cahiers de topologie et géométrie différentielle catégoriques

créés par CHARLES EHRESMANN en 1958
dirigés par Andrée CHARLES EHRESMANN

VOLUME LIX - 3, 3ème trimestre 2018

AMIENS

Cahiers de Topologie et Géométrie Différentielle Catégoriques

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33 rue Saint-Leu, F-80039 Amiens.

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES
VOLUME LIX-3 (2018)

A STUDY OF PENON WEAK n-CATEGORIES, PART 1: MONAD INTERLEAVING

Thomas COTTRELL

Résumé. L'article propose une construction alternative de la monade utilisée par Penon pour définir les n-catégories faibles. La monade de Penon ajoute deux éléments de structure supplémentaires à une structure d'ensemble n-globulaire : une structure de magma donnant une composition, et une structure de contraction donnant une cohérence. Ces deux structures sont ajoutées à l'aide d'une approche d'entrelacement, suivant la méthode utilisée par Cheng pour construire l'opérade de Leinster pour les ω-catégories faibles. Nous concluons en utilisant notre construction pour donner une description explicite de l'opérade n-globulaire pour les n-catégories faibles de Penon.
Abstract. We give an alternative construction of the monad used by Penon to define weak n-categories. Penon's monad adds two pieces of extra structure to an n-globular set: a magma structure, giving composition, and a contraction structure, giving coherence. We add these two structures using an interleaving approach, following the method used by Cheng to construct Leinster's operad for weak ω-categories. We conclude by using our construction to give an explicit description of the n-globular operad for Penon weak n-categories.
Keywords. n-category, higher-dimensional category, monad interleaving, operad.
Mathematics Subject Classification (2010). 18A40, 18C15, 18D05, 18D50.

1. Introduction

The main purpose of this paper is to give a new construction of the monad used by Penon to define weak n-categories. Penon weak n-categories, introduced in [15], are defined as the algebras for the monad induced by a certain non-monadic adjunction. The left adjoint of this adjunction, which Penon originally constructed using computads, freely adds two pieces of structure: a "magma structure", which gives composition in a weak n-category, and a "contraction structure", which gives coherence. In our construction we add these two structures by a process of monad interleaving. The use of this method to construct Penon's left adjoint was suggested by Cheng in [7], who used monad interleaving to construct Leinster's globular operad for weak ω-categories; thus we hope that this new construction should facilitate a comparison between the two definitions.

In Section 2 we recall the definition of Penon weak n-categories, along with the necessary preliminaries. In Section 3 we give the interleaving construction of the left adjoint in the definition of Penon weak n-categories. Finally, in Section 4 we show that our construction gives an explicit description of the n-globular operad whose algebras are Penon weak n-categories; the existence of this operad was proved by Batanin [3].

Throughout the paper we use a variant of Penon's definition of weak n-categories, given in [3, 9]. Penon defined weak n-categories as the algebras for a monad on the category of *reflexive* globular sets (globular sets in which each cell has a putative identity cell at the dimension above). In [9] Cheng and Makkai observed that, in the finite dimensional case, Penon's definition did not encompass certain well-understood examples of weak n-categories, such as braided monoidal categories, but that this could be remedied by using globular sets instead of reflexive globular sets. Note that Penon originally gave his definition in the case $n = \omega$, whereas we take n to be finite (this modification of the definition for finite n is standard, see [12, 9]). Throughout the paper, the letter n always denotes a fixed natural number. It is straightforward to adapt our construction to the case $n = \omega$; we explain how to do this at the end of Section 3.

Acknowledgements

This paper is adapted from material from my PhD thesis, and the research it contains was funded by a University of Sheffield studentship. I would like to thank my supervisor Eugenia Cheng for her invaluable guidance and support. I would also like to thank Nick Gurski, Roald Koudenburg, Jonathan Elliott, Tom Athorne, Alex Corner and Ben Fuller for many useful discussions.

2. Definition of Penon weak n-categories

In this section we recall the non-reflexive variant of Penon's definition of weak n-category [15, 3, 9]. The idea of Penon's definition is to weaken the well-understood notion of strict n-category by means of a "contraction". To do this Penon considers "n-magmas": n-globular sets equipped with binary composition operations that are not required to satisfy any axioms (apart from the usual source and target conditions). He then asks when an n-magma is "coherent enough" to be considered a weak n-category. To answer this question he uses the fact that every strict n-category has an underlying n-magma to compare n-magmas with strict n-categories by considering maps

$$X \xrightarrow{f} S,$$

where X is an n-magma, S is the underlying n-magma of a strict n-category, and f preserves the n-magma structure. Penon defines a notion of a contraction on such a map, which lifts identities in S to equivalences in X, ensuring that the axioms that hold in S hold up to equivalence in X; by analogy with contractions in the topological sense, we can think of the axioms as holding "up to homotopy" in X.

Penon then defines a category whose objects are maps $f \colon X \to S$ as above equipped with contractions; we denote this category by $\mathcal{Q}$, following the notation of Leinster [12]. An object of $\mathcal{Q}$ can be thought of as consisting of an n-magma X and a way of contracting it down to a strict n-category S. There is a forgetful functor $\mathcal{Q} \to n\text{-}\mathbf{GSet}$ sending an object of $\mathcal{Q}$ to the underlying n-globular set of its magma part. This functor has a left adjoint, which induces a monad on n-$\mathbf{GSet}$, and a Penon weak n-category is defined to be an algebra for this monad.

We begin by recalling the definition of an n-globular set, the underlying data for a Penon weak n-category.

Definition 2.1. The *n-globe category* $\mathbb{G}$ is defined as the category with

- objects: natural numbers $0, 1, \ldots, n-1, n$;
- morphisms generated by, for each $1 \leq m \leq n$, morphisms

$$\sigma_m, \tau_m \colon (m-1) \to m$$

  such that $\sigma_{m+1}\sigma_m = \tau_{m+1}\sigma_m$ and $\sigma_{m+1}\tau_m = \tau_{m+1}\tau_m$ for $m \geq 2$ (called the "globularity conditions").

An *n-globular set* is a presheaf on $\mathbb{G}$. We write n-$\mathbf{GSet}$ for the category of n-globular sets $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$.

For an n-globular set $X \colon \mathbb{G}^{\mathrm{op}} \to \mathbf{Set}$, we write s for $X(\sigma_m)$, and t for $X(\tau_m)$, regardless of the value of m, and refer to them as the source and target maps respectively. We denote the set $X(m)$ by X_m. We say that two m-cells x, $y \in X_m$ are *parallel* if $s(x) = s(y)$ and $t(x) = t(y)$; note that all 0-cells are considered to be parallel.

We now recall the definition of an n-magma, an n-globular set equipped with composition operations.

Definition 2.2. An *n-magma* (or simply *magma*, when n is fixed) consists of an n-globular set X equipped with, for each m, p, with $0 \leq p < m \leq n$, a binary composition function

$$\circ_p^m \colon X_m \times_{X_p} X_m \to X_m,$$

where $X_m \times_{X_p} X_m$ denotes the pullback

$$\begin{tikzcd} X_m \times_{X_p} X_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "s"] \\ X_m \arrow[r, "t"'] & X_p \end{tikzcd}$$

in $\mathbf{Set}$; these composition functions must satisfy the following source and target conditions:

- if $p = m - 1$, given $(a, b) \in X_m \times_{X_p} X_m$,

$$s(b \circ_p^m a) = s(a),\ t(b \circ_p^m a) = t(b);$$

- if $p < m - 1$, given $(a, b) \in X_m \times_{X_p} X_m$,

$$s(b \circ_p^m a) = s(b) \circ_p^{m-1} s(a),\ t(b \circ_p^m a) = t(b) \circ_p^{m-1} t(a).$$

A map of n-magmas $f\colon X \to Y$ is a map of the underlying n-globular sets such that, for all m, p, with $0 \leq p < m \leq n$, and for all $(a, b) \in X_m \times_{X_p} X_m$,

$$f(b \circ_p^m a) = f(b) \circ_p^m f(a).$$

We write n-$\mathbf{Mag}$ for the category whose objects are n-magmas and whose morphisms are maps of n-magmas.

Observe that every strict n-category has an underlying n-magma, and we have a forgetful functor

$$n\text{-}\mathbf{Cat} \longrightarrow n\text{-}\mathbf{Mag}.$$

We now recall the definition of a contraction on a map of n-globular sets $f\colon X \to S$, where S is the underlying n-globular set of a strict n-category. Note that this definition does not require a magma structure on X. We must treat dimension n slightly differently, since there is no dimension $n+1$; to do so, we define a notion of a "tame" map of n-globular sets (the terminology is due to Leinster [13, Definition 9.3.1]), which ensures that we have equalities between n-cells where we would normally expect contraction $(n + 1)$-cells.

It is common to express the definition of contraction in terms of lifting conditions [3, 4, 10]; however, we express the definition using pullbacks of sets since this approach allows for a straightforward construction of free contractions, which we describe in the next section.

In the following definition, X_{m+1}^c is the set of all pairs of m-cells requiring a contraction $(m + 1)$-cell, i.e. the set of all pairs of parallel m-cells on X_m which are mapped by f to the same m-cell in S_m. For any $(a, a) \in X_{m+1}^c$, we write $\gamma_m(a, a) = 1_a$, since it is these contraction cells that give us the identities in a Penon weak n-category.

Definition 2.3. Let $f\colon X \to S$ be a map of n-globular sets, where S is the underlying n-globular set of a strict n-category. The map f is said to be *tame* if, given a, $b \in X_n$, if $s(a) = s(b)$, $t(a) = t(b)$, and $f_n(a) = f_n(b)$, then $a = b$.

For each $0 \leq m < n$, define a set X^c_{m+1} by the following pullback:

$$\begin{tikzcd}
X^c_{m+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "{(s,t,f_m)}"] \\
X_m \arrow[r, "{(s,t,f_m)}"'] & X_{m-1} \times X_{m-1} \times S_m.
\end{tikzcd}$$

Note that when $m = 0$, we take X_{m-1} to be the terminal set.

A *contraction* γ on a tame map $f\colon X \to S$ consists of, for each $0 \leq m < n$, a map

$$\gamma_{m+1}\colon X^c_{m+1} \to X_{m+1}$$

such that, for all $(a, b) \in X^c_{m+1}$,

- $s(\gamma_{m+1}(a, b)) = a$;
- $t(\gamma_{m+1}(a, b)) = b$;
- $f_{m+1}(\gamma_{m+1}(a, b)) = 1_{f_m(a)} = 1_{f_m(b)}$.

Note that we only ever speak of a contraction on a tame map; thus, whenever we state that a map is equipped with a contraction, the map is automatically assumed to be tame. One way to think about this is to say that we do require a contraction $(n+1)$-cell for each pair of n-cells in X^c_n, and the only $(n+1)$-cells in X are equalities.

Penon does not use the term “contraction”; instead, he uses the word “stretching” (“étirement”). This may appear somewhat counterintuitive, as the two words seem antonymous. However, Penon’s terminology comes from viewing the same situation from a different point of view; rather than seeing S as a contracted version of X, Penon sees X as a stretched-out version of S. In the case in which X has a magma structure, Penon refers to a such a map as a “categorical stretching” (“étirement catégorique”). Categorical stretchings form a category $\mathcal{Q}$, which we now define.

Definition 2.4. The *category of n-categorical stretchings* $\mathcal{Q}$ is the category with

- objects: an object of $\mathcal{Q}$ consists of an n-magma X, a strict n-category S, and a map of n-magmas
$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$
equipped with a contraction γ;
- morphisms: a morphism in $\mathcal{Q}$ is a commuting square
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$
in n-$\mathbf{Mag}$ such that
  - v is a map of strict n-categories;
  - writing γ for the contraction on the map f and δ for the contraction on the map g, for all $0 \leq m < n$, and $(a, b) \in X^c_{m+1}$, we have
$$u(\gamma_m(a, b)) = \delta_m(u(a), u(b)).$$

  We denote such a morphism by (u, v).

For an object
$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

of $\mathcal{Q}$, we refer to X as its *magma part* and S as its *strict n-category part*. There is a forgetful functor

$$\begin{tikzcd} U\colon \mathcal{Q} \arrow[r] & n\text{-}\mathbf{GSet} \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[d, "f"'] & & \\ S & \arrow[r, mapsto] & X \end{tikzcd}$$

and this functor has a left adjoint $F\colon n\text{-}\mathbf{GSet} \to \mathcal{Q}$. Penon gives a construction of this left adjoint in the second part of [15].

Definition 2.5. Let P be the monad on $n\text{-}\mathbf{GSet}$ induced by the adjunction $F \dashv U$. A Penon weak n-category is defined to be an algebra for the monad P, and $P\text{-}\mathbf{Alg}$ is the category of Penon weak n-categories.

3. Construction of Penon's left adjoint

In [15] Penon gave a construction of the left adjoint F, mentioned above, using computads (which he called "polygraphs", terminology due to Burroni [6]). In this section we give a new, alternative construction of the functor F, using a monad interleaving construction similar to that used by Cheng to construct the operad for Leinster weak ω-categories [7] (see also [11], which describes a more general interleaving argument).

The first step of our construction is the same as that of Penon. There is a forgetful functor $U_T\colon n\text{-}\mathbf{Cat} \to n\text{-}\mathbf{GSet}$ (the notation U_T is used because $n\text{-}\mathbf{Cat} = T\text{-}\mathbf{Alg}$, where T is the free strict n-category monad on $n\text{-}\mathbf{GSet}$), and we write $\mathcal{R}$ for the comma category

$$n\text{-}\mathbf{GSet} \downarrow U_T.$$

Thus an object of $\mathcal{R}$ is a map of n-globular sets $f\colon X \to S$, where S is the underlying n-globular set of a strict n-category. Since an object of $\mathcal{Q}$ consists of an object $f\colon X \to S$ of $\mathcal{R}$ equipped with a magma structure on X and a contraction on f, we can factorise the forgetful functor $U\colon \mathcal{Q} \to n\text{-}\mathbf{GSet}$ as

$$\begin{tikzcd} \mathcal{Q} \arrow[rr, "U"] \arrow[dr, "W"'] & & n\text{-}\mathbf{GSet} \\ & \mathcal{R} \arrow[ur, "V"'] & \end{tikzcd}$$

where W forgets the magma and contraction structures, and V sends an object $f\colon X \to S$ of $\mathcal{R}$ to its n-globular set part X. To construct a left adjoint to U we construct left adjoints to V and W separately. Constructing a left adjoint to V is straightforward: it sends an n-globular set X to $\eta^T_X\colon\ X \to TX$.

We now explain the interleaving argument, which is used to construct the left adjoint to W; this is where our construction differs from that of Penon. In an object of $\mathcal{Q}$ the magma structure and contraction structure exist independently of one another, and there are no axioms governing their interaction. Thus, we can define categories

- $\mathbf{Mag}_n$, whose objects are objects $f\colon X \to S$ of $\mathcal{R}$, together with a magma structure on X, which is respected by f;
- $\mathbf{Contr}_{n+1}$, whose objects are objects $f\colon X \to S$ of $\mathcal{R}$, together with a contraction. (Note that $n+1$ in the superscript here indicates that we have contraction equality $(n+1)$-cells ensuring tameness.)

The maps in these categories are required to respect the magma and contraction structures respectively. We can write the category $\mathcal{Q}$ as the pullback

$$\begin{tikzcd}
\mathcal{Q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Mag}_n \arrow[d, "N"] \\
\mathbf{Contr}_{n+1} \arrow[r, "D"'] & \mathcal{R},
\end{tikzcd}$$

where N and D are the forgetful functors that forget the magma and contraction structures respectively. The functor N has a left adjoint M, which freely adds binary composites, and the functor D has a left adjoint C, which freely adds contraction cells. We wish to combine these left adjoints to obtain a left adjoint to $W\colon \mathcal{Q} \to \mathcal{R}$, which adds both the magma and contraction structures freely. However, we can't just add all of one structure, then all of the other, since with this approach we do not end up with enough cells. If we add a contraction structure first, followed by a magma structure, we do not get any contraction cells whose sources or targets are composites, such as unitors and associators. If we add a magma structure first, followed by a contraction structure, we do not get any composites involving contraction cells.

We therefore "interleave" the structures, one dimension at a time. To do so, we make the following observations:

- when we add contraction cells freely, the contraction m-cells depend only on cells at dimension $m - 1$;

- when we add composites freely, the composites of m-cells depend only on cells at dimensions m and below.

This means that we can add the contraction cells and composites one dimension at a time; starting with dimension 1, we first add contraction cells freely, then add composites freely; we then move up to the next dimension and repeat the process.

To formalise this, we give separate dimension-by-dimension constructions of both the free contraction structure and the free magma structure, then interleave these constructions by lifting them to the case in which we have both a magma structure and a contraction structure. Thus we obtain a left adjoint to the forgetful functor $W\colon \mathcal{Q} \to \mathcal{R}$; by composing this with the left adjoint to the functor $V\colon \mathcal{R} \to n\text{-}\mathbf{GSet}$ we obtain the left adjoint F to $U\colon \mathcal{Q} \to n\text{-}\mathbf{GSet}$.

Owing to the length of this construction, this section is divided into four subsections. In Subsection 3.1 we construct the left adjoint to V. In Subsections 3.2 and 3.3 we give dimension-by-dimension constructions of the left adjoints to D and N respectively; these describe the free contraction structure and free magma structure. Finally, in Subsection 3.4 we then interleave these constructions to give a left adjoint to W.

3.1 Left adjoint to V

We begin by describing $\mathcal{R}$ explicitly, in order to establish some terminology, and to make clear its connection with $\mathcal{Q}$.

Definition 3.1. Write $\mathcal{R}$ to denote the comma category $n\text{-}\mathbf{GSet} \downarrow U_T$; explicitly, $\mathcal{R}$ is the category with

- objects: an object of $\mathcal{R}$ consists of an n-globular set X, a strict n-

category S, and a map of n-globular sets

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

- morphisms: a morphism in $\mathcal{R}$ is a commuting square

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

  in n-$\mathbf{GSet}$ such that v is a map of strict n-categories. We denote such a morphism by (u, v).

As in the case of $\mathcal{Q}$, for an object

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

we refer to X as its *n-globular set part* and S as its *strict n-category part*.

We have a forgetful functor $W\colon \mathcal{Q} \to \mathcal{R}$, which forgets the contraction and n-magma structures but leaves the underlying map of n-globular sets unchanged, and a forgetful functor $V\colon \mathcal{R} \to n$-$\mathbf{GSet}$, defined by

$$V(\, X \xrightarrow{f} S \,) = X;$$

these compose to give $V \circ W = U$. We construct left adjoints to V and W separately, then compose these to obtain the left adjoint to U. We begin with the construction of the left adjoint to V; we do this in more generality than we require, since this construction is valid for any monad T.

Definition 3.2. Let T be a monad on a category $\mathcal{C}$, and write $U_T\colon T$-$\mathbf{Alg} \to \mathcal{C}$ for the forgetful functor that sends a T-algebra to its underlying object in $\mathcal{C}$. Define a functor $H\colon \mathcal{C} \to \mathcal{C}/U_T$ as follows:

- on objects: for $X \in n\text{-}\mathbf{GSet}$,

$$H(X) = (\ X \xrightarrow{\eta^T_X} TX\ ),$$

where TX has the structure of the free T-algebra on X;

- on morphisms: for $f\colon X \to Y$ in $\mathcal{C}$, $Hf = (f, Tf)$.

Proposition 3.3. *Write $V : \mathcal{C}/U_T \to \mathcal{C}$ for the forgetful functor defined by, for an object $f\colon X \to S$ of $\mathcal{C}/U_T$, where S has a T-algebra structure $\theta\colon TS \to S$,*

$$V(\ X \xrightarrow{f} S\ ) = X.$$

Then there is an adjunction $H \dashv V$.

Proof. First, we define the unit $\alpha\colon 1 \Rightarrow VH$ and the counit $\beta\colon HV \Rightarrow 1$. We have $VH = 1$, and we define $\alpha := \mathrm{id}$. To define β, let $f\colon X \to S$ in $\mathcal{C}/U_T$, where S has a T-algebra denoted by $\theta\colon TS \to S$. Observe that θ is a map of T-algebras since, by the algebra axioms, the diagram

$$\begin{tikzcd}
T^2S \arrow[r, "T\theta"] \arrow[d, "\mu^T_S"'] & TS \arrow[d, "\theta"] \\
TS \arrow[r, "\theta"'] & S
\end{tikzcd}$$

commutes. The component of β at $f\colon X \to S$, denoted β_f, is given by the commuting diagram

$$\begin{tikzcd}
X \arrow[rr, "\mathrm{id}_X"] \arrow[dr, "f"] \arrow[dd, "\eta^T_X"'] & & X \arrow[dd, "f"] \\
& S \arrow[d, "\eta^T_S"] \arrow[dr, "\mathrm{id}_S"] & \\
TX \arrow[r, "Tf"'] & TS \arrow[r, "\theta"'] & S,
\end{tikzcd}$$

as a map in $\mathcal{C}/U_T$. This diagram commutes since the left-hand square is a naturality square for η and the bottom-right triangle is the unit axiom for the algebra θ; the remaining square commutes trivially.

We now show that α and β satisfy the triangle identities. First, consider

$$\begin{tikzcd} V \arrow[r, "\alpha V"] \arrow[dr, "1_V"'] & VHV \arrow[d, "V\beta"] \\ & V. \end{tikzcd}$$

For $f\colon X \to S$ in $\mathcal{R}$,

$$V(\ X \xrightarrow{f} S\ ) = X = VHV(\ X \xrightarrow{f} S\ ),$$

$\alpha_X = 1_X$, and $U\beta_f = 1_X$, so this diagram commutes.

Now consider

$$\begin{tikzcd} H \arrow[r, "H\alpha"] \arrow[dr, "1_H"'] & HVH \arrow[d, "\beta H"] \\ & H. \end{tikzcd}$$

For $X \in \mathcal{C}$,

$$H(X) = (\ X \xrightarrow{\eta^T_X} TX\ ) = HVH(X),$$

$H\alpha_X = H\mathrm{id}_X = (\mathrm{id}_X, \mathrm{id}_{TX})$, and $\beta_{HX} = (\mathrm{id}, \mu_X \circ T\eta_X) = (\mathrm{id}_X, \mathrm{id}_{TX})$, so this diagram commutes. $\square$

This gives us the left adjoint to the functor $V\colon \mathcal{R} \to n$-$\mathbf{GSet}$.

3.2 Free contraction structure

We now construct the free contraction on an object of $\mathcal{R}$. In order to be able to use the construction in the interleaving argument in Section 3.4, we give the construction one dimension at a time. To do so, we define, for each $0 \leq k \leq n+1$, a category $\mathbf{Contr}_k$, an object of which consists of an object of $\mathcal{R}$ equipped with a contraction up to dimension k. (Observe that $\mathbf{Contr}_0 = \mathcal{R}$, and note that a "contraction at dimension $n+1$" refers to the tameness condition at dimension n.) We then have, for each $0 < k \leq n+1$, a forgetful functor

$$D_k\colon \mathbf{Contr}_k \to \mathbf{Contr}_{k-1}.$$

We construct a left adjoint to each D_k, which freely adds a contraction structure at dimension k, leaving all other dimensions unchanged.

Definition 3.4. Let $f\colon X \to S$ be a map of n-globular sets, where S is the underlying n-globular set of a strict n-category, and let $0 \leq k \leq n$. Recall from Definition 2.3 that, for each $0 \leq m < n$, the set X^c_{m+1} is defined by the pullback

$$\begin{tikzcd} X^c_{m+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "{(s,t,f_m)}"] \\ X_m \arrow[r, "{(s,t,f_m)}"'] & X_{m-1} \times X_{m-1} \times S_m . \end{tikzcd}$$

where, when $m = 0$, we take X_{m-1} to be the terminal set.

A k-contraction γ on the map f consists of, for each $0 \leq m < k$, a map

$$\gamma_{m+1}\colon X^c_{m+1} \to X_{m+1}$$

such that, for $(a, b) \in X^c_m$,

$$s(\gamma_{m+1}(a, b)) = a,$$

$$t(\gamma_{m+1}(a, b)) = b,$$

$$f_{m+1}(\gamma_{m+1}(a, b)) = \mathrm{id}_{f(a)}.$$

Note that having an n-contraction on a map f is not the same as having contraction on f; for a contraction on f, we also require that f is tame (see Definition 2.3).

Definition 3.5. For each $0 \leq k \leq n$, define a category $\mathbf{Contr}_k$, with

- objects: an object of $\mathbf{Contr}_k$ consists of an n-globular set X, a strict n-category S, and a map of n-globular sets

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

equipped with a k-contraction γ;

- morphisms: a morphism in $\mathbf{Contr}_k$ is a commuting square

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

  in n-$\mathbf{GSet}$ such that

  – v is a map of strict n-categories;

  – writing γ for the contraction on the map f and δ for the contraction on the map g, for all $0 < m \leq k$, and $(a, b) \in X^c_m$, we have
$$u(\gamma_m(a, b)) = \delta_m(u(a), u(b)).$$

Define a category $\mathbf{Contr}_{n+1}$, with

- objects: an object of $\mathbf{Contr}_{n+1}$ consists of an n-magma X, a strict n-category S, and a map of n-magmas

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

  equipped with a contraction γ;

- morphisms: a morphism in $\mathbf{Contr}_{n+1}$ is a commuting square

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

  in n-$\mathbf{GSet}$ such that

  – v is a map of strict n-categories;

- writing γ for the contraction on the map f and δ for the contraction on the map g, for all $0 < m \leq n$, and $(a, b) \in X^c_m$, we have
$$u(\gamma_m(a, b)) = \delta_m(u(a), u(b)).$$

For all $0 < k \leq n + 1$, we have a forgetful functor

$$D_k \colon \mathbf{Contr}_k \to \mathbf{Contr}_{k-1};$$

for $0 < k \leq n$, this functor forgets the contraction at dimension k, and for $k = n + 1$ it is the inclusion functor of the subcategory $\mathbf{Contr}_{n+1}$ into $\mathbf{Contr}_n$.

We now define a putative left adjoint C_k to the functor D_k; we will then prove that this functor is left adjoint to D_k in Proposition 3.7.

