheta(\varphi(x,y))$ and considering lax extensions $\hat{T}, \check{T}$ of $\mathbb{T}$ to $\mathcal{Q}-\mathbf{Rel}, \mathcal{R}-\mathbf{Rel}$, respectively, we call ϑ *compatible* with $\hat{T}, \check{T}$ if

$$\check{T}\tilde{B_\vartheta}\varphi \le \tilde{B_\vartheta}\hat{T}\varphi \qquad (\star)$$

for all $\varphi : X \longrightarrow\!\!\!\!\!\!\mapsto Y$ in $\mathcal{Q}-\mathbf{Rel}$. (Note that the two $\mathcal{R}$-relations in $(\star)$ are comparable since $B_{\vartheta_0}T = TB_{\vartheta_0}$.) If we describe the two lax extensions $\hat{T}, \check{T}$ equivalently by the monotone distributive laws λ, κ, respectively, using the natural lax natural transformation $\vartheta : B_{\vartheta_0}\mathsf{P}_{\mathcal{Q}} \longrightarrow \mathsf{P}_{\mathcal{R}}B_{\vartheta_0}$ (see the end of Section 3) and the easily verified rule $\overleftarrow{\tilde{B_\vartheta}\varphi} = \vartheta_X \cdot B_{\vartheta_0}\overleftarrow{\varphi}$, we see that $(\star)$ may equivalently be formulated as

$$\kappa B_{\vartheta_0} \cdot T\vartheta \le \vartheta T \cdot B_{\vartheta_0}\lambda \qquad (\star\star).$$

Now we can state the following proposition, which one may prove using lax extensions and transcribing the known proof for the quantale case (see [26], III.3.5); alternatively, one may proceed by using the monotone distributive laws and the lax monad inequalities of ϑ as stated at the end of Section 3.

Proposition 7.4. *Under hypothesis $(\star)$ one obtains the change-of-base functor*

$$B_\vartheta : (\mathbb{T}, \hat{T}, \mathcal{Q})-\mathbf{Cat} \longrightarrow (\mathbb{T}, \check{T}, \mathcal{R})-\mathbf{Cat},\ (X, \alpha) \mapsto (B_{\vartheta_0}X, \tilde{B_\vartheta}\alpha).$$

Under hypothesis $(\star\star)$ this functor is equivalently described as

$$B_\vartheta : (\lambda, \mathcal{Q})-\mathbf{Alg} \longrightarrow (\kappa, \mathcal{R})-\mathbf{Alg},\ (X, p) \mapsto (B_{\vartheta_0}X, \vartheta_X \cdot B_{\vartheta_0}p).$$

Example 7.5. For a commutative and (for simplicity) divisible quantale V, we consider the lax extensions of the list monad $\mathbb{L}$ to $\mathsf{V}-\mathbf{Rel}$ and $\mathsf{DV}-\mathbf{Rel}$ induced by the monotone distributive laws of Example 4.1(2),(3), which we may both denote by $\hat{L}$. In fact, for $\varphi : X \to\!\!\!\!\!\mapsto Y$ and $x_i \in X, y_j \in Y$ one has

$$\hat{L}\varphi((x_1, ..., x_n), (y_1, ..., y_m)) = \varphi(x_1, y_1) \otimes ... \otimes \varphi(x_m, y_m) \text{ if } m = n,$$

to be interpreted as an arrow $|x_1| \otimes ... \otimes |x_n| \rightsquigarrow |y_1| \otimes ... \otimes |y_n|$ in the DV-case, and the value is $\perp$ otherwise. For the homomorphism $\iota : \mathsf{V} \longrightarrow \mathsf{DV}$ and its retractions δ, γ as described in Section 2, one sees that $\tilde{B}_\iota$ embeds $\mathsf{V} - \mathbf{Rel}$ fully into $\mathsf{DV} - \mathbf{Rel}$, providing every set with the constant array function with value k, while its retractions $\tilde{B}_\delta$ and $\tilde{B}_\gamma$ are given by $\tilde{B}_\delta\varphi(x, y) = |y| \searrow \varphi(x, y)$ and $\tilde{B}_\gamma\varphi(x, y) = \varphi(x, y) \swarrow |x|$. Since the compatability condition $(\star)$ holds for all, ι, δ and γ (strictly so for ι), as "liftings" of the corresponding functors mentioned in Section 2, one obtains the full embedding $B_\iota : (\mathbb{L}, \mathsf{V})-\mathbf{Cat} \longrightarrow (\mathbb{L}, \mathsf{DV})-\mathbf{Cat}$ and its retractions B_δ, B_γ, which we describe explicitly here only in the case $\mathsf{V} = [0, \infty]$ using the notation of Example 6.9(2):

$$B_\delta, B_\gamma : \mathbf{ParMultMet} \longrightarrow \mathbf{MulMet}$$

$$B_\delta : (X, a) \mapsto (X, a_\delta),\ a_\delta((x_1, ..., x_n), y) = a((x_1, ..., x_n), y) - \sum_{i=1}^{n} a(x_i, x_i),$$

$$B_\gamma : (X, a) \mapsto (X, a_\gamma),\ a_\gamma((x_1, ..., x_n), y) = a((x_1, ..., x_n), y) - a(y, y).$$

