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EXTRIANGULATED CATEGORIES, HOVEY TWIN COTORSION PAIRS AND MODEL STRUCTURES.

Hiroyuki Nakaoka and Yann Palu

Résumé. Nous proposons une généralisation simultanée de la notion de catégorie exacte et de celle de catégorie triangulée, qui est bien adaptée à l'étude des paires de cotorsion, et que nous appelons catégorie extriangulée. Les sous-catégories pleines, stables par extension, d'une catégorie triangulée sont également des exemples de catégories extriangulées. Nous montrons une correspondance bijective entre certaines paires de cotorsion, que nous appelons paires de cotorsion jumelles de Hovey, et structures de modèle admissibles, généralisant ainsi la correspondance de Hovey. En passant à la catégorie homotopique, cette approche permet de relier certaines localisations à des quotients par des idéaux. Bien que ces structures de modèle ne soient généralement pas stables, nous montrons que la catégorie homotopique est triangulée. Nous donnons ainsi un cadre naturel pour formuler les notions de réduction et de mutation de paires de cotorsion, s'appliquant à la fois aux catégories exactes et aux catégories triangulées. Ces résultats peuvent être vus comme des arguments corroborant l'hypothèse que les catégories extriangulées sont une axiomatisation agréable pour écrire des démonstrations valables pour les catégories exactes et pour les catégories (stables par extension dans une catégorie) triangulée.

Abstract. We give a simultaneous generalization of exact categories and triangulated categories, which is suitable for considering cotorsion pairs, and which we call extriangulated categories. Extension-closed, full subcategories of triangulated categories are examples of extriangulated categories. We give a bijective correspondence between some pairs of cotorsion pairs which we call Hovey twin cotorsion pairs, and admissible model structures. As a consequence, these model structures relate certain localizations with certain ideal quotients, via the homotopy category which can be given a triangulated structure. This gives a natural framework to formulate reduction and mutation of cotorsion pairs, applicable to both exact categories and triangulated categories. These results can be thought of as arguments towards the view that extriangulated categories are a convenient setup for writing down proofs which apply to both exact categories and (extension-closed subcategories of) triangulated categories.
Keywords. Extriangulated categories, Quillen exact categories, triangulated categories, cotorsion pairs, model structures, localisation, mutation.
Mathematics Subject Classification (2010). 18E10, 18E30, 18G55, 18E35, 18G15, 13F60.
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Contents

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1. Introduction and Preliminaries

Cotorsion pairs, first introduced in [Sal], are defined on an exact category or a triangulated category, and are related to several homological structures, such as: t-structures [BBD], cluster tilting subcategories [KR, KZ], co-t-structures [Pau], functorially finite rigid subcategories. A careful look reveals that what is necessary to define a cotorsion pair on a category is the existence of an Ext^1 bifunctor with appropriate properties. In this article, we formalize the notion of an *extriangulated category* by extracting those properties of Ext^1 on exact categories and on triangulated categories that seem relevant from the point-of-view of cotorsion pairs.

The class of extriangulated categories not only contains exact categories and extension-closed subcategories of triangulated categories as examples, but it is also closed under taking some ideal quotients (Proposition 3.30). This will allow us to construct an extriangulated category which is not exact nor triangulated. Moreover, this axiomatization rams down the problem of the *non-existence of a canonical choice of the middle arrow in the axiom (TR3)* to the ambiguity of a representative of realizing sequences (Section 2.2) and the exactness of the associated sequences of natural transformations (Proposition 3.3).

Let us motivate a bit more the use of extriangulated categories. Many results which are homological in nature apply (after suitable adaptation) to both setups: exact categories and triangulated categories. In order to transfer a result known for triangulated categories to a result that applies to exact categories, the usual strategy is the following (non-chronological):

(1) Specify to the case of stable categories of Frobenius exact categories.

(2) Lift all definitions and statements from the stable category to the Frobenius category.

(3) Adapt the proof so that it applies to any exact categories (with suitable assumptions).

Conversely, a result known for exact categories might have an analog for triangulated categories proven as follows:

(1) Specify to the case of a Frobenius exact category.

(2) Descend all definitions and statements to the stable category.

(3) Adapt the proof so that it applies to any triangulated categories (with suitable assumptions).

Even though step (2) might be far from trivial, the main difficulty often lies in step (3) for both cases. The use of extriangulated categories somehow removes that difficulty. It is not that this difficulty has vanished into thin air, but that it has already been taken care of in the first results on extriangulated categories obtained in Section 3.

The term "extriangulated" stands for *externally triangulated* by means of a bifunctor. It can also be viewed as the mixing of *exact* and *triangulated*, or as an abbreviation of Ext-*triangulated*. The precise definition will be given in Section 2. Fundamental properties including several analogs of the octahedron axiom in an extriangulated category, will be given in Section 3.

On an extriangulated category, we can define the notion of a cotorsion pair, which generalizes that on exact categories [Ho1, Ho2, Liu, S] and on triangulated categories [AN, Na1]. Basic properties are stated in Section 4.

In Section 5, we give a bijective correspondence between *Hovey twin cotorsion pairs* and *admissible model structures*. This result is inspired from [Ho1, Ho2, G], where the case of exact categories is studied in more details. We note that an analog of Hovey's result in [Ho1, Ho2] has been proven for triangulated categories in [Y]. The use of extriangulated categories allows for a uniform proof.

As a result, we can realize the associated homotopy category by a certain ideal quotient, on which we can give a triangulated structure as in Section 6. This triangulation can be regarded as a simultaneous generalization of those given by Happel's theorem on stable categories of Frobenius exact categories and of Iyama-Yoshino reductions, and gives a link to the one given by the Verdier quotient. As a consequence, the homotopy category of any exact model structure on a weakly idempotent complete exact category is triangulated. This result was previously known in the case of hereditary exact model structures [G, Proposition 5.2].

With this view, in Section 7, we propose a natural framework to formulate reduction and mutation of cotorsion pairs, applicable to both exact categories and triangulated categories. Indeed, we establish a bijective correspondence between the class of *mutable cotorsion pairs* associated with a Hovey twin cotorsion pair and the class of all cotorsion pairs on the triangulated homotopy category.

2. Extriangulated category

In this section, we abstract the properties of extension-closed subcategory of triangulated or exact category, to formulate it in an internal way by means of an Ext^1 functor. This gives a simultaneous generalization of triangulated categories and exact categories, suitable for dealing with cotorsion pairs.

2.1 $\mathbb{E}$-extensions

Throughout this paper, let $\mathscr{C}$ be an additive category.

Definition 2.1. Suppose $\mathscr{C}$ is equipped with a biadditive functor $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$. For any pair of objects $A, C \in \mathscr{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension*. Thus formally, an $\mathbb{E}$-extension is a triplet (A, δ, C).

Remark 2.2. *Let (A, δ, C) be any $\mathbb{E}$-extension. Since $\mathbb{E}$ is a bifunctor, for any $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C', C)$, we have $\mathbb{E}$-extensions*

$$\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A') \ \textit{ and } \ \mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A).$$

We abbreviately denote them by $a_*\delta$ and $c^*\delta$. In this terminology, we have

$$\mathbb{E}(c, a)(\delta) = c^* a_* \delta = a_* c^* \delta \quad \textit{ in } \ \mathbb{E}(C', A').$$

Definition 2.3. Let $(A, \delta, C), (A', \delta', C')$ be any pair of $\mathbb{E}$-extensions. A *morphism* $(a, c)\colon (A, \delta, C) \to (A', \delta', C')$ of $\mathbb{E}$-extensions is a pair of morphisms $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C, C')$ in $\mathscr{C}$, satisfying the equality

$$a_*\delta = c^*\delta'.$$

We simply denote it as $(a, c)\colon \delta \to \delta'$.

We obtain the category $\mathbb{E}$-$\mathrm{Ext}(\mathscr{C})$ of $\mathbb{E}$-extensions, with composition and identities naturally induced from those in $\mathscr{C}$.

Remark 2.4. *Let (A, δ, C) be any $\mathbb{E}$-extension. We have the following.*

(1) Any morphism $a \in \mathscr{C}(A, A')$ gives rise to a morphism of $\mathbb{E}$-extensions

$$(a, \mathrm{id}_C)\colon \delta \to a_*\delta.$$

(2) Any morphism $c \in \mathscr{C}(C', C)$ gives rise to a morphism of $\mathbb{E}$-extensions

$$(\mathrm{id}_A, c)\colon c^*\delta \to \delta.$$

Definition 2.5. For any $A, C \in \mathscr{C}$, the zero element $0 \in \mathbb{E}(C, A)$ is called the *split* $\mathbb{E}$*-extension*.

Definition 2.6. Let $\delta = (A, \delta, C), \delta' = (A', \delta', C')$ be any $\mathbb{E}$-extensions. Let $C \xrightarrow{\iota_C} C \oplus C' \xleftarrow{\iota_{C'}} C'$ and $A \xleftarrow{p_A} A \oplus A' \xrightarrow{p_{A'}} A'$ be coproduct and product in $\mathscr{C}$, respectively. Remark that, by the biadditivity of $\mathbb{E}$, we have a natural isomorphism

$$\mathbb{E}(C \oplus C', A \oplus A') \cong \mathbb{E}(C, A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C', A').$$

Let $\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$ be the element corresponding to $(\delta, 0, 0, \delta')$ through this isomorphism.

If $A = A'$ and $C = C'$, then the sum $\delta + \delta' \in \mathbb{E}(C, A)$ of $\delta, \delta' \in \mathbb{E}(C, A)$ is obtained by

$$\delta + \delta' = \mathbb{E}(\Delta_C, \nabla_A)(\delta \oplus \delta'),$$

where $\Delta_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix} : C \to C \oplus C$, $\nabla_A = [1\ 1] \colon A \oplus A \to A$.

2.2 Realization of $\mathbb{E}$-extensions

Let $\mathscr{C}, \mathbb{E}$ be as before.

Definition 2.7. Let $A, C \in \mathscr{C}$ be any pair of objects. Two sequences of morphisms $A \xrightarrow{x} B \xrightarrow{y} C$ and $A \xrightarrow{x'} B' \xrightarrow{y'} C$ in $\mathscr{C}$ are said to be *equivalent* if there exists an isomorphism $b \in \mathscr{C}(B, B')$ which makes the following diagram commutative.

$$\begin{tikzcd}[row sep=small] & B \arrow[dr, "y"] \arrow[dd, "b", "\cong"'] & \\ A \arrow[ur, "x"] \arrow[dr, "x'"'] \arrow[r, phantom, "\circlearrowright"] & & C \arrow[l, phantom, "\circlearrowright"] \\ & B' \arrow[ur, "y'"'] & \end{tikzcd}$$

We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$.

Definition 2.8.

(1) For any $A, C \in \mathscr{C}$, we denote as $0 = [A \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} A \oplus C \xrightarrow{[0\ 1]} C]$.

(2) For any two classes $[A \xrightarrow{x} B \xrightarrow{y} C]$ and $[A' \xrightarrow{x'} B' \xrightarrow{y'} C']$, we denote as $[A \xrightarrow{x} B \xrightarrow{y} C] \oplus [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$ the class $[A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C']$.

Definition 2.9. Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. This $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$, if it satisfies the following condition $(*)$. In this case, we say that sequence $A \xrightarrow{x} B \xrightarrow{y} C$ *realizes* δ, whenever it satisfies $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$.

$(*)$ Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions, with $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. Then, for any morphism $(a, c) \in \mathbb{E}\text{-Ext}(\mathscr{C})(\delta, \delta')$, there exists $b \in \mathscr{C}(B, B')$ which makes the following diagram commutative.

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "c"] \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \end{tikzcd} \tag{1}$$

Remark that this condition does not depend on the choices of the representatives of the equivalence classes. In the above situation, we say that (1) (or the triplet (a, b, c)) *realizes* (a, c).

Definition 2.10. Let $\mathscr{C}, \mathbb{E}$ be as above. A realization of $\mathbb{E}$ is said to be *additive*, if it satisfies the following conditions.

(i) For any $A, C \in \mathscr{C}$, the split $\mathbb{E}$-extension $0 \in \mathbb{E}(C, A)$ satisfies
$$\mathfrak{s}(0) = 0.$$

(ii) For any pair of $\mathbb{E}$-extensions $\delta = (A, \delta, C)$ and $\delta' = (A', \delta', C')$,
$$\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta') \text{ holds.}$$

Remark 2.11. *If $\mathfrak{s}$ is an additive realization of $\mathbb{E}$, then the following holds.*

(1) For any $A, C \in \mathscr{C}$, if $0 \in \mathbb{E}(C, A)$ is realized by $A \xrightarrow{x} B \xrightarrow{y} C$, then there exist a retraction $r \in \mathscr{C}(B, A)$ of x and a section $s \in \mathscr{C}(C, B)$ of y which give an isomorphism $\begin{bmatrix} r \\ y \end{bmatrix} : B \xrightarrow{\cong} A \oplus C$.

(2) For any $f \in \mathscr{C}(A,B)$, the sequence

$$A \xrightarrow{\begin{bmatrix} 1 \\ -f \end{bmatrix}} A \oplus B \xrightarrow{[f\ 1]} B$$

realizes the split $\mathbb{E}$-extension $0 \in \mathbb{E}(B,A)$.

2.3 Definition of extriangulated categories

Definition 2.12. We call the pair $(\mathbb{E}, \mathfrak{s})$ an *external triangulation* of $\mathscr{C}$ if it satisfies the following conditions. In this case, we call $\mathfrak{s}$ an $\mathbb{E}$-*triangulation of* $\mathscr{C}$, and call the triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ an *externally triangulated category*, or for short, *extriangulated category*.

(ET1) $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ is a biadditive functor.

(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.

(ET3) Let $\delta \in \mathbb{E}(C,A)$ and $\delta' \in \mathbb{E}(C',A')$ be any pair of $\mathbb{E}$-extensions, realized as $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. For any commutative square in $\mathscr{C}$

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] & C \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \end{tikzcd} \tag{2}$$

there exists a morphism $(a,c)\colon \delta \to \delta'$ which is realized by (a,b,c).

(ET3)$^{\mathrm{op}}$ Let $\delta \in \mathbb{E}(C,A)$ and $\delta' \in \mathbb{E}(C',A')$ be any pair of $\mathbb{E}$-extensions, realized by $A \xrightarrow{x} B \xrightarrow{y} C$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C'$ respectively. For any commutative square in $\mathscr{C}$

$$\begin{tikzcd} A \arrow[r, "x"] & B \arrow[r, "y"] \arrow[d, "b"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "c"] \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \end{tikzcd}$$

there exists a morphism $(a,c)\colon \delta \to \delta'$ which is realized by (a,b,c).

(ET4) Let (A,δ,D) and (B,δ',F) be $\mathbb{E}$-extensions respectively realized by $A \xrightarrow{f} B \xrightarrow{f'} D$ and $B \xrightarrow{g} C \xrightarrow{g'} F$. Then there exist an object $E \in \mathscr{C}$, a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "f'"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & D \arrow[d, "d"] \\ A \arrow[r, "h"'] & C \arrow[r, "h'"'] \arrow[d, "g'"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[d, "e"] \\ & F \arrow[r, equal] & F \end{tikzcd} \tag{3}$$

in $\mathscr{C}$, and an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(E,A)$ realized by $A \xrightarrow{h} C \xrightarrow{h'} E$, which satisfy the following compatibilities.

(i) $D \xrightarrow{d} E \xrightarrow{e} F$ realizes $\mathbb{E}(F,f')(\delta')$,

(ii) $\mathbb{E}(d,A)(\delta'') = \delta$,

(iii) $\mathbb{E}(E,f)(\delta'') = \mathbb{E}(e,B)(\delta')$.

By (iii), $(f,e)\colon \delta'' \to \delta'$ is a morphism of $\mathbb{E}$-extensions, realized by

$$(f,\mathrm{id}_C,e)\colon [A \xrightarrow{h} C \xrightarrow{h'} E] \to [B \xrightarrow{g} C \xrightarrow{g'} F].$$

(ET4)$^{\mathrm{op}}$ Dual of (ET4) (see Remark 2.22).

Example 2.13. Exact categories (with a condition concerning the smallness) and triangulated categories are examples of extriangulated categories. See also Remark 2.18. We briefly show how an exact category can be viewed as an extriangulated category. As for triangulated categories, see the construction in Proposition 3.22. For the definition and basic properties of an exact category, see [Bu] and [Ke]. Let $A, C \in \mathscr{C}$ be any pair of objects. Remark that, as shown in [Bu, Corollary 3.2], for any morphism of short exact sequences (=*conflations* in [Ke]) of the form

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, equal] \\ A \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C \end{tikzcd},$$

the morphism b in the middle automatically becomes an isomorphism. Consider the same equivalence relation as in Definition 2.7, and define the class $\mathrm{Ext}^1(C,A)$ to be the collection of all equivalence classes of short exact sequences of the form $A \xrightarrow{x} B \xrightarrow{y} C$. We denote the equivalence class by $[A \xrightarrow{x} B \xrightarrow{y} C]$ as before.

This becomes a small set, for example in the following cases.

- $\mathscr{C}$ is skeletally small.
- $\mathscr{C}$ has enough projectives or injectives.

In such a case, we obtain a biadditive functor $\mathrm{Ext}^1\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$, as stated in [S, Definitions 5.1]. Its structure is given as follows.

- For any $\delta = [A \xrightarrow{x} B \xrightarrow{y} C] \in \mathrm{Ext}^1(C,A)$ and any $a \in \mathscr{C}(A,A')$, take a pushout in $\mathscr{C}$, to obtain a morphism of short exact sequences

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[r, "y"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, equal] \\ A' \arrow[r, "m"'] & M \arrow[r, "e"'] & C \end{tikzcd}.$$

  This gives $\mathrm{Ext}^1(C,a)(\delta) = a_*\delta = [A' \xrightarrow{m} M \xrightarrow{e} C]$.

- For any $c \in \mathscr{C}(C',C)$, the map $\mathrm{Ext}^1(c,A) = c^*\colon \mathrm{Ext}^1(C,A) \to \mathrm{Ext}^1(C',A)$ is defined dually by using pullbacks.

- The zero element in $\mathrm{Ext}^1(C,A)$ is given by the split short exact sequence

$$0 = [A \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} A \oplus C \xrightarrow{[0\ 1]} C].$$

  For any pair $\delta_1 = [A \xrightarrow{x_1} B_1 \xrightarrow{y_1} C], \delta_2 = [A \xrightarrow{x_2} B_2 \xrightarrow{y_2} C] \in \mathrm{Ext}^1(C,A)$, its sum $\delta_1 + \delta_2$ is given by the Baer sum

$$\Delta_C^*(\nabla_A)_*(\delta_1 \oplus \delta_2) = \Delta_C^*(\nabla_A)_*([A \oplus A \xrightarrow{x_1 \oplus x_2} B_1 \oplus B_2 \xrightarrow{y_1 \oplus y_2} C \oplus C]).$$

This shows (ET1). Define the realization $\mathfrak{s}(\delta)$ of $\delta = [A \xrightarrow{x} B \xrightarrow{y} C]$ to be δ itself. Then (ET2) is trivially satisfied. For (ET3) and (ET4), the following fact is useful.

Fact 2.14. *([Bu, Proposition 3.1]) For any morphism,* $(*)$ *below, of short exact sequences in* $\mathscr{C}$*, there exists a commutative diagram* $(**)$ *whose middle row is also a short exact sequence, the upper-left square is a pushout, and the lower-right square is a pullback.*

Remark that this means $a_*[A \xrightarrow{x} B \xrightarrow{y} C] = c^*[A' \xrightarrow{x'} B' \xrightarrow{y'} C']$.

$$
(*)\quad
\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "c"] \\
A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C'
\end{tikzcd}
\qquad
(**)\quad
\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[r, "y"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, equal] \\
A' \arrow[r, "m"'] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & {}^{\exists}M \arrow[r, "e"'] \arrow[d] \arrow[dr, phantom, "\mathrm{PB}"] & C \arrow[d, "c"] \\
A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C'
\end{tikzcd}
$$

By Fact 2.14, (ET3) follows immediately from the universality of cokernel. Similarly, (ET4) follows from [Bu, Lemma 3.5]. Dually for (ET3)$^{\mathrm{op}}$ and (ET4)$^{\mathrm{op}}$.

2.4 Terminology in an extriangulated category

To allow an argument with familiar terms, we introduce terminology from both exact categories and triangulated categories (cf. [Ke, Bu, Ne]).

Definition 2.15. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be a triplet satisfying (ET1) and (ET2).

(1) A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a *conflation* if it realizes some $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. For the ambiguity of such an $\mathbb{E}$-extension, see Corollary 3.8.

(2) A morphism $f \in \mathscr{C}(A, B)$ is called an *inflation* if it admits some conflation $A \xrightarrow{f} B \to C$. For the ambiguity of such a conflation, see Remark 3.10.

(3) A morphism $f \in \mathscr{C}(A, B)$ is called a *deflation* if it admits some conflation $K \to A \xrightarrow{f} B$.

Remark 2.16. *Condition* (ET4) *implies that inflations are closed under composition. Dually,* (ET4)$^{\mathrm{op}}$ *implies the composition-closedness of deflations.*

Definition 2.17. Let $\mathcal{D} \subseteq \mathscr{C}$ be a full additive subcategory, closed under isomorphisms. The subcategory $\mathcal{D}$ is said to be *extension-closed* if, for any conflation $A \to B \to C$ which satisfies $A, C \in \mathcal{D}$, then $B \in \mathcal{D}$.

The following can be checked in a straightforward way.

Remark 2.18. *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, and let $\mathcal{D} \subseteq \mathscr{C}$ be an extension-closed subcategory. If we define $\mathbb{E}_{\mathcal{D}}$ to be the restriction of $\mathbb{E}$ onto $\mathcal{D}^{\mathrm{op}} \times \mathcal{D}$, and define $\mathfrak{s}_{\mathcal{D}}$ by restricting $\mathfrak{s}$, then $(\mathcal{D}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}})$ becomes an extriangulated category.*

Definition 2.19. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be a triplet satisfying (ET1) and (ET2).

(1) If a conflation $A \xrightarrow{x} B \xrightarrow{y} C$ realizes $\delta \in \mathbb{E}(C, A)$, we call the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ an $\mathbb{E}$-*triangle*, and write it in the following way.

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow} \tag{4}$$

(2) Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ be any pair of $\mathbb{E}$-triangles. If a triplet (a, b, c) realizes $(a, c)\colon \delta \to \delta'$ as in (1), then we write it as

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, "c"] & {} \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \arrow[r, dashed, "\delta'"'] & {} \end{tikzcd}$$

and call (a, b, c) a *morphism of* $\mathbb{E}$-*triangles*.

Caution 2.20. *Although the abbreviated expression* (4) *looks superficially asymmetric, we remark that the definition of an extriangulated category is completely self-dual.*

Remark 2.21. (ET3) *means that any commutative square* (2) *bridging* $\mathbb{E}$-*triangles can be completed into a morphism of* $\mathbb{E}$-*triangles. Dually for* (ET3)$^{\mathrm{op}}$.

Condition $(*)$ *in Definition 2.9 means that any morphism of* $\mathbb{E}$-*extensions can be realized by a morphism of* $\mathbb{E}$-*triangles.*

In the above terminology, condition (ET4)$^{\mathrm{op}}$ can be stated as follows.

Remark 2.22. (*Paraphrase of* (ET4)$^{\mathrm{op}}$) *Let* $D \xrightarrow{f'} A \xrightarrow{f} B \overset{\delta}{\dashrightarrow}$ *and* $F \xrightarrow{g'} B \xrightarrow{g} C \overset{\delta'}{\dashrightarrow}$ *be* $\mathbb{E}$*-triangles. Then there exist an* $\mathbb{E}$*-triangle* $E \xrightarrow{h'} A \xrightarrow{h} C \overset{\delta''}{\dashrightarrow}$ *and a commutative diagram in* $\mathscr{C}$*, as on the left below, satisfying the compatibilities on the right:*

$$\begin{tikzcd}
D \arrow[r, "d"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[r, "e"] \arrow[d, "h'"] \arrow[dr, phantom, "\circlearrowright"] & F \arrow[d, "g'"] \\
D \arrow[r, "f'"'] & A \arrow[r, "f"'] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[d, "g"] \\
& C \arrow[r, equal] & C
\end{tikzcd}$$

(i) $D \xrightarrow{d} E \xrightarrow{e} F \overset{g'^*\delta}{\dashrightarrow}$ *is an* $\mathbb{E}$*-triangle*,

(ii) $\delta' = e_*\delta''$,

(iii) $d_*\delta = g^*\delta''$.

3. Fundamental properties

3.1 Associated exact sequence

In this section, we will associate exact sequences of natural transformations

$$\mathscr{C}(C,-) \overset{\mathscr{C}(y,-)}{\Longrightarrow} \mathscr{C}(B,-) \overset{\mathscr{C}(x,-)}{\Longrightarrow} \mathscr{C}(A,-) \overset{\delta^\sharp}{\Longrightarrow} \mathbb{E}(C,-) \overset{\mathbb{E}(y,-)}{\Longrightarrow} \mathbb{E}(B,-) \overset{\mathbb{E}(x,-)}{\Longrightarrow} \mathbb{E}(A,-)$$

$$\mathscr{C}(-,A) \overset{\mathscr{C}(-,x)}{\Longrightarrow} \mathscr{C}(-,B) \overset{\mathscr{C}(-,y)}{\Longrightarrow} \mathscr{C}(-,C) \overset{\delta_\sharp}{\Longrightarrow} \mathbb{E}(-,A) \overset{\mathbb{E}(-,x)}{\Longrightarrow} \mathbb{E}(-,B) \overset{\mathbb{E}(-,y)}{\Longrightarrow} \mathbb{E}(-,C)$$

to any $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ in an extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ (Corollary 3.12). Here, $\delta_\sharp$ and $\delta^\sharp$ are defined in the following.

Definition 3.1. Assume $\mathscr{C}$ and $\mathbb{E}$ satisfy (ET1). By Yoneda's lemma, any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$ induces natural transformations $\delta_\sharp \colon \mathscr{C}(-,C) \Rightarrow \mathbb{E}(-,A)$ and $\delta^\sharp \colon \mathscr{C}(A,-) \Rightarrow \mathbb{E}(C,-)$. For any $X \in \mathscr{C}$, these $(\delta_\sharp)_X$ and $\delta^\sharp_X$ are given as follows.

(1) $(\delta_\sharp)_X \colon \mathscr{C}(X,C) \to \mathbb{E}(X,A)\ ;\ f \mapsto f^*\delta$.

(2) $\delta^\sharp_X \colon \mathscr{C}(A,X) \to \mathbb{E}(C,X)\ ;\ g \mapsto g_*\delta$.

We abbreviately denote $(\delta_\sharp)_X(f)$ and $\delta^\sharp_X(g)$ by $\delta_\sharp f$ and $\delta^\sharp g$, when there is no confusion.

Lemma 3.2. Assume $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1),(ET2),(ET3),(ET3)$^{\mathrm{op}}$. Then for any $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, the following hold:
(1) $y \circ x = 0$, (2) $x_*\delta(= \delta^\sharp x) = 0$ and (3) $y^*\delta(= \delta_\sharp y) = 0$.

Proof. (1) By (ET2), the conflation $A \xrightarrow{\mathrm{id}} A \to 0$ realizes $0 \in \mathbb{E}(0, A)$. Applying (ET3) similarly as for triangulated categories gives the result.

(2) Similarly, applying (ET3) we obtain a morphism of $\mathbb{E}$-extensions $(x, 0) \colon \delta \to 0$. Especially we have $x_*\delta = 0$. (3) is dual to (2). $\square$

Proposition 3.3. Assume $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1),(ET2). Then the following are equivalent.

(1) $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET3) and (ET3)$^{\mathrm{op}}$.

(2) For any $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, the following sequences of natural transformations are exact:

(2-i) $\mathscr{C}(C, -) \overset{\mathscr{C}(y,-)}{\Longrightarrow} \mathscr{C}(B, -) \overset{\mathscr{C}(x,-)}{\Longrightarrow} \mathscr{C}(A, -) \overset{\delta^\sharp}{\Longrightarrow} \mathbb{E}(C, -) \overset{\mathbb{E}(y,-)}{\Longrightarrow} \mathbb{E}(B, -)$ in $\mathrm{Mod}(\mathscr{C})$. Here $\mathrm{Mod}(\mathscr{C})$ denotes the abelian category of additive functors from $\mathscr{C}$ to Ab.

(2-ii) $\mathscr{C}(-, A) \overset{\mathscr{C}(-,x)}{\Longrightarrow} \mathscr{C}(-, B) \overset{\mathscr{C}(-,y)}{\Longrightarrow} \mathscr{C}(-, C) \overset{\delta_\sharp}{\Longrightarrow} \mathbb{E}(-, A) \overset{\mathbb{E}(-,x)}{\Longrightarrow} \mathbb{E}(-, B)$ in $\mathrm{Mod}(\mathscr{C}^{\mathrm{op}})$.

Remark 3.4. *In the above* (2-i)*, the category* $\mathrm{Mod}(\mathscr{C})$ *is not locally small in general. The "exactness" of the sequence in* (2-i) *simply means that*

$$\mathscr{C}(C, X) \xrightarrow{\mathscr{C}(y,X)} \mathscr{C}(B, X) \xrightarrow{\mathscr{C}(x,X)} \mathscr{C}(A, X) \xrightarrow{\delta^\sharp_X} \mathbb{E}(C, X) \xrightarrow{\mathbb{E}(y,X)} \mathbb{E}(B, X)$$

is exact in Ab *for any* $X \in \mathscr{C}$*. Similarly for* (2-ii).

Proof of Proposition 3.3. First we assume (1). We only show the exactness of (2-i), since (2-ii) can be shown dually. By Lemma 3.2, composition of any consecutive morphisms in (2-i) is equal to 0. Let us show the exactness of $\mathscr{C}(C, X) \xrightarrow{\mathscr{C}(y,X)} \mathscr{C}(B, X) \xrightarrow{\mathscr{C}(x,X)} \mathscr{C}(A, X) \xrightarrow{\delta^\sharp_X} \mathbb{E}(C, X) \xrightarrow{\mathbb{E}(y,X)} \mathbb{E}(B, X)$ for any $X \in \mathscr{C}$.

Exactness at $\mathscr{C}(B, X)$ is shown similarly as for triangulated categories.

Exactness at $\mathscr{C}(A,X)$

Let $a\in\mathscr{C}(A,X)$ be any morphism satisfying $\delta_X^\sharp(a)=a_*\delta=0$. This means that $(a,0)\colon\delta\to 0$ is a morphism of $\mathbb{E}$-extensions. Since $\mathfrak{s}$ realizes $\mathbb{E}$, there exists $b\in\mathscr{C}(B,X)$ which gives the following morphism of $\mathbb{E}$-triangles.

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d] & {} \\ X \arrow[r, "\mathrm{id}_X"'] & X \arrow[r] & 0 \arrow[r, dashed, "0"'] & {} \end{tikzcd}$$

Especially we have $a=b\circ x=\mathscr{C}(x,X)(b)$.

Exactness at $\mathbb{E}(C,X)$

Let $\theta\in\mathbb{E}(C,X)$ be any $\mathbb{E}$-extension satisfying $\mathbb{E}(y,X)(\theta)=y^*\theta=0$. Realize them as $\mathbb{E}$-triangles $X\xrightarrow{f}Y\xrightarrow{g}C\overset{\theta}{\dashrightarrow}$ and $X\xrightarrow{m}Z\xrightarrow{e}B\overset{y^*\theta}{\dashrightarrow}$. Then the morphism $(\mathrm{id}_X,y)\colon y^*\theta\to\theta$ can be realized by $(*)$ below

$$(*)\quad\begin{tikzcd} X \arrow[r, "m"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[r, "e"] \arrow[d, "e'"] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, dashed, "y^*\theta"] \arrow[d, "y"] & {} \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & C \arrow[r, dashed, "\theta"'] & {} \end{tikzcd}
\qquad
(**)\quad\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "e'\circ s"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] & C \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & C \arrow[r, dashed, "\theta"'] & {} \end{tikzcd}$$

with some $e'\in\mathscr{C}(Z,Y)$. Since $y^*\theta$ splits by assumption, e has a section s. Applying (ET3)$^{\mathrm{op}}$ to $(**)$ above, we obtain $a\in\mathscr{C}(A,X)$ which gives a morphism $(a,\mathrm{id}_C)\colon\delta\to\theta$. This means $\theta=a_*\delta=\delta^\sharp a$.

Conversely, let us assume (2) and show (ET3). Let $A\xrightarrow{x}B\xrightarrow{y}C\overset{\delta}{\dashrightarrow}$ and $A'\xrightarrow{x'}B'\xrightarrow{y'}C'\overset{\delta'}{\dashrightarrow}$ be any pair of $\mathbb{E}$-triangles. Suppose that we are given a commutative diagram (α) below.

$$(\alpha)\quad\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] & C \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \end{tikzcd}
\qquad
(\beta)\quad\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b'"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, "c'"] & {} \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \arrow[r, dashed, "\delta'"'] & {} \end{tikzcd}$$

Remark that $\mathbb{E}(-,x)\circ\delta_\sharp=0$ is equivalent to $x_*\delta=0$ by Yoneda's lemma. Similarly, $y'^*\delta'$ vanishes. By the exactness of the sequence $\mathscr{C}(C,C')\xrightarrow{(\delta'_\sharp)_C}\mathbb{E}(C,A')\xrightarrow{\mathbb{E}(C,x')}\mathbb{E}(C,B')$ and the equality $\mathbb{E}(C,x')(a_*\delta)=x'_*a_*\delta=b_*x_*\delta=0$

there is $c' \in \mathscr{C}(C, C')$ satisfying $a_*\delta = \delta'_\sharp c' = c'^*\delta'$. Thus $(a, c)\colon \delta \to \delta'$ is a morphism of $\mathbb{E}$-extensions. Take its realization as in (β) above.

Then by the exactness of $\mathscr{C}(C, B') \xrightarrow{\mathscr{C}(y,B')} \mathscr{C}(B, B') \xrightarrow{\mathscr{C}(x,B')} \mathscr{C}(A, B')$ and the equality $(b - b') \circ x = x' \circ a - x' \circ a = 0$, there exists $c'' \in \mathscr{C}(C, B')$ satisfying $c'' \circ y = b - b'$. If we put $c = c' + y' \circ c''$, this satisfies

$$c \circ y = c' \circ y + y' \circ c'' \circ y = y' \circ b' + (y' \circ b - y' \circ b') = y' \circ b,$$
$$c^*\delta' = c'^*\delta' + c''^* y'^*\delta' = a_*\delta.$$

Dually, (2) implies (ET3)$^{\mathrm{op}}$. $\square$

The following two corollaries are direct consequences of the long exact sequences of Proposition 3.3.

Corollary 3.5. Assume $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1),(ET2),(ET3),(ET3)$^{\mathrm{op}}$. Let

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, "c"] & {} \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \arrow[r, dashed, "\delta'"'] & {} \end{tikzcd} \tag{5}$$

be any morphism of $\mathbb{E}$-triangles. Then a factors through x if and only if $a_*\delta = c^*\delta' = 0$ if and only if c factors through y'.

In particular, in the case $\delta = \delta'$ and $(a, b, c) = (\mathrm{id}, \mathrm{id}, \mathrm{id})$, we obtain

$$x \text{ has a retraction } \Leftrightarrow \delta \text{ splits } \Leftrightarrow y \text{ has a section.}$$

Corollary 3.6. Assume $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1),(ET2),(ET3),(ET3)$^{\mathrm{op}}$. The following holds for any morphism (a, b, c) of $\mathbb{E}$-triangles.

(1) If a and c are isomorphisms in $\mathscr{C}$ (equivalently, if (a, c) is an isomorphism in $\mathbb{E}\text{-}\mathrm{Ext}(\mathscr{C})$ in Definition 2.3), then so is b.

(2) If a and b are isomorphisms in $\mathscr{C}$, then so is c.

(3) If b and c are isomorphisms in $\mathscr{C}$, then so is a.

Proposition 3.7. Assume $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1),(ET2),(ET3),(ET3)$^{\mathrm{op}}$ and let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ be any $\mathbb{E}$-triangle. If $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C', C)$ are isomorphisms, then $A' \xrightarrow{x \circ a^{-1}} B \xrightarrow{c^{-1} \circ y} C' \overset{a_* c^* \delta}{\dashrightarrow}$ is again an $\mathbb{E}$-triangle.

Proof. Put $\mathfrak{s}(a_*c^*\delta) = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. Since $(c^{-1})^*\big(a_*c^*\delta\big) = (c \circ c^{-1})^*a_*\delta = a_*\delta$, we see that $(a, c^{-1})\colon \delta \to c^*a_*\delta$ is a morphism of $\mathbb{E}$-extensions. Take a morphism of $\mathbb{E}$-triangles (a, b, c^{-1}) realizing (a, c^{-1}). Then b is an isomorphism by Corollary 3.6. Thus $[A' \xrightarrow{x'} B' \xrightarrow{y'} C'] = [A' \xrightarrow{x\circ a^{-1}} B \xrightarrow{c^{-1}\circ y} C']$. □

Corollary 3.8. Assume $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1),(ET2),(ET3),(ET3)$^{\mathrm{op}}$. Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ be any $\mathbb{E}$-triangle. Then for any $\delta' \in \mathbb{E}(C, A)$, the following are equivalent.

(1) $\mathfrak{s}(\delta) = \mathfrak{s}(\delta')$.

(2) $\delta' = a_*\delta$ for some automorphism $a \in \mathscr{C}(A, A)$ satisfying $x \circ a = x$.

(3) $\delta' = c^*\delta$ for some automorphism $c \in \mathscr{C}(C, C)$ satisfying $c \circ y = y$.

(4) $\delta' = a_*c^*\delta$ for some automorphisms $a \in \mathscr{C}(A, A)$ and $c \in \mathscr{C}(C, C)$ satisfying $x \circ a = x$ and $c \circ y = y$.

Proof. $(2) \Rightarrow (4)$ is trivial. Similarly for $(3) \Rightarrow (4)$. Proposition 3.7 shows $(4) \Rightarrow (1)$. The implications $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$ follow from Corollary 3.6. □

Definition 3.9. Assume $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1),(ET2),(ET3),(ET3)$^{\mathrm{op}}$.

(1) For an inflation $f \in \mathscr{C}(A, B)$, take a conflation $A \xrightarrow{f} B \to C$, and denote this C by $\mathrm{Cone}(f)$.

(2) For a deflation $f \in \mathscr{C}(A, B)$, take a conflation $K \to A \xrightarrow{f} B$. We denote this K by $\mathrm{CoCone}(f)$.

$\mathrm{Cone}(f)$ is determined uniquely up to isomorphism by the following remark. They are not functorial in general, as the case of triangulated category suggests. Dually for $\mathrm{CoCone}(f)$.

Remark 3.10. *Let $f \in \mathscr{C}(A, B)$ be an inflation, and suppose*

$$A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}, \quad A \xrightarrow{f} B \xrightarrow{g'} C' \overset{\delta'}{\dashrightarrow}$$

are $\mathbb{E}$-triangles. Then, by (ET3) *applied to* $(\mathrm{id}_A, \mathrm{id}_B)\colon f \to f$, *there exists* $c \in \mathscr{C}(C, C')$ *which gives a morphism of* $\mathbb{E}$*-triangles* $(\mathrm{id}_A, \mathrm{id}_B, c)$*. By Corollary 3.6, this* c *is an isomorphism.*

Proposition 3.11. Assume $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1),(ET2),(ET3),(ET3)$^{\mathrm{op}}$. Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle. Then, we have the following.

(1) If $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET4), then $\mathbb{E}(-, A) \overset{\mathbb{E}(-,x)}{\Longrightarrow} \mathbb{E}(-, B) \overset{\mathbb{E}(-,y)}{\Longrightarrow} \mathbb{E}(-, C)$ is exact.

(2) If $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET4)$^{\mathrm{op}}$, then $\mathbb{E}(C, -) \overset{\mathbb{E}(y,-)}{\Longrightarrow} \mathbb{E}(B, -) \overset{\mathbb{E}(x,-)}{\Longrightarrow} \mathbb{E}(A, -)$ is exact.

Proof. (1) $\mathbb{E}(-, y) \circ \mathbb{E}(-, x) = 0$ follows from Lemma 3.2. Let $X \in \mathscr{C}$ be any object. Let $\theta \in \mathbb{E}(X, B)$ be any $\mathbb{E}$-extension, realized by an $\mathbb{E}$-triangle $B \xrightarrow{f} Y \xrightarrow{g} X \overset{\theta}{\dashrightarrow}$. By (ET4), there exist $E \in \mathscr{C}, \theta' \in \mathbb{E}(E, A)$ and a commutative diagram which satisfies

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "d"] \\
A \arrow[r, "h"'] & Y \arrow[r, "h'"'] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[d, "e"] \\
& X \arrow[r, equal] & X
\end{tikzcd}$$

$$\mathfrak{s}(y_*\theta) = [C \xrightarrow{d} E \xrightarrow{e} X],$$

$$\mathfrak{s}(\theta') = [A \xrightarrow{h} Y \xrightarrow{h'} E],$$

$$x_*\theta' = e^*\theta.$$

Thus if $\mathbb{E}(X, y)(\theta) = y_*\theta = 0$, then e has a section $s \in \mathscr{C}(X, E)$. If we put $\rho = s^*\theta'$, then this satisfies $\mathbb{E}(x, X)(\rho) = x_*s^*\theta' = s^*x_*\theta' = s^*e^*\theta = \theta$.

(2) is dual to (1). $\square$

Corollary 3.12. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. For any $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, the following sequences of natural transformations are exact.

$$\mathscr{C}(C, -) \overset{\mathscr{C}(y,-)}{\Longrightarrow} \mathscr{C}(B, -) \overset{\mathscr{C}(x,-)}{\Longrightarrow} \mathscr{C}(A, -) \overset{\delta^\sharp}{\Longrightarrow} \mathbb{E}(C, -) \overset{\mathbb{E}(y,-)}{\Longrightarrow} \mathbb{E}(B, -) \overset{\mathbb{E}(x,-)}{\Longrightarrow} \mathbb{E}(A, -)$$

$$\mathscr{C}(-, A) \overset{\mathscr{C}(-,x)}{\Longrightarrow} \mathscr{C}(-, B) \overset{\mathscr{C}(-,y)}{\Longrightarrow} \mathscr{C}(-, C) \overset{\delta_\sharp}{\Longrightarrow} \mathbb{E}(-, A) \overset{\mathbb{E}(-,x)}{\Longrightarrow} \mathbb{E}(-, B) \overset{\mathbb{E}(-,y)}{\Longrightarrow} \mathbb{E}(-, C)$$

Proof. This immediately follows from Propositions 3.3 and 3.11. □

The following lemma shows that the upper-right square of Diagram (3) obtained by (ET4) is a weak pushout.

