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CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE
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TOPOLOGY FROM ENRICHMENT: THE CURIOUS CASE OF PARTIAL METRICS

Dirk HOFMANN and Isar STUBBE

Résumé. Pour tout petit quantaloïde $\mathcal{Q}$, il y a un nouveau quantaloïde $\mathcal{D}(\mathcal{Q})$ de diagonaux dans $\mathcal{Q}$. Si $\mathcal{Q}$ est divisible alors il en est de même pour $\mathcal{D}(\mathcal{Q})$ (et vice versa), et il est alors particulièrement intéressant de comparer des catégories enrichies dans $\mathcal{Q}$ avec des catégories enrichies dans $\mathcal{D}(\mathcal{Q})$. Prenant le quantale des nombres réels positifs de Lawvere comme base, les $\mathcal{Q}$-categories sont les espaces métriques généralisés, alors que les $\mathcal{D}(\mathcal{Q})$-catégories sont les espaces métriques partiels généralisés, i.e. des espaces métriques dans lesquels la distance d'un point à lui-même ne doit pas être zéro et avec une inégalité triangulaire adaptée. Nous montrons comment toute catégorie enrichie dans un petit quantaloïde possède une fermeture canonique sur l'ensemble de ses objets: ceci constitue un foncteur des catégories enrichies dans un quantaloïde vers les espaces à fermeture. Sous de (faibles) conditions nécessaires-et-suffisantes sur le quantaloïde de base, ce foncteur prend ses valeurs dans la catégorie des espaces topologiques; et un quantaloïde involutif est Cauchy-bilatère (une propriété découverte auparavant dans le contexte des lois distributives) si et seulement si la fermeture sur toute catégorie enrichie est identique à la fermeture sur sa symétrisation. Puisque tout cela s'applique maintenant aussi bien aux espaces métriques qu'aux espaces métriques partiels, nous démontrons comment ces constructions catégoriques générales produisent les "bonnes" définitions de suite de Cauchy et de suite convergente dans les espaces métriques partiels. Finalement nous décrivons la Cauchy-complétion, la construction de Hausdorff et l'exponentiabilité d'un espace métrique partiel, une fois de plus en appliquant la théorie générale des catégories enrichies dans un quantaloïde.

Abstract. For any small quantaloid $\mathcal{Q}$, there is a new quantaloid $\mathcal{D}(\mathcal{Q})$ of diagonals in $\mathcal{Q}$. If $\mathcal{Q}$ is divisible then so is $\mathcal{D}(\mathcal{Q})$ (and vice versa), and then it is particularly interesting to compare categories enriched in $\mathcal{Q}$ with categories enriched in $\mathcal{D}(\mathcal{Q})$. Taking Lawvere's quantale of extended positive real numbers as base quantale, $\mathcal{Q}$-categories are generalised metric spaces, and $\mathcal{D}(\mathcal{Q})$-categories are generalised partial metric spaces, i.e. metric spaces in which self-distance need not be zero and with a suitably modified triangular inequality. We show how every small quantaloid-enriched category has a canonical closure operator on its set of objects: this makes for a functor from quantaloid-enriched categories to closure spaces. Under mild necessary-and-sufficient conditions on the base quantaloid, this functor lands in the category of topological spaces; and an involutive quantaloid is Cauchy-bilateral (a property discovered earlier in the context of distributive laws) if and only if the closure on any enriched category is identical to the closure on its symmetrisation. As this now applies to metric spaces and partial metric spaces alike, we demonstrate how these general categorical constructions produce the "correct" definitions of convergence and Cauchyness of sequences in generalised partial metric spaces. Finally we describe the Cauchy-completion, the Hausdorff contruction and exponentiability of a partial metric space, again by application of general quantaloid-enriched category theory.

Keywords. Quantaloid, divisibility, enriched category, topology, partial metric.
2010 Mathematics Subject Classification: 06A15, 06F07, 18D20, 54E35

1. Introduction

Following Fréchet [6], a metric space (X, d) is a set X together with a real-valued function d on $X \times X$ such that the following axioms hold:

[M0] $d(x, y) \geq 0$,
[M1] $d(x, y) + d(y, z) \geq d(x, z)$,
[M2] $d(x, x) = 0$,
[M3] if $d(x, y) = 0 = d(y, x)$ then $x = y$,
[M4] $d(x, y) = d(y, x)$,
[M5] $d(x, y) \neq +\infty$.

The categorical content of this definition, as first observed by Lawvere [17], is that the extended real interval $[0, \infty]$ underlies a quantale $([0, \infty], \bigwedge, +, 0)$, so that a "generalised metric space" (i.e. a structure as above, minus the axioms M3-M4-M5) is exactly a category enriched in that quantale.

More recently, see e.g. [18], the notion of a partial metric space (X, p) has been proposed to mean a set X together with a real-valued function p on $X \times X$ satisfying the following axioms:

[P0] $p(x, y) \geq 0$,
[P1] $p(x, y) + p(y, z) - p(y, y) \geq p(x, z)$,
[P2] $p(x, y) \geq p(x, x)$,
[P3] if $p(x, y) = p(x, x) = p(y, y) = p(y, x)$ then $x = y$,
[P4] $p(x, y) = p(y, x)$,
[P5] $p(x, y) \neq +\infty$.

The categorical content of *this* definition was discovered in two steps: first, Höhle and Kubiak [14] showed that there is a particular quantaloid of positive real numbers, such that categories enriched in that quantaloid correspond to ("generalised") partial metric spaces; and second, we realised in [22] that Höhle and Kubiak's quantaloid of real numbers is actually a universal construction on Lawvere's quantale of real numbers: namely, the quantaloid $\mathcal{D}[0,\infty]$ of diagonals in $[0,\infty]$.

It was shown in [13] that to any category enriched in a symmetric quantale one can associate a closure operator on its collection of objects. For a metric space (X,d), viewed as an $[0,\infty]$-enriched category, that "categorical closure" on X coincides precisely with the metric (topological) closure defined by d. And Lawvere [17] famously reformulated the Cauchy completeness of a metric space in terms of adjoint distributors. It is however not that complicated to extend the construction of the "categorical closure" to general quantaloid-enriched categories, thus making it applicable to partial metric spaces viewed as $\mathcal{D}[0,\infty]$-enriched categories. And then it is only natural to see if and how Lawvere's arguments for metric spaces go through in the case of partial metrics. This is what we set out to do in this paper—whence its title.

Here is a brief overview of the contents of this paper. Section 2 contains a compact presentation of well-known quantaloid-enriched category theory [20] that we shall need further on. In Section 3 we first explain the construction of the quantaloid $\mathcal{D}(\mathcal{Q})$ of *diagonals* in a given quantaloid, to then recall (and somewhat improve) the closely related notion of *divisible quantaloid* as it first appeared in [22]. For the sake of exposition, we shall say that a $\mathcal{D}(\mathcal{Q})$-enriched category is a *partial* $\mathcal{Q}$*-enriched category*, and at the end of Section 3 we explain how (generalised) *partial* metric spaces are, indeed, precisely the *partial* $[0,\infty]$-enriched categories. We start Section 4 by explaining how every quantaloid-enriched category determines a *categorical closure* on its set of objects (generalising results from [13]); we furthermore characterise those quantaloids for which the closure on any enriched category $\mathbb{C}$ is topological, and those involutive quantaloids for which the closure on any enriched category $\mathbb{C}$ is always identical to the closure on the symmetrised enriched category $\mathbb{C}_{\mathsf{s}}$. Viewing partial metric spaces as enriched categories, we identify in Section 5 the categorical topology induced by a (finitely typed) partial metric—and prove that it is always metrisable by means of a symmetric metric. We spell out what it means for a sequence to converge, resp. to be Cauchy, in such a partial metric space (X,p), and then show that all such Cauchy sequences converge in (X,p) if and only if all Cauchy distributors on (X,p) qua enriched category are representable. We end with some examples concerning Hausdorff distance in, and exponentiability of, partial metric spaces.

Of course, the study of partial metrics is not new. For example, in the survey paper [5] partial metrics are studied by analogy with metrics, and the reader will find there e.g. the definition of Cauchy sequence in a (symmetric) partial metric space (where $p(x,y)\neq\infty$). Let us also mention that [19] already adopts an enriched category point of view, and shows how those Cauchy sequences correspond with Cauchy distributors. However, none of the previously published papers have our purely categorical setup: we contruct a *topology* for any quantaloid-enrichment, so that – when applied to the quantaloid of diagonals in $[0, \infty]$ – the generic topological notions of convergence and Cauchyness of sequences reproduce those that were considered in a rather *ad hoc* manner before. So whereas our paper does not present many new results in partial metric spaces *per se*, it does propose a whole new categorical method to study partial metrics. That this method is benificial, can be seen in our treatment of a hitherto undiscovered subtlety involving the points with self-distance ∞, and in our results on Hausdorff distance and exponentiability!

2. Preliminaries, exemplified by metric spaces and ordered sets

A large part of the general theory of quantales and quantaloids is an instance of $\mathcal{V}$-*enriched category theory* [16], taking the base category $\mathcal{V}$ to be the category Sup of complete lattices and supremum-preserving functions. Indeed, a *quantaloid* $\mathcal{Q}$ is a Sup-enriched category (and quantales are exactly quantaloids with a single object, i.e. monoids in Sup), a *homomorphism* $H\colon \mathcal{Q} \to \mathcal{R}$ between quantaloids is a Sup-enriched functor, and so forth.

On the other hand, quantaloids are also (very particular) *bicategories* [1], so the general notions from bicategory theory apply as well. This point of view is important when defining lax morphisms between quantaloids, or adjunctions in a quantaloid, or quantaloid-enriched categories, because these concepts are not "naturally" catered for by Sup-enriched category theory alone.

In his seminal paper [17], Lawvere shows how both metric spaces and ordered sets are a guiding example of enriched categories—quantale-enriched, that is. In this section we shall reproduce some of his insightful examples, but we do explain the (slightly) more general case of quantaloid-enrichment, for in the next section it will be crucial to have this ready for the case of partial metric spaces (as will become clear there). For clarity's sake, and to fix our notations, we shall spell out some of these abstract categorical definitions in more elementary terms.

2.1 Quantaloids, quantales

A *quantaloid* $\mathcal{Q}$ is a category in which, for any two fixed objects A and B, the set $\mathcal{Q}(A, B)$ of morphisms from A to B is ordered and admits all suprema, in such a way that composition distributes on both sides over arbitrary suprema: whenever $f\colon A \to B$, $(g_i\colon B \to C)_{i \in I}$ and $h\colon C \to D$, then $h \circ (\bigvee_i g_i) \circ g = \bigvee_i (h \circ g_i \circ f)$. We write $1_A\colon A \to A$ for the identity morphism on an object A.

A crucial property of quantaloids is their so-called *closedness*. Precisely, for any morphism $f\colon A \to B$ in a quantaloid $\mathcal{Q}$ and any object X of $\mathcal{Q}$, both

$$-\circ f\colon \mathcal{Q}(B,X) \to \mathcal{Q}(A,X) \text{ and } f\circ -\colon \mathcal{Q}(X,A) \to \mathcal{Q}(X,B)$$

(*pre-* and *post-composition with* f) are supremum-preserving functions between complete lattices. Therefore these maps have right adjoints, called *lifting* and *extension through* f, and we shall write these as:

$$-\swarrow f\colon \mathcal{Q}(A,X) \to \mathcal{Q}(B,X) \text{ and } f\searrow -\colon \mathcal{Q}(X,B) \to \mathcal{Q}(X,A).$$

A *quantale* Q is, by definition, a one-object quantaloid. Equivalently, a quantale $Q = (Q, \bigvee, \circ, 1)$ is a sup-lattice $(Q, \bigvee)$ equipped with a monoidal structure $(\circ, 1)$ in such a way that multiplication distributes on both sides over suprema. Liftings and extensions in a quantale are often called *(left/right) residuations*, especially in the context of multi-valued logics.

The above says in particular that a quantaloid is a *locally complete and cocomplete closed bicategory* (and a quantale is a complete and cocomplete closed monoidal category). Importantly, we can therefore use all bicategorical notions in any quantaloid: adjoint pairs, monads, 2-dimensional universal properties, etc.

Example 2.1.1 Any locale (= complete Heyting algebra, cHa) H is a quantale $H = (H, \wedge, \top)$. In fact, cHa's are precisely those quantales which are integral (meaning that $1 = \top$) and idempotent (meaning that $f^2 = f$ for every $f \in Q$); they are of course also commutative. In particular shall we write $\mathbf{2} = (\mathbf{2}, \wedge, 1)$ for the 2-element Boolean algebra $\{0 < 1\}$ viewed as quantale.

Example 2.1.2 Writing $[0,\infty]^{\mathsf{op}}$ for the set of positive real numbers extended with $+\infty$, with the *opposite* of the natural order, $R = ([0,\infty]^{\mathsf{op}}, +, 0)$ is a (commutative and integral, but not idempotent) quantale; throughout this article we shall refer to it as *Lawvere's quantale of positive real numbers*, to honor its first appearence in [17].

In the remainder of this section, these quantales will be our main examples. It is however necessary to develop the general quantaloidal case for reasons that will become clear in the next section. To give but one example of a non-commutative, non-integral, non-idempotent quantale, consider the set $\mathsf{Sup}(L,L)$ of supremum-preserving functions on a complete lattice L. In fact, the category Sup of complete lattices and supremum-preserving morphisms itself is the example *par excellence* of a (large) quantaloid. Many more examples can be found in the references.

2.2 Quantaloid-enriched categories, functors, distributors

In all that follows, we fix a small quantaloid $\mathcal{Q}$; we shall write $\mathcal{Q}_0$ for its set of objects and $\mathcal{Q}_1$ for its set of morphisms.

A $\mathcal{Q}$-*enriched category* $\mathbb{C}$ (or $\mathcal{Q}$-*category* $\mathbb{C}$ for short) consists of

`(obj)` a set $\mathbb{C}_0$ of "objects",
`(typ)` a unary "type" predicate $t\colon \mathbb{C}_0 \to \mathcal{Q}_0 \colon x \mapsto tx$,
`(hom)` a binary "hom" predicate $\mathbb{C}\colon \mathbb{C}_0 \times \mathbb{C}_0 \to \mathcal{Q}_1 \colon (x, y) \mapsto \mathbb{C}(x, y)$,

such that the following conditions hold:

`[C0]` $\mathbb{C}(x, y)\colon ty \to tx$,
`[C1]` $\mathbb{C}(x, y) \circ \mathbb{C}(y, z) \leq \mathbb{C}(x, z)$,
`[C2]` $1_{tx} \leq \mathbb{C}(x, x)$.

Note how, when applied to a quantale Q (viewed as a one-object quantaloid $\mathcal{Q}$), the above definition symplifies: the "type" predicate becomes obsolete and condition `[C0]` trivialises. This is the case in our main examples (for now):

Example 2.2.1 Let $\mathbf{2} = (\mathbf{2}, \wedge, \top)$ be the 2-element Boolean algebra. A $\mathbf{2}$-category $\mathbb{A}$ is exactly an ordered set: for we may interpret that $\mathbb{A}(x, y) \in \{0, 1\}$ is 1 if and only if $x \leq y$. (So in this paper an *order* is a transitive and reflexive relation; if we want it to be anti-symmetric, then we shall explicitly mention so.)

Example 2.2.2 Considering the Lawvere quantale $R = ([0, \infty]^{\mathsf{op}}, +, 0])$, an R-category $\mathbb{X}$ consists of a set $X = \mathbb{X}_0$ together with a function $d = \mathbb{X}(-, -)\colon X \times X \to [0, \infty]$ such that $d(x, y) + d(y, z) \geq d(x, z)$ and $0 = d(x, x)$; all other data and conditions are trivially satisfied. Such an (X, d) is a *generalised metric space* [17]; adding symmetry ($d(x, y) = d(y, x)$), separatedness (if $d(x, y) = 0 = d(y, x)$ then $x = y$) and finiteness ($d(x, y) < \infty$) makes it a metric space in the sense of Fréchet [6].

A $\mathcal{Q}$-*functor* $F\colon \mathbb{C} \to \mathbb{D}$ between two $\mathcal{Q}$-categories is

`(map)` an "object map" $F\colon \mathbb{C}_0 \to \mathbb{D}_0 \colon x \mapsto Fx$

satisfying, for all $x, x' \in \mathbb{C}_0$,

`(F0)` $t(Fx) = tx$,
`(F1)` $\mathbb{C}(x', x) \leq \mathbb{D}(Fx', Fx)$.

Such $\mathcal{Q}$-functors $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{B} \to \mathbb{C}$ can be composed in the obvious way to produce a new functor $G \circ F\colon \mathbb{A} \to \mathbb{C}$, and the identity object map provides for the identity functor $1_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}$. Thus $\mathcal{Q}$-categories and $\mathcal{Q}$-functors are the objects and morphisms of a (large) category $\mathsf{Cat}(\mathcal{Q})$.

Example 2.2.3 There is no difficulty in proving that $\mathsf{Cat}(\mathbf{2})$ is exactly Ord, the category of ordered sets and order-preserving functions.

Example 2.2.4 Upon identifying two R-categories $\mathbb{X}$ and $\mathbb{Y}$ with two generalised metric spaces (X, d_X) and (Y, d_Y), it is straightforward to verify that an R-functor $F\colon \mathbb{X} \to \mathbb{Y}$ can be identified with a so-called 1-Lipschitz function $f\colon X \to Y$, i.e. $d_X(x', x) \geq d_Y(fx', fx)$. We shall write GMet for the category $\mathsf{Cat}(R)$.

To make $\mathcal{Q}$-enriched category theory really interesting, we need to introduce a second kind of morphism between $\mathcal{Q}$-categories: a $\mathcal{Q}$-*distributor* (also called 'module', 'bimodule' or 'profunctor' in the literature) $\Phi\colon \mathbb{C} \mathrel{-\!\circ\!\!\to} \mathbb{D}$ between two $\mathcal{Q}$-categories is

`(matr)` a "matrix" $\Phi\colon \mathbb{D}_0 \times \mathbb{C}_0 \to \mathcal{Q}_1\colon (y,x) \mapsto \Phi(y,x)$

satisfying, for all $x, x' \in \mathbb{C}_0$ and $y, y' \in \mathbb{D}_0$,

`(D0)` $\Phi(y,x)\colon tx \to ty$,

`(D1)` $\mathbb{D}(y',y) \circ \Phi(y,x) \le \Phi(y',x)$,

`(D2)` $\Phi(y,x) \circ \mathbb{C}(x,x') \le \Phi(y,x')$.

For two consecutive distributors $\Phi\colon \mathbb{C} \mathrel{-\!\circ\!\!\to} \mathbb{D}$ and $\Psi\colon \mathbb{D} \mathrel{-\!\circ\!\!\to} \mathbb{E}$, the composite distributor is written as $\Psi \otimes \Phi\colon \mathbb{C} \mathrel{-\!\circ\!\!\to} \mathbb{E}$ and computed as, for $x \in \mathbb{C}$ and $z \in \mathbb{E}$,

$$(\Psi \otimes \Phi)(z,x) = \bigvee_{y \in \mathbb{D}_0} \Psi(z,y) \circ \Phi(y,x).$$

The identity distributor $\mathsf{id}_{\mathbb{C}}\colon \mathbb{C} \mathrel{-\!\circ\!\!\to} \mathbb{C}$ has elements $\mathsf{id}_{\mathbb{C}}(x',x) = \mathbb{C}(x',x)$ for all $x, x' \in \mathbb{C}_0$. Two parallel distributors $\Phi, \Phi'\colon \mathbb{C} \mathrel{-\!\circ\!\!\to} \mathbb{D}$ are ordered 'elementwise':

$$\Phi \le \Phi' \overset{\text{def}}{\iff} \Phi(y,x) \le \Phi'(y,x) \text{ for all } (x,y) \in \mathbb{C}_0 \times \mathbb{D}_0,$$

and therefore the supremum of a family of parallel distributors, say $\Phi_i\colon \mathbb{C} \mathrel{-\!\circ\!\!\to} \mathbb{D}$, has elements

$$(\bigvee_i \Phi_i)(y,x) = \bigvee_i \Phi_i(y,x).$$

In this manner, distributors are the morphisms of a (large) quantaloid $\mathsf{Dist}(\mathcal{Q})$.

The importance of $\mathsf{Dist}(\mathcal{Q})$ being a *quantaloid* – instead of a mere *category* – cannot be overestimated: for it implies that $\mathsf{Dist}(\mathcal{Q})$ is closed, that we can speak of adjoint pairs of distributors, that we can perform 2-categorical constructions involving $\mathcal{Q}$-categories and distributors, and so on. For instance, it is not difficult to verify that we can compute liftings and extensions in $\mathsf{Dist}(\mathcal{Q})$ by the following formulas:

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, dotted, "\Psi \searrow \Phi", "\circ" marking] \arrow[dr, "\Phi"', "\circ" marking] & & \mathbb{D} \arrow[dl, "\Psi", "\circ" marking] \\
& \mathbb{E} &
\end{tikzcd} \qquad (\Psi \searrow \Phi)(y,x) = \bigwedge_{z \in \mathbb{E}_0} \Psi(z,y) \searrow \Phi(z,x) \qquad (1)$$

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, dotted, "\Psi \swarrow \Phi", "\circ" marking] & & \mathbb{D} \\
& \mathbb{E} \arrow[ul, "\Phi", "\circ" marking] \arrow[ur, "\Psi"', "\circ" marking] &
\end{tikzcd} \qquad (\Psi \swarrow \Phi)(y,x) = \bigwedge_{z \in \mathbb{E}_0} \Psi(y,z) \swarrow \Phi(x,z) \qquad (2)$$

In contrast, there is *a priori* no extra structure in $\mathsf{Cat}(\mathcal{Q})$—but luckily $\mathsf{Cat}(\mathcal{Q})$ embeds naturally in $\mathsf{Dist}(\mathcal{Q})$, and therefore inherits some of the latter's structure.

Indeed, every functor $F\colon \mathbb{A} \to \mathbb{B}$ determines an adjoint pair of distributors

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, bend left, "F_*", "\circ"{marking}] \arrow[rr, phantom, "\perp"] & & \mathbb{B} \arrow[ll, bend left, "F^*", "\circ"{marking}]
\end{tikzcd}$$

defined by $F_*(b, a) = \mathbb{B}(b, Fa)$ and $F^*(a, b) = \mathbb{B}(Fa, b)$. We shall say that the left adjoint F_* is the *graph* of the functor F, whereas the right adjoint F^* is its *cograph*. Taking graphs and cographs extends to a pair of functors, one covariant and the other contravariant:

$$\mathsf{Cat}(\mathcal{Q}) \to \mathsf{Dist}(\mathcal{Q})\colon \Big(F\colon \mathbb{A} \to \mathbb{B}\Big) \mapsto \Big(F_*\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{B}\Big), \tag{3}$$

$$\mathsf{Cat}(\mathcal{Q})^{\mathsf{op}} \to \mathsf{Dist}(\mathcal{Q})\colon \Big(F\colon \mathbb{A} \to \mathbb{B}\Big) \mapsto \Big(F^*\colon \mathbb{B} \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{A}\Big). \tag{4}$$

With this, we make $\mathsf{Cat}(\mathcal{Q})$ a *locally ordered category* by defining, for any parallel pair of functors $F, G\colon \mathbb{A} \to \mathbb{B}$,

$$F \leq G \overset{\text{def}}{\iff} F_* \leq G_* \iff F^* \geq G^*.$$

Whenever $F \leq G$ and $G \leq F$, we write $F \cong G$ and say that these functors are isomorphic.

With all this, we can now naturally speak of adjoint $\mathcal{Q}$-functors, fully faithful $\mathcal{Q}$-functors, equivalent $\mathcal{Q}$-categories, (co)monads on $\mathcal{Q}$-categories, etc.

2.3 Presheaves and completions

If X is an object of $\mathcal{Q}$, then we write $\mathbb{1}_X$ for the $\mathcal{Q}$-category defined by $(\mathbb{1}_X)_0 = \{*\}$, $t* = X$ and $\mathbb{1}_X(*, *) = 1_X$. Similarly, if $f\colon X \to Y$ is a morphism in $\mathcal{Q}$, then we write $(f)\colon \mathbb{1}_X \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{1}_Y$ for the distributor defined by $(f)(*, *) = f$. In doing so we get an injective homomorphism

$$i\colon \mathcal{Q} \to \mathsf{Dist}(\mathcal{Q})\colon \Big(f\colon X \to Y\Big) \mapsto \Big((f)\colon \mathbb{1}_X \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{1}_Y\Big) \tag{5}$$

which allows us to (tacitly) identify $\mathcal{Q}$ with its image in $\mathsf{Dist}(\mathcal{Q})$.

A *contravariant $\mathcal{Q}$-presheaf* ϕ of type $X \in \mathcal{Q}_0$ on a $\mathcal{Q}$-category $\mathbb{C}$ is, by definition, a distributor $\phi\colon \mathbb{1}_X \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{C}$. For two presheaves $\phi\colon \mathbb{1}_X \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{C}$ and $\psi\colon \mathbb{1}_Y \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{C}$, the lifting $(\psi \searrow \phi)\colon \mathbb{1}_X \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{1}_Y$ in the quantaloid $\mathsf{Dist}(\mathcal{Q})$ is a distributor with a single element, which can therefore be identified with an arrow from X to Y in $\mathcal{Q}$:

$$\begin{tikzcd}
& \mathbb{1}_Y \arrow[dr, "\psi", "\circ"{marking}] & \\
\mathbb{1}_X \arrow[ur, dotted, "\psi \searrow \phi", "\circ"{marking}] \arrow[rr, "\phi"', "\circ"{marking}] & & \mathbb{C}
\end{tikzcd}$$

We can thus define the *$\mathcal{Q}$-category $\mathcal{P}\mathbb{C}$ of contravariant presheaves* on $\mathbb{C}$ to have as objects the contravariant presheaves on $\mathbb{C}$ (of all possible types); the type of a presheaf $\phi\colon \mathbb{1}_X \looparrowright \mathbb{C}$ is X; and the hom $\mathcal{P}\mathbb{C}(\psi,\phi)$ is the (single element of the) lifting $\psi \searrow \phi$.

For $\Phi\colon \mathbb{C} \looparrowright \mathbb{D}$ it is easy to see that $\mathcal{P}\mathbb{C} \to \mathcal{P}\mathbb{D}\colon \psi \mapsto \Phi \otimes \psi$ is a $\mathcal{Q}$-functor. This action easily extends to form a 2-functor $\mathcal{P}\colon \mathsf{Dist}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$, and by composition with the inclusion 2-functor $\mathsf{Cat}(\mathcal{Q}) \to \mathsf{Dist}(\mathcal{Q})$ of (3) we find a functor $\mathcal{P}\colon \mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$. The latter turns out to be a KZ-doctrine (i.e. a 2-monad such that "algebras are adjoint to units"); its category of algebras is denoted $\mathsf{Cocont}(\mathcal{Q})$: its objects are so-called cocomplete $\mathcal{Q}$-categories, and its morphisms are the cocontinuous $\mathcal{Q}$-functors. The unit of the KZ-doctrine consists of the so-called (fully faithful) *Yoneda embeddings*

$$Y_{\mathbb{C}}\colon \mathbb{C} \to \mathcal{P}\mathbb{C}\colon x \mapsto \mathbb{C}(-,x).$$

The presheaves in the image of $Y_{\mathbb{C}}$ are said to be *representable* (by objects of $\mathbb{C}$); the Yoneda embedding $Y_{\mathbb{C}}$ exhibits $\mathcal{P}\mathbb{C}$ to be the *free cocompletion* of $\mathbb{C}$. And the *Yoneda Lemma* says that, for any $\phi \in \mathcal{P}\mathbb{C}$ and any $x \in \mathbb{C}$, we have $\mathcal{P}\mathbb{C}(Y_{\mathbb{C}}x,\phi) = \phi(x)$.

Dually, a covariant $\mathcal{Q}$-presheaf κ of type $X \in \mathcal{Q}_0$ on a $\mathcal{Q}$-category $\mathbb{C}$ is a distributor like $\kappa\colon \mathbb{C} \looparrowright \mathbb{1}_X$; they are the objects of a $\mathcal{Q}$-category $\mathcal{P}^{\dagger}\mathbb{C}$, in which $\mathcal{P}^{\dagger}\mathbb{C}(\lambda,\kappa) = \lambda \swarrow \kappa$. The obvious 2-functor $\mathcal{P}^{\dagger}\colon \mathsf{Dist}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})^{\mathsf{op}}$ composes with the inclusion 2-functor in (4) to form a co-KZ-doctrine $\mathcal{P}^{\dagger}\colon \mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$, whose category of algebras $\mathsf{Cont}(\mathcal{Q})$ consists of complete $\mathcal{Q}$-categories and continuous $\mathcal{Q}$-functors. The (fully faithful) Yoneda embeddings (also: free completions)

$$Y^{\dagger}_{\mathbb{C}}\colon \mathbb{C} \to \mathcal{P}^{\dagger}\mathbb{C}\colon x \mapsto \mathbb{C}(x,-),$$

form the unit of the co-KZ-doctrine.

Example 2.3.1 For an ordered set $(X,\leq)$, viewed as a $\mathbf{2}$-category, the free cocompletion $\mathcal{P}(X,\leq)$ – in the sense of $\mathbf{2}$-category theory – is precisely the free sup-lattice on $(X,\leq)$: the set of downclosed subsets ordered by inclusion; and the Yoneda embedding $Y_{(X,\leq)}\colon (X,\leq) \to \mathcal{P}(X,\leq)$ sends an element $x \in X$ to the principal downclosed set $\downarrow x$. Hence, upon identifying Ord with $\mathsf{Cat}(\mathbf{2})$, the KZ-doctrine $\mathcal{P}\colon \mathsf{Ord} \to \mathsf{Ord}$ is the free sup-lattice monad. In an entirely dual fashion, $\mathcal{P}^{\dagger}(X,\leq)$ is the free inf-lattice: its elements are the upclosed subsets of X, ordered by containment (the *opposite* of inclusion); and the co-KZ-doctrine $\mathcal{P}^{\dagger}\colon \mathsf{Ord} \to \mathsf{Ord}$ is the free inf-lattice monad.

A *Cauchy presheaf* on a $\mathcal{Q}$-category $\mathbb{C}$ is a (contravariant) presheaf $\phi\colon \mathbb{1}_X \looparrowright \mathbb{C}$ which – as morphism in $\mathsf{Dist}(\mathcal{Q})$ – has a right adjoint, which we shall then write as $\phi^*\colon \mathbb{C} \looparrowright \mathbb{1}_X$. The $\mathcal{Q}$-category $\mathbb{C}_{\mathsf{cc}}$ is, by definition, the full subcategory of $\mathcal{P}\mathbb{C}$

whose objects are the Cauchy presheaves. Furthermore, the Yoneda embedding $Y_{\mathbb{C}}\colon \mathbb{C} \to \mathcal{P}\mathbb{C}$ co-restricts to a functor $I_{\mathbb{C}}\colon \mathbb{C} \to \mathbb{C}_{\mathsf{cc}}$, which is now called the *Cauchy completion of* $\mathbb{C}$. Those Cauchy completions form the unit of a KZ-doctrine $(-)_{\mathsf{cc}}\colon \mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$; the category of algebras $\mathsf{Cat}(\mathcal{Q})_{\mathsf{cc}}$ contains the Cauchy complete $\mathcal{Q}$-categories and (all) $\mathcal{Q}$-functors between them. In fact, a $\mathcal{Q}$-category $\mathbb{C}$ is Cauchy complete if and only if, for every left adjoint distributor $\Phi\colon \mathbb{X} \to\!\!\!\!\!\circ\;\mathbb{C}$, there exists a (necessarily essentially unique) functor $F\colon \mathbb{X} \to \mathbb{C}$ such that $F_* = \Phi$, if and only if for every Cauchy presheaf $\phi\colon \mathbb{1}_X \to\!\!\!\!\!\circ\;\mathbb{C}$, there exists a (necessarily essentially unique) object $c \in \mathbb{C}_0$ such that $\mathbb{C}(-, c) = \phi$.

Example 2.3.2 The designations "Cauchy completion" and "Cauchy complete" are motivated by the interpretation of these concepts in generalised metric spaces [17], as follows. Every Cauchy sequence $(x_n)_n$ in a metric space (X, d) – suitably viewed as an R-category $\mathbb{X}$, cf. Example 2.2.2 – defines an adjoint pair $\phi \dashv \phi^*$ of R-distributors $\phi\colon \mathbb{1} \to\!\!\!\!\!\circ\;\mathbb{X}$ and $\phi^*\colon \mathbb{X} \to\!\!\!\!\!\circ\;\mathbb{1}$ by putting

$$\phi(y) = \lim_{n\to\infty} d(y, x_n) \quad \text{and} \quad \phi^*(y) = \lim_{n\to\infty} d(x_n, y)$$

for all $y \in X$. Moreover, $(x_n)_n$ converges to $x \in X$ precisely when $\phi = d(-, x)$. Conversely, a left adjoint R-distributor $\phi : \mathbb{1} \to\!\!\!\!\!\circ\; X$, with right adjoint $\phi^* : X \to\!\!\!\!\!\circ\;\mathbb{1}$, satisfies

$$\bigwedge_{x\in X} \phi^*(x) + \phi(x) = 0,$$

so that a Cauchy sequence $(x_n)_n$ can be built by choosing x_n with $\phi(x_n)+\phi^*(x_n) \leq \frac{1}{n}$. After identifying equivalent Cauchy sequences, these two processes turn out to be inverse to each other; and therefore *a generalised metric space is Cauchy complete in the traditional sense if and only if it is Cauchy complete in the sense of enriched category theory.*

Such (Cauchy) (co)complete $\mathcal{Q}$-categories can be studied and characterised in many different ways, and have a wealth of applications; we refer to [2, 7, 24] for examples.

