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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES *Vol. LVIII-3&4 (2017)*

TWO- AND ONE-DIMENSIONAL COMBINATORIAL EXACTNESS STRUCTURES IN KUROSH–AMITSUR RADICAL THEORY, I

by Marco GRANDIS, George JANELIDZE[1]
and László MÁRKI[2]

Résumé. Les auteurs proposent une nouvelle version non-pointée de structure d'exactitude combinatoire pour la théorie abstraite des radicaux de type Kurosh–Amitsur introduite par les deuxième et troisième auteurs en 2003, appelée ci-dessous structure 2-dimensionnelle. Elle est motivée par la notion de catégorie semi-exacte introduite par le premier auteur en 1992 et, brièvement, elle permet de définir un triplet radical-semisimple tel que, si (*R,r,S*) est un tel triplet, alors (*R,S*) est un couple radical-semisimple par rapport à la structure d'exactitude 1-dimensionnelle sous-jacente définie dans ce qui suit.

Abstract. We propose a new, non-pointed, version of combinatorial exactness structure for the abstract theory of Kurosh–Amitsur radicals introduced by the second and third author in 2003. We call it now 2-dimensional. It is motivated by the notion of semiexact category introduced by the first author in 1992, and, briefly, it allows us to define a radical-semisimple triple in such a way that if (*R,r,S*) is a radical-semisimple triple, then (*R,S*) is a radical-semisimple pair with respect to its underlying 1-dimensional exactness structure as defined below.

Key words. Adjoint functors, Kurosh-Amitsur radical, Non-pointed combinatorial exactness, Short exact sequence, Null morphism.

MS Classification. Primary: 18A40, Secondary: 18A20, 18A32, 18A99, 18G50, 18G55, 16N80, 06A15

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0. Introduction

Each of the papers [GrJM2013], [JM2003], and [JM2009] proposes a special combinatorial exactness structure as a framework for an abstract Kurosh–Amitsur type radical theory. We will call these three structures 1-, 2-, and 3-dimensional, respectively (although the 1-dimensional approach was, in a sense, known before: see Remark 1.3 in [GrJM2013]), and study the relationship between the resulting radical theories in a series of papers.

The structures introduced in [JM2003] and [JM2009] will be extended, in order to make them *non-pointed*. This is motivated by the following observation made in [GrJM2013]:

Surprisingly, the non-pointed context allows us to present the theory of closure operators as a special case of the theory of radicals by using semiexact categories in the sense of the first author.

In particular, in the present paper:

- In Section 1 we introduce our non-pointed counterpart of the 2-dimensional exactness structure (Definition 1.1), and its underlying 1-dimensional exactness structure (Definition 1.3). Example 1.6 explains how to associate such a structure to a semiexact category satisfying a mild additional condition.
- Section 2 briefly explains an obvious duality principle, in order to avoid various calculations that become dual to others.
- Section 3 introduces what we call radical-semisimple triples (Definition 3.1), that is, triples (R,r,S) consisting of a radical class R, its corresponding radical function r and semisimple class S; a list of counterparts of the first standard properties well known in Kurosh–Amitsur radical theory is then given.
- Section 4 is devoted to the First Comparison Theorem (Theorem 4.3), which says that if (R,r,S) is a radical-semisimple triple with respect to a given 2-dimensional exactness structure (satisfying a natural additional condition), then (R,S) is a radical-semisimple pair in the sense of [GrJM2013] with respect to the underlying 1-dimensional exactness structure.

- Section 5 briefly recalls the classical case of rings, and says a few words about the intermediate levels of generality. More about the pointed case can be found in [JM2003].
- Section 6 presents topological closure as a radical function. Unlike in [GrJM2013], we do not go to abstract-categorical closure operators here, because that would involve too much of additional material, e.g. from [DikT1995], and we are going to present this in a separate paper.
- Section 7 is devoted to a very simple example, not involving any kind of categorical exactness, showing that a ‘Naive Second Comparison Theorem’, converse to Theorem 4.3, would be obviously false. In fact, a Second Comparison Theorem should cover the classical result of Amitsur and Kurosh saying that the so-called Conditions (R1) and (R2) on a class *R* of rings characterize radical classes (see Theorem 2.15 in [GaW2004]). This will require, if not a ring-theoretic, at least a semi-abelian algebraic context.

1. 1- and 2-dimensional combinatorial exactness structures

The purpose of this section is to

- introduce (Definition 1.1) a non-pointed counterpart of pointed combinatorial exactness structure in the sense of [JM2003], which we shall call a 2-dimensional (combinatorial) exactness structure;
- define (Definition 1.3), for each such structure, its underlying 1-dimensional exactness structure in the sense of [GrJM2013];
- introduce (Definition 1.4) a new notion of a proper short exact sequence in a semiexact category in the sense of [Gr1992a], [Gr1992b], and [Gr2013], and use it to associate a 2-dimensional exactness structure to every semiexact category satisfying a certain completeness condition (Example 1.6).

Definition 1.1. A 2-dimensional (combinatorial) exactness structure is a diagram

$$\begin{tikzcd}
X_2 \arrow[r, "d^1_1"] \arrow[r, bend left=80, "d^1_0"] \arrow[r, bend right=80, "d^1_2"'] & X_1 \arrow[l, bend right=45, "s^1_0"] \arrow[l, bend left=45, "s^1_1"'] \arrow[r, bend left=45, "d^0_0"] \arrow[r, bend right=45, "d^0_1"'] & X_0, \arrow[l, "s^0_0"']
\end{tikzcd} \tag{1.1}$$

in the category of sets, satisfying the simplicial identities

$$d^0_0 s^0_0 = d^0_1 s^0_0 = 1, \tag{1.2}$$
$$d^0_0 d^1_1 = d^0_0 d^1_0,\ d^0_0 d^1_2 = d^0_1 d^1_0,\ d^0_1 d^1_2 = d^0_1 d^1_1, \tag{1.3}$$
$$s^1_1 s^0_0 = s^1_0 s^0_0, \tag{1.4}$$
$$d^1_0 s^1_1 = s^0_0 d^0_0, \tag{1.5}$$
$$d^1_0 s^1_0 = d^1_1 s^1_0 = d^1_1 s^1_1 = d^1_2 s^1_1 = 1, \tag{1.6}$$
$$d^1_2 s^1_0 = s^0_0 d^0_1, \tag{1.7}$$

and equipped with a complete lattice structure on each fibre $(d^1_1)^{-1}(a)$, for $a \in X_1$, such that $s^1_1(a)$ and $s^1_0(a)$ are, respectively, the smallest and the largest element in $(d^1_1)^{-1}(a)$.

Example 1.2. A pointed combinatorial exactness structure in the sense of Definition 2.1 of [JM2003] is nothing but a 2-dimensional exactness structure of Definition 1.1 in the case when X_0 is a one-element set. The notation we use here is, however, not the same; specifically:

- while X_1 and X_2 in the two definitions play the same role, X_0 being a one-element set is not mentioned in [JM2003], and so are the maps d^0_0, d^0_1, and s^0_0: instead, the element of X_1 corresponding to the unique element of X_0 under s^0_0 is denoted by 0 in [JM2003];
- the maps d^1_0, d^1_1, d^1_2, s^1_0, and s^1_1 of Definition 1.1 correspond, respectively, to the maps d_0, d_1, d_2, e_1, and e_0 of [JM2003].

Recall from [GJM2013] (slightly changing the notation) that a 1-*dimensional exactness structure* is a system $(A,Z,\lhd,\mathrel{-\mkern-6mu\rhd})$ in which A is a set, Z is a subset of A, and $\lhd$ and $\mathrel{-\mkern-6mu\rhd}$ are binary relations on A such that, for every a in A, there exist z and z' in Z with $z \lhd a$ and $a \mathrel{-\mkern-6mu\rhd} z'$.

Definition 1.3. Given a 2-dimensional exactness structure, we define its underlying 1-dimensional exactness structure as the system $(X_1,s^0_0(X_0),\lhd,\mathrel{-\mkern-6mu\rhd})$ in which $u \lhd v$ when there exists $x \in X_2$ with $d^1_0(x) = u$ and $d^1_1(x) = v$, and $v \mathrel{-\mkern-6mu\rhd} w$ when there exists $x \in X_2$ with $d^1_1(x) = v$ and $d^1_2(x) = w$.

Note that $(X_1, s^0_0(X_0),\lhd,\mathrel{-\mkern-6mu\rhd})$ constructed as in Definition 1.3 is indeed a 1-dimensional exactness structure, since, for every $v \in X_1$, we have

$$s^0_0d^0_0(v) \lhd v, \tag{1.8}$$

$$v \mathrel{-\mkern-6mu\rhd} s^0_0d^0_1(v). \tag{1.9}$$

Here (1.8) follows from $d^1_0s^1_1(v) = s^0_0d^0_0(v)$ and $d^1_1s^1_1(v) = v$, while (1.9) follows from $d^1_1s^1_0(v) = v$ and $d^1_2s^1_0(v) = s^0_0d^0_1(v)$.

Now, let us recall from [GrJM2013]:

A *semiexact* (=ex1-exact) *category* **C** in the sense of [Gr1992a] can be described as the data

$$\begin{tikzcd} \mathbf{C}_1 \arrow[r, bend left=60, "D"] \arrow[r, bend right=60, "C"'] & \mathbf{C}_0 \arrow[l, "E"'] \end{tikzcd}, \quad C \dashv E \dashv D, \tag{1.10}$$

in which:

- $\mathbf{C}_1$ is a category, $\mathbf{C}_0$ a full replete subcategory of $\mathbf{C}_1$, and E is the inclusion functor;
- D and C are a right adjoint left inverse and a left adjoint left inverse of E, respectively;
- all the counit components $\iota_A : D(A) \to A$ are monomorphisms that admit pullbacks along arbitrary morphisms into A, and all the unit

components $\pi_A : A \to C(A)$ are epimorphisms that admit pushouts along arbitrary morphisms from A.

Next, we need some discussion that will lead us to introducing the notion of proper short exact sequence which we are going to use:

One usually says that a diagram

$$U \to V \to W \tag{1.11}$$

in a category with a zero object is a short exact sequence if $U \to V$ is a kernel of $V \to W$ and $V \to W$ is a cokernel of $U \to V$, or, equivalently, if the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ 0 \arrow[r] & W \end{tikzcd} \tag{1.12}$$

is a pullback and a pushout at the same time. We shall refer to these equivalent conditions as the *kernel-cokernel condition* and the *pullback-pushout condition*.

In the semiexact context (with $U \to V \to W$ being a diagram in $\mathbf{C}_1$, where $\mathbf{C}_1$ is as in (1.10)), although the kernel-cokernel condition can be copied word for word using kernels and cokernels in the sense of [Gr1992a], there is a problem with the pullback-pushout condition, since:

- while $U \to V$ is a kernel of $V \to W$ if and only if $U \to V$ is a pullback of $D(W) \to W$ along $V \to W$,
- $V \to W$ is a cokernel of $U \to V$ if and only if $V \to W$ is a pushout of $U \to C(U)$ along $U \to V$.

That is, in order to copy the pullback-pushout condition we need $D(W)$ and $C(U)$, both of which will replace the zero object, to be canonically isomorphic.

In order to explain what "canonical" means, consider the commutative diagrams

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r] & U \arrow[rr, "f"] \arrow[d] & & W \arrow[r] & \mathrm{Coker}(f) \\ & \mathrm{Coker}(\ker(f)) \arrow[rr, "\bar{f}"'] & & \mathrm{Ker}(\mathrm{coker}(f)) \arrow[u] & \end{tikzcd} \tag{1.13}$$

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d] & W \\ C(U) \arrow[r, "\bar{f}"'] & D(W) \arrow[u] \end{tikzcd} \tag{1.14}$$

where f is the composite $U \to V \to W$ and $\bar{f}$ is induced by f. The existence and uniqueness of such $\bar{f}$ in (1.13) follows from:

- the universal property of a kernel and the fact that f is a null morphism in the semiexact context of [Gr1992a],
- or, equivalently, from the universal property of a cokernel and the fact that f is a null morphism in the semiexact context of [Gr1992a],

while the existence and uniqueness of such $\bar{f}$ in (1.14) follows from:

- the universal property of $D(W) \to W$ and the fact that f factors as $U \to C(U) \to W$,
- the universal property of $U \to C(U)$ and the fact that f factors as $U \to D(W) \to W$.

Moreover, the square part of diagram (1.13) is in fact the same as diagram (1.14). Indeed, since f is a null morphism in the sense of [Gr1992a], we can take Ker(f) = U and Coker(f) = W, and assume that Ker(f) $\to U$ and $W \to$ Coker(f) are the identity morphisms of U and W, respectively; this makes $U \to C(U)$ the cokernel of Ker(f) $\to U$ and makes $D(W) \to W$ the kernel of $W \to$ Coker(f).

It follows that there is a clear notion of the canonical morphism $C(U) \to D(W)$ for each short exact sequence $U \to V \to W$, namely, it is the morphism $\bar{f}$ above; and we introduce:

Definition 1.4. (a) A short exact sequence $U \to V \to W$ in a semiexact category (1.10) will be called proper if the canonical morphism $C(U) \to D(W)$ is an isomorphism.

(b) For two proper short exact sequences $U \to V \to W$ and $U' \to V' \to W'$, we shall write $(U \to V \to W) \leq (U' \to V' \to W')$ if $V = V'$ and there exist morphisms $U \to U'$ and $W \to W'$ making the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d, equal] & W \arrow[d] \\ U' \arrow[r] & V' \arrow[r] & W' \end{tikzcd} \tag{1.15}$$

commute.

(c) If $(U \to V \to W) \leq (U' \to V' \to W')$ and $(U' \to V' \to W') \leq (U \to V \to W)$, then we will say that $U \to V \to W$ and $U' \to V' \to W'$ are equivalent, and the equivalence class of $U \to V \to W$ will be denoted by $[U \to V \to W]$.

Remark 1.5. (a) Since a short exact sequence $U \to V \to W$ is determined, up to isomorphism, by each of the morphisms $U \to V$ and $V \to W$, Definition 1.4 also suggests us to define *proper normal monomorphisms* and *proper normal epimorphisms* as those normal monomorphisms and normal epimorphisms that appear as such $U \to V$ and $V \to W$, respectively, in proper short exact sequences.

(b) There are many situations where every short exact sequence is proper. For example, this is obviously the case if the ground semiexact category is pointed or satisfies axiom (ex3) of [Gr1992a], [Gr1992b], and [Gr2013].

Now we are ready to present our main example of a 2-dimensional exactness structure:

Example 1.6. Given a semiexact category (1.10) in which we assume $\mathbf{C}_1$ and $\mathbf{C}_0$ to be small skeletons, we would like to construct the associated 2-dimensional exactness structure (1.1) by saying that:

(a) X_0 and X_1 are the sets of objects of $\mathbf{C}_0$ and $\mathbf{C}_1$, respectively;

(b) X_2 is the set of equivalence classes of proper short exact sequences in the sense of Definition 1.4;

(b) the maps d_0^0, d_1^0, and s_0^0 are the object functions of the functors D, C, and E, respectively;

(c) the other maps involved in (1.1) are defined as follows:

$$\begin{gathered} d_0^1[U \to V \to W] = U,\ d_1^1[U \to V \to W] = V,\ d_2^1[U \to V \to W] = W, \\ s_0^1(U) = [U = U \to C(U)],\ s_1^1(U) = [D(U) \to U = U]; \end{gathered} \tag{1.16}$$

(d) the order on $(d_1^1)^{-1}(V)$ is defined according to Definition 1.4.

However, to do this we need an additional assumption on the data (1.10), namely that each $(d_1^1)^{-1}(V)$ be a complete lattice. We could briefly refer to this assumption by saying that our semiexact category *admits proper intersections*. Note also that the only reason of our restriction to *proper* short exact sequences in (b) is that the second equality of (1.3) should be satisfied.

2. Duality

Any 2-dimensional exactness structure (1.1) has its *opposite*, or *dual*, 2-dimensional exactness structure, in which:

- the sets X_i (i = 1, 2, 3) and the maps d_1^1 and s_0^0 are the same as in the original structure;
- the maps d_0^0, d_0^1, and s_0^1 of the original structure play the roles of the maps d_1^0, d_2^1, and s_1^1 of the opposite structure, and vice versa;
- for each $a \in X_1$, the order on $(d_1^1)^{-1}(a)$ in the opposite structure is opposite to the order in the original structure.

This gives the obvious *duality principle*, saying that every property that holds in all 2-dimensional exactness structures has an obvious dual, which also holds in all 2-dimensional exactness structures. For example, so are properties (1.8) and (1.9), and after proving (1.8) we could simply say: “dually, we obtain (1.9)”.

Similarly, the opposite category of any semiexact category is semiexact, and the data opposite to (1.10) is

$$\begin{tikzcd}
(\mathbf{C}_1)^{\mathrm{op}} \arrow[rr, bend left=50, "C^{\mathrm{op}}"] \arrow[rr, bend right=50, "D^{\mathrm{op}}"'] & & (\mathbf{C}_0)^{\mathrm{op}} \arrow[ll, "E^{\mathrm{op}}"']
\end{tikzcd}, \quad D^{\mathrm{op}} \dashv E^{\mathrm{op}} \dashv C^{\mathrm{op}}, \tag{2.1}$$

Moreover, the duality principals for the two types of data obviously agree with each other in the sense that the associated 2-dimensional exactness structure of the opposite semiexact category is opposite to the associated 2-dimensional exactness structure of the original semiexact category.

3. Radicals in terms of 2-dimensional exactness structures

The general approach to radicals developed in this section is almost a straightforward extension of the approach of Section 2 of [JM2003] from the context of a pointed combinatorial exactness structure recalled in Example 1.2 to the general context of Definition 1.1.

For a fixed 2-dimensional exactness structure (1.1) of Definition 1.3, consider the diagram

$$\begin{tikzcd}
K \arrow[r, shift right, "\Sigma_f"'] & L \arrow[l, shift right, "f"'] \arrow[r, shift left, "g"] & K \arrow[l, shift left, "\Pi_g"]
\end{tikzcd} \tag{3.1}$$

in which:

- $L = \{l : X_1 \to X_2 \mid d_1^1 l = 1_{X_1}\} = \prod_{a \in X_1} (d_1^1)^{-1}(a)$, considered as a complete lattice;
- K is the complete lattice of all subsets of X_1 containing the image of s_0^0;
- f and g are defined by $f(l) = d_0^1 l(X_1)$ and $g(l) = d_2^1 l(X_1)$;
- Σ_f and Π_g are defined by $\Sigma_f(k) = \vee\{l \in L \mid f(l) \leq k\}$ and $\Pi_g(k) = \wedge\{l \in L \mid g(l) \leq k\}$.

Note that, for each $a \in X_1$, since $s_1^1(a)$ and $s_0^1(a)$ are, respectively, the smallest and the largest element in $(d_1^1)^{-1}(a)$, we have:

- (1.4) implies that, for each $z \in X_0$, the lattice $(d^1_1)^{-1}(s^0_0(z))$ has only one element, namely $s^1_1 s^0_0(z) = s^1_0 s^0_0(z)$,
- and, in particular, $ls^0_0(z) = s^1_1 s^0_0(z) = s^1_0 s^0_0(z)$ for each $l \in L$;
- consequently, $d^1_0 l s^0_0(z) = s^0_0(z) = d^1_2 l s^0_0(z)$, and so $f(l)$ and $g(l)$ indeed belong to K.

Using this notation and extending Definition 2.5 of [JM2003], we introduce:

Definition 3.1. (a) A map $r \in L$ is said to be a radical function (with respect to the given 2-dimensional exactness structure) if $\Sigma_f f(r) = r = \Pi_g g(r)$.

(b) A subset R in X_1 is said to be a radical class if it corresponds to a radical function via f, that is, there exists a radical function r with $f(r) = R$.

(c) A subset S in X_1 is said to be a semisimple class if it corresponds to a radical function via g, that is, there exists a radical function r with $g(r) = S$.

(d) if (b) and (c) hold for the same radical function r, then we say that (R,r,S) is a radical-semisimple triple.

According to this definition, there are canonical bijections:

$$\text{Radical classes} \approx \text{Radical functions} \approx \text{Semisimple classes}. \qquad (3.2)$$

There is a number of standard properties of a radical-semisimple triple to be listed, to which the rest of this section is devoted.

Theorem 3.2. (R,r,S) *is a radical-semisimple triple with respect to a given 2-dimensional exactness structure if and only if* (S,r,R) *is a radical-semisimple triple with respect to the opposite 2-dimensional exactness structure.* □

In the rest of this section we are dealing with a given fixed 2-dimensional exactness structure (1.1), without further notice.

Theorem 3.3. *Let R and S be subsets of X_1, and $r : X_1 \to X_2$ be a map. Then the following conditions are equivalent:*

(a) (R,r,S) *is a radical-semisimple triple;*

(b) *for each a in X_1, $r(a)$ is the largest element x in the lattice $(d_1^1)^{-1}(a)$ with $d_0^1(x)$ in R, and, at the same time, is the smallest element y in the lattice $(d_1^1)^{-1}(a)$ with $d_2^1(y)$ in S.*

Proof. (a)⇒(b): Just note that, for each a in X_1, we have

$$\vee\{x \in (d_1^1)^{-1}(a) \mid d_0^1(x) \in R\} = r(a) = \wedge\{x \in (d_1^1)^{-1}(a) \mid d_2^1(x) \in S\}, \qquad (3.3)$$

$r(a)$ is in $(d_1^1)^{-1}(a)$, $d_0^1r(a)$ is in R (by 3.1(b) and 3.1(d)), and $d_2^1r(a)$ is in S (by 3.1(c) and 3.1(d)).

(b)⇒(a): According to Definition 3.1, (a) means:

$$R = f(r),\ S = g(r),\ \Sigma_f f(r) = r = \Pi_g g(r). \qquad (3.4)$$

The first two equalities of (3.4) are

$$R = d_0^1r(X_1),\ S = d_2^1r(X_1), \qquad (3.5)$$

respectively, while the last two are the same as (3.3) required for each a in X_1. We observe:

- The inclusions $d_0^1r(X_1) \subseteq R$ and $d_2^1r(X_1) \subseteq S$ follow from (b) trivially.
- For each $a \in X_1$, the largest element in the lattice $(d_1^1)^{-1}(a)$ is $s_0^1(a)$ (see Definition 1.1), and when a is in R we have $d_0^1s_0^1(a) = a \in R$ (see (1.6)). Therefore

  $$a \in R \Rightarrow r(a) = s_0^1(a) \qquad (3.6)$$

  by (b). This gives $a = d_0^1r(a)$, showing that every element a of R belongs to $d_0^1r(X_1)$. That is, $R \subseteq d_0^1r(X_1)$. The inclusion $S \subseteq d_2^1r(X_1)$ is dual to this inclusion.
- (3.3) immediately follows from (b). □

Corollary 3.4. *Let (R,r,S) be a radical-semisimple triple and a an element in X_1. Then $r(a)$ is the unique element x in $(d_1^1)^{-1}(a)$ with $d_0^1(x)$ in R and $d_2^1(x)$ in S.*

Proof. We know that $d_0^1r(a)$ is in R and $d_2^1r(a)$ is in S. On the other hand, if x is in $(d_1^1)^{-1}(a)$ with $d_0^1(x)$ in R and $d_2^1(x)$ in S, then, by 3.3(b), we have:

- $x \leq r(a)$ in $(d^1_1)^{-1}(a)$, since $d^1_0(x)$ in R;
- $r(a) \leq x$ in $(d^1_1)^{-1}(a)$, since $d^1_2(x)$ in S. □

Our next two propositions will partly use the following additional condition, which is self-dual since its parts (a) and (b) are dual to each other:

Condition 3.5. *For* $x \in X_2$,

(a) $d^1_0(x) = s^0_0 d^0_0 d^1_1(x) \Rightarrow x = s^1_1 d^1_1(x)$;

(b) $d^1_2(x) = s^0_0 d^0_1 d^1_1(x) \Rightarrow x = s^1_0 d^1_1(x)$.

Remark 3.6. There are several convenient equivalent ways to reformulate Condition 3.5. One of them is to replace the implications in (a) and (b) with equivalences. Indeed, $x = s^1_1 d^1_1(x)$ implies $d^1_0(x) = d^1_0 s^1_1 d^1_1(x) = s^0_0 d^0_0 d^1_1(x)$, where the second equality follows from (1.5); and dually, $x = s^1_0 d^1_1(x)$ implies $d^1_2(x) = s^0_0 d^0_1 d^1_1(x)$. Another equivalent way to express conditions 3.5(a) and 3.5(b), respectively, is to require:

(a) $d^1_0(x) \in s^0_0(X_0)$ if and only if x is the smallest element of the lattice $(d^1_1)^{-1}(d^1_1(x))$;

(b) $d^1_2(x) \in s^0_0(X_0)$ if and only if x is the largest element of the lattice $(d^1_1)^{-1}(d^1_1(x))$.