Lemma 3.13. Let (3) be a commutative diagram in $\mathscr{C}$, where

$$A \xrightarrow{f} B \xrightarrow{f'} D \overset{d^*\delta''}{\dashrightarrow}, \quad B \xrightarrow{g} C \xrightarrow{g'} F \overset{\delta'}{\dashrightarrow}$$
$$A \xrightarrow{h} C \xrightarrow{h'} E \overset{\delta''}{\dashrightarrow}, \quad D \xrightarrow{d} E \xrightarrow{e} F \overset{f'_*\delta'}{\dashrightarrow}$$

are $\mathbb{E}$-triangles, which satisfy $e^*\delta' = f_*\delta''$.

Then, for any commutative square $y \circ f' = x \circ g$ in $\mathscr{C}$, there exists $z \in \mathscr{C}(E, Y)$ which makes the following diagram commutative.

$$\begin{tikzcd}
B \arrow[r, "f'"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowright"] & D \arrow[d, "d"] \arrow[ddr, bend left, "y"] & \\
C \arrow[r, "h'"'] \arrow[drr, bend right, "x"'] & E \arrow[dr, dashed, "z"] & \\
& & Y
\end{tikzcd}$$

Proof. By $(f'_*\delta')^\sharp(y) = y_*f'_*\delta' = x_*g_*\delta' = 0$ and the exactness of

$$\mathscr{C}(E, Y) \xrightarrow{\mathscr{C}(d,Y)} \mathscr{C}(D, Y) \xrightarrow{(f'_*\delta')^\sharp_Y} \mathbb{E}(F, Y) \to 0,$$

there exists $z_1 \in \mathscr{C}(E, Y)$ satisfying $z_1 \circ d = y$. Then by $(x - z_1 \circ h') \circ g = y \circ f' - z_1 \circ d \circ f' = 0$ and the exactness of

$$\mathscr{C}(F, Y) \xrightarrow{\mathscr{C}(g',Y)} \mathscr{C}(C, Y) \xrightarrow{\mathscr{C}(g,Y)} \mathscr{C}(B, Y),$$

there exists $z_2 \in \mathscr{C}(F, Y)$ satisfying $z_2 \circ g' = x - z_1 \circ h'$. If we put $z = z_1 + z_2 \circ e$, this satisfies the desired commutativity. □

3.2 Shifted octahedrons

Condition (ET4) in Definition 2.12 is an analog of the octahedron axiom (TR4) for a triangulated category. As in the case of a triangulated category, we can make it slightly more rigid as follows.

Lemma 3.14. Let $(\mathscr{C},\mathbb{E},\mathfrak{s})$ be an extriangulated category. Let

$A \xrightarrow{f} B \xrightarrow{f'} D \overset{\delta_f}{\dashrightarrow}$, $B \xrightarrow{g} C \xrightarrow{g'} F \overset{\delta_g}{\dashrightarrow}$, $A \xrightarrow{h} C \xrightarrow{h_0} E_0 \overset{\delta_h}{\dashrightarrow}$

be any triplet of $\mathbb{E}$-triangles satisfying $h = g \circ f$. Then there are morphisms d_0, e_0 in $\mathscr{C}$ which make the diagram below commutative, with the following compatibilities:

(i) $D \overset{d_0}{\to} E_0 \overset{e_0}{\to} F \overset{f'_*(\delta_g)}{\dashrightarrow}$ is an $\mathbb{E}$-triangle,

(ii) $d_0^*(\delta_h) = \delta_f$,

(iii) $f_*(\delta_h) = e_0^*(\delta_g)$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "f'"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & D \arrow[d, "d_0"] \\ A \arrow[r, "h"'] & C \arrow[r, "h_0"'] \arrow[d, "g'"'] \arrow[dr, phantom, "\circlearrowright"] & E_0 \arrow[d, "e_0"] \\ & F \arrow[r, equal] & F \end{tikzcd} \tag{6}$$

Proof. By (ET4), there exist an object $E \in \mathscr{C}$, a commutative diagram (3) in $\mathscr{C}$, and an $\mathbb{E}$-triangle $A \xrightarrow{h} C \xrightarrow{h'} E \overset{\delta''}{\dashrightarrow}$, which satisfy the following compatibilities. (i′) $D \xrightarrow{d} E \xrightarrow{e} F \overset{f'_*(\delta_g)}{\dashrightarrow}$ is an $\mathbb{E}$-triangle, (ii′) $d^*(\delta'') = \delta_f$, (iii′) $f_*(\delta'') = e^*(\delta_g)$. By Remark 3.10, we obtain a morphism of $\mathbb{E}$-triangles

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, "h'"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[r, dashed, "\delta''"] \arrow[d, "u"] & {} \\ A \arrow[r, "h"'] & C \arrow[r, "h_0"'] & E_0 \arrow[r, dashed, "\delta_h"'] & {} \end{tikzcd}$$

in which u is an isomorphism. In particular we have $\delta'' = u^*(\delta_h)$. If we put $d_0 = u \circ d$ and $e_0 = e \circ u^{-1}$, then the commutativity of (6) follows from that of (3). By the definition of the equivalence relation, we have $[D \xrightarrow{d_0} E_0 \xrightarrow{e_0} F] = [D \xrightarrow{d} E \xrightarrow{e} F]$. It is straightforward to check that (i′),(ii′),(iii′) imply (i),(ii),(iii). □

Proposition 3.15. Let $(\mathscr{C},\mathbb{E},\mathfrak{s})$ be an extriangulated category. Then the following (and its dual) holds.

Let C be any object, and let $A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \overset{\delta_1}{\dashrightarrow}$, $A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \overset{\delta_2}{\dashrightarrow}$ be any pair of $\mathbb{E}$-triangles. Then there is a commutative diagram in $\mathscr{C}$

which satisfies:

$$\mathfrak{s}(y_2^*\delta_1) = [A_1 \overset{m_1}{\to} M \overset{e_1}{\to} B_2],$$
$$\mathfrak{s}(y_1^*\delta_2) = [A_2 \overset{m_2}{\to} M \overset{e_2}{\to} B_1],$$
$$m_{1*}\delta_1 + m_{2*}\delta_2 = 0.$$

$$\begin{tikzcd} & A_2 \arrow[r, equal] \arrow[d, "m_2"'] \arrow[dr, phantom, "\circlearrowright"] & A_2 \arrow[d, "x_2"] \\ A_1 \arrow[r, "m_1"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[r, "e_1"] \arrow[d, "e_2"] \arrow[dr, phantom, "\circlearrowright"] & B_2 \arrow[d, "y_2"] \\ A_1 \arrow[r, "x_1"'] & B_1 \arrow[r, "y_1"'] & C \end{tikzcd} \tag{7}$$

Proof. By the additivity of $\mathfrak{s}$, we have $\mathfrak{s}(\delta_1 \oplus \delta_2) = [A_1 \oplus A_2 \overset{x_1\oplus x_2}{\longrightarrow} B_1 \oplus B_2 \overset{y_1\oplus y_2}{\longrightarrow} C \oplus C]$. Let $A_1 \underset{p_1}{\overset{\iota_1}{\rightleftarrows}} A_1 \oplus A_2 \underset{p_2}{\overset{\iota_2}{\leftrightarrows}} A_2$ be a biproduct in $\mathscr{C}$. Put $\mu = (\Delta_C)^*(\delta_1 \oplus \delta_2)$ and take $\mathfrak{s}(\mu) = [A_1 \oplus A_2 \overset{j}{\longrightarrow} M \overset{k}{\longrightarrow} C]$. Then μ satisfies

$$p_{1*}\mu = \delta_1 \ \text{ and } \ p_{2*}\mu = \delta_2. \tag{8}$$

Applying (ET4) to $\mathfrak{s}(0) = [A_1 \overset{\iota_1}{\longrightarrow} A_1 \oplus A_2 \overset{p_2}{\longrightarrow} A_2]$ and $\mathfrak{s}(\mu) = [A_1 \oplus A_2 \overset{j}{\longrightarrow} M \overset{k}{\longrightarrow} C]$, we obtain $B_2' \in \mathscr{C}$, $\theta_1 \in \mathbb{E}(B_2', A_1)$ and a commutative diagram such that:

$$\begin{tikzcd} A_1 \arrow[r, "\iota_1"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & A_1 \oplus A_2 \arrow[r, "p_2"] \arrow[d, "j"] \arrow[dr, phantom, "\circlearrowright"] & A_2 \arrow[d, "x_2'"] \\ A_1 \arrow[r, "m_1"'] & M \arrow[r, "e_1'"'] \arrow[d, "k"] \arrow[dr, phantom, "\circlearrowright"] & B_2' \arrow[d, "y_2'"] \\ & C \arrow[r, equal] & C \end{tikzcd}
\qquad
\begin{aligned} &\mathfrak{s}(p_{2*}\mu) = [A_2 \overset{x_2'}{\longrightarrow} B_2 \overset{y_2'}{\longrightarrow} C], \\ &x_2^*\theta_1 = 0, \\ &\mathfrak{s}(\theta_1) = [A_1 \overset{m_1}{\longrightarrow} M \overset{e_1'}{\longrightarrow} B_2'], \end{aligned}$$

and $(\iota_1, \mathrm{id}_M, y_2')$ is a morphism of $\mathbb{E}$-triangles. Especially, we have an equality $y_2'^*\mu = \iota_{1*}\theta_1$. In particular, we have

$$[A_2 \overset{x_2'}{\longrightarrow} B_2 \overset{y_2'}{\longrightarrow} C] = \mathfrak{s}(\delta_2) = [A_2 \overset{x_2}{\longrightarrow} B_2 \overset{y_2}{\longrightarrow} C].$$

Thus there is an isomorphism $b_2 \in \mathscr{C}(B_2, B_2')$ satisfying $b_2 \circ x_2 = x_2'$ and $y_2' \circ b_2 = y_2$. If we put $e_1 = b_2^{-1} \circ e_1'$, then $\mathfrak{s}(b_2^*\theta_1) = [A_1 \overset{m_1}{\longrightarrow} M \overset{e_1}{\longrightarrow} B_2]$

by Proposition 3.7. Thus we obtain a commutative diagram (a) below

$$
(\mathrm{a})\quad
\begin{tikzcd}
A_1 \arrow[r, "\iota_1"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & A_1 \oplus A_2 \arrow[r, "p_2"] \arrow[d, "j"'] \arrow[dr, phantom, "\circlearrowright"] & A_2 \arrow[d, "x_2"] \\
A_1 \arrow[r, "m_1"'] & M \arrow[r, "e_1"'] \arrow[d, "k"'] \arrow[dr, phantom, "\circlearrowright"] & B_2 \arrow[d, "y_2"] \\
& C \arrow[r, equal] & C
\end{tikzcd}
\qquad
(\mathrm{b})\quad
\begin{tikzcd}
A_2 \arrow[r, "\iota_2"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & A_1 \oplus A_2 \arrow[r, "p_1"] \arrow[d, "j"'] \arrow[dr, phantom, "\circlearrowright"] & A_1 \arrow[d, "x_1"] \\
A_2 \arrow[r, "m_2"'] & M \arrow[r, "e_2"'] \arrow[d, "k"'] \arrow[dr, phantom, "\circlearrowright"] & B_1 \arrow[d, "y_1"] \\
& C \arrow[r, equal] & C
\end{tikzcd}
$$

which satisfies $y_2^*\delta_1 = b_2^* y_2'^* p_{1*}\mu = b_2^* p_{1*} y_2'^* \mu = b_2^* p_{1*}\iota_{1*}\theta_1 = b_2^*\theta_1$. Thus we obtain $\mathfrak{s}(y_2^*\delta_1) = [A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2]$. Similarly, from $\mathfrak{s}(0) = [A_2 \xrightarrow{\iota_2} A_1 \oplus A_2 \xrightarrow{p_1} A_1]$ and $\mathfrak{s}(\mu) = [A_1 \oplus A_2 \xrightarrow{j} M \xrightarrow{k} C]$, we obtain a commutative diagram (b) above, which satisfies $\mathfrak{s}(y_1^*\delta_2) = [A_2 \xrightarrow{m_2} M \xrightarrow{e_2} B_1]$. Since $e_2 \circ m_1 = e_2 \circ j \circ \iota_1 = x_1 \circ p_1 \circ \iota_1 = x_1$, $e_1 \circ m_2 = e_1 \circ j \circ \iota_2 = x_2 \circ p_2 \circ \iota_2 = x_2$ and $y_2 \circ e_1 = k = y_1 \circ e_2$, diagram (7) is commutative. Moreover, we have

$$m_{1*}\delta_1 + m_{2*}\delta_2 = j_*\big(\iota_{1*}\delta_1 + \iota_{2*}\delta_2\big) = j_*\big((\iota_1 \circ p_1)_* + (\iota_2 \circ p_2)_*\big)(\mu) = j_*\mu = 0$$

by (8) and Lemma 3.2. □

Corollary 3.16. Let $x \in \mathscr{C}(A,B)$, $f \in \mathscr{C}(A,D)$ be any pair of morphisms. If x is an inflation, then so is $\begin{bmatrix} f \\ x \end{bmatrix} \in \mathscr{C}(A, D \oplus B)$. Dually for deflations.

Proof. Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle. Realize $f_*\delta$ by an $\mathbb{E}$-triangle $D \xrightarrow{d} E \xrightarrow{e} C \overset{f_*\delta}{\dashrightarrow}$. By Proposition 3.15, we obtain a commutative diagram $(*)$, below, made of $\mathbb{E}$-triangles satisfying $m_*\delta + k_*f_*\delta = 0$. Since $y^*f_*\delta = f_*y^*\delta = 0$, we may assume $M = D \oplus B$, $k = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, $\ell = [0\ 1]$, and take $p \in \mathscr{C}(M,D)$, $i \in \mathscr{C}(B,M)$ which make $D \underset{p}{\overset{k}{\rightleftarrows}} M \underset{\ell}{\overset{i}{\leftrightarrows}} B$ a biproduct. By the exactness of $\mathscr{C}(B,M) \xrightarrow{\mathscr{C}(x,M)} \mathscr{C}(A,M) \xrightarrow{\delta^\sharp} \mathbb{E}(C,M)$ and the equality $\delta^\sharp(m + k \circ f) = m_*\delta + k_*f_*\delta = 0$, there exists $b \in \mathscr{C}(B,M)$ satisfying $b \circ x = m + k \circ f$. Modifying $A \xrightarrow{m} M \xrightarrow{e} E$ by the automorphism

$$n = \begin{bmatrix} -1 & p \circ b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \circ (\mathrm{id}_M - k \circ p \circ b \circ \ell) \colon M \xrightarrow{\cong} M,$$

we obtain a conflation $A \overset{n\circ m}{\longrightarrow} D \oplus B \overset{e\circ n^{-1}}{\longrightarrow} E$. Then, since

$$\begin{aligned} p\circ(n\circ m) &= -p\circ(\mathrm{id}_M - k\circ p\circ b\circ\ell)\circ m \\ &= p\circ k\circ p\circ b\circ\ell\circ m - p\circ m \\ &= p\circ b\circ x - p\circ m \;=\; p\circ k\circ f \;=\; f \end{aligned}$$

and $\ell\circ(n\circ m) = \ell\circ(\mathrm{id}_M - k\circ p\circ b\circ\ell)\circ m = \ell\circ m = x$, we have $n\circ m = \begin{bmatrix} f \\ x \end{bmatrix}$.

$$(*)\qquad \begin{tikzcd} & A \arrow[r, equal] \arrow[d, "m"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[d, "x"] & \\ D \arrow[r, "k"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[r, "\ell"'] \arrow[d, "e"] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, dashed, "y^*f_*\delta"] \arrow[d, "y"] & {} \\ D \arrow[r, "d"'] & E \arrow[r, "e"'] \arrow[d, dashed, "e^*\delta"'] & C \arrow[r, dashed, "f_*\delta"'] \arrow[d, dashed, "\delta"] & {} \\ & {} & {} & \end{tikzcd}$$

□

Proposition 3.17. Suppose we are given $\mathbb{E}$-triangles $D \xrightarrow{f} A \xrightarrow{f'} C \overset{\delta_f}{\dashrightarrow}$, $A \xrightarrow{g} B \xrightarrow{g'} F \overset{\delta_g}{\dashrightarrow}$, $E \xrightarrow{h} B \xrightarrow{h'} C \overset{\delta_h}{\dashrightarrow}$ satisfying $h'\circ g = f'$. Then there is an $\mathbb{E}$-triangle $D \xrightarrow{d} E \xrightarrow{e} F \overset{\theta}{\dashrightarrow}$ which makes diagram (9) commutative in $\mathscr{C}$ and satisfy the following equalities:

(i) $d_*(\delta_f) = \delta_h$,

(ii) $f_*(\theta) = \delta_g$,

(iii) $g'^*(\theta) + h'^*(\delta_f) = 0$.

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "d"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[r, "f'"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, equal] \\ E \arrow[r, "h"] \arrow[d, "e"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "h'"] \arrow[d, "g'"] & C \\ F \arrow[r, equal] & F & \end{tikzcd} \qquad (9)$$

Proof. By axiom (ET4), we have two $\mathbb{E}$-triangles $D \overset{g\circ f}{\longrightarrow} B \xrightarrow{a} G \overset{\mu}{\dashrightarrow}$ and $C \xrightarrow{b} G \xrightarrow{c} F \overset{\nu}{\dashrightarrow}$ which make the following diagram commutative

$$
\begin{tikzcd}
D \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[r, "f'"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "b"] \\
D \arrow[r, "g\circ f"'] & B \arrow[r, "a"'] \arrow[d, "g'"'] \arrow[dr, phantom, "\circlearrowright"] & G \arrow[d, "c"] \\
& F \arrow[r, equal] & F
\end{tikzcd}
\tag{10}
$$

and satisfy $f'_*(\delta_g) = \nu,\ b^*\mu = \delta_f,\ c^*(\delta_g) = f_*\mu$. It follows from Lemma 3.2 that $\nu = f'_*(\delta_g) = h'_*g_*(\delta_g) = 0$. Thus, up to equivalence, we may assume $G = C \oplus F$, $b = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $c = [0\ 1]$ from the start. Then $a = \left[\begin{smallmatrix}a_1\\a_2\end{smallmatrix}\right] : B \to G = C \oplus F$ satisfies $a_1 \circ g = f'$ and $a_2 = g'$ by the commutativity of (10). Since $h' - a_1 \in \mathscr{C}(B, C)$ satisfies $(h' - a_1) \circ g = f' - f' = 0$, there exists $z \in \mathscr{C}(F, C)$ satisfying $z \circ g' = h' - a_1$. Put $z' = \left[\begin{smallmatrix}-z\\1\end{smallmatrix}\right]$. Applying the dual of Lemma 3.14 to the following diagram $(*)$ made of $\mathbb{E}$-triangles,

$$
(*)\quad
\begin{tikzcd}
& E \arrow[d, "h"'] & F \arrow[d, "z'"] & \\
D \arrow[r, "g\circ f"'] & B \arrow[r, "a"] \arrow[d, "h'"'] \arrow[dr, phantom, "\circlearrowright"] & G \arrow[r, dashed, "\mu"] \arrow[d, "{[1\ z]}"] & {} \\
& C \arrow[r, equal] \arrow[d, dashed, "\delta_h"'] & C \arrow[d, dashed, "0"] & \\
& {} & {} &
\end{tikzcd}
\qquad
(**)\quad
\begin{tikzcd}
D \arrow[r, "d"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[r, "e"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowright"] & F \arrow[d, "z'"] \\
D \arrow[r, "g\circ f"'] & B \arrow[r, "a"'] \arrow[d, "h'"'] \arrow[dr, phantom, "\circlearrowright"] & G \arrow[d, "{[1\ z]}"] \\
& C \arrow[r, equal] & C
\end{tikzcd}
$$

we obtain an $\mathbb{E}$-triangle $D \xrightarrow{d} E \xrightarrow{e} F \overset{\theta}{\dashrightarrow}$ which makes diagram $(**)$ above commutative, and satisfies $\theta = z'^*\mu,\ d_*\mu = [1\ z]^*(\delta_h)$. Then the commutativity of (9) can be checked in a straightforward way. Let us show the equalities (i),(ii),(iii).

(i) follows from $d_*(\delta_f) = d_*b^*\mu = b^*[1\ z]^*(\delta_h) = ([1\ z] \circ b)^*(\delta_h) = \delta_h$.

(ii) follows from the injectivity of $\mathbb{E}(c, A) = c^*$ and

$$
\begin{aligned}
c^*f_*(\theta) &= c^*f_*z'^*\mu \;=\; f_*(z' \circ c)^*\mu \\
&= f_*\begin{bmatrix} 0 & -z \\ 0 & 1 \end{bmatrix}^*\mu \;=\; f_*(1 - b \circ [1\ z])^*\mu \\
&= f_*\mu - f_*[1\ z]^*(\delta_f) \;=\; f_*\mu - [1\ z]^*f_*(\delta_f) \;=\; f_*\mu \;=\; c^*(\delta_g).
\end{aligned}
$$

(iii) follows from

$$\begin{aligned} g'^*(\theta) + h'^*(\delta_f) &= g'^* z'^* \mu + h'^* b^* \mu \ = \ \left(\begin{bmatrix} -z \circ g' \\ g' \end{bmatrix} + \begin{bmatrix} h' \\ 0 \end{bmatrix} \right)^* \mu \\ &= \begin{bmatrix} a_1 \\ g' \end{bmatrix}^* \mu \ = \ a^* \mu \ = \ 0. \end{aligned}$$

□

As in Example 2.13, an exact category (with some smallness assumption) can be regarded as an extriangulated category, whose inflations are monomorphic and whose deflations are epimorphic. Conversely, the following holds.

Corollary 3.18. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, in which any inflation is monomorphic, and any deflation is epimorphic. If we let $\mathscr{S}$ be the class of conflations given by the $\mathbb{E}$-triangles (see Definition 2.15), then $(\mathscr{C}, \mathscr{S})$ is an exact category in the sense of [Bu].

Proof. By the exact sequences obtained in Proposition 3.3, for any conflation $A \xrightarrow{x} B \xrightarrow{y} C$, the pair (A, x) gives a weak kernel of y. Since x is monomorphic by assumption, it is a kernel of y. Dually (C, y) gives a cokernel of x, and $A \xrightarrow{x} B \xrightarrow{y} C$ becomes a kernel-cokernel pair.

Thus $\mathscr{S}$ consists of some kernel-cokernel pairs. Moreover, it is closed under isomorphisms. Indeed, let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ be any $\mathbb{E}$-triangle, let $A' \xrightarrow{x'} B' \xrightarrow{y'} C'$ be a kernel-cokernel pair, and suppose that there are isomorphisms $a \in \mathscr{C}(A, A'), b \in \mathscr{C}(B, B')$ and $\in \mathscr{C}(C, C')$ satisfying $x' \circ a = b \circ x$ and $y' \circ b = c \circ y$. By Proposition 3.7, we obtain an $\mathbb{E}$-triangle $A' \xrightarrow{x \circ a^{-1}} B \xrightarrow{c \circ y} C' \overset{(c^{-1})^* a_* \delta}{\dashrightarrow}$. This gives $\mathfrak{s}(\delta) = [A' \xrightarrow{x \circ a^{-1}} B \xrightarrow{c \circ y} C'] = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$, which means that $A' \xrightarrow{x'} B' \xrightarrow{y'} C'$ belongs to $\mathscr{S}$.

Let us confirm conditions [E0],[E1],[E2] in [Bu, Definition 2.1]. Since our assumptions are self-dual, the other conditions [E0$^{\mathrm{op}}$],[E1$^{\mathrm{op}}$],[E2$^{\mathrm{op}}$] can be shown dually.

[E0] For any object $A \in \mathscr{C}$, the split sequence $A \xrightarrow{\mathrm{id}_A} A \to 0$ belongs to $\mathscr{S}$ by (ET2).

[E1] The class of inflations (= admissible monics) is closed under composition by (ET4), as in Remark 2.16.

[E2] Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ be any $\mathbb{E}$-triangle, and let $a \in \mathscr{C}(A, A')$ be any morphism. By Corollary 3.16, there exists a conflation

$$A \xrightarrow{s} B \oplus A' \xrightarrow{{}^{\exists}[b\ x']} {}^{\exists}B',$$

where $s = \begin{bmatrix} x \\ -a \end{bmatrix}$. Since it becomes a kernel-cokernel pair by the above argument, it follows that (a) is a pushout square. By the dual of Proposition 3.17, we obtain a commutative diagram (b) made of conflations, which shows that x' is an inflation.

$$\text{(a)}\quad \begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "b"] \\ A' \arrow[r, "x'"'] & B' \end{tikzcd} \qquad \text{(b)}\quad \begin{tikzcd} & A' \arrow[r, equal] \arrow[d, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"'] \arrow[dr, phantom, "\circlearrowleft"] & A' \arrow[d, "x'"] \\ A \arrow[r, "s"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & B \oplus A' \arrow[r, "{[b\ x']}"'] \arrow[d, "{[1\ 0]}"] \arrow[dr, phantom, "\circlearrowleft"] & B' \arrow[d, "{}^{\exists}y'"] \\ A \arrow[r, "x"'] & B \arrow[r, "y"'] & C \end{tikzcd}$$

□

3.3 Relation with triangulated categories

In this section, let $\mathscr{C}$ be an additive category equipped with an equivalence $[1]\colon \mathscr{C} \xrightarrow{\simeq} \mathscr{C}$, and let $\mathbf{E}^1\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ be the bifunctor defined by $\mathbf{E}^1 = \mathrm{Ext}^1(-,-) = \mathscr{C}(-,-[1])$.

Remark 3.19. *As usual, we use notations like $X[1]$ and $f[1]$ for objects X and morphisms f in $\mathscr{C}$. The n-times composition of $[1]$ is denoted by $[n]$.*

We will show that, to give a triangulation of $\mathscr{C}$ with shift functor $[1]$, is equivalent to give an $\mathbf{E}^1$-triangulation of $\mathscr{C}$ (Proposition 3.22).

Remark 3.20. *Let $\mathscr{C}, [1], \mathbf{E}^1$ be as above. Then for any $\delta \in \mathbf{E}^1(C, A) = \mathscr{C}(C, A[1])$, we have the following.*

(1) $\delta_\sharp = \mathscr{C}(-,\delta)\colon \mathscr{C}(-,C) \Rightarrow \mathscr{C}(-,A[1])$.

(2) $\delta^\sharp$ *is given by* $\delta^\sharp_X\colon \mathscr{C}(A,X) \to \mathscr{C}(C,X[1])\ ;\ f \mapsto (f[1]) \circ \delta$, *for any* $X \in \mathscr{C}$.

Lemma 3.21. Let $\mathscr{C}, [1], \mathbf{E}^1$ be as above. Suppose that $\mathfrak{s}$ is an $\mathbf{E}^1$-triangulation of $\mathscr{C}$. Then for any $A \in \mathscr{C}$, the $\mathbf{E}^1$-extension $\mathbf{1} = \mathrm{id}_{A[1]} \in \mathbf{E}^1(A[1], A) = \mathscr{C}(A[1], A[1])$ can be realized as $\mathfrak{s}(\mathbf{1}) = [A \to 0 \to A[1]]$. Namely, $A \to 0 \to A[1] \overset{\mathbf{1}}{\dashrightarrow}$ is an $\mathbf{E}^1$-triangle.

Proof. Put $\mathfrak{s}(\mathbf{1}) = [A \xrightarrow{x} X \xrightarrow{y} A[1]]$. By Proposition 3.3, $\mathscr{C}(-, A) \overset{\mathscr{C}(-,x)}{\Longrightarrow} \mathscr{C}(-, X) \overset{\mathscr{C}(-,y)}{\Longrightarrow} \mathscr{C}(-, A[1]) \overset{\mathbf{1}_\sharp = \mathrm{id}}{\Longrightarrow} \mathscr{C}(-, A[1]) \overset{\mathscr{C}(-,x[1])}{\Longrightarrow} \mathscr{C}(-, X[1])$ is exact. In particular $\mathrm{id}_X \in \mathscr{C}(X, X)$ satisfies $y = (\mathbf{1}_\sharp)_X \circ \mathscr{C}(X, y)(\mathrm{id}_X) = 0$. Similarly $x[1] = 0$ implies $x = 0$. Thus $0 \Rightarrow \mathscr{C}(-, X) \Rightarrow 0$ becomes exact, which shows $X = 0$. $\square$

Proposition 3.22. As before, let $\mathscr{C}$ be an additive category equipped with an auto-equivalence $[1]$, and put $\mathbf{E}^1 = \mathscr{C}(-, -[1])$. Then we have the following.

(1) Suppose $\mathscr{C}$ is a triangulated category with shift functor $[1]$. For any $\delta \in \mathbf{E}^1(C, A) = \mathscr{C}(C, A[1])$, take a distinguished triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A[1]$$

and define as $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$. Remark that this $\mathfrak{s}(\delta)$ does not depend on the choice of the distinguished triangle above. With this definition, $(\mathscr{C}, \mathbf{E}^1, \mathfrak{s})$ becomes an extriangulated category.

(2) Suppose we are given an $\mathbf{E}^1$-triangulation $\mathfrak{s}$ of $\mathscr{C}$. Define that $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A[1]$ is a distinguished triangle if and only if it satisfies $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$. With this class of distinguished triangles, $\mathscr{C}$ becomes a triangulated category.

By construction, distinguished triangles correspond to $\mathbf{E}^1$-triangles by the above (1) and (2).

Proof. (1) is straightforward. For (2), all the axioms except for (TR2) are easily confirmed. Let us show (TR2).

Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ be any $\mathbf{E}^1$-triangle. Applying Proposition 3.15

to $A \to 0 \to A[1] \overset{\mathbf{1}}{\dashrightarrow}$ and δ, we obtain

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "m'"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, equal] \\ 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \exists M \arrow[r, "e"] \arrow[d, "e'"] & C \\ A[1] \arrow[r, equal] & A[1] & \end{tikzcd}$$

with
(i) $[0 \to M \xrightarrow{e} C] = 0_*\delta = 0$,
(ii) $e^*\delta + e'^*\mathbf{1} = 0$,
(iii) $\mathfrak{s}(x[1]) = \mathfrak{s}(x_*\mathbf{1}) = [B \xrightarrow{m'} M \xrightarrow{e'} A[1]]$.

Condition (i) shows that e is an isomorphism, by Remark 2.11 (1). Condition (ii) means $\delta \circ e + e' = 0$ in $\mathscr{C}(M, A[1])$, namely $e' \circ e^{-1} = -\delta$. Thus we have $\mathfrak{s}(x[1]) = [B \xrightarrow{y} C \xrightarrow{-\delta} A[1]]$ by condition (iii), which means that $B \xrightarrow{y} C \xrightarrow{-\delta} A[1] \xrightarrow{x[1]} B[1]$ is a distinguished triangle. This is isomorphic to $B \xrightarrow{y} C \xrightarrow{\delta} A[1] \xrightarrow{-x[1]} B[1]$. $\square$

3.4 Projectives and injectives

If $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ has enough "*projectives*", then the bifunctor $\mathbb{E}$ can be described in terms of them. Throughout this section, let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Duals of the results in this section hold true for "*injectives*".

Definition 3.23. An object $P \in \mathscr{C}$ is called *projective* if, for any $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and any morphism c in $\mathscr{C}(P, C)$, there exists $b \in \mathscr{C}(P, B)$ satisfying $y \circ b = c$.

We denote the full subcategory of projective objects in $\mathscr{C}$ by $\mathrm{Proj}(\mathscr{C})$. Dually, the full subcategory of injective objects in $\mathscr{C}$ is denoted by $\mathrm{Inj}(\mathscr{C})$.

Proposition 3.24. An object $P \in \mathscr{C}$ is projective if and only if it satisfies $\mathbb{E}(P, A) = 0$ for any $A \in \mathscr{C}$.

Proof. Sufficiency of $\mathbb{E}(P, A) = 0$ follows from the exact sequence of Proposition 3.3. Conversely, suppose P is projective. Let $A \in \mathscr{C}$ be any object, and let $\delta \in \mathbb{E}(P, A)$ be any element, with $\mathfrak{s}(\delta) = [A \xrightarrow{x} M \xrightarrow{y} P]$. Since P is projective, there exists $m \in \mathscr{C}(P, M)$ which makes the following diagram commutative.

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r, "\mathrm{id}_P"] \arrow[d, "m"'] \arrow[dr, phantom, "\circlearrowright"] & P \arrow[d, "\mathrm{id}_P"] \\ A \arrow[r, "x"'] & M \arrow[r, "y"'] & P \end{tikzcd}$$

By (ET3)$^{\mathrm{op}}$, the triplet $(0, m, \mathrm{id}_P)$ realizes the morphism $(0, \mathrm{id}_P)\colon 0 \to \delta$. Especially we have $\delta = \mathbb{E}(P, 0)(0) = 0$. □

Definition 3.25. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, as before. We say that it *has enough projectives* if, for any object $C \in \mathscr{C}$, there exists an $\mathbb{E}$-triangle $A \xrightarrow{x} P \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ satisfying $P \in \mathrm{Proj}(\mathscr{C})$.

Example 3.26. (1) If $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is an exact category, then these agree with the usual definitions.

(2) If $(\mathscr{C}, \mathbf{E}^1, \mathfrak{s})$ is a triangulated category as in the previous section, then $\mathrm{Proj}(\mathscr{C})$ consists of zero objects. Moreover it always has enough projectives.

(3) If $(\mathscr{C}, \mathbf{E}^1, \mathfrak{s})$ is a triangulated category with a rigid subcategory $\mathscr{R}$ (i.e. for all $R_1, R_2 \in \mathscr{R}, \mathrm{Ext}^1(R_1, R_2) = 0$), let $\mathscr{D}$ be its full subcategory whose objects are those objects X that satisfy $\mathrm{Ext}^1(R, X) = 0$ for all $R \in \mathscr{R}$. Then $\mathscr{D}$ is an additive and extension-closed subcategory of $\mathscr{C}$, which is thus extriangulated by Remark 2.18. We then have:

   (a) $\mathscr{R} \subseteq \mathrm{Proj}(\mathscr{D})$;

   (b) $\mathrm{Proj}(\mathscr{D}) = \mathscr{R}$ and $\mathscr{D}$ has enough projectives if and only if $\mathscr{R}$ is contravariantly finite.

Corollary 3.27. Assume that $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ has enough projectives. For any object $C \in \mathscr{C}$ and any $\mathbb{E}$-triangle $A \xrightarrow{x} P \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ with $P \in \mathrm{Proj}(\mathscr{C})$, the sequence $\mathscr{C}(P, -) \overset{\mathscr{C}(x,-)}{\Longrightarrow} \mathscr{C}(A, -) \overset{\delta^\sharp}{\Longrightarrow} \mathbb{E}(C, -) \Rightarrow 0$ is exact. Namely, we have a natural isomorphism $\mathbb{E}(C, -) \cong \mathrm{Cok}\big(\mathscr{C}(x, -)\big)$.

Proof. This immediately follows from Propositions 3.3 and 3.24. □

The isomorphism in Corollary 3.27 is natural in C, in the following sense.

Remark 3.28. *Assume that $(\mathscr{C},\mathbb{E},\mathfrak{s})$ has enough projectives. Let $c\colon C\to C'$ be any morphism, and let $A\xrightarrow{x}P\xrightarrow{y}C\overset{\delta}{\dashrightarrow}$, $A'\xrightarrow{x'}P'\xrightarrow{y'}C'\overset{\delta'}{\dashrightarrow}$ be any pair of $\mathbb{E}$-triangles satisfying $P,P'\in\operatorname{Proj}(\mathscr{C})$. By the projectivity of P and* (ET3)$^{\mathrm{op}}$*, we obtain a morphism of $\mathbb{E}$-triangles, and thus a morphism of exact sequences, as follows:*

$$
\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & P \arrow[r, "y"] \arrow[d, "p"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, "c"] & {} \\
A' \arrow[r, "x'"'] & P' \arrow[r, "y'"'] & C' \arrow[r, dashed, "\delta'"'] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{C}(P,-) \arrow[r, Rightarrow, "\mathscr{C}(x{,}-)"] \arrow[dr, phantom, "\circlearrowright"] & \mathscr{C}(A,-) \arrow[r, Rightarrow, "\delta^\sharp"] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{E}(C,-) \arrow[r, Rightarrow] & 0 \\
\mathscr{C}(P',-) \arrow[r, Rightarrow, "\mathscr{C}(x'{,}-)"'] \arrow[u, Rightarrow, "\mathscr{C}(p{,}-)"] & \mathscr{C}(A',-) \arrow[r, Rightarrow, "\delta'^\sharp"'] \arrow[u, Rightarrow, "\mathscr{C}(a{,}-)"'] & \mathbb{E}(C',-) \arrow[r, Rightarrow] \arrow[u, Rightarrow, "\mathbb{E}(c{,}-)"'] & 0.
\end{tikzcd}
$$

Lemma 3.29. Let $A\xrightarrow{x}B\xrightarrow{y}C\overset{\delta}{\dashrightarrow}$ be any $\mathbb{E}$-triangle, and let i in $\mathscr{C}(A,I)$ be any morphism with $I\in\operatorname{Inj}(\mathscr{C})$. Write p_C for the projection $C\oplus I\to C$. Then the $\mathbb{E}$-extension $p_C^*\delta$ is realized by an $\mathbb{E}$-triangle of the form

$$A\xrightarrow{x_I}B\oplus I\xrightarrow{y_I}C\oplus I\overset{p_C^*\delta}{\dashrightarrow},\quad \left(x_I=\begin{bmatrix}x\\ i\end{bmatrix},\ y_I=\begin{bmatrix}y & *\\ * & *\end{bmatrix}\right). \tag{11}$$

Proof. By Corollary 3.16, we have an $\mathbb{E}$-triangle $A\xrightarrow{x_I}B\oplus I\xrightarrow{d}D\overset{\nu}{\dashrightarrow}$. By the dual of Proposition 3.17, we obtain the following commutative diagram (α) made of $\mathbb{E}$-triangles

$$
(\alpha)\quad
\begin{tikzcd}
 & I \arrow[r, equal] \arrow[d, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"'] \arrow[dr, phantom, "\circlearrowright"] & I \arrow[d, "e"] & \\
A \arrow[r, "x_I"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B\oplus I \arrow[r, "d"'] \arrow[d, "{[1\ 0]}"] \arrow[dr, phantom, "\circlearrowright"] & D \arrow[r, dashed, "\nu"] \arrow[d, "f"] & {} \\
A \arrow[r, "x"'] & B \arrow[r, "y"'] \arrow[d, dashed, "0"'] & C \arrow[r, dashed, "\delta"'] \arrow[d, dashed, "\exists\theta"] & {} \\
 & {} & {} &
\end{tikzcd}
\qquad
(\beta)\quad
\begin{tikzcd}
A \arrow[r, "x_I"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B\oplus I \arrow[r, "n^{-1}\circ d"] \arrow[d, "{[1\ 0]}"] \arrow[dr, phantom, "\circlearrowright"] & C\oplus I \arrow[r, dashed, "n^*\nu"] \arrow[d, "{[1\ 0]=p_C}"] & {} \\
A \arrow[r, "x"'] & B \arrow[r, "y"'] & C \arrow[r, dashed, "\delta"'] & {}
\end{tikzcd}
$$

satisfying $f^*\delta=\nu$. Since $I\in\operatorname{Inj}(\mathscr{C})$, we have $\theta=0$. Thus there is some isomorphism $n\colon C\oplus I\xrightarrow{\cong}D$ satisfying $n\circ\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]=e$ and $f\circ n=[1\ 0]$. Then for $p_C=[1\ 0]\colon C\oplus I\to C$, diagram (β) above is a morphism of $\mathbb{E}$-triangles. Then $n^{-1}\circ d$ satisfies $p_C\circ n^{-1}\circ d\circ\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]=y\circ[1\ 0]\circ\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]=y$, and thus is of the form $\left[\begin{smallmatrix}y & *\\ * & *\end{smallmatrix}\right]$. $\square$

The following construction gives extriangulated categories which are not exact nor triangulated in general.

Proposition 3.30. Let $\mathcal{I} \subseteq \mathscr{C}$ be a full additive subcategory, closed under isomorphisms. If $\mathcal{I}$ satisfies $\mathcal{I} \subseteq \mathrm{Proj}(\mathscr{C}) \cap \mathrm{Inj}(\mathscr{C})$, then the ideal quotient $\mathscr{C}/\mathcal{I}$ has the structure of an extriangulated category, induced from that of $\mathscr{C}$. In particular, we can associate a "*reduced*" extriangulated category $\mathscr{C}' = \mathscr{C}/(\mathrm{Proj}(\mathscr{C}) \cap \mathrm{Inj}(\mathscr{C}))$ satisfying $\mathrm{Proj}(\mathscr{C}') \cap \mathrm{Inj}(\mathscr{C}') = 0$, to any extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$.

Proof. Put $\overline{\mathscr{C}} = \mathscr{C}/\mathcal{I}$. Let us confirm conditions (ET1),(ET2),(ET3),(ET4). The other conditions (ET3)$^{\mathrm{op}}$,(ET4)$^{\mathrm{op}}$ can be shown dually.

(ET1) Since $\mathbb{E}(\mathcal{I}, \mathscr{C}) = \mathbb{E}(\mathscr{C}, \mathcal{I}) = 0$, one can define the biadditive functor $\overline{\mathbb{E}} \colon \overline{\mathscr{C}}^{\mathrm{op}} \times \overline{\mathscr{C}} \to Ab$ given by

- $\overline{\mathbb{E}}(C, A) = \mathbb{E}(C, A) \quad (\forall A, C \in \mathscr{C})$,
- $\overline{\mathbb{E}}(\overline{c}, \overline{a}) = \mathbb{E}(c, a) \quad (\forall a \in \mathscr{C}(A, A'), c \in \mathscr{C}(C, C'))$. Here, $\overline{a}$ and $\overline{c}$ denote the images of a and c in $\mathscr{C}/\mathcal{I}$.

(ET2) For any $\overline{\mathbb{E}}$-extension $\delta \in \overline{\mathbb{E}}(C, A) = \mathbb{E}(C, A)$, define

$$\overline{\mathfrak{s}}(\delta) = \overline{\mathfrak{s}(\delta)} = [A \xrightarrow{\overline{x_0}} B \xrightarrow{\overline{y_0}} C],$$

using $\mathfrak{s}(\delta) = [A \xrightarrow{x_0} B \xrightarrow{y_0} C]$. Let us show that $\overline{\mathfrak{s}}$ is an additive realization of $\overline{\mathbb{E}}$.

