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THE HOMOTOPY THEORY OF EQUIVARIANT POSETS

by Peter MAY, Inna ZAKHAREVICH and Marc STEPHAN

Résumé. Soit G un groupe discret. Nous démontrons que la catégorie des G-ensembles ordonnés admet une structure de catégorie modèle qui est Quillen-équivalente à la structure de catégorie modèle standard sur les G-espaces. Comme dans le cas non-équivariant, les trois classes de morphismes qui constituent la structure de modèle ne sont pas bien comprises computationnellement. Nous illustrons ce fait avec quelques exemples d'ensembles ordonnés cofibrants et fibrants et un exemple d'un ensemble ordonné fini qui n'est pas cofibrant.

Abstract. Let G be a discrete group. We prove that the category of G-posets admits a model structure that is Quillen equivalent to the standard model structure on G-spaces. As is already true nonequivariantly, the three classes of maps defining the model structure are not well understood calculationally. To illustrate, we exhibit some examples of cofibrant and fibrant posets and an example of a non-cofibrant finite poset.
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1. Introduction

In [20], Thomason proved that categories model the homotopy theory of topological spaces by proving that the category **Cat** of (small) categories has a model structure that is Quillen equivalent to the standard model structure on the category **Top** of topological spaces. In [16], Raptis proved that the category of posets also models the homotopy theory of topological spaces by showing that the category **Pos** of posets has a model structure that is Quillen equivalent to the Thomason model structure on **Cat**. It is natural to expect this to hold since Thomason proved in [20, Proposition 5.7] that cofibrant categories in his model structure are posets. The first and third authors rediscovered this, observing that a geodesic proof, if not the statement, of that result is already contained in Thomason's paper. This implies that all of the algebraic topology of spaces can in principle be worked out in the category of posets. It can also be viewed as a bridge between the combinatorics of partial orders and algebraic topology.

In this paper we prove an analogous result for the category of G-spaces for a discrete group G. For a category $\mathcal{C}$, let $G\mathcal{C}$ denote the category of objects with a (left) action of G and maps that preserve the action. In [3], Bohmann, Mazur, Osorno, Ozornova, Ponto, and Yarnall proved in precise analogy to Thomason's result that $G\mathbf{Cat}$ models the homotopy theory of G-spaces. Here we prove the pushout of the results of Raptis and Bohmann, et al: the category $G\mathbf{Pos}$ of G-posets admits a model structure that is Quillen equivalent to the model structure on the category $G\mathbf{Cat}$ of G-categories and therefore also Quillen equivalent to the model structures on $G\mathbf{sSet}$ and $G\mathbf{Top}$. Just as the model structure on $\mathbf{Pos}$ is implicit in Thomason's paper [20], we shall see that the model structure on $G\mathbf{Pos}$ is implicit in the six author paper [3].

While the background makes this an expected result, it is perhaps surprising, at least psychologically. There is relatively little general study of equivariant posets in either the combinatorial or topological literature, especially not from a homotopy theoretic perspective. One thinks of group actions as permutations, as exemplified by the symmetric groups, and it does not come naturally to think of a general theory of groups acting by order-preserving maps of posets. However, our theorem says that group actions on posets abound: every G-space is weakly equivalent to the classifying G-space of a G-poset, where a map f of G-spaces is a weak equivalence if its fixed point maps f^H are weak equivalences for all subgroups H of G. The result can be viewed as a formal bridge between equivariant combinatorics and equivariant algebraic topology.

The combinatorial literature seems to start with Stanley's paper [17], which restricts to finite posets and focuses on the connection with representation theory. A paper of Babson and Kozlov [1] about G-posets X focuses on problems arising from the fact that the orbit category X/G is generally not a poset. There is considerable group theory literature about posets of subgroups of G with G acting by conjugation, starting from Quillen's paper [15]. That led Thévenaz and Webb to an equivariant generalization of Quillen's

Theorem A applicable to G-posets [19]. In turn, that led to Welker's paper [21], which considers the order G-complex associated to a G-poset, again with group theoretic applications in mind.

Let $\mathcal{O}_G$ denote the orbit category of G. Its objects are the G-sets G/H and its morphisms are the G-maps. Just as for G-spaces, G-simplicial sets (that is, simplicial G-sets), and G-categories, it is natural to start with the levelwise (or projective) model structure on the category $\mathcal{O}_G$-$\mathbf{Pos}$ of contravariant functors $\mathcal{O}_G \longrightarrow \mathbf{Pos}$. As a functor category, $\mathcal{O}_G$-$\mathbf{Pos}$ inherits a model structure from $\mathbf{Pos}$. Its fibrations and weak equivalences are defined levelwise. It is standard that this gives a compactly generated model structure (e.g. [10, 11.6.1]).[1]

Define the fixed point diagram functors

$$\Phi\colon G\mathbf{Pos} \longrightarrow \mathcal{O}_G\text{-}\mathbf{Pos} \quad \text{and} \quad \Phi\colon G\mathbf{Cat} \longrightarrow \mathcal{O}_G\text{-}\mathbf{Cat}$$

by

$$\Phi(X)(G/H) = X^H.$$

These functors Φ have left adjoints, denoted Λ; in both cases, Λ sends a contravariant functor Y defined on $\mathcal{O}_G$ to $Y(G/e)$.

We prove that $G\mathbf{Pos}$ inherits a model structure from $\mathcal{O}_G$-$\mathbf{Pos}$. The analogue for $G\mathbf{Cat}$ is [3, Theorem A]. After recalling details of the model structures already cited, we shall prove the following theorem.

Theorem 1.1. *The functor Φ creates a compactly generated proper model structure on $G\mathbf{Pos}$, so that a map of G-posets is a weak equivalence or fibration if it is so after applying Φ. The adjunction (Λ, Φ) is a Quillen equivalence between $G\mathbf{Pos}$ and $\mathcal{O}_G$-$\mathbf{Pos}$.*

Replacing $\mathbf{Pos}$ with $\mathbf{Cat}$ in Theorem 1.1 gives the statement of [3, Theorem A]. The strategy of proof in [3] is to verify general conditions on a

[1] Compactly generated is a variant of cofibrantly generated that applies when only countable colimits are needed in the small object argument, that is, when transfinite colimits are unnecessary and irrelevant, as they are in all of the model structures we shall consider. This variant is discussed in detail in [13, §15.2]. It seems reasonable to eliminate transfinite verbiage whenever possible, and that would shorten and simplify some of the work in the sources we shall cite.

model category $\mathcal{C}$ that ensure that $G\mathcal{C}$ inherits a model structure from $\mathcal{O}_G$-$\mathcal{C}$.[2] The cited general conditions are taken from a paper of the second author [18]. Our proof of Theorem 1.1 will proceed in the same way. The following result is a formal consequence of Theorem 1.1 and its analogue for $\mathbf{Cat}$.

Theorem 1.2. *The adjunction (P, U) between $G\mathbf{Cat}$ and $G\mathbf{Pos}$ is a Quillen equivalence. Therefore, $G\mathbf{Pos}$ is Quillen equivalent to $G\mathbf{sSet}$ and $G\mathbf{Top}$.*

The following diagram displays the relevant equivariant Quillen equivalences.

$$\begin{array}{ccccccc}
G\mathbf{Top} & \underset{S_*}{\overset{|-|}{\leftrightarrows}} & G\mathbf{sSet} & \underset{\mathrm{Ex}^2 N}{\overset{\Pi\,\mathrm{Sd}^2}{\rightleftarrows}} & G\mathbf{Cat} & \underset{U}{\overset{P}{\rightleftarrows}} & G\mathbf{Pos} \\
\Lambda \uparrow \downarrow \Phi & & \Lambda \uparrow \downarrow \Phi & & \Lambda \uparrow \downarrow \Phi & & \Lambda \uparrow \downarrow \Phi \\
\mathcal{O}_G\text{-}\mathbf{Top} & \underset{S_*}{\overset{|-|}{\leftrightarrows}} & \mathcal{O}_G\text{-}\mathbf{sSet} & \underset{\mathrm{Ex}^2 N}{\overset{\Pi\,\mathrm{Sd}^2}{\rightleftarrows}} & \mathcal{O}_G\text{-}\mathbf{Cat} & \underset{U}{\overset{P}{\rightleftarrows}} & \mathcal{O}_G\text{-}\mathbf{Pos}
\end{array}$$

The definitions of Π, Sd, Ex, and N are recalled in the next section.

All of the vertical adjunctions and the adjunctions on the bottom row are Quillen equivalences, hence so are all of the adjunctions on the top row. Applied to the righthand square, this gives the proof of Theorem 1.2. Applied to the middle square, this gives [3, Theorem B], which is the equivariant version of Thomason's comparison between $\mathbf{sSet}$ and $\mathbf{Cat}$.

Remark 1.3. Both equivariantly and nonequivariantly, replacing $\mathbf{Cat}$ by $\mathbf{Pos}$ ties in the Thomason model structure to more classical algebraic topology. The composite $N \circ U\colon \mathbf{Pos} \longrightarrow \mathbf{sSet}$ coincides with the composite of the functor that sends a poset to its order complex and the canonical functor from ordered simplicial complexes to simplicial sets, and the same is true equivariantly. It also ties in the Thomason model structure to finite T_0-spaces and, more generally T_0-Alexandroff spaces, or A-spaces, since the categories of posets and A-spaces are isomorphic.

[2]There are two slightly different ways to equip $G\mathcal{C}$ with a model structure, either transferring the model structure from $\mathcal{O}_G$-$\mathcal{C}$, as we shall do, or from copies of $\mathcal{C}$ via all of the fixed point functors, as in [3, 18].

An interesting and unfortunate feature of all of the model structures discussed in this paper is that the classes of weak equivalences, cofibrations, and fibrations are defined formally, using non-constructive arguments. In no case do we have a combinatorially accessible description of any of these classes of maps. Even in the case when G is trivial very little is known about the structure. In [6, Theorem 2.2.11], Cisinski gives a characterization of the subcategory of weak equivalences in $\mathbf{Cat}$ through a global characterization, but that does not allow us to determine whether or not a particular morphism is a weak equivalence.

The state of the art for fibrant and cofibrant objects is similarly sparse. The problem of determining the cofibrant posets has recently been studied by Bruckner and Pegel [4], who show in particular that every poset with at most five elements is cofibrant. In §6, we prove that all finite posets of dimension one are cofibrant and give an example of a six element poset that is *not* cofibrant.[3]

The problem of determining the fibrant categories has recently been studied by Meier and Ozornova [14]. In §7, we use work of Droz and the third author [8] to obtain a more concrete understanding of the posets that the main theorem of [14] shows to be fibrant.

Before turning to the equivariant generalizations, we review and reprove the nonequivariant theorems, giving some new details that streamline and clarify the key arguments.

2. Background

We recall as much as we need about the definitions of the nonequivariant versions of the functors in the diagram above and describe the relevant nonequivariant model structures. Of course, the nerve $N\mathcal{C}$ of a category $\mathcal{C}$ is the simplicial set with

$$(N\mathcal{C})_n = \{x_0 \longrightarrow \cdots \longrightarrow x_n \in \mathcal{C}\}.$$

Let $(\mathrm{Sd}\,\Delta)(n)$ be the nerve of the poset of nonempty subsets of $\{0, \cdots, n\}$. Then $\mathrm{Sd}\,\Delta$ is a covariant functor $\Delta \longrightarrow \mathbf{sSet}$. Let $K\colon \Delta^{op} \longrightarrow \mathbf{Set}$ be a

[3] Amusingly, when we found this example we did not know that it is the smallest possible one.

simplicial set. The subdivision $\operatorname{Sd} K$ is the simplicial set defined conceptually as the tensor product of functors

$$\operatorname{Sd} K = K \otimes_{\Delta} \operatorname{Sd} \Delta.$$

The functor Ex is the right adjoint of Sd; we will not need a description of it.

The fundamental category[4] ΠK has object set K_0 and morphism set freely generated by K_1, where $x \in K_1$ is viewed as a morphism $d_1 x \longrightarrow d_0 x$, subject to the relations

$$d_1 y = (d_0 y) \circ (d_2 y) \text{ for each } y \in K_2 \quad \text{and} \quad s_0 x = \mathrm{id}_x \text{ for each } x \in K_0.$$

The functor U is the full and faithful functor that sends a poset X to X regarded as the category with objects the elements of X and a morphism $x \longrightarrow y$ whenever $x \leq y$. The image of U consists of skeletal categories with at most one morphism $x \longrightarrow y$ for each pair of objects (x, y). The functor P sends a category $\mathcal{C}$ to the poset $P\mathcal{C}$ with points the equivalence classes $[c]$ of objects of $\mathcal{C}$, where $c \sim d$ if there are morphisms $c \longrightarrow d$ and $d \longrightarrow c$ in $\mathcal{C}$. The partial order $\leq$ is defined by $[c] \leq [d]$ if there is a morphism $c \longrightarrow d$ in $\mathcal{C}$, a condition independent of the choice of representatives in the equivalence classes. Note crucially that $P \circ U$ is the identity functor. We often drop the notation U, regarding posets as categories.