Definition 3.6. For each k, $0 < k \leq n$, we define a functor

$$C_k \colon \mathbf{Contr}_{k-1} \to \mathbf{Contr}_k.$$

We begin by giving the action of C_k on objects. Let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object of $\mathbf{Contr}_{k-1}$, and write γ for its $(k-1)$-contraction (assuming $k > 1$; if $k = 1$, we have $\mathbf{Contr}_{k-1} = \mathbf{Contr}_0 = \mathcal{R}$, so there is no contraction on f). We define an object

$$\begin{tikzcd} \tilde{X} \arrow[d, "\tilde{f}"'] \\ S \end{tikzcd}$$

of $\mathbf{Contr}_k$, with k-contraction $\tilde{\gamma}$. The n-globular set $\tilde{X}$ is defined by:

- $\tilde{X}_j = X_j$ for all $j \neq k$;

- $\tilde{X}_k = X_k \amalg X_k^c$,
- for $(x, y) \in X_k^c \subseteq \tilde{X}_k$, $s(x, y) = x$, $t(x, y) = y$,
- for all other cells, sources and targets are inherited from X.

The map $\tilde{f} \colon \tilde{X} \to S$ is defined by

- $\tilde{f}_j = f_j$ for all $j \neq k$;
- $\tilde{f}_k \colon \tilde{X}_k \to S_k$ is defined by
  - $\tilde{f}_k(\alpha) = f_k(\alpha)$ for $\alpha \in X_k \subseteq \tilde{X}_k$;
  - $\tilde{f}_k(x, y) = 1_{f_{k-1}(x)}$ for $(x, y) \in X_k^c \subseteq \tilde{X}_k$.

The k-contraction $\tilde{\gamma}$ on $\tilde{f}$ is defined by

- $\tilde{\gamma}_m = \gamma_m^{k-1}$ for all $m < k-1$;
- $\tilde{\gamma}_{k-1} \colon X_k^c \to \tilde{X}_k$ is the inclusion into the coproduct $\tilde{X}_k = X_k \amalg X_k^c$.

This defines the action of C_k on objects.

We now give the action of C_k on morphisms. Let

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

be a morphism in $\mathbf{Contr}_{k-1}$. Define a morphism

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{u}"] \arrow[d, "\tilde{f}"'] & \tilde{Y} \arrow[d, "\tilde{g}"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

in $\mathbf{Contr}_k$, where $\tilde{u}$ is defined by

- $\tilde{u}_j = u_j$ for all $j \neq k$;

- $\tilde{u}_k \colon \tilde{X}_k \to \tilde{Y}_k$ is given by
  - $\tilde{u}_k(\alpha) = u_k(\alpha)$ for $\alpha \in X_k \subseteq \tilde{X}_k$;
  - $\tilde{u}_k(x, y) = (u_{k-1}(x), u_{k-1}(y))$ for $(x, y) \in X_k^c \subseteq \tilde{X}_k$.

This defines the action of C_k on morphisms.

Proposition 3.7. *For all* $0 < k \leq n$, *there is an adjunction* $C_k \dashv D_k$.

Proof. We first define the unit $\eta \colon 1 \Rightarrow D_k C_k$, and counit $\epsilon : C_k D_k \Rightarrow 1$.

Let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object of $\mathbf{Contr}_{k-1}$, with $(k-1)$-contraction γ (assuming $k > 1$; if $k = 1$, we have $\mathbf{Contr}_{k-1} = \mathbf{Contr}_0 = \mathcal{R}$, so there is no contraction on f). Applying $D_k C_k$ gives

$$\begin{tikzcd} \tilde{X} \arrow[d, "\tilde{f}"'] \\ S \end{tikzcd}$$

in $\mathbf{Contr}_{k-1}$ with the same $(k-1)$-contraction. The corresponding component of the unit η is given by the map

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \tilde{X} \arrow[d, "\tilde{f}"] \\ S \arrow[r, "\mathrm{id}_S"'] & S \end{tikzcd}$$

where η_X is defined by

$$(\eta_X)_j = \begin{cases} 1_{X_j} & \text{if } j \neq k, \\ \text{the inclusion } X_j \hookrightarrow X_j \amalg X_j^c & \text{if } j = k. \end{cases}$$

Now let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object of $\mathbf{Contr}_k$, with k-contraction γ. Applying $C_k D_k$ gives

$$\begin{tikzcd} \tilde{X} \arrow[d, "\tilde{f}"'] \\ S \end{tikzcd}$$

in $\mathbf{Contr}_k$ with k-contraction $\tilde{\gamma}$, which is equal to γ at all dimensions except k. The corresponding component of the counit ϵ is given by the map

$$\begin{tikzcd} \tilde{X} \arrow[r, "\epsilon_X"] \arrow[d, "\tilde{f}"'] & X \arrow[d, "f"] \\ S \arrow[r, "\mathrm{id}_S"'] & S \end{tikzcd}$$

where ϵ_X is defined by

- $(\epsilon_X)_j = 1_{X_j}$ for all $j \neq k$;
- $(\epsilon_X)_k \colon \tilde{X}_k \to X_k$ is given by
  - $(\epsilon_X)_k(\alpha) = \alpha$ for $\alpha \in X_k \subseteq \tilde{X}_k$;
  - $(\epsilon_X)_k(x, y) = \tilde{\gamma}_k(x, y)$ for $(x, y) \in X_k^c \subseteq \tilde{X}_k$.

We now check that the triangle identities hold; consider the diagrams

$$\begin{tikzcd} D_k \arrow[r, "\eta D_k"] \arrow[dr, "1"'] & D_k C_k D_k \arrow[d, "D_k \epsilon"] \\ & D_k, \end{tikzcd} \qquad \begin{tikzcd} C_k \arrow[r, "C_k \eta"] \arrow[dr, "1"'] & C_k D_k C_k \arrow[d, "\epsilon C_k"] \\ & C_k. \end{tikzcd}$$

In all of the natural transformations in these diagrams, the components on the strict n-category parts are all identities, so to show that the diagrams commute we need only consider the components on the n-globular set parts. Since the components of the maps of n-globular sets are identities at every dimension except dimension k, we need only check that the corresponding diagrams of maps of sets of k-cells commute.

First, we must show that, given

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

in $\mathbf{Contr}_k$, the diagram

$$\begin{tikzcd} X_k \arrow[r, "(\eta_X)_k"] \arrow[dr, "1_{X_k}"'] & \tilde{X}_k = X_k \amalg X^c_{k-1} \arrow[d, "(\epsilon_X)_k"] \\ & X_k \end{tikzcd}$$

commutes; this is true, since given $\alpha \in X_k$, we have

$$(\epsilon_X)_k \circ (\eta_X)_k(\alpha) = (\epsilon_X)_k(\alpha) = \alpha.$$

Secondly, we must show that, given

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

in $\mathbf{Contr}_{k-1}$ with $(k-1)$-contraction γ, the diagram

$$\begin{tikzcd} \tilde{X}_k \arrow[r, "(\tilde{\eta}_X)_k"] \arrow[dr, "1_{\tilde{X}_k}"'] & \tilde{X}_k \amalg \tilde{X}^c_k \arrow[d, "(\epsilon_{\tilde{X}})_k"] \\ & \tilde{X}_k \end{tikzcd}$$

commutes. We have two kinds of freely added contraction cells in $\tilde{X}_k \amalg \tilde{X}^c_k$; we write (x, y) for the contraction cells in X^c_k, and $[x, y]$ for those in $\tilde{X}^c_k$ (the latter being the specified contraction cells in this case). Given $\alpha \in X_k \subseteq \tilde{X}_k$,

$$(\epsilon_{\tilde{X}})_k \circ (\tilde{\eta}_X)_k(\alpha) = (\epsilon_{\tilde{X}})_k(\alpha) = \alpha;$$

given $(x,y) \in X^c_k \subseteq \tilde{X}_k$,

$$(\epsilon_{\tilde{X}})_k \circ (\tilde{\eta}_X)_k(x,y) = (\epsilon_{\tilde{X}})_k[x,y] = (x,y);$$

hence the diagram commutes.

Thus the triangle identities hold, and we have an adjunction $C_k \dashv D_k$, with unit η and counit ϵ. □

We must also define C_{n+1} separately, since "adding contraction $(n+1)$-cells" consists of identifying certain n-cells rather than actually adding cells; we can think of this as adding equality $(n+1)$-cells between pairs of n-cells that would usually require a contraction cell between them.

Definition 3.8. We define a functor

$$C_{n+1} \colon \mathbf{Contr}_n \to \mathbf{Contr}_{n+1}.$$

We begin by giving the effect of C_{n+1} on objects. Let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object of $\mathbf{Contr}_n$, with n-contraction γ. Define a set X^c_{n+1} and maps $\pi_1, \pi_2 \colon X^c_{n+1} \to X_n$ by the following pullback:

$$\begin{tikzcd} X^c_{n+1} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, "{(s,t,f_n)}"] \\ X_n \arrow[r, "{(s,t,f_n)}"'] & X_{n-1} \times X_{n-1} \times S_n. \end{tikzcd}$$

The set X^c_{n+1} can be thought of as the set of pairs of n-cells to be identified, but note that there is some redundancy: for all $a \in X_n$, $(a,a) \in X^c_{n+1}$, and if we have $(a,b) \in X^c_{n+1}$ we also have $(b,a) \in X^c_{n+1}$.

We now define an object

$$\begin{tikzcd} \tilde{X} \arrow[d, "\tilde{f}"'] \\ S \end{tikzcd}$$

of $\mathbf{Contr}_{n+1}$ with contraction $\tilde{\gamma}$. For $0 \leq m < n$, define

$$\tilde{X}_m = X_m,$$

and define $\tilde{X}_n$ to be the coequaliser of the diagram

$$X^c_{n+1} \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X_n.$$

For $0 \leq m < n$, define

$$\tilde{f}_m = f_m,$$

and define $\tilde{f}_n \colon \tilde{X}_n \to S_n$ to be the unique map such that

$$\begin{tikzcd} X^c_{n+1} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X_n \arrow[r, "q"] \arrow[dr, "f_n"'] & \tilde{X}_n \arrow[d, dashed, "\tilde{f}_n"] \\ & & S_n \end{tikzcd}$$

commutes, where $q \colon X_n \to \tilde{X}_n$ is the coequaliser map. Finally, define $\tilde{\gamma}$ to be the n-contraction defined by

$$\tilde{\gamma}_m = \begin{cases} \gamma_m & \text{if } m < n, \\ q \circ \gamma_n & \text{if } m = n. \end{cases}$$

This defines the action of C_{n+1} on objects.

We now give the action of C_{n+1} on morphisms. Let

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

be a morphism in $\mathbf{Contr}_n$. Define a morphism

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{u}"] \arrow[d, "\tilde{f}"'] & \tilde{Y} \arrow[d, "\tilde{g}"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

in $\mathbf{Contr}_{n+1}$, where, for $0 \leq m < n$,

$$\tilde{u}_m = u_m,$$

and $\tilde{u}_n \colon \tilde{X}_n \to \tilde{Y}_n$ is defined to be the unique map such that the diagram

$$\begin{tikzcd} X^c_{n+1} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X_n \arrow[r, "q"] \arrow[d, "u_n"'] & \tilde{X}_n \arrow[d, dashed, "\tilde{u}_n"] \\ & Y_n \arrow[r, "p"'] & \tilde{Y}_n \end{tikzcd}$$

commutes, where p is the coequaliser map for $\tilde{Y}_n$. This defines the action of C_{n+1} on morphisms.

Proposition 3.9. *There is an adjunction $C_{n+1} \dashv D_{n+1}$.*

Proof. We first define the unit $\eta \colon 1 \Rightarrow D_{n+1}C_{n+1}$. Let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object of $\mathbf{Contr}_n$. Applying $D_{n+1}C_{n+1}$ gives

$$\begin{tikzcd} \tilde{X} \arrow[d, "\tilde{f}"'] \\ S \end{tikzcd}$$

in $\mathbf{Contr}_{n+1}$. The corresponding component of the unit η is given by the map

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \tilde{X} \arrow[d, "\tilde{f}"] \\ S \arrow[r, "\mathrm{id}_S"'] & S \end{tikzcd}$$

where η_X is defined by

$$(\eta_X)_j = \begin{cases} 1_{X_j} & \text{if } j < n, \\ \text{the coequaliser map } q\colon X_n \to \tilde{X}_n & \text{if } j = n. \end{cases}$$

For the counit, observe that $C_{n+1}D_{n+1} = \mathrm{id}$, and if

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

is in the image of D_{n+1}, $\tilde{X} = X$ and $q = \mathrm{id}_X$, so $\eta = \mathrm{id}$. We define the counit

$$\epsilon : C_{n+1}D_{n+1} \Rightarrow 1$$

to be the identity. Thus all maps appearing in the diagrams for the triangle identities are identity maps, so both diagrams commute. Hence there is an adjunction $C_{n+1} \dashv D_{n+1}$. □

Thus Definitions 3.6 and 3.8 give us a dimension-by-dimension construction of the free contraction on an object of $\mathcal{R}$.

3.3 Free magma structure

We now construct the free n-magma on the source of an object of $\mathcal{R}$. As with the construction of the free contraction in the previous subsection, in order to be able to use the construction in the interleaving argument in Subsection 3.4, we give the construction one dimension at a time. To do so we define, for each $0 \leq j \leq n$, a category $\mathbf{Mag}_j$, an object of which consists of an object of $\mathcal{R}$ in which the source is equipped with a magma structure up to dimension j. (Observe that $\mathbf{Mag}_0 = \mathcal{R}$.) We then have, for each $0 < j \leq n$, a forgetful functor

$$N_j \colon \mathbf{Mag}_j \to \mathbf{Mag}_{j-1}.$$

We construct a left adjoint to each N_j, which freely adds a magma structure at dimension j, leaving all other dimensions unchanged.

In order to define what it means for an n-globular set to have a j-magma structure, we use the j-truncation functor

$$\mathrm{Tr}_j \colon n\text{-}\mathbf{GSet} \longrightarrow j\text{-}\mathbf{GSet},$$

which forgets the sets of m-cells for all $m > j$, and, for $m \leq j$, leaves the sets of m-cells and their source and target maps unchanged; the action on maps is defined similarly.

Definition 3.10. Define a category $\mathbf{Mag}_j$, with

- objects: an object of $\mathbf{Mag}_j$ consists of an object

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

  in $\mathcal{R}$ such that $\mathrm{Tr}_j X$ is a j-magma, and $\mathrm{Tr}_j f$ is a map of j-magmas.

- morphisms: a morphism in $\mathbf{Mag}_j$ is a morphism

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

  in $\mathcal{R}$ such that $\mathrm{Tr}_j u$ is a map of j-magmas.

We can express the category $\mathbf{Mag}_j$ as a pullback. For any $j \in \mathbb{N}$ we have a commuting triangle of forgetful functors

$$\begin{tikzcd} j\text{-}\mathbf{Cat} \arrow[rr, "G"] \arrow[dr, "U_{T_j}"'] & & j\text{-}\mathbf{Mag} \arrow[dl, "E"] \\ & j\text{-}\mathbf{GSet} & \end{tikzcd}$$

in $\mathbf{CAT}$, where T_j is the free strict j-category monad (and thus $j\text{-}\mathbf{Cat} = T_J\text{-}\mathbf{Alg}$). We can then write $\mathbf{Mag}_j$ as the pullback

$$\begin{tikzcd} \mathbf{Mag}_j \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & n\text{-}\mathbf{GSet} \downarrow U_T \arrow[d, "\mathrm{Tr}_j"] \\ j\text{-}\mathbf{Mag} \downarrow G \arrow[r, "E"'] & j\text{-}\mathbf{GSet} \downarrow U_{T_j} \end{tikzcd}$$

For all $0 < j \leq n$, we have a forgetful functor

$$N_j \colon \mathbf{Mag}_j \to \mathbf{Mag}_{j-1},$$

which forgets the composition maps for j-cells. We will define, for each $0 < j \leq n$, a functor

$$M_j \colon \mathbf{Mag}_{j-1} \to \mathbf{Mag}_j$$

which freely adds binary composites at dimension j, taking an n-globular set equipped with a $(j-1)$-magma structure and adding a magma structure at dimension j to give an n-globular set equipped with a j-magma structure. We will then show that the functor M_j is left adjoint to the forgetful functor N_j.

Before defining M_j, we first fix some notation that will be used in the construction of the free binary composites. Let X be an n-globular set equipped with a $(j-1)$-magma structure. For each $0 \leq p < j$, we can form the set of pairs of j-cells that are composable along p-cells using the following pullback:

$$\begin{tikzcd} X_j \times_{X_p} X_j \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_j \arrow[d, "s"] \\ X_j \arrow[r, "t"'] & X_p. \end{tikzcd}$$

We view $X_j \times_{X_p} X_j$ as the set of freely generated binary composites of j-cells along p-cells. We can form the set of freely generated binary composites of j-cells along boundaries of all dimensions by taking the coproduct of these sets over p. As the notation will become somewhat complicated in the definition of the left adjoint to N_j, we use the following shorthand:

$$X_j^2 := \coprod_{0 \leq p < j} X_j \times_{X_p} X_j.$$

This set comes equipped with source and target maps into X_{j-1}, which are defined in analogy with the sources and targets of composites in a magma structure from Definition 2.2, as follows:

- if $p = m - 1$, given $(a, b) \in X_m \times_{X_p} X_m$,

$$s(a, b) = s(a),$$

$$t(a, b) = t(b)$$

- if $p < m-1$, given $(a, b) \in X_m \times_{X_p} X_m$,

$$s(a, b) = s(b) \circ_p^{m-1} s(a),$$
$$t(a, b) = t(b) \circ_p^{m-1} t(a).$$

The set X_j^2 contains only binary composites of "depth 1"; that is, it contains binary composites of pairs of j-cells in X, but it does not contain binary composites of binary composites, binary composites of binary composites of binary composites, etc. In order to obtain these composites of greater "depth", which we require in the free magma structure, we must iterate this process. To do so we define, for each $k \geq 0$, a set $X_j^{(k)}$ of composites of depth at most k. We have inclusion maps

$$X_j^{(k)} \hookrightarrow X_j^{(k+1)},$$

so this gives a sequence of sets; we take the colimit of this sequence to obtain the set of freely generated binary composites of all depths. We now describe and illustrate this iterative process for low depths of composite ($k \leq 2$).

When $k = 0$, we define

$$X_j^{(0)} = X_j,$$

with source and target maps $s, t \colon X_j^{(0)} \to X_{j-1}$ given by those in X.

When $k = 1$, we define

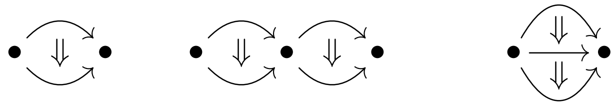
$$X_j^{(1)} := X_j + \left(X_j^{(0)}\right)^2 = X_j + X_j^2,$$

where the notation X_j^2 is shorthand, as described earlier. The set X_j^2 inherits source and target maps from $X_j^{(0)}$, so we have source and target maps

$$s, t \colon X_j^{(1)} \longrightarrow X_{j-1}$$

inherited from those for X_j and X_j^2. To see how this gives the set of composites of depth at most 1, we consider the case $j = 2$. By "expanding out" X_2^2, we see that $X_2^{(1)}$ contains the following shapes of composites:

$$X_2^{(1)} = \quad X_2 \quad + \quad X_2 \times_{X_0} X_2 \quad + \quad X_2 \times_{X_1} X_2$$



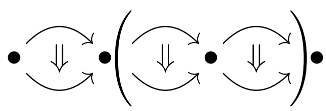
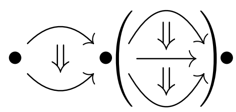
When $k = 2$, we define

$$X_j^{(2)} := X_j + \left(X_j^{(1)}\right)^2.$$

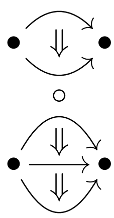
As in the case $k = 1$, this comes equipped with source and target maps. In the case $j = 2$, "expanding out" $\left(X_j^{(1)}\right)^2$ gives

$$\begin{aligned} X_2^{(2)} = \;& X_2 \;+\; X_2^2 \;+\; X_2 \times_{X_0} X_2^2 \;+\; X_2 \times_{X_1} X_2^2 \;+\; X_2^2 \times_{X_0} X_2 \\ &+\; X_2^2 \times_{X_1} X_2 \;+\; X_2^2 \times_{X_0} X_2^2 \;+\; X_2^2 \times_{X_1} X_2^2. \end{aligned}$$

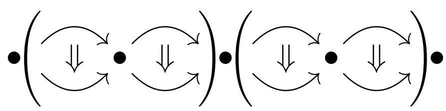
Thus $X_2^{(2)}$ contains the same shapes of composites that appear in $X_2^{(1)}$, as well as those composites of depth 2: in $X_2 \times_{X_0} X_2^2$ we have composites of the following shapes:

 and ;

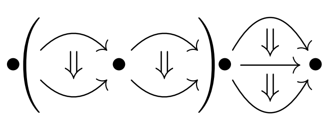
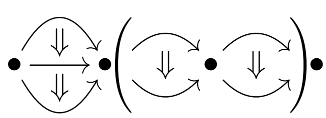
in $X_2 \times_{X_1} X_2^2$ we have composites of the following shape:

;

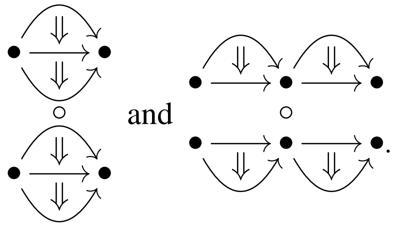
the shapes of composites in $X_2^2 \times_{X_0} X_2$ and $X_2^2 \times_{X_1} X_2$ are similar to those above; in $X_2^2 \times_{X_0} X_2^2$ we have composites of the following shapes:



and also

 and ;

and finally, in $X_2^2 \times_{X_1} X_2^2$ we have composites of the following shapes:



Thus $X_2^{(2)}$ contains all binary composites of 2-cells of depth at most 2.

Since the construction of the the free j-magma structure consists of taking pullbacks and filtered colimits of sets, in order to define the composition maps at dimension j we require the following lemma due to Mac Lane [14, Theorem IX.2.1], which states that finite limits commute with filtered colimits in $\mathbf{Set}$. Note that this theorem still holds if $\mathbf{Set}$ is replaced by any locally finitely presentable category; see [1, Proposition 1.59].

Lemma 3.11 (Mac Lane)**.** *Let $\mathbb{I}$ be a finite category, and let $\mathbb{J}$ be a small, filtered category. Then for any bifunctor*

$$F \colon \mathbb{I} \times \mathbb{J} \to \mathbf{Set}$$

the canonical arrow

$$\operatorname*{colim}_{j \in \mathbb{J}} \lim_{i \in \mathbb{I}} F(i,j) \longrightarrow \lim_{i \in \mathbb{I}} \operatorname*{colim}_{j \in \mathbb{J}} F(i,j)$$

is an isomorphism.

We now define a putative left adjoint M_j to the functor N_j; we will then prove that this functor is left adjoint to N_j in Proposition 3.13.

Definition 3.12. For each $0 < j \leq n$, we define a functor

$$M_j \colon \mathbf{Mag}_{j-1} \to \mathbf{Mag}_j.$$

We begin by giving the action of M_j on objects. Let

$$\begin{array}{c} X \\ \downarrow{\scriptstyle f} \\ S \end{array}$$

be an object of $\mathbf{Mag}_{j-1}$. We will define an object

$$\begin{tikzcd} \widehat{X} \arrow[d, "\hat{f}"'] \\ S \end{tikzcd}$$

of $\mathbf{Mag}_j$, where $\widehat{X}$ differs from X only at dimension j. The set $\widehat{X}_j$ of j-cells of $\widehat{X}$ is the set of freely generated binary composites of j-cells of X. We define this as the colimit of a sequence of sets $X_j^{(k)}$, where $X_j^{(k)}$ is the set of freely generated binary composites of j-cells of X of depth at most k. We define $X_j^{(k)}$ by induction over k, as follows: when $k = 0$, define

$$X_j^{(0)} = X_j,$$

with source and target maps $s, t\colon X_j^{(0)} \to X_{j-1}$ given by those in X. Now suppose that $k > 0$ and we have defined $X_j^{(k-1)}$, equipped with source and target maps

$$s, t\colon X_j^{(k-1)} \longrightarrow X_{j-1}.$$

We define $X_j^{(k)}$ by

$$X_j^{(k)} := X_j + \left(X_j^{(k-1)}\right)^2.$$

Recall that the notation used above is shorthand, defined by

$$\left(X_j^{(k-1)}\right)^2 := \coprod_{0 \le p < j} X_j^{(k-1)} \times_{X_p} X_j^{(k-1)},$$

and that this set inherits source and target maps from $X_j^{(k-1)}$. Thus we have source and target maps

$$s, t\colon X_j^{(k)} \longrightarrow X_{j-1}$$

inherited from those for X_j and $\left(X_j^{(k-1)}\right)^2$.

For each $k \ge 0$, we define a map

$$i^{(k)}\colon X_j^{(k)} \to X_j^{(k+1)},$$

which includes the freely generated composites in $X_j^{(k)}$ (those of depth at most k) into the set $X_j^{(k+1)}$ (which contains composites of depth at most $k+1$), and leaves the generating cells unchanged. The maps $i^{(k)}$ are defined by induction over k, as follows:

- for $k = 0$, $i^{(0)}$ is the coprojection map
$$i^{(0)}\colon X_j \to X_j + X_j^2;$$
- for $k \geq 1$, suppose we have defined $i^{(k-1)}\colon X_j^{(k-1)} \to X_j^{(k)}$. We define $i^{(k)}$ to be the map
$$i^{(k)} := 1_{X_j} + \coprod_{0 \leq p < j} \left(i^{(k-1)}, i^{(k-1)}\right) : X_j + \left(X_j^{(k-1)}\right)^2 \to X_j + \left(X_j^{(k)}\right)^2.$$

These sets and maps give us a diagram
$$X_j^{(0)} \xrightarrow{i^{(0)}} X_j^{(1)} \xrightarrow{i^{(1)}} X_j^{(2)} \xrightarrow{i^{(2)}} X_j^{(3)} \xrightarrow{i^{(3)}} \ldots$$
in $\mathbf{Set}$; we then define
$$\widehat{X}_j := \operatorname*{colim}_{k \geq 0} X_j^{(k)}.$$
For $m \neq j$, we define
$$\widehat{X}_m := X_m.$$
For $m \neq j, j+1$, the source and target maps
$$s, t\colon \widehat{X}_m \to \widehat{X}_{m-1}$$
are those inherited from X. Now write $c_j^{(k)}\colon X_j^{(k)} \to \widehat{X}_j$ for the coprojection maps. The source and target maps for $m = j+1$ are given by the composites
$$\widehat{X}_{j+1} = X_{j+1} \xrightarrow{s} X_j \xrightarrow{c_j^{(0)}} \widehat{X}_j$$
and
$$\widehat{X}_{j+1} = X_{j+1} \xrightarrow{t} X_j \xrightarrow{c_j^{(0)}} \widehat{X}_j$$

respectively. To define the source and target maps for $m = j$, recall that, for each k, we have source and target maps $s, t \colon X_j^{(k)} \to X_{j-1}$; we define $s, t \colon \widehat{X}_j \to X_{j-1}$ to be the unique maps induced by the colimit defining $\widehat{X}_j$ that make, for all $k \geq 1$, the diagrams

$$\begin{tikzcd} X_j^{(k)} \arrow[r, "c_j^{(k)}"] \arrow[dr, "s"'] & \widehat{X}_j \arrow[d, dashed, "s"] \\ & X_{j-1} \end{tikzcd} \qquad \begin{tikzcd} X_j^{(k)} \arrow[r, "c_j^{(k)}"] \arrow[dr, "t"'] & \widehat{X}_j \arrow[d, dashed, "t"] \\ & X_{j-1} \end{tikzcd}$$

commute respectively.