Proposition 3.7. *Let R be a radical class and r the corresponding radical function. Then, for $a \in X_1$, conditions (a), (b), (c) below are equivalent and imply (d), while (d) is equivalent to (e). Under Condition 3.5(b), condition (d) also implies the other conditions:*

(a) $a \in R$;

(b) $r(a) = s^1_0(a)$;

(c) $d^1_0 r(a) = a$;

(d) $d^1_2 r(a) = s^0_0 d^0_1(a)$;

(e) $d^1_2 r(a) \in s^0_0(X_0)$.

Proof. The arguments needed to prove (c)$\Rightarrow$(a)$\Leftrightarrow$(b)$\Rightarrow$(c) are in fact contained in the proof of Theorem 3.3. Nevertheless let us present them:

Since $s^1_0(a)$ is the largest in element in $(d^1_1)^{-1}(a)$, (a)$\Leftrightarrow$(b) follows from Theorem 3.3 (cf. (3.6)).

(b)$\Rightarrow$(c): Assuming (b), we have: $d^1_0r(a) = d^1_0s^1_0(a) = a$, where the last equality follows from (1.6).

(c)$\Rightarrow$(a): Assuming (c) and using (1.6) again, we have: $a = d^1_0r(a) \in d^1_0r(X_1) = f(r) = R$.

(b)$\Rightarrow$(d): Assuming (b), we have: $d^1_2r(a) = d^1_2s^1_0(a) = s^0_0d^0_1(a)$, where the last equality follows from (1.7).

(d)$\Rightarrow$(e) is trivial.

(e)$\Rightarrow$(d): If $d^1_2r(a) = s^0_0(z)$ for some $z \in X_0$, then

$d^1_2r(a) = s^0_0d^0_1d^1_2r(a)$ (by (1.2))
$= s^0_0d^0_1d^1_1r(a)$ (by the third equality in (1.3))
$= s^0_0d^0_1(a)$ (since $d^1_1r(a) = a$),

as desired.

(d)$\Rightarrow$(b) under Condition 3.5(b): Since $r(a)$ belongs to $(d^1_1)^{-1}(a)$, (d) gives $d^1_2r(a) = s^0_0d^0_1d^1_1r(a)$, and then Condition 3.5(b) gives $r(a) = s^1_0d^1_1r(a)$. But $d^1_1r(a) = a$, and so we obtain (b). □

Dually, we have:

Proposition 3.8. *Let S be a semisimple class and r the corresponding radical function. Then, for $a \in X_1$, conditions (a), (b), (c) below are equivalent to each other and imply (d), while (d) is equivalent to (e). Under Condition 3.5(a), condition (d) also implies the other conditions:*

(a) $a \in S$;

(b) $r(a) = s^1_1(a)$;

(c) $d^1_2r(a) = a$;

(d) $d_0^1 r(a) = s_0^0 d_0^0(a)$.

(e) $d_0^1 r(a) \in s_0^0(X_0)$. □

Proposition 3.9. *Let* (R,r,S) *be a radical-semisimple triple. Then* $R \cap S = s_0^0(X_0)$.

Proof. The inclusion $s_0^0(X_0) \subseteq R \cap S$ follows from the definition of K in (3.1). If a is in $R \cap S$, then $a = d_0^1 r(a)$ by 3.7(c) and $d_0^1 r(a) \in s_0^0(X_0)$ by 3.8(e), which implies that a is in $s_0^0(X_0)$. □

4. The First Comparison Theorem

The purpose of this section is to formulate and prove Theorem 4.3, which describes a situation where every radical-semisimple triple determines a radical-semisimple pair in the sense of [GrJM2013].

Let us recall from [GrJM2013]:

Given a 1-dimensional exactness structure $(A,Z,\triangleleft,\mathrel{-\!\triangleright},)$, and using the binary relations

$$\alpha = \{(a,b) \in A \times A \mid a \triangleleft b \Rightarrow a \in Z\}, \tag{4.1}$$
$$\beta = \{(a,b) \in A \times A \mid a \mathrel{-\!\triangleright} b \Rightarrow b \in Z\} \tag{4.2}$$

on A, we define maps α_* and β^* from the power set $\mathrm{P}(A)$ to itself by

$$\alpha_*(U) = \{b \in A \mid a \in U \Rightarrow a\alpha b\},\ \beta^*(U) = \{a \in A \mid b \in U \Rightarrow a\beta b\}. \tag{4.3}$$

Then a pair (R,S) of subsets of A is said to be a *radical-semisimple pair* (Definition 5.2(b) of [GrJM2013]), with respect to the given 1-dimensional exactness structure, if $R = \beta^*(S)$ and $S = \alpha_*(R)$. Accordingly, a subset U of A is said to be a radical class (semisimple class) if it occurs as the first (second) component in some radical-semisimple pair; that is, U is a radical class (semisimple class) if and only if $U = \beta^*\alpha_*(U)$ ($U = \alpha_*\beta^*(U)$).

As mentioned in [GrJM2013], the following two propositions are nothing but explicit reformulations of the definition above:

Proposition 4.1. (Proposition 5.3 of [GrJM2013]) *Let* $(A,Z,\triangleleft, \mathrel{-\!\triangleright})$ *be a 1-dimensional exactness structure. A subset R in A is a radical class if and only if satisfies the following conditions:*

(a) *if* a *is in* R, *then, for every* $b \in A \setminus Z$ *with* $a \mathrel{-\!\!\triangleright} b$, *there exists* $c \in R \setminus Z$ *with* $c \triangleleft b$;

(b) *given* a *in* A, *if, for every* $b \in A \setminus Z$ *with* $a \mathrel{-\!\!\triangleright} b$, *there exists* $c \in R \setminus Z$ *with* $c \triangleleft b$, *then* a *is in* R. □

Proposition 4.2. (Proposition 5.4 of [GrJM2013]) *Let* $(A,Z,\triangleleft,\mathrel{-\!\!\triangleright})$ *be a 1-dimensional exactness structure. A subset* S *in* A *is a semisimple class if and only if satisfies the following conditions:*

(a) *if* a *is in* S, *then, for every* $b \in A \setminus Z$ *with* $b \triangleleft a$, *there exists* $c \in S \setminus Z$ *with* $b \mathrel{-\!\!\triangleright} c$;

(b) *given* a *in* A, *if, for every* $b \in A \setminus Z$ *with* $b \triangleleft a$, *there exists* $c \in S \setminus Z$ *with* $b \mathrel{-\!\!\triangleright} c$, *then* a *is in* S. □

Our First Comparison Theorem, which compares radical-semisimple triples in the sense of Definition 3.1 with radical-semisimple pairs in the sense of [GrJM2013], is:

Theorem 4.3. *Let* (R,r,S) *be a radical-semisimple triple with respect to a given 2-dimensional exactness structure in the sense of Definition 1.1, satisfying Condition 3.5. Then* (R,S) *is a radical-semisimple pair in the sense of [GrJM2013] with respect to the underlying 1-dimensional exactness structure in the sense of Definition 1.3.*

Proof. First of all note that, for every $x \in X_2$, we have

$$d_0^1(x) \triangleleft d_1^1(x) \mathrel{-\!\!\triangleright} d_2^1(x), \tag{4.4}$$

which trivially follows from the definitions of $\triangleleft$ and $\mathrel{-\!\!\triangleright}$. In particular, for every $a \in X_1$ and every radical-semisimple triple (R,r,S), we have

$$d_0^1 r(a) \triangleleft a \mathrel{-\!\!\triangleright} d_2^1 r(a) \text{ with } d_0^1 r(a) \text{ in } R \text{ and } d_2^1 r(a) \text{ in } S, \tag{4.5}$$

obtained from (4.4) by taking $x = r(a)$.

What we have to prove are the equalities $R = \beta^*(S)$ and $S = \alpha_*(R)$.
To prove the inclusion $\beta^*(S) \subseteq R$, we take $a \in \beta^*(S)$ and observe:

- Since a is in $\beta^*(S)$ and $d_2^1 r(a)$ in S, we have $a\beta d_2^1 r(a)$ by the definition of $\beta^*(S)$.

- Since $a\beta d^1_2 r(a)$ and $a \mathrel{-\!\!\rhd} d^1_2 r(a)$, we know that $d^1_2 r(a)$ is in $s^0_0(X_0)$ by the definition of β.
- Since $d^1_2 r(a)$ is in $s^0_0(X_0)$ and Condition 3.5(b) holds, a is in R by the implication (e)$\Rightarrow$(a) in Proposition 3.7.

To prove the inclusion $R \subseteq \beta^*(S)$, we take $a \in R$ and $b \in S$ with $a \mathrel{-\!\!\rhd} b$, and we need to show that b is in $s^0_0(X_0)$. Indeed, $a \mathrel{-\!\!\rhd} b$ means that $d^1_1(x) = a$ and $d^1_2(x) = b$ for some $x \in X_2$, and we observe:

- By Theorem 3.3(b), $r(a)$ is the smallest element y in the lattice $(d^1_1)^{-1}(a)$ with $d^1_2(y)$ in S. By our assumptions on x, this gives $r(a) \leq x$.
- On the other hand, by the implication (a)$\Rightarrow$(b) in Proposition 3.7, we have $r(a) = s^1_0(a)$, which is the largest element in the lattice $(d^1_1)^{-1}(a)$. Together with the previous observation, this gives $x = r(a) = s^1_0(a)$.
- Since $x = r(a) = s^1_0(a)$, we have $b = d^1_2 s^1_0(a) = s^0_0 d^0_1(a) \in s^0_0(X_0)$, using (1.7).

This proves the equality $R = \beta^*(S)$, and the equality $S = \alpha_*(R)$ is dual to it. □

5. Classical contexts for Kurosh–Amitsur radicals

Ignoring the *problem of size* and the *difference between a category and its skeleton*, we take the ground 2-dimensional exactness structure (1.1) to be constructed as in Example 1.6 out of the category **Rings** of rings. The rings here are required to be associative but not required to be unital; in particular, the category **Rings** is pointed.

What are the radical-semisimple triples with respect to this structure and what are the radical-semisimple pairs with respect to its underlying 1-dimensional exactness structure?

The answers, as explained in [JM2003] and [GrJM2013], immediately come out of well-known results in the Kurosh–Amitsur radical theory, and they can be stated as:

Theorem 5.1. (a) (R,r,S) *is a radical-semisimple triple if and only if R, r, and S are a radical class, a radical function, and a semisimple class corresponding to each other in the classical sense.*

(b) (R,S) *is a radical-semisimple pair if and only if R and S are a radical class and a semisimple class corresponding to each other in the classical sense.* □

In particular:

- The assertion "if (R,r,S) is a radical-semisimple triple, then (R,S) is a radical-semisimple pair" of our Theorem 4.3 should be considered as well known in the present case.
- The converse assertion, namely "if (R,S) is a radical-semisimple pair, then (R,r,S) is a radical-semisimple triple for some r" should also be considered as well known in this case, although it is false in general, as a counter-example given in the next section will show.

Of course, Theorem 5.1 can be stated more generally, depending on what we mean by "classical sense". For instance, the category of rings can surely be replaced with any *semi-abelian* variety of universal algebras (in the sense of [JMT2002]; see also [BJ2003]), but even that would be far from the most general case. Various remarks on (more abstract) categorical contexts are made in [JM2003] and [GrJM2013], some referring to [MW1982]. However, full details can be found only in the case of rings: see [GaW2004] and [W1983], and references therein, especially [Div1973] and Section 2 in [FW1975].

Notice that a variant of Kurosh–Amitsur type radical theory, called connectednesses and disconnectednesses, has been developed for topological spaces and graphs and then for abstract relational structures in [AW1975], [FW1975] and [FW1982], respectively, also in a non-pointed setting. What we do here, however, is very different from their setting: we still have kernels while they have inverse images of all points ('connected components').

6. The topological closure operator

In this section, ignoring the problem of size, we take the ground 2-dimensional exactness structure (1.1) to be constructed as in Example 1.6 out of the semiexact category (1.10) in which:

- $\mathbf{C}_0$ is the category of topological spaces and inclusion maps of subspaces;
- $\mathbf{C}_1$ is the category of morphisms of $\mathbf{C}_0$ whose objects will be written as pairs (A,A'), where A' is a subspace of A.
- E is the inclusion functor and therefore C and D are defined by $C(A,A') = A$ and $D(A,A') = A'$, respectively.

In this context every short exact sequence is proper and it is just a diagram of the form

$$(A',A'') \to (A,A'') \to (A,A'), \tag{6.1}$$

where A' is any subspace of A and A'' is any subspace of A'. Using Definition 3.1 directly it is easy to prove:

Theorem 6.1. *Let* $r : X_1 \to X_2$ *be the map defined by*

$$r(A,A') = ((\bar{A}',A') \to (A,A') \to (A,\bar{A}')), \tag{6.2}$$

where $\bar{A}'$ *denotes the closure of* A' *in* A. *Then* r *is a radical function in the radical-semisimple triple* (R,r,S) *where*

$$R = \{(A,A') \in X_1 \mid A' \text{ is dense in } A\}, \tag{6.3}$$
$$S = \{(A,A') \in X_1 \mid A' \text{ is closed in } A\}. \;\square \tag{6.4}$$

This theorem obviously indicates the relationship between radicals and closure operators – a natural counterpart of what is done in [GJM2013] with radicals defined with respect to 1-dimensional exactness structures.

7. A simplified framework

Intuitively, the relations $\lhd$ and $\mathrel{-\!\rhd}$ are “almost order relations”: for example, in the usual radical theory of rings, $a \lhd b$ means that a is (isomorphic to) an ideal in a, while $a \mathrel{-\!\rhd} b$ means that b is (isomorphic to) a quotient ring of a. However, even in that example, both antisymmetry (of $\lhd$

and -▷) and transitivity (of ◁) fail. This suggests us to consider a simplified version of a 1-dimensional exactness structure of the form $(A,\{0\},\leq,\geq)$, in which $(A,\leq)$ is an ordered set with smallest element 0 (cf. Section 2 of [FW1975]). This will also give us a very simple counterexample (see Example 7.4) to the assertion “if (R,S) is a radical-semisimple pair, then (R,r,S) is a radical-semisimple triple for some r”, as mentioned in Section 5.

The following two propositions should be considered obvious after reading Section 2 of [FW1975], but since our proofs are very short and easy anyway, we do not discuss this connection.

Proposition 7.1. *If* $(A,\{0\},\leq,\geq)$ *is as above, then the following conditions on a subset* U *of* A *are equivalent:*

(a) *U is a radical class with respect to* $(A,\{0\},\leq,\geq)$;

(b) *U is a semisimple class with respect to* $(A,\{0\},\leq,\geq)$;

(c) *an element a of A is in U if and only if, for every non-zero $b \leq a$, there exists a non-zero $c \leq b$ which is in* U;

(d) *U is a down-closed subset of A such that an element a of A is in U whenever for every non-zero $b \leq a$, there exists a non-zero $c \leq b$ which is in U.*

Proof. The implications (a)⇔(b)⇔(c)⇐(d) immediately follow from the definitions, while (c)⇒(d) easily follows from the transitivity of ≤. □

Proposition 7.2. *If* $(A,\{0\},\leq,\geq)$ *is as above, then a pair* (R,S) *of subsets of A is a radical-semisimple pair if and only if*

$$R = \{b \in A \mid (a \in S \;\&\; a \leq b) \Rightarrow a = 0\}, \tag{7.1}$$

$$S = \{b \in A \mid (a \in R \;\&\; a \leq b) \Rightarrow a = 0\}. \tag{7.2}$$

Proof. Just note that ($\alpha_* = \beta^*$ and) the equalities above are nothing but $R = \beta^*(S)$ and $S = \alpha_*(R)$, respectively, where α and β are as in (4.3) in the case of $(A,\{0\},\leq,\geq)$. □

Continuing to develop our simplified counterpart of usual radical theory, what would be a reasonable 2-dimensional exactness structure whose underlying 1-dimensional exactness structure is $(A,\{0\},\leq,\geq)$? We propose the following one, requiring an additional condition on A; then, its underlying 1-dimensional exactness structure is indeed $(A,\{0\},\leq,\geq)$ under a further additional condition mentioned in Example 7.4(b) below.

Definition 7.3. Let A be an ordered set with smallest element 0 and such that, for every $b \in A$, the set

$$\{(a,c) \in A{\times}A \mid a{\wedge}c = 0 \ \& \ a{\vee}c = b\} \tag{7.3}$$

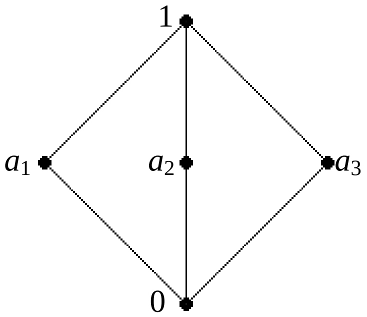
forms a complete lattice under the order defined by $(a,c) \leq (a',c') \Leftrightarrow (a \leq a' \ \& \ c' \leq c)$. The 2-dimensional exactness structure associated to A is

$$\begin{tikzcd}
A' \arrow[r, "d_1^1"] \arrow[r, bend left=80, "d_0^1"] \arrow[r, bend right=80, "d_2^1"'] & A \arrow[l, bend right=45, "s_0^1"'] \arrow[l, bend left=45, "s_1^1"] \arrow[r, bend left=45, "d_0^0"] \arrow[r, bend right=45, "d_1^0"'] & \{0\}, \arrow[l, "s_0^0"']
\end{tikzcd} \tag{7.4}$$

where $A' = \{(a,b,c) \in A{\times}A{\times}A \mid a{\wedge}c = 0 \ \& \ a{\vee}c = b\}$, $s_0^0(0) = 0$, $s_0^1(a) = (a,a,0)$, $s_1^1(a) = (0,a,a)$, $d_0^1(a,b,c) = a$, $d_1^1(a,b,c) = b$, $d_2^1(a,b,c) = c$, and the complete lattice structure on $(d_1^1)^{-1}(b)$ is defined via $(a,b,c) \leq (a',b,c') \Leftrightarrow (a \leq a' \ \& \ c' \leq c)$.

Although a further analysis of this 2-dimensional exactness structure, which always satisfies Condition 3.5, would be interesting, we will use it only in

Example 7.4. Consider the 2-dimensional exactness structure of Definition 7.3 where A is the lattice



(7.5)

and observe:

(a) this *non-distributive* lattice indeed satisfies the conditions required in Definition 4.3;

(b) the underlying 1-dimensional exactness structure is $(A,\{0\},\leq,\geq)$; more generally, this is true in the situation of Definition 7.3 whenever, for all $a \leq b$ in A, there exists c in A with (a,c) in the set (7.3);

(c) as follows from (b) and Proposition 7.2, $(\{0,a_1,a_2,\},\{0,a_3\})$ is a radical-semisimple pair.

Nevertheless there is no radical function r making $(\{0,a_1\},r,\{0,a_2,a_3\})$ a radical-semisimple triple. Indeed, having such an r, consider $r(1)$: by Theorem 3.3, it should be the largest element x in the lattice $(d_1^1)^{-1}(1)$ with $d_0^1(x)$ in $\{a_1\}$ – but such an element does not exist.

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GEOMETRIE DIFFERENTIELLE CATEGORIQUES

CONDITION FOR AN n-PERMUTABLE CATEGORY TO BE MALTSEV

by Sean TULL

Résumé. Nous améliorons la description des catégories n-permutables introduites par Carboni, Kelly et Pedicchio [2]. Cela donne une nouvelle caractérisation des catégories régulières de Maltsev parmi celles qui sont des catégories de Goursat ou, plus généralement, des catégories n-permutables.

Abstract. We give a strengthening of the description of an n-permutable category due to Carboni, Kelly and Pedicchio [2]. This provides a new characterisation of the regular Mal'tsev categories from among those which are Goursat categories, or more generally n-permutable.

Keywords. n-permutable category, Mal'tsev category, Goursat category

Mathematics Subject Classification (2010). 18B10, 08B05

Mal'tsev categories form an important and well-known class of categories in the study of universal algebra [3, 10]. In fact, many of their interesting properties extend to the broader class of Goursat categories [4, 6]. While most examples of Goursat categories are in fact Mal'tsev categories, no simple conditions for when this is the case have yet been presented. In [2], Carboni, Kelly and Pedicchio showed that both classes belong to the more general hierarchy of n-permutable categories. In this note, we give a strengthening of their original characterisation of n-permutable categories, leading to a condition for when a Goursat or n-permutable category is a Mal'tsev category. The condition, called *positive regularity*, is mild, satisfied even in logically well-structured categories such as the category $\mathbf{Set}$ of sets and functions.

1. A Condition for n-permutability

We work in a regular category $\mathbf{C}$. The internal logic of such categories allows one to reason using elements just as in $\mathbf{Set}$ [1, Metatheorem A.5.7], and we will use this technique throughout. For objects A, B in $\mathbf{C}$, recall that a *relation* $R\colon A \to B$ is a subobject $R \rightarrowtail A \times B$. Any such relation comes with a converse which we denote $R^\circ\colon B \to A$. Every pair of relations $R\colon A \to B$ and $S\colon B \to C$ have a composite $SR\colon A \to C$ defined by

$$SR = \{(a,c) \in A \times C \mid \exists b \; R(a,b) \wedge S(b,c)\}$$

Relations are partially ordered by the usual inclusion of subobjects. We call a relation $E\colon A \to A$ *reflexive* when $\mathrm{id}_A \leq E$, *symmetric* when $E = E^\circ$, *transitive* when $EE \leq E$, and an *equivalence relation* when all of these hold. Following [2], for any pair of relations $R\colon A \to B$, $S\colon B \to C$, we define a sequence of relations

$$(S,R)_1 = S,\ (S,R)_2 = SR,\ (S,R)_3 = SRS,\ (S,R)_4 = SRSR,\ \ldots$$

A regular category $\mathbf{C}$ is then *n-permutable* whenever $(E,E')_n = (E',E)_n$ for every pair of equivalence relations E, E' on the same object. In particular, a regular category is a *Mal'tsev category* when it is 2-permutable, and a *Goursat category* when it is 3-permutable. We begin by strengthening a characterisation from [2].

Theorem 1. *Let $\mathbf{C}$ be a regular category and $n \geq 2$. The following conditions are equivalent:*

(i) $\mathbf{C}$ *is n-permutable;*

(ii) *Every relation $R\colon A \to B$ in $\mathbf{C}$ satisfies $(R^\circ, R)_{n+1} = (R^\circ, R)_{n-1}$;*

(iii) *For every reflexive relation $E\colon A \to A$ in $\mathbf{C}$, $(E, E^\circ)_{n-1}$ is an equivalence relation;*

(iv) *For every reflexive relation $E\colon A \to A$ in $\mathbf{C}$, E^{n-1} is an equivalence relation;*

(v) *For every reflexive and symmetric relation $E\colon A \to A$ in $\mathbf{C}$, E^{n-1} is an equivalence relation.*

Proof. See [2] for the equivalence of (i), (ii) and (iii). (i) $\implies$ (iv) is also well-known: in an n-permutable category any reflexive relation E has that E^{n-1} is transitive [7, Theorem 1], and every reflexive and transitive relation is symmetric [9, Theorem 1]. Clearly (v) follows from (iii) or (iv).

We now show that (v) $\implies$ (ii). First note that by our assumption any reflexive and symmetric relation E satisfies $E^n = E^{n-1}$. Indeed, always $E^{n-1} \leq E^n$, while we now have $E^n \leq E^{2(n-1)} \leq E^{n-1}$ by transitivity. Now for any relation $R \rightarrowtail A \times B$, we always have $(R^\circ, R)_{n-1} \leq (R^\circ, R)_{n+1}$, so it suffices to show the converse holds. Define a new relation $S \rightarrowtail R \times R$ by

$$S((a,b),(a',b')) \iff R(a,b') \wedge R(a',b)$$

noting by definition that $S((a,b),(a',b'))$ also implies $R(a,b)$ and $R(a',b')$. Then S is reflexive since it is defined on R, and symmetric by definition. Hence $S^n = S^{n-1}$.

Suppose that $(R^\circ, R)_{n+1}(b,a)$, via a sequence of elements $(x_i)_{i=0}^{n+1}$ with $x_0 = a$, $x_{n+1} = b$ satisfying $R(x_i, x_{i-1})$ for even $i \geq 2$, and $R(x_i, x_{i+1})$ for even $i \leq n$. Then we have $S((x_i, x_{i+1}), (x_{i+2}, x_{i+1}))$ for all even $i \leq n-1$ and $S((x_{i+2}, x_{i+1}), (x_{i+2}, x_{i+3}))$, for all even $i \leq n-2$. So defining $(y,z) := (b, x_n)$ if n is odd, or $(y,z) := (x_n, b)$ if n is even, we have $S^n((a,x_1),(y,z))$. Hence $S^{n-1}((a,x_1),(y,z))$ also.