We recall the specification of the model structures that we are starting from.

Definition 2.1. A functor $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ between (small) categories is a fibration or weak equivalence if $\operatorname{Ex}^2 NF$ is a fibration or weak equivalence. An order preserving function $f \colon X \longrightarrow Y$ between posets is a fibration or weak equivalence if Uf is a fibration or weak equivalence; that is, f is a fibration or weak equivalence if it is so when considered as a functor.

As noted by Thomason [20, Proposition 2.4], F is a weak equivalence if and only if NF is a weak equivalence.

Notation 2.2. Let $\mathcal{I}$ denote the set of generating cofibrations $\partial\Delta[n] \longrightarrow \Delta[n]$ and let $\mathcal{J}$ denote the set of generating acyclic cofibrations $\Lambda^k[n] \longrightarrow \Delta[n]$ for the standard model structure on $\mathbf{sSet}$.

[4]Following [20], the functor Π is generally denoted c, or sometimes cat, in the literature.

Theorem 2.3 (Thomason)**.** *With these fibrations and weak equivalences,* $\mathbf{Cat}$ *is a compactly generated proper model category whose sets of generating cofibrations and generating acyclic cofibrations are* $\Pi\,\mathrm{Sd}^2\,\mathcal{I}$ *and* $\Pi\,\mathrm{Sd}^2\,\mathcal{J}$*. Via the adjunction* $(\Pi\,\mathrm{Sd}^2, \mathrm{Ex}^2\,N)$*, this model structure is Quillen equivalent to the standard model structure on* $\mathbf{sSet}$*.*

Remark 2.4. In contrast to more recent papers, which use but do not always need transfinite colimits, Thomason's paper preceded the formal introduction of cofibrantly generated model categories, and he neither used nor needed such colimits; our statement is a reformulation of what he actually proved.

Theorem 2.5 (Raptis)**.** *With these fibrations and weak equivalences,* $\mathbf{Pos}$ *is a compactly generated proper model category whose sets of generating cofibrations and generating acyclic cofibrations are* $P\Pi\,\mathrm{Sd}^2\,\mathcal{I}$ *and* $P\Pi\,\mathrm{Sd}^2\,\mathcal{J}$*. Via the adjunction* (P, U)*, this model structure is Quillen equivalent to the Thomason model structure on* $\mathbf{Cat}$*.*

3. The proofs of Theorems 2.3 and 2.5

The proofs of the model axioms in [20, 16] can be streamlined by use of a slight variant of Kan's transport theorem [10, Theorem 11.3.2]. It is proven in [13, 16.2.5].

Theorem 3.1 (Kan)**.** *Let* $\mathcal{C}$ *be a compactly generated model category with generating cofibrations* $\mathcal{I}$ *and generating acyclic cofibrations* $\mathcal{J}$*. Let* $\mathcal{D}$ *be a bicomplete category, and let* $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : U$ *be a pair of adjoint functors. Assume that*

(i) all objects in the sets $F\mathcal{I}$ *and* $F\mathcal{J}$ *are compact and*

(ii) the functor U *takes relative* $F\mathcal{J}$*-cell complexes to weak equivalences.*

Then there is a compactly generated model structure on $\mathcal{D}$ *such that* $F\mathcal{I}$ *is the set of generating cofibrations,* $F\mathcal{J}$ *is the set of generating acyclic cofibrations, and the weak equivalences and fibrations are the morphisms* f *such that* Uf *is a weak equivalence or fibration. Moreover,* $(F \dashv U)$ *is a Quillen pair.*

Remark 3.2. It is clear that if $\mathcal{C}$ is right proper then so is $\mathcal{D}$. Since the standard model structure on **sSet** is right proper, so are the model structures on **Cat** and **Pos** described below. It is less clear that they are left proper, as we shall discuss.

Compactly generated makes sense when the generating sets are compact in the sense of [13, 15.1.6], as we require in condition (i). In Theorem 2.5, the domain posets of all maps in $P\Pi\,\mathrm{Sd}^2\,\mathcal{I}$ and $P\Pi\,\mathrm{Sd}^2\,\mathcal{J}$ are finite since they are obtained from simplicial sets with only finitely many 0-simplices. Therefore they are compact relative to all of **Cat** and in particular are compact relative to $P\Pi\,\mathrm{Sd}^2\,\mathcal{I}$ and $P\Pi\,\mathrm{Sd}^2\,\mathcal{J}$. This shows that (i) holds, and we need only prove (ii) to complete the proof of the model axioms in Theorem 2.5.

Since we are working with compact generation, a relative $P\Pi\,\mathrm{Sd}^2\,\mathcal{J}$-complex $i\colon A \longrightarrow X = \operatorname{colim} X_n$ is the colimit of a sequence of maps of posets $X_n \longrightarrow X_{n+1}$, where $X_0 = A$ and X_{n+1} is a pushout

$$\begin{array}{ccc} K_n & \xrightarrow{f} & X_n \\ {\scriptstyle j}\downarrow & & \downarrow \\ L_n & \longrightarrow & X_{n+1} \end{array} \tag{1}$$

in **Pos** in which j is a coproduct of maps in $P\Pi\,\mathrm{Sd}^2\,\mathcal{J}$. We must prove that such a map i, or rather Ui, is a weak equivalence in **Cat**. The only subtlety in the proof of Theorem 2.5 is that pushouts in **Cat** between maps in **Pos** are generally not posets. Rather, pushouts in **Pos** are constructed by taking pushouts in **Cat** and then applying the left adjoint P. However, results already in [20] show that we do not encounter that problem when constructing relative $P\Pi\,\mathrm{Sd}^2\,\mathcal{J}$-complexes, as we now explain.

To deal with pushouts when proving Theorem 2.3, Thomason introduced the notion of a Dwyer map.

Definition 3.3. Let $\mathcal{S}$ be a subcategory of a category $\mathcal{C}$. Then $\mathcal{S}$ is called a *sieve in* $\mathcal{C}$ if for every morphism $f\colon c \longrightarrow s$ in $\mathcal{C}$ with $s \in \mathcal{S}$, c and f are in $\mathcal{S}$. Dually, $\mathcal{S}$ is a *cosieve* if for every morphism $f\colon s \longrightarrow c$ in $\mathcal{C}$ with $s \in \mathcal{S}$, c and f are in $\mathcal{S}$. In either case, $\mathcal{S}$ must be a full subcategory of $\mathcal{C}$. Observe that if a sieve factors as a composite of inclusions $\mathcal{S} \longrightarrow \mathcal{T} \longrightarrow \mathcal{C}$, then $\mathcal{S} \longrightarrow \mathcal{T}$ is again a sieve.

Definition 3.4. A functor $k\colon \mathcal{S} \longrightarrow \mathcal{C}$ in $\mathbf{Cat}$ or in $\mathbf{Pos}$ is a Dwyer map if k is the inclusion of a sieve and k factors as a composite

$$\mathcal{S} \xrightarrow{i} \mathcal{T} \xrightarrow{j} \mathcal{C},$$

where j is the inclusion of a cosieve and i is an inclusion with a right adjoint $r\colon \mathcal{T} \longrightarrow \mathcal{S}$ such that the unit $\mathrm{id} \longrightarrow r \circ i$ of the adjunction is the identity.

The following sequence of results shows that Theorem 2.5 is directly implied by details in Thomason's paper [20] that he used to prove Theorem 2.3. Except that we add in the trivial statement about coproducts, the first is [20, Lemma 5.6].

Lemma 3.5. *The following statements about posets hold.*

(i) For any simplicial set K, $\Pi \operatorname{Sd}^2 K$ is a poset.

(ii) Any subcategory of a poset is a poset.

(iii) Any coproduct of posets in $\mathbf{Cat}$ is a poset.

(iv) If $j\colon K \longrightarrow L$ is a Dwyer map between posets and $f\colon K \longrightarrow X$ is a map of posets, then the pushout Y in $\mathbf{Cat}$ of j and f is a poset.

(v) The (directed) colimit in $\mathbf{Cat}$ of any sequence of maps of posets is a poset.

The second is [20, Proposition 4.2].

Lemma 3.6. *Let $K \subset L$ be an inclusion of simplicial sets that arises from an inclusion of ordered simplicial complexes. Then the induced map $\Pi \operatorname{Sd}^2 K \longrightarrow \Pi \operatorname{Sd}^2 L$ is a Dwyer map in $\mathbf{Cat}$ and thus, by Lemma 3.5(i), in $\mathbf{Pos}$.*

For completeness, we state an analogue to Lemma 3.5 about Dwyer maps in $\mathbf{Cat}$. It combines part of [20, Proposition 4.3] with the correct parts of [20, Lemma 5.3]. We again add in a trivial statement about coproducts.

Lemma 3.7. *The following statements about Dwyer maps in $\mathbf{Cat}$ hold.*

(i) Any composite of Dwyer maps is a Dwyer map.

(ii) *Any coproduct of Dwyer maps is a Dwyer map.*

(iii) *If $j\colon \mathcal{K} \longrightarrow \mathcal{L}$ is a Dwyer map and $f\colon \mathcal{K} \longrightarrow \mathcal{C}$ is a functor, then the pushout $k\colon \mathcal{C} \longrightarrow \mathcal{D}$ of j along f is a Dwyer map.*

(iv) *For Dwyer maps $\mathcal{C}_n \longrightarrow \mathcal{C}_{n+1}$, $n \geq 0$, the induced map $\mathcal{C}_0 \longrightarrow \operatorname{colim} \mathcal{C}_n$ is a Dwyer map.*

Therefore the same statements hold for Dwyer maps in $\mathbf{Pos}$.

Corollary 3.8. *If A is a poset and $i\colon A \longrightarrow X$ is a relative $\Pi \operatorname{Sd}^2 \mathcal{J}$-complex in $\mathbf{Cat}$, then X is a poset and i is both a Dwyer map and a relative $\Pi \operatorname{Sd}^2 \mathcal{J}$-complex in $\mathbf{Pos}$. The same statement holds for relative $\Pi \operatorname{Sd}^2 \mathcal{I}$-complexes.*

Remark 3.9. Once the model structures on $\mathbf{Pos}$ and $\mathbf{Cat}$ are in place, the results above imply that a map f between posets is a cofibration in $\mathbf{Pos}$ if and only if f is a cofibration in $\mathbf{Cat}$.

The real force of the introduction of Dwyer maps comes from the following result. It combines Thomason's [20, Proposition 4.3 and Corollary 4.4].

Proposition 3.10. *If $j\colon \mathcal{K} \longrightarrow \mathcal{L}$ is a Dwyer map in $\mathbf{Cat}$, $f\colon \mathcal{K} \longrightarrow \mathcal{C}$ is a functor, and $\mathcal{D}$ is their pushout, then the canonical map*

$$N\mathcal{L} \cup_{N\mathcal{K}} N\mathcal{C} \longrightarrow N(\mathcal{L} \cup_{\mathcal{K}} \mathcal{C}) = N\mathcal{D}$$

is a weak equivalence. The same statement holds in $\mathbf{Pos}$. Therefore, if f is a weak equivalence, then so is the pushout $g\colon \mathcal{L} \longrightarrow \mathcal{D}$ of f along j.

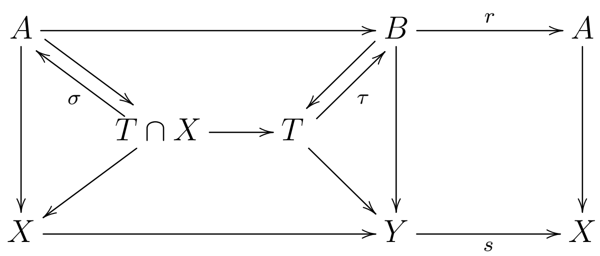
The last statement is inherited from the corresponding statement in $\mathbf{sSet}$.

Remark 3.11. The incorrect part of [20, Lemma 5.3] states that a retract of a Dwyer map is a Dwyer map. As noticed by Cisinski [5], that is not true. He gave an example to show that a retract of a cofibration in $\mathbf{Cat}$ need not be a Dwyer map, which invalidates the proof that $\mathbf{Cat}$ is left proper given in [20, Corollary 5.5]. He introduced the slightly more general notion of a pseudo Dwyer map to get around this. He proved that a retract of a pseudo Dwyer map is a pseudo Dwyer map, so that any cofibration in $\mathbf{Cat}$ is a pseudo Dwyer map. He then used that to give a correct proof that $\mathbf{Cat}$ is left proper, and he observed that our Lemmas 3.6 and 3.7 remain true with Dwyer maps replaced by pseudo Dwyer maps.

The problem discussed in the remark does not arise when dealing with $\mathbf{Pos}$, where Dwyer maps and pseudo Dwyer maps coincide, as follows directly from the definition of the latter. Since we are omitting that definition, we give a simple direct proof of the following result. Once the model structure is in place, it gives that cofibrations in $\mathbf{Pos}$ are Dwyer maps. This highlights the technical convenience of posets, as compared with general categories.