We now define the j-magma structure on $\widehat{X}$. For all $m < j$, and for all $0 \leq p < m$, the composition map

$$\circ_p^m \colon \widehat{X}_m \times_{\widehat{X}_p} \widehat{X}_m = X_m \times_{X_p} X_m \to X_m$$

is the corresponding composition map from the $(j-1)$-magma structure on X. To define the composition map $\circ_p^j$ for $0 \leq p < j$, we begin by observing that, by Lemma 3.11, we have an isomorphism

$$\operatorname*{colim}_{k,l \geq 0} \left(X_j^{(k)} \times_{X_p} X_j^{(l)} \right) \cong \left(\operatorname*{colim}_{k \geq 0} X_j^{(k)} \right) \times_{X_p} \left(\operatorname*{colim}_{l \geq 0} X_j^{(l)} \right) = \widehat{X}_j \times_{X_p} \widehat{X}_j.$$

Thus, to define the composition maps at dimension j, we define, for each $k, l > 0$, $0 \leq p < j$, a map

$$\circ_p^j \colon X_j^{(k)} \times_{X_p} X_j^{(l)} \to \widehat{X}_j.$$

To do so, observe that, in the case $k = l$, the source of the composition map above includes in $X_j^{(k+1)}$, which in turn includes in $\widehat{X}_j$; thus in this case we define the composition map to be the composite:

$$X_j^{(k)} \times_{X_p} X_j^{(k)} \hookrightarrow X_j^{(k+1)} \xrightarrow{c_j^{(k+1)}} \widehat{X}_j.$$

Now suppose that $k < l$; in this case we first include the source of the composition map in

$$X_j^{(l)} \times_{X_p} X_j^{(l)},$$

and we can then follow the same method as for $k = l$. Write

$$i^{(k,l)} := i^{(l)} \circ i^{(l-1)} \circ \cdots \circ i^{(k)} \colon X_j^{(k)} \longrightarrow X_j^{(l)},$$

and define $\circ_p^j$ to be the composite

$$X_j^{(k)} \times_{X_p} X_j^{(l)} \xrightarrow{(i^{(k,l)},\mathrm{id})} X_j^{(l)} \times_{X_p} X_j^{(l)} \hookrightarrow X_j^{(l+1)} \xrightarrow{c_j^{(l+1)}} \widehat{X}_j,$$

where the second map is the coprojection into the coproduct defining $X_j^{(l+1)}$. Similarly, for $l > k$, we define $\circ_p^j$ to be the composite

$$X_j^{(k)} \times_{X_p} X_j^{(l)} \xrightarrow{(\mathrm{id},i^{(k,l)})} X_j^{(k)} \times_{X_p} X_j^{(k)} \hookrightarrow X_j^{(k+1)} \xrightarrow{c_j^{(k+1)}} \widehat{X}_j,$$

Then $\circ_p^j \colon \widehat{X}_j \times_{X_p} \widehat{X}_j \to \widehat{X}_j$ is defined to be the unique map induced by universal property of

$$\widehat{X}_j \times_{X_p} \widehat{X}_j$$

as a colimit (using Lemma 3.11) such that, for all k, $l > 0$, the diagram

$$\begin{tikzcd}
X_j^{(k)} \times_{X_p} X_j^{(l)} \arrow[r, "\left(c_j^{(k)},c_j^{(l)}\right)"] \arrow[dr, "\circ_p^j"'] & \widehat{X}_j \times_{X_p} \widehat{X}_j \arrow[d, dashed, "\circ_p^j"] \\
& \widehat{X}_j
\end{tikzcd}$$

commutes. This defines a j-magma structure on $\widehat{X}$.

We now define the map $\hat{f} \colon \widehat{X} \to S$. At dimension j, $\hat{f}$ acts on a freely generated composite in $\widehat{X}_j$ by first applying f to each individual generating j-cell in the composite, then evaluating this composite via the magma structure on S; at all other dimensions it is the same as the map f.

For $m \neq j$, define

$$\hat{f}_m = f_m \colon \widehat{X}_m = X_m \to S_m.$$

To define $\hat{f}_m$ for $m = j$, we first define, for each $k \geq 0$, a map

$$f_j^{(k)} \colon X_j^{(k)} \to S_j.$$

When $k = 0$, define
$$f_j^{(k)} = f_j \colon X_j^{(0)} \to S_j.$$
Now let $k \geq 1$ and suppose we have defined the map
$$f_j^{(k-1)} \colon X_j^{(k-1)} \to S_j;$$
we define the map
$$f_j^{(k)} \colon X_j^{(k)} \to S_j$$
as follows: for each $0 \leq p < j$ there is a map
$$\left(f_j^{(k-1)}, f_j^{(k-1)}\right) : X_j^{(k-1)} \times_{X_p} X_j^{(k-1)} \to S_j \times_{S_p} S_j$$
induced by the universal property of $S_j \times_{S_p} S_j$. We compose each of these with the composition map $\circ_p^j$, and define $f_j^{(k)} \colon X_j^{(k)} \to S_j$ to be a coproduct of these composites, as follows:
$$\begin{aligned} f_j^{(k)} :={}& f_j^{(0)} + \coprod_{0 \leq p < j} \left((\circ_p^j) \circ \left(f_j^{(k-1)}, f_j^{(k-1)}\right)\right) : \\ & X_j^{(k)} = X_j + \coprod_{0 \leq p < j} X_j^{(k-1)} \times_{X_p} X_j^{(k-1)} \to S_j. \end{aligned}$$

We then define $\hat{f}_j$ to be the unique map such that, for all $k \geq 1$, the diagram

$$\begin{tikzcd} X_j^{(k)} \arrow[r, "c_j^{(k)}"] \arrow[dr, "f_j^{(k)}"'] & \widehat{X}_j \arrow[d, dashed, "\hat{f}_j"] \\ & S_j \end{tikzcd}$$

commutes.

Thus we have defined an object

$$\begin{tikzcd} \widehat{X} \arrow[d, "\hat{f}"'] \\ S \end{tikzcd}$$

of $\mathbf{Mag}_j$; this gives the action of M_j on objects.

We now give the action of M_j on morphisms. Let

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

be a morphism in $\mathbf{Mag}_{j-1}$. We define a morphism

$$\begin{tikzcd} \hat{X} \arrow[r, "\hat{u}"] \arrow[d, "\hat{f}"'] & \hat{Y} \arrow[d, "\hat{g}"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

in $\mathbf{Mag}_j$. At dimension j, the map $\hat{u}$ acts on a freely generated composite in $\hat{X}$ by applying u to each individual generating j-cell in the composite, thus giving a freely generated composite of j-cells in $\hat{Y}_j$; at all other dimensions it is the same as the map u. The construction of $\hat{u}$ is very similar to that of $\hat{f}$.

For $m \neq j$, we define $\hat{u}_m = u_m$. To define $\hat{u}_m$ for $m = j$, first we define, for each $k \geq 1$, a map

$$u_j^{(k)} \colon X_j^{(k)} \to Y_j^{(k)}.$$

When $k = 0$, define

$$u_j^{(1)} := u_j \colon X_j^{(1)} \to Y_j^{(1)}.$$

Now let $k \geq 1$ and suppose we have defined

$$u_j^{(k-1)} := u_j \colon X_j^{(k-1)} \to Y_j^{(k-1)};$$

we define $u_j^{(k)}$ as follows:

$$u_j^{(k)} := u_j^{(k-1)} + \coprod_{0 \leq p < j} \left(u_j^{(k-1)}, u_j^{(k-1)} \right) \colon X_j^{(k)} \to Y_j^{(k)}.$$

We then define $\hat{u}_j$ to be the unique map such that, for all $k \geq 1$, the diagram

$$\begin{tikzcd} X_j^{(k)} \arrow[r, "c_j^{(k)}"] \arrow[d, "u_j^{(k)}"'] & \widehat{X}_j \arrow[d, dashed, "\hat{u}_j"] \\ Y_j^{(k)} \arrow[r, "c_j^{(k)}"'] & \hat{Y}_j \end{tikzcd}$$

commutes. This gives the action of the functor M_j on morphisms.

Proposition 3.13. *For all $0 < j \leq n$, there is an adjunction $M_j \dashv N_j$.*

Proof. We first define the unit $\eta \colon 1 \Rightarrow N_j M_j$ and counit $\epsilon \colon M_j N_j \Rightarrow 1$.

Let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object in $\mathbf{Mag}_{j-1}$. Then the corresponding component of the unit map η is

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \widehat{X} \arrow[d, "\hat{f}"] \\ S \arrow[r, "\mathrm{id}_S"'] & S, \end{tikzcd}$$

where η_X is defined by

$$(\eta_X)_k = \begin{cases} \mathrm{id}_{X_k} & \text{if } k \neq j, \\ \text{the coprojection map } c_j^{(0)} \colon X_j \to \widehat{X}_j & \text{if } k = j. \end{cases}$$

Naturality of η is immediate at dimensions $k \neq j$, and follows from the definition of the action of M_j on maps when $k = j$.

Now let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object in $\mathbf{Mag}_j$. The corresponding component of the counit map ϵ should be a map of the form

$$\begin{tikzcd} \widehat{X} \arrow[r, "\epsilon_X"] \arrow[d, "\hat{f}"'] & X \arrow[d, "f"] \\ S \arrow[r, "\mathrm{id}_S"'] & S. \end{tikzcd}$$

To define the map ϵ_X, recall that

$$\widehat{X}_j := \operatorname*{colim}_{k \geq 0} X_j^{(k)};$$

thus for each $k \geq 0$, we define a map

$$\epsilon_X^{(k)} \colon X_j^{(k)} \to X_j,$$

by induction over k.

When $k = 0$, $X_j^{(k)} = X_j$, and we define

$$\epsilon_X^{(0)} := \mathrm{id}_{X_j}.$$

Now suppose we have defined $\epsilon_X^{(k)}$ for some $k = l$. Recall that

$$X_j^{(l+1)} := X_j + \coprod_{0 \leq p < j} X_j^{(l)} \times_{X_p} X_j^{(l)}.$$

We define $\epsilon_X^{(l+1)}$ by

$$\epsilon_X^{(l+1)} := \mathrm{id}_{X_j} + \coprod_{0 \leq p < j} \Big((\circ_p^j) \circ \big(\epsilon_X^{(l)}, \epsilon_X^{(l)}\big) \Big) \colon X_j^{(l+1)} \longrightarrow X_j,$$

where $\circ_p^j$ is the composition map from the j-magma structure on X. We then define $(\epsilon_X)_j \colon \widehat{X}_j \to X_j$ to be the unique map such that, for all $k \geq 0$, the diagram

$$\begin{tikzcd} X_j^{(k)} \arrow[r, "c_j^{(k)}"] \arrow[dr, "\epsilon_X^{(k)}"'] & \widehat{X}_j \arrow[d, dashed, "(\epsilon_X)_j"] \\ & X_j \end{tikzcd}$$

commutes. This defines the counit $\epsilon\colon M_jN_j \Rightarrow 1$. We now check naturality of ϵ. Let

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$

be a morphism in $\mathbf{Mag}_j$; since the components of ϵ are identities on strict n-category parts, and at all dimensions other than dimension j, to show that ϵ is natural we need only show that the diagram

$$\begin{tikzcd} \widehat{X}_j \arrow[r, "\hat{u}_j"] \arrow[d, "(\epsilon_X)_j"'] & \hat{Y}_j \arrow[d, "(\epsilon_Y)_j"] \\ X_j \arrow[r, "u_j"'] & Y_j \end{tikzcd}$$

commutes. By definition of $\widehat{X}$ as a colimit, this diagram commutes if, for each $k \geq 0$, the diagram

$$\begin{tikzcd} X_j^{(k)} \arrow[r, "u_j^{(k)}"] \arrow[d, "\epsilon_X^{(k)}"'] & Y_j^{(k)} \arrow[d, "\epsilon_Y^{(k)}"] \\ X_j \arrow[r, "u_j"'] & Y_j \end{tikzcd}$$

commutes; we prove this by induction over k. It is immediate when $k = 0$, since $\epsilon_X^{(0)} = \mathrm{id}_{X_j}$ and $\epsilon_Y^{(0)} = \mathrm{id}_{Y_j}$. Now suppose we have shown that the diagram commutes for some $k = l$; then we have

$$\begin{aligned} u \circ \epsilon_X^{(l+1)} &= u_j + \coprod_{0 \leq p < j} \Big(u_j \circ (\circ_p^j) \circ \big(\epsilon_X^{(l)}, \epsilon_X^{(l)}\big) \Big) \\ &= u_j + \coprod_{0 \leq p < j} \Big((\circ_p^j) \circ \big(u_j\epsilon_X^{(l)}, u_j\epsilon_X^{(l)}\big) \Big) \\ &= u_j + \coprod_{0 \leq p < j} \Big((\circ_p^j) \circ \big(\epsilon_X^{(l)}u_j^{(l)}, \epsilon_X^{(l)}u_j^{(l)}\big) \Big) = \epsilon_j^{(l+1)}u_j^{(l+1)}, \end{aligned}$$

so the diagram commutes for $k = l + 1$. Thus, by induction, the diagram commutes for all $k \geq 0$. Hence ϵ is natural.

We now check that η and ϵ satisfy the triangle identities, i.e. that the diagrams

$$\begin{tikzcd} N_k \arrow[r, "\eta N_k"] \arrow[dr, "1"'] & N_k M_k N_k \arrow[d, "N_k \epsilon"] \\ & N_k, \end{tikzcd} \qquad \begin{tikzcd} M_k \arrow[r, "M_k \eta"] \arrow[dr, "1"'] & M_k N_k M_k \arrow[d, "\epsilon M_k"] \\ & M_k \end{tikzcd}$$

commute. In all of the natural transformations in these diagrams, the components on strict n-category parts are all identities, so to show that the diagrams commute we need only consider the components on n-globular set parts. Since the components of the maps of n-globular sets are identities at every dimension except dimension j, we need only check that the corresponding diagrams of maps of sets of j-cells commute.

For the first triangle identity, let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object of $\mathbf{Mag}_j$. Then the diagram

$$\begin{tikzcd} X_j \arrow[r, "(\eta_X)_j = c_j^{(0)}"] \arrow[dr, "\epsilon_j^{(0)}"'] & \widehat{X}_j \arrow[d, "(\epsilon_X)_j"] \\ & X_j \end{tikzcd}$$

commutes by the universal property of $(\epsilon_X)_j$, so this triangle identity is satisfied.

Similarly, for the second triangle identity, let

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

be an object of $\mathbf{Mag}_{j-1}$. Then the diagram

$$\begin{tikzcd}
\widehat{X}_j \arrow[r, "(\eta_{\widehat{X}})_j = c_j^{(0)}"] \arrow[dr, "\epsilon_j^{(0)}"'] & \widehat{\widehat{X}}_j \arrow[d, "(\epsilon_{\widehat{X}})_j"] \\
& \widehat{X}_j
\end{tikzcd}$$

commutes by the universal property of $(\epsilon_{\widehat{X}})_j$, so this triangle identity is satisfied.

Thus we have an adjunction $M_j \dashv N_j$, as required. $\square$

3.4 Interleaving the contraction and magma structures

We now explain the interleaving argument and show that we can interleave the constructions of Subsections 3.2 and 3.3 to give a construction of the left adjoint to the functor

$$W : \mathcal{Q} \to \mathcal{R}.$$

To do so we add the contraction and magma structures one dimension at a time, starting with dimension 1 and working upwards. At dimension m we first add free contraction cells using the functor C_m, then add free composites using the functor M_m, and then move up to the next dimension. Finally, we add "contraction $(n+1)$-cells" using the functor C_{n+1}, which identifies the appropriate cells at dimension n. Note that the method we use very closely follows the method used by Cheng in [7].

This construction is possible because of the dimensional dependencies of the functors C_k and M_j defined in Subsections 3.2 and 3.3; the contraction k-cells added by C_k only depend on the $(k-1)$-cells, and the composites added by the M_j only depend on the j-cells.

In order to describe this interleaving process formally, we define, for each $0 \leq j, k \leq n$, a category whose objects are objects of $\mathcal{R}$ equipped with both a j-magma structure and a k-contraction.

Definition 3.14. For each $0 \leq j \leq n$, $0 \leq k \leq n+1$, define a category $\mathcal{R}_{j,k}$ with

- objects: an object of $\mathcal{R}_{j,k}$ consists of an n-globular set X equipped with a j-magma structure, a strict n-category S, and a map of n-globular sets
$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$
that preserves the j-magma structure of X, equipped with a k-contraction γ;

- morphisms: a morphism in $\mathcal{R}_{j,k}$ is a commuting square
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, "v"'] & R \end{tikzcd}$$
in n-$\mathbf{GSet}$ such that
  - v is a map of strict n-categories;
  - u preserves the j-magma structure of X;
  - writing γ for the contraction on the map f and δ for the contraction on the map g, for all $0 < m \leq n$, and $(a,b) \in X^c_m$, we have
$$u(\gamma_m(a,b)) = \delta_m(u(a), u(b)).$$

For $0 < j \leq n$, $0 < k \leq n+1$, we have forgetful functors
$$D_{j,k} \colon \mathcal{R}_{j,k} \to \mathcal{R}_{j,k-1},$$
which forgets the contraction structure at dimension k, and
$$N_{j,k} \colon \mathcal{R}_{j,k} \to \mathcal{R}_{j-1,k},$$
which forgets the magma structure at dimension j. Thus we can write the forgetful functor
$$W \colon \mathcal{Q} \to \mathcal{R}$$

as the composite

$$\mathcal{Q} = \mathcal{R}_{n,n+1} \xrightarrow{D_{n,n+1}} \mathcal{R}_{n,n} \xrightarrow{N_{n,n}} \mathcal{R}_{n-1,n} \xrightarrow{D_{n-1,n}} \cdots \xrightarrow{N_{1,1}} \mathcal{R}_{0,1} \xrightarrow{D_{0,1}} \mathcal{R}_{0,0} = \mathcal{R}.$$

In order to construct the left adjoint to W, we construct a left adjoint to each of the factors in the composite above, by lifting the constructions of C_k and M_j from Subsections 3.2 and 3.3 in a way that interacts properly with the forgetful functors

$$\mathcal{R}_{j,k} \to \mathbf{Mag}_j,$$

which forget the k-contraction structure entirely, and

$$\mathcal{R}_{j,k} \to \mathbf{Contr}_j,$$

which forget the j-magma structure entirely.

Lemma 3.15. *For all $0 < k \leq n+1$, the adjunction*

$$\begin{tikzcd} \mathbf{Contr}_{k-1} \arrow[r, shift left, "C_k"] \arrow[r, phantom, "\perp"] & \mathbf{Contr}_k \arrow[l, shift left, "D_k"] \end{tikzcd}$$

lifts to an adjunction

$$\begin{tikzcd} \mathcal{R}_{k-1,k-1} \arrow[r, shift left, "C_{k-1,k}"] \arrow[r, phantom, "\perp"] & \mathcal{R}_{k-1,k} \arrow[l, shift left, "D_{k-1,k}"] \end{tikzcd}$$

making the diagram

$$\begin{tikzcd} \mathcal{R}_{k-1,k-1} \arrow[r, shift left, "C_{k-1,k}"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathcal{R}_{k-1,k} \arrow[l, shift left, "D_{k-1,k}"] \arrow[d] \\ \mathbf{Contr}_{k-1} \arrow[r, shift left, "C_k"] \arrow[r, phantom, "\perp"] & \mathbf{Contr}_k \arrow[l, shift left, "D_k"] \end{tikzcd}$$

commute serially.

Proof. We need to show that, given an object of $\mathbf{Contr}_{k-1}$ with n-globular set part X, if X is equipped with a $(k-1)$-magma structure, then this $(k-1)$-magma structure is "stable" under C_k; this is immediate since, by construction, C_k adds only k-cells to X, so the underlying $(k-1)$-globular set of X remains stable under C_k. $\square$

Lemma 3.16. *For all $0 < j \leq n$, the adjunction*

$$\begin{tikzcd} \mathbf{Mag}_{j-1} \arrow[r, shift left, "M_j"] \arrow[r, phantom, "\perp"] & \mathbf{Mag}_j \arrow[l, shift left, "N_j"] \end{tikzcd}$$

lifts to an adjunction

$$\begin{tikzcd} \mathcal{R}_{j-1,j} \arrow[r, shift left, "M_{j,j}"] \arrow[r, phantom, "\perp"] & \mathcal{R}_{j,j} \arrow[l, shift left, "N_{j,j}"] \end{tikzcd}$$

making the diagram

$$\begin{tikzcd} \mathcal{R}_{j-1,j} \arrow[r, shift left, "M_{j,j}"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathcal{R}_{j,j} \arrow[l, shift left, "N_{j,j}"] \arrow[d] \\ \mathbf{Mag}_{j-1} \arrow[r, shift left, "M_j"] \arrow[r, phantom, "\perp"] & \mathbf{Mag}_j \arrow[l, shift left, "N_j"] \end{tikzcd}$$

commute serially.

Proof. We need to show that, given an object

$$\begin{tikzcd} X \arrow[d, "f"'] \\ S \end{tikzcd}$$

of $\mathbf{Mag}_{j-1}$, if f is equipped with a j-contraction γ, this j-contraction structure is "stable" under M_j. By construction, M_j adds only j-cells to X, so $\mathrm{Tr}_{j-1}X$ remains stable under M_j. The required contraction j-cells depend only on the $(j-1)$-cells of $\widehat{X}$, and we have $\widehat{X}^c_{j-1} = X^c_{j-1}$, so the contraction j-cells in $\widehat{X}$ are given by

$$X^c_{j-1} \xrightarrow{\gamma_{j-1}} X_j \xrightarrow{c_j} \widehat{X}_j.$$

For $m < j$, we have $\widehat{X}^c_{m-1} = X^c_{m-1}$, $\widehat{X}_m = X_m$, and the contraction m-cells are given by $\gamma_{m-1}\colon X^c_{m-1} \to X_m$. Hence the j-contraction structure is stable under M_j. $\square$

Combining Lemmas 3.15 and 3.16, we obtain a chain of adjunctions

$$\mathcal{R} = \mathcal{R}_{0,0} \underset{D_{0,1}}{\overset{C_{0,1}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{R}_{0,1} \underset{N_{1,1}}{\overset{M_{1,1}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \cdots \underset{D_{n-1,n}}{\overset{C_{n-1,n}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{R}_{n-1,n} \underset{N_{n,n}}{\overset{M_{n,n}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{R}_{n,n} \underset{D_{n,n+1}}{\overset{C_{n,n+1}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{R}_{n,n+1} = \mathcal{Q}.$$

Composing these, we obtain an adjunction

$$\mathcal{R} \underset{W}{\overset{J}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{Q},$$

where $J = C_{n,n+1} \circ M_{n,n} \circ C_{n-1,n} \circ \cdots \circ M_{1,1} \circ C_{0,1}$. We then have

$$n\text{-}\mathbf{GSet} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{Q},$$

where $F = J \circ H$. Thus U has a left adjoint, so Penon weak n-categories are indeed well-defined, and moreover we have an explicit description of this left adjoint.

We now explain how to apply this construction in the case $n = \omega$. In this case, for each natural number k we have a composite adjunction

$$\mathcal{R} \underset{W_k}{\overset{J_k}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{R}_{k,k}$$

(i.e. $J_k = M_{k,k} \circ C_{k-1,k} \circ \cdots \circ M_{1,1} \circ C_{0,1}$). We define the functor $J\colon \mathcal{R} \longrightarrow \mathcal{Q}$ as follows: for an object A in $\mathcal{R}$, and for each natural number k,

$$(JA)_k := (J_k A)_k,$$

with magma structure, map of magmas, and contraction structure at dimension k given by those of $J_k A$. We then define $F := J \circ H$ as before, and we have

$$\omega\text{-}\mathbf{GSet} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{Q}.$$

4. The operad for Penon weak n-categories

In [3], Batanin proved that there is an n-globular operad whose algebras are Penon weak n-categories, and that this operad can be equipped with a contraction and system of compositions. In this section we give a new, alternative proof of this fact using the construction of Penon's left adjoint from Section 3. Although it is not a new result, our proof is more direct than that of Batanin, offering an alternative point of view in a way that elucidates the structure of the operad, and makes clear the fact that the contraction and system of compositions arise naturally from the contraction and magma structure in the original definition of the monad P.

Throughout this section, we write T for the free strict n-category monad on n-$\mathbf{GSet}$. This is the monad induced by the adjunction

$$\begin{tikzcd} n\text{-}\mathbf{GSet} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & n\text{-}\mathbf{Cat}, \arrow[l, shift left] \end{tikzcd}$$

where n-$\mathbf{Cat}$ is the category of strict n-categories, and the right adjoint is the forgetful functor sending a strict n-category to its underlying n-globular set. We write $\eta^T\colon 1 \Rightarrow T$ for the unit of the monad T, and $\mu^T\colon T^2 \Rightarrow T$ for its multiplication. Similarly, for any monad P, we denote its unit by $\eta^P\colon 1 \Rightarrow P$ and its multiplication by $\mu^P\colon P^2 \Rightarrow P$.

We begin by recalling the definition of n-globular operad. These were introduced by Batanin [2]; as it is technically convenient for our purposes, we use a form of the definition that describes an n-globular operad as a cartesian map of monads (see [13, Corollary 6.2.4]).