Let $(\overline{a}, \overline{c}) \colon \delta = (A, \delta, C) \to \delta' = (A', \delta', C')$ be any morphism of $\overline{\mathbb{E}}$-extensions. By definition, this is equivalent to that $(a, c) \colon \delta \to \delta'$ is a morphism of $\mathbb{E}$-extensions. Put $\overline{\mathfrak{s}}(\delta) = [A \xrightarrow{\overline{x}} B \xrightarrow{\overline{y}} C]$, $\overline{\mathfrak{s}}(\delta') = [A' \xrightarrow{\overline{x'}} B' \xrightarrow{\overline{y'}} C']$. Since the condition in Definition 2.9 does not depend on the representatives of the equivalence classes of sequences in $\overline{\mathscr{C}}$, we may assume $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. Then there is $b \in \mathscr{C}(B, B')$ with which (a, b, c) realizes (a, c). It follows that $(\overline{a}, \overline{b}, \overline{c})$ realizes $(\overline{a}, \overline{c})$.

As for the additivity, the equality $\overline{\mathfrak{s}}(0) = 0$ is trivially satisfied. Since $\overline{\mathfrak{s}}(\delta) \oplus \overline{\mathfrak{s}}(\delta')$ only depends on the equivalence classes $\overline{\mathfrak{s}}(\delta)$ and $\overline{\mathfrak{s}}(\delta')$, the equality $\overline{\mathfrak{s}}(\delta \oplus \delta') = \overline{\mathfrak{s}}(\delta) \oplus \overline{\mathfrak{s}}(\delta')$ follows from $\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$.

(ET3) Suppose we are given $\overline{\mathfrak{s}}(\delta)=[A\xrightarrow{\overline{x}}B\xrightarrow{\overline{y}}C]$, $\overline{\mathfrak{s}}(\delta')=[A'\xrightarrow{\overline{x'}}B'\xrightarrow{\overline{y'}}C']$, and morphisms $\overline{a}\in\overline{\mathscr{C}}(A,A'),\overline{b}\in\overline{\mathscr{C}}(B,B')$ satisfying $\overline{x'}\circ\overline{a}=\overline{b}\circ\overline{x}$. As in the proof of (ET2), we may assume $\mathfrak{s}(\delta)=[A\xrightarrow{x}B\xrightarrow{y}C]$, $\mathfrak{s}(\delta')=[A'\xrightarrow{x'}B'\xrightarrow{y'}C']$. By $\overline{x'}\circ\overline{a}=\overline{b}\circ\overline{x}$, there exist $I\in\mathcal{I},i\in\mathscr{C}(A,I),j\in\mathscr{C}(I,B')$ which satisfy $x'\circ a=b\circ x+j\circ i$. By Lemma 3.29, we obtain an $\mathbb{E}$-triangle (11). This gives the following isomorphism of $\overline{\mathbb{E}}$-triangles.

$$\begin{tikzcd}
A \arrow[r, "\overline{x_I}"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B\oplus I \arrow[r, "\overline{y_I}"] \arrow[d, "\overline{p_B}"] \arrow[dr, phantom, "\circlearrowright"] & C\oplus I \arrow[r, dashed, "p_C^*\delta"] \arrow[d, "\overline{p_C}"] & {} \\
A \arrow[r, "\overline{x}"'] & B \arrow[r, "\overline{y}"'] & C \arrow[r, dashed, "\delta"'] & {}
\end{tikzcd} \tag{12}$$

On the other hand, by axiom (ET3) for $(\mathscr{C},\mathbb{E},\mathfrak{s})$, we have a morphism of $\mathbb{E}$-triangles as follows.

$$\begin{tikzcd}
A \arrow[r, "x_I"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B\oplus I \arrow[r, "y_I"] \arrow[d, "{[b\ j]}"] \arrow[dr, phantom, "\circlearrowright"] & C\oplus I \arrow[r, dashed, "p_C^*\delta"] \arrow[d, "c"] & {} \\
A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \arrow[r, dashed, "\delta'"'] & {}
\end{tikzcd} \tag{13}$$

From the two diagrams (12) and (13), we obtain a morphism of $\overline{\mathbb{E}}$-extensions $(\overline{a},\overline{c}\circ\overline{p_C}^{-1})\colon\delta\to\delta'$ which satisfies $(\overline{c}\circ\overline{p_C}^{-1})\circ\overline{y}=\overline{y'}\circ\overline{b}$.

(ET4) Let $A\xrightarrow{\overline{f}}B\xrightarrow{\overline{f'}}D\overset{\delta}{\dashrightarrow}$ and $B\xrightarrow{\overline{g}}C\xrightarrow{\overline{g'}}F\overset{\delta'}{\dashrightarrow}$ be $\overline{\mathbb{E}}$-triangles. As in the above arguments, we may assume $A\xrightarrow{f}B\xrightarrow{f'}D\overset{\delta}{\dashrightarrow}$ and $B\xrightarrow{g}C\xrightarrow{g'}F\overset{\delta'}{\dashrightarrow}$ are $\mathbb{E}$-triangles. Then by (ET4) for $(\mathscr{C},\mathbb{E},\mathfrak{s})$, we obtain a commutative diagram (3) made of conflations, satisfying $\mathfrak{s}(f'_*\delta')=[D\xrightarrow{d}E\xrightarrow{e}F]$, $d^*\delta''=\delta$ and $f_*\delta''=e^*\delta'$. The image of this diagram in $\overline{\mathscr{C}}$ shows (ET4) for $(\overline{\mathscr{C}},\overline{\mathbb{E}},\overline{\mathfrak{s}})$. $\square$

Remark 3.31. *Proposition 3.30 applied to an exact category, together with Corollary 3.18, gives an another proof*[3] *of [DI, Theorem 3.5].*

Corollary 3.32. Let $\mathcal{I}\subseteq\mathscr{C}$ be a strictly full additive subcategory, satisfying $\mathbb{E}(\mathcal{I},\mathcal{I})=0$. Let $\mathcal{Z}\subseteq\mathscr{C}$ be the full subcategory of those $Z\in\mathscr{C}$ satisfying $\mathbb{E}(Z,\mathcal{I})=\mathbb{E}(\mathcal{I},Z)=0$. Then, $\mathcal{Z}/\mathcal{I}$ is extriangulated.

[3]The first author wishes to thank Professor Osamu Iyama for informing this to him.

Proof. This follows from Remark 2.18 and Proposition 3.30. □

4. Cotorsion pairs

In the rest of this article, let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category.

4.1 Cotorsion pairs

Definition 4.1. Let $\mathcal{U}, \mathcal{V} \subseteq \mathscr{C}$ be a pair of full additive subcategories, closed under isomorphisms and direct summands. The pair $(\mathcal{U}, \mathcal{V})$ is called a *cotorsion pair* on $\mathscr{C}$ if it satisfies the following conditions.

(1) $\mathbb{E}(\mathcal{U}, \mathcal{V}) = 0$.

(2) For any $C \in \mathscr{C}$, there exists a conflation $V^C \to U^C \to C$ satisfying $U^C \in \mathcal{U}, V^C \in \mathcal{V}$.

(3) For any $C \in \mathscr{C}$, there exists a conflation $C \to V_C \to U_C$ satisfying $U_C \in \mathcal{U}, V_C \in \mathcal{V}$.

Definition 4.2. Let $\mathcal{X}, \mathcal{Y} \subseteq \mathscr{C}$ be any pair of full subcategories closed under isomorphisms. Define full subcategories $\mathrm{Cone}(\mathcal{X}, \mathcal{Y})$ and $\mathrm{CoCone}(\mathcal{X}, \mathcal{Y})$ of $\mathscr{C}$ as follows. These are closed under isomorphisms.

(i) $C \in \mathscr{C}$ belongs to $\mathrm{Cone}(\mathcal{X}, \mathcal{Y})$ if and only if it admits a conflation $X \to Y \to C$ satisfying $X \in \mathcal{X}, Y \in \mathcal{Y}$.

(ii) $C \in \mathscr{C}$ belongs to $\mathrm{CoCone}(\mathcal{X}, \mathcal{Y})$ if and only if it admits a conflation $C \to X \to Y$ satisfying $X \in \mathcal{X}, Y \in \mathcal{Y}$.

If $\mathcal{X}$ and $\mathcal{Y}$ are additive subcategories of $\mathscr{C}$, then so are $\mathrm{Cone}(\mathcal{X}, \mathcal{Y})$ and $\mathrm{CoCone}(\mathcal{X}, \mathcal{Y})$, by condition (ET2).

Remark 4.3. *In the case of exact categories, cotorsion pairs satisfying (2) and (3) are often called* complete *cotorsion pairs. Since all cotorsion pairs considered in this article are complete, this adjective is omitted. Also remark that completeness is equivalent to the equalities* $\mathscr{C} = \mathrm{Cone}(\mathcal{V}, \mathcal{U}) = \mathrm{CoCone}(\mathcal{V}, \mathcal{U})$*, in the notation of Definition 4.2.*

Remark 4.4. *Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair on $(\mathscr{C},\mathbb{E},\mathfrak{s})$. By Remark 2.11, for any $C\in\mathscr{C}$: $C\in\mathcal{U}\Leftrightarrow\mathbb{E}(C,\mathcal{V})=0$ and $C\in\mathcal{V}\Leftrightarrow\mathbb{E}(\mathcal{U},C)=0$.*

Corollary 4.5. Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair on $(\mathscr{C},\mathbb{E},\mathfrak{s})$. Let $C\in\mathscr{C}$, $U\in\mathcal{U}$ be any pair of objects. If there exists a section $C\to U$ or a retraction $U\to C$, then C also belongs to $\mathcal{U}$. Similarly for $\mathcal{V}$.

Proof. In either case, there are $s\in\mathscr{C}(C,U)$ and $r\in\mathscr{C}(U,C)$ satisfying $r\circ s=\mathrm{id}_C$. This gives the following commutative diagram of natural transformations.

$$\begin{tikzcd} & \mathbb{E}(U,-) \arrow[dr, Rightarrow, "\mathbb{E}(s,-)"] & \\ \mathbb{E}(C,-) \arrow[ur, Rightarrow, "\mathbb{E}(r,-)"] \arrow[rr, Rightarrow, "\mathbb{E}(\mathrm{id}_C,-)=\mathrm{id}"'] & \circlearrowright & \mathbb{E}(C,-) \end{tikzcd}$$

Thus $\mathbb{E}(U,\mathcal{V})=0$ implies $\mathbb{E}(C,\mathcal{V})=0$, and thus $C\in\mathcal{U}$ by Remark 4.4. $\square$

Remark 4.6. *Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair on $\mathscr{C}$. By Proposition 3.11, the subcategories $\mathcal{U}$ and $\mathcal{V}$ are extension-closed in $\mathscr{C}$.*

Remark 4.7. *By Proposition 3.24 and Remark 4.4, we have: $(\mathcal{X},\mathscr{C})$ is a cotorsion pair for some subcategory $\mathcal{X}\subseteq\mathscr{C}$ if and only if $(\mathrm{Proj}(\mathscr{C}),\mathscr{C})$ is a cotorsion pair, if and only if $\mathscr{C}$ has enough projectives. A dual remark concerning injectives holds.*

4.2 Associated adjoint functors

Definition 4.8. For a cotorsion pair $(\mathcal{U},\mathcal{V})$ on $\mathscr{C}$, put $\mathcal{I}=\mathcal{U}\cap\mathcal{V}$, and call it the *core* of $(\mathcal{U},\mathcal{V})$. For any full additive subcategory $\mathcal{X}\subseteq\mathscr{C}$ containing $\mathcal{I}$, let $\mathcal{X}/\mathcal{I}$ denote the ideal quotient. The image of a morphism f in the ideal quotient is denoted by $\overline{f}$.

Lemma 4.9. For any cotorsion pair $(\mathcal{U},\mathcal{V})$, we have $(\mathscr{C}/\mathcal{I})(\mathcal{U}/\mathcal{I},\mathcal{V}/\mathcal{I})=0$. Namely, for any $U\in\mathcal{U}$ and $V\in\mathcal{V}$, any morphism $f\in\mathscr{C}(U,V)$ factors through some $I\in\mathcal{I}$.

Proof. By Proposition 3.3 a proof similar as that for triangulated categories apply. $\square$

Proposition 4.10. Let $C \in \mathscr{C}$ be any object, and let λ be an $\mathbb{E}$-extension with $\mathfrak{s}(\lambda) = [V^C \xrightarrow{v^C} U^C \xrightarrow{u^C} C]$, where $U^C \in \mathcal{U}$ and $V^C \in \mathcal{V}$. Then for any $U \in \mathcal{U}$, the map $\overline{u}^C \circ -$ is bijective:

$$\overline{u}^C \circ - \colon (\mathscr{C}/\mathcal{I})(U, U^C) \to (\mathscr{C}/\mathcal{I})(U, C) \tag{14}$$

Proof. By exactness of $\mathscr{C}(U, U^C) \xrightarrow{\mathscr{C}(U,u^C)} \mathscr{C}(U, C) \xrightarrow{(\lambda_\sharp)_U} \mathbb{E}(U, V) = 0$, the map $\mathscr{C}(U, U^C) \to \mathscr{C}(U, C)$ is surjective. This implies the surjectivity of (14). Let us show the injectivity of (14). Let $g \in \mathscr{C}(U, U^C)$ be any morphism which satisfies $\overline{u}^C \circ \overline{g} = \overline{u^C \circ g} = 0$. By definition, there exist $I \in \mathcal{I}$, $i_1 \in \mathscr{C}(U, I)$ and $i_2 \in \mathscr{C}(I, C)$ which makes the following diagram commutative.

$$\begin{tikzcd} & U \arrow[r, "i_1"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowright"] & I \arrow[d, "i_2"] \\ V^C \arrow[r, "v^C"'] & U^C \arrow[r, "u^C"'] & C \end{tikzcd}$$

Since $\mathbb{E}(I, V^C) = 0$, $\mathscr{C}(I, V^C) \xrightarrow{\mathscr{C}(I,v^C)} \mathscr{C}(I, U^C) \xrightarrow{\mathscr{C}(I,u^C)} \mathscr{C}(I, C) \to 0$ is exact. Thus there exists $j \in \mathscr{C}(I, U^C)$ satisfying $u^C \circ j = i_2$. Then by $u^C \circ (g - j \circ i_1) = 0$, we obtain $h \in \mathscr{C}(U, V^C)$ satisfying $v^C \circ h = g - j \circ i_1$.

By Lemma 4.9, this h factors through some $I' \in \mathcal{I}$. It follows that $g = v^C \circ h + j \circ i_1$ factors through $I \oplus I' \in \mathcal{I}$. $\square$

Proposition 4.10 means that $(U^C, \overline{u}^C)$ is coreflection of $C \in \mathscr{C}/\mathcal{I}$ along the inclusion functor $\mathcal{U}/\mathcal{I} \hookrightarrow \mathscr{C}/\mathcal{I}$ ([Bo, Definition 3.1.1]). We thus obtain:

Corollary 4.11. The inclusion functor $\mathcal{U}/\mathcal{I} \hookrightarrow \mathscr{C}/\mathcal{I}$ has a right adjoint $\omega_\mathcal{U} \colon \mathscr{C}/\mathcal{I} \to \mathcal{U}/\mathcal{I}$, which assigns $\omega_\mathcal{U}(C) = U^C$ for any $C \in \mathscr{C}/\mathcal{I}$, where $V^C \xrightarrow{v^C} U^C \xrightarrow{u^C} C$ is a conflation with $U^C \in \mathcal{U}, V^C \in \mathcal{V}$. Moreover $\varepsilon_\mathcal{U} = \{\overline{u}^C\}_{C \in \mathrm{Ob}(\mathscr{C}/\mathcal{I})}$ gives the counit of this adjoint pair.

4.3 Concentric twin cotorsion pairs

Definition 4.12. Let $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{U}, \mathcal{V})$ be cotorsion pairs on $\mathscr{C}$. Then the pair $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ is called a *twin cotorsion pair* if it satisfies $\mathbb{E}(\mathcal{S}, \mathcal{V}) = 0$. (Pairs of cotorsion pairs are considered in abelian/exact categories in [Ho1, Ho2], and in triangulated categories in [Na2, Na3].)

If moreover it satisfies $\mathcal{S}\cap\mathcal{T} = \mathcal{U}\cap\mathcal{V}(= \mathcal{I})$, then $\mathcal{P}$ is called a *concentric* twin cotorsion pair similarly as in the triangulated case [Na4]. In this case, we put $\mathcal{Z} = \mathcal{T} \cap \mathcal{U}$.

Remark 4.13. *Let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a concentric twin cotorsion pair. By Corollary 4.11, the inclusion $\mathcal{U}/\mathcal{I} \hookrightarrow \mathscr{C}/\mathcal{I}$ has a right adjoint $\omega_{\mathcal{U}}$. Dually the inclusion $\mathcal{T}/\mathcal{I} \hookrightarrow \mathscr{C}/\mathcal{I}$ has a left adjoint $\sigma_{\mathcal{T}}$.*

These restrict to yield the following.

- *Left adjoint σ of the inclusion $\mathcal{Z}/\mathcal{I} \hookrightarrow \mathcal{U}/\mathcal{I}$.*
- *Right adjoint ω of the inclusion $\mathcal{Z}/\mathcal{I} \hookrightarrow \mathcal{T}/\mathcal{I}$.*

Definition 4.14. Let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a twin cotorsion pair on $\mathscr{C}$. We define full subcategories $\mathcal{N}^i, \mathcal{N}^f$ of $\mathscr{C}$ as follows.

$$\mathcal{N}^i = \mathrm{Cone}(\mathcal{V}, \mathcal{S}) \text{ and } \mathcal{N}^f = \mathrm{CoCone}(\mathcal{V}, \mathcal{S}).$$

Remark 4.15. *The notation $\mathcal{N}^i$, $\mathcal{N}^f$ is motivated by Section 5. Morally, an object X belongs to $\mathcal{N}^i$ if and only if the morphism $0 \to X$ from the initial object is a weak equivalence. For a precise statement, see Proposition 5.7.*

Remark 4.16. *If $\mathcal{P}$ is concentric, then for any $C \in \mathscr{C}$, we have:*

$$C \in \mathcal{N}^i \Leftrightarrow \omega_{\mathcal{U}}(C) \in \mathcal{S}/\mathcal{I} \text{ and } C \in \mathcal{N}^f \Leftrightarrow \sigma_{\mathcal{T}}(C) \in \mathcal{V}/\mathcal{I}.$$

Remark 4.17. *Let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a twin cotorsion pair. Then the following holds:*

$$\mathcal{S} \subseteq \mathcal{N}^i, \mathcal{V} \subseteq \mathcal{N}^f, \mathcal{U} \cap \mathcal{N}^i = \mathcal{S} \text{ and } \mathcal{T} \cap \mathcal{N}^f = \mathcal{V}.$$

If moreover $\mathcal{P}$ is concentric, then $\mathcal{S} \subseteq \mathcal{N}^f$ and $\mathcal{V} \subseteq \mathcal{N}^i$.

Lemma 4.18. Let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a concentric twin cotorsion pair. Then the following holds:

(1) $\mathrm{Cone}(\mathcal{V}, \mathcal{N}^i) \subseteq \mathcal{N}^i$ and (2) $\mathrm{CoCone}(\mathcal{N}^f, \mathcal{S}) \subseteq \mathcal{N}^f$.

Proof. We only prove (1) since (2) is dual. By definition, $C \in \mathrm{Cone}(\mathcal{V}, \mathcal{N}^i)$ admits a conflation $V \to N \to C$, where $V \in \mathcal{V}$ and $N \in \mathcal{N}^i$. Resolve N by a conflation $V^N \to S^N \to N$ with $S^N \in \mathcal{S}$, $V^N \in \mathcal{V}$.

By (ET4)$^{\mathrm{op}}$, we obtain a commutative diagram in $\mathscr{C}$

$$\begin{tikzcd} V^N \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & V^N \arrow[d] & \\ {}^{\exists}E \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & S^N \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, equal] \\ V \arrow[r] & N \arrow[r] & C \end{tikzcd}$$

in which $V^N \to E \to V$ and $E \to S^N \to C$ are conflations. Since $\mathcal{V} \subseteq \mathscr{C}$ is extension-closed, it follows that $E \in \mathcal{V}$. This means $C \in \mathcal{N}^i$. $\square$

Lemma 4.19. Let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a concentric twin cotorsion pair. Let $U \in \mathcal{U}$ be any object. Assume there is a conflation $M \to U \to S$ satisfying $M \in \mathcal{N}^f$ and $S \in \mathcal{S}$. Then U belongs to $\mathcal{N}^f$.

Dually, if $T \in \mathcal{T}$ appears in a conflation $V \to T \to N$ satisfying $V \in \mathcal{V}, N \in \mathcal{N}^i$, then T belongs to $\mathcal{N}^i$.

Proof. By definition, M admits a conflation $M \to V_M \to S_M$ with $V_M \in \mathcal{V}, S_M \in \mathcal{S}$. By Proposition 3.15, we obtain a commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & U \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & S \arrow[d, equal] \\ V_M \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & {}^{\exists}X \arrow[r] \arrow[d] & S \\ S_M \arrow[r, equal] & S_M & \end{tikzcd}$$

consisting of conflations. Since $\mathcal{U}$ is extension-closed, it follows that $X \in \mathcal{U}$. Since $\mathbb{E}(S, V_M) = 0$, the $\mathbb{E}$-extension realized by $V_M \to X \to S$ splits. Especially V_M is a direct summand of X, and thus it follows that $V_M \in \mathcal{U} \cap \mathcal{V} = \mathcal{I}$. By the extension-closedness of $\mathcal{S}$, we obtain $X \in \mathcal{S}$. Thus $U \in \mathcal{N}^f$ follows from Remark 4.17 and Lemma 4.18 (2). $\square$

Lemma 4.20. Let $\mathcal{P}$ be as in Lemma 4.19. Let $T \in \mathcal{T}, M \in \mathcal{N}^f$ be any pair of objects. If there is a section $T \to M$ or a retraction $M \to T$, then T belongs to $\mathcal{V}$.

Proof. In either case, we have morphisms $s \in \mathscr{C}(T, M)$ and $r \in \mathscr{C}(M, T)$ satisfying $r \circ s = \mathrm{id}$. By definition, M admits a conflation $M \xrightarrow{v} V \to S$. By $\mathbb{E}(S, T) = 0$, the morphism r factors through v. Then $v \circ s \in \mathscr{C}(T, V)$ becomes a section, and thus it follows from Corollary 4.5 that $T \in \mathcal{V}$. $\square$

5. Bijective correspondence with model structures

In the rest, let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category.

In this section, we give a bijective correspondence between *Hovey twin cotorsion pairs* and *admissible model structures* which we will soon define. This gives a unification of the following preceding works.

- For an abelian category, Hovey has shown their correspondence in [Ho1, Ho2] (*abelian model structure*). This has been generalized to an exact category by Gillespie [G] (*exact model structure*), and investigated by Šťovíček [S].
- For a triangulated category, Yang [Y] has introduced an analogous notion of *triangulated model structure* and showed its correspondence with cotorsion pairs.

5.1 Hovey twin cotorsion pair

We recall that $\mathcal{N}^i$ (resp. $\mathcal{N}^f$) is the collection of all objects $X \in \mathscr{C}$ which are part of a conflation $V \longrightarrow S \longrightarrow X$ (resp. $X \longrightarrow V \longrightarrow S$), for some $V \in \mathcal{V}$ and $S \in \mathcal{S}$.

Definition 5.1. Let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a twin cotorsion pair. We call $\mathcal{P}$ a *Hovey twin cotorsion pair* if it satisfies $\mathcal{N}^f = \mathcal{N}^i$. We denote this subcategory by $\mathcal{N}$.

Remark 5.2. *Any Hovey twin cotorsion pair is concentric. In fact, we have* $\mathcal{U} \cap \mathcal{V} = \mathcal{U} \cap (\mathcal{N}^f \cap \mathcal{T}) = (\mathcal{U} \cap \mathcal{N}^i) \cap \mathcal{T} = \mathcal{S} \cap \mathcal{T}$ *by Remark 4.17.*

For any Hovey twin cotorsion pair, the subcategory $\mathcal{N} \subseteq \mathscr{C}$ is extension-closed. More strongly, it satisfies the following.

Proposition 5.3. Let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a Hovey twin cotorsion pair. For any conflation $A \xrightarrow{x} B \xrightarrow{y} C$, if two out of A, B, C belong to $\mathcal{N}$, then so does the third.

Proof. We prove (1) $A, C \in \mathcal{N} \Rightarrow B \in \mathcal{N}$, (2) $A, B \in \mathcal{N} \Rightarrow C \in \mathcal{N}$ and (3) $B, C \in \mathcal{N} \Rightarrow A \in \mathcal{N}$.

(1) Resolve A, C by conflations $A \to V_A \to S_A$ and $V^C \to S^C \to C$ with $S_A, S^C \in \mathcal{S}$, $V_A, V^C \in \mathcal{V}$ respectively. By Proposition 3.15, we obtain the two commutative diagrams on the left of:

$$
\begin{tikzcd}
 & V^C \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & V^C \arrow[d] \\
A \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & {}^{\exists}X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & S^C \arrow[d] \\
A \arrow[r] & B \arrow[r] & C
\end{tikzcd}
,\quad
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & S^C \arrow[d, equal] \\
V_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & {}^{\exists}Y \arrow[r] \arrow[d] & S^C \\
S_A \arrow[r, equal] & S_A &
\end{tikzcd}
,\quad
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, equal] \\
V_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & {}^{\exists}G \arrow[r] \arrow[d] & C \\
S_A \arrow[r, equal] & S_A &
\end{tikzcd}
$$

which are made of conflations. Since $\mathbb{E}(S^C, V_A) = 0$, the conflation $V_A \to Y \to S^C$ realizes the split $\mathbb{E}$-extension. It follows that $Y \cong V_A \oplus S^C \in \mathcal{N}$. We obtain $X \in \mathcal{N}$ by Lemma 4.18 (2), and thus $B \in \mathcal{N}$ by Lemma 4.18 (1).

(2) Resolve A by a conflation $A \to V_A \to S_A$ with $S_A \in \mathcal{S}$, $V_A \in \mathcal{V}$. Then by Proposition 3.15, we obtain the rightmost commutative diagram above, made of conflations. Since $B, S_A \in \mathcal{N}$, we have $G \in \mathcal{N}$ by (1). From Lemma 4.18 (1), it follows that $C \in \mathcal{N}$.

(3) is dual to (2). $\square$

Remark 5.4. *As a corollary, $\mathcal{N}$ becomes an extriangulated category by Remark 2.18. Almost by definition, the pair $(\mathcal{S}, \mathcal{V})$ is a cotorsion pair on $\mathcal{N}$.*

5.2 From admissible model structure to Hovey twin cotorsion pair

Throughout this section, let $\mathcal{M} = (\mathit{Fib}, \mathit{Cof}, \mathbb{W})$ be a model structure on $\mathscr{C}$, where $\mathit{Fib}, \mathit{Cof}, \mathbb{W}$ are the classes of fibrations, cofibrations, and weak equivalences. Let $w\mathit{Fib} = \mathit{Fib} \cap \mathbb{W}$ and $w\mathit{Cof} = \mathit{Cof} \cap \mathbb{W}$ denote the classes of acyclic fibrations and acyclic cofibrations, respectively. Associate full subcategories $\mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V} \subseteq \mathscr{C}$ as follows.

$$
\begin{aligned}
C \in \mathcal{S} &\Leftrightarrow (0 \to C) \in w\mathit{Cof}, \\
C \in \mathcal{T} &\Leftrightarrow (C \to 0) \in \mathit{Fib}, \\
C \in \mathcal{U} &\Leftrightarrow (0 \to C) \in \mathit{Cof}, \\
C \in \mathcal{V} &\Leftrightarrow (C \to 0) \in w\mathit{Fib}.
\end{aligned}
$$

Remark that these are full additive subcategories of $\mathscr{C}$, closed under isomorphisms and direct summands. In particular, the definition below makes sense.

Definition 5.5. $\mathcal{M}$ is called an *admissible model structure* if it satisfies the following conditions for any morphism $f \in \mathscr{C}(A,B)$.

(1) $f \in wCof$ if and only if it is an inflation with $\mathrm{Cone}(f) \in \mathcal{S}$.

(2) $f \in Fib$ if and only if it is a deflation with $\mathrm{CoCone}(f) \in \mathcal{T}$.

(3) $f \in Cof$ if and only if it is an inflation with $\mathrm{Cone}(f) \in \mathcal{U}$.

(4) $f \in wFib$ if and only if it is a deflation with $\mathrm{CoCone}(f) \in \mathcal{V}$.

We note that the model structures which might appear in [Pal] are not admissible.

Proposition 5.6. Let $\mathcal{M}$ be an admissible model structure. Then $\mathcal{P} = ((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ is a twin cotorsion pair on $(\mathscr{C},\mathbb{E},\mathfrak{s})$.

Proof. $\mathcal{S} \subseteq \mathcal{U}$ is obvious from the definition. Since a similar argument works for $(\mathcal{S},\mathcal{T})$, we show that $(\mathcal{U},\mathcal{V})$ is a cotorsion pair. Let us confirm the conditions (1) and (2) in Definition 4.1 since (3) is dual to (2).

(1) Let $(U,V) \in \mathcal{U} \times \mathcal{V}$ be any pair of objects, and let $\delta \in \mathbb{E}(U,V)$ be any element. Realize it as an $\mathbb{E}$-triangle $V \xrightarrow{v} B \xrightarrow{u} U \overset{\delta}{\dashrightarrow}$. Since $U \in \mathcal{U}$ and $u \in wFib$, there exists a section $s \in \mathscr{C}(U,B)$ of u. Thus δ splits by Corollary 3.5.

(2) Let $C \in \mathscr{C}$ be any object. Factorize the zero morphism $0\colon 0 \to C$ as a cofibration $i \in Cof$ followed by an acyclic fibration $u \in wFib$. Since $\mathcal{M}$ is admissible, we have conflations $0 \xrightarrow{i} D \xrightarrow{j} U$ and $V \to D \xrightarrow{u} C$ with $U \in \mathcal{U}, V \in \mathcal{V}$. This shows that j is an isomorphism, and thus we obtain a conflation $V \to U \to C$. $\square$

Proposition 5.7. Let $\mathcal{M}$ be an admissible model structure as above. Then the associated twin cotorsion pair $\mathcal{P}$ obtained in Proposition 5.6 is a Hovey twin cotorsion pair. Indeed, if we let $\mathcal{N}^i, \mathcal{N}^f \subseteq \mathscr{C}$ be as in Definition 4.14, then the following are equivalent for any object $N \in \mathscr{C}$:

(1) $N \in \mathcal{N}^i$; (2) $(0 \to N) \in \mathbb{W}$; (3) $(N \to 0) \in \mathbb{W}$; (4) $N \in \mathcal{N}^f$.

Proof. (1) $\Rightarrow$ (2) If $N \in \mathcal{N}^i$, there is a conflation $V \to S \xrightarrow{s} N$ with $V \in \mathcal{V}, S \in \mathcal{S}$ by definition. Thus $0 \to N$ can be factorized as an acyclic cofibration $0 \to S$ followed by an acyclic fibration $S \xrightarrow{s} N$.

It follows that $(0 \to N) \in wFib \circ wCof = \mathbb{W}$.

$(2) \Rightarrow (1)$ Factorize $0 \to N$ as an acyclic cofibration $0 \xrightarrow{i} D$ followed by an acyclic fibration $D \xrightarrow{u} N$. A similar argument as in the proof (2) of Proposition 5.6 gives a conflation $V \to S \to N$.

$(2) \Leftrightarrow (3)$ follows from the 2-out-of-3 property of $\mathbb{W}$.

$(3) \Leftrightarrow (4)$ is dual to $(1) \Leftrightarrow (2)$. □

5.3 From Hovey twin cotorsion pair to admissible model structure

Throughout this section, let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a Hovey twin cotorsion pair on $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$. In addition, we assume the following condition, analogous to the weak idempotent completeness ([Bu, Proposition 7.6]).

Condition 5.8 (WIC)**.** *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Consider the following conditions.*

(1) Let $f \in \mathscr{C}(A, B)$, $g \in \mathscr{C}(B, C)$ be any composable pair of morphisms. If $g \circ f$ is an inflation, then so is f.

(2) Let $f \in \mathscr{C}(A, B)$, $g \in \mathscr{C}(B, C)$ be any composable pair of morphisms. If $g \circ f$ is a deflation, then so is g.

With the assumption of Condition (WIC), we have the following analog of the nine lemma.

Lemma 5.9. Assume $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category satisfying Condition (WIC). Let

$$\begin{tikzcd}
K \arrow[d, "k"'] & K' \arrow[d, "k'"] & & \\
A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[r, dashed, "\delta"] & {} \\
A' \arrow[r, "x'"'] \arrow[d, dashed, "\kappa"'] & B' \arrow[r, "y'"'] \arrow[d, dashed, "\kappa'"] & C' \arrow[r, dashed, "\delta'"] & {} \\
{} & {} & &
\end{tikzcd}$$

be a diagram made of $\mathbb{E}$-triangles. Then for some $X \in \mathscr{C}$, we obtain $\mathbb{E}$-triangles

$$K \xrightarrow{m} K' \xrightarrow{n} X \overset{\nu}{\dashrightarrow} \quad \text{and} \quad X \xrightarrow{i} C \xrightarrow{c} C' \overset{\tau}{\dashrightarrow}$$

which make the following diagram commutative,

$$
\begin{tikzcd}
K \arrow[r, "m"] \arrow[d, "k"'] \arrow[dr, phantom, "\circlearrowright"] & K' \arrow[r, "n"] \arrow[d, "k'"] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[r, dashed, "\nu"] \arrow[d, "i"] & {} \\
A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, "c"] & {} \\
A' \arrow[r, "x'"'] \arrow[d, dashed, "\kappa"'] & B' \arrow[r, "y'"'] \arrow[d, dashed, "\kappa'"] & C' \arrow[r, dashed, "\delta'"] \arrow[d, dashed, "\tau"] & {} \\
{} & {} & {} &
\end{tikzcd}
\tag{15}
$$

in which, those (k, k', i), (a, b, c), (m, x, x'), (n, y, y') are morphisms of $\mathbb{E}$-triangles.

Proof. By (ET4)$^{\mathrm{op}}$, we obtain an $\mathbb{E}$-triangle $E \xrightarrow{f} B' \xrightarrow{y'\circ b} C' \overset{\theta}{\dashrightarrow}$ and a commutative diagram (α) as in:

$$
(\alpha)\quad
\begin{tikzcd}
K' \arrow[r, "d"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[r, "e"] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowright"] & A' \arrow[d, "x'"] \\
K' \arrow[r, "k'"'] & B \arrow[r, "b"'] \arrow[d, "y'\circ b"'] \arrow[dr, phantom, "\circlearrowright"] & B' \arrow[d, "y'"] \\
& C' \arrow[r, equal] & C'
\end{tikzcd}
\qquad
(\beta)\quad
\begin{tikzcd}
A \arrow[dr, "g"] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "a"'] & & \\
& E \arrow[r, "f"'] \arrow[d, "e"] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[d, "b"] \\
& A' \arrow[r, "x'"'] & B'
\end{tikzcd}
$$

in $\mathscr{C}$, satisfying the following compatibilities:
(i) $K' \xrightarrow{d} E \xrightarrow{e} A' \overset{x'^{*}\kappa'}{\dashrightarrow}$ is an $\mathbb{E}$-triangle, (ii) $\delta' = e_{*}\theta$, and (iii) $d_{*}\kappa' = y'^{*}\theta$.

By the dual of Lemma 3.13, the upper-right square $x' \circ e = b \circ f$ is a weak pullback. Thus there exists a morphism $g \in \mathscr{C}(A, E)$ which makes the diagram (β) commutative. By Condition (WIC), this g becomes an inflation. Complete it into an $\mathbb{E}$-triangle $A \xrightarrow{g} E \xrightarrow{h} X \overset{\mu}{\dashrightarrow}$. By Lemma 3.14, we obtain a commutative diagram (γ)

$$
(\gamma)\quad
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[r, "h"] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d, "i"] \\
A \arrow[r, "x"'] & B \arrow[r, "y"'] \arrow[d, "y' \circ b"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "c"] \\
 & C' \arrow[r, equal] & C'
\end{tikzcd}
\qquad
(\delta)\quad
\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "m"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[r, "a"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & A' \arrow[d, equal] \\
K' \arrow[r, "d"] \arrow[d, "n"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[r, "e"] \arrow[d, "h"] & A' \\
X \arrow[r, equal] & X &
\end{tikzcd}
$$

made of conflations, which satisfies:
(iv) $X \xrightarrow{i} C \xrightarrow{c} C' \overset{h_*\theta}{\dashrightarrow}$ is an $\mathbb{E}$-triangle, (v) $\mu = i^*\delta$, (vi) $g_*\delta = c^*\theta$.

By Proposition 3.17, we obtain an $\mathbb{E}$-triangle $K \xrightarrow{m} K' \xrightarrow{n} X \xrightarrow{\nu}$ which makes the diagram (δ) commutative in $\mathscr{C}$, and satisfies:
(vii) $m_*\kappa = x'^*\kappa'$, (viii) $\mu = k_*\nu$, (ix) $h^*\nu + e^*\kappa = 0$.

Put $\tau = h_*\theta$. It is straightforward to show that the diagram (15) is indeed commutative. Moreover, (v) and (viii) show $k_*\nu = i^*\delta$, (ii) and (vi) show $a_*\delta = c^*\delta'$, (vii) shows $m_*\kappa = x'^*\kappa'$, (iii) shows $n_*\kappa' = y'^*\tau$. $\square$

Remark 5.10. *In the proof of Lemma 5.9, we have obtained an extra compatibility* (ix)*. This can be interpreted by the following analog of* Ext^2*-group.*

Let $A, D \in \mathscr{C}$ be any pair of objects. We denote triplet of $X \in \mathscr{C}, \sigma \in \mathbb{E}(D,X), \tau \in \mathbb{E}(X,A)$ by (σ, X, τ). For any pair of such triplets (σ, X, τ) and (σ', X', τ'), we write as

$$(\sigma, X, \tau) \underset{x}{\rightsquigarrow} (\sigma', X', \tau') \quad (\textit{or simply } (\sigma, X, \tau) \rightsquigarrow (\sigma', X', \tau'))$$

if and only if there exists $x \in \mathscr{C}(X, X')$ satisfying $x_*\sigma = \sigma'$ and $\tau = x^*\tau'$.

Let $\sim$ be the equivalence relation generated by $\rightsquigarrow$, and denote the equivalence class of (σ, X, τ) by $\tau \underset{X}{\circ} \sigma$. Let us denote their collection by

$$\mathbb{E}^2(D, A) = \Big(\coprod_{X \in \mathscr{C}} \mathbb{E}(D,X) \times \mathbb{E}(X,A) \Big) / \sim .$$

The proof of Lemma 5.9 shows

θ E h) $(\delta', A', -\kappa)$, , $*$ $_e$ ($\leftsquigarrow$

h

Definition 5.11. Define classes of morphisms $\mathit{Fib}, w\mathit{Fib}, \mathit{Cof}, w\mathit{Cof}$ and $\mathbb{W}$ in $\mathscr{C}$ as follows.

(1) $f \in \mathit{Fib}$ if it is a deflation with $\mathrm{CoCone}(f) \in \mathcal{T}$.

(2) $f \in w\mathit{Fib}$ if it is a deflation with $\mathrm{CoCone}(f) \in \mathcal{V}$.

(3) $f \in \mathit{Cof}$ if it is an inflation with $\mathrm{Cone}(f) \in \mathcal{U}$.

(4) $f \in w\mathit{Cof}$ if it is an inflation with $\mathrm{Cone}(f) \in \mathcal{S}$.

(5) $\mathbb{W} = w\mathit{Fib} \circ w\mathit{Cof}$.

Claim 5.12.

(1) If a conflation $A \xrightarrow{f} B \to N$ satisfies $N \in \mathcal{N}$, then f belongs to $\mathbb{W}$.

(2) If a conflation $N \to A \xrightarrow{f} B$ satisfies $N \in \mathcal{N}$, then f belongs to $\mathbb{W}$.

Proof. This follows from Proposition 3.15. □

Proposition 5.13. $\mathit{Fib}, w\mathit{Fib}, \mathit{Cof}, w\mathit{Cof}$ are closed under composition.

Proof. For Fib, this follows from (ET4) and the extension-closedness of $\mathcal{T}$. Similarly for the others. □

Proposition 5.14. We have the following.

(1) $w\mathit{Cof}$ satisfies the left lifting property against Fib.

(2) $w\mathit{Fib}$ satisfies the right lifting property against Cof.

Proof. (1) Suppose we are given a commutative square

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "g"] \\ B \arrow[r, "b"'] & D \end{tikzcd} \tag{16}$$

in $\mathscr{C}$, satisfying $f \in w\mathit{Cof}$ and $g \in \mathit{Fib}$. By definition, there are $\mathbb{E}$-triangles $A \xrightarrow{f} B \xrightarrow{s} S \overset{\delta}{\dashrightarrow}$ and $T \xrightarrow{t} C \xrightarrow{g} D \overset{\kappa}{\dashrightarrow}$. By Corollary 3.12,

$$\mathscr{C}(B,T) \xrightarrow{\mathscr{C}(f,T)} \mathscr{C}(A,T) \to 0 \to \mathbb{E}(B,T) \xrightarrow{\mathbb{E}(f,T)} \mathbb{E}(A,T), \tag{17}$$

$$\mathscr{C}(A,T) \xrightarrow{\mathscr{C}(A,t)} \mathscr{C}(A,C) \xrightarrow{\mathscr{C}(A,g)} \mathscr{C}(A,D), \tag{18}$$

$$\mathscr{C}(B,C) \xrightarrow{\mathscr{C}(B,g)} \mathscr{C}(B,D) \xrightarrow{(\kappa_\sharp)_B} \mathbb{E}(B,T) \tag{19}$$

are exact. By the commutativity of (16), we have $\mathbb{E}(f,T)(b^*\kappa) = f^*b^*\kappa = a^*g^*\kappa = 0$ by Lemma 3.2. Exactness of (17) shows $\kappa_\sharp b = b^*\kappa = 0$. Thus by the exactness of (19), there exists $c \in \mathscr{C}(B,C)$ satisfying $g \circ c = b$. Then $a - c \circ f \in \mathscr{C}(A,C)$ satisfies $g \circ (a - c \circ f) = g \circ a - b \circ f = 0$. By the exactness of (18), there is $c' \in \mathscr{C}(A,T)$ satisfying $t \circ c' = a - c \circ f$. By the exactness of (17), there is $c'' \in \mathscr{C}(A,T)$ satisfying $c'' \circ f = c'$. If we put $h = c + t \circ c'' \in \mathscr{C}(B,C)$, it satisfies $h \circ f = c \circ f + t \circ c'' \circ f = c \circ f + t \circ c' = a$ and $g \circ h = g \circ c + g \circ t \circ c'' = b$.