Lemma 3.12. *A retract of a Dwyer map in* $\mathbf{Pos}$ *is a Dwyer map. Therefore retracts in* $\mathbf{Pos}$ *of relative* $\Pi\,\mathrm{Sd}^2\,\mathcal{I}$*-complexes are Dwyer maps.*

Proof. Consider the following diagram of posets, which commutes with σ and τ omitted. All unlabeled arrows are inclusions.



We assume that r restricts to the identity on A and s restricts to the identity on X. We also assume that $B \longrightarrow Y$ is a sieve, $T \longrightarrow Y$ is a cosieve, and τ is right adjoint to the inclusion $B \longrightarrow T$ with unit the identity, so that τ restricts to the identity on B. We define σ to be the restriction of $r \circ \tau$ to $T \cap X$. The following observations prove that $A \longrightarrow X$ is a Dwyer map.
(i) The restriction $T \cap X \longrightarrow X$ of the cosieve $T \longrightarrow Y$ is again a cosieve.
Proof. If $w \in T \cap X$ and $w \leq x$ in X, then $x \in T$, hence $x \in T \cap X$.
(ii) The restriction $A \longrightarrow X$ of the sieve $B \longrightarrow Y$ is again a sieve.
Proof. If $a \in A$, $x \in X$, and $x \leq a$, then $x \in B$ since $B \longrightarrow Y$ is a sieve, and then $x = s(x) = r(x) \leq r(a) = a$ in A.
(iii) σ is right adjoint to the inclusion $A \longrightarrow T \cap X$, with unit the identity map.
Proof. σ restricts to the identity on A since if $a \in A$, then

$$\sigma(a) = (r \circ \tau)(a) = r(a) = a.$$

For the adjunction, we must show that if $a \in A$ and $x \in T \cap X$, then $a \leq x$ if and only if $a \leq \sigma(x)$. If $a \leq x$, then $a = \sigma(a) \leq \sigma(x)$. Suppose $a \leq \sigma(x)$

and note that $\sigma(x) = (r \circ \tau)(x) = (s \circ \tau)(x)$. Since τ is right adjoint to $B \longrightarrow T$, the counit of the adjunction gives that $\tau(y) \leq y$ for any $y \in T$. Thus $(s \circ \tau)(x) \leq s(x) = x$. □

Proof of Theorems 2.3 and 2.5. The key idea of Thomason's proof of Theorem 2.3 is the verification of condition (ii) of Theorem 3.1. Since coproducts and colimits of weak equivalences are weak equivalences, this reduces to showing that the pushouts in the construction of relative $\mathcal{J}$-complexes are weak equivalences. But that is immediate from Proposition 3.10. Since a relative $P\Pi\,\mathrm{Sd}^2\,\mathcal{J}$-complex in $\mathbf{Pos}$ is a special case of a relative $\Pi\,\mathrm{Sd}^2\,\mathcal{J}$-complex in $\mathbf{Cat}$, condition (ii) of Theorem 3.1 holds in $\mathbf{Pos}$ since it is a special case of the condition in $\mathbf{Cat}$. This proves that $\mathbf{Cat}$ and $\mathbf{Pos}$ are compactly generated model categories. In view of Lemma 3.12, Proposition 3.10 also implies that $\mathbf{Pos}$ is left proper and therefore proper. As pointed out in Remark 3.11, Cisinski [5] proves that $\mathbf{Cat}$ is left proper and therefore proper.

It remains to show that the adjunctions $(\Pi\,\mathrm{Sd}^2, \mathrm{Ex}^2\,N)$ and (P, U) are Quillen equivalences. To show that $(\Pi\,\mathrm{Sd}^2, \mathrm{Ex}^2\,N)$ is a Quillen equivalence, it suffices to show that the composite $\mathrm{Ex}^2\,N$ induces an equivalence between the homotopy categories of $\mathbf{Cat}$ and $\mathbf{sSet}$. Quillen [11, Ch. VI, Corollaire 3.3.1] proved that the nerve N induces an equivalence. Kan [9, Ch. III, Theorem 4.6] proved that Ex and therefore Ex^2 induces an equivalence by showing that there is a natural weak equivalence $K \longrightarrow \mathrm{Ex}\,K$ for simplicial sets K.

To show that (P, U) is a Quillen equivalence, it suffices to show that for all cofibrant categories $\mathcal{C} \in \mathbf{Cat}$ and all fibrant posets $X \in \mathbf{Pos}$, a functor $f\colon \mathcal{C} \longrightarrow UX$ is a weak equivalence if and only if its adjunct $\widetilde{f}\colon P\mathcal{C} \longrightarrow X$ is a weak equivalence. Since $\mathcal{C}$ is cofibrant, it is a poset, hence $\mathcal{C} = UY$ for a poset Y. But then $U\widetilde{f} = f$ and the conclusion holds by the definition of weak equivalences in $\mathbf{Pos}$. □

Remark 3.13. The fact that $\Pi\,\mathrm{Sd}^2\,K$ is a poset for any simplicial set K is closely related to the less well-known fact that $\mathrm{Sd}^2\,\mathcal{C}$ is a poset for any category $\mathcal{C}$. However, the subdivision functor on $\mathbf{Cat}$ plays no role in Thomason's work or ours. The relation between these subdivision functors is studied in [7] and [12].

4. Equivariant Dwyer maps and cofibrations

To mimic the arguments just given equivariantly, we introduce equivariant Dwyer maps and relate them to cofibrations in $\mathbf{Pos}$.

Definition 4.1. A functor $k\colon \mathcal{S} \longrightarrow \mathcal{C}$ in $G\mathbf{Cat}$ or in $G\mathbf{Pos}$ is a Dwyer G-map if k is the inclusion of a sieve and k factors in $G\mathbf{Cat}$ as a composite

$$\mathcal{S} \xrightarrow{i} \mathcal{T} \xrightarrow{j} \mathcal{C},$$

where j is the inclusion of a cosieve and i is an inclusion with a right adjoint $r\colon \mathcal{T} \longrightarrow \mathcal{S}$ in $G\mathbf{Cat}$ such that the unit $\mathrm{id} \longrightarrow r \circ i$ of the adjunction is the identity.[5]

The following two lemmas are immediate from the definition.

Lemma 4.2. *If k is a Dwyer G-map, then k^H is a Dwyer map for any subgroup H of G.*

Regard the G-set G/H as a discrete G-category (identity morphisms only).

Lemma 4.3. *If $j\colon K \subset L$ is a Dwyer map and H is a subgroup of G, then $\mathrm{id} \times j\colon G/H \times K \longrightarrow G/H \times L$ is a Dwyer G-map.*

We have the equivariant analogues of Lemma 3.7 and Corollary 3.8, with the same proofs.

Lemma 4.4. *The following statements about Dwyer G-maps in $G\mathbf{Cat}$ hold.*

(i) Any composite of Dwyer G-maps is a Dwyer G-map.

(ii) Any coproduct of Dwyer G-maps is a Dwyer G-map.

(iii) If $j\colon \mathcal{K} \longrightarrow \mathcal{L}$ is a Dwyer G-map and $f\colon \mathcal{K} \longrightarrow \mathcal{C}$ is a G-map, then the pushout $k\colon \mathcal{C} \longrightarrow \mathcal{D}$ of j along f is a Dwyer G-map.

(iv) For Dwyer G-maps $\mathcal{C}_n \longrightarrow \mathcal{C}_{n+1}$, $n \geq 0$, the induced map $\mathcal{C}_0 \longrightarrow \operatorname{colim} \mathcal{C}_n$ is a Dwyer G-map.

[5] Since the unit is the identity, the pair (i, r) is automatically an adjunction in the 2-category of G-objects in $\mathbf{Cat}$, equivariant functors, and equivariant natural transformations.

Therefore the same statements hold for Dwyer G-maps in $G\mathbf{Pos}$.

Let $G\amalg\operatorname{Sd}^2\mathfrak{I}$ and $G\amalg\operatorname{Sd}^2\mathfrak{J}$ denote the sets of all G-maps that are of the form $\operatorname{id}\times j\colon G/H\times K\longrightarrow G/H\times L$, where j is in $\amalg\operatorname{Sd}^2\mathfrak{I}$ or $\amalg\operatorname{Sd}^2\mathfrak{J}$. These are the generating cofibrations and generating acyclic cofibrations in $G\mathbf{Cat}$.

Corollary 4.5. *If A is a G-poset and $i\colon A\longrightarrow X$ is a relative $G\amalg\operatorname{Sd}^2\mathfrak{J}$-complex in $G\mathbf{Cat}$, then X is a G-poset and i is both a Dwyer G-map and a relative $G\amalg\operatorname{Sd}^2\mathfrak{J}$-complex in $G\mathbf{Pos}$. The same statement holds for relative $G\amalg\operatorname{Sd}^2\mathfrak{I}$-complexes.*

We also have the equivariant analogue of Lemma 3.12.

Lemma 4.6. *A retract of a Dwyer G-map in $G\mathbf{Pos}$ is a Dwyer G-map. Therefore all cofibrations in $G\mathbf{Pos}$ are Dwyer G-maps.*

We require a description of pushouts inside $G\mathbf{Pos}$. The following is a simplification of [3, Lemma 2.5].

Lemma 4.7. *Let $j\colon K\longrightarrow L$ be a sieve of G-posets and $f\colon K\longrightarrow X$ be a map of G-posets. Consider the set $Y=(L\setminus K)\amalg X$ with the order relation given by restriction on $L\setminus K$ and on X, with the additional relation that for $x\in X$ and $y\in L\setminus K$, $x\leq y$ if there exists $w\in K$ such that $x\leq f(w)$ and $j(w)\leq y$. Then Y is a G-poset and the following diagram is a pushout in $G\mathbf{Pos}$, where k is the inclusion of the summand X and g is the sum of f on K and the identity on $L\setminus K$.*

$$\begin{array}{ccc} K & \xrightarrow{f} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ L & \xrightarrow[g]{} & Y \end{array} \tag{2}$$

Moreover, if j is a Dwyer map with factorization $K\xrightarrow{\iota}S\xrightarrow{\nu}L$ and retraction $r\colon S\longrightarrow K$, then for $x\in X$ and $y\in L\setminus K$, $x\leq y$ if and only if $y=\nu(z)$ for some $z\in S$ such that $x\leq(f\circ r)(z)$.

Proof. First, note that Y is well-defined, since $L\setminus K$ is a G-subposet of L. Indeed, if $y\in L\setminus K$ and $gy\in K$ then $y=g^{-1}gy\in K$, a contradiction. The relation $\leq$ on Y is reflexive and anti-symmetric since L and

X are posets. Transitivity requires a straightforward verification in the two non-trivial cases when $x \le y$ and $y \le z$ with either $x, y \in X$ and $z \in L \setminus K$ or $x \in X$ and $y, z \in L$. Thus Y is a poset.

Clearly the map k is order-preserving. Using that j is a sieve, we see that g is order-preserving by the definition of the order on Y. The square (2) is clearly a pushout of sets. Thus to show that it is a pushout of posets it suffices to show that for any commutative square

$$\begin{array}{ccc} K & \xrightarrow{f} & X \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle \ell} \\ L & \xrightarrow[h]{} & Z \end{array}$$

of posets, the induced map $Y \longrightarrow Z$ is order-preserving. The only case that is non-trivial to check is when $x \le y$ with $x \in X$ and $y \in L \setminus K$. We must show that $\ell(x) \le h(y)$. By assumption, there is an element $w \in K$ such that $x \le f(w)$ and $j(w) \le y$. It follows that

$$\ell(x) \le (\ell \circ f)(w) = (h \circ j)(w) \le h(y),$$

as desired.

For the last statement of the lemma, if $y = \nu(z)$ where $z \in S$ and $x \le (f \circ r)(z)$, let $w = r(z)$. Then $x \le f(w)$ and $j(w) = (\nu \circ \iota \circ r)(z) \le \nu(z) = y$ by the counit of the adjunction (ι, r). Conversely, let $j(w) \le y$ and $x \le f(w)$. Since ν is a cosieve, $j(w) = (\nu \circ \iota)(w) \le y$ implies $y = \nu(z)$ for some $z \in S$ with $\iota(w) \le z$, and then $w = (r \circ \iota)(w) \le r(z)$ so that $x \le f(w)$ implies $x \le (f \circ r \circ \iota)(w) \le (f \circ r)(z)$. $\square$

Using this description we can show that pushouts along Dwyer G-maps are preserved when taking H-fixed points for any subgroup H of G. The statement about fixed points is a modification of [3, Proposition 2.4].

Lemma 4.8. *Let $j\colon K \longrightarrow L$ be a Dwyer G-map of G-posets, such as a retract of a relative $G\Pi\,\mathrm{Sd}^2\,\mathfrak{I}$-cell complex, and let $f\colon K \longrightarrow X$ be any map of G-posets. Form the pushout diagram*

$$\begin{array}{ccc} K & \xrightarrow{f} & X \\ {\scriptstyle j}\big\downarrow & & \big\downarrow \\ L & \longrightarrow & Y \end{array}$$

in $G\mathbf{Cat}$. Then Y is a G-poset and the diagram remains a pushout after taking H-fixed points for any subgroup H of G.