Letting $(y_0, z_0) = (a, x_1)$ and $(y_{n-1}, z_{n-1}) = (y,z)$, this means there is a sequence of pairs $(y_i, z_i)_{i=0}^{n-1}$ satisfying $S((y_i,z_i),(y_{i+1},z_{i+1}))$ for all $i \leq n-2$. In particular, we have $R(y_i, z_{i-1})$ for even $i \geq 2$ and $R(y_i, z_{i+1})$ for even $i \leq n-2$. Hence via the sequence $a = y_0, z_1, y_2, z_3, \ldots$ of length n ending in b, we have $(R^\circ, R)_{n-1}(b,a)$, as desired. □

2. Positively regular and Mal'tsev categories

We now turn to classifying the Mal'tsev categories as the n-permutable categories with a special property. Let us call a relation $E\colon A \to A$ *positive* when it is of the form $E = R^\circ R$ for some relation $R\colon A \to B$. The following notion first appeared in [5].

Proposition 2. *For a regular category* $\mathbf{C}$*, the following are equivalent:*

(i) *A relation* $E\colon A \to A$ *in* $\mathbf{C}$ *is positive if and only if it satisfies:*

$$E(a,b) \implies E(a,a) \wedge E(b,a) \tag{$*$}$$

(ii) *Any reflexive and symmetric relation in* $\mathbf{C}$ *is positive.*

We call a regular category satisfying either of these equivalent conditions positively regular[1].

Proof. For (i) $\implies$ (ii), and the 'only if' in (ii) $\implies$ (i), note that any reflexive, symmetric relation in a regular category automatically satisfies $(*)$, as does any positive relation. Conversely, if (ii) holds and $E\colon A \to A$ satisfies $(*)$, define

$$I = \{a \in A \mid \exists b \; E(a,b)\} \rightarrowtail A$$

writing $i\colon I \to A$ for the inclusion. Then it's easy to see that $E = Eii^\circ = ii^\circ E$. Further, $i^\circ Ei$ is a reflexive, symmetric relation on I, and hence is positive, say equal to $R^\circ R$. Then we have $E = ii^\circ Eii^\circ = iR^\circ Ri^\circ = (Ri^\circ)^\circ(Ri^\circ)$ and so E is positive. □

Example 3. $\mathbf{Set}$ is positively regular. More generally so is any regular coherent category, coming with unions of subobjects. To see this, for any relation $E \rightarrowtail A \times A$ satisfying $(*)$, define

$$R = \{(a,(a,b)) \mid E(a,b)\} \vee \{(a,(b,a)) \mid E(b,a)\} \rightarrowtail A \times E$$

Then $E = R^\circ R$, making E positive.

Theorem 4. *For a regular category* $\mathbf{C}$*, the following are equivalent:*

(i) $\mathbf{C}$ *is a Mal'tsev category;*

(ii) *Every reflexive relation in* $\mathbf{C}$ *is an equivalence relation;*

(iii) *Every reflexive and symmetric relation in* $\mathbf{C}$ *is an equivalence relation;*

(iv) $\mathbf{C}$ *is a Goursat category and every reflexive relation in* $\mathbf{C}$ *is positive;*

(v) $\mathbf{C}$ *is a Goursat category and positively regular;*

(vi) $\mathbf{C}$ *is* n*-permutable, for some* $n \geq 2$*, and positively regular.*

---

[1] Not to be confused with the notion of a positive coherent category [8].

Proof. The equivalence of (i), (ii) and (iii) is in Theorem 1, and clearly we have (iv) $\implies$ (v) $\implies$ (vi). For (iii) $\implies$ (iv), any reflexive relation E in **C** is an equivalence relation, and therefore positive since $E = E^{\circ}E$. Hence by Proposition 2, **C** is positively regular. Further, since **C** is a Mal'tsev category, it is a Goursat category.

It remains to show that (vi) $\implies$ (i). Let **C** be positively regular. First suppose **C** is $(2m+1)$-permutable, for some $m \geq 1$. Let $E\colon A \to A$ be a reflexive and symmetric relation. By positive regularity, $E = R^{\circ}R$ for some relation $R\colon A \to B$, and so:

$$E^{2m} = (R^{\circ}, R)_{4m} = (R^{\circ}, R)_{(2m+2)+2(m-1)} = (R^{\circ}, R)_{2m} = E^{m}$$

where we repeatedly applied $(R^{\circ}, R)_{2m+2} = (R^{\circ}, R)_{2m}$ from Theorem 1, condition (ii). Hence E^m is an equivalence relation. By condition (v) of Theorem 1, **C** is then in fact $(m+1)$-permutable.

Now if **C** is n-permutable, there is some k with $n \leq 2^k + 1$ so that **C** is $(2^k + 1)$-permutable. Then the above argument shows that **C** is in fact $(2^{k-1} + 1)$-permutable, and hence inductively that **C** is 2-permutable, *i.e.* a Mal'tsev category. $\square$

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CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

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A MULTIPLE CATEGORY OF MULTIPLE LAX CATEGORIES

by Marco GRANDIS and Robert PARE

Résumé. On construit une catégorie multiple, utile dans l'étude des adjunctions multiples. Les objets sont les catégories multiples 'laxes'. Les flèches transversales sont les foncteurs multiples stricts tandis que les flèches en direction positive sont des foncteurs multiples de 'laxité mixte', qui varient des foncteurs laxes (en direction 1) aux colaxes (en direction ∞).

Abstract. We construct a multiple category which occurs in the study of multiple adjunctions. The objects are all the 'lax' multiple categories. The transversal arrows are their strict multiple functors while the arrows in a positive direction are multiple functors of a 'mixed laxity', varying from the lax ones (in direction 1) to the colax ones (in direction ∞).

Keywords. Multiple category, weak double category, cubical set.

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0. Introduction

This note is about strict, weak and lax multiple categories, an extension of double categories that we have studied in the articles [5] – [9]. The first two of them are about the 3-dimensional case, where *intercategories* (a kind of lax triple category) cover and combine diverse structures like duoidal categories [1, 3, 12], Gray categories [10], Verity double bicategories [13] and monoidal double categories [11]. The other papers [7] – [9] are about weak and lax infinite-dimensional multiple categories, an extension of the strict case introduced by Bastiani – Ehresmann [2].

A weak multiple category has objects, i-directed arrows in each direction $i \in \mathbb{N}$, ij-cells of dimension two for all $i < j$, and so on. Composition is strict in the *transversal* direction $i = 0$ and weak in each direction $i > 0$, i.e. associative and unitary up to invertible transversal comparisons. The transversal composition has a strict interchange with all the geometric ones, while the latter have invertible ij-interchangers; more generally, *chiral multiple categories* and *intercategories* have directed ij-interchangers, for $i < j$. A (weak or lax) n-tuple category has indices in the ordinal $\mathbf{n} = \{0, 1, ..., n-1\}$.

Here we investigate the different sorts of morphisms that can link chiral multiple categories. We know, from [9], that in a general multiple adjunction $F \dashv G$ the left adjoint is a *colax* (multiple) functor, while the right adjoint is *lax*; the adjunction lives in a double category $\mathbb{C}\mathrm{mc}$ of chiral multiple categories, where the horizontal arrows are lax functors and the vertical ones are colax functors.

But we have already seen in [5] that – in dimension three – there exists an intermediate sort, called a *colax-lax morphism*, which is colax in direction 1 and lax in direction 2 (and of course strict in the transversal direction 0). Also this case is important in concrete situations, when a triple adjunction $F \dashv G$ has a *colax-pseudo* left adjoint and a *pseudo-lax* right adjoint, so that the composites GF and FG are colax-lax morphisms, forming a monad GF and a comonad FG. Higher dimensional examples present higher dimensional cases of 'mixed laxity functors'.

With these motivations, we construct here a multiple category Cmc of chiral multiple categories, indexed by the ordinal $\boldsymbol{\omega} + \mathbf{1} = \{0, 1, ..., \infty\}$. Its transversal arrows are the strict multiple functors while, in direction p (for $1 \leqslant p \leqslant \infty$), the *$p$-morphisms* are 'multiple functors of mixed laxity', that vary from the lax ones (in direction 1) to the colax ones (in direction ∞). The double category $\mathbb{C}\mathrm{mc}$ is embedded in Cmc, with indices in $\{1, \infty\}$. Similar frameworks are concerned with intercategories, and the n-dimensional case.

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1. Notation

We mainly follow the notation of [7] – [9]. The symbol $\subset$ denotes weak inclusion. Categories and 2-categories are generally denoted as $\mathbf{A}, \mathbf{B}, ...$; weak double categories as $\mathbb{A}, \mathbb{B}, ...$; weak or lax multiple categories as $\mathsf{A}, \mathsf{B}, ...$

The definitions of weak and chiral multiple categories can be found in [7], or – briefly reviewed – in [8], Section 1. Here we only give a sketch of them, while recalling the notation we are using.

The two-valued index α (or β) varies in the set $2 = \{0, 1\}$, also written as $\{-, +\}$.

A *multi-index* $\mathbf{i}$ is a finite subset of $\mathbb{N}$, possibly empty. Writing $\mathbf{i} \subset \mathbb{N}$ it is understood that $\mathbf{i}$ is finite; writing $\mathbf{i} = \{i_1, ..., i_n\}$ it is understood that $\mathbf{i}$ has n distinct elements, written in the natural order $i_1 < i_2 < ... < i_n$; the integer $n \geqslant 0$ is called the *dimension* of $\mathbf{i}$. We write:

$$\begin{aligned} \mathbf{i}j = j\mathbf{i} &= \mathbf{i} \cup \{j\} && (\text{for } j \in \mathbb{N} \setminus \mathbf{i}), \\ \mathbf{i}|j &= \mathbf{i} \setminus \{j\} && (\text{for } j \in \mathbf{i}). \end{aligned} \tag{1}$$

For a weak multiple category A, the set of $\mathbf{i}$-cells $A_\mathbf{i}$ is written as A_*, A_i, A_{ij} when $\mathbf{i}$ is $\varnothing$, $\{i\}$ or $\{i, j\}$ respectively. Faces and degeneracies, satisfying the *multiple relations* (cf. [7], Section 2.2), are denoted as

$$\partial_j^\alpha \colon A_\mathbf{i} \to A_{\mathbf{i}|j}, \qquad e_j \colon A_{\mathbf{i}|j} \to A_\mathbf{i} \qquad\qquad (\text{for } \alpha = \pm,\ j \in \mathbf{i}). \tag{2}$$

The *transversal direction* $i = 0$ is set apart from the positive, or *geometric*, directions. For a *positive multi-index* $\mathbf{i} = \{i_1, ..., i_n\} \subset \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, the *augmented multi-index* $0\mathbf{i} = \{0, i_1, ..., i_n\}$ has dimension $n + 1$, but both $\mathbf{i}$ and $0\mathbf{i}$ are said to have *degree* n. An $\mathbf{i}$-cell $x \in A_\mathbf{i}$ of A is also called an $\mathbf{i}$-*cube*, while a $0\mathbf{i}$-cell $f \in A_{0\mathbf{i}}$ is viewed as an $\mathbf{i}$-*map* $f \colon x \to_0 y$, where $x = \partial_0^- f$ and $y = \partial_0^+ f$. Composition in direction 0 is categorical (and generally realised by ordinary composition of mappings); it is written as $gf = f +_0 g$, with identities $1_x = \mathrm{id}(x) = e_0(x)$.

The *transversal category* $\mathrm{tv}_\mathbf{i}(\mathsf{A})$ consists of the $\mathbf{i}$-cubes and $\mathbf{i}$-maps of A, with transversal composition and identities. Their family forms a multiple object in $\mathbf{Cat}$, indexed by the positive multi-indices.

Composition of $\mathbf{i}$-cubes and $\mathbf{i}$-maps in a *positive* direction $i \in \mathbf{i}$ (often realised by pullbacks, pushouts, tensor products, etc.) is written in additive notation

$$\begin{aligned} & x +_i y && (\partial_i^+ x = \partial_i^- y), \\ & f +_i g \colon x +_i y \to x' +_i y' && (f \colon x \to x',\ g \colon y \to y',\ \partial_i^+ f = \partial_i^- g). \end{aligned} \tag{3}$$

The transversal composition has a strict interchange with each of the positive operations. The latter satisfy the unitarity, associativity and interchange laws up to transversally invertible comparisons (for $0 < i < j$)

$$\begin{aligned}
&\lambda_i x\colon (e_i\partial_i^- x) +_i x \to_0 x && (\textit{left i-unitor}),\\
&\rho_i x\colon x +_i (e_i\partial_i^+ x) \to_0 x && (\textit{right i-unitor}),\\
&\kappa_i(x,y,z)\colon x +_i (y +_i z) \to_0 (x +_i y) +_i z && (\textit{i-associator}),\\
&\chi_{ij}(x,y,z,u)\colon (x +_i y) +_j (z +_i u) \to_0 (x +_j z) +_i (y +_j u)\\
& && (\textit{ij-interchanger}).
\end{aligned} \tag{4}$$

The comparisons are natural with respect to transversal maps; λ_i, ρ_i and κ_i are special in direction i (i.e. their i-faces are transversal identities) while χ_{ij} is special in both directions i, j; all of them commute with ∂_k^α for $k \neq i$ (or $k \neq i, j$ in the last case). Finally the comparisons must satisfy various conditions of coherence, listed in [7], Sections 3.3 and 3.4.

More generally for a *chiral multiple category* A the ij-interchangers χ_{ij} are not assumed to be invertible (see [7], Section 3.7).

Even more generally, in an *intercategory* we also have ij-interchangers $\mu_{ij}, \delta_{ij}, \tau_{ij}$ involving the units; this extension is studied in [5, 6] for the 3-dimensional case, the really important one. Infinite dimensional intercategories have been introduced in [7], Section 5, and mentioned marginally in [8] and [9], but a further study must await good examples.

While a chiral multiple category A is a multiple object of ordinary categories $\mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ indexed by positive multi-indices $\mathbf{i} = \{i, j, k...\} \subset \mathbb{N}^*$, the structure Cmc that we shall construct will be indexed by 'extended' positive multi-indices $\mathbf{p} = \{p, q, r...\} \subset \{1, 2, ..., \infty\}$.

2. Lax and colax multiple functors

We want to analyse which sorts of 'morphisms' $\mathsf{A} \to \mathsf{B}$ between chiral multiple categories are of interest.

Two main kinds stand out:

(a) a *lax* (multiple) functor $F\colon \mathsf{A} \to \mathsf{B}$ is equipped with *comparison* $\mathbf{i}$-maps $\underline{F}_{\mathbf{i}}$, for the i-directed composition (for $t \in \mathsf{A}_{\mathbf{i}|i}$ and i-consecutive cubes x, y in $A_{\mathbf{i}}$)

$$\underline{F}_i(t)\colon e_iF(t) \to_0 F(e_it), \quad \underline{F}_i(x,y)\colon F(x) +_i F(y) \to_0 F(x +_i y), \tag{5}$$

(b) a *colax* (multiple) functor $F\colon \mathsf{A} \to \mathsf{B}$ has comparisons in the opposite direction.

The definitions of such 'functors', with the *transversal transformations* of both sorts, can be found in [7], Section 3.9 (or here, in a more general form, in Sections 4 and 5.)

A *pseudo* functor is a lax multiple functor with (transversally) invertible comparisons, and is made colax by the inverse comparisons. It is *strict* when the comparisons are identities, so that the whole structure is strictly preserved.

In a general multiple adjunction (defined and studied in [9]) these two sorts appear together: *the left adjoint* $F\colon \mathsf{A} \to \mathsf{B}$ *is colax while the right adjoint* $G\colon \mathsf{B} \to \mathsf{A}$ *is lax*; many natural situations are of this type, with non-invertible comparisons. We do not want to compose F and G, since this would destroy their comparisons; yet we must give a unit and a counit.

This point was solved in [9], Section 2, where we constructed a (strict) double category $\mathbb{C}\mathrm{mc}$ of chiral multiple categories. The lax and colax multiple functors form the horizontal and vertical arrows, respectively. They are not to be composed, but linked by suitable double cells.

Finally, a *colax-lax (multiple) adjunction* $F \dashv G$ is a pair of adjoint arrows in this double category. This means a colax functor $F\colon \mathsf{A} \to \mathsf{B}$, a lax functor $G\colon \mathsf{B} \to \mathsf{A}$ and two double cells of $\mathbb{C}\mathrm{mc}$, called a *unit* and a *counit*

$$\begin{tikzcd}
\mathsf{A} \arrow[r, equal] \arrow[d, "F"'] \arrow[dr, phantom, "\eta"] & \mathsf{A} \arrow[d, equal] \\
\mathsf{B} \arrow[r, "G"'] & \mathsf{A}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathsf{B} \arrow[r, "G"] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & \mathsf{A} \arrow[d, "F"] \\
\mathsf{B} \arrow[r, equal] & \mathsf{B}
\end{tikzcd}
\qquad (6)$$

They have components $\eta x\colon x \to_0 GF(x)$ and $\varepsilon y\colon FG(y) \to_0 y$ (for any cube x in A and y in B), whose coherence conditions are based – separately – on the comparisons of F and G. The triangular laws state that the composites $(\varepsilon \,|\, \eta)$ and $(\frac{\eta}{\varepsilon})$ are identities.

If F is a pseudo functor, this is the same as an adjunction in the 2-category $\mathrm{Lx}\mathbf{Cmc}$ of chiral multiple categories, lax functors and their transversal transformations (as proved in [9], Section 5). Symmetrically, *if G is a pseudo functor*, this is the same as an adjunction in the 2-category $\mathrm{Cx}\mathbf{Cmc}$, whose arrows are the colax multiple functors.

These two particular cases, a *pseudo-lax* and a *colax-pseudo* adjunction, do not cover the examples of [9]; furthermore, composing adjunctions of these two kinds we come back to the general case.

Yet the particular cases are important, since the first gives a *lax* (multiple) *monad* $GF\colon \mathsf{A} \to \mathsf{A}$ and a *lax comonad* $FG\colon \mathsf{B} \to \mathsf{B}$, while the second case gives a *colax monad* and a *colax comonad.*

3. Examples

Some examples, from [9], Section 1.7, will lead to new morphisms, intermediate between the two previous kinds, and also important in adjunctions. For the sake of simplicity, we begin by working in dimension 3.

For a category $\mathbf{C}$ with (a choice of) pullbacks, we have a weak triple category $\mathbf{3}\mathsf{Span}(\mathbf{C})$ of 'spans of spans'. A 12-cube is a functor $x\colon \vee\times\vee \to \mathbf{C}$ (where $\vee$ is the formal-span category) and a 12-map $f\colon x \to_0 y$ is a natural transformation of such functors (a 3-dimensional item $f\colon \vee\times\vee\times\mathbf{2} \to \mathbf{C}$).

Dually, if $\mathbf{C}$ has pushouts, there is a weak triple category $\mathbf{3}\mathsf{Cosp}(\mathbf{C})$ whose highest cubes are 'cospans of cospans' $x\colon \wedge\times\wedge \to \mathbf{C}$.

When $\mathbf{C}$ has both pullbacks and pushouts, we can form a chiral triple category $\mathsf{SC}(\mathbf{C}) = \mathsf{S}_1\mathsf{C}_1(\mathbf{C})$ where a 12-cube is a functor $x\colon \vee\times\wedge \to \mathbf{C}$; the 1-directed composition is by pullbacks, the 2-directed one by pushouts.

An ordinary functor $F\colon \mathbf{X} \to \mathbf{A}$ between categories with pullbacks and pushouts produces:

(a) *a colax (triple) functor* $\mathbf{3}\mathsf{Span}(F)\colon \mathbf{3}\mathsf{Span}(\mathbf{X}) \to \mathbf{3}\mathsf{Span}(\mathbf{A})$ of weak triple categories,

(b) a *lax (triple) functor* $\mathbf{3}\mathsf{Cosp}(F)\colon \mathbf{3}\mathsf{Cosp}(\mathbf{X}) \to \mathbf{3}\mathsf{Cosp}(\mathbf{A})$ of weak triple categories,

(c) a *colax-lax morphism* $\mathsf{SC}(F)\colon \mathsf{SC}(\mathbf{X}) \to \mathsf{SC}(\mathbf{A})$ of chiral triple categories.

We have thus a new morphism of an intermediate sort: $\mathsf{SC}(F)$ is colax for the 1-directed composition, realised by pullbacks, and lax for the 2-composition, realised by pushouts; the precise definition can be found in [5], Section 5. Moreover, if F preserves pushouts, $\mathbf{3}\mathsf{Cosp}(F)$ is a pseudo functor and $\mathsf{SC}(F)$ is a colax-pseudo morphism; and so on.

Now, an ordinary adjunction between categories with pullbacks and pushouts

$$F\colon\ \mathbf{X} \rightleftarrows \mathbf{A}\ :G, \qquad\qquad F \dashv G, \tag{7}$$

has three natural extensions to colax-lax triple adjunctions:

$$\mathbf{3}\mathsf{Span}(F) \dashv \mathbf{3}\mathsf{Span}(G),\ \ \mathbf{3}\mathsf{Cosp}(F) \dashv \mathbf{3}\mathsf{Cosp}(G),\ \ \mathsf{SC}(F) \dashv \mathsf{SC}(G). \tag{8}$$

The first is actually a *colax-pseudo adjunction* (because G preserves pullbacks), and gives a colax triple monad on $\mathbf{3}\mathsf{Span}(\mathbf{X})$. The second is *pseudo-lax*, and gives a lax triple monad on $\mathbf{3}\mathsf{Cosp}(\mathbf{X})$.

In the last, $F' = \mathsf{SC}(F)$ is a *colax-pseudo morphism* while $G' = \mathsf{SC}(G)$ is a *pseudo-lax morphism*; their composites $G'F' = \mathsf{SC}(GF)$ and $F'G' = \mathsf{SC}(FG)$ make sense: they are colax-lax morphisms, and we still have a triple monad on $\mathsf{SC}(X)$, where $T = G'F'$ is a colax-lax morphism. (Multiple monads will be studied elsewhere.)

All this can be extended to higher dimensions, for the weak multiple categories $\mathsf{Span}(\mathbf{C})$, $\mathsf{Cosp}(\mathbf{C})$ and the chiral multiple categories $\mathsf{S}_p\mathsf{C}_q(\mathbf{C})$, $\mathsf{S}_p\mathsf{C}_\infty(\mathbf{C})$, $\mathsf{S}_{-\infty}\mathsf{C}_\infty(\mathbf{C})$ (see [9], Section 1.3). We get thus morphisms of 'mixed laxity', colax up to a certain degree and lax above. (The reverse case cannot occur, as we shall see below.)

Finally we recall from [9], Sections 1.5 – 1.6, a colax-lax adjunction of weak triple categories, based on an ordinary category $\mathbf{C}$ with pullbacks and pushouts:

$$F\colon\ \mathsf{Span}(\mathbf{C}) \rightleftarrows \mathsf{Cosp}(\mathbf{C})\ :G, \qquad F \dashv G, \tag{9}$$

F works by pushouts and G by pullbacks. None of them is pseudo (in general, of course), and we do not have an associated multiple monad (nor comonad).

4. Mixed-laxity functors

We are now ready to begin the construction of a multiple category Cmc containing different morphisms in different directions, that vary from the lax case to the colax one.

In degree 0, the objects of Cmc are the (small) chiral multiple categories, and the transversal arrows (or 0-morphisms) are the strict multiple functors $F\colon \mathsf{A} \to_0 \mathsf{B}$.

In degree 1 and direction p (for $1 \leqslant p \leqslant \infty$), a *$p$-morphism* $R\colon \mathsf{A} \to_p \mathsf{B}$ between chiral multiple categories will be a *mixed-laxity functor* which is colax in all positive directions $i < p$ and lax in all directions $i \geqslant p$. In particular, this is a lax functor for $p = 1$ and a colax functor for $p = \infty$.

Basically, R has components $R_{\mathbf{i}} = \mathrm{tv}_{\mathbf{i}}(R)\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{B})$, for all positive multi-indices $\mathbf{i}$, that are ordinary functors and commute with faces: $\partial_i^\alpha.R_{\mathbf{i}} = R_{\mathbf{i}|i}.\partial_i^\alpha$ (for $i \in \mathbf{i}$).

Moreover R is equipped with *comparison* $\mathbf{i}$-maps $\underline{R}_i$ (for $t \in A_{\mathbf{i}|i}$ and x, y i-consecutive in $A_{\mathbf{i}}$), either in the lax direction for $i \geqslant p$

$$\underline{R}_i(t)\colon e_iR(t) \to_0 R(e_it), \;\; \underline{R}_i(x,y)\colon R(x) +_i R(y) \to_0 R(x +_i y), \qquad (10)$$

or in the colax direction for $0 < i < p$

$$\underline{R}_i(t)\colon R(e_it) \to_0 e_iR(t), \;\; \underline{R}_i(x,y)\colon R(x +_i y) \to_0 R(x) +_i R(y). \qquad (11)$$

All these comparisons are i-special, i.e. their two i-faces are transversal identities, and must commute with the other faces ∂_j^α (for $j \neq i$ in $\mathbf{i}$)

$$\partial_j^\alpha \underline{R}_i(t) = \underline{R}_i(\partial_j^\alpha t), \qquad \partial_j^\alpha \underline{R}_i(x,y) = \underline{R}_i(\partial_j^\alpha x, \partial_j^\alpha y). \qquad (12)$$

Then they have to satisfy the axioms of naturality and coherence (see [7], Section 3.9), either in the lax form (lmf.1 – 4) for $i \geqslant p$, or in the transversally dual form for $i < p$.