(2) is dual to (1). $\square$

Proposition 5.15. $\mathrm{Mor}(\mathscr{C}) = wFib \circ Cof = Fib \circ wCof$.

Proof. We only show $\mathrm{Mor}(\mathscr{C}) = wFib \circ Cof$. Let $f \in \mathscr{C}(A,B)$ be any morphism. Resolve A by a conflation $A \xrightarrow{v_A} V_A \xrightarrow{u_A} U_A$ with $U_A \in \mathcal{U}$, $V_A \in \mathcal{V}$, and put $f' = \begin{bmatrix} f \\ v_A \end{bmatrix} : A \to B \oplus V_A$. By Corollary 3.16, it admits some conflation

$$A \xrightarrow{f'} B \oplus V_A \to C.$$

Resolve C by a conflation $V^C \to U^C \to C$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Then by Proposition 3.15, we obtain a diagram made of conflations as follows.

$$\begin{tikzcd}
 & V^C \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & V^C \arrow[d] \\
A \arrow[r, "m"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[r] \arrow[d, "e"] \arrow[dr, phantom, "\circlearrowright"] & U^C \arrow[d] \\
A \arrow[r, "f'"'] & B \oplus V_A \arrow[r] & C
\end{tikzcd}$$

We have $m \in Cof$. Moreover, for $p_B = [1\ 0] \in \mathscr{C}(B \oplus V_A, B)$, we have $p_B \circ e \in wFib \circ wFib = wFib$. Thus $f = (p_B \circ e) \circ m$ gives the desired factorization. $\square$

Proposition 5.16. $Fib, wFib, Cof, wCof$ are closed under retraction.

Proof. We only show the result for Fib. Suppose we are given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "a"'] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, "c"'] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[d, "f"] \\ B \arrow[r, "b"] \arrow[rr, bend right, "\mathrm{id}"'] & D \arrow[r, "d"] & B \end{tikzcd}$$

in $\mathscr{C}$, satisfying $g \in Fib$. By definition, there is an $\mathbb{E}$-triangle $T \xrightarrow{t} C \xrightarrow{g} D \overset{\theta}{\dashrightarrow}$ with $T \in \mathcal{T}$. By Condition (WIC), $d \circ b = \mathrm{id}$ implies that d is a deflation. Thus $d \circ g$ becomes a deflation by (ET4)$^{\mathrm{op}}$. Again by Condition (WIC), it follows that f is a deflation. Thus there exists an $\mathbb{E}$-triangle $X \xrightarrow{x} A \xrightarrow{f} B \overset{\delta}{\dashrightarrow}$. By (ET3)$^{\mathrm{op}}$, we obtain the following two morphisms of $\mathbb{E}$-triangles (on the left below).

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "k"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[r, "f"] \arrow[d, "a"] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, dashed, "\delta"] \arrow[d, "b"] & {} \\ T \arrow[r, "t"'] & C \arrow[r, "g"'] & D \arrow[r, dashed, "\theta"'] & {} \end{tikzcd} \quad \begin{tikzcd} T \arrow[r, "t"] \arrow[d, "\ell"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, "g"] \arrow[d, "c"] \arrow[dr, phantom, "\circlearrowright"] & D \arrow[r, dashed, "\theta"] \arrow[d, "d"] & {} \\ X \arrow[r, "x"'] & A \arrow[r, "f"'] & B \arrow[r, dashed, "\delta"'] & {} \end{tikzcd} \quad \begin{tikzcd} X \arrow[r, "x"] \arrow[d, "\ell\circ k"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[r, "f"] \arrow[d, "\mathrm{id}"] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, dashed, "\delta"] \arrow[d, "\mathrm{id}"] & {} \\ X \arrow[r, "x"'] & A \arrow[r, "f"'] & B \arrow[r, dashed, "\delta"'] & {} \end{tikzcd}$$

Composing them, we obtain a morphism of $\mathbb{E}$-triangles (on the right above). By Corollary 3.6, it follows that $\ell \circ k$ is an isomorphism. Especially k is a section, and thus $X \in \mathcal{T}$. This means that f belongs to Fib. $\square$

Lemma 5.17. Suppose that two morphisms $f \in \mathscr{C}(A, B)$ and $g \in \mathscr{C}(B, C)$ satisfy $f \in wCof$, $g \in Fib$ and $g \circ f \in wCof$. Then g belongs to $wFib$.

Proof. Let $h = g \circ f$. By assumption, there are conflations $A \xrightarrow{f} B \xrightarrow{s_1} S_1$, $T \xrightarrow{t} B \xrightarrow{g} C$, $A \xrightarrow{h} C \xrightarrow{s_2} S_2$ where $S_1 \in \mathcal{S}$, $T \in \mathcal{T}$ and $S_2 \in \mathcal{S}$. By the dual of Lemma 3.17, we obtain the following commutative diagram made of conflations.

$$\begin{tikzcd}
 & T \arrow[r, equal] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowright"] & T \arrow[d] \\
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "s_1"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & S_1 \arrow[d] \\
A \arrow[r, "h"'] & C \arrow[r, "s_2"'] & S_2
\end{tikzcd}$$

By Lemma 4.18 (2) and Remark 4.17, we obtain $T \in \mathcal{T} \cap \mathcal{N} = \mathcal{V}$. This means $g \in wFib$. $\square$

Proposition 5.18. $\mathbb{W}$ is closed under composition.

Proof. It suffices to show that $wCof \circ wFib \subseteq \mathbb{W}$. Let $a \in wFib$ and $x' \in wCof$. By Proposition 5.15, there are some $x \in wCof$ and $b \in Fib$ such that $b \circ x = x' \circ a$. It is thus enough to show that b belongs to $wFib$. By definition, there is a commutative diagram of $\mathbb{E}$-triangles:

$$\begin{tikzcd}
V \arrow[d, "k"'] & T \arrow[d, "k'"] & & \\
A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] & S \arrow[r, dashed, "\delta"] & {} \\
A' \arrow[r, "x'"'] \arrow[d, dashed, "\kappa"'] & B' \arrow[r, "y'"'] \arrow[d, dashed, "\kappa'"] & S' \arrow[r, dashed, "\delta'"] & {} \\
{} & {} & &
\end{tikzcd}$$

with $V \in \mathcal{V}$, $T \in \mathcal{T}$ and $S, S' \in \mathcal{S}$. Applying Lemma 5.9 gives some $X \in \mathscr{C}$ and two conflations $X \xrightarrow{i} S \xrightarrow{c} S'$ and $V \xrightarrow{m} T \xrightarrow{n} X$. The existence of the first conflation (and Lemma 4.18(2)) shows that X belongs to $\mathcal{N}$; that of the latter conflation and the dual of Lemma 4.19 imply that T belongs to $\mathcal{V}$, and therefore that $b \in wFib$. $\square$

Lemma 5.19. Suppose that two morphisms $f \in \mathscr{C}(A, B)$ and $g \in \mathscr{C}(B, C)$ satisfy $f \in wCof$, $g \in Fib$ and $g \circ f \in wFib$. Then g belongs to $wFib$.

Proof. Let $h = g \circ f$. By assumption, there are conflations $A \xrightarrow{f} B \xrightarrow{s} S$, $T \xrightarrow{t} B \xrightarrow{g} C$, $V \xrightarrow{v} A \xrightarrow{h} C$ with $S \in \mathcal{S}$, $T \in \mathcal{T}$ and $V \in \mathcal{V}$.

By Proposition 3.17, we obtain a conflation $V \to T \to S$. Thus from Proposition 5.3 and Remark 4.17, it follows $T \in \mathcal{T} \cap \mathcal{N} = \mathcal{V}$. This means $g \in wFib$. $\square$

Lemma 5.20. Suppose that two morphisms $f \in \mathscr{C}(A,B)$ and $g \in \mathscr{C}(B,C)$ satisfy $f \in wFib$, $g \in Fib$ and $g \circ f \in wFib$. Then g belongs to $wFib$.

Proof. Let $h = g \circ f$. By assumption, there are conflations $V_f \to A \xrightarrow{f} B$, $T \to B \xrightarrow{g} C$, $V_h \to A \xrightarrow{h} C$ with $V_f \in \mathcal{V}$, $T \in \mathcal{T}$ and $V_h \in \mathcal{V}$. By (ET4)$^{\mathrm{op}}$, we obtain a conflation $V_f \to V_h \to T$. Thus from Lemma 4.18 (1) and Remark 4.17, it follows that $T \in \mathcal{T} \cap \mathcal{N} = \mathcal{V}$. This means $g \in wFib$. $\square$

Proposition 5.21. Suppose that $f \in \mathscr{C}(A,B)$ and $g \in \mathscr{C}(B,C)$ satisfy $f \in \mathbb{W}$ and $g \circ f \in \mathbb{W}$. Then g also belongs to $\mathbb{W}$.

Proof. Let $h = g \circ f$. By definition and the dual of Proposition 5.15, the morphisms f, g, h can be factorized as

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "f_1"'] & \arrow[d, phantom, "\circlearrowright"] & B \\
& X_f \arrow[ur, "f_2"'] &
\end{tikzcd},
\quad
\begin{tikzcd}
B \arrow[rr, "g"] \arrow[dr, "g_1"'] & \arrow[d, phantom, "\circlearrowright"] & C \\
& X_g \arrow[ur, "g_2"'] &
\end{tikzcd},
\quad
\begin{tikzcd}
A \arrow[rr, "h"] \arrow[dr, "h_1"'] & \arrow[d, phantom, "\circlearrowright"] & C \\
& X_h \arrow[ur, "h_2"'] &
\end{tikzcd},
$$

with $f_1, g_1, h_1 \in wCof$, $f_2, h_2 \in wFib$ and $g_2 \in Fib$.

By Proposition 5.18, the morphism $g_1 \circ f_2$ belongs to $\mathbb{W}$ and can thus be factorized as an acyclic cofibration $w_1 \colon X_f \to X$ followed by an acyclic fibration $w_2 \colon X \to X_g$. By Proposition 5.14, there is $k \in \mathscr{C}(X_h, X)$ which makes $(*)$ commutative in $\mathscr{C}$. By Proposition 5.15, we can factorize k as an acyclic cofibration $k_1 \colon X_h \to X_k$ followed by a fibration $k_2 \colon X_k \to X$. Thus we obtain the following commutative diagram on the right

$$
(*)\quad
\begin{tikzcd}
A \arrow[r, "w_1 \circ f_1"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\circlearrowright", near start] & X \arrow[d, "g_2 \circ w_2"] \\
X_h \arrow[ur, "k"] \arrow[r, "h_2"'] \arrow[ur, phantom, "\circlearrowright", near end] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "w_1 \circ f_1"] \arrow[dd, "h_1"'] & & X \arrow[dr, "w_2"] & \\
& X_k \arrow[ur, "k_2"] & & X_g \arrow[d, "g_2"] \\
X_h \arrow[ur, "k_1"] \arrow[rrr, "h_2"'] & & & C
\end{tikzcd}
\qquad
\begin{array}{l}
(k_1, h_1, w_1 \circ f_1 \in wCof) \\
(w_2, h_2 \in wFib) \\
(k_2, g_2 \in Fib)
\end{array}
$$

Lemma 5.17 shows $k_2 \in wFib$. On the other hand, Lemma 5.19 shows $g_2 \circ w_2 \circ k_2 \in wFib$. Thus Lemma 5.20 shows $g_2 \in wFib$. $\square$

Corollary 5.22. The class $\mathbb{W}$ satisfies the 2-out-of-3 condition.

Proof. This follows from Proposition 5.18, Proposition 5.21 and its dual. $\square$

When a category has enough pull-backs or enough push-outs, the fact that weak equivalences are stable under retracts follows from the other axioms (e.g. [Joy, Proposition E.1.3], attributed to Joyal–Tierney). However, that proof does not carry over to the setup of extriangulated categories[4]. The following lemma will thus be used for proving that the class $\mathbb{W}$ is closed under retracts.

Lemma 5.23. Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and $A \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ be $\mathbb{E}$-triangles. Suppose that $b \in \mathscr{C}(B, B')$ belongs to $\mathbb{W}$ and satisfies $b \circ x = x'$. Then there is $c \in \mathscr{C}(C, C')$ which belongs to $\mathbb{W}$ and gives a morphism of $\mathbb{E}$-triangles as follows.

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, "c"] & {} \\
A \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \arrow[r, dashed, "\delta'"'] & {}
\end{tikzcd}$$

Proof. By definition, b can be factorized as $b = v \circ s$, using $\mathbb{E}$-triangles $B \xrightarrow{s} P \to S \overset{\theta}{\dashrightarrow}$ and $V \to P \xrightarrow{v} B' \overset{\tau}{\dashrightarrow}$ with $S \in \mathcal{S}, V \in \mathcal{V}$. By (ET4), and then by the dual of Proposition 3.17, we obtain the following commutative diagrams made of $\mathbb{E}$-triangles,

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "s"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, "\exists c_1"] & {} \\
A \arrow[r, "s\circ x"'] & P \arrow[r, "\exists p"'] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \exists Q \arrow[r, dashed, "\exists \nu"] \arrow[d] & {} \\
& S \arrow[r, equal] \arrow[d, dashed, "\theta"'] & S \arrow[d, dashed, "y_*\theta"] & \\
& {} & {} &
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
& A \arrow[r, equal] \arrow[d, "s\circ x"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[d, "x'"] & \\
V \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & P \arrow[r, "v"'] \arrow[d, "p"] \arrow[dr, phantom, "\circlearrowright"] & B' \arrow[r, dashed, "\tau"] \arrow[d, "y'"] & {} \\
V \arrow[r] & Q \arrow[r, "\exists c_2"'] \arrow[d, dashed, "\nu"'] & C' \arrow[r, dashed] \arrow[d, dashed, "\delta'"] & {} \\
& {} & {} &
\end{tikzcd}$$

[4]It turns out that mere existence of finite products and finite coproducts is enough, as follows from [Eg] and the fact that weak equivalences are precisely the morphisms that become isomorphisms in the localisation $\mathscr{C}[\mathbb{W}^{-1}]$. This is explained in detail in Pierre Cagne's PhD Thesis [Ca, Section 2.2]. We nonetheless include a different proof.

in which $c_1^*\nu = \delta$ holds. Then $c = c_2 \circ c_1$ belongs to $\mathbb{W}$, satisfies $c \circ y = c_2 \circ p \circ s = y' \circ v \circ s = y' \circ b$ and $c^*\delta' = c_1^* c_2^* \delta' = c_1^*\nu = \delta$. $\square$

Proposition 5.24. The class $\mathbb{W}$ is closed under retracts.

Proof. Suppose we are given a commutative diagram (a) in $\mathscr{C}$

$$
\text{(a)}\quad
\begin{tikzcd}
A \arrow[r, "a"'] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "f"'] & C \arrow[r, "c"'] \arrow[d, "g"] \arrow[u, phantom, "\circlearrowleft"] & A \arrow[d, "f"] \\
B \arrow[r, "b"] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[ur, phantom, "\circlearrowleft"] & D \arrow[r, "d"] \arrow[ur, phantom, "\circlearrowleft"] \arrow[d, phantom, "\circlearrowleft"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r, "c"] \arrow[d, "j"] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, "i"] \\
M \arrow[r, "m"'] \arrow[d, "x"'] \arrow[dr, phantom, "\circlearrowleft"] & N \arrow[r, "n"'] \arrow[d, "y"] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[d, "x"] \\
B \arrow[r, "b"'] & D \arrow[r, "d"'] & B
\end{tikzcd}
\quad\text{(b)}
$$

in which $g \in \mathbb{W}$. Let us show that $f \in \mathbb{W}$. If we decompose f and g as

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "i"'] & \circlearrowleft & B \\
& M \arrow[ur, "x"'] &
\end{tikzcd}
,\quad
\begin{tikzcd}
C \arrow[rr, "g"] \arrow[dr, "j"'] & \circlearrowleft & D \\
& N \arrow[ur, "y"'] &
\end{tikzcd}
\qquad
\left(\begin{array}{c} i \in \mathit{Cof},\ j \in w\mathit{Cof}, \\ x, y \in w\mathit{Fib} \end{array}\right)
$$

by Proposition 5.15, then there exist morphisms m, n which make diagram (b) commutative by Proposition 5.14.

By Corollary 5.22 applied to the lower half, it follows $n \circ m \in \mathbb{W}$. By definition of Cof and $w\mathit{Cof}$, there are $\mathbb{E}$-triangles $A \xrightarrow{i} M \xrightarrow{p} U \overset{\rho}{\dashrightarrow}$, $C \xrightarrow{j} N \xrightarrow{q} S \overset{\tau}{\dashrightarrow}$ with $U \in \mathcal{U}$, $S \in \mathcal{S}$. It suffices to show $U \in \mathcal{S}$. Realize $c_*\tau$ by an $\mathbb{E}$-triangle $A \xrightarrow{j'} N' \xrightarrow{q'} S \overset{c_*\tau}{\dashrightarrow}$. Put $c' = \begin{bmatrix} -c \\ j \end{bmatrix} \colon C \to A \oplus N$. Then by an argument similar to that of the proof of Corollary 3.16, we can find a morphism of $\mathbb{E}$-triangles as on the left of (20) below which gives an $\mathbb{E}$-triangle $C \xrightarrow{c'} A \oplus N \xrightarrow{[j'\ n_1]} N' \overset{q'^*\tau}{\dashrightarrow}$ (cf. [LN, Proposition 1.20]). Since we have $[i\ n] \circ c' = n \circ j - i \circ c = 0$, there is $n' \in \mathscr{C}(N', M)$ which satisfies $n' \circ [j'\ n_1] = [i\ n]$, namely $n' \circ j' = i$ and $n' \circ n_1 = n$.

Put $m' = n_1 \circ m$. This satisfies $n' \circ m' = n \circ m \in \mathbb{W}$ and $m' \circ i = j'$. Resolve N' by an $\mathbb{E}$-triangle $T' \to S' \xrightarrow{s'} N' \overset{\theta}{\dashrightarrow}$, with $S' \in \mathcal{S}$, $T' \in \mathcal{T}$. Then by the dual of Corollary 3.16, the morphism $[m'\ s'] \colon M \oplus S' \to N'$ can be completed into an $\mathbb{E}$-triangle ${}^{\exists}L \to M \oplus S' \xrightarrow{[m'\ s']} N' \dashrightarrow$. By the dual of Proposition 3.17, we obtain the following commutative diagram (right) made of conflations.

$$
\begin{tikzcd}
C \arrow[r, "j"] \arrow[d, "c"'] \arrow[dr, phantom, "\circlearrowright"] & N \arrow[r, "q"] \arrow[d, "n_1"] \arrow[dr, phantom, "\circlearrowright"] & S \arrow[r, dashed, "\tau"] \arrow[d, equal] & {} \\
A \arrow[r, "j'"'] & N' \arrow[r, "q'"'] & S \arrow[r, dashed, "c_*"'] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[r, equal] \arrow[d, "{\left[\begin{smallmatrix} i \\ 0 \end{smallmatrix}\right]}"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[d, "j'"] \\
L \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & M \oplus S' \arrow[r, "{[m'\ s']}"] \arrow[d, dashed, "p \oplus \mathrm{id}"] \arrow[dr, phantom, "\circlearrowright"] & N' \arrow[d, "q'"] \\
L \arrow[r, "\exists \ell"'] & U \oplus S' \arrow[r, "\exists k"'] & S
\end{tikzcd}
\tag{20}
$$

If we put $m_0 = n' \circ [m'\ s']$, then since $n' \circ m' = m_0 \circ \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, we see that $m_0 \in \mathbb{W}$ follows from $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \in wCof$ and $n' \circ m' \in \mathbb{W}$ by Corollary 5.22. Applying Lemma 5.23 to

$$
\begin{tikzcd}
A \arrow[r, "{\left[\begin{smallmatrix} i \\ 0 \end{smallmatrix}\right]}"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & M \oplus S' \arrow[r, "p \oplus \mathrm{id}"] \arrow[d, "m_0"] & U \oplus S' \arrow[r, dashed] & {} \\
A \arrow[r, "i"'] & M \arrow[r, "p"'] & U \arrow[r, dashed] & {}
\end{tikzcd}
$$

we obtain $u \in \mathscr{C}(U \oplus S', U)$ which belongs to $\mathbb{W}$ satisfying $u \circ (p \oplus \mathrm{id}_{S'}) = p \circ m_0$. Then since $u \circ \ell = 0$, we see that u factors through S, in the bottom $\mathbb{E}$-triangle in (20). Thus if we apply the functor $\sigma \colon \mathcal{U}/\mathcal{I} \to \mathcal{Z}/\mathcal{I}$, it follows $\sigma(\underline{u}) = 0$. On the other hand, it can be easily seen that $u \in \mathbb{W}$ implies that u can be written as composition of $U \oplus S' \xrightarrow{u'} U \oplus I \xrightarrow{[1\ 0]} U$, with $u' \in wCof$ and $I \in \mathcal{I}$. By (ET4), we can show that $\sigma(\underline{u'})$ is an isomorphism, and thus $\sigma(\underline{u})$ is an isomorphism. This means $\sigma(U) = 0$ in $\mathcal{Z}/\mathcal{I}$, which shows $U \in \mathcal{U} \cap \mathrm{CoCone}(\mathcal{I}, \mathcal{S}) \subseteq \mathcal{U} \cap \mathcal{N} = \mathcal{S}$. $\square$

By the argument so far, admissible model structures and Hovey twin cotorsion pairs on $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ correspond bijectively. Remark that, a model structure induces an equivalence

$$\mathscr{C}_{cf}/_{\sim} \xrightarrow{\simeq} \mathscr{C}[\mathbb{W}^{-1}].$$

Here, the right hand side is the localization $\ell \colon \mathscr{C} \to \mathscr{C}[\mathbb{W}^{-1}]$. The left hand side is the category of fibrant-cofibrant objects modulo homotopies. Let us describe it in terms of the corresponding Hovey twin cotorsion pair $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$.

- $X \in \mathscr{C}$ is fibrant if and only if $(X \to 0) \in Fib$, if and only if $X \in \mathcal{T}$. Dually, X is cofibrant if and only if $X \in \mathcal{U}$. Thus the full subcategory of fibrant-cofibrant objects in $\mathscr{C}$ agrees with $\mathcal{Z} \subseteq \mathscr{C}$.

- For any $X, Y \in \mathcal{Z}$, morphisms $f, g \in \mathcal{Z}(X, Y)$ satisfy $f \sim g$ if and only if $f - g$ factors through some object $I \in \mathcal{I}$.

Thus we have $\mathscr{C}_{cf}/_{\sim} = \mathcal{Z}/\mathcal{I}$. In summary, we obtain the following. This gives an explanation for the equivalence in [Na4, Proposition 6.10] and [IYa, Theorem 4.1].

Corollary 5.25. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, and let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a Hovey twin cotorsion pair. Then for $\mathbb{W} = wFib \circ wCof$ defined as above, we have an equivalence $\mathcal{Z}/\mathcal{I} \xrightarrow{\simeq} \mathscr{C}[\mathbb{W}^{-1}]$ which makes the following diagram commutative up to natural isomorphism.

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "\text{inclusion}"] \arrow[d, "\substack{\text{ideal}\\ \text{quotient}}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathscr{C} \arrow[d, "\ell"] \\
\mathcal{Z}/\mathcal{I} \arrow[r, "\simeq"'] & \mathscr{C}[\mathbb{W}^{-1}]
\end{tikzcd}$$

In particular, the map

$$(\mathcal{Z}/\mathcal{I})(X, Y) \to \mathscr{C}[\mathbb{W}^{-1}](X, Y)\ ;\ \overline{f} \mapsto \ell(f)$$

is an isomorphism for any $X, Y \in \mathcal{Z}$.

Remark 5.26. *By the generality of a model structure, we can also deduce that*

$$(\mathscr{C}/\mathcal{I})(U, T) \to \mathscr{C}[\mathbb{W}^{-1}](U, T)\ ;\ \overline{f} \mapsto \ell(f)$$

is an isomorphism for any $U \in \mathcal{U}$ and $T \in \mathcal{T}$. (This also follows from the adjoint property given in Remark 4.13.)

Remark 5.27. *If $\mathscr{C}$ is abelian, then we can show easily that $\mathbb{W}$ agrees with the class of morphisms f satisfying $\mathrm{Ker}(f) \in \mathcal{N}$ and $\mathrm{Cok}(f) \in \mathcal{N}$. Remark that $\mathcal{N} \subseteq \mathscr{C}$ becomes a Serre subcategory only when $\mathcal{N} = \mathscr{C}$. Indeed, if $\mathcal{N} \subseteq \mathscr{C}$ is a Serre subcategory, then the localization $\ell \colon \mathscr{C} \to \mathscr{C}[\mathbb{W}^{-1}]$ becomes an exact functor between abelian categories. Since any $C \in \mathscr{C}$ admits an inflation $C \to V$ to some $V \in \mathcal{V}$, this shows that $C = 0$ holds in $\mathscr{C}[\mathbb{W}^{-1}]$, which means $C \in \mathcal{N}$.*

6. Triangulation of the homotopy category

In this section, we assume $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ to be a Hovey twin cotorsion pair. Put $\widetilde{\mathscr{C}} = \mathscr{C}[\mathbb{W}^{-1}]$ and let $\ell\colon \mathscr{C} \to \widetilde{\mathscr{C}}$ be the localization functor. We will show that $\widetilde{\mathscr{C}}$ is triangulated (Theorem 6.20).

Remark 6.1. *We note that the category $\mathscr{C}$ is usually not complete and cocomplete, so that the model structure is not stable. However, axiom* (ET4) *gives specific choices of weak bicartesian squares which will compensate for the lack of stability.*

6.1 Shift functor

We first aim at defining a shift functor on the category $\widetilde{\mathscr{C}}$.

Definition 6.2. Let us fix a choice, for any object $A \in \mathscr{C}$, of an $\mathbb{E}$-triangle $A \xrightarrow{v_A} V_A \xrightarrow{u_A} U_A \overset{\rho_A}{\dashrightarrow}$, with $V_A \in \mathcal{V}$ and $U_A \in \mathcal{U}$. The functor $[1]\colon \mathscr{C} \to \widetilde{\mathscr{C}}$ is defined on objects by $A[1] = U_A$, and on morphisms as follows: Let $A \xrightarrow{f} B$ be a morphism in $\mathscr{C}$. Then there exists $u_f \in \mathscr{C}(U_A, U_B)$ which gives a morphism of $\mathbb{E}$-extensions $(f, u_f)\colon \rho_A \to \rho_B$. Indeed, since $\mathbb{E}(U_A, V_B) = 0$, one shows, by using the long exact sequence of Proposition 3.3 that there is a morphism $V_A \xrightarrow{v_f} V_B$ such that $v_f \circ v_A = v_B \circ f$. There is an induced morphism of $\mathbb{E}$-triangles as follows.

$$\begin{tikzcd}
A \arrow[r, "v_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & V_A \arrow[r, "u_A"] \arrow[d, "v_f"] \arrow[dr, phantom, "\circlearrowright"] & U_A \arrow[r, dashed, "\rho_A"] \arrow[d, "u_f"] & {} \\
B \arrow[r, "v_B"'] & V_B \arrow[r, "u_B"'] & U_B \arrow[r, dashed, "\rho_B"'] & {}
\end{tikzcd}$$

Define $f[1]$ to be the image $\ell(u_f)$ of u_f in $\widetilde{\mathscr{C}}$. Since $f_*\rho_A = u_f^*\rho_B$, the morphism u_f is uniquely defined in $\mathscr{C}/\mathcal{V}$ by Corollary 3.5. This implies that $f[1] = \ell(u_f)$ is well-defined.

Claim 6.3. *The functor* $[1]$ *does not essentially depend on the choices made. More precisely, fix any other choice of $\mathbb{E}$-triangles $A \xrightarrow{v'_A} V'_A \xrightarrow{u'_A} U'_A \overset{\rho'_A}{\dashrightarrow}$ and let $\{1\}$ be the functor defined as above by means of these $\mathbb{E}$-triangles. Then* $[1]$ *and* $\{1\}$ *are naturally isomorphic.*

Proof. The identity on A induces two morphisms of $\mathbb{E}$-extensions

$$(\mathrm{id}, {}^{\exists}t_A)\colon \rho_A \to \rho'_A \quad \text{and} \quad (\mathrm{id}, {}^{\exists}t'_A)\colon \rho'_A \to \rho_A.$$

Then, since both $(\mathrm{id}, t'_A \circ t_A)$, $(\mathrm{id}, \mathrm{id})\colon \rho_A \to \rho_A$ are morphisms of $\mathbb{E}$-extensions, we have $\ell(t'_A \circ t_A) = \ell(\mathrm{id}) = \mathrm{id}$ as in the argument in Definition 6.2. Similarly we have $\ell(t_A \circ t'_A) = \mathrm{id}$, and thus $\ell(t_A)$ is an isomorphism. Put $\tau_A = \ell(t_A)$, and let us show the naturality of $\tau = \{\tau_A \in \widetilde{\mathscr{C}}(A[1], A\{1\})\}_{A \in \mathscr{C}}$. Let $f \in \mathscr{C}(A, B)$ be any morphism. By definition, $f[1] = \ell(u)$ and $f\{1\} = \ell(u')$ are given by morphisms of $\mathbb{E}$-extensions $(f, u)\colon \rho_A \to \rho_B$ and $(f', u')\colon \rho'_A \to \rho'_B$. Then, since both $(f, t_B \circ u)$ and $(f, u' \circ t_A)$ are morphisms $\rho_A \to \rho'_B$, we obtain $\ell(t_B \circ u) = \ell(u' \circ t_A)$, namely $\tau_B \circ f[1] = f\{1\} \circ \tau_A$. □

We would like to show that $[1]\colon \mathscr{C} \to \widetilde{\mathscr{C}}$ induces an endofunctor of $\widetilde{\mathscr{C}}$. For this, it is enough to show that $[1]$ sends weak equivalences to isomorphisms. Since weak equivalences are compositions of a morphism in $w\mathit{Cof}$ followed by a morphism in $w\mathit{Fib}$, it is enough to show that $[1]$ inverts all morphisms in $w\mathit{Cof}$ and in $w\mathit{Fib}$.

Lemma 6.4. Let $j \in w\mathit{Cof}$. Then $j[1]$ is an isomorphism in $\widetilde{\mathscr{C}}$.

Proof. Let $A \xrightarrow{j} B$ be a morphism in $w\mathit{Cof}$. There is an $\mathbb{E}$-triangle $A \xrightarrow{j} B \to S \dashrightarrow$, with $S \in \mathcal{S}$. Then (ET4) gives morphisms $(*)$ of $\mathbb{E}$-triangles:

$$(*) \quad \begin{tikzcd} A \arrow[r, "j"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r] \arrow[d, "v_B"] \arrow[dr, phantom, "\circlearrowright"] & S \arrow[r, dashed] \arrow[d] & {} \\ A \arrow[r, "v'_A"'] & V_B \arrow[r, "u'_A"'] \arrow[d, "u_B"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\rho'"] \arrow[d, "w"] & {} \\ & U_B \arrow[r, equal] \arrow[d, dashed, "\rho_B"] & U_B \arrow[d, dashed] & \\ & {} & {} & \end{tikzcd} \qquad (**) \quad \begin{tikzcd} A \arrow[r, "v'_A"] \arrow[d, "j"'] \arrow[dr, phantom, "\circlearrowright"] & V_B \arrow[r, "u'_A"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\rho'_A"] \arrow[d, "w"] & {} \\ B \arrow[r, "v_B"'] & V_B \arrow[r, "u_B"'] & U_B \arrow[r, dashed, "\rho_B"'] & {} \end{tikzcd}$$

Since S and U_B belong to $\mathcal{U}$, so does C. Moreover, w is a weak equivalence by Claim 5.12 (2). Claim 6.3 allows to conclude that $j[1]$ is an isomorphism in $\widetilde{\mathscr{C}}$, since $(**)$ is a morphism of $\mathbb{E}$-triangles. □

Lemma 6.5. Let $q \in wFib$. Then $q[1]$ is an isomorphism in $\widetilde{\mathscr{C}}$.

Proof. Let $X \xrightarrow{q} Y$ be a morphism in $wFib$. It induces a morphism of $\mathbb{E}$-triangles:

$$\begin{tikzcd} X \arrow[r, "v_X"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowright"] & V_X \arrow[r, "u_X"] \arrow[d, "n"] \arrow[dr, phantom, "\circlearrowright"] & U_X \arrow[r, dashed, "\rho_X"] \arrow[d, "q[1]"] & {} \\ Y \arrow[r, "v_Y"'] & V_Y \arrow[r, "u_Y"'] & U_Y \arrow[r, dashed, "\rho_Y"'] & {} \end{tikzcd}$$

Since $V_X \to 0$ and $V_Y \to 0$ are acyclic fibrations, in particular they are weak equivalences. By the 2-out-of-3 property, this implies that $V_X \xrightarrow{n} V_Y$ is a weak equivalence. Factor n as an acyclic cofibration j followed by an acyclic fibration p. Then (ET4) gives a diagram (α) of $\mathbb{E}$-triangles:

$$(\alpha)\quad \begin{tikzcd} X \arrow[r, "v_X"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & V_X \arrow[r, "u_X"] \arrow[d, "j"] \arrow[dr, phantom, "\circlearrowright"] & U_X \arrow[d] \\ X \arrow[r, "j \circ v_X"'] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d] \\ & S \arrow[r, equal] & S \end{tikzcd} \qquad (\beta)\quad \begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & V' \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & D \arrow[r, dashed] \arrow[d] & {} \\ X \arrow[r, "j \circ v_X"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed] \arrow[d, "q'"] & {} \\ Y \arrow[r, "v_Y"] \arrow[d, dashed] & V_Y \arrow[r, "u_Y"] \arrow[d, dashed] & U_Y \arrow[r, dashed, "\rho_Y"] \arrow[d, dashed] & {} \\ {} & {} & {} \end{tikzcd}$$

where $S \in \mathcal{S}$. We have $B \in \mathcal{N}$ (since $V_X, S \in \mathcal{N}$) and $C \in \mathcal{U}$ (since $U_X, S \in \mathcal{U}$). The nine Lemma 5.9 gives morphisms of $\mathbb{E}$-triangles as in (β), with $V, V' \in \mathcal{V}$, and thus $B \in \mathcal{V}$. This implies $D \in \mathcal{N}$ and thus q' is a weak equivalence by Claim 5.12 (2). We conclude that $q[1]$ is an isomorphism in $\widetilde{\mathscr{C}}$ in the same manner as in the end of the proof of Lemma 6.4. $\square$

Corollary 6.6. The functor $[1]\colon \mathscr{C} \to \widetilde{\mathscr{C}}$ induces an endofunctor of $\widetilde{\mathscr{C}}$, which we denote by the same symbol [1]. Thus $f[1]$ for a morphism f in $\mathscr{C}$ will be denoted also by $\ell(f)[1]$ in the rest.

Proof. By Lemma 6.4 and Lemma 6.5, the functor $[1]\colon \mathscr{C} \to \widetilde{\mathscr{C}}$ inverts all weak equivalences. By the universal property of the localization of a category, it induces a functor $[1]\colon \widetilde{\mathscr{C}} \to \widetilde{\mathscr{C}}$. $\square$

Definition 6.7. By using $\mathbb{E}$-triangles $T^C \xrightarrow{t^C} S^C \xrightarrow{s^C} C \overset{\lambda^C}{\dashrightarrow}$ one can define dually a functor $[-1] \colon \mathscr{C} \to \widetilde{\mathscr{C}}$ with $C[-1] = T^C$, which induces an endofunctor $[-1]$ of $\widetilde{\mathscr{C}}$.

6.2 Connecting morphism

Define the bifunctor $\widetilde{\mathbb{E}} \colon \widetilde{\mathscr{C}}^{\mathrm{op}} \times \widetilde{\mathscr{C}} \to \widetilde{\mathscr{C}}$ by $\widetilde{\mathbb{E}} = \widetilde{\mathscr{C}}(-,-[1])$. In this section, we will construct a homomorphism $\widetilde{\ell} = \widetilde{\ell}_{C,A} \colon \mathbb{E}(C,A) \to \widetilde{\mathbb{E}}(C,A)$ for each pair $A, C \in \mathscr{C}$. For any $A \in \mathscr{C}$, we continue to use the $\mathbb{E}$-triangle

$$A \xrightarrow{v_A} V_A \xrightarrow{u_A} U_A \overset{\rho_A}{\dashrightarrow} \tag{21}$$

chosen to define the shift functor $[1]$.

Lemma 6.8. Let $A, C \in \mathscr{C}$ be any pair of objects. If $f_1, f_2 \in \mathscr{C}(X, U_A)$ satisfy $f_1^* \rho_A = f_2^* \rho_A$, then $\ell(f_1) = \ell(f_2)$ holds.

Proof. This immediately follows from the exactness of
$\mathscr{C}(X, V_A) \xrightarrow{u_A \circ -} \mathscr{C}(X, U_A) \xrightarrow{(\rho_A)_\sharp} \mathbb{E}(X, A)$ shown in Proposition 3.3. □

Definition 6.9. For any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C,A)$, define $\widetilde{\ell}(\delta) \in \widetilde{\mathbb{E}}(C,A)$ by the following. Take a span of morphisms $(C \xleftarrow{w} D \xrightarrow{d} U_A)$ from some $D \in \mathscr{C}$, which satisfy

$$w \in wFib \quad \text{and} \quad w^* \delta = d^* \rho_A. \tag{22}$$

Then, define as $\widetilde{\ell}(\delta) = \ell(d) \circ \ell(w)^{-1}$.

Claim 6.10. *For any $\delta \in \mathbb{E}(C,A)$, the morphism $\widetilde{\ell}(\delta)$ in Definition 6.9 is well-defined. More precisely, the following holds.*

(1) Take any $D \in \mathscr{C}$, $w \in \mathscr{C}(D,C)$. If both $d_1, d_2 \in \mathscr{C}(D, U_A)$ satisfy $w^* \delta = d_i^* \rho_A$ $(i = 1, 2)$, then $\ell(d_1) = \ell(d_2)$ holds.

(2) If both spans $(C \xleftarrow{w_1} D_1 \xrightarrow{d_1} U_A)$ and $(C \xleftarrow{w_2} D_2 \xrightarrow{d_2} U_A)$ satisfy (22), then $\ell(d_1) \circ \ell(w_1)^{-1} = \ell(d_2) \circ \ell(w_2)^{-1}$ holds.

(3) There exists at least one span $(C \xleftarrow{w} D \xrightarrow{d} U_A)$ satisfying (22).

Proof. (1) This immediately follows from Lemma 6.8.

(2) Let $V_i \xrightarrow{v_i} D_i \xrightarrow{w_i} C$ $(i = 1, 2)$ be conflations. By Proposition 3.15, we have a commutative diagram made of conflations as follows.

$$\begin{tikzcd}
 & V_2 \arrow[r, equal] \arrow[d, "m_2"'] \arrow[dr, phantom, "\circlearrowright"] & V_2 \arrow[d, "v_2"] \\
V_1 \arrow[r, "m_1"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & {}^{\exists}D \arrow[r, "e_1"] \arrow[d, "e_2"] \arrow[dr, phantom, "\circlearrowright"] & D_2 \arrow[d, "w_2"] \\
V_1 \arrow[r, "v_1"'] & D_1 \arrow[r, "w_1"'] & C
\end{tikzcd}$$

If we put $w = w_1 \circ e_2 = w_2 \circ e_1$, then we have $w \in wFib \circ wFib = wFib$. If we put $k_1 = d_2 \circ e_1$ and $k_2 = d_1 \circ e_2$, then they give

$$\ell(k_1) \circ \ell(w)^{-1} = \ell(d_2 \circ e_1) \circ \ell(w_2 \circ e_1)^{-1} = \ell(d_2) \circ \ell(w_2)^{-1},$$
$$\ell(k_2) \circ \ell(w)^{-1} = \ell(d_1 \circ e_2) \circ \ell(w_1 \circ e_2)^{-1} = \ell(d_1) \circ \ell(w_1)^{-1}.$$

Since both k_1, k_2 satisfy $k_1^*\delta = e_1^*d_2^*\delta = e_1^*w_2^*\delta = w^*\delta$, $k_2^*\delta = e_2^*d_1^*\delta = e_2^*w_1^*\delta = w^*\delta$, we obtain $\ell(k_1) = \ell(k_2)$ by (1). Thus it follows that $\ell(d_2) \circ \ell(w_2)^{-1} = \ell(d_1) \circ \ell(w_1)^{-1}$.

(3) Realize δ by an $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$. Then Proposition 3.15 gives a commutative diagram made of $\mathbb{E}$-triangles

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "v_A"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
V_A \arrow[r] \arrow[d, "u_A"'] \arrow[dr, phantom, "\circlearrowright"] & {}^{\exists}D \arrow[r, "w"'] \arrow[d, "e"] & C \arrow[r, dashed, "(v_A)_*\delta"'] & {} \\
U_A \arrow[r, equal] \arrow[d, dashed, "\rho_A"'] & U_A \arrow[d, dashed, "x_*\rho_A"] & & \\
{} & {} & &
\end{tikzcd}$$

satisfying $w^*\delta + e^*\rho_A = 0$. Thus the span $(C \xleftarrow{w} D \xrightarrow{-e} U_A)$ satisfies (22). $\square$

Proposition 6.11. For any $A, C \in \mathscr{C}$, the map $\widetilde{\ell}\colon \mathbb{E}(C, A) \to \widetilde{\mathbb{E}}(C, A)$ is an additive homomorphism.