2.4 Involution and symmetry

In this subsection we shall suppose that $\mathcal{Q}$ is a quantaloid equipped with an *involution*: a function $\mathcal{Q}_1 \to \mathcal{Q}_1 : f \mapsto f^\circ$ on the morphisms of $\mathcal{Q}$ such that $f \leq g$ implies $f^\circ \leq g^\circ$, $(g \circ f)^\circ = f^\circ \circ g^\circ$, and $f^{\circ\circ} = f$. It is easy to check that this automatically extends to a (necessarily invertible) homomorphism $(-)^\circ\colon \mathcal{Q}^{\mathsf{op}} \to \mathcal{Q}$ which is the identity on objects and satisfies $f^{\circ\circ} = f$ for any morphism f in $\mathcal{Q}$. The pair $(\mathcal{Q}, (-)^\circ)$ is said to form an *involutive quantaloid*, but we leave the notation for the involution understood when no confusion can arise.

A $\mathcal{Q}$-category $\mathbb{A}$ is *symmetric* if we have, for all $x, y \in \mathbb{A}_0$,

`[C4]` $\mathbb{A}(x,y) = \mathbb{A}(y,x)^\circ$.

We shall write $\mathsf{SymCat}(\mathcal{Q})$ for the full sub-2-category of $\mathsf{Cat}(\mathcal{Q})$ determined by the symmetric $\mathcal{Q}$-categories. The full embedding $\mathsf{SymCat}(\mathcal{Q}) \hookrightarrow \mathsf{Cat}(\mathcal{Q})$ has a right adjoint functor[1],

$$\begin{tikzcd}\mathsf{SymCat}(\mathcal{Q}) \arrow[r, bend left, "\text{incl.}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}(\mathcal{Q}), \arrow[l, bend left, "(-)_\mathsf{s}"]\end{tikzcd}$$

which sends a $\mathcal{Q}$-category $\mathbb{C}$ to the symmetric $\mathcal{Q}$-category $\mathbb{C}_\mathsf{s}$ whose objects (and types) are those of $\mathbb{C}$, but for any two objects x, y the hom-arrow is $\mathbb{C}_\mathsf{s}(y,x) := \mathbb{C}(y,x) \wedge \mathbb{C}(x,y)^\circ$. A functor $F \colon \mathbb{C} \to \mathbb{D}$ is sent to $F_\mathsf{s} \colon \mathbb{C}_\mathsf{s} \to \mathbb{D}_\mathsf{s} \colon x \mapsto Fx$. The counit of this adjunction has components $S_\mathbb{C} \colon \mathbb{C}_\mathsf{s} \to \mathbb{C} \colon x \mapsto x$.

Example 2.4.1 A commutative quantale is the same thing as a quantale for which the identity map is an involution; in particular can we thus consider $\mathbf{2} = \{0,1\}$ and $R = [0,\infty]^\mathsf{op}$ to be involutive. For both ordered sets and generalised metric spaces it is straightforward to interpret the symmetry axiom: an order $(X, \leq)$ is symmetric qua $\mathbf{2}$-enriched category if and only if the order-relation $\leq$ is symmetric (and so it is an *equivalence relation* on X); and a generalised metric space (X,d) is symmetric qua R-enriched category if and only if the distance function d is symmetric (and so (X,d) is an *écart* in the sense of [3]).

Composing right and left adjoint, we find a comonad $(-)_\mathsf{s} \colon \mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$ whose coalgebras are exactly the symmetric $\mathcal{Q}$-categories. In the previous subsection we had the important monad $(-)_\mathsf{cc} \colon \mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$ whose algebras are exactly the Cauchy complete $\mathcal{Q}$-categories. Because both arise from (co)reflexive subcategories, there can be at most one distributive law of the Cauchy monad over the symmetrisation comonad; here is a sufficient condition for its existence:

Proposition 2.4.2 ([10]) *If an involutive quantaloid is* Cauchy-bilateral*, that is to say, for each family* $(f_i \colon X \to X_i, g_i \colon X_i \to X)_{i \in I}$ *of morphisms in* $\mathcal{Q}$,

$$\left.\begin{array}{r}\forall j,k \in I: \ f_k \circ g_j \circ f_j \leq f_k \\ \forall j,k \in I: \ g_j \circ f_j \circ g_k \leq g_k \\ 1_X \leq \bigvee_{i \in I} g_i \circ f_i\end{array}\right\} \Longrightarrow 1_X \leq \bigvee_{i \in I} (g_i \wedge f_i^\circ) \circ (g_i^\circ \wedge f_i),$$

then there is a distributive law L of $(-)_\mathsf{cc} \colon \mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$ *over* $(-)_\mathsf{s} \colon \mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$ *with components* $L_\mathbb{C} \colon (\mathbb{C}_\mathsf{s})_\mathsf{cc} \to (\mathbb{C}_\mathsf{cc})_\mathsf{s} \colon \phi \mapsto (S_\mathbb{C})_* \otimes \phi$. *The category of*

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[1] But the right adjoint is not a 2-functor, so this is not a 2-adjunction!

L-bialgebras contains precisely those $\mathcal{Q}$-categories which are both symmetric and Cauchy complete, and all $\mathcal{Q}$-functors between these.

This means that, for such a Cauchy-bilateral $\mathcal{Q}$, the Cauchy completion of a symmetric $\mathcal{Q}$-category is again symmetric, and that the symmetrisation of a Cauchy complete $\mathcal{Q}$-category is again Cauchy complete; so the category of L-bialgebras can be computed, either by Cauchy-completing all symmetric $\mathcal{Q}$-categories, or by symmetrising all Cauchy complete $\mathcal{Q}$-categories. For more details we refer to [10].

Example 2.4.3 Every locale H is Cauchy-bilateral (for the identity involution); in particular so is $\mathbf{2}$. But in the $\mathbf{2}$-enriched case, every $\mathbf{2}$-category is Cauchy complete!

Example 2.4.4 The Lawvere quantale $R = ([0,\infty]^{\mathsf{op}}, +, 0)$ is Cauchy-bilateral (again, for the identity involution), so the Cauchy completion of a symmetric generalised metric space is again a symmetric generalised metric space. (Perhaps this motivated Fréchet [6] to include the symmetry axiom in his definition of 'metric space'?)

The above example generalises, as follows:

Example 2.4.5 Any linearly ordered, integral, commutative quantale Q is Cauchy-bilateral (for the identity involution). Indeed, the condition to be Cauchy-bilateral reduces to

$$1 \leq \bigvee_i g_i \circ f_i \implies 1 \leq \bigvee_i (g_i \wedge f_i)^2$$

for any family $(f_i, g_i)_{i \in I}$ of pairs of elements of Q. But integrality of Q assures that $g_i \circ f_i \leq g_i \wedge f_i$, so the hypothesis implies that

$$1 \leq \bigvee_i g_i \circ f_i \leq \bigvee_i g_i \wedge f_i,$$

and therefore – taking squares on both ends of this inequality – also

$$1 \leq (\bigvee_i g_i \wedge f_i)^2 = \bigvee_{i,j} (g_i \wedge f_i)(g_j \wedge f_j).$$

Now it is linearity of Q which makes $(g_i \wedge f_i)(g_j \wedge f_j) \leq (g_i \wedge f_i)^2 \vee (g_j \wedge f_j)^2$, so that from the previous line we easily find the desired result. A *left-continuous t-norm* [8] is exactly an integral commutative quantale structure on the (linearly ordered) real unit interval; so here we find in particular all these to be Cauchy-bilateral.

2.5 Homomorphisms, lax functors, change of base

A *homomorphism* $H\colon \mathcal{Q} \to \mathcal{R}$ between quantaloids (and in particular quantales) is a functor, mapping $f\colon A \to B$ in $\mathcal{Q}$ to $Hf\colon HA \to HB$ in $\mathcal{R}$ and preserving composition and identities in the usual manner, which furthermore preserves local suprema: whenever $(f_i\colon A \to B)_{i\in I}$ in $\mathcal{Q}$, then $H(\bigvee_i f_i) = \bigvee_i Hf_i$. Homomorphisms $H\colon \mathcal{Q} \to \mathcal{R}$ and $K\colon \mathcal{R} \to \mathcal{S}$ compose to produce a new homomorphism $K \circ H\colon \mathcal{Q} \to \mathcal{S}$, and on each quantaloid $\mathcal{Q}$ there is an identity homomorphism $1_{\mathcal{Q}}$; so (small) quantaloids and homomorphisms themselves are the objects and morphisms of a (large) category Quant.

A *lax functor*[2] $F\colon \mathcal{Q} \to \mathcal{R}$ maps $f\colon A \to B$ in $\mathcal{Q}$ to $Ff\colon FA \to FB$ in $\mathcal{R}$ in such a way that

- if $f \le f'$ in $\mathcal{Q}(A,B)$ then $Ff \le Ff'$ in $\mathcal{R}(FA,FB)$,
- if $f\colon A \to B$ and $g\colon B \to C$ in $\mathcal{Q}$, then $Fg \circ Ff \le F(g \circ f)$ in $\mathcal{R}$,
- for any A in $\mathcal{Q}$, $1_{FA} \le F1_A$ in $\mathcal{R}$.

Lax functors compose in the obvious manner, so there is a (large) category $\mathsf{Quant}_{\mathsf{lax}}$ of (small) quantaloids and lax morphisms, containing Quant. If a lax functor preserves all identities, then it is said to be *normal*; the composite of such is again a normal lax functor, so these are the morphisms of a category containing Quant and contained in $\mathsf{Quant}_{\mathsf{lax}}$.

Now suppose that $F\colon \mathcal{Q} \to \mathcal{R}$ is a lax functor. If $\mathbb{C}$ is any $\mathcal{Q}$-category, then it is straightforward to define an $\mathcal{R}$-category $F\mathbb{C}$ with the same object set as $\mathbb{C}$ but with homs given by $F\mathbb{C}(y,x) = F(\mathbb{C}(y,x))$. This construction extends to distributors and functors, producing a 2-functor $\mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{R})$ and a lax morphism $\mathsf{Dist}(\mathcal{Q}) \to \mathsf{Dist}(\mathcal{R})$, both referred to as *change of base* functors.

For any quantaloid $\mathcal{Q}$ we can define the lax morphism $\mathcal{Q} \to \mathbf{2}$ which (obviously) sends every object of $\mathcal{Q}$ to the single object of $\mathbf{2}$, every arrow bigger or equal to an identity in $\mathcal{Q}$ to the non-zero arrow in $\mathbf{2}$, and all other arrows to zero. The change of base $\mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathbf{2}) = \mathsf{Ord}$ thus associates to any $\mathcal{Q}$-category its *underlying ordered set* (and sends $\mathcal{Q}$-functors to monotone functions); precisely, it sends a $\mathcal{Q}$-category $\mathbb{C}$ to the order $(\mathbb{C}_0, \le)$ where

$$x \le y \text{ exactly when } tx = ty \text{ and } 1_{tx} \le \mathbb{C}(x,y).$$

The $\mathcal{Q}$-category $\mathbb{C}$ is said to be *skeletal* (or *separated*) when its underlying order is anti-symmetric. Even when $\mathbb{C}$ is not skeletal, $\mathcal{P}\mathbb{C}$ (and hence its full subcategory $\mathbb{C}_{\mathsf{cc}}$) is.

Example 2.5.1 Tautologically, an order $(X,\le)$ is skeletal qua $\mathbf{2}$-enriched category if and only if the order-relation $\le$ is anti-symmetric; in other words, $(X,\le)$ is a *partially ordered set* (but we will avoid that terminology, leaving the adjective ‘partial’ available for something quite different—see Subsection 3.3).

Example 2.5.2 Applied to the Lawvere quantale $R = ([0,\infty]^{\mathsf{op}}, +, 0)$, the change of base into $\mathbf{2}$ becomes the functor $\mathsf{GMet} \to \mathsf{Ord}$ which sends a generalised metric

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[2] What we really define here, is a *lax* Ord-*functor*, i.e. a lax functor between quantaloids qua Ord-enriched categories.

space (X, d) to the ordered set $(X, \leq)$ in which $x \leq y$ precisely when $d(x, y) = 0$. It follows that a generalised metric space (X, d) is skeletal qua R-enriched category if and only if the distance function is separating in the sense that $d(x, y) = 0 = d(y, x)$ implies $x = y$. A symmetric and skeletal (X, d) is thus the same thing as a metric $d\colon X \times X \to [0, \infty]$ (allowing ∞). Clearly, if the metric is symmetric or separated then its underlying order is so too.

Example 2.5.3 Up to now we have considered the ordered set $[0, \infty]^{\mathsf{op}}$ as a quantale for the addition; and we saw that $([0, \infty]^{\mathsf{op}}, +, 0)$-enriched categories are generalised metric spaces. But we can also consider the locale $([0, \infty]^{\mathsf{op}}, \vee, 0)$—so it is the same underlying order, but now with binary supremum as binary operation. It is straightforward to check that a $([0, \infty]^{\mathsf{op}}, \vee, 0)$-enriched category is exactly a *generalised ultrametric space* (X, d), i.e. a distance function $d\colon X \times X \to [0, \infty]$ satisfying $d(x, x) = 0$ and $d(x, y) \vee d(y, z) \geq d(x, z)$. (As for generalised metrics, also generalised ultrametrics can be symmetric or skeletal.) Because for any $a, b \in [0, \infty]^{\mathsf{op}}$ we obviously have $a + b \geq a \vee b$, the identity function is a lax morphism from $([0, \infty]^{\mathsf{op}}, \vee, 0)$ to $([0, \infty]^{\mathsf{op}}, +, 0)$; the induced change of base functor is simply the inclusion of generalised ultrametric spaces into generalised metric spaces.

In Example 3.2.6 and further we shall come back to this example.

3. Partial metric spaces as enriched categories

3.1 Diagonals

It often happens in practice that quantaloids arise from quantales by one or another universal construction. We shall describe one such case, which will turn out to be crucial to describe the categorical content of partial metric spaces.

First we recall a definition from [22]:

Definition 3.1.1 *Fixing two morphisms $f\colon A \to B$ and $g\colon C \to D$ in a quantaloid $\mathcal{Q}$, we say that a third morphism $d\colon A \to D$ in $\mathcal{Q}$ is a* diagonal from f to g *if any (and thus both) of the following equivalent conditions holds:*

i. there exist $x\colon A \to C$ and $y\colon B \to D$ in $\mathcal{Q}$ such that $y \circ f = d = g \circ x$,

ii. $g \circ (g \searrow d) = d = (d \swarrow f) \circ f$.

Proof of the equivalence. Obviously $(ii \Rightarrow i)$ is trivial. Conversely, $d = g \circ x \leq g \circ (g \searrow d) \leq d$ holds because $g \circ x = d$ implies $x \leq g \searrow d$; similarly $d = (d \swarrow f) \circ f$ follows from $y \circ f = d$. □

The reason for the term “diagonal” is clear from a picture to accompany the first condition in the above definition: given the solid morphisms in the diagram

$$\begin{tikzcd} A \arrow[r, dotted, "x"] \arrow[d, "f"'] \arrow[dr, "d"] & C \arrow[d, "g"] \\ B \arrow[r, dotted, "y"'] & D \end{tikzcd}$$

in $\mathcal{Q}$, one seeks to add the dotted morphisms, to form a commutative diagram. The equivalent second condition then adds that, whenever such x and y exist, then there is a *canonical choice* for them, namely $x = g \searrow d$ and $y = d \swarrow f$.

Proposition 3.1.2 ([22]) *For any (small) quantaloid $\mathcal{Q}$, a new (small) quantaloid $\mathcal{D}(\mathcal{Q})$ of* diagonals in $\mathcal{Q}$ *is built as follows:*

- *the objects of $\mathcal{D}(\mathcal{Q})$ are the morphisms of $\mathcal{Q}$,*
- *a morphism from f to g in $\mathcal{D}(\mathcal{Q})$ is a diagonal from f to g in $\mathcal{Q}$,*
- *the composition of two diagonals $d\colon f \to g$ and $e\colon g \to h$ is defined to be*

$$e \circ_g d := (e \swarrow g) \circ g \circ (g \searrow d),$$

- *the identity on f is $f\colon f \to f$ itself,*
- *and the supremum of a set of diagonals $(d_i\colon f \to g)_{i \in I}$ is computed "as in $\mathcal{Q}$".*

Remark 3.1.3 Regarding composition of diagonals, it is useful to point out that the formula given above for $e \circ_g d$ is really just one of many equivalent expressions for the composite arrow from the upper left corner to the lower right corner in the following commutative diagram:

$$\begin{tikzcd} \cdot \arrow[r, dotted, "x"] \arrow[d, "f"'] \arrow[dr, "d"] & \cdot \arrow[r, dotted, "u"] \arrow[d, "g"] \arrow[dr, "e"] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, dotted, "y"'] & \cdot \arrow[r, dotted, "v"'] & \cdot \end{tikzcd}$$

Particularly, in doing so, one can *choose any x, y, u, v that make the diagram commute*, not just the canonical $x = g \searrow d$, $y = d \swarrow f$, $u = h \searrow e$, $v = e \swarrow g$.

There is an obvious full and faithful inclusion *homomorphism*[3] of $\mathcal{Q}$ in $\mathcal{D}(\mathcal{Q})$:

$$I\colon \mathcal{Q} \to \mathcal{D}(\mathcal{Q})\colon \Big(f\colon A \to B\Big) \mapsto \Big(f\colon 1_A \to 1_B\Big). \tag{6}$$

---

[3] Better still, this embedding enjoys a powerful universal property: it is the *splitting-of-everything* in $\mathcal{Q}$; and consequently it is the unit of a 2-monad on the category Quant of small quantaloids. This has been described by Grandis [9] for small categories.

It is thus a natural problem to study how properties of a given quantaloid $\mathcal{Q}$ extend (or not) to the larger quantaloid $\mathcal{D}(\mathcal{Q})$. For later use we record a simple example:

Example 3.1.4 Say that a quantaloid $\mathcal{Q}$ is *symmetric* whenever the identity function $\mathcal{Q}_1 \to \mathcal{Q}_1$ is an involution on $\mathcal{Q}$; explicitly, this means that $\mathcal{Q}(A,B) = \mathcal{Q}(B,A)$ and $f \circ g = g \circ f$ for all objects A, B and all morphisms f, g of $\mathcal{Q}$. It is then a simple fact that $\mathcal{Q}$ is symmetric if and only if $\mathcal{D}(\mathcal{Q})$ is. Note that a 'symmetric quantaloid with a single object' is precisely a commutative quantale. So as a particular case we find here that, for any commutative quantale Q, the quantaloid $\mathcal{D}(Q)$ is symmetric.

In the next subsection we shall study a particular class of quantales and quantaloids – the so-called *divisible* ones – whose diagonals behave particularly well; thereafter we shall be interested in categories enriched in $\mathcal{D}(Q)$ whenever Q is a divisible quantale.

3.2 Divisible quantaloids

In [22] we first introduced our notion of *divisible quantaloid*, but that first definition contained some redundancies—so here we spell it out again, in a more optimal form.

Definition 3.2.1 *A quantaloid $\mathcal{Q}$ is* divisible[4] *if it satisfies any (and thus all) of the following equivalent conditions:*

i. for all $d, e \colon A \to B$ in $\mathcal{Q}$: $d \le e$ if and only if there exist $x \colon A \to A$ and $y \colon B \to B$ such that $e \circ x = d = y \circ e$,

ii. for all $d, e \colon A \to B$ in $\mathcal{Q}$: $d \le e$ if and only if $e \circ (e \searrow d) = d = (d \swarrow e) \circ e$,

iii. for all $d, e \colon A \to B$ in $\mathcal{Q}$: $e \circ (e \searrow d) = d \wedge e = (d \swarrow e) \circ e$.

iv. for all $e \colon A \to B$ in $\mathcal{Q}$: $\mathcal{D}(\mathcal{Q})(e,e) = \downarrow e$ (as sublattices of $\mathcal{Q}(A,B)$),

v. for all $d, e \colon A \to B$ in $\mathcal{Q}$: $\mathcal{D}(\mathcal{Q})(d,e) = \downarrow(d \wedge e)$ (as sublattices of $\mathcal{Q}(A,B)$).

Proof of the equivalences. The implications $(i \Leftarrow ii \Leftarrow iii)$ hold trivially, as do $(ii \Leftrightarrow iv \Leftarrow v)$; and to see that $(i \Rightarrow ii)$ one merely needs to adapt slightly the argument given for the equivalence of the conditions in Definition 3.1.1.

Now assume (ii); putting $e = 1_A$ in the condition and using that $\mathcal{Q}(A,A) = \mathcal{D}(\mathcal{Q})(1_A, 1_A)$, shows that $\mathcal{Q}$ is necessarily integral (meaning that, for every object A, the identity morphism 1_A is the top element of $\mathcal{Q}(A,A)$). From $d \wedge e \le e$ we get $d \wedge e = e \circ (e \searrow (d \wedge e))$; but $e \searrow (d \wedge e) = (e \searrow d) \wedge (e \searrow e)$ (because right adjoints preserve infima) and $e \searrow e = 1_A$ (by integrality of $\mathcal{Q}$) so $(e \searrow d) \wedge (e \searrow e) = (e \searrow d)$; altogether, we find $d \wedge e = e \circ (e \searrow d)$. A similar computation proves that $d \wedge e = (d \swarrow e) \circ e$, so in all we proved $(ii \Rightarrow iii)$.

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[4] Note how the notion of *divisibility* is self-dual: $\mathcal{Q}$ is divisible if and only if $\mathcal{Q}^{\mathsf{op}}$ is. Put differently, formally it makes sense to define $\mathcal{Q}$ to be *semi-divisible* if, for all $d, e \colon A \to B$ in $\mathcal{Q}$, $e \circ (e \searrow d) = d \wedge e$; and then $\mathcal{Q}$ is divisible if and only if both $\mathcal{Q}$ and $\mathcal{Q}^{\mathsf{op}}$ are semi-divisible.

To complete the proof we shall show that $(ii \Rightarrow v)$. First, we use again that $\mathcal{Q}$ is necessarily integral to find, for any $f \in \mathcal{D}(\mathcal{Q})(d,e)$, that $f = (f \swarrow d) \circ d \leq 1_B \circ d = d$; and similar for $f \leq e$. Conversely, if $f \leq d \wedge e$ then $e \circ (e \searrow f) = f$ and $(f \swarrow d) \circ d = f$ both follow directly from the assumption, and show that $f \in \mathcal{D}(\mathcal{Q})(d,e)$. □

To stress that the definition in [22] agrees with Definition 3.2.1 above, we record an observation made in the proof above[5]:

Proposition 3.2.2 *A divisible quantaloid $\mathcal{Q}$ is always integral.*

We also observe:

Proposition 3.2.3 *A divisible quantaloid $\mathcal{Q}$ is always locally localic*[6].

Proof. For $(f_i)_{i\in I}, g$ be in $\mathcal{Q}(A,B)$ we certainly have $\bigvee_i (f_i \wedge g) \leq (\bigvee_i f_i) \wedge g$. Assuming $\mathcal{Q}$ to be divisible, we can furthermore compute that

$$\begin{aligned}(\bigvee_i f_i) \wedge g = (\bigvee_i f_i) \circ \Big((\bigvee_j f_j) \searrow g\Big) &= \bigvee_i \Big(f_i \circ \Big(\bigwedge_j (f_j \searrow g)\Big)\Big) \\ &\leq \bigvee_i \Big(f_i \circ (f_i \searrow g)\Big) = \bigvee_i (f_i \wedge g)\end{aligned}$$

which leads to the conclusion: $\bigvee_i (f_i \wedge g) = (\bigvee_i f_i) \wedge g$. □

The very definition of divisibility already shows a link with the diagonal construction; the next proposition adds to that:

Proposition 3.2.4 *A quantaloid $\mathcal{Q}$ is divisible if and only if $\mathcal{D}(\mathcal{Q})$ is divisible.*

Proof. As we may regard $\mathcal{Q}$ as a full subquantaloid of $\mathcal{D}(\mathcal{Q})$, if the latter is divisible then so must be the former.

Now suppose that $\mathcal{Q}$ is divisible. By Condition (iv) in Definition 3.2.1 we find that $\mathcal{D}(\mathcal{Q})$ is integral, so one implication in Condition (i) is trivial for $\mathcal{D}(\mathcal{Q})$. For the other implication, consider two diagonals

$$\begin{tikzcd}
X_0 \arrow[d, "f"'] \arrow[dr, shift left, "d"] \arrow[dr, shift right, "d'"'] & Y_0 \arrow[d, "g"] \\
X_1 & Y_1
\end{tikzcd}$$

---

[5] This Proposition 3.2.3, and also Proposition 3.2.4, help to show that there are very many *non-divisible* quantaloids. For instance, if $(M, \circ, 1)$ is any monoid, then the free quantale $(\mathcal{P}(M), \circ, \{1\})$ is divisible if and only if $M = \{1\}$. The quantaloid $\mathsf{Dist}(\mathcal{Q})$ is not divisible in general, even for divisible $\mathcal{Q}$—because it is not integral. The quantale of sup-endomorphisms on a sup-lattice is not divisible in general. And so forth.

[6] This means that each lattice $\mathcal{Q}(A,B)$ of morphisms in $\mathcal{Q}$ with the same domain and the same codomain is a locale. This does *not* mean that composition in $\mathcal{Q}$ preserves finite infima in each variable (e.g. the empty infimum is hardly ever preserved).

such that $d \leq d'$ in $\mathcal{D}(\mathcal{Q})(f,g)$. Because we necessarily have $d \leq d'$ in $\mathcal{Q}(X_0, Y_1)$ too, it follows from the assumptions on $\mathcal{Q}$ that there exist $x\colon X_0 \to X_0$ and $y\colon Y_1 \to Y_1$ such that $y \circ d' = d = d' \circ x$ in $\mathcal{Q}$. Furthermore, $f \circ x \leq f$ in $\mathcal{Q}(X_0, X_1)$ (because $\mathcal{Q}$ is integral), so that there exists an $x'\colon X_1 \to X_1$ such that $\xi := f \circ x = x' \circ f$; and for similar reasons there is an $y'\colon Y_0 \to Y_0$ such that $\eta := y \circ g = g \circ y'$. Displaying all these morphisms in the diagram

$$\begin{tikzcd}
X_0 \arrow[r, dotted, "x"] \arrow[d, "f"'] \arrow[dr, "\xi"] & X_0 \arrow[d, "f"'] \arrow[drr, "d"] \arrow[drr, "d'"'] & Y_0 \arrow[r, dotted, "y'"] \arrow[d, "g"] \arrow[dr, "\eta"] & Y_0 \arrow[d, "g"] \\
X_1 \arrow[r, dotted, "x'"'] & X_1 & Y_1 \arrow[r, dotted, "y"'] & Y_1
\end{tikzcd}$$

shows that ξ and η are diagonals too, and from Remark 3.1.3 it is clear that $d' \circ_f \xi = d = \eta \circ_g d'$. So we showed that d' divides d in $\mathcal{D}(\mathcal{Q})$, as required. □

An important class of examples is provided by:

Example 3.2.5 Any locale H is (commutative and) divisible: for any $a, b \in H$ we have $a \wedge (a \Rightarrow b) \leq b$ by the universal property of the "implication", and $a \wedge (a \Rightarrow b) \leq a$ holds trivially, so we already find $a \wedge (a \Rightarrow b) \leq a \wedge b$; and conversely, $b \leq (a \Rightarrow b)$ holds by its equivalence to the trivial $a \wedge b \leq b$, and therefore also $a \wedge b \leq a \wedge (a \Rightarrow b)$ holds. Via the argument in Example 3.1.4 and the above Proposition 3.2.4 it follows that also the quantaloid $\mathcal{D}(H)$ is symmetric and divisible[7]. Both H and $\mathcal{D}(H)$ are Cauchy-bilateral (for the identity involution), see [10, Example 4.5]. In particular is this all true for $H = ([0,\infty]^{\mathsf{op}}, \vee, 0)$.

We hasten to point out our *other* main example:

Example 3.2.6 The Lawvere quantale $R = ([0,\infty]^{\mathsf{op}}, +, 0)$ is (commutative and) divisible. The "implication" in this quantale is given by $a \rightsquigarrow b = 0 \vee (b - a)$ (truncated substraction), and so it is easily seen that $a + (a \rightsquigarrow b) = a \vee b$, as required. It thus follows that its quantaloid of diagonals $\mathcal{D}(R)$ is symmetric and divisible. In [10, Example 4.4] it is shown that R is Cauchy-bilateral; we shall now prove the stronger fact that also $\mathcal{D}(R)$ is Cauchy-bilateral.

Because the Lawvere quantale is divisible, we know by Condition (iv) in Definition 3.2.1 that its quantaloid of diagonals is integral. Therefore, as explained in [10, Definition 4.2], the latter is Cauchy-bilateral if and only if the following holds[8]:

---

[7] Note that – because every element of H is idempotent – the quantaloid $\mathcal{D}(H)$ is exactly the universal *splitting-of-idempotents* in H; that is to say, any homomorphism $H \to \mathcal{R}$ into a quantaloid in which all idempotents split, extends essentially uniquely to a homomorphism $\mathcal{D}(H) \to \mathcal{R}$.

[8] This expresses exactly that, whenever $f_i\colon a \to b_i$ and $g_i\colon b_i \to a$ are diagonals so that the supremum of the composites $g_i \circ_{b_i} f_i$ is bigger than the identity diagonal on a, then so is the supremum

for any index-set I and elements $a, b_i, f_i, g_i \in [0,\infty]$,

$$\text{if } f_i \wedge g_i \geq a \vee b_i \text{ and } a \geq \bigwedge_i (g_i + f_i - b_i) \text{ then } a \geq \bigwedge_i (2(f_i \vee g_i) - b_i).$$

Because R is linearly ordered and the formulas are symmetric in f_i's and g_i's, we may suppose that $g_i \geq f_i \geq b_i$ for all $i \in I$. Under this harmless extra assumption we can compute that

$$\bigwedge_i g_i \geq \bigwedge_i f_i \geq a \geq \bigwedge_i (g_i + f_i - b_i) \geq \bigwedge_i g_i$$

and furthermore

$$\bigwedge_i (2(f_i \vee g_i) - b_i) = \bigwedge_i (2g_i - b_i) \geq \bigwedge_i g_i \geq \bigwedge_i f_i.$$

It is thus sufficient to prove that: for any index-set I and elements $b_i, f_i, g_i \in [0,\infty]$,

$$\text{if } g_i \geq f_i \geq b_i \text{ and } \bigwedge_i f_i \geq \bigwedge_i (g_i + f_i - b_i) \text{ then } \bigwedge_i f_i \geq \bigwedge_i (2g_i - b_i).$$

But from $\bigwedge_i (g_i + f_i - b_i) \leq \bigwedge_i f_i$ we know that for any $\varepsilon > 0$ there exists $k \in I$ such that for any $j \in I$: $g_k + f_k - b_k < f_j + \varepsilon$; and upon putting $j = k$ it follows that $g_k - b_k < \varepsilon$. Secondly, since $\bigwedge_i g_i \leq \bigwedge_i f_i$ we find for any $\varepsilon > 0$ some $i \in I$ such that for any $j \in I$, $g_i < f_j + \varepsilon$. Summing up, for any $\varepsilon > 0$ there is an $i \in I$ such that for any $j \in I$, $2g_i - b_i < f_j + 2\varepsilon$; and this means exactly that $\bigwedge_i (2g_i - b_i) \leq \bigwedge_i f_i$, as wanted.

The close relationship between the two previous examples, $([0,\infty]^{\mathsf{op}}, \vee, 0)$ and $([0,\infty]^{\mathsf{op}}, +, 0)$, can be traced back to divisibility, as follows.

Let $Q = (Q, \circ, 1)$ be any divisible quantale, and write $Q_H = (Q, \wedge, 1)$ for the underlying locale; because Q is integral it follows that the identity function is a lax morphism from Q_H to Q. We must distinguish between the quantaloid $\mathcal{D}(Q)$ of diagonals in Q and the quantaloid $\mathcal{D}(Q_H)$ of diagonals in Q_H. However, both these divisible quantaloids have the same objects, and – as spelled out above – for fixed $f, g \in Q$ we also find that

$$\mathcal{D}(Q)(f,g) = {\downarrow}(f \wedge g) = \mathcal{D}(Q_H)(f,g).$$

Furthermore, the identity on an object $f \in Q$, in both $\mathcal{D}(Q)$ and $\mathcal{D}(Q_H)$, is the greatest element of $\mathcal{D}(Q) = \mathcal{D}(Q_H)$, *viz.* f itself. So the only (but crucial) difference between both these quantaloids, is the composition law:

---

of the composites of $(g_i \vee f_i^\circ) \circ_{b_i} (g_i^\circ \vee f_i)$; but the involution on $\mathcal{D}(Q)$ is the identity – stemming from Q's commutativity – and composition is computed as $g \circ_b f = g + f - b$.

- the composite of $d\colon f \to g$ and $e\colon g \to h$ in $\mathcal{D}(Q)$ is $e \circ_g d = (e \swarrow g) \circ g \circ (g \searrow d)$,
- the composite of $d\colon f \to g$ and $e\colon g \to h$ in $\mathcal{D}(Q_H)$ is $e \wedge d$.

However, these expressions compare: because $e \circ_g d = e \circ (g \searrow d) \leq e$, and similarly $e \circ_g d \leq d$, so $e \circ_g d \leq e \wedge d$. In other words, the identity function on objects and arrows defines a normal lax morphism from $\mathcal{D}(Q_H)$ to $\mathcal{D}(Q)$. Furthermore, the lax morphisms $Q_H \to Q$ and $\mathcal{D}(Q_H) \to \mathcal{D}(Q)$ commute with the full embeddings $Q \to \mathcal{D}(Q)$ and $Q_H \to \mathcal{D}(Q_H)$ to make the following square commute:

$$\begin{tikzcd}
Q_H \arrow[r] \arrow[d] & Q \arrow[d] \\
\mathcal{D}(Q_H) \arrow[r] & \mathcal{D}(Q)
\end{tikzcd} \tag{7}$$

When applying the above constructions to $R = ([0,\infty]^{\mathsf{op}}, +, 0)$, we already know that categories enriched in R are generalised metric spaces (Example 2.2.2); we also know that categories enriched in R_H are generalised ultrametric spaces and that the change of base induced by the lax morphism from R_H to R encodes precisely the inclusion of ultrametrics into metrics (Example 2.5.3). In the next section we study the two *other* bases of enrichement made available in Diagram (7).