Furthermore there is an axiom of coherence with the interchanger χ_{ij} (for $0 < i < j$) which has three forms (where (a) corresponds to (lmf.5), (c) corresponds to its dual and (b) is an intermediate case):

(a) for $p \leqslant i < j$ (so that R is i- and j-lax), we have commutative diagrams of transversal maps:

$$\begin{tikzcd}
(Rx +_i Ry) +_j (Rz +_i Ru) \arrow[r, "\chi_{ij}R"] \arrow[d, "\underline{R}_i +_j \underline{R}_i"'] & (Rx +_j Rz) +_i (Ry +_j Ru) \arrow[d, "\underline{R}_j +_i \underline{R}_j"] \\
R(x +_i y) +_j R(z +_i u) \arrow[d, "\underline{R}_j"'] & R(x +_j y) +_i R(z +_j u) \arrow[d, "\underline{R}_i"] \\
R((x +_i y) +_j (z +_i u)) \arrow[r, "R\chi_{ij}"'] & R((x +_j z) +_i (y +_j u))
\end{tikzcd} \qquad (13)$$

(b) for $0 < i < p \leqslant j$ (so that R is i-colax and j-lax), we have commutative diagrams:

$$\begin{tikzcd}
& (Rx +_i Ry) +_j (Rz +_i Ru) \arrow[r, "\chi_{ij}R"] & (Rx +_j Rz) +_i (Ry +_j Ru) \arrow[dr, "\underline{R}_j +_i \underline{R}_j"] & \\
R(x +_i y) +_j R(z +_i u) \arrow[ur, "\underline{R}_i +_j \underline{R}_i"] \arrow[dr, "\underline{R}_j"'] & & & R(x +_j y) +_i R(z +_j u) \\
& R((x +_i y) +_j (z +_i u)) \arrow[r, "R\chi_{ij}"'] & R((x +_j z) +_i (y +_j u)) \arrow[ur, "\underline{R}_i"'] &
\end{tikzcd} \tag{14}$$

(c) for $0 < i < j < p$ (so that R is i- and j-colax), we have commutative diagrams as in (13), with all vertical arrows reversed.

The composition of p-morphisms $R'R = R +_p R'$ is easily defined: their comparisons are separately composed.

Finally, a transversal map $(F, G) \colon R \to_0 S$ of p-arrows will be a commutative square

$$\begin{tikzcd}
\bullet \arrow[r, "F"] \arrow[d, "R"', "\bullet" marking] \arrow[dr, phantom, "="] & \bullet \arrow[d, "S", "\bullet" marking] \\
\bullet \arrow[r, "G"'] & \bullet
\end{tikzcd} \qquad SF = GR \qquad \begin{tikzcd} \bullet \arrow[r, "0"] \arrow[d, "p"'] & {} \\ {} & \end{tikzcd} \tag{15}$$

with strict functors F, G and p-morphisms R, S. Commutativity means that $SF = GR$ *as* p-morphisms, including comparisons.

(As already remarked in [5], the 'lax-colax' case makes no sense: modifying diagram (13) by reversing all arrows $\underline{R}_j$ would lead to a diagram where no pairs of arrows compose.)

We have thus defined the double category $\mathrm{dbl}_{0p}(\mathsf{Cmc})$ of chiral multiple categories, strict functors and p-morphisms.

5. Two-dimensional cubes

To define a pq-cube (for $1 \leqslant p < q \leqslant \infty$) we have to adapt the axioms of transversal transformation (again in [7], Section 3.9).

A *pq-cube* $\varphi\colon (U\ {}^{R}_{S}\ V)$ will be a 'generalised quintet' consisting of two p-morphisms R, S, two q-morphisms U, V, together with – roughly speaking – a 'transversal transformation' $\varphi\colon VR \dashrightarrow SU$

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "R", "\bullet" marking] \arrow[d, "U"', "\bullet" marking] & \bullet \arrow[d, "V", "\bullet" marking] \arrow[ld, dashed, "\varphi"] \\
\bullet \arrow[r, "S"', "\bullet" marking] & \mathsf{B}
\end{tikzcd} \qquad \varphi\colon VR \dashrightarrow SU. \qquad \begin{tikzcd}
\bullet \arrow[r, "p"] \arrow[d, "q"] & {} \\
{} &
\end{tikzcd} \tag{16}$$

This is an abuse of notation since there are no composites VR and SU in our structure: the coherence conditions of φ are based on the four morphisms R, S, U, V and all their comparison maps. Precisely, the cell φ consists of a face-consistent family of transversal maps in B

$$\begin{array}{ll} \varphi(x) = \varphi_{\mathbf{i}}(x)\colon VR(x) \to_0 SU(x), & \text{(for every } \mathbf{i}\text{-cube } x \text{ of } \mathsf{A}), \\ \partial_i^\alpha.\varphi_{\mathbf{i}} = \varphi_{\mathbf{i}|i}.\partial_i^\alpha & \text{(for } i \in \mathbf{i}), \end{array} \tag{17}$$

so that each component $\varphi_{\mathbf{i}}\colon V_{\mathbf{i}}R_{\mathbf{i}} \to S_{\mathbf{i}}U_{\mathbf{i}}\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{B})$ is a natural transformation of ordinary functors:

(nat) for all $f\colon x \to_0 y$ in A, we have a commutative diagram of transversal maps in B

$$\begin{tikzcd}
VR(x) \arrow[r, "\varphi x"] \arrow[d, "VRf"'] \arrow[rd, phantom, "="] & SU(x) \arrow[d, "SUf"] \\
VR(y) \arrow[r, "\varphi y"'] & SU(y)
\end{tikzcd} \tag{18}$$

Moreover φ has to satisfy the following coherence conditions (coh.a), (coh.b), (coh.c) with the comparisons of R, S, U, V, for a degenerate cube $e_i(t)$ (with $t \in A_{\mathbf{i}|i}$) and a composite $z = x +_i y$ in $A_{\mathbf{i}}$.

(coh.a) If $p < q \leqslant i$ (so that R, S, U, V are lax in direction i), we have commutative diagrams (with $\varphi = \varphi x +_i \varphi y$):

$$\begin{tikzcd}
e_iVR(t) \arrow[r, "e_i(\varphi t)"] \arrow[d, "\underline{V}_i(Rt)"'] & e_iSU(t) \arrow[d, "\underline{S}_i(Ut)"] \\
V(e_iRt) \arrow[d, "V\underline{R}_i(t)"'] & S(e_iUt) \arrow[d, "S\underline{U}_i(t)"] \\
VR(e_it) \arrow[r, "\varphi(e_it)"'] & SU(e_it)
\end{tikzcd}
\qquad
\begin{tikzcd}
VRx +_i VRy \arrow[r, "\varphi"] \arrow[d, "\underline{V}_i(Rx,Ry)"'] & SUx +_i SUy \arrow[d, "\underline{S}_i(Ux,Uy)"] \\
V(Rx +_i Ry) \arrow[d, "V\underline{R}_i(x,y)"'] & S(Ux +_i Uy) \arrow[d, "S\underline{U}_i(x,y)"] \\
VR(z) \arrow[r, "\varphi(z)"'] & SU(z)
\end{tikzcd}
\qquad (19)$$

(coh.b) If $p \leqslant i < q$ (so that R, S are lax and U, V are colax in direction i), we have commutative diagrams:

$$\begin{tikzcd}
e_iVR(t) \arrow[r, "e_i(\varphi t)"] & e_iSU(t) \arrow[d, "\underline{S}_i(Ut)"] \\
V(e_iRt) \arrow[u, "\underline{V}_i(Rt)"] \arrow[d, "V\underline{R}_i(t)"'] & S(e_iUt) \\
VR(e_it) \arrow[r, "\varphi(e_it)"'] & SU(e_it) \arrow[u, "S\underline{U}_i(t)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
VRx +_i VRy \arrow[r, "\varphi"] & SUx +_i SUy \arrow[d, "\underline{S}_i(Ux,Uy)"] \\
V(Rx +_i Ry) \arrow[u, "\underline{V}_i(Rx,Ry)"] \arrow[d, "V\underline{R}_i(x,y)"'] & S(Ux +_i Uy) \\
VR(z) \arrow[r, "\varphi(z)"'] & SU(z) \arrow[u, "S\underline{U}_i(x,y)"']
\end{tikzcd}
\qquad (20)$$

(coh.c) If $i < p < q$ (so that R, S, U, V are colax in direction i), we have commutative diagrams as in (19), with all vertical arrows reversed.

The p- and q-composition of these cubes are both defined using componentwise the transversal composition of a chiral multiple category. Namely, for a consistent matrix of pq-cubes and $x \in \mathsf{A}$

$$\begin{tikzcd}
\bullet \arrow[r, "R"] \arrow[d, "U"'] \arrow[dr, phantom, "\varphi"] & \bullet \arrow[r, "R'"] \arrow[d, "V"] \arrow[dr, phantom, "\psi"] & \bullet \arrow[d, "W"] \\
\bullet \arrow[r, "S"] \arrow[d, "U'"'] \arrow[dr, phantom, "\sigma"] & \bullet \arrow[r, "S'"] \arrow[d, "V'"] \arrow[dr, phantom, "\tau"] & \bullet \arrow[d, "W'"] \\
\bullet \arrow[r, "T"'] & \bullet \arrow[r, "T'"'] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "p"] \arrow[d, "q"] & {} \\
{} &
\end{tikzcd}
\qquad (21)$$

$$\begin{aligned}(\varphi +_p \psi)(x) &= \psi(Rx) +_0 S'(\varphi x)\colon WR'Rx \to S'VRx \to S'SUx,\\ (\varphi +_q \sigma)(x) &= V'(\varphi x) +_0 \sigma(Ux)\colon V'VRx \to V'SUx \to TU'Ux.\end{aligned} \qquad (22)$$

The main technical points of the whole construction of Cmc are concerned with these composition laws. We shall prove, in Theorem 10, that they are well-defined, i.e. the cells above do satisfy the previous coherence conditions. We also prove that these laws strictly satisfy unitarity, associativity and the middle-four interchange law.

6. Transversal maps of degree two

Given two pq-cubes

$$\varphi\colon (U \, {}^{R}_{S} \, V), \qquad \varphi'\colon (U' \, {}^{R'}_{S'} \, V') \qquad (23)$$

a transversal pq-map $(F, G, F', G')\colon \varphi \to_0 \varphi'$ (of degree two and dimension three) is a quadruple of strict functors forming four transversal maps of degree 1

$$\begin{aligned}(F, G)\colon R \to_0 R', &\qquad (F', G')\colon S \to_0 S',\\ (F, F')\colon U \to_0 U', &\qquad (G, G')\colon V \to_0 V',\end{aligned} \qquad (24)$$

$$\begin{tikzcd}
\mathsf{A} \arrow[rr, "R"] \arrow[dr, "F"] \arrow[dd, "U"'] & & \bullet \arrow[dr, "G"] & \\
& \bullet \arrow[rr, "R'"] \arrow[dd, "U'"] \arrow[ur, phantom, "="] & \arrow[dl, dashed, "\varphi'"] & \bullet \arrow[dd, "V'"] \\
\bullet \arrow[dr, "F'"] \arrow[ur, phantom, "="] & & & \\
& \bullet \arrow[rr, "S'"'] & & \mathsf{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{A} \arrow[rr, "R"] \arrow[dd, "U"'] & \arrow[dl, dashed, "\varphi"] & \bullet \arrow[dr, "G"] \arrow[dd, "V"] & \\
& & & \bullet \arrow[dd, "V'"] \\
\bullet \arrow[rr, "R'"] \arrow[dr, "F'"'] & & \bullet \arrow[dr, "G'"'] \arrow[ur, phantom, "="] & \\
& \bullet \arrow[rr, "S'"'] \arrow[ur, phantom, "="] & & \mathsf{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, "0"] & {} \\
{} & {}
\end{tikzcd}$$

and such that 'the cube commutes', in the sense that, for every $\mathbf{i}$-cube x of A, the following transversal maps of B coincide

$$G'(\varphi x)\colon G'VR(x) \to G'SU(x), \quad \varphi'(Fx)\colon V'R'F(x) \to S'U'F(x). \quad (25)$$

We have thus defined the triple category $\mathrm{trp}_{0pq}(\mathsf{Cmc})$ of chiral multiple categories, with strict functors and p- and q-morphisms (for $0 < p < q \leqslant \infty$). Its indices vary in the pointed ordered set $\{0, p, q\}$.

7. Three-dimensional cubes

A *pqr-cube* (for $0 < p < q < r \leqslant \infty$) will be a 'commutative cube' Π determined by its six faces:

– two pq-cubes φ, ψ (the faces $\partial_r^\alpha \Pi$),

– two pr-cubes π, ρ (the faces $\partial_q^\alpha \Pi$),

– two qr-cubes ω, ζ (the faces $\partial_p^\alpha \Pi$),

$$\begin{tikzcd}
\mathsf{A} \arrow[rr, "R"] \arrow[dr, "U"] \arrow[dd, "X"'] & & \bullet \arrow[dr, "V"] & \\
& \bullet \arrow[rr, "S"] \arrow[dd, "Y"] & & \bullet \arrow[dd, "Y'"] \\
\bullet \arrow[dr, "U'"'] & & & \\
& \bullet \arrow[rr, "S'"'] & & \mathsf{B}
\arrow[from=1-1, to=2-4, phantom, "\varphi"]
\arrow[from=1-1, to=4-2, phantom, "\omega"]
\arrow[from=2-2, to=4-4, phantom, "\rho"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{A} \arrow[rr, "R"] \arrow[dd, "X"'] & & \bullet \arrow[dr, "V"] \arrow[dd, "X'"'] & \\
& & & \bullet \arrow[dd, "Y'"] \\
\bullet \arrow[rr, "R'"] \arrow[dr, "U'"'] & & \bullet \arrow[dr, "V'"] & \\
& \bullet \arrow[rr, "S'"'] & & \mathsf{B}
\arrow[from=1-1, to=3-3, phantom, "\pi"]
\arrow[from=1-3, to=4-4, phantom, "\zeta"]
\arrow[from=3-1, to=4-4, phantom, "\psi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "p"] \arrow[d, "r"'] \arrow[dr, "q"] & {} \\
{} & {}
\end{tikzcd}
\qquad (26)$$

The commutativity condition means that, for every $\mathbf{i}$-cube x of A, the following composed transversal arrows in B coincide

$$S'\omega x.\rho Ux.Y'\varphi x \colon Y'VR(x) \to Y'SU(x) \to S'YU(x) \to S'U'X(x)$$
$$\psi Xx.V'\pi x.\zeta Rx : Y'VR(x) = V'X'R(x) = V'R'X(x) = S'U'X(x).$$

These cubes are composed in direction p, q, or r, by pasting cubes (with the operations of 2-dimensional cubes). Again, these operations are associative, unitary and satisfy the middle-four interchange by pairs.

8. Higher items

A transversal pqr-map $F \colon \Pi \to_0 \Pi'$ between pqr-cubes is determined by its boundary, a face-consistent family of six transversal maps of degree two (and dimension three)

$$\partial_j^\alpha F \colon \partial_j^\alpha \Pi \to_0 \partial_j^\alpha \Pi' \qquad\qquad (\alpha = \pm,\ j \in \{p, q, r\}), \qquad (27)$$

under no other conditions. Their operations are computed on such faces.

We have thus defined a quadruple category of chiral multiple categories, with strict functors and p-, q-, r-morphisms (for extended positive integers $p < q < r$). The indices vary in the pointed ordered set $\{0, p, q, r\}$.

Finally, we have the multiple category Cmc (indexed by the ordinal $\boldsymbol{\omega} + \mathbf{1}$), where each cell of dimension $\geqslant 4$ (starting with the transversal maps of degree 3 considered above and the cubes of dimension 4, not yet considered) is coskeletally determined by a face-consistent family of all its iterated faces of dimension 3.

In the truncated case we have the $(n+1)$-dimensional multiple category $\mathsf{Cmc}_\mathbf{n}$ of (small) chiral $\mathbf{n}$-multiple categories, where the objects are indexed by the ordinal $\mathbf{n} = \{0, ..., n-1\}$, while $\mathsf{Cmc}_\mathbf{n}$ is indexed by $\mathbf{n} + \mathbf{1}$ (the previous ∞ being replaced by n). But one should note that $\mathsf{Cmc}_\mathbf{n}$ is *not* an ordinary truncation of Cmc, as its objects too are truncated.

Cmc is a substructure of the – similarly defined – multiple category Inc of small infinite dimensional intercategories, and $\mathsf{Cmc}_\mathbf{n}$ is a substructure of the $(n+1)$-dimensional multiple category $\mathsf{Inc}_\mathbf{n}$ of small $\mathbf{n}$-intercategories.

9. Comments

These multiple categories are related to various double or triple categories previously constructed.

(a) A chiral 1-multiple category is just a category, and $\mathsf{Cmc}_\mathbf{1}$ is the double category of small categories, with commutative squares of functors as double cells.

(b) A chiral 2-multiple category is a weak double category. We have studied in [4], Section 2, the double category $\mathbb{D}\mathrm{bl}$ of weak double categories, with lax and colax functors – where double adjunctions live. Later $\mathbb{D}\mathrm{bl}$ was extended to a triple category $\mathrm{S}\mathbb{D}\mathrm{bl}$ of weak double categories, with strict, lax and colax functors (see [7], Section 1); in the latter all 2-dimensional cells are inhabited by possibly non-trivial transformations, while in Cmc_2 the 01- and 02-cells are 'commutative squares', inhabited by identities. Thus $\mathsf{Cmc}_\mathbf{2}$ extends $\mathbb{D}\mathrm{bl}$ but is a triple subcategory of $\mathrm{S}\mathbb{D}\mathrm{bl}$.

(c) As we have recalled, multiple adjunctions live in the double category $\mathbb{C}\mathrm{mc}$ of chiral multiple categories, with lax and colax multiple functors ([9], Section 2). This can be extended to a triple category $\mathrm{S}\mathbb{C}\mathrm{mc}$ of chiral multiple categories, with strict, lax and colax functors, where again all 2-dimensional cells are inhabited by possibly non-trivial transformations. Then $\mathrm{S}\mathbb{C}\mathrm{mc}$ contains the triple category obtained from Cmc by restricting to the multi-indices $\mathbf{i} \subset \{0, 1, \infty\}$.

(d) The quadruple category Inc_3 of 3-dimensional intercategories is an extension of the triple category ICat of [9], Section 6, obtained by adding strict functors in the transversal direction and 'commutative transversal cells'.

10. Theorem

The structure Cmc *constructed above is indeed a strict multiple category.*

Proof. We prove the non-obvious points, listed at the end of Section 5.

(a) First we prove that the the composition law $\varphi +_p \psi$ of pq-cubes is well-defined by the formulas (22)

$$(\varphi +_p \psi)(x) = \psi(Rx) +_0 S'(\varphi x) \colon WR'Rx \to S'VRx \to S'SUx, \quad (28)$$

in the sense that this family of transversal maps does satisfy the conditions (coh.a) – (coh.c) of Section 5.

The argument is an extension of a similar one for the double category $\mathbb{D}\mathrm{bl}$ in [4], Section 2, or for the double category $\mathbb{C}\mathrm{mc}$ in [9], Section 2, taking into account the mixed laxity of the present 'functors'. We prove the three coherence axioms with respect to a composed cube $z = x +_i y$ in $A_{\mathbf{i}}$; one would work in a similar way for a degenerate cube $e_i(t)$, with $t \in A_{\mathbf{i}|i}$.

First we prove (coh.a), letting $p < q \leqslant i$, so that all our functors R, R', S, S', U, V, W are lax in direction i. This amounts to the commutativity of the outer diagram below, formed of transversal maps (the index i being omitted in $+_i$ and in all comparisons $\underline{R}_i, \underline{R}'_i$, etc.)

$$\begin{tikzcd}
WR'Rz \arrow[r, "\psi Rz"] & S'VRz \arrow[r, "S'\varphi z"] & S'SUz \\
WR'(Rx+Ry) \arrow[u, "WR'\underline{R}"] \arrow[r, "\psi(Rx+Ry)"] & S'V(Rx+Ry) \arrow[u, "S'V\underline{R}"'] & S'S(Ux+Uy) \arrow[u, "S'S\underline{U}"'] \\
W(R'Rx+R'Ry) \arrow[u, "W\underline{R'}R"] & S'(VRx+VRy) \arrow[u, "S'\underline{V}R"'] \arrow[r, "S'(\varphi x+\varphi y)"] & S'(SUx+SUy) \arrow[u, "S'\underline{S}U"'] \\
WR'Rx+WR'Ry \arrow[u, "\underline{W}R'R"'] \arrow[r, "\psi Rx+\psi Ry"'] & S'VRx+S'VRy \arrow[u, "\underline{S'}VR"'] \arrow[r, "S'\varphi x+S'\varphi y"'] & S'SUx+S'SUy \arrow[u, "\underline{S'}SU"']
\end{tikzcd}$$

Indeed, the two hexagons commute by (coh.a), applied to φ and ψ, respectively. The upper rectangle commutes by naturality of ψ on $\underline{R}_i(x, y)$. The lower rectangle commutes by axiom (lmf.2) (in [7], Section 3.9), on the lax functor S', with respect to the transversal **i**-maps $\varphi x\colon VR(x) \to_0 SU(x)$ and $\varphi y\colon VR(y) \to_0 SU(y)$

$$S'(\varphi x +_i \varphi y).\underline{S}'_i(VR(x), VR(y)) = \underline{S}'_i(SU(x), SU(y)).(S'(\varphi x) +_i S'(\varphi y)).$$

The proof of (coh.c) is transversally dual to the previous one. To prove (coh.b) we let $p \leqslant i < q$, so that R, R', S, S' are lax, while U, V, W are colax in direction i. We reverse the comparisons $\underline{U}_i, \underline{V}_i, \underline{W}_i$ in the diagram above

$$\begin{tikzcd}
WR'Rz \arrow[r, "\psi Rz"] & S'VRz \arrow[r, "S'\varphi z"] & S'SUz \arrow[d, "S'S\underline{U}"] \\
WR'(Rx+Ry) \arrow[u, "WR'\underline{R}"] \arrow[r, "\psi(Rx+Ry)"] & S'V(Rx+Ry) \arrow[u, "S'V\underline{R}"'] \arrow[d, "S'\underline{V}R"'] & S'S(Ux+Uy) \\
W(R'Rx+R'Ry) \arrow[u, "W\underline{R}'R"] \arrow[d, "\underline{W}R'R"'] & S'(VRx+VRy) \arrow[r, "S'(\varphi x+\varphi y)"] & S'(SUx+SUy) \arrow[u, "S'\underline{S}U"'] \\
WR'Rx+WR'Ry \arrow[r, "\psi Rx+\psi Ry"'] & S'VRx+S'VRy \arrow[u, "\underline{S}'VR"'] \arrow[r, "S'\varphi x+S'\varphi y"'] & S'SUx+S'SUy \arrow[u, "\underline{S}'SU"']
\end{tikzcd}$$

and note that the two hexagons commute, by (coh.b) on φ and ψ, while the rectangles are unchanged.

(b) The composition law $\varphi +_p \psi$ has been defined via the composition of transversal maps, and therefore is strictly unitary and associative.

(c) Finally, to verify the middle-four interchange law on the four double cells of diagram (21), we compute the composites $(\varphi +_p \psi) +_q (\sigma +_p \tau)$ and $(\varphi +_q \sigma) +_p (\psi +_q \tau)$ on an **i**-cube x, and we obtain the two transversal maps $W'WR'Rx \to_0 T'TU'Ux$ of the upper or lower path in the following diagram

$$\begin{tikzcd}
W'WR'Rx \arrow[r, "W'\psi Rx"] & W'S'VRx \arrow[r, "W'S'\varphi x"] \arrow[d, "\tau VRx"'] \arrow[dr, phantom, "="] & W'S'SUx \arrow[d, "\tau SUx"] & \\
& T'V'VRx \arrow[r, "T'V'\varphi x"'] & T'V'SUx \arrow[r, "T'\sigma Ux"'] & T'TU'Ux
\end{tikzcd}$$

The square commutes, by naturality of the double cell τ (with respect to the transversal map $\varphi x\colon VR(x) \to_0 SU(x)$), so that the two composites coincide. □

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ARITHMETIC UNIVERSES AND CLASSIFYING TOPOSES

by Steven VICKERS

Résumé. Cet article utilise la structure de $\mathfrak{Con}$ (la 2-catégorie des esquisses pour les univers arithmétiques (AU) de l'auteur) pour obtenir des résultats constructifs, indépendants de la base pour les topos de Grothendieck ($\mathcal{S}$-topos bornés) comme espaces généralisés. Le principal résultat montre comment une application extension $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ peut être vue comme un fibré, transformant les points de base (modèles de $\mathbb{T}_0$ dans un topos $\mathcal{S}$ avec objet des nombres naturels) en fibres (espaces généralisés au-dessus de $\mathcal{S}$). Parmi les caractéristiques de ce travail, on notera : une comparaison entre modèles stricts ou non-stricts, utilisant les propriétés des objets de $\mathfrak{Con}$; l'utilisation des produits tensoriels de Gray pour relier la transformation syntactique de modèles par des 1-cellules de $\mathfrak{Con}$ et les transformations sémantiques par des AU-foncteurs non stricts ; et l'utilisation de 2-fibrations pour indexer au-dessus d'une 2-catégorie de topos de base $\mathcal{S}$.