Proof. Let $\delta_1, \delta_2 \in \mathbb{E}(C, A)$ be any pair of elements. By (3) of Claim 6.10, we can find spans $(C \overset{w_1}{\longleftarrow} D_1 \overset{d_1}{\longrightarrow} U_A)$, $(C \overset{w_2}{\longleftarrow} D_2 \overset{d_2}{\longrightarrow} U_A)$ which give $\widetilde{\ell}(\delta_1)$, $\widetilde{\ell}(\delta_2)$. As in the proof of (2) in Claim 6.10, replacing (w_i, D_i) by a common (w, D), we may assume $D_1 = D_2 = D$ and $w_1 = w_2 = w$ from the start. Then by $w^*\delta_1 = d_1^*\rho_A$ and $w^*\delta_2 = d_2^*\rho_A$, we have $w^*(\delta_1 + \delta_2) = (d_1 + d_2)^*\rho_A$. Thus the span $(C \overset{w}{\longleftarrow} D \overset{d_1+d_2}{\longrightarrow} U_A)$ satisfies (22) for $\delta_1 + \delta_2$. This shows $\widetilde{\ell}(\delta_1 + \delta_2) = \ell(d_1 + d_2) \circ \ell(w)^{-1} = \ell(d_1) \circ \ell(w)^{-1} + \ell(d_2) \circ \ell(w)^{-1} = \widetilde{\ell}(\delta_1) + \widetilde{\ell}(\delta_2)$. $\square$

Lemma 6.12. For any $U \in \mathcal{U}$ and $T \in \mathcal{T}$, the map $\widetilde{\ell} \colon \mathbb{E}(U, T) \to \widetilde{\mathbb{E}}(U, T)$ is monomorphic.

Proof. Let $\delta \in \mathbb{E}(U, T)$ be any $\mathbb{E}$-extension. Realize it by an $\mathbb{E}$-triangle $T \overset{x}{\longrightarrow} A \overset{y}{\longrightarrow} U \overset{\delta}{\dashrightarrow}$. Let $T \overset{v_T}{\longrightarrow} V_T \overset{u_T}{\longrightarrow} U_T \overset{\rho_T}{\dashrightarrow}$ be the chosen $\mathbb{E}$-triangle, as before. By Proposition 3.15, we obtain a diagram made of $\mathbb{E}$-triangles

$$\begin{tikzcd}
T \arrow[r, "x"] \arrow[d, "v_T"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[r, "y"] \arrow[d, "m"] \arrow[dr, phantom, "\circlearrowright"] & U \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
V_T \arrow[r, "m'"] \arrow[d, "u_T"'] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[r, "e'"'] \arrow[d, "e"] & U \arrow[r, dashed, "(v_T)_*\delta"'] & {} \\
U_T \arrow[r, equal] \arrow[d, dashed, "\rho_T"'] & U_T \arrow[d, dashed, "x_*(\rho_T)"] & & \\
{} & {} & &
\end{tikzcd}$$

satisfying $e^*(\rho_T) + e'^*\delta = 0$. As the proof of (3) of Claim 6.10 suggests, we have $\widetilde{\ell}(\delta) = -\ell(e) \circ \ell(e')^{-1}$. By $\mathbb{E}(U, V_T) = 0$, we have $(v_T)_*\delta = 0$. Thus, replacing M by an isomorphic object, we may assume

$$M = V_T \oplus U, \ \ m' = \iota_{V_T} \colon V_T \to M, \ \ e' = p_U \colon M \to U,$$

where

$$V_T \underset{p_{V_T}}{\overset{\iota_{V_T}}{\rightleftarrows}} V_T \oplus U \underset{p_U}{\overset{\iota_U}{\leftrightarrows}} U$$

is a biproduct. Put $q = -e \circ \iota_U \colon U \to U_T$. Then we have $e = u_T \circ p_{V_T} - q \circ e'$. Since $\ell(p_{V_T}) = 0$, this gives $\ell(e) = -\ell(q) \circ \ell(e')$, namely

$$\widetilde{\ell}(\delta) = \ell(q). \tag{23}$$

On the other hand, since the morphism e'^* is a monomorphism, the equality

$$e'^*\delta = -e^*(\rho_T) = -p_{V_T}^* u_T^*(\rho_T) + e'^* q^*(\rho_T) = e'^* q^*(\rho_T)$$

implies

$$\delta = q^*(\rho_T). \tag{24}$$

By (23) and (24), it suffices to show $\ell(q) = 0 \ \Rightarrow\ q^*(\rho_T) = 0$. Assume $\ell(q) = 0$. Take $\mathbb{E}$-triangles $U_T \xrightarrow{z} Z \to S \dashrightarrow$ $(Z \in \mathcal{Z},\, S \in \mathcal{S})$, $T' \to I \xrightarrow{i} Z \dashrightarrow$ $(I \in \mathcal{I},\, T' \in \mathcal{T})$. Then by $\ell(z \circ q) = \ell(z) \circ 0 = 0$, the morphism $z \circ q \in \mathscr{C}(U, Z)$ factors through i by Remark 5.26. Namely, there exists $k \in \mathscr{C}(U, I)$ such that $z \circ q = i \circ k$. By (ET4)$^{\mathrm{op}}$, we obtain a diagram (a) made of conflations in which, N belongs to $\mathcal{N}$ by Lemma 4.18 (2). By the dual of Lemma 3.13, we obtain $j \in \mathscr{C}(U, N)$ which makes diagram (b) commutative. By $N \in \mathcal{N} = \mathrm{Cone}(\mathcal{V}, \mathcal{S})$ and $\mathbb{E}(U, \mathcal{V}) = 0$, this j factors through some $S_0 \in \mathcal{S}$, as in (c).

(a)
$$\begin{tikzcd}
T' \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & {}^{\exists}N \arrow[r, "{}^{\exists}p"] \arrow[d, "{}^{\exists}n"'] \arrow[dr, phantom, "\circlearrowright"] & U_T \arrow[d, "z"] \\
T' \arrow[r] & I \arrow[r, "i"'] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[d] \\
& S \arrow[r, equal] & S
\end{tikzcd}$$

(b)
$$\begin{tikzcd}
U \arrow[dr, "j"] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "k"'] & & \\
& N \arrow[r, "p"'] \arrow[d, "n"] \arrow[dr, phantom, "\circlearrowright"] & U_T \arrow[d, "z"] \\
& I \arrow[r, "i"'] & Z
\end{tikzcd}$$

(c)
$$\begin{tikzcd}
U \arrow[dr, "j"] \arrow[drr, bend left, "q"] \arrow[dd] & & \\
& N \arrow[r, "p"'] & U_T \\
S_0 \arrow[ur, "{}^{\exists}r"'] & &
\end{tikzcd}$$

By $\mathbb{E}(S_0, T) = 0$, the morphism $p \circ r$ factors through u_T. This implies $q^*(\rho_T) = 0$ by Lemma 3.2. $\square$

Lemma 6.13. Let (A, δ, C), (A', δ', C') be $\mathbb{E}$-extensions, and let $(a, c)\colon \delta \to \delta'$ be a morphism of $\mathbb{E}$-triangles. Then $(\ell(a), \ell(c))\colon \widetilde{\ell}(\delta) \to \widetilde{\ell}(\delta')$ is a morphism of $\widetilde{\mathbb{E}}$-extensions. Namely,

$$\begin{tikzcd}
C \arrow[r, "\widetilde{\ell}(\delta)"] \arrow[d, "\ell(c)"'] \arrow[dr, phantom, "\circlearrowright"] & U_A = A[1] \arrow[d, "\ell(a)[1]"] \\
C' \arrow[r, "\widetilde{\ell}(\delta')"'] & U_{A'} = A'[1]
\end{tikzcd}$$

is commutative in $\widetilde{\mathscr{C}}$.

Proof. Take spans $(C \xleftarrow{w} D \xrightarrow{d} U_A)$ and $(C' \xleftarrow{w'} D' \xrightarrow{d'} U_{A'})$ satisfying $w, w' \in wFib,\ w^*\delta = d^*\rho_A,\ w'^*\delta' = d'^*\rho_{A'}$. By definition, we have $\widetilde{\ell}(\delta) = \ell(d) \circ \ell(w)^{-1},\ \widetilde{\ell}(\delta') = \ell(d') \circ \ell(w')^{-1}$. Remark that $\ell(a)[1] = \ell(u)$ is given by a morphism of $\mathbb{E}$-triangles (α) below.

$$(\alpha)\quad \begin{tikzcd} A \arrow[r, "v_A"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & V_A \arrow[r, "u_A"] \arrow[d, "v"] \arrow[dr, phantom, "\circlearrowright"] & U_A \arrow[r, dashed, "\rho_A"] \arrow[d, "u"] & {} \\ A' \arrow[r, "v_{A'}"'] & V_{A'} \arrow[r, "u_{A'}"'] & U_{A'} \arrow[r, dashed, "\rho_{A'}"'] & {} \end{tikzcd} \qquad (\beta)\quad \begin{tikzcd} V' \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & \exists D'' \arrow[r, "\exists w''"] \arrow[d, "\exists f"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "c^*\nu'"] \arrow[d, "c"] & {} \\ V' \arrow[r] & D' \arrow[r, "w'"'] & C' \arrow[r, dashed, "\nu'"'] & {} \end{tikzcd}$$

Since $w' \in wFib$, there exists an $\mathbb{E}$-triangle $V' \to D' \xrightarrow{w'} C' \overset{\nu'}{\dashrightarrow}$ with $V' \in \mathcal{V}$. By realizing $c^*\nu'$, we obtain a morphism (β) of $\mathbb{E}$-triangles. Then both spans $(C \xleftarrow{w} D \xrightarrow{u \circ d} U_{A'})$ and $(C \xleftarrow{w''} D'' \xrightarrow{d' \circ f} U_{A'})$ satisfy

$$\begin{aligned} w^*(c^*\delta') &= w^*a_*\delta \ = \ a_*w^*\delta \\ &= a_*d^*\rho_A \ = \ d^*u^*\rho_{A'} \ = \ (u \circ d)^*\rho_{A'}, \\ w''^*(c^*\delta') = f^*w'^*\delta' &= f^*d'^*\rho_{A'} = (d' \circ f)^*\rho_{A'}. \end{aligned}$$

Thus by Claim 6.10 (2), we obtain

$$\begin{aligned} \ell(u \circ d) \circ \ell(w)^{-1} &= \ell(d' \circ f) \circ \ell(w'')^{-1} \\ &= \ell(d') \circ (\ell(f) \circ \ell(w'')^{-1}) \ = \ \ell(d') \circ (\ell(w')^{-1} \circ \ell(c)), \end{aligned}$$

which means $\ell(u) \circ \widetilde{\ell}(\delta) = \widetilde{\ell}(\delta') \circ \ell(c)$. □

Proposition 6.14. The functor $[1]\colon \widetilde{\mathscr{C}} \to \widetilde{\mathscr{C}}$ is an auto-equivalence, with quasi-inverse $[-1]$.

Proof. By the definitions of $[-1]$ and $[1]$, for each $C \in \mathscr{C}$, there are $\mathbb{E}$-triangles $C[-1] \to S^C \to C \overset{\lambda^C}{\dashrightarrow}$ and $C[-1] \to V_{C[-1]} \to (C[-1])[1] \overset{\rho_{C[-1]}}{\dashrightarrow}$, where $S^C \in \mathcal{S}$, $V_{C[-1]} \in \mathcal{V}$. Then by Proposition 3.15, we have a commutative diagram $(*)$ made of conflations

$$(*)\quad \begin{tikzcd} C[-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & S^C \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, equal] \\ V_{C[-1]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \exists D \arrow[r, "w"] \arrow[d, "e"] & C \\ (C[-1])[1] \arrow[r, equal] & (C[-1])[1] & \end{tikzcd} \qquad (**)\quad \begin{tikzcd} C \arrow[r, "\widetilde{\ell}(\lambda^C)"] \arrow[d, "\ell(f)"'] \arrow[dr, phantom, "\circlearrowright"] & (C[-1])[1] \arrow[d, "{\ell(g)[1] = (\ell(f)[-1])[1]}"] \\ C' \arrow[r, "\widetilde{\ell}(\lambda^{C'})"'] & (C'[-1])[1] \end{tikzcd}$$

which gives $\widetilde{\ell}(\lambda^C) = -\ell(e) \circ \ell(w)^{-1}$ as in the proof of (3) in Claim 6.10. Since $e \in \mathbb{W}$, it follows that $\widetilde{\ell}(\lambda^C)$ is an isomorphism in $\widetilde{\mathscr{C}}$. Let us show the naturality of $\{\widetilde{\ell}(\lambda^C)\colon C \to (C[-1])[1]\}_{C\in\widetilde{\mathscr{C}}}$. For this purpose, it suffices to show the naturality with respect to the morphisms in $\mathscr{C}$. For any morphism $f \in \mathscr{C}(C, C')$, the morphism $\ell(f)[-1] = \ell(g) \in \widetilde{\mathscr{C}}(C[-1], C'[-1])$ is given by a morphism of $\mathbb{E}$-extensions $(g, f)\colon \lambda^C \to \lambda^{C'}$, dually to Definition 6.2. Thus by Lemma 6.13, the diagram $(**)$ becomes commutative. This shows $[1] \circ [-1] \cong \mathrm{Id}$. The isomorphism $[-1] \circ [1] \cong \mathrm{Id}$ can be shown dually. □

6.3 Triangulation

Definition 6.15. For an $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, its associated *standard triangle* in $\widetilde{\mathscr{C}}$ is defined to be $A \xrightarrow{\ell(x)} B \xrightarrow{\ell(y)} C \xrightarrow{\widetilde{\ell}(\delta)} A[1]$. A *distinguished triangle* in $\widetilde{\mathscr{C}}$ is a triangle isomorphic to some standard triangle.

Proposition 6.16. Any morphism of $\mathbb{E}$-triangles (left, below) gives the following morphism between standard triangles (right, below):

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, "c"] & {} \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \arrow[r, dashed, "\delta'"'] & {} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\ell(x)"] \arrow[d, "\ell(a)"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "\ell(y)"] \arrow[d, "\ell(b)"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, "\ell(\delta)"] \arrow[d, "\ell(c)"] \arrow[dr, phantom, "\circlearrowright"] & A[1] \arrow[d, "\ell(a)[1]"] \\ A' \arrow[r, "\ell(x')"'] & B' \arrow[r, "\ell(y')"'] & C' \arrow[r, "\ell(\delta')"'] & A'[1] \end{tikzcd}$$

Proof. This immediately follows from Lemma 6.13. □

This gives a cofibrant replacement of a standard triangle, as follows.

Corollary 6.17. Assume $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ satisfies Condition (WIC) as before. Any standard triangle is isomorphic to a standard triangle associated to an $\mathbb{E}$-triangle $U \to U' \to U'' \dashrightarrow$ whose terms satisfy $U, U', U'' \in \mathcal{U}$.

Proof. Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ be any $\mathbb{E}$-triangle. Resolve A by an $\mathbb{E}$-triangle $V \xrightarrow{v} U \xrightarrow{a} A \overset{\lambda}{\dashrightarrow}$ satisfying $U \in \mathcal{U}$ and $V \in \mathcal{V}$. By Proposition 5.15, we have $x \circ a \in wFib \circ Cof$. Namely, there are $\mathbb{E}$-triangles $U \xrightarrow{x'} B' \xrightarrow{y'} U_0 \overset{\delta'}{\dashrightarrow}$ and $V_0 \xrightarrow{v'} B' \xrightarrow{b} B \dashrightarrow$ satisfying $U_0 \in \mathcal{U}$, $V_0 \in \mathcal{V}$ and $x \circ a = b \circ x'$.

Since $\mathcal{U}$ is extension-closed, it follows that $B' \in \mathcal{U}$. Moreover, by Lemma 5.9 and Lemma 4.18 (1), we obtain a morphism (α) of $\mathbb{E}$-triangles

$$(\alpha)\quad \begin{tikzcd} U \arrow[r, "x'"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B' \arrow[r, "y'"] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & U_0 \arrow[r, dashed, "\delta'"] \arrow[d, "c"] & {} \\ A \arrow[r, "x"'] & B \arrow[r, "y"'] & C \arrow[r, dashed, "\delta"'] & {} \end{tikzcd} \qquad (\beta)\quad \begin{tikzcd} U \arrow[r, "\ell(x')"] \arrow[d, "\ell(a)"', "\cong"] \arrow[dr, phantom, "\circlearrowright"] & B' \arrow[r, "\ell(y')"] \arrow[d, "\ell(b)", "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & U_0 \arrow[r, "\widetilde{\ell}(\delta')"] \arrow[d, "\ell(c)", "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & U[1] \arrow[d, "\ell(a)[1]", "\cong"'] \\ A \arrow[r, "\ell(x)"'] & B \arrow[r, "\ell(y)"'] & C \arrow[r, "\widetilde{\ell}(\delta)"'] & A[1] \end{tikzcd}$$

which admits an $\mathbb{E}$-triangle $N \to U_0 \xrightarrow{c} C \dashrightarrow$ with some $N \in \mathcal{N}$. By Claim 5.12, it follows that $c \in \mathbb{W}$. By Proposition 6.16, we obtain an isomorphism (β) of standard triangles in $\widetilde{\mathscr{C}}$. $\square$

Remark 6.18. *Similarly, for any $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, we can construct a morphism of $\mathbb{E}$-triangles*

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, "c"] & {} \\ T_A \arrow[r] & T_B \arrow[r] & T_C \arrow[r, dashed] & {} \end{tikzcd}$$

which satisfies $T_A, T_B, T_C \in \mathcal{T}$ and $a, b, c \in \mathbb{W}$.

Lemma 6.19. Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ be any $\mathbb{E}$-triangle. From the $\mathbb{E}$-triangle $A \xrightarrow{v_A} V_A \xrightarrow{u_A} U_A \overset{\rho_A}{\dashrightarrow}$, we obtain an $\mathbb{E}$-triangle

$$A \xrightarrow{\begin{bmatrix} x \\ v_A \end{bmatrix}} B \oplus V_A \to X \overset{\theta}{\dashrightarrow} \tag{25}$$

by Corollary 3.16. Then the standard triangles associated to the $\mathbb{E}$-triangles (x, y, δ), and (25) are isomorphic. Moreover we have $V_A \in \mathcal{I}$ if $A \in \mathcal{U}$.

Proof. By the dual of Proposition 3.17, we obtain a commutative diagram made of $\mathbb{E}$-triangles, as follows.

$$\begin{tikzcd} & V_A \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & V_A \arrow[d] & \\ A \arrow[r, "\begin{bmatrix} x \\ v_A \end{bmatrix}"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \oplus V_A \arrow[r] \arrow[d, "{[1\ 0]}"] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[r, dashed, "\theta"] \arrow[d, "{}^{\exists}e"] & {} \\ A \arrow[r, "x"'] & B \arrow[r, "y"'] \arrow[d, dashed, "0"'] & C \arrow[r, dashed, "\delta"'] \arrow[d, dashed, "0"] & {} \\ & {} & {} & \end{tikzcd}$$

Remark that $\ell([1\ \ 0])\colon B \to B \oplus V_A$ and $\ell(e)\colon C \to Z$ are isomorphisms in $\widetilde{\mathscr{C}}$. Thus Lemma 6.19 follows from Proposition 6.16. $\square$

Theorem 6.20. *The shift functor in Definition 6.2 and the class of distinguished triangles in Definition 6.15 give a triangulation of* $\widetilde{\mathscr{C}}$.

Proof. (TR1) By definition, the class of distinguished triangles is closed under isomorphisms. From the $\mathbb{E}$-triangle $A \xrightarrow{\mathrm{id}_A} A \to 0 \dashrightarrow$, we obtain a distinguished triangle $A \xrightarrow{\mathrm{id}_A} A \to 0 \to A[1]$.

Let $\alpha \in \widetilde{\mathscr{C}}(A, B)$ be any morphism, and let us show the existence of a distinguished triangle of the form $A \xrightarrow{\alpha} B \to C \to A[1]$. Up to isomorphism in $\widetilde{\mathscr{C}}$, we may assume $A \in \mathcal{U}, B \in \mathcal{T}$ from the start. As in Remark 5.26, then there is a morphism $f \in \mathscr{C}(A, B)$ satisfying $\ell(f) = \alpha$.

By Corollary 3.16, we have an $\mathbb{E}$-triangle $A \to B \oplus V_A \overset{g}{\to} C \overset{\delta}{\dashrightarrow}$, where the first morphism is $\left[\begin{smallmatrix} f \\ v_A \end{smallmatrix}\right]$, which gives a standard triangle compatible with $\ell(f)$ as follows.

$$\begin{tikzcd}
A \arrow[r, "\begin{bmatrix} \ell(f) \\ \ell(v_A) \end{bmatrix}"] \arrow[dr, "\ell(f)"'] & B \oplus V_A \arrow[r, "\ell(g)"] \arrow[d, "\cong"] \arrow[dl, phantom, "\circlearrowright"] & C \arrow[r, "\widetilde{\ell}(\delta)"] & A[1] \\
& B & &
\end{tikzcd}$$

(TR2) It suffices to show this axiom for standard triangles. Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle, and let $A \xrightarrow{\ell(x)} B \xrightarrow{\ell(y)} C \xrightarrow{\widetilde{\ell}(\delta)} A[1]$ be its associated standard triangle. Let us show that $B \xrightarrow{\ell(y)} C \xrightarrow{\widetilde{\ell}(\delta)} A[1] \xrightarrow{-\ell(x)[1]} B[1]$ is distinguished. By Proposition 3.15, we obtain a commutative diagram made of $\mathbb{E}$-triangles

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "v_A"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "m"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
V_A \arrow[r] \arrow[d, "u_A"'] \arrow[dr, phantom, "\circlearrowright"] & \exists D \arrow[r, "w"'] \arrow[d, "e"] & C \arrow[r, dashed, "(v_A)_*\delta"'] & {} \\
U_A \arrow[r, equal] \arrow[d, dashed, "\rho_A"'] & U_A \arrow[d, dashed, "x_*\rho_A"] & & \\
{} & {} & &
\end{tikzcd}$$

satisfying $w^*\delta + e^*\rho_A = 0$. In particular, we obtain a distinguished triangle $B \xrightarrow{\ell(m)} D \xrightarrow{\ell(e)} A[1] \xrightarrow{\widetilde{\ell}(x_*\rho_A)} B[1]$. Remark that we have the following equality $\widetilde{\ell}(\delta) = -\ell(e) \circ \ell(w)^{-1}$, as the proof of (3) of Claim 6.10 suggests. Thus it remains to show the commutativity in $\widetilde{\mathscr{C}}$ of the right-most square of (a).

$$\text{(a)}\quad \begin{tikzcd} B \arrow[r, "\ell(m)"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & D \arrow[r, "\ell(e)"] \arrow[d, "\cong", "\ell(w)"'] \arrow[dr, phantom, "\circlearrowright"] & A[1] \arrow[r, "\widetilde{\ell}(x_*\rho_A)"] \arrow[d, "\cong", "-1"'] \arrow[dr, phantom, "\circlearrowright"] & B[1] \arrow[d, equal] \\ B \arrow[r, "\ell(y)"'] & C \arrow[r, "\widetilde{\ell}(\delta)"'] & A[1] \arrow[r, "-\ell(x)[1]"'] & B[1] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "v_A"] \arrow[d, "x"'] \arrow[dr, phantom, "\circlearrowright"] & V_A \arrow[r, "u_A"] \arrow[d, "v"] \arrow[dr, phantom, "\circlearrowright"] & U_A \arrow[r, dashed, "\rho_A"] \arrow[d, "u"] & {} \\ B \arrow[r, "v_B"'] & V_B \arrow[r, "u_B"'] & U_B \arrow[r, dashed, "\rho_B"'] & {} \end{tikzcd} \quad \text{(b)}$$

Let (b) be a morphism of $\mathbb{E}$-triangles, which gives $\ell(x)[1] = \ell(u)$. Then, since the span $(U_A \xleftarrow{\mathrm{id}} U_A \xrightarrow{u} U_B)$ satisfies $\mathrm{id}^*(x_*\rho_A) = x_*\rho_A = u^*\rho_B$, it follows that $\widetilde{\ell}(x_*\rho_A) = \ell(u) = \ell(x)[1]$.

(TR3) Up to isomorphism of triangles, it suffices to show this axiom for standard triangles. Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ be $\mathbb{E}$-triangles, and suppose we are given a commutative diagram $(*)$

$$(*)\quad \begin{tikzcd} A \arrow[r, "\ell(x)"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[d, "\beta"] \\ A' \arrow[r, "\ell(x')"'] & B' \end{tikzcd} \qquad\qquad (**)\quad \begin{tikzcd} A \arrow[r, "x_0"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \oplus V_A \arrow[d, "{[b\ j]}"] \\ A' \arrow[r, "x'"'] & B' \end{tikzcd}$$

in $\widetilde{\mathscr{C}}$. By Corollary 6.17 and Remark 6.18, we may assume $A, B, C \in \mathcal{U}$ and $A', B', C' \in \mathcal{T}$ from the start. In that case, α and β can be written as $\alpha = \ell(a)$, $\beta = \ell(b)$ for some $a \in \mathscr{C}(A, A')$ and $b \in \mathscr{C}(B, B')$ by Remark 5.26. Moreover, the commutativity of $(*)$ means that $b \circ x - x' \circ a$ factors through some $I \in \mathcal{I}$. By exactness of $\mathscr{C}(V_A, I) \xrightarrow{-\circ v_A} \mathscr{C}(A, I) \to \mathbb{E}(U_A, I) = 0$, this shows that there exists $j \in \mathscr{C}(V_A, B')$ which makes $(**)$ commutative in $\mathscr{C}$, where $x_0 = \left[\begin{smallmatrix} x \\ v_A \end{smallmatrix}\right]$. By Lemma 6.19, replacing $A \xrightarrow{x} B$ by $A \xrightarrow{x_0} B \oplus V_A$, we may assume $b \circ x = x' \circ a$ in $\mathscr{C}$ from the start. Now (TR3) follows from (ET3) and Proposition 6.16.

(TR4) Let $A \xrightarrow{\alpha} B \to D \to A[1]$, $B \xrightarrow{\beta} C \to F \to B[1]$ and $A \xrightarrow{\gamma} C \to E \to A[1]$ be any distinguished triangles in $\widetilde{\mathscr{C}}$ satisfying $\beta \circ \alpha = \gamma$. In a similar way as in the proof of (TR3), we may assume that they are standard triangles associated to $\mathbb{E}$-triangles $A \overset{f}{\to} B \overset{f'}{\to} D \overset{\delta_f}{\dashrightarrow}$,

$B \xrightarrow{g} C \xrightarrow{g'} F \overset{\delta_g}{\dashrightarrow}$, $A \xrightarrow{h} C \xrightarrow{h'} E \overset{\delta_h}{\dashrightarrow}$, satisfying $g \circ f = h$. By Lemma 3.14, we obtain a commutative diagram (α) made of $\mathbb{E}$-triangles:

$$(\alpha)\quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "f'"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & D \arrow[r, dashed, "\delta_f"] \arrow[d, "\exists d"] & {} \\
A \arrow[r, "h"'] & C \arrow[r, "h'"'] \arrow[d, "g'"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[r, dashed, "\delta_h"] \arrow[d, "\exists e"] & {} \\
& F \arrow[r, equal] \arrow[d, dashed, "\delta_g"'] & F \arrow[d, dashed, "f'_*\delta_g"] & \\
& {} & {} &
\end{tikzcd}
\qquad
(\beta)\quad
\begin{tikzcd}
A \arrow[r, "\ell(f)"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "\ell(f')"] \arrow[d, "\ell(g)"] \arrow[dr, phantom, "\circlearrowright"] & D \arrow[r, "\widetilde{\ell}(\delta_f)"] \arrow[d, "\ell(d)"] \arrow[dr, phantom, "\circlearrowright"] & A[1] \arrow[d, equal] \\
A \arrow[r, "\ell(h)"'] & C \arrow[r, "\ell(h')"] \arrow[d, "\ell(g')"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[r, "\widetilde{\ell}(\delta_h)"] \arrow[d, "\ell(e)"] \arrow[dr, phantom, "\circlearrowright"] & A[1] \arrow[d, "\ell(f)[1]"] \\
& F \arrow[r, equal] \arrow[d, "\widetilde{\ell}(\delta_g)"'] \arrow[dr, phantom, "\circlearrowright"] & F \arrow[r, "\widetilde{\ell}(\delta_g)"'] \arrow[d, "\widetilde{\ell}(f'_*\delta_g)"] & B[1] \\
& B[1] \arrow[r, "\ell(x')[1]"'] & D[1] &
\end{tikzcd}$$

Thus by Proposition 6.16, we obtain a diagram (β) made of distinguished triangles, as desired. □

The following argument ensures the dual arguments concerning distinguished triangles, in the following sections. Recall that the endofunctor $[-1]$ of $\widetilde{\mathscr{C}}$ induced from the chosen $\mathbb{E}$-triangle $T^C \xrightarrow{t^C} S^C \xrightarrow{s^C} C \overset{\lambda^C}{\dashrightarrow}$ with $S^C \in \mathcal{S}$, $T^C \in \mathcal{T}$, $T^C = C[-1]$, for each C, gives a quasi-inverse of $[1]$ by Proposition 6.14. Its proof shows that the isomorphisms $\widetilde{\ell}(\lambda^C)\colon C \to T^C[1]$ give a natural isomorphism $\mathrm{Id} \overset{\cong}{\Longrightarrow} [1] \circ [-1]$.

The dual construction of Definition 6.9 goes as follows.

Definition 6.21. For any $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, take a cospan of morphisms

$$(T^C \xrightarrow{m} E \xleftarrow{n} A) \tag{26}$$

to some $T \in \mathscr{C}$ satisfying

$$n \in wCof \quad \text{and} \quad n_*\delta = m_*\lambda^C. \tag{27}$$

Then, $\ell^\dagger(\delta) = \ell(n)^{-1} \circ \ell(m) \in \widetilde{\mathscr{C}}(C[-1], A)$ is well-defined.

With this definition, we can give a triangulation of $\widetilde{\mathscr{C}}$ by requiring the diagram $T^C \xrightarrow{\ell^\dagger(\delta)} A \xrightarrow{\ell(x)} B \xrightarrow{\ell(y)} C$ to be a left triangle. The following proposition (and its dual) shows that the resulting triangulation is the same as that defined in Definition 6.15.

Proposition 6.22. For any $\mathbb{E}$-triangle $A \overset{x}{\longrightarrow} B \overset{y}{\longrightarrow} C \overset{\delta}{\dashrightarrow}$, the diagram $T^C \overset{\ell^\dagger(\delta)}{\longrightarrow} A \overset{\ell(x)}{\longrightarrow} B \overset{\widetilde{\ell}(\lambda^C)\circ\ell(y)}{\longrightarrow} T^C[1]$ becomes a distinguished triangle in $\widetilde{\mathscr{C}}$, with respect to the triangulation given in Definition 6.15.

Proof. Take the standard triangle $A \overset{\ell(x)}{\longrightarrow} B \overset{\ell(y)}{\longrightarrow} C \overset{\widetilde{\ell}(\delta)}{\longrightarrow} A[1]$. Since $\widetilde{\mathscr{C}}$ is triangulated, by the converse of (TR2), it suffices to show the commutativity of the following diagram.

$$\begin{tikzcd}
A \arrow[r, "\ell(x)"] & B \arrow[r, "\widetilde{\ell}(\lambda^C)\circ\ell(y)"] \arrow[dr, "\ell(y)"'] \arrow[dr, phantom, shift left=3, "\circlearrowright"] & T^C[1] \arrow[r, "-\ell^\dagger(\delta)[1]"] \arrow[d, phantom, shift left=3, "\circlearrowright"] & A[1] \\
 & & C \arrow[u, "\cong"', "\widetilde{\ell}(\lambda^C)"] \arrow[ur, "\widetilde{\ell}(\delta)"'] &
\end{tikzcd}$$

As $\ell^\dagger(\delta)$ does not depend on the choice of a cospan (26), we may take it in the following way: By Proposition 3.15, we obtain a commutative diagram made of $\mathbb{E}$-triangles (a) below

$$\text{(a)}\quad
\begin{tikzcd}
 & T^C \arrow[r, equal] \arrow[d, "k"] \arrow[dr, phantom, "\circlearrowright"] & T^C \arrow[d, "t^C"] & \\
A \arrow[r, "n"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & \exists E \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & S^C \arrow[r, dashed, "(s^C)^*\delta"] \arrow[d, "s^C"] & {} \\
A \arrow[r, "x"'] & B \arrow[r, "y"'] \arrow[d, dashed, "y^*\lambda^C"'] & C \arrow[r, dashed, "\delta"'] \arrow[d, dashed, "\lambda^C"] & {} \\
 & {} & {} &
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "\widetilde{\ell}(\lambda^C)"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, equal] \arrow[d, "\widetilde{\ell}(\theta)"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "\widetilde{\ell}(\delta)"] \\
T^C[1] \arrow[r, "-\ell(k)[1]"'] & E[1] & A[1] \arrow[l, "\ell(n)[1]"]
\end{tikzcd}\quad\text{(b)}$$

satisfying $n_*\delta + k_*\lambda^C = 0$. Then the cospan $(T^C \overset{-k}{\longrightarrow} E \overset{n}{\longleftarrow} A)$ satisfies the desired property (27), and thus gives $\ell^\dagger(\delta) = -\ell(n)^{-1} \circ \ell(k)$. If we put $\theta = n_*\delta = -k_*\lambda^C$, then $(n, \mathrm{id}_C)\colon \delta \to \theta$ and $(-k, \mathrm{id}_C)\colon \lambda^C \to \theta$ are morphisms of $\mathbb{E}$-extensions. Thus (b) becomes commutative by Lemma 6.13. This shows $(\ell^\dagger(\delta)[1]) \circ \widetilde{\ell}(\lambda^C) = (-\ell(n)^{-1}[1] \circ \ell(k)[1]) \circ \widetilde{\ell}(\lambda^C) = \widetilde{\ell}(\delta)$. $\square$

7. Reduction and mutation via localization

7.1 Happel and Iyama-Yoshino's construction

Definition 7.1. An extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is said to be *Frobenius* if it satisfies the following conditions.

(1) $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ has enough injectives and enough projectives.

(2) $\mathrm{Proj}(\mathscr{C}) = \mathrm{Inj}(\mathscr{C})$.

Example 7.2. (1) If $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is an exact category, then this agrees with the usual definition ([Ha, section I.2]).

(2) Suppose that $\mathcal{T}$ is a triangulated category and $(\mathcal{Z}, \mathcal{Z})$ is an $\mathcal{I}$-mutation pair in the sense of [IYo, Definition 2.5]. Then $\mathcal{Z}$ becomes a Frobenius extriangulated category, with the extriangulated structure given in Remark 2.18.

Remark 7.3. *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, as before. By Remark 4.7, $((\mathcal{I}, \mathscr{C}), (\mathscr{C}, \mathcal{I}))$ is a twin cotorsion pair for some subcategory $\mathcal{I} \subseteq \mathscr{C}$ if and only if $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is Frobenius. Moreover, in that case $\mathcal{I} = \mathrm{Proj}(\mathscr{C}) = \mathrm{Inj}(\mathscr{C})$.*

The following can be regarded as a generalization of the constructions by Happel [Ha] and Iyama-Yoshino [IYo]. See also [Li] for the triangulated case.

Corollary 7.4. Let $(\mathscr{F}, \mathbb{E}, \mathfrak{s})$ be a Frobenius extriangulated category satisfying Condition (WIC), with $\mathcal{I} = \mathrm{Inj}(\mathscr{F})$. Then its stable category, namely the ideal quotient $\mathscr{F}/\mathcal{I}$, becomes triangulated.

Proof. Since $((\mathcal{I}, \mathscr{F}), (\mathscr{F}, \mathcal{I}))$ becomes a Hovey twin cotorsion pair with $\mathrm{Cone}(\mathcal{I}, \mathcal{I}) = \mathrm{CoCone}(\mathcal{I}, \mathcal{I}) = \mathcal{I}$, this follows from Corollary 5.25 and Theorem 6.20. □

Remark 7.5. *A direct proof for Corollary 7.4 is not difficult either, by imitating the proofs by [Ha] or [IYo], even without assuming Condition (WIC).*

Corollary 7.6. Let $\mathscr{C}$ be a category. Then $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is triangulated, as in Proposition 3.22 if and only if $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is a Frobenius extriangulated category and $\mathrm{Proj}(\mathscr{C}) = \mathrm{Inj}(\mathscr{C}) = 0$.

Proof. This follows from Corollary 7.4. □

7.2 Mutable cotorsion pairs

Lemma 7.7.

(1) For any weak equivalence $f \in \mathscr{C}(U,U')$ between $U,U' \in \mathcal{U}$, there exist $I \in \mathcal{I}$ and $i \in \mathscr{C}(U,I)$, with which $\left[\begin{smallmatrix} f \\ i \end{smallmatrix}\right] : U \to U' \oplus I$ becomes an acyclic cofibration.

(2) Dually, for any weak equivalence $g \in \mathscr{C}(T,T')$ between $T,T' \in \mathcal{T}$, there exist $J \in \mathcal{I}$ and $j \in \mathscr{C}(J,T')$, with which $[g\ j] \colon T \oplus J \to T'$ becomes an acyclic fibration.

Proof. We only show (1). Since $f \in \mathbb{W} = wFib \circ wCof$, there are $\mathbb{E}$-triangles

$$U \xrightarrow{m} E \to S \dashrightarrow, \tag{28}$$

$$V \to E \xrightarrow{e} U' \overset{\delta}{\dashrightarrow} \tag{29}$$

satisfying $S \in \mathcal{S}$, $V \in \mathcal{V}$ and $e \circ m = f$. By $\mathbb{E}(U',V) = 0$, we have $\delta = 0$. Thus we may assume $E = U' \oplus V$ and $e = [1\ 0]$ in (29), from the start.

By the extension-closedness of $\mathcal{U} \subseteq \mathscr{C}$, the $\mathbb{E}$-triangle (28) gives $U' \oplus V = E \in \mathcal{U}$, which implies $V \in \mathcal{I}$. Moreover by $e \circ m = f$, the acyclic cofibration $m \colon U \to U' \oplus V$ should be of the form $m = \left[\begin{smallmatrix} f \\ i \end{smallmatrix}\right]$, with some $i \in \mathscr{C}(U,V)$. $\square$

The following is an immediate consequence of the existence of the model structure.

Remark 7.8. *For any morphism $f \in \mathscr{C}(A,B)$ in $\mathscr{C}$, we have $f \in \mathbb{W}$ if and only if $\ell(f)$ is an isomorphism in $\widetilde{\mathscr{C}}$.*

For any extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$, let $\mathfrak{CP}(\mathscr{C})$ denote the class of cotorsion pairs on $\mathscr{C}$. Since $\widetilde{\mathscr{C}}$ is triangulated as shown in Theorem 6.20, we may use the usual notation $\mathrm{Ext}^1_{\widetilde{\mathscr{C}}}$ for $\widetilde{\mathbb{E}}$.

Definition 7.9. Let $\mathcal{P} = ((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ be a Hovey twin cotorsion pair on $\mathscr{C}$ and let $\ell \colon \mathscr{C} \to \widetilde{\mathscr{C}}$ be the associated localization functor as before. Define the class of *mutable cotorsion pairs on $\mathscr{C}$ with respect to $\mathcal{P}$* by

$$\mathfrak{M}_{\mathcal{P}} = \left\{ (\mathcal{A},\mathcal{B}) \in \mathfrak{CP}(\mathscr{C}) \;\middle|\; \begin{array}{c} \mathcal{S} \subseteq \mathcal{A} \subseteq \mathcal{U} \\ \mathcal{V} \subseteq \mathcal{B} \subseteq \mathcal{T} \end{array},\ \mathrm{Ext}^1_{\widetilde{\mathscr{C}}}(\ell(\mathcal{A}),\ell(\mathcal{B})) = 0 \right\}.$$

Here, $\ell(\mathcal{A}), \ell(\mathcal{B}) \subseteq \widetilde{\mathscr{C}}$ denote the essential images of $\mathcal{A}, \mathcal{B}$ under ℓ. Remark that $\mathcal{S} \subseteq \mathcal{A}$ is equivalent to $\mathcal{B} \subseteq \mathcal{T}$, and $\mathcal{A} \subseteq \mathcal{U}$ is equivalent to $\mathcal{V} \subseteq \mathcal{B}$, for any $(\mathcal{A}, \mathcal{B}) \in \mathfrak{CP}(\mathscr{C})$.

Theorem 7.10. *For any Hovey twin cotorsion pair $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ on $\mathscr{C}$, we have mutually inverse bijective correspondences*

$$\mathbb{R} = \mathbb{R}_{\mathcal{P}} \colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{CP}(\widetilde{\mathscr{C}}), \qquad \mathbb{I} = \mathbb{I}_{\mathcal{P}} \colon \mathfrak{CP}(\widetilde{\mathscr{C}}) \to \mathfrak{M}_{\mathcal{P}}$$

given by

$$\mathbb{R}((\mathcal{A}, \mathcal{B})) = (\ell(\mathcal{A}), \ell(\mathcal{B})), \quad \mathbb{I}((\mathcal{L}, \mathcal{R})) = (\mathcal{U} \cap \ell^{-1}(\mathcal{L}), \mathcal{T} \cap \ell^{-1}(\mathcal{R})).$$

Proof. It suffices to show the following. (1) For any mutable $(\mathcal{A}, \mathcal{B}) \in \mathfrak{M}_{\mathcal{P}}$, we have $\mathbb{R}((\mathcal{A}, \mathcal{B})) \in \mathfrak{CP}(\widetilde{\mathscr{C}})$. (2) For any $(\mathcal{L}, \mathcal{R}) \in \mathfrak{CP}(\widetilde{\mathscr{C}})$, we have $\mathbb{I}((\mathcal{L}, \mathcal{R})) \in \mathfrak{M}_{\mathcal{P}}$. (3) $\mathbb{I} \circ \mathbb{R} = \mathrm{id}$. (4) $\mathbb{R} \circ \mathbb{I} = \mathrm{id}$.

To distinguish, in this proof, let $\ell(X) \in \widetilde{\mathscr{C}}$ denote the image under ℓ of an object $X \in \mathscr{C}$.

(1) Since $\ell(\mathcal{A})$ is the essential image of $\mathcal{A}$ under ℓ, it is closed under isomorphisms and finite direct sums. The equality $\mathscr{C} = \mathrm{Cone}(\mathcal{B}, \mathcal{A})$ implies $\widetilde{\mathscr{C}} = \ell(\mathcal{A}) * \ell(\mathcal{B})[1]$, by Definition 6.15. Moreover $\mathrm{Ext}^1_{\widetilde{\mathscr{C}}}(\ell(\mathcal{A}), \ell(\mathcal{B})) = 0$ follows from the definition of $\mathfrak{M}_{\mathcal{P}}$.

It remains to show that $\ell(\mathcal{A}), \ell(\mathcal{B}) \subseteq \widetilde{\mathscr{C}}$ are closed under direct summands. To show that $\ell(\mathcal{A}) \subseteq \widetilde{\mathscr{C}}$ is closed under direct summands, it suffices to show $\ell(\mathcal{A}) = {}^{\perp}\ell(\mathcal{B})[1]$. Take any $X \in \mathscr{C}$, and suppose it satisfies $\mathrm{Ext}^1_{\widetilde{\mathscr{C}}}(\ell(X), \ell(\mathcal{B})) = 0$.