Proof. Ignoring the G-action, the left vertical arrow is a Dwyer map of posets. Therefore Y is a poset by Lemma 3.5(iv) and is thus a G-poset. Fix a subgroup H of G; by Lemma 4.2 j^H is a Dwyer map, and thus the description from Lemma 4.7 can be used for $X^H \cup_{K^H} L^H$. □

Example 4.9. Let G be the cyclic group of order two. Let L be the three object G-poset depicted by $0 \longrightarrow 2 \longleftarrow 1$ equipped with the action that interchanges 0 and 1, but fixes 2. Let K be the G-subposet that consists of the elements 0 and 1. Then the inclusion $K \longrightarrow L$ is a sieve but *not* a Dwyer G-map. If $X = *$ is the terminal G-poset and $K \longrightarrow X$ is the unique map, then the pushout $L \cup_K X$ in $G\mathbf{Pos}$ is the G-poset depicted by $* \longrightarrow 2$, with trivial G-action. Thus its G-fixed point poset is also $* \longrightarrow 2$. However, the pushout $L^G \amalg_{K^G} X^G$ is the discrete poset with two elements $*$ and 2.

5. The proof of Theorem 1.1

For our equivariant model structures, we start with the following general result, which puts together results of the second author [18, Proposition 2.6, Theorem 2.10] with augmentations of those results due to Bohmann, et al [3, Propositions 1.4, 1.5, and 1.6], all reformulated in our simpler compactly generated setting. Recall that $\mathcal{O}_G$ denotes the orbit category of G.

Definition 5.1. For a category $\mathcal{C}$, let $G\mathcal{C}$ denote the category of G-objects in $\mathcal{C}$ and let $\mathcal{O}_G$-$\mathcal{C}$ denote the category of contravariant functors $\mathcal{O}_G \longrightarrow \mathcal{C}$. Assuming that $\mathcal{C}$ has coproducts, define a functor

$$\otimes\colon G\mathbf{Set} \times \mathcal{C} \longrightarrow G\mathcal{C}$$

by $S \otimes X = \amalg_S X$, the coproduct of copies of X indexed by elements of S, with G-action induced from the action of G on S by permutation of the copies of X.

We have an adjunction (Λ, Φ) between $G\mathcal{C}$ and $\mathcal{O}_G$-$\mathcal{C}$. The left adjoint Λ sends a functor $\mathcal{O}_G \longrightarrow \mathcal{C}$ to its value on G/e and the right adjoint Φ sends a G-object to its fixed point functor.

Theorem 5.2. *Let $\mathcal{C}$ be a compactly generated model category. Assume that for each subgroup H of G, the H-fixed point functor $(-)^H\colon G\mathcal{C} \longrightarrow \mathcal{C}$ satisfies the following properties.*

(i) It preserves colimits of sequences of maps $i_n\colon X_n \longrightarrow X_{n+1}$ in $G\mathcal{C}$, where each i_n is a cofibration in $\mathcal{C}$.

(ii) It preserves coproducts.

(iii) It preserves pushouts of diagrams in which one leg is given by a coproduct of maps of the form

$$\mathrm{id} \otimes j\colon G/J \otimes X \longrightarrow G/J \otimes Y,$$

where j is a generating cofibration (or generating acyclic cofibration) of $\mathcal{C}$ and J is a subgroup of G.[6]

(iv) For any object X of $\mathcal{C}$, the natural map

$$(G/J)^H \otimes X \longrightarrow (G/J \otimes X)^H$$

is an isomorphism in $\mathcal{C}$.

Then $G\mathcal{C}$ admits a compactly generated model structure, where a map f in $G\mathcal{C}$ is a fibration or weak equivalence if each fixed point map f^H is a fibration or weak equivalence, so that $\Phi(f)$ is a fibration or weak equivalence in $\mathcal{O}_G$-$\mathcal{C}$. The generating (acyclic) cofibrations are the G-maps $\mathrm{id} \otimes j\colon G/J \otimes K \longrightarrow G/J \otimes L$, where the maps $j\colon K \longrightarrow L$ are the generating (acyclic) cofibrations of $\mathcal{C}$. Moreover, (Λ, Φ) is then a Quillen equivalence between $G\mathcal{C}$ and $\mathcal{O}_G$-$\mathcal{C}$. Further, if $\mathcal{C}$ is left or right proper, then so is $G\mathcal{C}$.

By [3, 1.3], the model structure is functorial with respect to Quillen pairs.

Theorem 5.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be compactly generated model categories satisfying the assumptions of Theorem 5.2 and let (L, R) be a Quillen pair between them. Then there is an induced Quillen pair between $G\mathcal{C}$ and $G\mathcal{D}$, and it is a Quillen equivalence if (L, R) is a Quillen equivalence.*

[6] We don't need to assume the condition for acyclic cofibrations, but we do so for convenience.

Proof of Theorem 1.1. It suffices to verify conditions (i)-(iv) of Theorem 5.2 when $\mathcal{C} = \mathbf{Pos}$. Cofibrations in $\mathbf{Pos}$ are inclusions and if $x \in X = \operatorname{colim} X_n$, then $x \in X^H$ if and only if $x \in X_n^H$ for a large enough n; thus condition (i) holds. Condition (ii) holds by the definition of coproducts in $\mathbf{Cat}$. Since the action of G on $G/J \otimes X$ comes from the action of G on G/J, condition (iv) holds as well.

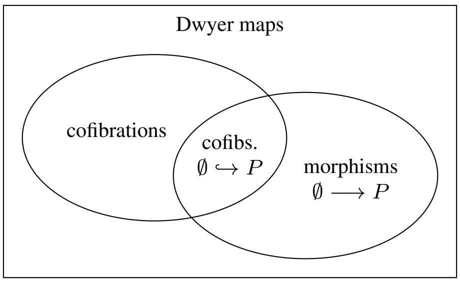
It remains to check condition (iii). By Lemma 3.6, the generating cofibrations in $\mathbf{Pos}$ are Dwyer maps. Consider a pushout diagram in $G\mathbf{Cat}$

$$\begin{array}{ccc} \coprod_{i\in I} G/J_i \otimes K_i & \xrightarrow{\amalg f_i} & X \\ {\scriptstyle \amalg \mathrm{id} \otimes j_i}\downarrow & & \downarrow \\ \coprod_{i\in I} G/J_i \otimes L_i & \longrightarrow & Y \end{array}$$

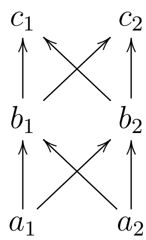
where each $j_i : K_i \longrightarrow L_i$ is a Dwyer map and $f_i : G/J_i \otimes K_i \longrightarrow X$ is a map of G-posets. Condition (iii) holds if, for any such diagram, Y is a G-poset (hence Y^H is also a poset) and the diagram remains a pushout after passage to H-fixed points. This is a special case of Lemma 4.8. □

6. Cofibrant posets

Since every cofibrant object in $\mathbf{Cat}$ is a poset and, by Remark 3.9, a poset is cofibrant in $\mathbf{Pos}$ if and only if it is cofibrant in $\mathbf{Cat}$, it follows that $\mathbf{Pos}$ and $\mathbf{Cat}$ have the same cofibrant objects. We have an explicit cofibrant replacement functor for $\mathbf{Pos}$, namely double subdivision. While this does give a large class of cofibrant objects, it does not help to determine whether or not a given poset is cofibrant. By Lemma 3.12, any cofibration in $\mathbf{Pos}$ is a Dwyer map and it follows immediately from the definition of Dwyer maps that the map $\emptyset \longrightarrow P$ is a Dwyer map for any poset P. Our understanding is summarized in the following picture:



It is not difficult to show that most of the sections in this Venn diagram are nonempty; the only difficulty is to show that there exist morphisms $\emptyset \longrightarrow P$ which are not cofibrations. As the referee pointed out to us, it is not hard to find infinite posets that are not cofibrant, such as the natural numbers with its reverse ordering. However, as far as we know ours is the first example of a finite poset that is not cofibrant. Specifically, in Proposition 6.2 we show that the following model of the 2-sphere, which is a finite poset A whose classifying space is homeomorphic to S^2, is not cofibrant in $\mathbf{Pos}$.



This example of a finite, non-cofibrant poset is minimal in dimension and in cardinality. We prove in Proposition 6.4 that every one-dimensional finite poset is cofibrant, and Bruckner and Pegel [4] have shown that every poset with at most five elements is cofibrant.

We first give a tool for showing that posets are not cofibrant.

Lemma 6.1. *Let A be a nonempty finite poset. Suppose that A satisfies the following condition: for any pushout square*

$$\begin{array}{ccc} \Pi\,\mathrm{Sd}^2\,\partial\Delta[n] & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Pi\,\mathrm{Sd}^2\,\Delta[n] & \longrightarrow & Y \end{array}$$

if A is a retract of Y then it is also a retract of X. Then A is not cofibrant in $\mathbf{Pos}$.

Proof. Assume that A is cofibrant. We prove that A must be empty, a contradiction. Since $\mathbf{Pos}$ is compactly generated and A is cofibrant, A is a retract of a sequential colimit $\operatorname{colim}_n X_n$, where $X_0 = \emptyset$ and $X_i \longrightarrow X_{i+1}$ is a pushout of a coproduct of generating cofibrations for $i \geq 0$. Since A is finite, the inclusion $A \longrightarrow \operatorname{colim}_n X_n$ factors through some X_n, and then A is a retract of X_n. Assume $n > 0$. Since A is finite, the inclusion $A \longrightarrow X_n$ factors through a pushout Y_n obtained by attaching only finitely many generating cofibrations to X_{n-1}, and then A is a retract of Y_n. We can now use the assumed condition on A to induct downwards one generating cofibration at a time; our condition ensures that A is a retract of X_{n-1}. Iterating, we deduce that A is a retract of $X_0 = \emptyset$ and thus $A = \emptyset$. $\square$

We will also need the following explicit description of the generating cofibrations

$$\Pi \operatorname{Sd}^2 \partial\Delta[n] \longrightarrow \Pi \operatorname{Sd}^2 \Delta[n].$$

An element of the poset $\Pi \operatorname{Sd}^2 \Delta[n]$ is a sequence of strict inclusions

$$S_0 \subset \ldots \subset S_k$$

of nonempty subsets of $\mathbf{n} = \{0, \ldots, n\}$. We can identify such a sequence with the totally ordered set $\{S_0, \ldots, S_k\}$. With this identification the order relation on $\Pi \operatorname{Sd}^2 \Delta[n]$ is given by subset inclusion. The poset $\Pi \operatorname{Sd}^2 \partial\Delta[n]$ is the subposet of $\Pi \operatorname{Sd}^2 \Delta[n]$ given by the sequences $S_0 \subset \ldots \subset S_k$ with $S_k \neq \mathbf{n}$.

We are now ready to show that our model A of the 2-sphere is not cofibrant.

Proposition 6.2. *The finite poset A is not cofibrant in* $\mathbf{Pos}$.

Proof. We will show that A satisfies the condition in Lemma 6.1; since A is nonempty, this implies that A is not cofibrant.

Let Y be the pushout of a diagram of the form

$$\Pi \operatorname{Sd}^2 \Delta[n] \longleftarrow \Pi \operatorname{Sd}^2 \partial\Delta[n] \longrightarrow X,$$

where X is any poset. We use the explicit description of the pushout from Lemma 4.7. Suppose that A is a retract of Y, so that id_A admits a factorization $A \xrightarrow{i} Y \xrightarrow{r} A$.

Consider the map $(\Pi\,\mathrm{Sd}^2\,\Delta[n]) \setminus \{\mathbf{n}\} \longrightarrow \Pi\,\mathrm{Sd}^2\,\partial\Delta[n]$ defined by

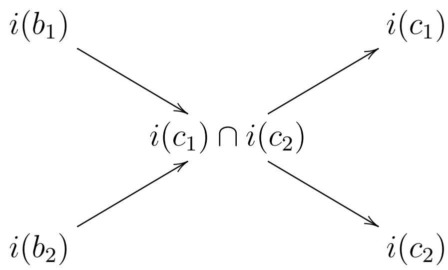
$$S_0 \subset \ldots \subset S_k \quad \longmapsto \begin{cases} S_0 \subset \ldots \subset S_{k-1} & \text{if } S_k = \mathbf{n}, \\ S_0 \subset \ldots \subset S_k & \text{otherwise.} \end{cases}$$

This induces a map $p\colon Y \setminus \{\mathbf{n}\} \longrightarrow X$. We show that $\mathbf{n} \notin i(A)$, and that the composite

$$A \xrightarrow{i} Y \setminus \{\mathbf{n}\} \xrightarrow{p} X \longrightarrow Y \xrightarrow{r} A \tag{3}$$

is the identity on A. From this we can conclude that A is a retract of X.