Abstract. The paper uses structures in $\mathfrak{Con}$, the author's 2-category of sketches for arithmetic universes (AUs), to provide constructive, base-independent results for Grothendieck toposes (bounded $\mathcal{S}$-toposes) as generalized spaces.

The main result is to show how an extension map $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ can be viewed as a bundle, transforming base points (models of $\mathbb{T}_0$ in any elementary topos $\mathcal{S}$ with nno) to fibres (generalized spaces over $\mathcal{S}$).

Features of the work include comparison of strict and non-strict models, using properties of the objects of $\mathfrak{Con}$; the use of Gray tensor products to relate syntactic transformation of models by 1-cells in $\mathfrak{Con}$ and semantic transformations by non-strict AU-functors; and the use of 2-fibrations to index over a 2-category of base toposes $\mathcal{S}$.

Keywords. geometric theory, 2-fibration, sketch, Gray tensor

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1. Introduction

If $\mathbb{T}$ is a geometric theory, then the generalized topological space – in Grothendieck's sense – of models of $\mathbb{T}$ is realized mathematically as its category of sheaves, the classifying topos $\mathcal{S}[\mathbb{T}]$.

$\mathcal{S}$ here, the base, could be any elementary topos with nno that is able to support the infinite disjunctions appearing in $\mathbb{T}$, and if those disjunctions are countable then any such $\mathcal{S}$ will do. So which topos $\mathcal{S}[\mathbb{T}]$ is the true incarnation of the generalized space?

[12] developed a 2-category $\mathfrak{Con}$ whose objects are, in sketch form, such theories; and whose 1-cells are the maps got if one replaces the classifying topos $\mathcal{S}[\mathbb{T}]$ by a classifying *arithmetic universe* $\mathbf{AU}\langle\mathbb{T}\rangle$, which can thus be understood as a base-independent incarnation of the space.

The present paper shows how to recover the base-dependent topos theory, but in an indexed way, using 2-fibrations, that allows for change of base.

As a significant generalization of the indexed construction $\mathcal{S} \mapsto \mathcal{S}[\mathbb{T}]$, we also relativize by looking at certain maps $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ in $\mathfrak{Con}$ considered as *bundles* – that is to say, transformations from base point M (model of $\mathbb{T}_0$) to space (fibre of U over M). If M is in an elementary topos $\mathcal{S}$, then we construct an $\mathcal{S}$-geometric theory $\mathbb{T}_1/M$ of models of $\mathbb{T}_1$ that are reduced to M by U, and then the fibre, as generalized space in the topos sense, is $\mathcal{S}[\mathbb{T}_1/M]$.

Our main result, Theorem 5.12, is that the whole construction $(\mathcal{S}, M) \mapsto \mathcal{S}[\mathbb{T}_1/M]$ is indexed over pairs $(\mathcal{S}, M)$. This is formalized 2-fibrationally using a new notion (Definition 4.4) of *local representability*.

Throughout this paper, *every elementary topos will be assumed to have a natural numbers object*. We write $\mathfrak{eTop}$ for the 2-category of elementary toposes with nno, geometric morphisms (not necessarily bounded), and natural transformations.

1.1 Generalized spaces and their categories of sheaves

Let us elaborate on the underlying question. Grothendieck discovered a huge generalization of the notions of topology and continuity, with a generalized space represented concretely by its category of sheaves (continuous set-valued maps).

This is point-free topology, analogous to representing a space X by its frame ΩX of opens, albeit on a much grander scale.

[10] made an explicit attempt to make the analogous notational distinction, writing X for the generalized space and $\mathcal{S}X$ for its category of sheaves. If $[\mathbb{T}]$ is written for the space of models of a geometric theory $\mathbb{T}$, then $\mathcal{S}[\mathbb{T}]$ can be read either as "$\mathcal{S}$heaves over the space $[\mathbb{T}]$" or as "the (geometric) mathematics generated over the category $\mathcal{S}$ of sets by adjoining a generic model of $\mathbb{T}$".

That paper was applied to domain theory, and in particular the ideal completion of information systems (the compact bases) for SFP-domains. These were studied using a generic SFP-domain, a geometric morphism $[\mathrm{IS}][\mathrm{idl}] \to [\mathrm{IS}]$, where $[\mathrm{IS}]$ classifies SFP information systems and the fibre over one of them is its ideal completion. (We shall see a more general account of such bundles in Section 5.2.)

But what is this category $\mathcal{S}$ of sets, within which one constructs the sheaves, and over which one constructs $\mathcal{S}[\mathbb{T}]$? To Grothendieck it would have been classical set theory $\mathbf{Set}$. With the subsequent discovery of elementary toposes, it was found that any elementary topos $\mathcal{S}$ with nno could be used as base for a notion of geometric theory and for constructing generalized spaces (bounded geometric morphisms into $\mathcal{S}$) as classifying toposes. $\mathcal{S}$-indexed categories are used to capture the idea that an object of $\mathcal{S}$ can be used as an indexing set for a colimit diagram (see [7, B1.4]).

That relieves the classical dependency, but unfortunately creates a problem of its own: even if (as in [10]) the working is foundationally robust, one still has to choose a base $\mathcal{S}$ in order to have a mathematical incarnation $\mathcal{S}[\mathbb{T}]$ of the generalized space $[\mathbb{T}]$.

In its conclusions, [10] proposed that $\mathcal{S}$ might be dispensed with if all the working could be reduced to that of arithmetic universes (AUs), with finite colimits and list objects instead of "$\mathcal{S}$-indexed" colimits. By [8], every elementary topos with nno is an AU, and for any geometric morphism f between them, the inverse image part f^* is a (non-strict) AU-functor. Then the infinities of geometric logic, supplied extrinsically by $\mathcal{S}$, would be replaced where possible by intrinsic infinities supplied by the list object construction.

The ultimate ambition would be to develop an entirely "arithmetic" account of generalized topology, using AUs $\mathbf{AU}\langle\mathbb{T}\rangle$, to replace the present geometric account using Grothendieck toposes. How far that can be carried through remains to be tested. The more modest aim of the present paper is to show how arithmetic techniques can give base-independent results in the existing topos theory.

1.2 Outline

Section 2 summarizes the background of AUs, their sketches, and the 2-category $\mathfrak{Con}$ [12]; and of geometric theories and classifying toposes largely as presented in [7, B4.2].

Section 3 discusses the models of AU-sketches in AUs in general, and elementary toposes in particular. A particular issue is whether the models should be *strict* or not. We need both, and the *contexts*, the AU-sketches appearing as objects of $\mathfrak{Con}$, have the special property that every non-strict model has a canonical strict isomorph. We describe two interacting actions on models: one by context maps between theories, and one by non-strict AU-functors between the AUs where the models are found.

Section 4 collects miscellaneous remarks on the 2-fibrational background that allows us to vary the base elementary topos $\mathcal{S}$, and includes (Definition 4.4) a notion of *local representability* that captures, 2-fibrationally, the idea of classifying toposes behaving in an indexed way under pseudopullback along change of base topos. In essence this is the idea of "geometricity" as expressed in [11].

Section 5 examines classifying toposes for contexts. In fact, we deal with a relativized version, with a context extension map $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ (given by $\mathbb{T}_0 \subset \mathbb{T}_1$). If each context represents "the space of its models", then we wish to view U as a bundle: over each model M of $\mathbb{T}_0$, the fibre over it is the "space of models of $\mathbb{T}_1$ that restrict to M". We shall show how these fibres can be represented as classifying toposes.

Now we fibre over pairs $(\mathcal{S}, M)$, where M is a strict model of $\mathbb{T}_0$ in $\mathcal{S}$. We find a geometric (though not arithmetic in general) theory $\mathbb{T}_1/M$ of models of $\mathbb{T}_1$ restricting to M, and it has a classifying topos $\mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ (with its generic model).

Our main result, Theorem 5.12, is that this construction is locally representable, in other words that it is geometric – preserved by pseudopullback along arbitrary geometric morphisms. A corollary is the "geometricity of presentations" result of [11, Section 5].

2. Background

2.1 Sketches for arithmetic universes

We summarize the sketch approach to arithmetic universes as set out in [12]. The sketches are roughly as in [3], with a reflexive graph of nodes and edges for objects and morphisms, a set of "commutativities" to specify commutative triangles, and "universals" (the cones and cocones) for finite limits and finite colimits – specifically: terminals, pullbacks, initials, pushouts. In addition they have universals to specify list objects, thus gaining an nno as $\mathsf{List}\,1$.

In our *sketch extensions* $\mathbb{T} \subset \mathbb{T}'$ such universals may be introduced only for fresh objects, and hence in a definitional way. A *context* is then an extension of the empty sketch $\mathbb{1}$.

In *equivalence extensions* $\mathbb{T} \Subset \mathbb{T}'$, everything fresh that is introduced must have been implicitly present already. This includes composites of composable pairs of edges; commutativities deducible from existing ones (e.g. by unit laws or associativities); universals, fillins for universals and uniqueness of fillins; and inverses for certain edges that must be isomorphisms because of the categorical properties of AUs such as balance, stability and exactness.

Homomorphisms $\mathbb{T} \lessdot \mathbb{T}'$ are structure-preserving homomorphisms for the algebraic theory of sketches. They translate nodes to nodes, edges to edges, commutativities to commutativities and universals to universals. The two kinds of extensions are special cases of this.

Next, we have a notion of *object equalities* between nodes, certain edges that include all identity edges but can also arise as fillins when the same universal construction is applied to equal data. We extend this to object equalities between edges, when their domains have an object equality and so do the codomains, and there are appropriate commutativities to make a commutative square; and then we extend to object equalities between homomorphism of models, using object equalities between corresponding nodes and edges in the image.

Putting these together we get a category $\mathfrak{Con}$ whose objects are contexts. Its morphisms, context *maps*, are the dual of context homomorphisms, but subject to (i) those for equivalence extensions are invertible, and (ii) object equalities become identity morphisms between actually equal objects. Every map $\mathbb{T}_0 \to \mathbb{T}_1$ is an equivalence class of opspans of homomorphisms $\mathbb{T}_0 \Subset \mathbb{T}'_0 \gtrdot \mathbb{T}_1$.

Notice that, for each of the special symbols $\subset$, $\Subset$ and $\lessdot$, the narrow end is at the codomain for the corresponding reduction *map*.

For each context $\mathbb{T}$ there is also a context $\mathbb{T}^{\to}$ for which a model is a pair of models of $\mathbb{T}$, together with a $\mathbb{T}$-homomorphism between them. These enable us to define 2-cells between maps, using maps $\mathbb{T}_0 \to \mathbb{T}_1^{\to}$, and $\mathfrak{Con}$ becomes a 2-category. It has finite PIE-limits (Product, Inserter, Equifier) and pullbacks of extension maps (the duals of the homomorphisms corresponding to extensions).

There is a full and faithful 2-functor from $\mathfrak{Con}$ to the category $\mathbf{AU}_s$ of AUs and strict AU-functors, contravariant on 1-cells, that takes $\mathbb{T} \mapsto \mathbf{AU}\langle\mathbb{T}\rangle$, the AU presented using $\mathbb{T}$ as generators and relations.

A central issue for models of sketches is that of *strictness*. The standard sketch-theoretic notion is non-strict: for a universal, such as a pullback of some given opspan, the pullback cone can be interpreted as any pullback of the opspan. However, we could also seek strict models that use the canonical pullbacks (in categories where they exist). Strictness is essential for the universal algebra that generates $\mathbf{AU}\langle\mathbb{T}\rangle$, but in general it is inconvenient. Significant parts of the present paper are concerned with relating the strict and the non-strict.

Contexts are designed to give us good control over strictness, as summarized by the following proposition.

Proposition 2.1. *Let $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ be an extension map in $\mathfrak{Con}$, that is to say one deriving from an extension $\mathbb{T}_0 \subset \mathbb{T}_1$. Suppose in some AU $\mathcal{A}$ we have a model M_1 of $\mathbb{T}_1$, a strict model M'_0 of $\mathbb{T}_0$, and an isomorphism $\phi_0\colon M'_0 \cong M_1U$ (the restriction of M_1 to $\mathbb{T}_0$).*

$$\begin{tikzcd}
\mathbb{T}_1 \arrow[d, "U"'] & M'_1 \arrow[r, dotted, "\phi_1", "\cong"'] \arrow[d, mapsto] & M_1 \arrow[d, mapsto] \\
\mathbb{T}_0 & M'_0 \arrow[r, "\phi_0", "\cong"'] & M_1U
\end{tikzcd}$$

Then there is a unique model M'_1 of $\mathbb{T}_1$ and isomorphism $\phi_1\colon M'_1 \cong M_1$ such that

1. *M'_1 is strict,*
2. *$M'_1U = M'_0$,*
3. *$\phi_1 U = \phi_0$, and*
4. *ϕ_1 is equality on all the primitive nodes for the extension $\mathbb{T}_0 \subset \mathbb{T}_1$.*

The proof can be deduced from the strictness results in [12]. In brief, it is reduced by induction to the case of simple extension steps in $\mathbb{T}_0 \subset \mathbb{T}_1$. Adjoining a primitive node, M'_1 and ϕ_1 are determined by (4). Adjoining a primitive edge, M'_1 and ϕ_1 are determined by the need to make ϕ_1 an isomorphism. Adjoining a universal, M'_1 is determined by (1) and ϕ_1 by (3), as the unique fillin consistent with ϕ_0.

In the case where $\mathbb{T}_0$ is the empty context $\mathbb{1}$, we see the important corollary that for a context $\mathbb{T}$ *every model is uniquely isomorphic to a unique strict model with which it agrees on all primitive nodes.* We call this its *canonical strict isomorph.*

Thus in topos theory, where non-strict AU-functors are liable to transform strict models into non-strict ones, we can regain strictness of models.

Example 2.2. The Proposition does not hold for arbitrary context maps $H\colon \mathbb{T}_1 \to \mathbb{T}_0$. Let $\mathbb{O}, \mathbb{O}^2$ be the contexts that have, respectively, one and two nodes, and nothing else. Consider the diagonal $\Delta\colon \mathbb{O} \to \mathbb{O}^2$ given by the context homomorphism that takes both nodes in $\mathbb{O}^2$ to the node in $\mathbb{O}$. If X is a model of $\mathbb{O}$, then its Δ-reduct $X\Delta = (X, X)$. If we can find $X_1 \cong X \cong X_2$ with $X_1 \neq X_2$, then $(X_1, X_2) \cong X\Delta$ without itself being a Δ-reduct.

2.2 Elephant theories

Here we briefly summarize the account in [7, B4.2] of classifying toposes, over a fixed base elementary topos $\mathcal{S}$.

Central to its treatment is the 2-category $\mathfrak{BTop}/\mathcal{S}$. A 0-cell is a bounded geometric morphism $p\colon \mathcal{E} \to \mathcal{S}$, a *Grothendieck topos over $\mathcal{S}$*. In Definition 4.1 these will appear in the fibre of our $\mathfrak{GTop}$ over $\mathcal{S}$. A 1-cell f is a pair $(\overline{f}, f{\Downarrow})$, where $\overline{f}$ is a bounded geometric morphism and $f{\Downarrow}$ is a specified isomorphism in the triangle over $\mathcal{S}$.

Any logical description of a theory does implicitly describe the models, but one can also try to use the category of models as a direct semantic description of the theory. Unfortunately this does not work for geometric theories, which may be incomplete – there are not enough models for semantic entailment to agree with the syntactic entailment got from the rules of geometric logic.

The semantic description used to get round this in [7, B4.2] is to describe all the models in *all Grothendieck toposes*. For narrative purposes in the present paper, to make a clear distinction from the logical theories, I shall refer to such an "all model" description as an "elephant theory". Of course that acknowledges their use in [7], but I also want to convey something of the sheer quantity of data encapsulated in one of these theories.

Definition 2.3. *An* elephant theory over $\mathcal{S}$ *is an indexed category* $\mathbb{T}$ *over* $\mathfrak{BTop}/\mathcal{S}$*. Then an object of* $\mathbb{T}(\mathcal{E})$ *is a "model of* $\mathbb{T}$ *in* $\mathcal{E}$*".*

In our applications derived from AU-sketches, the elephant theories will be strict, 2-functors to $\mathfrak{CAT}$.

A particularly important example is the context $\mathbb{O}$, the object classifier, with $\mathbb{O}(\mathcal{E}) = \mathcal{E}$.

Given an elephant theory $\mathbb{T}$ over $\mathcal{S}$, a *geometric construct* on $\mathbb{T}$ is an indexed functor from $\mathbb{T}$ to $\mathbb{O}$.

Definition 2.4. *Let* $\mathbb{T}_0$ *be an elephant theory over* $\mathcal{S}$*. A* geometric extension *of* $\mathbb{T}_0$ *is a theory built, starting from* $\mathbb{T}_0$*, by a finite sequence of the following "simple" steps from* $\mathbb{T}$ *to* $\mathbb{T}'$*.*

- Simple functional extension: *Let* $H_0, H_1 \colon \mathbb{T} \to \mathbb{O}$ *be two geometric constructs. Define the theory* $\mathbb{T}'$ *whose models in* $\mathcal{E}$ *are pairs* (M, u) *where* M *is a model of* $\mathbb{T}$ *in* $\mathcal{E}$ *and* $u \colon MH_0 \to MH_1$ *is a morphism. A morphism from* (M, u) *to* (M', u') *is morphism* $\phi \colon M \to M'$ *such that that following diagram commutes.*

$$\begin{tikzcd} MH_0 \arrow[r, "u"] \arrow[d, "\phi H_0"'] & MH_1 \arrow[d, "\phi H_1"] \\ M'H_0 \arrow[r, "u'"'] & M'H_1 \end{tikzcd} .$$

- Simple geometric quotient: *Let* $\phi\colon H_0 \to H_1$ *be a morphism of geometric constructs on* $\mathbb{T}$. $\mathbb{T}'$ *is the theory whose models in* $\mathcal{E}$ *are those models of* $\mathbb{T}$ *for which* ϕ *is an isomorphism; its morphisms are all* $\mathbb{T}$*-morphisms.*

- Simple extension by primitive object: *We define* $\mathbb{T}'(\mathcal{E}) = \mathbb{T}(\mathcal{E}) \times \mathcal{E}$. *In other words, we may write* $\mathbb{T}' = \mathbb{T} \times \mathbb{O}$.

Then a geometric theory *over* $\mathcal{S}$ *is a geometric extension of* $\mathbb{1}$*, the trivial theory for which every* $\mathbb{1}(\mathcal{E})$ *is the category with one object* $*$ *and its identity morphism.*

Note that [7] does not define the general notion of geometric extension, but simply that of geometric theory as an extension of $\mathbb{O}^n$ (for some finite n) by simple functional extensions and simple geometric quotients. The two are equivalent, because no harm is done if the primitive sorts are all adjoined at the start, and doing this n times to $\mathbb{1}$ gives $\mathbb{O}^n$.

If $\mathbb{T}_1$ is a geometric extension of $\mathbb{T}_0$, then there is a theory morphism from $\mathbb{T}_1$ to $\mathbb{T}_0$ given by model reduction.

For future reference we prove the following result that does not appear to be in [7].

Proposition 2.5. *In the category of elephant theories over* $\mathcal{S}$ *and indexed functors between them, geometric extensions can be pulled back along any morphism.*

Proof. The point is that we have a pullback, not a pseudopullback.

Let $H\colon \mathbb{T}'_0 \to \mathbb{T}_0$ be an indexed functor between elephant theories over $\mathcal{S}$, and let $\mathbb{T}_1$ be a geometric extension of $\mathbb{T}_0$ with indexed functor $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ defined by model reduction. We define the elephant theory $\mathbb{T}'_1$ by argumentwise pullback of categories.

$$\begin{tikzcd}
\mathbb{T}'_1(\mathcal{E}) \arrow[r] \arrow[d] & \mathbb{T}_1(\mathcal{E}) \arrow[d, "U(\mathcal{E})"] \\
\mathbb{T}'_0(\mathcal{E}) \arrow[r, "H(\mathcal{E})"'] & \mathbb{T}_0(\mathcal{E})
\end{tikzcd}$$

Thus a model of $\mathbb{T}'_1$ is a pair (M_0, M_1) of models of $\mathbb{T}'_0$ and $\mathbb{T}_1$ for which $M_0 H = M_1 U$.

For reindexing along $f\colon \mathcal{F} \to \mathcal{E}$ (over $\mathcal{S}$), the naive attempt to define $f^*(M_0, M_1)$ as (f^*M_0, f^*M_1) fails because we only have

$$(f^*M_0)H \cong f^*(M_0H) = f^*(M_1U) = (f^*M_1)U.$$

(The last equality can be readily checked for different kinds of simple geometric extension.) The trick then is to define $f^*(M_0, M_1)$ as (f^*M_0, N_1) for some $N_1 \cong f^*M_1$ whose $\mathbb{T}_0$-reduct is $(f^*M_0)H \cong (f^*M_1)U$.

It suffices to check the three kinds of simple geometric extension. For extension by primitive sort, $\mathbb{T}_1 = \mathbb{T}_0 \times \mathbb{O}$, we find that $\mathbb{T}'_1$ as defined by pullback is $\mathbb{T}'_0 \times \mathbb{O}$. For the reindexing question, we have M_1 of the form (M_0H, X) and define $N_1 = ((f^*M_0)H, f^*X)$.

The next case is when $\mathbb{T}_1$ is a simple functional extension of $\mathbb{T}_0$ for two geometric constructs $G_0, G_1\colon \mathbb{T}_0 \to \mathbb{O}$. We find that $\mathbb{T}'_1$, as defined by pullback, is a simple functional extension of $\mathbb{T}'_0$ for HG_0 and HG_1. For the reindexing, we have M_1 of the form $(M_0H, u\colon M_0HG_0 \to M_0HG_1)$. Then we take N_1 to be $((f^*M_0)H, u')$, where u' is so as to make the following diagram commute.

$$\begin{tikzcd}
(f^*M_0)HG_0 \arrow[r, "\cong"] \arrow[d, "u'"'] & (f^*(M_0H))G_0 \arrow[r, "\cong"] & f^*(M_0HG_0) \arrow[d, "f^*u"] \\
(f^*M_0)HG_1 \arrow[r, "\cong"] & (f^*(M_0H))G_1 \arrow[r, "\cong"] & f^*(M_0HG_1)
\end{tikzcd}$$

For the final case, $\mathbb{T}_1$ is an extension of $\mathbb{T}_0$ by simple geometric quotient for a morphism $\phi\colon G_0 \to G_1$ of two geometric constructs on $\mathbb{T}_0$. Now $\mathbb{T}'_1$ is an extension of $\mathbb{T}'_0$ by simple geometric quotient for a morphism $H\phi\colon HG_0 \to HG_1$. $\square$

Definition 2.6. *Let $\mathbb{T}$ be an elephant theory over $\mathcal{S}$. A* classifying topos *for $\mathbb{T}$ is a bounded $\mathcal{S}$-topos $p\colon \mathcal{S}[\mathbb{T}] \to \mathcal{S}$, equipped with a "generic" $\mathbb{T}$-model N_G, such that, for each bounded $\mathcal{S}$-topos $\mathcal{E}$, the functor*

$$\mathfrak{BTop}/\mathcal{S}[\mathcal{E}, \mathcal{S}[\mathbb{T}]] \to \mathbb{T}(\mathcal{E}), \quad f \mapsto f^*N_G,$$

is one half of an equivalence of categories.

In other words, the pseudofunctor $\mathbb{T}\colon \mathfrak{BTop}/\mathcal{S} \to \mathfrak{CAT}$ is representable.

Since all our elementary toposes have nno, [7, Theorem B4.2.9] tells us that *every geometric theory has a classifying topos.*

3. Indexed categories of models

In this section we deal with categories of models of AU-contexts from $\mathfrak{Con}$. For each AU $\mathcal{A}$ and AU-context $\mathbb{T}$ we have a category $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathbb{T}$ of models of $\mathbb{T}$ in $\mathcal{A}$, and a full subcategory $\mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}$ of strict models.

We shall show that $\mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}$ is acted on strictly (on the right) by $\mathfrak{Con}$, and strictly (on the left) by $\mathbf{AU}$, the category of AUs and *non-strict* AU-functors. This strict left action arises because $\mathbb{T}$, a context, has the strict model corollary of Proposition 2.1: applying a non-strict AU-functor gives us a non-strict model, but we can then replace it by its canonical strict isomorph.[1] The left and right actions commute up to isomorphism, which we express in Theorem 3.6 as a category strictly indexed over the Gray tensor product. However, right action by extension maps commutes *up to equality* with the left actions (Lemma 3.7), and this will be important for us.