Definition 4.1. An *n-globular operad* consists of a monad K on n-$\mathbf{GSet}$, and a cartesian map of monads $k\colon K \Rightarrow T$ (by which we mean a cartesian natural transformation $k\colon K \Rightarrow T$ respecting the monad structure). Given operads $k\colon K \Rightarrow T$, $k'\colon K' \Rightarrow T$, a *map of operads* $f\colon K \Rightarrow K'$ is a map of monads such that the diagram

$$\begin{tikzcd} K \arrow[rr, Rightarrow, "f"] \arrow[dr, Rightarrow, "k"'] & & K' \arrow[dl, Rightarrow, "k'"] \\ & T & \end{tikzcd}$$

commutes. The category of algebras for an operad $k\colon K \Rightarrow T$ is the category K-$\mathbf{Alg}$ of algebras for the monad K.

It is a straightforward and enlightening exercise to prove that the monad K is necessarily cartesian. We leave this to the reader.

In Definition 4.1, replacing n-$\mathbf{GSet}$ with $\mathbf{Set}$ and T with the free monoid monad yields a definition equivalent to that of classical non-symmetric operads. Both are examples of the more general notion of T-operads, introduced by Burroni [5] and described in detail by Leinster [13, Section 4.2]. Symmetric operads can then be obtained by equipping classical non-symmetric operads with a symmetric group action, but there is no such action in the case of n-globular operads. For the remainder of the paper "operad" is taken to mean "n-globular operad", since they are the only type of operads we use.

To prove that there is an operad whose algebras are Penon weak n-categories using Proposition 4.1 we must prove three facts: that there is a natural transformation

$$p \colon P \Longrightarrow T,$$

that this natural transformation is cartesian, and that it is a map of monads. Note that we know that the source of this natural transformation must be P to ensure that the algebras for the resulting operad are indeed P-algebras.

Proposition 4.2. *Recall from Definition 2.5 that $P \colon n\text{-}\mathbf{GSet} \to n\text{-}\mathbf{GSet}$ is the monad induced by the adjunction*

$$n\text{-}\mathbf{GSet} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{Q}.$$

There is a natural transformation $p \colon P \Rightarrow T$ whose component p_X at an object X of n-$\mathbf{GSet}$ is given by

$$F(X) = (\ PX \xrightarrow{p_X} TX\ ),$$

an object of $\mathcal{Q}$.

Proof. Recall that there is a forgetful functor

$$U_T \colon n\text{-}\mathbf{Cat} \longrightarrow n\text{-}\mathbf{GSet}$$

that sends a strict n-category to its underlying n-globular set, and that the category $\mathcal{R}$ can be considered as the comma category

$$n\text{-}\mathbf{GSet} \downarrow U_T.$$

Write

$$\pi_1\colon n\text{-}\mathbf{GSet} \downarrow U_T \to n\text{-}\mathbf{GSet}$$

and

$$\pi_2\colon n\text{-}\mathbf{GSet} \downarrow U_T \to n\text{-}\mathbf{Cat}$$

for the projection maps, and consider the following diagram:

$$\begin{tikzcd}
 & n\text{-}\mathbf{GSet} \arrow[d, "F"] & \\
 & \mathcal{Q} \arrow[d, "W"] & \\
 & n\text{-}\mathbf{GSet} \downarrow G \arrow[dl, bend right, "\pi_1"'] \arrow[dr, bend left, "\pi_2"] \arrow[d, phantom, "\Rightarrow"] & \\
n\text{-}\mathbf{GSet} & & n\text{-}\mathbf{Cat}. \arrow[ll, "G"]
\end{tikzcd}$$

Then the universal property of $n\text{-}\mathbf{GSet} \downarrow U_T$ as a 2-limit (see [16]) induces a unique natural transformation $p\colon P \Rightarrow T$ such that

$$\begin{tikzcd}
 & n\text{-}\mathbf{GSet} \arrow[dl, bend right, "P"'] \arrow[dr, bend left, "F_T"] \arrow[d, phantom, "\Rightarrow p"] & \\
n\text{-}\mathbf{GSet} & & n\text{-}\mathbf{Cat} \arrow[ll, "G"]
\end{tikzcd}
=
\begin{tikzcd}
 & n\text{-}\mathbf{GSet} \arrow[d, "F"] \arrow[ddddl, bend right, "P"'] \arrow[ddddr, bend left, "F_T"] & \\
 & \mathcal{Q} \arrow[d, "W"] & \\
 & n\text{-}\mathbf{GSet} \downarrow U_T \arrow[ddl, bend right, "\pi_1"'] \arrow[ddr, bend left, "\pi_2"] \arrow[d, phantom, "\Rightarrow"] & \\
 & & \\
n\text{-}\mathbf{GSet} & & n\text{-}\mathbf{Cat} \arrow[ll, "G"]
\end{tikzcd}$$

where F_T is the free strict n-category functor. □

Proposition 4.3. *The natural transformation $p\colon P \Rightarrow T$ is cartesian.*

To prove this, we must show that each naturality square for p is a pullback square. To do so, we use the construction of the adjunction

$$\begin{tikzcd}
n\text{-}\mathbf{GSet} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{Q}. \arrow[l, shift left, "U"]
\end{tikzcd}$$

from Section 2. Recall that this adjunction can be decomposed as

$$n\text{-}\mathbf{GSet} \underset{V}{\overset{H}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{R} \underset{W}{\overset{J}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{Q}.$$

Given a map $f\colon X \to Y$ in n-$\mathbf{GSet}$, the corresponding naturality square is obtained by applying the functor $J\colon \mathcal{R} \to \mathcal{Q}$ to the map

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta^T_X"'] & Y \arrow[d, "\eta^T_Y"] \\ TX \arrow[r, "Tf"'] & TY. \end{tikzcd}$$

in $\mathcal{R}$, which is a pullback square in n-$\mathbf{GSet}$, since the free strict n-category monad T is cartesian [13, 4.1.18 and F.2.2]. Thus we prove that p is cartesian by proving that the functor J sends maps that are pullback squares to maps that are pullback squares (in fact, we do so only for a certain class of such maps). Recall that the adjunction $J \dashv W$ can be decomposed as the following chain of adjunctions:

$$\mathcal{R} = \mathcal{R}_{0,0} \underset{D_{0,1}}{\overset{C_{0,1}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{R}_{0,1} \underset{N_{1,1}}{\overset{M_{1,1}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \cdots \underset{D_{n-1,n}}{\overset{C_{n-1,n}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{R}_{n-1,n} \underset{N_{n,n}}{\overset{M_{n,n}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{R}_{n,n} \underset{D_{n,n+1}}{\overset{C_{n,n+1}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{R}_{n,n+1} = \mathcal{Q},$$

where the functor $C_{m,m+1}$ freely adds the contraction structure at dimension $m+1$, and the functor $M_{m,m}$ freely adds the magma structure at dimension m. We now prove three lemmas to show that each of these functors sends maps that are pullback squares to maps that are pullback squares, thus showing that their composite J does so as well. There are three lemmas since the functor $C_{n,n+1}$ must be treated separately from the functors $C_{m,m+1}$ for $0 \leq m \leq n-1$.

Note that we only consider maps whose the strict n-category part is a map in the image of T between free strict n-categories; this is as general as we need it to be to prove Proposition 4.3, and it allows us to use the fact that T is cartesian in the proofs of the lemmas.

Lemma 4.4. *Let $0 \leq m \leq n-1$ and suppose we have a morphism*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "x"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "y"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

in $\mathcal{R}_{m,m}$ that is a pullback square in n-$\mathbf{GSet}$. Then its image under the functor

$$C_{m,m+1} \colon \mathcal{R}_{m,m} \longrightarrow \mathcal{R}_{m,m+1}$$

is also a pullback square in n-$\mathbf{GSet}$.

Proof. The idea of the proof is as follows: the functor $C_{m,m+1}$ freely adds contraction $(m+1)$-cells to X and Y. These contraction cells are obtained by taking pullbacks in $\mathbf{Set}$, and then added to the sets of $(m+1)$-cells X_{m+1} and Y_{m+1} by taking coproducts in $\mathbf{Set}$. The action of $C_{m,m+1}$ on the map itself is then induced by the universal properties of these pullbacks and coproducts. Thus the image of this map under the functor $C_{m,m+1}$ is a coproduct of pullback squares (with some adjustments at the bottom to ensure that the strict n-category parts TX and TY remain unchanged). Since pullbacks commute with coproducts in $\mathbf{Set}$ [14, IX.2 Exercise 3], this coproduct of pullback squares is itself a pullback square.

Recall from Definition 2.3 that we have

$$\begin{tikzcd} X^c_{m+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "{(s,t,x_m)}"] \\ X_m \arrow[r, "{(s,t,x_m)}"'] & X_{m-1} \times X_{m-1} \times TA_m \end{tikzcd}$$

For $k \neq m+1$, we have $C_{m,m}(u, Tf)_k = (u, Tf)_k$, and since pullbacks in n-$\mathbf{GSet}$ are computed pointwise, we need only check that $C_{m,m}(u, Tf)_{m+1}$ is a pullback square, i.e. that

$$\begin{tikzcd} X_{m+1} \amalg X^c_{m+1} \arrow[r, "u_{m+1} \amalg u^c_{m+1}"] \arrow[d, "x_{m+1} \amalg x^c_{m+1}"'] & Y_{m+1} \amalg X^c_{m+1} \arrow[d, "y_{m+1} \amalg y^c_{m+1}"] \\ TA_{m+1} \arrow[r, "Tf_{m+1}"'] & TB_{m+1} \end{tikzcd}$$

is a pullback square. Since coproducts commute with pullbacks in $\mathbf{Set}$ [14, IX.2, exercise 3], this is true if the squares

$$\begin{tikzcd} X_{m+1} \arrow[r, "u_{m+1}"] \arrow[d, "x_{m+1}"'] & Y_{m+1} \arrow[d, "y_{m+1}"] \\ TA_{m+1} \arrow[r, "Tf_{m+1}"'] & TB_{m+1} \end{tikzcd} \qquad \begin{tikzcd} X^c_{m+1} \arrow[r, "u^c_{m+1}"] \arrow[d, "x^c_{m+1}"'] & Y^c_{m+1} \arrow[d, "y^c_{m+1}"] \\ TA_{m+1} \arrow[r, "Tf_{m+1}"'] & TB_{m+1} \end{tikzcd}$$

are both pullback squares. The left-hand square is a pullback square by hypothesis. For the right-hand square, suppose we have a cone

$$\begin{tikzcd} V \arrow[r, "v_1"] \arrow[d, "v_2"'] & Y^c_{m+1} \arrow[d, "y^c_{m+1}"] \\ TA_{m+1} \arrow[r, "Tf_{m+1}"'] & TB_{m+1} \end{tikzcd}$$

in $\mathbf{Set}$. Recall that we have source and target maps $s, t \colon Y^c_{m+1} \to Y_m$ given by the projections from the pullback defining Y^c_{m+1}. Composing with these, and source and target maps for TA and TB, induces maps

$$\begin{tikzcd} V \arrow[dr, "!\sigma"] \arrow[drr, bend left, "sv_1"] \arrow[ddr, bend right, "sv_2"'] & & \\ & X_m \arrow[r, "u_m"] \arrow[d, "x_m"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_m \arrow[d, "y_m"] \\ & TA_m \arrow[r, "Tf_m"'] & TB_m, \end{tikzcd} \qquad \begin{tikzcd} V \arrow[dr, "!\tau"] \arrow[drr, bend left, "tv_1"] \arrow[ddr, bend right, "tv_2"'] & & \\ & X_m \arrow[r, "u_m"] \arrow[d, "x_m"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_m \arrow[d, "y_m"] \\ & TA_m \arrow[r, "Tf_m"'] & TB_m. \end{tikzcd}$$

The maps σ and τ give us a cone over the pullback square defining X^c_{m+1}; commutativity of this cone comes from the globularity conditions and the fact that every cell in the image of v_2 is an identity, so has the same source and target. Thus the universal property of X^c_{m+1} induces a unique map such that the diagram

$$\begin{tikzcd}
V \arrow[dr, "v"] \arrow[drr, bend left, "\sigma"] \arrow[ddr, bend right, "\tau"'] & & \\
& X^c_{m+1} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & X_m \arrow[d, "{(s,t,x_m)}"] \\
& X_m \arrow[r, "{(s,t,x_m)}"'] & X_{m-1} \times X_{m-1} \times TA_m.
\end{tikzcd}$$

commutes.

We now check that v makes the diagram

$$\begin{tikzcd}
V \arrow[dr, "v"] \arrow[drr, bend left, "v_1"] \arrow[ddr, bend right, "v_2"'] & & \\
& X^c_{m+1} \arrow[r, "u^c_{m+1}"] \arrow[d, "x^c_{m+1}"] & Y^c_{m+1} \arrow[d, "y^c_{m+1}"] \\
& TA_{m+1} \arrow[r, "Tf_{m+1}"'] & TB_{m+1}
\end{tikzcd}$$

commute. To show that the top triangle commutes, observe that the map $v_1 = u^c_{m+1} \circ v$ makes the following diagram commute:

$$\begin{tikzcd}
V \arrow[dr, "v"] \arrow[drrr, bend left, "\sigma"] \arrow[dddr, bend right, "\tau"'] & & & & \\
& X^c_{m+1} \arrow[rr, "s"] \arrow[dr, "u^c_{m+1}"] \arrow[dd, "t"] & & X_m \arrow[dr, "u_m"] & \\
& & Y^c_{m+1} \arrow[rr, "s"] \arrow[dd, "t"] \arrow[ddrr, phantom, very near start, "\lrcorner"] & & Y_m \arrow[dd, "{(s,t,y_m)}"] \\
& X_m \arrow[dr, "u_m"'] & & & \\
& & Y_m \arrow[rr, "{(s,t,y_m)}"'] & & Y_{m-1} \times Y_{m-1} \times TB_m.
\end{tikzcd}$$

Since $u_m\sigma = sv_1$ and $u_m\tau = tv_1$, by the universal property of Y^c_{m+1}, we have $u^c_{m+1} \circ v = v_1$.

To show that the left-hand triangle commutes, write $i\colon TA_m \to TA_{m+1}$ for the map that sends an m-cell to its identity $(m+1)$-cell, and consider that we can factorise $x^c_{m+1} \circ v$ as

$$\begin{tikzcd} V \arrow[r, "v"] \arrow[rr, bend right, "\sigma"'] \arrow[rrr, bend right, "sv_2"'] & X^c_{m+1} \arrow[r, "s"] \arrow[rrr, bend left, "x^c_{m+1}"] & X_m \arrow[r, "x_m"] & TA_m \arrow[r, "i"] & TA_{m+1}. \end{tikzcd}$$

Thus we have $x^c_{m+1} \circ v = isv_2 = v_2$, since all cells in the image of v_2 are identities.

Finally, uniqueness of v comes from the universal property of X^c_{m+1}. Hence

$$\begin{tikzcd} X^c_{m+1} \arrow[r, "u^c_{m+1}"] \arrow[d, "x^c_{m+1}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y^c_{m+1} \arrow[d, "y^c_{m+1}"] \\ TA_{m+1} \arrow[r, "Tf_{m+1}"'] & TB_{m+1} \end{tikzcd}$$

is a pullback square, so $C_{m,m+1}(u, Tf)$ is a pullback square. □

We must treat the case $m = n$ separately.

Lemma 4.5. *Suppose we have a morphism*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "x"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "y"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

in $\mathcal{R}_{n,n}$ that is a pullback square in n-$\mathbf{GSet}$. Then its image under the functor

$$C_{n,n+1} \colon \mathcal{R}_{n,n} \longrightarrow \mathcal{R}_{n,n+1} = \mathcal{Q}$$

is also a pullback square in n-$\mathbf{GSet}$.

Proof. Recall from Definition 3.8 that we have

$$\begin{tikzcd} X^c_{n+1} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X_n \arrow[d, "(s,t,x_n)"] \\ X_n \arrow[r, "(s,t,x_n)"'] & X_{n-1} \times X_{n-1} \times TA_n, \end{tikzcd}$$

and that $\tilde{X}_n$ is defined to be the coequaliser of the diagram

$$\begin{tikzcd} X^c_{n+1} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X_n \end{tikzcd}$$

in $\mathbf{Set}$. We write $q\colon X_n \to \tilde{X}_n$ for the coprojection. The set $\tilde{Y}_n$ is defined similarly, and we write $r\colon Y_n \to \tilde{Y}_n$ for the coprojection. For all $0 \le m < n$ we have

$$C_{n,n+1}(u, Tf)_m = (u, Tf)_m,$$

and for $m = n$, we have that $C_{n,n+1}(u, Tf)_n$ is given by

$$\begin{tikzcd} \tilde{X}_n \arrow[r, "\tilde{u}_n"] \arrow[d, "\tilde{x}_n"'] & \tilde{Y}_n \arrow[d, "\tilde{y}_n"] \\ TA_n \arrow[r, "Tf_n"'] & TB_n, \end{tikzcd}$$

so we only need to check that this is a pullback square in $\mathbf{Set}$.

Write w for the unique map making the diagram

$$\begin{tikzcd} X_n \arrow[r, "u_n"] \arrow[dr, dashed, "w"] \arrow[ddr, bend right, "x_n"'] & Y_n \arrow[dr, "r"] & \\ & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{Y}_n \arrow[d, "\tilde{y}_n"] \\ & TA_n \arrow[r, "Tf_n"'] & TB_n \end{tikzcd}$$

commute. We will show that, for a, $b \in X_n$, $w(a) = w(b)$ if and only if $(a, b) \in X^c_{n+1}$, and also that w is surjective; and thus $C_{n,n+1}(u, Tf)_n$ is a pullback square and $w = q$.

Let $(a, b) \in X^c_{n+1}$, so $x_n(a) = x_n(b)$, $s(a) = s(b)$, $t(a) = t(b)$. We have $(u_n(a), u_n(b)) \in Y^c_{n+1}$, so $ru_n(a) = ru_n(b)$. Thus

$$w(a) = (x_n(a), ru_n(a)) = (x_n(b), ru_n(b)) = w(b).$$

Now let a, $b \in X_n$ with $w(a) = w(b)$, so $x_n(a) = x_n(b)$, $ru_n(a) = ru_n(b)$. The source map $s\colon X_n \to X_{n-1}$ is the unique map making the diagram

$$\begin{tikzcd}
X_n \arrow[rr, "u_n"] \arrow[dr, "s"] \arrow[dd, "x_n"'] & & Y_n \arrow[dr, "s"] & \\
& X_{n-1} \arrow[rr, "u_{n-1}"] \arrow[dd, "x_{n-1}"] \arrow[drr, phantom, "\lrcorner", very near start] & & Y_{n-1} \arrow[dd, "y_{n-1}"] \\
TA_n \arrow[dr, "s"'] & & & \\
& TA_{n-1} \arrow[rr, "Tf_{n-1}"'] & & TB_{n-1}
\end{tikzcd}$$

commute. Thus, since $su_n(a) = su_n(b)$ and $sx_n(a) = sx_n(b)$, we have $s(a) = s(b)$. Similarly, $t(a) = t(b)$. Hence $(a, b) \in X^c_{n+1}$.

Now let $\pi \in TA_n$, $c \in \tilde{Y}_n$, with $Tf_n(\pi) = \tilde{y}_n(c)$. We wish to show that there is some $a \in X_n$ with $w(a) = (\pi, c)$, and thus that w is surjective. Since r is surjective, there exists $c' \in Y_n$ with $r(c') = c$. Since X_n is given by the pullback

$$\begin{tikzcd}
X_n \arrow[r, "u_n"] \arrow[d, "x_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_n \arrow[d, "y_n"] \\
TA_n \arrow[r, "Tf_n"'] & TB_n
\end{tikzcd}$$

and $yr(c') = Tf_n(\pi)$, we have $a \in X_n$ with $x_n(a) = \pi$, $u_n(a) = c'$. Thus $w(a) = (\pi, c)$, so w is surjective. Hence

$$\begin{tikzcd}
\tilde{X}_n \arrow[r, "\tilde{u}_n"] \arrow[d, "\tilde{x}_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{Y}_n \arrow[d, "\tilde{y}_n"] \\
TA_n \arrow[r, "Tf_n"'] & TB_n
\end{tikzcd}$$

is a pullback square. $\square$

Thus we have shown that the functors adding the free contraction cells send maps that are pullback squares to maps that are pullback squares. We now do the same for the functors adding the free magma structure.

Lemma 4.6. *Let* $0 < m \le n$ *and suppose we have a morphism*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "x"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "y"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

in $\mathcal{R}_{m-1,m}$ *that is a pullback square in* n-$\mathbf{GSet}$. *Then its image under the functor*

$$M_{m,m} \colon \mathcal{R}_{m-1,m} \longrightarrow \mathcal{R}_{m,m}$$

is also a pullback square in n-$\mathbf{GSet}$.

Proof. The idea of this proof is similar to that of the proof of Lemma 4.4, but is slightly more complicated since the construction of $M_{m,m}$ uses filtered colimits as well as coproducts. The functor $M_{m,m}$ freely adds binary composites of m-cells to X and Y. These composites are added through a process of taking pullbacks, coproducts, and filtered colimits in $\mathbf{Set}$. The action of $M_{m,m}$ on the map itself is then induced by the universal properties of these pullbacks, coproducts, and filtered colimits. Thus the image of this map under the functor $M_{m,m}$ is a filtered colimit of coproducts of pullback squares (with some adjustments at the bottom to ensure that the strict n-category parts TX and TY remain unchanged). Since pullbacks commute with both coproducts and filtered colimits in $\mathbf{Set}$ [14, IX.2, Exercise 3 and Theorem 1], this filtered colimit of coproducts of pullback squares is itself a pullback square.

Recall the notation from Definition 3.12: we write

$$M_{m,m}(\, X \xrightarrow{x} TA \,) = \hat{X} \xrightarrow{\hat{x}} TA\ ,$$

$$M_{m,m}(\, Y \xrightarrow{y} TB \,) = \hat{Y} \xrightarrow{\hat{y}} TB\ .$$

Since $M_{m,m}$ changes only dimension m, and since pullbacks in n-$\mathbf{GSet}$ are computed pointwise, we just need to check that

$$\begin{tikzcd} \hat{X}_m \arrow[r, "\hat{u}_m"] \arrow[d, "\hat{x}_m"'] & \hat{Y}_m \arrow[d, "\hat{y}_m"] \\ TA_m \arrow[r, "Tf_m"'] & TB_m \end{tikzcd}$$

is a pullback square in $\mathbf{Set}$. Recall that $\hat{X}_m$ and $\hat{Y}_m$ are defined as filtered colimits in $\mathbf{Set}$, with

$$\hat{X}_m := \operatorname*{colim}_{j \geq 1} X_m^{(j)}, \; \hat{Y}_m := \operatorname*{colim}_{j \geq 1} Y_m^{(j)}.$$

Since pullbacks commute with filtered colimits in $\mathbf{Set}$, we can prove that the above diagram is a pullback square by proving that, for each $j \geq 1$, the diagram

$$\begin{tikzcd} X_m^{(j)} \arrow[r, "u_m^{(j)}"] \arrow[d, "x_m^{(j)}"'] & Y_m^{(j)} \arrow[d, "y_m^{(j)}"] \\ TA_m \arrow[r, "Tf_m"'] & TB_m \end{tikzcd}$$

is a pullback square in $\mathbf{Set}$. We do this by induction. When $j = 1$, we have $X_m^{(j)} = X_m$, $Y_m^{(j)} = Y_m$, and the square above becomes is a pullback square by hypothesis.

Now suppose that $j > 1$, and we have shown that

$$\begin{tikzcd} X_m^{(j-1)} \arrow[r, "u_m^{(j-1)}"] \arrow[d, "x_m^{(j-1)}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y_m^{(j-1)} \arrow[d, "y_m^{(j-1)}"] \\ TA_m \arrow[r, "Tf_m"'] & TB_m \end{tikzcd}$$

is a pullback square; we will show that

$$\begin{tikzcd} X_m^{(j)} \arrow[r, "u_m^{(j)}"] \arrow[d, "x_m^{(j)}"'] & Y_m^{(j)} \arrow[d, "y_m^{(j)}"] \\ TA_m \arrow[r, "Tf_m"'] & TB_m \end{tikzcd}$$

is a pullback square. Recall that $X_m^{(j)}$ is defined by

$$X_m^{(j)} := X_m \amalg \coprod_{0 \leq p < m} X_m^{(j-1)} \times_{X_p} X_m^{(j-1)},$$

and similarly for $Y_m^{(j)}$. Since pullbacks commute with coproducts in $\mathbf{Set}$, the above diagram is a pullback square if, for all $0 \leq p < m$, the diagram

$$\begin{tikzcd}
X_m^{(j-1)} \times_{X_p} X_m^{(j-1)} \arrow[r, "(u_m^{(j-1)}, u_m^{(j-1)})"] \arrow[d] & Y_m^{(j-1)} \times_{Y_p} Y_m^{(j-1)} \arrow[d] \\
TA_m \arrow[r, "Tf_m"'] & TB_m
\end{tikzcd}$$

is a pullback square. We can write this as

$$\begin{tikzcd}
X_m^{(j-1)} \times_{X_p} X_m^{(j-1)} \arrow[r, "(u_m^{(j-1)}, u_m^{(j-1)})"] \arrow[d, "(x_m^{(j,1)}, x_m^{(j,1)})"'] & Y_m^{(j-1)} \times_{Y_p} Y_m^{(j-1)} \arrow[d, "(y_m^{(j,1)}, y_m^{(j,1)})"] \\
TA_m \times_{TA_p} TA_m \arrow[r, "(Tf_m, Tf_m)"'] \arrow[d, "\circ_p^m"'] & TB_m \times_{TB_p} TB_m \arrow[d, "\circ_p^m"] \\
TA_m \arrow[r, "Tf_m"'] & TB_m.
\end{tikzcd}$$

The top square is a pullback of pullback squares, and hence is itself a pullback square. The fact that the bottom square is a pullback square is left as a straightforward exercise to the reader; it is an application of the fact that T is a cartesian monad [13, Example 4.1.18 and Theorem F.2.2], so the naturality squares for its multiplication μ^T are pullbacks squares, and the fact that T^2A and T^2B can be constructed via a series a pullbacks in n-$\mathbf{GSet}$ (see [13, F.1] and [8], which give constructions of T using this method).

