cahiers de topologie et géométrie différentielle catégoriques

créés par CHARLES EHRESMANN en 1958
dirigés par Andrée CHARLES EHRESMANN

VOLUME LIX - 2, 2ème trimestre 2018

AMIENS

Cahiers de Topologie et Géométrie Différentielle Catégoriques

Directeur de la publication: Andrée C. EHRESMANN,
Faculté des Sciences, Mathématiques LAMFA
33 rue Saint-Leu, F-80039 Amiens.

Les "*Cahiers*" comportent un Volume par an, divisé en 4 fascicules trimestriels. Ils publient des articles originaux de Mathématiques, de préférence sur la Théorie des Catégories et ses applications, e.g. en Topologie, Géométrie Différentielle, Géométrie ou Topologie Algébrique, Algèbre homologique…
Les manuscrits soumis pour publication doivent être envoyés à l'un des Rédacteurs comme fichiers .pdf.

A partir de 2018, les "*Cahiers*" ont une **Edition Numérique en Libre Accès,** sans charge pour l'auteur : le fichier pdf d'un fascicule trimestriel sera, dès parution, librement téléchargeable sur :

The "*Cahiers*" are a quarterly Journal with one Volume a year (divided in 4 issues). They publish original papers in Mathematics, the center of interest being the Theory of categories and its applications, e.g. in topology, differential geometry, algebraic geometry or topology, homological algebra…
Manuscripts submitted for publication should be sent to one of the Editors as pdf files.

From 2018 on, the "*Cahiers*" have also a **Full Open Access Edition (**without Author Publication Charge): the pdf file of each quarterly issue will be instantly freely downloadable on:

http://ehres.pagesperso-orange.fr/Cahiers/Ctgdc.htm
and, split in separate articles, on: http://cahierstgdc.com/

cahiers de topologie et géométrie différentielle catégoriques

créés par CHARLES EHRESMANN en 1958
dirigés par Andrée CHARLES EHRESMANN
VOLUME LIX-2, 2ème trimestre 2018

SOMMAIRE

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES
VOLUME LIX-2 (2018)

SMOOTH LOOP STACKS OF DIFFERENTIABLE STACKS AND GERBES

David Michael ROBERTS and Raymond F. VOZZO

Résumé. Nous définissons un groupoïde de Fréchet-Lie $\mathrm{Map}(S^1, X)$ d'ana-foncteurs du cercle vers un groupoïde de Lie X. Ceci fournit une présentation du Hom-champ $\underline{\mathcal{H}\mathrm{om}}(S^1, \mathcal{X})$, où $\mathcal{X}$ est le champ différentiable associé à X. Nous appliquons cette construction au groupoïde de Lie sous-jacent au 'fibré-gerbe' (= "bundle gerbe") d'une variété différentiable M; le résultat est un fibré-gerbe au-dessus de l'espace des lacets LM de M.
Abstract. We define a Fréchet–Lie groupoid $\mathrm{Map}(S^1, X)$ of anafunctors from the circle into a Lie groupoid X. This provides a presentation of the Hom-stack $\underline{\mathcal{H}\mathrm{om}}(S^1, \mathcal{X})$, where $\mathcal{X}$ is the differentiable stack associated to X. We apply this construction to the Lie groupoid underlying a bundle gerbe on a manifold M; the result is a bundle gerbe on the loop space LM of M.
Keywords. Differentiable stacks, Lie groupoids, Hom-stacks, loop stacks, gerbes, bundle gerbes
Mathematics Subject Classification (2010). Primary 22A22; Secondary 58B25, 58D15, 14A20, 18F99, 53C08.

1. Introduction

The notion of smooth loop space of a manifold is useful in a variety of areas of geometry, while at the same time being just outside the usual sphere of study, namely finite-dimensional manifolds. While it is naturally a topological space, it carries a very well-behaved smooth structure as an infinite-dimensional manifold. While recent progress on generalised smooth spaces means that any mapping space, in particular a loop space, is easily a smooth space, and the general study of smooth spaces is advancing in leaps and bounds (see e.g. the lengthy book [IZ13] on diffeological spaces), the fact that the loop space is a manifold with well-understood charts is extremely useful.

The area of geometry has in recent years expanded to include what is becoming known as 'higher geometry', where, loosely speaking, the geometric objects of study have a categorical or higher categorical aspect. One example of such objects are differentiable, or *Lie*, groupoids, which are known [Pro96] to be incarnations of *differentiable stacks*: stacks that look locally like manifolds, but with internal symmetries captured by Lie groupoids. A rather well-known simple case is that of *orbifolds*. Other examples that are still stacks on manifolds but which are still akin to Lie groupoids, are groupoids built from infinite-dimensional manifolds, or from smooth spaces. Clearly these objects can become locally less well-behaved as one becomes more general; an arbitrary diffeological space, for instance, may have rather terrible topological and homotopical properties.

The construction that this short paper wishes to address is that of the loop stack of a differentiable stack. This was introduced in the special case of orbifolds in [LU02], and then considered in full generality for the purposes of studying string topology in [BGNX12]. All of these are special cases of the loop stack of the underlying *topological stack*, a special case of the topological mapping stack studied in [Noo10].[1] In other words, the end result is only a *topological* stack, rather than a differentiable stack.

One (quite reasonable) approach is to consider if we can find a loop stack on manifolds that arises from a diffeological groupoid (i.e. a diffeological stack). This is not too difficult, and the parts of our construction that do not require special handling due to the nature of manifolds are performed for diffeological stacks. The novelty here is that this construction can be lifted so that it becomes a stack arising from what we call a *Fréchet–Lie* groupoid:

[1]The paper [Car12] considers the more general problem of a cartesian closed bicategory of stacks, whereas [Noo10] considers the special case with compactness conditions on the domain.

a groupoid in the category of Fréchet manifolds. This is the optimal result, since the construction applied to a manifold returns (a stack equivalent to) the usual loop space of that manifold, which is an infinite-dimensional Fréchet manifold in general. This is in contrast to the case of algebraic Hom-stacks, for instance [Ols06], which are again algebraic stacks.

The benefits of having a smooth version of the loop stack is that one can start to do actual geometry on it, rather than just topological constructions (such as the string topology in [BGNX12]). Moreover, while one can perform smooth geometric constructions on diffeological spaces, as *spaces*, unlike manifolds there is little control over the local structures. So, for instance, our construction provides a *smoothly paracompact* groupoid, admitting partitions of unity on object and arrow manifolds.

Another example, which was the original impetus for this article, are the loop stacks of *bundle gerbes*. Bundle gerbes over manifolds are higher geometric objects analogous to line bundles, and as such can support structures analogous to connections. One can form the construction given below to the groupoid underlying a bundle gerbe and then the resulting groupoid is in fact still a gerbe, now over a loop space, and this should again carry a connective structure of the appropriate sort. Of particular interest is the bundle gerbe underlying the *String 2-group*, which will be the subject of future work.

We consider in this paper various categories of smooth objects, groupoids in those categories and corresponding smooth stacks. Figure 1 summarises these, as well as the relations between them. The first row consists of categories, the remaining rows consist of 2-categories, and the inclusions denote *full* subcategories and sub-2-categories. The vertical arrows of type $\longrightarrow\!\!\!\!\!\!\rightarrow$ denote surjective-on-objects 2-functors. We use $\mathbf{Stack}_X$ to denote the 2-category of stacks of groupoids on the site X.

The paper outline is as follows:

- Section 2—Gives background on sites, internal groupoids, anafunctors (a type of generalised morphism between internal groupoids) and stacks presented by groupoids internal to the base site.
- Section 3—We construct a diffeological groupoid $\mathrm{Map}(S^1, X)$ of anafunctors and transformations.
- Section 4—Proves that $\mathrm{Map}(S^1, X)$ is indeed a presentation over the site of diffeological spacs, making $\underline{\mathcal{H}\mathrm{om}}(S^1, \mathcal{X})$ a diffeological stack.

$$\begin{array}{ccccc}
\mathcal{M} = \{\text{manifolds}\} & \subset & \{\text{Fréchet manifolds}\} & \subset & \{\text{diffeological spaces}\} = \mathcal{D} \\
\cap & & \cap & & \cap \\
\{\text{Lie groupoids}\} & \subset & \{\text{Fréchet Lie groupoids}\} & \subset & \{\text{diffeological groupoids}\} \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
\{\text{Differentiable stacks}\} & \subset & \{\text{Fréchet differentiable stacks}\} & \subset & \{\text{Diffeological stacks}\} \\
\cap & & \cap & & \cap \\
\mathbf{Stack}_{\mathcal{M}} & = & \mathbf{Stack}_{\mathcal{M}} & \xrightarrow{\simeq} & \mathbf{Stack}_{\mathcal{D}}
\end{array}$$

Figure 1: Categories and 2-categories appearing in the paper

- Section 5—We show that the construction of $\mathrm{Map}(S^1, X)$ actually lands in the sub-2-category of Fréchet–Lie groupoids, and that this gives a (weak) presentation of $\underline{\mathcal{H}\mathrm{om}}(S^1, \mathcal{X})$. This is our first main result.

- Section 6—Gives a treatment of the theory of gerbes on the site of manifolds presented by (Fréchet–)Lie groupoids, including establishing stability of various properties under forming the mapping groupoid.

- Section 7—We prove our second main result, namely that given a bundle gerbe (a special sort of abelian gerbe), the mapping groupoid is again a bundle gerbe.

Acknowledgements. This research was supported under the Australian Research Council's Discovery Projects funding scheme (projects numbered DP120100106 & DP130102578). This project was born in 'Coffee Spot': thanks to the staff for uninterrupted and secluded working time. A big thanks to Andrew Stacey for writing the paper [Sta13], in order to prove theorem 5.1, after discussions with the first named author; this theorem was crucial to the success of the current paper. Thanks also to Alexander Schmeding for side discussions about possible extensions to the infinite-dimensional setting. The authors thank the anonymous referee for their careful reading and helpful suggestions, which helped us find a small error in the original version of Lemma 6.5.

2. Background and preliminaries

2.1 Sites

We will be interested in stacks over sites where the Grothendieck topology, arises from a *coverage* (see e.g. [Joh02, Section C.2.1]), rather than the more familiar data of a pretopology. In this paper we will work only with a coverage and not the Grothendieck topology generated by it.

Definition 2.1. *Let $\mathcal{C}$ be a category. A* coverage *J on $\mathcal{C}$ is a collection $J(x)$, for each object x, of families of arrows $\{u_i \to x \mid i \in I\}$ (called* covering families*) with the property that for each covering family $\{u_i \to x \mid i \in I\} \in$*

$J(x)$ and $f\colon y \to x$ there is a covering family $\{v_k \to y \mid k \in K\} \in J(y)$ such that for all k there is an $i \in I$ and lift as shown

$$\begin{array}{ccc} v_k & \dashrightarrow & u_i \\ \downarrow & & \downarrow \\ y & \xrightarrow[f]{} & x \end{array}$$

A site $(\mathcal{C}, J)$ *is then a category $\mathcal{C}$ equipped with a coverage J, and sites with the same underlying category are* equivalent *if their coverages generate the same sieves.*

It will be the case that the coverages we consider satisfy the saturation condition that composites of coverages are again coverages, but not still not necessarily that pullbacks of covering families are covering families.

If we have a pair of covering families $\mathcal{U} = \{u_i \to x \mid i \in I\}$ and $\mathcal{V} = \{v_j \to x \mid j \in J\}$ then we say $\mathcal{V}$ *refines* $\mathcal{U}$ if for every $j \in J$ there is an $i \in I$ and a lift of $v_j \to x$ through u_i. We can say that a coverage J_1 refines the coverage J_2 if every covering family in J_1 refines a covering family in J_2. If J_1 refines J_2 and J_2 refines J_1 then they give rise to equivalent sites.

A coverage is called a *singleton* coverage if all covering families consist of single maps, in which case covering families will be referred to as *covering maps*. An example of a singleton coverage is a class of maps containing identity arrows, closed under composition and pullback along arbitrary maps; such a class will be called a *singleton pretopology*

A *superextensive* coverage (on an extensive category, see [CLW93]) is one that is generated by a singleton coverage and the coverage where covering families are inclusions of summands $\{u_i \to \coprod_{i \in I} u_i \mid i \in I\}$. For all intents and purposes, a superextensive coverage J can be reduced to considering just the singleton coverage $\amalg J$ it gives rise to: $\amalg J$-covering maps are of the form $\coprod_i u_i \to x$, for $\{u_i \to x \mid i \in I\}$ a covering family in the original superextensive coverage. We shall abuse terminology slightly and say that a superextensive coverage J_1 and another singleton coverage J_2 give rise to equivalent sites when the *singleton coverage associated to* J_1 is equivalent to J_2. We shall also abuse notation and refer to a covering map in $\amalg J$ as being in J when no confusion shall arise.

A site is called *subcanonical* if all representable presheaves are in fact sheaves. For a singleton coverage this is implied by all covering maps being regular epimorphisms, and for a subcanonical superextensive coverage J, the singleton coverage $\amalg J$ is subcanonical. In fact all of the coverages we consider in this paper will be subcanonical.

We will need the following examples over the course of the paper.

Example 2.2. Consider the category $\mathbf{Cart}$ with objects $\mathbb{R}^n$ for $n = 0, 1, 2, \ldots$ and $\mathbf{Cart}(\mathbb{R}^n, \mathbb{R}^m) = C^\infty(\mathbb{R}^n, \mathbb{R}^m)$. This has a coverage where a covering family $\{\phi_i \colon \mathbb{R}^n \hookrightarrow \mathbb{R}^n \mid i \in I\}$ is an open cover in the usual sense.

For the purposes of the current paper, we consider manifolds to be finite dimensional unless otherwise specified.

Example 2.3. The category $\mathcal{M}$ of smooth manifolds has the following coverages:

- the coverage O of open covers in the usual sense;
- the coverage C, where covering families $C(X)$ are covers of X by regular closed compact neighbourhoods, such that the interiors also cover;
- the singleton pretopology Subm where covering maps are surjective submersions.

All these coverages give equivalent sites, the first two because manifolds are locally compact and regular[2] and the first and last because surjective submersions have local sections. The first two coverages are superextensive, and we will be considering their associated singleton coverages.

Recall that a (smooth) *Fréchet* manifold is a smooth manifold locally modelled on Fréchet spaces (a good reference is [Ham82]). The definition does not assume second-countability, so that the category of Fréchet manifolds admits small coproducts. A submersion between Fréchet manifolds is a map for which there are charts on which the map looks locally like a projection out of a direct sum: $V \oplus W \to V$ (it is not enough to ask that this is surjective, or even split surjective, on tangent spaces).

[2]In fact there is a coverage on the category of locally compact spaces consisting of compact neighbourhoods, and a coverage on the category of regular spaces consisting of closed neighbourhoods.

Example 2.4. The category $\mathcal{F}$ of Fréchet manifolds has a coverage given by open covers, and also a singleton pretopology given by surjective submersions. The first is superextensive, and these give rise to equivalent sites.

Our last example needs some preliminaries. The following definition is quite different to that which appears in the original article [Sou80], but is in fact equivalent by work of Baez–Hoffnung [BH11]. An extensive reference is the book [IZ13].

Definition 2.5. *A* diffeological space *is a sheaf X on $\mathbf{Cart}$ that is a subsheaf of $\mathbb{R}^n \mapsto \mathbf{Set}(\mathbb{R}^n, \underline{X})$, where $\underline{X} = X(\mathbb{R}^0)$ is the* set of points of X. *A smooth map of diffeological spaces is just a map between the underlying sheaves. We denote the category of diffeological spaces by $\mathcal{D}$.*

We can think of cartesian spaces $\mathbb{R}^n$ as diffeological spaces via the Yoneda embedding, and for X a diffeological space, the elements of $X(\mathbb{R}^n)$ as maps $\mathbb{R}^n \to X$ in $\mathcal{D}$. The category of diffeological spaces is a Grothendieck quasitopos [BH11], in particular is complete, cocomplete, extensive and cartesian closed.

A map $X \to Y$ of diffeological spaces is a *subduction* if for every $f\colon \mathbb{R}^n \to Y$ there is a covering family $\phi_i\colon \mathbb{R}^n \hookrightarrow \mathbb{R}^n$ such that each map $f \circ \phi_i\colon \mathbb{R}^n \to Y$ lifts to X. Note that there are fully faithful inclusions $\mathcal{M} \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{D}$. Surjective submersions of manifolds and also of Fréchet manifolds are subductions.

Example 2.6. The category of diffeological spaces has a singleton pretopology Subd given by subductions.

The following facts about subductions will be useful.

- Every subduction $A \to B$ is refined by a subduction with domain a coproduct of Euclidean spaces;
- Every subduction $A \to M$ with M a manifold is refined by an open cover of M.

The astute reader will have noticed that almost all of the examples are in fact pretopologies or singleton pretopologies. The important fact is that we need to use the singleton coverage $\amalg C$ which is *not* a pretopology, but *refines* a singleton pretopology.

2.2 Internal groupoids

We will be dealing with internal groupoids that satisfy extra conditions, due to the fact that the ambient categories of manifolds are not finitely complete. To that end, *Lie groupoids* are groupoids internal to $\mathcal{M}$ where the source and target maps are submersions, and *Fréchet–Lie groupoids* are groupoids internal to $\mathcal{F}$ where again the source and target maps are submersions of Fréchet manifolds. We will also consider *diffeological* groupoids, which are just groupoids internal to $\mathcal{D}$; their source and target maps are automatically subductions.

Functors between internal groupoids, be they Lie, Fréchet–Lie or diffeological groupoids, will be assumed to be smooth. The same will be true for natural transformations between such functors. We denote, for a category $\mathcal{C}$, the 2-category of groupoids internal to $\mathcal{C}$ by $\mathbf{Gpd}(\mathcal{C})$, with the above caveats for $\mathcal{C} = \mathcal{M},\ \mathcal{F}$. Since the inclusions $\mathcal{M} \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{D}$ are full, we have full inclusions of 2-categories $\mathbf{Gpd}(\mathcal{M}) \hookrightarrow \mathbf{Gpd}(\mathcal{F}) \hookrightarrow \mathbf{Gpd}(\mathcal{D})$.

It is a well-known problem that there are just not enough morphisms between internal groupoids, in particular Lie groupoids and their cousins. One approach to this problem is through the use of *internal anafunctors*. These were introduced in Bartels' thesis [Bar06], inspired by work of Makkai on foundational issues surrounding the Axiom of Choice in category theory. We do not need the full theory of internal anafunctors, the basic definitions are enough for the present paper, for the special case where we only consider internal groupoids. We have also generalised the notion ever so slightly, by using singleton *coverages*; the fragment of the theory we need here does not lose out by considering this more general setting.

Definition 2.7 ([Bar06]). *Let J be a singleton coverage on $\mathcal{C}$ and let Y and X be groupoids in $\mathcal{C}$. An* anafunctor $Y \nrightarrow X$ *is a span of internal functors*

$$Y \xleftarrow{j} Y' \xrightarrow{f} X$$

where the object component $j_0 \colon Y'_0 \to Y_0$ of j is a J-cover, and the following square is a pullback

$$\begin{array}{ccc} Y'_1 & \xrightarrow{j_1} & Y_1 \\ \downarrow & & \downarrow \\ Y'_0 \times Y'_0 & \xrightarrow[j_0 \times j_0]{} & Y_0 \times Y_0 \end{array}$$

Of primary interest to us is the case when the groupoid Y has no nontrivial arrows, that is, it is just an object of $\mathcal{C}$, say M. In that case, any functor $j\colon Y' \to M$ satisfying the conditions is determined by the map on objects and the groupoid Y is what is known as a *Čech groupoid* of the covering map j_0 (or by abuse of notation, of its domain). If we let $U = Y'_0$, then $Y'_1 = U \times_M U$, and we denote Y' by $\check{C}(U)$. Thus any anafunctor from M to an internal groupoid X is of the form $M \xleftarrow{j} \check{C}(U) \xrightarrow{f} X$.

Assume for the moment that J is a singleton pretopology, so that we have pullbacks of covering maps. Given a pair of anafunctors $M \leftarrow \check{C}(U_1) \xrightarrow{f} X$ and $M \leftarrow \check{C}(U_2) \xrightarrow{g} X$, we want to define what it means to have a transformation between them. Let $U_{12} = U_1 \times_M U_2$. Then a transformation is a diagram

$$\begin{array}{ccccc} & & \check{C}(U_1) & & \\ & \swarrow & \uparrow & \overset{f}{\searrow} & \\ M & & \check{C}(U_{12}) & \Downarrow\alpha & X\,, \\ & \nwarrow & \downarrow & \underset{g}{\nearrow} & \\ & & \check{C}(U_2) & & \end{array}$$

where the two functors $\check{C}(U_{12}) \to \check{C}(U_i)$ are induced by the projections $U_{12} \to U_i$. The picture one should keep in mind here is a coboundary between X-valued Čech cocycles that lives over a common refinement.

For a singleton *coverage*, such as the coverage $\coprod C$ of compact neighbourhoods on manifolds, we can define a transformation to be a diagram as above, where instead of considering the pullback, which does not necessarily exist (or if it does, may not be a covering map), one considers a refinement U_{12}, equipped with maps to U_1 and U_2. One of the lessons that can be gleaned from [Rob16] is that when working with anafunctors nothing is lost by considering a coverage that is cofinal in a pretopology, rather than the pretopology itself (as in [Bar06]).

Using the notion of anafunctor with respect to a pretopology, internal groupoids, anafunctors and transformations form a bicategory [Bar06]. We will not use this bicategory structure directly, but it is relied on implicitly to take advantage of Theorem 2.10 below.

2.3 Stacks

We are considering stacks on the category $\mathcal{M}$ of manifolds using the coverage O of open covers. A standard reference is [BX11], and we point the reader to the detailed discussion of stacks in section 2.2 therein. We give the definition we need and then mention without proof some standard facts.

Definition 2.8. *Let $\mathcal{X}\colon \mathcal{M}^{op} \to \mathbf{Gpd}$ be a weak 2-functor. We say $\mathcal{X}$ is a* stack *if the following conditions are satisfied for every covering family $\{\phi_i\colon U_i \to M \mid i \in I\}$:*

1. *For any pair of objects x, y of $\mathcal{X}(M)$ and any family of isomorphisms $\sigma_i\colon x|_{U_i} \to y|_{U_i}$ in $\mathcal{X}(U_i)$, $i \in I$, there is a* unique *isomorphism $\sigma\colon x \to y$ in $\mathcal{X}(M)$ such that $\sigma|_{U_i} = \sigma_i$.*

2. *For every family of objects $x_i \in \mathcal{X}(U_i)$, $i \in I$, and collection of isomorphisms $\sigma_{ij}\colon x_i|_{U_{ij}} \to x_j|_{U_{ij}}$ in $\mathcal{X}(U_{ij})$, $i, j \in I$ satisfying $\sigma_{jk} \circ \sigma_{ik} = \sigma_{ik}$ in $\mathcal{X}(U_{ijk})$ (leaving the restrictions implicit), then there is an object x of $\mathcal{X}(M)$ and isomorphisms $\rho_i\colon x|_{U_i} \to x_i$ for all $i \in I$ such that $\sigma_{ij} \circ \rho_i = \rho_j$ (in $\mathcal{X}(U_{ij})$) for all $i, j \in I$ (where as usual, we write $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$).*

If only the first point is satisfied, then we say $\mathcal{X}$ is a prestack.

A morphism of stacks is given by a transformation of weak 2-functors, and there is a 2-category $\mathbf{Stack}_{\mathcal{M}}$ of stacks on $(\mathcal{M}, O)$. The relevant points we need are as follows:

- Any manifold M gives rise to a stack, also denoted by M (as O is subcanonical). Also, any diffeological space is a stack. The Yoneda embedding ensures that any map of stacks between manifolds or diffeological spaces is just a smooth map in the usual sense. A stack equivalent to a manifold is called *representable*.

- Any Lie groupoid gives rise to a *prestack*, by sending the groupoid X to the presheaf of groupoids $\mathcal{M}(-, X)\colon \mathcal{M}^{\mathrm{op}} \to \mathbf{Gpd}$, and this prestack can be 'stackified'. More generally, any Fréchet–Lie or diffeological groupoid gives rise to a prestack and hence a stack.

- The 2-category of stacks $\mathbf{Stack}_{\mathcal{D}}$ on $(\mathcal{D}, \mathrm{Subd})$ is equivalent to $\mathbf{Stack}_{\mathcal{M}}$. This follows from a stack version of the "lemme de comparaison" [SGA4.1, Esposé III, Théorème 4.1]; see discussion at [Car13].

The correct notion of 'pullback' for stacks is a *comma object*.[3] For a cospan $G \xrightarrow{f} H \xleftarrow{g} K$ of groupoids, the comma object $G \downarrow_H K$ (or sometimes $f \downarrow g$) can be computed as the *strict* limit $G \times_H H^{\mathbb{2}} \times_H K$ where $H^{\mathbb{2}}$ is the arrow groupoid of H. The comma object of a cospan of stacks is calculated pointwise, that is, $(\mathcal{X} \downarrow_{\mathcal{Z}} \mathcal{Y})(M) = \mathcal{X}(M) \downarrow_{\mathcal{Z}(M)} \mathcal{Y}(M)$. The comma object fits into a 2-commuting square called a *comma square*,

$$\begin{array}{ccc} \mathcal{X} \downarrow_{\mathcal{Z}} \mathcal{Y} & \longrightarrow & \mathcal{Y} \\ \downarrow & \swarrow & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Z} \end{array}$$

which is universal among such 2-commuting squares.

A stack is said to be *presentable* if it is the stackification of an internal groupoid. In this case, there is extra structure that the stack admits, from which we can recover the groupoid up to weak equivalence [Pro96, BX11].

First, we say a map of stacks $\mathcal{Y} \to \mathcal{X}$ is *representable* (resp. representable by diffeological spaces) if for every manifold M and map $M \to \mathcal{X}$, the comma object $M \downarrow_{\mathcal{X}} \mathcal{Y}$ is representable by a manifold (resp. a diffeological space). We can talk about properties of representable maps arising from properties of maps in $\mathcal{M}$ or $\mathcal{D}$; if P is a property of maps of manifolds (or diffeological spaces) that is stable under pullback and local on the target in a given coverage J, then we say a representable map of stacks $\mathcal{Y} \to \mathcal{X}$ has property P if for every $M \to \mathcal{X}$ the projection $M \downarrow_{\mathcal{X}} \mathcal{Y} \to M$ has property P.

Definition 2.9. *A stack $\mathcal{X}$ on $(\mathcal{M}, O)$ is* presentable *(resp. presentable by a diffeological groupoid) if there is a manifold (resp. diffeological space) X_0 and a representable epimorphism $p\colon X_0 \to \mathcal{X}$ that is a submersion (resp. subduction).*

[3]This is sometimes called a weak pullback, or even just a pullback, in the stack literature. However the definition usually given is clearly that of a comma object.

It follows from the definition that the comma object $X_1 := X_0 \downarrow_{\mathcal{X}} X_0$ is representable (by a manifold or diffeological space), the two projection maps $X_1 \to X_0$ are submersions (or subductions) and $X_1 \rightrightarrows X_0$ is an internal groupoid. This internal groupoid is said to *present* the stack $\mathcal{X}$. Then $\mathcal{X}$ is the stackification of the prestack arising from this internal groupoid. Note that this definition also works if we ask for presentability by a Fréchet–Lie groupoid: one asks for a representable submersion from a Fréchet manifold.

The usual name for a stack presentable by a Lie groupoid is *differentiable stack*, and we will call stacks presentable by diffeological groupoids, *diffeological stacks*. Stacks presented by a Fréchet–Lie groupoid shall be called *Fréchet differentiable stacks*.

The main result we need here is the following, and follows from the combination of the general theory of [Pro96] and [Rob12, Theorem 7.2] in the case of Lie groupoids, and uses an adaptation of Pronk's argument for the case of diffeological groupoids.

Theorem 2.10. *The 2-category of differentiable stacks (resp. diffeological stacks) is equivalent to the bicategory of Lie groupoids (resp. diffeological groupoids), anafunctors and transformations.*

What this means in practice is that we can pass between maps between presentable stacks and anafunctors between the presenting groupoids, and we shall use this below.

If we have an epimorphism $p\colon X_0 \to \mathcal{X}$ from a representable stack X_0 such that merely the comma object $X_0 \downarrow_{\mathcal{X}} X_0$ is representable and the projections are surjective submersions, then we call p a *weak presentation*. For certain sites a weak presentation gives a strong presentation: this is true for instance for presentations by diffeological spaces. This relies on the following lemma adapted from [BX11, Lemma 2.2], which works in the framework of stacks on the site of *not-necessarily-Hausdorff* (finite-dimensional) manifolds.