The full reflective embedding $E_{\mathsf{V}} : \mathsf{DV}-\mathbf{Cat} \longrightarrow \mathsf{V}-\mathbf{Cat}/\mathsf{V}$ of Section 2 may be "lifted" along the algebraic functors $(\mathbb{L}, \mathsf{DV})-\mathbf{Cat} \longrightarrow \mathsf{DV}-\mathbf{Cat}$ and $(\mathbb{L}, \mathsf{V})-\mathbf{Cat}/\mathsf{V} \longrightarrow \mathsf{V}-\mathbf{Cat}/\mathsf{V}$ to obtain a full reflective embedding

$$E = E_{\mathbb{L},\mathsf{V}} : (\mathbb{L}, \mathsf{DV})-\mathbf{Cat} \longrightarrow (\mathbb{L}, \mathsf{V})-\mathbf{Cat}/\mathsf{V},$$

which we briefly describe next, always assuming that V be commutative and divisible. First, in accordance with the general setting of III.5.3 of [26], we combine the monoid structure of V with its internal hom and regard V as an $(\mathbb{L}, \mathsf{V})$-category (V, h) with $h : L\mathsf{V} \to\!\!\!\!\!\mapsto \mathsf{V}$ (see Remark 6.7) given by $h((v_1, ..., v_n), u) = (v_1 \otimes ... \otimes v_n) \swarrow u$. Now E provides an $(\mathbb{L}, \mathsf{DV})$-category (X, a) with the $(\mathbb{L}, \mathsf{V})$-category structure d defined by $d((x_1, ..., x_n), y) =$

$a((x_1,...,x_n),,y) \swarrow a(y,y)$ and considers it an $(\mathbb{L},\mathsf{V})$-category over V via $tx = a(x,x)$. Conversely, the reflector provides an $(\mathbb{L},\mathsf{V})$-category (X,d) that comes equipped with an $(\mathbb{L},\mathsf{V})$-functor $t : X \longrightarrow \mathsf{V}$, with the $(\mathbb{L},\mathsf{DV})$-category structure a defined by $a((x_1,...,x_n),y) = d((x_1,...,x_n),y) \otimes ty$.

In the case $\mathsf{V} = [0,\infty]$ the functor E becomes an isomorphism of categories, so that in the notation of Example 6.9(2) one has

$$\mathbf{ParMulMet} \cong \mathbf{MulMet}/[0,\infty].$$

Therefore, just as described in Section 2 in the "non-multi" case, the standard construction of a right adjoint to the functor

$$\Sigma : \mathbf{MulMet}/[0,\infty] \longrightarrow \mathbf{MulMet}$$

therefore gives a right adjoint to $B_\gamma : \mathbf{ParMulMet} \longrightarrow \mathbf{MulMet}$.

8. Comparison with Hofmann's topological theories

In [24], for a $\mathbf{Set}$-monad $\mathbb{T} = (T,m,e)$ and a *commutative* quantale $\mathsf{V} = ((\mathsf{V},\otimes,\mathsf{k})$, Hofmann considers maps $\xi : T\mathsf{V} \longrightarrow \mathsf{V}$ satisfying the following conditions:

1. $1_\mathsf{V} \le \xi \cdot e_\mathsf{V}, \quad \xi \cdot T\xi \le \xi \cdot m_\mathsf{V}$;
2*. $\mathsf{k} \cdot \zeta \le \xi \cdot T\mathsf{k}, \quad \otimes \cdot (\xi \times \xi) \cdot \mathrm{can} \le \xi \cdot T(\otimes)$;
3. $\forall f,g : Y \longrightarrow \mathsf{V}$ in $\mathbf{Set}$ $\quad (f \le g \Rightarrow \xi \cdot Tf \le \xi \cdot Tg)$;
4. $\xi_X(\sigma) := \xi \cdot T\sigma \quad (\sigma \in \mathsf{P_V}X = \mathsf{V}^X)$ gives a nat. transf. $\mathsf{P_V} \to \mathsf{P_V}T$.

Here k and $\otimes$ are considered as maps $1 \longrightarrow \mathsf{V}$ and $\mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V}$, respectively; $\mathrm{can} : T(\mathsf{V}\times\mathsf{V}) \longrightarrow T\mathsf{V}\times T\mathsf{V}$ is the canonical map with components $T\pi_1, T\pi_2$, where π_1, π_2 are product projections, and (in accordance with the notation introduced in Proposition 5.1) $\zeta : T1 \longrightarrow 1$ is the unique map onto a singleton set 1. Note that Hofmann [24] combined conditions 3 and 4 to a single axiom; however, the separation as given above (and in [16]) is easily seen to be equivalent with Hofmann's combined axiom and will make the comparison with the conditions of Definition 5.4 more transparent.