Since $\mathbf{n} \in Y$ is not a codomain of a non-identity arrow, the only elements of A that i could send to $\mathbf{n}$ are a_1 and a_2. We show more generally, that $i(a_1), i(a_2) \in X$. If $i(a_1) \in Y \setminus X$ or $i(a_2) \in Y \setminus X$, then $i(b_1), i(b_2), i(c_1), i(c_2) \in Y \setminus X$. Considering $i(c_1)$ and $i(c_2)$ as totally ordered sets of nonempty subsets of $\mathbf{n}$, the intersection $i(c_1) \cap i(c_2)$ is an element of $Y \setminus X$ and we have a diagram

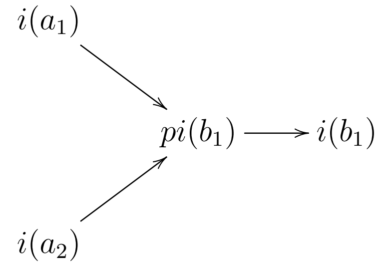


in Y. Applying the retraction $r\colon Y \longrightarrow A$ to this diagram yields an arrow between b_1 and b_2 or an arrow between c_1 and c_2. Both cases are impossible. We have shown that $i(a_1), i(a_2) \in X$ and thus that $\mathbf{n} \notin i(A)$.

We can also show by the same argument as above that $i(b_1)$ and $i(b_2)$ cannot both belong to $Y \setminus X$.

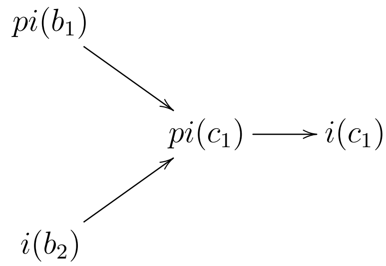
It remains to show that the composite (3) is the identity. Recall that $i(a_1)$, $i(a_2)$ and at least one of $i(b_1)$, $i(b_2)$ belong to X. By symmetry we can assume that $i(b_2) \in X$. We need to show that $rpi(b_1) = b_1$, $rpi(c_1) = c_1$ and $rpi(c_2) = c_2$. Implicitly, we will use that any arrow in Y from an element in

X to an element z in $Y \setminus X$ factors through $p(z)$. Since $i(a_1) \leq i(b_1)$ and $i(a_2) \leq i(b_1)$, we have a diagram



in Y. By applying r to this diagram, we deduce that $rpi(b_1) = b_1$ since there is no arrow between a_1 and a_2.

Applying r to the diagram



in Y, we deduce that $rpi(c_1) = c_1$. By symmetry, we also have $rpi(c_2) = c_2$. We have shown that A is a retract of X. □

Corollary 6.3. *Not all finite posets in Thomason's model structure on* **Cat** *are cofibrant.*

The above proof used many special properties of A and thus cannot be used in general to determine which objects are cofibrant. However, there is one class of posets that we can prove are cofibrant: the one-dimensional finite ones. We say that a poset P is (at most) one-dimensional if in any pair of composable morphisms at least one is an identity morphism.

Proposition 6.4. *Every one-dimensional finite poset X is cofibrant.*

Proof. We proceed by induction on the number m of elements of X. If $m = 0$, then $X = \emptyset$ and is thus cofibrant. Now suppose that $m \geq 1$. If X

has no non-identity morphisms (is zero-dimensional), then X can be built up by attaching singleton sets $\Pi\,\mathrm{Sd}^2\,\Delta[0]$ to $\emptyset$ and is thus cofibrant.

Otherwise, let a be the domain of a non-identity morphism. Set $A = X \setminus \{a\}$. By the induction hypothesis A is cofibrant. Let $Y = \{y_0, \ldots, y_n\}$ be the set of elements $y \in X$ such that there exists a non-identity morphism $a \longrightarrow y$ in X.

Let CY denote the cone on Y obtained by adding a least element $*$ to Y. Note that $Y \longrightarrow CY$ is an inclusion of a cosieve. Thus $X \cong A \cup_Y CY$ by a dual version of Lemma 4.7.

We distinguish the two cases $n = 0$ and $n > 0$. If $n = 0$, we glue $\Pi\,\mathrm{Sd}^2\,\Delta[1]$ to A along a cofibration in such a way that X is a retract of the resulting pushout, and therefore cofibrant. The inclusion of the vertex 0 into $\Delta[1]$ is a cofibration. Applying $\Pi\,\mathrm{Sd}^2$ to this cofibration yields the inclusion of the poset $\{\{0\}\}$ into $\Pi\,\mathrm{Sd}^2\,\Delta[1]$. Identifying the element $\{0\}$ with y_0, we show that X is a retract of the pushout $A \cup_Y \Pi\,\mathrm{Sd}^2\,\Delta[1]$. Let $X \longrightarrow A \cup_Y \Pi\,\mathrm{Sd}^2\,\Delta[1]$ be the map

$$x \mapsto \begin{cases} \{0\} \subset \mathbf{1} & \text{if } x = y_0 \\ \mathbf{1} & \text{if } x = a \\ x & \text{otherwise} \end{cases}$$

The map $\Pi\,\mathrm{Sd}^2\,\Delta[1] \longrightarrow CY$,

$$S_0 \subset \ldots \subset S_k \quad \mapsto \begin{cases} y_0 & \text{if } S_0 = \{0\} \\ * & \text{otherwise} \end{cases}$$

induces a retraction $A \cup_Y \Pi\,\mathrm{Sd}^2\,\Delta[1] \longrightarrow X$ of the map $X \longrightarrow A \cup_Y \Pi\,\mathrm{Sd}^2\,\Delta[1]$ above. Thus X is cofibrant if $n = 0$ and we now assume that $n > 0$.

Similarly to the case $n = 0$, we glue $\Pi\,\mathrm{Sd}^2\,\Delta[n]$ to A along a cofibration in such a way that X is a retract of the resulting pushout, and therefore cofibrant.

The inclusion of the set of vertices of $\Delta[n]$ into $\Delta[n]$ is a cofibration. Applying $\Pi\,\mathrm{Sd}^2$ to this cofibration yields the inclusion of the discrete poset $\{\{i\} \mid 0 \leq i \leq n\}$ into $\Pi\,\mathrm{Sd}^2\,\Delta[n]$. Identifying the element $\{i\}$ with y_i, let Z denote the pushout $A \cup_Y (\Pi\,\mathrm{Sd}^2\,\Delta[n])$. We claim that X is a retract of Z.

Indeed, let $j\colon X \longrightarrow Z$ be the map

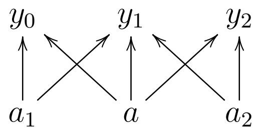
$$x \mapsto \begin{cases} \{i\} \subset \mathbf{n} & \text{if } x = y_i \\ \mathbf{n} & \text{if } x = a \\ x & \text{otherwise} \end{cases}$$

The map $\Pi \operatorname{Sd}^2 \Delta[n] \longrightarrow CY$,

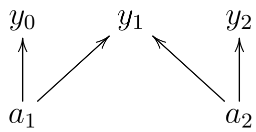
$$S_0 \subset \ldots \subset S_k \quad \mapsto \begin{cases} y_i & \text{if } S_0 = \{i\} \\ * & \text{otherwise} \end{cases}$$

induces a map $r\colon Z \longrightarrow X$ such that $rj = \mathrm{id}_X$ as desired. □

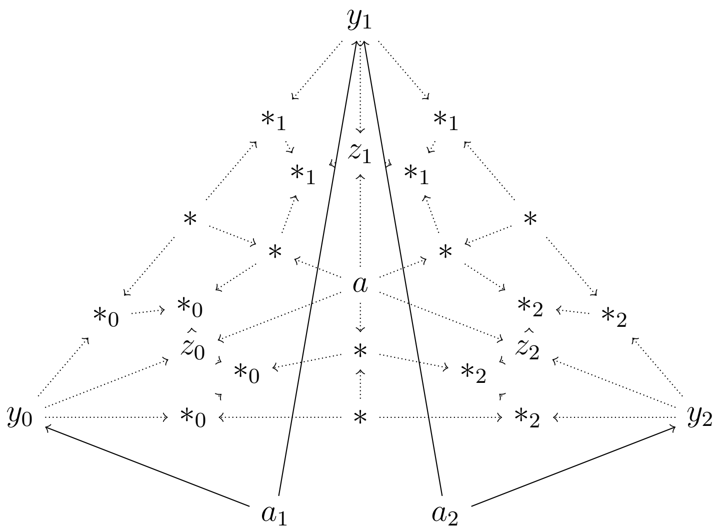
We illustrate the induction step of this proof using the following poset X:



After removing a we obtain the following poset A, which by induction hypothesis is cofibrant.



The poset Z in the proof above can be pictured as follows.



Here each vertex is a distinct object of Z (although we have not given the objects distinct names), and the edges give all of the non-identity morphisms of Z. The inclusion $j\colon X \longrightarrow Z$ maps a_i to a_i, y_k to z_k and a to a. The retraction r is defined by

$$r(z_k) = r(y_k) = r(*_k) = y_k \qquad r(a_i) = a_i \qquad r(a) = r(*) = a.$$

The essential point is that, even in such simple cases as in this section, proving that a poset is or is not cofibrant is a non-trivial exercise.

7. Fibrant posets

In this section we give a class of examples of fibrant posets. Before we begin we give several easy lemmas needed in the proofs. First, we show that when proving a category is fibrant it suffices to consider its connected components. Here, a category is connected if any two objects are connected by a finite zigzag of morphisms. A component of a category is a maximal connected full subcategory, and any category is the disjoint union of its components.

Lemma 7.1. *Let $\mathcal{C} \in$ **Cat** or **Pos**. Then $\mathcal{C}$ is fibrant if and only if all of its components are so.*

Proof. The image of a connected category under a functor lies in a single component. Since each $\Pi\,\mathrm{Sd}^2\,\Lambda^k[n]$ (resp. $\Pi\,\mathrm{Sd}^2\,\Delta[n]$) is connected, any functor $\Pi\,\mathrm{Sd}^2\,\Lambda^k[n] \longrightarrow \mathcal{C}$ lands in a single component. A category $\mathcal{C}$ is fibrant if and only if for every functor $f : \Pi\,\mathrm{Sd}^2\,\Lambda^k[n] \longrightarrow \mathcal{C}$, there exists a functor $h\colon \Pi\,\mathrm{Sd}^2\,\Delta[n] \longrightarrow \mathcal{C}$ such that the diagram

$$\begin{array}{ccc} \Pi\,\mathrm{Sd}^2\,\Lambda^k[n] & \xrightarrow{\quad f \quad} & \mathcal{C} \\ \big\downarrow & \nearrow_{h} & \\ \Pi\,\mathrm{Sd}^2\,\Delta[n] & & \end{array}$$

commutes, and this holds if and only if it holds with $\mathcal{C}$ replaced by each of its components. $\square$

Second, we record the following result relating pullbacks and pushouts to binary products $\times$ and binary coproducts $\cup$ inside a poset P. Its proof is an exercise using that there is at most one morphism between any two objects of P.

Lemma 7.2. *If the pullback of a given pair of maps $x \longrightarrow a \longleftarrow y$ exists, it is the product $x \times y$, and if the product $x \times y$ exists, it is the pullback of any pair of maps $x \longrightarrow a \longleftarrow y$. Dually, if the pushout of a given pair of maps $x \longleftarrow a \longrightarrow y$ exists, it is the coproduct $x \cup y$, and if the coproduct $x \cup y$ exists, it is the pushout of any pair of maps $x \longleftarrow a \longrightarrow y$.*

The following addendum implies that a poset with binary products or coproducts is contractible, meaning that its classifying space is contractible.

Lemma 7.3. *If P is a poset containing an object c such that either $c \times x$ exists for any $x \in P$ or $c \cup x$ exists for any $x \in P$, then P is contractible.*

Proof. We prove the lemma in the first case; the second case follows by duality. Let P/c be the poset of all elements x over c; this means that $x \leq c$, or, thinking of P and P/c as categories, that there is a morphism $x \longrightarrow c$; it is contractible since it has the terminal object $c \longrightarrow c$. Since P is a poset, there is at most one morphism $x \longrightarrow c$ for any object x and the functor $P \longrightarrow P/c$ that sends an object y to $c \times y \longrightarrow c$ is right adjoint to the forgetful functor that sends $x \longrightarrow c$ to x. Therefore the classifying space of P is homotopy equivalent to that of P/c. $\square$

In [14], Meier and Ozornova construct examples of fibrant categories. They start from the notion of a partial model category, which is a weakening of the notion of a model category. Recall that a homotopical category $(\mathcal{C}, \mathcal{W})$ is a category $\mathcal{C}$ together with a subcategory $\mathcal{W}$, whose maps we call weak equivalences, such that every object of $\mathcal{C}$ is in $\mathcal{W}$ and $\mathcal{W}$ satisfies the 2 out of 6 property: if morphisms $h \circ g$ and $g \circ f$ are in $\mathcal{W}$, then so are f, g, h, and $h \circ g \circ f$.

Definition 7.4 ([2, §1.1])**.** A *partial model category* is a homotopical category $(\mathcal{C}, \mathcal{W})$ such that $\mathcal{W}$ contains subcategories $\mathcal{U}$ and $\mathcal{V}$ that satisfy the following properties.

(i) $\mathcal{U}$ is closed under pushouts along morphisms in $\mathcal{C}$ and $\mathcal{V}$ is closed under pullbacks along morphisms in $\mathcal{C}$.