Thus the diagram

$$\begin{tikzcd}
\hat{X}_m \arrow[r, "\hat{u}_m"] \arrow[d, "\hat{x}_m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \hat{Y}_m \arrow[d, "\hat{y}_m"] \\
TA_m \arrow[r, "Tf_m"'] & TB_m
\end{tikzcd}$$

is a pullback square in n-$\mathbf{GSet}$. Hence $M_{m,m}$ sends maps that are pullback squares to maps that are pullback squares, as required. $\square$

We now combine these results to prove that $p\colon P \Rightarrow T$ is cartesian.

Proof of Proposition 4.3. Combining the above results, and using the fact that $J\colon \mathcal{R} \to \mathcal{Q}$ is defined as the composite

$$J = C_{n,n+1} \circ M_{n,n} \circ C_{n-1,n} \circ \cdots \circ M_{1,1} \circ C_{0,1},$$

we see that, given a map (u, Tf) in $\mathcal{R}$ such that

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "x"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "y"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

is a pullback square in n-$\mathbf{GSet}$, the map $J(u, Tf)$ in $\mathcal{Q}$ is also a pullback square in n-$\mathbf{GSet}$. Take (u, Tf) to be

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta^T_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\eta^T_B"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

for any $f\colon A \to B$ in n-$\mathbf{GSet}$, which is a pullback square since T is cartesian. Applying J gives us that

$$\begin{tikzcd} PA \arrow[r, "Pf"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & PB \arrow[d, "p_B"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

is a pullback square in n-$\mathbf{GSet}$. Thus $p\colon P \Rightarrow T$ is a cartesian natural transformation. $\square$

Thus the natural transformation $p\colon P \Rightarrow T$ satisfies one of the conditions in Proposition 4.1; to prove that it is an operad, we now only need to prove the following:

Proposition 4.7. *The natural transformation $p\colon P \Rightarrow T$ is a map of monads.*

Proof. We need to check that p satisfies the monad map axioms. To do so, recall that P is the monad induced by the adjunction

$$\begin{tikzcd} n\text{-}\mathbf{GSet} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{Q} \arrow[l, shift left, "U"] \end{tikzcd}$$

defined in Section 3, and that this adjunction can be decomposed as

$$\begin{tikzcd} n\text{-}\mathbf{GSet} \arrow[r, shift left, "H"] \arrow[r, phantom, "\perp"] & \mathcal{R} \arrow[l, shift left, "V"] \arrow[r, shift left, "J"] \arrow[r, phantom, "\perp"] & \mathcal{Q}. \arrow[l, shift left, "W"] \end{tikzcd}$$

Write α, β for the unit and counit of $H \dashv V$, and write κ, ζ for the unit and counit of $J \dashv W$. Then the unit $\eta = \eta^P$ of the adjunction $F \dashv U$ is given by the composite

$$1 \xrightarrow{\alpha} VH \xrightarrow{V\kappa H} VWJH = UF$$

and the counit ϵ of $F \dashv U$ is given by the composite

$$FU = JHVW \xrightarrow{J\beta W} JW \xrightarrow{\zeta} 1.$$

To show that p satisfies the axioms for a monad map we consider the unit η^P and counit ϵ for the adjunction $F \dashv U$. By Proposition 3.3, $\alpha = \mathrm{id}$, so $\eta^P = V\kappa H$. For all $X \in n\text{-}\mathbf{GSet}$, κ_{HX} is the map

$$\begin{tikzcd} X \arrow[r, "\eta^P_X"] \arrow[d, "\eta^T_X"'] & PX \arrow[d, "p_X"] \\ TX \arrow[r, "\mathrm{id}_{TX}"'] & TX \end{tikzcd}$$

in $\mathcal{R}$. Commutativity of this diagram shows that p satisfies the first axiom for a monad map.

For all $X \in n\text{-}\mathbf{GSet}$, ϵ_{FX} is the map

$$\begin{tikzcd} P^2X \arrow[rr, "\mu^P_X"] \arrow[d, "p_{PX}"'] & & PX \arrow[d, "p_X"] \\ TPX \arrow[r, "Tp_X"'] & T^2X \arrow[r, "\mu^T_X"'] & TX \end{tikzcd}$$

in $\mathcal{Q}$. Commutativity of this diagram shows p satisfies the second axiom for a monad map.

Thus $p\colon P \Rightarrow T$ is a monad map. □

Combining Propositions 4.3 and 4.7 gives us the following theorem:

Theorem 4.8. *There is an operad whose algebras are Penon weak n-categories, given by the cartesian map of monads $p\colon P \Rightarrow T$.*

Proof. The natural transformation $p\colon P \Rightarrow T$ is cartesian by Proposition 4.3, and is a monad map by Proposition 4.7. Thus it is an operad, and its category of algebras is P-**Alg**, the category of Penon weak n-categories. □

In [2] Batanin uses two pieces of extra structure to identify which n-globular operads give sensible notions of weak n-category: a system of compositions, which gives the operad binary composition operations, and a contraction, which gives coherence. Both pieces of extra structure are defined on the "underlying collection" of an operad – its component on the terminal n-globular set. We now recall the necessary definitions, then show that the operad $p\colon P \Rightarrow T$ can be equipped with both structures in a way that arises naturally from the n-magma and contraction structures in the definition of P.

Definition 4.9. Given an n-globular operad $k\colon K \Rightarrow T$, its *underlying collection* is the component of k at the terminal n-globular set 1, that is $k_1\colon K1 \to T1$.

Definition 4.10. A *contraction* on an n-globular operad $k\colon K \Rightarrow T$ consists of a contraction (in the sense of Definition 2.3) on its underlying collection.

Definition 4.11. Let $0 \leq m \leq n$, and write $\eta_m := \eta^T_m(1)$, the single m-cell in the image of the unit map $\eta^T : 1 \to T1$. Define, for $0 \leq p \leq m \leq n$,

$$\beta^m_p = \begin{cases} \eta_m & \text{if } p = m, \\ \eta_m \circ^m_p \eta_m & \text{if } p < m. \end{cases}$$

Define an n-globular set S, in which

$$S_m := \{\beta^m_p \mid 0 \leq p \leq m \leq n\} \subseteq T1_m.$$

Write $s\colon S \to T1$ for the inclusion, and define the "unit" map $\eta^S\colon\ 1 \to S$ by $\eta^S_m(1) = \beta^m_m$.

A *system of compositions* on an n-globular operad $k\colon K \Rightarrow T$ consists of a map $\sigma\colon S \to K1$ in n-$\mathbf{GSet}$ such that the diagrams

$$\begin{tikzcd} S \arrow[rr, "\sigma"] \arrow[dr, "s"'] & & K1 \arrow[dl, "k_1"] \\ & T1 & \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[r, "\eta^S"] \arrow[rr, bend right, "\eta^K_1"'] & S \arrow[r, "\sigma"] & K1 \end{tikzcd}$$

commute.

Proposition 4.12. *The operad P for Penon weak n-categories can be equipped with a contraction and system of compositions which arise naturally from the contraction on $p_1\colon P1 \to T1$ and the magma structure on $P1$ respectively.*

Proof. The presence of the contraction is immediate, since

$$\begin{tikzcd} P1 \arrow[d, "p_1"] \\ T1 \end{tikzcd}$$

is an object of $\mathcal{Q}$, so is equipped with a contraction as constructed in Section 3. Similarly, $P1$ is equipped with a magma structure; we use this to define a system of compositions

$$\begin{tikzcd} S \arrow[rr, "\sigma"] \arrow[dr, "s"'] & & P1 \arrow[dl, "p_1"] \\ & T1 & \end{tikzcd}$$

as follows: for all $0 \leq m \leq n$, writing 1 for the unique m-cell in the terminal n-globular set,

- $\sigma_m(\beta^m_m) := (\eta^P_1)_m(1) = 1$;
- for $0 \leq l \leq m$, $\sigma_m(\beta^m_l) := 1 \circ^m_l 1$.

From the definition of the magma structure on $P1$ given in Definition 3.12, this satisfies the source and target conditions for a map of n-globular sets, and the commutativity conditions required to be a map of collections. By definition of $\sigma_m(\beta^m_m)$,

$$\begin{tikzcd} 1 \arrow[r, "\epsilon^S"] \arrow[rr, bend right, "\eta^P_1"'] & S \arrow[r, "\sigma"] & P1 \end{tikzcd}$$

commutes. Thus, σ is a system of compositions on $p\colon P \Rightarrow T$. $\square$

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

VOLUME LIX-3 (2018)

CENTRAL EXTENSIONS OF MAPPING CLASS GROUPS FROM CHARACTERISTIC CLASSES

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Résumé. Les structures tangentielles sur les variétés lisses, et l'extension du *'mapping class'* groupe qu'elles induisent, admettent une formulation naturelle en termes de géométrie différentielle supérieure (stratifiée). C'est la traduction littérale d'une construction, classique en topologie différentielle, en un langage sophistiqué, mais elle a l'avantage de souligner comment toute la construction émerge naturellement de l'idée de base de travailler avec des *'slice'* catégories. Nous caractérisons, pour chaque champ lisse supérieur muni d'une structure tangentielle, l'extension induite du groupe supérieur de la réalisation géométrique de son champ d'automorphismes supérieur. Nous montrons que lorsque l'on se restreint à des variétés lisses équipées de structures topologiques de degré supérieur, cela produit des extensions supérieures de types homotopiques de groupes de difféomorphismes. Passant aux groupes de composantes connexes, nous obtenons des extensions abéliennes des groupes de classes de difféomorphismes et nous en déduisons des conditions suffisantes pour qu'elles soient centrales. Nous montrons, à titre d'example, que ceci fournit une reconstruction élégante de l'approche de Segal des extensions par $\mathbb{Z}$ du 'mapping class' groupe de surfaces qui fournissent une annulation d'anomalie du foncteur modulaire dans la théorie de Chern-Simons. Notre construction généralise l'approche de Segal des extensions centrales supérieures du 'mapping class' groupe de variétés de dimension supérieure avec des structures tangentielles supérieures, qui devraient fournir une annulation d'anomalie analogue pour les TQFT en dimension

supérieure.

Abstract. Tangential structures on smooth manifolds, and the extension of mapping class groups they induce, admit a natural formulation in terms of higher (stacky) differential geometry. This is the literal translation of a classical construction in differential topology to a sophisticated language, but it has the advantage of emphasizing how the whole construction naturally emerges from the basic idea of working in slice categories. We characterize, for every higher smooth stack equipped with tangential structure, the induced higher group extension of the geometric realization of its higher automorphism stack. We show that when restricted to smooth manifolds equipped with higher degree topological structures, this produces higher extensions of homotopy types of diffeomorphism groups. Passing to the groups of connected components, we obtain abelian extensions of mapping class groups and we derive sufficient conditions for these being central. We show as a special case that this provides an elegant re-construction of Segal's approach to $\mathbb{Z}$-extensions of mapping class groups of surfaces that provides the anomaly cancellation of the modular functor in Chern-Simons theory. Our construction generalizes Segal's approach to higher central extensions of mapping class groups of higher dimensional manifolds with higher tangential structures, expected to provide the analogous anomaly cancellation for higher dimensional TQFTs.
Keywords. Mapping class groups; diffeomorphism groups; characteristic classes; higher categories.
Mathematics Subject Classification (2010). 57R50, 57R56.

"Everything in its right place"
Kid A, Radiohead

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1. Introduction

We review the construction of higher automorphism groups of smooth manifolds equipped with higher tangential structure from [GMTW06, GR-W10, Lu09] reformulating it into the language of higher smooth stacks [Sc13]. In the final part we use this to provide a clear and natural construction of central extensions of mapping class groups, such as demanded by Segal's discussion of conformal field theory [Se04].

In higher (stacky) geometry, there is a general and fundamental class of higher (stacky) group extensions: for $\psi : Y \to B$ any morphism between higher stacks, the automorphism group stack of Y over B extends the automorphisms of Y itself by the loop object of the mapping stack $[Y, B]$ based at ψ. Schematically this extension is of the following form

$$
\left\{ \begin{tikzcd} Y \arrow[d, bend right, "\psi"'] \arrow[d, bend left, "\psi"] \arrow[d, phantom, "\swArrow"] \\ B \end{tikzcd} \right\} \longrightarrow \left\{ \begin{tikzcd} Y \arrow[rr, "\simeq"] \arrow[dr, "\psi"'] & \arrow[d, phantom, "\swArrow"] & Y \arrow[dl, "\psi"] \\ & B & \end{tikzcd} \right\} \longrightarrow \left\{ Y \xrightarrow{\simeq} Y \right\}
$$

but the point is that all three items here are themselves realized "internally" as higher group stacks. This is not hard to prove [Sc13, prop. 3.6.16], but as a general abstract fact it has many non-trivial incarnations. Here we are concerned with a class of examples of these extensions for the case of smooth higher stacks, i.e. higher stacks over the site of all smooth manifolds.

In [FRS13] it was shown that for the choice that $B = \mathbf{B}^n U(1)_{\mathrm{conn}}$ is the universal moduli stack for degree $n+1$ ordinary differential cohomology, then these extensions reproduce and generalize the Heisenberg-Kirillov-Kostant-Souriau-extension from prequantum line bundles to higher "prequantum gerbes" which appear in the local (or "extended") geometric quantization of higher dimensional field theories.

Here we consider a class of examples at the other extreme: we consider the case in which Y is a smooth manifold (regarded as the stack that it presents), but B is geometrically discrete (i.e., it is a locally constant ∞-stack), and particularly the case that B is the homotopy type of the classifying space of the general linear group. This means that the slice automorphism group (the middle term above) becomes a smooth group stack that extends the smooth diffeomorphism group of Y (the item on the right above) by a locally constant higher group stack (the item on the left).

We are interested in the homotopy type of this higher stacky extension of the diffeomorphism group, that is in the geometric realization of the smooth slice group stack. In general, geometric realization of higher smooth group stacks will not preserve the above extension, but here it does, due to the fact that B is assumed to be geometrically discrete. This resulting class of extensions is our main Theorem 4.1 below. It uses that geometric realization of smooth ∞-stacks happens to preserve homotopy fibers over geometrically discrete objects [Sc13, thm. 3.8.19]. Hence, where the internal extension theorem gives extensions of smooth diffeomorphism groups by higher homotopy types, after geometric realization we obtain higher extensions of the homotopy type of diffeomorphism groups, and in particular of mapping class groups.

We emphasize that it is the interplay between smooth higher stacks and their geometric realization that makes this work: one does not see diffeomorphism groups, nor their homotopy types, when forming the above extension in the plain homotopy theory of topological spaces. So, even though the group extensions that we study are geometrically discrete, they encode information about smooth diffeomorphism groups.

A key application where extensions of the mapping class group traditionally play a role is anomaly cancellation in 3-dimensional topological field theories, e.g., in 3d Chern-Simons theory, see, e.g., [Wi89].

Our general extension result reduces to a new and elegant construction of the anomaly cancellation construction for modular functors in 3d Chern-Simons theory, and naturally generalizes this to higher extensions relevant for higher dimensional topological quantum field theories (TQFTs).

In more detail, by functoriality, a 3d TQFT associates to any connected oriented surface Σ a vector space V_Σ which is a linear representation of the oriented mapping class group $\Gamma^{\mathrm{or}}(\Sigma)$ of Σ. However, if the 3d theory has an "anomaly", then the vector space V_Σ fails to be a genuine representation of $\Gamma^{\mathrm{or}}(\Sigma)$, and it rather is only a projective representation. One way to think of this phenomenon is to look at anomalous theories as relative theories, that intertwine between the trivial theory and an invertible theory, namely the anomaly. See, e.g. [FT12, FV14]. In particular, for an anomalous TQFT of the type obtained from modular tensor categories with nontrivial central charge [Tu94, BK01], the vector space V_Σ can be naturally realised as a genuine representation of a $\mathbb{Z}$-central extension

$$0 \to \mathbb{Z} \to \widehat{\Gamma(\Sigma)} \to \Gamma(\Sigma) \to 1 \tag{1}$$

of the mapping class group $\Gamma(\Sigma)$. As suggested in Segal's celebrated paper on conformal field theory [Se04], these data admit an interpretation as a genuine functor where one replaces 2-dimensional and 3-dimensional manifolds by suitable "enriched" counterparts, in such a way that the automorphism group of an enriched connected surface is the relevant $\mathbb{Z}$-central extension of the mapping class group of the underlying surface. Moreover, the set of (equivalence classes of) extensions of a 3-manifold with prescribed (connected) boundary behaviour is naturally a $\mathbb{Z}$-torsor. In [Se04] the extension consists in a "rigging" of the 3-manifold, a solution which is not particularly simple, and which is actually quite ad hoc for the 3-dimensional case. Namely, riggings are based on the contractibility of Teichmüller spaces, and depend on the properties of the η-invariant for Riemannian metrics on 3-manifolds with boundary. On the other hand, in [Se04] it is suggested that simpler variants of this construction should exist, the *leitmotiv* being that of associating functorially to any connected surface a space with fundamental group $\mathbb{Z}$. Indeed, there is a well known realization of extended surfaces as surfaces equipped with a choice of a Lagrangian subspace in their first real cohomology group. This is the point of view adopted, e.g., in [BK01]. The main problem with this approach is the question of how to define a corresponding notion for an extended 3-manifold.

In the present work we describe a natural way of defining enrichments of 2-and-3-manifolds, which are topological (or better homotopical) in nature, and in particular do not rely on special features of the dimensions 2 and 3. Moreover, they have the advantage of being immediately adapted to a general TQFT framework. Namely, we consider enriched manifolds as (X, ξ)-framed manifolds in the sense of [Lu09]. In this way, we in particular recover the fact that the simple and natural notion of p_1-structure, i.e. a trivialization of the first Pontryagin class, provides a very simple realization

of Segal's prescription by showing how it naturally drops out as a special case of the "higher modularity" encoded in the (∞, n)-category of framed cobordisms. This is discussed in detail in Section 5.2 below.
Finally, if one is interested in higher dimensional Chern-Simons theories, the notable next case being 7-dimensional Chern-Simons theory [FSaS12], then the above discussion gives general means for determining and constructing the relevant higher extensions of diffeomorphism groups of higher dimensional manifolds.
More on this is going to be discussed elsewhere.

The present paper is organised as follows. In section 2 we discuss the ambient homotopy theory $\mathbf{H}^\infty$ of smooth higher stacks, and we discuss how smooth manifolds and homotopy actions of ∞-groups can be naturally regarded as objects in its slice ∞-category over the homotopy type $\mathscr{B}GL(n;\mathbb{R})$ of the mapping stack $\mathbf{B}GL(n;\mathbb{R})$ of principal $GL(n;\mathbb{R})$-bundles.
In section 3 we introduce the notion of a ρ-framing (or ρ-structure) over a smooth manifold, and study extensions of their automorphism ∞-group. We postpone the proof of the extension result to the Appendix.
In section 4 we discuss the particular but important case of ρ-structures arising from the homotopy fibers of morphisms of ∞-stacks, which leads to Theorem 4.1, the main result of the present paper. In this section we also consider the case of manifolds with boundaries.
In section 5, we apply the abstract machinery developed in the previous sections to the concrete case of the mapping class group usually encountered in relation to topological quantum field theories.
The Appendix contains a proof of the extension result in section 4.

Throughout, we freely use the language of ∞-categories, as developed in [Lu06]. There are various equivalent models for these, such as by simplicially enriched categories as well as by quasi-categories, but since these are equivalent, we mostly do not specify the model, and the reader is free to think of whichever model they prefer.

Acknowledgements. The authors would like to thank Oscar Randal-Williams and Chris Schommer-Pries for useful discussions.

2. Framed manifolds

2.1 From framed cobordism to (X, ξ)-manifolds

The principal player in the celebrated constructions of [GMTW06, GR-W10, Lu09] are manifolds with exotic "tangential structure" or "framing". These framings come in various flavours, from literal n-framings, i.e., trivialisations of the (stabilized) tangent bundle to more general and exotic framings called (X, ξ)-structures in [Lu09]. Here we make explicit that these structures are most naturally understood in the slice of a suitable smooth ∞-topos over $\mathbf{H}^\infty$ over $\mathscr{B}GL(n; \mathbb{R})$. This will allow us not only to see Lurie's framings from a unified perspective, but also to consider apparently more exotic (but actually completely natural) framings given by characteristic classes for the orthogonal group.

2.1.1 Homotopies, homotopies, homotopies everywhere

The ∞-topos of ∞-stacks over the site of all smooth manifolds, or equivalently just over the site of Cartesian spaces among these, we denote by

$$\mathbf{H} := \mathrm{Sh}_\infty(\mathrm{SmthMfd}) \simeq \mathrm{Sh}_\infty(\mathrm{CartSp})$$

[FScS12, def. 3.1.4].

This is a *cohesive ∞-topos* [Sc13, prop. 4.4.8], which in particular means ([Sc13, def. 3.4.1] following [Law07]) that the locally constant ∞-stack functor $\mathrm{LConst} : \infty\mathrm{Grp} \longrightarrow \mathbf{H}$ is fully faithful and has a left adjoint $|-|$ that preserves products

$$(|-| \dashv \mathrm{LConst} \dashv \Gamma) \ : \ \mathbf{H} \underset{\Gamma}{\overset{\overset{|-|}{\times\!\!\longrightarrow}}{\hookleftarrow\!\mathrm{LConst}\!\!-\!\!\longrightarrow}} \infty\mathrm{Grpd} \ .$$

This extra left adjoint $|-|$ is the operation of sending a smooth ∞-stack to its topological geometric realization, thought of as an ∞-groupoid [Sc13, cor. 4.4.28],[Car15, thm. 1.1]. In particular a smooth manifold is sent to its homotopy type.

Notice that the hom-∞-groupoids of any ∞-topos $\mathbf{H}$ may be expressed in terms of the internal hom (mapping ∞-stack) construction $[-,-]$ as

$$\mathbf{H}(\Sigma_1, \Sigma_2) \simeq \Gamma([\Sigma_1, \Sigma_2]) \ .$$

But now since the left adjoint $|-|$ exists and preserves products, this means that there naturally exists an alternative ∞-category, which we denote by $\mathbf{H}^\infty$, with the same objects as $\mathbf{H}$, but with hom-∞-groupoids defined by[1]

$$\mathbf{H}^\infty(\Sigma_1, \Sigma_2) := \big|[\Sigma_1, \Sigma_2]\big| \tag{2}$$

Accordingly, we write $\mathbf{Aut}^\infty(\Sigma)$ for the sub-∞-groupoid on the invertible elements in $\mathbf{H}^\infty(\Sigma, \Sigma)$.

The reason we pass to $\mathbf{H}^\infty$ is that $\mathbf{H}$ itself is too rigid (or, in other words, the homotopy type of its hom-spaces is too simple) for our aims. For instance, given two smooth manifolds Σ_1 and Σ_2, the ∞-groupoid $\mathbf{H}(\Sigma_1, \Sigma_2)$ is 0-truncated, i.e., it is just a set. Namely, $\mathbf{H}(\Sigma_1, \Sigma_2)$ is the set of smooth maps from Σ_1 and Σ_2 and there are no nontrivial morphisms between smooth maps in $\mathbf{H}(\Sigma_1, \Sigma_2)$. In other words, two smooth maps between Σ_1 and Σ_2 either are equal or they are different: in this hom-space there's no such thing as "a smooth map can be smoothly deformed into another smooth map", which however is a kind of relation that geometry naturally suggests. To take it into account, we make the topology (or, even better, the smooth structure) of Σ_1 and Σ_2 come into play, and we use it to informally define $\mathbf{H}^\infty(\Sigma_1, \Sigma_2)$ as the ∞-groupoid whose objects are smooth maps between Σ_1 and Σ_2, much as for $\mathbf{H}(\Sigma_1, \Sigma_2)$, but whose 1-morphisms are the smooth homotopies between smooth maps, and we also have 2-morphisms given by homotopies between homotopies, 3-morphisms given by homotopies between homotopies between homotopies, and so on.

Here is another example. For G a Lie group, we will write $\mathbf{B}G$ for the smooth stack of principal G-bundles. This means that for Σ a smooth manifold, a morphism $f\colon \Sigma \to \mathbf{B}G$ is precisely a G-principal bundle over Σ. So, in particular, $\mathbf{B}GL(n;\mathbb{R})$ is the smooth stack of principal $GL(n;\mathbb{R})$-bundles. Identifying a principal $GL(n;\mathbb{R})$-bundle with its associated rank n real vector bundle, $\mathbf{B}GL(n;\mathbb{R})$ is equivalently the smooth stack of rank n real vector

[1] This construction of an ∞-category $\mathbf{H}^\infty$ from a cohesive ∞-topos $\mathbf{H}$ is the direct ∞-category theoretic analog of what for cohesive 1-categories is called their "canonical extensive quality" in [Law07, thm. 1]. In fact $|-|$ is the derived functor of a left Quillen functor on the local projective model structure of simplicial presheaves over smooth manifolds which preserves 1-categorical products [Sc13, proof of prop. 3.4.18]. Since this is a Cartesian monoidal model category, any functorial cofibrant replacement functor is compatible with products, as is Kan's fibrant replacement functor on simplicial sets. This means that $\mathbf{H}^\infty$ may be represented by a Kan-enriched category.

bundles and their isomorphisms. In particular, a map $\Sigma \to \mathbf{B}GL(n;\mathbb{R})$ is precisely the datum of a rank n vector bundle on the smooth manifold Σ. Again, for a given smooth manifold Σ, the homotopy type of $\mathbf{H}(\Sigma, \mathbf{B}G)$ is too rigid for our aims: the ∞-groupoid $\mathbf{H}(\Sigma, \mathbf{B}G)$ is actually a 1-groupoid. This means that we have objects, which are the principal G-bundles over Σ, and 1-morphisms between these objects, which are isomorphisms of principal G-bundles, and then nothing else: we do not have nontrivial morphisms between the morphisms, and there's no such a thing like "a morphism can be smoothly deformed into another morphism", which again is something very natural to consider from a geometric point of view. Making the smooth structure of the group G come into play we get the following description of the ∞-groupoid $\mathbf{H}^\infty(\Sigma, \mathbf{B}G)$: its objects are the principal G-bundles over Σ and its 1-morphism are the isomorphisms of principal G-bundles, much as for $\mathbf{H}(\Sigma, \mathbf{B}G)$, but then we have also 2-morphisms given by isotopies between isomorphisms, 3-morphisms given by isotopies between isotopies, and so on. Notice that we have a canonical ∞-functor[2]

$$\mathbf{H}(\Sigma, \mathbf{B}G) \longrightarrow \mathbf{H}^\infty(\Sigma, \mathbf{B}G). \tag{3}$$

This is nothing but saying that for $j \geq 2$, the j-morphisms in $\mathbf{H}(\Sigma, \mathbf{B}G)$ are indeed very special j-morphisms in $\mathbf{H}^\infty(\Sigma, \mathbf{B}G)$, namely the identities. Moreover, when G happens to be a discrete group, this embedding is actually an equivalence of ∞-groupoids.