3.3 Partial categories, partial metrics

When $\mathcal{Q}$ is a small quantaloid, then so is $\mathcal{D}(\mathcal{Q})$; hence the theory of enriched categories applies to $\mathcal{D}(\mathcal{Q})$ as much as it does to $\mathcal{Q}$. The full embedding $I\colon \mathcal{Q} \to \mathcal{D}(\mathcal{Q})$ induces a change of base $\mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{D}(\mathcal{Q}))$ which shows how $\mathcal{Q}$-categories fit into $\mathcal{D}(\mathcal{Q})$-categories. For the sake of exposition, we introduce the following terminology and notation:

Definition 3.3.1 *A* **partial $\mathcal{Q}$-enriched category (functor, distributor)** *is a* $\mathcal{D}(\mathcal{Q})$*-enriched category (functor, distributor). We write* $\mathsf{PCat}(\mathcal{Q}) := \mathsf{Cat}(\mathcal{D}(\mathcal{Q}))$ *and* $\mathsf{PDist}(\mathcal{Q}) := \mathsf{Dist}(\mathcal{D}(\mathcal{Q}))$.

Explicitly, a partial $\mathcal{Q}$-category $\mathbb{C}$ consists of

`(obj)` a set $\mathbb{C}_0$,
`(typ)` a function $t\colon \mathbb{C}_0 \to \mathcal{Q}_1$,
`(hom)` a function $\mathbb{C}\colon \mathbb{C}_0 \times \mathbb{C}_0 \to \mathcal{Q}_1$,

such that, in the quantaloid $\mathcal{Q}$, we have that

`(PC0)` $(\mathbb{C}(y,x) \swarrow tx) \circ tx = \mathbb{C}(y,x) = ty \circ (ty \searrow \mathbb{C}(y,x))$,
`(PC1)` $tx \leq \mathbb{C}(x,x)$,
`(PC2)`[9] $(\mathbb{C}(z,y) \swarrow ty) \circ ty \circ (ty \searrow \mathbb{C}(y,x)) \leq \mathbb{C}(z,x)$.

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[9] Or any of the equivalent expressions obtained by replacing the left hand side, thanks to `(PC0)`, with either $(\mathbb{C}(z,y) \swarrow ty) \circ \mathbb{C}(y,x)$ or $\mathbb{C}(z,y) \circ (ty \searrow \mathbb{C}(y,x))$.

Similarly one can express the notions of $\mathcal{D}(\mathcal{Q})$-enriched functor and distributor to avoid explicit references to the diagonal construction, and speak of 'partial $\mathcal{Q}$-functor' and 'partial $\mathcal{Q}$-distributor' between partial $\mathcal{Q}$-categories.

Upon identification of $\mathcal{Q}$ with its image in $\mathcal{D}(\mathcal{Q})$ along the full embedding $I\colon \mathcal{Q} \to \mathcal{D}(\mathcal{Q})$, it is clear that ("total") $\mathcal{Q}$-categories (and functors between them) are exactly the same thing as partial $\mathcal{Q}$-categories for which all object-types are identity morphisms (and partial functors between them). Indeed, the change of base $\mathsf{Cat}(\mathcal{Q}) \to \mathsf{PCat}(\mathcal{Q})$ induced by the full embedding $I\colon \mathcal{Q} \to \mathcal{D}(\mathcal{Q})$ is precisely the full inclusion of $\mathcal{Q}$-categories (and functors) into partial $\mathcal{Q}$-categories (and partial functors).

As a converse to the inclusion of $\mathcal{Q}$ into $\mathcal{D}(\mathcal{Q})$, we can observe that any diagonal $d\colon f \to g$ can be "projected" onto its "domain" and onto its "codomain":

Proposition 3.3.2 *For any quantaloid $\mathcal{Q}$, both*

$$J_0\Big(d\colon f \to g\Big) = \Big(g \searrow d\colon \mathsf{dom}(f) \to \mathsf{dom}(g)\Big)$$

$$J_1\Big(d\colon f \to g\Big) = \Big(d \swarrow f\colon \mathsf{cod}(f) \to \mathsf{cod}(g)\Big)$$

are lax morphisms. The induced change of base functors $J_0, J_1\colon \mathsf{PCat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$, send a partial $\mathcal{Q}$-category $\mathbb{C}$ to:

- *the $\mathcal{Q}$-category $J_0\mathbb{C}$ with object set $\mathbb{C}_0$, type function $\mathbb{C}_0 \to \mathcal{Q}_0\colon x \mapsto \mathsf{dom}(tx)$, and hom function $\mathbb{C}_0 \times \mathbb{C}_0 \to \mathcal{Q}_1\colon (y,x) \mapsto ty \searrow \mathbb{C}(y,x)$,*
- *the $\mathcal{Q}$-category $J_1\mathbb{C}$ with object set $\mathbb{C}_0$, type function $\mathbb{C}_0 \to \mathcal{Q}_0\colon x \mapsto \mathsf{cod}(tx)$, and hom function $\mathbb{C}_0 \times \mathbb{C}_0 \to \mathcal{Q}_1\colon (y,x) \mapsto \mathbb{C}(y,x) \swarrow tx$.*

Proof. For morphisms f, g in $\mathcal{Q}$, the map $\mathcal{D}(\mathcal{Q})(f,g) \to \mathcal{Q}(\mathsf{dom}(f), \mathsf{dom}(g))\colon d \mapsto g \searrow d$ preserves order. Given diagonals $d\colon f \to g$ and $e\colon g \to h$, we know that $h \circ (h \searrow e) \circ (g \searrow d) = e \circ_g d$, from which it follows by lifting through h that $(h \searrow e) \circ (g \searrow d) \leq h \searrow (e \circ_g d)$, or in other words, $J_0(e) \circ J_0(d) \leq J_0(e \circ_g d)$. Finally, for any morphism f in $\mathcal{Q}$ we have that $1_{\mathsf{dom}(f)} \leq (f \searrow f) = J_0(1_f)$. The proof for J_1 is entirely dual. $\square$

In a somewhat different context [23], these change of base functors have been called the *forward* and *backward globalisation* of a partial $\mathcal{Q}$-category. It can be remarked that, since $J_0\colon \mathcal{D}(\mathcal{Q}) \to \mathcal{Q}$ is a left inverse to $I\colon \mathcal{Q} \to \mathcal{D}(\mathcal{Q})$ (that is, $J_0 \circ I$ is the identity on $\mathcal{Q}$), the same is true for the induced functors $J_0\colon \mathsf{PCat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$ and $I\colon \mathsf{Cat}(\mathcal{Q}) \to \mathsf{PCat}(\mathcal{Q})$ (and similar for J_1).

Even though it could be an interesting topic to compare partial $\mathcal{Q}$-categories with "total" $\mathcal{Q}$-categories for a general base quantaloid $\mathcal{Q}$, we shall narrow our study down to a more specific situation: in the rest of this section we shall be concerned only with *commutative and divisible quantales*—in keeping with our main example, the Lawvere quantale $R = ([0,\infty]^{\mathsf{op}}, +, 0)$.

Let us first note that, whenever $Q = (Q, \circ, 1)$ is a commutative quantale, the function $Q \times Q \to Q\colon (f,g) \mapsto f \circ g$ is a homomorphism of quantales, so that composition with the lax morphism $(J_0, J_1)\colon \mathcal{D}(Q) \to Q \times Q$ (whose components J_0 and J_1 are those of Proposition 3.3.2) produces yet another lax morphism from $\mathcal{D}(\mathcal{Q})$ to $\mathcal{Q}$:

Proposition 3.3.3 *If Q is a commutative quantale*[10]*, then*

$$K\colon \mathcal{D}(Q) \to Q\colon (d\colon f \to g) \mapsto ((g \rightsquigarrow d) \circ (f \rightsquigarrow d))$$

is a lax morphism. The induced change of base $K\colon \mathsf{PCat}(Q) \to \mathsf{Cat}(Q)$ sends a partial Q-category $\mathbb{C}$ to the symmetric Q-category $K\mathbb{C}$ with object set $\mathbb{C}_0$ and hom function $\mathbb{C}_0 \times \mathbb{C}_0 \to Q\colon (y,x) \mapsto (ty \rightsquigarrow \mathbb{C}(y,x)) \circ (tx \rightsquigarrow \mathbb{C}(y,x))$.

Unlike J_0 and J_1, the lax morphism K is *not* a left (or right) inverse to $I\colon Q \to \mathcal{D}(Q)$.

Secondly, let us narrow down the definition of partial $\mathcal{Q}$-category [22]:

Proposition 3.3.4 *If $Q = (Q, \bigvee, \circ, 1)$ is a divisible quantale, then a partial Q-category $\mathbb{C}$ is determined by a set $\mathbb{C}_0$ together with a function $\mathbb{C}\colon \mathbb{C}_0 \times \mathbb{C}_0 \to Q\colon (y,x) \mapsto \mathbb{C}(y,x)$ satisfying*

$$\mathbb{C}(y,x) \le \mathbb{C}(x,x) \wedge \mathbb{C}(y,y) \text{ and } (\mathbb{C}(z,y) \swarrow \mathbb{C}(y,y)) \circ \mathbb{C}(y,x) \le \mathbb{C}(z,x).$$

A partial functor $F\colon \mathbb{C} \to \mathbb{D}$ between partial Q-categories is a function $F\colon \mathbb{C}_0 \to \mathbb{D}_0$ satisfying

$$\mathbb{C}(x,x) = \mathbb{D}(Fx,Fx) \text{ and } \mathbb{C}(y,x) \le \mathbb{D}(Fy,Fx).$$

And a partial distributor $\Phi\colon \mathbb{C} \looparrowright \mathbb{D}$ is a function $\Phi\colon \mathbb{D}_0 \times \mathbb{C}_0 \to Q$ satisfying

$$\Phi(y,x) \le \mathbb{C}(x,x) \wedge \mathbb{D}(y,y), \quad (\mathbb{D}(y',y) \swarrow \mathbb{D}(y,y)) \circ \Phi(y,x) \le \Phi(y',x)$$

$$\text{and} \quad \Phi(y,x) \circ (\mathbb{C}(x,x) \searrow \mathbb{C}(x,x')) \le \Phi(y,x').$$

Sketch of proof. Take the explicit description, below Definition 3.3.1, of a partial $\mathcal{Q}$-category $\mathbb{C}$, and weed out the redundancies due to the particularities of the divisible quantale Q (and the therefore also divisible quantaloid $\mathcal{D}(Q)$): because both the set of objects and the set of arrows of $\mathcal{D}(Q)$ are equal to Q, we find that both the type function and the hom function take values in Q; $\mathcal{D}(Q)$ is integral and $tx = 1_{tx}$ by construction, so the reflexivity of the hom function becomes $tx =$

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[10] The commutativity of the multiplication implies, by uniqueness of adjoints, that liftings and extensions are the same thing; so in this case we shall write $x \rightsquigarrow y$ instead of $x \searrow y = y \swarrow x$. We reserve the notation $x \Rightarrow y$ for the case where the multiplication is given by binary infimum, i.e. when the quantale considered is actually a locale.

$\mathbb{C}(x,x)$, making the type function implicit in the hom function and `[PC1]` obsolete; divisibility of Q makes `[PC0]` equivalent to $\mathbb{C}(x,y) \leq \mathbb{C}(x,x) \wedge \mathbb{C}(y,y)$; and formulating the composition in $\mathcal{D}(Q)$ back into terms proper to Q, `[PC2]` is exactly $(\mathbb{C}(z,y) \swarrow \mathbb{C}(y,y)) \circ \mathbb{C}(y,x) \leq \mathbb{C}(z,x)$. Similar simplifications apply to functors and distributors. □

Finally, we can fully develop – as we set out to do – the notion of 'partial metric space':

Example 3.3.5 For Lawvere's quantale $R = ([0,\infty]^{\mathsf{op}}, +, 0)$, and adopting common notations, a partial R-category $\mathbb{X}$ is precisely a set $X := \mathbb{X}_0$ together with a function $p := \mathbb{X} : X \times X \to [0,\infty]$ satisfying

$$p(y,x) \geq p(x,x) \vee p(y,y) \text{ and } p(z,y) - p(y,y) + p(y,x) \geq p(z,x).$$

In line with Example 2.2.2 we call such a structure (X,p) a *generalised partial metric space*—indeed, upon imposing *finiteness*, *symmetry* and *separatedness*, we recover exactly the partial metric spaces of [18], whose definition we recalled in the Introduction. A partial functor $f\colon (X,p) \to (Y,q)$ between such spaces is a non-expansive map $f\colon X \to Y : x \mapsto fx$ satisfying furthermore $p(x,x) = q(fx,fx)$; these objects and morphisms thus form the (locally ordered) category $\mathsf{PMet} := \mathsf{PCat}(R) = \mathsf{Cat}(\mathcal{D}(R))$.

Furthermore, the underlying locale $R_H = ([0,\infty]^{\mathsf{op}}, \vee, 0)$ of the Lawvere quantale is also a divisible quantale. A partial R_H-enriched category $\mathbb{X}$ is a set $X := \mathbb{X}_0$ together with a function $u := \mathbb{X}\colon X \times X \to [0,\infty]$ satisfying

$$u(y,x) \geq u(x,x) \vee u(y,y) \ \text{ and } \ u(z,y) \vee u(y,x) \geq u(z,x).$$

For all the obvious reasons we shall call such a (X,u) a *generalised partial ultrametric space*. These are the objects of a (locally ordered) category $\mathsf{GPUMet} := \mathsf{PCat}(R_H) = \mathsf{Cat}(\mathcal{D}(R_H))$.

The commutative Diagram (7) of lax morphisms induces a commutative diagram

$$\begin{tikzcd}
\mathsf{GUMet} \arrow[r] \arrow[d] & \mathsf{GMet} \arrow[d] \\
\mathsf{GPUMet} \arrow[r] & \mathsf{GPMet}
\end{tikzcd}$$

in which all arrows are full embeddings. When restricting to symmetric, finite and separating distance functions in all four categories in this square, one finds the appropriate categories of "non-generalised" (partial) (ultra)metric spaces.

On the other hand, as a corollary of Propositions 3.3.2 and 3.3.3 (which apply to R as well as R_H!), we have three ways to compute a "total" (generalised) (ultra)metric from a partial one: given (X,p) we find

- $p_0(y,x) := p(y,x) - p(y,y)$ via the lax morphism $J_0\colon \mathcal{D}(R) \to R$,
- $p_1(y,x) := p(y,x) - p(x,x)$ via the lax morphism $J_1\colon \mathcal{D}(R) \to R$,
- $p_K(y,x) := 2p(y,x) - p(x,x) - p(y,y)$ via the lax morphism $K\colon \mathcal{D}(R) \to R$.

These constructions will be useful in the next Section.

To end this Section, we insist on the fact that partial functors between (generalised) partial (ultra)metrics are non-expansive maps *that preserve self-distance*. At first sight this may seem too strong a requirement—would it not be more natural to allow (non-expansive) functions $f\colon (X,p) \to (Y,q)$ to *decrease* the self-distances too? But for our later purposes (namely, to canonically associate a topology to every quantaloid-enriched category, and therefore also to each partial metric space) the latter type of map is not suitable (it does not give rise to continuous maps). However, there is also a simple algebraic argument in favour of maps that do not decrease self-distances (apart from their origin as functors in the appropriate categorical setting, *viz.* as $\mathcal{D}(R)$-enriched functors). Consider the one-element partial metric space $\mathbb{1}_a$ whose single element has self-distance $a \in [0,\infty]$. General non-expansive maps $f\colon \mathbb{1}_a \to (X,p)$ are in 1-1 correspondence with elements of X whose self-distance is *at most* a; if we impose f to preserve self-distance, then it picks out an element of X whose self-distance is *exactly* a. The second situation is thus to be preferred, if one wants to be able to identify each element of (X,p) with *precisely one* map defined on a singleton partial metric.

4. Topology from enrichment

4.1 Density and closure

A functor $F\colon \mathbb{C} \to \mathbb{D}$ between $\mathcal{Q}$-categories is fully faithful when $\mathbb{C}(y,x) = \mathbb{D}(Fy,Fx)$ for every $x \in \mathbb{C}_0$ and $y \in \mathbb{D}$; equivalently, this says that the unit of the adjunction of distributors $F_* \dashv F^*$ is an equality (instead of a mere inequality). The complementary notion to full faithfulness will be of importance to us in this section:

Definition 4.1.1 *A functor $F\colon \mathbb{C} \to \mathbb{D}$ between $\mathcal{Q}$-categories is* **fully dense** *if the counit of the adjunction of distributors $F_* \dashv F^*$ is an equality (instead of a mere inequality); explicitly, we have for all $x, y \in \mathbb{D}_0$ that*

$$\mathbb{D}(y,x) = \bigvee_{c \in \mathbb{C}_0} \mathbb{D}(y,Fc) \circ \mathbb{D}(Fc,x).$$

It is clear that an *essentially surjective* $F\colon \mathbb{C} \to \mathbb{D}$ (meaning that for every $y \in \mathbb{D}$ there exists an $x \in \mathbb{C}$ such that $Fx \cong y$) is always fully dense; but the converse need not hold.

Proposition 4.1.2 *A functor $F\colon \mathbb{C} \to \mathbb{D}$ between $\mathcal{Q}$-categories is fully dense if and only if it is essentially epimorphic, i.e. for every $H, K\colon \mathbb{D} \to \mathbb{E}$, if $H \circ F \cong K \circ F$ then $H \cong K$.*

Proof. If F is fully dense and $H \circ F \cong K \circ F$, then – looking at the represented distributors – we can precompose both sides of $\mathbb{E}(-, H-) \otimes \mathbb{D}(-, F-) = \mathbb{E}(-, K-) \otimes \mathbb{D}(-, F-)$ with $\mathbb{D}(F-, -)$ to find $\mathbb{E}(-, H-) = \mathbb{E}(-, K-)$, which means precisely that $H \cong K$.

To see the converse, consider the Yoneda embedding $Y_{\mathbb{D}}\colon \mathbb{D} \to \mathcal{P}\mathbb{D}\colon d \mapsto \mathbb{D}(-, d)$ alongside the functor $Z\colon \mathbb{D} \to \mathcal{P}\mathbb{D}\colon d \mapsto \mathbb{D}(-, F-) \otimes \mathbb{D}(F-, d)$. Because $Y_{\mathbb{D}}(Fc) = Z(Fc)$ holds for all $c \in \mathbb{C}$, the assumed essential epimorphic F provides that $Y_{\mathbb{D}}d \cong Zd$ for all $d \in \mathbb{D}$—but since $\mathcal{P}\mathbb{D}$ is a skeletal $\mathcal{Q}$-category (isomorphic objects are necessarily equal), we actually have that $Y_{\mathbb{D}}d = Zd$ for all $d \in \mathbb{D}$. This says precisely that F is fully dense. □

Whenever $\mathbb{C}$ is a $\mathcal{Q}$-category, any $S \subseteq \mathbb{C}_0$ determines a full subcategory $\mathbb{S} \hookrightarrow \mathbb{C}$. In particular, two subsets $S \subseteq T \subseteq \mathbb{C}_0$ determine an inclusion of full subcategories $\mathbb{S} \hookrightarrow \mathbb{T} \hookrightarrow \mathbb{C}$. Slightly abusing terminology we shall say that S is fully dense in T whenever the canonical inclusion $\mathbb{S} \hookrightarrow \mathbb{T}$ is fully dense. Fixing S, we now want to compute the largest T in which S is fully dense.

Lemma 4.1.3 *If subsets $S, (T_i)_{i \in I}$ of $\mathbb{C}_0$ are such that S is fully dense in each T_i, then S is fully dense in $\bigcup_i T_i$.*

Proof. Let us write respectively $\mathbb{S}$, $\mathbb{T}_i$ and $\mathbb{T}$ for the full subcategories of $\mathbb{C}$ determined by $S \subseteq \mathbb{C}_0$, $T_i \subseteq \mathbb{C}_0$ and $\bigcup_i T_i \subseteq \mathbb{C}_0$. Suppose that functors $F, G\colon \mathbb{T} \to \mathbb{D}$ agree (to within isomorphism) on S, then density of S in each T_i makes them agree on each T_i, and therefore on $\bigcup_i T_i$. That is, the inclusion of S in $\bigcup_i T_i$ is fully dense, according to Proposition 4.1.2. □

The above lemma allows for the following definition:

Definition 4.1.4 *Let $\mathbb{C}$ be a $\mathcal{Q}$-category. The* **categorical closure** *of a subset $S \subseteq \mathbb{C}_0$ is the largest subset $\overline{S} \subseteq \mathbb{C}_0$ in which S is fully dense; that is to say,*

$$\overline{S} = \bigcup \{T \subseteq \mathbb{C}_0 \mid S \textit{ is fully dense in } T\}.$$

To explicitly compute the closure of a subset S of objects of $\mathbb{C}$, we can use:

Proposition 4.1.5 *Let $\mathbb{C}$ be a $\mathcal{Q}$-category and for $S \subseteq \mathbb{C}_0$ write $i\colon \mathbb{S} \hookrightarrow \mathbb{C}$ for the corresponding full embedding. For an object $x \in \mathbb{C}$ the following are equivalent:*

i. $x \in \overline{S}$,

ii. $\mathbb{C}(i-, x) \dashv \mathbb{C}(x, i-)$, *or explicitly:* $1_{tx} \leq \bigvee_{s \in S} \mathbb{C}(x, s) \circ \mathbb{C}(s, x)$,

iii. $\mathbb{C}(x, x) = \mathbb{C}(x, i-) \otimes \mathbb{C}(i-, x)$, *or explicitly:* $\mathbb{C}(x, x) = \bigvee_{s \in S} \mathbb{C}(x, s) \circ \mathbb{C}(s, x)$,

iv. for every $F, G\colon \mathbb{C} \to \mathbb{D}$, *if* $F_{|S} \cong G_{|S}$ *then* $Fx \cong Gx$.

Proof. (i $\Rightarrow$ iv) By density of S in $\overline{S}$, whenever F and G agree (up to isomorphism) on S then they necessarily do so on $\overline{S}$ too. In particular $Fx \cong Gx$ whenever $x \in \overline{S}$.

(iv $\Rightarrow$ iii) For the functors

$$F\colon \mathbb{C} \to \mathcal{P}\mathbb{C}\colon c \mapsto \mathbb{C}(-,i-) \otimes \mathbb{C}(i-,c) \text{ and } G = Y_{\mathbb{C}}\colon \mathbb{C} \to \mathcal{P}\mathbb{C}\colon c \mapsto \mathbb{C}(-,c)$$

we have (much as in the proof of Proposition 4.1.2) for any $s \in S$ that $Fs = \mathbb{C}(-,i-) \otimes \mathbb{C}(i-,s) = \mathbb{C}(-,s) = Gs$. So $F_{|S} \cong G_{|S}$, and therefore $F \cong G$ by assumption, from which $\mathbb{C}(x,x) = (Gx)(x) = (Fx)(x) = \bigvee_{s\in S} \mathbb{C}(x,s) \circ \mathbb{C}(s,x)$ follows.

(iii $\Rightarrow$ ii) Is trivial.

(ii $\Rightarrow$ i) For $T = \{x \in \mathbb{C}_0 \mid 1_{tx} \leq \bigvee_{s\in S} \mathbb{C}(x,s) \circ \mathbb{C}(s,x)\}$ we surely have $S \subseteq T$; so let $j\colon \mathbb{S} \hookrightarrow \mathbb{T}$ be the corresponding full embedding. For any $x, y \in T$ we have $\mathbb{T}(y,x) = \mathbb{C}(y,x)$, so we can use the composition inequality in $\mathbb{T}$ to compute that

$$\begin{aligned}\mathbb{T}(y,x) \geq \bigvee_{s\in S} \mathbb{T}(y,js) \circ \mathbb{T}(js,x) &\geq \bigvee_{s\in S} \mathbb{T}(y,x) \circ \mathbb{T}(x,js) \circ \mathbb{T}(js,x) \\ &\geq \mathbb{T}(y,x) \circ \bigvee_{s\in S} \mathbb{T}(x,js) \circ \mathbb{T}(js,x) \geq \mathbb{T}(y,x) \circ 1_{tx} = \mathbb{T}(y,x)\end{aligned}$$

This shows S to be fully dense in T, and therefore $T \subseteq \overline{S}$. □

Next we prove that the term 'closure' is well-chosen:

Proposition 4.1.6 *For every $\mathcal{Q}$-category $\mathbb{C}$, $(\mathbb{C}_0, \overline{(\cdot)})$ is a closure space, and for every functor $F\colon \mathbb{C} \to \mathbb{D}$, $F\colon (\mathbb{C}_0, \overline{(\cdot)}) \to (\mathbb{D}_0, \overline{(\cdot)})$ is a continuous function. This makes for a functor* $\mathsf{Cat}(\mathcal{Q}) \to \mathsf{Clos}$.

Proof. It is straightforward to check that $S \mapsto \overline{S}$ is a monotone and increasing operation on the subsets of $\mathbb{C}_0$. As S is fully dense in $\overline{S}$, which itself is fully dense in $\overline{\overline{S}}$, and the composition of two fully dense functors is again fully dense, it follows easily that S is fully dense in $\overline{\overline{S}}$, so $\overline{\overline{S}} \subseteq \overline{S}$. This makes $(\mathbb{C}_0, \overline{(\cdot)})$ a closure space.

Now fix $S \subseteq \mathbb{C}_0$, and suppose that $x \in \overline{S}$. Functoriality of $F\colon \mathbb{C} \to \mathbb{D}$ implies that

$$\begin{aligned}1_{tFx} = 1_{tx} \leq \bigvee_{s\in S} \mathbb{C}(x,s) \circ \mathbb{C}(s,x) \leq \bigvee_{s\in S} \mathbb{D}(Fx,Fs) \circ \mathbb{D}(Fs,Fx) \\ = \bigvee_{t\in FS} \mathbb{D}(Fx,t) \circ \mathbb{D}(t,Fx),\end{aligned}$$

which goes to show that $Fx \in \overline{FS}$. This makes $F\colon (\mathbb{C}_0, \overline{(\cdot)}) \to (\mathbb{D}_0, \overline{(\cdot)})$ a continuous function.

The functoriality of these constructions is a mere triviality. □

The following example nicely relates to Subsection 2.3.

Example 4.1.7 Via the Yoneda embedding $Y_{\mathbb{C}}\colon \mathbb{C} \to \mathcal{P}\mathbb{C}\colon x \mapsto \mathbb{C}(-,x)$ we may consider any $\mathcal{Q}$-category $\mathbb{C}$ as a full subcategory of the presheaf $\mathcal{Q}$-category $\mathcal{P}\mathbb{C}$: so $Y_{\mathbb{C}}(\mathbb{C})$ is precisely the full subcategory of representable presheaves. For any presheaf $\phi\colon \mathbb{1}_X \looparrowright \mathbb{C}$ we may compute – using the Yoneda Lemma – that

$$\begin{aligned}\phi \in \overline{Y_{\mathbb{C}}(\mathbb{C})} &\iff 1_X \leq \bigvee_{x\in\mathbb{C}_0} \mathcal{P}\mathbb{C}(\phi, Y_{\mathbb{C}}x) \circ \mathcal{P}\mathbb{C}(Y_{\mathbb{C}}x, \phi) \\ &\iff 1_X \leq \bigvee_{x\in\mathbb{C}_0} \mathcal{P}\mathbb{C}(\phi, Y_{\mathbb{C}}x) \circ \phi(x).\end{aligned}$$

On the other hand, in $\mathsf{Dist}(\mathcal{Q})$ we have (as in any quantaloid) that $\phi\colon \mathbb{1}_X \looparrowright \mathbb{C}$ is a left adjoint if and only if its lifting through the identity, namely $\phi \searrow \mathbb{C} : \mathbb{C} \looparrowright \mathbb{1}_X$, is its right adjoint, if and only if

$$1_X \leq \bigvee_{x\in\mathbb{C}_0} (\phi \searrow \mathbb{C})(x) \circ \phi(x)$$

holds. Because $(\phi \searrow \mathbb{C})(x) = \phi \searrow \mathbb{C}(-,x) = \mathcal{P}\mathbb{C}(\phi, Y_{\mathbb{C}}(x))$ we thus find that $\phi \in \overline{Y_{\mathbb{C}}(\mathbb{C})}$ exactly when ϕ is a left adjoint; or in words: *the Cauchy completion* $\mathbb{C}_{\mathsf{cc}}$ *of* $\mathbb{C}$ *is the categorical closure of* $\mathbb{C}$ *in the free completion* $\mathcal{P}\mathbb{C}$.

4.2 Strong Cauchy bilaterality—revisited

Suppose now that $\mathcal{Q}$ is an involutive quantaloid (and, as usual, write $f \mapsto f^{\circ}$ for the involution). When $\mathbb{C}$ is a $\mathcal{Q}$-category and $S \subseteq \mathbb{C}_0$ determines the full subcategory $\mathbb{S} \hookrightarrow \mathbb{C}$, then that same set S also determines a full subcategory $\mathbb{S}_{\mathsf{s}} \hookrightarrow \mathbb{C}_{\mathsf{s}}$ of the symmetrisation $\mathbb{C}_{\mathsf{s}}$ of $\mathbb{C}$. Thus we may compute *two* closures of S: for notational convenience, let us write $\overline{S}$ for its closure in $\mathbb{C}$, and $\widehat{S}$ for its closure in $\mathbb{C}_{\mathsf{s}}$. We can then spell out that, for any $x \in \mathbb{C}_0$,

$$x \in \overline{S} \iff 1_{tx} \leq \bigvee_{s\in S} \mathbb{C}(x,s) \circ \mathbb{C}(s,x) \tag{8}$$

whereas

$$\begin{aligned}x \in \widehat{S} &\iff 1_{tx} \leq \bigvee_{s\in S} \mathbb{C}_{\mathsf{s}}(x,s) \circ \mathbb{C}_{\mathsf{s}}(s,x) \\ &\iff 1_{tx} \leq \bigvee_{s\in S} (\mathbb{C}(x,s) \wedge \mathbb{C}(s,x)^{\circ}) \circ (\mathbb{C}(s,x) \wedge \mathbb{C}(x,s)^{\circ}).\end{aligned} \tag{9}$$

It is straightforward that the second condition implies the first (without any further condition on $\mathcal{Q}$), so that $\widehat{S} \subseteq \overline{S}$. This inclusion can be strict—but we have that:

Proposition 4.2.1 *For any involutive quantaloid $\mathcal{Q}$, the following conditions are equivalent:*

i. for every $\mathcal{Q}$-category $\mathbb{C}$ and every subset $S \subseteq \mathbb{C}_0$, the closure of S in $\mathbb{C}$ coincides with the closure of S in $\mathbb{C}_{\mathsf{s}}$,

ii. $\mathcal{Q}$ is strongly Cauchy bilateral: for every family $(f_i\colon X \to Y_i, g_i\colon Y_i \to X)_{i\in I}$ of morphisms in $\mathcal{Q}$, $1_X \leq \bigvee_i g_i \circ f_i$ implies $1_X \leq \bigvee_i (g_i \wedge f_i^{\circ}) \circ (g_i^{\circ} \wedge f_i)$.

Proof. We continue with the notations introduced before the statement of this Proposition. If we apply the second condition to the family

$$(\mathbb{C}(s,x)\colon tx \to ts, \mathbb{C}(x,s)\colon ts \to tx)_{s\in S}$$

then we obtain immediately that $x \in \widehat{S}$ whenever $x \in \overline{S}$, so $\overline{S} = \widehat{S}$.

Conversely, given the family of morphisms in the second condition, define the $\mathcal{Q}$-category $\mathbb{C}$ with object set $\mathbb{C}_0 = I \uplus \{x\}$, types given by $tx = X$ and $ti = Y_i$, and homs given by

$$\mathbb{C}(i,x) = f_i,\ \mathbb{C}(x,i) = g_i,\ \mathbb{C}(x,x) = 1_X, \text{ and } \mathbb{C}(j,i) = \begin{cases} 0_{Y_j,Y_i} \text{ when } i \neq j, \\ 1_{Y_i} \text{ when } i = j. \end{cases}$$

By assumption we must have $\overline{I} = \widehat{I}$ for the subset $I \subseteq \mathbb{C}_0$, so in particular $x \in \overline{I}$ must imply $x \in \widehat{I}$. Spelling this out with the aid of Equations (8) and (9) reveals the required formulas. □

In [10], the notion of a 'strongly Cauchy bilateral' quantaloid $\mathcal{Q}$ was introduced as a purely formal stronger version of ("ordinary") Cauchy bilaterality, because in several examples the stronger version holds, and it is easier to verify. Here now, in the context of closures on $\mathcal{Q}$-categories, we have an explanation for the strong Cauchy bilaterality of $\mathcal{Q}$ as encoding precisely that "closures can be symmetrised". (But we do repeat that, for an integral quantaloid, strong Cauchy bilaterality and ('ordinary') Cauchy bilaterality are equivalent.) Whereas the Cauchy bilaterality of an involutive quantaloid $\mathcal{Q}$ implies that there is a distributive law of the Cauchy monad over the symmetrisation comonad on $\mathsf{Cat}(\mathcal{Q})$ [10, Corollary 3.9], we can express strong Cauchy bilaterality of $\mathcal{Q}$ to mean that the functor $\mathsf{Cat}(\mathcal{Q}) \to \mathsf{Clos}$ is invariant under composition with the symmetrisation comonad $(-)_{\mathsf{s}}\colon \mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$.

4.3 Groundedness and additivity

The final issue we wish to address here in full generality, concerns the topologicity of the closure associated with any $\mathcal{Q}$-category—and this turns out to be a rather subtle point. Recall that a closure is said to be *topological* when it is both *grounded* (i.e. $\overline{\emptyset} = \emptyset$) and *additive* (i.e. $\overline{S \cup T} = \overline{S} \cup \overline{T}$). Especially when considering (convergence of) sequences in a closure space – as we shall wish to do in the next section in the case of partial metric spaces – it is problematic if that closure is non-grounded: for then any sequence converges to every point in $\overline{\emptyset}$.