(ii) The morphisms of $\mathcal{W}$ admit a functorial factorization into a morphism in $\mathcal{U}$ followed by a morphism in $\mathcal{V}$.

In (i), it is implicitly required that the cited pushouts and pullbacks exist in $\mathcal{C}$. For example, if $\mathcal{C}$ has a model structure with weak equivalences $\mathcal{W}$ then it has a partial model structure, with $\mathcal{U}$ being the subcategory of acyclic cofibrations and $\mathcal{V}$ being the subcategory of acyclic fibrations.

Theorem 7.5 ([14, Main Theorem])**.** *If $(\mathcal{C}, \mathcal{W})$ is a homotopical category that admits a partial model structure, then $\mathcal{W}$ is fibrant in the Thomason model structure on* $\mathbf{Cat}$.

In the present context, it is very natural to consider those partial model structures such that $\mathcal{C}$ is a poset. In [8], Droz and Zakharevich classified all of the model structures on posets.

Theorem 7.6 ([8, Theorem B])**.** *Let P be a poset which contains all finite products and coproducts, and let $\mathcal{W}$ be a subcategory that contains all objects of P. Then P has a model structure with $\mathcal{W}$ as its subcategory of weak equivalences if and only if the following two properties hold.*

(i) If a composite gf of morphisms in P is in $\mathcal{W}$, then both f and g are in $\mathcal{W}$.

(ii) *There is a functor $\chi\colon P \longrightarrow P$ that takes all maps in $\mathcal{W}$ to identity maps and has the property that for every object $x \in P$, the four canonical maps of the diagram*

$$\begin{array}{ccc} \chi(x) \times x & \longrightarrow & \chi(x) \\ \downarrow & & \downarrow \\ x & \longrightarrow & \chi(x) \cup x \end{array}$$

in P are weak equivalences.

These two results have the following consequence.

Proposition 7.7. *Let P be a poset satisfying the following conditions:*

(i) *P contains an object c such that $c \times x$ and $c \cup x$ exist in P for any other object $x \in P$.*

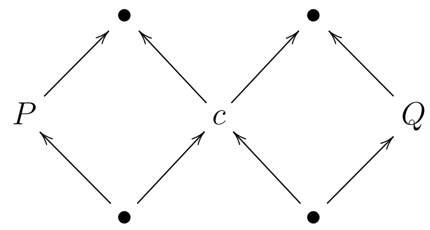
(ii) *For any two objects $a, b \in P$, either $a \times b$ exists or there does not exist an $x \in P$ such that $x \leq a$ and $x \leq b$. Dually, either $a \cup b$ exists or there does not exist an $x \in P$ such that $x \geq a$ and $x \geq b$.*

Then P is a component of the weak equivalences in a model category and is therefore fibrant in **Pos**. *Moreover, P is contractible.*

Proof. Consider the poset $\widetilde{P}$ whose objects are those of P and two further objects, $\emptyset$ and $*$. The morphisms are those of P and those dictated by requiring $\emptyset$ to be an initial object and $*$ to be a terminal object (so there is no morphism $* \longrightarrow \emptyset$). Condition (ii) ensures that $\widetilde{P}$ has all finite products and coproducts. Indeed, if $a, b \in P$ and $a \times b$ does not exist in P, then $a \times b = \emptyset$ in $\widetilde{P}$ and dually for coproducts. For all $x \in \widetilde{P}$, $x \times * = x$, $x \times \emptyset = \emptyset$, $x \cup \emptyset = x$, and $x \cup * = *$.

Let $\mathcal{W}$ be the union of P and the discrete subcategory $\{\emptyset, *\}$ of P. Although $\widetilde{P}$ is connected, P is one of the three components of $\mathcal{W}$, the other two being the discrete components $\{\emptyset\}$ and $\{*\}$ (which are clearly fibrant). Theorem 7.6 implies that $\widetilde{P}$ has a model structure with $\mathcal{W}$ as its subcategory of weak equivalences. Indeed, condition (i) is clear and, for condition (ii), we define $\chi\colon \widetilde{P} \longrightarrow \widetilde{P}$ by mapping all of P to c (and its identity morphism), mapping $\emptyset$ to $\emptyset$, and mapping $*$ to $*$. Therefore $\mathcal{W}$ is fibrant by Theorem 7.5, hence P is fibrant by Lemma 7.1; P is contractible by Lemma 7.3. $\square$

For example, if P and Q are any posets satisfying condition (ii) of Proposition 7.7 then the following poset is fibrant:



Finally, we prove a partial converse to Proposition 7.7 which shows that in many cases the connected fibrant posets constructed by Theorem 7.5 are contractible.

Definition 7.8. A map $f\colon a \longrightarrow b$ in a poset P is *maximal* if there do not exist any non-identity morphisms $z \longrightarrow a$ or $b \longrightarrow z$.

For example, the composition of a sequence of maximal length in P is maximal.

Proposition 7.9. *Let $(\mathcal{W},\mathcal{U},\mathcal{V})$ be a partial model structure on a poset P and let Q be a connected component of $\mathcal{W}$ that contains a maximal map. Then Q contains an object c such that $c \times x$ and $c \cup x$ exist in Q for any other object $x \in Q$. Therefore Q is contractible.*

Proof. Let $f\colon a \longrightarrow b$ be a maximal map in Q and factor it as a map $a \longrightarrow c$ in $\mathcal{U}$ followed by a map $c \longrightarrow b$ in $\mathcal{V}$, using the functorial factorization. Since Q is a connected component of $\mathcal{W}$, c is in Q.

First, we claim that any morphism $g\colon z \longrightarrow c$ in Q is in $\mathcal{U}$. Factor g as a morphism $z \longrightarrow w$ in $\mathcal{U}$ followed by a morphism $w \longrightarrow c$ in $\mathcal{V}$. Since $\mathcal{V}$ is closed under pullbacks, $a \times_c w \longrightarrow a$ exists and is in $\mathcal{V}$. However, since f is maximal in $\mathcal{W}$, we must have $a \times_c w = a$, so there exists a morphism $a \longrightarrow w$. By Lemma 7.2, the pushout $c \cup_a w$ of $a \longrightarrow c$ along $a \longrightarrow w$ is $w \cup c$, and $w \cup c = c$ since P is a poset and there is a map $w \longrightarrow c$. But then $w \longrightarrow c$ is the pushout of a morphism in $\mathcal{U}$, so it is also in $\mathcal{U}$. Thus g is the composite of two morphisms in $\mathcal{U}$, so it is also in $\mathcal{U}$, as claimed. Dually, any morphism $c \longrightarrow z$ in Q is in $\mathcal{V}$.

Now let x be any object in Q. Since Q is connected, there is a finite zigzag of morphisms of Q connecting x to c. If the zigzag ends with

$$w \xrightarrow{h} y \xleftarrow{i} z \xrightarrow{j} c,$$

then j is in $\mathcal{U}$, so $y \cup_z c$ exists and we can shorten the zigzag via the diagram

$$\begin{array}{ccccccc} w & \xrightarrow{h} & y & \xleftarrow{i} & z & \xrightarrow{j} & c \\ & \searrow & & \searrow & & \swarrow & \\ & & & & y \cup_z c. & & \end{array}$$

The dual argument applies to shorten the zigzag if it ends with

$$w \xleftarrow{h} y \xrightarrow{i} z \xleftarrow{j} c.$$

Inductively, we can shorten any zigzag to one of either of the forms

$$x \longleftarrow z \longrightarrow c \qquad \text{or} \qquad x \longrightarrow z \longleftarrow c.$$

We show that $c \cup x$ and $c \times x$ exist in the first case; the same is true in the second case by symmetry. Since $z \longrightarrow c$ is in $\mathcal{U}$, $c \cup_z x$ exists, and it is $c \cup x$ by Lemma 7.2. Since $c \longrightarrow c \cup x$ is in $\mathcal{V}$, $c \times_{c \cup x} x$ also exists, and it is $c \times x$ by Lemma 7.2 again.

Thus Q contains an object c such that $c \times x$ and $c \cup x$ exists for any object $x \in Q$, as claimed, and it follows from Lemma 7.3 that Q is contractible. $\square$

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AFFINE COMBINATIONS IN AFFINE SCHEMES

by Anders KOCK

Résumé. Nous prouvons que la notion géométrique de *points voisins*, dérivée du "premier voisinage de la diagonale" en géométrie algébrique, a la propriété que toute combinaison affine d'un n-tuple quelconque de points mutuellement voisins a un sens invariant, dans tout schema affine. La preuve est obtenue par des considérations d'algèbre commutative élémentaire.

Abstract. The geometric notion of *neighbour points*, as derived from the "first neighbourhood of the diagonal" in algebraic geometry, is shown to have the property that affine combinations of any n-tuple of mutual neighbour points make invariant sense, in any affine scheme. The proof is a piece of elementary commutative algebra.

Keywords. First neighbourhood of the diagonal, neighbour points, affine schemes, affine combinations.

Mathematics Subject Classification (2010). 14B10, 14B20, 51K10.

Introduction

The notion of "neighbour points" in algebraic geometry is a geometric rendering of the notion of nilpotent elements in commutative rings, and was developed since the time of Study, Hjelmslev, later by Kähler, and notably, since the 1950s, by French algebraic geometry (Grothendieck, Weil et al.). The latter school introduced it via what they call *the first neighbourhood of the diagonal*.

In [4], [5] and [8] the neighbour notion was considered on an axiomatic basis, essentially for finite dimensional manifolds; one of the aims was to describe a combinatorial theory of differential forms.

In the specific context of algebraic geometry, such theory of differential forms was also developed in [2], where it applies not only to manifolds, but to arbitrary schemes.

One aspect, present in [5] and [8], but not in [2], is the possibility of forming *affine combinations* of finite sets of mutual (1st order) neighbour points. The present note completes this aspect, by giving the construction of such affine combinations, at least in the category of *affine* schemes[1] (the dual of the category of commutative rings or k-algebras).

The interest in having the possibility of such affine combinations is documented in several places in [8], and is in [5] the basis for constructing, for any manifold, a simplicial object, whose cochain complex is the deRham complex of the manifold.

One may say that the possibility of having affine combinations, for sets of mutual neighbour points, expresses in a concrete way the idea that spaces are "infinitesimally like affine spaces".

1. Neighbour maps between algebras

Let k be a commutative ring. Consider commutative k-algebras B and C and two algebra maps f and $g : B \to C$.[2] We say that they are *neighbours*, or more completely, *(first order) infinitesimal neighbours*, if

$$(f(a) - g(a)) \cdot (f(b) - g(b)) = 0 \text{ for all } a, b \in B, \tag{1}$$

or equivalently, if

$$f(a) \cdot g(b) + g(a) \cdot f(b) = f(a \cdot b) + g(a \cdot b) \text{ for all } a, b \in B. \tag{2}$$

(Note that this latter formulation makes no use of "minus".) When this holds, we write $f \sim g$ (or more completely, $f \sim_1 g$). The relation $\sim$ is a reflexive and symmetric relation (but not transitive). If the element $2 \in k$ is invertible, a third equivalent formulation of $f \sim g$ goes

$$(f(a) - g(a))^2 = 0 \text{ for all } a \in B. \tag{3}$$

[1] Added in proof: Since the construction is local in nature, it is not surprising that it may be extended to more general schemes, and also to the C^∞-context. These issues are dealt with in [1].

[2] Henceforth, "algebra" means throughout "commutative k-algebra", and "algebra map" (or just "map") means k-algebra homomorphism; and "linear" means k-linear. By $\otimes$, we mean $\otimes_k$.

For, it is clear that (1) implies (3). Conversely, assume (3), and let $a, b \in B$ be arbitrary, and apply (3) to the element $a + b$. Then by assumption, and using that f and g are algebra maps,

$$0 = (f(a+b) - g(a+b))^2 = [(f(a) - g(a)) + (f(b) - g(b))]^2$$

$$= (f(a) - g(a))^2 + (f(b) - g(b))^2 - 2(f(a) - g(a)) \cdot (f(b) - g(b)).$$

The two first terms are 0 by assumption, hence so is the third. Now divide by 2.

Note that if C has no zero-divisors, then $f \sim g$ is equivalent to $f = g$.

It is clear that the relation $\sim$ is stable under precomposition:

$$\text{if } h : B' \to B \text{ and } f \sim g : B \to C, \text{ then } f \circ h \sim g \circ h : B' \to C, \quad (4)$$

and (by a small calculation), it is also stable under postcomposition:

$$\text{if } k : C \to C' \text{ and } f \sim g : B \to C, \text{ then } k \circ f \sim k \circ g : B \to C'. \quad (5)$$

Also, if $h : B' \to B$ is a surjective algebra map, precomposition by h not only preserves the neighbour relation, it also reflects it, in the following sense

$$\text{if } f \circ h \sim g \circ h \text{ then } f \sim g. \quad (6)$$

This is immediate from (1); the a and b occurring there is of the form $h(a')$ and $h(b')$ for suitable a' and b' in B', by surjectivity of h.