2.2 Geometrically discrete ∞-stacks and the homotopy type $\mathscr{B}GL(n)$

The following notion will be of great relevance for the results of this note. Recall from above the full inclusion

$$\mathrm{LConst} : \infty\mathrm{Grpd} \to \mathbf{H} \tag{4}$$

given by regarding an ∞-groupoid $\mathcal{G}$ as a constant presheaf over Cartesian spaces. We will say that an object in $\mathbf{H}$ is a *geometrically discrete* ∞-stack if it belongs to the essential image of LConst. An example of a geometrically discrete object in $\mathbf{H}$ is given by the 1-stack $\mathbf{B}G$, with G a discrete

[2]In terms of cohesion this is a component of the canonical points-to-pieces-transform $\Gamma[\Sigma, \mathbf{B}G] \to [\Sigma, \mathbf{B}G] \to |[\Sigma, \mathbf{B}G]|$.

group. More generally, for A an abelian discrete group the (higher) stacks $\mathbf{B}^n A$ of principal A-n-bundles are geometrically discrete. The importance of considering geometrically discrete ∞-stacks is that the geometric realization functor $|-|$ introduced before is left adjoint to LConst. In particular, denoting by $\Pi\colon \mathbf{H} \to \mathbf{H}$ the composition $\mathrm{LConst} \circ |-|$, we have a canonical unit morphism

$$\mathrm{id}_{\mathbf{H}} \to \Pi \tag{5}$$

which is the canonical morphism from a smooth stack to its homotopy type (and which corresponds to looking at points of a smooth manifold Σ as constant paths into Σ). In particular, for G a group, we will write $\mathscr{B}G$ for the homotopy type of $\mathbf{B}G$, i.e., we set $\mathscr{B}G := \Pi\mathbf{B}G$. This precisely encodes the traditional classifying space BG for the group G (or rather of its principal bundles) within $\mathbf{H}^\infty$. Namely, for Σ a smooth manifold we have, by the very definition of adjunction

$$\mathbf{H}(\Sigma, \mathscr{B}G) = \infty\mathrm{Grpd}(|\Sigma|, |\mathbf{B}G|).$$

A model for the classifying space BG is precisely given by the topological realization of $\mathbf{B}G$, while $|\Sigma|$ is nothing but the topological space underlying the smooth manifold Σ (so that by a little abuse of notation, we will simply write Σ for Σ). Moreover, since by definition $\mathscr{B}G$ is geometrically discrete we also have $\mathbf{H}^\infty(\Sigma, \mathscr{B}G) \cong \mathbf{H}(\Sigma, \mathscr{B}G)$, so that in the end we have a natural equivalence

$$\mathbf{H}^\infty(\Sigma, \mathscr{B}G) = \infty\mathrm{Grpd}(\Sigma, BG).$$

Under the equivalence between (nice) topological spaces and ∞-groupoids, on the right we have the ∞-groupoid of *continuous* maps from Σ to the classifying space BG. Notice how this example precisely shows how $\mathbf{H}^\infty$ is a setting where we can talk on the same footing of smooth and continuous phenomena. For instance, smooth maps from a smooth manifold Σ to another smooth manifold M and their smooth homotopies are encoded into $\mathbf{H}^\infty(\Sigma, M)$, while continuous maps between Σ and M and their continuous homotopies are encoded into $\mathbf{H}^\infty(\Sigma, \Pi(M))$.

The unit $\mathrm{id}_{\mathbf{H}} \to \Pi$ gives a canonical morphism

$$\mathbf{B}G \to \mathscr{B}G, \tag{6}$$

which is an equivalence for a discrete group G. This tells us in particular that any object over $\mathbf{B}G$ is naturally also an object over $\mathscr{B}G$. For instance (and this example will be the most relevant for what follows), a choice of a rank n vector bundle over a smooth manifold Σ realises Σ as an object over $\mathscr{B}GL(n;\mathbb{R})$.

Notice how we have a canonical morphism

$$\mathbf{H}(\Sigma, \mathbf{B}G) \longrightarrow \mathbf{H}^\infty(\Sigma, \mathscr{B}G) \tag{7}$$

obtained by composing the canonical morphism $\mathbf{H}(\Sigma, \mathbf{B}G) \to \mathbf{H}^\infty(\Sigma, \mathbf{B}G)$ mentioned in the previous section with the push forward morphism $\mathbf{H}^\infty(\Sigma, \mathbf{B}G) \to \mathbf{H}^\infty(\Sigma, \mathscr{B}G)$, The main reason to focus on geometrically discrete stacks is that, though $|-|$ preserves finite products, it does *not* in general preserve homotopy pullbacks. Nevertheless, $|-|$ does indeed preserve homotopy pullbacks of diagrams whose tip is a geometrically discrete object in $\mathbf{H}$ [Sc13, thm. 3.8.19].

2.2.1 Working in the slice

Let now n be a fixed nonnegative integer and let $0 \le k \le n$. Any k-dimensional smooth manifold M_k comes canonically equipped with a rank n real vector bundle given by the stabilized tangent bundle $T^{\mathrm{st}}M_k = TM_k \oplus \underline{\mathbb{R}}^{n-k}_{M_k}$, where $\underline{\mathbb{R}}^{n-k}_{M_k}$ denotes the trivial rank $(n-k)$ real vector bundle over M_k. We can think of the stabilised tangent bundle[3] as a morphism

$$M_k \xrightarrow{T^{\mathrm{st}}} \mathscr{B}GL(n) \tag{8}$$

where $GL(n)$, as in the following, denotes $GL(n;\mathbb{R})$.

Namely, we can regard any smooth manifold of dimension at most n as an object *over* $\mathscr{B}GL(n)$. This suggests that a natural setting to work in is the slice topos $\mathbf{H}^\infty_{/\mathscr{B}GL(n)}$, which in the following we will refer to simply as "the slice": in other words, all objects involved will be equipped with morphisms to $\mathscr{B}GL(n)$, and a morphism between $X \xrightarrow{\varphi} \mathscr{B}GL(n)$ and $Y \xrightarrow{\psi} \mathscr{B}GL(n)$

[3]To be precise, T^{st} is the map of stacks induced by the frame bundle of the stabilised tangent bundle to M_k.

will be a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\varphi"'] \arrow[rr, phantom, shift right=4ex, "\swarrow\!\!\!\!\swarrow_{\eta}"] & & Y \arrow[dl, "\psi"] \\ & \mathscr{B}GL(n) & \end{tikzcd} . \tag{9}$$

More explicitly, if we denote by E_φ and E_ψ the rank n real vector bundles over X and Y corresponding to the morphisms φ and ψ, respectively, then we see that a morphism in the slice between $X \xrightarrow{\varphi} \mathscr{B}GL(n)$ and $Y \xrightarrow{\psi} \mathscr{B}GL(n)$ is precisely the datum of a morphism $f\colon X \to Y$ together with an *isomorphism* of vector bundles over X,

$$\eta : f^*E_\psi \xrightarrow{\simeq} E_\varphi. \tag{10}$$

Notice that these are precisely the same objects and morphisms as if we were working in the slice over $\mathbf{B}GL(n)$ in $\mathbf{H}$. Nevertheless, as we will see in the following sections, where the use of $\mathbf{H}^\infty$ makes a difference is precisely in allowing nontrivial higher morphisms. Also, the use of the homotopy type $\mathscr{B}GL(n)$ in place of the smooth stack $\mathbf{B}GL(n)$ will allow us to make all constructions work "up to homotopy", and to identify, for instance, $\mathscr{B}GL(n)$ with $\mathscr{B}O(n)$.

Example 2.1. The inclusion of the trivial group into $GL(n)$ induces a natural morphism $* \to \mathscr{B}GL(n)$, corresponding to the choice of the trivial bundle. If M_k is a k-dimensional manifold, then a morphism

$$\begin{tikzcd} M_k \arrow[rr] \arrow[dr, "T^{\mathrm{st}}"'] \arrow[rr, phantom, shift right=4ex, "\swarrow\!\!\!\!\swarrow_{\eta}"] & & * \arrow[dl] \\ & \mathscr{B}GL(n) & \end{tikzcd} \tag{11}$$

is precisely a trivialisation of the stabilised tangent bundle of M_k, i.e., an n-framing of M.

Example 2.2. Let X be a smooth manifold, and let ζ be a rank n real vector bundle over X, which we can think of as a morphism $\rho_\zeta\colon X \to \mathscr{B}GL(n)$.

Then a morphism

$$\begin{tikzcd} M_k \arrow[rr, "f"] \arrow[dr, "T^{\mathrm{st}}"'] & \arrow[d, phantom, "\swArrow\eta"] & X \arrow[dl, "\rho_\zeta"] \\ & \mathscr{B}GL(n) & \end{tikzcd} \tag{12}$$

is precisely the datum of a smooth map $f\colon M_k \to X$ and of an isomorphism $\eta\colon f^*\zeta \to TM \oplus \mathbb{R}^{n-k}_{M_k}$. These are the data endowing M_k with a (X, ζ)-structure in the terminology of [Lu09].

The examples above suggest to allow X to be not only a smooth manifold, but a smooth ∞-stack. While choosing such a general target (X, ζ) could at first seem like a major abstraction, this is actually what one commonly encounters in everyday mathematics. For instance a lift through $\mathbf{B}O(n) \to \mathbf{B}GL(n)$ is precisely a (n-stable) Riemannian structure. Generally, for $G \hookrightarrow GL(n)$ any inclusion of Lie groups, or even more generally for $G \to GL(n)$ any morphism of Lie groups, then a lift through $\mathbf{B}G \to \mathbf{B}GL(n)$ is a (n-stable) *G-structure*, e.g., an almost symplectic structure, an almost complex structure, etc. (one may also phrase integrable G-structures in terms of slicing, using more of the axioms of cohesion than we need here). For instance, the inclusion of the connected component of the identity $GL^+(n) \hookrightarrow GL(n)$ corresponds to a morphism of higher stacks $\iota\colon \mathbf{B}GL^+(n) \to \mathbf{B}GL(n)$, and a morphism in the slice from (M_k, T^{st}) to $(\mathbf{B}GL^+(n), \iota)$ is precisely the choice of a (stabilised) orientation on M_k. For G a higher connected cover of $O(n)$ then lifts through $\mathbf{B}G \to \mathbf{B}O(n) \to \mathbf{B}GL(n)$ correspond to spin structures, string structures, etc.
On the other hand, since $\mathscr{B}O(n) \to \mathscr{B}GL(n)$ is an equivalence, a lift through $\mathscr{B}O(n) \to \mathscr{B}GL(n)$ is no additional structure on a smooth manifold M_k, and the stabilized tangent bundle of M_k can be equally seen as a morphism to $\mathscr{B}O(n)$. Similarly, for $G \to GL(n)$ any morphism of Lie groups, lifts of T^{st} through $\mathscr{B}G \to \mathscr{B}GL(n)$ correspond to (n-stable) *topological G*-structures.

2.3 From homotopy group actions to objects in the slice

We will mainly be interested in objects of $\mathbf{H}^\infty_{/\mathscr{B}GL(n)}$ obtained as a homotopy group action of a smooth (higher) group G on some stack X, when

G is equipped with a ∞-group morphism to $GL(n)$. We consider then the following

Definition 2.3. *A homotopy action of a smooth ∞-group G on X is the datum of a smooth ∞-stack $X//_h G$ together with a homotopy pullback diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X//_h G \arrow[d, "\rho"'] \\ * \arrow[r] & \mathscr{B}G \end{tikzcd} \tag{13}$$

Unwinding the definition, one sees that a homotopy action of G is nothing but an action of the homotopy type of G and that $X//_h G$ is realised as the stack quotient $X//\Pi G$. See [NSS12a] for details. Since G is equipped with a smooth group morphism to $GL(n)$, and since this induces a morphism of smooth stacks $\mathscr{B}G \to \mathscr{B}GL(n)$, the stack $X//_h G$ is naturally an object over $\mathscr{B}GL(n)$. In particular, when X is a deloopable object, i.e., when there exists a stack Y such that $\Omega Y \cong X$, then one obtains a homotopy G-action out of any morphism $c\colon \mathscr{B}G \to Y$. Indeed, in this situation one can define $X//_h G \to \mathscr{B}G$ by the homotopy pullback

$$\begin{tikzcd} X//_h G \arrow[r] \arrow[d, "\rho_c"'] & * \arrow[d] \\ \mathscr{B}G \arrow[r, "c"] & Y \end{tikzcd} \tag{14}$$

By using the pasting law for homotopy pullbacks, we can see that X, $X//_h G$, and the morphism ρ_c fit in a homotopy pullback diagram as in (13).

Example 2.4. Let c be a degree $d+1$ characteristic class for the group $SO(n)$. Then c can be seen as the datum of a morphism of stacks $c\colon \mathscr{B}SO(n) \to \mathscr{B}^{d+1}\mathbb{Z} \cong \mathbf{B}^{d+1}\mathbb{Z}$, where $\mathbf{B}^{d+1}\mathbb{Z}$ is the smooth stack associated by the Dold-Kan correspondence to the chain complex with $\mathbb{Z}$ concentrated in degree $d+1$, i.e., the stack (homotopically) representing degree $d+1$ integral cohomology. Notice how the discreteness of the abelian group $\mathbb{Z}$ came into play to give the equivalence $\mathscr{B}^{d+1}\mathbb{Z} \cong \mathbf{B}^{d+1}\mathbb{Z}$. Since we have $\Omega\mathbf{B}^{d+1}\mathbb{Z} \cong \mathbf{B}^{d}\mathbb{Z}$, the characteristic class c defines a homotopy action

$$\rho_c\colon \mathbf{B}^d\mathbb{Z}//_h SO(n) \to \mathscr{B}SO(n) \tag{15}$$

and so it realises $\mathbf{B}^d\mathbb{Z}//_h SO(n)$ as an object in the slice $\mathbf{H}^\infty_{/\mathscr{B}GL(n)}$. For instance, the first Pontryagin class p_1 induces a homotopy action

$$\rho_{p_1}\colon \mathbf{B}^3\mathbb{Z}//_h SO(n) \to \mathscr{B}SO(n). \tag{16}$$

3. ρ-framed manifolds and their automorphisms ∞-group

We can now introduce the main definition in the present work.

Definition 3.1. *Let M be a k-dimensional manifold, and let $\rho\colon X \to \mathscr{B}GL(n)$ be a morphism of smooth ∞-stacks, with $k \leq n$. Then a ρ-*framing *(or ρ-*structure*) on M is a lift of the stabilised tangent bundle seen as a morphism $T^{\mathrm{st}}\colon M \to \mathscr{B}GL(n)$ to a morphism $\sigma\colon M \to X$, namely a homotopy commutative diagram of the form*

$$\begin{tikzcd} M \arrow[rr, "\sigma"] \arrow[dr, "T^{\mathrm{st}}"'] & \arrow[d, phantom, "\swArrow_{\eta}"] & X \arrow[dl, "\rho"] \\ & \mathscr{B}GL(n) & \end{tikzcd} \tag{17}$$

By abuse of notation, we will often say that the morphism σ is the ρ-framing, omitting the explicit reference to the homotopy η, which is, however, always part of the data of a ρ-framing.
Since the morphism $\rho\colon X \to \mathscr{B}GL(n)$ is an object in the slice $\mathbf{H}^\infty_{/\mathscr{B}GL(n)}$, we can consider the slice over ρ: $(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\rho}$. Although this double slice may seem insanely abstract at first, it is something very natural. Its objects are homotopy commutative diagrams, namely 2-simplices

$$\begin{tikzcd} Y \arrow[rr, "a"] \arrow[dr, "\tilde{\rho}"'] & \arrow[d, phantom, "\swArrow_{\eta}"] & X \arrow[dl, "\rho"] \\ & \mathscr{B}GL(n) & \end{tikzcd} \tag{18}$$

while its morphisms are homotopy commutative 3-simplices

$$\begin{tikzcd}
Y \arrow[rr, "a"] \arrow[dr, "f"] \arrow[ddr, "\tilde{\rho}"'] & & X \arrow[ddl, "\rho"] \\
& Z \arrow[ur, "b"'] \arrow[d, "\hat{\rho}"] & \\
& \mathscr{B}GL(n) &
\end{tikzcd} \tag{19}$$

where for readability we have omitted the homotopies decorating the faces and the interior of the 3-simplex, and similarly, additional data must be provided for higher morphisms.
In particular we see that a ρ-framing σ on M is naturally an object in the double slice $(\mathbf{H}_{/\mathscr{B}GL(n)})_{/\rho}$. Moreover, the collection of all k-dimensional ρ-framed manifolds has a natural ∞-groupoid structure which is compatible with the forgetting of the framing, and with the fact that any ρ-framed manifold is in particular an object in the double slice $(\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)})_{/\rho}$. More precisely, let $\mathscr{M}_k$ denote the ∞-groupoid whose objects are k-dimensional smooth manifolds, whose 1-morphisms are diffeomorphisms of k-dimensional manifolds whose 2-morphisms are isotopies of diffeomorphisms, and so on[4]. There is then an ∞-groupoid $\mathscr{M}_k^{\rho}$ of ρ-framed k-dimensional manifolds which is a ∞-subcategory of $(\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)})_{/\rho}$, and comes equipped with a forgetful ∞-functor

$$\mathscr{M}_k^{\rho} \to \mathscr{M}_k. \tag{20}$$

Namely, since the differential of a diffeomorphism between k-dimensional manifolds M and N can naturally be seen as an invertible 1-morphism between M and N as objects over $\mathbf{B}GL(n)$, we have a natural (not full) embedding

$$\mathscr{M}_k \hookrightarrow \mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}. \tag{21}$$

Consider then the forgetful functor

$$(\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)})_{/\rho} \to \mathbf{H}^{\infty}_{/\mathscr{B}GL(n)} \tag{22}$$

We have then the following important

[4]The ∞-groupoid $\mathscr{M}_k$ can be rigorously defined as $\Omega(\mathrm{Cob_t(k)})$, where $\mathrm{Cob_t(k)}$ is the $(\infty,1)$-category defined in [Lu09] in the context of topological field theory.

Definition 3.2. *Let $\rho\colon X \to \mathscr{B}GL(n)$ be an object in $\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}$. The ∞-groupoid $\mathscr{M}_k^{\rho}$ is then defined as the homotopy pullback diagram*

$$\begin{tikzcd} \mathscr{M}_k^{\rho} \arrow[r] \arrow[d] & (\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)})_{/\rho} \arrow[d] \\ \mathscr{M}_k \arrow[r] & \mathbf{H}^{\infty}_{/\mathscr{B}GL(n)} \end{tikzcd} \tag{23}$$

Given two ρ-framed k-dimensional manifolds (M, σ, η) and (N, τ, ϑ), the ∞-groupoid $\mathscr{M}_k^{\rho}((M, \sigma, \eta), (N, \tau, \vartheta))$ is the homotopy pullback

$$\begin{tikzcd} \mathscr{M}_k^{\rho}((M, \sigma, \eta), (N, \tau, \vartheta)) \arrow[r] \arrow[d] & (\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)})_{/\rho}(\sigma, \tau) \arrow[d] \\ \mathscr{M}_k(M, N) \arrow[r] & \mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}(T_M^{\mathrm{st}}, T_N^{\mathrm{st}}) \end{tikzcd} \tag{24}$$

In particular, if we denote with $\mathrm{Diff}(M)$ the ∞-groupoid of diffeomorphisms of M, namely the automorphism ∞-group of M as an object in $\mathscr{M}_k$, and we accordingly write $\mathrm{Diff}^{\rho}(M, \sigma)$ for the automorphisms ∞-group of (M, σ) as an object in $\mathscr{M}_k^{\rho}$ (where to simplify notation we suppress the dependence on η) , then we have a homotopy pullback

$$\begin{tikzcd} \mathrm{Diff}^{\rho}(M, \sigma) \arrow[r] \arrow[d] & \mathbf{Aut}^{\infty}_{/\rho}(\sigma) \arrow[d] \\ \mathrm{Diff}(M) \arrow[r] & \mathbf{Aut}^{\infty}_{/\mathscr{B}GL(n)}(T_M^{\mathrm{st}}) \end{tikzcd} \tag{25}$$

where $\mathbf{Aut}^{\infty}_{(-)}(-)$ denotes the homotopy type of the relevant $\mathbf{H}$-internal automorphisms ∞-group. In particular, to abbreviate the notation, we will denote with $\mathbf{Aut}^{\infty}_{/\rho}(\sigma)$ the automorphism ∞-group of σ in $(\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)})_{/\rho}$. More explicitly, an element in $\mathrm{Diff}^{\rho}(M, \sigma)$ is a diffeomorphism $\varphi\colon M \to M$

together with an isomorphism $\alpha\colon \varphi^*\sigma \xrightarrow{\simeq} \sigma$, and a filler β for the 3-simplex

$$\begin{tikzcd} M \arrow[rr, bend left, "\sigma"] \arrow[r, "\varphi"] \arrow[dr, "T^{st}"'] \arrow[r, phantom, shift left=3, "\Nearrow\alpha"] \arrow[dr, phantom, shift left=2, "\Downarrow d\varphi"] & M \arrow[r, "\sigma"] \arrow[d, "T^{st}"] \arrow[dr, phantom, "\Downarrow\eta"] & X \arrow[dl, "\rho"] \\ & \mathscr{B}GL(n) & \end{tikzcd} \tag{26}$$

3.1 Functoriality and homotopy invariance of $\mathscr{M}_k^\rho$

In this section we will explore some of the properties of $\mathscr{M}_k^\rho$, which will be useful in the following.
It immediately follows from the definition that the forgetful functor $\mathscr{M}_k^\rho \to \mathscr{M}_k$ is an equivalence for $\rho\colon X \to \mathscr{B}GL(n)$ an equivalence in $\mathbf{H}^\infty(X, \mathscr{B}GL(n))$. In particular, if ρ is the identity morphism of $\mathscr{B}GL(n)$ and we write $\mathscr{M}_k^{GL(n)}$ for $\mathscr{M}_k^{\mathrm{id}_{\mathscr{B}GL(n)}}$ then we have $\mathscr{M}_k^{GL(n)} \cong \mathscr{M}_k$. Less trivially, if $X = \mathscr{B}O(n)$, and ρ is the natural morphism

$$\iota_{O(n)}\colon \mathscr{B}O(n) \to \mathscr{B}GL(n) \tag{27}$$

induced by the inclusion of $O(n)$ in $GL(n)$, then ρ is again an equivalence, and we get $\mathscr{M}_k^{O(n)} \cong \mathscr{M}_k$, where we have denoted $\mathscr{M}_k^{\iota_{O(n)}}$ with $\mathscr{M}_k^{O(n)}$.
More generally, if ρ and $\tilde{\rho}$ are equivalent objects in the slice $\mathbf{H}^\infty_{/\mathscr{B}GL(n)}$, then we have equivalent ∞-groupoids $\mathscr{M}_k^\rho$ and $\mathscr{M}_k^{\tilde{\rho}}$. For instance, the inclusion of $SO(n)$ into $GL(n)^+$ induces an equivalence between $\mathscr{B}SO(n)$ and $\mathscr{B}GL(n)^+$ over $\mathscr{B}GL(n)$, and so we have a natural equivalence $\mathscr{M}_k^{SO(n)} \cong \mathscr{M}_k^{GL(n)^+}$. Since the objects in the ∞-groupoid $\mathscr{M}_k^{GL(n)^+}$ are k-dimensional manifolds whose stabilised tangent bundle is equipped with a lift to an $SO(n)$-bundle, the objects of $\mathscr{M}_k^{GL(n)^+}$ are oriented k-manifolds. Moreover the pullback defining $\mathscr{M}_k^{GL(n)^+}$ precisely picks up oriented diffeomorphisms, hence the forgetful morphism $\mathscr{M}_k^{GL(n)^+} \to \mathscr{M}_k$ induces an equivalence between $\mathscr{M}_k^{GL(n)^+}$ and the ∞-groupoid $\mathscr{M}_k^{\mathrm{or}}$ of oriented k-dimensional manifolds with orientation preserving diffeomorphisms between them. As a con-

sequence, one has a natural equivalence

$$\mathscr{M}_k^{SO(n)} \cong \mathscr{M}_k^{\mathrm{or}} \tag{28}$$

Let $\psi\colon \rho \to \tilde{\rho}$ be a morphism in the slice $\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}$ between $\rho\colon X \to \mathscr{B}GL(n)$ and $\tilde{\rho}\colon Y \to \mathscr{B}GL(n)$. Then one has an induced push-forward morphism

$$\psi_*\colon \mathscr{M}_k^{\rho} \to \mathscr{M}_k^{\tilde{\rho}}, \tag{29}$$

which (by (24), and using the pasting law) fits into the homotopy pullback diagram

$$\begin{tikzcd} \mathscr{M}_k^{\rho} \arrow[r] \arrow[d, "\psi_*"'] & (\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)})_{/\rho} \arrow[d, "\Psi_*"] \\ \mathscr{M}_k^{\tilde{\rho}} \arrow[r] & (\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)})_{/\tilde{\rho}} \end{tikzcd} \tag{30}$$

where Ψ_* denotes the base changing ∞-functor on the slice topos.
The homotopy equivalences illustrated above are particular cases of this functoriality: indeed, when ψ is invertible, then ψ_* is invertible as well (up to coherent homotopies, clearly).
Recall from Example 2.4 that for any characteristic class c of $SO(n)$ we obtain an object ρ_c in the slice $\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}$. In this way we obtain natural morphisms $\mathscr{M}_k^{\rho_c} \to \mathscr{M}_k^{SO(n)}$. In particular, by considering the first Pontryagin class $p_1\colon \mathscr{B}SO(n) \to \mathbf{B}^4\mathbb{Z}$, we obtain a canonical morphism

$$\mathscr{M}_k^{\rho_{p_1}} \to \mathscr{M}_k^{\mathrm{or}}. \tag{31}$$

3.2 Extensions of ρ-diffeomorphism groups

We are now ready for the extension theorem, which is the main result of this note. Not to break the flow of the exposition, we will postpone the details of the proof to the Appendix.
Let