Lemma 2.11. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism in $\mathbf{Stack}_{\mathcal{M}}$. If M is a diffeological space, $M \to \mathcal{X}$ an epimorphism of stacks, and the comma object $M \downarrow_{\mathcal{X}} \mathcal{Y}$ is a diffeological space, then f is representable as a map of stacks considered in the equivalent 2-category $\mathbf{Stack}_{\mathcal{D}}$.*

The analogous result is not true for stacks on the category $\mathcal{M}$, but it *is* true (following [BX11]) if we allow ourselves to use possibly non-Hausdorff manifolds. In practice, one often finds that the stack is weakly presented by a Lie groupoid, which is made up of (Hausdorff) manifolds, which then can be used without reference to non-Hausdorff manifolds. The same can be said for weak presentations by Fréchet–Lie groupoids, an example of which will arise in our main construction.

While it is not always the case that the 2-category of internal groupoids has internal homs, the 2-category of stacks *does* have internal homs, namely for a pair of stacks $\mathcal{X}, \mathcal{Y}$, there is a stack $\underline{\mathcal{H}\mathrm{om}}(\mathcal{Y}, \mathcal{X})$ and an evaluation map $\mathcal{Y} \times \underline{\mathcal{H}\mathrm{om}}(\mathcal{Y}, \mathcal{X}) \to \mathcal{X}$ with the necessary properties.

Definition 2.12. *The* Hom-stack $\underline{\mathcal{H}\mathrm{om}}(\mathcal{Y}, \mathcal{X})$ *is defined by taking the value on the object M to be the groupoid* $\mathbf{Stack}_{\mathcal{M}}(\mathcal{Y} \times M, \mathcal{X})$.

Thus we have a Hom-stack for any pair of stacks on $\mathcal{M}$. The case we are interested in is where we have a stack $\mathcal{X}$ associated to an internal groupoid X in $\mathcal{M}$ or $\mathcal{D}$, and the Hom-stack $\underline{\mathcal{H}\mathrm{om}}(S^1, \mathcal{X})$.

3. Construction of the diffeological loop groupoid

We will now describe the construction of the loop groupoid of a diffeological groupoid X. This will naturally be a groupoid also internal to $\mathcal{D}$, and we shall show in the next section that it in fact presents the Hom-stack $\underline{\mathcal{H}\mathrm{om}}(S^1, \mathcal{X})$, for $\mathcal{X}$ the stack associated to X.

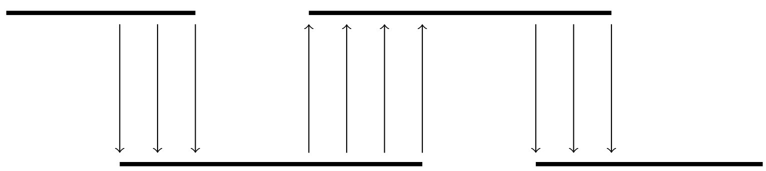
The objects of the diffeological mapping groupoid are anafunctors $S^1 \to\!\!\!\!\!\!\!\!+\;\; X$, using the compact neighbourhood coverage C of Example 2.3.

As the category $\mathcal{D}$ of diffeological spaces is cartesian closed and finitely complete, results of Bastiani–Ehresmann [BE72] imply that the category $\mathbf{Gpd}(\mathcal{D})$ of diffeological groupoids is also cartesian closed and finitely complete. Therefore the set $\mathbf{Gpd}(\mathcal{D})(\check{C}(V), X)_0$ of objects of the internal hom—a groupoid—is in fact a diffeological space. We shall, for the sake of saving space, write $X^{\check{C}(V)} := \mathbf{Gpd}(\mathcal{D})(\check{C}(V), X)_0$. The category $\mathcal{D}$ is also cocomplete (in fact extensive) and so we define the object space $\mathrm{Map}(S^1, X)_0$ to be the diffeological space

$$\coprod_{V \in C(S^1)} X^{\check{C}(V)}.$$

Remark 3.1. This mirrors the construction of the topological loop groupoid as in [LU02, BGNX12], even though for the purposes of diffeological groupoids it is not necessary to focus only on *compact* neighbourhoods; the diffeological groupoid of functors would exist using ordinary open covers. This would even give an equivalent mapping diffeological groupoid in the end. However, for the Lie groupoids in section 5 we *do* need to use compact covers to get the appropriate Fréchet topology on mapping spaces.

The picture we keep in mind for the elements of the object space is a sequence like:



where the horizontal lines are paths in X_0 and the vertical arrows, varying smoothly, are given by a path in X_1.

Next we move on to the arrow space of $\mathrm{Map}(S^1, X)$. Recall that a *transformation* $t\colon f \to g$ of anafunctors $f, g\colon S^1 \to\!\!\!\!\to X$ is a diagram

$$\begin{array}{ccc} \check{C}(V_{12}) & \longrightarrow & \check{C}(V_1) \\ \downarrow & \swArrow & \downarrow{\scriptstyle f} \\ \check{C}(V_2) & \xrightarrow[g]{} & X \end{array}$$

where V_{12} is the chosen refinement of $V_1 \times_{S^1} V_2$ as discussed in section 2.2. Note that t is necessarily a natural isomorphism as X is a groupoid.

For arbitrary f and g with domains $\check{C}(V_1)$ and $\check{C}(V_2)$, respectively, the diffeological space of all transformations is

$$X^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} (X^{\mathbf{2}})^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)}$$

where the two maps

$$X^{\check{C}(V_i)} \to X^{\check{C}(V_{12})}$$

are given by precomposition with the canonical functors $\check{C}(V_{12}) \to \check{C}(V_i)$. Here the groupoid $X^{\mathbb{2}}$ is the arrow groupoid of X and we are pulling back along the maps

$$(X^{\mathbb{2}})^{\check{C}(V_{12})} \to X^{\check{C}(V_{12})}$$

which are given by postcomposition with the functors $S, T\colon X^{\mathbb{2}} \to X$ (on objects these are the usual source and target maps).

The space of arrows $\mathrm{Map}(S^1, X)_1$ is then

$$\coprod_{V_1, V_2 \in C(S^1)} X^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} (X^{\mathbb{2}})^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)}$$

The source and target maps are projections on to the first and last factors. These are automatically smooth maps, and are both split by the unit map and hence are quotient maps; quotient maps in $\mathcal{D}$ are subductions and hence the source and target maps are subductions.

Composition of transformations of anafunctors [Bar06, Proposition 12] (or [Rob12, Section 5] for a description closer to what is given here) is a little involved, but is essentially induced by the composition in $X^{\mathbb{2}}$, which is smooth. This implies that composition in $\mathrm{Map}(S^1, X)$ is smooth, and hence that $\mathrm{Map}(S^1, X)$ is a diffeological groupoid.

4. Presentation by a diffeological groupoid

For X a diffeological groupoid, to give a presentation over $\mathcal{D}$ of the Hom-stack $\underline{\mathcal{H}\mathrm{om}}(S^1, X)$ we will need to define a map from some diffeological space A, considered as a stack, to $\underline{\mathcal{H}\mathrm{om}}(S^1, X)$. Such a map is determined by a map of stacks $A \times S^1 \to \mathcal{X}$. Since these are all stacks arising from diffeological groupoids, this a map can be specified by constructing an anafunctor $A \times S^1 \to\!\!\!\!\!\to X$ in the category of diffeological spaces as per Theorem 2.10.

Consider then the covers $V \to S^1$ used in the construction of the diffeological mapping groupoid, which are subductions since they admit local sections over open sets. The product of subductions is again a subduction, so we can, for each $V \in C(S^1)$ define the anafunctor

$$S^1 \times X^{\check{C}(V)} \leftarrow \check{C}(V) \times X^{\check{C}(V)} \xrightarrow{\mathrm{ev}} X$$

where the right-pointing arrow is just the evaluation map for diffeological groupoids. This gives us, via the preceeding argument, a map

$$X^{\check{C}(V)} \to \underline{\mathcal{H}\mathrm{om}}(S^1, \mathcal{X})$$

of stacks, and hence a map $q\colon\ \mathrm{Map}(S^1, X)_0 \to \underline{\mathcal{H}\mathrm{om}}(S^1, \mathcal{X})$.

Proposition 4.1. *For X a diffeological groupoid, the map q is an epimorphism of stacks.*

Proof. It is enough to show that for any $f\colon \mathbb{R}^n \to \underline{\mathcal{H}\mathrm{om}}(S^1, \mathcal{X})$ there is an open cover of $\mathbb{R}^n$ and local sections of q over it. The map f is determined by a map $\mathbb{R}^n \times S^1 \to \mathcal{X}$ of stacks, and hence an anafunctor $F\colon \mathbb{R}^n \times S^1 \mathrel{+\!\!\!\!\to} X$ of diffeological groupoids. Any subduction with codomain a manifold is refined by an open cover of the manifold, so we can replace the anafunctor by an isomorphic one of the form $\mathbb{R}^n \times S^1 \leftarrow \check{C}(U) \to X$, where $U = \coprod_i U_i \to \mathbb{R}^n \times S^1$ is an open cover. We can further repeat the argument from [Noo10, Proof of Theorem 4.2] to construct an open cover $\coprod_j W_j \to \mathbb{R}^n$ and for each j an open cover $V_j^o \to S^1$ with closure $V_j \to S^1$ an element of $C(S^1)$. Then the restrictions $W_j \times \check{C}(V_j) \to X$ of F give maps of diffeological spaces $W_j \to X^{\check{C}(V_j)}$, i.e. local sections of q over the open cover $\{W_j\}$. $\square$

To show that the groupoid $\mathrm{Map}(S^1, X)$ presents the Hom-stack, we need to show that the comma object of the map q with itself is the arrow space $\mathrm{Map}(S^1, X)_1$ of our diffeological groupoid.

First, we do indeed have a 2-commuting square

$$\begin{array}{ccc}
\mathrm{Map}(S^1, X)_1 & \xrightarrow{\ s\ } & \mathrm{Map}(S^1, X)_0 \\
{\scriptstyle t}\downarrow & \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle q} \\
\mathrm{Map}(S^1, X)_0 & \xrightarrow[\ q\ ]{} & \underline{\mathcal{H}\mathrm{om}}(S^1, \mathcal{X})
\end{array}$$

which we can see by considering a component of the top left corner labelled by $V_1, V_2 \in C(S^1)$. The projection

$$X^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} (X^2)^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} \to (X^2)^{\check{C}(V_{12})}$$

can be unwound to give a natural transformation between the maps $q \circ s$ and $q \circ t$.

To show the above diagram is indeed a comma square, we shall show that for any Euclidean space $\mathbb{R}^n$ the diagram of groupoids

$$\begin{array}{ccc}
X^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} (X^{\mathbf{2}})^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)}(\mathbb{R}^n) & \longrightarrow & X^{\check{C}(V_1)}(\mathbb{R}^n) \\
\downarrow & \swArrow & \downarrow{\scriptstyle q} \\
X^{\check{C}(V_2)}(\mathbb{R}^n) & \xrightarrow[q]{} & \underline{\mathcal{H}\mathrm{om}}(S^1, \mathcal{X})(\mathbb{R}^n)
\end{array}$$

is a comma square. It is immediate that all but the bottom right corner are sets, so we need to show the canonical map

$$\begin{aligned}
c\colon X^{\check{C}(V_1)} &\times_{X^{\check{C}(V_{12})}} (X^{\mathbf{2}})^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)}(\mathbb{R}^n) \\
&\longrightarrow X^{\check{C}(V_1)} \downarrow_{\underline{\mathcal{H}\mathrm{om}}(S^1,\mathcal{X})(\mathbb{R}^n)} X^{\check{C}(V_2)}(\mathbb{R}^n)
\end{aligned}$$

is a bijection. These sets are as follows:

$$\begin{aligned}
&X^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} (X^{\mathbf{2}})^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)}(\mathbb{R}^n) \\
&\qquad \simeq \left\{ (f, \alpha, g) \;\middle|\; \begin{array}{ccc}
\mathbb{R}^n \times \check{C}(V_{12}) & \longrightarrow & \mathbb{R}^n \times \check{C}(V_1) \\
\downarrow & \overset{\alpha}{\swArrow} & \downarrow{\scriptstyle f} \\
\mathbb{R}^n \times \check{C}(V_2) & \xrightarrow[g]{} & X
\end{array} \right\}
\end{aligned}$$

and

$$\begin{aligned}
&X^{\check{C}(V_1)} \downarrow_{\underline{\mathcal{H}\mathrm{om}}(S^1,\mathcal{X})(\mathbb{R}^n)} X^{\check{C}(V_2)}(\mathbb{R}^n) \\
&\qquad \simeq \left\{ (\tilde{f}, \tilde{\alpha}, \tilde{g}) \;\middle|\; \begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{\tilde{f}} & X^{\check{C}(V_1)} \\
{\scriptstyle \tilde{g}}\downarrow & \overset{\tilde{\alpha}}{\swArrow} & \downarrow{\scriptstyle q} \\
X^{\check{C}(V_2)} & \xrightarrow[q]{} & \underline{\mathcal{H}\mathrm{om}}(S^1, \mathcal{X})
\end{array} \right\}.
\end{aligned}$$

It is not difficult to see that c must be injective; in particular f and g correspond to $\tilde{f}$ and $\tilde{g}$, respectively. Unravelling the description of $\tilde{\alpha}$ we can see it must arise from some α as in the first set, and so the map is bijective, and hence $X^{\check{C}(V_1)} \downarrow_{\underline{\mathcal{H}\mathrm{om}}(S^1,\mathcal{X})(\mathbb{R}^n)} X^{\check{C}(V_2)}$ is representable, by the component of $\mathrm{Map}(S^1,X)_1$ labelled by V_1, V_2.

Since q is an epimorphism, we then see that q is representable (by Lemma 2.11) and hence

Theorem 4.2. *For X a diffeological groupoid, the Hom-stack $\underline{\mathcal{H}\mathrm{om}}(S^1,X)$ is presented by the diffeological groupoid $\mathrm{Map}(S^1,X)$.*

Note that there was nothing special about S^1 in this argument: we only required S^1 to be a manifold in order for the proof of Proposition 4.1 to work. However for the next section the analysis is more delicate and so we have only treated the case of S^1.

5. Presentation by a Fréchet–Lie groupoid

The diffeological groupoid $\mathrm{Map}(S^1,X)$ can also be considered in the case that X is a Lie groupoid. In this section we will show that whenever X is a Lie groupoid, the diffeological groupoid $\mathrm{Map}(S^1,X)$ defined in section 3 is in fact a Fréchet–Lie groupoid (Theorem 5.9) and that it also weakly presents the Hom-stack $\underline{\mathcal{H}\mathrm{om}}(S^1,\mathcal{X})$ over the site of manifolds (Theorem 5.12).

From now on we will work with the coverage C as in section 3 but we will always use *minimal* covers of S^1 (those such that triple intersections are empty), which are cofinal in $C(S^1)$. We denote the set of these minimal covers by $C(S^1)_{\mathrm{min}}$ The object space is then

$$\coprod_{V\in C(S^1)_{\mathrm{min}}} X^{\check{C}(V)},$$

where again each component $X^{\check{C}(V)}$ is the space of (smooth) functors $\check{C}(V)\to X$. This is naturally described as the iterated pullback

$$X_0^{I_1}\times_{X_0^{J_1}} X_1^{J_1}\times_{X_0^{J_1}} X_0^{I_2}\times_{X_0^{J_2}} X_1^{J_2}\times_{X_0^{J_2}}\ldots\times_{X_0^{J_{n-1}}} X_0^{I_n}$$

where I_i are closed subintervals of S^1, $V = \coprod_{i=1}^n I_i$ and $J_i = I_i \cap I_{i+1}$, the maps

$$X_0^{I_i} \to X_0^{J_i} \leftarrow X_0^{I_{i+1}}$$

are given by restriction, and the maps $X_1^{J_i} \to X_0^{J_i}$ are induced by the source and target maps alternately. Here the I_i and J_i are intervals so that a functor $\check{C}(V) \to X$ consists of a series of paths $I_i \to X_0$ and a series of paths $J_i \to X_1$ that "patch together using source and target".

Recall that a pullback of a submersion in the category of Fréchet manifolds exists, and is again a submersion. Our strategy is to show that the maps above are all submersions, which will imply that the object space is a Fréchet manifold.

The following result of Stacey [Sta13] guarantees that the maps $X_1^{J_i} \to X_0^{J_i}$ are submersions; see also [AS17, Lemma 2.4].

Theorem 5.1 (Stacey)**.** *Let $M \to N$ be a submersion of finite-dimensional manifolds and K a compact manifold. Then the induced map of Fréchet manifolds $M^K \to N^K$ is a submersion.*

For the maps $X_0^{I_i} \to X_0^{J_i} \leftarrow X_0^{I_{i+1}}$ we will need the following theorem, which may be derived from the result in [See64] (see also [Mit61, §7], which essentially proves Corollary 5.3 directly).

Theorem 5.2 (Seely)**.** *The Fréchet space $(\mathbb{R}^n)^{\mathbb{R}_+}$ is a direct summand of $(\mathbb{R}^n)^{\mathbb{R}}$, where we take the topology of uniform convergence of all derivatives on compact subsets.*

Corollary 5.3. *The Fréchet space $(\mathbb{R}^n)^{[0,1]}$ is a direct summand of $(\mathbb{R}^n)^{[-1,1]}$, hence the restriction map $(\mathbb{R}^n)^{[-1,1]} \to (\mathbb{R}^n)^{[0,1]}$ is a submersion of Fréchet spaces. The same is true with $[0,1] \subset [-1,1]$ replaced with any inclusion $J \subset I$ of compact intervals.*

This allows us to prove

Proposition 5.4. *Let M be an n-dimensional manifold and $J \subset I$ two compact intervals. Then the restriction map $M^I \to M^J$ is a submersion of Fréchet manifolds.*

Proof. Let $f\colon I \to M$ be a smooth function, and denote by $f_J\colon J \to M$ its restriction along the inclusion. To show that $M^I \to M^J$ is a submersion, we need to find charts around f and f_J such that the map is a submersion of Fréchet spaces on those charts. Recall [Ham82, §I.4.1] that a chart around f is a neighbourhood of the zero section in $\Gamma(I, I \times_{f,M} TM)$, and similarly for f_J. Clearly $I \times_{f,M} TM \simeq I \times \mathbb{R}^n$, and given such an isomorphism we get an induced isomorphism $J \times_{f_J,M} TM \simeq J \times \mathbb{R}^n$ that is compatible with the restriction map. The induced map on spaces of sections,

$$(\mathbb{R}^n)^I = \Gamma(I, I \times \mathbb{R}^n) \to \Gamma(J, J \times \mathbb{R}^n) = (\mathbb{R}^n)^J,$$

is just the obvious restriction map, and this map is locally the same, after unwinding the isomorphisms just given, to the restriction map. But Corollary 5.3 says that this map is a submersion, as we needed. □

Proposition 5.4 implies that the maps $X_0^{I_i} \to X_0^{J_i} \leftarrow X_0^{I_{i+1}}$ are submersions and hence we have

Proposition 5.5. *For X a Lie groupoid, the object space $\mathrm{Map}(S^1, X)_0$ is a Fréchet manifold.*

To see that the set of arrows has a manifold structure as well, recall that this set is given by

$$\coprod_{V_1, V_2 \in C(S^1)_{\mathrm{min}}} X^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} (X^2)^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)}$$

where the chosen refinement V_{12} is also a minimal cover of S^1. To use the same reasoning as above we need to know that the maps

$$(X^2)^{\check{C}(V_{12})} \to X^{\check{C}(V_{12})},$$

induced by $S, T\colon X^2 \to X$, and

$$X^{\check{C}(V_i)} \to X^{\check{C}(V_{12})}$$

$(i = 1, 2)$ are submersions.

Now for $M \to N$ a map of finite-dimensional manifolds, and $C \to D$ a map of compact manifolds with boundary, the two induced maps

$$M^C \to N^C, \quad \text{and} \quad M^D \to M^D$$

have a rather nice property in that on certain canonical charts they are actually *linear* maps (recall that these maps above look locally like maps between spaces of sections induced by vector bundle maps). More generally one can consider larger diagrams, all of whose maps have this local linearity, and further the charts exhibiting this local behaviour can all be chosen compatibly. Such a diagram will be called be called *locally linear*.

An example of such a diagram is one where all the objects are mapping spaces as above, and all arrows are induced by pre- or post-composition as above. A much simpler and familiar example would be in the finite-dimensional setting, where the exponential map is a local diffeomorphism. The induced diagram on tangent spaces, for any compatible system of basepoints, is then a diagram of vector spaces.

We have the following Lemma:

Lemma 5.6. *Let*

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longleftarrow & A_3 & \longrightarrow & \cdots & \longleftarrow & A_n \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
B_1 & \longrightarrow & B_2 & \longleftarrow & B_3 & \longrightarrow & \cdots & \longleftarrow & B_n
\end{array}$$

be a diagram of submersions that is locally linear. Then the natural map

$$\lim A_i \to \lim B_i,$$

where the limits are iterated fibre products, is also a submersion.

Proof. The local linearity of the diagram means that we can find a diagram of the same shape in the category of Fréchet *spaces* and linear maps, and in fact *split* linear maps, since all of the maps are submersions, hence locally split. Then the proof that the induced map is a split submersion of Fréchet spaces proceeds exactly as one would in the finite-dimensional case. One can induct on the length of the zig-zags and so reduce to the case of a diagram

$$\begin{array}{ccccc} A_1 & \longrightarrow & A_2 & \longleftarrow & A_3 \\ \downarrow & & \downarrow & & \downarrow \\ B_1 & \longrightarrow & B_2 & \longleftarrow & B_3 \end{array}$$

in the category of Fréchet spaces and linear maps and then show that one can find a section of the linear map $A_1 \times_{A_2} A_3 \to B_1 \times_{B_2} B_3$. □

Let $X \to Y$ be a functor between Lie groupoids such that the object and arrow components are submersions. We call such a functor *submersive*. We have the following result.

Lemma 5.7.

1. *Let $X \to Y$ be a submersive functor between Lie groupoids. Then the induced map*
$$X^{\check{C}(V)} \to Y^{\check{C}(V)}$$
is a submersion.
2. *Let X be a Lie groupoid and $V_1 \to V_2$ be a refinement of minimal covers. Then the induced map*
$$X^{\check{C}(V_2)} \to X^{\check{C}(V_1)}$$
is a submersion.

Proof. The first part follows from Theorem 5.1 and Lemma 5.6 and the second follows from Proposition 5.4 and Lemma 5.6. □

Lemma 5.7 implies that the maps above are submersions and so we have

Proposition 5.8. *For X a Lie groupoid, the arrow space $\mathrm{Map}(S^1, X)_1$ is a Fréchet manifold.*

Now happily, the source and target map for our Fréchet–Lie groupoid are given, on each component of the arrow Fréchet manifold, by the two projections

$$X^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} \left(X^{\mathbf{2}}\right)^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} \to X^{\check{C}(V_i)}$$

where $i = 1, 2$, which are submersions. Therefore

Theorem 5.9. *For X a Lie groupoid,* $\mathrm{Map}(S^1, X)$ *is a Fréchet–Lie groupoid.*

Observe that $\mathbb{L}X$ is built by taking disjoint unions of pullbacks of smooth path spaces, and smooth path spaces are metrisable and smoothly paracompact (as they are nuclear Fréchet spaces). By a combination of Lemma 27.9 and the comments in §27.11 of [KM97], the pullback $M_1 \times_N M_2$, where M_1, M_2 are metrisable smoothly paracompect and where at least one of $M_i \to N$ is a submersion, is smoothly paracompact. Thus by induction the iterated pullback that defines $X^{\check{C}(V)}$ is a smoothly paracompact manifold, and so the object and arrow manifolds of $\mathbb{L}X$ are smoothly paracompact. This means that every open cover admits subordinate *smooth* partitions of unity, and so any geometric constructions with smooth objects (differential forms and so on) can be built locally.

In fact the spaces $\mathbb{L}X_n$ of sequences of n composable arrows are also paracompact, so that $\mathbb{L}X$ is a paracompact groupoid in the terminology of Gepner–Henriques. As a result we know that the fat geometric realisation $||\mathbb{L}X||$ of the nerve of $\mathbb{L}X$ is a paracompact space [GH07, Lemma 2.25].

The following Proposition means that the endo-2-functor on stacks on $\mathcal{M}$ lifts to a 2-functor on *presentations* of stacks. It is thus a kind of rigidification of the loop stack functor.

Proposition 5.10. *The assignment $X \mapsto \mathbb{L}X$ extends to a 2-functor*

$$\mathbb{L}\colon\ \mathbf{Gpd}(\mathcal{M}) \to \mathbf{Gpd}(\mathcal{F}).$$

Proof. Given a functor $f\colon X \to Y$ between Lie groupoids, we clearly get a functor $\mathbb{L}f\colon \mathbb{L}X \to \mathbb{L}Y$ between Fréchet–Lie groupoids, by composing everything in sight with f. Moreover, given a second functor $k\colon Y \to Z$, we clearly have $\mathbb{L}(kf) = \mathbb{L}k\,\mathbb{L}f$.

Assume now that we have a natural transformation $\alpha\colon f \Rightarrow g\colon X \to Y$, or in other words a functor $X \to Y^{\mathbf{2}}$. We need to show that this induces a natural transformation $\mathbb{L}f \Rightarrow \mathbb{L}g$, which is determined by the data of a smooth map

$$\mathrm{Map}(S^1, X)_0 \to \mathrm{Map}(S^1, Y)_1.$$

We first need to describe this map on the level of underlying sets. Let $S^1 \leftarrow \check{C}(V) \xrightarrow{h} X$ be an anafunctor. The value of the natural transformation

$\mathbb{L}\alpha\colon \mathbb{L}f \Rightarrow \mathbb{L}g$ at h is a transformation of anafunctors

$$\mathbb{L}\alpha(h)\colon (\check{C}(V) \xrightarrow{fh} Y) \Rightarrow (\check{C}(V) \xrightarrow{gh} Y)$$

and so lives in the component

$$Y^{\check{C}(V)} \times_{Y^{\check{C}(V)}} (Y^{\mathbf{2}})^{\check{C}(V)} \times_{Y^{\check{C}(V)}} Y^{\check{C}(V)} \simeq (Y^{\mathbf{2}})^{\check{C}(V)}$$

Moreover, the transformation $\mathbb{L}\alpha(h)$ is simply the left whiskering of α by the functor h. Thus $\mathbb{L}\alpha$ is given (on one component) by the map

$$X^{\check{C}(V)} \to (Y^{\mathbf{2}})^{\check{C}(V)},$$

induced by composition with the given $X \to Y^{\mathbf{2}}$, hence the component map of the natural transformation $\mathbb{L}f \Rightarrow \mathbb{L}g$ is smooth.

Now it remains to show firstly that $\mathbb{L}\alpha$ is natural, and secondly that this is functorial for both compositions of 2-cells. Naturality follows from the proof that anafunctors are 1-cells in a bicategory, and that functors are 1-cells in the locally full sub-bicategory $\mathbf{Gpd}(\mathcal{M})$. Functoriality follows from the fact whiskering is a functorial process. □

Remark 5.11. The 2-functor $\mathbb{L}\colon \mathbf{Gpd}(\mathcal{M}) \to \mathbf{Gpd}(\mathcal{F})$ preserves products up to weak equivalence. This follows formally using the equivalence between differentiable stacks and Lie groupoids and anafunctors, and the fact that the product of differentiable stacks is presented by the product of Lie groupods. However we actually have a slightly more rigid result, with the coherence functor (in one direction) being the canonical inclusion

$$\mathbb{L}(X \times Y) \hookrightarrow \mathbb{L}X \times \mathbb{L}Y,$$

rather than some comparison anafunctor. This has a quasi-inverse *functor* that takes a pair of objects, in summands indexed by the covers V_1 and V_2 respectively, to the isomorphic pair indexed by the same cover V_{12}, the chosen common refinement of V_1 and V_2.

Now the construction of the map q from section 4 is identical, we need to additionally show that it is a submersion. There is a small subtlety here, in that we haven't been able to show directly that q is a representable map of stacks, rather we will rely on (a submersion variant of) the weaker notion of presentation from [Pro96, §6.2.0.1], which only requires that the comma object of q with itself gives a submersion between manifolds. Since we know the comma object $q \downarrow q$ is already a manifold, namely the arrow space $\mathrm{Map}(S^1, X)_1$, and the projections are the source and target maps, which are submersions, then we have our first main result.

Theorem 5.12. *For X a Lie groupoid, the Hom-stack $\underline{\mathcal{H}\mathrm{om}}(S^1, \mathcal{X})$ is weakly presented by the Fréchet–Lie groupoid $\mathrm{Map}(S^1, X)$.*

As the stack $\underline{\mathcal{H}\mathrm{om}}(M, \mathcal{X})$ is presented by a paracompact groupoid it is well-behaved homotopically. Proposition 8.5 in [Noo12] ensures that since $\mathbb{L}X$ has object and arrow manifolds metrisable, $\underline{\mathcal{H}\mathrm{om}}(M, \mathcal{X})$ has a *hoparacompact* underlying topological stack. Then the classifying space of $\underline{\mathcal{H}\mathrm{om}}(M, \mathcal{X})$ (as defined in [Noo12]) is well-defined up to homotopy equivalence, rather than weak homotopy equivalence.