Let us now compare these conditions with conditions 0–3 for a topological theory as given in Definition 5.4, in the case that $\mathcal{Q} = \mathsf{V}$ is a commutative quantale. First we give a direct comparison of condition 2* with condition 2 of Definition 5.4 which, in the current context, reads as follows:

2. $\mathsf{y}_1 \cdot \zeta \leq \xi \cdot T\mathsf{y}_1, \;\; \mathsf{s}_1 \cdot \theta \leq \xi \cdot T\mathsf{s}_1;$

here $\zeta : T1 \longrightarrow 1$ is trivial, and $\theta = \xi_! \cdot (\zeta \cdot Tt)^! \cdot \xi \cdot T(t_!)$, for $t : \mathsf{V} \longrightarrow 1$. Indeed, the latter condition implies the former, as we show first.

Proposition 8.1. *Every map $\xi : T\mathsf{V} \longrightarrow \mathsf{V}$ satisfying Condition 2 satisfies Condition 2*.*

Proof. Since $\mathsf{k} = \mathsf{y}_1$, the first inequality of Condition 2 actually coincides with the first inequality of Condition 2*. The crucial ingredient to comparing the second inequalities in both conditions is the map

$$\chi : \mathsf{V}\times\mathsf{V} \longrightarrow \mathsf{P}_{\mathsf{V}}\mathsf{V} = \mathsf{V}^{\mathsf{V}}, \quad \chi(u,v)(w) = u \otimes (\mathsf{y}_{\mathsf{V}} v)(w) = \left\{ \begin{array}{ll} u & \text{if } w = v, \\ \bot & \text{else} \end{array} \right\},$$

since, as one easily verifies, $\mathsf{s}_1 \cdot \chi = \otimes$. It now suffices to show

$$(*) \qquad\qquad \chi \cdot (\xi \times \xi) \cdot \mathrm{can} \leq \theta \cdot T\chi;$$

indeed, one can then conclude from $\mathsf{s}_1 \cdot \theta \leq \xi \cdot T\mathsf{s}_1$ the desired inequality, as follows:

$$\otimes \cdot (\xi \times \xi) \cdot \mathrm{can} = \mathsf{s}_1 \cdot \chi \cdot (\xi \times \xi) \cdot \mathrm{can} \leq \mathsf{s}_1 \cdot \theta \cdot T\chi \leq \xi \cdot T\mathsf{s}_1 \cdot T\chi = \xi \cdot T(\otimes).$$

In order to check $(*)$, let $\mathfrak{w} \in T(\mathsf{V} \times \mathsf{V})$ and $z \in \mathsf{V}$. On one hand, with $\mathfrak{x} := T\pi_1(\mathfrak{w}), \mathfrak{y} := T\pi_2(\mathfrak{w})$, one obtains

$$(\chi \cdot (\xi \times \xi) \cdot \mathrm{can}(\mathfrak{w}))(z) = \chi(\xi(\mathfrak{x}), \xi(\mathfrak{y}))(z) = \left\{ \begin{array}{ll} \xi(\mathfrak{x}) & \text{if } z = \xi(\mathfrak{y}), \\ \bot & \text{else} \end{array} \right\},$$

and on the other, with $\mathfrak{z} := T\chi(\mathfrak{w})$, and since $t_! \cdot \chi = \pi_1$, one obtains

$$\begin{aligned} (\theta \cdot T\chi(\mathfrak{w}))(z) &= (\xi_! \cdot (\zeta \cdot Tt)^! \cdot \xi \cdot T(t_!)(\mathfrak{z}))(z) \\ &= \textstyle\bigvee_{\mathfrak{a} \in T\mathsf{V}, \xi(\mathfrak{a}) = z} ((\zeta \cdot Tt)^! \cdot \xi \cdot T(t_!)(\mathfrak{z}))(z) \\ &= \textstyle\bigvee_{\mathfrak{a} \in T\mathsf{V}, \xi(\mathfrak{a}) = z} \xi(T(t_!)(\mathfrak{z})) \\ &= \left\{ \begin{array}{ll} \xi(\mathfrak{x}) & \text{if } \exists\, \mathfrak{a} \in T\mathsf{V}\ (\xi(\mathfrak{a}) = z) \\ \bot & \text{else} \end{array} \right\}, \end{aligned}$$

which shows $(*)$. ☐

Next we will show that, in the presence of conditions 1, 3, 4, conditions 2 and 2* become equivalent, provided that the $\mathbf{Set}$-functor T of $\mathbb{T}$ satisfies the *Beck-Chevalley condition (BC)*, that is: if T transforms (weak) pullback diagrams in $\mathbf{Set}$ into weak pullback diagrams (see [24, 26]). Note that the $\mathbf{Set}$-functors of both $\mathbb{L}$ and $\mathbb{U}$ satisfy BC.