Let us show $\ell(X) \in \ell(\mathcal{A})$. By a cofibrant replacement, we may assume X belongs to $\mathcal{U}$. Resolve X by an $\mathbb{E}$-triangle in $\mathscr{C}$: $B \to A \to X \overset{\delta}{\dashrightarrow}$, with $A \in \mathcal{A}, B \in \mathcal{B}$. Since $\ell(\delta) = 0$ by assumption, we obtain $\delta = 0$ by Lemma 6.12. Thus X is a direct summand of A, which implies that X itself belongs to $\mathcal{A}$. Similarly for $\ell(\mathcal{B}) \subseteq \widetilde{\mathscr{C}}$.

(2) Put $\mathcal{A} = \mathcal{U} \cap \ell^{-1}(\mathcal{L})$, $\mathcal{B} = \mathcal{T} \cap \ell^{-1}(\mathcal{R})$. Since both $\mathcal{U}$ and $\ell^{-1}(\mathcal{L})$ are closed under isomorphisms, finite direct sums and direct summands, so is their intersection $\mathcal{A}$. Similarly for $\mathcal{B}$. By $\ell(\mathcal{S}) \subseteq \ell(\mathcal{N}) = 0$, we have $\mathcal{S} \subseteq \mathcal{A} \subseteq \mathcal{U}$. By Lemma 6.12, $\mathrm{Ext}^1_{\widetilde{\mathscr{C}}}(\ell(\mathcal{A}), \ell(\mathcal{B})) = 0$ implies $\mathbb{E}(\mathcal{A}, \mathcal{B}) = 0$. It remains to show that $\mathscr{C} = \mathrm{Cone}(\mathcal{B}, \mathcal{A}) = \mathrm{CoCone}(\mathcal{B}, \mathcal{A})$. Let us show that $\mathscr{C} = \mathrm{Cone}(\mathcal{B}, \mathcal{A})$.

Let $X \in \mathscr{C}$ be any object. By assumption, there exist $R \in \ell^{-1}(\mathcal{R})$, $L \in \ell^{-1}(\mathcal{L})$ and a distinguished triangle $\ell(R) \to \ell(L) \to \ell(X) \to \ell(R)[1]$ in $\widetilde{\mathscr{C}}$. By definition, it is isomorphic to the standard triangle associated to an $\mathbb{E}$-triangle, which we may assume to be of the form

$$R_0 \xrightarrow{x} L_0 \xrightarrow{y} Z \overset{\delta}{\dashrightarrow} \tag{30}$$

satisfying $R_0, L_0, Z \in \mathcal{Z}$, by a fibrant-cofibrant replacement (Corollary 6.17 and Remark 6.18). Thus we have an $\mathbb{E}$-triangle (30) satisfying $R_0 \in \mathcal{Z} \cap \ell^{-1}(\mathcal{R})$, $L_0 \in \mathcal{Z} \cap \ell^{-1}(\mathcal{L})$ and $Z \in \mathcal{Z}$, together with an isomorphism $\zeta \colon \ell(Z) \xrightarrow{\cong} \ell(X)$ in $\widetilde{\mathscr{C}}$. Resolve X by an $\mathbb{E}$-triangle $X \overset{t_X}{\to} T_X \overset{s_X}{\to} S_X \overset{\rho_X}{\dashrightarrow}$, with $T_X \in \mathcal{T}$, $S_X \in \mathcal{S}$. Then there exists a morphism $z \in \mathscr{C}(Z, T_X)$ which satisfies $\zeta = \ell(t_X)^{-1} \circ \ell(z)$. Since ζ is an isomorphism, it follows that $z \in \mathbb{W}$. By Lemma 7.7 (2), there exists an $\mathbb{E}$-triangle $V \to Z \oplus I \xrightarrow{[z\ i]} T_X \dashrightarrow$, with $V \in \mathcal{V}$, $I \in \mathcal{I}$. On the other hand by (ET2), we have an $\mathbb{E}$-triangle $R_0 \xrightarrow{x_0} L_0 \oplus I \xrightarrow{y_0} Z \oplus I \dashrightarrow$ from (30), where $x_0 = \begin{bmatrix} x \\ 0 \end{bmatrix}$, $y_0 = y \oplus \mathrm{id}_I$. Thus by (ET4)$^{\mathrm{op}}$, we obtain a diagram $(*)$

$$(*)\quad \begin{tikzcd} R_0 \arrow[r, "\exists r"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & \exists E \arrow[r] \arrow[d, "\exists e"] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d] \\ R_0 \arrow[r, "x_0"'] & L_0 \oplus I \arrow[r, "y_0"'] \arrow[d, "d"'] \arrow[dr, phantom, "\circlearrowright"] & Z \oplus I \arrow[d, "{[z\ i]}"] \\ & T_X \arrow[r, equal] & T_X \end{tikzcd} \qquad (**)\quad \begin{tikzcd} E \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & \exists F \arrow[r] \arrow[d, "\exists f"] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d, "t_X"] \\ E \arrow[r, "e"'] & L_0 \oplus I \arrow[r, "d"'] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & T_X \arrow[d, "s_X"] \\ & S_X \arrow[r, equal] & S_X \end{tikzcd}$$

made of conflations, where $d = [z\ \ i] \circ y_0 = [z \circ y\ \ i]$. Since $\ell(r)$ is an isomorphism in $\widetilde{\mathscr{C}}$, we have $E \in \ell^{-1}(\mathcal{R})$. Besides, $R_0, V \in \mathcal{T}$ implies $E \in \mathcal{T}$. By (ET4)$^{\mathrm{op}}$, we obtain a diagram $(**)$ made of conflations. Since $\ell(f)$ is an isomorphism in $\widetilde{\mathscr{C}}$, this shows $F \in \ell^{-1}(\mathcal{L})$. Resolve F by an $\mathbb{E}$-triangle $V^F \to U^F \to F \dashrightarrow$, with $U^F \in \mathcal{U}$, $V^F \in \mathcal{V}$. By (ET4)$^{\mathrm{op}}$, we obtain a diagram (A), below, made of conflations. Then in the $\mathbb{E}$-triangle $G \to U^F \to X \dashrightarrow$, we have the equalities $U^F \in \mathcal{U} \cap \ell^{-1}(\mathcal{L}) = \mathcal{A}$ and $G \in \mathcal{T} \cap \ell^{-1}(\mathcal{R}) = \mathcal{B}$.

$$(A)\qquad \begin{tikzcd} V^F \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & {}^{\exists}G \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[d] \\ V^F \arrow[r] & U^F \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & F \arrow[d] \\ & X \arrow[r, equal] & X \end{tikzcd}$$

(3) For any $(\mathcal{A},\mathcal{B})\in\mathfrak{M}_{\mathcal{P}}$, we have

$$\mathbb{I}\circ\mathbb{R}((\mathcal{A},\mathcal{B}))=(\mathcal{U}\cap\ell^{-1}(\ell(\mathcal{A})),\mathcal{T}\cap\ell^{-1}(\ell(\mathcal{B}))).$$

Obviously, $\mathcal{A}\subseteq\mathcal{U}\cap\ell^{-1}(\ell(\mathcal{A}))$ and $\mathcal{B}\subseteq\mathcal{T}\cap\ell^{-1}(\ell(\mathcal{B}))$ hold. Since both $(\mathcal{A},\mathcal{B})$ and $\mathbb{I}\circ\mathbb{R}((\mathcal{A},\mathcal{B}))$ are cotorsion pairs, this means $(\mathcal{A},\mathcal{B})=\mathbb{I}\circ\mathbb{R}((\mathcal{A},\mathcal{B}))$.

(4) For any $(\mathcal{L},\mathcal{R})\in\mathfrak{CP}(\widetilde{\mathscr{C}})$, we have

$$\mathbb{R}\circ\mathbb{I}((\mathcal{L},\mathcal{R}))=(\ell(\mathcal{U}\cap\ell^{-1}(\mathcal{L})),\ell(\mathcal{T}\cap\ell^{-1}(\mathcal{R}))),$$

which obviously satisfies $\ell(\mathcal{U}\cap\ell^{-1}(\mathcal{L}))\subseteq\mathcal{L}$ and $\ell(\mathcal{T}\cap\ell^{-1}(\mathcal{R}))\subseteq\mathcal{R}$. Similarly as in (3), it follows that $(\mathcal{L},\mathcal{R})=\mathbb{R}\circ\mathbb{I}((\mathcal{L},\mathcal{R}))$. $\square$

Claim 7.11. *For any $(\mathcal{A},\mathcal{B})\in\mathfrak{CP}(\mathscr{C})$ satisfying $\mathcal{S}\subseteq\mathcal{A}\subseteq\mathcal{U}$ (or equivalently $\mathcal{V}\subseteq\mathcal{B}\subseteq\mathcal{T}$), we have*

$$\mathcal{U}\cap\ell^{-1}\ell(\mathcal{A})=\mathcal{U}\cap\operatorname{CoCone}(\mathcal{A},\mathcal{S}),$$
$$\mathcal{T}\cap\ell^{-1}\ell(\mathcal{B})=\mathcal{T}\cap\operatorname{Cone}(\mathcal{V},\mathcal{B}).$$

Proof. We only show $\mathcal{U}\cap\ell^{-1}\ell(\mathcal{A})=\mathcal{U}\cap\operatorname{CoCone}(\mathcal{A},\mathcal{S})$.

$\mathcal{U}\cap\operatorname{CoCone}(\mathcal{A},\mathcal{S})\subseteq\mathcal{U}\cap\ell^{-1}\ell(\mathcal{A})$ is obvious. For the converse, let $U\in\mathcal{U}$ be any object satisfying $\ell(U)\cong\ell(A)$ in $\widetilde{\mathscr{C}}$ for some $A\in\mathcal{A}$. Resolve U by an $\mathbb{E}$-triangle $U\to Z\to S \dashrightarrow$, with $Z\in\mathcal{Z}$, $S\in\mathcal{S}$. Then $\ell(A)\cong\ell(Z)$ holds in $\widetilde{\mathscr{C}}$. Since $A\in\mathcal{U}, Z\in\mathcal{T}$, there is a morphism $f\in\mathscr{C}(A,Z)$ which gives the isomorphism $\ell(f)\colon\ell(A)\to\ell(Z)$ by Remark 5.26. Factorize this $f\in\mathbb{W}$ as $f=h\circ g$, with $g\in wCof$, $h\in wFib$. By definition, we have $\mathbb{E}$-triangles $V_0\to E\xrightarrow{h}Z\dashrightarrow$ and $A\xrightarrow{g}E\to S_0\dashrightarrow$, where $V_0\in\mathcal{V}$, $S_0\in\mathcal{S}$. Since $\mathbb{E}(Z,V_0)=0$, it follows that $V_0\oplus Z\cong E\in\mathcal{A}$, which implies $Z\in\mathcal{A}$. $\square$

The class $\mathfrak{M}_{\mathcal{P}}$ can be rewritten as follows.

Corollary 7.12. Let $\mathcal{P}$ be a Hovey twin cotorsion pair on $\mathscr{C}$. For any $(\mathcal{A},\mathcal{B})\in\mathfrak{CP}(\mathscr{C})$ satisfying $\mathcal{S}\subseteq\mathcal{A}\subseteq\mathcal{U}$ (or equivalently $\mathcal{V}\subseteq\mathcal{B}\subseteq\mathcal{T}$), the following are equivalent.

(1) $(\mathcal{A},\mathcal{B})\in\mathfrak{M}_{\mathcal{P}}$ i.e., it satisfies $\mathrm{Ext}^1_{\widetilde{\mathscr{C}}}(\ell(\mathcal{A}),\ell(\mathcal{B}))=0$.

(2) $\mathcal{U}\cap\ell^{-1}\ell(\mathcal{A})=\mathcal{A}$. (2′) $\mathcal{U}\cap\ell^{-1}\ell(\mathcal{A})\subseteq\mathcal{A}$.

(3) $\mathcal{T}\cap\ell^{-1}\ell(\mathcal{B})=\mathcal{B}$. (3′) $\mathcal{T}\cap\ell^{-1}\ell(\mathcal{B})\subseteq\mathcal{B}$.

Thus by Claim 7.11, we have

$$\begin{aligned}\mathfrak{M}_{\mathcal{P}} &= \{(\mathcal{A},\mathcal{B})\in\mathfrak{CP}(\mathscr{C})\mid \mathcal{S}\subseteq\mathcal{A}\subseteq\mathcal{U},\ \mathcal{U}\cap\mathrm{CoCone}(\mathcal{A},\mathcal{S})\subseteq\mathcal{A}\}\\ &= \{(\mathcal{A},\mathcal{B})\in\mathfrak{CP}(\mathscr{C})\mid \mathcal{V}\subseteq\mathcal{B}\subseteq\mathcal{T},\ \mathcal{T}\cap\mathrm{Cone}(\mathcal{V},\mathcal{B})\subseteq\mathcal{B}\}.\end{aligned}$$

Proof. We only show $(1)\Leftrightarrow(2)\Leftrightarrow(2)'$. The implication $(1)\Rightarrow(2)$ follows from Theorem 7.10 and $(2)\Rightarrow(2)'$ is obvious. Suppose that $(2)'$ is satisfied. Let us show $\mathrm{Ext}^1_{\widetilde{\mathscr{C}}}(\ell(A),\ell(B))=0$ for any pair of objects $A\in\mathcal{A}$, $B\in\mathcal{B}$. Resolve A by an $\mathbb{E}$-triangle $T^A\xrightarrow{t}S^A\to A\dashrightarrow$, with $T^A\in\mathcal{T}$, $S^A\in\mathcal{S}$ and T_A by $V^T\to Z^T\xrightarrow{z}T^A\dashrightarrow$, with $Z^T\in\mathcal{Z}$, $V^T\in\mathcal{V}$. Then we have $\ell(A)[-1]\cong\ell(T^A)\cong\ell(Z^T)$, and thus

$$\mathrm{Ext}^1_{\widetilde{\mathscr{C}}}(\ell(A),\ell(B))\cong\widetilde{\mathscr{C}}(\ell(Z^T),\ell(B))\cong(\mathscr{C}/\mathcal{I})(Z^T,B)$$

by Remark 5.26. Let us show that $(\mathscr{C}/\mathcal{I})(Z^T,B)=0$. Factorize $t\circ z$ as $t\circ z=h\circ g$, where $g\in\mathit{Cof}$, $h\in\mathit{wFib}$, to obtain a diagram

$$\begin{tikzcd} V^T \arrow[d] & {}^{\exists}V_0 \arrow[d] & \\ Z^T \arrow[r, "g"] \arrow[d, "z"'] \arrow[dr, phantom, "\circlearrowright"] & {}^{\exists}E \arrow[r] \arrow[d, "h"] & {}^{\exists}U_0 \\ T^A \arrow[r, "t"'] & S^A \arrow[r] & A \end{tikzcd} \qquad (U_0\in\mathcal{U},\ V_0\in\mathcal{V})$$

made of conflations. Since $\mathbb{E}(S^A,V_0)=0$, we have $E\cong S^A\oplus V_0$. Besides, by the extension-closedness of $\mathcal{U}\subseteq\mathscr{C}$, we have $E\in\mathcal{U}$, which shows that $V_0\in\mathcal{V}\cap\mathcal{U}=\mathcal{I}$. Thus it follows that $E\in\mathcal{S}$.

By Lemma 5.9, we obtain $X \in \mathscr{C}$ and conflations $V^T \to V_0 \to X$ and $X \to U_0 \xrightarrow{u} A$. By Lemma 4.18 (1), we have $X \in \mathcal{N}$, and thus $u \in \mathbb{W}$ by Claim 5.12. This shows $\ell(U_0) \cong \ell(A)$, which means that $U_0 \in \mathcal{U} \cap \ell^{-1}\ell(\mathcal{A}) \subseteq \mathcal{A}$ by assumption. Thus we obtain an exact sequence

$$\mathscr{C}(E,B) \xrightarrow{\mathscr{C}(g,B)} \mathscr{C}(Z^T,B) \to \mathbb{E}(U_0,B) = 0.$$

This shows that any morphism $f \in \mathscr{C}(Z^T,B)$ factors through $E \in \mathcal{S}$, and thus $\overline{f} = 0$ holds in $(\mathscr{C}/\mathcal{I})(Z^T,B)$. $\square$

The above arguments allow us to define the mutation of cotorsion pairs as follows.

Definition 7.13. Let $\mathcal{P} = ((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ be a Hovey twin cotorsion pair on $\mathscr{C}$. We define *mutation with respect to* $(\mathcal{S},\mathcal{V})$ as a $\mathbb{Z}$-action on $\mathfrak{M}_{\mathcal{P}}$ given, for any $n \in \mathbb{Z}$, by $\mu_n = \mathbb{I} \circ [n] \circ \mathbb{R}$, i.e.,

$$\mu_n \colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}\ ;\ (\mathcal{A},\mathcal{B}) \mapsto \big(\mathcal{U} \cap \ell^{-1}(\ell(\mathcal{A})[n]), \mathcal{T} \cap \ell^{-1}(\ell(\mathcal{B})[n])\big).$$

References

[AN] Abe, N.; Nakaoka, H.: *General heart construction on a triangulated category (II): associated homological functor*, Appl. Categ. Structures, **20** (2012) no.2, 161–174.

[BBD] Beĭlinson, A. A.; Bernstein, J.; Deligne, P.: *Faisceaux pervers* (French) [Perverse sheaves] Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, **100**, Soc. Math. France, Paris, 1982.

[Bo] Borceux, F.: *Handbook of categorical algebra 1, Basic category theory*, Encyclopedia of Mathematics and its Applications, **50**. Cambridge University Press, Cambridge, 1994. xvi+345 pp.

[Bu] Bühler, T.: *Exact Categories*, Expo. Math. **28** (2010) 1–69.

[Ca] Cagne, P.: *Towards a homotopical algebra of dependent types*, PhD Thesis (2018), Université Paris Diderot.

[DI] Demonet, L.; Iyama, O.: *Lifting preprojective algebras to orders and categorifying partial flag varieties*, Algebra Number Theory, **10** (2016) no. 7, 1527–1580.

[Eg] Egger, J.M.: *Quillen model categories without equalisers or coequalisers*, arXiv:0609808v1.

[G] Gillespie, J.: *Model structures on exact categories*. J. Pure Appl. Algebra **215** (2011) no. 12, 2892–2902.

[Ha] Happel, D.: *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, **119**. Cambridge University Press, Cambridge, 1988. x+208 pp.

[Ho1] Hovey, M.: *Cotorsion pairs, model category structures, and representation theory*, Math. Z. **241** (2002) no. 3, 553–592.

[Ho2] Hovey, M.: *Cotorsion pairs and model categories. Interactions between homotopy theory and algebra*, 277–296, Contemp. Math., **436**, Amer. Math. Soc., Providence, RI, 2007.

[IYa] Iyama, O.; Yang, D.: *Silting reduction and Calabi–Yau reduction of triangulated categories*, Trans. Amer. Math. Soc., **370** (2018) no. 11, 7861–7898.

[IYo] Iyama, O.; Yoshino, Y.: *Mutation in triangulated categories and rigid Cohen-Macaulay modules*, Invent. Math. **172** (2008) no. 1, 117–168.

[Joy] Joyal, A.: *The theory of quasi-categories and its applications*, available online at https://web.archive.org/web/20110706214636/http://www.crm.cat/HigherCategories/hc2.pdf.

[Ke] Keller, B.: *Derived categories and their uses*. Handbook of algebra, Vol. 1, 671–701, North-Holland, Amsterdam, 1996.

[KR] Keller, B.; Reiten, I.: *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, Adv. Math. **211** (2007) no. 1, 123–151.

[KZ] Koenig, S.; Zhu, B.: *From triangulated categories to abelian categories: cluster tilting in a general framework*, Math. Z. **258** (2008) no. 1, 143–160.

[Li] Li, Z.: *The triangulation of the subfactor categories of additive categories with suspensions*, arXiv:1510.02258.

[Liu] Liu, Y.: *Hearts of twin cotorsion pairs on exact categories*, J. Algebra, **394** (2013) 245–284.

[LN] Liu, Y; Nakaoka, H.: *Hearts of twin Cotorsion pairs on extriangulated categories*. arXiv:1702.00244.

[MP] Marsh, R.; Palu, Y.: *Nearly Morita equivalences and rigid objects*, Nagoya Math. J., **225** (2017), 64–99.

[Na1] Nakaoka, H.: *General heart construction on a triangulated category (I): unifying t-structures and cluster tilting subcategories*, Appl. Categ. Structures, **19** (2011) no.6, 879–899.

[Na2] Nakaoka, H.: *General heart construction for twin torsion pairs on triangulated categories*, J. Algebra, **374** (2013) 195–215.

[Na3] Nakaoka, H.: *Equivalence of hearts of twin cotorsion pairs on triangulated categories*, Comm. Algebra, **44** (2016) no. 12, 5105–5148.

[Na4] Nakaoka, H.: *A simultaneous generalization of mutation and recollement on a triangulated category*, Appl. Categ. Structures, **26** (2018) no. 3, 491–544.

[Ne] Neeman, A.: *Triangulated categories*. Annals of Mathematics Studies, 148. Princeton University Press, Princeton, NJ, 2001. viii+449 pp.

[Pal] Palu, Y.: *From triangulated categories to module categories via homotopical algebra*, arXiv:1412.7289.

[Pau] Pauksztello, D.: *Compact corigid objects in triangulated categories and co-t-structures*. Cent. Eur. J. Math. **6** (2008) no. 1, 25–42.

[Sal] Salce, L.: *Cotorsion theories for abelian groups* in Symposia Mathematica, Vol. XXIII (Conf. Abelian Groups and their Relationship to the Theory of Modules, INDAM, Rome, 1977) (1979), 11–32.

[S] Šťovíček, J.: *Exact model categories, approximation theory, and cohomology of quasi-coherent sheaves*. Advances in representation theory of algebras, 297–367, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2013.

[Y] Yang, X.: *Model structures on triangulated categories*. Glasg. Math. J. **57** (2015) no. 2, 263–284.

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

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THE SERRE AUTOMORPHISM VIA HOMOTOPY ACTIONS AND THE COBORDISM HYPOTHESIS FOR ORIENTED MANIFOLDS

Jan Hesse *Alessandro Valentino*

Résumé. Nous construisons explicitement une action de $SO(2)$ sur une version squelettique de la bicatégorie des cobordismes à bords à deux dimensions. Par l'hypothèse du cobordisme bidimensionnel pour les variétés à bords, nous obtenons une action de $SO(2)$ sur le noyau des objets complètement dualisables de la bicatégorie cible. Cette action coïncide avec celle donnée par l'automorphisme de Serre. Nous donnons une description explicite de la bicatégorie des points fixes homotopiques de cette action, et discutons de sa relation avec la classification des théories quantiques des champs topologiques en 2 dimensions.

Abstract. We explicitly construct an $SO(2)$-action on a skeletal version of the 2-dimensional framed bordism bicategory. By the 2-dimensional Cobordism Hypothesis for framed manifolds, we obtain an $SO(2)$-action on the core of fully-dualizable objects of the target bicategory. This action is shown to coincide with the one given by the Serre automorphism. We give an explicit description of the bicategory of homotopy fixed points of this action, and discuss its relation to the classification of oriented 2d topological quantum field theories.

Keywords. Serre Automorphism, Cobordism Hypothesis, Topological Quantum Field Theory, Group actions, Bicategory, Homotopy Fixed Point.

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1. Introduction

As defined by Atiyah in [Ati88] and Segal in [Seg04], an n-dimensional Topological Quantum Field Theory (TQFT) consists of a functor between two symmetric monoidal categories, namely a category of n-cobordisms, and a category of algebraic objects. This definition was introduced to axiomatize the locality properties of the path integral, and has given rise to a fruitful interplay between mathematics and physics in the last 30 years. A prominent example is given by a quantum-field-theoretic interpretation of the Jones polynomial by Witten in [Wit89].

More recently, there has been a renewed interest in the study of TQFTs, due in great part to the Baez-Dolan Cobordism Hypothesis and its proof by Lurie, whose main objects of investigation are *fully extended* TQFTs. These are a generalization of the notion of n-dimensional TQFTs, where data is assigned to manifolds of codimension up to n. The Baez-Dolan Cobordism Hypothesis, originally stated in [BD95], and proved by Lurie in [Lur09] in an ∞-categorical version, can be stated as follows: fully extended *framed* TQFTs are classified by their value on a point, which must be a fully dualizable object in the target symmetric monoidal (∞, n)-category $\mathcal{C}$. Moreover, the ∞-groupoid $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ given by the core of fully dualizable objects of $\mathcal{C}$ carries a homotopy $O(n)$-action induced by the "rotation of the framing" on the framed (∞, n)-cobordism category [Lur09, Corollary 2.4.10]. The inclusion $SO(n) \hookrightarrow O(n)$ then induces an $SO(n)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. By the Cobordism Hyothesis for manifolds whose tangent bundle is equipped with an additional G-structure, homotopy fixed-points for this action classify fully extended *oriented* TQFTs. It is relevant to notice that in [Lur09] the homotopy $O(n)$-action on the framed (∞, n)-category of cobordisms is not explicitly constructed, or even briefly sketched. For an extensive introduction to extended TQFTs and the Cobordism Hypothesis, we refer the reader to [Fre12].

Blurring the distinction between $(\infty, 2)$-categories and bicategories, in [FHLT10] it is argued that in the case where the target is given by the bicategory Alg_2 of algebras, bimodules, and intertwiners, the fully dualizable objects are semisimple finite-dimensional algebras, and that the additional $SO(2)$-fixed-points structure should correspond to the structure of a symmetric Frobenius algebra. Via a direct construction, in [SP09] it is showed that the bigroupoid Frob of Frobenius algebras, Morita contexts and intertwiners indeed classifies fully extended oriented 2-dimensional TQFTs valued in Alg_2. In [Dav11], it is observed that the $SO(2)$-action given by the Serre automorphism on the core of fully-dualizable objects of Alg_2 is trivializable. In a purely bicategorical setting, in [HSV17] the homotopy-fixed-point bigroupoid of the $SO(2)$-action on Alg_2 is computed, and it is shown that it coincides with Frob.

In the present paper we provide an explicit $SO(2)$-action on the framed bordism bicategory, and show that the $SO(2)$-action induced on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ for any symmetric monoidal bicategory $\mathcal{C}$ is given by the Serre automorphism, regarded as a pseudo-natural isomorphism of the identity functor. More precisely, we make use of a presentation of the framed bordism bicategory provided in [Pst14] to construct such an $SO(2)$-action.

By the Cobordism Hypothesis for framed manifolds, which has been proven in the setting of bicategories in [Pst14], there is an equivalence of bicategories

$$\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{fr}}_{2,1,0}, \mathcal{C}) \cong \mathscr{K}(\mathcal{C}^{\mathrm{fd}}). \tag{1}$$

This equivalence allows us to transport the $SO(2)$-action on the framed bordism bicategory to the core of fully-dualizable objects of $\mathcal{C}$. We then prove that this induced $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ is given precisely by the Serre automorphism, showing that the Serre automorphism has indeed a geometric origin, as expected from [Lur09].

Along the way, we also provide results concerning monoidal homotopy actions which are useful in determining when such actions are trivializable. The relevance for TQFT is the following: in the case of a trivializable $SO(2)$-action, *any* framed fully extended 2d TQFT can be promoted to an oriented one by providing the appropriate structure of a homotopy fixed point. In particular, we apply these results to the case of invertible 2d TQFTs, which have recently attracted interest for their application to condensed matter physics, more specifically to the study of topological insulators [Fre14a, Fre14b, FH16]. Namely, fully extended invertible TQFTs have been proposed as the low energy limit of short-range entanglement systems; see [Fre14b] for a discussion of these topics.

First defintions of monoidal bicategories appear in [KV94], [BN96] and [DS97], with a first full definition of a symmetric monoidal bicategory in [McC00]. We will refer to [SP09] for technical details. In section 5, we use the wire-diagram calculus developed in [Bar14].

It is worth noticing that the study of actions of groups on higher categories and their homotopy fixed points is also of independent interest, see for instance [EGNO15, BGM17] for the case of finite groups.

The paper is organized as follows.
In Section 2 we recall the notion of a fully-dualizable object in a symmetric monoidal bicategory $\mathcal{C}$. For each such an object X, we define the Serre automorphism as a certain 1-endomorphism of X. We show that the Serre automorphism is a pseudo-natural transformation of the identity functor on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, which is moreover monoidal. This suffices to define an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$.

Section 3 investigates when a group action on a bicategory $\mathcal{C}$ is equivalent to the trivial action. We obtain a general criterion for when such an action is trivializable.

In Section 4, we compute the bicategory of homotopy fixed points of an $SO(2)$-action coming from a pseudo-natural transformation of the identity functor of an arbitrary bicategory $\mathcal{C}$. This generalizes the main result in [HSV17], which computes homotopy fixed points of the trivial $SO(2)$-action on $\mathrm{Alg}_2^{\mathrm{fd}}$. Our more general theorem allows us to give an explicit description of the bicategory of homotopy fixed points of the Serre automorphism.

In Section 5, we introduce a skeletal version of the framed bordism bicategory by generators and relations, and define a non-trivial $SO(2)$-action on this bicategory. By the framed Cobordism Hypothesis, as in Equation (1), we obtain an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, which we prove to coincide with the one given by the Serre automorphism.

In Section 6 we discuss invertible 2d TQFTs, providing a general criterion for the trivialization of the $SO(2)$-action in this case.

In Section 7, we give an outlook on *homotopy co-invariants* of the $SO(2)$-action, and argue about their relation to the Cobordism Hypothesis for oriented manifolds.

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2. Fully-dualizable objects and the Serre automorphism

The aim of this section is to introduce the main objects of the present paper. On the algebraic side, these are fully-dualizable objects in a symmetric monoidal bicategory $\mathcal{C}$, and the Serre automorphism. Though some of the following material has already appeared in the literature, we recall the relevant definitions in order to fix notation. For details, we refer the reader to [Pst14].

Definition 2.1. *A dual pair in a symmetric monoidal bicategory $\mathcal{C}$ consists of an object X, an object X^*, two 1-morphisms*

$$\begin{aligned} \mathrm{ev}_X &: X \otimes X^* \to 1 \\ \mathrm{coev}_X &: 1 \to X^* \otimes X \end{aligned} \tag{2}$$

and two invertible 2-morphisms α and β in the diagrams below.

$$\begin{tikzcd}
 & & X \otimes (X^* \otimes X) \arrow[r, "a"] & (X \otimes X^*) \otimes X \arrow[dr, "\mathrm{ev}_X \otimes \mathrm{id}_X"] & & \\
 & X \otimes 1 \arrow[ur, "\mathrm{id}_X \otimes \mathrm{coev}_X"] & {} \arrow[r, phantom, "\Downarrow \alpha"] & {} & 1 \otimes X \arrow[dr, "l"] & \\
X \arrow[ur, "r"] \arrow[rrrrr, "\mathrm{id}_X"'] & & & & & X
\end{tikzcd} \tag{3}$$

$$\begin{tikzcd}
 & & (X^* \otimes X) \otimes X^* \arrow[r, "a"] & X^* \otimes (X \otimes X^*) \arrow[dr, "\mathrm{id}_{X^*} \otimes \mathrm{ev}_X"] & & \\
 & 1 \otimes X^* \arrow[ur, "\mathrm{coev}_X \otimes \mathrm{id}_{X^*}"] & {} \arrow[r, phantom, "\Downarrow \beta"] & {} & X^* \otimes 1 \arrow[dr, "r"] & \\
X^* \arrow[ur, "l"] \arrow[rrrrr, "\mathrm{id}_{X^*}"'] & & & & & X^*
\end{tikzcd} \tag{4}$$

We call an object X of $\mathcal{C}$ dualizable if it can be completed to a dual pair. A dual pair is said to be coherent if the "swallowtail" equations are satisfied, as in [Pst14, Def. 2.6].

Remark 2.2. Given a dual pair, it is always possible to modify the 2-cell β in such a way that the swallowtail are fulfilled, cf. [Pst14, Theorem 2.7].

Dual pairs can be organized into a bicategory by defining appropriate 1- and 2-morphisms between them, cf. [Pst14, Section 2.1]. The bicategory of dual pairs turns out to be a 2-groupoid. Moreover, the bicategory of coherent dual pairs is equivalent to the core of dualizable objects in $\mathcal{C}$. In particular, this shows that any two coherent dual pairs over the same dualizable object are equivalent.

We now come to the stronger concept of fully-dualizability.

Definition 2.3. *An object X in a symmetric monoidal bicategory is called fully-dualizable if it can be completed into a dual pair and the evaluation and coevaluation maps admit both left- and right adjoints.*

Note that if left- and right adjoints exists, the adjoint maps will have adjoints themselves, since we work in a bicategorical setting, cf. [Pst14]. Note that if left- and right adjoints for the 1-morphisms ev and coev exist, these adjoint 1-morphisms will in turn have additional adjoints themselves. Thus, Definition 2.3 agrees with the definition of [Lur09] in the special case of bicategories.

2.1 The Serre automorphism

Recall that by definition, the evaluation morphism for a fully dualizable object X admits both a right-adjoint ev_X^R and a left adjoint ev_X^L. We use these adjoints to define the Serre-automorphism of X:

Definition 2.4. *Let X be a fully-dualizable object in a symmetric monoidal bicategory. The Serre automorphism of X is the following composition of 1-morphisms:*

$$S_X : X \cong X \otimes 1 \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X^R} X \otimes X \otimes X^* \xrightarrow{\tau_{X,X} \otimes \mathrm{id}_{X^*}} X \otimes X \otimes X^* \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X \otimes 1 \cong X. \tag{5}$$

Notice that the Serre automorphism is actually a 1-equivalence of X, since an inverse is given by the 1-morphism

$$S_X^{-1} = (\mathrm{id}_X \circ \mathrm{ev}_X) \circ (\tau_{X,X} \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_X \otimes \mathrm{ev}_X^L), \tag{6}$$

cf. [Lur09, DSS13].
The next lemma is well-known [Lur09, Pst14], and is straightforward to show graphically.

Lemma 2.5. *Let X be fully-dualizable in $\mathcal{C}$. Then, there are 2-isomorphisms*

$$\begin{aligned} \mathrm{ev}_X^R &\cong \tau_{X^*,X} \circ (\mathrm{id}_{X^*} \otimes S_X) \circ \mathrm{coev}_X \\ \mathrm{ev}_X^L &\cong \tau_{X^*,X} \circ (\mathrm{id}_{X^*} \otimes S_X^{-1}) \circ \mathrm{coev}_X\,. \end{aligned} \tag{7}$$

Next, we show that the Serre automorphism is actually a pseudo-natural transformation of the identity functor on the maximal subgroupoid of $\mathcal{C}$, as suggested in [Sch13]. To the best of our knowledge, a proof of this statement has not appeared in the literature so far, hence we illustrate the details in the following. We begin by showing that the evaluation 1-morphism is "dinatural".

Lemma 2.6. *Let X be dualizable in $\mathcal{C}$. The evaluation 1-morphism ev_X is "dinatural": for every 1-morphism $f : X \to Y$ between dualizable objects, there is a natural 2-isomorphism ev_f in the diagram below.*

$$\begin{tikzcd} X \otimes Y^* \arrow[r, "\mathrm{id}\otimes f^*"] \arrow[d, "f\otimes\mathrm{id}"'] & X \otimes X^* \arrow[d, "\mathrm{ev}_X"] \arrow[dl, Rightarrow, "\mathrm{ev}_f"] \\ Y \otimes Y^* \arrow[r, "\mathrm{ev}_Y"'] & 1 \end{tikzcd} \tag{8}$$

By "di-naturality", we explicitly mean that for every 2-morphism $\alpha : f \Rightarrow g$ in $\mathcal{C}$, the following diagram commutes

$$\begin{tikzcd} X \otimes Y^* \arrow[r, "\mathrm{id}\otimes f^*"'] \arrow[r, bend left=60, "\mathrm{id}\otimes g^*"] \arrow[r, phantom, bend left=30, "\Downarrow \mathrm{id}\otimes\alpha"] \arrow[d, "f\otimes\mathrm{id}"] \arrow[d, bend right=80, "g\otimes\mathrm{id}"'] \arrow[d, phantom, bend right=40, "\Leftarrow \alpha\otimes\mathrm{id}"] & X \otimes X^* \arrow[d, "\mathrm{ev}_X"] \arrow[dl, Rightarrow, "\mathrm{ev}_f"] \\ Y \otimes Y^* \arrow[r, "\mathrm{ev}_Y"'] & 1 \end{tikzcd} = \begin{tikzcd} X \otimes Y^* \arrow[r, "\mathrm{id}\otimes g^*"'] \arrow[d, "g\otimes\mathrm{id}"] & X \otimes X^* \arrow[d, "\mathrm{ev}_X"] \arrow[dl, Rightarrow, "\mathrm{ev}_g"] \\ Y \otimes Y^* \arrow[r, "\mathrm{ev}_Y"'] & 1 \end{tikzcd} \tag{9}$$

$$(\mathrm{ev}_X \circ f^{-1} \otimes f^*)^R = f \otimes (f^*)^{-1} \circ \mathrm{ev}_X^R \, .$$

Indeed, let

$$\eta_X : \mathrm{id}_{X \otimes X^*} \to \mathrm{ev}_X^R \circ \mathrm{ev}_X$$
$$\varepsilon_X : \mathrm{ev}_X \circ \mathrm{ev}_X^R \to \mathrm{id}_1$$

Proof. We explicitly write out the definition of f^* and define ev_f to be the composition of the 2-morphisms in the diagram below.

$*$ $X1$ $X\,1\,Y^*\;X$ $X\,X^*\,X\,Y^*$ $X\,X^*\,Y\,Y^*$ $X\,X$

$\mathrm{ev}_f :=$

$*$

be the unit and counit of the right-adjunction of ev_X and its right adjoint ev_X^R. We construct unit and counit for the adjunction in Equation (12). Let

$$\begin{aligned}
&\tilde{\varepsilon} : \mathrm{ev}_X \circ (f^{-1} \otimes f^*) \circ (f \otimes (f^*)^{-1}) \circ \mathrm{ev}_X^R \cong \mathrm{ev}_X \circ \mathrm{ev}_X^R \xrightarrow{\varepsilon_X} \mathrm{id}_1 \\
&\tilde{\eta} : \mathrm{id}_{Y \otimes Y^*} \cong (f \otimes (f^*)^{-1}) \circ (f^{-1} \otimes f^*) \\
&\xrightarrow{\mathrm{id} * \eta_X * \mathrm{id}} (f \otimes (f^*)^{-1}) \circ \mathrm{ev}_X^R \circ \mathrm{ev}_X \circ (f^{-1} \otimes f^*).
\end{aligned} \tag{14}$$

Now, one checks that the quadruple

$$(\mathrm{ev}_X \circ (f^{-1} \otimes f^*),\ (f \otimes (f^*)^{-1}) \circ \mathrm{ev}_X^R,\ \tilde{\varepsilon}, \tilde{\eta}) \tag{15}$$

fulfills indeed the axioms of an adjunction. This follows from the fact that the quadruple $(\mathrm{ev}_X, \mathrm{ev}_X^R, \varepsilon_X, \eta_X)$ is an adjunction. This shows Equation (12).

Now, notice that due to the dinaturality of the evaluation in Lemma 2.6, we have a natural 2-isomorphism

$$\mathrm{ev}_Y \cong \mathrm{ev}_X \circ (f^{-1} \otimes f^*). \tag{16}$$

Combining this 2-isomorphism with Equation (12) shows that the right adjoint of ev_Y is given by $f \otimes (f^*)^{-1} \circ \mathrm{ev}_X^R$. Since all right-adjoints are isomorphic the 1-morphism $f \otimes (f^*)^{-1} \circ \mathrm{ev}_X^R$ is isomorphic to ev_Y^R, as desired. □

We can now prove the following proposition.