We note that with minor modifications, one can repeat the above analysis for the case of the mapping stack $\underline{\mathcal{H}\mathrm{om}}([0, 1], \mathcal{X})$, but we leave that to the interested reader.

6. Recap on differentiable gerbes

Definition 6.1. *A (Fréchet-)Lie groupoid $X \to M$ is a* gerbe *if $\pi\colon X_0 \to M$ and $(s, t)\colon X_1 \to X_0^{[2]}$ are surjective submersions. The stack on $\mathcal{M}$ that such a groupoid (weakly) presents will be called a (Fréchet-)differentiable gerbe.*

Equivalently, we can require that $X \to M$ and $X \to \check{C}(X_0)$ are submersive functors that are surjective on objects and arrows. We rephrase these properties in terms of functors rather than component maps because later we wish to prove stability of these properties under forming mapping groupoids.

Remark 6.2. In this section the results also apply to general Fréchet–Lie groupoids, even though we have only stated them for Lie groupoids for brevity.

Because (s, t) is a submersion the pullback $\Lambda X := \Delta^* X_1 \to X_0$, for $\Delta\colon X_0 \to X_0^{[2]}$ the diagonal, is a bundle of Lie groups. We shall call ΛX

the *inertia bundle*. If the fibre $\Lambda X_x \simeq G$ for every $x \in X_0$ then this gives a G*-gerbe* in the sense of [LGSX09], that is, an extension of groupoids

$$\Lambda X \to X \to \check{C}(X_0).$$

However, we wish to use a mental picture as close to bundle gerbes [Mur96] as possible, so offer the following diagram encoding a gerbe $X \to M$:

$$\begin{array}{ccc} X_1 & & \Lambda X \\ \downarrow & & \downarrow \\ X_0^{[2]} & \rightrightarrows & X_0 \\ & & \downarrow \\ & & M \end{array}$$

We have left and right actions of ΛX on X_1, or rather a left action of $\Lambda X_L := \mathrm{pr}_1^* \Lambda X$ and a right action of $\Lambda X_R := \mathrm{pr}_2^* \Lambda X$ on X_1, preserving the fibres of (s,t), by composition in the groupoid X. This makes $X_1 \to X_0^{[2]}$ a principal ΛX_L-ΛX_R-bibundle. Notice that X_1 is locally isomorphic to ΛX_L and to ΛX_R (as spaces over $X_0^{[2]}$) using local sections of (s,t).

There is also an action of X as a groupoid on the family $\Lambda X \to X_0$, covering the action of X on X_0. This is by conjugation in the groupoid: if $f\colon x \to y \in X_1$ and $\alpha \in \Lambda X_x$, then $f^{-1}\alpha f \in \Lambda X_y$, where we are using the diagrammatic (or algebraic) composition order. This defines a smooth map

$$\Lambda X \times_{X_0,s} X_1 \to \Lambda X$$

over X_0, using the target map composed with the second projection on the domain. We also want to think of this in the equivalent form of

$$\Lambda X_L \times_{X_0^{[2]}} X_1 \to \Lambda X_R,$$

a map over $X_0^{[2]}$. This action defines an action groupoid $\Lambda X//X$ with objects ΛX and morphisms $\Lambda X_L \times_{X_0^{[2]}} X_1$. This groupoid will become important for calculations in the next section. We will denote an object and an arrow of $\Lambda X//X$ by

$$\alpha \circlearrowright x \qquad \text{and} \qquad \alpha \circlearrowright x \xrightarrow{f} y \ ,$$

respectively. The action of X on ΛX, that is, the target map of $\Lambda X//X$, is

$$\alpha \circlearrowright x \xrightarrow{f} y \qquad \longmapsto \qquad f^{-1}\alpha f \circlearrowright y \tag{1}$$

For the purposes of being confident that various pullbacks exist in what follows, we record some trivial consequences of the conditions on the definition of a gerbe. Note that the surjectivity requirements are superfluous at this point, but will become important later.

Lemma 6.3. *For a Lie groupoid X with submersive functors $X \to M$ and $X \to \check{C}(X_0)$, the following functors are also submersive:*

1. $(S,T)\colon X^{\mathbf{2}} \to X \times_M X$
2. $\mathrm{pr}_i\colon X \times_M X \to X$ *for* $i = 1, 2$
3. $S, T\colon X^{\mathbf{2}} \to X$

While there may be some utility in maintaining extra generality at this point, our results will ultimately be applied in the case that ΛX is a bundle of *abelian* Lie groups. Thus from now on we make this assumption. Note however that an *abelian gerbe* in the sense of [Bre94, Definition 2.9] is more restrictive than simply demanding ΛX_x is abelian for every $x \in X_0$. We will get to this type of gerbe soon (see Definition 6.6 below)

We are also interested primarily in the case that $\Lambda X \to X_0$ descends to M. This means that there is an isomorphism

$$\phi\colon \Lambda X_L \xrightarrow{\simeq} \Lambda X_R$$

over $X_0^{[2]}$ which satisfies the cocycle condition over $X_0^{[3]}$. We will refer to ϕ as the *descent isomorphism* for ΛX. We can denote this isomorphism by

$$\begin{array}{ccc} \alpha \circlearrowright x & & x \\ & \overset{\simeq}{\longmapsto} & \\ y & & \phi(\alpha) \circlearrowright y \end{array}$$

where $(x, y) \in X_0 \times_M X_0$. There is then a bundle of groups $\mathcal{A} \to M$ such that $\pi^*\mathcal{A} \simeq \Lambda X$. Another way to phrase this is that there is an action

$\Lambda X \times_{X_0,\mathrm{pr}_1} X_0^{[2]} = \Lambda X_L \to \Lambda X$ of the groupoid $\check{C}(X_0)$ on ΛX, and hence we have an action groupoid $\Lambda X//\check{C}(X_0)$ with arrows ΛX_L. This has a projection map to $\check{C}(X_0)$ making $\Lambda X//\check{C}(X_0) \to \check{C}(X_0)$ a bundle of groups object in the category of Lie groupoids.

Lemma 6.4. *If ΛX descends to $\mathcal{A}$ on M, the following square is a pullback of Lie groupoids*

$$\begin{array}{ccc} \Lambda X//\check{C}(X_0) & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \\ \check{C}(X_0) & \longrightarrow & M\,. \end{array}$$

Proof. We can verify this by looking at the level of objects and arrows, individually. The object manifold of $\Lambda X//\check{C}(X_0)$ is ΛX, and by assumption this is isomorphic to $X_0 \times_M \mathcal{A}$, as needed. The square at the level of arrow manifolds is

$$\begin{array}{ccc} \Lambda X \times_{X_0,\mathrm{pr}_1} (X_0 \times_M X_0) & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \\ X_0 \times_M X_0 & \longrightarrow & M \end{array}$$

and so we need to show that the induced map

$$\Lambda X \times_{X_0,\mathrm{pr}_1} (X_0 \times_M X_0) \to (X_0 \times_M X_0) \times_M \mathcal{A} \tag{2}$$

is an isomorphism. But $\Lambda X \simeq X_0 \times_M \mathcal{A}$, so (2) is just the canonical isomorphism rearranging the factors of a iterated pullback. □

We can hence talk about *$\mathcal{A}$-gerbes on M* for a fixed bundle of abelian groups $\mathcal{A} \to M$, and we will restrict attention to this case from now on. The bundle $\mathcal{A}$ will be referred to as the *structure group bundle*.

To go further and talk about *abelian* $\mathcal{A}$-gerbes we need to say what it means for the left and right actions of ΛX_L and ΛX_R on X_1 to agree. In the special case that $\Lambda X = X_0 \times A$, then $\Lambda X_L = X_0^{[2]} \times A = \Lambda X_R$, and we could ask that the A-A-bibundle X_1 is in fact just an A-bundle, with the right action equal to the left action.

In the case that $\Lambda X = \pi^*\mathcal{A}$ is non-trivial, the best we can do is identify ΛX_L with ΛX_R via the given descent isomorphism, and ask that *relative to* *this identification*, the left and right actions agree. There are two ways to look at this agreement, from the point of view of the actions of ΛX_L, ΛX_R on X_1, or the action of X on ΛX. However, we want to also introduce a third way, that uses a more global, groupoid-based approach to be used in the next section.

Recall that $X^{\mathbb{2}}$ is the arrows of a groupoid object in Lie groupoids—that is, a double groupoid. There is a groupoid action *in the category of Lie groupoids*

$$\Lambda X//X \times_{X,S} X^{\mathbb{2}} \to \Lambda X//X$$

with the object component of this functor given by equation (1). The arrow component is given by

$$\left(\alpha \circlearrowright x \xrightarrow{g} y \ , \ \begin{array}{ccc} x & \xrightarrow{g} & y \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ z & \xrightarrow[h]{} & w \end{array} \right) \quad \longmapsto \quad f^{-1}\alpha f \circlearrowright z \xrightarrow{h} w$$

We remind the reader that here notation for the conjugation action is using the diagrammatic order for composition.

While this seems to iterate our data to another level of complexity, this allows us to consider stability of structures under the functor

$$(-)^{\check{C}(V)} \colon \mathbf{Gpd}(\mathcal{M}) \to \mathcal{F}.$$

In particular, since $(-)^{\check{C}(V)}$ preserves products and even pullbacks of submersive functors, for a bundle of groups $\mathcal{G} \to X$ in $\mathbf{Gpd}(\mathcal{M})$ (considered as a 1-category), $\mathcal{G}^{\check{C}(V)} \to X^{\check{C}(V)}$ is a bundle of Fréchet–Lie groups. This will allow a calculation of the structure group bundle of the (putative) gerbe $\mathbb{L}X$, once we prove that it is in fact a gerbe.

Lemma 6.5. *For X an $\mathcal{A}$-gerbe, the following are equivalent:*

1. The diagram

$$\begin{array}{ccccc} \Lambda X_L \times_{X_0^{[2]}} X_1 & \xrightarrow{\simeq} & \Lambda X \times_{X_0,s} X_1 & & \\ \Big\downarrow{\scriptstyle \phi\times \mathrm{id}_{X_1}} & & & \searrow & \\ & & & & X_1 \\ & & & \nearrow & \\ \Lambda X_R \times_{X_0^{[2]}} X_1 & \xrightarrow{\simeq} & X_1 \times_{t,X_0} \Lambda X & & \end{array}$$

sitting over $X_0^{[2]}$ commutes ("the right and left actions of ΛX on X_1 agree");

2. The conjugation action of X on ΛX factors through the action of $\check{C}(X_0)$ on ΛX, via the projection $X \to \check{C}(X_0)$;

3. The action of $X^{\mathbf{2}}$ on $\Lambda X//X$ factors through an action

$$\Lambda X//X \times_{X,\mathrm{pr}_1} (X \times_M X) \to \Lambda X//X, \tag{3}$$

of the double groupoid $X \times_M X \rightrightarrows X$ on $\Lambda X//X$, in the category of Lie groupoids, whose object component is the descent isomorphism ϕ for ΛX, via the functor $(S,T)\colon X^{\mathbf{2}} \to X \times_M X$.

Proof. We will first prove that 1. and 2. are equivalent. The implication 3.⇒2. is immediate because 2. is merely the object component of 3. We will then show how 3. follows from 2.

The diagram in 1. commuting means that for all $(\alpha, f) \in \Lambda X_L \times_{X_0^{[2]}} X_1$, $\alpha f = f\phi(\alpha)$. In other words, that $\phi(\alpha) = f^{-1}\alpha f$, but this is precisely what it means for the action of X on ΛX to factor through the action of $\check{C}(X_0)$ on ΛX, and so 1.⇔2.

To prove that 2. implies 3., we need first to describe an action as in (3) with object component $\Lambda X_L = \Lambda X \times_{X_0,\mathrm{pr}_2} X_0^{[2]} \xrightarrow{\phi} \Lambda X_R \xrightarrow{\mathrm{pr}} \Lambda X$. If 2. holds then

$$\begin{array}{ccccc} {\scriptstyle\alpha}\circlearrowright x & & x & & x \\ & \overset{\simeq}{\longmapsto} & & = & \\ y & & {\scriptstyle\phi(\alpha)}\circlearrowright y & & {\scriptstyle f^{-1}\alpha f}\circlearrowright y \end{array} \tag{4}$$

for any $f\colon x \to y \in X_1$. Hence we can define the arrow component of (3) by

$$\left(\alpha \circlearrowleft x \xrightarrow{g} y \;,\; \begin{array}{c} x \xrightarrow{g} y \\ \\ z \xrightarrow[h]{} w \end{array} \right) \qquad \longmapsto \qquad \phi(\alpha) \circlearrowleft z \xrightarrow{h} w$$

which is indeed a functor by virtue of (4), and the fact it is an action follows from the cocycle identity for ϕ. The action of X^2 on $\Lambda X//X$ factors through this action by construction.

The arrow component of the action in 3. is in fact determined uniquely, rather than merely being 'an' action, since 3. implies 2. and then one can construct the required arrow component of the action functor. $\square$

Note that in particular that if the conditions of the lemma are satsified there is a functor $\Lambda X//X \to \Lambda X//\check{C}(X_0)$ sitting over $X \to \check{C}(X_0)$.

Definition 6.6. *We call an $\mathcal{A}$-gerbe $X \to M$* abelian *if the equivalent conditions of Lemma 6.5 hold.*

Lemma 6.7. *For an abelian $\mathcal{A}$-gerbe $X \to M$, the left and hence all squares below are pullbacks of Lie groupoids*

$$\begin{array}{ccccc} \Lambda X//X & \longrightarrow & \Lambda X//\check{C}(X_0) & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \check{C}(X_0) & \longrightarrow & M\,. \end{array}$$

In particular, $\Lambda X//X \to X$ is a bundle of groups object in the category of Lie groupoids.

Proof. Since X is an abelian $\mathcal{A}$-gerbe, and hence an $\mathcal{A}$-gerbe, the right square is a pullback by Lemma 6.4. By the pullback pasting lemma, the left square is a pullback if and only if the outer rectangle is a pullback; we shall prove the former. The object components of the top and bottom horizontal functors in the left square are identity maps $\mathrm{id}_{\Lambda X}$ and id_{X_0} respectively, and the left and right vertical maps are both $\Lambda X \to X_0$, hence on objects the left square is a pullback. The morphism components of the left square give the square

$$\begin{array}{ccc} \Lambda X_L \times_{X_0^{[2]}} X_1 & \xrightarrow{\mathrm{id}\times(s,t)} & \Lambda X_L \times_{X_0^{[2]}} X_0^{[2]} \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ X_1 & \xrightarrow[(s,t)]{} & X_0^{[2]}, \end{array}$$

which is manifestly a pullback. □

Example 6.8. Let A be an abelian Lie group. An A-bundle gerbe on M in the sense of [Mur96] is an abelian $M \times A$ gerbe $X \to M$. Most often one just considers the case[4] of $A = U(1)$ or $\mathbb{C}^\times$. Note that the local triviality of the A-bundle $X_1 \to X_0 \times_M X_0$ follows from the rest of the definition, as it is a surjective submersion, hence has local sections, and has an action by A that is free and transitive on fibres.

Example 6.9. In [HMSV13] the second-named author and collaborators considered 'bundle gerbes with non-constant structure group bundle'. This is a case intermediate between gerbes as defined here and ordinary bundle gerbes as in [Mur96], requiring that $\mathcal{A}$ is a locally trivial bundle of groups, and $X_1 \to X_0^{[2]}$ is locally trivial in a way compatible with the induced local trivialisations of $\pi^*\mathcal{A}$. The main nontrivial example of [HMSV13] is however infinite-dimensional, meaning the results of the present paper can only be applied if we consider it as a diffeological groupoid.

In fact, assuming $\mathcal{A}$ is a locally trivial bundle of groups has consequences for the structure of abelian $\mathcal{A}$-gerbes.

Lemma 6.10. *Let $\mathcal{A} \to M$ be a locally trivial bundle of abelian groups. Then for any abelian $\mathcal{A}$-gerbe $X \to M$, the map $(s,t)\colon X_1 \to X_0 \times_M X_0$ is a locally trivial bundle.*

[4]There is also a version of bundle gerbes where $X_1 \to X_0^{[2]}$ is a *line bundle*, rather than a principal bundle. This is captured in our framework if we allow for Lie groupoids that are enriched over a monoidal category of smooth objects, in this case the category $\mathrm{Lines}_{\mathbb{C}}$ of complex lines with the usual tensor product. Asking that X is enriched over $\mathrm{Lines}_{\mathbb{C}}$ in this internal setting is nothing other than asking that $X_1 \to X_0 \times X_0$ is a line bundle over its image.

Proof. The locally trivial bundles of groups ΛX_L and ΛX_R—pullbacks of $\mathcal{A}$—act principally on X_1 (that is: freely, and transitively on the fibres of (s,t)), and (s,t) admits local sections as it is a submersion. From these local sections and local trivialisations of, say ΛX_L, we can construct local trivialisations of $X_1 \to X_0 \times_M X_0$ □

We end with a final technical lemma used in the next section, but of independent interest.

Lemma 6.11. *The bundle of groups $\Lambda X//X \to X$ (internal to Lie groupoids) is the pullback of $(S,T)\colon X^{\mathbb{2}} \to X\times X$ along the diagonal $\Delta\colon X \to X\times X$.*

Proof. On the level of objects this says that ΛX is the pullback of X_1 along $X_0 \to X_0 \times X_0$, which is true by definition. The arrow manifold of $X^{\mathbb{2}}$ can be described as the pullback of $(s,t)\colon X_1 \to X_0^2$ along $(s,s)\colon X_1^2 \to X_0^2$. In this description, the (arrow component of the) source functor S projects on the first factor of X_1^2, and the (arrow component of the) target functor T projects on the other factor. Thus the pullback of $X_1 \times_{X_0^2} X_1^2$ along the diagonal $X_1 \to X_1 \times X_1$ forces the last two components to be equal, and hence that the middle factor must be ΛX, and the pullback is $X_1 \times_{s,X_0} \Lambda X$ which is the arrow manifold of $\Lambda X//X$. □

7. The loop stack of a gerbe

This section shows that given a differentiable gerbe $\mathcal{X}$ on a manifold M presented by a Lie groupoid X satisfying (a) a connectedness property for its automorphism groups $X(x,x)$ and (b) a weak form of local triviality of $X_1 \to X_0 \times_M X_0$; then the loop stack is again a (Fréchet) differentiable gerbe. An example of such a groupoid is a bundle gerbe (see below), in which case (s,t) is the projection map for a principal bundle.

In the following, denote $\mathrm{Map}(S^1, X)$ by $\mathbb{L}X$. We will also denote $(\mathbb{L}X)_i$, i.e. the object and arrow manifolds, simply by $\mathbb{L}X_i$.

Proposition 7.1. *Let X be a Lie groupoid with a submersive functor $X \to \mathrm{disc}(M)$ such that the resulting map $X_1 \to X_0 \times_M X_0$ is a submersion. Then $\mathbb{L}X \to \mathrm{disc}(LM)$ is submersive and $(s,t)^{\mathbb{L}X}\colon \mathbb{L}X_1 \to \mathbb{L}X_0 \times_{LM} \mathbb{L}X_0$ is a submersion.*

Note that we do not need to assume that X presents a gerbe on M, so that the result will be applicable to more general bundles of groupoids, in particular those whose fibres are not necessarily transitive.

Proof. Firstly, as $X \to \mathrm{disc}(M)$ is submersive we have the composite map $X^{\check{C}(V)} \to \mathrm{disc}(M)^{\check{C}(V)} \to LM$ a submersion (Lemma 5.7 parts 1 and 3), and so applying Lemma 5.6 we get that $\mathbb{L}X_0 \to LM$ is a submersion. Thus we know $\mathbb{L}X_0 \times_{LM} \mathbb{L}X_0$ is a Fréchet manifold.

The crux of the proof to show $(s,t)^{\mathbb{L}X}$ is a submersion is in finding an isomorph[5] of the map $(s,t)^{\mathbb{L}X}$ in such a way that Lemmata 5.6 and 5.7 can be applied.

Firstly notice that we can work with $(s,t)^{\mathbb{L}X}$ over each component of its domain and codomain, which are indexed by pairs V_1, V_2 of covers of S^1. This is because the disjoint union of submersions is again a submersion. Hence we are only dealing with the map

$$X^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} (X^{\mathbf{2}})^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} \longrightarrow X^{\check{C}(V_1)} \times_{LM} X^{\check{C}(V_2)} \quad (5)$$

which is projection on the first and third factors of the domain. There are isomorphisms

$$\begin{aligned} X^{\check{C}(V_1)} \times_{LM} X^{\check{C}(V_2)} &\simeq X^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} \left(X^{\check{C}(V_{12})} \times_{LM} X^{\check{C}(V_{12})} \right) \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} \\ &\simeq X^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} (X \times_M X)^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} \end{aligned}$$

which arise from the isomorphisms

$$X^{\check{C}(V_{12})} \times_{LM} X^{\check{C}(V_{12})} \simeq (X \times_M X)^{\check{C}(V_{12})}$$

and $\mathrm{disc}(M)^{\check{C}(V_{12})} \simeq LM$.

Now we have the following isomorph of (5):

$$\begin{array}{c} X^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} (X^{\mathbf{2}})^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} \\ \downarrow \\ X^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} (X \times_M X)^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} \end{array} \quad (6)$$

[5] An *isomorph* of a map $f\colon A \to B$ is a map $g\colon A' \to B'$ such that there are isomorphisms $A \simeq A'$ and $B \simeq B'$ making the resulting square commute. It is obvious that the isomorph of a submersion is a submersion.

which is the map induced from the map

$$(X^{\mathbb{2}})^{\check{C}(V_{12})} \longrightarrow (X \times_M X)^{\check{C}(V_{12})} \tag{7}$$

by interated pullback. This is, in turn, induced by applying the functor $(-)^{\check{C}(V_{12})}$ to the internal functor

$$(S, T)\colon X^{\mathbb{2}} \to X \times_M X,$$

which is submersive by Lemma 6.3. We can then apply Lemma 5.7.1 to see that the map (7) is a submersion.

Now notice that we can apply Lemma 5.7.2 to the maps $X^{\check{C}(V_i)} \to X^{\check{C}(V_{12})}$ $(i = 1, 2)$ to see they are submersions. It also follows from Lemma 6.3 together with Lemma 5.7 that the two maps $(X^{\mathbb{2}})^{\check{C}(V_{12})} \to X^{\check{C}(V_{12})}$ induced by $S, T\colon X^{\mathbb{2}} \to X$, and the two maps $(X \times_M X)^{\check{C}(V_{12})} \to X^{\check{C}(V_{12})}$ induced by the two projections are submersions. Now we can apply Lemma 5.6, as the diagram giving the iterated pullback defining the map (6) to get the desired result, namely that (6) is a submersion. □

Let us say a gerbe has *connected stabilisers* if $\Lambda X \to X_0$ is a bundle of connected groups. It then follows that $X_1 \to X_0 \times_M X_0$ has connected fibres, and the group ΛX_x acts simply transitively on all fibres $(s,t)^{-1}(x, y)$.

Call a submersion $E \to B$ *curvewise trivial* if for every map $\eta\colon [a, b] \to B$, the projection $\eta^* E \to [a, b]$ is isomorphic to a trivial bundle $[a, b] \times F \to [a, b]$. For a gerbe X that has connected stabilisers, if (s, t) is curvewise trivial then the manifold F is connected.

A gerbe X that has (s, t) curvewise trivial satisfies the property that a lift, as shown in the diagram

$$\begin{array}{ccc} [a, c] & \longrightarrow & X_1 \\ \downarrow & \nearrow & \downarrow{\scriptstyle (s,t)} \\ [a, b] & \longrightarrow & X_0 \times_M X_0 \end{array}$$

always exists, for $c \in [a, b)$. If the gerbe additionally has connected stabilisers, then there is always a lift as in this diagram:

$$\begin{array}{ccc} [a,c]\coprod[d,b] & \longrightarrow & X_1 \\ \downarrow & \nearrow & \downarrow{\scriptstyle (s,t)} \\ [a,b] & \longrightarrow & X_0 \times_M X_0 \end{array}$$

for $a < c < d < b$. Both of these follow from the ability to extend functions $[a,c] \to F$ (respectively $[a,c]\coprod[d,b] \to F$) to $[a,b]$, using Corollary 5.3 (and the fact F is connected in the latter case).

Lemma 7.2. *Let M be a finite-dimensional manifold and X a Lie groupoid that is a gerbe on M with (s,t) curvewise trivial, then $\mathbb{L}X_0 \to LM$ is a surjective submersion. If additionally X has connected stabilisers then $(s,t)^{\mathbb{L}X} \colon \mathbb{L}X_1 \to \mathbb{L}X_0 \times_{LM} \mathbb{L}X_0$ is a surjective submersion.*

Proof. For the first statement, note that it is immediate that $\mathbb{L}\check{C}(X_0) \to LM$ is surjective, because since $X_0 \to M$ is a surjective submersion, it has local sections which can be used to lift locally any loop $\gamma\colon S^1 \to M$. Then to show that $\mathbb{L}X \to \mathbb{L}\check{C}(X_0)$ is surjective, we need to use the first assumption on (s,t).

Note that we only need to show we can lift paths $[a,b] \to X_0 \times_M X_0$ through $(s,t)\colon X_1 \to X_0 \times_m X_0$, where $[a,b] \subset \check{C}(V)_1$; there are no compatibility conditions. But note that since X_1 trivialises after pulling back to $[a,b]$, one can just use a section to lift paths as needed. Thus $\mathbb{L}X \to \mathbb{L}\check{C}(X_0)$ is surjective, and so the first claim follows.

For the second claim we only need to prove that $(s,t)^{\mathbb{L}X}$ is surjective (it is already a submersion), so consider a single component $X^{\check{C}(V_1)} \times_{LM} X^{\check{C}(V_2)} \subset \mathbb{L}X_0 \times_{LM} \mathbb{L}X_0$. It suffices to prove that $(X^{\mathbf{2}})^{\check{C}(V_{12})} \to X^{\check{C}(V_{12})} \times_{LM} X^{\check{C}(V_{12})} \simeq (X \times_M X)^{\check{C}(V_{12})}$ is surjective, since $(s,t)^{\mathbb{L}X}$ is a disjoint union of pullbacks of such maps. Write $V = V_{12}$, and consider $\gamma = (\gamma_1,\gamma_2)\colon \check{C}(V) \to X \times_M X$. We need to find a lift $\widehat{\gamma}$ as in the diagram:

$$\begin{array}{ccc} & & X^{\mathbf{2}} \\ & \overset{\widehat{\gamma}}{\nearrow} & \downarrow \\ \check{C}(V) & \underset{\gamma}{\longrightarrow} & X \times_M X \end{array}$$

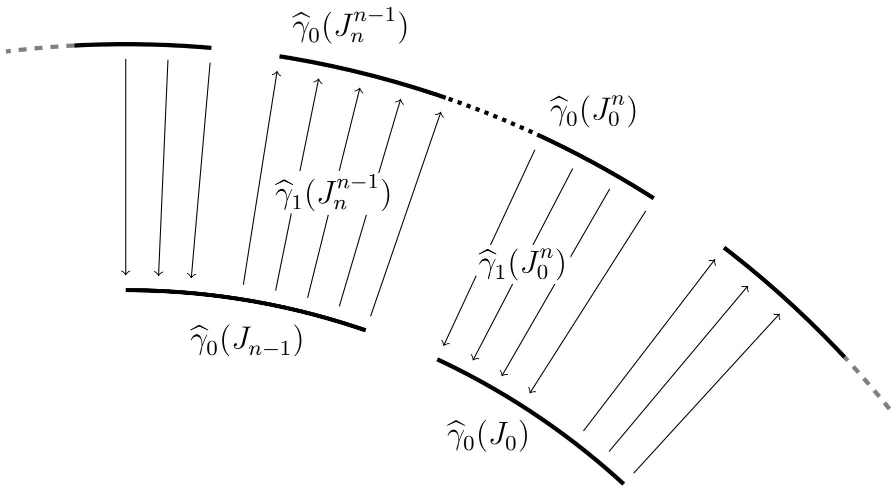


Figure 2: Defining a lift to $X^{\mathbb{2}}$

We will iterate through the connected components of V to define $\widehat{\gamma}$ on both objects and arrows. The functor γ has an underlying object component a map $\coprod_{i=0}^n J_i \to X_0^{[2]}$, and starting with $J_0 = [a_0, b_0]$ we can find an arrow $b\colon \gamma_1(a_0) \to \gamma_2(a_0) \in X_1$. This uses the fact $X_1 \to X_0^{[2]}$ is surjective. Since (s,t) is curvewise trivial, we can find a section over J_0, and hence a map $J_0 \to X_1 = (X^{\mathbb{2}})_0$.