Calling a topological theory ξ (as defined in Definition 5.4) *natural* if ξ satisfies condition 4 above, we can show:

Theorem 8.2. *For a commutative quantale* V *and a* $\mathbf{Set}$*-monad* $\mathbb{T}$ *with* T *satisfying the Beck-Chevalley condition, the natural topological theories for* $\mathbb{T}$ *and* V *are characterized as the maps* ξ *satisfying Hofmann's conditions* 1, 2*, 3, 4.

Proof. From Proposition 8.1 we know that every natural topological theory satisfies Hofmann's conditions. Conversely, having ξ satisfying Hofmann's conditions, since T satisfies BC, one can define the induced *lax Barr-Hofmann extension* T_ξ of $\mathbb{T}$, as given in Definition 3.4 of [24]:

$$(T_\xi\varphi)(\mathfrak{x},\mathfrak{y}) = \bigvee\{\xi\cdot(T|\varphi|)(\mathfrak{w}) \mid \mathfrak{w}\in T(X\times Y), T\pi_1(\mathfrak{w})=\mathfrak{x}, T\pi_2(\mathfrak{w})=\mathfrak{y}\}, \qquad (\dagger)$$

for all V-relations $\varphi : X \relbar\joinrel\mapsto Y$, $\mathfrak{x}\in TX, \mathfrak{y}\in TY$, with $|\varphi| : X\times Y \longrightarrow \mathsf{V}$ denoting the map giving the values of φ. Let $\lambda := \Psi(T_\xi)$ be the corresponding monotone distributive law (see Theorem 6.5), and $\overline{\xi} = \xi^\lambda$ the induced topological theory (see Proposition 5.2), *i.e.*,

$$\overline{\xi} = \zeta_! \cdot \lambda_1 = \zeta_! \cdot \overleftarrow{T_\xi\varepsilon_1},$$

with $\varepsilon_1 : 1 \relbar\joinrel\mapsto \mathsf{V}$ the counit at 1 of the discrete presheaf adjunction. Since $|\varepsilon_1| : 1\times\mathsf{V}\longrightarrow\mathsf{V}$ and $\pi_2 : 1\times\mathsf{V}\longrightarrow\mathsf{V}$ may both be identified with the identity map on V, this formula gives, for all $\mathfrak{a}\in T\mathsf{V}$,

$$\begin{aligned}\overline{\xi}(\mathfrak{a}) &= \bigvee\{(T_\xi\varepsilon_1)(\mathfrak{b},\mathfrak{a}) \mid \mathfrak{b}\in T1\}\\ &= \bigvee\{\xi(\mathfrak{w}) \mid \mathfrak{w}\in T\mathsf{V}, T\pi_2(\mathfrak{w})=\mathfrak{a}\} = \xi(\mathfrak{a}),\end{aligned}$$

Consequently, since $\overline{\xi}$ is induced by a monotone distributive law, $\xi = \overline{\xi}$ is a topological theory, with naturality given by hypothesis. □

In [16] we showed that, when T satisfies BC, the assignment $\xi \mapsto T_\xi$ of ($\dagger$) defines a bijective correspondence between the maps ξ satisfying conditions 1, 2*, 3, 4 and those lax extensions $\hat{T}$ of $\mathbb{T}$ that are

- *left-whiskering*, that is: $\hat{T}(g_\circ \circ \varphi) = (Tg)_\circ \circ \hat{T}\varphi$ for all V-relations $\varphi : X \longrightarrow\!\!\!\!\!\mapsto Y$ and maps $g : Y \longrightarrow Z$; and
- *algebraic*, that is: $\hat{T}\varphi(\mathfrak{x}, \mathfrak{y}) =$

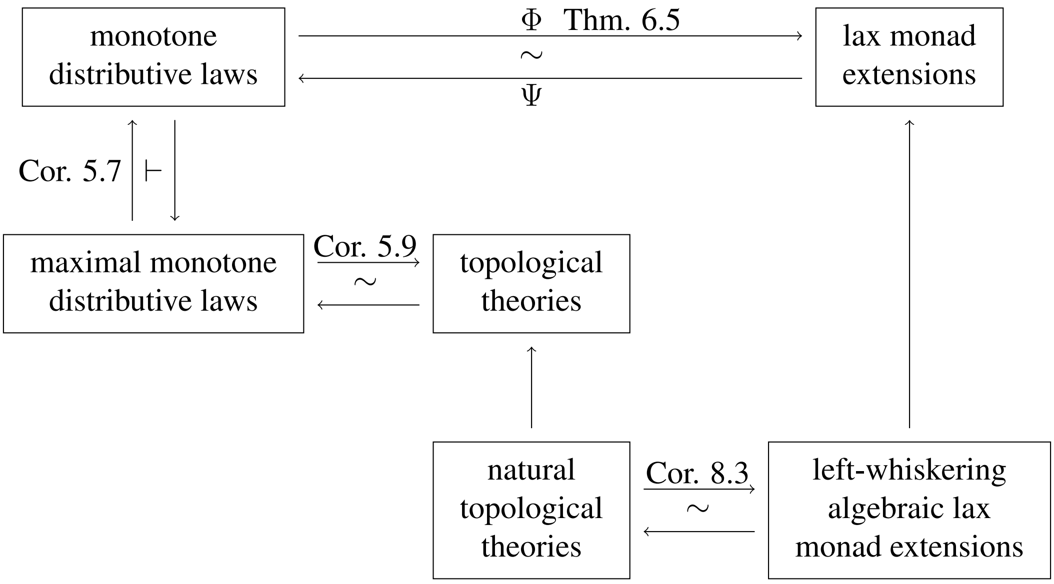
  $$\bigvee\{\hat{T}(\varphi^1)(\mathfrak{b}, \mathfrak{w}) \mid \mathfrak{b} \in T1, \mathfrak{w} \in T(X \times Y), T\pi_1(\mathfrak{w}) = \mathfrak{x}, T\pi_2(\mathfrak{w}) = \mathfrak{y}\},$$