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "\rho"'] & \arrow[d, phantom, "\swarrow\Psi"] & Y \arrow[dl, "\tilde{\rho}"] \\ & \mathscr{B}GL(n) & \end{tikzcd} \tag{32}$$

be a morphism in the slice over $\mathscr{B}GL(n)$, as at the end of the previous section, and let

$$\begin{tikzcd} M \arrow[rr, "\tau"] \arrow[dr, "T_M^{\mathrm{st}}"'] & \arrow[d, phantom, "\swarrow T"] & Y \arrow[dl, "\tilde{\rho}"] \\ & \mathscr{B}GL(n) & \end{tikzcd} \tag{33}$$

be a $\tilde{\rho}$-structure on M. Then, arguing as in Section 3, associated to any lift

$$\begin{tikzcd} M \arrow[rrr, "\sigma"] \arrow[dr, "\tau"] \arrow[ddr, "T^{st}"'] & & \arrow[d, phantom, "\Leftarrow \alpha"] & X \arrow[dll, "\psi"'] \arrow[ddll, "\rho"] \\ & Y \arrow[d, "\tilde{\rho}"] \arrow[l, phantom, "\swarrow T"] \arrow[r, phantom, "\Rightarrow \Psi"] & & \\ & \mathscr{B}GL(n) & & \end{tikzcd} \tag{34}$$

(where we are not displaying the label Σ on the back face, nor the filler β of the 3-simplex) of T to a ρ-structure Σ on M, we have a homotopy pullback diagram

$$\begin{tikzcd} \mathrm{Diff}^{\rho}(M,\Sigma) \arrow[r] \arrow[d, "\psi_*"'] & \mathbf{Aut}^{\infty}_{/\rho}(\Sigma) \arrow[d, "\psi_*"] \\ \mathrm{Diff}^{\tilde{\rho}}(M,T) \arrow[r] & \mathbf{Aut}^{\infty}_{/\tilde{\rho}}(T) \end{tikzcd} \tag{35}$$

By the pasting law for homotopy pullbacks, we have the following homotopy diagram (see Appendix for the proof)

$$\begin{tikzcd} \Omega_{\beta}(\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)})_{/\tilde{\rho}}(T,\Psi) \arrow[r] \arrow[d] & \Omega_{\Sigma}\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}(T_M^{\mathrm{st}},\rho) \arrow[r] \arrow[d] & \mathbf{Aut}^{\infty}_{/\rho}(\Sigma) \arrow[d, "\psi_*"] \\ * \arrow[r] & \Omega_{T}\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}(T_M^{\mathrm{st}},\tilde{\rho}) \arrow[r] \arrow[d] & \mathbf{Aut}^{\infty}_{/\tilde{\rho}}(T) \arrow[d] \\ & * \arrow[r] & \mathbf{Aut}^{\infty}_{/\mathscr{B}GL(n)}(T_M^{\mathrm{st}}) \end{tikzcd} \tag{36}$$

We therefore obtain the homotopy pullback diagram

$$\begin{tikzcd} \Omega_\beta(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\tilde{\rho}}(T,\Psi) \arrow[r] \arrow[d] & \mathrm{Diff}^\rho(M,\Sigma) \arrow[d, "\psi_*"] \\ * \arrow[r] & \mathrm{Diff}^{\tilde{\rho}}(M,T) \end{tikzcd} \tag{37}$$

presenting $\mathrm{Diff}^\rho(M,\Sigma)$ as an extension of $\mathrm{Diff}^{\tilde{\rho}}(M,T)$ by the ∞-group $\Omega_\beta(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\tilde{\rho}}(T,\Psi)$, i.e., by the loop space (at a given lift β) of the space $(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\tilde{\rho}}(T,\Psi)$ of lifts of the $\tilde{\rho}$-structure T on M to a ρ-structure Σ. Now notice that, by the Kan condition, we have a natural homotopy equivalence

$$(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\tilde{\rho}}(T,\Psi) \cong \mathbf{H}^\infty_{/Y}(\tau,\psi). \tag{38}$$

Namely, since T and Ψ are fixed, the datum of the filler α is homotopically equivalent to the datum of the full 3-simplex, as T,Ψ and α together give the datum of the horn at the vertex Y. As a consequence we see that the space of lifts of the $\tilde{\rho}$-structure T to a ρ-structure Σ is homotopy equivalent to the space of lifts

$$\begin{tikzcd} & X \arrow[d, "\psi"] \\ M \arrow[ur, "\sigma"] \arrow[r, "\tau"'] \arrow[r, phantom, shift left=3, "\Swarrow\alpha"] & Y \end{tikzcd} \tag{39}$$

of τ to a morphism $\sigma\colon M \to X$. We refer the reader to the Appendix for a rigorous proof of equivalence (38).
The arguments above lead directly to

Proposition 3.3. *Let $\rho : X \to \mathscr{B}GL(n)$ and $\tilde{\rho} : Y \to \mathscr{B}GL(n)$ be morphisms of ∞-stacks, and let $(\psi,\Psi) : \rho \to \tilde{\rho}$ be a morphism in $\mathbf{H}^\infty_{/\mathscr{B}GL(n)}$. Let (M,T) be a $\tilde{\rho}$-framed manifold, and let Σ be a ρ-structure on M lifting T through (α,β). We have then the following homotopy pullback*

$$\begin{tikzcd} \Omega_\alpha \mathbf{H}^\infty_{/Y}(\tau,\psi) \arrow[r] \arrow[d] & \mathrm{Diff}^\rho(M,\Sigma) \arrow[d, "\psi_*"] \\ * \arrow[r] & \mathrm{Diff}^{\tilde{\rho}}(M,T) \end{tikzcd} \tag{40}$$

Proof. Combine diagram (37) with equivalence (38), which preserves homotopy pullbacks. □

Remark 3.4. Proposition 3.3 gives a presentation of $\mathrm{Diff}^{\rho}(M, \Sigma)$ as an extension of $\mathrm{Diff}^{\tilde{\rho}}(M, T)$ by the ∞-group $\Omega_\alpha \mathbf{H}^\infty_{/Y}(\tau, \psi)$. Notice how, for (T, τ) the identity morphism, i.e.

$$\begin{tikzcd} Y \arrow[rr, "\mathrm{id}_Y"] \arrow[dr, "\tilde{\rho}"'] & \arrow[d, phantom, "\swarrow\mathrm{Id}"] & Y \arrow[dl, "\tilde{\rho}"] \\ & \mathscr{B}GL(n) & \end{tikzcd} \tag{41}$$

the space $\mathbf{H}^\infty_{/Y}(\tau, \mathrm{id}_Y)$ is contractible since id_Y is the terminal object in the slice $\mathbf{H}^\infty_{/Y}$ and so one finds that the extension of $\mathrm{Diff}^{\tilde{\rho}}(M, T)$ is the trivial one in this case, as expected.

4. Lifting ρ-structures along homotopy fibres

In this section we will investigate a particularly simple and interesting case of the lifting procedure of ρ-structures, and of extensions of ρ-diffeomorphisms ∞-groups, namely the case when $\psi \colon X \to Y$ is the homotopy fibre in $\mathbf{H}^\infty$ of a morphism $c \colon Y \to Z$ from Y to some pointed stack Z.
In this case, by the universal property of the homotopy pullback, the space $\mathbf{H}^\infty_{/Y}(\tau, \psi)$ of lifts of the $\tilde{\rho}$-structure τ to a ρ-structure σ is given by the space of homotopies between the composite morphism $c \circ \tau$ and the trivial morphism $M \to Z$ given by the constant map on the marked point of Z:

$$\begin{tikzcd} M \arrow[dr, dashed, "\sigma"] \arrow[drr, bend left] \arrow[ddr, bend right, "\tau"'] & & \\ & X \arrow[r] \arrow[d, "\psi"'] & * \arrow[d] \\ & Y \arrow[r, "c"] & Z \end{tikzcd} \tag{42}$$

This fact has two important consequences:

- a lift σ of τ exists if and only if the class of $c \circ \tau$ in $\pi_0 \mathbf{H}^\infty(M, Z)$ is the trivial class (the class of the constant map on the marked point z of Z);

- when a lift exists, the space $\mathbf{H}^{\infty}_{/Y}(\tau,\psi)$ is a torsor for the ∞-group of self-homotopies of the constant map $M \to Z$, i.e., for the ∞-group object $\Omega\mathbf{H}^{\infty}(M,Z)$. In particular, as soon as $\mathbf{H}^{\infty}_{/Y}(\tau,\psi)$ is nonempty, any lift σ of τ induces an equivalence of ∞-groupoids $\mathbf{H}^{\infty}_{/Y}(\tau,\psi) \cong \Omega\mathbf{H}^{\infty}(M,Z)$ and so an equivalence

$$\Omega_{\alpha}\mathbf{H}^{\infty}_{/Y}(\tau,\psi) \cong \Omega^2\mathbf{H}^{\infty}(M,Z). \tag{43}$$

Moreover, as soon as (Z,z) is a geometrically discrete pointed ∞-stack, we have $\Omega\mathbf{H}^{\infty}(M,Z) \cong \mathbf{H}^{\infty}(M,\Omega Z)$, where ΩZ denotes the loop space of Z in $\mathbf{H}$ at the distinguished point z. In other words, for a geometrically discrete ∞-stack Z, the loop space of Z in $\mathbf{H}$ also provides a loop space object for Z in $\mathbf{H}^{\infty}$. Namely, by definition of $\mathbf{H}^{\infty}$, showing that

$$\begin{tikzcd} \mathbf{H}^{\infty}(W,\Omega Z) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \mathbf{H}^{\infty}(W,Z) \end{tikzcd} \tag{44}$$

is a homotopy pullback of ∞-groupoids for any ∞-stack W amounts to showing that

$$\begin{tikzcd} \big|[W,\Omega Z]\big| \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \big|[W,Z]\big| \end{tikzcd} \tag{45}$$

is a homotopy pullback, and this in turn follows from the fact that $[W,-]$ preserves homotopy pullbacks and geometrical discreteness, and that $|-|$ preserves homotopy pullbacks along morphisms of geometrically discrete stacks [Sc13, thm. 3.8.19]. If the pointed stack (Z,z) is geometrically discrete, then so is the stack ΩZ (pointed at the constant loop at z), and so

$$\Omega^2\mathbf{H}^{\infty}(M,Z) \cong \Omega\mathbf{H}^{\infty}(M,\Omega Z) \cong \mathbf{H}^{\infty}(M,\Omega^2 Z). \tag{46}$$

Therefore, we can assemble the general considerations of the previous section in the following

Theorem 4.1. *Let $\psi\colon X \to Y$ be the homotopy fibre of a morphism of smooth ∞-stacks $Y \to Z$, where Z is pointed and geometrically discrete. For any $\tilde{\rho}$-structured manifold (M, τ), we have a sequence of natural homotopy pullbacks*

$$\begin{tikzcd} \mathbf{H}^\infty(M, \Omega^2 Z) \arrow[r] \arrow[d] & \mathrm{Diff}^\rho(M, \sigma) \arrow[r] \arrow[d, "\psi_*"] & * \arrow[d] \\ * \arrow[r] & \mathrm{Diff}^{\tilde{\rho}}(M, \tau) \arrow[r] & \mathbf{H}^\infty(M, \Omega Z) \end{tikzcd} \tag{47}$$

whenever a lift to of τ to a ρ-structure σ exists.

4.1 The case of manifolds with boundary

With an eye to topological quantum field theories, it is interesting to consider also the case of k-dimensional manifolds with boundary $(M, \partial M)$. Since the boundary ∂M comes with a collar in M, i.e. with a neighbourhood in M diffeomorphic to $\partial M \times [0, 1)$ the restriction of the tangent bundle of M to ∂M splits as[5] $TM|_{\partial M} \cong T\partial M \oplus \mathbb{R}_{\partial M}$ and this gives a natural homotopy commutative diagram

$$\begin{tikzcd} \partial M \arrow[rr, "\iota"] \arrow[dr, "T^{\mathrm{st}}"'] & & M \arrow[dl, "T^{\mathrm{st}}"] \\ & \mathscr{B}GL(n) & \end{tikzcd} \tag{48}$$

for any $n \geq k$. In other words, the embedding of the boundary, $\iota\colon \partial M \to M$ is naturally a morphism in the slice over $\mathscr{B}GL(n)$. This means that any $\tilde{\rho}$-framing on M can be pulled back to a $\tilde{\rho}$-framing on ∂M:

$$\iota^*\colon \mathbf{H}^\infty_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}, \tilde{\rho}) \to \mathbf{H}^\infty_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}\big|_{\partial M}, \tilde{\rho}). \tag{49}$$

[5] See section 2.2.1 for notation.

That is, for any $\tilde{\rho}$-framing on M we have a natural homotopy commutative diagram

$$\begin{tikzcd}
\partial M \arrow[rr, "\iota"] \arrow[dr, "\tau|_{\partial M}"] \arrow[ddr, "T^{\mathrm{st}}|_{\partial M}"'] & & M \arrow[dl, "\tau"'] \arrow[ddl, "T^{\mathrm{st}}"] \\
& Y \arrow[d] & \\
& \mathscr{B}GL(n) &
\end{tikzcd} \tag{50}$$

realizing ι as a morphism in the slice over Y. Therefore we have a further pullback morphism

$$\iota^*\colon \mathbf{H}^{\infty}_{/Y}(\tau,\psi) \to \mathbf{H}_{/Y}(\tau|_{\partial M},\psi) \tag{51}$$

for any morphism $\psi\colon (X,\rho) \to (Y,\tilde{\rho})$ in the slice over $\mathscr{B}GL(n)$. For any fixed ρ-framing ж on ∂M we can then form the space of ρ-framings on the $\tilde{\rho}$-framed manifold M extending ж. This is the homotopy fibre of ι^* at ж:

$$\begin{tikzcd}
\mathbf{H}^{\infty,\text{ж}}_{/Y}((M,\partial M,\tau),(X,\psi)) \arrow[r] \arrow[d] & * \arrow[d, "\text{ж}"] \\
\mathbf{H}_{/Y}(\tau,\psi) \arrow[r, "\iota^*"] & \mathbf{H}_{/Y}(\tau|_{\partial M}),\psi)
\end{tikzcd} \tag{52}$$

Reasoning as in Section 4, when the morphism $\psi\colon X \to Y$ is the homotopy fibre of a morphism $c\colon Y \to Z$ one sees that, as soon as the ρ-structure ж on ∂M can be extended to a ρ-structure on M, then the space $\mathbf{H}^{\infty,\text{ж}}_{/Y}((M,\partial M,\tau),(X,\psi))$ of such extensions is a torsor for the ∞-group $\mathbf{H}^{\infty,\mathrm{rel}}(M,\partial M;\Omega Z)$ defined by the homotopy pullback

$$\begin{tikzcd}
\mathbf{H}^{\infty,\mathrm{rel}}(M,\partial M;\Omega Z) \arrow[r] \arrow[d] & * \arrow[d, "\mathbf{0}"] \\
\mathbf{H}^{\infty}(M,\Omega Z) \arrow[r, "\iota^*"] & \mathbf{H}^{\infty}(\partial M,\Omega Z)
\end{tikzcd} \tag{53}$$

In particular, for $Z = \mathbf{B}^n A$ for some discrete abelian group A, the space $\mathbf{H}^{\infty,\mathrm{rel}}(M,\partial M;\mathbf{B}^{n-1}A)$ is the space whose set of connected components is the $(n-1)$-th relative cohomology group of $(M,\partial M)$:

$$\pi_0\mathbf{H}^{\infty,\mathrm{rel}}(M,\partial M;\mathbf{B}^{n-1}A) \cong H^{n-1}(M,\partial M;A). \tag{54}$$

Moreover, since $\mathbf{B}^n A$ is $(n-1)$-connected, we see that any homotopy from $c \circ \tau|_{\partial M} \colon \partial M \to \mathbf{B}^n A$ to the trivial map can be extended to a homotopy from $c \circ \tau \colon M \to \mathbf{B}^n A$ to the trivial map, as soon as $\dim M < n$. In other words, for $Z = \mathbf{B}^n A$, if $k < n$ every ρ-structure on ∂M can be extended to a ρ-structure on M.

The space $\mathbf{H}^{\infty;ж}_{/Y}((M, \partial M, \tau), (X, \psi))$ has a natural interpretation in terms of ρ-framed cobordism: it is the space of morphisms from the empty manifold to the ρ-framed manifold $(\partial M, ж)$, whose underlying non-framed cobordism is M. As such, it carries a natural action of the ∞-group of ρ-framings on the cylinder $\partial M \times [0,1]$ which restrict to the ρ-framing ж both on $\partial M \times \{0\}$ and on $\partial M \times \{1\}$. These are indeed precisely the ρ-framed cobordisms lifting the trivial non-framed cobordism. Geometrically this action is just the glueing of such a ρ-framed cylinder along ∂M, as a collar in M. On the other hand, by the very definition of $\mathbf{H}^\infty$, this ∞-group of ρ-framed cylinders is nothing but the loop space $\Omega_{ж}(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\rho}(T^{\mathrm{st}}\big|_{\partial M}, \psi)$, i.e., the loop space at ж of the space of ρ-structures on ∂M lifting the $\tilde{\rho}$-structure $\tau|_{\partial M}$. Comparing this to the diagram (37), we see that the space of ρ-structures on M extending a given ρ-structure on ∂M comes with a natural action of the ∞-group which is the centre of the extension $\mathrm{Diff}^{\tilde{\rho}}(\partial M, ж)$ of $\mathrm{Diff}^{\tilde{\rho}}(M, \tau|_{\partial M})$.[6] In the case $\psi \colon X \to Y$ is the homotopy fibre of a morphism $c \colon Y \to \mathbf{B}^n A$, passing to equivalence classes we find the natural action of $H^{n-2}(\partial M, A)$ on the relative cohomology group $H^{n-1}(M, \partial M; A)$ given by the suspension isomorphism $H^{n-2}(\partial M, A) \cong H^{n-1}(\partial M \times [0,1], \partial M \times \{0,1\}, A)$ combined with the natural translation action

$$H^{n-1}(M, \partial M; A) \times H^{n-1}(\partial M \times [0,1], \partial M \times \{0,1\}, A) \to H^{n-1}(M, \partial M; A). \tag{55}$$

For instance, if M is a connected oriented 3-manifold with connected boundary ∂M and we choose $n = 4$ and $A = \mathbb{Z}$, then we get the translation action of $\mathbb{Z}$ on itself.[7]

[6] This should be compared to Segal's words in [Se04]: "An oriented 3-manifold Y whose boundary ∂Y is rigged has itself a set of riggings which form a principal homogeneous set under the group $\mathbb{Z}$ which is the centre of the central extension of $\mathrm{Diff}(\partial Y)$."

[7] Again, compare to Segal's prescription on the set of riggings on a oriented 3-manifold.

5. Mapping class groups of ρ-framed manifolds

In this final section, we consider an application of the general notion of ρ-structure developed in the previous sections to investigate extensions of the mapping class group of smooth manifolds.
Inspired by the classical notion of mapping class group, see for instance [Ha12], we consider the following

Definition 5.1. *Let M be a k-dimensional manifold, and let $\rho\colon X \to \mathscr{B}GL(n)$ be a morphisms of smooth ∞-stacks, with $k \leq n$. The* mapping class group $\Gamma^\rho(M,\sigma)$ *of a ρ-framed manifold (M,σ) is the group of connected components of the ρ-diffeomorphism ∞-group of (M,σ), namely*

$$\Gamma^\rho(M,\sigma) := \pi_0\mathrm{Diff}^\rho(M,\sigma) \tag{56}$$

In the setting of the Section 4, we consider the case in which the ∞-stack X is the homotopy fiber of a morphism $Y \to Z$, with Z a geometrically discrete ∞-stack. Then, induced by diagram (47), we have the following long exact sequence in homotopy

$$\begin{aligned}\cdots \to \pi_1\mathrm{Diff}^\rho(M,\sigma) \to \pi_1\mathrm{Diff}^{\tilde\rho}(M,\tau) \to \pi_2\mathbf{H}^\infty(M,Z) \to \\ \to \Gamma^\rho(M,\sigma) \to \Gamma^{\tilde\rho}(M,\tau) \to \pi_1\mathbf{H}^\infty(M,Z).\end{aligned} \tag{57}$$

Notice that the morphism

$$\Gamma^{\tilde\rho}(M,\tau) \to \pi_1\mathbf{H}^\infty(M,Z) \tag{58}$$

is a homomorphism at the π_0 level, so it is only a morphism of pointed sets and *not* a morphism of groups. It is the morphism that associates with a ρ-diffeomorphism f the pullback of the lift σ of τ. In other words, it is the morphism of pointed sets from the set of isotopy classes of ρ-diffeomorphisms to the set of equivalence classes of lifts induced by the natural action

$$\begin{aligned}\Gamma^{\tilde\rho}(M,\tau)\times\{\text{(equivalence classes of) lifts of } \tau\} \to \\ \to \{\text{(equivalence classes of) lifts of } \tau\}\end{aligned} \tag{59}$$

once one picks a distinguished element σ in the set (of equivalence classes of) of lifts and uses it to identify this set with $\pi_0\mathbf{H}^\infty(M,\Omega Z) \cong \pi_1\mathbf{H}^\infty(M,Z)$.

A particularly interesting situation is the case when c is a degree d characteristic class for Y, i.e., when $c\colon Y \to \mathbf{B}^d A$ for some discrete abelian group A, and M is a closed manifold. Since $\mathbf{B}^d A$ is a geometrically discrete ∞-stack, we have that $\mathbf{H}^\infty(M, \mathbf{B}^d A)$ is equivalent, as an ∞-groupoid, to $\mathbf{H}(M, \mathbf{B}^d A)$. Consequently, we obtain that $\pi_k \mathbf{H}^\infty(M, \mathbf{B}^d A) = H^{d-k}(M, A)$ for $0 \leq k \leq d$ (and zero otherwise): in particular, the obstruction to lifting a $\tilde{\rho}$-framing τ on M to a ρ-framing σ is given by an element in $H^d(M, A)$. When this obstruction vanishes, hence when a lift σ of τ does exist, the long exact sequence above reads as

$$\begin{aligned} \cdots \to \pi_1 \mathrm{Diff}^{\tilde{\rho}}(M, \tau) \to H^{d-2}(M,A) \to \Gamma^\rho(M, \sigma) \to \\ \to \Gamma^{\tilde{\rho}}(M, \tau) \to H^{d-1}(M, A) \end{aligned} \tag{60}$$

for $d \geq 2$, and simply as

$$\cdots \to \pi_1 \mathrm{Diff}^{\tilde{\rho}}(M, \tau) \to 1 \to \Gamma^\rho(M, \sigma) \to \Gamma^{\tilde{\rho}}(M, \tau) \to H^0(M, A) \tag{61}$$

for $d = 1$.

Remark 5.2. The long exact sequences (60) and (61) are a shadow of Theorem 4.1, which is a more general extension result for the *whole* ∞-group $\mathrm{Diff}^\rho(M, \sigma)$.

The morphism of pointed sets $\Gamma^{\tilde{\rho}}(M, \tau) \to H^{d-1}(M, A)$ is easily described: once a lift σ for τ has been chosen, the space of lifts is identified with $\mathbf{H}^\infty(M, \mathbf{B}^{d-1} A)$ and the natural pullback action of the $\tilde{\rho}$-diffeomorphism group of M on the space of maps from M to $\mathbf{B}^{d-1} A$ induces the morphism

$$\begin{aligned} \mathrm{Diff}^{\tilde{\rho}}(M, \tau) &\to \mathbf{H}^\infty(M, \mathbf{B}^{d-1} A) \\ f &\mapsto f^*\sigma - \sigma \end{aligned} \tag{62}$$

where we have written $f^*\sigma - \sigma$ for the element in $\mathbf{H}^\infty(M, \mathbf{B}^{d-1} A)$ which represents the "difference" between $f^*\sigma$ and σ in the space of lifts of τ seen as a $\mathbf{H}^\infty(M, \mathbf{B}^{d-1} A)$-torsor. The morphism $\Gamma^{\tilde{\rho}}(M, \tau) \to H^{d-1}(M, A)$ is obtained by passing to π_0's and so we see in particular from the long exact sequence (60) that the image of $\Gamma^\rho(M, \tau)$ into $\Gamma^{\tilde{\rho}}(M, \tau)$ consist of precisely the isotopy classes of those $\tilde{\rho}$-diffeomorphisms of $(M, \tilde{\rho})$ which fix the ρ-structure σ up to homotopy.

Similarly, for $d \geq 2$, the morphism of groups $\pi_1 \mathrm{Diff}^{\tilde{\rho}}(M,\tau) \to H^{d-2}(M,A)$ in sequence (60) can be described explicitly as follows. A closed path γ based at the identity in $\mathrm{Diff}^{\tilde{\rho}}(M,\tau)$ defines then a morphism $\gamma^{\#}\colon M \times [0,1] \to \mathbf{B}^{d-1}A$, as the composition

$$M \times [0,1] \to M \xrightarrow{\mathbf{0}} \mathbf{B}^{d-1}A, \tag{63}$$

where the first arrow is the homotopy from the identity of M to itself and where $\mathbf{0}\colon M \to \mathbf{B}^{d-1}A$ is the *collapsing* morphism, namely the morphism obtained as the composition $M \to * \to \mathbf{B}^{d-1}A$ (here we are using that $\mathbf{B}^{d-1}A$ comes naturally equipped with a base point). The image of $[\gamma]$ in $H^{d-2}(M,A)$ is then given by the element $[\gamma^{\#}]$ in the relative cohomology group

$$H^{d-1}(M \times [0,1], M \times \{0,1\}, A) \cong H^{d-1}(\mathbf{\Sigma} M, A) \cong H^{d-2}(M,A)\,. \tag{64}$$

By construction, $[\gamma^{\#}]$ is the image in $H^{d-1}(M \times [0,1], M \times \{0,1\}, A) \cong H^{d-2}(M,A)$ of the zero class in $H^{d-1}(M,A)$ via the pullback morphism $M \times [0,1] \to M$, so it is the zero class in $H^{d-1}(M \times [0,1], M \times \{0,1\}, A)$. That is, the morphism $\pi_1 \mathrm{Diff}^{\tilde{\rho}}(M,\tau) \to H^{d-2}(M,A)$ is the zero morphism, and we obtain the short exact sequence

$$1 \to H^{d-2}(M,A) \to \Gamma^{\rho}(M,\sigma) \to \Gamma^{\tilde{\rho}}(M,\tau) \to H^{d-1}(M,A) \tag{65}$$

showing that $\Gamma^{\rho}(M,\sigma)$ is a $H^{d-2}(M,A)$-extension of a subgroup of $\Gamma^{\tilde{\rho}}(M,\tau)$: namely, the subgroup is the $\Gamma^{\tilde{\rho}}(M,\tau)$-stabilizer of the element of $H^{d-1}(M,A)$ corresponding to the lift σ of τ. The action of this stabiliser on $H^{d-2}(M,A)$ is the pullback action of $\tilde{\rho}$-diffeomorphisms of M on the $(d-2)$-th cohomology group of M with coefficients in A. Since this action is not necessarily trivial, the $H^{d-2}(M,A)$-extension $\Gamma^{\rho}(M,\sigma)$ of the stabiliser of σ is not a central extension in general.