If we denote $J_{i-1} \cap J_i$ by J_i^{i-1} (working mod $n+1$), then by naturality the object component $\widehat{\gamma}_0$ of the lift γ on $J_i^{i-1} \subset J_i$ is determined by its value on $J_i^{i-1} \subset J_{i-1}$. By this we mean that for $\widehat{\gamma}$ to be a functor to $X^{\mathbb{2}}$, or in other words a natural transformation $\gamma_1 \Rightarrow \gamma_2$, it must for every point in $J_i^{i-1} \subset \check{C}(V)_1$ satistfy naturality. Thus from the lift on J_0 we can define the lift on $J_1^0 \subset J_1$, and then again use the fact (s,t) is curvewise trivial to continue the lift on the rest of J_1.

So starting from J_0 we can work through the indexing set for the cover until we have defined $\widehat{\gamma}_0$ on J_{n-1}, and hence on $J_n^{n-1} \subset J_n$. The first lift, on J_0 defines $\widehat{\gamma}_0$ on $J_0^n \subset J_n$, and so we need to be able to define a map $J_n \to X_1$ extending both of these partial maps. This is the situation as in Figure 7, where we need to define the dotted portion of upper central arc.

It is here we use the hypothesis that X has connected stabilisers, since if we pull back $X_1 \to X_0^{[2]}$ along $\gamma\big|_{J_n}$, we can trivialise to $J_n \times A$. Then

A is necessarily a connected Lie group, so we can extend the map $J_n \supset J_n^{n-1} \coprod J_0^n \to A$ to all of J_n, completing the lift $\widehat{\gamma}\colon \check{C}(V) \to X^{\mathbf{2}}$, and the proof. $\square$

Thus we get the first main result of this section.

Theorem 7.3. *For a differentiable gerbe $\mathcal{X}$ presented by a Lie groupoid X with connected stabilisers such that (s,t) is curvewise trivial, then $\underline{\mathcal{H}\mathrm{om}}(S^1, \mathcal{X})$ is a Fréchet differentiable gerbe.*

We have the following additional result if we know some more about the gerbe X.

Proposition 7.4. *Let $X \to M$ be an abelian $\mathcal{A}$-gerbe where $\mathcal{A}$ is a locally trivial bundle of connected (abelian Lie) groups. Then $\Lambda\mathbb{L}X \simeq \mathbb{L}X_0 \times_{LM} L\mathcal{A}$.*

Proof. The assumptions on X mean that $\mathbb{L}X$ is a gerbe. The definition of $\Lambda\mathbb{L}X$ is that it is the pullback

$$\begin{array}{ccc} \Lambda\mathbb{L}X & \longrightarrow & \mathbb{L}X_1 \\ \downarrow & & \downarrow{\scriptstyle (s,t)^{\mathbb{L}X}} \\ \mathbb{L}X_0 & \xrightarrow[\Delta]{} & (\mathbb{L}X_0)^{[2]} \end{array}$$

and on the component $X^{\check{C}(V)} \subset \mathbb{L}X_0$ this is precisely the pullback

$$\begin{array}{ccc} (\Lambda\mathbb{L}X)_V & \longrightarrow & (X^{\mathbf{2}})^{\check{C}(V)} \\ \downarrow & & \downarrow \\ X^{\check{C}(V)} & \longrightarrow & X^{\check{C}(V)} \times_{LM} X^{\check{C}(V)} \quad \simeq \quad (X \times_M X)^{\check{C}(V)}. \end{array}$$

But since $(-)^{\check{C}(V)}$ preserves strict pullbacks of submersive functors, we have $(\Lambda\mathbb{L}X)_V \simeq (X \times_{X\times_M X} X^{\mathbf{2}})^{\check{C}(V)}$. By Lemma 6.11, $X \times_{X\times_M X} X^{\mathbf{2}} \simeq \Lambda X/\!/X$, and since X presents an abelian $\mathcal{A}$-gerbe, $\Lambda X/\!/X \simeq \mathcal{A} \times_M X$, by Lemma 6.7. Thus the summand $(\Lambda\mathbb{L}X)_V$ of $\Lambda\mathbb{L}X$ over $X^{\check{C}(V)}$ is isomorphic to $(\mathcal{A} \times_M X)^{\check{C}(V)} \simeq L\mathcal{A} \times_{LM} X^{\check{C}(V)}$ (where we have implicitly identified $M^{\check{C}(V)}$ with LM and similarly for $L\mathcal{A}$). $\square$

This gives us the final main result, and in fact the original motivation for this paper.

Theorem 7.5. *Let M be a finite-dimensional smooth manifold, $\mathcal{A}$ be a locally trivial bundle of connected abelian Lie groups on M and $X \to M$ a finite-dimensional abelian $\mathcal{A}$-gerbe. Then $\mathbb{L}X$ is an abelian $L\mathcal{A}$-gerbe on LM.*

Proof. Theorem 7.3 ensures that $\mathbb{L}X$ is again a gerbe, and from Proposition 7.4 we know that $\Lambda\mathbb{L}X$ descends to a bundle of groups $L\mathcal{A} \to LM$. Hence we know $\mathbb{L}X$ is an $L\mathcal{A}$-gerbe, and we thus need to show that $\mathbb{L}X$ is an *abelian* gerbe. This will be done by showing condition 2 of Lemma 6.5 holds for $\mathbb{L}X$, given that condition 3 of Lemma 6.5 holds for X. That is, we need to show the diagram

$$\begin{array}{ccc} \Lambda\mathbb{L}X \times_{\mathbb{L}X_0} \mathbb{L}X_1 & \longrightarrow & \Lambda\mathbb{L}X \\ {\scriptstyle \mathrm{id}_{\Lambda\mathbb{L}X} \times (s,t)^{\mathbb{L}X}} \downarrow & & \| \\ \Lambda\mathbb{L}X \times_{\mathbb{L}X_0} \mathbb{L}X_0 \times_{LM} \mathbb{L}X_0 & \longrightarrow & \Lambda\mathbb{L}X \end{array}$$

commutes. This reduces (using Lemma 6.11) to showing the following diagram commutes, for all V_1, V_2:

$$\begin{array}{ccc} (\Lambda X/\!/X)^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} (X^{\mathbf{2}})^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} & \xrightarrow{\mathrm{act}_{\mathbb{L}}} & (\Lambda X/\!/X)^{\check{C}(V_2)} \\ {\scriptstyle \mathrm{pr}_{124}} \downarrow & ? & \| \\ (\Lambda X/\!/X)^{\check{C}(V_1)} \times_{X^{\check{C}(V_1)}} \left(X^{\check{C}(V_1)} \times_{LM} X^{\check{C}(V_2)}\right) & \longrightarrow & (\Lambda X/\!/X)^{\check{C}(V_2)} \end{array}$$

Using the isomorphism $(\Lambda X/\!/X)^{\check{C}(V)} \simeq (\mathcal{A} \times_M X)^{\check{C}(V)} \simeq L\mathcal{A} \times_{LM} X^{\check{C}(V)}$, we can rewrite the desired diagram as

$$\begin{array}{ccc} L\mathcal{A} \times_{LM} X^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} (X^{\mathbf{2}})^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} & \xrightarrow{\mathrm{pr}_{14}} & L\mathcal{A} \times_{LM} X^{\check{C}(V_2)} \\ {\scriptstyle \mathrm{pr}_{124}} \downarrow & & \| \\ L\mathcal{A} \times_{LM} X^{\check{C}(V_1)} \times_{LM} X^{\check{C}(V_2)} & \xrightarrow[\mathrm{pr}_{13}]{} & L\mathcal{A} \times_{LM} X^{\check{C}(V_2)} \end{array}$$

in other words, we need to prove that the action map $\mathrm{act}_{\mathbb{L}}$ above (defined using conjugation of transformations of anafunctors) is, up to isomorphism, the projection pr_{14} in the top row of the preceeding diagram.

Now note that condition 2 in Lemma 6.5 for X (which holds since we are assuming X is an abelian $\mathcal{A}$-gerbe) can be rewritten as

$$\begin{array}{ccc}
\mathcal{A} \times_M X^{\mathbb{2}} & \xrightarrow{\ \mathrm{id} \times T\ } & \mathcal{A} \times_M X \\
{\scriptstyle \simeq}\uparrow & & \downarrow{\scriptstyle \simeq} \\
\Lambda X /\!/ X \times_X X^{\mathbb{2}} & \xrightarrow{\ \mathrm{act}\ } & \Lambda X /\!/ X \\
{\scriptstyle \mathrm{id} \times (S,T)}\downarrow & & \| \\
\Lambda X /\!/ X \times_X (X \times_M X) & \longrightarrow & \Lambda X /\!/ X \\
{\scriptstyle \simeq}\downarrow & & \uparrow{\scriptstyle \simeq} \\
\mathcal{A} \times_M (X \times_M X) & \xrightarrow[\ \mathrm{pr}_{13}\ ]{} & \mathcal{A} \times_M X
\end{array}$$

In other words, the action of $X^{\mathbb{2}}$ on $\Lambda X/\!/X$ is, up to isomorphism, essentially given by the target functor $T\colon X^{\mathbb{2}} \to X$. We will in particular use the top square of this diagram for the next step of the proof.

The map $\mathrm{act}_{\mathbb{L}}$ is defined (using the incorporated simplifications) to be the composite of the left column of arrows in the diagram on the following page.

$$\begin{array}{ccc}
(\Lambda X/\!/X)^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} (X^{2})^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} & \xrightarrow{\simeq} & L\mathcal{A} \times_{LM} X^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} (X^{2})^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} \\
\downarrow & & \downarrow \\
(\Lambda X/\!/X)^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} (X^{2})^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} & \xrightarrow{\simeq} & L\mathcal{A} \times_{LM} (X^{2})^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} \\
\downarrow{\simeq} & & \downarrow{\simeq} \\
\left(\Lambda X/\!/X \times_X X^{2}\right)^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} & \xrightarrow{\simeq} & \left(\mathcal{A} \times_M X^{2}\right)^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} \\
\downarrow{\mathrm{act}^{\check{C}(V_{12})}\times \mathrm{id}} & & \downarrow{(\mathrm{id}\times T)^{\check{C}(V_{12})}\times \mathrm{id}} \\
(\Lambda X/\!/X)^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} & \xrightarrow{\simeq} & (\mathcal{A} \times_M X)^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} \\
\downarrow{\simeq} & & \downarrow{\simeq} \\
(\Lambda X/\!/X)^{\check{C}(V_2)} & \xrightarrow{\simeq} & L\mathcal{A} \times_{LM} X^{\check{C}(V_2)}
\end{array}$$

Note that the square second from the bottom commutes because of the assumption that X is an abelian $\mathcal{A}$-gerbe. The composite of the right column of arrows is just pr_{14}, and hence condition 2 of Lemma 6.5 holds, and so $\mathbb{L}X$ is an abelian $L\mathcal{A}$-gerbe, as we needed to prove. $\square$

Corollary 7.6. *If A is a connected abelian Lie group and X is an A-bundle gerbe on M, then $\mathbb{L}X$ is an LA-bundle gerbe.*

Proof. An A-bundle gerbe $X \to M$ is an abelian $A \times M$-gerbe and (s,t) is the projection for a locally trivial bundle, so $\mathbb{L}X$ is an abelian $L(A \times M) \simeq LA \times LM$ gerbe. Thus $\mathbb{L}X$ is an LA-bundle gerbe. $\square$

Remark 7.7. We would like to apply this result to the basic gerbe on a Lie group, since then we get a gerbe over the free loop group that is *multiplicative*. This is fine if we use one of the finite-dimensional models, but it would be useful if we could also use the infinite-dimensional strict model $\mathrm{String}_G^{\mathrm{BCSS}}$ described in [BCSS07]. The results from Section 4 show that $\mathbb{L}\,\mathrm{String}_G^{\mathrm{BCSS}}$ is at worst a diffeological groupoid. Since $\mathbb{L}$ preserves products up to equivalence this in fact a coherent diffeological 2-group (see eg [BL04]). We conjecture, based on private discussion with Alexander Schmeding, that the results of this paper should apply to $\mathrm{String}_G^{\mathrm{BCSS}}$, and in fact Fréchet–Lie groupoids with (adapted) local additions and possibly also smoothly locally regular[6] source and target more generally.

References

[AS17] Habib Amiri and Alexander Schmeding, *A differentiable monoid of smooth maps on Lie groupoids*, preprint, `arXiv:1706.04816`, 2017.

[BCSS07] John C. Baez, Alissa S. Crans, Danny Stevenson, and Urs Schreiber, *From Loop Groups to 2-Groups*, Homology Homotopy Appl. **9** (2007), no. 2, 101–135, `arXiv:math/0504123`.

[6]See [Sta13, Definition 3.7] for the formal definition: a smoothly locally regular map is one that is equipped with the analogue of a local addition, smoothly parameterising submersion charts in the domain.

[BH11] John C. Baez and Alex Hoffnung, *Convenient categories of smooth spaces*, Trans. Amer. Math. Soc. **363** (2011), no. 11, 5789–5825, arXiv:0807.1704.

[BL04] John C. Baez and Aaron D. Lauda, *Higher dimensional algebra V: 2-groups*, Theory and Appl. Categ. **12** (2004), no. 14, 423–491, arXiv:math/0307200.

[Bar06] Toby Bartels, *Higher gauge theory I: 2-Bundles*, Ph.D. thesis, University of California Riverside, 2006, arXiv:math/0410328.

[BE72] Andrée Bastiani and Charles Ehresmann, *Categories of sketched structures*, Cahiers Topologie Géom. Différentielle Catég. **13** (1972), no. 2, 104–214, http://www.numdam.org/item?id=CTGDC_1972__13_2_104_0.

[BGNX12] Kai Behrend, Grégory Ginot, Behrang Noohi, and Ping Xu, *String topology for stacks*, Astérisque (2012), no. 343, xiv+169, arXiv:0712.3857.

[BX11] Kai Behrend and Ping Xu, *Differentiable stacks and gerbes*, J. Symplectic Geom. **9** (2011), no. 3, 285–341, arXiv:math/0605694.

[Bre94] Lawrence Breen, *On the classification of* 2*-gerbes and* 2*-stacks*, Astérisque (1994), no. 225, 160.

[CLW93] Aurelio Carboni, Stephen Lack, and Robert F. C. Walters, *Introduction to extensive and distributive categories*, J. Pure Appl. Algebra **84** (1993), 145–158, DOI:10.1016/0022-4049(93)90035-R.

[Car12] David Carchedi, *Compactly generated stacks: a Cartesian closed theory of topological stacks*, Adv. Math. **229** (2012), no. 6, 3339–3397, arXiv:0907.3925.

[Car13] David Carchedi (https://mathoverflow.net/users/4528/david-carchedi), *Stacks over diffeologies*, MathOverflow, 2013, https://mathoverflow.net/q/147425 (version: 2013-11-09).

[GH07] David Gepner and André Henriques, *Homotopy theory of orbispaces*, preprint, arXiv:math/0701916, 2007.

[Ham82] Richard S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. AMS (new series) **7** (1982), no. 1, 65–222, DOI:10.1090/S0273-0979-1982-15004-2.

[HMSV13] Pedram Hekmati, Michael K. Murray, Danny Stevenson, and Raymond F. Vozzo, *The Faddeev–Mickelsson–Shatashvili anomaly and lifting bundle gerbes*, Comm. Math. Phys. **319** (2013), no. 2, 379–393, arXiv:1112.1752.

[IZ13] Patrick Iglesias-Zemmour, *Diffeology*, Mathematical Surveys and Monographs, vol. 185, Amer. Math. Soc., Providence, RI, 2013.

[Joh02] Peter Johnstone, *Sketches of an elephant: a topos theory compendium*, Oxford Logic Guides, vol. 43 and 44, The Clarendon Press Oxford University Press, 2002.

[KM97] Andreas Kriegl and Peter W. Michor, *The convenient setting of global analysis*, Mathematical Surveys and Monographs, vol. 53, Amer. Math. Soc., Providence, RI, 1997, available from http://www.mat.univie.ac.at/~michor/apbookh-ams.pdf.

[LGSX09] Camille Laurent-Gengoux, Mathieu Stiénon, and Ping Xu, *Non-abelian differentiable gerbes*, Adv. Math. **220** (2009), no. 5, 1357–1427, arXiv:math/0511696.

[LU02] Ernesto Lupercio and Bernardo Uribe, *Loop groupoids, gerbes, and twisted sectors on orbifolds*, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, arXiv:math/0110207, pp. 163–184.

[Mit61] Boris S. Mitjagin, *Approximate dimension and bases in nuclear spaces*, Uspehi Mat. Nauk **16** (1961), no. 4, 63–132, Available from `https://mathnet.ru/eng/umn6643` (=Russian Math. Surveys **16** (1961), no. 4, 59–127, `DOI:10.1070/RM1961v016n04ABEH004109`).

[Mur96] Michael K. Murray, *Bundle gerbes*, J. London Math. Soc. **54** (1996), no. 2, 403–416, `arXiv:dg-ga/9407015`.

[Noo10] Behrang Noohi, *Mapping stacks of topological stacks*, J. Reine Angew. Math. **646** (2010), 117–133, `arXiv:0809.2373`.

[Noo12] ———, *Homotopy types of topological stacks*, Adv. Math. **230** (2012), 2014–2047, `arXiv:0808.3799`.

[Ols06] Martin C. Olsson, Hom-*stacks and restriction of scalars*, Duke Math. J. **134** (2006), no. 1, 139–164, available from `https://math.berkeley.edu/~molsson/homstackfinal.pdf`.

[Pro96] Dorette Pronk, *Etendues and stacks as bicategories of fractions*, Compositio Math. **102** (1996), no. 3, 243–303, `http://www.numdam.org/item?id=CM_1996__102_3_243_0`.

[Rob12] David Michael Roberts, *Internal categories, anafunctors and localisation*, Theory Appl. Categ. **26** (2012), no. 29, 788–829, `arXiv:1101.2363`.

[Rob16] ———, *On certain 2-categories admitting localisation by bi-categories of fractions*, Appl. Categ. Structures **24** (2016), no. 4, 373–384, `arXiv:1402.7108`.

[See64] Robert T. Seeley, *Extension of C^∞ functions defined in a half space*, Proc. Amer. Math. Soc. **15** (1964), 625–626, `DOI:10.2307/2034761`.

[SGA4.1] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*, Lecture Notes in Mathematics, Vol. 269, Springer-Verlag, Berlin-New York, 1972, Séminaire de

Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[Sou80] Jean-Marie Souriau, *Groupes différentiels*, Differential geometrical methods in mathematical physics (Proc. Conf., Aix-en-Provence/Salamanca, 1979), Lecture Notes in Math., vol. 836, Springer, Berlin-New York, 1980, DOI:10.1007/BFb0089728, pp. 91–128.

[Sta13] Andrew Stacey, *Yet more smooth mapping spaces and their smoothly local properties*, preprint, arXiv:1301.5493, 2013.

David Michael Roberts, david.roberts@adelaide.edu.au

Raymond F. Vozzo, raymond.vozzo@adelaide.edu.au

School of Mathematical Sciences, University of Adelaide
Adelaide 5005, Australia

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

VOLUME LIX-2 (2018)

SUPREMA OF EQUIVALENCE RELATIONS AND NON-REGULAR GOURSAT CATEGORIES

Dominique BOURN

Résumé. Nous caractérisons intrinsèquement l'existence des suprema de paires de relations d'équivalence dans une catégorie finiment complète. De là, une caractérisation catégorique de la *congruence formula* ainsi que des conditions suivant lesquelles la catégorie $Equ\mathbb{E}$ des relations d'équivalence dans $\mathbb{E}$ est régulière. Puis, entre autres applications, nous introduisons une définition des catégories de Goursat valide dans un contexte non régulier qui coïncide avec celle de Carboni-Kelly-Pedicchio dans le contexte régulier.

Abstract. Given any finitely complete category $\mathbb{E}$, we characterize intrinsically the existence of suprema of pairs of equivalence relations. From this follow a categorical characterization of the modular formula and of the conditions under which the category $Equ\mathbb{E}$ of equivalence relations in $\mathbb{E}$ is a regular one. Among other applications, we introduce, in the non-regular context, a new definition of Goursat categories, which, in the regular context, coincides with the notion introduced by Carboni-Kelly-Pedicchio (namely congruence 3-permutable ones) and is such that when $\mathbb{T}$ is an algebraic theory giving rise to a Goursat variety $\mathbb{V}(\mathbb{T})$, the category $\mathbb{T}(\mathbb{E})$ of internal $\mathbb{T}$-algebras in $\mathbb{E}$ is itself a Goursat category.

Keywords. Regular categories, internal equivalence relations, Goursat and Mal'tsev varieties, Goursat and Mal'tsev categories, congruence n-permutable varieties and categories, congruence modularity, congruence regularness.

Mathematics Subject Classification (2010). 08B10, 08C05, 18A20, 18A32, 18C05, 18E10, 20Jxx.

Introduction

The Mal'tsev [8], [9] and Goursat [7] regular categories (resp. varieties in the sense of Universal Algebra) are such that the direct images of equivalence relations are equivalence relations as well. From that, it is clear that these categories (resp. varieties) $\mathbb{E}$ are such that the category $Equ\mathbb{E}$ of internal equivalences relations (resp. congruences) in $\mathbb{E}$ is itself regular, and that the regular epimorphisms in $Equ\mathbb{E}$ are levelwise regular epimorphisms in $\mathbb{E}$.

So a question arises: is it the case that any variety $\mathbb{V}$ is such that $Equ\mathbb{V}$ is regular? And more generally is it possible to find the conditions under which the category $Equ\mathbb{E}$ of equivalence relations in a regular category $\mathbb{E}$ is itself a regular category?

In this article we show how the existence of regular epimorphisms in $Equ\mathbb{E}$ is closely related to the existence of suprema of equivalence relations (Proposition 2.4) and how the modular formula is then characterized by the stability of these regular epimorphisms under pullbacks along a certain class of morphisms in $Equ\mathbb{E}$ (Proposition 2.7). Finally we give the conditions on $\mathbb{E}$ under which the category $Equ\mathbb{E}$ is a regular category (Theorem 3.9). These conditions include the modular formula. Accordingly any variety $\mathbb{V}$ which is not congruence modular is such that the category $Equ\mathbb{V}$ is certainly not regular.

Doing the first part of this work, the author will incidentally achieve the old project of understanding the main properties of the regular categories regarding the equivalence relations without any resort to Metatheorems [1], and of uniquely dealing with diagrammatic proofs, see Corollary 3.3 via Lemma 1.14 and Proposition 3.5.

From these investigations and thanks to a result of Raftery [23], it will derive in particular, in Section 3.4, that there are ideal determined varieties $\mathbb{V}$ such that the category $Equ\mathbb{V}$ is regular and the regular epimorphisms in $Equ\mathbb{V}$ are not levelwise in $\mathbb{V}$.

In [16], Gumm characterized the varieties of Universal Algebra satisfying the congruence formula by the validity of the *Shifting Lemma*. The categorical description of this Shifting Lemma in any finitely complete category (see Section 2.3) is given in [6], where it is called the *shifting property*. Here we show that the categorical congruence modular formula implies the categorical shifting property.

We will then be able to introduce a definition of Goursat categories valid in the non-regular context and such that: 1) of course, in the regular context, it coincides with the pioneering notion of Carboni-Kelly-Pedicchio, 2) any (non-regular) Mal'tsev category satisfies this definition, 3) this definition is characterized by a property of the fibration of points $¶_{\mathbb{E}} : Pt(\mathbb{E}) \to \mathbb{E}$, as it is the case for Mal'tsev categories [3] and for Goursat regular ones [15], 4) when $\mathbb{T}$ is an algebraic theory giving rise to a Goursat variety $\mathbb{V}(\mathbb{T})$, the category $\mathbb{T}(\mathbb{E})$ of internal $\mathbb{T}$-algebras in $\mathbb{E}$ is itself a Goursat category, for any finitely complete category $\mathbb{E}$.

Since Mal'tsev and Goursat regular categories are respectively the congruence 2-permutable and 3-permutable ones, the question similarly arises of a possible definition of the congruence n-permutable categories in the non-regular context, opening to the wider question of characterizing those algebraic theories which can be freed of the classical varietal context of Universal Algebra to make sense under the unique assumption of finite completion.

We did not yet reach that point, however, these considerations unexpectedly lead us to 1) a characterization of the regular epimorphisms in $Equ\mathbb{E}$ when $\mathbb{E}$ is a n-permutable regular category, and to 2) a new characterization of the regular categories which are congruence $(2n+1)$-permutable, this characteriziation being surprinsingly not valid for the congruence $2n$-permutable ones.

The article is organized along the following lines. Section 1 is devoted to recalling basic facts about extremal epimorphims and internal equivalence relations; it gives a characterization of extremal epimorphisms in $Pt\mathbb{E}$ and $Equ\mathbb{E}$. Section 2 is devoted to the existence of suprema of equivalence relations and to the validity of the modular formula. Section 3 makes explicit the conditions under which $Equ\mathbb{E}$ is a regular category. Section 4 introduces a notion of Goursat category in the non-regular context which satisfies the four conditions described above. Finally Section 5 enlarges the applications of the results of Section 3 to congruence n-permutable regular categories.

1. Extremal and regular epimorphisms in $Equ\mathbb{E}$

In this article any category $\mathbb{E}$ is supposed to be finitely complete. The aim of this section is mainly to characterize the extremal and regular epimorphisms in the category $Equ\mathbb{E}$ of equivalence relations in $\mathbb{E}$.

1.1 Preliminaries

Recall that an *extremal epimorphism* is a map $f : X \to Y$ such that any decomposition $f = \bar{f}.m$ with a monomorphism m implies that m is an isomorphism. A regular epimorphism (i.e. a map which is the coequalizer of its kernel equivalence relation) is an extremal epimorphism. Both classes of maps are preserved by functors having a right adjoint; in particular they are stable under pushout along any map. The class of extremal epimorphisms is stable under composition and such that g is an extremal epimorphism as soon as so is any composite $g.f$.

It is straightforward that the underlying functor of a fibration $U : \mathbb{F} \to \mathbb{E}$ is faithful if and only if the fibers are preorders. In this case, any map in a fibre is a monomorphism; any square with x and y in the fibers is commutative if and only if we have $U(f) = U(f')$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

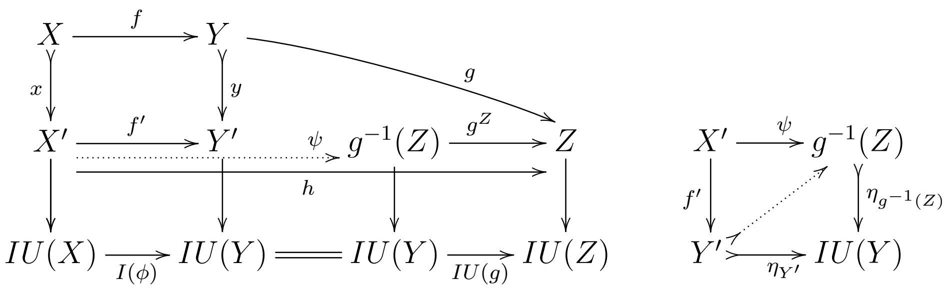
When any fibre above Z has a greatest element $I(Z)$ which is stable under cartesian maps, it defines a right adjoint $I : \mathbb{E} \to \mathbb{F}$ to the functor U; we shall denote by $\eta_X : X \rightarrowtail IU(X)$ the induced unique map.

Theorem 1.1. *Let $U : \mathbb{F} \to \mathbb{E}$ be a faithful fibration for which each fiber has a greatest element stable under cartesian maps. Given any commutative square in $\mathbb{F}$ where f is an extremal epimorphism:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

and the maps x and y are inside a fibre, it is a pushout if and only if f' is an extremal epimorphism. When it is the case, f' is a regular epimorphism as soon as so is f.

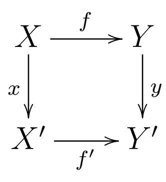
Proof. We just recalled it is a necessary condition. Let us show it is sufficient. For that, consider the following diagram in $\mathbb{F}$, where we have $h.x = g.f$ $(*)$ and $U(f) = \phi = U(f')$:



From (*), we get $U(h) = U(g).\phi$. Let g^Z be the cartesian map above $U(g)$. Then we get a unique factorization ψ such that $g^Z.\psi = h$ and $U(\psi) = \phi$. Accordingly the right hand side square commutes, and we get the dotted desired factorization, since f' is an extremal epimorphism and $\eta_{g^{-1}(Z)}$ is a monomorphism. $\square$

We have also the following very easy and useful:

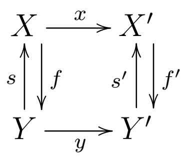
Lemma 1.2. *Suppose $U : \mathbb{F} \to \mathbb{E}$ is a left exact fibration. Given any commutative square in $\mathbb{F}$ where f' is a cartesian map:*



and its image by U is a pullback in $\mathbb{E}$, it is a pullback if and only if the map f is cartesian. A map f is cocartesian if and only if it is hypercocartesian.