  for all V-relations $\varphi : X \longrightarrow\!\!\!\!\!\mapsto Y$; here φ^1 has the same values as φ but is considered as a V-relation $1 \longrightarrow\!\!\!\!\!\mapsto X \times Y$.

(The proof of this characterization is easily reconstructed by following the proof of Theorem 8.5 below.) We therefore obtain with Theorem 8.2:

Corollary 8.3. *Under the hypotheses of* Theorem 8.2*, the assignment* $\xi \mapsto T_\xi$ *of* $(\dagger)$ *defines a bijective correspondence between the natural topological theories for* $\mathbb{T}$ *and* V *and the left-whiskering and algebraic lax extensions of* $\mathbb{T}$ *to* V$-$**Rel**.

The following chart summarizes the correspondences described in this paper; up-directed vertical arrows are full embeddings:



Caution is needed when reading this chart as a diagram, as it commutes only in a limited way. The following remark and theorem shed light on this cautionary note.

Remark 8.4. (1) The proof of Theorem 8.2 shows that, starting with a natural topological theory and, under the provisions of Theorem 8.2 on V and $\mathbb{T}$, chasing it counterclockwise all around the chart, one arrives at the same topological theory.

(2) However, under the assumptions of Theorem 8.2 on V and $\mathbb{T}$, chasing a natural topological theory ξ upwards on the two possible paths one obtains very different types of lax monad extensions; their typical properties appear to be almost disjoint. Most remarkably, assigning to ξ the *maximal* monotone distributive law λ^{ξ} and then the lax monad extension $\hat{T} = \Phi(\lambda^{\xi})$, one observes easily that, for $\varphi : X \relbar\!\!\!\mapsto Y,\ \mathfrak{x} \in TX, \mathfrak{y} \in TY$ and $a : X \longrightarrow 1$,

$$\hat{T}\varphi(\mathfrak{x}, \mathfrak{y}) = \xi(T(a_! \cdot \overleftarrow{\varphi})(\mathfrak{y}))$$

does not depend on $\mathfrak{x}$! But also the other path up ($\xi \mapsto T_\xi$) leads to quite special monad extensions, since being left-whiskering and algebraic are restrictive properties which, for example, exclude all extensions $\hat{T}$ that fail to satisfy the symmetry condition $\hat{T}(\varphi^\circ) = (\hat{T}\varphi)^\circ$ (see Remark 6.7), in particular the important extensions first considereded by Seal [49]. In fact, in the following theorem we give a context in which T_ξ is described as *minimal* among extension families inducing ξ.

For a commutative quantale V and a $\mathbf{Set}$-monad $\mathbb{T} = (T, m, e)$, continuing to use the notations $\zeta : T1 \longrightarrow 1$ and $\varphi^1 : 1 \relbar\!\!\!\mapsto X \times Y$ whenever $\varphi : X \relbar\!\!\!\mapsto Y$ in $\mathsf{V}\text{-}\mathbf{Rel}$, let us call an extension family $\hat{T} = (\hat{T}\varphi)_\varphi$ (see Section 6) *admissible* if, for all φ,

$$(\hat{T}\varphi)^1 \geq (\mathrm{can}_{X,Y})_\circ \circ \hat{T}(\varphi^1) \circ \zeta^\circ,$$

and *algebraic*, if " $\geq$ " may always be replaced by "="; here $\mathrm{can}_{X,Y} : T(X \times Y) \longrightarrow TX \times TY$ is the canonical map. (Note that this definition of algebraicity is just an element-free rendering of the definition given above in a narrower context.) Denoting by $(\mathbb{T},\mathsf{V})\!-\!\mathrm{EXT}_{\mathrm{adm}}$ the conglomerate of all admissible, left-op-whiskering and monotone extension families of T (see Proposition 6.2), one has a monotone map