Note that the context maps, between contexts $\mathbb{T}$, correspond to *strict* AU-functors between the classifying AUs $\mathbf{AU}\langle\mathbb{T}\rangle$. What we have done, therefore, is in effect to have strict and non-strict AU-functors acting on the right and left respectively, with the Gray tensor action representing the interplay between strict and non-strict.

One might wonder whether we could instead have focused on the non-strict models $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathbb{T}$. There is an obvious action on the left by $\mathbf{AU}$, and an action on the right, by model reduction, by the context maps that correspond to context homomorphisms. Those left and right actions commute up to equality. However, the right action does not extend strictly to arbitrary context maps: this is because the maps for context equivalence extensions, which are invertible in $\mathfrak{Con}$, give only equivalences between model categories, not isomorphisms. We prefer to work with the strict action on strict models.

In any case, the non-strict models of a context $\mathbb{T}$ are the strict models of an extension $\mathbb{T}'$. For each node X in $\mathbb{T}$ introduced by a universal, adjoin another copy X' with edges and commutativities to make $X' \cong X$.

Definition 3.1. *Let $\mathcal{A}$ be an AU and $\mathbb{T}$ a context. Then $\mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}$ is the category of strict models of $\mathbb{T}$ in $\mathcal{A}$.*

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[1] In fact, the definitions of extension and context in [12] were made in anticipation of these results.

Lemma 3.2. *For each arithmetic universe $\mathcal{A}$, we can define a 2-functor*

$$\mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\bullet\colon \mathfrak{Con} \to \mathfrak{CAT}$$

for which $\mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\bullet(\mathbb{T}) = \mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}$.

Proof. Since those models are in bijection with strict AU-functors from $\mathbf{AU}\langle\mathbb{T}\rangle$ to $\mathcal{A}$, and we have a (full and faithful) 2-functor from $\mathfrak{Con}$ to $\mathbf{AU}_s^{op}$, this extends to a 2-functor $\mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\bullet$ as desired. ☐

If M is a strict model in $\mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}_0$ and $H\colon \mathbb{T}_0 \to \mathbb{T}_1$ is a context map, then we write MH for $\mathcal{A}\text{-}\mathbf{Mod}_s\text{-}H(M)$. If H is the dual of a context homomorphism then MH is got by model reduction. If H is the inverse of the dual for an equivalence extension $\mathbb{T}_0 \Subset \mathbb{T}_1$, then MH is got by interpreting all the adjoined ingredients of $\mathbb{T}_1$ in the unique strict way.

Now we fix $\mathbb{T}$ and let $\mathcal{A}$ vary.

Definition 3.3. *Let $f\colon \mathcal{A}_0 \to \mathcal{A}_1$ be an AU-functor, $\mathbb{T}$ a context and M a model in $\mathcal{A}_0\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}$. Then we define $f^*M = f\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}(M)$ as follows. We first define $f \cdot M$ as the non-strict model got by applying f to M. Then f^*M is (using Proposition 2.1) the canonical strict isomorph of $f \cdot M$.*

We extend this to 2-cells $\alpha\colon f_0 \to f_1$ by treating them as AU-functors from $\mathcal{A}_0$ to the comma AU $\mathcal{A}_1 \downarrow \mathcal{A}_1$. $\alpha^*M\colon f_0^*M \to f_1^*M$ is then calculated by pasting the following diagram.

$$\begin{tikzcd}
\mathcal{A}_1 & \mathcal{A}_0 \arrow[l, bend right=30, "f_0"'] \arrow[l, bend left=30, "f_1"] \arrow[l, phantom, "\Downarrow\alpha"] & \mathbf{AU}\langle\mathbb{T}\rangle \arrow[l, "M"', "\cong"] \arrow[ll, bend right=40, "f_0^*M"'] \arrow[ll, bend left=40, "f_1^*M"] \arrow[l, phantom, shift right=3, "\cong"]
\end{tikzcd}$$

Proposition 3.4. *For each context $\mathbb{T}$ we have a 2-functor*

$$\bullet\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}\colon \mathbf{AU} \to \mathfrak{CAT}$$

for which $\bullet\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}(\mathcal{A}) = \mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}$ *and* $\bullet\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}(f)(M) = f^*(M)$.

Proof. The main point is that it is strictly functorial on 1-cells f. Suppose we have AU-functors

$$\begin{tikzcd}
\mathcal{A}_2 & \mathcal{A}_1 \arrow[l, "f_1"'] & \mathcal{A}_0 \arrow[l, "f_0"']
\end{tikzcd} .$$

Then $f_1^* f_0^* M$ and $(f_0 f_1)^* M$ are both the canonical strict isomorph of $f_1 \cdot f_0 \cdot M$.

After this, the rest follows by pasting diagrams. $\square$

The equation $f_1^* f_0^* M = (f_0 f_1)^* M$ will seem notationally perverse for morphisms in $\mathbf{AU}$, composed diagrammatically, but it looks more natural for geometric morphisms, where the AU-functor for f is f^*.

Definition 3.5. *Suppose we have 1-cells* $f\colon \mathcal{A}_0 \to \mathcal{A}_1$ *in* $\mathbf{AU}$ *and* $H\colon \mathbb{T}_0 \to \mathbb{T}_1$ *in* $\mathfrak{Con}$. *Then we define a natural isomorphism* $\Sigma_{f,H}$ *as follows.*

$$\begin{tikzcd}
\mathcal{A}_0\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}_0 \arrow[r, "\mathcal{A}_0\text{-}\mathbf{Mod}_s\text{-}H"] \arrow[d, "f\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}_0"'] & \mathcal{A}_0\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}_1 \arrow[d, "f\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}_1"] \arrow[dl, phantom, "\Sigma_{f,H}\Downarrow"] \\
\mathcal{A}_1\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}_0 \arrow[r, "\mathcal{A}_1\text{-}\mathbf{Mod}_s\text{-}H"'] & \mathcal{A}_1\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}_1
\end{tikzcd} \tag{1}$$

For each M *in* $\mathcal{A}_0\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}_0$, *we define the isomorphism*

$$\Sigma_{f,H}(M)\colon f^*(MH) \cong (f^*M)H$$

by pasting the following diagram.

$$\begin{tikzcd}
\mathcal{A}_1 & \mathcal{A}_0 \arrow[l, "f"] & \mathbf{AU}\langle\mathbb{T}_0\rangle \arrow[l, "M"'] \arrow[ll, bend left, "f^*M", "\cong"'] & \mathbf{AU}\langle\mathbb{T}_1\rangle \arrow[l, "\mathbf{AU}\langle H\rangle"] \arrow[lll, bend right, "f^*(MH)"', "\cong"]
\end{tikzcd}$$

Naturality is clear.

Theorem 3.6. *The two actions on* $\bullet\text{-}\mathbf{Mod}_s\text{-}\bullet$ *by* $\mathbf{AU}$ *and* $\mathfrak{Con}$, *together with the pseudo-naturality isomorphisms* $\Sigma_{f,H}$, *make up a "cubical functor" from* $\mathbf{AU} \times \mathfrak{Con}$ *to* $\mathfrak{CAT}$ *in the sense of [5], and hence a 2-functor from the Gray tensor product* $\mathbf{AU} \otimes \mathfrak{Con}$ *to* $\mathfrak{CAT}$.

Proof. There are three conditions to be checked. The first two are that the squares (1) paste together correctly, either horizontally or vertically, for composition of 1-cells in either $\mathfrak{Con}$ or $\mathbf{AU}$. The third is that it pastes correctly with 2-cells in $\mathfrak{Con}$ and $\mathbf{AU}$. All are clear by pasting the appropriate isomorphisms from the definition of f^*. □

Lemma 3.7.

1. *If U is an extension map (for $\mathbb{T}_0 \subset \mathbb{T}_1$) then $(f^*M)U = f^*(MU)$ for every f and M, and $\Sigma_{f,U}(M)$ is the identity morphism.*
2. *If U is an equivalence extension map ($\mathbb{T}_0 \Subset \mathbb{T}_1$), then $(f^*M)U^{-1} = f^*(MU^{-1})$, and $\Sigma_{f,U^{-1}}(M)$ is the identity morphism.*

Proof. (1) $f^*(MU)$ is the canonical strict isomorph of $f \cdot (MU)$.

On the other hand $(f^*M)U \cong (f \cdot M)U = f \cdot (MU)$ and they are equal on all the primitive nodes of $\mathbb{T}_0$ because they are also primitive in the extension $\mathbb{T}_1$.

(2) Apply part (1) to MU^{-1}. □

Example 3.8. Equality in Lemma 3.7 can fail for a map $H\colon \mathbb{T}_1 \to \mathbb{T}_0$ involving a context homomorphism that maps primitive nodes to non-primitives. Consider the context $\mathbb{T}$ with a single node T, declared terminal, and $H\colon \mathbb{T} \to \mathbb{O}$ given by the sketch homomorphism that takes the single node X in $\mathbb{O}$ to T.

If M is the unique strict model of $\mathbb{T}$ in $\mathcal{A}$, then MH simply picks out the canonical terminal object, and $(f^*M)H$ does the same in $\mathcal{A}'$. $f^*(MH)$ picks out the image under f of the canonical terminal in $\mathcal{A}$.

Finally, we can translate these results to elementary toposes. For each AU-context $\mathbb{T}$ we have a 2-category $\bullet$-$\mathbf{Mod}_s$-$\mathbb{T}$, strictly indexed over $\mathfrak{eTop}$, and it restricts to $\mathfrak{BTop}/\mathcal{S}$, with the geometric morphisms $p\colon \mathcal{E} \to \mathcal{S}$ playing no role in the reindexing. Thus it gives a strict elephant theory over $\mathcal{S}$ for $\mathbb{T}$. Also, each context map $H\colon \mathbb{T}_0 \to \mathbb{T}_1$ gives a corresponding indexed functor from $\mathbb{T}_0$ to $\mathbb{T}_1$ as elephant theories.

4. Remarks on 2-fibrations

In the 2-functor $\bullet\text{-}\mathbf{Mod}_s\text{-}\mathbb{T} \colon \mathbf{AU} \to \mathfrak{CAT}$ we have already seen a category strictly indexed over the 2-category $\mathbf{AU}^{op}$. As we proceed, however, we shall encounter non-strict indexations, with pseudofunctors, and for these we shall prefer a fibrational approach. Thus we avoid confronting coherence conditions for indexed 2-categories.

For the appropriate notion of 2-fibration we shall follow Buckley's account [4], which in turn was based on earlier work of Hermida [6] and Baković [1]. Definitions are given for fibrations both between 2-categories and between bicategories. Note that, although we deal only with 2-categories, and 2-functors between them, we shall still need to use the bicategorical notion of fibration once we go beyond strictly indexed categories. The essential difference, for a 2-functor $P \colon \mathcal{E} \to \mathcal{B}$, is that the properties characterizing a cartesian 1-cell $f \colon x \to y$ in $\mathcal{E}$ are weaker. Given $g \colon z \to y$ and $h \colon Pz \to Px$ with $h(Pf) = Pg$, we can lift h to $\hat{h} \colon z \to x$ but the corresponding triangle in $\mathcal{E}$ commutes only up to isomorphism.

$$\begin{tikzcd}
& z \arrow[dl, dotted, "\hat{h}"'] \arrow[dr, "g"] \arrow[d, phantom, "\cong"] & \\
x \arrow[rr, "f"'] & {} & y
\end{tikzcd}
\qquad
\begin{tikzcd}
& Pz \arrow[dl, "h"'] \arrow[dr, "Pg"] \arrow[d, phantom, "="] & \\
Px \arrow[rr, "Pf"'] & {} & Py
\end{tikzcd}$$

To summarize Buckley's definitions, –

- A 1-cell f in $\mathcal{E}$ is cartesian if it lifts 1-cells up to isomorphism, and lifts 2-cells coherently with the lifted isos. The uniqueness of lifted 2-cells implies that lifted 1-cells are unique up to a coherent isomorphism.

- A 2-cell $\alpha \colon f \Rightarrow g \colon x \to y$ in $\mathcal{E}$ is cartesian if it is cartesian as a 1-cell for the functor $P_{xy} \colon \mathcal{E}(x, y) \to \mathcal{B}(Px, Py)$.

- P is a fibration if for every $f \colon b \to Pe$ in $\mathcal{B}$, there is a cartesian $h \colon a \to e$ with $Ph = f$; each P_{xy} is a fibration of categories; and the cartesian 2-cells are closed under whiskering on both sides.

4.1 The fibred 2-category of Grothendieck toposes

By “Grothendieck topos”, we mean a *bounded* geometric morphism from some elementary topos $\mathcal{E}$ to some, understood, base elementary topos $\mathcal{S}$.[2] The 2-category of Grothendieck toposes over $\mathcal{S}$ is studied in [7, B4] as $\mathfrak{BTop}/\mathcal{S}$.

A notable property of $\mathfrak{BTop}/\mathcal{S}$ is that any geometric theory $\mathbb{T}$ (geometric, that is, with respect to $\mathcal{S}$) has a classifying topos $\mathcal{S}[\mathbb{T}]$ that behaves in many respect as “the space of models of $\mathbb{T}$”; indeed, the whole of $\mathfrak{BTop}/\mathcal{S}$ may then be viewed as the 2-category of generalized spaces relative to $\mathcal{S}$: 0-cells are spaces, 1-cells are (continuous) maps, and 2-cells are generalized specializations (morphisms, not order).

Our interest in using arithmetic universes is to deal with theories $\mathbb{T}$ that depend on the base $\mathcal{S}$ only to the extent that nnos are required to exist. Our aim here will be to prove results about Grothendieck toposes that are fibred over choice of base.

From the point of view of indexed categories, the key result [7, B3.3.6] is that bounded geometric morphisms can be pseudo-pulled-back along arbitrary geometric morphisms.[3] Thus for any geometric morphism $f\colon \mathcal{S}_0 \to \mathcal{S}_1$ we get a reindexing $f^*\colon \mathfrak{BTop}/\mathcal{S}_1 \to \mathfrak{BTop}/\mathcal{S}_0$. This does not extend to arbitrary natural transformations $\alpha\colon f \to g$ unless the Grothendieck toposes are restricted to fibrations or opfibrations over $\mathcal{S}$, so instead we restrict the αs at the base level to be isomorphisms.

We write $\mathfrak{eTop}_{\cong}$ for the 2-category of elementary toposes (with nno), geometric morphisms and natural isomorphisms.

We now express $\mathcal{S} \mapsto \mathfrak{BTop}/\mathcal{S}$ as a fibred 2-category $\mathfrak{GTop}$ of Grothendieck toposes.

Definition 4.1. *The data for the 2-category $\mathfrak{GTop}$ is defined as follows.*

A 0-cell is a bounded geometric morphism $p\colon \mathcal{E} \to \mathcal{S}$.

---

[2] As always for us, our elementary toposes are assumed to have nnos.

[3] Beware that, in 2-categorical contexts, [7] consistently omits “pseudo-” – see B1.1.

A 1-cell $f = (\overline{f}, f\Downarrow, \underline{f})$ *from* $\mathcal{E}_0 \xrightarrow{p_0} \mathcal{S}_0$ *to* $\mathcal{E}_1 \xrightarrow{p_1} \mathcal{S}_1$ *is a square*

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r, "\overline{f}"] \arrow[d, "p_0"'] \arrow[dr, phantom, "f\Downarrow"] & \mathcal{E}_1 \arrow[d, "p_1"] \\
\mathcal{S}_0 \arrow[r, "\underline{f}"'] & \mathcal{S}_1
\end{tikzcd}$$

in which $f\Downarrow\colon \overline{f}p_1 \to p_0\underline{f}$ *is an isomorphism.*

Given two such 1-cells, f *and* f' *from* p_0 *to* p_1*, a 2-cell* $\alpha\colon f \to f'$ *is a pair of natural transformations* $\overline{\alpha}\colon \overline{f} \to \overline{f}'$ *and* $\underline{\alpha}\colon \underline{f} \to \underline{f}'$

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\mathcal{E}_0 \arrow[r, bend left, "\overline{f}"] \arrow[r, bend right, "\overline{f}'"'] \arrow[r, phantom, "\Downarrow\overline{\alpha}"] \arrow[d, "p_0"'] \arrow[dr, phantom, "f\Downarrow \quad f'\Downarrow"] & \mathcal{E}_1 \arrow[d, "p_1"] \\
\mathcal{S}_0 \arrow[r, bend left, "\underline{f}"] \arrow[r, bend right, "\underline{f}'"'] \arrow[r, phantom, "\Downarrow\underline{\alpha}"] & \mathcal{S}_1
\end{tikzcd}$$

such that the obvious diagram of 2-cells commutes. Moreover, as mentioned earlier, we require $\underline{\alpha}$ *to be an isomorphism.*

It is clear that $\mathfrak{GTop}$ is a 2-category

Proposition 4.2. *There is a 2-functor* $\mathfrak{GTop}^{co} \to \mathfrak{cTop}^{co}_{\cong}$ *that forgets all but the downstairs part. Although it is strict, we consider it as a homomorphsm of* bicategories *for the purposes of [4, 3.1].*

1. *A 1-cell is cartesian iff it is a pseudopullback square in* $\mathfrak{cTop}$.
2. *A 2-cell* α *is cartesian iff* $\overline{\alpha}$ *is an isomorphism.*
3. *The 2-functor is a fibration of bicategories.*

Proof. (1): This is essentially the same as the proof of the result for 1-categories, that for the codomain fibration $\mathrm{cod}\colon \mathcal{C}^{\to} \to \mathcal{C}$, a morphism for $\mathcal{C}^{\to}$ is cartesian iff it is a pullback square in $\mathcal{C}$. The conditions for pseudopullbacks and cartesian 1-cells both bring in the 2-cells in the same way. For the "$\Rightarrow$" direction, note that an arbitrary elementary topos $\mathcal{E}$ can be treated as a 0-cell in $\mathfrak{GTop}$ using the identity geometric morphism.

(2): If $\overline{\alpha}$ is an isomorphism then so is the 2-cell α, and it is then clearly cartesian. For the converse, suppose $\alpha\colon f \to g$ is a cartesian 2-cell. (Note that because we are going to dualize, α is really cocartesian in $\mathfrak{GTop}$.) Downstairs, $\underline{\alpha}$ is invertible and so by lifting $\underline{\alpha}^{-1}$ we get $\alpha'\colon g \to f$, with $\alpha\alpha' = \mathsf{Id}_f$. By considering Id_g and $\alpha'\alpha$ as lifts of $\mathsf{Id}_{\underline{g}}$ we see that they are equal.

(3) Cartesian lifting of 1-cells arises because, in $\mathfrak{eTop}$, pseudopullbacks of bounded geometric morphisms along arbitrary geometric morphisms always exist [7, B3.3.6].

Cartesian lifting of 2-cells is easy – in fact we can ensure that the upstairs part of the lifted 2-cell is an identity. $\square$

Of course, $\mathfrak{eTop}_{\cong}^{co} \cong \mathfrak{eTop}_{\cong}$, so we could equally well consider $\mathfrak{GTop}^{co}$ as fibred over $\mathfrak{eTop}_{\cong}$.

4.2 Representability

In Definition 2.6, "classifying topos" is *defined* in terms of representability of an indexed category, a pseudofunctor $\mathbb{T}\colon (\mathfrak{BTop}/\mathcal{S})^{op} \to \mathfrak{CAT}$. We now look at how this appears in terms of fibrations.

To work abstractly, suppose $\mathcal{C}$ is a 2-category, and $F\colon \mathcal{C}^{coop} \to \mathfrak{CAT}$ a pseudofunctor. We shall describe the Grothendieck construction for it. In our applications, for elephant theories deriving from AU-contexts, F will be strict and the Grothendieck construction is described in [4, 2.2] as a fibration of 2-categories. For the present section, however, we shall not assume strictness: thus we retain the connection with general elephant theories. Because of this we need to use [4, 3.3.3], which describes the Grothendieck construction as a fibration of bicategories. Nonetheless, our situation is somewhat simpler than Buckley's. We have not allowed $\mathcal{C}$ to be a bicategory, and we have taken each $F(X)$ to be a category, not a bicategory. Because of this, our fibred bicategory $\mathcal{E}$ is actually a 2-category, though not fibred as such. It has –

0-cells are pairs (x, x_-) of objects of $\mathcal{C}$ and Fx.

1-cells are pairs $(f, f_-)\colon (x, x_-) \to (y, y_-)$ where

$$f\colon x \to y \text{ and } f_-\colon x_- \to Ff(y_-).$$

2-cells $(f, f_-) \to (g, g_-)\colon (x, x_-) \to (y, y_-)$ are 2-cells $\alpha\colon f \to g$ such that the following diagram commutes.

$$\begin{tikzcd}
x_- \arrow[rr, "f_-"] \arrow[dr, "g_-"'] & \arrow[d, phantom, "="] & Ff(y_-) \\
& Fg(y_-) \arrow[ur, "F\alpha_{y_-}"'] &
\end{tikzcd}$$

Then the 1-cell (f, f_-) is cartesian iff f_- is an isomorphism. Every 2-cell α is cartesian.

In the following proposition we characterize representability of the pseudofunctor F in a purely fibrational way, independent of F as choice of cleavage.

Proposition 4.3. *Let* $F\colon \mathcal{C}^{coop} \to \mathfrak{CAT}$ *be a pseudofunctor as above, and let* $P\colon \mathcal{E} \to \mathcal{C}$ *be its Grothendieck construction. Then* F *is representable iff there is an object* (x, x_-) *in* $\mathcal{E}$ *(a* representing object*) with the following properties.*

1. *For each* (y, y_-) *in* $\mathcal{E}$*, there is a cartesian 1-cell* $(f, f_-)\colon (y, y_-) \to (x, x_-)$.
2. *Each cartesian 1-cell* $(f, f_-)\colon (y, y_-) \to (x, x_-)$ *is terminal in the category* $\mathcal{E}((y, y_-), (x, x_-))$.

Proof. By definition, F is represented by (x, x_-) iff for every y the functor $K_y\colon \mathcal{C}(y, x)^{op} \to Fy$, given by $f \mapsto Ff(x_-)$, is an equivalence.

Condition (1) says that each K_y is essentially surjective. It remains to show that, for each y, K_y is full and faithful iff condition (2) holds.

Suppose K_y is full and faithful and, for a given y_-, we have

$$(f, f_-), (g, g_-)\colon (y, y_-) \to (x, x_-)$$

with (f, f_-) cartesian, i.e. f_- an isomorphism. Then there is a unique $\alpha\colon g \to f$ such that $F\alpha_{x_-} = f_-^{-1}; g_-$, in other words a unique 2-cell from (g, g_-) to (f, f_-).

Conversely, suppose condition (2) holds for a given y, and suppose we have $f, g\colon y \to x$ and $g_-\colon Ff(x_-) \to Fg(x_-)$. We then have two 1-cells

$$(f, \mathsf{Id}), (g, g_-)\colon (y, Ff(x_-)) \to (x, x_-).$$

Since (f, Id) is cartesian we get a unique 2-cell $\alpha\colon (g, g_-) \to (f, \mathsf{Id})$, in other words, a unique $\alpha\colon g \to f$ such that $K_y(\alpha) = g_-$. $\square$

By the usual means, one can show that if x is a representing object for P, then for any object x' in $\mathcal{E}$ we have that x' is a representing object iff it is equivalent to x.

We now extend the above discussion to a situation where $\mathcal{C}$ too is fibred: we have fibrations

$$\begin{tikzcd} \mathcal{E} \arrow[r, "P"] & \mathcal{C} \arrow[r, "Q"] & \mathcal{B}\,. \end{tikzcd}$$

In our applications, P will again be got from a pseudofunctor (in fact a 2-functor) $\mathcal{C}^{coop} \to \mathfrak{CAT}$, but Q will be more general. The paradigm example for Q is $\mathfrak{GTop}^{co}$ fibred over $\mathfrak{eTop}^{co}_{\cong}$.

We also assume (as in the paradigm) that all 2-cells in $\mathcal{B}$ are isomorphisms.

Note that $f\colon x \to y$ in $\mathcal{E}$ is cartesian for $P;Q$ iff it is cartesian for P and Pf is cartesian for Q. For the "$\Leftarrow$" direction, we just lift in two stages. For "$\Rightarrow$", consider cartesian lifts $\hat{f}\colon \hat{x} \to Py$ of $Q(Pf)$, and then $\hat{\hat{f}}\colon \hat{\hat{x}} \to y$ of $\hat{f}$. We get an equivalence $x \simeq \hat{\hat{x}}$ and deduce the result from that.

Now each object w of $\mathcal{B}$ has a fibre over it, a fibration $P_w\colon \mathcal{E}_w \to \mathcal{C}_w$: it comprises the 0-cells of $\mathcal{C}$ and $\mathcal{E}$ that map to w, and the 1- and 2-cells that map to identities at w. We are now interested in the situation where each P_w is representable, and in how the representing objects transform under 1-cells in $\mathcal{B}$.

Since we are assuming P arises from a pseudofunctor, it is easy to see that a 1-cell or 2-cell in $\mathcal{E}_w$ is cartesian for P_w iff it is cartesian for P.