1.2 Extremal and regular epimorphisms in $Pt\mathbb{E}$

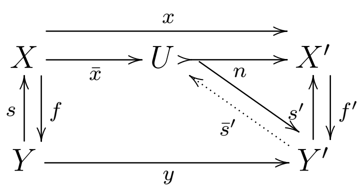
We denote by $Pt(\mathbb{E})$ the category whose objects are the split epimorphisms and whose maps are the commutative squares of split epimorphisms:



and by $¶_{\mathbb{E}} : Pt(\mathbb{E}) \to \mathbb{E}$ the functor associating with any split epimorphism (f, s) its codomain Y. It is a fibration whose cartesian maps are the pullbacks of split epimorphisms. It is called the *fibration of points* and the fibre above Y is denoted by $Pt_Y(\mathbb{E})$, see [3].

Lemma 1.3. *A morphism (y, x) in $Pt(\mathbb{E})$ as above is an extremal (resp. regular) epimorphism if and only if both y and x are extremal (resp. regular) epimorphisms in $\mathbb{E}$.*

Proof. It is clear that if x and y are extremal (resp. regular) epimorphisms in $\mathbb{E}$, the morphism $(y, x) : (f, s) \to (f', s')$ is an extremal (resp. regular) epimorphism in $Pt(\mathbb{E})$. Conversely suppose it is an extremal (resp. regular) epimorphism in $Pt(\mathbb{E})$. Then $y = ¶_{\mathbb{E}}(y, x)$ is an extremal (resp. regular) epimorphism since $¶_{\mathbb{E}}$ has a right adjoint. Suppose (y, x) is an extremal epimorphism and $x = \bar{x}.n$ with a monomorphism n:



Since y is an extremal epimorphism and n a monomorphism, we get a factorization $\bar{s}'$ such that $n.\bar{s}' = s'$ and $\bar{s}'.y = \bar{x}.s$. Whence a decomposition $(y, x) = (1_Y, n).(y, \bar{x})$ in $Pt(\mathbb{E})$, so that n is an isomorphism in $\mathbb{E}$, and x an extremal epimorphism. Suppose now (y, x) is a regular epimorphism and we have a map $\phi : X \to T$ in $\mathbb{E}$ which coequalizes the two legs of the kernel equivalence relation $R[x]$, namely such that $R[x] \subset R[\phi]$. Since y is a regular epimorphism, there is a unique map $\sigma : Y' \to T$ such that $\sigma.y = \phi.s$.

Now, consider the following commutative diagram:

$$\begin{array}{ccccc}
 & & \overset{(f'.x,\phi)}{\frown} & & \\
X & \xrightarrow{x} & X' & \overset{(f',\tau)}{\dashrightarrow} & Y' \times T \\
s \uparrow\downarrow f & & s' \uparrow\downarrow f' & & (1_{Y'},\sigma) \uparrow\downarrow p_{X'} \\
Y & \xrightarrow[y]{} & Y' & = = = & Y'
\end{array}$$

We have $R[x] \subset R[f'.x] \cap R[\phi] = R[(f'.x, \phi)]$, and since (y, x) is a regular epimorphism, we get a unique factorization (f', τ) in $Pt(\mathbb{E})$ which provides the desired factorization $\tau : X' \to T$ in $\mathbb{E}$. □

1.3 Reflexive and equivalence relations

We shall denote by $Ref\mathbb{E}$ (resp. $Equ\mathbb{E}$) the category of internal reflexive relations (resp. equivalence relations) in $\mathbb{E}$ and by $\mathcal{O}_{\mathbb{E}} : Ref\mathbb{E} \to \mathbb{E}$ associating with any reflexive relation its underlying object. This functor has a fully faithful right adjoint $\nabla : \mathbb{E} \to Ref\mathbb{E}$ associating with any object X, the undiscrete equivalence relation:

$$X \times X \underset{p_1^X}{\overset{p_0^X}{\rightrightarrows}} X \qquad (s_0^X : X \to X \times X)$$

which makes $\mathcal{O}_{\mathbb{E}}$ a left exact fibration: given any reflexive relation S on Y and any map $f : X \to Y$, the cartesian map above f is given by the following pullback in $Ref\mathbb{E}$:

$$\begin{array}{ccc}
f^{-1}(S) & \xrightarrow{\tilde{f}^S} & S \\
{\scriptstyle (\tilde{d}_0^S,\tilde{d}_1^S)} \downarrow & & \downarrow {\scriptstyle (d_0^S,d_1^S)} \\
\nabla_X & \xrightarrow[\nabla_f]{} & \nabla_Y
\end{array}$$

The fibres of $\mathcal{O}_{\mathbb{E}} : Ref\mathbb{E} \to \mathbb{E}$ are preorders; we noticed that it is equivalent to the fact that the functor $\mathcal{O}_{\mathbb{E}}$ is faithful. Accordingly, given any diagram where R and S are reflexive relations:

$$\begin{array}{ccc} R & \overset{\hat{f}}{\dashrightarrow} & S \\ d_0^R \downarrow \uparrow \downarrow d_1^R & & d_0^S \downarrow \uparrow \downarrow d_1^S \\ X & \xrightarrow[f]{} & Y \end{array}$$

there exists at most one factorization $\hat{f}$, i.e. at most one map in $Ref\mathbb{E}$, above f; it is the case if and only if $R \subset f^{-1}(S)$. Given any map $f : X \to Y$ and any reflexive relation S on X, the unique map $(f, \check{f}) : S \to \nabla_Y$ above f will be called the *paraterminal map* above f. Since the equivalence relation ∇_X is the greatest element in the fiber above X, both $\mathcal{O}_{\mathbb{E}} : Ref\mathbb{E} \to \mathbb{E}$ and its restriction $\mathcal{O}_{\mathbb{E}} : Equ\mathbb{E} \to \mathbb{E}$ satisfy the conditions of Theorem 1.1 and Lemma 1.2. The subcategory $Equ\mathbb{E}$ is stable in $Ref\mathbb{E}$ under finite limits and cartesian maps.

The functor $\mathcal{O}_{\mathbb{E}} : Ref\mathbb{E} \to \mathbb{E}$ has a left adjoint functor Δ as well, associating, with any object X, the discrete equivalence relation:

$$X \; \underset{1_X}{\overset{1_X}{\rightrightarrows}} \; X \quad (\text{with } 1_X : X \leftarrow X)$$

The *kernel equivalence relation* $R[f]$ of a map $f : X \to Y$ is the domain of the cartesian map with codomain Δ_Y above f:

$$\begin{array}{ccc} R[f] & \xrightarrow{\tilde{f}^Y} & \Delta_Y \\ {\scriptstyle (d_0^f, d_1^f)} \downarrow & & \downarrow {\scriptstyle (1_Y, 1_Y)} \\ \nabla_X & \xrightarrow[\nabla_f]{} & \nabla_Y \end{array}$$

An equivalence relation R on X is said to be *effective* when there is some map $f : X \to Y$ such that $R = R[f]$.

In the category Set of sets, a reflexive relation R on X is an equivalence relation if and only if it satisfies the *horn-filler condition*, namely: xRx' and xRx'' imply $x'Rx''$, $\forall (x, x', x'') \in X^3$. Or, equivalently, if and only if we have $R[d_0^R] \subset d_1^{-1}(R)$, or equivalently if and only if there is a morphism $(d_1^R, d_2^R) : R[d_0^R] \to R$ above d_1^R in Ref defined by $d_2^R(xRx', xRx'') =$

$x'Rx''$:

$$\begin{array}{ccc} R[d_0^R] & \overset{d_2^R}{\dashrightarrow} & R \\ {\scriptstyle d_0^R}\downarrow\uparrow\downarrow{\scriptstyle d_1^R} & & {\scriptstyle d_0^R}\downarrow\uparrow\downarrow{\scriptstyle d_1^R} \\ R & \underset{d_1^R}{\longrightarrow} & X \end{array}$$

The same characterization holds in any category $\mathbb{E}$. We shall denote by $U_0 : Ref\mathbb{E} \to Pt\mathbb{E}$ the left exact functor associating with any morphism of reflexive relations its underlying square indexed by 0. A morphism $(f, \hat{f}) : R \to R'$ in $Equ\mathbb{E}$ is called *fibrant* when $U_0(f, \hat{f})$ is cartesian in $Pt\mathbb{E}$. The following observation is straightforward:

Proposition 1.4. *The functor U_0 has a left adjoint associating its kernel equivalence relation $R[f]$ with any split epimorphism (f, s).*

The map (d_1^R, d_2^R) described above is precisely the counit of this adjonction and *it is is necessarily a fibrant morphism.*

1.4 Inverse image along split epimorphims

Given a pair (R, S) of equivalence relations on an object X in a category $\mathbb{E}$, we denote by $R\square S$ the inverse image of the equivalence relation $S \times S$ on $X \times X$ along the inclusion $(d_0^R, d_1^R) : R \rightarrowtail X \times X$. This defines a double equivalence relation:

$$\begin{array}{ccc} R\square S & \overset{\delta_1^S}{\underset{\delta_0^S}{\rightleftarrows\!\rightarrow}} & S \\ {\scriptstyle \delta_0^R}\downarrow\uparrow\downarrow{\scriptstyle \delta_1^R} & & {\scriptstyle d_0^S}\downarrow\uparrow\downarrow{\scriptstyle d_1^S} \\ R & \overset{d_1^R}{\underset{d_0^R}{\rightleftarrows\!\rightarrow}} & X \end{array}$$

which is actually the largest double equivalence relation on R and S. In set-theoretical terms, this double relation $R\square S$ is the subset of elements (u, v, u', v') of X^4 such that the set of relations $uRu', vRv', uSv, u'Sv'$ holds:

$$\begin{array}{ccc} u & \overset{S}{\dashrightarrow} & v \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\ u' & \underset{S}{\dashrightarrow} & v' \end{array}$$

As an equivalence relation on R, we get $R\Box S = (d_0^R)^{-1}(S) \cap (d_1 R)^{-1}(S)$. Accordingly we have:

Lemma 1.5. *Let (R, S) be any pair of equivalence relations on an object X in a category $\mathbb{E}$. The following commutative diagram, where ξ_R^S is the natural inclusion:*

$$\begin{array}{ccc}
R\Box S & \xrightarrow{\delta_1^S} & S \\
{\scriptstyle \xi_R^S}\downarrow & & \downarrow{\scriptstyle (d_0^S, d_1^S)} \\
(d_0^R)^{-1}(S) & \xrightarrow[\bar{d}_1^R]{} & X \times X
\end{array}$$

is a pullback in $Equ\mathbb{E}$, where the lower map is the paraterminal one above d_1^R.

The following observation is straightforward and nevertheless important:

Lemma 1.6. *Given any map $f : X \to Y$ in $\mathbb{E}$, and any equivalence relation S on X, the kernel equivalence relation in $Equ\mathbb{E}$ of the paraterminal map $(f, \check{f}) : S \to \nabla_Y$ above f is given by the following diagram:*

$$R[f]\Box S \underset{\delta_0^S}{\overset{\delta_1^S}{\rightleftarrows}} S \xrightarrow{(f,\check{f})} \nabla_Y$$

Moreover it is the kernel equivalence relation of any other map with domain S above f in $Equ\mathbb{E}$. Accordingly this equivalence relation is the unique effective equivalence relation on S in $Equ\mathbb{E}$ above $R[f]$.

If any equivalence relation is effective in $\mathbb{E}$, the only effective equivalence relations on S in $Equ\mathbb{E}$ are the equivalence relations $R\Box S \rightrightarrows S$.

Now, the following characterization will appear to be very useful:

Lemma 1.7. *Let $\mathbb{E}$ be a category and (R, S) any pair of equivalence relations on X. The following morphism of equivalence relations:*

$$\begin{array}{ccc}
R\Box S & \underset{\delta_0^S}{\overset{\delta_1^S}{\rightleftarrows}} & S \\
{\scriptstyle (\delta_0^R, \delta_1^R)}\downarrow & & \downarrow{\scriptstyle (d_0^S, d_1^S)} \\
R \times R & \underset{d_0^R \times d_0^R}{\overset{d_1^R \times d_1^R}{\rightleftarrows}} & X \times X
\end{array}$$

is fibrant in $Equ\mathbb{E}$ if and only if we have $R \subset S$.

Proof. By the Yoneda embedding, it is enough to check our assertion in Set. Suppose $R \subset S$. Then from the following left hand side diagram:

$$\begin{array}{ccc} x & \overset{S}{\dashrightarrow} & y \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\ x' & & y' \end{array} \qquad\qquad \begin{array}{ccc} x & \overset{S}{\dashrightarrow} & y \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ x' & & y' \end{array}$$

we can deduce the right hand side one, so that $x'Sy'$. Consequently the square indexed by 0 is a pullback (which means that the morphism of equivalence relations in question is fibrant). Conversely suppose this square is a pullback. From the following left hand side diagram drawn from xRx', we can deduce the right hand side one:

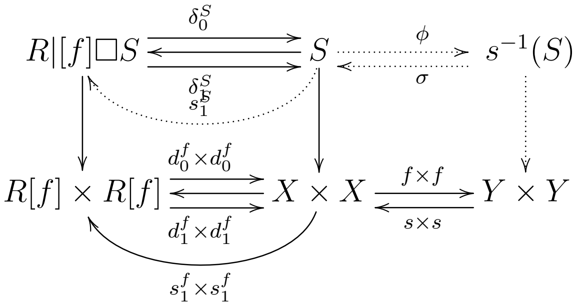
$$\begin{array}{ccc} x & \overset{S}{\dashrightarrow} & x \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\ x & & x' \end{array} \qquad\qquad \begin{array}{ccc} x & \overset{S}{\dashrightarrow} & x \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\ x & \underset{S}{\dashrightarrow} & x' \end{array}$$

so that we get xSx' and $R \subset S$. $\square$

In this case, the equivalence relation $R\square S$ on R is both the inverse image of S along d_0^R and d_1^R.

Theorem 1.8. *Given any split epimorphism $(f, s) : X \rightleftarrows Y$, the inverse image $f^{-1} : Equ_Y\mathbb{E} \to Equ_X\mathbb{E}$ induces a preorder bijection between the equivalence relations on Y and the equivalence relations on X containing $R[f]$. Its inverse mapping is given by the restriction of s^{-1}.*

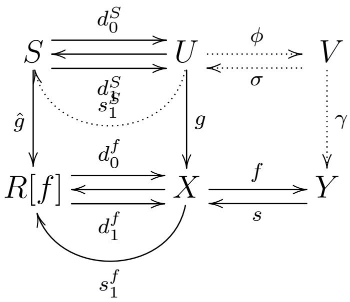
Proof. It is clear that f^{-1} takes values among the equivalence relations on X containing $R[f]$ and is a preorder homomorphism. Now if S is an equivalence relation on X containing $R[f]$, consider the following diagram:



Its left hand side part is a fibrant morphism of equivalence relations by Lemma 1.7. Lemma 1.9 below produces the right hand side pullback. □

Recall that an equivalence relation S is said to be split, when the leg d_1^S is split by a map s_1^S such that the map $d_0^S.s_1^S$ coequalizes the pair (d_0^S, d_1^S).

Lemma 1.9 (Lemma 2.2 in [4])**.** *Consider any vertical fibrant morphism of equivalence relations in $\mathbb{E}$ where f is split by s:*



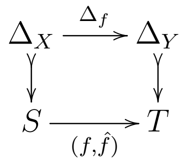
Then the unique section s_1^S of d_1^S above s_1^f makes S a split equivalence relation on U. Moreover the split epimorphism (ϕ, σ) associated with the idempotent $d_0^S.s_1^S$ produces the dotted right hand side leftward pullback.

1.5 Extremal and regular epimorphisms in $Equ\mathbb{E}$

We have now the following characterization:

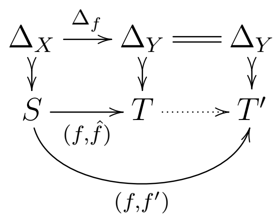
Proposition 1.10. *Given any extremal epimorphism $(f, \hat{f}) : S \to T$ in $Ref\mathbb{E}$ (resp. $Equ\mathbb{E}$), its underlying map f in $\mathbb{E}$ is an extremal epimorphism.*

Suppose f is an extremal epimorphism in $\mathbb{E}$. The following conditions are equivalent in $Ref\mathbb{E}$ (resp. $Equ\mathbb{E}$):
1) the map $(f, \hat{f}) : S \to T$ is extremal
2) the following diagram is a pushout:



3) the map $(f, \hat{f}) : S \to T$ is hypercocartesian.
Consider now the condition: 4) the map $\hat{f}$ is an extremal epimorphism in $\mathbb{E}$.
We get 4)⇒1) in $Equ\mathbb{E}$ and 4) $\iff$ 1) in $Ref\mathbb{E}$.
The extremal epimorphism $(f, \hat{f}) : S \to T$ is a regular epimorphism in $Ref\mathbb{E}$ (resp. $Equ\mathbb{E}$) if and only if its underlying map $f : X \to Y$ is a regular epimorphism in $\mathbb{E}$.

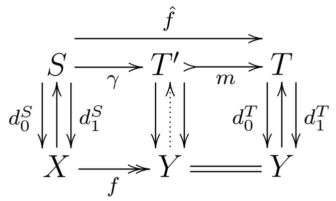
Proof. The first point is a consequence of the fact that the functor $\mathcal{O}_\mathbb{E}$ has a right adjoint. We have then 1)⇒2) by Theorem 1.1 and the fact that Δ_f is an extremal epimorphism when so is f since the functor Δ has a right adjoint. As for 2)⇒3), it is enough to show that $(f, \hat{f})$ is cocartesian since the fibration $\mathcal{O}_\mathbb{E}$ is left exact. Given any map $(f, f') : S \to T'$ above f, the following commutative diagram induces the desired dotted factorization:



Now 3)⇒1) is straightforward. On the other hand, it is clear that when the map $\hat{f}$ is an extremal epimorphism in $\mathbb{E}$, the map $(f, \hat{f})$ is an extremal epimorphism in $Ref\mathbb{E}$ (resp. $Equ\mathbb{E}$), namely we have 4)⇒1).

Let us show 1)⇒4) in $Ref\mathbb{E}$. So, suppose that $(f, \hat{f})$ is extremal in $Ref\mathbb{E}$, and consider the following commutative diagram where we have $\hat{f} =$

$m.\gamma$ in $\mathbb{E}$ with a monomorphism m:



The fact that f is an extremal epimorphism and m a monomorphism produces a dotted factorization from the commutation of the upward rectangle determined by the subdiagonal maps s_0. Accordingly the middle part is a reflexive relation and produces a decomposition in $Ref\mathbb{E}$. Since $(f, \hat{f})$ is extremal in $Ref\mathbb{E}$, the map m is an isomorphism. Accordingly $\hat{f}$ is extremal in $\mathbb{E}$.

In the same way, if $(f, \hat{f})$ is a regular epimorphism in $Ref\mathbb{E}$ (resp. $Equ\mathbb{E}$), so is f in $\mathbb{E}$ since $\mathcal{O}_{\mathbb{E}}$ has a right adjoint. Suppose now f is a regular epimorphism, and $(f, \hat{f}) : S \to T$ an extremal epimorphism in $Ref\mathbb{E}$ (resp. $Equ\mathbb{E}$), then $(f, \hat{f})$ is a regular epimorphism by the above condition 2) and Theorem 1.1. □

By taking $\mathbb{E} = Set$, the following proposition shows that there are extremal epimorphisms in $Equ\mathbb{E}$ which are not levelwise epimorphic in $\mathbb{E}$:

Proposition 1.11. *Given any extremal epimorphism $(h, g) : (f, s) \to (f', s')$ in $Pt\mathbb{E}$, the morphism $(g, R(g))$ is an extremal epimorphism in $Equ\mathbb{E}$.*

Proof. It is a consequence of Proposition 1.4, since the right adjoints preserve the extremal epimorphism. □

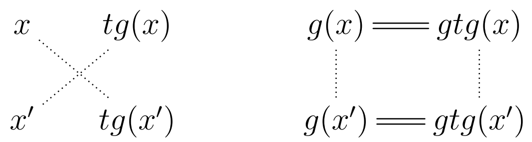
In any category $\mathbb{E}$, here is a large class of regular epimorphisms in $Equ\mathbb{E}$:

Proposition 1.12. *Any fibrant morphism $(g, \bar{g}) : R \to S$ in $Equ\mathbb{E}$ above a split epimorphism $(g, t) : X \rightleftarrows Y$ is an extremal epimorphism in $Equ\mathbb{E}$ above g. Suppose moreover that the supremum $S \bigvee T$ does exist in $Equ\mathbb{E}$; then we have $g^{-1}(S \bigvee T) = R \bigvee g^{-1}(T)$.*

Proof. Since g is split in $\mathbb{E}$ and $(g, \bar{g}) : R \to S$ is fibrant, the map $\bar{g}$ is split in $\mathbb{E}$ (just consider $U_0(g, \hat{g})$), so $(g, \bar{g})$ is an extremal epimorphism by the above condition 4) or equivalently is cocartesian.

Suppose now that the supremum $S\bigvee T$ does exist in $Equ\mathbb{E}$. Then we get $R[g] \subset g^{-1}(S\bigvee T)$ and $R \subset g^{-1}(S) \subset g^{-1}(S\bigvee T)$. Now let W be any equivalence relation on X containing $g^{-1}(T)$ and R. The first inclusion implies that we have $R[g] \subset g^{-1}(T) \subset W$, so that we get $W = g^{-1}(t^{-1}(W))$ and $T \subset t^{-1}(W)$ by Theorem 1.8. Then the second inclusion gives us $S \subset t^{-1}(W)$ since $(g, \hat{g})$ is cocartesian. Whence $S\bigvee T \subset t^{-1}(W)$, and $g^{-1}(S\bigvee T) \subset g^{-1}(t^{-1}(W)) = W$. Accordingly we get $g^{-1}(S\bigvee T) = R\bigvee g^{-1}(T)$. □

There are fibrant morphisms in $Equ\mathbb{E}$ above split epimorphisms in $\mathbb{E}$ which are not split epimorphisms in $Equ\mathbb{E}$ as shown by the following diagram where the left hand side part describes the equivalence relation R on the four elements set X, while the right hand side one describes the equivalence relation S on the two elements set Y:



1.6 Hyperextremal epimorphisms in $Pt\mathbb{E}$ and $Ref\mathbb{E}$

The following tools and observations will appear to be very discriminating:

Definition 1.13. *A morphism (y, x) in $Pt\mathbb{E}$ is said to be a hyperextremal (resp. hyperregular) epimorphism when it is an extremal (resp. regular) epimorphism in $Pt\mathbb{E}$ such that the factorization $R(x) : R[f] \to R[f']$ is an extremal (resp. regular) epimorphism in $\mathbb{E}$ as well. A morphism $(f, \hat{f}) : R \to S$ in $Ref\mathbb{E}$ is said to be a hyperextremal (resp. hyperregular) epimorphism when $U_0(f, \hat{f})$ is hyperextremal (resp. hyperregular) in $Pt\mathbb{E}$.*

It is clear that, in both categories $Pt(\mathbb{E})$ and $Ref\mathbb{E}$, the class of hyperextremal (resp. hyperregular) epimorphisms is an intermediate class between the class of extremal (resp. regular) epimorphisms and the class of split epimorphisms. This class has the same kind of stability properties in $Pt\mathbb{E}$ (resp. $Ref\mathbb{E}$) as the class of extremal epimorphims in $\mathbb{E}$. When $\mathbb{E}$ is regular (see Section 3.1) extremal epimorphisms and regular epimorphisms coincide in

$\mathbb{E}$, so that hyperextremal epimorphisms and hyperregular epimorphisms coincide in $Pt(\mathbb{E})$ (resp. $Ref\mathbb{E}$), and they are stable under pullbacks in these categories. The main justification of the previous definition is the following:

Proposition 1.14. *Given any category $\mathbb{E}$, the hyperextremal epimorphisms in $Ref\mathbb{E}$ reflect the equivalence relations.*

Proof. We have to show that a hyperextremal epimorphism $(f, \hat{f})$ in $Ref\mathbb{E}$ whose domain R is an equivalence relation makes its codomain S an equivalence relation as well:

$$\begin{array}{ccc} R & \xrightarrow{\hat{f}} & S \\ {\scriptstyle d_0^R}\downarrow\uparrow\downarrow{\scriptstyle d_1^R} & & {\scriptstyle d_0^S}\downarrow\uparrow\downarrow{\scriptstyle d_1^S} \\ X & \xrightarrow[f]{} & Y \end{array}$$

For that, consider the following commutative square in $\mathbb{E}$:

$$\begin{array}{ccccc} R[d_0^R] & \xrightarrow{R(\hat{f})} & R[d_0^S] & \rightarrowtail & Y \times Y \times Y \\ {\scriptstyle d_2^R}\downarrow & & \vdots\,{\scriptstyle d_2^S} & & \downarrow{\scriptstyle p_2^Y} \\ R & \xrightarrow[\hat{f}]{} & S & \underset{(d_0^S, d_1^S)}{\rightarrowtail} & Y \times Y \end{array}$$

where the upper unlabelled horizontal map associates (a, b, c) with (aSb, aSc). Since $R(\hat{f})$ is an extremal epimorphism in $\mathbb{E}$ and (d_0^S, d_1^S) is a monomorphism, we get the dotted desired factorization d_2^S. $\square$

Proposition 1.12 shows that any fibrant morphism $(g, \hat{g}) : R \to S$ above a split epimorphism (g, t) is an hyperregular epimorphism in $Equ\mathbb{E}$ without being a split epimorphism, in general.

2. Categorical congruence modular formula

2.1 Suprema of pairs of equivalence relations

Now let us point out a characterization relating extremal epimorphisms in $Equ\mathbb{E}$ and suprema of pairs of equivalence relations:

Proposition 2.1. *A map $(g, \hat{g}) : R \to T$ is an extremal epimorphism in $Equ\mathbb{E}$ above the split epimorphism $(g, t) : X \rightleftarrows Z$ in $\mathbb{E}$ if and only if we have $g^{-1}(T) = R[g] \bigvee R$.*

Proof. Being an extremal epimorphism above the split epimorphism g is equivalent to being cocartesian above g by Lemma 1.10. Accordingly $(g, \hat{g})$ is characterized by the fact that, for any other morphism $(g, \bar{g}) : R \to S$ above g, we have $T \subset S$, or equivalently by the fact that, for any equivalence relation S on Z, we have:

$$R \subset g^{-1}(S) \iff T \subset S \iff g^{-1}(T) \subset g^{-1}(S)$$

the second equivalence being given by Theorem 1.8. Now, the bijection of this same theorem characterizes T by: for any equivalence U above X containing $R[g]$, we have: $R \subset U \iff g^{-1}(T) \subset U$ or, in other words, by $g^{-1}(T) = R[g] \bigvee R$. □

Whence, now, a very simple situation which produces a supremum of equivalence relations without any condition and which will be the main point for the definition of Goursat categories in a non-regular setting:

Corollary 2.2. *Consider any commutative square of split epimorphisms in a category $\mathbb{E}$:*

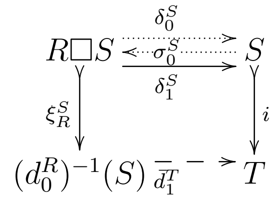
$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} & & {\scriptstyle s'}\uparrow\downarrow{\scriptstyle f'} \\ Y & \xrightarrow[h]{} & Y' \end{array}$$

then the equivalence relation $g^{-1}(R[f'])$ is the supremum $R[g] \bigvee R[f]$ of the equivalence relations $R[f]$ and $R[g]$.

Proof. The split epimorphism $(R(f), R(s)) : R[g] \rightleftarrows R[h]$ is an extremal epimorphism in the category $Equ\mathbb{E}$. According to the previous proposition, we get $f^{-1}(R[h]) = R[f] \bigvee R[g]$. Now clearly we have $f^{-1}(R[h]) = g^{-1}(R[f'])$. □

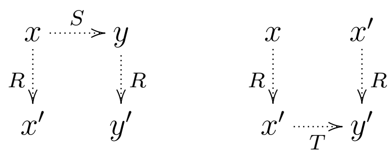
Here are now the main results of the article:

Proposition 2.3. *Let (R, S) be any pair of equivalence relations on X in $\mathbb{E}$. Suppose that T is another equivalence relation on X such that $S \subset T$. We get $R \subset T$ if and only if we have $(d_0^R)^{-1}(S) \subset (d_1^R)^{-1}(T)$ or, equivalently, if and only if there is a morphism $(d_1^R, \bar{d}_1^T) : (d_0^R)^{-1}(S) \to T$ in $Equ\mathbb{E}$ making the following square commute, where ξ_R^S is the natural inclusion:*

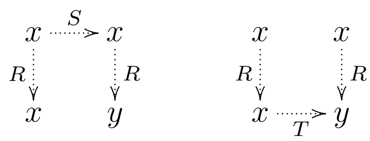


This square is then necessarily a pullback.