$$\Xi : (\mathbb{T},\mathsf{V})\!-\!\mathrm{EXT}_{\mathrm{adm}} \longrightarrow \{\xi \in \mathbf{Set}(T\mathsf{V},\mathsf{V}) \mid \xi\mathrm{monotone}\}$$
$$\hat{T} \mapsto \zeta_! \cdot \overleftarrow{\hat{T}\varepsilon_1} = \overleftarrow{\hat{T}\varepsilon_1 \circ \zeta^\circ},$$

with monotonicity of arbitrary maps $T\mathsf{V} \longrightarrow \mathsf{V}$ to be understood as in condition 3 above, and with their order given pointwise as in V. The following Theorem shows that this map is an order embedding and has a right adjoint, given by

$$\xi \mapsto T_\xi, \text{ with } (T_\xi\varphi)^1 = (\mathrm{can}_{X,Y})_\circ \circ (T|\varphi|)^\circ \circ \xi^\circ \circ \varepsilon_1$$

and $|\varphi| = \overleftarrow{\varphi^1} : X \times Y \longrightarrow \mathsf{V}$ as used in (†); in fact, the formula above is just an element-free rendering of the formula (†) of Theorem 8.2.

Theorem 8.5. *Let $T : \mathbf{Set} \longrightarrow \mathbf{Set}$ satisfy BC, V be a commutative quantale and the map $\xi : T\mathsf{V} \longrightarrow \mathsf{V}$ be monotone. Then T_ξ is the least of all admissible, left-op-whiskering and monotone extension families $\hat{T}$ with $\Xi(\hat{T}) = \xi$.*

Proof. First we verify that T_ξ is left-op-whiskering, so that it satisfies condition (0) of Proposition 6.2. Indeed, for $\varphi : X \relbar\joinrel\mapsto Y$ and $h : Z \longrightarrow Y$, with $|h^\circ \circ \varphi| = |\varphi| \cdot (1_X \times h)$ one obtains

$$\begin{aligned}(T_\xi(h^\circ \circ \varphi))^1 &= (\mathrm{can}_{X,Z})_\circ \circ (T|h^\circ \circ \varphi|)^\circ \circ \xi^\circ \circ \varepsilon_1\\ &= (\mathrm{can}_{X,Z})_\circ \circ (T(1 \times h))^\circ \circ (T|\varphi|)^\circ \circ \xi_\circ \circ \varepsilon_1.\end{aligned}$$

Since the satisfaction of BC by T forces

$$(\mathrm{can}_{X,Z})_\circ \circ (T(1 \times h))^\circ = (1_{TX} \times Th)^\circ \circ (\mathrm{can}_{X,Y})_\circ$$

(see Proposition 1.4.3 of [16]), the previous identity gives $(T_\xi(h^\circ \circ \varphi))^1 = ((Th)^\circ \circ T_\xi\varphi)^1$, as desired.

For admissibility of T_ξ, first an easy inspection shows $\xi^\circ \circ \varepsilon_1 = T_\xi\varepsilon_1 \circ \zeta^\circ$. Since T_ξ is left-op-whiskering, this identity and $\varphi^1 = |\varphi|^\circ \circ \varepsilon_1$ in fact confirm even its algebraicity:

$$\begin{aligned}(T_\xi\varphi)^1 &= (\mathrm{can}_{X,Y})_\circ \circ (T|\varphi|)^\circ \circ T_\xi\varepsilon_1 \circ \zeta^\circ\\ &= (\mathrm{can}_{X,Y})_\circ \circ T_\xi(|\varphi|^\circ \circ \varepsilon_1) \circ \zeta^\circ = (\mathrm{can}_{X,Y})_\circ \circ T_\xi(\varphi^1) \circ \zeta^\circ.\end{aligned}$$

For an arbitrary admissible, left-op-whiskering and monotone $\hat{T}$ with $\Xi(\hat{T}) = \xi$, we first use the left-op-whiskering property to obtain $\hat{T}(\varphi^1) = (T|\varphi|)^\circ \circ \hat{T}\varepsilon_1$ and then

$$\overleftarrow{\hat{T}(\varphi^1) \circ \zeta^\circ} = \zeta_! \cdot \overleftarrow{\hat{T}(\varphi^1)} = \zeta_! \cdot \overleftarrow{\hat{T}\varepsilon_1} \cdot T|\varphi| = \xi \cdot T|\varphi|.$$

Consequently, the admissibility of $\hat{T}$ gives the desired inequality

$$\begin{aligned} (T_\xi\varphi)^1 &= (\mathrm{can}_{X,Y})_\circ \circ (T|\varphi|)^\circ \circ \xi^\circ \circ \varepsilon_1 \\ &= (\mathrm{can}_{X,Y})_\circ \circ (\overleftarrow{\hat{T}(\varphi^1) \circ \zeta^\circ})^\circ \circ \varepsilon_1 \\ &= (\mathrm{can}_{X,Y})_\circ \circ \hat{T}(\varphi^1) \circ \zeta^\circ \leq (\hat{T}\varphi)^1, \end{aligned}$$

which confirms the minimality of T_ξ. □

References

[1] J. Adámek, H. Herrlich, and G.E. Strecker. *Abstract and Concrete Categories: The Joy of Cats*. Wiley, New York, 1990.