Definition 4.4. *P is* locally representable *(over Q) iff*

1. *Each fibre P_w is representable.*
2. (Geometricity) *Suppose P_w is represented by x_w, $f\colon w' \to w$ in $\mathcal{B}$, and $h\colon y \to x_w$ is $P;Q$-cartesian over f. Then y is a representing object for $P_{w'}$.*

We call condition (2) "geometricity" in line with [11], because it concerns a property that is preserved by pseudopullback in $\mathfrak{eTop}$. Note that it suffices to verify it for *some* x_w and *some* h. This is because representing objects are equivalent, and so too are cartesian liftings.

As defined, local representability focuses on the fibres P_w. We can express the property in a way that says more about the interaction with change of base.

Proposition 4.5. *P is locally representable over Q iff, for each object w of $\mathcal{B}$, we have an object x_w of $\mathcal{E}$ over it that satisfies the following conditions.*

1. *For every object y of $\mathcal{E}$, and 1-cell $f\colon Q(Py) \to w$ in $\mathcal{B}$, there is some $\hat{f}\colon y \to x_w$ over f that is cartesian with respect to P.*

2. *Suppose $h_0, h_1\colon y \to x_w$ in $\mathcal{E}$, with h_1 being P-cartesian.*

   If $\alpha\colon Q(Ph_0) \to Q(Ph_1)$, then there is a unique $\hat{\alpha}\colon h_0 \to h_1$ over α.

Proof. $\Leftarrow$: Clearly any x_w satisfying the conditions must be a representing object for P_w. It remains to show that the representing objects transform correctly under base 1-cells $f\colon w' \to w$.

Suppose x_w and $x_{w'}$ satisfy the conditions. By the conditions for x_w we have P-cartesian $g\colon x_{w'} \to x_w$ over f. Suppose also that $h\colon y \to x_w$ is $P;Q$-cartesian over f. By the conditions on $x_{w'}$ we get P-cartesian $u\colon y \to x_{w'}$ over $\mathsf{Id}_{w'}$, and by cartesianness of h we get $v\colon x_{w'} \to y$ over $\mathsf{Id}_{w'}$ with an isomorphism $\alpha\colon vh \to g$ over Id_f.

$$\begin{tikzcd}
x_{w'} \arrow[d, "v"] \arrow[dr, "g"] & \\
y \arrow[u, bend left, "u"] \arrow[r, "h"'] \arrow[ur, phantom, "\alpha\Uparrow"] & x_w
\end{tikzcd}$$

Since both g and h are P-cartesian, so is v. It follows by the conditions on $x_{w'}$ that there is a unique isomorphism $vu \cong \mathsf{Id}_{x_{w'}}$ in $P_{w'}$. Also, by the $P;Q$-cartesian property of h, there is a unique isomorphism $uv \cong \mathsf{Id}_y$ in $P_{w'}$. Hence y is equivalent to $x_{w'}$, and so represents $P_{w'}$ as required.

$\Rightarrow$: Let x_w be a representing object for P_w. We show it has the two properties stated.

Suppose y is an object in $\mathcal{E}$, and $f\colon w' = Q(Py) \to w$ a 1-cell in $\mathcal{B}$. Let $g\colon x_{w'} \to x_w$ be $P;Q$-cartesian over f, so that $x_{w'}$ is a representing object for $P_{w'}$. Then there is a P-cartesian 1-cell $u\colon y \to x_{w'}$ in $P_{w'}$, and $ug\colon y \to x_w$ is P-cartesian (because u and g are) over f.

Now suppose $h_0, h_1\colon y \to x_w$ are two 1-cells, with h_1 cartesian for P, and with $f_i = Q(Ph_i)\colon w' \to w$, and $\alpha\colon f_0 \to f_1$. Recall our assumption that all 2-cells in $\mathcal{B}$ are isomorphisms. Let $g_i\colon z_i \to x_w$ be a $P;Q$-cartesian lifting of f_i, with $u_i\colon y \to z_i$ in $P_{w'}$ and $\beta_i\colon u_i g_i \cong h_i$ over f_i. By [4, 3.1.15], there is an equivalence $k\colon z_0 \simeq z_1$ with isomorphism $kg_1 \cong g_0$ over α, and the pair is unique up to unique isomorphism between ks in $P_{w'}$. Thus 2-cells $h_0 \to h_1$ over α are in bijection with 2-cells $u_0 k g_1 \to u_1 g_1$ over f_1, and hence (because g_1 is $P;Q$-cartesian) with 2-cells $u_0 k \to u_1$ in $P_{w'}$. Since z_1 is a representing object for $P_{w'}$, and u_1 is P-cartesian (because h_1 and g_1 are), and hence cartesian in $P_{w'}$, we get a unique 2-cell $u_0 k \to u_1$ in $P_{w'}$. $\square$

5. Context extensions as bundles

In this Section we gather together the previous remarks to get results on classifying toposes in a form that is fibred over a category of bases.

This is most easily understood in the simple case of a single context $\mathbb{T}$. For each Grothendieck topos $p\colon \mathcal{E} \to \mathcal{S}$ we have a category $\mathcal{E}$-$\mathbf{Mod}_s$-$\mathbb{T}$ of strict models of $\mathbb{T}$ in $\mathcal{E}$. This extends to a 2-functor from $\mathfrak{GTop}^{op} = (\mathfrak{GTop}^{co})^{coop}$ to $\mathfrak{CAT}$, and its Grothendieck construction can be written as $P\colon (\mathfrak{GTop}\text{-}\mathbb{T})^{co} \to \mathfrak{GTop}^{co}$.

In constructing that fibration we ignored the parts $\xrightarrow{p} \mathcal{S}$, but when we bring in $\mathcal{S}$ we find that the classifying topos $\mathcal{S}[\mathbb{T}]$ provides a representing object for $P_{\mathcal{S}}$.

The main novelty here is that those representing objects transform according to Definition 4.4: that (Theorem 5.11) the pseudopullback along any $\underline{f}\colon \mathcal{S}_0 \to \mathcal{S}_1$ preserves classifiers. Our proof is non-trivial, and shows that the steps constructing the classifier are preserved under pseudopullback.

As mentioned in Section 1.2, we shall prove local representability more generally, dealing not just with a single context $\mathbb{T}$, but in the relativized situation for an extension $\mathbb{T}_0 \subset \mathbb{T}_1$.

Why extensions, and not arbitrary $H\colon \mathbb{T}_1 \to \mathbb{T}_0$? The main reason is the repeated use of Proposition 2.1, sometimes via Lemma 3.7.

5.1 Models for a context extension

Definition 5.1. *Let $\mathbb{T}_0 \subset \mathbb{T}_1$ be an extension of contexts, with corresponding extension map $U\colon \mathbb{T}_1 \to \mathbb{T}_0$, and let $p\colon \mathcal{E} \to \mathcal{S}$ be a bounded geometric morphism. A* strict model of U in p *is a pair (M, N) where M is a strict model of $\mathbb{T}_0$ in $\mathcal{S}$, N a strict model of $\mathbb{T}_1$ in $\mathcal{E}$, and $NU = p^*M$.*

A morphism from one such strict model, (M, N), to another, (M', N'), is a pair $\phi = (\phi_-, \phi^-)$ where $\phi_-\colon M \to M'$ and $\phi^-\colon N \to N'$ are homomorphisms and $\phi_- U = p^*\phi^-$.

For given U we thus get, for each p, a category p-$\mathbf{Mod}_s$-U. It is strictly indexed over $\mathfrak{GTop}$ in the following way.

First suppose $f\colon p_0 \to p_1$ is a 1-cell in $\mathfrak{GTop}$, as in Definition 4.1. If (M, N) is a strict model in p_1, then we define a strict model $f^*(M, N) = (\underline{f}^*M, f^*N)$

$$\begin{tikzcd}
f^*N \arrow[d, mapsto] & \overline{f}^*N \arrow[l, "\cong"'] \arrow[d, mapsto] \\
p_0^*\underline{f}^*M & \overline{f}^*p_1^*M \arrow[l, "(f\Downarrow)^*M"]
\end{tikzcd}$$

where the upstairs isomorphism is the canonical one obtained from Proposition 2.1. The action extends to morphisms between strict models of U, and we obtain a functor f-$\mathbf{Mod}_s$-$U\colon p_1$-$\mathbf{Mod}_s$-$U \to p_0$-$\mathbf{Mod}_s$-U.

If $\alpha\colon f \to f'$ is a 2-cell in $\mathfrak{GTop}$, then it gives a natural transformation from f-$\mathbf{Mod}_s$-U to f'-$\mathbf{Mod}_s$-U. We obtain a strict 2-functor from $\mathfrak{GTop}^{op}$ to $\mathfrak{CAT}$. Its Grothendieck construction is a fibration $(\mathbf{Mod}_s\text{-}U)^{co} \to \mathfrak{GTop}^{co}$.

Definition 5.2. *The data for the 2-category $\mathbf{Mod}_s$-U is defined as follows. In each case, a 0-, 1- or 2-cell is the corresponding item for $\mathfrak{GTop}$, equipped with extra structure in the form of models of U.*

A 0-cell is a bounded geometric morphism $p\colon \mathcal{E} \to \mathcal{S}$, equipped with a strict model (M, N) of U.

A 1-cell from (p_0, M_0, N_0) to (p_1, M_1, N_1) is a 1-cell $f\colon p_0 \to p_1$ from $\mathfrak{GTop}$, equipped with a homomorphism $(f_-, f^-)\colon (M_0, N_0) \to f^*(M_1, N_1)$. (Note that the letter f is highly decorated: we have $\underline{f}$, $f\Downarrow$, $\overline{f}$, f_- and f^-.)

Given 1-cells (f, f_-, f^-) and (f', f'_-, f'^-), with the same domain and codomain, a 2-cell from one to the other is a 2-cell $\alpha\colon f \to f'$ in $\mathfrak{GTop}$ such *that*

$$(f_-, f^-)(\alpha^*(M_1, N_1)) = (f'_-, f'^-).$$

It is clear that $\mathbf{Mod}_s$-U is a 2-category, with a functor $F' \colon \mathbf{Mod}_s\text{-}U \to \mathfrak{GTop}$ that forgets the model, and by construction F'^{co} is a split fibration. Note that –

1. A 1-cell (f, f_-, f^-) is cartesian iff f_- and f^- are isomorphisms.

2. Every 2-cell α is (co-)cartesian.

Note the special case of a trivial extension $\mathbb{T}_0 = \mathbb{T}_0$. A model of this in p is simply a model M of $\mathbb{T}_0$ in $\mathcal{S}$, since the corresponding model in $\mathcal{E}$ has to be p^*M. In this case we write $\mathbf{Mod}_s$-$(\mathbb{T}_0 \subset \mathbb{T}_0)$.

We have an obvious forgetful functor from $\mathbf{Mod}_s$-U to $\mathbf{Mod}_s$-$(\mathbb{T}_0 \subset \mathbb{T}_0)$, which (or its co-dual) is almost, but not quite, a fibration. The problem is that $\mathbb{T}_0$-homomorphisms $\phi_- \colon M \to M'$ do not lift to functors for the categories of U-models over them. To rectify this, we restrict to isomorphisms downstairs.

Definition 5.3. *$\mathfrak{GTop}$-U is the sub-2-category of $\mathbf{Mod}_s$-U with all the 0-cells, but with only the 1-cells (f, f_-, f^-) for which f_- is an isomorphism. It is full on 2-cells.*

Proposition 5.4. *We write $P^{co} \colon \mathfrak{GTop}\text{-}U \to \mathfrak{GTop}\text{-}(\mathbb{T}_0 \subset \mathbb{T}_0)$ for the forgetful functor. Then $P \colon (\mathfrak{GTop}\text{-}U)^{co} \to (\mathfrak{GTop}\text{-}(\mathbb{T}_0 \subset \mathbb{T}_0))^{co}$ is a split fibration. A 1-cell (f, f_-, f^-) is cartesian iff its f^- is an isomorphism. Every 2-cell is cartesian.*

Proof. It is the Grothendieck construction for the evident 2-functor from $(\mathfrak{GTop}\text{-}(\mathbb{T}_0 \subset \mathbb{T}_0))^{op}$ to $\mathfrak{CAT}$. ☐

We now fibre over pairs $(\mathcal{S}, M)$.

Definition 5.5. *The 2-category $\mathfrak{eTop}_{\cong}$-$\mathbb{T}$ has structure as follows. A* 0-cell *is a pair $(\mathcal{S}, M)$ where $\mathcal{S}$ is an elementary topos and M a model of $\mathbb{T}$ in $\mathcal{S}$. A* 1-cell *from $(\mathcal{S}_0, M_0)$ to $(\mathcal{S}_1, M_1)$ is a pair $(\underline{f}, f_-)$ where $\underline{f} \colon \mathcal{S}_0 \to \mathcal{S}_1$ is a geometric morphism and $f_- \colon M_0 \to \underline{f}^* M_1$ is an isomorphism. A* 2-cell *from $(\underline{f}, f_-)$ to $(\underline{g}, g_-)$ is a natural isomorphism $\underline{\alpha} \colon \underline{f} \to \underline{g}$ such that $f_-; \underline{\alpha}^* M_1 = g_-$.*

The 2-category $\mathfrak{GTop}\text{-}(\mathbb{T} \subset \mathbb{T})$ *is made from* $\mathfrak{GTop}$ *by adding components* M *and* f_-*, and the condition on* $\underline{\alpha}$*, in the same way as* $\mathfrak{eTop}_{\cong}\text{-}\mathbb{T}$ *is made from* $\mathfrak{eTop}_{\cong}$*.*

Proposition 5.6. *Let* $Q^{co}\colon \mathfrak{GTop}\text{-}(\mathbb{T} \subset \mathbb{T}) \to \mathfrak{eTop}_{\cong}\text{-}\mathbb{T}$ *be the evident forgetful functor. Then* $Q = (Q^{co})^{co}$ *is a fibration of bicategories.*

Proof. Much as in Proposition 4.2. □

We now get a diagram of 2-functors as follows, where the Ps and Qs are fibrations. The left hand tower is for the relativized situation $\mathbb{T}_0 \subset \mathbb{T}_1$, while the right hand tower is the special case $\mathbb{T}_0 = \mathbb{1}$.

$$\begin{tikzcd}
(\mathfrak{GTop}\text{-}U)^{co} \arrow[d, "P"'] \arrow[dr] & \\
(\mathfrak{GTop}\text{-}(\mathbb{T}_0 \subset \mathbb{T}_0))^{co} \arrow[d, "Q"'] \arrow[dr] & (\mathfrak{GTop}\text{-}\mathbb{T}_1)^{co} \arrow[d, "P"] \\
(\mathfrak{eTop}_{\cong}\text{-}\mathbb{T}_0)^{co} \arrow[dr] & \mathfrak{GTop}^{co} \arrow[d, "Q"] \\
 & \mathfrak{eTop}_{\cong}^{co}
\end{tikzcd} \tag{2}$$

5.2 Context extensions fibred over models

Our aim now is to show that, in diagram (2), each P is locally representable over its Q. (Note that the right hand one is a special case of the left hand, for when $\mathbb{T}_0 = \mathbb{1}$.) The existence of the representing objects (as classifying toposes) is straightforward; what seems more novel is their preservation by pseudopullback.

Proposition 5.7. *Let* $\mathbb{T}_0 \subset \mathbb{T}_1$ *be a context extension. Then, over any elementary topos* $\mathcal{S}$*, it is also a geometric extension of elephant theories.*

Proof. It suffices to check the different kinds of simple context extension. Note that any node X in $\mathbb{T}_0$ gives a context homomorphism $\mathbb{O} \lessdot \mathbb{T}_0$, so a map $\mathbb{T}_0 \to \mathbb{O}$, and hence a geometric construct on $\mathbb{T}_0$. Likewise, any edge or composite of edges gives a map $\mathbb{T}_0 \to \mathbb{O}^{\to}$, and hence a morphism between geometric constructs.

An extension by primitive node is a geometric extension by primitive sort.

A simple functional extension of contexts (adjoining a primitive edge) is also a simple functional extension of geometric theories.

An extension by a universal is essentially no geometric extension at all, as the categories of (strict) models are isomorphic.

An extension by commutativities is a simple geometric quotient, as imposing an equality between morphisms is equivalent to requiring the equalizer to be an isomorphism. □

Proposition 5.8. *Let* $\mathbb{T}_0$ *be a context, and* M *a strict model of* $\mathbb{T}_0$ *in an elementary topos* $\mathcal{S}$*. Then there is an elephant morphism* $M\colon \mathbb{1} \to \mathbb{T}_0$ *that, on bounded* $\mathcal{S}$*-topos* $(\mathcal{E}, p)$*, takes* $*$ *to* p^*M*.*

Proof. Note that, although the elephant theories for both $\mathbb{1}$ and $\mathbb{T}_0$ are strictly indexed, M is not a strict morphism. Consider a morphism of $\mathcal{S}$-toposes

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "\overline{f}"] \arrow[dr, "q"'] & \arrow[d, phantom, "f\Downarrow"] & \mathcal{E}, \arrow[dl, "p"] & \mathbb{1}(\mathcal{F}) \arrow[r, equal] \arrow[d, "M(\mathcal{F})"'] \arrow[dr, phantom, "\cong"] & \mathbb{1}(\mathcal{E}) \arrow[d, "M(\mathcal{E})"] \\
& \mathcal{S} & & \mathbb{T}_0(\mathcal{F}) & \mathbb{T}_0(\mathcal{E}) \arrow[l, "\mathbb{T}_0(\overline{f})"]
\end{tikzcd}$$

On the right is a pseudo-naturality square, subject to the isomorphism

$$(f\Downarrow)^*M\colon \overline{f}^*p^*M \cong q^*M.$$

□

Of course, $M\colon \mathbb{1} \to \mathbb{T}_0$ is not a map of contexts in general.

Definition 5.9. *Let* $\mathbb{T}_0 \subset \mathbb{T}_1$ *be a context extension and* M *a strict model of* $\mathbb{T}_0$ *in an elementary topos* $\mathcal{S}$*. By Proposition 2.5 we can pull back the geometric extension for* $\mathbb{T}_0 \subset \mathbb{T}_1$ *along* $M\colon \mathbb{1} \to \mathbb{T}_0$*, getting a geometric theory* $\mathbb{T}_1/M$ *over* $\mathcal{S}$*. Its models in* $(\mathcal{E}, p)$ *are the strict models of* $\mathbb{T}_1$ *whose* $\mathbb{T}_0$*-reducts are equal to* p^*M*. It has a classifying topos* $\mathcal{S}[\mathbb{T}_1/M]$*.*

Proposition 5.10. *Let* $\mathbb{T}_0 \subset \mathbb{T}_1 \subset \mathbb{T}_2$ *be a sequence of context extensions, with extension maps* $\mathbb{T}_2 \xrightarrow{U'} \mathbb{T}_1 \xrightarrow{U} \mathbb{T}_0$ *.*

Let M *be a strict model of* $\mathbb{T}_0$ *in an elementary topos* $\mathcal{S}$*, and consider the classifying toposes* $p\colon \mathcal{S}' = \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ *with generic model* N_G*, and* $p'\colon \mathcal{S}'' = \mathcal{S}'[\mathbb{T}_2/N_G] \to \mathcal{S}'$ *with generic model* N'_G*.*

Then $(\mathcal{S}'', p'p)$ *serves as classifier for* $\mathbb{T}_2/M$*, with generic model* N'_G*.*

Proof. Note that, using Lemma 3.7, $N'_G U'U = p'^* N_G U = (p'p)^* M$.

For the "essential surjectivity" part, suppose N is a model of $\mathbb{T}_2/M$ in $(\mathcal{F}, q)$. Then NU' is a model of $\mathbb{T}_1/M$, so we get $g = (\overline{g}, g\Downarrow)\colon (\mathcal{F}, q) \to (\mathcal{S}', p)$ with $NU' \cong g^* N_G$ as models of $\mathbb{T}_1/M$; also $g^* N_G \cong \overline{g}^* N_G$ as models of $\mathbb{T}_1$. Now using Proposition 2.1 we can find a strict model $N' \cong N$ of $\mathbb{T}_2$ with $N'U' = \overline{g}^* N_G$, so N' is a model of $\mathbb{T}_2/N_G$ in $(\mathcal{F}, \overline{g})$. Hence there is a morphism

$$f' = (\overline{f'}, f'\Downarrow)\colon (\mathcal{F}, \overline{g}) \to (\mathcal{S}'', p')$$

such that $f'^* N'_G \cong N'$. Now define

$$f = (\overline{f'}, ((f'\Downarrow) \cdot p; g\Downarrow))\colon (\mathcal{F}, q) \to (\mathcal{S}'', p'p) \qquad \begin{tikzcd} \mathcal{F} \arrow[r, "\overline{f'}"] \arrow[dr, "\overline{g}"'] \arrow[ddr, bend right, "q"'] & \mathcal{S}'' \arrow[d, "p'"] \arrow[dl, phantom, "f'\Downarrow"] \\ & \mathcal{S}' \arrow[d, "p"] \arrow[l, phantom, "g\Downarrow"] \\ & \mathcal{S} \end{tikzcd} \quad .$$

As models of $\mathbb{T}_2$, we have $f^* N'_G \cong \overline{f'}^* N'_G \cong f'^* N'_G \cong N' \cong N$; and we see from the following diagram that this composite isomorphism restricts under

$U'U$ to the identity – it is an isomorphism for $\mathbb{T}_2/M$.

$$\begin{tikzcd}
& f^*N'_G \arrow[dl, dotted, "\cong"'] \arrow[dd, mapsto] & & f'^*N'_G \arrow[dl, "\cong"'] \arrow[ddl, mapsto] & \overline{f'}^*N'_G \arrow[lll, bend right, "\cong"'] \arrow[l, "\cong"] \arrow[dd, mapsto] \\
N \arrow[d, mapsto] & & N' \arrow[ll, "\cong"'] \arrow[d, mapsto] & & \\
NU' \arrow[d, mapsto] & g^*N_G \arrow[l, "\cong"'] \arrow[d, mapsto] & \overline{g}^*N_G \arrow[l, "\cong"'] \arrow[d, mapsto] & & \overline{f'}^*p'^*N_G \arrow[ll, "\cong", "(f\Downarrow)^*N_G"'] \arrow[d, mapsto] \\
NU'U \arrow[r, equal] & q^*M & \overline{g}^*p^*M \arrow[l, "\cong", "(g\Downarrow)^*M"'] & & \overline{f'}^*p'^*p^*M \arrow[ll, "\cong", "(f\Downarrow)^*p^*M"']
\end{tikzcd}$$

Now suppose we have two morphisms $f_i = (\overline{f_i}, f_i\Downarrow)\colon (\mathcal{F}, q) \to (\mathcal{S}'', p'p)$ $(i = 0, 1)$. Let us write $\overline{g_i} = \overline{f_i}p'$ and

$$g_i = (\overline{g_i}, f_i\Downarrow)\colon (\mathcal{F}, q) \to (\mathcal{S}', p) \qquad \begin{tikzcd}
\mathcal{F} \arrow[r, "\overline{f_i}"] \arrow[dr, "\overline{g_i}"'] \arrow[ddr, bend right, "q"'] \arrow[dr, phantom, shift left=3, "="] \arrow[ddr, phantom, shift right=1, "f_i\Downarrow"] & \mathcal{S}'' \arrow[d, "p'"] \\
& \mathcal{S}' \arrow[d, "p"] \\
& \mathcal{S}
\end{tikzcd} \qquad .$$

This makes $\mathcal{F}$ two separate toposes $(\mathcal{F}, \overline{g_i})$ over $\mathcal{S}'$.

Suppose also we have a $\mathbb{T}_2/M$-morphism $\theta\colon f_0^*N'_G \to f_1^*N'_G$. Our aim is to show that there is a unique 2-cell $\alpha\colon f_0 \to f_1$ such that $\alpha^*N'_G = \theta$. Consider the diagram

$$\begin{tikzcd}
f_i^*N'_G \arrow[dd, mapsto, bend right] & \overline{f_i}^*N'_G \arrow[l, "\cong"'] \arrow[d, mapsto] & \mathbb{T}_2 \arrow[d, "U'"] \\
g_i^*N_G \arrow[d, mapsto] & \overline{f_i}^*p'^*N_G = \overline{g_i}^*N_G \arrow[l, "\cong"] \arrow[d, mapsto] & \mathbb{T}_1 \arrow[d, "U"] \\
q^*M & \overline{f_i}^*p'^*p^*M \arrow[l, "f_i\Downarrow"', "\cong"] & \mathbb{T}_0.
\end{tikzcd}$$

We find that $(f_i^* N'_G)U' = g_i^* N_G$, as it has the correct properties according to Proposition 2.1. Hence we have $\theta U' \colon g_0^* N_G \to g_1^* N_G$, and there is a unique $\beta \colon g_0 \to g_1$ such that $\theta U' = \beta^* N_G$. (This is modulo the appropriate isomorphisms, for β is actually a natural transformation from $\overline{g_0}$ to $\overline{g_1}$.)