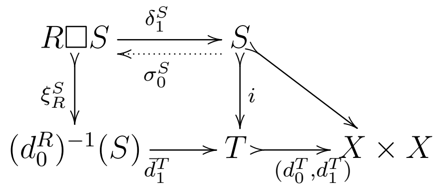
Proof. By the Yoneda embedding it is enough to check the assertion in Set. The elements of $(d_0^R)^{-1}(S)$ are given by the left hand side diagrams while the elements of $(d_1^R)^{-1}(T)$ are given by the right hand side diagrams:



It is clear that if we have $S \subset T$ and $R \subset T$, the left hand side diagram implies $x'Ty'$ and the validity of the right hand side diagram. Conversely if we have the inclusion $(d_0^R)^{-1}(S) \subset (d_1^R)^{-1}(T)$, the following left hand side diagram, drawn from xRy, implies the right hand side one which implies that $R \subset T$:

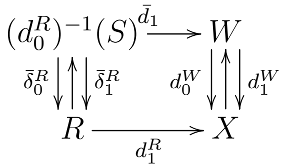


Now, saying $(d_0^R)^{-1}(S) \subset (d_1^R)^{-1}(T)$ is equivalent to the existence of a morphism $\bar{d}_1^T : (d_1^R)^{-1}(S) \to T$ in $Equ\mathbb{E}$ above d_1^R, i.e. making commutative the diagram in question in $Equ\mathbb{E}$. It fits inside the following commutative diagram:



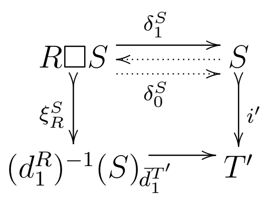
The quadrangle is a pullback by Lemma 1.5. Its squared part is a pullback as well because of the monomorphic factorization (d_0^T, d_1^T). $\square$

Proposition 2.4. *Let (R, S) be any pair of equivalence relations on X in the category $\mathbb{E}$. The following conditions are equivalent:*
1) the supremum $R \bigvee S$ does exist in $Equ\mathbb{E}$
2) there is an extremal epimorphism $(d_1^R, \bar{d}_1)$ in $Equ\mathbb{E}$ above the split epimorphism (d_1^R, s_0^R):



In this case we get $W = R \bigvee S$.

Proof. Given any equivalence relation T containing R and S. Let us show that the square asserted by the previous proposition is a pushout if and only if $T = R \bigvee S$. If T' is another equivalence relation containing R and S, we get another commutative diagram:



If the diagram with T provides us with a pushout in $Equ\mathbb{E}$, we get the inclusion $T \subset T'$ and $T = R \bigvee S$. Conversely if we have $T = R \bigvee S$, we get the inclusion $T \subset T'$ which shows that the square with T is a pushout in the category $Equ\mathbb{E}$. Accordingly, the equivalence 1) $\iff$ 2) is a consequence of Theorem 1.1. $\square$

Theorem 2.5. *The category* $\mathbb{E}$ *is such that any pair* (R, S) *of equivalence relations has a supremum* $R \bigvee S$ *if and only if* $Equ\mathbb{E}$ *has extremal epimorphisms with any domain above the split epimorphisms in* $\mathbb{E}$*. Under these assumptions, a morphism of equivalence relation is an extremal epimorphism above the split epimorphism* (f, s) *(and therefore a regular one) in* $Equ\mathbb{E}$*:*

$$\begin{array}{ccc} S & \xrightarrow{\hat{f}} & V \\ d_0^S \downarrow\uparrow\downarrow d_1^S & & d_0^V \downarrow\uparrow\downarrow d_1^V \\ X & \underset{f}{\overset{s}{\rightleftarrows}} & Y \end{array}$$

if and only if we have $f^{-1}(V) = R[f] \bigvee S$.

Proof. By the previous theorem, if $Equ\mathbb{E}$ has regular epimorphisms with any domain above a split epimorphism, then the regular epimorphism with domain $(d_0^R)^{-1}(S)$ above the split epimorphism (d_1^R, s_0^R) provides us with $R \bigvee S$. Conversely suppose that $\mathbb{E}$ is such that any pair (R, S) of equivalence relations has a supremum $R \bigvee S$. Let $(f, s) : X \rightleftarrows Y$ be a split epimorphism in $\mathbb{E}$ and S an equivalence relation on X. Then $R[f] \subset R[f] \bigvee S$, so that, by Theorem 1.8 we know that there is a split epimorphism $R[f] \bigvee S \to s^{-1}(R[f] \bigvee S)$ in $Equ\mathbb{E}$ above (f, s). Then the morphism $S \rightarrowtail R[f] \bigvee S \rightleftarrows s^{-1}(R \bigvee S)$ is the regular epimorphism above f with codomain S; indeed, given any other map $(f, \hat{f}) : S \to T$ above f, we get $S \subset f^{-1}(T)$ and $R[f] \subset f^{-1}(T)$, whence $R[f] \bigvee S \subset f^{-1}(T)$ and:

$$s^{-1}(R[f] \bigvee S) \subset s^{-1}(f^{-1}(T)) = T$$

which produces the desired factorization. □

When $\mathbb{E}$ fulfils any of the equivalent conditions of the previous theorem we shall denote by $(f, f^\sharp) : S \to f^\sharp(S)$ the extremal epimorphism in $Equ\mathbb{E}$ above the split epimorphism (f, s). The following proposition will provide us with a large class of categories in which $Equ\mathbb{E}$ has regular epimorphisms with any domain above split epimorphisms.

Proposition 2.6. *Suppose that* $\mathbb{E}$ *is such that the partial order associated with the preorder determined by any fibre* $Equ_Y\mathbb{E}$ *has infima. Then the fibration* $\mathcal{O}_\mathbb{E} : Equ\mathbb{E} \to \mathbb{E}$ *is a cofibration as well. Accordingly it has regular* *epimorphisms with any domain above regular epimorphisms and a fortiori above split epimorphisms.*

Proof. Let $f : X \to Y$ be any morphism in $\mathbb{E}$ and S an equivalence relation on X. Then the cocartesian map with domain S above it is determined by the infimum W of the family of the equivalence relations V on Y such that $S \subset f^{-1}(V)$. Indeed, by commutation of limits, the infima are preserved by inverse image, and we get $S \subset f^{-1}(W)$, whence a map $(f, f^\sharp) : S \to W$ above f. Let us show that it is the cocartesian map above f. Since $\mathcal{O}_\mathbb{E}$ is a fibration, it is enough to prove the universal property for the maps above f. Given any other map $(f, \hat{f}) : S \to T$, we get $S \subset f^{-1}(T)$, and then we have $W \subset T$ which produces the desired factorization. The last assertion is a consequence of Lemma 1.10. □

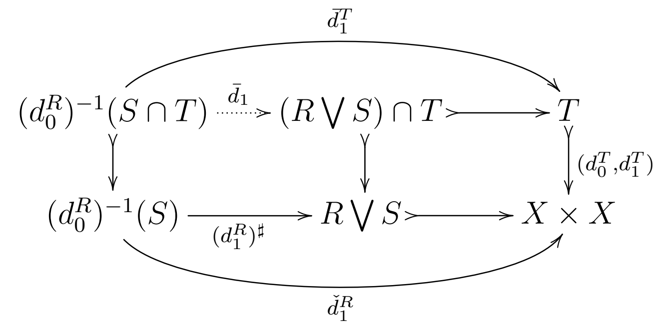
As we know, any variety $\mathbb{V}$ of universal algebras obviously satisfies the condition of the previous proposition.

2.2 The modular formula

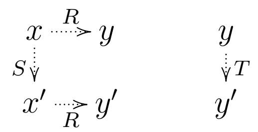
Now that we know when any pair of equivalence relations has a supremum, the aim of this section is to give a categorical characterization of the so-called *congruence modularity*. Recall that the *modular formula* for equivalence relations is: given any triple (R, S, T) of equivalence relations on X, then we get $(R \bigvee S) \cap T = R \bigvee (S \cap T)$ provided that we have $R \subset T$.

Proposition 2.7. *Suppose that $Equ\mathbb{E}$ is such that any pair (R, S) of equivalence relations has a supremum $R \bigvee S$ or, equivalently, that it has regular epimorphisms with any domain above split epimorphisms in $\mathbb{E}$. Then the modular formula holds if and only if these regular epimorphisms are stable under pullbacks along maps in the fibers of $\mathcal{O}_\mathbb{E}$.*

Proof. Suppose that the regular epimorphisms in question are stable under pullbacks along maps in the fibres and that we have $R \subset T$. Then consider the following commutative diagram:



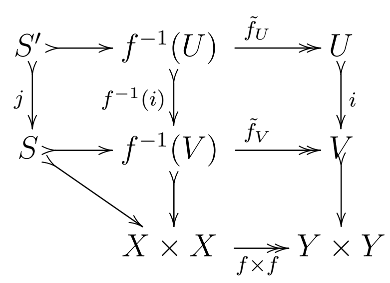
The right hand side square is a pullback with vertical maps in a fibre. The whole diagram is a pullback in any category $\mathbb{E}$: consider the following diagram:



since we have $R \subset T$, we get xTx' and consequently $x(T \cap S)x'$.

So there is a dotted factorization $\bar{d}_1$ making the left hand side square a pullback. Since $(d_1^R)^\sharp$ provides us with a regular epimorphism above d_1^R, so does this dotted factorization. Accordingly its codomain $(R \bigvee S) \cap T$ is $R \bigvee (S \cap T)$.

Conversely suppose the modular formula holds. Consider the following diagram where f is a split epimorphism and any square is a pullback:



The map $S \to V$ is a regular epimorphism in $Equ\mathbb{E}$ if and only if we get $f^{-1}(V) = R[f] \bigvee S$. We have $f^{-1}(U) \subset f^{-1}(V) = R[f] \bigvee S$, and so we get:

$$f^{-1}(U) = (R[f] \bigvee S) \cap f^{-1}(U) = R[f] \bigvee (S \cap f^{-1}(U)) = R[f] \bigvee S'$$

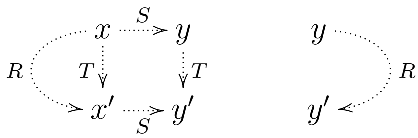
the second equality being given by the modular law since $R[f] \subset f^{-1}(U)$, the third one since the left hand side upper square is a pullback. Accordingly the map $S' \to U$ is a regular epimorphism in $Equ\mathbb{E}$. $\square$

The previous characterization induces the following:

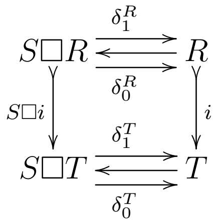
Definition 2.8. *A category $\mathbb{E}$ is said to be categorically congruence modular (cc-modular for short) when $Equ\mathbb{E}$ has regular epimorphisms with any domain above split epimorphisms in $\mathbb{E}$ and these regular epimorphisms are stable under pullbacks along maps in the fibers of $\mathcal{O}_{\mathbb{E}}$.*

2.3 The categorical shifting property

In [16], Gumm characterized the varieties of Universal Algebra satisfying the congruence modular formula by the validity of the *Shifting Lemma*: given any triple of equivalence relations (R, S, T) such that $S \cap T \subset R$ the following left hand side situation implies the right hand side one:



The categorical description of the shifting property was given in [6] and is the following one: given any triple of equivalence relations (R, S, T) on X such that $S \cap T \subset R \subset T$, the following morphism of equivalence relations is fibrant:



In this section, we shall investigate what is remaining of the varietal equivalence described by Gumm in the categorical setting.

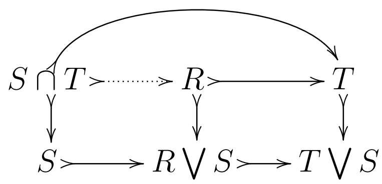
Definition 2.9. *[4][6] A category $\mathbb{E}$ is said to be a Gumm category when the categorical shifting property holds.*

In [4], it is shown that any regular Mal'tsev category is a Gumm one. In any exact category with coequalizers of reflexive pairs, it is possible to construct the supremum of any pair of equivalence relations; if in addition, the congruence modular formula holds, then the category in question is a Gumm one, see also [4]. In [6], it was noticed that, since the conservative functors which preserve pullbacks reflect them as well, any conservative functor $U : \mathbb{F} \to \mathbb{E}$ preserving pullbacks is such that the category $\mathbb{F}$ is a Gumm category as soon as so is $\mathbb{E}$. In particular it was noticed that this notion is stable under passage to slice categories $\mathbb{E}/Y$, coslice categories $Y/\mathbb{E}$ and fibres $Pt_Y(\mathbb{E})$. Now, let us enlarge a bit the range of examples of Gumm categories.

First, we can add the following important observation. Let $\mathbb{T}$ be an algebraic theory in the sense of Universal algebra and $\mathbb{V}(\mathbb{T})$ the corresponding variety of $\mathbb{T}$-algebras. Since the pullbacks in the functor category $\mathcal{F}(\mathbb{E}, \mathbb{V}(\mathbb{T}))$ are componentwise, *any functor category* $\mathcal{F}(\mathbb{E}, \mathbb{V}(\mathbb{T}))$ *is a Gumm category as soon as* $\mathbb{V}(\mathbb{T})$ *is a congruence modular variety*. Now let $\mathbb{T}(\mathbb{E})$ be the category of internal $\mathbb{T}$-algebras in $\mathbb{E}$; the corresponding Yoneda embedding $Y^{\mathbb{T}} : \mathbb{T}(\mathbb{E}) \to \mathcal{F}(\mathbb{E}^{op}, \mathbb{V}(\mathbb{T}))$ being left exact and conservative, *the category* $\mathbb{T}(\mathbb{E})$ *is a Gumm category as soon as* $\mathbb{V}(\mathbb{T})$ *is a congruence modular variety*.

Let us introduce now the following categorical description of the modular formula:

Lemma 2.10. *Suppose that* $Equ\mathbb{E}$ *has supremum of equivalence relations. The modular formula holds if and only if, considering any commutative diagram in the fibre* $Equ_X\mathbb{E}$ *where we have* $R \subset T$ *and where the whole rectangle is a pullback:*

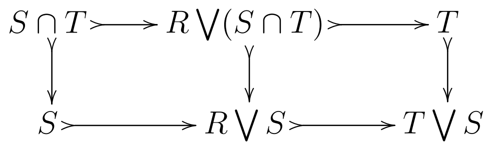


the right hand side square is a pullback if and only if there is a dotted factorization.

Proof. Suppose the modular law holds. We get:

$$(R \bigvee S) \cap T = R \bigvee (S \cap T) = R$$

when we have $S \cap T \subset R$. Conversely suppose that we have $R \subset T$ and consider the following commutative diagram in $Equ_X\mathbb{E}$:

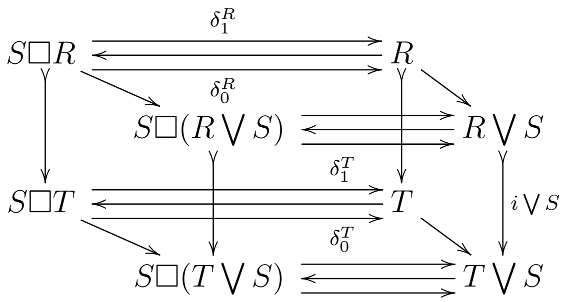


where the upper right hand side horizontal map is determined by the inclusion $R \subset T$. Then notice that $R \bigvee (S \cap T) \bigvee S = R \bigvee S$ since $S \cap T \subset R \bigvee S$. Accordingly the right hand side square is a pullback and we get $R \bigvee (S \cap T) = (R \bigvee S) \cap T$. □

In the categorical setting, it remains one part of the varietal equivalence described by Gumm:

Proposition 2.11. *Any cc-modular category $\mathbb{E}$ is a Gumm category.*

Proof. Suppose that $\mathbb{E}$ is cc-modular and we have $S \cap T \subset R \subset T$; then consider the following commutative diagram:



The right hand side vertical square is a pullback by the modular law, and so is the left hand side vertical one since inverse image preserves intersections. The front morphism of equivalence relations is fibrant since both $R \bigvee S$ and $T \bigvee S$ contain S. So that the back morphism of equivalence relations is fibrant as well. □

Now it would remain the difficult task to understand why the equational context of Universal Algebra implies the converse. Finally, we have also the following examples without any reference to the supremum of pairs of equivalence relations, see [3] for the definition of a protomodular category:

Proposition 2.12. *Any protomodular category $\mathbb{E}$ is a Gumm category.*

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
S\square(S\cap T) & \overset{S\square j}{\rightarrowtail} & S\square R & \overset{S\square i}{\rightarrowtail} & S\square T \\
\delta_0^{S\cap T}\downarrow\uparrow\downarrow\delta_1^{S\cap T} & & \delta_0^{R}\downarrow\uparrow\downarrow\delta_1^{R} & & \delta_0^{T}\downarrow\uparrow\downarrow\delta_1^{T} \\
S\cap T & \underset{j}{\rightarrowtail} & R & \underset{i}{\rightarrowtail} & T
\end{array}$$

Clearly, the whole rectangle indexed by 0 is a pullback (check it in Set). Since $S\square i$ is a monomorphism, so is the left hand side square. Accordingly, when $\mathbb{E}$ is protomodular, so is the right hand side one as well. □

2.4 Pullbacks of extremal epimorphisms along cartesian maps in $Equ\mathbb{E}$

In the previous section, we characterized the stability of the class of the extremal epimorphisms in $Equ\mathbb{E}$ above split epimorphisms in $\mathbb{E}$ under pullbacks along maps in the fibres of the fibration $\mathcal{O}_{\mathbb{E}} : Equ\mathbb{E} \to \mathbb{E}$. In this section we shall characterize the stability of this class along cartesian maps in $Equ\mathbb{E}$. For that let us begin with the following observation:

Theorem 2.13. *Suppose that $Equ\mathbb{E}$ has extremal epimorphisms with any domain above split epimorphisms. Then these extremal epimorphisms are stable under pullbacks along cartesian maps in $Equ\mathbb{E}$ if and only if any fibrant morphism $(g,\hat{g}) : R \to R'$ of equivalence relations is such that, given any equivalence relation T on the codomain X' of g, we get: $g^{-1}(R' \bigvee T) = R \bigvee g^{-1}(T)$. So, these extremal epimorphisms are stable under any pullback in $Equ\mathbb{E}$ if and only if the category $\mathbb{E}$ is cc-modular and the previous condition on fibrant morphisms holds.*

Proof. Suppose our condition holds and we have a pullback in $Equ\mathbb{E}$ as on the left hand side, above the right hand side pullback in $\mathbb{E}$:

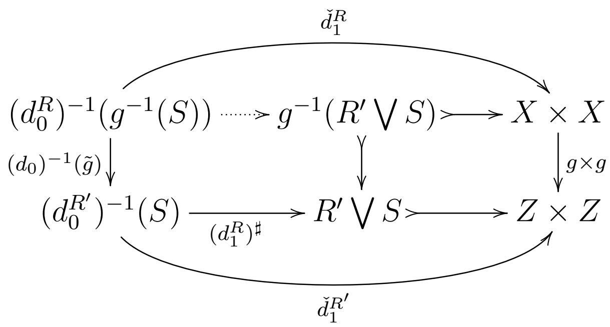
$$\begin{array}{ccc}
g^{-1}(S) & \xrightarrow{(f,\hat{f})} & h^{-1}(V) \\
{\scriptstyle (g,\tilde{g})}\downarrow & & \downarrow{\scriptstyle (h,\tilde{h})} \\
S & \xrightarrow[(f',(f')^{\sharp})]{} & V
\end{array}
\qquad\qquad
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\
X' & \xrightarrow[f']{} & Y'
\end{array}$$

Suppose moreover that the right hand side vertical map $(h, \tilde{h})$ is cartesian in $Equ\mathbb{E}$. So is the left hand side vertical one. Moreover the morphism $R(g) : R[f] \to R[f']$ is fibrant since the right hand side square is a pullback in $\mathbb{E}$. By our condition we have $g^{-1}(R[f'] \bigvee S) = R[f] \bigvee g^{-1}(S)$. On the other hand, we have $(f')^{-1}(V) = R[f'] \bigvee S$ since the lower map is a regular epimorphism above the split epimorphism (f', s'). So we get:

$$f^{-1}(h^{-1}(V)) = g^{-1}((f')^{-1}(V)) = g^{-1}(R[f'] \bigvee S) = R[f] \bigvee g^{-1}(S)$$

which means, by Theorem 2.5 that $(f, \hat{f})$ is a regular epimorphism above the split epimorphism (f, s).

Conversely suppose that the regular epimorphisms in $Equ\mathbb{E}$ above split epimorphisms in $\mathbb{E}$ are stable under pullbacks along cartesian maps. Consider the following diagram:



where $(g, \hat{g}) : R \to R'$ is a fibrant morphism of equivalence relations and S an equivalence relation on the codomain X' of g. The right hand side square is a pullback by definition of the inverse image. The whole diagram is a pullback by the following lemma. Accordingly there is a dotted factorization making the left hand side square a pullback; it is a pullback along the vertical middle cartesian map. Accordingly this dotted arrow provides us with a regular epimorphism above the split epimorphism d_1^R in $Equ\mathbb{E}$ and its codomain $g^{-1}(R' \bigvee S)$ is $R \bigvee g^{-1}(S)$ according to the same Theorem 2.5 . Now, these regular epimorphisms in $Equ\mathbb{E}$ are stable under any pullback if and only if they are stable under pullbacks along maps in the fibres and pullbacks along cartesian maps; the first point is equivalent to the cc-modularity while the second one is equivalent to our condition on fibrant morphisms. □

The lemma we needed in the above proof is rather technical:

Lemma 2.14. *Given any fibrant morphism $(g, \hat{g}) : R \to R'$ of equivalence relations in a category $\mathbb{E}$, for any equivalence relation T on the codomain X' of g, the following square is a pullback:*

$$\begin{array}{ccc} (d_0^R)^{-1}(g^{-1}(T)) & \xrightarrow{\check{d}_1^R} & X \times X \\ {\scriptstyle (d_0)^{-1}(\tilde{g})}\downarrow & & \downarrow{\scriptstyle g\times g} \\ (d_0^{R'})^{-1}(T) & \xrightarrow[\check{d}_1^{R'}]{} & X' \times X' \end{array}$$

or, in other words, given any cartesian map $(g, \tilde{g}) : \Sigma \to T$, the unique factorization $(\hat{g}, (d_0)^{-1}(\tilde{g})) : (d_0^{R'})^{-1}(\Sigma) \to (d_0^{R'})^{-1}(T)$ is cartesian as well.

Proof. For that consider the following diagram:

$$\begin{array}{ccccc} (d_0^R)^{-1}(g^{-1}(T)) & \rightarrowtail & R \times R & \underset{d_1^R \times d_1^R}{\overset{d_0^R \times d_0^R}{\rightleftarrows\!\!\rightarrow}} & X \times X \\ {\scriptstyle \tilde{g}}\downarrow & & {\scriptstyle \hat{g}\times\hat{g}}\downarrow & & \downarrow{\scriptstyle g\times g} \\ (d_0^{R'})^{-1}(T) & \rightarrowtail & R' \times R' & \underset{d_1^{R'} \times d_1^{R'}}{\overset{d_0^{R'} \times d_0^{R'}}{\rightleftarrows\!\!\rightarrow}} & X' \times X' \end{array}$$

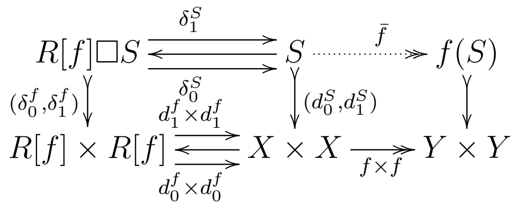
Any of the right hand side squares is a pullback since $(g, \hat{g})$ is fibrant while the left hand side one is a pullback as well since the underlying morphism of equivalence relations $(d_0^R)^{-1}(g^{-1}(T)) \to (d_0^{R'})^{-1}(T)$ is cartesian as the pullback of the cartesian morphism $g^{-1}(T) \to T$ along $(d_0^{R'})^{-1}(T) \to T$. $\square$

3. When is $Equ\mathbb{E}$ a regular category?

The characterization given by the previous theorem deals with the stability under pullbacks of a large class of regular epimorphisms in $Equ\mathbb{E}$. So we are not very far from the situation where $Equ\mathbb{E}$ is a regular category. The aim of this section is to characterize such a situation.

3.1 Basic facts on regular categories

A category $\mathbb{E}$ is regular [1] when the regular epimorphisms are stable under pullbacks and any effective equivalence relation has a coequalizer. In this case regular epimorphisms coincide with extremal epimorphisms. The *direct image* along a regular epimorphism $f : X \twoheadrightarrow Y$ of the reflexive relation S on X is given by the codomain of the coequalizer $\bar{f}$ in $\mathbb{E}$ of the upper row of the left hand side part of the following diagram which, as we noticed in Lemma 1.6, is an effective equivalence relation in the category $\mathbb{E}$:

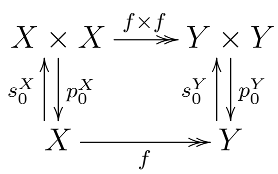


It gives rise to a reflexive relation $f(S)$ on Y. According to Proposition 1.10, the morphism $(f, \bar{f})$ is a regular epimorphism in $Ref\mathbb{E}$. Accordingly we get the very well known observation:

Proposition 3.1. *When $\mathbb{E}$ is a regular category, so is $Ref\mathbb{E}$.*

When S is an equivalence relation $f(S)$ is reflexive and symmetric, but not transitive in general. The following proposition is very important; the point (v) and the corollary are classical, but, here, we do not use Metatheorems and introduce very straightforward categorical proofs:

Proposition 3.2. *Let $\mathbb{E}$ be a regular category:*
(i) any hyperextremal epimorphism in $Pt(\mathbb{E})$ is hyperregular
(ii) any cartesian extremal epimorphism in $Pt(\mathbb{E})$ is hyperextremal
(iii) the hyperextremal epimorphisms are stable under pullbacks in the categories $Pt(\mathbb{E})$ and $Ref\mathbb{E}$
(iv) if f is a regular epimorphism in $\mathbb{E}$, the following morphism in $Pt(\mathbb{E})$:



is a hyperextremal epimorphism; accordingly ∇_f is a hyperextremal epimorphism in $Ref\mathbb{E}$ and any cartesian extremal epimorphism in $Ref\mathbb{E}$ is hyperextremal
(v) the inverse image f^{-1} of reflexive relations along a regular epimorphism $f : X \twoheadrightarrow Y$ reflects the equivalence relations.

Proof. The first point is straightforward. The second and third ones are direct consequences of the fact that pullbacks in $Pt(\mathbb{E})$ and $Ref\mathbb{E}$ are levelwise and that extremal (=regular) epimorphisms are stable under pullbacks. The fourth one is a direct consequence of the fact that regular epimorphisms are stable under products.