First, for any $\mathcal{Q}$-category $\mathbb{C}$ it is easy to check that $\overline{\emptyset} = \{x \in \mathbb{C}_0 \mid 1_{tx} = 0_{tx}\}$; but for an object Z in $\mathcal{Q}$ we have that $0_Z = 1_Z$ if and only if Z is a zero object (both terminal and initial); therefore $\overline{\emptyset} = \emptyset$ if and only $\mathbb{C}_0$ has no element whose type is a zero object in $\mathcal{Q}$. Conversely, if $\mathcal{Q}$ has a zero object Z, then quite obviously the categorical closure of the $\mathcal{Q}$-category $\mathbb{1}_Z$ does not satisfy $\overline{\emptyset} = \emptyset$. That is to say, the functor $\mathsf{Cat}(\mathcal{Q}) \to \mathsf{Clos}$ of Proposition 4.1.6 factors through the full subcategory $\mathsf{Clos}_{\mathsf{gd}}$ of grounded closure spaces if and only if $\mathcal{Q}$ does not have a zero object.

Example 4.3.1 Any non-trivial quantale – viewed as a one-object quantaloid – does not have a zero object, and therefore the categorical closure on such a quantale-enriched category is always grounded. However, every quantaloid of diagonals (our main concern in this paper) has zero objects: indeed, every zero morphism in a quantaloid $\mathcal{Q}$ determines a zero object in $\mathcal{D}(\mathcal{Q})$. In particular, even when $\mathcal{Q}$ is a non-trivial quantale, $\mathcal{D}(\mathcal{Q})$ will still have exactly one zero object. The categorical closure on a $\mathcal{D}(\mathcal{Q})$-enriched category may thus very well be ungrounded—and thus we must be a little bit more careful when studying (convergence of) sequences in such an enriched category. The case that springs to mind is Lawvere's quantale of positive reals, $R = ([0,\infty]^{\mathsf{op}}, +, 0)$, where R-categories (generalised metric spaces, cf. Example 2.2.2) have a grounded closure, but $\mathcal{D}(R)$-categories (generalised partial metric spaces, cf. Example 3.3.3) may have an ungrounded closure.

However, if $\mathcal{Q}$ does have a (unique[11]) zero object Z, we can always "discard" the elements of type Z from any given $\mathcal{Q}$-category $\mathbb{C}$: more precisely, if we define its full subcategories $\mathbb{C}_{\mathsf{z}}$ and $\mathbb{C}_{\mathsf{nz}}$ to have as elements

$$(\mathbb{C}_{\mathsf{z}})_0 = \{x \in \mathbb{C}_0 \mid tx = Z\} \quad \text{and} \quad (\mathbb{C}_{\mathsf{nz}})_0 = \{x \in \mathbb{C}_0 \mid tx \neq Z\}$$

then $\mathbb{C}$ is exactly their categorical sum (coproduct): $\mathbb{C} = \mathbb{C}_{\mathsf{nz}} + \mathbb{C}_{\mathsf{z}}$.

Because any $\mathcal{Q}$-functor $F \colon \mathbb{C} \to \mathbb{D}$ preserves types, it restricts to elements of non-zero type as $F_{\mathsf{nz}} \colon \mathbb{C}_{\mathsf{nz}} \to \mathbb{D}_{\mathsf{nz}}$. It follows easily that the canonical injection $i \colon \mathbb{C}_{\mathsf{nz}} \to \mathbb{C}$ is the counit for a (strictly) idempotent comonad on $\mathsf{Cat}(\mathcal{Q})$, whose category of coalgebras $\mathsf{Cat}(\mathcal{Q})_{\mathsf{nz}}$ is exactly the full coreflective subcategory of those $\mathcal{Q}$-categories that do not have elements of type Z:

$$\begin{tikzcd} \mathsf{Cat}(\mathcal{Q})_{\mathsf{nz}} \arrow[r, hook, bend right] \arrow[r, phantom, "\top"] & \mathsf{Cat}(\mathcal{Q}) \arrow[l, dotted, bend right, "(-)_{\mathsf{nz}}"'] \end{tikzcd}$$

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[11] A similar reasoning holds when $\mathcal{Q}$ has several (necessarily uniquely isomorphic) zero objects, but we shall not need encounter that situation further on; indeed, our main concern is $\mathcal{Q} = \mathcal{D}(R)$.

Furthermore, if we write $\mathcal{Q}_{\mathsf{nz}}$ for the (smaller) quantaloid obtained from $\mathcal{Q}$ by discarding its zero object Z, then $\mathsf{Cat}(\mathcal{Q})_{\mathsf{nz}} = \mathsf{Cat}(\mathcal{Q}_{\mathsf{nz}})$ (and the full embedding $\mathsf{Cat}(\mathcal{Q})_{\mathsf{nz}} = \mathsf{Cat}(\mathcal{Q}_{\mathsf{nz}}) \hookrightarrow \mathsf{Cat}(\mathcal{Q})$ is actually the change of base determined by the homomorphism $\mathcal{Q}_{\mathsf{nz}} \hookrightarrow \mathcal{Q}$). This goes to show that we always have a factorisation

$$\begin{tikzcd}
\mathsf{Cat}(\mathcal{Q})_{\mathsf{nz}} = \mathsf{Cat}(\mathcal{Q}_{\mathsf{nz}}) \arrow[r, hook] \arrow[d, dotted] & \mathsf{Cat}(\mathcal{Q}) \arrow[d] \\
\mathsf{Clos}_{\mathsf{gd}} \arrow[r, hook] & \mathsf{Clos}
\end{tikzcd}$$

The study of (convergence of) sequences of elements in a $\mathcal{Q}$-category $\mathbb{C}$ (for the categorical closure) is most useful, not in the whole of $\mathbb{C}$, but in its "non-zero coreflection" $\mathbb{C}_{\mathsf{nz}}$.

In Section 5 we shall consider (convergence and Cauchyness of) sequences in (the non-zero coreflection of) a generalised partial metric space, and we shall want to relate it to the categorical Cauchy completion. To prepare the ground, we make here a few general observations regarding the Cauchy completion of a $\mathcal{Q}$-category $\mathbb{C}$ in case the quantaloid $\mathcal{Q}$ has a (unique) zero object Z. For any $\mathcal{Q}$-category $\mathbb{C}$ there is a unique Cauchy distributor from $\mathbb{1}_Z$ to $\mathbb{C}$, namely $\phi\colon \mathbb{1}_Z \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{C}$ with, for all $x \in \mathbb{C}_0$, the $\phi(x)\colon Z \to tx$ being the unique element of $\mathcal{Q}(tx, Z)$. In other words, the $\mathcal{Q}$-category $\mathbb{C}_{\mathsf{cc}}$ contains exactly one element of type Z, which means that

$$\mathbb{C}_{\mathsf{cc}} \cong (\mathbb{C}_{\mathsf{cc}})_{\mathsf{nz}} + \mathbb{1}_Z.$$

On the other hand, a Cauchy presheaf on $\mathbb{C}_{\mathsf{nz}}$ *as $\mathcal{Q}_{\mathsf{nz}}$-category* is exactly a Cauchy presheaf on $\mathbb{C}_{\mathsf{nz}}$ *as $\mathcal{Q}$-category whose type is not zero.* That is to say, the following square commutes:

$$\begin{tikzcd}
\mathsf{Cat}(\mathcal{Q}) \arrow[r, "(-)_{\mathsf{nz}}"] \arrow[d, "(-)_{\mathsf{cc}}"'] & \mathsf{Cat}(\mathcal{Q}_{\mathsf{nz}}) = \mathsf{Cat}(\mathcal{Q})_{\mathsf{nz}} \arrow[d, "(-)_{\mathsf{cc}}"] \\
\mathsf{Cat}(\mathcal{Q}) \arrow[r, "(-)_{\mathsf{nz}}"'] & \mathsf{Cat}(\mathcal{Q}_{\mathsf{nz}}) = \mathsf{Cat}(\mathcal{Q})_{\mathsf{nz}}
\end{tikzcd}$$

(where on the right hand side we do the Cauchy completion qua $\mathcal{Q}_{\mathsf{nz}}$-enriched category!). As a consequence, we find:

Proposition 4.3.2 *For $\mathcal{Q}$ a quantaloid with a unique zero object Z and $\mathbb{C}$ any $\mathcal{Q}$-category, we have that*

$$\mathbb{C}_{\mathsf{cc}} \cong (\mathbb{C}_{\mathsf{nz}})_{\mathsf{cc}} + \mathbb{1}_Z \textit{ in } \mathsf{Cat}(\mathcal{Q})$$

where $\mathbb{C}_{\mathsf{cc}}$ is the $\mathcal{Q}$-enriched Cauchy completion of $\mathbb{C}$ and $(\mathbb{C}_{\mathsf{nz}})_{\mathsf{cc}}$ is the $\mathcal{Q}_{\mathsf{nz}}$-enriched Cauchy completion of $\mathbb{C}_{\mathsf{nz}}$ (whose elements are in fact the $\mathcal{Q}$-enriched Cauchy presheaves on $\mathbb{C}_{\mathsf{nz}}$ whose type is not Z).

Finally, we end with a comment on the additivity of the categorical closure on $\mathbb{C}$. As for any closure, it is always true that $\overline{S} \cup \overline{T} \subseteq \overline{S \cup T}$ for any $S, T \subseteq \mathbb{C}_0$, but this inclusion need not be an equality. Indeed, for an $x \in \mathbb{C}_0$ we have that

$$\begin{aligned} x \in \overline{S \cup T} &\iff 1_{tx} \leq \bigvee_{r \in S \cup T} \mathbb{C}(x,r) \circ \mathbb{C}(r,x) \\ &\iff 1_{tx} \leq (\bigvee_{s \in S} \mathbb{C}(x,s) \circ \mathbb{C}(s,x)) \vee (\bigvee_{t \in T} \mathbb{C}(x,t) \circ \mathbb{C}(t,x)), \end{aligned} \tag{10}$$

whereas

$$x \in \overline{S} \cup \overline{T} \iff 1_{tx} \leq \bigvee_{s \in S} \mathbb{C}(x,s) \circ \mathbb{C}(s,x), \quad 1_{tx} \leq \bigvee_{t \in T} \mathbb{C}(x,t) \circ \mathbb{C}(t,x). \tag{11}$$

It is now straightforward to identify a sufficient condition for the closure of any $\mathcal{Q}$-category to be topological (i.e. grounded and additive), which turns out to be also necessary when $\mathcal{Q}$ is integral). Admittedly this is not the most elegant condition—but it serves our purposes in the upcoming subsections.

Proposition 4.3.3 *For any quantaloid $\mathcal{Q}$, if every identity arrow is finitely join-irreducible*[12] *then the closure associated to any $\mathcal{Q}$-category $\mathbb{C}$ is topological. For any integral quantaloid $\mathcal{Q}$ the converse holds too.*

Proof. For any $\mathcal{Q}$-category $\mathbb{C}$ it is easy to check that $\overline{\emptyset} = \{x \in \mathbb{C}_0 \mid 1_{tx} = 0_{tx}\}$; therefore $\overline{\emptyset} = \emptyset$ if and only if none of the identities in $\mathcal{Q}$ is a bottom element. It is furthermore clear from the comparison of (10) and (11) that finite join-irreducibility of identities in (any) $\mathcal{Q}$ suffices for closures to be topological. Conversely, and under the extra assumption that $\mathcal{Q}$ is integral, for any $f, g \in \mathcal{Q}(X,X)$ there is a $\mathcal{Q}$-category $\mathbb{C}$ with three objects of type X, say x, y, z, and hom-arrows

$$\mathbb{C}(x,y) = f,\ \mathbb{C}(y,z) = g,\ \mathbb{C}(x,z) = f \circ g, \text{ and all others are } 1_X.$$

It is easy to compute with the formula in Proposition 4.1.5–ii that

$$y \in \overline{\{x,z\}} \Leftrightarrow 1_X \leq f \vee g, \quad y \in \overline{\{x\}} \Leftrightarrow 1_X \leq f, \quad y \in \overline{\{z\}} \Leftrightarrow 1_X \leq g.$$

Thus, if this closure is topological then 1_X must be finitely join-irreducible. □

If a quantaloid $\mathcal{Q}$ has a (unique) zero object, then it can never satisfy the condition in the above proposition; but removing that zero object from $\mathcal{Q}$ may very well produce a quantaloid $\mathcal{Q}_{\mathsf{nz}}$ that does satisfy the condition above.

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[12] We mean here that, for any object X of $\mathcal{Q}$, if $1_X \leq f_1 \vee ... \vee f_n$ ($n \in \mathbb{N}$) then $1_X \leq f_i$ for some $i \in \{1, ..., n\}$. In other words, $1_X \neq 0_X$ and for any $1_X \leq f \vee g$ we have $1_X \leq f$ or $1_X \leq g$.

5. Topology from partial metrics

5.1 Finitely typed partial metric spaces

From now on we shall apply the previous material to the particular case where the base quantaloid is the quantaloid of diagonals in the (divisible, commutative) Lawvere quantale $R = ([0,\infty]^{\mathsf{op}}, +, 0)$. As before, we shall write an R-enriched category as (X,d), and a $\mathcal{D}(R)$-enriched category as (X,p), to insist on their understanding as generalised (partial) metric spaces—even though we shall of course use the fully general theory of quantaloid-enriched categories where we see fit.

The quantale R has its identity finitely join-irreducible (because the order is linear). The quantaloid $\mathcal{D}(R)$ has a unique zero object, namely ∞ (an unfortunate notational clash, due to the reversal of the natural order on $[0,\infty]$), but once we remove this zero object, the resulting quantaloid $\mathcal{D}(R)_{\mathsf{nz}}$ has all its identities finitely join-irreducible (because the local order is linear). We saw in Examples 3.2.5 and 3.2.6 that R and $\mathcal{D}(R)$ are both strongly Cauchy bilateral; *a fortiori* the same is true for the subquantaloid $\mathcal{D}(R)_{\mathsf{nz}}$.

The categorical closure on a generalised partial metric space (X,p) is, as indicated in the previous section, non-grounded as soon as there exists an $x \in X$ such that $p(x,x) = \infty$. Excluding the points of self-distance[13] ∞ from (X,p) (that is, those elements which are of type ∞ in (X,p) qua $\mathcal{D}(R)$-enriched category), we make sure that the categorical closure on that *finitely typed part of* (X,p) is topological.

Restricting our attention now to *finitely typed* generalised partial metric spaces – by which we mean of course those partial metrics such that $p(x,x) < \infty$, so that in effect we consider categories enriched in $\mathcal{D}(R)_{\mathsf{nz}}$ – we may infer from Propositions 4.2.1 and 4.3.3 that:

Proposition 5.1.1 *The categorical closure on a finitely typed generalised partial metric space (X,p) is topological, and is identical to the closure on the associated symmetric finitely typed generalised partial metric space (X,p_{s}) (where $p_{\mathsf{s}}(y,x) = p(y,x) \vee p(x,y)$).*

Now, for a finitely typed generalised partial metric space (X,p), we find from Proposition 4.1.5 that, for any subset $S \subseteq X$ and any $x \in X$,

$$\begin{aligned} x \in \overline{S} \quad &\iff \quad p(x,x) \geq \bigwedge_{s \in S} p(x,s) - p(s,s) + p(s,x) \\ &\iff \quad 0 \geq \bigwedge_{s \in S} p(x,s) - p(s,s) + p(s,x) - p(x,x) \end{aligned} \tag{12}$$

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[13] But we insist that for $x \neq y$ in X it may still happen that $p(x,y) = \infty$.

The expression under the infimum is thus precisely equal to $p_0(x,s)+p_0(x,s)$ for the generalised metric p_0 associated with the partial metric p through the change of base $J_0\colon \mathcal{D}(R)\to R$ (see below Example 3.3.5). That is to say:

Proposition 5.1.2 *The categorical topology on a finitely typed generalised partial metric space (X,p) is identical to the topology on the generalised metric space (X,p_0) (where $p_0(y,x):=p(y,x)-p(x,x)$).*

Putting both previous Propositions together, we can conclude that the categorical topology on a finitely typed generalised partial metric space *is always metrisable by means of a symmetric generalised metric*. And for such a *symmetric* generalised metric space (X,d), Proposition 4.1.5 says that

$$\begin{aligned} x\in\overline{S} \iff 0\geq\bigwedge_{s\in S} d(x,s)+d(s,x) &\iff 0\geq\bigwedge_{s\in S} 2\cdot d(x,s)\\ \iff 0\geq\bigwedge_{s\in S} d(x,s) &\iff \forall\varepsilon>0\,\exists s\in S: d(x,s)<\varepsilon \end{aligned}$$

Thus the categorical topology on (X,d) is exactly the usual metric topology—with a basis given by the collection of open balls

$$\{B(x,\varepsilon):=\{y\in X\mid d(x,y)<\varepsilon\}\mid x\in X,\varepsilon>0\},$$

with its usual notion of convergent sequences, etc.

One could consider this a disappointment: there are not more "partially metrisable topologies" then there are metrisable ones. Still, one must realise that it is not always trivial to interpret topological and/or metric phenomena in a given finitely typed partial metric (X,p) by passing to some metric (X,d) which just happens to define the same topology. The next subsection is entirely devoted to the study of convergent sequences in finitely typed partial metrics.

5.2 Convergence . . .

A fundamental use of topology is its inherent notion of convergence for sequences: $(x_n)_n\to x$ in a topological space $(X,\mathcal{T})$ when for every $x\in U\in\mathcal{T}$ there exists an n_0 such that $x_n\in U$ for every $n\geq n_0$. When the topology stems from a symmetric generalised metric d on X, it is sufficient to consider open balls centered in x, and so

$$(x_n)_n\to x \iff \forall\varepsilon>0\,\exists n_0\,\forall n\geq n_0: d(x_n,x)<\varepsilon.$$

A convergent sequence is necessarily a Cauchy sequence, meaning that

$$\forall\varepsilon>0\,\exists n_0\,\forall m,n\geq n_0: d(x_n,x_m)<\varepsilon,$$

and a symmetric generalised metric space is said to be (sequentially) Cauchy complete precisely when every Cauchy sequence converges. But note that the definition of Cauchy sequence is symmetric in x_n and x_m *even when the generalised metric* d *is not symmetric*—and so it makes perfect sense for *any* generalised metric space. As recalled in Example 2.3.2, Lawvere [17] proved that a generalised metric space (X, d) is sequentially Cauchy complete if and only if every left adjoint distributor into (X, d) (now viewed as an R-enriched category) is representable.

Now consider a finitely typed generalised partial metric space (X, p); its categorical topology is equivalently described by the symmetric generalised metric

$$(p_0)_{\mathsf{s}}(y,x) = p_0(y,x) \vee p_0(x,y) = (p(y,x) - p(x,x)) \vee (p(x,y) - p(y,y)), \quad (13)$$

and therefore a sequence $(x_n)_n$ in (X, p) converges to $x \in X$ precisely when

$$\forall \varepsilon > 0\, \exists n_0\, \forall n \geq n_0 : (p(x,x_n) - p(x_n,x_n)) \vee (p(x_n,x) - p(x,x)) < \varepsilon. \quad (14)$$

In what follows we shall first try to improve on our understanding of this formula, and thus of convergence in (X, p), before we look at Cauchy sequences and completion.

For any quantaloid $\mathcal{Q}$, the terminal object $\mathbb{T}$ in $\mathsf{Cat}(\mathcal{Q})$ (exists and) has the following description: its object set is $\mathbb{T}_0 = \mathcal{Q}_0$, the type function is the identity, and the hom function is $\mathbb{T}_0 \times \mathbb{T}_0 \to \mathcal{Q}_1 \colon (Y, X) \mapsto \top_{X,Y}$ (where $\top_{X,Y}$ is the top element in $\mathcal{Q}(X, Y)$). The unique functor from a $\mathcal{Q}$-category $\mathbb{C}$ to $\mathbb{T}$ is $\mathbb{C}_0 \to \mathbb{T}_0 \colon x \mapsto tx$, that is, it is $\mathbb{C}$'s type function. From Proposition 4.1.6 we deduce that the type function of a $\mathcal{Q}$-category is continuous—but, of course, the use of this statement depends on the categorical topology of $\mathbb{T}$. If we work over the quantaloid $\mathcal{D}(R)_{\mathsf{nz}}$ (so that $\mathsf{Cat}(\mathcal{D}(R)_{\mathsf{nz}}) = \mathsf{Cat}(\mathcal{D}(R))_{\mathsf{nz}}$ is exactly the category of *finitely typed* partial metric spaces), then things are as follows:

Proposition 5.2.1 *The terminal finitely typed generalised partial metric space* (T, p) *is defined by* $T = [0, \infty[$ *and* $p(a, b) = a \vee b$*; its categorical topology is the usual metric topology.*

Proof. The general construction of the terminal $\mathcal{D}(R)_{\mathsf{nz}}$-category $\mathbb{T}$, which we now write as a generalised partial metric space (T, p), says that

$$\begin{cases} T = \text{objects of } \mathcal{D}(R)_{\mathsf{nz}} = [0, \infty[\\ p(a,b) = \text{top element of } \mathcal{D}(R)_{\mathsf{nz}}(a,b) = a \vee b \end{cases}$$

From the above discussion, the categorical topology on the partial generalised metric (T, p) is equivalently described by the ("total") generalised metric (T, p_0), which in turn is equivalently described by its symmetrisation $(X, (p_0)_{\mathsf{s}})$. A simple computation leads to

$$(p_0)_{\mathsf{s}}(a,b) = p_0(a,b) \vee p_0(b,a) = ((a \vee b) - a) \vee (a \vee b) - b) = |a - b|.$$

That is to say, the categorical topology on (T, p) (qua partial metric space) is precisely the usual metric topology. $\square$

Corollary 5.2.2 *For every finitely typed generalised partial metric space* (X, p)*, equipped with its categorical topology,*

- *i. the function* $X \to [0, \infty[\colon x \mapsto p(x, x)$ *is continuous (for the usual topology on* $[0, \infty[$*),*
- *ii. if* $(x_n)_n \to x$ *in* (X, p) *then* $\lim_{n\to\infty} p(x_n, x_n) = p(x, x)$*.*

We can now prove a practical characterisation of convergence in a finitely typed partial metric space, which subsumes the definition of convergence from [5] (in the case p is symmetric, separated and never takes the value ∞) and improves the one given in [19] (in that it eliminates all double limits[14]):

Proposition 5.2.3 *In a finitely typed generalised partial metric space* (X, p)*, equipped with its categorical topology, we have a convergent sequence* $(x_n)_n \to x$ *if and only if all three limits*

$$\lim_{n\to\infty} p(x, x_n),\ \lim_{n\to\infty} p(x_n, x_n) \text{ and } \lim_{n\to\infty} p(x_n, x)$$

(exist and) are equal to $p(x, x)$*.*

Proof. Suppose first that $(x_n)_n \to x$ in (X, p). Because $p(x, x_n) \wedge p(x_n, x) \geq p(x, x) \vee p(x_n, x_n)$, the expression in (14) is equivalent to

$$\forall \varepsilon > 0\ \exists n_0\ \forall n \geq n_0 : \left\{ \begin{array}{l} |p(x, x_n) - p(x_n, x_n)| < \varepsilon \\ |p(x_n, x) - p(x, x)| < \varepsilon \end{array} \right.$$

and so in particular $\lim_{n\to\infty} p(x_n, x) = p(x, x)$. Corollary 5.2.2 assures that $\lim_{n\to\infty} p(x_n, x_n) = p(x, x)$, that is,

$$\forall \varepsilon > 0\ \exists n_1\ \forall n \geq n_1 :\ |p(x_n, x_n) - p(x, x)| < \varepsilon,$$

and so for any $n \geq n_0 \vee n_1$ also

$$|p(x, x_n) - p(x, x)| \leq |p(x, x_n) - p(x_n, x_n)| + |p(x_n, x_n) - p(x, x)| < 2\varepsilon.$$

Therefore $\lim_{n\to\infty} p(x, x_n) = p(x, x)$ too. Hence we proved the necessity of the three limits.

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[14] Precisely, in [19] it is required that $\lim_{m,n\to\infty} p(x_m, x_n) = p(x, x)$ instead of our condition $\lim_{n\to\infty} p(x_n, x_n) = p(x, x)$. Conceptually, our simple limit expresses that the type of x_n should converge to the type of x (but nothing more); therefore our notion of convergence is directly applicable to the typed sequences of Definition 5.3.1 further on.

Conversely, knowing that $\lim_{n\to\infty} p(x, x_n) = p(x, x) = \lim_{n\to\infty} p(x_n, x_n)$, we find also

$$0 \leq |p(x, x_n) - p(x_n, x_n)| \leq |p(x, x_n) - p(x, x)| + |p(x, x) - p(x_n, x_n)|,$$

whence $\lim_{n\to\infty}(p(x, x_n) - p(x_n, x_n)) = 0$. Together with $\lim_{n\to\infty} p(x_n, x) = p(x, x)$ this shows the sufficiency of the three limits. □
The middle limit in the above proposition is crucial, as the following example indicates:

Example 5.2.4 For A a (non-empty) finite alphabet, let X be the union of all non-empty words and all sequences in that alphabet: it is a finitely typed generalised partial metric space if we put $p(x, y) = (\frac{1}{2})^k$ where k is the position of the first letter in which x and y do not agree. Now consider a sequence $(x_n)_n$ with $x_0 \in A$ and each x_{n+1} is equal to x_n concatenated with one extra letter: we then have that $\lim_{n\to\infty} p(x_0, x_n) = p(x_0, x_0) = \lim_{n\to\infty} p(x_n, x_0)$, but it is against all intuition to say that $(x_n)_n$ converges to x_0! Precisely because $\lim_{n\to\infty} p(x_n, x_n) \neq p(x_0, x_0)$ this pathological behaviour is excluded.

Note that Proposition 5.2.3 contains the usual convergence criterion in an ordinary metric space, where we would have $p(x, x) = 0 = p(x_n, x_n)$ and $p(x, x_n) = p(x_n, x)$.

5.3 … and completeness

We now turn to the study of Cauchy sequences in, and completion of, finitely typed partial metric spaces (for the categorical topology).

Recall that a finitely typed generalised partial metric space (X, p) is a $\mathcal{D}(R)_{\mathsf{nz}}$-category $\mathbb{X}$ with object set $\mathbb{X}_0 = X$, type function $tx = p(x, x)$ and hom-arrows $\mathbb{X}(y, x) = p(y, x)$. The crucial rôle of the type function as "indicator of partialness" was already apparent in the previous subsection. To facilitate our discussion of sequences in (X, p) we find it useful to introduce some further terminology:

Definition 5.3.1 *A sequence $(x_n)_n$ in a finitely typed generalised partial metric space (X, p) is **typed** whenever $\lim_{n\to\infty} p(x_n, x_n)$ exists in $[0, \infty[$; that limit is then called the type of $(x_n)_n$.*

Because we only consider sequences in a *finitely typed* (X, p) (for the reasons explained in Subsection 4.3), any typed sequence is in fact of finite type too.

Lemma 5.3.2 *For any finitely typed generalised partial metric space (X, p), the following defines an equivalence relation on the set of all typed sequences in (X, p):*

$$\begin{aligned}(x_n)_n \sim (y_n)_n \overset{\text{def.}}{\iff} \lim_{n\to\infty} p(x_n, y_n) &= \lim_{n\to\infty} p(x_n, x_n) \\ &= \lim_{n\to\infty} p(y_n, y_n) = \lim_{n\to\infty} p(y_n, x_n).\end{aligned}$$

Proof. Reflexivity and symmetry are obvious. If $(x_n)_n \sim (y_n)_n$ and $(y_n)_n \sim (z_n)_n$ then all three must have the same type, say q, and since both extremes of the double inequality

$$p(x_n, x_n) \vee p(z_n, z_n) \leq p(x_n, z_n) \leq p(x_n, y_n) - p(y_n, y_n) + p(y_n, z_n)$$

converge to q when n goes to ∞, so does the middle term. Similarly we can compute that $\lim_{n\to\infty} p(z_n, x_n) = q$. □

Let us stress that the equivalence relation only pertains to typed sequences, and that equivalent sequences necessarily have the same type. As was also done in [18], we furthermore define:

Definition 5.3.3 *A sequence $(x_n)_n$ in a finitely typed generalised partial metric space (X, p) is* **Cauchy** *if $(p(x_n, x_m))_{(n,m)}$ is a Cauchy net in $[0, \infty]$.*

Here we regard $[0, \infty]$ canonically as a generalised metric space: $d(a, b) = \max(a - b, 0)$. By our general considerations, its categorical topology is metrisable by the symmetric distance function

$$d_s(a, b) = d(a, b) \vee d(b, a) = \begin{cases} 0 \text{ if } a = \infty = b \\ |a - b| \text{ if } a \neq \infty \neq b \\ \infty \text{ otherwise} \end{cases}$$

If a net $(a_{(m,n)})_{(m,n)}$ is Cauchy in $[0, \infty]$ then it is either eventually constant ∞ or eventually finite. Since (X, p) is finitely typed by assumption, the former cannot happen for $a_{(m,n)} = p(x_n, x_m)$, so every such Cauchy net lies eventually[15] in $[0, \infty[$. As this is a complete space, the Cauchy net $(p(x_n, x_m))_{(n,m)}$ converges to the "double" limit $\lim_{m,n\to\infty} p(x_n, x_m)$ in the usual sense (see [15] for details on sequences and nets).

The following results use the equivalence relation on typed sequences to express the expected interplay between convergent sequences and Cauchy sequences in partial metric spaces:

Proposition 5.3.4 *In a finitely typed generalised partial metric space (X, p),*

i. any constant sequence $(x)_n$ is typed, with type $p(x, x)$,

ii. $(x_n)_n \to x$ if and only if $(x_n)_n$ (is typed and) $(x_n)_n \sim (x)_n$,

iii. if $(x_n)_n \to x$ and $(y_n)_n$ is typed, then $(x_n)_n \sim (y_n)_n$ if and only if $(y_n)_n \to x$,

iv. any Cauchy sequence $(x_n)_n$ is typed, with type $\lim_{m,n\to\infty} p(x_n, x_m)$,

v. if $(x_n)_n \sim (y_n)_n$ then either one is Cauchy if and only if the other one is too,

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[15] Thus this notion of Cauchyness narrows down to the one in [5, 18] which is only concerned with partial metrics satisfying $p(x, y) < \infty$ for all $x, y \in X$.

vi. every convergent sequence is Cauchy.

Proof. (1) Is trivial. (2) This is a reformulation of Proposition 5.2.3. (3) Follows from the previous assertion and transitivity of $\sim$. (4) If the net $(p(x_n, x_m))_{(n,m)}$ converges in $[0, \infty]$, then the subnet $(p(x_n, x_n))_n$ converges to the same value. (5) Assume that $(x_n)_n$ and $(y_n)_n$ have type q and that $(y_n)_n$ is a Cauchy sequence. Then, for all natural numbers n and m,

$$\begin{aligned} p(x_n, x_n) &\leq p(x_n, x_m) \\ &\leq p(x_n, y_n) - p(y_n, y_n) + p(y_n, y_m) - p(y_m, y_m) + p(y_m, x_m); \end{aligned}$$

and since both extremes of this double inequality converge to q when (n, m) goes to (∞, ∞), so does $p(x_n, x_m)$. (6) If $(x_n)_n \to x$, then $(x_n)_n \sim (x)_n$; since $(x)_n$ is Cauchy, $(x_n)_n$ is so too. □

To convince the categorically inclined that Definition 5.3.3 makes perfect sense, we shall show that there is an essentially bijective correspondence between Cauchy sequences in a finitely typed partial metric space (X, p) on the one hand, and Cauchy distributors on the $\mathcal{D}(R)_{\mathsf{nz}}$-category $\mathbb{X}$ (still defined by $\mathbb{X}_0 = X$, $tx = p(x, x)$ and $\mathbb{X}(y, x) = p(y, x)$, of course). Recall that a $\mathcal{D}(R)_{\mathsf{nz}}$-distributor $\phi : \mathbb{1}_q \xrightarrow{\circ} \mathbb{X}$ is (in terms of the partial metric) defined by a number $q \in [0, \infty[$ together with a function $\phi \colon X \to [0, \infty]$ such that

$$q \vee p(y, y) \leq \phi(y) \leq p(y, x) - p(x, x) + \phi(x) \tag{15}$$

for all $x, y \in X$. Similarly, a $\mathcal{D}(R)_{\mathsf{nz}}$-distributor $\psi \colon \mathbb{X} \xrightarrow{\circ} \mathbb{1}_q$ is a number $q \in [0, \infty[$ together with a function $\psi \colon X \to [0, \infty]$ such that

$$q \vee p(y, y) \leq \psi(y) \leq \psi(x) - p(x, x) + p(x, y) \tag{16}$$

for all $x, y \in X$. Such distributors form an adjoint pair $\phi \dashv \psi$ (and so ϕ is a Cauchy presheaf, and then we rather write $\phi^* = \psi$) if and only if $\mathbb{1}_q \leq \psi \otimes \phi$ and $\phi \otimes \psi \leq \mathbb{X}$ in $\mathsf{Dist}(\mathcal{D}(R)_{\mathsf{nz}})$, that is, for all $x, y \in X$,

$$\bigwedge_{z \in X} \psi(z) - p(z, z) + \phi(z) \leq q \quad \text{and} \quad p(y, x) \leq \phi(y) - q + \psi(x). \tag{17}$$

Fixing $x \in X$, the representables $\mathbb{X}(-, x) \colon \mathbb{1}_{tx} \xrightarrow{\circ} \mathbb{X}$ and $\mathbb{X}(x, -) \colon \mathbb{X} \xrightarrow{\circ} \mathbb{1}_{tx}$ always form an adjoint pair; they correspond to the functions $p(-, x) \colon X \to [0, \infty]$ and $p(x, -) \colon X \to [0, \infty]$, with $q = p(x, x)$.