An alternative "element-free" formulation of the neighbour relation (Proposition 1.2 below) comes from a standard piece of commutative algebra. Recall that for commutative k-algebras A and B, the tensor product $A \otimes B$ carries structure of commutative k-algebra ($A \otimes B$ is in fact a coproduct of A and B); the multiplication map $m : B \otimes B \to B$ is a k-algebra homomorphism; so the kernel is an ideal $J \subseteq B \otimes B$.

The following is a classical description of the ideal $J \subseteq B \otimes B$; we include it for completeness.

Proposition 1.1. *The kernel J of $m : B \otimes B \to B$ is generated by the expressions $1 \otimes b - b \otimes 1$, for $b \in B$. Hence the ideal J^2 is generated by the expressions $(1 \otimes a - a \otimes 1) \cdot (1 \otimes b - b \otimes 1)$. Equivalently, J^2 is generated by the expressions*

$$1 \otimes ab \; + \; ab \otimes 1 \; - \; a \otimes b \; - \; b \otimes a.$$

Proof. It is clear that $1 \otimes b - b \otimes 1$ is in J. Conversely, assume that $\sum_i a_i \otimes b_i$ is in J, i.e. that $\sum_i a_i \cdot b_i = 0$. Rewrite the ith term $a_i \otimes b_i$ as follows:

$$a_i \otimes b_i = a_i b_i \otimes 1 + (a_i \otimes 1) \cdot (1 \otimes b_i - b_i \otimes 1)$$

and sum over i; since $\sum_i a_i b_i = 0$, we are left with $\sum_i (a_i \otimes 1) \cdot (1 \otimes b_i - b_i \otimes 1)$, which belongs to the $B \otimes B$-module generated by elements of the form $1 \otimes b - b \otimes 1$. – The second assertion follows, since $ab \otimes 1 + 1 \otimes ab - a \otimes b - b \otimes a$ is the product of the two generators $1 \otimes a - a \otimes 1$ and $1 \otimes b - b \otimes 1$, except for sign. (Note that the proof gave a slightly stronger result, namely that J is generated already as a B-module, by the elements $1 \otimes b - b \otimes 1$, via the algebra map $i_0 : B \to B \otimes B$, where $i_0(a) = a \otimes 1$). □

From the second assertion in this Proposition immediately follows that $f \sim g$ iff $\binom{f}{g} : B \otimes B \to C$ factors across the quotient map $B \otimes B \to (B \otimes B)/J^2$ (where $\binom{f}{g} : B \otimes B \to C$ denotes the map given by $a \otimes b \mapsto f(a) \cdot g(b)$); equivalently:

Proposition 1.2. *For $f, g : B \to C$, we have $f \sim g$ if and only if $\binom{f}{g}$: $B \otimes B \to C$ annihilates J^2.*

The two natural inclusion maps i_0 and $i_1 : B \to B \otimes B$ (given by $b \mapsto b \otimes 1$ and $b \mapsto 1 \otimes b$, respectively) are not in general neighbours, but when postcomposed with the quotient map $\pi : B \otimes B \to (B \otimes B)/J^2$, they are:

$$\pi \circ i_0 \sim \pi \circ i_1,$$

and this is in fact the universal pair of neighbour algebra maps with domain B.

2. Neighbours for polynomial algebras

We consider the polynomial algebra $B := k[X_1, \ldots, X_n]$. Identifying $B \otimes B$ with $k[Y_1, \ldots, Y_n, Z_1, \ldots, Z_n]$, the multiplication map m is the algebra map given by $Y_i \mapsto X_i$ and $Z_i \mapsto X_i$, so it is clear that the kernel J of m contains the n elements $Z_i - Y_i$. The following Proposition should be classical:

Proposition 2.1. *The ideal $J \subseteq B \otimes B$, for $B = k[X_1, \ldots, X_n]$, is generated (as a $B \otimes B$-module) by the n elements $Z_i - Y_i$.*

Proof. From Proposition 1.1, we know that J is generated by elements $P(\underline{Z}) - P(\underline{Y})$, for $P \in k[\underline{X}]$ (where $\underline{X}$ denotes $X_1, \ldots, X_n$, and similarly for $\underline{Y}$ and $\underline{Z}$). So it suffices to prove that $P(\underline{Z}) - P(\underline{Y})$ is of the form

$$\sum_{i=1}^{n} (Z_i - Y_i) Q_i(\underline{Y}, \underline{Z}).$$

This is done by induction in n. For $n = 1$, it suffices, by linearity, to prove this fact for each monomial X^s. And this follows from the identity

$$Z^s - Y^s = (Z - Y) \cdot (Z^{s-1} + Z^{s-2}Y + \ldots + ZY^{s-2} + Y^{s-1}) \quad (7)$$

(for $s \geq 1$; for $s = 0$, we get 0). For the induction step: Write $P(\underline{X})$ as a sum of increasing powers of X_1,

$$P(X_1, X_2, \ldots) = P_0(X_2, \ldots) + X_1 P_1(X_2, \ldots) + X_1^2 P_2(X_2, \ldots) + \ldots.$$

Apply the induction hypothesis to the first term. The remaining terms are of the form $X_1^s P_s(X_2, \ldots)$ with $s \geq 1$; then for this term, the difference to be considered is

$$Y_1^s P_s(Y_2, \ldots) - Z_1^s P_s(Z_2, \ldots)$$

which we may write as

$$Y_1^s (P_s(Y_2, \ldots) - P_s(Z_2, \ldots)) + P_s(Z_2, \ldots)(Y_1^s - Z_1^s).$$

The first term in this sum is taken care of by the induction hypothesis, the second term uses the identity (7) which shows that this term is in the ideal generated by $(Z_1 - Y_1)$. □

From this follows immediately

Proposition 2.2. *The ideal $J^2 \subseteq B \otimes B$, for $B = k[X_1, \ldots, X_n]$, is generated (as a $B \otimes B$-module) by the elements $(Z_i - Y_i)(Z_j - Y_j)$ (for $i, j = 1, \ldots, n$) (identifying $B \otimes B$ with $k[Y_1, \ldots, Y_n, Z_1, \ldots, Z_n]$).*

(The algebra $(B \otimes B)/J^2$ is the algebra representing the affine scheme "first neighbourhood of the diagonal" for the affine scheme represented by B, alluded to in the introduction.)

An algebra map $\underline{a} : k[X_1, \ldots, X_n] \to C$ is completely given by an n-tuple of elements $a_i := \underline{a}(X_i) \in C$ ($i = 1, \ldots, n$). Let $\underline{b} : k[X_1, \ldots, X_n] \to C$ be similarly given by the n-tuple $b_i \in C$. The decision when $\underline{a} \sim \underline{b}$ can be expressed equationally in terms of these two n-tuples of elements in C, i.e. as a purely equationally described condition on elements $(a_1, \ldots, a_n, b_1, \ldots, b_n) \in C^{2n}$:

Proposition 2.3. *Consider two algebra maps $\underline{a}$ and $\underline{b} : k[X_1, \ldots, X_n] \to C$. Let $a_i := \underline{a}(X_i)$ and $b_i := \underline{b}(X_i)$. Then we have $\underline{a} \sim \underline{b}$ if and only if*

$$(b_i - a_i) \cdot (b_j - a_j) = 0 \tag{8}$$

for all $i, j = 1, \ldots, n$.

Proof. We have that $\underline{a} \sim \underline{b}$ iff the algebra map $\binom{\underline{a}}{\underline{b}}$ annihilates the ideal J^2 for the algebra $k[X_1, \ldots, X_n]$; and this in turn is equivalent to that it annihilates the set of generators for J^2 described in Proposition 2.2. But $\binom{\underline{a}}{\underline{b}}((Z_i - Y_i) \cdot (Z_j - Y_j)) = (b_i - a_i) \cdot (b_j - a_j)$, and then the result is immediate. □

We therefore also say that the pair of n-tuples of elements in C

$$\begin{bmatrix} a_1 & \ldots & a_n \\ b_1 & \ldots & b_n \end{bmatrix}$$

are *neighbours* if (8) holds.

For brevity, we call an n-tuple $(c_1, \ldots, c_n)$ of elements in C^n a *vector*, and denote it $\underline{c}$. Thus a vector $(c_1, \ldots, c_n)$ is neighbour of the “zero” vector $\underline{0} = (0, \ldots, 0)$ iff $c_i \cdot c_j = 0$ for all i and j.

Remark. Even when $2 \in k$ is invertible, one cannot conclude that $(b_i - a_i)^2 = 0$ for all $i = 1, \ldots, n$ implies $\underline{a} \sim \underline{b}$. For, consider $C := k[\epsilon_1, \epsilon_2] = k[\epsilon] \otimes k[\epsilon]$ (where $k[\epsilon]$ is the “ring of dual numbers over k”, so $\epsilon^2 = 0$). Then the pair of n-tuples ($n = 2$ here) given by $(a_1, a_2) = (\epsilon_1, \epsilon_2)$ and $(b_1, b_2) := (0, 0)$ has $(a_i - b_i)^2 = \epsilon_i^2 = 0$ for $i = 1, 2$, but $(a_1 - b_1) \cdot (a_2 - b_2) = \epsilon_1 \cdot \epsilon_2$, which is not 0 in C.

We already have the notion of when two algebra maps f and $g : B \to C$ are neighbours. We also say that the pair (f, g) form an *infinitesimal 1-simplex* (with f and g as *vertices*). Also, we have with (8) the derived notion of when two vectors $\underline{a}$ and $\underline{b}$ in C^n are neighbours, or form an infinitesimal 1-simplex. This terminology is suited for being generalized to defining the notion of *infinitesimal p-simplex* of algebra maps $B \to C$, or of *infinitesimal p-simplex* of vectors in C^n (for $p = 1, 2, \ldots$), namely a $(p+1)$-tuple of mutual neighbouring algebra maps, resp. neighbouring vectors.

Proposition 2.3 generalizes immediately to infinitesimal p-simplices (where the Proposition is the special case of $p = 1$):

Proposition 2.4. *Consider $p+1$ algebra maps $\underline{a}_i : k[X_1, \ldots, X_n] \to C$ (for $i = 0, \ldots, p$), and let $a_{ij} \in C$ be $\underline{a}_i(X_j)$, for $j = 1, \ldots n$. Then the $\underline{a}_i$ form an infinitesimal p-simplex iff for all $i, i' = 0, \ldots p$ and $j, j' = 1, \ldots, n$*

$$(a_{ij} - a_{i'j}) \cdot (a_{ij'} - a_{i'j'}) = 0. \tag{9}$$

3. Affine combinations of mutual neighbours

Let C be a k-algebra. An *affine* combination in a C-module means here a linear combination in the module, with coefficients from C, and where the sum of the coefficients is 1. We consider in particular the C-module $\mathrm{Lin}_k(B, C)$ of k-linear maps $B \to C$, where B is another k-algebra. Linear combinations of algebra maps are linear, but may fail to preserve the multiplicative structure (including 1). However

Theorem 3.1. *Let $f_0, \ldots, f_p$ be a $p+1$-tuple of mutual neighbour algebra maps $B \to C$, and let $t_0, \ldots, t_p$ be elements of C with $t_0 + \ldots + t_p = 1$. Then the affine combination*

$$\sum_{i=0}^{p} t_i \cdot f_i : B \to C$$

is an algebra map. The construction is natural in B and in C.

Proof. Since the sum is a k-linear map, it suffices to prove that it preserves the multiplicative structure. It clearly preserves 1. To prove that it preserves products $a \cdot b$, we should compare $\sum t_i f_i(a \cdot b)$. with

$$(\sum_i t_i f_i(a)) \cdot (\sum_j t_j f_j(b)) = \sum_{i,j} t_i t_j f_i(a) \cdot f_j(b).$$

Now use that $\sum_j t_j = 1$; then $\sum t_i f_i(a \cdot b)$ may be rewritten as

$$\sum_{ij} t_i t_j f_i(a \cdot b).$$

Compare the two displayed double sums: the terms with $i = j$ match since each f_i preserves multiplication. Consider a pair of indices $i \neq j$; the terms with index ij and ji from the first sum contribute $t_i t_j$ times

$$f_i(a) \cdot f_j(b) + f_j(a) \cdot f_i(b), \tag{10}$$

and the terms terms with index ij and ji from the second sum contribute $t_i t_j$ times

$$f_i(a \cdot b) + f_j(a \cdot b), \tag{11}$$

and the two displayed contributions are equal, since $f_i \sim f_j$ (use the formulation (2)). The naturality assertion is clear. □

Theorem 3.2. *Let Let $f_0, \ldots, f_p$ be a $p+1$-tuple of mutual neighbour algebra maps $B \to C$. Then any two affine combinations (with coefficients from C) of these maps are neighbours.*

Proof. Let $\sum_i t_i f_i$ and $\sum_j s_j f_j$ be two such affine combinations. To prove that they are neighbours means (using (2)) to prove that for all a and b in B,

$$(\sum_i t_i f_i(a)) \cdot (\sum_j s_j f_j(b)) + (\sum_j s_j f_j(a)) \cdot (\sum_i t_i f_i(b)) \tag{12}$$

equals

$$\sum_i t_i f_i(a \cdot b) + \sum_j s_j f_j(a \cdot b). \tag{13}$$

Now (12) equals

$$\sum_{ij} t_i s_j f_i(a) \cdot f_j(b) + \sum_{ij} t_i s_j f_j(a) \cdot f_i(b) = \sum_{ij} t_i s_j [f_i(a) \cdot f_j(b) + f_j(a) \cdot f_i(b)]$$

For (13), we use $\sum_j s_j = 1$ and $\sum_i t_i = 1$, to rewrite it as the left hand expression in

$$\sum_{ij} t_i s_j f_i(a \cdot b) + \sum_{ij} t_i s_j f_j(a \cdot b) = \sum_{ij} t_i s_j [f_i(a \cdot b) + f_j(a \cdot b)].$$

For each ij, the two square bracket expression match by (2), since $f_i \sim f_j$. □

Combining these two results, we have

Theorem 3.3. *Let $f_0, \ldots, f_p$ be a $p+1$-tuple of mutual neighbour algebra maps $B \to C$. Then in the C-module of k-linear maps $B \to C$, the affine subspace $\mathrm{Aff}_C(f_0, \ldots, f_p)$ of affine combinations (with coefficients from C) of the f_is consists of algebra maps, and they are mutual neighbours.*

Note that these two Theorems are also vaild for commutative *rigs*, i.e. no negatives are needed for the notions or the theorems.