Proposition 2.8. *Let $\mathcal{C}$ be a symmetric monoidal bicategory. Denote by $\mathscr{K}(\mathcal{C})$ the maximal sub-bigroupoid of $\mathcal{C}$. The Serre automorphism S is a pseudo-natural isomorphism of the identity functor on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$.*

Proof. Let $f : X \to Y$ be a 1-morphism in $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. We need to provide a natural 2-isomorphism in the diagram

$$\begin{tikzcd}
X \arrow[r, "S_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \arrow[dl, Rightarrow, "S_f"] \\
Y \arrow[r, "S_Y"'] & Y
\end{tikzcd} \tag{17}$$

By spelling out the definition of the Serre automorphism, we see that this is equivalent to filling the following diagram with natural 2-cells:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & X\,1 \arrow[r, "\mathrm{id}_X\ \mathrm{ev}_X^R"] \arrow[d, "f\,\mathrm{id}"] & X\,X\,X^* \arrow[r, "\tau_{X,X}\ \mathrm{id}_{X^*}"] \arrow[d, "f\,f\,(f^*)^{-1}"] & X\,X\,X^* \arrow[r, "\mathrm{id}_X\ \mathrm{ev}_X"] \arrow[d, "f\,f\,(f^*)^{-1}"] & X\,1 \arrow[r] \arrow[d, "f\,\mathrm{id}"] & X \arrow[d, "f"] \\
Y \arrow[r] & Y\,1 \arrow[r, "\mathrm{id}_Y\ \mathrm{ev}_Y^R"'] & Y\,Y\,Y^* \arrow[r, "\tau_{Y,Y}\ \mathrm{id}_{Y^*}"'] & Y\,Y\,Y^* \arrow[r, "\mathrm{id}_Y\ \mathrm{ev}_Y"'] & Y\,1 \arrow[r] & Y
\end{tikzcd} \tag{18}$$

The first, the last and the middle square can be filled with a natural 2-cell due to the fact that $\mathcal{C}$ is a symmetric monoidal bicategory. The square involving the evaluation commutes up to a 2-cell using the mate of the 2-cell of Lemma 2.6, while the square involving the right adjoint of the evaluation commutes up a 2-cell using the mate of the 2-cell of Lemma 2.7. □

2.2 Monoidality of the Serre automorphism

In this section we show that the Serre automorphism respects the monoidal structure. We will show that the Serre-automorphism is a *monoidal* pseudo-natural transformation of the identity functor. We begin with the following two lemmas:

Lemma 2.9. *Let $\mathcal{C}$ be a monoidal bicategory. Let X and Y be dualizable objects of $\mathcal{C}$. Then, there is a 1-equivalence $\xi_{X,Y} : (X \otimes Y)^* \cong Y^* \otimes X^*$. Furthermore, this 1-equivalence ξ is pseudo-natural: suppose that $f : X \to X'$ and $g : Y \to Y'$ are two 1-morphisms in $\mathcal{C}$. Then, there is a pseudo-natural 2-isomorphism in the diagram in equation* (19).

$$\begin{tikzcd}
(X \otimes Y)^* \arrow[r, "\xi_{X,Y}"] & Y^* \otimes X^* \arrow[dl, Rightarrow, "\xi_{f,g}"] \\
(X' \otimes Y')^* \arrow[u, "(f\otimes g)^*"] \arrow[r, "\xi_{X',Y'}"'] & Y'^* \otimes X'^* \arrow[u, "g^*\otimes f^*"']
\end{tikzcd} \tag{19}$$

Proof. Define a 1-morphism $(X \otimes Y)^* \to Y^* \otimes X^*$ in $\mathcal{C}$ by the composition

$$(\mathrm{id}_{Y^*} \otimes \mathrm{id}_{X^*} \otimes \mathrm{ev}_{X\otimes Y}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_Y \otimes \mathrm{id}_{(X\otimes Y)^*}) \circ (\mathrm{coev}_Y \otimes \mathrm{id}_{(X\otimes Y)^*}) \tag{20}$$

and define another 1-morphism $Y^* \otimes X^* \to (X \otimes Y)^*$ in $\mathcal{C}$ by the composition

$$(\mathrm{id}_{(X\otimes Y)^*} \otimes \mathrm{ev}_X) \circ (\mathrm{id}_{(X\otimes Y)^*} \otimes \mathrm{id}_X \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{X^*}) \circ (\mathrm{coev}_{X\otimes Y} \otimes \mathrm{id}_Y^* \otimes \mathrm{id}_{X^*}). \tag{21}$$

These two 1-morphisms are (up to invertible 2-cells) inverse to each other. This shows the first claim. The existence and the pseudo-naturality of the 2-isomorphism $\xi_{f,g}$ now follows from the definition of ξ and lemma 2.6. $\square$

Now, we show that the evaluation 1-morphism respects the monoidal structure:

Lemma 2.10. *For a dualizable object X of a symmetric monoidal bicategory $\mathcal{C}$, the evaluation 1-morphism ev_X is a* monoidal *pseudo-dinatural transformation: namely, the following diagram commutes up to 2-isomorphism.*

$$\begin{tikzcd}
(X\otimes Y)\otimes(X\otimes Y)^* \arrow[rr, "\mathrm{ev}_{X\otimes Y}"] \arrow[d, "\mathrm{id}_{X\otimes Y}\otimes\xi"'] & & 1 \arrow[d] \\
(X\otimes Y)\otimes Y^*\otimes X^* \arrow[r, "\mathrm{id}_X\otimes\tau_{Y\otimes Y^*,X^*}"'] & X\otimes X^*\otimes Y\otimes Y^* \arrow[r, "\mathrm{ev}_X\otimes\mathrm{ev}_Y"'] & 1\otimes 1
\end{tikzcd} \tag{22}$$

Here, the 1-equivalence ξ is due to Lemma 2.9.

Proof. Let us construct a 2-isomorphism in the diagram in Equation (22). Consider the diagram in figure 1 on page 41: here, the composition of the horizontal arrows at the top, together with the two arrows on the vertical right are exactly the 1-morphism in Equation (22). The other arrow is given by $\mathrm{ev}_{X\otimes Y}$. We have not written down the tensor product, and left out isomorphisms of the form $1\otimes X \cong X \cong X \otimes 1$ for readability. $\square$

We can now establish the monoidality of the right adjoint of the evaluation via the following lemma.

Lemma 2.11. *Let $\mathcal{C}$ a symmetric monoidal bicategory, and let X and Y be fully-dualizable objects. Then, the right adjoint of the evaluation is monoidal. More precisely: if $\xi : (X\otimes Y)^* \to Y^* \otimes X^*$ is the 1-equivalence of Lemma*

2.9, the following diagram commutes up to 2-isomorphism.

$$\begin{tikzcd} 1 \arrow[r, "\mathrm{ev}^R_{X\otimes Y}"] \arrow[d, "\mathrm{ev}^R_X \otimes \mathrm{ev}^R_Y"'] & X \otimes Y \otimes (X\otimes Y)^* \arrow[d, "\mathrm{id}_{X\otimes Y}\otimes \xi"] \\ X\otimes X^* \otimes Y \otimes Y^* \arrow[r, "\mathrm{id}_X \otimes \tau_{X^*,Y\otimes Y^*}"'] & X\otimes Y \otimes Y^* \otimes X^* \end{tikzcd} \tag{23}$$

Proof. In a first step, we show that the right adjoint of the 1-morphism

$$(\mathrm{ev}_X \otimes \mathrm{ev}_Y) \circ (\mathrm{id}_X \otimes \tau_{Y\otimes Y^*, X^*}) \circ (\mathrm{id}_{X\otimes Y} \otimes \xi) \tag{24}$$

is given by the 1-morphism

$$(\mathrm{id}_{X\otimes Y} \otimes \xi^{-1}) \circ (\mathrm{id}_X \circ \tau_{X^*, Y\otimes Y^*}) \circ (\mathrm{ev}^R_X \otimes \mathrm{ev}^R_Y). \tag{25}$$

Indeed, if

$$\begin{aligned} \eta_X &: \mathrm{id}_{X\otimes X^*} \to \mathrm{ev}^R_X \circ \mathrm{ev}_X \\ \varepsilon_X &: \mathrm{ev}_X \circ \mathrm{ev}^R_X \to \mathrm{id}_1 \end{aligned} \tag{26}$$

are the unit and counit of the right-adjunction of ev_X and its right adjoint ev^R_X, we construct adjunction data for the adjunction in equations (24) and (25) as follows. Let $\tilde{\varepsilon}$ and $\tilde{\eta}$ be the following 2-morphisms:

$$\begin{aligned} \tilde{\varepsilon} : (\mathrm{ev}_X \otimes \mathrm{ev}_Y) &\circ (\mathrm{id}_X \otimes \tau_{Y\otimes Y^*, X^*}) \circ (\mathrm{id}_{X\otimes Y} \otimes \xi) \circ (\mathrm{id}_{X\otimes Y} \otimes \xi^{-1}) \\ &\qquad \circ (\mathrm{id}_X \otimes \tau_{X^*, Y\otimes Y^*}) \circ (\mathrm{ev}^R_X \otimes \mathrm{ev}^R_Y) \\ \cong (\mathrm{ev}_X \otimes \mathrm{ev}_Y) &\circ (\mathrm{id}_X \otimes \tau_{Y\otimes Y^*, X^*}) \circ (\mathrm{id}_X \otimes \tau_{X^*, Y\otimes Y^*}) \circ (\mathrm{ev}^R_X \otimes \mathrm{ev}^R_Y) \\ \cong (\mathrm{ev}_X \otimes \mathrm{ev}_Y) &\circ (\mathrm{ev}^R_X \otimes \mathrm{ev}^R_Y) \xrightarrow{\varepsilon_X \otimes \varepsilon_Y} \mathrm{id}_1 \end{aligned} \tag{27}$$

and

$$\begin{aligned} \tilde{\eta} :&\mathrm{id}_{X\otimes Y \otimes (X\otimes Y)^*} \cong (\mathrm{id}_{X\otimes Y} \otimes \xi^{-1}) \circ (\mathrm{id}_{X\otimes Y} \otimes \xi) \\ &\cong (\mathrm{id}_{X\otimes Y} \otimes \xi^{-1}) \circ (\mathrm{id}_X \otimes \tau_{X^*, Y\otimes Y^*}) \circ (\mathrm{id}_X \otimes \tau_{Y\otimes Y^*, X^*}) \circ (\mathrm{id}_{X\otimes Y} \otimes \xi) \\ &\xrightarrow{\mathrm{id}\otimes \eta_X \otimes \eta_Y \otimes \mathrm{id}} (\mathrm{id}_{X\otimes Y} \otimes \xi^{-1}) \circ (\mathrm{id}_X \otimes \tau_{X^*, Y\otimes Y^*}) \circ (\mathrm{ev}^R_X \otimes \mathrm{ev}^R_Y) \\ &\qquad \circ (\mathrm{ev}_X \otimes \mathrm{ev}_Y) \circ (\mathrm{id}_X \otimes \tau_{Y\otimes Y^*, X^*}) \circ (\mathrm{id}_{X\otimes Y} \otimes \xi) \end{aligned} \tag{28}$$

One now shows that the two 1-morphisms in Equation (24) and (25), together with the two 2-morphisms $\tilde{\varepsilon}$ and $\tilde{\eta}$ form an adjunction. This gives that the two 1-morphisms in Equations (24) and (25) are adjoint.

Next, notice that the 1-morphism in Equation (24) is isomorphic to the 1-morphism $\mathrm{ev}_{X\otimes Y}$ by Lemma 2.10. Thus, the right adjoint of $\mathrm{ev}_{X\otimes Y}$ is given by the right adjoint of the 1-morphism in Equation (24), which is the 1-morphism in Equation (25) by the argument above. Since all adjoints are equivalent, this shows the lemma. □

We are now ready to prove that the Serre automorphism is a *monoidal* pseudo-natural transformation.

Proposition 2.12. *Let $\mathcal{C}$ be a symmetric monoidal bicategory. Then, the Serre automorphism is a* monoidal *pseudo-natural transformation of* $\mathrm{Id}_{\mathscr{K}(\mathcal{C}^{\mathrm{fd}})}$.

Proof. By definition (cf. [SP09, Definition 2.7]), we have to provide invertible 2-cells

$$\begin{aligned} \Pi_{X,Y} &: S_{X\otimes Y} \to S_X \otimes S_Y \\ M &: S_1 \to \mathrm{id}_1, \end{aligned} \tag{29}$$

satisfying suitable coherence equations. By the definition of the Serre automorphism in Definition 2.4, it suffices to show that the evaluation and its right adjoint are monoidal, since the braiding τ will be monoidal by definition. The monoidality of the evaluation is proven in Lemma 2.10, while the monoidality of its right adjoint follows from Lemma 2.11. These two lemmas thus provide an invertible 2-cell $S_{X\otimes Y} \cong S_X \otimes S_Y$. The second 2-cell $\mathrm{id}_1 \to S_1$ can be constructed in a similar way, by noticing that $1 \cong 1^*$.

The three coherence equations for a pseudo-natural transformation now read

$$\begin{aligned} \Pi_{X\otimes Y,Z} \circ (\Pi_{X\otimes Y} \otimes \mathrm{id}_{S_Z}) &= \Pi_{X,Y\otimes Z} \circ (\mathrm{id}_{S_X} \otimes \Pi_{Y,Z}) \\ \Pi_{1,X} &= M \otimes \mathrm{id}_{S_X} \\ \Pi_{X,1} &= \mathrm{id}_{S_X} \otimes M \end{aligned} \tag{30}$$

and can be checked directly by hand. □

3. Monoidal homotopy actions

In this section, we investigate homotopy actions on symmetric monoidal bicategories. In particular, we are interested in the case when the group

action is compatible with the monoidal structure. By a (homotopy) action of a topological group G on a bicategory $\mathcal{C}$, we mean a weak monoidal 2-functor $\rho : \Pi_2(G) \to \mathrm{Aut}(\mathcal{C})$, where $\Pi_2(G)$ is the fundamental 2-groupoid of G, and $\mathrm{Aut}(\mathcal{C})$ is the bicategory of auto-equivalences of $\mathcal{C}$. For details on homotopy actions of groups on bicategories, we refer the reader to [HSV17].
In order to simplify the exposition, we introduce the following

Definition 3.1. *Let G be a topological group. We will say that G is 2-truncated if $\pi_2(G, x)$ is trivial for every base point $x \in G$.*

Moreover, we will need also the following definition.

Definition 3.2. *Let $\mathcal{C}$ be a symmetric monoidal bicategory. We will say that $\mathcal{C}$ is algebraically 1-connected if it is monoidally equivalent to B^2H, for some abelian group H.*

In the following, we denote by $\mathrm{Aut}_{\otimes}(\mathcal{C})$ the monoidal bicategory of invertible monoidal weak 2-functors of $\mathcal{C}$, invertible monoidal pseudo-natural transformations, and invertible monoidal modifications. Details of the construction can be found in [Hes17, Appendix A].

Definition 3.3. *Let $\mathcal{C}$ be a symmetric monoidal category and G be a topological group. A* monoidal *homotopy action of G on $\mathcal{C}$ is a monoidal morphism $\rho : \Pi_2(G) \to \mathrm{Aut}_{\otimes}(\mathcal{C})$.*

We now prove a general criterion for when monoidal homotopy actions are trivializable.

Proposition 3.4. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and let G be a path connected topological group. Assume that G is 2-truncated, and that $\mathrm{Aut}_{\otimes}(\mathcal{C})$ is algebraically 1-connected, with abelian group H. If the second cohomology group $H^2_{grp}(\pi_1(G, e), H) \simeq 0$, then any monoidal homotopy action of G on $\mathcal{C}$ is pseudo-naturally isomorphic to the trivial action.*

Proof. Let $\rho : \Pi_2(G) \to \mathrm{Aut}_{\otimes}(\mathcal{C})$ be a weak monoidal 2-functor. Since $\mathrm{Aut}_{\otimes}(\mathcal{C})$ was assumed to be monoidally equivalent to B^2H for some abelian group H, the group action ρ is equivalent to a weak monoidal 2-functor $\rho : \Pi_2(G) \to B^2H$. Due to the fact that G is path connected and 2-truncated, we have that $\Pi_2(G) \simeq B\pi_1(G, e)$, where $\pi_1(G, e)$ is regarded as a discrete monoidal category. Thus, the monoidal homotopy action ρ is monoidally equivalent to a weak monoidal 2-functor $B\pi_1(G,e) \to B^2H$.

We claim that such functors are classified by $H^2_{grp}(\pi_1(G,e),H)$ up to monoidal pseudo-natural isomorphism. Indeed, let $F : B\pi_1(G,e) \to B^2H$ be a weak monoidal 2-functor. It is easy to see that F is trivial as a weak 2-functor, since we must have $F(*) = *$ on objects, $F(\gamma) = \mathrm{id}_*$ on 1-morphisms, and $B\pi_1(G)$ only has identity 2-morphisms. Thus, the only non-trivial data of F can come from the monoidal structure on F. The 1-dimensional components of the pseudo-natural transformations $\chi_{a,b} : F(a) \otimes F(b) \to F(a \otimes b)$ must be trivial since there are only identity 1-morphisms in B^2H. The 2-dimensional components of this pseudo-natural transformation consists of a 2-morphism $\chi_{\gamma,\gamma'}$ in B^2H for every pair of 1-morphisms $\gamma : a \to b$ and $\gamma' : a' \to b'$ in $B\pi_1(G)$ in the diagram in equation (31) below.

$$\begin{tikzcd} F(a)\otimes F(a') \arrow[r, "\chi_{a,a'}"] \arrow[d, "F(\gamma)\otimes F(\gamma')"'] & F(a\otimes a') \arrow[d, "F(\gamma\otimes\gamma')"] \arrow[dl, Rightarrow, "\chi_{\gamma,\gamma'}"] \\ F(b)\otimes F(b') \arrow[r, "\chi_{b,b'}"'] & F(b\otimes b') \end{tikzcd} \tag{31}$$

Hence, we obtain a 2-cochain $\pi_1(G) \times \pi_1(G) \to H$, which obeys the cocycle condition due to the coherence equations of a monoidal 2-functor, cf. [SP09, Definition 2.5].

One now checks that a monoidal pseudo-natural transformation between two such functors is exactly a 2-coboundary, which shows the claim. Since we assumed that $H^2_{grp}(\pi_1(G,e),H) \simeq 0$, the original action ρ must be trivializable. □

Next, we show that the bicategory $\mathrm{Alg}_2^{\mathrm{fd}}$ of finite-dimensional, semi-simple algebras, bimodules and intertwiners, equipped with the monoidal structure given by the *direct sum* fulfills the conditions of Proposition 3.4.

Lemma 3.5. *Let $\mathbb{K}$ be an algebraically closed field. Let $\mathcal{C} = \mathrm{Alg}_2^{\mathrm{fd}}$ be the bicategory where objects are given by finite-dimensional, semi-simple algebras, equipped with the monoidal structure given by the direct sum. By viewing $\mathcal{C}$ with the monoidal structure equipped by the direct sum, $\mathcal{C}$ turns into a* linear *bicategory. Then, $\mathrm{Aut}_\otimes(\mathcal{C})$ and $B^2\mathbb{K}^*$ are equivalent as symmetric monoidal bicategories.*

Proof. Let $F : \mathrm{Alg}_2^{\mathrm{fd}} \to \mathrm{Alg}_2^{\mathrm{fd}}$ be a weak monoidal 2-equivalence, and let A be a finite-dimensional, semi-simple algebra. Then A is isomorphic to a direct sum of matrix algebras. Calculating up to Morita equivalence and using that F has to preserve the single simple object $\mathbb{K}$ of Alg_2, we have

$$\begin{aligned} F(A) &\cong F\left(\bigoplus_i M_{n_i}(\mathbb{K})\right) \cong \bigoplus_i F\left(M_{n_i}(\mathbb{K})\right) \cong \bigoplus_i F(\mathbb{K}) \\ &\cong \bigoplus_i \mathbb{K} \cong \bigoplus_i M_{n_i}(\mathbb{K}) \cong A. \end{aligned} \tag{32}$$

A straightforward calculation using basic linear algebra confirms that these isomorphisms are even pseudo-natural. Thus, the functor F is pseudo-naturally isomorphic to the identity functor on $\mathrm{Alg}_2^{\mathrm{fd}}$.

Now, let $\eta : F \to G$ be a monoidal pseudo-natural isomorphism between two endofunctors of Alg_2. Since both F and G are pseudo-naturally isomorphic to the identity, we may consider instead a pseudo-natural isomorphism $\eta : \mathrm{id}_{\mathrm{Alg}_2^{\mathrm{fd}}} \to \mathrm{id}_{\mathrm{Alg}_2^{\mathrm{fd}}}$. We claim that up to an invertible modification, the 1-equivalence $\eta_A : A \to A$ must be given by the bimodule ${}_AA_A$, which is the identity 1-morphism on A in Alg_2. Indeed, since η_A is assumed to be linear, it suffices to consider the case of $A = M_n(\mathbb{K})$ and to take direct sums. It is well-known that the only simple modules of A are given by $\mathbb{K}^n$. Thus,

$$\eta_A = (\mathbb{K}^n)^\alpha \otimes_{\mathbb{K}} (\mathbb{K}^n)^\beta, \tag{33}$$

where α and β are multiplicities. Now, [HSV17, Lemma 2.6] ensures that these multiplicities are trivial, and thus we have $\eta_A = {}_AA_A$ up to an invertible intertwiner. This shows that up to invertible modifications, all 1-morphisms in $\mathrm{Aut}_\otimes(\mathrm{Alg}_2^{\mathrm{fd}})$ must be identities.

Now, let m be a monoidal invertible endo-modification of the pseudo-natural transformation $\mathrm{id}_{\mathrm{id}_{\mathrm{Alg}_2^{\mathrm{fd}}}}$. Then, the component $m_A : {}_AA_A \to {}_AA_A$ is an element of $\mathrm{End}_{(A,A)}(A) \cong \mathbb{K}$. As the modification square commutes automatically, this show that the 2-morphisms of $\mathrm{Aut}_\otimes(\mathrm{Alg}_2^{\mathrm{fd}})$ stand in bijection to $\mathbb{K}^*$. □

Remark 3.6. Notice that the symmetric monoidal structure on $\mathrm{Alg}_2^{\mathrm{fd}}$ considered above is *not* the standard one, which is instead the one induced by the tensor product of algebras, and which is the monoidal structure relevant for the remainder of the paper.

The last lemmas imply the following

Lemma 3.7. *Any monoidal $SO(2)$-action on $\mathrm{Alg}_2^{\mathrm{fd}}$ equipped with the monoidal structure given by the direct sum is trivial.*

Proof. Since $\pi_1(SO(2), e) \simeq \mathbb{Z}$, and $H^2_{grp}(\mathbb{Z}, \mathbb{K}^*) \simeq H^2(S^1, \mathbb{K}^*) \simeq 0$, Proposition 3.4 and Lemma 3.5 ensure that any monoidal $SO(2)$-action on $\mathrm{Alg}_2^{\mathrm{fd}}$ is trivializable. □

Recall that we regarded $\mathcal{C} = \mathrm{Alg}_2^{\mathrm{fd}}$ as a monoidal bicategory with the monoidal structure given by direct sums.

Corollary 3.8. *Since $\mathrm{Alg}_2^{\mathrm{fd}}$ and $\mathrm{Vect}_2^{\mathrm{fd}}$ are equivalent as additive categories, any $SO(2)$-action on $\mathrm{Vect}_2^{\mathrm{fd}}$ via linear morphisms is trivializable.*

Remark 3.9. The last two results rely on the fact that $\mathrm{Aut}_\otimes(\mathrm{Alg}_2^{\mathrm{fd}})$ and $\mathrm{Aut}_\otimes(\mathrm{Vect}_2^{\mathrm{fd}})$ are 1-connected as additive categories. This is due to the fact that fully-dualizable part of either Alg_2 or Vect_2 is semi-simple. An example in which the conditions in Proposition 3.4 do *not* hold is provided by the bicategory of Landau-Ginzburg models.

4. Computing homotopy fixed points

In this Section, we explicitly compute the bicategory of homotopy fixed points of an $SO(2)$-action which is induced by an arbitrary pseudo-natural equivalence of the identity functor of an arbitrary bicategory $\mathcal{C}$. Recall that a G-action on a bicategory $\mathcal{C}$ is a monoidal 2-functor $\rho : \Pi_2(G) \to \mathrm{Aut}(\mathcal{C})$, or equivalently a trifunctor $\rho : B\Pi_2(G) \to \mathrm{Bicat}$ with $\rho(*) = \mathcal{C}$. The bicategory of homotopy fixed points $\mathcal{C}^G$ is then given by the tri-limit of this trifunctor.

In Bicat, the tricategory of bicategory, this trilimit can be computed as follows: if $\Delta : B\Pi_2(G) \to \mathrm{Bicat}$ is the constant functor assigning to the one object $*$ the terminal bicategory with one object, the trilimit of the action functor ρ is given by

$$\mathcal{C}^G := \lim \rho = \mathrm{Nat}(\Delta, \rho), \tag{34}$$

the bicategory of tri-transformations between ρ and Δ. This definition is explicitly spelled out in [HSV17, Remark 3.11]. We begin by defining an

$SO(2)$-action on an arbitrary symmetric monoidal bicategory, starting from a pseudo-natural transformation of the identity functor on $\mathcal{C}$.

Definition 4.1. *Since $\Pi_2(SO(2))$ is equivalent to the bicategory with one object, $\mathbb{Z}$ worth of morphisms, and only identity 2-morphisms, we may define an $SO(2)$-action $\rho : \Pi_2(SO(2)) \to \mathrm{Aut}_{\otimes}(\mathcal{C})$ by the following data:*

- *For every group element $g \in SO(2)$, we assign the identity functor of $\mathcal{C}$.*
- *For the generator $1 \in \mathbb{Z}$, we assign the pseudo-natural transformation of the identity functor given by α. Due to the monoidality, this determines the value of ρ on an arbitrary integer.*
- *Since there are only identity 2-morphisms in $\mathbb{Z}$, we have to assign these to identity 2-morphisms in $\mathcal{C}$.*
- *For composition of 1-morphisms, we assign the invertible modification $\rho(a + b) \cong \rho(a) \circ \rho(b)$ coming from the fact that α is a* monoidal *pseudo-natural transformation with respect to composition, which is the monoidal product in $\mathrm{Aut}_{\otimes}(\mathcal{C})$.*
- *In order to make ρ into a monoidal 2-functor, we have to assign additional data which we can choose to be trivial. In detail, we set $\rho(g \otimes h) := \rho(g) \otimes \rho(h)$, and $\rho(e) := \mathrm{id}_{\mathcal{C}}$. Finally, we choose ω, γ and δ as in [HSV17, Remark 3.8] to be identities.*

For a proof that this defines indeed a weak 2-functor, we refer to [Dav11, Lemma 3.2.3].

Our main example is the action of the Serre automorphism on the core of fully-dualizable objects:

Example 4.2. If $\mathcal{C}$ is a symmetric monoidal bicategory, consider $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, the core of the fully-dualizable objects of $\mathcal{C}$. By Proposition 2.8, the Serre automorphism defines a pseudo-natural equivalence of the identity functor on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. By Definition 4.1, we obtain an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, which we denote by ρ^S.

The next theorem computes the bicategory of homotopy fixed points $\mathcal{C}^{SO(2)}$ of the action in Definition 4.1. This theorem generalizes [HSV17, Theorem 4.1], which only computes the bicategory of homotopy fixed points of the *trivial* $SO(2)$-action.

Theorem 4.3. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and let $\alpha : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ be a monoidal pseudo-natural equivalence of the identity functor on $\mathcal{C}$. Let ρ be the $SO(2)$-action on $\mathcal{C}$ as in Definition 4.1. Then, the bicategory of homotopy fixed points $\mathcal{C}^G$ is equivalent to the bicategory with*

- *objects: (c, λ) where c is an object of $\mathcal{C}$ and $\lambda : \alpha_c \to \mathrm{id}_c$ is a 2-isomorphism,*
- *1-morphisms $(c, \lambda) \to (c', \lambda')$ in $\mathcal{C}^G$ are given by 1-morphisms $f : c \to c'$ in $\mathcal{C}$, so that the diagram*

$$\begin{tikzcd} \alpha_{c'} \circ f \arrow[d, "\lambda' * \mathrm{id}_f"'] & f \circ \alpha_c \arrow[l, "\alpha_f"'] \arrow[r, "\mathrm{id}_f * \lambda"] & f \circ \mathrm{id}_c \arrow[d] \\ \mathrm{id}_c \circ f \arrow[rr] & & f \end{tikzcd} \tag{35}$$

  commutes,
- *2-morphisms of $\mathcal{C}^G$ are given by 2-morphisms in $\mathcal{C}$.*

Proof. In order to prove the theorem, we need to explicitly unpack the definition of the bicategory of homotopy fixed points $\mathcal{C}^G$. This is done in [HSV17, Remark 3.11 - 3.14]. In the following, we will use the notation introduced in [HSV17].

The idea of the proof is to show that the forgetful functor which on objects of $\mathcal{C}^G$ forgets the data Θ, Π and M is an equivalence of bicategories. In order to show this, we need to analyze the bicategory of homotopy fixed points. We start with the objects of $\mathcal{C}^G$.

By definition, a homotopy fixed point of this action consists of

- An object $c \in \mathcal{C}$,
- A 1-equivalence $\Theta : c \to c$,

- For every $n \in \mathbb{Z}$, an invertible 2-morphism $\Theta_n : \alpha_c^n \circ \Theta \to \Theta \circ \mathrm{id}_c$ so that (Θ, Θ_n) fulfill the axioms of a pseudo-natural transformation,
- A 2-isomorphism $\Pi : \Theta \circ \Theta \to \Theta$ which obeys the modification square,
- Another 2-isomorphism $M : \Theta \to \mathrm{id}_c$

so that the following equations hold: Equation 3.18 of [HSV17] demands that

$$\Pi \circ (\mathrm{id}_\Theta * \Pi) = \Pi \circ (\Pi * \mathrm{id}_\Theta) \tag{36}$$

whereas Equation 3.19 of [HSV17] demands that Π equals the composition

$$\Theta \circ \Theta \xrightarrow{\mathrm{id}_\Theta * M} \Theta \circ \mathrm{id}_c \cong \Theta \tag{37}$$

and finally Equation 3.20 of [HSV17] tells us that Π must also be equal to the composition

$$\Theta \circ \Theta \xrightarrow{M * \mathrm{id}_\Theta} \mathrm{id}_c \circ \Theta \cong \Theta. \tag{38}$$

Hence Π is fully specified by M. An explicit calculation using the two equations above then confirms that Equation (36) is automatically fulfilled. Indeed, by composing with Π^{-1} from the right, it suffices to show that $\mathrm{id}_\Theta * \Pi = \Pi * \mathrm{id}_\Theta$. Suppose for simplicity that $\mathcal{C}$ is a strict 2-category. Then,

$$\begin{aligned} \mathrm{id}_\Theta * \Pi &= \mathrm{id}_\Theta * (M * \mathrm{id}_\Theta) && \text{by equation (38)} \\ &= (\mathrm{id}_\Theta * M) * \mathrm{id}_\Theta \\ &= \Pi * \mathrm{id}_\Theta && \text{by equation (37).} \end{aligned} \tag{39}$$

Adding appropriate associators shows that this is true in a general bicategory.

Note that by using the modification M, the 2-morphism $\Theta_n : \alpha_n^c \to \Theta \circ \mathrm{id}_c$ can be regarded as a 2-morphism $\lambda_n : \alpha_c \to \mathrm{id}_c$. Here, α_c^n is the n-times composition of 1-morphism α_c. Indeed, define λ_n by setting

$$\lambda_n := \left(\alpha_c \cong \alpha_c \circ \mathrm{id}_c \xrightarrow{\mathrm{id}_{\alpha_c} * M^{-1}} \alpha_c \circ \Theta \xrightarrow{\Theta_n} \Theta \circ \mathrm{id}_c \cong \Theta \xrightarrow{M} \mathrm{id}_c \right). \tag{40}$$

In a strict 2-category, the fact that Θ is a pseudo-natural transformation requires that $\lambda_0 = \mathrm{id}_c$ and that $\lambda_n = \lambda_1 * \cdots * \lambda_1$. In a bicategory, similar equations hold by adding coherence morphisms. Thus, λ_n is fully determined by λ_1. In order to simplify notation, we set $\lambda := \lambda_1 : \alpha_c \to \mathrm{id}_c$.

A 1-morphism of homotopy fixed points $(c, \Theta, \Theta_n, \Pi, M) \to (c', \Theta', \Theta'_n, \Pi', M')$ consists of:

- a 1-morphism $f : c \to c'$,
- an invertible 2-morphism $m : f \circ \Theta \to \Theta' \circ f$ which fulfills the modification square. Note that m is equivalent to a 2-isomorphism $m : f \to f'$ which can be seen by using the 2-morphism M.

The condition due to Equation 3.24 of [HSV17] demands that the following 2-isomorphism

$$f \circ \Theta \xrightarrow{\mathrm{id}_f * M} f \circ \mathrm{id}_c \cong f \tag{41}$$

is equal to the 2-isomorphism

$$f \circ \Theta \xrightarrow{m} \Theta' \circ f \xrightarrow{M' * \mathrm{id}_f} \mathrm{id}_{c'} \circ f \cong f \tag{42}$$

and thus is equivalent to the equation

$$m = \left(f \circ \Theta \xrightarrow{\mathrm{id}_f * M} f \circ \mathrm{id}_c \cong f \cong \mathrm{id}_{c'} \circ f \xrightarrow{M'^{-1} * \mathrm{id}_f} \Theta' \circ f \right) \tag{43}$$

Thus, m is fully determined by M and M'. The condition due to Equation 3.23 of [HSV17] reads

$$m \circ (\mathrm{id}_f * \Pi) = (\Pi' * \mathrm{id}_f) \circ (\mathrm{id}_{\Theta'} * m) \circ (m * \mathrm{id}_\Theta) \tag{44}$$

and is automatically satisfied, as an explicit calculation in [HSV17] confirms. Now, it suffices to look at the modification square of m, in Equation 3.25 of [HSV17]. This condition is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
\alpha_{c'} \circ f \circ \Theta \arrow[d, "\mathrm{id}_{\alpha_{c'}} * m"'] & f \circ \alpha_c \circ \Theta \arrow[l, "\alpha_f * \mathrm{id}_\Theta"'] \arrow[r, "\mathrm{id}_f * \Theta_1"] & f \circ \Theta \arrow[d, "m"] \\
\alpha_{c'} \circ \Theta' \circ f \arrow[rr, "\Theta'_1 * \mathrm{id}_f"'] & & \Theta' \circ f
\end{tikzcd} \tag{45}$$

Substituting m as in Equation (43) and Θ_1 for $\lambda := \lambda_1$ as defined in Equation (40), one confirms that this diagram commutes if and only if the diagram in Equation (35) commutes.

If (f, m) and (g, n) are 1-morphisms of homotopy fixed points, a 2-morphism of homotopy fixed points consists of a 2-isomorphism $\beta : f \to g$ in $\mathcal{C}$. The condition coming from Equation 3.26 of [HSV17] then demands that the diagram

$$\begin{array}{ccc} f \circ \Theta & \xrightarrow{m} & \Theta' \circ f \\ {\scriptstyle \beta * \mathrm{id}_\Theta} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{id}_{\Theta'} * \beta} \\ g \circ \Theta & \xrightarrow[n]{} & \Theta' \circ g \end{array} \tag{46}$$

commutes. Using the fact that both m and n are uniquely specified by M and M', one quickly confirms that this diagram commutes automatically.

Our detailed analysis of the bicategory $\mathcal{C}^G$ shows that the forgetful functor which forgets the data Θ, M, and Π on objects and assigns Θ_1 to λ, which forgets the data m on 1-morphisms, and which is the identity on 2-morphisms is an equivalence of bicategories. $\square$

Corollary 4.4. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and consider the $SO(2)$-action of the Serre automorphism on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ as in Example 4.2. Then, the bicategory of homotopy fixed points $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}$ is equivalent to a bicategory where*

- *objects are given by pairs (X, λ_X) with X a fully-dualizable object of $\mathcal{C}$ and $\lambda_X : S_X \to \mathrm{id}_X$ is a 2-isomorphism which trivializes the Serre automorphism,*
- *1-morphisms are given by 1-equivalences $f : X \to Y$ in $\mathcal{C}$, so that the diagram*

$$\begin{array}{ccccc} S_Y \circ f & \xleftarrow{S_f} & f \circ S_X & \xrightarrow{\mathrm{id}_f * \lambda_X} & f \circ \mathrm{id}_X \\ {\scriptstyle \lambda_Y * \mathrm{id}_f} \Big\downarrow & & & & \Big\downarrow \\ \mathrm{id}_X \circ f & & \xrightarrow{\qquad\qquad} & & f \end{array} \tag{47}$$

  commutes, and
- *2-morphisms are given by 2-isomorphisms in $\mathcal{C}$.*

Remark 4.5. Recall that we have defined the bicategory of homotopy fixed points $\mathcal{C}^G$ as the tri-limit of the action considered as a trifunctor $\rho : B\Pi_2(G) \to$

Bicat. Since we only consider symmetric monoidal bicategories, we actually obtain an action with values in $\mathrm{SymMonBicat}$, the tricategory of symmetric monoidal bicategories. It would be interesting to compute the limit of the action in this tricategory. We expect that this trilimit computed in $\mathrm{SymMonBicat}$ is given by $\mathcal{C}^G$ as a bicategory, with the symmetric monoidal structure induced by the symmetric monoidal structure of $\mathcal{C}$.

Remark 4.6. By [Dav11], the action via the Serre automorphism on $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$ is trivializable. The category of homotopy fixed points $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})^{SO(2)}$ is then equivalent to the bigroupoid of symmetric, semi-simple Frobenius algebras.

Similarly, the action of the Serre automorphism on Vect_2 is trivializable. The bicategory of homotopy fixed points of this action is equivalent to the bicategory of finite Calabi-Yau categories, cf. [HSV17].

5. The 2-dimensional framed bordism bicategory

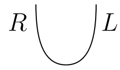
In this Section, we introduce a stricter version of the framed bordism bicategory $\mathrm{Cob}_{2,1,0}^{\mathrm{fr}}$: this symmetric monoidal bicategory $\mathbb{F}_{cfd}$ is the free bicategory of a coherent fully-dual pair as introduced in [Pst14, Definition 3.13].

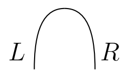
In order to efficiently work with this symmetric monoidal bicategory $\mathbb{F}_{cfd}$, we use a strictification result for symmetric monoidal bicategories as proven in [Bar14, Proposition 13]: any symmetric monoidal bicategory is equivalent to a *stringent* symmetric monoidal 2-category, which can be completely described in terms of a wire diagram calculus introduced in [Bar14]. In the following, we apply this strictification result to the symmetric monoidal bicategory $\mathbb{F}_{cfd}$, and provide a description using the wire diagram calculus developed in [Bar14], which we also refer to for the definition of a stringent symmetric monoidal 2-category.

Using this description, we define a non-trivial $SO(2)$-action on $\mathbb{F}_{cfd}$. If $\mathcal{C}$ is an arbitrary symmetric monoidal bicategory, the action on $\mathbb{F}_{cfd}$ will induce an action on the functor bicategory $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$ of symmetric monoidal functors. Using the Cobordism Hypothesis for framed manifolds, which has been proven in the bicategorical framework in [Pst14], we obtain an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. We show that this induced action coming from the framed bordism bicategory is exactly the action given by the Serre automorphism.

We begin by recasting the definition of $\mathbb{F}_{cfd}$ in terms of the wire diagram calculus.

Definition 5.1. *The symmetric monoidal bicategory $\mathbb{F}_{cfd}$ consists of*

- *2 generating objects L and R,*
- *4 generating 1-morphisms, given by*
  - *a 1-morphism* $\mathrm{coev} : 1 \to R \otimes L$*, which we write as* 

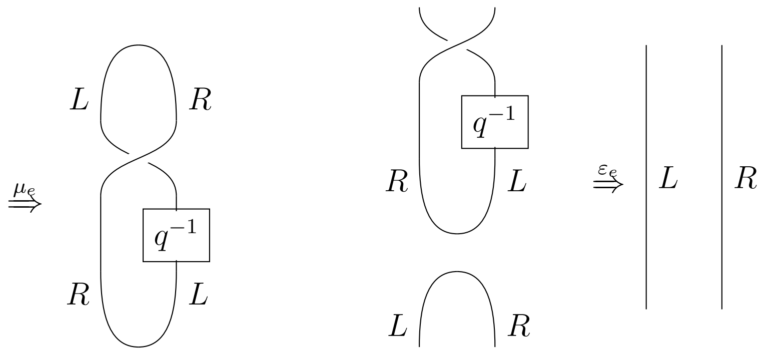
  - $\mathrm{ev} : L \otimes R \to 1$ *which we write as* 

  - *a 1-morphism $q : L \to L$,*
  - *another 1-morphism $q^{-1} : L \to L$,*
- *12 generating 2-cells given by*
  - *isomorphisms α, β, α^{-1} and β^{-1} as in Definition 2.1, which in pictorial form are given as follows:*

(48)

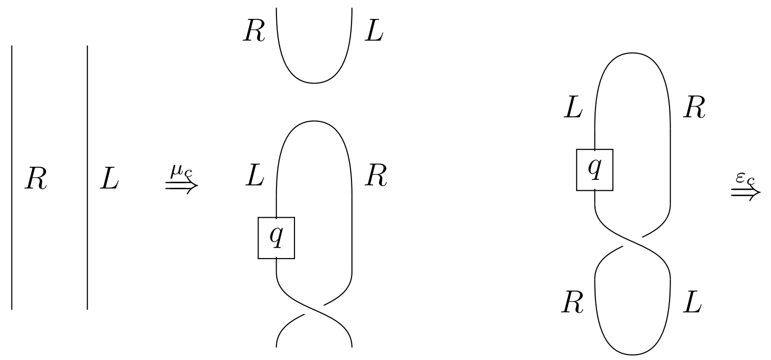
  - *isomorphisms $\psi : qq^{-1} \cong \mathrm{id}_L : \psi^{-1}$ and $\phi : q^{-1}q \cong \mathrm{id}_L : \phi^{-1}$*
  - *2-cells $\mu_e : \mathrm{id}_1 \to \mathrm{ev} \circ \mathrm{ev}^L$ and $\varepsilon_e : \mathrm{ev}^L \circ \mathrm{ev} \to \mathrm{id}_{L\otimes R}$, where* $\mathrm{ev}^L := \tau \circ (\mathrm{id}_R \otimes q^{-1}) \circ \mathrm{coev}$ *which in pictorial form are given*

as follows:



(49)

– *2-cells* $\mu_c : \mathrm{id}_{R\otimes L} \to \mathrm{coev} \circ \mathrm{coev}^L$ *and* $\varepsilon_c : \mathrm{coev}^L \circ \mathrm{coev} \to \mathrm{id}_1$, *where* $\mathrm{coev}^L := \mathrm{ev} \circ (q \otimes \mathrm{id}_R) \circ \tau$ *which in pictorial form are given as follows:*

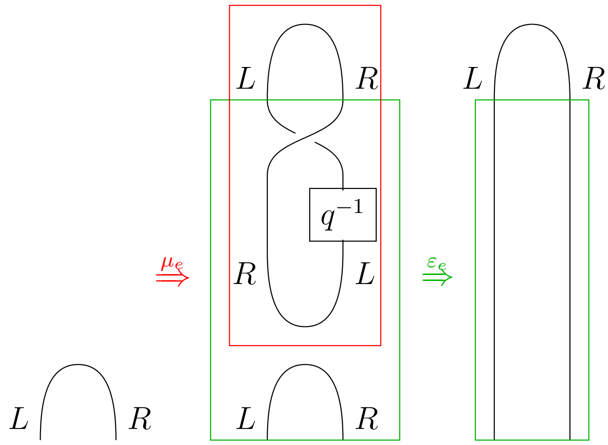


(50)

so that the following relations hold:

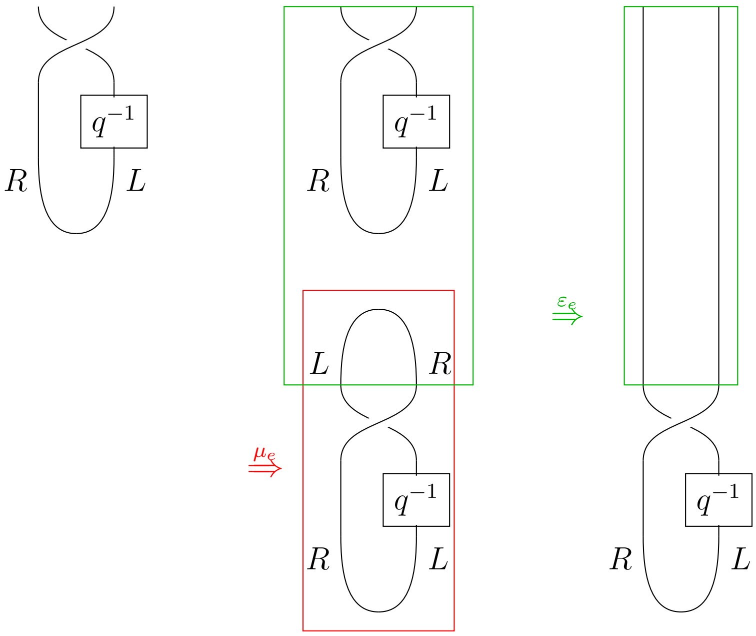
- α *and* α^{-1}, β *and* β^{-1}, ϕ *and* ϕ^{-1}, ψ *and* ψ^{-1} *are inverses to each other,*
- μ_e *and* ε_e *satisfy the two Zorro equations, which in pictorial form*

demand that the following composition of 2-morphisms



(51)

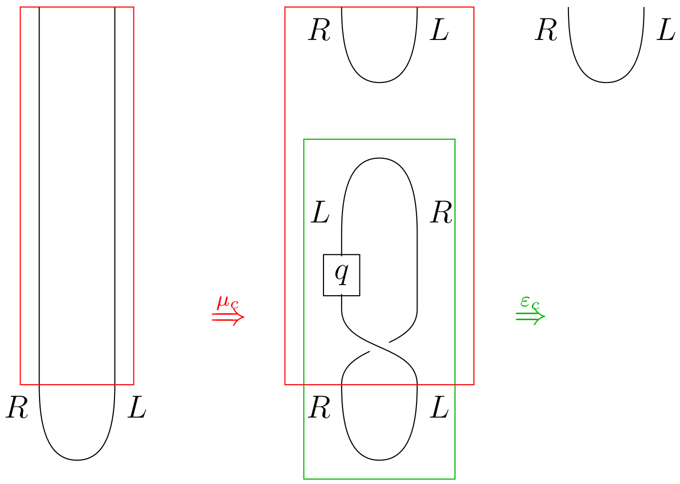
is equal to $\mathrm{id}_{\mathrm{ev}}$, *and that the following composition of 2-morphisms*



(52)

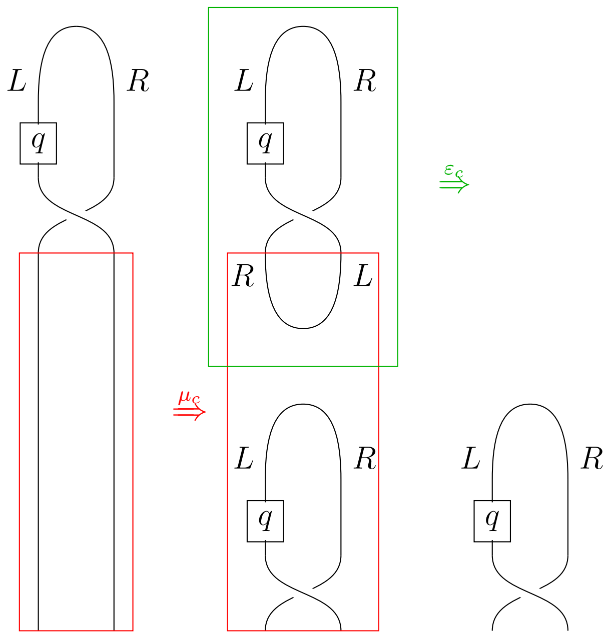
is equal to $\mathrm{id}_{\mathrm{ev}^L}$.