Lemma 5.3.5 *If $(x_n)_n$ and $(y_n)_n$ are Cauchy sequences in a finitely typed generalised partial metric space (X, p) then $(p(x_n, y_m))_{n,m}$ is a Cauchy net in $[0, \infty]$.*

Proof. Let $\varepsilon > 0$. Since $(x_n)_n$ and $(y_n)_n$ are Cauchy sequences in (X, p), there is a natural number N so that for all $n, m, n', m' \geq N$,

$$p(x_n, x_{n'}) - p(x_{n'}, x_{n'}) \leq \varepsilon, \quad p(y_{m'}, y_m) - p(y_{m'}, y_{m'}) \leq \varepsilon,$$

$$p(x_{n'}, x_n) - p(x_n, x_n) \leq \varepsilon, \quad p(y_m, y_{m'}) - p(y_m, y_m) \leq \varepsilon.$$

From these inequalities (and the triangular inequality for p) we obtain

$$\begin{aligned}
p(x_n, y_m) &- p(x_{n'}, y_{m'}) \\
&\leq (p(x_n, x_{n'}) - p(x_{n'}, x_{n'}) + p(x_{n'}, y_m)) - p(x_{n'}, y_{m'}) \\
&\leq \varepsilon + p(x_{n'}, y_m) - p(x_{n'}, y_{m'}) \\
&\leq \varepsilon + (p(x_{n'}, y_{m'}) - p(y_{m'}, y_{m'}) + p(y_{m'}, y_m)) - p(x_{n'}, y_{m'}) \\
&\leq \varepsilon + p(x_{n'}, y_{m'}) + \varepsilon - p(x_{n'}, y_{m'}) \\
&\leq 2\varepsilon;
\end{aligned}$$

similarly (and simultaneously) $p(x_{n'}, y_{m'}) - p(x_n, y_m) \leq 2\varepsilon$ too. This tells us that $|p(x_n, y_m) - p(x_{n'}, y_{m'})| \leq 2\varepsilon$, for all $n, m, n', m' \geq N$, which establishes Cauchyness of the net. □

In particular, if $(x_n)_n$ is a Cauchy sequence in a finitely typed generalised partial metric space (X, p) then, for every $y \in X$, both $(p(y, x_n))_n$ and $(p(x_n, y))_n$ are Cauchy sequences in $[0, \infty]$, and therefore converge. This guarantees the existence of the limits in the statement of the next theorem.

Theorem 5.3.6 *Let (X, p) be a finitely typed generalised partial metric space, and $\mathbb{X}$ the corresponding $\mathcal{D}(R)_{\mathsf{nz}}$-category (with $\mathbb{X}_0 = X$, $tx = p(x, x)$ and $\mathbb{X}(y, x) = p(x, x)$, as always). If $(x_n)_n$ is a Cauchy sequence in (X, p), and we put $q = \lim_{n,m\to\infty} p(x_n, x_m)$, then*

$$\phi : \mathbb{1}_q \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{X} \text{ with elements } \phi(y) = \lim_{n\to\infty} p(y, x_n)$$

is a Cauchy presheaf of finite type, whose right adjoint is

$$\psi : \mathbb{X} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{1}_q \text{ with elements } \psi(y) = \lim_{n\to\infty} p(x_n, y).$$

This correspondence is bijective between equivalence classes of Cauchy sequences on the one hand, and Cauchy distributors of finite type on the other. Moreover, a Cauchy sequence converges (to $x \in X$) if and only if the corresponding Cauchy distributor is representable (by $x \in X$).

Proof. First we verify (15) to make sure that $\phi\colon \mathbb{1}_q \mathrel{\circ\!\!\to} \mathbb{X}$ is a well-defined presheaf on $\mathbb{X}$. Because p is a partial metric we certainly have $p(y,y) \vee p(x_n,x_n) \leq p(y,x_n) \leq p(y,x) - p(x,x) + p(x,x_n)$ for all n. Letting n go to ∞, we therefore find that $p(y,y) \vee q \leq \phi(y) \leq p(y,x) - p(x,x) + \phi(x)$, as required. A similar reasoning holds to verify (16) for ψ.

To show that $\phi \dashv \psi$, we have to verify (17); applied to the case at hand, this means that

$$\bigwedge_{z\in X} \lim_{n\to\infty} p(x_n,z) - p(z,z) + \lim_{n\to\infty} p(z,x_n) \leq q$$

$$\text{and} \quad p(y,x) \leq \lim_{n\to\infty} p(y,x_n) - q + \lim_{n\to\infty} p(x_n,x)$$

for all $x, y \in X$. To see the first inequality, let $\varepsilon > 0$. Since $(x_n)_n$ is a Cauchy sequence of type q in (X,p), there is a natural number N so that for any $n \geq N$

$$p(x_n,x_N) \leq q+\varepsilon, \quad p(x_N,x_n) \leq q+\varepsilon \quad \text{and} \quad q-\varepsilon \leq p(x_N,x_N).$$

Therefore

$$p(x_n,x_N) - p(x_N,x_N) + p(x_N,x_n) \leq q + 3\varepsilon,$$

and the assertion follows by choosing ε arbitrarily small and letting n go to ∞. To show the second inequality, for $\varepsilon > 0$ let N be a natural number so that, for all $n, m \geq N$,

$$p(x_n,x_m) - p(x_n,x_n) \leq \varepsilon \quad \text{and} \quad q-\varepsilon \leq p(x_m,x_m)$$

for all $n, m \geq N$. It then follows that

$$\begin{aligned} p(y,x) &\leq p(y,x_n) - p(x_n,x_n) + p(x_n,x_m) - p(x_m,x_m) + p(x_m,y) \\ &\leq p(y,x_n) + \varepsilon - p(x_m,x_m) + p(x_m,y) \\ &\leq p(y,x_n) + \varepsilon - (q-\varepsilon) + p(x_m,y) \\ &\leq p(y,x_n) - q + p(x_m,y) + 2\varepsilon. \end{aligned}$$

Choosing ε arbitrary small and letting n and m go to ∞, this proves the point.

If $(y_n)_n$ is a Cauchy sequence with $(y_n)_n \sim (x_n)_n$, then we have in particular that $\lim_{n\to\infty} p(y_n,y_n) = \lim_{n\to\infty} p(y_n,x_n)$. For any $z \in X$ we know that

$$p(z,x_n) \leq p(z,y_n) - p(y_n,y_n) + p(y_n,x_n),$$

and therefore $\lim_{n\to\infty} p(z,x_n) \leq \lim_{n\to\infty} p(z,y_n)$. A similar argument shows the reverse inequality, which proves that both sequences define the same Cauchy distributor. Conversely, if two Cauchy sequences $(x_n)_n$ and $(y_n)_n$ induce the same Cauchy distributor $\phi : \mathbb{1}_q \mathrel{\circ\!\!\to} \mathbb{X}$, with right adjoint $\psi : \mathbb{X} \mathrel{\circ\!\!\to} \mathbb{1}_q$, then they are of the same type q. Moreover, for every $\varepsilon > 0$, there exist some natural number N so that, for all $n \geq N$

$$q \leq \phi(x_n) = \lim_{m\to\infty} p(x_n, x_m) \leq q + \varepsilon \text{ and } q \leq \psi(y_n) = \lim_{m\to\infty} p(y_m, y_n) \leq q + \varepsilon.$$

Hence, $p(x_n, x_n) \leq p(x_n, y_n) \leq \phi(x_n) - q + \psi(y_n) \leq q + 2\varepsilon$, for all $n \geq N$, which proves $(x_n)_n \sim (y_n)_n$.

Let now $\phi \dashv \psi$ with $\phi : \mathbb{1}_q \mathrel{-\!\!\circ\!\!\!\to} \mathbb{X}$ and $\psi : \mathbb{X} \mathrel{-\!\!\circ\!\!\!\to} \mathbb{1}_q$ (for some $q \in [0, \infty[$). Thanks to the first inequality in (17) we can pick, for every natural number n, an element $x_n \in X$ so that

$$\phi(x_n) - p(x_n, x_n) + \psi(x_n) \leq q + \frac{1}{n}.$$

But (15) and (16) say that $p(x_n, x_n) \vee q \leq \phi(x_n) \wedge \psi(x_n)$, so we find

$$\begin{aligned} &p(x_n, x_n) \leq q + \frac{1}{n} \quad \text{and} \quad q \leq p(x_n, x_n) + \frac{1}{n}, \\ &p(x_n, x_n) \leq \phi(x_n) \leq p(x_n, x_n) + \frac{1}{n}, \\ &\text{and} \quad p(x_n, x_n) \leq \psi(x_n) \leq p(x_n, x_n) + \frac{1}{n}, \end{aligned}$$

which implies that

$$q = \lim_{n\to\infty} p(x_n, x_n) = \lim_{n\to\infty} \phi(x_n) = \lim_{n\to\infty} \psi(x_n). \tag{18}$$

By the second inequality in (17) we know that

$$p(x_n, x_n) \leq p(x_n, x_m) \leq \phi(x_n) - q + \psi(x_m)$$

for all n and m, so with (18) we obtain $\lim_{n,m\to\infty} p(x_n, x_m) = q$, and we proved $(x_n)_n$ to be a Cauchy sequence in (X, p). Finally, this Cauchy sequence in turn determines the Cauchy presheaf it was constructed from: because from (15) and (16) we get

$$\phi(x) \leq p(x, x_n) - p(x_n, x_n) + \phi(x_n) \text{ and } \psi(x) \leq \psi(x_n) - p(x_n, x_n) + p(x_n, x)$$

for all $x \in X$ and all natural numbers n, and with (18) we find that

$$\phi \leq \lim_{n\to\infty} p(-, x_n) \quad \text{and} \quad \psi \leq \lim_{n\to\infty} p(x_n, -)$$

too; and these inequalities are equalities because $\phi \dashv \psi$ and (as attested by the first part of this proof) $\lim_{n\to\infty} p(-, x_n) \dashv \lim_{n\to\infty} p(x_n, -)$. □

Combining the above Theorem 5.3.6 with the remarks in Subsection 4.3, we arrive at the following conclusions.

Corollary 5.3.7 *A generalised partial metric space (X,p) is categorically Cauchy complete (meaning that every Cauchy distributor on (X,p) qua $\mathcal{D}(R)$-enriched category is representable) if and only if the finitely typed part of (X,p) is sequentially Cauchy complete (meaning that every Cauchy sequence in (X,p) converges) and (X,p) has at least one point of type ∞.*

Especially Proposition 4.3.2 helps us with:

Example 5.3.8 The Cauchy completion of a generalised partial metric space (X,p) viewed as a $\mathcal{D}(R)$-category $\mathbb{X}$, is the sum in $\mathsf{Cat}(\mathcal{D}(R))$ of the Cauchy completion *qua* $\mathcal{D}(R)_{\mathsf{nz}}$*-enriched category* of $\mathbb{X}_{\mathsf{nz}}$ plus a singleton of type ∞:

$$\mathbb{X}_{\mathsf{cc}} = (\mathbb{X}_{\mathsf{nz}})_{\mathsf{cc}} + \mathbb{1}_{\infty}.$$

But $\mathbb{X}_{\mathsf{nz}}$ is exactly the finitely typed part of (X,p), and we know by Theorem 5.3.6 that the finitely typed Cauchy presheaves on the finitely typed part of (X,p) are in one-to-one correspondence with equivalence classes of Cauchy sequences. Therefore, the Cauchy completion of (X,p) has as elements the equivalence classes of Cauchy sequences in the finitely typed part of (X,p), plus an extra point which we shall denote by ∞, and comes with the partial metric defined by

$$p(\infty,[(x_n)_n]) = p([(x_n)_n],\infty) = p(\infty,\infty) = \infty$$

and

$$p([(x_n)_n],[(y_n)_n]) = \bigwedge_{z\in X} \lim_{n\to\infty} p(x_n,z) - p(z,z) + \lim_{n\to\infty} p(z,y_n) = \lim_{n\to\infty} p(x_n,y_n).$$

Indeed, the first equality in the line above is exactly the formula for the hom-arrow in $\mathbb{X}_{\mathsf{cc}}$ between the corresponding Cauchy distributors; the second equality can be proven as follows. Thanks to Lemma 5.3.5 we know that $(p(x_n,y_m)_{(n,m)})$ is a Cauchy net in $[0,\infty]$, so it converges, and therefore so does the subnet $(p(x_n,y_n)_n)$; so we may put $q = \lim_{n\to\infty} p(x_n,y_n)$. Since we always have

$$p(x_n,y_n) \leq p(x_n,z) - p(z,z) + p(z,y_n)$$

we can let n go to ∞, and then take the infimum over z, to see that the "$\geq$" in the second equality always holds. For the "$\leq$", let $\varepsilon > 0$. Since both $(p(x_n,x_m))_{(n,n)}$ and $(p(x_n,y_m))_{(n,m)}$ are Cauchy nets in $[0,\infty]$ (as, again, attested by Lemma 5.3.5), there is some natural number N so that, for all $n \geq N$,

$$p(x_n,n_N) - p(x_N,x_N) \leq \varepsilon \quad \text{and} \quad p(x_N,y_n) \leq q+\varepsilon.$$

Therefore $\lim_{n\to\infty} p(x_n,x_N) - p(x_N,x_N) + \lim_{n\to\infty} p(x_N,y_n) \leq q + 2\varepsilon$, and the assertion follows.

The above results for partial metric spaces of course apply to metric spaces too—and *almost* produce the "usual" results. Indeed, a Cauchy sequence in a (generalised) metric space (X, d) in the sense of Definition 5.3.3 is exactly a Cauchy sequence in the usual sense; and it converges in (X, d) qua partial metric if and only if it does so in (X, d) qua metric. Put differently, a Cauchy distributor on (X, d) qua R-category is neither more nor less than a Cauchy distributor on (X, d) qua $\mathcal{D}(R)$-category of type 0 (because the type of a Cauchy presheaf $\phi = \lim_{n\to\infty} d(-, x_n)$ on $\mathbb{X} = (X, d)$ is necessarily $\lim_{n\to\infty} d(x_n, x_n) = 0$); and it is representable qua R-enriched distributor if and only if it is qua $\mathcal{D}(R)$-enriched distributor. However, the Cauchy completion of (X, d) qua metric space does not create that "extra point at infinity", which the Cauchy completion of (X, d) qua *partial* metric space always does!

5.4 Hausdorff distance, exponentiability

In [21] we developed a general theory of 'Hausdorff distance' for quantaloid-enriched categories; applied to the quantaloid $\mathcal{D}(R)$ this produces the following results for partial metrics.

Example 5.4.1 The **Hausdorff space** $\mathcal{H}(X, p) = (\mathcal{H}X, p_{\mathcal{H}})$ of a generalised partial metric space (X, p) is the new generalised partial metric space with elements

$$\mathcal{H}X = \{S \subseteq X \mid \forall x, x' \in S\colon p(x, x) = p(x', x')\}$$

(i.e. the **typed subsets** of X) and partial distance

$$p_{\mathcal{H}}(T, S) = \bigvee_{t\in T} \bigwedge_{s\in S} p(t, s). \tag{19}$$

The inclusion $(X, p) \to \mathcal{H}(X, p)\colon x \mapsto \{x\}$ is the unit for the so-called **Hausdorff doctrine** $\mathcal{H}\colon \mathsf{GPMet} \to \mathsf{GPMet}$, and as such enjoys a universal property: it is the universal conical cocompletion (see [21, Section 5]).

The naive extension of the formula in (19) to *arbitrary* subsets of (X, p) fails to produce a partial metric, for the following reason. Suppose a and b are elements of a partial metric space (X, p), with $p(a, a) < p(b, b)$. Then $\{a, b\}$ is *not* a typed subset of X, but if we nevertheless use the sup-inf formula we find in particular that

$$p_{\mathcal{H}}(\{a\}, \{a, b\}) = p(a, a),\ p_{\mathcal{H}}(\{a, b\}, \{b\}) = p(a, b),$$

$$p_{\mathcal{H}}(\{a\}, \{b\}) = p(a, b),\ p_{\mathcal{H}}(\{a, b\}, \{a, b\}) = p(b, b).$$

We have $p_{\mathcal{H}}(\{a\}, \{a, b\}) - p_{\mathcal{H}}(\{a, b\}, \{a, b\}) + p_{\mathcal{H}}(\{a, b\}, \{b\}) \not\geq p_{\mathcal{H}}(\{a\}, \{b\})$, so that $p_{\mathcal{H}}$ fails to be a partial metric.

We gave a general characterisation of exponentiable quantaloid-enriched categories and functors in [4]; this specialises to the case of partial metric spaces as follows.

Example 5.4.2 A generalised partial metric space (X, p) is **exponentiable** in the (cartesian) category GPMet if and only if

$$\begin{aligned}&\text{for all } x_0, x_2 \in X \text{ and } u, v, w \in [0, \infty] \text{ such that}\\ &p(x_0, x_0) \vee v \leq u \text{ and } p(x_2, x_2) \vee v \leq w :\\ &\bigwedge \Big\{ (u \vee p(x_0, x_1)) - v + (w \vee p(x_1, x_2)) \mid x_1 \in X, p(x_1, x_1) = v \Big\}\\ &\qquad\qquad = (u - v + w) \vee p(x_0, x_2).\end{aligned} \tag{20}$$

This literal application of the very general Theorem 1.1 of [4] (but see also Section 5 of that paper) to the specific quantaloid $\mathcal{D}(R)$ can be simplified somewhat. First, using the triangular inequality for the partial metric, it is straightforward to verify that the "$\geq$" in (20) always holds. Second, the "$\leq$" is trivially satisfied whenever either of $p(x_0, x_2)$, u or w is ∞ (because the right hand side is then ∞); because $p(x_0, x_0) \vee p(x_2, x_2) \leq p(x_0, x_2)$ we may also exclude the cases where either $p(x_0, x_0)$ or $p(x_2, x_2)$ is ∞; and because $v \leq u \wedge w$ (in the hypotheses) we may exclude the case $v = \infty$. The above condition thus becomes:

$$\begin{aligned}&\text{for all } x_0, x_2 \in X \text{ and } u, v, w \in [0, \infty[\text{ such that}\\ &p(x_0, x_2) < \infty,\ p(x_0, x_0) \vee v \leq u \text{ and } p(x_2, x_2) \vee v \leq w :\\ &\bigwedge \Big\{ (u \vee p(x_0, x_1)) - v + (w \vee p(x_1, x_2)) \mid x_1 \in X, p(x_1, x_1) = v \Big\}\\ &\qquad\qquad \leq (u - v + w) \vee p(x_0, x_2).\end{aligned} \tag{21}$$

It actually suffices to check *this* condition only when $p(x_0, x_2) \leq u - v + w$. Indeed, whenever $u - v + w < p(x_0, x_2)$ we may apply this (hypothetically valid) condition on $u' - v + w = p(x_0, x_2)$ for the appropriate $u' \geq u$ in the first inequality below, to find that

$$\begin{aligned}(u - v + w) \vee p(x_0, x_2) &= (u' - v + w) \vee p(x_0, x_2)\\ &\geq \bigwedge \{ (u' \vee p(x_0, x_1)) - v + (w \vee p(x_1, x_2)) \mid x_1 \in X, p(x_1, x_1) = v \}\\ &\geq \bigwedge \{ (u \vee p(x_0, x_1)) - v + (w \vee p(x_1, x_2)) \mid x_1 \in X, p(x_1, x_1) = v \}\end{aligned}$$

anyway. But for $p(x_0, x_2) \leq u - v + w$, the inequality in (21) is further equivalent to

$$\bigwedge \{ (u \vee p(x_0, x_1)) + (w \vee p(x_1, x_2)) \mid x_1 \in X, p(x_1, x_1) = v \} \leq u + w$$

since $v \leq u + w < \infty$ and $\{x_1 \in X \mid p(x_1, x_1) = v\}$ cannot be empty. Therefore we finally find that a generalised partial metric space (X, p) is exponentiable in GPMet if and only if

$$\begin{array}{l}\text{for all } x_0, x_2 \in X,\ u, v, w \in [0, \infty[\text{ and } \varepsilon > 0 \text{ such that} \\ p(x_0, x_2) \leq u - v + w,\ p(x_0, x_0) \vee v \leq u \text{ and } p(x_2, x_2) \vee v \leq w \\ \text{there exists } x_1 \in X \text{ such that} \\ p(x_1, x_1) = v,\ p(x_0, x_1) \leq u + \varepsilon \text{ and } p(x_1, x_2) \leq w + \varepsilon.\end{array} \tag{22}$$

This immediately implies that an exponentiable partial metric space is either empty, or has all distances equal to ∞, or has for every $r \in [0, \infty[$ at least one element with self-distance r. In particular a generalised metric space (X, d) is exponentiable in GPMet if and only if it is empty (even though a non-empty (X, d) may still be exponentiable in GMet!).

Furthermore, with the same proof as in [11, Theorem 5.3 and Corollary 5.4], we obtain that every injective partial metric space (in particular, every partial metric obtained from the presheaf construction in $\mathsf{GPMet} = \mathsf{Cat}(\mathcal{D}(R))$, see Subsection 2.3) is exponentiable; moreover, the full subcategory of GPMet defined by all injective partial metric spaces is Cartesian closed.

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ON ENRICHED FIBRATIONS

Christina VASILAKOPOULOU

Résumé. Nous introduisons la notion de fibration enrichie, à savoir une fibration dont la catégorie totale et la catégorie de base sont enrichies d'une manière appropriée dans celles d'une fibration. De plus, nous proposons un moyen d'obtenir une telle structure, à partir des actions de catégories monoïdales avec des adjoints paramétrés. L'objectif est de capturer certaines classes d' exemples, comme la fibration des modules sur des algèbres enrichie dans l'opfibration de comodules sur des coalgèbres.

Abstract. We introduce the notion of an enriched fibration, i.e. a fibration whose total category and base category are enriched in those of a monoidal fibration in an appropriate way. Furthermore, we provide a way to obtain such a structure, starting from actions of monoidal categories with parameterized adjoints. The motivating goal is to capture certain example cases, like the fibration of modules over algebras enriched in the opfibration of comodules over coalgebras.

Keywords. monoidal category, fibration, enriched category, parameterized adjunction.

Mathematics Subject Classification (2010). 18D10,18D20,18D30.

1. Introduction

Enriched category theory [Kel05], as well as the theory of fibrations [Gro61], have both been of central importance to developments in many contexts. Both are classical theories for formal category theory; however, they do not seem to 'go together' in some evident way.

The goal of the present work is to introduce a notion of an *enriched fibration*. This should combine elements of both concepts in an appropriate and natural way; the enriched structure of a category cannot really be internalized in order to provide a definite answer. In any case, 'being enriched in' and 'being internal to' are two major but separate generalizations of ordinary category theory, whereas 'being fibred over' is often considered as a third one.

More explicitly, we would like to characterize a fibration as being enriched in some special kind of fibration, serving similar purposes as the monoidal base of usual enriched categories; this has already been identified as a *monoidal fibration* [Shu08]. For the desired enriched fibration definition, there are two main factors that determine its relevance. First of all, it should be able to adequately capture certain cases that first arose in [Vas14] and furthermore studied in [HLFV17a, HLFV17b, Vas17], and in fact motivated these explorations. Further details of these examples and how they ultimately fit in the described framework can be found in Section 4. The original driving example case is the enrichment of algebras in coalgebras via Sweedler's *measuring coalgebra* construction [Swe69], together with the enrichment of a global category of modules in comodules; the latter categories are respectively fibred and opfibred over algebras and coalgebras. This also extends to their many-object generalizations, namely (enriched) categories and cocategories and their (enriched) modules and comodules. These cases can be roughly depicted as

$$\begin{tikzcd}
\mathbf{Mod} \arrow[r, dashed, "\text{enriched}"] \arrow[d, "\text{fibred}"'] & \mathbf{Comod} \arrow[d, "\text{opfibred}"] \\
\mathbf{Alg}(\mathcal{V}) \arrow[r, dashed, "\text{enriched}"'] & \mathbf{Coalg}(\mathcal{V})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Mod} \arrow[r, dashed, "\text{enriched}"] \arrow[d, "\text{fibred}"'] & \mathcal{V}\text{-}\mathbf{Comod} \arrow[d, "\text{opfibred}"] \\
\mathcal{V}\text{-}\mathbf{Cat} \arrow[r, dashed, "\text{enriched}"'] & \mathcal{V}\text{-}\mathbf{Cocat}.
\end{tikzcd}$$

Secondly, the introduced enriched fibration concept should theoretically constitute an as-close-as-possible fibred analogue of the usual enrichment of categories. In order to initiate such an effort, we provide a theorem which under certain assumptions ensures the existence of such a structure. This theorem lifts a standard result, which combines the theory of *actions* of monoidal categories and parameterized adjunctions to produce an enrichment [GP97, JK02], to the fibred context.

Notably, a strongly related notion called *enriched indexed category* has been studied, from a slightly different point of view, originally in [Bun13] and also independently in [Shu13]. However, the main definitions and constructions diverge from the ones presented here. We postpone a short discussion on these differences until the very end of the paper, Section 4.3.

Finally, it should be indicated that this paper deliberately includes only what is necessary to first of all sufficiently describe the examples at hand. It elaborates on and extends a sketched narrative from [Vas14, §8.1], and provides the first steps in such a research direction. Future work may build on the current development, towards a theory of enriched fibrations and related structures.

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2. Background

In this section, we recall some basic definitions and known results which serve as background material in what follows, and we also fix terminology.

2.1 Monoidal categories, actions and enrichment

We assume familiarity with the basics of monoidal categories, see [JS93, ML98]. A monoidal category is denoted by $(\mathcal{V}, \otimes, I)$ with associator and left and right unit constraints a, ℓ, r. A lax monoidal structure on a functor $F\colon \mathcal{V} \to \mathcal{W}$ between monoidal categories is denoted by (ϕ, ϕ_0), with components $\phi_{AB}\colon FX \otimes FY \to F(X \otimes Y)$ and $\phi_0\colon I \to FI$ satisfying usual axioms. If these are isomorphisms/identities, this is a strong/strict monoidal structure.

A (left) monoidal closed category is one where the functor $(-\otimes X)$ has a right adjoint $[X, -]$, for all objects X. This induces the internal hom functor

$[-,-]\colon \mathcal{V}^{\mathrm{op}}\times\mathcal{V}\to\mathcal{V}$, as a result of the classic *parameterized adjunctions* theorem [ML98, §IV.7.3]:

Theorem 2.1. *Suppose that, for a functor of two variables $F:\mathcal{A}\times\mathcal{B}\to\mathcal{C}$, there exists an adjunction*

$$\begin{tikzcd}\mathcal{A} \arrow[r, shift left, "{F(-,B)}"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left, "{G(B,-)}"]\end{tikzcd}$$

for each $B\in\mathcal{B}$, with an isomorphism $\mathcal{C}(F(A,B),C)\cong\mathcal{A}(A,G(B,C))$ natural in A and C. Then, there is a unique way to make G into a functor of two variables $\mathcal{B}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{A}$ for which the isomorphism is natural also in B.

The functor G is called the *(right) parameterized adjoint* of F, and we denote this as $F\dashv_p G$. In particular, $\otimes\dashv_p[-,-]$ in any monoidal (left) closed category. We could also decide to fix the other parameter, and have that $F(A,-)\dashv H(A,-)$ for $H\colon\mathcal{A}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{B}$. For a 2-categorical proof and generalizations, see [CGR14].

We now recall some basics of the theory of actions of monoidal categories, [JK02].

Definition 2.2. *A* (left) action *of a monoidal category $\mathcal{V}$ on a category $\mathcal{D}$ is given by a functor $*\colon\mathcal{V}\times\mathcal{D}\to\mathcal{D}$ along with two natural isomorphisms χ,ν with components*

$$\chi_{XYD}\colon(X\otimes Y)*D\xrightarrow{\sim}X*(Y*D),\qquad \nu_D\colon I*D\xrightarrow{\sim}D \tag{1}$$

satisfying the commutativity of

$$\begin{tikzcd}((X\otimes Y)\otimes Z)*D \arrow[r, "\chi"] \arrow[d, "a*1"'] & (X\otimes Y)*(Z*D) \arrow[r, "\chi"] & X*(Y*(Z*D)) \\ (X\otimes(Y\otimes Z))*D \arrow[rr, "\chi"'] & & X*((Y\otimes Z)*D) \arrow[u, "1*\chi"']\end{tikzcd} \tag{2}$$

$$\begin{tikzcd}(I\otimes X)*D \arrow[rr, "\chi"] \arrow[dr, "l*1"'] & & I*(X*D) \arrow[dl, "\nu"] \\ & X*D & \end{tikzcd}\qquad\begin{tikzcd}(X\otimes I)*D \arrow[rr, "\chi"] \arrow[dr, "r*1"'] & & X*(I*D) \arrow[dl, "1*\nu"] \\ & X*D & \end{tikzcd}$$

The category $\mathcal{D}$ is called a $\mathcal{V}$-representation*, or a* $\mathcal{V}$-actegory *[McC00].*

For example, every monoidal category has a canonical action on itself via its tensor product, $\otimes = *\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, and $\chi = a, \nu = \ell$; it is called the *regular* $\mathcal{V}$-representation. Moreover, for any monoidal closed category, its internal hom constitutes an action of the monoidal $\mathcal{V}^{\mathrm{op}}$ (with the same tensor product $\otimes^{\mathrm{op}}$) on $\mathcal{V}$, via the standard natural isomorphisms

$$\chi_{XYZ} : [X \otimes Y, D] \xrightarrow{\sim} [X, [Y, Z]], \quad \nu_D : [I, D] \xrightarrow{\sim} D$$

which satisfy (2) using the transpose diagrams under the tensor-hom adjunction.

Familiarity with enrichment theory is also assumed, see [Kel05]. We denote the 2-category of $\mathcal{V}$-enriched categories, along with enriched functors and enriched natural transformations, $\mathcal{V}$-$\mathbf{Cat}$; we call $\mathcal{V}$ the monoidal base of the enrichment. If $\mathcal{A}$ is a $\mathcal{V}$-enriched category with hom-objects $\mathcal{A}(A, B) \in \mathcal{V}$, we will write $j_A\colon I \to \mathcal{A}(A, A)$ for its identites and $M_{ABC}\colon \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \to \mathcal{A}(B, C)$ for the composition. Its *underlying category* $\mathcal{A}_0$ has the same objects, while morphisms $f\colon A \to B$ in $\mathcal{A}_0$ are just 'elements' $f\colon I \to \mathcal{A}(A, B)$ in $\mathcal{V}$, i.e. $\mathcal{A}_0(A, B) = \mathcal{V}(I, \mathcal{A}(A, B))$ as sets. In fact, we can define a functor

$$\mathcal{A}(-,-)\colon \mathcal{A}_0^{\mathrm{op}} \times \mathcal{A}_0 \to \mathcal{V} \tag{3}$$

called the *enriched hom-functor*, which maps (A, B) to $\mathcal{A}(A, B)$, and a pair of arrows $(A' \xrightarrow{f} A, B \xrightarrow{g} B')$ in $\mathcal{A}_0^{\mathrm{op}} \times \mathcal{A}_0$ to the top arrow

$$\begin{tikzcd}
\mathcal{A}(A,B) \arrow[rr, dashed, "\mathcal{A}(f,g)"] \arrow[d, "r^{-1}"'] & & \mathcal{A}(A',B') \\
\mathcal{A}(A,B) \otimes I \arrow[d, "1 \otimes f"'] & & \mathcal{A}(B,B') \otimes \mathcal{A}(A',B) \arrow[u, "M"'] \\
\mathcal{A}(A,B) \otimes \mathcal{A}(A',A) \arrow[r, "M"'] & \mathcal{A}(A',B) \arrow[r, "l^{-1}"'] & I \otimes \mathcal{A}(A',B) \arrow[u, "g \otimes 1"']
\end{tikzcd}$$

Speaking loosely, we say that an ordinary category $\mathcal{C}$ is enriched in a monoidal category $\mathcal{V}$ when we have a $\mathcal{V}$-enriched category $\mathcal{A}$ (often denoted $\underline{\mathcal{C}}$) and an isomorphism $\mathcal{A}_0 \cong \mathcal{C}$. Consequently, *to be enriched in* $\mathcal{V}$ is not a property, but additional structure. Of course, a given ordinary category may be enriched in more than one monoidal category; this is evident in view of Proposition 2.3. But also, a category $\mathcal{C}$ may be enriched in $\mathcal{V}$ in more than one way.

Proposition 2.3 (Change of Base)**.** *Suppose $F:\mathcal{V}\to\mathcal{W}$ is a lax monoidal functor between two monoidal categories. There is an induced 2-functor*

$$\widetilde{F}\colon \mathcal{V}\text{-}\mathbf{Cat}\longrightarrow\mathcal{W}\text{-}\mathbf{Cat}$$

between the 2-categories of $\mathcal{V}$ and $\mathcal{W}$-enriched categories, which maps any $\mathcal{V}$-category $\mathcal{A}$ to a $\mathcal{W}$-category with the same objects as $\mathcal{A}$ and hom-objects $F\mathcal{A}(A,B)$.

Sketch of proof. On the level of objects, the composition and identities are given by

$$\begin{tikzcd}
F\mathcal{A}(B,C)\otimes F\mathcal{A}(A,B) \arrow[r, dashed] \arrow[d, "\phi_{\mathcal{A}(B,C),\mathcal{A}(A,B)}"'] & F\mathcal{A}(A,C) \\
F(\mathcal{A}(B,C)\otimes\mathcal{A}(A,B)) \arrow[ur, "FM_{ABC}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I_{\mathcal{W}} \arrow[r, dashed] \arrow[d, "\phi_0"'] & F\mathcal{A}(A,A) \\
FI_{\mathcal{V}} \arrow[ur, "Fj_A"'] &
\end{tikzcd}$$

□

A crucial result for what follows is that given a category $\mathcal{D}$ with an action from a monoidal category $\mathcal{V}$ with a parameterized adjoint, we obtain a $\mathcal{V}$-enriched category.