Let us first deal with the case where θ is an isomorphism, and β likewise. We thus have two morphisms $f'_i \colon (\mathcal{F}, \overline{g_1}) \to (\mathcal{S}'', p')$, given by $f'_0 = (\overline{f_0}, \beta)$ and $f'_1 = (\overline{f_1}, \mathsf{Id})$. In the diagrams below, three levels are for $\mathbb{T}_2$, $\mathbb{T}_1$ and $\mathbb{T}_0$, successively reduced by U' and U. The horizontal isomorphisms '$\cong$' come from Proposition 2.1, and the vertical ones are defined to make their outer squares commute. We then find a unique $\alpha \colon f'_0 \to f'_1$ (which is the same as saying $\alpha \cdot p' = \beta$) such that $\alpha^* N'_G$ is the isomorphism $f'^*_0 N'_G \cong f'^*_1 N'_G$ at top right in the diagram. Then $\alpha \colon f_0 \to f_1$ and is unique such that $\theta = \alpha^* N'_G$.

$$\begin{tikzcd}
\mathbb{T}_2 \arrow[dd, "U'"'] & f_0^* N'_G \arrow[d, "\theta"'] & \overline{f_0}^* N'_G \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, dotted, "\alpha^* N'_G"] & f'^*_0 N'_G \arrow[d, dotted, "\cong"] \\
 & f_1^* N'_G & \overline{f_1}^* N'_G \arrow[l, "\cong"'] \arrow[r, equal] & f'^*_1 N'_G \\
\mathbb{T}_1 \arrow[dd, "U"'] & g_0^* N_G \arrow[d, "\theta U'"'] & \overline{g_0}^* N_G \arrow[l, "\cong"'] \arrow[r, "\beta^* N_G"] \arrow[d, dotted, "\beta^* N_G"] & g'^*_1 N_G \arrow[d, equal] \\
 & g_1^* N_G & \overline{g_1}^* N_G \arrow[l, "\cong"'] \arrow[r, equal] & g'^*_1 N_G \\
\mathbb{T}_0 & q^* M \arrow[d, equal] & \overline{g_0}^* p^* M \arrow[l, "(f_0 \Downarrow)^* M"'] \arrow[d, dotted, "\cong"] & \\
 & q^* M & \overline{g_1}^* p^* M \arrow[l, "(f_1 \Downarrow)^* M"] &
\end{tikzcd}$$

We now generalize to arbitrary morphisms θ. Let $(\mathcal{G}, q')$ be the cocomma object in $\mathfrak{BTop}/\mathcal{S}$ of the identity on $(\mathcal{F}, q)$ against itself, with cocomma injections $h_i \colon (\mathcal{F}, q) \to (\mathcal{G}, q')$ and $\eta \colon h_0 \to h_1$. By [7, B3.4.7], $\mathcal{G}$ as a category is just the comma category $\mathcal{F} \downarrow \mathcal{F}$. It follows from [9] that there is a bijection between, on the one hand, morphisms $\theta \colon N_0 \to N_1$ between strict $\mathbb{T}_2/M$-models in $\mathcal{F}$, and, on the other, strict $\mathbb{T}_2/M$-models in $\mathcal{G}$. Applying the essential surjectivity property (already proved) for $\mathcal{S}''$, in relation to $\mathcal{G}$, we see that for every such θ there is a morphism $f'\colon (\mathcal{G}, q') \to (\mathcal{S}'', p'p)$, hence a pair of morphisms $f'_i\colon (\mathcal{F}, q) \to (\mathcal{S}'', p'p)$, and a 2-cell α' between them, with a commuting diagram

$$\begin{tikzcd}
N_0 \arrow[r, "\cong"] \arrow[d, "\theta"'] & f_0'^* N'_G \arrow[d, "\alpha'^* N'_G"] \\
N_1 \arrow[r, "\cong"] & f_1'^* N'_G
\end{tikzcd}$$

We return to the case of interest, where $N_i = f_i^* N'_G$. By the restricted case, with θ an isomorphism, we find 2-cell isomorphisms $\beta_i\colon f_i \to f'_i$ that, applied to N'_G, give the horizontal isomorphisms above. Then, taking $\alpha = \beta_0; \alpha'; \beta_1^{-1}$, we get $\theta = \alpha^* N'_G$. This proves fullness.

Finally we must prove faithfulness. Suppose we have f_0 and f_1 as before, and 2-cells $\alpha, \alpha'\colon f_0 \to f_1$ with $\alpha^* N'_G = \alpha'^* N'_G$. We deduce that $\alpha \cdot p' = \alpha' \cdot p'$ because $\mathcal{S}'$ is a classifier. Hence we have two geometric morphisms $g = (\overline{f_0}, \alpha, \overline{f_1})$ and $g' = (\overline{f_0}, \alpha', \overline{f_1})$ from $\mathcal{G}$ to $\mathcal{S}''$, with $gp' = g'p'$. We have $g^* N'_G = g'^* N'_G$, so from the properties of $\mathcal{S}''$ as classifying topos we get a unique 2-cell $\beta\colon g \to g'$ such that $\beta^* N'_G$ is the identity. This gives two 2-cells $\beta_i\colon \overline{f_i} \to \overline{f_i}$, making a commutative square with α and α', with $\beta_i^* N'_G$ the identity on $\overline{f_i}^* N'_G$. We deduce that each β_i is an identity, and it follows that $\alpha = \alpha'$. ☐

Theorem 5.11. *Let $\mathbb{T}_0 \subset \mathbb{T}_1$ be a context extension and M a strict model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}_1$. Let the following diagram be a cartesian 1-cell f in $\mathfrak{GTop}$ over $\mathfrak{eTop}_{\cong}$, hence a pseudopullback in $\mathfrak{eTop}$.*

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r, "\overline{f}"] \arrow[d, "p_0"'] \arrow[dr, phantom, "f\Downarrow"] & \mathcal{E}_1 = \mathcal{S}_1[\mathbb{T}_1/M] \arrow[d, "p_1"] \\
\mathcal{S}_0 \arrow[r, "\underline{f}"'] & \mathcal{S}_1
\end{tikzcd}$$

Then $p_0\colon \mathcal{E}_0 \to \mathcal{S}_0$ serves as a classifying topos $\mathcal{S}_0[\mathbb{T}_1/\underline{f}^* M]$.

If N_G is a generic model for $\mathbb{T}_1/M$, then $f^* N_G$ serves as generic model for $\mathbb{T}_1/\underline{f}^* M$.

Proof. First, pseudopullback squares are preserved under composition with equivalences over $\mathcal{S}_0$ and $\mathcal{S}_1$, so it suffices to show that there is *some* pseudopullback square whose vertical maps are classifiers as stated.

By Proposition 5.10 we can reduce to the case where the extension is simple.

For extension by primitive node, we have the task of constructing an object classifier, and this is a special case of classifying torsors (internal flat presheaves) over an internal category $\mathcal{C}$, here the category of finite sets: objects are natural numbers, morphisms defined in the appropriate way.

For extension by commutativity, we have already remarked that this is equivalent to inverting a morphism.

For a simple functional extension, adjoining a morphism from X to Y, we can decompose the classification problem into two steps of the above kinds. First, we adjoin a subjobject of $X \times Y$ for the graph of the morphism, and this is equivalent to adjoining a torsor (ideal) for the poset $\mathcal{F}(X \times Y)$, the Kuratowski finite powerobject (free semilattice). Next we impose some axioms for single-valuedness and totality, and this is equivalent to making some morphism invertible.

It follows that we reduce to two cases over $\mathbb{T}_0$: adjoining a torsor for an internal category $\mathcal{C}$, and forcing the invertibility of some morphism. (Although these are not simple extensions of contexts, we can still work with them as single steps.) We show that our classifiers $\mathcal{S}_1[\mathbb{T}_1/M]$ can be found in a way that is preserved under pseudopullback. The argument parallels that of [7, B3.3.6].

In one case, $\mathbb{T}_1$ adjoins a torsor (flat presheaf) for an internal category $\mathcal{C}$ in $\mathcal{S}_1$. Here we can take the classifier to be $[\mathcal{C}, \mathcal{S}_1]$ by Diaconescu's Theorem, and this can be pulled back along any $\underline{f}\colon \mathcal{S}_0 \to \mathcal{S}_1$ to $[\underline{f}^*\mathcal{C}, \mathcal{S}_0]$ over $\mathcal{S}_0$. (See [7, B3.2.7, B3.2.14].)

For any geometric theory $\mathbb{T}$, the models of $\mathbb{T}$ in $[\mathcal{C}, \mathcal{S}_1]$ are the internal $\mathcal{C}$-indexed families of models of $\mathbb{T}$ in $\mathcal{S}_1$, and in the particular case of $\mathcal{C}$-torsors the generic model is the Yoneda embedding $\mathcal{Y}$, with the representable torsor $\mathcal{Y}(j)$ for each object j of $\mathcal{C}$. To express this more concretely as a $(p_1^*\mathcal{C})$-indexed family of $\mathcal{C}$-torsors in $[\mathcal{C}, \mathcal{S}_1]$, use the morphism

$$\mathcal{C}_1 \xrightarrow{\langle d,c\rangle} \mathcal{C}_0 \times \mathcal{C}_0 \xrightarrow{\pi_2} \mathcal{C}_0$$

It becomes the object part of an internal presheaf over $p_1^*\mathcal{C}$, and is the generic torsor. Its construction is geometric (arithmetic, even) and so is preserved by f^*.

In the other case, $\mathbb{T}_1$ imposes invertibility for a morphism $u\colon X \to Y$ in $\mathcal{S}_1$. Here $p_1\colon \mathcal{E}_1 \to \mathcal{S}_1$ is an inclusion, and by [7, A4.3.11] it can be taken to be the topos of sheaves for the smallest local operator for which $\mathrm{im}(u) \rightarrowtail Y$ and $X \rightarrowtail \mathrm{kp}(u)$, the kernel pair, are both dense. Inverting both of these monomorphisms will make u invertible. By [7, A4.5.14(e)] its pseudopullback along $\underline{f}$ is also an inclusion, in fact for the smallest local operator that makes $\underline{f}^*u$ an isomorphism. The generic model is p_1^*M, so $f^*p_1^*M \cong p_0^*\underline{f}^*M$ is a generic model in $\mathcal{E}_0$. □

Putting together these results, we now obtain our main *Local Representability Theorem* –

Theorem 5.12. *In diagram* (2)*, the left hand fibration P is locally representable (Definition 4.4) over its Q.*

Proof. Given $(\mathcal{S}, M)$, then, as noted in Definition 5.9, the classifying topos $\mathcal{S}[\mathbb{T}_1/M]$ exists, and, by Proposition 4.3, this gives condition (1) of Definition 4.4. The geometricity condition (2) is Theorem 5.11. □

As we have already mentioned, by taking $\mathbb{T}_0 = \mathbb{1}$ we get that the right hand P in diagram (2) is also locally representable. This tells us that the assignment $\mathcal{S} \mapsto \mathcal{S}[\mathbb{T}_1]$ is preserved under change of base $\mathcal{S}$.

After the main theorem, Proposition 4.5 now provides us with ways to use the classifying toposes $\mathcal{S}[\mathbb{T}_1/M]$ in places beyond $\mathfrak{BTop}/\mathcal{S}$. In particular, –

Corollary 5.13. *Let $\mathbb{T}_0 \subset \mathbb{T}_1$ be a context extension and M a strict model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$. Then $\mathcal{E} = \mathcal{S}[\mathbb{T}_1/M]$ has the classifying topos property for arbitrary $\underline{q}\colon \mathcal{F} \to \mathcal{S}$, not necessarily bounded.*

Proof. Apply Proposition 4.5 to models of U in $\mathsf{Id}\colon \mathcal{F} \to \mathcal{F}$ for which the $\mathbb{T}_0$ part is $\underline{q}^*M$. □

Example 5.14. Let $\mathbb{T}_0$ be the context whose models are "GRD-systems" as in [11]. It has three nodes G, R, D, together with (amongst other ingredients) a further node $\mathcal{F}G$ constrained to be the Kuratowski finite powerset of G.

(For instance, it can be constructed as a quotient of the list object $\mathsf{List}\, G$.) Finally, it has edges

$$\begin{tikzcd}
 & D \arrow[dl, "\rho"'] \arrow[d, "\pi"] \\
\mathcal{F}G & R \arrow[l, "\lambda"]
\end{tikzcd}$$

This can be used to present a frame, with generators $g \in G$ subject to relations (for $r \in R$)

$$\bigwedge \lambda(r) \leq \bigvee \{ \bigwedge \rho(d) \mid \pi(d) = r \}.$$

The points of the corresponding locale, the subsets $F \subseteq G$ respecting the relations, are models of a context $\mathbb{T}_1$ that extends $\mathbb{T}_0$. It has a node for F, with an edge $F \to G$ constrained to be monic, nodes for $X = \{r \in R \mid \lambda(r) \subseteq F\}$ and $Y = \{r \in R \mid (\exists d)(\pi(d) = r \wedge \rho(d) \subseteq F)\}$ (which can be constructed in the AU-sketches) and an edge $X \subseteq Y$.

Then the local representability Theorem 5.12 implies [11, Corollary 5.4], the geometricity of presentations.

6. Conclusion

What our main result has done is to elaborate the idea that a map $U \colon \mathbb{T}_1 \to \mathbb{T}_0$, a $\mathbb{T}_0$-valued map on $\mathbb{T}_1$, may also be a *bundle:* that is to say, a *space*-valued map on the *co*domain $\mathbb{T}_0$, transforming points to the corresponding fibres.

This interpretation is often tacit in a morphism in a category, and is particularly important in type theory. We have made it concrete in the particular case of a morphism U in $\mathfrak{Con}$ that arises from a context extension.

Note that U certainly is a "$\mathbb{T}_0$-valued map on $\mathbb{T}_1$", if we think of the points of a context as its strict models. This is shown in Section 3 and does not need toposes – the models can be taken in any AU.

To get U as a bundle, we interpret "space" as Grothendieck topos and look for the classifying toposes for the fibres. However, the base toposes are now allowed to vary, and in Theorem 5.11 we showed the geometricity property that when you change the base, and the corresponding base point of $\mathbb{T}_0$, the classifier (representing the fibre) transforms by pseudopullback.

This result, which I have not been able to find in the literature, relies on a difference between the "arithmetic" theories of $\mathfrak{Con}$ and the geometric theories that are classified. An arithmetic theory depends only on the existence of an nno, whereas a geometric theory depends on the choice of some base topos $\mathcal{S}$.

To avoid the intricacies of coherence for the choices made in indexed 2-categories, we have adopted a fibrational approach to classifiers. As part of that, the definition of classifier as representing object for an indexed 2-category has been reformulated in terms of fibrations. Then their indexed behaviour was formulated (our "local representability", Definition 4.4) in terms of towers of two 2-fibrations. It may be that the formulation in Proposition 4.5 has broader usefulness. I sense that the classifying objects x_w may be trying to fulfill some notion of "cartesian 0-cell", though I have not been able to make that idea any more precise.

The results here are a piece in the broad programme of using AU techniques to prove base-independent, geometric results for toposes in those situations that do not need the full power of $\mathcal{S}$-indexed colimits for some $\mathcal{S}$. One already mentioned is the "geometricity of presentations", Example 5.14.

On the other hand, the results also provide clues to how one might seek a self-standing arithmetic logic of spaces, developing [9]. They suggest that the extension maps might be the correct analogues of bounded geometric morphisms.

A final comment regards the word "topos" itself, which Grothendieck chose to suggest "those things of which topology is the study". The very word "topos" should strongly carry the idea of generalized space, but with the advent of the elementary topos this inherent meaning has become obscure. It is not so much the elementary toposes themselves that are the generalized spaces, as the bounded geometric morphisms between them, and those are what are called "Grothendieck toposes" in the present paper. One might dream that the true toposes, the generalized spaces, the subjects of topology, are arithmetic universes, and [9, 12] were written with that in mind. All the same, there are significant gaps between that and Grothendieck's vision, which was partly of "those categories with the structure needed to do sheaf cohomology". Much as one might hope that, suitably formulated, the basic results of algebraic topology are foundationally robust enough to work with AUs, we have no idea at present as to how to do that.

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TAC: THEORY AND APPLICATIONS OF CATEGORIES

Hereafter we give some information about the electronic Journal:
Theory and Applications of Categories **(TAC),** ISSN 1201-561X

Contents of Volume 31, 2016

14. Construction of categorical bundles from local data, Saikat Chatterjee, Amitabha Lahiri and Ambar N. Sengupta, 388-417
15. Partial linearity and partial natural Mal'tsevness, Dominique Bourn, 418-443
16. Relative internal actions, James Richard Andrew Gray and Tamar Janelidze-Gray, 444-461
17. Right delocalization of model categories, Bruce R Corrigan-Salter, 462-476
18. The morphism axiom for *n*-angulated categories, Emilie Arentz-Hansen, Petter Andreas Bergh and Marius Thaule, 477-483
19. The snail lemma, Enrico M. Vitale, 484-501
20. Linear structures on locales, Pedro Resende and Joao Paulo Santos, 502-541
21. Representation and character theory of finite categorical groups, Nora Ganter and Robert Usher, 542-570
22. Classical and relative realizability, Jaap van Oosten and Tingxiang Zou, 571-593
23. The linearity of traces in monoidal categories and bicategories, Kate Ponto and Michael Shulman, 594-689
24. The local Joyal model structure, Nicholas J. Meadows, 690-711
25. Homotopy locally presentable enriched categories, Stephen Lack and Jiri Rosicky, 712-754
26. Compact closed bicategories, Michael Stay, 755-798
27. Strict ω-categories are monadic over polygraphs, Francois Metayer, 799-806
28. The Fermat functors, Enxin Wu, 807-832
29. Enriched Yoneda lemma, Vladimir Hinich, 833-838
30. Projectivity, continuity and adjointness: quantales, Q-posets and Q-modules, Susan Niefield, 839-851
31. Tannaka theory over sup-lattices and descent for topoi, Eduardo J. Dubuc and Martin Szyld, 852-906
32. On the representations of 2-groups in Baez-Crans 2-vector spaces, Benjamin A. Heredia and Josep Elgueta, 907-927
33. Compositories and gleaves, Cecilia Flori and Tobias Fritz, 928-988
34. On Evrard's homotopy fibrant replacement of a functor, Boris Shoikhet, 989-1015
35. Monads on dagger categories, Chris Heunen and Martti Karvonen, 1016-1043

36. Products of families of types and (Π, λ)-structures on C-systems, Vladimir Voevodsky, 1044-1094
37. A functorial approach to Dedekind completions and the representation of vector lattices and l-algebras by normal functions, G. Bezhanishvili, P. J. Morandi, B. Olberding, 1095-1133
38. Cosheafification, Andrei V. Prasolov, 1134-1175

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RESUMES DES ARTICLES PUBLIES
dans le Volume LVIII (2017)

GRANDIS & PARE, Adjoints for multiple categories (on weak and lax multiple categories, III), 3-48.

Cet article est la suite de 2 articles précédents de cette même série, parus dans les *Cahiers* . Ici les auteurs étudient les foncteurs adjoints entre catégories multiples de dimension infinie. Le cadre général est constitué par les *catégories multiples chirales* – une forme faible partiellement laxe ayant des interchangeurs dirigés entre les compositions faibles.

M. BARR, On certain topological *-autonomous categories, 49-66.

Etant donné une catégorie additive équationnelle, munie d'une structure fermée monoïdale symétrique ainsi que d'un objet dualisateur potentiel, on trouve des conditions suffisantes pour que la catégorie des objets topologiques sur cette catégorie admette une bonne notion de sous-catégories pleines qui contiennent des objets fortement et faiblement topologisés. On montre que chacune de ces sous-catégories est équivalente à la catégorie chu de la catégorie originale par rapport à l'objet dualisateur.

R. GUITART, Autocategories: III. Representations, and expansions of previous examples, 67-80.

Cet article est le troisième d'une série d'articles sur la notion d'autographe. Un *autographe* est un ensemble A équipé d'une action du monoïde libre à deux générateurs ; une *algèbre autographique* est une algèbre d'une monade sur le topos des autographes. Dans deux articles précédents l'auteur a montré que les diagrammes de nœuds et les 2-graphes sont des exemples, et que les algèbres graphiques basiques sont autographiques.

Dans ce troisième article, il ajoute trois résultats nouveaux. Il montre comment représenter concrètement les autographes, et réciproquement comment collecter une représentation en un autographe ; il explique précisément comment les noeuds, les entrelacs, les diagrammes de grilles, et aussi les catégories doubles, sont des exemples d'autographes, et il identifie les algèbres graphiques générales avec des algèbres autographiques.

MAY, ZAKHAREVICH AND STEPHAN, The homotopy theory of equivariant posets, 82-114.

Soit G un groupe discret. Les auteurs démontrent que la catégories des G-ensembles ordonnés admet une structure de catégorie modèle qui est Quillen-équivalente à la structure de catégorie modèle standard sur les G-espaces. Comme dans le cas non-équivariant, les trois classes de morphismes qui constituent la structure de modèle ne sont pas bien comprises computationnellement. Ce fait est illustré par quelques exemples d'ensembles ordonnés cofibrants et fibrants et un exemple d'un ensemble ordonné fini qui n'est pas cofibrant.

A. KOCK, Affine combinations in affine schemes, 115-130.

L'auteur prouve que la notion géométrique de *points voisins*, dérivée du "premier voisinage de la diagonale" en géométrie algébrique, a la propriété que toute combinaison affine d'un *n*-tuple quelconque de points mutuellement voisins a un sens invariant, dans tout schéma affine. La preuve est obtenue par des considérations d'algèbre commutative élémentaire.

AOKI & KURIBAYASHI, On the category of stratifolds, 131-160

Les auteurs étudient les espaces stratifiés de Kreck (stratifolds) d'un point de vue catégorique. Ils montrent entre autres que la catégorie des espaces stratifiés de Kreck admet un plongement pleinement fidèle dans la catégorie des R-algèbres tout comme la catégorie des variétés lisses. Ils établissent une variante du théorème de Serre-Swan pour les espaces stratifiés de Kreck. En particulier, ils montrent que les fibrés vectoriels sur un espace stratifié de Kreck forment une catégorie equivalente à celle formée par les fibrés vectoriels sur un schéma affine qui est canoniquement associé à, mais en géneeral plus grand que, l'espace stratifié lui-même.

GRANDIS, G. JANELIDZE & MARKI, Two- and one-dimensional combinatorial exactness structures in Kurosh–Amitsur radical theory, I, 165-188**.**

Les auteurs proposent une nouvelle version non-pointée de structure d'exactitude combinatoire pour la théorie abstraite des radicaux de type Kurosh–Amitsur introduite par les deuxième et troisième auteurs en 2003, appelée ci-dessous structure 2-dimensionnelle. Elle est motivée par la notion de catégorie semi-exacte introduite par le premier auteur en 1992 et, brièvement, elle permet de définir un triplet radical-semisimple tel que, si (R,r,S) est un tel triplet, alors (R,S) est un couple radical-semisimple par rapport à la structure d'exactitude 1-dimensionnelle sous-jacente définie dans ce qui suit.

Sean TULL, Condition for an *n*-permutable category to be Mal'tsev, 189-194.

L'auteur améliore la description des catégories *n*-permutables introduites par Carboni, Kelly et Pedicchio. Ceci donne une nouvelle caractérisation des catégories régulières de Mal'tsev parmi celles qui sont des catégories de Goursat ou, plus gnéralement, des catégories *n*-permutables.

GRANDIS & PARE, A multiple category of multiple lax-categories, 195-212.

Les auteurs construisent une catégorie multiple, utile dans l'étude des adjonctions multiples. Les objets sont les catégories multiples 'laxes'. Les flèches transversales sont les foncteurs multiples stricts tandis que les flèches en direction positive sont des foncteurs multiples de 'laxité mixte', qui varient des foncteurs laxes (en direction 1) aux foncteurs colaxes (en direction 1).

S. VICKERS, Arithmetic universes and classifying toposes, 213-248.
Cet article utilise la structure de *Con* (la 2-catègorie des esquisses pour les univers arithmétiques (AU) de l'auteur) pour obtenir des rèsultats constructifs, indépendants de la base pour les topos de Grothendieck (S-topos bornés) comme espaces généralisés. Le principal résultat montre comment une application extension U : $T_1 \rightarrow T_0$ peut être vue comme un fibré, transformant les points de base (modèles de T_0 dans un topos S avec objet des nombres naturels) en fibres (espaces généralisés au-dessus de S). Parmi les caractéristiques de ce travail, on notera : une comparaison entre modèles stricts ou non-stricts, utilisant les propriétés des objets de *Con* ; l'utilisation des produits tensoriels de Gray pour relier la transformation syntactique de modèles par des 1-cellules de *Con* et les transformations sémantiques par des AU-foncteurs non stricts ; et l'utilisation de 2-fibrations pour indexer au-dessus d'une 2-catégorie de topos de base S.

TABLE DES MATIERES DU VOLUME LVIII (2017)

Fascicule 1

Fascicule 2

Fascicules 3 and 4