- μ_c *and* ε_c *satisfy the two Zorro equations, which in pictorial form* *demand that the composition*



(53)

is equal to $\mathrm{id}_{\mathrm{coev}}$*, and the composition of the following 2-morphisms*



(54)

is equal to $\mathrm{id}_{\mathrm{coev}^L}$.

- ϕ *and* ψ *satisfy triangle identities,*

- *the cusp-counit equations in figure 5 and 6 on p.33 of [Pst14] are satisfied,*
- *the swallowtail equations in figure 3 and 4 on p.15 of [Pst14] are satisfied.*

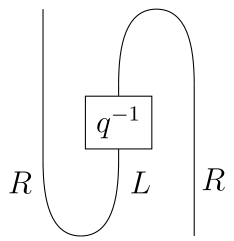
5.1 Action on the framed bordism bicategory

We can now proceed to construct an $SO(2)$-action on $\mathbb{F}_{cfd}$. This action will be vital for the remainder of the paper.

By Definition 4.1 it suffices to construct a pseudo-natural equivalence of the identity functor on $\mathbb{F}_{cfd}$ in order to construct an $SO(2)$-action. This pseudo-natural transformation is given as follows:

Definition 5.2. *Let $\mathbb{F}_{cfd}$ be the free symmetric monoidal bicategory on a coherent fully-dual object as in Definition 5.1. We construct a pseudo-natural equivalence $\alpha : \mathrm{id}_{\mathbb{F}_{cfd}} \to \mathrm{id}_{\mathbb{F}_{cfd}}$ of the identity functor on $\mathbb{F}_{cfd}$ as follows:*

- *For every object c of $\mathbb{F}_{cfd}$, we need to provide a 1-equivalence $\alpha_c : c \to c$.*
  - *For the object L of $\mathbb{F}_{cfd}$, we define $\alpha_L := q : L \to L$,*
  - *for the object R of $\mathbb{F}_{cfd}$, we set $\alpha_R := (q^{-1})^*$, which in pictorial form is given by*

$(q^{-1})^* :=$ 

(55)

- *for every 1-morphism $f : c \to d$ in $\mathbb{F}_{cfd}$, we need to provide a 2-isomorphism*

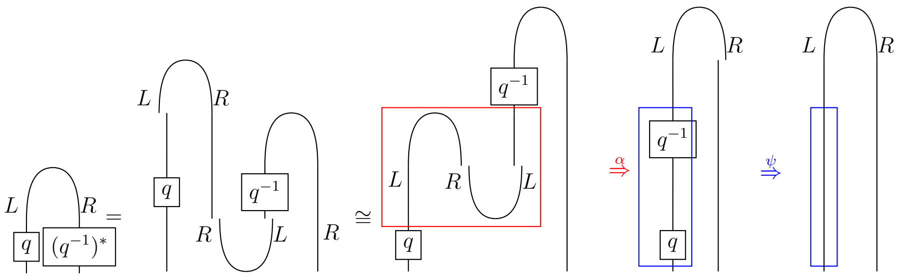
$$\alpha_f : f \circ \alpha_c \to \alpha_d \circ f. \tag{56}$$

  - *For the 1-morphism $q : L \to L$ of $\mathbb{F}_{cfd}$ we define the 2-isomorphism $\alpha_q := \mathrm{id}_{q \circ q}$.*

– *For the 1-morphism $q^{-1} : L \to L$ we define the 2-isomorphism*

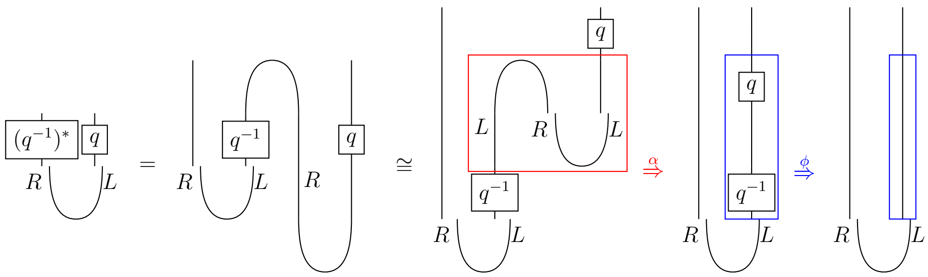
$$\alpha_{q^{-1}} := \left(q^{-1} \circ q \xrightarrow{\phi} \mathrm{id}_L \xrightarrow{\psi^{-1}} q \circ q^{-1} \right). \tag{57}$$

– *For the evaluation* $\mathrm{ev} : L \otimes R \to 1$*, we define the 2-isomorphism* α_{ev} *to be the following composition:*



(58)

– *For the coevaluation* $\mathrm{coev} : 1 \to R \otimes L$*, we define the 2-isomorphism* α_{coev} *to be the composition*



(59)

One now checks that this defines a pseudo-natural transformation of $\mathrm{id}_{\mathbb{F}_{cfd}}$. Using Definition 4.1 gives us a non-trivial $SO(2)$-action on $\mathbb{F}_{cfd}$.

Remark 5.3. Note that the $SO(2)$-action on $\mathbb{F}_{cfd}$ does *not* send generators to generators: for instance, the 1-morphism $(q^{-1})^*$ in Equation (55) is not part of the generating data of $\mathbb{F}_{cfd}$.

Remark 5.4. Notice that the pseudo-natural equivalence $\alpha : \mathrm{id}_{\mathbb{F}_{cfd}} \to \mathrm{id}_{\mathbb{F}_{cfd}}$ constructed in Definition 5.2 is a *monoidal* pseudo-natural transformation.

This follows from the fact that we have defined α via generators and relations. In detail, we set

$$\begin{aligned} \alpha_X \otimes \alpha_Y &:= \alpha_{X \otimes Y} \\ \alpha_1 &:= \mathrm{id}_1. \end{aligned} \tag{60}$$

Thus, we can choose the additional data Π and M of a monoidal pseudo-natural transformation to be trivial, and we obtain an $SO(2)$-action on $\mathbb{F}_{cfd}$ via symmetric monoidal morphisms.

5.2 Induced action on functor categories

Starting from the action defined on $\mathbb{F}_{cfd}$, we induce an action on the bicategory of functors $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$ for an arbitrary bicategory $\mathcal{C}$. The construction of the induced action on the bicategory of functors is a general construction. We provide details in the following.

Definition 5.5. *Let $\rho : \Pi_2(G) \to \mathrm{Aut}(\mathcal{C})$ be a G-action on a bicategory $\mathcal{C}$, and let $\mathcal{D}$ be another bicategory. The G-action $\tilde{\rho} : \Pi_2(G) \to \mathrm{Aut}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ induced by ρ is defined as follows:*

- *On objects $g \in G$, we define an endofunctor $\tilde{\rho}(g)$ of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ on objects F on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ by $\tilde{\rho}(g)(F) := F \circ \rho(g^{-1})$. If $\alpha : F \to G$ is a 1-morphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, we define*

$$\tilde{\rho}(g)(\alpha) := \quad \begin{tikzcd} F\rho(g^{-1})c \arrow[r, "\alpha_{\rho(g^{-1})(c)}"] \arrow[d, "F\rho(g^{-1})(f)"'] & G\rho(g^{-1})c \arrow[d, "G\rho(g^{-1})(f)"] \arrow[dl, Rightarrow, "\alpha_{\rho(g^{-1})(f)}"] \\ F\rho(g^{-1})d \arrow[r, "\alpha_{\rho(g^{-1})(d)}"'] & G\rho(g^{-1})d \end{tikzcd} \tag{61}$$

  If $m : \alpha \to \beta$ is a 2-morphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, the value of $\tilde{\rho}(\gamma)$ is given by

$$\tilde{\rho}(\gamma)(m)_x := m_{\rho(g^{-1})(x)}. \tag{62}$$

- *on 1-morphisms $\gamma : g \to h$ of $\Pi_2(G)$, we define a 1-morphism $\tilde{\rho}(\gamma)$ in $\mathrm{Aut}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ between the two endofunctors $F \mapsto F \circ \rho(g^{-1})$ and $F \mapsto F \circ \rho(h^{-1})$ of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.*

  Explicitly, this means:

– *For each 2-functor $F : \mathcal{C} \to \mathcal{D}$, we need to provide a pseudo-natural transformation $\tilde{\rho}(\gamma)_F : F \circ \rho(g^{-1}) \to F \circ \rho(h^{-1})$ which we define via the diagram*

$$\begin{tikzcd} F\rho(g^{-1})x \arrow[r, "F(\rho(\gamma^{-1})_x)"] \arrow[d, "F\rho(g^{-1})(f)"'] & F\rho(h^{-1})x \arrow[d, "F\rho(h^{-1})(f)"] \arrow[dl, Rightarrow, "F(\rho(\gamma^{-1})_f)"] \\ F\rho(g^{-1})y \arrow[r, "F(\rho(\gamma^{-1})_y)"'] & F\rho(h^{-1})y \end{tikzcd} \tag{63}$$

Here, γ^{-1} is the "inverse" path of γ given by $t \mapsto \gamma(t)^{-1}$, and $f : x \to y$ is a 1-morphism in $\mathcal{C}$.

– *For every pseudo-natural transformation $\alpha : F \to G$, we need to provide a modification $\tilde{\rho}(\gamma)_\alpha$ in the diagram*

$$\begin{tikzcd} \tilde{\rho}(g)(F) \arrow[r, "\tilde{\rho}(\gamma)_F"] \arrow[d, "\tilde{\rho}(g)(\alpha)"'] & \tilde{\rho}(h)(F) \arrow[d, "\tilde{\rho}(h)(\alpha)"] \arrow[dl, Rightarrow, "\tilde{\rho}(\gamma)_\alpha"] \\ \tilde{\rho}(g)(G) \arrow[r, "\tilde{\rho}(\gamma)_G"'] & \tilde{\rho}(h)(G) \end{tikzcd} \tag{64}$$

which we define by

$$\tilde{\rho}(\gamma)_\alpha := \alpha^{-1}_{\rho(\gamma^{-1})_x}. \tag{65}$$

- *For the 2-morphisms in $\mathrm{Aut}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ we proceed in a similar fashion: if $m : \gamma \to \gamma'$ is a 2-track, we have to provide a 2-morphism $\tilde{\rho}(m) : \tilde{\rho}(\gamma) \to \tilde{\rho}(\gamma')$ which can be done by explicitly writing down diagrams as above.*

The rest of the data of a monoidal functor $\tilde{\rho}$ is induced from the data of the monoidal functor ρ.

For $\mathcal{C}$ and $\mathcal{D}$ symmetric monoidal bicategories, the bicategory of symmetric monoidal functors $\mathrm{Fun}_\otimes(\mathcal{C}, \mathcal{D})$ acquires a monoidal structure by "pointwise evaluation" of functors. Such a monoidal structure is also symmetric, see [SP09]. The following result is straightforward.

Lemma 5.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal bicategories, and let ρ be a monoidal action of a group G on $\mathcal{C}$. Then ρ induces a monoidal action $\tilde{\rho} : \Pi_2(G) \to \mathrm{Aut}_{\otimes}(\mathrm{Fun}_{\otimes}(\mathcal{C}, \mathcal{D}))$.*

Example 5.7. Our main example for induced actions is the $SO(2)$-action on $\mathbb{F}_{cfd}$ as in Definition 5.2. This action only depends on a pseudo-natural equivalence α of the identity functor on $\mathrm{id}_{\mathbb{F}_{cfd}}$. Consequently, the induced action on $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$ also only depends on a pseudo-natural equivalence of the identity functor on $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$. Using the definition above, we construct this induced pseudo-natural equivalence $\tilde{\alpha}$ as follows.

- For every 2-functor $F : \mathcal{C} \to \mathcal{D}$, we need to provide a pseudo-natural equivalence $\tilde{\alpha}_F : F \to F$, which is given by the diagram

$$\tilde{\alpha}_F := \qquad \begin{tikzcd} Fx \arrow[r, "F(\alpha_x^{-1})"] \arrow[d, "F(f)"'] & Fx \arrow[d, "F(f)"] \arrow[dl, Rightarrow, "F(\alpha_f^{-1})"] \\ Fy \arrow[r, "F(\alpha^{-1})_y"'] & Fy \end{tikzcd} \tag{66}$$

- for every pseudo-natural transformation $\beta : F \to G$, we need to give a modification $\tilde{\alpha}_\beta$, which we define by the diagram

$$\begin{tikzcd} Fx \arrow[r, "F(\alpha_x^{-1})"] \arrow[d, "\beta_x"'] & Fx \arrow[d, "\beta_x"] \arrow[dl, Rightarrow, "\beta^{-1}_{(\alpha_x^{-1})}"] \\ Gx \arrow[r, "G(\alpha^{-1})_x"'] & Gx \end{tikzcd} \tag{67}$$

This defines a pseudo-natural equivalence of the identity functor on $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$. By Definition 4.1, we obtain an $SO(2)$-action on $\mathrm{Fun}(\mathbb{F}_{cfd,}\mathcal{C})$. Note that $\mathbb{F}_{cfd}$ is even a *symmetric monoidal* bicategory. The $SO(2)$-action on $\mathbb{F}_{cfd}$ of Definition 5.2 is via symmetric monoidal homomorphisms by Remark 5.4. Hence, if $\mathcal{C}$ is also symmetric monoidal, then Lemma 5.6 provides a monoidal action on $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$.

5.3 Induced action on the core of fully-dualizable objects

In this subsection, we compute the $SO(2)$-action on the core of fully-dualizable objects coming from the $SO(2)$-action on $\mathbb{F}_{cfd}$. Starting from the $SO(2)$-action on $\mathbb{F}_{cfd}$ as by Definition 5.2, we have shown in the previous subsection how to induce an $SO(2)$-action on the bicategory of symmetric monoidal functors $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$ for $\mathcal{C}$ some symmetric monoidal bicategory. By the Cobordism Hypothesis for framed manifolds, we obtain an induced $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. More precisely, denote by

$$\begin{aligned} \mathrm{ev}_L : \mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C}) &\to \mathscr{K}(\mathcal{C}^{\mathrm{fd}}) \\ Z &\mapsto Z(L) \end{aligned} \tag{68}$$

the evaluation map. The Cobordism Hypothesis for framed manifolds in two dimensions [Pst14, Lur09] states that ev_L is an equivalence of symmetric monoidal bicategories. Hence, the composition of the $SO(2)$-action on $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$ and (the inverse of) ev_L provides an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. The next proposition shows that this action is equivalent to the action ρ^S induced by the Serre automorphism which is illustrated in Example 4.2.

Proposition 5.8. *Let ρ be the $SO(2)$-action on $\mathbb{F}_{cfd}$ given in Definition 5.2, and let $\mathcal{C}$ be a symmetric monoidal bicategory. By Definition 5.5, we obtain a monoidal $SO(2)$-action on* $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$*. Then, the monoidal $SO(2)$-action induced by the evaluation in Equation* (68) *on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ is equivalent to ρ^S.*

Proof. Let

$$\rho : \Pi_2(SO(2)) \to \mathrm{Aut}_{\otimes}(\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})) \tag{69}$$

be the $SO(2)$-action on the bicategory of symmetric monoidal functors $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$ as in Example 5.7. This action only depends on a monoidal pseudo-natural transformation α on the identity functor on $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$. By [Pst14], the 2-functor in Equation (68) which evaluates a framed field theory on the object L is an equivalence of bicategories. Thus, we obtain an $SO(2)$-action ρ' on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. This action is given as follows. By Definition 4.1, we only need to provide a monoidal pseudo-natural transformation of the identity functor of $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. In order to write down this monoidal pseudo-natural transformation, note that the functor

$$\begin{aligned} \mathrm{Aut}_\otimes(\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})) &\to \mathrm{Aut}_\otimes(\mathscr{K}(\mathcal{C}^{\mathrm{fd}})) \\ F &\mapsto \mathrm{ev}_L \circ F \circ \mathrm{ev}_L^{-1} \end{aligned} \tag{70}$$

is a monoidal equivalence. Hence, the induced pseudo-natural transformation of $\mathrm{id}_{\mathscr{K}(\mathcal{C}^{\mathrm{fd}})}$ is given as follows:

- For each fully-dualizable object c of $\mathcal{C}$, we assign the 1-morphism $\alpha'_c : c \to c$ defined by

$$\alpha'_c := \mathrm{ev}_L \left(\alpha_{\left(\mathrm{ev}_L^{-1}(c)\right)} \right) \tag{71}$$

- for each 1-equivalence $f : c \to d$ between fully-dualizable objects of $\mathcal{C}$, we define a 2-isomorphism $\alpha'_f : f \circ \alpha'_c \to \alpha'_d \circ f$ by the formula

$$\alpha'_f := \mathrm{ev}_L \left(\alpha_{\left(\mathrm{ev}_L^{-1}(f)\right)} \right). \tag{72}$$

Here, α is the pseudo-natural transformation as in Example 5.7. In order to see that α'_c is given by the Serre automorphism of the fully-dualizable object c, note that the 1-morphism $q : L \to L$ of $\mathbb{F}_{cfd}$ is mapped to the Serre automorphism $S_{Z(L)}$ by the equivalence in Equation (68). □

Corollary 5.9. *Let ρ be the $SO(2)$-action on $\mathbb{F}_{cfd}$ given in Definition 5.2, and let $\mathcal{C}$ be a symmetric monoidal bicategory. Consider the $SO(2)$-action ρ^S on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ induced by the Serre automorphism. Then the evaluation morphism ev_L induces an equivalence of bicategories*

$$\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \to \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}. \tag{73}$$

Proof. By Proposition 5.8, the equivalence of Equation (68) is $SO(2)$-equivariant. Thus, it induces an equivalence on homotopy fixed points, cf. [Hes16, Definition 5.3] for an explicit description. It is also possible to construct this equivalence directly: by theorem 4.3, the bicategory of homotopy fixed points $\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)}$ is equivalent to the bicategory where

- objects are given by symmetric monoidal functors $Z : \mathbb{F}_{cfd} \to \mathcal{C}$, together with a modification $\lambda_Z : \tilde{\alpha}_Z \to \mathrm{id}_Z$. Explicitly, this means: if α is the endotransformation of the identity functor of $\mathbb{F}_{cfd}$ as in Definition 5.2, we obtain two 2-isomorphisms in $\mathcal{C}$:

$$\begin{aligned} &\lambda_L : Z(q^{-1}) \to \mathrm{id}_{Z(L)} \\ &\lambda_R : Z(((q^{-1})^*)^{-1}) \to \mathrm{id}_{Z(R)} \end{aligned} \tag{74}$$

  which are compatible with evaluation and coevaluation,

- 1-morphisms are given by symmetric monoidal pseudo-natural transformations $\mu : Z \to Z'$, so that the analogue of the diagram in Equation (35) commutes,

- 2-morphisms are given by symmetric monoidal modifications.

Now notice that $Z(q)$ is precisely the Serre automorphism of the object $Z(L)$. Thus, λ_L provides a trivialization of (the inverse of) the Serre automorphism. Applying theorem 4.3 again to the action of the Serre automorphism on the core of fully-dualizable objects shows that the functor $Z \mapsto (Z(L), \lambda_L)$ is an equivalence of homotopy fixed point bicategories. □

Remark 5.10. Note that in Corollary 5.9 we have proven that the evaluation induces an equivalence of bicategories $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \to \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}$. We expect that this equivalence is an equivalence of *monoidal* bicategories. In order to prove this, one would have to explicitly work out the monoidal structure of $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}$ which is induced from the monoidal structure of $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$.

6. Invertible Field Theories

In the section, we consider the case of 2-dimensional oriented *invertible* topological field theories: such theories are in many ways easier to describe than arbitrary TQFTs, and play an important role in condensed matter physics and homotopy theory, as suggested in [Fre14a, Fre14b].

Denote with $\mathrm{Pic}(\mathcal{C})$ the *Picard groupoid* of a symmetric monoidal bicategory $\mathcal{C}$: it is defined as the maximal subgroupoid of $\mathcal{C}$ where the objects are invertible with respect to the monoidal structure of $\mathcal{C}$. Notice that $\mathrm{Pic}(\mathcal{C})$ inherits the symmetric monoidal structure from $\mathcal{C}$. Recall that $\mathrm{Fun}_{\otimes}(\mathrm{Cob}_{2,1,0}, \mathcal{C})$ is equipped with a monoidal structure which is defined pointwise.

Definition 6.1. *An invertible framed TQFT with values in $\mathcal{C}$ is an invertible object in $\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{fr}}_{2,1,0}, \mathcal{C})$. The space of invertible framed TQFTs with values in $\mathcal{C}$ is given by $\mathrm{Pic}(\mathrm{Fun}_{\otimes}(\mathrm{Cob}_{2,1,0}, \mathcal{C}))$.*

Remark 6.2. Equivalently, an invertible TQFT assigns to the point in $\mathrm{Cob}_{2,1,0}$ an invertible object in $\mathcal{C}$, and to any 1- and 2-dimensional manifold it assigns invertible 1- and 2-morphisms.

Since the Cobordism Hypothesis provides a *monoidal* equivalence between $\mathrm{Fun}_{\otimes}(\mathrm{Cob}_{2,1,0}, \mathcal{C})$ and $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, the space of invertible framed TQFTs is given by $\mathrm{Pic}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}}))$, since taking the Picard groupoid behaves well with respect to monoidal equivalences.

We begin by proving the following:

Lemma 6.3. *Let $\mathcal{C}$ be a symmetric monoidal bicategory. Then, there is an equivalence of symmetric monoidal bicategories*

$$\mathrm{Pic}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}})) \cong \mathrm{Pic}(\mathcal{C}). \tag{75}$$

Proof. First note that $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ is a monoidal 2-groupoid, so there is an equivalence of bicategories $\mathrm{Pic}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}})) \cong \mathrm{Pic}(\mathcal{C}^{\mathrm{fd}})$. Now, it suffices to show that every object X in $\mathrm{Pic}(\mathcal{C})$ is already fully-dualizable. Indeed, denote the tensor-inverse of X by X^{-1}. By definition, we have 1-equivalences $X \otimes X^{-1} \cong 1$ and $1 \cong X^{-1} \otimes X$, which serve as evaluation and coevaluation. These maps may be promoted to adjoint 1-equivalences by [SP09, Proposition A.27]. Thus, the evaluation and coevaluation also admit adjoints, which suffices for fully-dualizability. $\square$

Notice that given a monoidal bicategory $\mathcal{C}$, any monoidal auto-equivalence of $\mathcal{C}$ preserves the Picard groupoid of $\mathcal{C}$, since it preserves invertibility of objects and (higher) morphisms. In particular, we have a monoidal 2-functor

$$\mathrm{Aut}_{\otimes}(\mathcal{C}) \to \mathrm{Aut}_{\otimes}(\mathrm{Pic}(\mathcal{C})) \tag{76}$$

obtained by restriction. Since the $SO(2)$-action induced by the action on $\mathrm{Cob}_{2,1,0}$ is monoidal, it induces an action on $\mathrm{Pic}(\mathcal{C})$. To proceed, we need the following

Lemma 6.4. *Let $\mathcal{C}$ be a symmetric monoidal bicategory such that* $\mathrm{Pic}(\mathcal{C})$ *is monoidally equivalent to* $B^2\mathbb{K}^*$. *Then*

$$\mathrm{Aut}_{\otimes}(\mathrm{Pic}(\mathcal{C})) \simeq \mathrm{Iso}(\mathbb{K}^*) \tag{77}$$

where the category on the right hand side is regarded as a discrete symmetric monoidal bicategory.

Proof. Since $\mathrm{Pic}(\mathcal{C}) \simeq B^2\mathbb{K}^*$ monoidally, we have to describe the Picard groupoid of the category of monoidal functors from $B^2\mathbb{K}^*$ to $B^2\mathbb{K}^*$. First, notice that the monoidal bicategory $B^2\mathbb{K}^*$ is the strict symmetric monoidal bicategory with a single object $\bullet$, and $B\mathbb{K}^*$ as the strict symmetric monoidal category of 1- and 2-morphisms. The bicategory of symmetric monoidal functors from $B^2\mathbb{K}^*$ to itself is then equivalent to the category $\mathrm{Fun}_{\otimes}(B\mathbb{K}^*, B\mathbb{K}^*)$ regarded as a bicategory with only identity 2-cells; see [CG07] for details.

By direct investigation, $\mathrm{Fun}_{\otimes}(B\mathbb{K}^*, B\mathbb{K}^*)$ is equivalent as a symmetric monoidal category to $\mathrm{Hom}(\mathbb{K}^*, \mathbb{K}^*)$ regarded as a discrete category. Indeed, any monoidal functor $F : B\mathbb{K}^* \to B\mathbb{K}^*$ is determined by a group homomorphism $\phi^F : \mathbb{K}^* \to \mathbb{K}^*$, and monoidality ensures that any natural transformation must correspond to the identity element in $\mathbb{K}^*$. Notice that the composition of monoidal functors $F' \circ F$ corresponds to $\phi^{F'} \circ \phi^F$. In follows then that the Picard groupoid of $\mathrm{Fun}_{\otimes}(B^2\mathbb{K}^*, B^2\mathbb{K}^*)$ is given by $\mathrm{Iso}(\mathbb{K}^*)$, which correspond to the invertible elements in the monoid $\mathrm{Hom}(\mathbb{K}^*, \mathbb{K}^*)$. □

Examples of symmetric monoidal bicategories satisfying the assumption of Lemma 6.4 are $\mathrm{Alg}_2^{\mathrm{fd}}$ and $\mathrm{Vect}_2^{\mathrm{fd}}$. In general cases, we have the following

Lemma 6.5. *Let $\mathcal{C}$ be a symmetric monoidal bicategory such that* $\mathrm{Pic}(\mathcal{C})$ *is monoidally equivalent to* $B^2\mathbb{K}^*$. *Then any monoidal* $SO(2)$*-action on* $\mathrm{Pic}(\mathcal{C})$ *is trivializable.*

Proof. Since we have equivalences of monoidal bicategories $\Pi_2(SO(2)) \simeq B\mathbb{Z}$ and $\mathrm{Aut}_{\otimes}(\mathrm{Pic}(\mathcal{C})) \simeq \mathrm{Iso}(\mathbb{K}^*)$, monoidal actions correspond to monoidal 2-functors $B\mathbb{Z} \to \mathrm{Iso}(\mathbb{K}^*)$: here we regard $B\mathbb{Z}$ as a symmetric monoidal bicategory with a single object, and the group $\mathrm{Iso}(\mathbb{K}^*)$ as a discrete symmetric monoidal bicategory, i.e. all 1- and 2-cells are identities. Monoidality implies that the single object of $B\mathbb{Z}$ is sent to the identity isomorphism of $\mathbb{K}^*$, which correspond to the identity functor on $\mathrm{Pic}(\mathcal{C})$. This forces the functor to be trivial on objects. It is clear that the action is also trivial on 1-

and 2-morphisms. Since there are no nontrivial morphisms in $\mathrm{Iso}(\mathbb{K}^*)$, the monoidal structure on the action ρ must also be trivial. □

Finally, we need the following

Lemma 6.6. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and let ρ_S be the $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ by the Serre automorphism as in Example 4.2. Since this action is monoidal, it induces an action on $\mathrm{Pic}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}})) \cong \mathrm{Pic}(\mathcal{C})$ by Lemma 6.3. We have then an equivalence of monoidal bicategories*

$$\mathrm{Pic}\left((\mathscr{K}(\mathcal{C}^{\mathrm{fd}}))^{SO(2)}\right) \cong \mathrm{Pic}(\mathcal{C})^{SO(2)}. \tag{78}$$

Proof. Theorem 4.3 allows us to compute the two bicategories of homotopy fixed points explicitly: we see that both bicategories have invertible objects X of $\mathcal{C}$, together with the choice of a trivialization of the Serre automorphism as objects. The 1-morphisms of both bicategories are given by 1-equivalences between invertible objects of $\mathcal{C}$, so that the diagram in equation (47) commutes, while 2-morphisms are given by 2-isomorphisms in $\mathcal{C}$. □

The implication of the above lemmas is the following: when $\mathcal{C}$ is a symmetric monoidal bicategory with $\mathrm{Pic}(\mathcal{C}) \cong B^2\mathbb{K}^*$, the action of the Serre-automorphism on framed, invertible field theories with values in $\mathcal{C}$ is trivializable. Thus *all* framed invertible 2d TQFTs with values in $\mathcal{C}$ can be turned into orientable ones.

7. Comments on Homotopy Orbits

So far, we have constructed an $SO(2)$-action on the bicategory $\mathbb{F}_{cfd}$. We have shown how the action on $\mathbb{F}_{cfd}$ induces an action on the bicategory of symmetric monoidal functors $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$, and that via the (framed) Cobordism Hypothesis the induced action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ for framed manifolds agrees with the action of the Serre automorphism. As a consequence, we are able to provide an equivalence of bicategories

$$\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \to \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)} \tag{79}$$

in Corollary 5.9. We could then in principle deduce the Cobordism Hypothesis for oriented manifolds from 79, once we provide an equivalence of bicategories

$$\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \cong \mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}). \tag{80}$$

The above equivalence can be proven directly by using a presentation of the oriented bordism bicategory via generators and relations, given in [SP09], and the notion of a Calabi-Yau object internal to a bicategory. The details appear in [Hes17].

Here, we want instead to comment on an alternative approach. Namely, in order to provide an equivalence as in (80), it suffices to identify the oriented bordism bicategory with the *colimit* of the $SO(2)$-action on $\mathbb{F}_{cfd}$. Indeed, recall that one may define a G-action on a bicategory $\mathcal{C}$ to be a trifunctor $\rho : B\Pi_2(G) \to \mathrm{Bicat}$ with $\rho(*) = \mathcal{C}$. The tricategorical colimit of this functor will then be the bicategory of co-invariants or *homotopy orbits* of the G-action, denoted by $\mathcal{C}_G$. By Definition of the tricategorical colimit, and the fact that colimits are sent to limits by the Hom functor, we then obtain an equivalence of bicategories

$$\mathrm{Fun}_{\otimes}(\mathcal{C}_G, \mathcal{D}) \cong \mathrm{Fun}_{\otimes}(\mathcal{C}, \mathcal{D})^G. \tag{81}$$

The following conjecture is then natural:

Conjecture 7.1. *The bicategory of co-invariants of the $SO(2)$-action on $\mathbb{F}_{cfd}$ is monoidally equivalent to the oriented bordism bicategory, i.e. we have a monoidal equivalence*

$$(\mathbb{F}_{cfd})_{SO(2)} \cong \mathrm{Cob}^{\mathrm{or}}_{2,1,0}\,. \tag{82}$$

Furthermore, the colimit is compatible with the monoidal structure.

Remark 7.2. We believe that this is not an isolated phenomenon, in the sense that any higher bordism category equipped with additional tangential structure should be obtained by taking an appropriate colimit of a G-action on the framed bordism category.

Given Conjecture 7.1 and Equation 81, we obtain the following sequence of monoidal equivalences

$$\begin{aligned}\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}) &\cong \mathrm{Fun}_{\otimes}((\mathbb{F}_{cfd})_{SO(2)}, \mathcal{C}) \\ &\cong \mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \cong \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}.\end{aligned} \tag{83}$$

Hence Conjecture 7.1 implies the Cobordism Hypothesis for oriented 2-manifolds. Notice that the chain of equivalences in 83 is natural in $\mathcal{C}$.
On the other hand, the Cobordism Hypothesis for oriented manifolds in 2-dimensions implies Conjecture 7.1 . Indeed, by using a tricategorical version of the Yoneda Lemma, as developed for instance in [Buh15], the chain of equivalences

$$\begin{aligned} \mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}) &\cong \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)} \\ &\cong \mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{fr}}_{2,1,0}, \mathcal{C})^{SO(2)} \\ &\cong \mathrm{Fun}_{\otimes}((\mathbb{F}_{cfd})_{SO(2)}, \mathcal{C}) \end{aligned} \tag{84}$$

implies that $\mathrm{Cob}^{\mathrm{or}}_{2,1,0}$ is equivalent to $(\mathbb{F}_{cfd})_{SO(2)}$, due to the uniqueness of representable objects.
We summarize the above arguments in the following

Lemma 7.3. *The Cobordism Hypothesis for oriented 2-dimensional manifolds is equivalent to Conjecture 7.1.*

It would then be of great interest to develop concrete constructions of homotopy co-invariants of actions of groups on bicategories, in the same spirit of [HSV17] and the present work, in order to verify directly the equivalence in Conjecture 7.1, and to extend the above arguments to general tangential G-structures.

$$\begin{tikzcd}
XY(XY)^* \arrow[r, "\mathrm{id}_{XY}\ \mathrm{coev}_Y\ \mathrm{id}_{(XY)^*}"] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{id}_{\mathrm{id}_X}\ \alpha_Y\ \mathrm{id}_{\mathrm{id}_{(XY)^*}}"] & XYY^*Y(XY)^* \arrow[r, "\mathrm{id}_{XYY^*}\ \mathrm{coev}_X\ \mathrm{id}_{Y(XY)^*}"] \arrow[d, "\mathrm{id}_X\ \mathrm{ev}_Y\ \mathrm{id}_{Y(XY)^*}"] \arrow[dr, phantom, "\cong"] & XYY^*X^*XY(XY)^* \arrow[r, "\mathrm{id}_{XYY^*X^*}\ \mathrm{ev}_{XY}"] \arrow[d, "\mathrm{id}_X\ \mathrm{ev}_Y\ \mathrm{id}_{X^*XY(XY)^*}"] \arrow[dr, phantom, "\cong"] & XYY^*X^* \arrow[r, "\mathrm{id}_X\ \tau_{YY^*,X^*}"] \arrow[d, "\mathrm{id}_X\ \mathrm{ev}_Y\ \mathrm{id}_{X^*}"] \arrow[dr, phantom, "\cong"] & XX^*YY^* \arrow[d, "\mathrm{ev}_X\ \mathrm{ev}_Y"] \\
XY(XY)^* \arrow[r, "\mathrm{id}_{XY(XY)^*}"'] \arrow[drr, equal, bend right] & XY(XY)^* \arrow[r, "\mathrm{id}_X\ \mathrm{coev}_X\ \mathrm{id}_{Y(XY)^*}"'] \arrow[d, phantom, "\alpha_X\ \mathrm{id}_{\mathrm{id}_{(XY)^*}}"] & XX^*XY(XY)^* \arrow[r, "\mathrm{id}_{XX^*}\ \mathrm{ev}_{XY}"'] \arrow[d, "\mathrm{ev}_X\ \mathrm{id}_{XY(XY)^*}"] \arrow[dr, phantom, "\cong"] & XX^* \arrow[r, "\mathrm{ev}_X"'] & 1 \arrow[d, equal] \\
& & XY(XY)^* \arrow[rr, "\mathrm{ev}_{XY}"] & & 1
\end{tikzcd}$$

Figure 1: Diagram for the proof of Lemma 2.10

References

[Ati88] M. Atiyah, *Topological quantum field theories*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 68, 175–186.

[Bar14] B. Bartlett, *Quasistrict symmetric monoidal 2-categories via wire diagrams*, ArXiv e-prints (2014). arXiv:1409.2148

[BD95] J. C. Baez and J. Dolan, *Higher-dimensional algebra and topological quantum field theory*, J. Math. Phys. **36** (1995), no. 11, 6073–6105.

[BGM17] E. Bernaschini, C. Galindo, and M. Mombelli, *Group actions on 2-categories*, ArXiv e-prints (2017). arXiv:1702.02627

[BN96] J. Baez and M. Neuchl, *Higher dimensional algebra: I. braided monoidal 2-categories*, Adv. Math. **121** (1996), no. 2, 196 – 244.

[Buh15] L. Buhné, *Topics In Three-Dimensional Descent Theory*, available at `http://ediss.sub.uni-hamburg.de/volltexte/2015/7346/pdf/Dissertation.pdf`, 2015, Ph.D. thesis, Universität Hamburg.

[CG07] E. Cheng and N. Gurski, *The periodic table of n-categories for low dimensions. I. Degenerate categories and degenerate bicategories*, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 143–164. arXiv:0708.1178

[Dav11] O. Davidovich, *State sums in two dimensional fully extended topological field theories*, Ph.D. thesis, University Texas at Austin (2011).

[DS97] B. Day and R. Street, *Monoidal bicategories and Hopf algebroids*, Adv. Math. **129** (1997), no. 1, 99 – 157.

[DSS13] C. L. Douglas, C. Schommer-Pries, and N. Snyder, *Dualizable Tensor Categories*, ArXiv e-prints (2013). arXiv:1312.7188

[EGNO15] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*, Mathematical Surveys and Monographs, vol. 205, American Mathematical Society, Providence, RI, 2015. MR 3242743

[FH16] D. S. Freed and M. J. Hopkins, *Reflection positivity and invertible topological phases*, ArXiv e-prints (2016). arXiv:1604.06527

[FHLT10] D. S. Freed, M. J. Hopkins, J. Lurie, and C. Teleman, *Topological quantum field theories from compact Lie groups*, A celebration of the mathematical legacy of Raoul Bott, CRM Proc. Lecture Notes, vol. 50, Amer. Math. Soc., Providence, RI, 2010, pp. 367–403. arXiv:0905.0731

[Fre12] D. S. Freed, *The cobordism hypothesis*, ArXiv e-prints (2012). arXiv:1210.5100

[Fre14a] D. S. Freed, *Anomalies and Invertible Field Theories*, Proc. Symp. Pure Math. **88** (2014), 25–46. arXiv:1404.7224

[Fre14b] ______, *Short-range entanglement and invertible field theories*. arXiv:1406.7278

[Hes16] J. Hesse, *An equivalence between Frobenius algebras and Calabi-Yau categories*, ArXiv e-prints (2016). arXiv:1609.06475

[Hes17] J. Hesse, *Group Actions on Bicategories and Topological Quantum Field Theories*, available at `http://ediss.sub.uni-hamburg.de/volltexte/2017/8655/pdf/Dissertation.pdf`, 2017, Ph.D. thesis, Universität Hamburg.

[HSV17] J. Hesse, C. Schweigert, and A. Valentino, *Frobenius algebras and homotopy fixed points of group actions on bicategories*, Theory Appl. Categ. **32** (2017), no. 18, 652–681. arXiv:1607.05148

[KV94] M. M. Kapranov and V. A. Voevodsky, *2-categories and Zamolodchikov tetrahedra equations*, Proc. Sympos. Pure Math., vol. 56, p. 177–259, Amer. Math. Soc., Providence, RI, 1994.

[Lur09] J. Lurie, *On the classification of topological field theories*, Current developments in mathematics, Int. Press, Somerville, MA, 2009, pp. 129–280.

[McC00] P. McCrudden, *Balanced coalgebroids*, Theory Appl. Categ. **7** (2000), no. 6, 71 – 147.

[Pst14] P. Pstrągowski, *On dualizable objects in monoidal bicategories, framed surfaces and the Cobordism Hypothesis*, ArXiv e-prints (2014). arXiv:1411.6691

[Sch13] C. Schommer-Pries, *Dualizability in Low-Dimensional Higher Category Theory*, ArXiv e-prints (2013). arXiv:1308.3574

[Seg04] G. Segal, *The definition of conformal field theory*, Topology, geometry and quantum field theory, London Math. Soc. Lecture Note Ser., vol. 308, Cambridge Univ. Press, Cambridge, 2004, pp. 421–577.

[SP09] C. Schommer-Pries, *The classification of two-dimensional extended topological field theories*, ProQuest LLC, Ann Arbor, MI, 2009, Thesis (Ph.D.)–University of California, Berkeley. arXiv:1112.1000

[Wit89] E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), no. 3, 351–399.

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