Theorem 2.4. *Suppose that $\mathcal{V}$ is a monoidal category which acts on a category $\mathcal{D}$ via a functor $*:\mathcal{V}\times\mathcal{D}\to\mathcal{D}$, such that $-*D$ has a right adjoint $F(D,-)$ for every $D\in\mathcal{D}$. Then we can enrich $\mathcal{D}$ in $\mathcal{V}$, with hom-objects $\underline{\mathcal{D}}(A,B)=F(A,B)$.*

The proof and further details can be found in [JK02] or [Vas14, § 4.3]. Briefly, due to the adjunction $-*D\dashv F(D,-)$, we have natural isomorphisms

$$\mathcal{D}(X*D,E)\cong\mathcal{V}(X,F(D,E)) \tag{4}$$

which give rise to a functor $F\colon\mathcal{D}^{\mathrm{op}}\times\mathcal{D}\to\mathcal{V}$ by Theorem 2.1. This serves as the enriched hom-functor of the induced enrichment of $\mathcal{D}$ in $\mathcal{V}$: we can define a composition law $F(B,C) \otimes F(A,B) \to F(A,C)$ as the adjunct of the composite

$$\begin{tikzcd}
(F(B,C) \otimes F(A,B)) * A \arrow[r, "\chi"] \arrow[ddr, dashed] & F(B,C) * (F(A,B) * A) \arrow[d, "1*\varepsilon_B"] \\
& F(B,C) * B \arrow[d, "\varepsilon_C"] \\
& C
\end{tikzcd} \tag{5}$$

and identities $I \to F(A,A)$ as the adjuncts of

$$I * A \xrightarrow{\nu} A \tag{6}$$

where χ and ν give the action structure (1) and ε is the counit of the adjunction. The associativity and identity axioms for the enrichment can be established using the action axioms. Finally, $\underline{\mathcal{D}}_0 \cong \mathcal{D}$ since they have the same objects, and

$$\underline{\mathcal{D}}_0(A,B) = \mathcal{V}(I, F(A,B)) \overset{(4)}{\cong} \mathcal{D}(I * A, B) \overset{\nu}{\cong} \mathcal{D}(A,B).$$

In fact, Theorem 2.4 gives part of one direction of an equivalence

$$\mathcal{V}\text{-Rep}_{\text{cl}} \simeq \mathcal{V}\text{-}\mathbf{Cat}_{\otimes}$$

between *closed* $\mathcal{V}$-representations (those with action equipped with a parameterized adjoint) and *tensored* $\mathcal{V}$-categories (those with specific weighted limits), for $\mathcal{V}$ a monoidal closed category. This equivalence, discussed in [JK02, §6], is a special case of the much more general [GP97, Theorem 3.7] for bicategory-enriched categories.

Remark 2.5. When $\mathcal{V}$ is monoidal closed, the regular action $\otimes\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ has a parameterized adjoint $[-,-]\colon \mathcal{V}^{\text{op}} \times \mathcal{V} \to \mathcal{V}$. We thus re-discover the well-known enrichment of a monoidal closed category in itself via the internal hom, as a direct application of Theorem 2.4.

2.2 Pseudomonoids and pseudomodules

Recall that a *monoidal 2-category* $(\mathcal{K},\otimes,I)$ is a 2-category equipped with a pseudofunctor $\otimes\colon \mathcal{K}\times\mathcal{K}\to\mathcal{K}$ and a unit $I\colon \mathbf{1}\to\mathcal{K}$ which are associative and unital up to coherence equivalence, see [GPS95].

Definition 2.6. *[DS97, §3] A* pseudomonoid *A in $\mathcal{K}$ is an object equipped with multiplication $m\colon A\otimes A\to A$ and unit $j\colon I\to A$ along with invertible 2-cells satisfying coherence conditions.*

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "1\otimes m"] \arrow[d, "m\otimes 1"'] \arrow[dr, phantom, "\overset{a}{\cong}"] & A\otimes A \arrow[d, "m"] \\
A\otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes I \arrow[r, "1\otimes j"] \arrow[dr, "\sim"'] \arrow[dr, phantom, "\overset{\ell}{\cong}", near end, shift left=3] & A\otimes A \arrow[d, "m"] & I\otimes A \arrow[l, "j\otimes 1"'] \arrow[dl, "\sim"] \arrow[dl, phantom, "\overset{r}{\cong}", near end, shift right=3] \\
 & A &
\end{tikzcd}
\tag{7}$$

The notion of a pseudomodule for a pseudomonoid in a monoidal 2-category (or bicategory) can be found in similar contexts [Mar97, Lac00]; conceptually, as it is the case for modules for monoids in a monoidal category, it arises as a pseudoalgebra for the pseudomonad $(A\otimes -)$ induced by a pseudomonoid A in $\mathcal{K}$.

Definition 2.7. *A (left) A-*pseudomodule *is an object M in $(\mathcal{K},\otimes,I)$ equipped with $\mu\colon A\otimes M\to M$ (the* pseudoaction*) and invertible 2-cells*

$$\begin{tikzcd}
A\otimes A\otimes M \arrow[r, "1\otimes\mu"] \arrow[d, "m\otimes 1"'] \arrow[dr, phantom, "\overset{\chi}{\cong}"] & A\otimes M \arrow[d, "\mu"] \\
A\otimes M \arrow[r, "\mu"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
I\otimes M \arrow[r, "j\otimes 1"] \arrow[dr, "\sim"'] \arrow[dr, phantom, "\overset{\nu}{\cong}", near end, shift left=3] & A\otimes M \arrow[d, "\mu"] \\
 & A
\end{tikzcd}
\tag{8}$$

satisfying coherence conditions.

Example 2.8. As a fundamental example of a (cartesian) monoidal 2-category, consider $\mathbf{Cat}$ equipped with the 2-functor $\times : \mathbf{Cat}\times\mathbf{Cat}\to\mathbf{Cat}$ and $\mathcal{I}$ the unit category. It is a standard fact that a pseudomonoid therein is precisely a monoidal category $(\mathcal{V},\otimes,I,a,\ell,r)$. Moreover, an action of a monoidal category $\mathcal{V}$ on an ordinary category $\mathcal{A}$ as defined in Definition 2.2 is precisely a $\mathcal{V}$-pseudoaction inside $(\mathbf{Cat},\times,\mathcal{I})$, exhibiting $\mathcal{A}$ as a $\mathcal{V}$-pseudomodule.

2.3 Fibrations and adjunctions

We now briefly recall some basic concepts and constructions from the theory of fibred categories. Relevant references for what follows are [Bor94, Her94].

Consider a functor $P : \mathcal{A} \to \mathbb{X}$. A morphism $\phi : A \to B$ in $\mathcal{A}$ over a morphism $f = P(\phi) : X \to Y$ in $\mathbb{X}$ is called *cartesian* if and only if, for all $g : X' \to X$ in $\mathbb{X}$ and $\theta : A' \to B$ in $\mathcal{A}$ with $P\theta = f \circ g$, there exists a unique arrow $\psi : A' \to A$ such that $P\psi = g$ and $\theta = \phi \circ \psi$:

$$\begin{tikzcd}
A' \arrow[rrd, "\theta"] \arrow[rd, dashed, "\exists!\psi"'] \arrow[dd, dotted] & & & \\
& A \arrow[r, "\phi"'] \arrow[dd, dotted] & B \arrow[dd, dotted] & \text{in } \mathcal{A} \\
X' \arrow[rrd, "f\circ g=P\theta"] \arrow[rd, "g"'] & & & \\
& X \arrow[r, "f=P\phi"'] & Y & \text{in } \mathbb{X}
\end{tikzcd}$$

For $X \in \mathrm{ob}\mathbb{X}$, the *fibre of P over X* written $\mathcal{A}_X$, is the subcategory of $\mathcal{A}$ which consists of objects A such that $P(A) = X$ and morphisms ϕ with $P(\phi) = 1_X$, called *vertical* morphisms. The functor $P : \mathcal{A} \to \mathbb{X}$ is called a *fibration* if and only if, for all $f : X \to Y$ in $\mathbb{X}$ and $B \in \mathcal{A}_Y$, there is a cartesian morphism ϕ with codomain B above f; it is called a *cartesian lifting* of B along f. The category $\mathbb{X}$ is then called the *base* of the fibration, and $\mathcal{A}$ its *total* category.

Dually, the functor $U : \mathcal{C} \to \mathbb{X}$ is an *opfibration* if U^{op} is a fibration, *i.e.* for every $C \in \mathcal{C}_X$ and $g : X \to Y$ in $\mathbb{X}$, there is a cocartesian morphism with domain C above g, the *cocartesian lifting* of C along g.

If $P : \mathcal{A} \to \mathbb{X}$ is a fibration, assuming the axiom of choice we may select a cartesian arrow over each $f : X \to Y$ in $\mathbb{X}$ and $B \in \mathcal{A}_Y$, denoted by $\mathrm{Cart}(f, B) : f^*(B) \to B$. Such a choice of cartesian liftings is called a *cleavage* for P, which is then called a *cloven* fibration; any fibration is henceforth assumed to be cloven. Dually, if U is an opfibration, for any $C \in \mathcal{C}_X$ and $g : X \to Y$ in $\mathbb{X}$ we can choose a cocartesian lifting of C along g, $\mathrm{Cocart}(g, C) : C \longrightarrow g_!(C)$. The choice of (co)cartesian liftings in an (op)fibration induces a so-called *reindexing functor* between the fibre categories

$$f^* : \mathcal{A}_Y \to \mathcal{A}_X \quad \text{ and } \quad g_! : \mathcal{C}_X \to \mathcal{C}_Y$$

respectively, for each morphism $f : X \to Y$ and $g\colon X \to Y$ in the base category, mapping each object to the (co)domain of its lifting.

An *oplax morphism of fibrations* (or oplax fibred 1-cell) (S, F) between $P : \mathcal{A} \to \mathbb{X}$ and $Q : \mathcal{B} \to \mathbb{Y}$ is given by a commutative square of categories and functors

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "S"] \arrow[d, "P"'] & \mathcal{B} \arrow[d, "Q"] \\
\mathbb{X} \arrow[r, "F"'] & \mathbb{Y}
\end{tikzcd} \tag{9}$$

as in [Shu08, Def. 3.5]. If moreover S preserves cartesian arrows, meaning that if ϕ is P-cartesian then $S\phi$ is Q-cartesian, the pair (S, F) is called a *fibred 1-cell* or *strong morphism of fibrations*. Dually, we have the notion of an *lax morphism of opfibrations* (K, F), and *opfibred 1-cell* when K is cocartesian. Notice that any oplax fibred 1-cell (S, F) determines a collection of functors between the fibres $S_X\colon \mathcal{A}_X \to \mathcal{B}_{FX}$ as the restriction of S to the corresponding subcategories.

A *fibred 2-cell* between oplax fibred 1-cells (S, F) and (T, G) is a pair of natural transformations ($\alpha : S \Rightarrow T, \beta : F \Rightarrow G$) with α above β, *i.e.* $Q(\alpha_A) = \beta_{PA}$ for all $A \in \mathcal{A}$, displayed

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "S"{name=U}] \arrow[r, bend right, "T"'{name=D}] \arrow[d, "P"'] \arrow[phantom, from=U, to=D, "\Downarrow\alpha"] & \mathcal{B} \arrow[d, "Q"] \\
\mathbb{X} \arrow[r, bend left, "F"{name=U2}] \arrow[r, bend right, "G"'{name=D2}] \arrow[phantom, from=U2, to=D2, "\Downarrow\beta"] & \mathbb{Y}.
\end{tikzcd} \tag{10}$$

Notice that if the 1-cells are strong, the definition of a 2-cell between them remains the same. Dually, we have the notion of an *opfibred 2-cell* between (lax) opfibred 1-cells.

We obtain 2-categories $\mathbf{Fib}_{\text{opl}}$ and $\mathbf{Fib}$ of fibrations over arbitrary base categories, (oplax) fibred 1-cells and fibred 2-cells. Evidently, these are both subcategories of $\mathbf{Cat}^{\mathbf{2}}$. $\mathbf{Fib}_{\text{opl}}$ is a full sub-2-category of those objects which are fibrations, and $\mathbf{Fib}$ is the non-full sub-2-category whose morphism are commutative squares where the top functor is cartesian. Dually,

$\mathbf{OpFib} \subset \mathbf{OpFib}_{\text{lax}} \subset_{\text{full}} \mathbf{Cat}^{\mathbf{2}}$. These 2-categories are monoidal, inheriting the tensor product from $\mathbf{Cat}^{\mathbf{2}}$: the cartesian product of two fibrations is still a fibration. The unit is $1_{\mathcal{I}} \colon \mathcal{I} \to \mathcal{I}$, the identity on the terminal category.

Notice that the terminology for oplax morphisms of fibrations and lax morphisms of opfibrations is justified by a relaxed version of the fundamental equivalence between fibrations and pseudofunctors (Grothendieck construction). For more details, see [Shu08, Prop. 3.6].

We now turn to notions of adjunctions between fibrations, internally to any of the above 2-categories of (op)fibrations.

Definition 2.9. *Given fibrations $P : \mathcal{A} \to \mathbb{X}$ and $Q : \mathcal{B} \to \mathbb{Y}$, a* general (oplax) fibred adjunction $(L, F) \dashv (R, G)$ *is given by a pair of (oplax) fibred 1-cells $(L, F) : P \to Q$ and $(R, G) : Q \to P$ together with fibred 2-cells $(\zeta, \eta) : (1_{\mathcal{A}}, 1_{\mathbb{X}}) \Rightarrow (RL, GF)$ and $(\xi, \varepsilon) : (LR, FG) \Rightarrow (1_{\mathcal{B}}, 1_{\mathbb{Y}})$ such that $L \dashv R$ via ζ, ξ and $F \dashv G$ via η, ε. This is displayed as*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] \arrow[d, "P"'] & \mathcal{B} \arrow[l, shift left, "R"] \arrow[d, "Q"] \\
\mathbb{X} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{Y} \arrow[l, shift left, "G"]
\end{tikzcd}$$

Notice that by definition, ζ is above η and ξ is above ε, hence (P, Q) is an ordinary map between adjunctions. Dually, we have the notion of *general (lax) opfibred adjunction* in $\mathbf{OpFib}_{(\text{lax})}$.

The following result establishes certain (co)cartesian properties of adjoints.

Lemma 2.10. *[Win90, 4.5] Right adjoints in the 2-category $\mathbf{Cat}^{\mathbf{2}}$ preserve cartesian morphisms; dually left adjoints preserve cocartesian morphisms.*

Finally, in [HLFV17b, §3.2], conditions under which a fibred 1-cell has an adjoint are investigated in detail, and that proves very useful in determining enrichment relations in conjunction with Theorem 2.4. Here we recall a main result providing a general lax opfibred adjunction, with regards to the applications of Section 4.

Theorem 2.11. *Suppose $(K,F):U\to V$ is an opfibred 1-cell and $F\dashv G$ is an adjunction with counit ε between the bases of the opfibrations, as in*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "K"] \arrow[d, "U"'] & \mathcal{D} \arrow[d, "V"] \\
\mathbb{X} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{Y}. \arrow[l, shift left, "G"]
\end{tikzcd}$$

If, for each $Y\in\mathbb{Y}$, the composite functor $\mathcal{C}_{GY}\xrightarrow{K_{GY}}\mathcal{D}_{FGY}\xrightarrow{(\varepsilon_Y)_!}\mathcal{D}_Y$ between the fibres has a right adjoint for each $Y\in\mathbb{Y}$, then K has a right adjoint R between the total categories and $(K,F)\dashv(R,G)$ is an general oplax adjunction.

3. Enriched fibrations

This section's goal is to introduce a notion of an enriched fibration. It will do so in a way that an adjusted version of Theorem 2.4, instead of providing an enrichment of an ordinary category in a monoidal category, will give an enrichment of an ordinary fibration in a *monoidal* one. The key idea is to shift all necessary structure (Example 2.8) from the context of categories to fibrations, moving from $(\mathbf{Cat},\times,\mathcal{I})$ to the monoidal 2-category $(\mathbf{Fib},\times,1_{\mathcal{I}})$.

First of all, a pseudomonoid (Definition 2.6) in the 2-category of fibrations, which will serve as the base of the enrichment, is a fibration $T:\mathcal{V}\to\mathbb{W}$ equipped with a multiplication $m:T\times T\to T$ and unit $j:1_{\mathcal{I}}\to T$, along with 2-isomorphisms a,ℓ,r as in (7). More explicitly, the multiplication and unit are fibred 1-cells $m=(\otimes_{\mathcal{V}},\otimes_{\mathbb{W}})$ and $j=(I_{\mathcal{V}},I_{\mathbb{W}})$ (9), displayed as

$$\begin{tikzcd}
\mathcal{V}\times\mathcal{V} \arrow[r, "\otimes_{\mathcal{V}}"] \arrow[d, "T\times T"'] & \mathcal{V} \arrow[d, "T"] \\
\mathbb{W}\times\mathbb{W} \arrow[r, "\otimes_{\mathbb{W}}"'] & \mathbb{W}
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\mathcal{I} \arrow[r, "I_{\mathcal{V}}"] \arrow[d, "1"'] & \mathcal{V} \arrow[d, "T"] \\
\mathcal{I} \arrow[r, "I_{\mathbb{W}}"'] & \mathbb{W}
\end{tikzcd} \tag{11}$$

where $\otimes_{\mathcal{V}}$ and $I_{\mathcal{V}}$ are cartesian, and invertible fibred 2-cells $a=(a^{\mathcal{V}},a^{\mathbb{W}})$,

$r=(r^{\mathcal{V}},r^{\mathbb{W}})$, $\ell=(\ell^{\mathcal{V}},\ell^{\mathbb{W}})$ (10), displayed as

$$\begin{tikzcd}
\mathcal{V}\times\mathcal{V}\times\mathcal{V} \arrow[rr, bend left, "\otimes(\otimes\times1)"] \arrow[rr, bend right, "\otimes(1\times\otimes)"'] \arrow[rr, phantom, "\Downarrow a^{\mathcal{V}}"] \arrow[d, "T\times T\times T"'] & & \mathcal{V} \arrow[d, "T"] \\
\mathbb{W}\times\mathbb{W}\times\mathbb{W} \arrow[rr, bend left, "\otimes(\otimes\times1)"] \arrow[rr, bend right, "\otimes(1\times\otimes)"'] \arrow[rr, phantom, "\Downarrow a^{\mathbb{W}}"] & & \mathbb{W}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{V}\times1 \arrow[r, bend left, "\otimes(1\times I)"] \arrow[r, bend right, "\sim"'] \arrow[r, phantom, "\Downarrow r^{\mathcal{V}}"] \arrow[d, "T\times1"'] & \mathcal{V} \arrow[d, "T"] \\
\mathbb{W}\times1 \arrow[r, bend left, "\otimes(1\times I)"] \arrow[r, bend right, "\sim"'] \arrow[r, phantom, "\Downarrow r^{\mathbb{W}}"] & \mathbb{W}
\end{tikzcd}
\qquad
\begin{tikzcd}
1\times\mathcal{V} \arrow[r, bend left, "\otimes(I\times1)"] \arrow[r, bend right, "\sim"'] \arrow[r, phantom, "\Downarrow \ell^{\mathcal{V}}"] \arrow[d, "1\times T"'] & \mathcal{V} \arrow[d, "T"] \\
1\times\mathbb{W} \arrow[r, bend left, "\otimes(I\times1)"] \arrow[r, bend right, "\sim"'] \arrow[r, phantom, "\Downarrow \ell^{\mathbb{W}}"] & \mathbb{W}
\end{tikzcd}$$

where by definition $a^{\mathcal{V}},r^{\mathcal{V}},l^{\mathcal{V}}$ lie above $a^{\mathbb{W}},r^{\mathbb{W}},l^{\mathbb{W}}$. The coherence axioms they satisfy turn out to give the usual axioms which make $(\mathcal{V},\otimes_{\mathcal{V}},I_{\mathcal{V}})$ and $(\mathbb{W},\otimes_{\mathbb{W}},I_{\mathbb{W}})$ into monoidal categories with the respective associativity, left and right unit constraints.

Remark 3.1. The latter can be deduced also by the fact that the domain and codomain 2-functors $dom,\ cod:\mathbf{Fib}\subset\mathbf{Cat}^{\mathbf{2}}\to\mathbf{Cat}$ are in fact strict monoidal, i.e. preserve the cartesian structure on the nose. In other words, the equality of pasted diagrams of 2-cells in $\mathbf{Fib}$ breaks down into equalities $\mathbf{Cat}$ for the two (ordinary) natural transformations they consist of.

Moreover, the strict commutativity of the diagrams (11) implies that T strictly preserves the tensor product and the unit object between $\mathcal{V}$ and $\mathbb{W}$, *i.e.*

$$TA\otimes_{\mathbb{W}}TB=T(A\otimes_{\mathcal{V}}B),\quad I_{\mathbb{W}}=T(I_{\mathcal{V}}).$$

Along with the conditions that $T(a^{\mathcal{V}})=a^{\mathbb{W}}$, $T(l^{\mathcal{V}})=l^{\mathbb{W}}$ and $T(r^{\mathcal{V}})=r^{\mathbb{W}}$, these data define a strict monoidal structure on T; we obtain the following definition, which coincides with [Shu08, 12.1].

Definition 3.2. *A* monoidal fibration *is a fibration $T:\mathcal{V}\to\mathbb{W}$ such that*

(i) $\mathcal{V}$ and $\mathbb{W}$ are monoidal categories,

(ii) T *is a strict monoidal functor,*

(iii) the tensor product $\otimes_{\mathcal{V}}$ *of* $\mathcal{V}$ *preserves cartesian arrows.*

If $\mathcal{V}$ and $\mathcal{W}$ are symmetric monoidal categories and T is a symmetric strict monoidal functor, we call T a *symmetric monoidal fibration*. In a dual way, we can define a (*symmetric*) *monoidal opfibration* to be an opfibration which is a (symmetric) strict monoidal functor, where the tensor product of the total category preserves cocartesian arrows. Notice that a monoidal opfibration is still a pseudomonoid (and not a pseudocomonoid), this time in **OpFib**. Finally, a *monoidal bifibration* is one where the tensor product of the total category preserves both cartesian and cocartesian liftings.

We now describe a pseudomodule for a pseudomonoid in $(\mathbf{Fib}, \times, 1_{\mathcal{I}})$; in analogy to Theorem 2.4, this will be the object which will eventually have the enriched structure. According to Definition 2.7, a *pseudoaction* of a monoidal fibration $T : \mathcal{V} \to \mathbb{W}$ on an ordinary fibration $P : \mathcal{A} \to \mathbb{X}$ is a fibred 1-cell $\mu = (\mu^{\mathcal{A}}, \mu^{\mathbb{X}}) : T \times P \to P$

$$\begin{tikzcd}
\mathcal{V}\times\mathcal{A} \arrow[r, "\mu^{\mathcal{A}}"] \arrow[d, "T\times P"'] & \mathcal{A} \arrow[d, "P"] \\
\mathbb{W}\times\mathbb{X} \arrow[r, "\mu^{\mathbb{X}}"'] & \mathbb{X}
\end{tikzcd} \tag{12}$$

where $\mu^{\mathcal{A}}$ is cartesian, along with 2-isomorphisms χ, ν as in (8) in **Fib**. Explicitly, these are invertible fibred 2-cells $\chi = (\chi^{\mathcal{A}}, \chi^{\mathbb{X}})$, $\nu = (\nu^{\mathcal{A}}, \nu^{\mathbb{X}})$ represented by

$$\begin{tikzcd}[column sep=small]
& \mathcal{V}\times\mathcal{A} \arrow[dr, "\mu"] & \\
\mathcal{V}\times\mathcal{V}\times\mathcal{A} \arrow[ur, "M\times1"] \arrow[dr, "1\times\mu"'] \arrow[ddd, "T\times T\times P"'] & \Downarrow\chi^{\mathcal{A}} & \mathcal{A} \arrow[ddd, "P"] \\
& \mathcal{V}\times\mathcal{A} \arrow[ur, "\mu"'] & \\
& \mathbb{W}\times\mathbb{X} \arrow[dr, "\mu"] & \\
\mathbb{W}\times\mathbb{W}\times\mathbb{X} \arrow[ur, "M\times1"] \arrow[dr, "1\times\mu"'] & \Downarrow\chi^{\mathbb{X}} & \mathbb{X} \\
& \mathbb{W}\times\mathbb{X} \arrow[ur, "\mu"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& \mathcal{V}\times\mathcal{A} \arrow[dr, "\mu"] & \\
1\times\mathcal{A} \arrow[ur, "I\times1"] \arrow[rr, bend right, "\sim"'] \arrow[ddd, "1\times P"'] & \Downarrow\nu^{\mathcal{A}} & \mathcal{A} \arrow[ddd, "P"] \\
& & \\
& \mathbb{W}\times\mathbb{X} \arrow[dr, "\mu"] & \\
1\times\mathbb{X} \arrow[ur, "I\times1"] \arrow[rr, bend right, "\sim"'] & \Downarrow\nu^{\mathbb{X}} & \mathbb{X} \\
& &
\end{tikzcd}$$

where $\chi^{\mathcal{A}}, \nu^{\mathcal{A}}$ are above $\chi^{\mathbb{X}}, \nu^{\mathbb{X}}$ with respect to the appropriate fibrations. These data are subject to certain axioms, which in fact again split up in two sets of commutative diagrams for the the two natural isomorphisms that χ and ν consist of; these coincide with the action of a monoidal category axioms (Definition 2.2).

Definition 3.3. *A* T-representation *for a monoidal fibration* $T\colon \mathcal{V}\to\mathbb{W}$ *is a fibration* $P\colon \mathcal{A}\to\mathbb{X}$ *equipped with a* T*-pseudoaction* $\mu=(\mu^{\mathcal{A}},\mu^{\mathbb{X}})$*. This amounts to two actions*

$$\mu^{\mathcal{A}}=*:\mathcal{V}\times\mathcal{A}\longrightarrow\mathcal{A}$$
$$\mu^{\mathbb{X}}=\diamond:\mathbb{W}\times\mathbb{X}\longrightarrow\mathbb{X}$$

of the monoidal categories $\mathcal{V}$, $\mathbb{W}$ *on the categories* $\mathcal{A}$ *and* $\mathbb{X}$ *respectively, satisfying the commutativity of (12) where* $\mu^{\mathcal{A}}$ *preserves cartesian arrows, such that for all* $X,Y\in\mathcal{V}$ *and* $A\in\mathcal{A}$ *the following conditions hold:*

$$P\chi^{\mathcal{A}}_{XYA}=\chi^{\mathbb{X}}_{(TX)(TY)(PA)},\quad P\nu^{\mathcal{A}}_{A}=\nu^{\mathbb{X}}_{PA}. \tag{13}$$

The compatibility conditions of the above definition are natural, since by (12)

$$P(X*A)=TX\diamond PA$$

for any $X\in\mathcal{V}$, $A\in\mathcal{A}$, hence the isomorphisms $\chi^{\mathcal{A}}_{XYA}:X*(Y*A)\cong(X\otimes_{\mathcal{V}}Y)*A$ in $\mathcal{A}$ lie above certain isomorphisms

$$P\chi^{\mathcal{A}}_{XYA}:TX\diamond(TY\diamond PA)\xrightarrow{\sim}(TX\otimes_{\mathbb{W}}TY)\diamond PA \tag{14}$$

in $\mathbb{X}$, due to the strict monoidality of T. Similarly, $\nu^{\mathcal{A}}_{A}:I*A\cong A$ is mapped to

$$P\nu^{\mathcal{A}}_{A}:I_{\mathbb{X}}\diamond PA\xrightarrow{\sim}PA \tag{15}$$

since $P(I_{\mathcal{V}}*A)=T(I_{\mathcal{V}})\diamond PA=I_{\mathbb{W}}\diamond PA$. Thus (13) demand that (14) and (15) coincide with the components of $\chi^{\mathbb{X}}$ and $\nu^{\mathbb{X}}$, from the $\mathbb{W}$-representation $\mathbb{X}$.

The last step in modifying Theorem 2.4 to obtain a correspondence between representations of a monoidal fibration and the desired enriched fibrations, is to introduce a notion of a parameterized adjunction in $\mathbf{Fib}$. For that, we first re-formulate the 'adjunctions with a parameter' Theorem 2.1 in the context of $\mathbf{Cat^2}$.

Theorem 3.4. *Suppose we have a morphism (F,G) of two variables in $\mathbf{Cat^2}$, given by a commutative square of categories and functors*

$$\begin{tikzcd}
\mathcal{A}\times\mathcal{B} \arrow[r, "F"] \arrow[d, "H\times J"'] & \mathcal{C} \arrow[d, "K"] \\
\mathbb{X}\times\mathbb{Y} \arrow[r, "G"'] & \mathbb{Z}.
\end{tikzcd} \tag{16}$$

Assume that, for every $B\in\mathcal{B}$ and $Y\in\mathbb{Y}$, there exist adjunctions $F(-,B)\dashv R(B,-)$ and $G(-,Y)\dashv S(Y,-)$, such that $(F(-,B),G(-,JB))$ has a right adjoint $(R(B,-),S(JB,-))$ in $\mathbf{Cat^2}$. This is represented by

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift left, "F(-{,}B)"] \arrow[r, phantom, "\perp"] \arrow[d, "H"'] & \mathcal{C} \arrow[l, shift left, "R(B{,}-)"] \arrow[d, "K"] \\
\mathbb{X} \arrow[r, shift left, "G(-{,}JB)"] \arrow[r, phantom, "\perp"] & \mathbb{Z} \arrow[l, shift left, "S(JB{,}-)"]
\end{tikzcd} \tag{17}$$

where (H,K) is a map of adjunctions (both squares commute and $\varepsilon_K = K\varepsilon$, $H\eta=\eta_H$). Then, there is a unique way to define a morphism of two variables

$$\begin{tikzcd}
\mathcal{B}^{\mathrm{op}}\times\mathcal{C} \arrow[r, "R"] \arrow[d, "J^{\mathrm{op}}\times K"'] & \mathcal{A} \arrow[d, "H"] \\
\mathbb{Y}^{\mathrm{op}}\times\mathbb{Z} \arrow[r, "S"'] & \mathbb{X}
\end{tikzcd} \tag{18}$$

in $\mathbf{Cat}^2$, for which $\mathcal{C}(F(A,B),C)\cong\mathcal{A}(A,R(B,C)),\mathbb{Z}(G(X,Y),Z)\cong\mathbb{X}(X,S(Y,Z))$ are natural in all three variables.

Proof. The result clearly follows from ordinary parameterized adjunctions. The fact that $(R(B,-),S(JB,-))$ is an arrow in $\mathbf{Cat^2}$ for all B's ensures that the diagram (18) commutes on the second variable, and also on the first variable on objects, since $HR(B,C)=S(JB,KC)$. On arrows, commutativity follows from the unique way of defining $R(h,1)$ and $S(Jh,1)$ for any $h:B\to B'$ under these assumptions. $\square$

We call (R,S) the *parameterized adjoint* of (F,G) in $\mathbf{Cat^2}$, written $(F,G)\dashv_p (R,S)$.

Remark 3.5. Although the notion of an adjunction can be internalized in any bicategory, its parameterized version seems to be much more involved. In any monoidal bicategory with duals, we could ask for 1-cells $A \cong A\otimes I \xrightarrow{1\times b} A \otimes B \xrightarrow{t} C$ to have adjoints $g_b\colon C \to A$, for every $b\colon I \to B$. For the cartesian 2-monoidal case at least, with 'category-like' objects like in $\mathbf{Fib}$, the 2-categorical approach of [CGR14, Thm. 2.4] clarifies things.

Restricting to fibrations, consider a morphism of two variables in $\mathbf{Fib} \subset \mathbf{Cat}^2$ *i.e.* a fibred 1-cell (F, G) as in (16) with F cartesian, with the property that (17) is a general fibred adjunction as in Definition 2.9, *i.e.* the partial right adjoint $R(B, -)$ is also cartesian. Dually, in $\mathbf{OpFib}$ we request both F and $R(B, -)$ to be cocartesian. Notice that in both cases, the parameterized adjoint of two variables (R, S) can neither be a fibred nor an opfibred 1-cell 'wholly', since at (18) the vertical $J^{\mathrm{op}} \times K$ is a product of a fibration with an opfibration, hence neither of the two.

If we lift the (co)cartesian requirements, we end up with the (op)lax version of these adjunctions. Since those cases are the most relevant to our examples, we abuse notation as to call (op)fibred parameterized adjunctions the (op)lax ones. Based on the remark that follows, this abuse is in fact only fractional.

Remark 3.6. There exists an interesting asymmetry regarding the (co)cartesianness requirement of the left/right partial adjoints, due to Lemma 2.10. Since right adjoints always preserve cartesian arrows in $\mathbf{Cat}^2$ and dually left adjoints always preserve cocartesian ones, we can deduce that any fibred 1-cell (F, G) has a (right) fibred parameterized adjoint as long as it has a $\mathbf{Cat^2}$-parameterized adjoint. Dually, an opfibred 1-cell has a (left) opfibred parameterized adjoint as long as it has it in $\mathbf{Cat^2}$.

Definition 3.7. *Suppose H, K are fibrations. A* fibred parameterized adjunction *is a parameterized adjunction $(F, G) \underset{p}{\dashv} (R, S)$ in $\mathbf{Cat^2}$, between two 1-cells*

$$\begin{tikzcd}
\mathcal{A}\times\mathcal{B} \arrow[r, "F"] \arrow[d, "H\times J"'] & \mathcal{C} \arrow[d, "K"] \\
\mathbb{X}\times\mathbb{Y} \arrow[r, "G"'] & \mathbb{Z}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B}^{\mathrm{op}}\times\mathcal{C} \arrow[r, "R"] \arrow[d, "J^{\mathrm{op}}\times K"'] & \mathcal{A} \arrow[d, "H"] \\
\mathbb{Y}^{\mathrm{op}}\times\mathbb{Z} \arrow[r, "S"'] & \mathbb{X}
\end{tikzcd}$$

where $R(B,-)$ is by default cartesian. Dually, an opfibred parameterized adjunction *is as above, where $F(-,B)$ is by default cocartesian.*

The proposed definition of an enriched fibration is justified by the subsequent Theorem 3.11 which fulfills our initial goal, i.e. to generalize Theorem 2.4 to the context of (op)fibrations. In Remark 3.9 we give an equivalent formulation in terms of enriched functors. The enriched hom-functor is defined as in (3), writing $\mathcal{A}$ for both the enriched and the underlying category.