5.1 Oriented and spin manifolds, and r-spin surfaces

Before discussing p_1-structures and their modular groups, which is the main goal of this note, let us consider two simpler but instructive examples: oriented manifolds and spin curves.

Since the ∞-stack $\mathscr{B}SO(n)$ is the homotopy fibre of the first Stiefel-Whitney class

$$w_1\colon \mathscr{B}O(n) \to \mathbf{B}\mathbb{Z}/2\mathbb{Z} \tag{66}$$

an n-dimensional manifold can be oriented if and only if $[w_1 \circ T_M]$ is the trivial element in $\pi_0\mathbf{H}^\infty(M, \mathbf{B}\mathbb{Z}/2\mathbb{Z}) = H^1(M, \mathbb{Z}/2\mathbb{Z})$. When this happens, the space of possible orientations on M is equivalent to $\mathbf{H}^\infty(M, \mathbb{Z}/2\mathbb{Z})$, so when M is connected it is equivalent to a 2-point set. For a fixed orientation on M, we obtain from (61) with $A = \mathbb{Z}/2\mathbb{Z}$ the exact sequence

$$1 \to \Gamma^{\mathrm{or}}(M) \to \Gamma(M) \to \mathbb{Z}/2\mathbb{Z} \tag{67}$$

where $\Gamma^{\mathrm{or}}(M)$ denotes the mapping class group of oriented diffeomorphisms of M, and where the rightmost morphism is induced by the action of the diffeomorphism group of M on the set of its orientations. The oriented mapping class group of M is therefore seen to be a subgroup of order 2 in $\Gamma(M)$ in case there exists at least an orientation reversing diffeomorphism of M, and to be the whole $\Gamma(M)$ when such a orientation reversing diffeomorphism does not exist (e.g., for $M = \mathbb{P}^{n/2}\mathbb{C}$, for $n \equiv 0 \mod 4$).

Consider now the ∞-stack $\mathscr{B}\mathrm{Spin}(n)$ for $n \geq 3$. It can be realised as the homotopy fibre of the second Stiefel-Whitney class

$$w_2\colon \mathscr{B}SO(n) \to \mathbf{B}^2\mathbb{Z}/2\mathbb{Z}. \tag{68}$$

An oriented n-dimensional manifold M will then admit a spin structure if and only if $[w_2 \circ T_M]$ is the trivial element in $\pi_0\mathbf{H}^\infty(M, \mathbf{B}^2\mathbb{Z}/2\mathbb{Z}) = H^2(M, \mathbb{Z}/2\mathbb{Z})$. When this happens, the space of possible orientations on M is equivalent to $\mathbf{H}^\infty(M, \mathbf{B}\mathbb{Z}/2\mathbb{Z})$, and we obtain, for a given spin structure σ on M lifting the orientation of M, the exact sequence

$$1 \to H^0(M, \mathbb{Z}/2\mathbb{Z}) \to \Gamma^{\mathrm{Spin}}(M, \sigma) \to \Gamma^{\mathrm{or}}(M) \to H^1(M, \mathbb{Z}/2\mathbb{Z}). \tag{69}$$

In particular, if M is connected, we get the exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \Gamma^{\mathrm{Spin}}(M, \sigma) \to \Gamma^{\mathrm{or}}(M) \to H^1(M, \mathbb{Z}/2\mathbb{Z}). \tag{70}$$

Since, for a connected M, the pullback action of oriented diffeomorphisms on $H^0(M, \mathbb{Z}/2\mathbb{Z})$ is trivial, we see that in this case the group $\Gamma^{\mathrm{Spin}}(M, \sigma)$ is a $\mathbb{Z}/2\mathbb{Z}$-central extension of the subgroup of $\Gamma^{\mathrm{or}}(M)$ consisting of (isotopy classes of) orientation preserving diffeomorphisms of M which fix the spin structure σ (up to homotopy). The group $\Gamma^{\mathrm{Spin}}(M,\sigma)$ and its relevance to Spin TQFTs are discussed in detail in [Ma96].

For $n = 2$, the homotopy fibre of $w_2 \colon \mathscr{B}SO(2) \to \mathbf{B}^2\mathbb{Z}/2\mathbb{Z}$ is again $\mathscr{B}SO(2)$ with the morphism $\mathscr{B}SO(2) \to \mathscr{B}SO(2)$ induced by the group homomorphism

$$\begin{aligned} SO(2) &\to SO(2) \\ x &\mapsto x^2 \end{aligned} \tag{71}$$

Since the second Stiefel-Whitney class of an oriented surface M is the mod 2 reduction of the first Chern class of the holomorphic tangent bundle of M (for any choice of a complex structure compatible with the orientation), and $\langle c_1(T^{\mathrm{hol}})M|[M]\rangle = 2-2g$, where g is the genus of M, one has that $[w_2 \circ T_M]$ is always the zero element in $H^2(M,\mathbb{Z}/2\mathbb{Z})$ for a compact oriented surface, and so the orientation of M can always be lifted to a spin structure. More generally, one can consider the group homomorphism $SO(2) \to SO(2)$ given by $x \mapsto x^r$, with $r \in \mathbb{Z}$. We have then a homotopy fibre sequence

$$\begin{tikzcd} \mathscr{B}SO(2) \arrow[r] \arrow[d, "\rho_{1/r}"'] & {*} \arrow[d] \\ \mathscr{B}SO(2) \arrow[r, "c_{(x\mapsto x^r)}"] & \mathbf{B}^2\mathbb{Z}/2\mathbb{Z} \end{tikzcd} \tag{72}$$

In this case one sees that an *r-spin structure* on an oriented surface M, i.e. a lift of the orientation of M through $\rho_{1/r}$, exists if and only if $2-2g \equiv 0 \mod r$. When this happens, one obtains the exact sequence

$$1 \to \mathbb{Z}/r\mathbb{Z} \to \Gamma^{1/r}(M,\sigma) \to \Gamma^{\mathrm{or}}(M) \to H^1(M,\mathbb{Z}/r\mathbb{Z}), \tag{73}$$

which exhibits the r-spin mapping class group $\Gamma^{1/r}(M,\sigma)$ as a $\mathbb{Z}/r\mathbb{Z}$-central extension of the subgroup of $\Gamma^{\mathrm{or}}(M)$ consisting of isotopy classes of orientation preserving diffeomorphisms of M fixing the r-spin structure σ (up to homotopy). The group $\Gamma^{1/r}(M,\sigma)$ appears as the fundamental group of the moduli space of r-spin Riemann surfaces, see [R-W12, R-W14].

5.2 p_1-structures on oriented surfaces

Let us now finally specialise the general construction above to the case of p_1-structures on closed oriented surfaces, to obtain the $\mathbb{Z}$-central extensions considered in [Se04] around page 476. In particular we will see, how p_1-structures provide a simple realisation of Segal's idea of extended surfaces and 3-manifolds (see also [BN09, CHMV95]).[8]To this aim, our stack Y will be the stack $\mathscr{B}SO(n)$ for some $n \geq 3$, the stack Z will be $\mathbf{B}^4\mathbb{Z}$ and the morphism c will be the first Pontryagin class $p_1 \colon \mathscr{B}SO(n) \to \mathbf{B}^4\mathbb{Z}$. the stack X will be the homotopy fiber of p_1, and so the morphism ψ will be the morphism

$$\rho_{p_1} \colon \mathbf{B}^3\mathbb{Z}//_h SO(n) \to \mathscr{B}SO(n). \tag{74}$$

of example 2.4. A lift σ of an orientation on a manifold M of dimension at most 3 to a morphism $M \to \mathbf{B}^3\mathbb{Z}//_h SO(n)$ over $\mathscr{B}O(n)$ will be called a p_1-struture on M. That is, a pair (M, σ) is the datum of a smooth oriented manifold M together with a trivialisation of its first Pontryagin class. Note that, since p_1 is a degree four cohomology class, it can always be trivialised on manifolds of dimension at most 3. In particular, when M is a closed connected oriented 3-manifold, we see that the space of lifts of the orientation of M to a p_1 structure, is equivalent to the space $\mathbf{H}(M, \mathbf{B}^3\mathbb{Z})$ and so its set of connected components is

$$\pi_0\mathbf{H}(M, \mathbf{B}^3\mathbb{Z}) = H^3(M, \mathbb{Z}) \cong \mathbb{Z}. \tag{75}$$

In other words, there is a $\mathbb{Z}$-torsor of equivalence classes of p_1-strctures on a connected oriented 3-manifold. Similarly, in the relative case, i.e., when M is a connected oriented 3-manifold with boundary, the set of equivalence classes of p_1-strctures on M extending a given p_1-structure on ∂M is nonempty and is a torsor for the relative cohomology group

$$H^3(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}, \tag{76}$$

[8] In [Se04], the extension is defined in terms of "riggings", a somehow ad hoc construction depending on the contractiblity of Teichm̈uller spaces and on properties of the η-invariant of metrics on 3-manifolds. Segal says: "I've not been able to think of a less sophisticated definition of a rigged surface, although there are many possible variants. The essential idea is to associate *functorially* to a smooth surface a space -such as $\mathcal{P}_X$- which has fundamental group $\mathbb{Z}$."

in perfect agreement with the prescription in [Se04, page 480].[9]
We can now combine the results of the previous section in the following

Proposition 5.3. *Let M be a connected oriented surface, and let σ be a p_1-structure on M. We have then the following central extension*

$$1 \to \mathbb{Z} \to \Gamma^{p_1}(M, \sigma) \to \Gamma^{\mathrm{or}}(M) \to 1, \tag{77}$$

where Γ^{p_1} as a shorthand notation for $\Gamma^{\rho_{p_1}}$.

Proof. Since M is oriented, we have a canonical isomorphism $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$ induced by Poincaré duality. Moreover, since M is connected, from (65) we obtain the following short exact sequence

$$1 \to \mathbb{Z} \to \Gamma^{\rho_{p_1}}(M, \sigma) \to \Gamma^{\mathrm{or}}(M) \to 1 \tag{78}$$

Finally, since the oriented diffeomorphisms action on $H^2(M, \mathbb{Z})$ is trivial for a connected oriented surface M, this short exact sequence is a $\mathbb{Z}$-central extension. $\square$

Appendix: proof of the extension theorem

Here we provide the details for proof of the existence of the homotopy fibre sequence (36), which is the extension theorem this note revolves around. All the notations in this Appendix are taken from Section 3.2.

Lemma A.1. *We have a homotopy pullback diagram*

$$\begin{tikzcd}
\mathrm{Diff}^{\rho}(M, \Sigma) \arrow[r] \arrow[d, "\psi_*"'] & \mathbf{Aut}^{\infty}_{/\rho}(\sigma) \arrow[d, "\psi_*"] \\
\mathrm{Diff}^{\tilde{\rho}}(M, T) \arrow[r] & \mathbf{Aut}^{\infty}_{/\tilde{\rho}}(\tau)
\end{tikzcd} \tag{79}$$

[9] The naturality of the appearance of this $\mathbb{Z}$-torsor here should be compared to Segal's words in [Se04]: "An oriented 3-manifold Y whose boundary ∂Y is rigged has itself a set of riggings which form a principal homogeneous set under the group $\mathbb{Z}$ which is the centre of the central extension of $\mathrm{Diff}(\partial Y)$. I do not know an altogether straightforward way to define a rigging of a 3-manifold." Rigged 3-manifolds are then introduced by Segal in terms of the space of metrics on the 3-manifold Y and of the η-invariant of these metrics.

Proof. By definition of (equation (25)), we have homotopy pullback diagrams

$$\begin{tikzcd} \mathrm{Diff}^{\rho}(M,\Sigma) \arrow[r] \arrow[d] & \mathbf{Aut}^{\infty}_{/\rho}(\sigma) \arrow[d] \\ \mathrm{Diff}(M) \arrow[r] & \mathbf{Aut}^{\infty}_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M) \end{tikzcd} \tag{80}$$

and

$$\begin{tikzcd} \mathrm{Diff}^{\tilde\rho}(M,T) \arrow[r] \arrow[d] & \mathbf{Aut}^{\infty}_{/\tilde\rho}(\tau) \arrow[d] \\ \mathrm{Diff}(M) \arrow[r] & \mathbf{Aut}^{\infty}_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M) \end{tikzcd} \tag{81}$$

By pasting them together as

$$\begin{tikzcd} \mathrm{Diff}^{\rho}(M,\Sigma) \arrow[r] \arrow[d, "\psi_*"'] & \mathbf{Aut}^{\infty}_{/\rho}(\sigma) \arrow[d, "\psi_*"] \\ \mathrm{Diff}^{\tilde\rho}(M,T) \arrow[r] \arrow[d] & \mathbf{Aut}^{\infty}_{/\tilde\rho}(\tau) \arrow[d] \\ \mathrm{Diff}(M) \arrow[r] & \mathbf{Aut}^{\infty}_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M) \end{tikzcd} \tag{82}$$

and by the 2-out-of-3 law for homotopy pullbacks the claim follows. □

We need the following basic fact [Lu06, Lemma 5.5.5.12]:

Lemma A.2. *Let $\mathbf{C}$ be an ∞-category, $\mathbf{C}_{/x}$ its slice over an object $x \in \mathbf{C}$, and let $f\colon a \to x$ and $g\colon b \to x$ be two morphisms into x. Then the hom space $\mathbf{C}_{/x}(f,g)$ in the slice is expressed in terms of that in $\mathbf{C}$ by the fact that there is a homotopy pullback (in $\infty\mathrm{Grpd}$) of the form*

$$\begin{tikzcd} \mathbf{C}_{/x}(f,g) \arrow[r] \arrow[d] & \mathbf{C}(a,b) \arrow[d, "g\circ(-)"] \\ * \arrow[r, "{[f]}"] & \mathbf{C}(a,x) \end{tikzcd}$$

where the right morphism is composition with g, and where the bottom morphism picks f regarded as a point in $\mathbf{C}(a,x)$.

Lemma A.3. *We have homotopy pullback diagrams*

$$\begin{tikzcd} \Omega_T \mathbf{H}^\infty_{/\mathscr{B}GL(n)}(T_M^{\mathrm{st}}, \tilde\rho) \arrow[r] \arrow[d] & \mathbf{Aut}^\infty_{/\tilde\rho}(T) \arrow[d] \\ * \arrow[r] & \mathbf{Aut}^\infty_{/\mathscr{B}GL(n)}(T_M^{\mathrm{st}}) \end{tikzcd} \tag{83}$$

and

$$\begin{tikzcd} \Omega_\Sigma \mathbf{H}^\infty_{/\mathscr{B}GL(n)}(T_M^{\mathrm{st}}, \rho) \arrow[r] \arrow[d] & \mathbf{Aut}^\infty_{\rho}(\Sigma) \arrow[d] \\ * \arrow[r] & \mathbf{Aut}^\infty_{/\mathscr{B}GL(n)}(T_M^{\mathrm{st}}) \end{tikzcd} \tag{84}$$

Proof. Let $\mathbf{C}$ be an $(\infty,1)$-category, and let $f\colon x \to y$ be a morphism in $\mathbf{C}$. Then by Lemma A.2 and using 2-out-of-3 for homotopy pullbacks, the forgetful morphism $\mathbf{C}_{/y} \to \mathbf{C}$ from the slice over y to $\mathbf{C}$ induces a morphism of ∞-groups $\mathbf{Aut}_{\mathbf{C}_{/y}}(f) \to \mathbf{Aut}_{\mathbf{C}}(x)$ sitting in a pasting of homotopy pullbacks like this:

$$\begin{tikzcd} \Omega_f \mathbf{C}(x,y) \arrow[r] \arrow[d] & \mathbf{Aut}_{\mathbf{C}_{/y}}(f) \arrow[r] \arrow[d] & * \arrow[d, "{[f]}"] \\ * \arrow[r, "{[\mathrm{id}]}"] \arrow[rr, bend right, "{[f]}"'] & \mathbf{Aut}_{\mathbf{C}}(x) \arrow[r, "f\circ(-)"] & \mathbf{C}(x,y) \end{tikzcd} \tag{85}$$

By taking here $\mathbf{C} = \mathbf{H}^\infty_{/\mathscr{B}GL(n)}$, $x = T_M^{\mathrm{st}}$, $y = \tilde\rho$ (resp., $y = \rho$), and $f = T$ (resp., $f = \Sigma$), the left square yields the first (resp., the second) diagram in the statement of the lemma. $\square$

Lemma A.4. *We have a homotopy pullback diagram*

$$\begin{tikzcd} \Omega_\beta (\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\tilde\rho}(T, \Psi) \arrow[r] \arrow[d] & \Omega_\Sigma \mathbf{H}^\infty_{/\mathscr{B}GL(n)}(T_M^{\mathrm{st}}, \rho) \arrow[d] \\ * \arrow[r] & \Omega_T \mathbf{H}^\infty_{/\mathscr{B}GL(n)}(T_M^{\mathrm{st}}, \tilde\rho) \end{tikzcd} \tag{86}$$

Proof. If we take $\mathbf{C} = \mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}$, $g = (\psi, \Psi)$, $a = T^{\mathrm{st}}_M$, $f = T$, $b = \rho$ and $x = \tilde{\rho}$ in Lemma A.2, we find the homotopy fibre sequence

$$\begin{tikzcd} (\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)})_{/\tilde{\rho}}(T, \Psi) \arrow[r] \arrow[d] & \mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}, \rho) \arrow[d, "\psi_*"] \\ * \arrow[r] & \mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}, \tilde{\rho}) \end{tikzcd} \tag{87}$$

By looping the above diagram, the claim follows. □

Lemma A.5. *We have an equivalence of* $(\infty, 1)$*-categories*

$$(\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)})_{/\tilde{\rho}} \cong \mathbf{H}^{\infty}_{/Y}. \tag{88}$$

Proof. Let $\mathbf{C}$ be an $(\infty, 1)$-category, and let $f : b \to x$ be a 1-morphism in $\mathbf{C}$. By abuse of notation, we can regard f as a diagram $f : \Delta^1 \to \mathbf{C}$. We have then a morphism

$$\varphi : (\mathbf{C}_{/x})_{/f} \to \mathbf{C}_{/b} \tag{89}$$

induced by the ∞-functor $\Delta^0 \hookrightarrow \Delta^1$ induced by sending 0 to 1. Since 1 is an initial object in Δ^1, the opposite ∞-functor is a cofinal map. By noticing that $\mathbf{C}^{op}_{x/}$ is canonically equivalent to $\mathbf{C}_{/x}$, then by [Lu06, Proposition 4.1.1.8] we have that φ is an equivalence of ∞-categories. Therefore, if we take $\mathbf{C} = \mathbf{H}^{\infty}$, and $f = \tilde{\rho} : Y \to \mathscr{B}GL(n)$, we have that the claim follows. □

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

VOLUME LIX-3 (2018)

NEW BOOKS

Andrée EHRESMANN

Résumé. Cette rubrique présente de brèves revues de livres qui peuvent avoir un intérêt pour les lecteurs des "Cahiers".

Abstract. This "New Books" entry presents brief reviews of books which might be of interest for the readers of the "Cahiers".

Key words. Topology, Differential Geometry, Anticipation. Categories.

MS Classification. 00-01, 00-02, 97-02.

1. "*Synthetic Differential Topology", by Marta Bunge, Felipe Gago and Ana María San Luis*

This book, published by Cambridge University Press (in the London Mathematical Society Lecture Notes Series 448, 2018), gives a comprehensive coverage of Synthetic Differential Topology (SDT) as an extension of Synthetic Differential Geometry (SDG), after a clear recall (Part I) of the main notions of Topos theory and SDG necessary for the sequel.

In the fifties, the introduction of local and infinitesimal jet bundles by Charles Ehresmann and of 'near points' by André Weil lead to a new foundation of Differential Geometry. In the seventies, F. William Lawvere, followed by Anders Kock, developed SDG by 'synthetically' transposing these notions in a non-boolean topos *E* with a commutative unitary ring R 'of line type' which satisfies adequate axioms, so that the infinitesimal jets are representable by algebraic 'tiny' objects (nilpotents).

Part II shows how to extend to SDG the theory of connections and sprays, with results extending the classical Ambrose-Palais-Singer Theorem. The

Calculus of variations is also extended. In these 2 extensions, the differences with the classical case are well stressed, explaining how they influence the results; for instance in the calculus of variations there is no need of the notion of variation, with local being replaced by infinitesimals.

Part III introduces the main concepts of SDT. While the representability of jets by tiny objects of an algebraic nature is at the basis of SDG, in SDT what is essential is the representability of germs (of smooth mappings) by tiny objects of a logical sort introduced and studied by Jacques Penon in the eighties. The intrinsic topology on any object of a topos, introduced by Penon, also plays a main role.

The strength of SDT is revealed in Part IV which gives applications of SDT to the theory of stable germs of smooth mappings including Mather's Theorem, and Morse Theory on the classification of singularities

Parts V discusses what would be a well-adapted model for SDT, in relation with what is a well-adapted model for SDG. The existence of such a model of SDT is important to recover several classical results on smooth manifolds, but also to extend their generality and their conceptual simplicity. An application to unfoldings is given.

Finally, Part VI describes such a model, namely the Dubuc topos G which Dubuc introduced in 1979 as a well-adapted model of SDG. It is a Grothendieck topos with a ring R of line type, in which the category of smooth manifolds is fully embedded, the embedding sending **R** to R, preserving the existing limits and some other constructions.

This seminal book is very important because it opens the way to the relatively new and important domain of SDT, by giving a unified clear exposition of its main results, up to now dispersed in more or less accessible papers. Moreover, its parallel treatment of SDG makes visible the main points of both theories.
This book is well written and only requires a knowledge of the basic notions of Category Theory; for instance its parts on topos represent a plain and concise approach to Topos Theory. Not being a specialist of SDT, I have much appreciated the numerous Remarks which insist on specific

points, for instance allowing to avoid some risk of confusion.
I recommend this book both to mathematicians who just look for an overview of SDG and SDT, and to researchers who want to acquire an advanced knowledge of these domains, with full proofs of fundamental theorems. And its different parts could be used as a basis for advanced courses or seminars.

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2. *"Transforming the Future: Anticipation in the XXIst century", edited by Riel Miller*

This book, published by Routledge (2018) presents the results of a UNESCO project, directed by Riel Miller from 2013 to 2018, on Anticipation and "Futures Literacy" (FL), the motivating assumption being that FL should enable a better grasp of complexity. It is reviewed in these "Cahiers" to illustrate how category theory may provide a theoretical support for such a research.

Anticipation is fundamental for futures-thinking and decision-making. However, in these domains there is still a dominant deterministic and reductionist paradigm. To counter it, there is need of a "Discipline of Anticipation" (Chapter 2) which takes account of the complexity and impredicativity of the world to imagine 'novel' futures, instead of 'colonizing' the future by an extrapolation of the present and past.

In Chapter 1, Riel Miller describes Futures Literacy as "an emergent and evolving human capability to identify, design, target and deploy anticipatory assumptions" and to use them to better distinguish constraints, dependencies and even weak signals in the present. The conjecture is that FL development increases awareness for new opportunities emerging in the present and ways to exploit them, thus improving strategic decision-making in the face of uncertainty.

A new general-purpose research tool, called a "Futures Literacy Knowledge Laboratory", has been designed to experimentally test this conjecture. Such an FLL (Chapter 4) consists in gathering a group of people to collectively exchange ideas about the future in relation to a specific problem (for example in Case Studiy 3, Chapter 5: using the future for local labor markets in Bogotà). The exchange generates collective intelligence knowledge creation processes which lead to share a number of 'archetypal' anticipatory assumptions of different kinds: some concern closed 'optimisation' or 'contingent' futures, others concern more creative futures seizing new opportunities in the present.

The more theoretical Chapter 3 analyses how category theory can give a conceptual support to the preceding results. Already in the eighties, Robert Rosen[1] had promoted the use of category theory to study anticipation. Here the performance of different specific FLL designs is assessed by using the Memory Evolutive Systems (MES) methodology[2] which provides a 'dynamic' categorical framework to study evolutionary multi-level and multi-agent impredicative complex systems.
A FLL is modelled by a (FL-)MES in which its members act as co-regulator agents. The collective knowledge creation processes are modelled by complexification/decomplexification processes (CDP). They lead to the formation of a shared "FL-Archetypal Pattern" as follows:

(i) Each agent G has its own record A_G of a concept A, e.g. an anticipatory assumption, in its landscape and it is reflected to other agents through their exchanges. (ii) A shared multi-faceted 'archetypal' record A*, encompassing the diverse meanings of A, is constructed by a CDP process that adds A* as colimit of the (A_G). (iii) The construction implies that the A*s are connected by 'complex links' (Emergence Theorem) which play the role of "changes in the conditions of change".

The shared FL-archetypal pattern AP so obtained acts as an engine to construct an evolving macro-landscape in which the agents acting together develop a retrospection and a prospection process to select strategies. These strategies can "use the future in the present" by recognizing weak signals and new opportunities and exploit them in an innovative way.

The Futures Literacy Framework developed in this book proposes a new approach to anticipation and decision-making which takes into account the complexity and impredicativity of the world. The FL-laboratories offer both an analytical tool and an experimental ground to verify its assumptions; and the several Cas**e** Studies around the world studied in Chapter 5 illustrate the power of Futures Literacy to cope with uncertainty and to reach creative decision-making.

[1] Rosen, R. 1985, *Anticipatory Systems, philosophical, mathematical and methodological foundations*. New York: Pergamon.
[2] Ehresmann, A. & Vanbremeersch, J.-P., 2007, *Memory Evolutive Systems: Hierarchy, Emergence, Cognition*, Elsevier.

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(There is an Open Access version of this book available at https://www.taylorfrancis.com/books/e/9781351047999)

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Commission paritaire n° 58964.
ISSN: 1245-530X

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