[2] M. Barr. Relational algebras. In: *Lecture Notes in Mathematics* 170, pp. 39–55. Springer, Berlin-Heidelberg-New York, 1970.

[3] M. Barr and C. Wells. *Toposes, Triples and Theories*. Springer, New York, 1985

[4] J. Beck. Distributive laws. In: *Lecture Notes in Mathematics* 80, pp. 119-140. Springer, Berlin-Heidelberg-New York, 1969. Reprinted in: *Reprints in Theory and Applications of Categories*, 18:95–112, 2008.

[5] R. Betti, A. Carboni, R. Street, and R.F.C. Walters. Variation through enrichment. *Journal of Pure and Applied Algebra*, 29:109–127, 1981.

[6] G. Böhm. The weak theory of monads. *Advances in Mathematics*, 225:1-32, 2010.

[7] F. Borceux. *Handbook of Categorical Algebra 1, 2, 3*. Cambridge University Press, 1994.

[8] P. Brock and D.C. Kent. Approach spaces, limit tower spaces, and probabilistic convergence spaces. *Applied Categorical Structures*, 5:99-110, 1997.

[9] M. Bukatin, R. Kopperman, S. Matthews and H. Pajoohesh. Partial metric spaces. *American Mathematical Monthly*, 116:708-718, 2009.

[10] A. Burroni. T-catégories (catégories dans un triple). *Cahiers de Topologie et Géométrie Différentielle*, 12:215–321, 1971.

[11] A. Carboni, G.M. Kelly, and R.J. Wood. A 2-categorical approach to change of base and geometric morphisms, I. *Cahiers de Topologie et Géométrie Différentielle*, 32:47–95, 1991.

[12] M.M. Clementino and D. Hofmann. Topological features of lax algebras. *Applied Categorical Structures*, 11(3):267–286, 2003.

[13] M.M. Clementino and D. Hofmann. On extensions of lax monads. *Theory and Applications of Categories*, 13(3):41–60, 2004.

[14] M.M. Clementino, D. Hofmann, and W. Tholen. One setting for all: metric, topology uniformity, approach structure. *Applied Categorical Structures*, 12(2):127–154, 2004.

[15] M.M. Clementino and W. Tholen. Metric, topology and multicategory– a common approach. *Journal of Pure and Applied Algebra*, 179(1–2):13–47, 2003.

[16] M.M. Clementino and W. Tholen. From lax monad extensions to topological theories. *Textos Matematicos (Universidade de Coimbra)*, 46:99–123, 2014.

[17] M.P. Fourman and D.S. Scott. Sheaves and logic. *Lecture Notes in Mathematics*, 753:302–401, 1979.

[18] P.J. Freyd. Stable homotopy. In: S. Eilenberg, D.K. Harrison, S. Mac Lane, and H. Röhrl, editors, *Proceedings of the Conference on Categorical Algebra, La Jolla 1965*, pp. 121–176. Springer, Berlin-Heidelberg, 1966.

[19] R. Garner. Topological functors as total categories. *Theory and Applications of Categories*, 29(15):406–421, 2014.

[20] M. Grandis. On the monad of proper factorization systems in categories. *Journal of Pure and Applied Algebra*, 171(1):17–26, 2002.

[21] C. Hermida. Representable multicategories. *Advances in Mathematics*, 151(2):164–225, 2000.

[22] H. Herrlich. Topological functors. *General Topology and its Applications*, 4(2):125–142, 1974.

[23] H. Heymans. *Sheaves on quantales as generalized metric spaces*. Ph.D thesis, Universiteit Antwerpen, 2010.

[24] D. Hofmann. Topological theories and closed objects. *Advances in Mathematics*, 215:789–824, 2007.

[25] D. Hofmann and C.D. Reis. Probabilistic metric spaces as enriched categories. *Fuzzy Sets and Systems*, 210:1-21, 2013.

[26] D. Hofmann, G.J. Seal, and W. Tholen, editors. *Monoidal Topology: A Categorical Approach to Order, Metric, and Topology*. Cambridge University Press, Cambridge, 2014.

[27] D. Hofmann and I. Stubbe. Topology from enrichment: the curious case of partial metrics. *Cahiers de Topologie et Géométrie Différentielle Catégoriques* (to appear).

[28] U. Höhle. *Many-Valued Topology and its Applications* Kluwer, Boston-Dordrecht-London, 2001.

[29] U. Höhle and T. Kubiak. A non-commutative and non-idempotent theory of quantale sets. *Fuzzy Sets and Systems*, 166(1):1–43, 2011.

[30] G. Jäger. A convergence theory for probabilistic metric spaces, *Quaestiones Mathematicae*, 38(4):587–599, 2015.

[31] A. Joyal and M. Tierney. An extension of the Galois theory of Grothendieck. *Memoirs of the American Mathematical Society*, 51(309), 1984.

[32] G.M. Kelly. *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1982.