Definition 3.8 (*Enriched Fibration*)**.** *Suppose $T:\mathcal{V}\to\mathbb{W}$ is a monoidal fibration. A fibration $P:\mathcal{A}\to\mathbb{X}$ is* enriched *in T when the following conditions are satisfied:*

- *the total category $\mathcal{A}$ is enriched in the total monoidal category $\mathcal{V}$ and the base category $\mathbb{X}$ is enriched in the base monoidal category $\mathbb{W}$, in such a way that the following commutes:*

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{op}}\times\mathcal{A} \arrow[r, "\mathcal{A}(-,-)"] \arrow[d, "P^{\mathrm{op}}\times P"'] & \mathcal{V} \arrow[d, "T"] \\
\mathbb{X}^{\mathrm{op}}\times\mathbb{X} \arrow[r, "\mathbb{X}(-,-)"'] & \mathbb{W}
\end{tikzcd} \tag{19}$$

- *the composition law and the identities of the enrichments are compatible, in the sense that*

$$\begin{gathered} TM^{\mathcal{A}}_{A,B,C}=M^{\mathbb{X}}_{PA,PB,PC} \\ Tj^{\mathcal{A}}_{A}=j^{\mathbb{X}}_{PA} \end{gathered} \tag{20}$$

The compatibilities (20) only state that the composition and identities

$$M^{\mathcal{A}}_{A,B,C}:\mathcal{A}(B,C)\otimes_{\mathcal{V}}\mathcal{A}(A,B)\to\mathcal{A}(A,C),\quad j^{\mathcal{A}}_{A}:I_{\mathcal{V}}\to\mathcal{A}(A,A)$$

of the $\mathcal{V}$-enriched $\mathcal{A}$ are mapped, under T, exactly to those of the $\mathbb{W}$-enriched $\mathbb{X}$:

$$\begin{gathered} M^{\mathbb{X}}_{PA,PB,PC}:\mathbb{X}(PB,PC)\otimes_{\mathbb{W}}\mathbb{X}(PA,PB)\to\mathbb{X}(PA,PC) \\ j^{\mathbb{X}}_{PA}:I_{\mathbb{W}}\to\mathbb{X}(PA,PA) \end{gathered}$$

where the domains and codomains already coincide by strict monoidality of T and the commutativity of (19).

For the above definition, it could be argued that some sort of cartesian condition for the enriched hom-functor $\mathcal{A}(-,-)$ should be asked; notice however that for P a fibration, the product $P^{\mathrm{op}} \times P$ has neither a fibration not an opfibration structure. If we required that the partial functor $\mathcal{A}(A,-)\colon \mathcal{A} \to \mathcal{V}$ is cartesian, all results below would still be valid with minor adjustments. Since the examples (Section 4) so far do not seem to satisfy this extra condition, for the moment we adhere to this more general definition.

Remark 3.9 (*Enriched fibrations as enriched functors*)**.** For a T-enriched fibration P as above, the strict monoidal structure of T induces a 2-functor $\widetilde{T}\colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathbb{W}\text{-}\mathbf{Cat}$ by Proposition 2.3. Hence we can make the $\mathcal{V}$-category $\mathcal{A}$ into a $\mathbb{W}$-category $\widetilde{T}\mathcal{A}$, with the same set of objects $\mathrm{ob}\mathcal{A}$ and hom-objects $T\mathcal{A}(A,B) = \mathbb{X}(PA,PB)$. Then $P\colon \mathcal{A} \to \mathbb{X}$ can be verified to have the structure of a $\mathbb{W}$-enriched functor between the $\mathbb{W}$-categories $\widetilde{T}\mathcal{A}$ and $\mathbb{X}$, with hom-objects mapping $T\mathcal{A}(A,B) \xrightarrow{=} \mathbb{X}(PA,PB)$. The compatibility with the composition and the identities is ensured by (20).

From this perspective, the definition of a $(T\colon \mathcal{V} \to \mathbb{W})$-enriched fibration between a $\mathcal{V}$-category $\mathcal{A}$ and a $\mathbb{W}$-category $\mathbb{X}$ could be reformulated as a strictly fully faithful $\mathbb{W}$-enriched functor $P\colon \widetilde{T}\mathcal{A} \to \mathbb{X}$, whose underlying ordinary functor $P_0\colon \mathcal{A}_0 \to \mathbb{X}_0$ is a fibration (the commutativity of (19) follows).

Dually, we have the notion of an *enriched opfibration*, as well as the following combined version.

Definition 3.10. *Suppose that $T : \mathcal{V} \to \mathbb{W}$ is a symmetric monoidal opfibration. We say that a fibration $P : \mathcal{A} \to \mathbb{X}$ is* enriched *in T if the opfibration $P^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathbb{X}^{\mathrm{op}}$ is an enriched T-opfibration.*

Finally, we prove that to give a fibration with an action $(*, \diamond)$ of a monoidal fibration T (Definition 3.3) with a fibred parameterized adjoint (Definition 3.7), is to give a T-enriched fibration (Definition 3.8).

Theorem 3.11. *Suppose that $T : \mathcal{V} \to \mathbb{W}$ is a monoidal fibration, which acts on an (ordinary) fibration $P : \mathcal{A} \to \mathbb{X}$ via the fibred 1-cell*

$$\begin{tikzcd}
\mathcal{V} \times \mathcal{A} \arrow[r, "*"] \arrow[d, "T\times P"'] & \mathcal{A} \arrow[d, "P"] \\
\mathbb{W} \times \mathbb{X} \arrow[r, "\diamond"'] & \mathbb{X}.
\end{tikzcd}$$

If this action has a parameterized adjoint (R, S) as in

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \arrow[r, "R"] \arrow[d, "P^{\mathrm{op}}\times P"'] & \mathcal{V} \arrow[d, "T"] \\
\mathbb{X}^{\mathrm{op}} \times \mathbb{X} \arrow[r, "S"'] & \mathbb{W}
\end{tikzcd}$$

we can enrich the fibration P in the monoidal fibration T.

Proof. Recall by Definition 3.3 that the T-action in particular consists of two actions $*$ and $\diamond$ of the monoidal categories $\mathcal{V}$ and $\mathbb{W}$ on the categories $\mathcal{A}$ and $\mathbb{X}$ respectively. Since $(*, \diamond) \dashv_{p} (R, S)$, by Theorem 3.4 we have two ordinary adjunctions

$$\begin{tikzcd}
\mathcal{V} \arrow[r, shift left, "-*A"] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift left, "{R(A,-)}"]
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\mathbb{W} \arrow[r, shift left, "-\diamond X"] \arrow[r, phantom, "\perp"] & \mathbb{X} \arrow[l, shift left, "{S(X,-)}"]
\end{tikzcd}$$

for all $A \in \mathcal{A}$ and $X \in \mathbb{X}$. By Theorem 2.4, there exists a $\mathcal{V}$-category $\underline{\mathcal{A}}$ with underlying category $\mathcal{A}$ and hom-objects $\underline{\mathcal{A}}(A, B) = R(A, B)$ and also a $\mathbb{W}$-category $\underline{\mathbb{X}}$ with underlying category $\mathbb{X}$ and hom-objects $\underline{\mathbb{X}}(X, Y) = S(X, Y)$. Also, the enriched hom-functors satisfy the required commutativity $TS(-, -) = R(P-, P-)$ by (18).

Finally, we need to show that the composition and identity laws of the enrichments are compatible as in (20), i.e. $TM^{\mathcal{A}}_{A,B,C} = M^{\mathbb{X}}_{PA,PB,PC}$ and $Tj^{\mathcal{A}}_{A} = j^{\mathbb{X}}_{PA}$. For that, it is enough to confirm that their adjuncts under $(- \diamond X) \dashv S(X, -)$ coincide. The latter ones are explicitly given by (5) and (6), i.e.

$$\begin{tikzcd}
(S(PB,PC)\otimes S(PA,PB))\diamond PA \arrow[rr, dashed] \arrow[d, "\chi^{\mathbb{X}}"'] & & PC \\
S(PB,PC)\diamond(S(PA,PB)\diamond PA) \arrow[r, "1\diamond\varepsilon_{PB}"] & S(PB,PC)\diamond PB \arrow[ur, "\varepsilon_{PC}"'] &
\end{tikzcd}$$

$$I\diamond PA \xrightarrow{\nu^{\mathbb{X}}} PA$$

For the former ones, since (P,T) is a map of adjunctions

$$\begin{tikzcd}
\mathcal{V} \arrow[r, shift left, "-*A"] \arrow[r, phantom, "\perp"] \arrow[d, "T"'] & \mathcal{A} \arrow[l, shift left, "R(A,-)"] \arrow[d, "P"] \\
\mathbb{W} \arrow[r, shift left, "-\diamond PA"] \arrow[r, phantom, "\perp"] & \mathbb{X}, \arrow[l, shift left, "S(PA,-)"]
\end{tikzcd}$$

taking the images of $M^{\mathcal{A}}_{A,B,C}$ and $j^{\mathcal{A}}_{A}$ under T and translating under the adjunction $(-\diamond X)\dashv S(X,-)$ is the same as first translating under $(-*A)\dashv R(A,-)$ and then applying P. That produces

$$\begin{tikzcd}
P\left(R(B,C)\otimes R(A,B)\right)*A) \arrow[r, "P\chi^{\mathcal{A}}"] \arrow[ddr, dashed] & P(R(B,C)*R(A,B)*A) \arrow[d, "P(1*\varepsilon_B)"] \\
& P(R(B,C)*B) \arrow[d, "P(\varepsilon_C)"] \\
& PC
\end{tikzcd}$$

$$P(I*A) \xrightarrow{P(\nu^{\mathbb{A}})} PA$$

Since $P\chi^{\mathcal{A}}=\chi^{\mathbb{X}}$ and $P\nu^{\mathcal{A}}=\nu^{\mathbb{X}}$ from (13), and also $P\varepsilon=\varepsilon_P$ as a map of adjunctions, the above composites coincide and the proof is complete. ☐

An important first example that should fit this setting of an action-induced enrichment is that of a *closed* monoidal fibration. Just like a monoidal closed category $\mathcal{V}$ is one where $\otimes\colon\mathcal{V}\times\mathcal{V}\to\mathcal{V}$ has a (right) parameterized adjoint via $-\otimes X\dashv[X,-]$ for every object X, we can consider the following notion based on Definition 3.7.

Definition 3.12. *A monoidal fibration* $T\colon \mathcal{V} \to \mathbb{W}$ *is (right)* closed *when its tensor product fibred 1-cell* (11)

$$\begin{tikzcd}
\mathcal{V} \times \mathcal{V} \arrow[r, "\otimes_{\mathcal{V}}"] \arrow[d, "T\times T"'] & \mathcal{V} \arrow[d, "T"] \\
\mathbb{W} \times \mathbb{W} \arrow[r, "\otimes_{\mathbb{W}}"'] & \mathbb{W}
\end{tikzcd}$$

has a parameterized adjoint

$$\begin{tikzcd}
\mathcal{V}^{\mathrm{op}} \times \mathcal{V} \arrow[r, "{[-,-]_{\mathcal{V}}}"] \arrow[d, "T^{\mathrm{op}}\times T"'] & \mathcal{V} \arrow[d, "T"] \\
\mathbb{W}^{\mathrm{op}} \times \mathbb{W} \arrow[r, "{[-,-]_{\mathbb{W}}}"'] & \mathbb{W}
\end{tikzcd}$$

Equivalently, by Theorem 3.4, T *is monoidal closed when*

(i) $\mathcal{V}$ *and* $\mathbb{W}$ *are monoidal closed categories,*

(ii) T *is a strict closed functor,*

(iii) $T\varepsilon = \varepsilon_T$ *and* $\eta_T = T\eta$ *for the respective units and counits of the adjunctions.*

Notice that by Lemma 2.10, the right adjoint $[V,-]_{\mathcal{V}}$ between the total categories automatically preserves cartesian liftings. On the other hand, for the dual notion of a *monoidal closed opfibration*, the right adjoint is not cocartesian by default.

Remark 3.13. In [Shu08, §13], definitions of an *internally closed* monoidal fibration over a cartesian monoidal base, as well as *externally closed* monoidal fibration over an arbitrary monoidal base are given. These are equivalent to each other under certain hypotheses; none, however, guarantee that the total category is closed on its own right. The external definition gives some, but not all, conditions in terms of the fibres and the reindexing functors for a fibred adjoint to exist, in the spirit of results such as Theorem 2.11.

Applying Theorem 3.11 we can deduce the enrichment of a monoidal closed fibration in itself, analogously to the ordinary case (Remark 2.5).

Proposition 3.14. *A monoidal closed fibration $T\colon \mathcal{V} \to \mathbb{W}$ is T-enriched. Dually, a monoidal closed opfibration is enriched in itself.*

Proof. All clauses of Definition 3.3 are satisfied, since the functor $-\otimes_{\mathcal{V}}-$ is cartesian in both variables by Definition 3.2, and also $Ta^{\mathcal{V}}_{XYZ} = a^{\mathcal{W}}_{TXTYTZ}$ and $T\ell^{\mathcal{V}}_X = \ell^{\mathbb{W}}_{TX}$ for the respective associator and the left unitor since T is a strict monoidal functor. Therefore $(\otimes_{\mathcal{V}}, \otimes_{\mathbb{W}})$ is indeed a T-action, just like the regular representation of a monoidal category earlier. Since this action has a parameterized adjoint, by definition of a monoidal closed fibration, the result follows. ☐

Finally, there is a dual version to Theorem 3.11, characterizing the enrichment of an opfibration in a monoidal opfibration.

Theorem 3.15. *Suppose that $T : \mathcal{V} \to \mathbb{W}$ is a monoidal opfibration, which acts on an (ordinary) opfibration $U : \mathcal{B} \to \mathbb{Y}$ via the opfibred 1-cell*

$$\begin{tikzcd}
\mathcal{V}\times\mathcal{B} \arrow[r, "*"] \arrow[d, "T\times U"'] & \mathcal{B} \arrow[d, "U"] \\
\mathbb{W}\times\mathbb{Y} \arrow[r, "\diamond"'] & \mathbb{Y}.
\end{tikzcd}$$

If this action has a parameterized adjoint (R, S) as in

$$\begin{tikzcd}
\mathcal{B}^{\mathrm{op}}\times\mathcal{B} \arrow[r, "R"] \arrow[d, "U^{\mathrm{op}}\times U"'] & \mathcal{V} \arrow[d, "T"] \\
\mathbb{Y}^{\mathrm{op}}\times\mathbb{Y} \arrow[r, "S"'] & \mathbb{W}
\end{tikzcd}$$

we can enrich the opfibration U in the monoidal opfibration T.

Remark 3.16. The asymmetry between cartesian and cocartesian functors with regards to fibred and opfibred adjunctions is still apparent when comparing Theorems 3.11 and 3.15. For the former, Lemma 2.10 ensures that the right parameterized adjoint will be, at least partially as $R(A,-)$, cartesian; as a result, the whole parameterized adjunction lifts from $\mathbf{Cat}^{\mathbf{2}}$ to $\mathbf{Fib}$. On the other hand, for the latter dual theorem, the assumptions cannot ensure that the enriched hom R will be partially cocartesian. One reason for this discrepancy is that even if we change our setting from $\mathbf{Fib}_{(\mathrm{opl})}$ to $\mathbf{OpFib}_{(\mathrm{lax})}$, the enrichment is given in both cases by the existence of a *right* adjoint (and not of a left one in the dual setting).

4. Applications

In this final chapter, we exhibit a few examples of the enriched fibration notion. In these cases, Theorems 3.11 and 3.15 seem to be the easiest way to deduce the enrichment, due to the fact that the enrichments on the level of bases and total categories are themselves obtained by the similarly-flavored Theorem 2.4. In what follows, we do not present all the relevant theory as it would take up many pages; instead we provide the appropriate references, in the hope that the interested reader will look for the details therein.

4.1 (Co)modules over (co)monoids

In the context of a locally presentable symmetric monoidal closed category $\mathcal{V}$, previous work [HLFV17a] establishes an enrichment of the category of monoids $\mathbf{Mon}(\mathcal{V})$ in the symmetric monoidal category of comonoids $\mathbf{Comon}(\mathcal{V})$, via Theorem 2.4. The action of comonoids on monoids is induced by the internal hom of $\mathcal{V}$: for any coalgebra C and algebra B, $[C,B]$ has always the structure of an algebra via the convolution product. Its (right) parametrized adjoint $P\colon \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}\times\mathbf{Mon}(\mathcal{V})\to\mathbf{Comon}(\mathcal{V})$ which is the enriched hom-functor is called the *Sweedler hom*, since the original notion of a *measuring coalgebra* $P(A,B)$ goes back to [Swe69].

Furthermore, in [HLFV17b] a similarly action-induced enrichment is established for the *global* category of modules in the symmetric monoidal global category of comodules, i.e. the category of all (co)modules over any (co)monoid in $\mathcal{V}$. The action again comes from the internal hom of the monoidal category, and its parametrized adjoint $Q\colon \mathbf{Mod}^{\mathrm{op}}\times\mathbf{Mod}\to\mathbf{Comod}$ maps an A-module M and a B-comodule N to their *measuring comodule* $Q(M,N)$ [Bat00]. This parameterized adjunction is obtained itself using the theory of fibred adjunctions, since the functor $U\colon \mathbf{Mod}\to\mathbf{Mon}(\mathcal{V})$ which gives the 'underlying' algebra of a module is a fibration and dually $V\colon \mathbf{Comod}\to\mathbf{Comon}(\mathcal{V})$ is an opfibration. Therefore the very enrichment on the level of the total categories is accomplished via Theorem 2.11, producing a general (lax) opfibred adjunction

$$\begin{tikzcd}
\mathbf{Mod}^{\mathrm{op}} \arrow[r, bend left=10, "Q(-{,}N_B)"] \arrow[r, phantom, "\top"] \arrow[d, "U^{\mathrm{op}}"'] & \mathbf{Comod} \arrow[l, bend left=10, "{[-{,}N_B]^{\mathrm{op}}}"] \arrow[d, "V"] \\
\mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \arrow[r, bend left=10, "P(-{,}B)"] \arrow[r, phantom, "\top"] & \mathbf{Comon}(\mathcal{V}) \arrow[l, bend left=10, "{[-{,}B]^{\mathrm{op}}}"]
\end{tikzcd} \tag{21}$$

To establish an enriched opfibration structure, we apply Theorem 3.15. First of all, $V\colon \mathbf{Comod} \to \mathbf{Comon}(\mathcal{V})$ can be shown to be a monoidal opfibration, by definition of the monoidal product in $\mathbf{Comod}$. Moreover, since both actions on the level of total and base categories are in fact the internal hom of $\mathcal{V}$ restricted to the appropriate subcategories, compatibility (13) also follows. Finally, the top action is cartesian [HLFV17b, (20)] hence the opfibration U^{op} is enriched in V, as in Definition 3.10.

Proposition 4.1. *Suppose $\mathcal{V}$ is a locally presentable symmetric monoidal closed category. The fibration* $\mathbf{Mod} \to \mathbf{Mon}(\mathcal{V})$ *is enriched in the monoidal opfibration* $\mathbf{Comod} \to \mathbf{Comon}(\mathcal{V})$.

An example of such a monoidal category $\mathcal{V}$, which also motivated this whole development, is the category of modules over a commutative ring, $\mathbf{Mod}_R$. Both enriching functors arise as adjoints to the linear maps space functor $\mathbf{Mod}_R(-,-)$ restricted to the respective subcategories. In particular, for two arbitrary modules M and N over R-algebras A and B, the measuring comodule $Q(M,N)$ which provides the enrichment of modules in comodules has its coaction over the Sweedler's measuring R-coalgebra $P(A,B)$ which provides the enrichment of algebras in coalgebras. Similarly, the comodule composition maps $Q(N,S) \otimes_R Q(M,N) \to Q(M,S)$ are above the coalgebra composition maps $P(B,C) \otimes_R P(A,B) \to P(A,B)$. This is a substantial step exhibiting the tight relations between these dual-flavored, standard (op)fibrations.

Furthermore, the forgetful $\mathbf{Comod} \to \mathbf{Comon}(\mathcal{V})$ is in fact an example of a monoidal closed opfibration, Definition 3.12. First of all, it is the case that the category of comonoids is monoidal closed, if $\mathcal{V}$ is locally presentable and symmetric monoidal closed, as was already proved in [Por08, 3.2]. In [HLFV17b, 4.5], it is shown in detail how the opfibred 1-cell

$$\begin{tikzcd}
\mathbf{Comod} \times \mathbf{Comod} \arrow[r, "(-\otimes-)"] \arrow[d] & \mathbf{Comod} \arrow[d] \\
\mathbf{Comon}(\mathcal{V}) \times \mathbf{Comon}(\mathcal{V}) \arrow[r, "(-\otimes-)"'] & \mathbf{Comon}(\mathcal{V})
\end{tikzcd}$$

has a parameterized right adjoint, so the result follows by Proposition 3.14.

Proposition 4.2. *Suppose $\mathcal{V}$ is a locally presentable symmetric monoidal closed category. The monoidal opfibration $\mathbf{Comod} \to \mathbf{Comon}(\mathcal{V})$ is closed, therefore enriched in itself.*

Notice that this does not dualize for $\mathbf{Mod} \to \mathbf{Mon}(\mathcal{V})$, since in general the category of monoids is not monoidal closed, e.g. rings or R-algebras.

4.2 Enriched (co)modules over enriched (co)categories

The above study on enrichment relations between monoids and comonoids, as well as modules and comodules, can be appropriately extended to their many-object generalizations, in the sense that a monoid can be thought of as a one-object category.

For a detailed exposition of the notions and constructions that follow, see [Vas14, §7] or from a double categorical perspective [Vas17]. Briefly, for a symmetric monoidal category with colimits preserved by $\otimes$, we can consider the category of $\mathcal{V}$-enriched categories $\mathcal{V}$-$\mathbf{Cat}$, whose objects are monads in the bicategory of *enriched matrices* $\mathcal{V}$-$\mathbf{Mat}$ [BCSW83]. In a dual way, considering comonads therein, we can construct the category $\mathcal{V}$-$\mathbf{Cocat}$ of *enriched cocategories*, serving as a many object generalization of comonoids in $\mathcal{V}$. A $\mathcal{V}$-cocategory $\mathcal{C}_X$ with set of objects X comes equipped with cocomposition and coidentity arrows

$$\Delta_x, z\colon \mathcal{C}(x,z) \to \sum_{y\in X} \mathcal{C}(x,y) \otimes \mathcal{C}(y,z), \quad \epsilon_x\colon \mathcal{C}(x,x) \to I$$

in $\mathcal{V}$, satisfying coassociativity and counitality axioms. Both categories $\mathcal{V}$-$\mathbf{Cat}$ and $\mathcal{V}$-$\mathbf{Cocat}$ are in fact fibred and opfibred, respectively, over the category of sets, via the usual forgetful functors that give the set of objects of the (co)categories. They also both live inside $\mathcal{V}$-**Grph**, the category of enriched graphs, which is bifibred over **Set**. All these (op)fibrations are monoidal in the sense of Definition 3.2: for two $\mathcal{V}$-graphs (or categories, cocategories) G_X and H_Y, their tensor product is a graph $G \otimes H$ with set of objects $X \times Y$, and $(G \otimes H)\left((x,y),(z,w)\right) = G(x,z) \otimes H(y,w)$.

When $\mathcal{V}$ is moreover monoidal closed with products and coproducts, $\mathcal{V}$-**Grph** is monoidal closed: for two graphs G_X and H_Y, their internal hom is the graph $\mathrm{Hom}(G,H)$ with set of objects Y^X, given by the collection of $\mathcal{V}$-objects

$$\mathrm{Hom}(G,H)(k,s) = \prod_{x',x} [G(x',x), H(kx',sx)] \text{ for } k,s \in Y^X$$

By definition of these structures, the following diagram

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Grph} \arrow[r, shift left, "-\otimes G_X"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathcal{V}\text{-}\mathbf{Grph} \arrow[l, shift left, "\mathrm{Hom}(G_X,-)"] \arrow[d] \\
\mathbf{Set} \arrow[r, shift left, "-\times X"] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, shift left, "(-)^X"]
\end{tikzcd}$$

is a map of adjunctions, therefore all three clauses of Definition 3.12 are satisfied.

Proposition 4.3. *Suppose $\mathcal{V}$ is a symmetric monoidal closed category with products and coproducts. The bifibration $\mathcal{V}$-**Grph** $\to$ **Set** mapping a $\mathcal{V}$-graph to its set of objects is monoidal closed, therefore it is enriched in itself.*

Similarly to how the internal hom of $\mathcal{V}$ was lifted to an action of comonoids on monoids in Section 4.1, the internal hom of $\mathcal{V}$-**Grph** induces an action

$$\begin{tikzcd}
K : \mathcal{V}\text{-}\mathbf{Cocat}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Cat} \arrow[r] & \mathcal{V}\text{-}\mathbf{Cat} \\
(\ \mathcal{C}_X\ ,\ \mathcal{B}_Y\ ) \arrow[r, mapsto] & \mathrm{Hom}(\mathcal{C},\mathcal{B})_{Y^X}
\end{tikzcd}$$

Its opposite has a parameterized adjoint, again by Theorem 2.11,

$$T : \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Cocat}$$

called the *generalized Sweedler hom*. We get the following (lax) opfibred parameterized adjunction

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \arrow[r, bend left, "T(-{,}\mathcal{B}_Y)"] \arrow[r, phantom, "\top"] \arrow[d] & \mathcal{V}\text{-}\mathbf{Cocat} \arrow[l, bend left, "K(-{,}\mathcal{B}_Y)^{\mathrm{op}}"] \arrow[d] \\
\mathbf{Set}^{\mathrm{op}} \arrow[r, bend left, "Y^{(-)}"] \arrow[r, phantom, "\top"] & \mathbf{Set} \arrow[l, bend left, "Y^{(-)^{\mathrm{op}}}"]
\end{tikzcd}$$

thus Theorem 3.15 applies again, see [Vas17, 4.38].

Proposition 4.4. *Suppose $\mathcal{V}$ is a locally presentable, monoidal closed category. The fibration $\mathcal{V}$-$\mathbf{Cat} \to \mathbf{Set}$ is enriched in the monoidal opfibration $\mathcal{V}$-$\mathbf{Cocat} \to \mathbf{Set}$, where both functors send the enriched structure to its set of objects.*

Finally, we can consider many object generalizations of modules and comodules, namely $\mathcal{V}$-modules for $\mathcal{V}$-categories and $\mathcal{V}$-comodules for $\mathcal{V}$-cocategories, see [Vas14, 7.6]. The former are quite standard: an $\mathcal{A}_X$-module Ψ can also be thought as a $\mathcal{V}$-profunctor $\Psi\colon \mathcal{I} \nrightarrow \mathcal{A}$ for $\mathcal{I}$ the unit category. Objects are the same as $\mathcal{A}$, and hom-objects are $\Psi(x) \in \mathcal{V}$ equipped with $(\sum_{x,x'})\mathcal{A}(x,x') \otimes \Psi(x') \to \Psi(x')$ satisfying appropriate axioms. The notion of comodules is dual, and these form global categories much like before, $\mathcal{V}$-$\mathbf{Mod}$ and $\mathcal{V}$-$\mathbf{Comod}$. The internal hom of enriched graphs further restricts to these categories, giving an action of $\mathcal{V}$-$\mathbf{Comod}$ on $\mathcal{V}$-$\mathbf{Mod}$ via a functor

$$\begin{aligned}
\overline{K}:\ \mathcal{V}\text{-}\mathbf{Comod}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Mod} &\longrightarrow \mathcal{V}\text{-}\mathbf{Mod} \\
(\ \Phi_{\mathcal{C}}\ ,\ \Psi_{\mathcal{B}}\ ) &\longmapsto \mathrm{Hom}(\Phi,\Psi)_{\mathrm{Hom}(\mathcal{C},\mathcal{B})}
\end{aligned}$$

where $\mathrm{Hom}(\Phi,\Psi)(t) = \prod_x [\Phi(x), \Psi(tx)]$. It has a parameterized adjoint

$$\overline{T} : \mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Mod} \to \mathcal{V}\text{-}\mathbf{Comod}$$

by Theorem 2.11 which once more heavily relies on the fact that $\mathcal{V}$-$\mathbf{Mod}$ is fibred over $\mathcal{V}$-$\mathbf{Cat}$ and $\mathcal{V}$-$\mathbf{Comod}$ is opfibred $\mathcal{V}$-$\mathbf{Cocat}$, and there exists an adjunction between the base categories:

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}} \arrow[r, bend left=10, "\overline{T}(-{,}\Psi_{\mathcal{B}})"] \arrow[r, phantom, "\top"] \arrow[d] & \mathcal{V}\text{-}\mathbf{Comod} \arrow[l, bend left=10, "\overline{K}(-{,}\Psi_{\mathcal{B}})^{\mathrm{op}}"] \arrow[d] \\
\mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \arrow[r, bend left=10, "T(-{,}\mathcal{B}_Y)"] \arrow[r, phantom, "\top"] & \mathcal{V}\text{-}\mathbf{Cocat} \arrow[l, bend left=10, "K(-{,}\mathcal{B}_Y)^{\mathrm{op}}"]
\end{tikzcd} \tag{22}$$

Proposition 4.5. *If $\mathcal{V}$ is a locally presentable symmetric monoidal closed category, the fibration $\mathcal{V}$-$\mathbf{Mod} \to \mathcal{V}$-$\mathbf{Cat}$ is enriched in the opfibration $\mathcal{V}$-$\mathbf{Comod} \to \mathcal{V}$-$\mathbf{Cocat}$.*

Proof. Theorem 3.15 applies, to first establish the enrichment of the opfibration $\mathcal{V}$-$\mathbf{Mod}^{\mathrm{op}} \to \mathcal{V}$-$\mathbf{Cat}^{\mathrm{op}}$. First of all, $\mathcal{V}$-$\mathbf{Comod} \to \mathcal{V}$-$\mathbf{Cocat}$ is a monoidal opfibration by definition of the respective products and cartesianness of $\otimes_{\mathcal{V}\text{-}\mathbf{Comod}}$, [Vas14, 7.7.6]. The commutative square of categories and functors

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Comod} \times \mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}} \arrow[r, "\overline{K}^{\mathrm{op}}"] \arrow[d] & \mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}} \arrow[d] \\
\mathcal{V}\text{-}\mathbf{Cocat} \times \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \arrow[r, "K^{\mathrm{op}}"'] & \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}}
\end{tikzcd}$$

constitutes an opfibred action, since both K and $\overline{K}$ are actions, $\overline{K}$ preserves cartesian arrows by [Vas14, 7.7.3] and the action axioms are the one above each other as per their definitions. Finally, this opfibred 1-cell has an oplax opfibred parameterized adjoint by [Vas14, 7.7.5], and the proof is complete. ☐

4.3 Comparison with existing notions

The above examples were the ones that motivated the proposed enriched fibration notion – although more should be identified in future work. In this final section, we would like to discuss why other existing approaches were not applicable, due to the nature of these cases.

Recall that assuming the axiom of choice, one can construct an equivalence between fibrations $\mathcal{A} \to \mathbb{X}$ and indexed categories, i.e. pseudofunctors $\mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$ via the classic *Grothendieck construction* [Gro61]. More recently [Shu08, MV18] this correspondence has been lifted between the respective monoidal structures; we believe that a (global) enriched version of the Grothendieck construction in the future, which in a fibrewise sense appears in [BW18], will shed more light to the tight connections between our enriched fibration notion and the ones that follow. For the moment, we only sketch some of the main relevant theory and differences.

M. Bunge in [Bun13] first introduced the notion of an $\mathbb{S}$-*indexed* V-*category*, for $\mathbb{S}$ an elementary topos and V an $\mathbb{S}$-indexed *monoidal* category $V\colon \mathbb{S}^{\mathrm{op}} \to \mathbf{MonCat}$. The goal of this work was to provide a general context in order to compare as well as clarify certain misconceptions regarding different completions on 2-categories, such as the Karoubi, Grothendieck, Cauchy and Stack completion.

Independently, M. Shulman in [Shu13] also develops a theory of *enriched indexed categories* over base categories $\mathbb{S}$ with finite products. The motivation in that paper was to capture and study 'mixed' fibred, indexed and internal structures in various contexts, such as Parameterized and Equivariant Homotopy Theory, abelian sheaves and many more.

Briefly, for $\mathbb{S}$ cartesian monoidal, take V to be an $\mathbb{S}$-indexed monoidal category, equivalently viewed as a monoidal fibration $\int V\colon \mathcal{V} \to \mathbb{S}$. A V-enriched indexed category A is simultaneously indexed (or fibred) over the same $\mathbb{S}$ and also 'fibrewise' enriched in $\mathcal{V}$: every category (or fiber) $A(s)$ for $s \in \mathbb{S}$ is $V(s)$-enriched, and the reindexing functors are fully faithful enriched under the appropriate change of base. Although this formulation employs the same notion of a monoidal fibration (Definition 3.2) as the base of the enrichment, there are some crucial differences resulting in two separate definitions, [Bun13, 2.4] - [Shu13, 4.1] and Definition 3.8.

First of all, Bunge's and Shulman's approach only concerns enrichment in fibrations over monoidal categories whose tensor product is the cartesian product. This is fundamental for the development and definitions, and not a special case of something more general; of course this was relevant to their examples at hand. On the contrary, for our examples this is evidently not the case: in (21) and (22) the base monoidal categories of the monoidal fibrations, $\mathbf{Comon}(\mathcal{V})$ and $\mathcal{V}$-$\mathbf{Cocat}$, are non-cartesian.

Moreover, the notion of an enriched indexed category roughly expressed in the fibred world, essentially refers to a fibration 'enriched' in another fibration over the same base, approximately depicted as

$$\begin{tikzcd}
\mathcal{A} \arrow[r, dashed, "\text{‘fibrewise' enriched}"] \arrow[dr, "\text{fibred}"'] & \mathcal{V} \arrow[d, "\text{fibred}"] \\
& \mathbb{S}.
\end{tikzcd}$$

In our examples, this fibrewise enrichment is certainly not the case: the fibre categories of our monoidal fibrations, like $\mathbf{Comod}_{\mathcal{V}}(C)$, do not even have a monoidal structure themselves in order to serve as enriching bases. Furthermore, even if in [Shu13, §7] there is a short treatment of changing the indexed monoidal enriching base, and the development in [Bun13] is a special case of this via the identity functor on $\mathbb{S}$, in our context the enriched fibration concept involves simultaneous enrichments between both the total and the base categories of the two fibrations as essential building blocks of the structure:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, dashed, "\text{enriched}"] \arrow[d, "\text{fibred}"'] & \mathcal{V} \arrow[d, "\text{fibred}"] \\
\mathbb{X} \arrow[r, dashed, "\text{enriched}"] & \mathbb{W}
\end{tikzcd}$$

In conclusion, even if there are strong conceptual similarities between the two definitions of an enriched fibration and indexed $\mathcal{V}$-category, our definition does not seem to even restrict in a straightforward way to the case of fibrations over the same base, since the monoidal category $\mathbb{W}$ is not in principle enriched over itself, nor via some sort of an identity or projection functor. As mentioned earlier, future work would aim to clarify how these two theories compare in more detail and depth. What is admittedly striking though is that several different goals and motivations have separately led to the need for a theory that combines fibred structure over a base topos or (cartesian) monoidal category, and enriched structure.