As for the fifth point: given any reflexive relation T on Y, the inverse image along the regular epimorphism f is given by the following pullback in $Ref\mathbb{E}$:

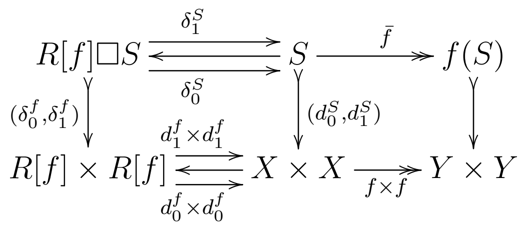
$$\begin{array}{ccc} f^{-1}(T) & \xrightarrow{\tilde{f}^T} & T \\ {\scriptstyle (d_0,d_1)}\big\downarrow & & \big\downarrow{\scriptstyle (d_0^T,d_1^T)} \\ \nabla_X & \xrightarrow[\nabla_f]{} & \nabla_Y \end{array}$$

According to the point (iv), the upper horizontal morphism is a hyperextremal epimorphism in $Ref\mathbb{E}$; by Lemma 1.14, the reflexive relation T is then an equivalence relation as soon as so is $f^{-1}(T)$. Whence (v). □

Whence the following collorary which extends Theorem 1.8 to any regular epimorphism f:

Corollary 3.3. *Let $\mathbb{E}$ be a regular category, $f : X \twoheadrightarrow Y$ a regular epimorphism and S any equivalence relation on X. Then we have $R[f] \subset S$ if and only if $S = f^{-1}(f(S))$. In this case, the direct image $f(S)$ is actually an equivalence relation. In other words, when f is a regular epimorphism, the mapping f^{-1} induces a preorder bijection between the equivalence relations on Y and the equivalence relations on X containing $R[f]$.*

Proof. Consider the following diagram:

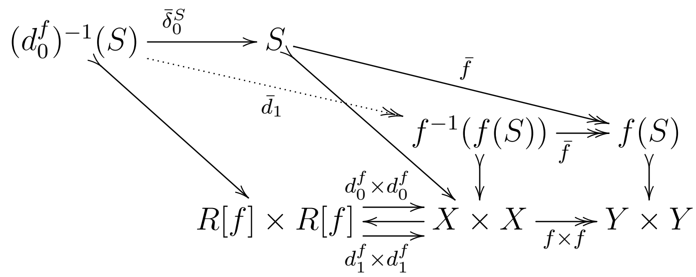


By Lemma 1.7, we know that we have $R[f] \subset S$ if and only if the left hand side morphism of equivalence relations is fibrant. By the Barr-Kock theorem, since $\bar{f}$ is a regular epimorphism, this is the case if and only if the right hand side square is a pullback, i.e. if and only if $S = f^{-1}(f(S))$. The last assertion is a consequence of the point (v) of the previous proposition. □

We shall close this section by the following observations on $Equ\mathbb{E}$ of which the first one brings some precisions about the first part of the previous corollary:

Proposition 3.4. *Let $\mathbb{E}$ be a regular category, $f : X \twoheadrightarrow Y$ a regular epimorphism and S an equivalence relation on X. The direct image $f(S)$ is an equivalence relation if and only if $f^{-1}(f(S))$ is the supremum $R[f] \bigvee S$ in $Equ\mathbb{E}$. If it is the case, $(f, \bar{f}) : S \to f(S)$ is a regular epimorphism in $Equ\mathbb{E}$ above f.*

Proof. Suppose that $f(S)$ is an equivalence relation and consider the following diagram in $\mathbb{E}$:



There is a dotted factorization $\bar{d}_1$ making the upper quadrangle a pullback. Since $\bar{f}$ is a regular epimorphism, so is $\bar{d}_1$. Accordingly it produces a levelwise regular epimorphic map in $\mathbb{E}$, and, as such, a regular epimorphism $(d_1^f, \bar{d}_1) : (d_0^f)^{-1}(S) \to f^{-1}(f(S))$ in $Equ\mathbb{E}$ above d_1^f. By Proposition 2.4, we get $f^{-1}(f(S)) = R[f] \bigvee S$. Now, the morphism $(f, \bar{f}) : S \to f(S)$ in $Equ\mathbb{E}$ is a levelwise regular epimorphic map in $\mathbb{E}$, and, as such, a regular epimorphism in $Equ\mathbb{E}$ above the regular epimorphism f.

Conversely, if we have $f^{-1}(f(S)) \simeq R[f] \bigvee S$, the reflexive relation $f^{-1}(f(S))$ is an equivalence relation, and so is $f(S)$ by Proposition 3.2. □

Proposition 3.5. *Let $\mathbb{E}$ be a regular category, $f : X \twoheadrightarrow Y$ a regular epimorphism and (T, S) any pair of equivalence relations on X such that $R[f] \subset T$ and $f(S)$ is an equivalence relation. Then we have $f(S) \cap f(T) = f(S \cap T)$, which, according to the previous proposition is equivalent to the modular formula $(R[f] \bigvee S) \cap T = R[f] \bigvee (S \cap T)$.*

Proof. Since we have $R[f] \subset T$, the direct image $f(T)$ is an equivalence relation as well. Now, consider the following diagram:

$$\begin{array}{ccccc}
T \cap S & \rightarrowtail & T & \overset{\tilde{f}^T}{\twoheadrightarrow} & f(T) \\
\downarrow & & \downarrow & & \downarrow \\
S & \rightarrowtail & X \times X & \underset{f \times f}{\twoheadrightarrow} & Y \times Y
\end{array}$$

where the left hand side square is a pullback by definition and the right hand side one since we have $R[f] \subset T$. The following whole rectangle is the same as the previous one and consequently is a pullback:

$$\begin{array}{ccccc}
T \cap S & \overset{\bar{f}^{T \cap S}}{\twoheadrightarrow} & f(T \cap S) & \rightarrowtail & f(T) \\
\downarrow & & \downarrow & & \downarrow \\
S & \underset{\bar{f}^S}{\twoheadrightarrow} & f(S) & \rightarrowtail & Y \times Y
\end{array}$$

Since the upper right hand side horizontal map is a monomorphism, the left hand side square is a pullback as well. Since it has horizontal regular epimorphisms and $\mathbb{E}$ is regular, the right hand side square is a pullback as well, which means that $f(T \cap S) = f(T) \cap f(S)$. □

3.2 cc-regular categories

First, if we wish the category $Equ\mathbb{E}$ to be a regular category, it must have coequalizers of effective equivalence relations, and consequently so has $\mathbb{E}$.

Proposition 3.6. *Suppose $\mathbb{E}$ has coequalizers of effective relations. The following conditions are equivalent:*
1) $Equ\mathbb{E}$ has coequalizers of effective relations
2) $Equ\mathbb{E}$ has regular epimorphism with any domain above a regular epimorphism in the category $\mathbb{E}$
In particular, the category $Equ\mathbb{E}$ certainly has suprema of equivalence relations.

Proof. Starting with an equivalence relation R on X, its image by Δ has a coequalizer which is preserved by $\mathcal{O}_{\mathbb{E}}$ and consequently provides us with a coequalizer of R. Suppose now that $\mathbb{E}$ has coequalizers of effective relations. Suppose 1) and consider any regular epimorphism $f : X \twoheadrightarrow Y$ in $\mathbb{E}$. Let S be any equivalence relation on X and consider the following equivalence relation in $Equ\mathbb{E}$, see Lemma 1.6:

$$R[f]\square S \underset{\delta_0^S}{\overset{\delta_1^S}{\rightleftarrows\!\!\rightarrow}} S \xrightarrow{(f,\check{f})} \nabla_Y$$

It has a coequalizer $(q, \hat{q})$; since it is preserved by $\mathcal{O}_{\mathbb{E}}$ we have $q = f$, and being above f, its kernel equivalence relation is $R[f]\square S$. Accordingly $(f, \hat{q})$ is a regular epimorphism.

Conversely, suppose that $Equ\mathbb{E}$ has regular epimorphism with any domain above a regular epimorphsm in $\mathbb{E}$. Given any effective equivalence relation in $Equ\mathbb{E}$, it is preserved by $\mathcal{O}_{\mathbb{E}}$ and is necessarily of the previous kind according to Lemma 1.6. Let $q : X \twoheadrightarrow Q$ be the coequalizer of $R[f]$; then q is a regular epimorphism in $\mathbb{E}$. Let $(q, \hat{q}) : S \to W$ the regular epimorphism above q in $Equ\mathbb{E}$. Let us show it is a coequalizer of $R[f]\square S$. It coequalizes (δ_0^S, δ_0^S) since its image in $\mathbb{E}$ coequalizes the two legs of $R[f]$. Let $(h, \hat{h}) : S \to V$ be any map in $Equ\mathbb{E}$ coequalizing the pair (δ_0^S, δ_0^S). The map h necessarily factors through q; and since $\mathcal{O}_{\mathbb{E}}$ is a left exact fibration, we can now reduce our attention to the case $h = q$. The morphism $(q, \hat{q})$ being a regular epimorphism, it is cocartesian in $Equ\mathbb{E}$, and we get the desired factorization $W \rightarrowtail V$. □

We get a large class of examples of such categories $Equ\mathbb{E}$ with Proposition 2.6. Secondly, if we wish $Equ\mathbb{E}$ to be a regular category, the category $\mathbb{E}$ must be itself regular:

Proposition 3.7. *If the category $Equ\mathbb{E}$ is a regular category, so is $\mathbb{E}$.*

Proof. Since the "discrete" functor $\Delta_{\mathbb{E}} : \mathbb{E} \rightarrowtail Equ\mathbb{E}$ is fully faithful, left exact and has a right adjoint, it makes $\mathbb{E}$ equivalent to a fully faithful subcategory of $Equ\mathbb{E}$, stable under pullbacks and subobjects and regular epimorphisms. Whence the assertion. □

From that, we shall step to the further observation:

Proposition 3.8. *Let $\mathbb{E}$ be a regular category. The following conditions are equivalent:*
(i) $Equ\mathbb{E}$ has coequalizers of effective relations
(ii) $Equ\mathbb{E}$ has regular epimorphisms with any domain above regular epimorphisms in $\mathbb{E}$
(iii) $Equ\mathbb{E}$ has suprema of equivalence relations.
In this case a morphism $(f, \hat{f}) : S \to T$ is regular in $Equ\mathbb{E}$ if and only if f is regular in $\mathbb{E}$ and $f^{-1}(T) = R[f] \bigvee S$. The modular formula holds if and only if the regular epimorphisms in $Equ\mathbb{E}$ are stable under pullbacks along maps in the fibres.

Proof. We already noticed that when $\mathbb{E}$ has coequalizers of equivalence relations we have (i) $\iff$ (ii) and (ii)$\Rightarrow$(iii). Now suppose (iii). Given any regular epimorphism $f : X \twoheadrightarrow Y$ and any equivalence relation S on X, take the supremum $R[f] \bigvee S$. We have $R[f] \subset R[f] \bigvee S$, so that, when $\mathbb{E}$ is regular, the direct image $f(R[f] \bigvee S)$ is an equivalence relation on Y. Then the map:

$$S \rightarrowtail R[f] \bigvee S \twoheadrightarrow f(R[f] \bigvee S)$$

is the regular epimorphism above f by the same proof as the one of Proposition 2.5. So we get (ii).

Now by Proposition 2.7 it is clear that if the regular epimorphisms in $Equ\mathbb{E}$ are stable under pullbacks along maps in the fibres, the modular formula holds. The proof of the converse is obtained in the same way as in Proposition 2.7. □

Whence the following characterization:

Theorem 3.9. *Given any category $\mathbb{E}$, the following conditions are equivalent:*
(i) the category $Equ\mathbb{E}$ is regular
(ii) the category $\mathbb{E}$ is regular, cc-modular and such that:
(*) for any fibrant morphism $(g,\hat{g}) : R \to R'$ and any equivalence relation S on the codomain Y of g we get: $g^{-1}(R' \bigvee S) = R \bigvee g^{-1}(S)$.

Proof. We know that (i) implies that $\mathbb{E}$ is regular. On the other hand (i) implies that there are regular epimorphisms with any domain above regular epimorphisms in $\mathbb{E}$, so $Equ\mathbb{E}$ has suprema of equivalence relations, and since these regular epimorphisms are stable under pullbacks along any morphism, this is the case, in particular, along maps in the fibres of $Equ\mathbb{E}$. Accordingly $\mathbb{E}$ is cc-modular. And by Theorem 2.13, since the regular epimorphisms with any domain above split epimorphisms are stable under pullbacks along cartesian maps in $Equ\mathbb{E}$, given any fibrant morphism $(g,\hat{g}) : R \to R'$ of equivalence relations and any equivalence relation S on the codomain Y of g, we get: $g^{-1}(R' \bigvee S) = R \bigvee g^{-1}(S)$, namely we get the condition (*).

Conversely suppose (ii). By the previous proposition, we know that $Equ\mathbb{E}$ has coequalizers of effective relations and that these regular epimorphisms are stable under pullbacks along maps in the fibres. It remains to show they are stable under pullbacks along cartesian maps in $Equ\mathbb{E}$ when the condition (*) holds. Suppose this condition holds and that we have a pullback in $Equ\mathbb{E}$ as on the left hand side, above the right hand side pullback in $\mathbb{E}$:

$$\begin{array}{ccc} g^{-1}(S) & \xrightarrow{(f,\hat{f})} & h^{-1}(V) \\ {\scriptstyle (g,\tilde{g})}\downarrow & & \downarrow{\scriptstyle (h,\tilde{h})} \\ S & \xrightarrow[(f',(f')^{\sharp})]{} & V \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

where the right hand side vertical map is cartesian in $Equ\mathbb{E}$. So is the left hand side vertical one. Moreover the morphism $R(g) : R[f] \to R[f']$ is fibrant since the right hand side square is a pullback. By the condition (*) we have $g^{-1}(R[f'] \bigvee S) = R[f] \bigvee g^{-1}(S)$. On the other hand, we have $(f')^{-1}(V) = R[f'] \bigvee S$ since the lower map is a regular epimorphism. So we get

$$f^{-1}(h^{-1}(V)) = g^{-1}((f')^{-1}(V)) = g^{-1}(R[f'] \bigvee S) = R[f] \bigvee g^{-1}(S)$$

which means that $(f, \hat{f})$ is a regular epimorphism in $Equ\mathbb{E}$. □

The previous characterization induces the following:

Definition 3.10. *A category $\mathbb{E}$ is said to be cc-regular when any of the previous equivalent conditions holds.*

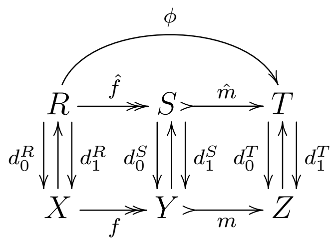
3.3 On the condition (*)

There is now a natural question: how far is a regular category $\mathbb{E}$ from satisfying the condition (*). First notice the following extension of Proposition 1.12:

Proposition 3.11. *Let $\mathbb{E}$ be a regular category with suprema of pairs of equivalence relations, and $(g, \hat{g}) : R \to R'$ a fibrant morphism where g is a regular epimorphism. Then for any equivalence relation S on the codomain Z of g, we get $g^{-1}(R' \bigvee S) = R \bigvee g^{-1}(S)$.*

Proof. We have $R[g] \subset g^{-1}(S) \subset R \bigvee g^{-1}(S)$. By Corollary 3.3, since g is a regular epimorphism and $\mathbb{E}$ a regular category, we get an equivalence relation W on Z such that $g^{-1}(W) = R \bigvee g^{-1}(S)$. Let us show that $W = R' \bigvee S$. Let V be any equivalence relation on Z containing R' and S. So $g^{-1}(V)$ contains $g^{-1}(S)$ and $g^{-1}(R')$, and consequently it contains R as well. Accordingly $g^{-1}(V)$ contains $R \bigvee g^{-1}(S) = g^{-1}(W)$. According to the same Corollary 3.3, we get $W \subset V$. □

Proposition 3.12. *Let $\mathbb{E}$ be a regular category. When m is a monomorphism, f a regular epimorphism and the morphism $(m.f, \phi) : R \to T$ in $Equ\mathbb{E}$ is fibrant:*



then both morphisms $(m, \hat{m})$ and $(f, \hat{f})$ are so, where $\phi = \hat{m}.\hat{f}$ is the canonical decomposition.

Proof. It is clear that S is a reflexive relation on Y. Since the whole rectangle indexed by 0 is a pullback and $\hat{m}$ is a monomorphism, so is the left hand side square. Accordingly we get a hyperextremal epimorphism $(f, \hat{f}) : R \to S$ in $Ref\mathbb{E}$ and, since R is an equivalence relation, so is S by Lemma 1.14, and the left hand side morphism is fibrant. Now, the category $\mathbb{E}$ is regular and the whole rectangle indexed by 0 is a pullback, accordingly the two squares determined by the decomposition of the horizontal maps produce two pullbacks indexed by 0; therefore the two morphisms in question are fibrant. □

Whence the following:

Corollary 3.13. *Let $\mathbb{E}$ be a regular and cc-modular category. It is cc-regular if and only if the condition (*) holds for any monomorphic fibrant morphism $(m, \hat{m}) : R \rightarrowtail R'$ in $Equ\mathbb{E}$.*

3.4 The case of varieties

Any variety $\mathbb{V}$ of algebras is an exact category, and consequently a regular one, where the regular epimorphisms are the homomorphic surjections. On the other hand any pair of congruences has a supremum. Then Lemma 1.6 implies that the only effective equivalence relations on an object S in the category $Equ\mathbb{V}$ are of the kind $R\square S \rightrightarrows S$, and Proposition 3.8 guarantees that $Equ\mathbb{V}$ has regular epimorphisms above regular epimorphisms in $\mathbb{V}$; they are characterized in the following way: $(f, \hat{f})$ is a regular epimorphism above the surjective homomorphism f

$$\begin{array}{ccccc} & & \overset{\hat{f}}{\frown} & & \\ S & \rightarrowtail & f^{-1}(T) & \overset{\tilde{f}}{\twoheadrightarrow} & T \\ d_0^S \downarrow\uparrow\downarrow d_1^S & & d_0 \downarrow\uparrow\downarrow d_1 & & d_0^T \downarrow\uparrow\downarrow d_1^T \\ X & = & Y & \underset{f}{\twoheadrightarrow} & Y \end{array}$$

if and only if $f^{-1}(T) = R[f] \bigvee S$. How far are we now from the stability under pullbacks of these regular epimorphisms in $Equ\mathbb{V}$, namely from the fact that the category $Equ\mathbb{V}$ is itself a regular category?

We know that Mal'tsev varieties $\mathbb{V}$ are such that the category $Equ\mathbb{V}$ is a regular category in which the regular epimorphisms are levelwise. More generally a variety fulfils this last condition if and only if it is a Goursat variety, namely a congruence 3-permutable one, see for instance Theorem 4.3 below.

Since any variety $\mathbb{V}$ is an exact category, the category $Equ\mathbb{V}$ is equivalent to the category $Reg\mathbb{V}$ whose objects are the regular epimorphisms and morphisms are the commutative squares between regular epimorphisms. It is shown in [11], thanks to a result of [18], that, when $\mathbb{V}$ is an ideal determined variety, the category $Reg\mathbb{V}$, and thus the category $Equ\mathbb{V}$, is regular. But, in [23], it is shown that ideal determined varieties need not be congruence 3-permutable. Accordingly there are ideal determined varieties $\mathbb{V}$ such that $Equ\mathbb{V}$ is a regular category in which the regular epimorphisms are not levelwise regular epimorphisms in $\mathbb{V}$.

On the other hand, it is shown in [10] that the *Polin variety* is congruence 4-permutable, but not congruence modular. Accordingly, in this variety $\mathbb{V}$, the regular epimorphisms in $Equ\mathbb{V}$ are not stable under pullback along maps in the fibres of $\mathcal{O}_{\mathbb{V}}$ and therefore the category $Equ\mathbb{V}$ is not regular.

Certainly, it would be very interesting to be able to characterize the varieties $\mathbb{V}$ which are such that $Equ\mathbb{V}$ is regular, namely those varieties $\mathbb{V}$ which are such that the condition (*) holds for any monomorphic fibrant morphism of congruences $(m, \hat{m}) : R \rightarrowtail R'$.

4. Goursat condition in non-regular context

4.1 Composition of relations in the regular context

One of the major interests of regular categories is that the relations $U \rightarrowtail X \times Y$, understood as morphims $X - - > Y$, are composable and that this composition is associative up to isomorphism [8]. The reflexive relations are clearly stable under composition. It is not the case neither for reflexive and symmetric relations nor for equivalence relations. However, if R and S are reflexive relations, the reflexive $R \circ S \circ R$ is necessarily symmetric as soon as R and S are symmetric, since we have $(R \circ S \circ R)^{op} = R^{op} \circ S^{op} \circ R^{op} \simeq R \circ S \circ R$. This remark leads us to the following very general key result:

Proposition 4.1. *Let $\mathbb{E}$ be a regular category and (R, S) a pair of equivalence relations on X. Then the reflexive and symmetric relation $d_1^R((d_0^R)^{-1}(S))$ is nothing but $R \circ S \circ R$. So, the following conditions are equivalent:*
1) $d_1^R((d_0^R)^{-1}(S))$ is an equivalence relation
2) $R \circ S \circ R$ is an equivalence relation
3) $R \circ S \circ R = S \circ R \circ S$.
In this case, we get: $d_1^R((d_0^R)^{-1}(S)) = R \circ S \circ R = R \bigvee S$. On the other hand, given any other equivalence relation T such that $R \subset T$, the modular formula holds: $(R \bigvee S) \cap T = R \bigvee (S \cap T)$.

Proof. We noticed that the objects of $(d_0^R)^{-1}(S)$ are given by the following diagrams in $\mathbb{E}$:

$$\begin{array}{ccc} x & \overset{S}{\dashrightarrow} & y \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\ x' & & y' \end{array}$$

Accordingly, when $\mathbb{E}$ is a regular category, the reflexive and symmetric relation $d_1^R((d_0^R)^{-1}(S))$ is nothing but $R \circ S \circ R^{op} \simeq R \circ S \circ R$. Whence the first point and 1) $\iff$ 2).

Suppose 1), then $(d_1^R, \bar{d}_1^R) : (d_0^R)^{-1}(S) \to R \circ S \circ R$ is a levelwise regular epimorphism in $\mathbb{E}$, and, as such, a regular epimorphism in $Equ\mathbb{E}$ above d_1^R. According to Proposition 2.4, we get $R \circ S \circ R = R \bigvee S$, and then $R \circ S \circ R = S \circ R \circ S$. Whence 3). Now suppose 3). We know already that $R \circ S \circ R$ is reflexive and symmetric. Then $R \circ S \circ R$ becomes transitive since we have $(R \circ S \circ R) \circ (R \circ S \circ R) = R \circ S \circ R \circ R \circ S \circ R = R \circ S \circ R \circ S \circ R = R \circ R \circ S \circ R \circ R = R \circ S \circ R$. Whence 2).

Suppose now we have an equivalence relation T on X such that $R \subset T$. By Lemma 1.7, since $R \subset T$, we know that $(d_0^R)^{-1}(T) = (d_1^R)^{-1}(T)$, so that we get $R[d_1^R] \subset (d_0^R)^{-1}(T)$. Then apply Proposition 3.5 to the split (and thus regular) epimorphism $d_1^R : R \to X$ together with the pair $((d_0^R)^{-1}(S), (d_0^R)^{-1}(T))$:

$$(d_1^R)((d_0^R)^{-1}(S \cap T)) = (d_1^R)((d_0^R)^{-1}(S) \cap (d_0^R)^{-1}(T))$$
$$= (d_1^R)((d_0^R)^{-1}(S)) \cap (d_1^R)((d_0^R)^{-1}(T))) = (d_1^R)((d_0^R)^{-1}(S)) \cap T$$

namely: $R \bigvee (S \cap T) = (R \bigvee S) \cap T$. $\square$

4.2 Goursat regular category according to Carboni-Kelly-Pedicchio

In [7], a regular category was said to be a *Goursat category* (from an old result of 1889, see [13], concerning relations in the category of groups), when the composition of equivalence relations is congruence 3-permutable, namely when, for any pair (R, S) of equivalence relation on X, we get $R \circ S \circ R = S \circ R \circ S$.

In the varietal context, Mitschke [22] gave an example of a congruence 3-permutable variety which is not congruence 2-permutable with the notion of *implication algebra*, namely a set X equipped with a binary operation such that:

$$(x * y) * x = x \ , \quad (x * y) * y = (y * x) * x \ , \quad x * (y * z) = y * (x * z)$$

Later on, Hagemann-Mitschke [17] gave a characterization of congruence 3-permutable varieties, and another example of this notion with the *right-complemented semi-groups*: a variety is congruence 3-permutable if and only if its algebraic theory contains two ternary operations r and s such that $r(x, y, y) = x$, $r(x, x, y) = s(x, y, y)$ and $s(x, x, y) = y$.

Many further investigations about the notion of Goursat regular category can be found in [20], [5], [19], [14], [15]. From Proposition 4.1, we know that $R \circ S \circ R = d_1((d_0^R)^{-1}(S))$. So, from it and Proposition 2.11, we get immediately:

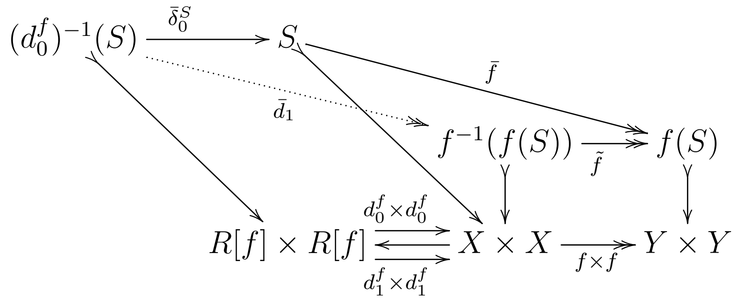
Lemma 4.2. *Let $\mathbb{E}$ be a Goursat regular category. Then:*
(i) given any pair (R, S) of equivalence relations on X, the reflexive relation $d_1((d_0^R)^{-1}(S))$ is the equivalence relation $R \bigvee S$
(ii) it is a cc-modular category
(iii) it is a Gumm category.

The point (iii) is all the more important that in a Gumm category there is a notion of centrentralization of equivalence relations, via the notion of pseudogroupoid and quasi-connector, see [6]. Now we have the following characterizations:

Theorem 4.3. *Let $\mathbb{E}$ be a regular category. The following conditions are equivalent:*
(i) $\mathbb{E}$ is a Goursat regular category
(ii) for any pair (R, S) of equivalence relations on X, $d_1^R((d_0^R)^{-1}(S))$ is an

equivalence relation
(iii) the direct image along a regular epimorphism in $\mathbb{E}$ *of any equivalence relation is an equivalence relation*
(iv) $\mathbb{E}$ *is a cc-regular category such that the regular epimorphims in* $Equ\mathbb{E}$ *are the levelwise regular epimorphisms in* $\mathbb{E}$*.*

Proof. We get (i) $\iff$ (ii) from the Proposition 4.1. Let us check (ii)⇒(iii). Let $f : X \twoheadrightarrow Y$ be a regular epimorphism and S an equivalence relation on X. Then, considering the following diagram in $\mathbb{E}$, there is a dotted factorization $\bar{d}_1$ making the upper quadrangle a pullback, and since $\bar{f}$ is a regular epimorphism, so is this factorization $\bar{d}_1$:



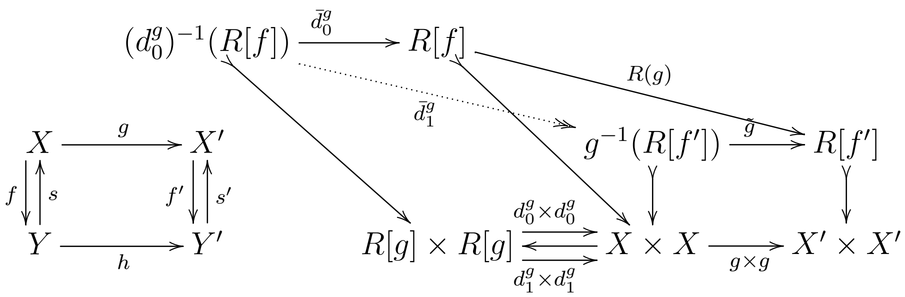
So, $f^{-1}(f(S))$ is the direct image of the equivalence relation $(d_0^f)^{-1}(S)$ along d_1^f, namely $d_1^f((d_0^f)^{-1}(S))$, and thus $f^{-1}(f(S))$ is an equivalence relation which implies, by Corollary 3.3 that $f(S)$ is an equivalence relation as well. On the other hand (ii) is clearly a particular case of (iii). Finally (iii) is equivalent to the fact that $Equ\mathbb{E}$ is a regular category in which the regular epimorphisms are levelwise regular in $\mathbb{E}$ which is (iv). □

The point (ii) of the previous lemma and the characterization (i) $\iff$ (iii) of the previous theorem are already given by Proposition 3.2 and Theorem 6.8 in the pioneering paper [7] where both results were proved by means of calculus of relations (namely using Metatheorems and internal logic, see [1]); we introduced here direct diagrammatic proofs.

4.3 Goursat condition in a non-regular context

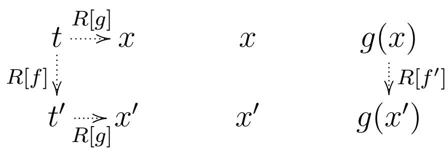
The Mal'tsev categories were first introduced in the regular context [8], before having been freed of this restriction [9]. In the same way, we shall introduce in this section a categorical Goursat condition valid in the non-regular context. For that let us begin by recalling Corollary 2.2: given any commutative square of split epimorphisms in a category $\mathbb{E}$ as on the left hand side of the diagram below: the equivalence relation $g^{-1}(R[f'])$ is the supremum $R[g] \bigvee R[f]$ of the equivalence relations $R[f]$ and $R[g]$. So, according to Theorem 4.3, a Goursat regular category obviously satisfies the following:

Definition 4.4. *Given any category $\mathbb{E}$, it is said to be a Goursat category when, given any morphism in $Pt(\mathbb{E})$ as in the left hand side diagram:*



the unique dotted factorization $\bar{d}_1^g$ is an extremal epimorphism in $\mathbb{E}$.