[33] S. Lack and R. Street. The formal theory of monads II. *Journal of Pure and Applied Algebra* (Special volume celebrating the 70th birthday of Professor Max Kelly), 175(1-3): 243–265, 2002.

[34] H. Lai and W. Tholen. Quantale-valued topological spaces via closure and convergence. *Topology and its Applications*, 230:599-620, 2017.

[35] H. Lai, L. Shen, and W. Tholen. Lax distributive laws for topology, II. *Theory and Applications of Categories*, 32(21):736-768, 2017.

[36] J. Lambek. Deductive systems and categories, II. Standard constructions and closed categories. In: *Lecture Notes in Mathematics* 86, pp. 76–122. Springer, Berlin-Heidelberg-New York, 1969.

[37] F.W. Lawvere. Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano*, 43:135–166. Reprinted in: *Reprints in Theory and Applications of Categories*, 1:1–37, 2002.

[38] R. Lowen. *Approach Spaces, The Missing Link in the Topology-Uniformity-Metric Triad*. Oxford University Press, New York , 1997.

[39] S. Mac Lane. *Categories for the Working Mathematician*, Second Edition. Springer, New York, 1998.

[40] E.G. Manes. A triple-theoretc construction of compact algebras. In: *Lecture Notes in Mathematics* 80, pp. 91–118. Springer, Berlin-Heidelberg-New York, 1969. Reprinted in: *Reprints in Theory and Applications of Categories*, 18:73–94, 2008.

[41] E.G. Manes. *Algebraic Theories*. Springer, New York-Heidelberg-Berlin, 1976.

[42] E.G. Manes. Monads in topology. *Topology and its Applications*, 157:961–989, 2010.

[43] S.G. Matthews. Partial metric topology. In: S. Andima (editor), Papers on general topology and applications. *Annals of the New York Academy of Sciences*, 728:183–197, 1994.

[44] A. Möbus. *Relational-Algebren*. Doctoral dissertation, Universität Düsseldorf, 1981.

[45] Q. Pu and D. Zhang. Preordered sets valued in a GL-monoid. *Fuzzy Sets and Systems*, 187(1):1–32, 2012.

[46] K.I. Rosenthal. *Quantales and Their Applications*. Addison Wesley Longman, Harlow, 1990.

[47] K.I. Rosenthal. *The Theory of Quantaloids*. Addison Wesley Longman, Harlow, 1996.

[48] C. Schubert. *Lax algebras – a scenic approach*. Doctoral dissertation. Universität Bremen, 2006.

[49] G.J. Seal. Canonical and op-canonical lax algebras. *Theory and Applications of Categories*, 14:221–243, 2005.

[50] G.J. Seal. A Kleisli-based approach to lax algebras. *Applied Categorical Structures*, 17(1):75-89, 2009.

[51] L. Shen and W. Tholen. Limits and colimits of quantaloid-enriched categories and their distributors. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 56(3):209-231, 2015.

[52] L. Shen and W. Tholen. Topological categories, quantaloids and Isbell adjunctions. *Topology and its Applications*, 200:212–236, 2016.

[53] L. Shen and D. Zhang. Categories enriched over a quantaloid: Isbell adjunctions and Kan adjunctions. *Theory and Applications of Categories*, 28(20):577–615, 2013.

[54] R. Street. The formal theory of monads. *Journal of Pure and Applied Algebra*, 2:149–168, 1972.

[55] R. Street and R.F.C. Walters. Yoneda structures on 2-categories. *Journal of Algebra*, 50(2):350–379, 1978.

[56] I. Stubbe. Categorical structures enriched in a quantaloid: categories, distributors and functors. *Theory and Applications of Categories*, 14(1):1–45, 2005.

[57] I. Stubbe. Categorical structures enriched in a quantaloid: tensored and cotensored categories. *Theory and Applications of Categories*, 16(14):283–306, 2006.

[58] I. Stubbe. An introduction to quantaloid-enriched categories. *Fuzzy Sets and Systems* (Special Issue on Enriched Category Theory and Related Topics), 256:95-116, 2014.

[59] Y. Tao, H. Lai, and D. Zhang. Quantale-valued preorders: Globalization and cocompleteness. *Fuzzy Sets and Systems* (Special Issue on Enriched Category Theory and Related Topics), 256:236–251, 2014.

[60] W. Tholen. Ordered topological structures. *Topology and its Applications*, 156:2148–2157, 2009.

[61] W. Tholen. Quantalic topological theories. *Tbilisi Mathematical Journal*. 10(3):223–237, 2017.

[62] R.F.C. Walters. Sheaves and Cauchy-complete categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 22(3):283–286, 1981.

[63] R.J. Wood. Ordered sets via adjunction. In: *Categorical Foundations* (edited by M.C. Pedicchio and W. Tholen), pp. 5–47, Cambridge University Press, 2004.

[64] D. Zhang. Tower extensions of topological constructs. *Commentationes Mathematicae Universitatis Carolinae*, 41(1):41-51, 2000.

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