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COOPERATIVE PROPERTIES AND CONNECTED SUM

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Résumé. Un *cycle* est un graphe connexe 2-régulier. Une propriété relative aux cycles est coopérative si elle est valable pour tout cycle qui est la somme mod-2 de deux cycles se croisant dans un chemin nontrivial lorsque les deux sommands ont la propriété. Une telle propriété vaut pour tous les cycles si elle est valable pour les cycles dans une base à sommes connectées (CS), et tous les graphes ont des bases CS. Nous montrons que la commutativité "équivalence naturelle prés" est une propriété coopérative pour les cycles d'un diagramme dans un groupoide et que le critére du cycle de Kolmogorov est coopératif pour les cycles dans les chaînes de Markov.

Abstract. A *cycle* in a graph is a 2-regular connected subgraph. A property of cycles is *cooperative* if it holds for any cycle which is the mod-2 sum of two cycles intersecting in a nontrivial path when both summands have the property. Cooperative properties hold for all cycles when they hold for the cycles in a *connected sum* (CS) basis, and all graphs have CS bases. It is shown that cooperative properties include commutativity up to natural equivalence for cycles in a groupoid diagram and the Kolmogorov cycle criterion for reversibility of an irreducible, stationary, aperiodic Markov chain.

Keywords. Groupoid diagram, commutative up to a natural equivalence, Kolmogorov criterion, reversibility of a Markov chain, robust cycle basis.

Mathematics Subject Classification (2010). 05C38, 18A10, 60J10.

1. Introduction

We apply some concepts from graph theory to commutativity up to natural equivalence for diagrams in a groupoid category and to reversibility of Markov chains. See Harary [5] for undefined graph terminology below.

Properties for the cycles in a graph are exhibited which need only be checked *for the cycles in a basis* vs. *all* cycles in the graph. This avoids a combinatorial explosion. For instance, the 5-dimensional (binary) hypercube Q_5, with 32 vertices and 80 edges, contains more than 51 *billion* distinct cycle-subgraphs but has a basis with 49 elements.

While our program works for some interesting properties, not every cycle basis will do. One needs a *connected sum basis* (CS basis). This will enable construction of cycles in a system which involves topology, order, and hierarchy. Every graph has a CS basis; these bases are defined using the concept of *connected sum of cycles*.

The *connected sum of two cycles* Z_1 and Z_2 in a graph G is defined precisely when the intersection of the cycles is a nontrivial path, and, in that case, it is the symmetric difference of the edge sets (that is, the mod-2 sum).

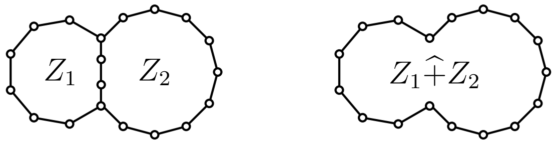


Figure 1: The connected sum of two cycles.

We write $Z_1 \widehat{+} Z_2$ for connected sum. The connected sum of two cycles is always a cycle, but their ordinary mod-2 sum is only guaranteed to have all vertices of even degree. Connected sum is commutative but not associative. The *connected sum of a sequence of cycles*, when it is defined, uses left-most parenthesization. So the sequence of cycles (Z_1, Z_2, Z_3) has a connected sum iff $Z_1 \cap Z_2$ and $(Z_1 \widehat{+} Z_2) \cap Z_3$ are nontrivial paths. Starting with a set $\mathcal{S}$ of cycles, one can form all possible connected sums for sequences from the set, and we call the resulting family of cycles the *robust closure* $\rho(S)$ of $\mathcal{S}$.

The edge sets of the even-degree subgraphs of G determine its *cycle space*, an $\mathbb{F}_2$-vector space, usually denoted $\mathbb{Z}(G)$, where addition mod-2 is symmetric difference. As every even degree graph has an edge-disjoint de-composition into cycles, there are bases for $\mathbb{Z}(G)$ consisting only of cycles. These are called *cycle bases*. See, e.g., [1, 6, 13, 17]. We introduced the following concepts in [7].

A cycle basis for a graph G is a *connected sum basis* if it can be used to construct every cycle in G by iteratively taking robust closure. Note that *topology* is involved in the definition of connected sum of two cycles, *order* in the choice of a sequence of cycles whose connected sum is defined giving the desired cycle, and *hierarchy* in the recursive construction of cycles.

Formally, we define a *property* of cycles to be a subset $\mathcal{P}$ of a graph's cycles. A *cooperative property* is one for which

$$Z_1, Z_2 \in \mathcal{P} \implies Z_1 \hat{+} Z_2 \in \mathcal{P}. \tag{1}$$

Properties that hold for the cycles in a CS basis, and that are cooperative, will hold for all cycles. In contrast, for less carefully controlled cycle sums, where partial summands need not even be connected and where intersections of cycles can be arbitrary, properties holding for the cycles in a basis may not spread to the other cycles. Commutativity of cycles in a diagram turns out to be cooperative.

Cooperative properties involve additional structure superimposed on the graph. For diagram commutativity, this structure consists of a suitable diagram in a groupoid. Later we consider the structure of a Markov chain.

It may seem unnecessary to have a property for all cycles guaranteed by the members of a *special* cycle basis when it is almost a default assumption that properties of a graph related to cycles need only be checked for members of an *arbitrary* cycle basis. This belief could be due to two well-known examples:

Kirchoff's voltage law (*the sum of the voltages around any cycle is zero.*) By a linear-algebra argument, one need only check for the cycles in any basis.

A graph is bipartite if and only if each cycle has even length. Counting shows this holds for all cycles if it holds for the cycles in any basis.

A third example might come to mind. Many of the diagrams arising in elementary category theory and also in homological algebra are planar.

Plane diagrams commute iff the region boundaries commute. The region boundaries (of all the bounded regions) do constitute a cycle basis.

However, this last example, considered more carefully, shows that not all cycle bases will suffice. We provided a nonplanar diagram and a non-cs cycle basis where the cycles of the basis commute but some other cycles do *not* commute; see [4] and Figure 3 below. Further, the region-boundaries basis is a connected sum basis [10]. We showed in [4] that commutativity is a cooperative property so diagrams commute if (and only if) all cycles in a connected sum basis commute.

In this paper, the applicability of cooperative properties is demonstrated with two more examples: *commutativity up to a natural equivalence* is a cooperative property of cycles in a groupoid diagram, and the *Kolmogorov cycle criterion* (KCC) is a cooperative property of cycles in a Markov chain.

In §2 below, we review CS bases and §3 extends the machinery to directed graphs (digraphs). The results are applied in §4 to groupoid diagrams and in §5 to Markov chains; we conclude with a brief discussion.

2. Background on connected sum

For any graph H, we write $E(H)$ for the edge-set. The **connected sum** $Z_1 \widehat{+} Z_2$ of two cycle subgraphs of a graph G is just the usual mod-2 sum (i.e., symmetric difference of edge sets) but it is only defined when $Z_1 \cap Z_2$ is a path containing at least one edge.

Let $\mathrm{Cyc}(G)$ denote the set of all cycle-subgraphs of a graph G and let $\emptyset \neq \mathcal{S} \subseteq \mathrm{Cyc}(G)$. A sequence of not necessarily distinct cycles from $\mathcal{S}$

$$(Z_1, Z_2, \ldots, Z_k) \tag{2}$$

is called **$\mathcal{S}$-admissable** and its connected sum is defined by

$$\widehat{+}(Z_1, Z_2, \ldots, Z_k) := (\cdots((Z_1 \widehat{+} Z_2) \widehat{+} Z_3) \cdots) \widehat{+} Z_k. \tag{3}$$

provided that its members have pairwise intersections as specified so that all of the partial sums on the RHS of (3) are connected sums. Hence, the connected sum of a sequence is defined iff the sequence is $\mathcal{S}$-admissible.

The **robust closure** of $\mathcal{S}$ is the set of all cycles in G which are connected sums of $\mathcal{S}$-admissable sequences

$$\rho(\mathcal{S}) := \Big\{ Z : \exists\, \ell \geq 1, Z_i \in \mathcal{S},\ 1 \leq i \leq \ell,\ Z = \widehat{+}(Z_1, Z_2, \ldots, Z_\ell) \Big\}. \tag{4}$$

By definition, $\mathcal{S} \subseteq \rho(\mathcal{S}) \subseteq \mathrm{Cyc}(G)$ and ρ preserves inclusion. If $\mathcal{S} = \mathcal{B}$ is a cycle basis for which $\rho^k(\mathcal{B}) := \rho(\rho^{k-1}(\mathcal{B})) = \mathrm{Cyc}(G)$ for a positive integer k, then $\mathcal{B}$ is a **connected sum (CS) basis**; the **depth** of $\mathcal{B}$ is the least such k, the number of iterated robust closures needed to generate all cycles.

Examples of CS bases of depths 1 and 2 are given in [10] and [7]. The depth 1 case (called *robust* bases) includes plane graphs and complete graphs.

It was shown in [3] that for $n \geq 8$, the complete bipartite graph $K_{n,n}$ does not have *any* robust basis, but the basis consisting of all 4-cycles through a fixed edge from [7] is a CS basis. In [4], we gave a general method for constructing cs bases, involving *ear decompositions* [14], based on a theorem of Whitney [19], .

Recall that a **property** of cycles is a subset $\mathcal{P} \subseteq \mathrm{Cyc}(G)$. A property $\mathcal{P}$ is **cooperative** provided $Z_1, Z_2 \in \mathcal{P} \implies Z_1 \widehat{+} Z_2 \in \mathcal{P}$. The following is shown in [10], see also [7], [4].

Theorem 2.1. *If $\mathcal{P}$ is any cooperative property and $\mathcal{P}$ holds for all cycles in a connected sum basis for a graph G, then $\mathcal{P}$ holds for every cycle in G.*

3. Connected sum of directed cycles

It will be convenient to describe a directed versions of connected sum and cooperativity. We collect a few related definitions.

A *digraph* D is an ordered pair (V, A), where $V \neq \emptyset$ is a finite set of vertices and a set of arcs $A \subseteq V \times V$. The *underlying graph* $U(D)$ of D has the same vertex set with $vw \in E(U(D))$ iff $(v, w) \in A$ or $(w, v) \in A$. We also write $a \in A$ with $s(a) = v$ (v is the *source*) $t(a) = w$ (w is the *target*). Let $\deg_+(v)$ denote the *in-degree* of a vertex v which is the number of arcs a with $t(a) = v$ and let $\deg_-(v)$ denote *out-degree* of v, the number of arcs a with $s(a) = v$.

A *quiver* is an ordered pair (V, A), where $V \neq \emptyset$ is a set of vertices and A is a multiset, allowing for each $(v, w) \in V \times V$ a family $a_j, j \in J(v, w)$, of arcs, all with $s(a_j) = v, t(a_j) = w$. Quivers have an underlying multigraph and can be infinite. We write $D = (V, A)$ for both digraphs and quivers.

If D is a digraph (or quiver), then $U(D)$ will denote the *underlying* graph (or multigraph) obtained by discarding the direction of the arcs, replacing them by the corresponding edges.

A *directed walk (diwalk)* in D of length $\ell \geq 0$ is a sequence of vertices $(v_0, v_1, \dots, v_\ell)$ and a sequence of arcs $(a_1, a_2, \dots, a_\ell)$ such that

$$t(a_i) = v_i = s(a_{i+1}),\ 1 \leq i \leq \ell - 1,\ \ s(a_1) = v_0,\ \ t(a_\ell) = v_\ell.$$

If all vertices are distinct, the diwalk is called a *dipath.* If $v_0 = v_\ell$, the diwalk is called a *closed diwalk.* A *dicycle* is a closed diwalk where $v_i = v_j$ for $i < j$ implies $i = 0$, $j = \ell$, and $\ell \geq 1$. Loops are dicycles of length 1.

Given any graph G, one can form an *orientation digraph* by choosing for each edge of G exactly one of the two possible arcs.

Note that a digraph is a dicycle iff it is an orientation of a cycle such that

$$\deg_+(v) = 1 = \deg_-(v)$$

for all vertices v. Given a graph G, one forms the *symmetric digraph induced by* G, denoted $D(G)$, by replacing every edge of G by *both* possible arcs, so $D(G)$ is the union of the set of possible orientations. A digraph is *strongly connected* if every ordered pair of vertices is joined by a dipath from the first to the second. Hence, a dicycle is a minimal strongly connected digraph.

Two orientation digraphs that have underlying graphs sharing at least one edge will be in exactly one of the following relations with respect to their common edges: *consistently oriented* (in agreement on all); *oppositely oriented* (disagreeing on all); or *variably oriented.* We use this mostly for dicycles.

Define the **connected sum** $D_1 \widehat{+} D_2$ of two dicycles D_1, D_2 when both of the following hold: the underlying cycles $U(D_1)$ and $U(D_2)$ meet in a non-trivial path and the dicycles are oppositely oriented. In this case, $D_1 \widehat{+} D_2$ is the unique dicycle orientation of the cycle $U(D_1) \widehat{+} U(D_2)$ such that $D_1 \widehat{+} D_2$ is consistently oriented with D_1 and D_2. See Figure 2. We extend connected sum to sequences of dicycles, using leftmost parenthesization analogously with the undirected case.

Note that in any connected sum $\widehat{+}(D_1, D_2, D_3)$ of dicycles for which $U(D_3) \in \{U(D_1), U(D_2)\}$, the repeated cycle appears with both dicycle orientations.

A property of dicycles is *cooperative* if it holds for the connected sum of two dicycles whenever it holds for the summands.

For each cycle Z of G, we define

$$Z^{\pm} := \{Z^+, Z^-\},$$

where Z^+, Z^- are the oppositely-oriented dicycle orientations of Z.

If $\mathcal{S}$ is a set of cycles, we write the corresponding set of dicycles as

$$\mathcal{S}^{\pm} := \bigcup_{Z \in \mathcal{S}} Z^{\pm}$$

The orientation of a dicycle is determined on any nontrivial path subgraph. Because of the constrained intersections of connected sum sequences, one can choose the orientations so that each successive pairwise connected sum is the sum of two *oppositely oriented* dicycles. When this is done, the connected sum of the two underlying cycles is oriented consistently with the summands. The following theorem is sufficient to confirm this.

Theorem 3.1. *Let C be any cycle in a graph G with $C = \widehat{+}(Z_1, Z_2, \ldots, Z_k)$ and let C^ε be any orientation of C. Then there exists a unique sequence $(\varepsilon_1, \ldots, \varepsilon_k) \in \{+, -\}^k$ such that*

$$C^\varepsilon = \widehat{+}(Z_1^{\varepsilon_1}, Z_2^{\varepsilon_2}, \ldots, Z_k^{\varepsilon_k}). \tag{5}$$

Proof. By induction on k. The claim is trivial for $k = 1$ where $C = Z_1$ and ε_1 is determined by the orientation of C. Assume the result for $k - 1$ and suppose that C has a connected sum (3). Let $Z_k^{\varepsilon_k}$ be the unique orientation of Z_k consistent with C^ε. Put $C_k^\varepsilon := C^\varepsilon \widehat{+} Z_k^{-\varepsilon_k}$. Then

$$C_k = \widehat{+}(Z_1, \ldots, Z_{k-1}).$$

By the inductive hypothesis, we have unique $\varepsilon_1, \ldots, \varepsilon_{k-1}$ and (5) holds. □

The results here show that, as 1-chains over the integers, dicycles can be built so that all coefficients are 0, 1, or -1, using a connected sum basis. Further, a cooperative property holds for all dicycles in a digraph when it holds for all the dicycles in $\mathcal{B}^{\pm}$ where $\mathcal{B}$ is a CS basis for $U(D)$.

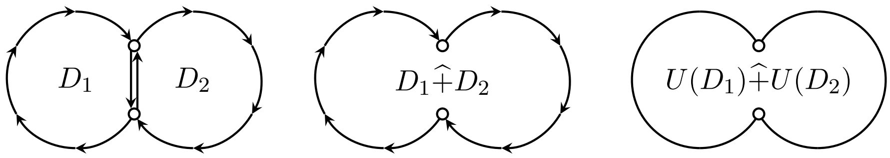


Figure 2: Two compatible dicycles (left), their connected sum (center) and the connected sum of their underlying graphs (right).

4. Diagrams in groupoids

In this section, we review our previous result on cooperativity of commutativity in groupoid diagrams and show how to extend this to groupoid diagrams which are only commutative up to a natural transformation.

Let D be a digraph and let $\mathcal{C}$ be a category. A *diagram* δ *of shape* D in $\mathcal{C}$ is a homomorphism from D to the underlying quiver of $\mathcal{C}$. For example, the digraph with vertices a, b, c, d and arcs $(a, b), (b, c), (a, d), (d, c)$ could be mapped by sending all vertices to a fixed object X in $\mathcal{C}$, with the arcs associated with various morphisms $X \to X$. See Mac Lane [15, p. 8].

Two dipaths in a digraph with a common source and a common target vertex are called *parallel*; in the extreme case, they are *internally disjoint* but this is not required. A diagram δ of shape D in $\mathcal{C}$ is *parallel-commutative* if for any two parallel dipaths in D, the corresponding dipaths in $\mathcal{C}$ give the same composite morphism. But any diagram parallel-commutes if its shape has no two distinct parallel paths (e.g., a cycle in which arcs alternate in direction).

Instead, we shall consider a stronger type of commutativity which, however, is only defined when the morphisms of the diagram are all invertible; that is, when the category $\mathcal{C}$ is a *groupoid* $\mathcal{G}$ [15, p. 20], [18, pp. 45, 134].

For a diagram

$$\delta : D \to \mathcal{G}$$

in a groupoid $\mathcal{G}$, we say that δ *groupoid-commutes* (g-commutes) if the composition around any cycle of the underlying graph of D induces an identity morphism in $\mathcal{G}$ [7], [4]. We assume that arc $x = (v, w)$ of D, traversed in proper order while going around the cycle, produces the morphism $\delta(x)$ but traversed in reverse, produces the morphism $\delta(x)^{-1}$ from $\delta(w)$ to $\delta(v)$.

It is easy to check that the groupoid commutativity of a cycle is independent of which traversal is chosen (i.e., of starting point and of clockwise vs counterclockwise orientation). But the particular identity morphism may depend on starting point.

Indeed, consider the following case, which is sufficient. Let the diagram have two objects X and Y with morphisms $a : X \to Y$ and $b : Y \to X$. Suppose that (i) $ba = 1_X$, where we write composition from right to left as usual. Hence, (ii) $ab = 1_Y$. Also, (i) implies (iii) $a^{-1}b^{-1} = 1_X$.

The following was shown in [10].

Theorem 4.1. *Groupoid commutativity is a cooperative property.*

The significance of the result follows from the fact that in [4], we exhibited a diagram $\delta : D \to \mathbb{C}^*$, where D is the orientation of the complete bipartite graph $K_{3,3}$ shown below, $\mathbb{C}^*$ is the group of nonzero complex numbers, and each morphism is rotation by $\frac{2\pi}{3}$, together with a particular basis $\mathcal{B}$ of $U(D)$, indicated on the right, such that δ commutes on the members of $\mathcal{B}$ (and indeed on any 6-cycle but does *not* commute on the 4-cycles. See Figure 3.

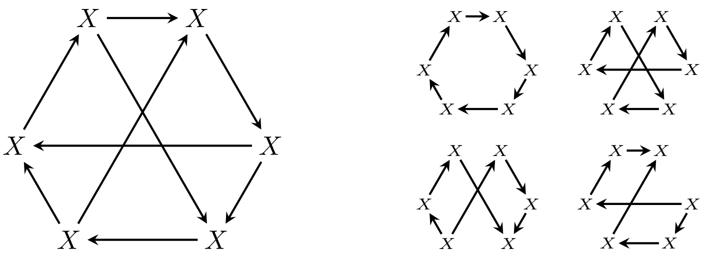


Figure 3: A noncommutative diagram (left) that commutes on a cycle basis (right). This basis is not a CS basis because no two of its members are compatible.

Assume where necessary that categories are *small* with only a *set* of objects. A *natural transformation* $\nu : F \Rightarrow G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is a family of $\mathcal{D}$-morphisms indexed by the objects of $\mathcal{C}$

$$\{\nu_x : Fx \to Gx\}_{x \in Obj(C)}$$

such that for every $\mathcal{C}$-morphism $\alpha : x \to y$, we have

$$\nu_y \circ F(\alpha) = G(\alpha) \circ \nu_x;$$

that is, all the associated squares commute. A *natural equivalence* is a natural transformation all of whose arrows are equivalences. A natural transformation is an equivalence iff it is invertible as a natural transformation.

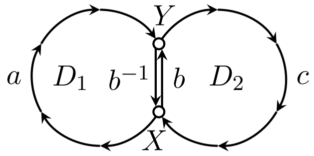
Let CAT be a small subcategory of the category of all small categories with functors as morphisms. We shall consider a fixed groupoid subcategory **G** of CAT. Let D be an orientation digraph and let $\delta : D \to \mathbf{G}$ be any diagram. Then one can extend δ in a unique way to a diagram $\widehat{\delta}$ on the symmetric digraph which D induces

$$\widehat{\delta}: \ \widehat{D} := D(U(D)) \to \mathbf{G} \tag{6}$$

Commutativity up to natural equivalence for diagrams in $\mathbf{G}$ means that the composition functor around any dicycle in $\widehat{D}$ is naturally equivalent to the appropriate identity functor.

Theorem 4.2. *Commutativity up to natural equivalence is cooperative.*

Proof. Without loss of generality, we take two dicycles D_1 and D_2 which are oppositely oriented with $P := U(D_1) \cap U(D_2)$ a nontrivial path. Let P^+ be the orientation of P consistent with D_1 and let Y and X, resp., denote the first and last vertex of P^+. We write a for the composition of the morphisms along the path $D_1 - P^+$ from X to Y, and b for the composition along P^+ from Y to X. For the other dicycle, we do the same thing: let c denote the composition of the morphisms along $D_2 - P^-$ from Y to X. By definition, the composition of the morphisms in P^- from X to Y is b^{-1}.



Suppose now that both dicycles commute up to natural equivalences; let

$$\nu : b \circ a \Rightarrow 1_X \quad \text{and} \quad \mu : c \circ b^{-1} \Rightarrow 1_X. \tag{7}$$

where ν and μ are natural equivalences. We define a composition $\tau := \mu \Box \nu$ which is both a natural transformation and an equivalence; for every $x \in X$,

$$\tau_x := \mu_x \circ \Big((c \circ b^{-1})(\nu_x) \Big), \tag{8}$$

which is an X-morphism from $\Big((c \circ b^{-1}) \circ (b \circ a)\Big)(x) = (c \circ a)(x)$ to x. Hence, $D_1 \widehat{+} D_2$ commutes up to natural equivalence. This is illustrated in the following four commutative squares.

The first square expresses the fact that ν is a natural transformation; the second applies the functor cb^{-1}; the third expresses the naturality of μ; and the fourth is the (vertical) composition of the second and third squares.

$$\begin{tikzcd}
ba(x) \arrow[r, "ba(\alpha)"] \arrow[d, "\nu_x"'] & ba(x') \arrow[d, "\nu_{x'}"] \\
x \arrow[r, "\alpha"'] & x'
\end{tikzcd}$$

$$\begin{tikzcd}
ca(x) \arrow[r, "ca(\alpha)"] \arrow[d] & ca(x') \arrow[d] \\
cb^{-1}(x) \arrow[r, "cb^{-1}(\alpha)"'] & cb^{-1}(x')
\end{tikzcd}$$

$$\begin{tikzcd}
cb^{-1}(x) \arrow[r, "cb^{-1}(\alpha)"] \arrow[d, "\mu_x"'] & cb^{-1}(x') \arrow[d, "\mu_{x'}"] \\
x \arrow[r, "\alpha"'] & x'
\end{tikzcd}$$

$$\begin{tikzcd}
ca(x) \arrow[r, "ca(\alpha)"] \arrow[d, "\tau_x"'] & ca(x') \arrow[d, "\tau_{x'}"] \\
x \arrow[r, "\alpha"'] & x'
\end{tikzcd}$$

This completes the proof. □

5. Application to Markov chains

In this section, we consider the *Kolmogorov cycle criterion* (KCC) for the reversibility of discrete Markov chains. See, e.g., Kelly [11, chap. 1].

Let $(X(t), t \in T)$ be a Markov chain with a finite or countable state-space $\mathcal{S} \subseteq \mathbb{N}_+$. A Markov chain is *stationary* if for all $n \in \mathbb{N}_+$ and all $\tau, t_1, \ldots, t_n \in T$,

$$\Big(X(t_1), X(t_2), \ldots, X(t_n)\Big) \sim \Big(X(\tau + t_1), X(\tau + t_2), \ldots, X(\tau + t_n)\Big),$$

where $A \sim B$ denotes the relation of equality of distributions.

A Markov chain is *reversible* if for all $n \in \mathbb{N}_+$ and all $\tau, t_1, \ldots, t_n \in T$,

$$\Big(X(t_1), X(t_2), \ldots, X(t_n)\Big) \sim \Big(X(\tau - t_1), X(\tau - t_2), \ldots, X(\tau - t_n)\Big).$$

Reversibilty implies stationarity [11, p. 5]. Also, a stationary Markov chain $X(t)$ is *time-homogeneous*; i.e., for all $\tau, t_1, t_2 \in T$ and all $j, k \in \mathcal{S}$,

$$P(X(t_1 + \tau) = k | X(t_1) = j) = P(X(t_2 + \tau) = k | X(t_2) = j).$$

Write $p(j,k)$ for $P\Big(X(t+1) = k \,\Big|\, X(t) = j\Big)$ as it is independent of t and for every state j the state transitions describe all events, so $\sum_{k \in \mathcal{S}} p(j,k) = 1$.

We define the *communications digraph* $D(X)$ of a Markov chain X to be the digraph with vertex set $\mathcal{S}$, where (j,k) is an arc iff $p(j,k) > 0$. A Markov chain is *irreducible* if and only if its communications digraph is *strongly connected* (there is a positive probability of a dipath joining each pair of states).

A Markov chain is *periodic* if there exists $d > 1$, $d \in \mathbb{N}_+$, such that

$$P\left(X(t+\tau) = j \,\middle|\, X(t) = j\right) > 0 \quad \Rightarrow \quad d|\tau$$

($d|\tau$ means d divides τ). If the chain is not periodic, it is called *aperiodic*.

The following result is well-known (e.g., [11, p. 6]).

Theorem 5.1. *A stationary, irreducible, and aperiodic Markov chain $X(t)$ is reversible if and only if there exists a function μ on $\mathcal{S}$ with $\mu(j) > 0$ for all j and $\sum_{j \in \mathcal{S}} \mu(j) = 1$ such that, for all $j, k \in \mathcal{S}$,* detailed balance *holds:*

$$\mu(j)p(j,k) = \mu(k)p(k,j). \tag{9}$$

As Kelly [11, p. 21] puts it, "... *it is natural to ask whether we can establish the reversibility of a process directly from the transition rates alone.*"

The *Kolmogorov Cycle Criterion* (KCC) makes this possible. (There are also extensions to continuous-time Markov processes which we omit here.) The KCC for a closed diwalk

$$\omega := (j_1, j_2, \ldots, j_n, j_1)$$

asserts that

$$P(\omega) = P(\omega^{op}), \tag{10}$$

where ω^{op} denotes the diwalk oppositely orientated to ω,

$$\omega^{op} = (j_1, j_n, \ldots, j_2, j_1)$$

and the probability of a diwalk is the product of the probabilities of its arcs,

$$P(\omega) := p(j_1, j_2) \ldots p(j_{n-1}, j_n)\, p(j_n, j_1).$$

We sketch Kelly's argument [11, p. 22] for the original result of [12].

Theorem 5.2. *A stationary, irreducible, and aperiodic Markov chain $X(t)$ is reversible if and only if the KCC* (10) *holds for all closed walks ω in $D(X)$.*

Proof. If X is reversible, then by the previous theorem, there exists a positive measure on $\mathcal{S}$ which satisfies detailed balance (9) for each oppositely oriented pair of arcs in $D(X)$. Take the product of the set of detailed-balance equations corresponding to the arcs in ω and divide by the product of the (positive!) measures of the states which occur (in reverse order) for the two opposing diwalk orientations. The result is equation (10).

Conversely, suppose that (10) holds for every closed walk. One defines a positive measure μ as follows. Select an arbitrary base-point j_0 in $\mathcal{S}$ and let $j \in \mathcal{S}$. As X is irreducible, there exists a diwalk (in fact, a dipath) ω from j to j_0 in the communications digraph $D(X)$. Define $\mu(j)$ by the following equation,

$$\mu(j) \;=\; B\,\frac{P(\omega^{op})}{P(\omega)}, \tag{11}$$

where B is an arbitrary positive constant that can later be adjusted to give a probability measure.

To see that $\mu(j)$ does not depend on the path from j to j_0, let ζ denote another j-j_0-dipath in $D(X)$. Using "$*$" to concatenate diwalks, the equation

$$\frac{P(\omega^{op})}{P(\omega)} = \frac{P(\zeta^{op})}{P(\zeta)} \tag{12}$$

holds by (10) as $P(\omega)P(\zeta^{op}) = P(\omega * \zeta^{op}) = P(\zeta * \omega^{op}) = P(\omega^{op})P(\zeta)$.

Also, $\mu(j) > 0$; indeed, as there is a diwalk η in $D(X)$ from j_0 to j, the concatenation $\omega * \eta$ is a closed diwalk of positive probability, so by (10), $P(\eta^{op}) > 0$; hence, with η^{op} instead of ω in (11), $\mu(j) > 0$.

It is routine to show that each arc satisfies (9). □

The following result applies our theory to obtain a more efficient characterization of reversibility.

Theorem 5.3. *A stationary, irreducible, and aperiodic Markov chain* $X(t)$ *is reversible if and only if the KCC* (10) *holds for all dicycles in* $\mathcal{B}^{\pm}$, *where* $\mathcal{B}$ *is any* CS *basis of* $U(D(X))$

Proof. One direction is trivial in view of Kolmogorov's theorem. In the opposite direction, suppose that his criteria hold for all the dicycles of a CS basis. Once we've established the next theorem, it follows that the KCC holds for all dicycles and hence X is reversible. □

Theorem 5.4. *The KCC is a cooperative property for dicycles in the communications digraph of a stationary, irreducible, and aperiodic Markov chain.*

Proof. We follow the same outline as in the proof of Theorem 4.2. Take oppositely oriented dicycles D_1 and D_2 with $P := U(D_1) \cap U(D_2)$ a nontrivial path. Let P^+ be the orientation of P consistent with D_1 and let Y and X, resp., denote the first and last vertex of P^+. We write α for the dipath $D_1 - P^+$ from X to Y, and β for the dipath P^+ from Y to X. Let γ denote the dipath $D_2 - P^-$ from Y to X. As D_1 and D_2 satisfy the KCC, we have the equations

$$P(\alpha)P(\beta) = P(\alpha^{op})P(\beta^{op})$$

$$P(\gamma)P(\beta^{op}) = P(\gamma^{op})P(\beta)$$

Multiplying the two equations and cancelling the positive term $P(\beta)P(\beta^{op})$ gives the KCC for $D_1 \widehat{+} D_2$, $P(\alpha)P(\gamma) = P(\alpha^{op})P(\gamma^{op})$. □

We now apply connected sum theory to an exercise [11, Ex. 1.5.2, p. 24].

Proposition 5.5 (Kelly). *Let $X(t)$ be a stationary, irreducible, and aperiodic Markov chain. If $j_0 \in \mathcal{S}$ is such that for every $j \in \mathcal{S}$, we have $p(j, j_0) > 0$, then X is reversible iff the KCC holds for all 3-cycles through j_0; that is, for all $j_1 \neq j_2 \in \mathcal{S} \setminus \{j_0\}$,*

$$p(j_0, j_1)\, p(j_1, j_2)\, p(j_2, j_0) \;=\; p(j_0, j_2)\, p(j_2, j_1)\, p(j_1, j_0).$$

Proof. The bouquet of triangles centered at j_0 is a CS basis for K_n for every $n \geq 3$ by [7, Proposition 1], and any dicycle in $D(X)$ is contained in some $D(K_n)$. □

6. Discussion

The implications of cooperativity for commutativity are interesting from the perspective of *the information theory of mathematics*. For example, one can define the structure of a group by a set of commutative diagrams and it is well-known that only a subset need to be checked. As with application of the Cube Lemma [16, p. 43], savings are modest. However, in a more complex situation, savings might be substantial, cf. [7], [9].

Perhaps the theory of diagrams which commute or commute up to natural equivalence could decrease the complexity of verifying commutativity for the groupoid diagrams involved in higher category theory and adjointness.

Indeed, the only result in the literature, of which we are aware, with a similar direction to ours is in Gray [2], who proved that a hypercube diagram in a 2-category is 2-commutative if and only if all its Q_3-subgraphs are 2-commutative.

Are there applications of connected sum theory to natural processes in biology and physics? The notion that cycles can be generated in a hierarchical fashion so that one must first prepare the ingredients in a previous stage before combining them in a connected sum could be a desirable feature.

Also, are there implications for the random spread of cooperative properties? Given a fixed probability that any one cycle will have the property, if the number of cs bases grows sufficiently rapidly as a function of the order of a graph family, then we might expect that there is a threshold number of vertices above which the property almost surely holds.

What other properties of cycles are cooperative?

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