In set-theoretical terms, the map $\bar{d}_1^g$ associates (x, x') with any left hand side data:



Since the extremal epimorphisms are reflected by the conservative functors which preserve pullbacks, *given any conservative functor $U : \mathbb{F} \to \mathbb{E}$ preserving pullbacks, the category $\mathbb{F}$ is a Goursat category as soon as so is $\mathbb{E}$.* In particular this notion is stable under passage to slice categories $\mathbb{E}/Y$, coslice categories $Y/\mathbb{E}$ and fibres $Pt_Y(\mathbb{E})$.

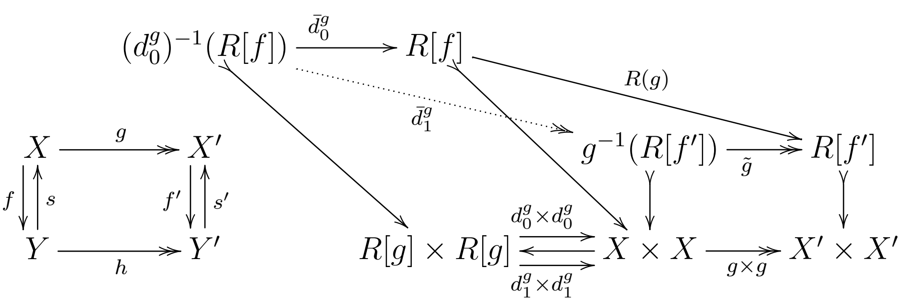
Let $\mathbb{T}$ be an algebraic theory in the sense of Universal algebra and $\mathbb{V}(\mathbb{T})$ the corresponding variety of $\mathbb{T}$-algebras. Any functor category $\mathcal{F}(\mathbb{E}, \mathbb{V}(\mathbb{T}))$ being regular, its extremal epimorphims are those natural transformations which are componentwise extremal epimorphisms. So $\mathcal{F}(\mathbb{E}, \mathbb{V}(\mathbb{T}))$ is clearly a Goursat category in our sense as soon as $\mathbb{V}(\mathbb{T})$ is a Goursat variety. Now let $\mathbb{T}(\mathbb{E})$ be the category of internal $\mathbb{T}$-algebras in $\mathbb{E}$. Then consider the corresponding Yoneda embedding $Y^{\mathbb{T}} : \mathbb{T}(\mathbb{E}) \to \mathcal{F}(\mathbb{E}^{op}, \mathbb{V}(\mathbb{T}))$; it is left exact and conservative. Accordingly *the category $\mathbb{T}(\mathbb{E})$ is a Goursat category is our sense as soon as $\mathbb{V}(\mathbb{T})$ is a Goursat variety*. For instance the category $Imp\mathbb{E}$ of internal implication algebras in any category $\mathbb{E}$ is a Goursat category in our sense.

First we have to check that, in the regular context, our definition coincides with the Carboni-Kelly-Pedicchio one.

Proposition 4.5. *Let $\mathbb{E}$ be a regular category. Then it is a Goursat one according to the previous definition if and only if it is a Goursat regular category in the sense of [7].*

Proof. We just noticed before our definition that any Goursat regular category in the sense of [7] satisfies it.

To get the converse we shall use the characterization given by Theorem 1 in [14]: a regular category is a Goursat category in the sense of [7] if and only if any regular epimorphism (h, g) in $Pt\mathbb{E}$ is hyperregular. Suppose now $\mathbb{E}$ is Goursat in our sense and regular. Now consider the following diagram where both h and g are regular epimorphisms:

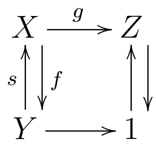


Since g is a regular epimorphism, so are $g \times g$ and $\tilde{g}$. Accordingly so is $R(g).\bar{d}_0^g = \tilde{g}.\bar{d}_1^g$, and the morphism $R(g)$ as well. □

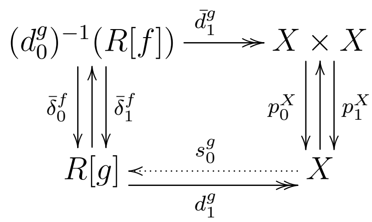
4.4 Goursat categories and fibration of points

In this section we shall show that, in the same way as Mal'tsev categories [3], and Gumm categories [4], Goursat categories are classified by a property of the fibration of points. This was already observed in the Goursat regular case in [15].

Definition 4.6. *Let $\mathbb{E}$ be a pointed category. It is said to be a punctually Goursat category when, given any morphism in $Pt(\mathbb{E})$ above the terminal map:*

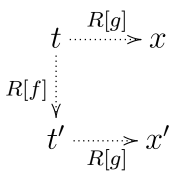


the paraterminal map:



is such that $\bar{d}_1^g$ is an extremal epimorphism in $\mathbb{E}$.

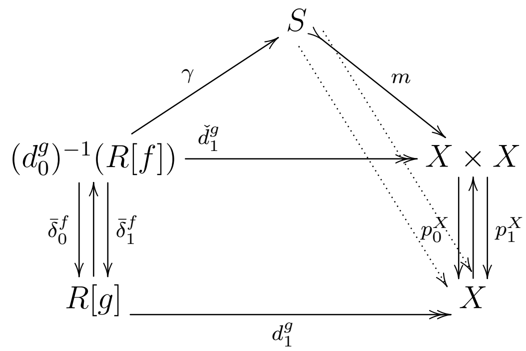
In set theoretical terms, this means that, for any pair of elements (x, x') in $X \times X$, there is a pair $(t, t') \in X \times X$ satisfying the following data:



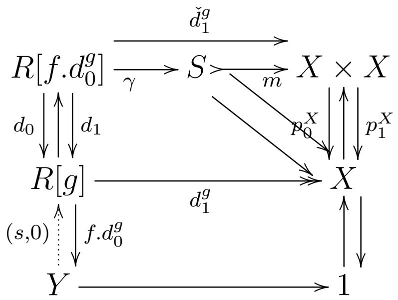
Similarly to above, *given any left exact conservative functor $U : \mathbb{F} \to \mathbb{E}$, the pointed category $\mathbb{F}$ is a punctually Goursat category as soon as so is the pointed category $\mathbb{E}$.*

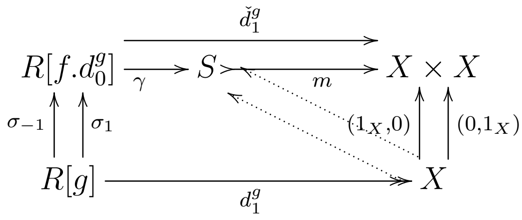
Proposition 4.7. *Any unital category $\mathbb{E}$ is a punctually Goursat category.*

Proof. Suppose $\mathbb{E}$ is unital and consider any decomposition $\check{d}_1^g = m.\gamma$ in $\mathbb{E}$ with a monomorphism m. It produces a relation S on X. We have to show that S coincides with ∇_X.



For that consider the following diagram where $R[f.d_0^g] = (d_0^g)^{-1}(R[f])$:





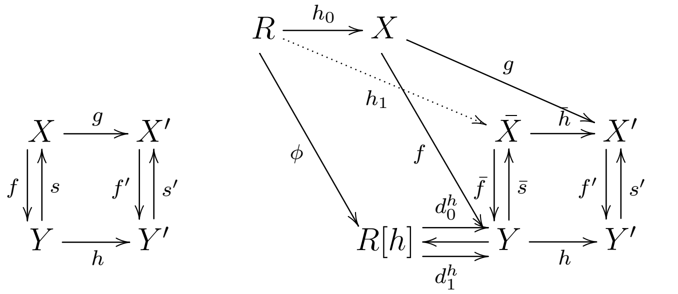
The morphism $f.d_0^g$ is split by $(s,0)$ since $g.s = 0$ and makes the lower square a morphism in $Pt(\mathbb{E})$. In turn, this splitting produces a pair of morphims $(\sigma_{-1}, \sigma_1) : R[g] \rightrightarrows R[f.d_0^g]$ commuting with the pair $((1_X, 0), (0, 1_X))$. Since the map d_1^g is a split epimorphism, the commutative square induced by the pair $(\sigma_{-1}, (1_X, 0))$ produces a factorization of $(1_X, 0)$ through m, while the commutative square induced by the pair $(\sigma_1, (0, 1_X))$ produces a factorization of $(0, 1_X)$ through m. Since, in a unital category, the pair $(1_X, 0), (0, 1_X)$ is jointly extremally epimorphic, the monomorphism m is an isomorphism, and $\check{d}_1^g$ is an extremal epimorphism in the category $\mathbb{E}$. $\square$

Theorem 4.8. *Let $\mathbb{E}$ be any category. It is a Goursat category if and only if any fibre $Pt_Y(\mathbb{E})$ is a punctually Goursat category. Accordingly any Mal'tsev category is a Goursat category in our sense.*

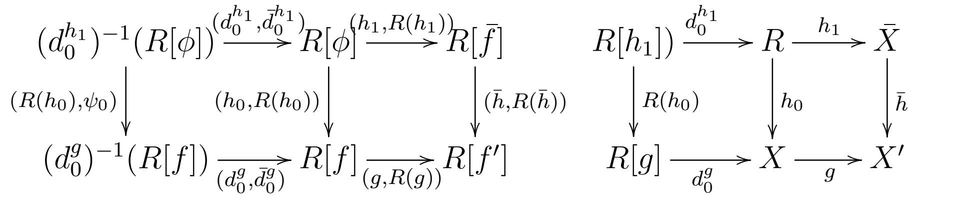
Proof. It is clear that the punctual Goursat axiom for the fibre $Pt_{Y'}(\mathbb{E})$ is the particular case of the Goursat axiom for $\mathbb{E}$ where the lower map h in

Definition 4.4 is split. Accordingly when $\mathbb{E}$ is a Goursat category, any fibre $Pt\mathbb{E}$ is a punctually Goursat one.

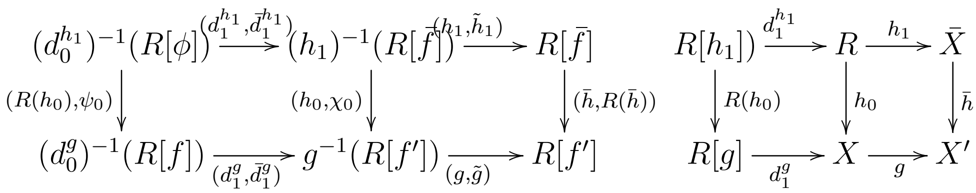
Conversely, starting with any morphism $(h, g) : (f, s) \to (f', s')$ in $Pt(\mathbb{E})$:



complete the diagram by the pullbacks $(\bar{f}, \bar{s})$ of (f', s') along h and (ϕ, σ) of (f, s) along d_0^h. Then there is the dotted factorization h_1 making the upper quadrangle (*) a pullback. As soon as the fibre $Pt_Y(\mathbb{E})$ is a punctually Goursat category, the morphism $(d_1^{h_1}, \bar{d}_1^{h_1}) : (d_0^{h_1})^{-1}(R[\phi]) \to (h_1)^{-1}(R[\bar{f}])$ in $Equ\mathbb{E}$ is a levelwise extremal epimorphism. Now consider the following left hand side commutative diagram in $Equ\mathbb{E}$ above the right hand side diagram in $\mathbb{E}$:



The right hand side square of the left hand side diagram is a pullback since so is the square (*); its left hand side square is a pullback as well since so is its image by the functor $\mathcal{O}_{\mathbb{E}}$ and the parallel horizontal maps are cartesian in $Equ\mathbb{E}$. This defines ψ_0 as the pullback of $R(h_0)$ along $\bar{d}_0^g$. The previous whole rectangles above are the following ones as well:



The right hand side square of the left hand side diagram is a pullback since so is its image by the functor $\mathcal{O}_{\mathbb{E}}$ and the parallel horizontal maps are cartesian in $Equ\mathbb{E}$. This defines χ_0 as the pullback of $R(\bar{h})$ along $\tilde{g}$, and makes the left hand side square of left hand side diagram a pullback as well. Since the vertical right hand side map is clearly fibrant, as produced from a cartesian map in $Pt(\mathbb{E})$, so is the middle vertical one. Then, certainly, χ_0 is a split epimorphism in $\mathbb{E}$, since so is the morphism h_0. Now $\bar{d}_1^g.\psi_0 = \chi_0.\bar{d}_1^{h_1}$ is an extremal epimorphism in $\mathbb{E}$ as a composition of extremal epimorphisms. Accordingly the map $\bar{d}_1^g$ is an extremal epimorphism in $\mathbb{E}$, and $\mathbb{E}$ is a Goursat category in our sense. The last assertion of the theorem comes from the previous proposition and the fact that a category $\mathbb{E}$ is a Mal'tsev one if and only any fibre $Pt_Y(\mathbb{E})$ is unital [3]. □

4.5 Hyperextremal categories

We just introduced a notion of Goursat category in a non-regular context which, in a regular one, coicindes with the one introduced by Carboni-Kelly-Pedicchio. We obviously can ask wether it is the only way to get to this point. Actually Proposition 1.11 here and Theorem 1 in [14] suggest another way:

Definition 4.9. *A category $\mathbb{E}$ is said to be hyperextremal, when any extremal epimorphism in $Pt\mathbb{E}$ is hyperextremal (see definition 1.13).*

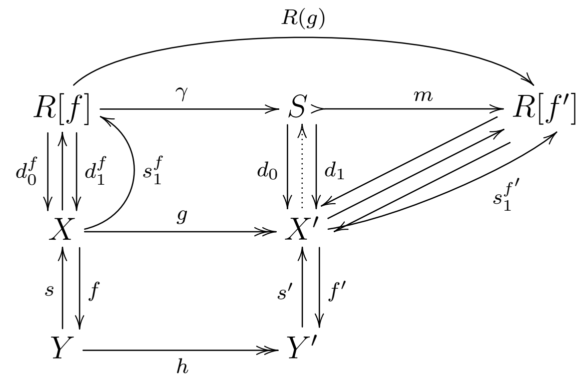
Proposition 4.10. *Given any regular category $\mathbb{E}$, the following conditions are equivalent:*
(i) $\mathbb{E}$ is a Goursat category in our sense
(ii) $\mathbb{E}$ is hyperextremal.

Proof. Suppose (i). Start with an extremal epimorphism $(h, g) : (f, s) \twoheadrightarrow (f', s')$ in $Pt\mathbb{E}$. By Proposition 1.11 the morphism $(g, R(g)) : R[f] \to R[f']$ is a regular epimorphism in $Equ\mathbb{E}$. According to condition (iv) in Theorem 4.3, the morphism $R(g)$ is a regular epimorphism in $\mathbb{E}$, and $(h, g) : (f, s) \twoheadrightarrow (f', s')$ is hyperextremal in $Pt\mathbb{E}$, whence (ii). Conversely suppose (ii). We shall show condition (iii) of Theorem 4.3. Take the direct image $(f, \bar{f}) : S \twoheadrightarrow f(S)$ of any equivalence relation S along the regular epimorphism f. By (ii) $(f, \bar{f})$ is hyperextremal in $Ref\mathbb{E}$, and since S is an equivalence relation, so is $f(S)$ by Proposition 1.14. This, actually, gave us a way of proving Theorem 1 from [14] without resort to Metatheorems. □

This definition is all the more interesting since, in the non-regular context, we get in a very simple way:

Proposition 4.11. *Any Mal'tsev category $\mathbb{E}$ is hyperextremal.*

Proof. Start with an extremal epimorphism (h, g) in $Pt\mathbb{E}$. As observed in Proposition 1.11 the induced morphism $(g, R(g)) : R[f] \to R[f']$ of equivalence relations is an extremal epimorphism in $Equ\mathbb{E}$. Now consider any decomposition $R(g) = \gamma.m$ with a monomorphism m in $\mathbb{E}$:



When g is an extremal epimorphism, the dotted factorization produces a reflexive relation S, which is an equivalence relation since $\mathbb{E}$ is a Mal'tsev category, so that (g, γ) becomes a morphism in $Equ\mathbb{E}$. Now since $(g, R(g))$ is extremal in $Equ\mathbb{E}$, then m is an isomorphism, and $R(g)$ is an extremal epimorphism in $\mathbb{E}$. □

Accordingly, in the case of hyperextremal categories, the two first conditions of the following list of four, taken from the introduction, about the desired notion of non-regular Goursat category are fulfilled:
1) in the regular context, it coincides with the pioneering notion of Carboni-Kelly-Pedicchio
2) any Mal'tsev category satisfies this definition
3) this definition is characterized by a property of the fibration of points $¶_{\mathbb{E}}$
4) when $\mathbb{T}$ is an algebraic theory giving rise to a Goursat variety $\mathbb{V}(\mathbb{T})$, the category $\mathbb{T}(\mathbb{E})$ of internal $\mathbb{T}$-algebras $\mathbb{E}$ is itself a Goursat category.
We do not believe that the point 3) holds, and we are no more in position to prove the point 4). However remain the following questions:

a) in the non-regular context, what are the link between hyperextremal categories and non-regular Goursat categories in our sense?
b) when $\mathbb{T}$ is an algebraic theory giving rise to a Goursat variety $\mathbb{V}(\mathbb{T})$, what are the left exact conditions for a non-regular category $\mathbb{E}$ to guarantee that the category $\mathbb{T}(\mathbb{E})$ of internal $\mathbb{T}$-algebras $\mathbb{E}$ is a hyperextremal category?

5. The case of congruence n-permutable regular categories

Following [7], given any pair (R, S) of reflexive relations on an object X in a regular category $\mathbb{E}$, let us denote by $(R, S)_n, n \geq 2$ the alternate composition $R \circ S \circ R \circ S...$ of length n which is a reflexive relation as well. Clearly we have: $(R, S)_n \subset (R, S)_{n+1}$ and $(S, R)_n \subset (R, S)_{n+1}$. Then call congruence *$n$-permutable* a regular category satisfying $(R, S)_n = (S, R)_n$ for all pairs (R, S) of equivalence relations. Let us recall also a direct consequence of the Theorem 3.1 in [7]:

Theorem 5.1. *Given a regular category $\mathbb{E}$, the following conditions are equivalent:*
(i) the regular category $\mathbb{E}$ is congruence n-permutable
(ii) for any pair (R, S) of equivalence relations in $\mathbb{E}$, $(R, S)_n$ is an equivalence relation as well.
In this case we get $(R, S)_n = R \bigvee S$ in $Equ\mathbb{E}$.

Since regular Mal'tsev and Goursat categories are respectively the congruence 2-permutable and 3-permutable ones, the question similarly arises of a possible definition of the congruence n-permutable categories in the non-regular context. We shall not yet reach that point. However we shall be able to extract some pertinent observations.

Since a congruence n-permutable regular category $\mathbb{E}$ admits suprema of pairs of equivalence relations, Proposition 3.8 guarantees that $Equ\mathbb{E}$ has regular epimorphisms above regular epimorphisms in $\mathbb{E}$. Let us investigate what they look like.

Proposition 5.2. *Given any congruence $2n$-permutable or $(2n+1)$-permutable regular category $\mathbb{E}$, any regular epimorphism $f : X \twoheadrightarrow Y$ and any equivalence relation S on X, then the reflexive relation $f(S)^n$ is an equivalence relation in $\mathbb{E}$. The morphisms $S \to f(S)^n$ in $Equ\mathbb{E}$ are the regular* *epimorphisms above f.*

Proof. In the first case, we have $(R[f], S)_{2n} = R[f] \bigvee S$, while, in the second we have $(R[f], S)_{2n+1} = R[f] \bigvee S$. Accordingly, by Corollary 3.3, in the first case $f((R[f], S)_{2n})$ is an equivalence relation, while so is $f((R[f], S)_{2n+1})$ in the second one. That, in both cases $m = 2n$ and $m = 2n+1$, we have $f((R[f], S)_m) = f(S)^n$ is a direct consequence of the following lemma by: $f((R[f], S)_{2n}) \subset (\Delta_Y, f(S))_{2n} = f(S)^n$
$= f((S \circ R[f])_{2n-1}) \subset f((R[f], S)_{2n})$; and by:
$f((R[f], S)_{2n+1}) \subset (\Delta_Y, f(S))_{2n+1} = f(S)^n = f((S \circ R[f])_{2n-1})$ with $f((S \circ R[f])_{2n-1}) \subset f((R[f], S)_{2n+1})$.

Suppose we have a morphism $(f, \hat{f}) : S \to T$ in $Equ\mathbb{E}$ above f. Then we have $S \subset f^{-1}(T)$, and therefore $f(S) \subset f(f^{-1}(T)) = T$. Accordingly we get $f(S)^n \subset T^n = T$. So the morphism $S \to f(S)^n$ is cocartesian above f and, according to Proposition 1.10, a regular epimorphism. □

Lemma 5.3. *Let $\mathbb{E}$ be a regular category and $f : X \twoheadrightarrow Y$ any regular epimorphism:*
(i) given any pair (S, T) of reflexive relations on X, we have $f(S) \circ f(T) = f((S \circ R[f]) \circ T)$.
(ii) for any reflexive relation S on X, we have $f(S)^n = f((S, R[f])_{2n-1})$.

Proof. (i) In any regular category, we have $f(S \circ T) \subset f(S) \circ f(T)$ for any pair (S, T) of reflexive relations; so we get: $f((S \circ R[f]) \circ T) \subset f(S) \circ f(T)$.

By the Metatheorems [1], it is enough to check the converse in Set. Suppose that we have $yf(S) \circ f(T)y'$ in the set Y. So there is an element $t \in Y$ such that $yf(T)t$ and $tf(S)y'$. This means that, in the set X, there are pairs of elements $(x, u), (v, x')$ such that we have: $f(x) = y, f(u) = t$ and xTu, $f(v) = t, f(x') = y'$ and vSx'. Accordingly we have $xTuR[f]vSx'$, namely $xS \circ R[f] \circ Tx'$ and $yf(S \circ R[f] \circ T)y'$.

(ii) In particular we have $f(S)^2 = f((S, R[f])_3)$. The end of the proof is made by induction. Suppose $f(S)^k = f((R[f], S)_{2k-1}), \forall k < n$. Then: $f(S)^n = f(S)^{n-1} \circ f(S) = f((S, R[f])_{2n-3}) \circ f(S) = f((S, R[f])_{2n-3} \circ R[f] \circ S) = f((S, R[f])_{2n-1})$. □

Theorem 5.4. *Given any regular category $\mathbb{E}$, the following conditions are equivalent:*
(i) the regular category $\mathbb{E}$ is congruence (2n+1)-permutable

(ii) for any regular epimorphism $f : X \twoheadrightarrow Y$ in $\mathbb{E}$ and any equivalence relation S on X, the reflexive relation $f(S)^n$ is an equivalence relation.

Proof. It remains to show (ii)⇒(i). So suppose (ii). We noticed in Proposition 4.1 that the direct image $d_1^R((d_0^R)^{-1}(S))$ is $R \circ S \circ R$. In presence of (ii), $(d_1^R((d_0^R)^{-1}(S))^n = (R \circ S \circ R)^n = (R, S)_{2n+1}$ is an equivalence relation. Accordingly the regular category $\mathbb{E}$ is congruence $(2n+1)$-permutable. □

This characterization holds, obviously, for any variety of Universal Algebra. As it is showed by the case $n = 1$, namely the case of Mal'tsev and Goursat categories, this characterization is surprinsingly not valid for the congruence $2n$-permutable ones. Of course, it would be extremely interesting to have for them a characterization of the same type, which would allow us to understand the nature of the conceptual gap between the odd and even cases.

References

[1] M. Barr – *Exact categories*, Springer L.N. in Math., **236** (1971), 1-120.

[2] D. Bourn – *Lower dimensional geometry of the notion of choice*, CMS Conférence Proc., **13**, Category Theory (1991), 55-73.

[3] D. Bourn – *Mal'cev categories and fibration of pointed objects*, Appl. Categ. Struct. **4** (1996), 307-327.

[4] D. Bourn – *Fibration of points and congruence modularity*, Algebra Universalis **52** (2005), 403-429.

[5] D. Bourn – *Congruence distributivity in Goursat and Mal'cev categories*, Appl. Categ. Struct. **13** (2005), 101-111.

[6] D. Bourn and M. Gran, – *Categorical aspects of modularity*, in :Galois theory, Hopf algebras, and Semiabelian categories, G.Janelidze, B.Pareigis, W.Tholen editors, Fields Institute Communications, **43**, Amer. Math. Soc., 2004, 77-100.

[7] A. Carboni, G. M. Kelly and M. C. Pedicchio – *Some remarks on Mal'cev and Goursat categories*, Appl. Categ. Struct. **1** (1993), 385-421.

[8] A. Carboni, J. Lambek, M.C. Pedicchio – *Diagram chasing in Mal'cev categories*, J. Pure Appl. Algebra **69** (1991), 271-284.

[9] A. Carboni, M.C. Pedicchio, N. Pirovano – *Internal graphs and internal groupoids in Mal'cev categories*, CMS Conference Proceedings **13** (1992), 97-109.

[10] A. Day and R. Freese – *A characterization of identities implying congruence modularity I*, Canad. J. Math., **32** (1980), 1140-1167.

[11] T. Everaert – *Effective descent morphisms of regular epimorphisms*, J. Pure Appl. Algebra. **216** (2012), 1896-1904.

[12] T.H. Fay – *On categorical conditions for congruence to commute*, Algebra Universalis, **8** (1978), 173-179.

[13] E. Goursat – *Sur les substitutions orhtogonales*, Ann. Sci. Ec. Norm. Sup. **3** (1889), 9-102.

[14] M. Gran and D. Rodelo – *A new characterization of Goursat categories*, Appl. Categ. Struct. **20** (2012), 229-238.

[15] M. Gran and D. Rodelo – *Beck-Chevalley condition and Goursat categories*, J. Pure Appl. Algebra. **221** (2017), 2445-2457.

[16] H.P. Gumm – *Geometrical methods in congruence modular algebras*, Mem. Amer. Math. Soc. **45** (1983).

[17] J. Hagemann and A. Mitschke – *On n-permutable congruences*, Algebra Universalis **3** (1973), 8-12.

[18] G. Janelidze, L. Márki, W. Tholen and A. Ursini – *Ideal determined categories*, Cah. Topol. Géom. Diff. Catég. **51** (2010), 115-125.

[19] G. Janelidze and M. Sobral – *Descent for regular epimorphims in Barr exact Goursat categories*, Appl. Categ. Struct. **11** (2011), 271-276.

[20] S. Lack – *The 3 by 3 lemma for Goursat regular categories*, Homology, Homotopy and Applications **6** (2004), 1-3.

[21] J. Lambek – *Goursat's theorem and homological algebra*, Can. Math. Bull. **7** (1964), 597-608.

[22] A. Mitschke – *Implication algebras are 3-permutable and 2-distributive*, Algebra Universalis **1** (1971), 182-186.

[23] J.G. Raftery – *Ideal determined varieties need not be congruence* 3-*permutable*, Algebra Universalis **31** (1994), 293-297.

[24] J. D. H. Smith – *Mal'cev varieties*, Springer Verlag, Lect. Notes Math. **554**, Springer Verlag (1976).

Laboratoire de Mathématiques Pures et Appliquées
Université du Littoral
Bat. H. Poincaré, 50 Rue F. Buisson BP 699
62228 Calais - France
bourn@lmpa.univ-littoral.fr

Backsets and Open Access

Tous les articles publiés dans les "*Cahiers*" depuis leur création sont librement téléchargeables sur les sites

All the papers published in the "*Cahiers*" since their creation are freely downloadable online

Volumes I – VII : http://www.numdam.org/actas/SE
Volumes VIII – LII : http://www.numdam.org/journals/CTGDC

Volumes ≥ L : http://ehres.pagesperso-orange.fr/Cahiers/Ctgdc.htm
et http://cahierstgdc.com

Charles Ehresmann: Œuvres Complètes et Commentées

Online : http://ehres.pagesperso-orange.fr/C.E.WORKS_fichiers/C.E_Works.htm

Part I: 1-2. *Topologie et Géométrie Différentielle*
Part II: 1. *Structures locales*
2. *Catégories ordonnées; Applications en Topologie*
Part III: 1. *Catégories structurées et Quotients*
2. *Catégories internes ett Fibrations*
Part IV: 1. *Esquisses et Complétions.*
2. *Esquisses et structures monoïdales fermées*

De 1980 à 1983, les "*Cahiers"* ont publié des *Suppléments* formés de 7 volumes (édités et commentés par Andrée Ehresmann) réunissant tous les articles du mathématicien Charles Ehresmann (1905-1979) ; ces articles sont suivis de longs commentaires (en Anglais) indiquant leur genèse et les replaçant dans l'histoire. Ces volumes sont aussi librement téléchargeables.

From 1980 to 1983, the "*Cahiers*" have also published *Supplements* consisting of 7 volumes (edited and commented by Andrée Ehresmann) which collect all the articles published by the mathematician Charles Ehresmann (1905-1979); these articles are followed by long comments (in English) to update and complement them. The 7 volumes are freely downloadable.

Mme Ehresmann, ehres@u-picardie.fr

Commission paritaire n° 58964.
ISSN: 1245-530X

.