In [2], the authors describe an ideal $J_{0p}^{(2)}$. It is the sum of ideals J_{rs}^2 in the $p+1$-fold tensor product $B \otimes \ldots \otimes B$, where J_{rs} is the ideal generated by $i_s(b) - i_r(b)$ for $b \in B$ and $r < s$ (with i_k the kth inclusion map). We shall here denote $J_{0p}^{(2)}$ just by $\overline{J}^{(2)}$ for brevity; it has the property that the $p+1$ inclusions $B \to B \otimes \ldots \otimes B$ become mutual neighbours, when composed with the quotient map $\pi : B \otimes \ldots \otimes B \to (B \otimes \ldots \otimes B)/\overline{J}^{(2)}$, and this is in fact the universal $p+1$ tuple of mutual neighbour maps with domain B.

We may, for any given k-algebra B, encode the construction of Theorem 3.1 into one single canonical map which does not mention any individual $B \to C$. This we do by using the universal $p+1$-tuple of neighbour elements, and the generic $p+1$ tuple of elements (to be used as coefficients) with sum 1, meaning $(X_0, X_1, \ldots, X_p) \in k[X_1, \ldots, X_p]$ (where X_0 denotes $1-(X_1 + \ldots + X_p)$). We shall construct a k-algebra map

$$B \to (B^{\otimes p+1}/\overline{J}^{(2)}) \otimes k[X_1, \ldots, X_p]. \tag{14}$$

By the Yoneda Lemma, this is equivalent to giving a (set theoretical) map, natural in C,

$$\hom((B^{\otimes p+1}/\overline{J}^{(2)}) \otimes k[X_1, \ldots, X_p], C) \to \hom(B, C),$$

(where $\hom$ denotes the set of k-algebra maps). An element on the left hand side is given by a $p+1$-tuple of mutual neighbouring algebra maps $f_i : B \to C$, together with a p-tuple $(t_1, \ldots, t_p)$ of elements in C. With $t_0 := 1 - \sum_1^p t_i$, such data produce an element $\sum_0^p t_i \cdot f_i$ in $\hom(B, C)$, by Theorem 3.1, and the construction is natural in C by the last assertion in the Theorem.

The affine scheme defined by the algebra $B^{\otimes p+1}/\overline{J}^{(2)}$ is (essentially) called $\Delta_B^{(p)}$ in [2], and, (in axiomatic context, for manifolds, in a suitable sense), the corresponding object is called $M_{[p]}$ in [5] and $M_{(1,1,\ldots,1)}$ in [4] I. 18 (for suitable M).

4. Affine combinations in a k-algebra C

The constructions and results of the previous Section concerning infinitesimal p-simplices of algebra maps $B \to C$, specialize (by taking $B = k[X_1, \ldots, X_n]$, as in Section 2) to infinitesimal p-simplices of vectors in C^n; such a p-simplex is conveniently exhibited in a $(p+1) \times n$ matrix with entries a_{ij} from C:

$$\begin{bmatrix} a_{01} & \ldots & a_{0n} \\ a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{p1} & \ldots & a_{pn} \end{bmatrix}$$

in which the rows (the "vertices" of the simplex) are mutual neighbours. We may of course form affine (or even linear) combinations, with coefficients from C, of the rows of this matrix, whether or not the rows are mutual neighbours. But the same affine combination of the corresponding algebra maps is in general only a k-linear map, not an algebra map. However, if the rows are mutual neighbours in C^n, and hence the corresponding algebra maps are mutual neighbours $k[X_1, \ldots, X_n] \to C$, we have, by Theorem 3.1 that the affine combinations of the rows of the matrix corresponds to the similar affine combination of the algebra maps. For, it suffices to check their equality on the X_is, since the X_is generate $k[X_1, \ldots, X_n]$ as an algebra. Therefore, the Theorems 3.2 and 3.3 immediately translate into theorems about $p + 1$-tuples of mutual neighbouring n-tuples of elements in the algebra C; recall that such a $p + 1$-tuple may be identified with the rows of a $(p + 1) \times n$ matrix with entries from C, satisfying the equations (9). We therefore have (cf. also [6])

Theorem 4.1. *Let the rows of a $(p + 1) \times n$ matrix with entries from C be mutual neighbours. Then any two affine combinations (with coefficients from C) of these rows are neighbours. The set of all such affine combinations form an affine subspace of the C-module C^n.*

Let us consider in particular the case where the 0th row of a $(p+1) \times n$ matrix is the zero vector $(0, \ldots, 0)$. Then the following is an elementary calculation:

Proposition 4.2. *Consider a $(p+1) \times n$ matrix $\{a_{ij}\}$ as above, but with $a_{0j} = 0$ for $j = 1, \ldots n$. Then the rows form an infinitesimal p-simplex iff*

$$a_{ij} \cdot a_{i'j'} + a_{i'j} \cdot a_{ij'} = 0 \text{ for all } i, i' = 1, \ldots p, j, j' = 1, \ldots n \tag{15}$$

and

$$a_{ij} \cdot a_{ij'} = 0 \text{ for all } i = 1, \ldots, p, j = 1, \ldots n \tag{16}$$

hold. If 2 is invertible in C, the equations (16) follow from (15).

Proof. The last assertion follows by putting $i = i'$ in (15), and dividing by 2. Assume that the rows of the matrix form an infinitesimal p-simplex. Then (16) follows from $\underline{a}_i \sim \underline{0}$ by (8). The equation which asserts that $\underline{a}_i \sim \underline{a}_{i'}$ (for $i, i' = 1, \ldots, p$) is

$$(a_{ij} - a_{i'j}) \cdot (a_{ij'} - a_{i'j'}) = 0 \text{ for all } j, j' = 1, \ldots n.$$

Multiplying out gives four terms, two of which vanish by virtue of (16), and the two remaining add up to (minus) the sum on the left of (15). For the converse implication, (16) give that the last p rows are $\sim \underline{0}$; and (16) and (15) jointly give that $\underline{a}_i \sim \underline{a}_{i'}$, by essentially the same calculation which we have already made. □

When $\underline{0}$ is one of the vectors in a $p+1$-tuple, any linear combination of the remaining p vectors has the same value as a certain affine combination of all $p+1$ vectors, since the coefficient for $\underline{0}$ may be chosen arbitrarily without changing the value of the linear combination. Therefore the results on *affine* combinations of the rows in the $(p+1) \times n$ matrix with $\underline{0}$ as top row immediately translate to results about *linear* combinations of the remaining rows, i.e. they translate into results about $p \times n$ matrices, satisfying the equations (15) and (16); even the equations (15) suffice, if 2 is invertible. In this form, the results were obtained in the preprint [6], and are stated here for completeness. We assume that $2 \in k$ is invertible.

We use the notation from [4] I.16 and I. 18, where set of $p \times n$ matrices $\{a_{ij}\}$ satisfying (15) was denoted $\tilde{D}(p, n) \subseteq C^{p \cdot n}$ (we there consider algebras C over $k = \mathbb{Q}$, so (16) follows). In particular $\tilde{D}(2,2)$ consists of matrices of the form

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{with} \quad a_{11} \cdot a_{22} + a_{12} \cdot a_{21} = 0.$$

Note that the determinant of such a matrix is 2 times the product of the diagonal entries. And also note that $\tilde{D}(2,2)$ is stable under transposition of matrices.

The notation $\tilde{D}(p,n)$ may be consistently augmented to the case where $p = 1$; we say $(a_1, \ldots, a_n) \in \tilde{D}(1,n)$ if it is neighbour of $\underline{0} \in C^n$, i.e. if $a_j \cdot a_{j'} = 0$ for all $j, j' = 1, \ldots n$. (In [4], $\tilde{D}(1,n)$ is also denoted $D(n)$, and $D(1)$ is denoted D.)

It is clear that a $p \times n$ matrix belongs to $\tilde{D}(p,n)$ precisely when all its 2×2 sub-matrices do; this is just a reflection of the fact that the defining equations (15) only involve two row indices and two column indices at a time. From the transposition stability of $\tilde{D}(2,2)$ therefore follows that transposition $p \times n$ matrices takes $\tilde{D}(p,n)$ into $\tilde{D}(n,p)$.

Note that each of the rows of a matrix in $\tilde{D}(p,n)$ is a neighbour of $\underline{0} \in C^n$.

The results about affine combinations now has the following corollary in terms of linear combinations of the rows of matrices in $\tilde{D}(p,n)$:

Theorem 4.3. *Given a matrix $X \in \tilde{D}(p,n)$. Let a $(p+1) \times n$ matrix X' be obtained by adjoining to X a row which is a linear combination of the rows of X. Then X' is in $\tilde{D}(p+1,n)$.*

5. Geometric meaning

Commutative rings often come about as rings $O(M)$ of scalar valued functions on some space M, and this gives some geometric aspects (arising from the space M) into the algebra $O(M)$. Does *every* commutative ring (or k-algebra) come about this way? This depends of course what "space" is supposed to mean, and what the "scalars" and "functions" are. What could they be?

Algebraic geometry has over time developed a radical, almost self-referential answer. The first thing is to define the *category* $\mathcal{E}$ of spaces, and among these, a commutative ring object $R \in \mathcal{E}$ of *scalars*. The radical answer consists in taking the category $\mathcal{E}$ of spaces and functions to be the dual of the category $\mathcal{A}$ of commutative rings, and *any commutative ring B to be the ring $O(\overline{B})$ of scalar valued functions on the space $\overline{B}$ which it defines*. This will come about by letting the ring of scalars $R \in \mathcal{E}$ be the free ring in one generator: the polynomial ring in one variable, cf. (17) below.

To fix terminology, we elaborate this viewpoint. For flexibility and generality, we consider a commutative base ring k, and consider the category $\mathcal{A}$ of commutative k-algebras (the "absolute" case comes about by taking $k = \mathbb{Z}$).

So $\mathcal{E}$ is $\mathcal{A}^{op}$; for $B \in \mathcal{A}$, the corresponding object in $\mathcal{E}$ is denoted $\overline{B}$ or $\mathrm{Spec}(B)$; for $M \in \mathcal{E}$, the corresponding object in $\mathcal{A}$ is denoted $O(M)$. Thus, $B = O(\overline{B})$ and $M = \overline{O(M)}$.

We have in particular $k[X] \in \mathcal{A}$, the free k-algebra in one generator, and we put $R := \overline{k[X]}$. Then for any $M \in \mathcal{E}$,

$$\hom_{\mathcal{E}}(M, R) = \hom_{\mathcal{E}}(M, \overline{k[X]}) = \hom_{\mathcal{A}}(k[X], O(M)) \cong O(M), \quad (17)$$

(the last isomorphism because $k[X]$ is the free k-algebra in one generator X), and since the right hand side is a k-algebra, (naturally in M), we have that R is a k-algebra object in $\mathcal{E}$. And (17) documents that $O(M)$ is indeed canonically isomorphic as a k-algebra to the k-algebra of R-valued functions on M.

If $\mathcal{E}$ is a category with finite products, algebraic structure on an object R in $\mathcal{E}$ may be described in diagrammatic terms, but it is equivalent to description of the same kind of structure on the *sets* $\hom_{\mathcal{E}}(M, R)$ (naturally in M), thus is a description in terms of *elements*.

It is useful to think of, and speak of, such an element (map) $a : M \to R$ as an "element of R, defined at stage M", or just as a "generalized element (or generalized point[3]) of R defined at stage M". We may write $a \in R$, or $a \in_M R$, if we need to remember the "stage" at which the element a of R is defined; and we may drop the word "generalized".

If $f : R \to S$ is a map in $\mathcal{E}$, then for an $a : M \to R$, we have the composite $f \circ a : M \to S$; viewing a and $f \circ a$ as generalized elements of

[3]Grothendieck called a map $M \to R$ "an M-valued point of R", extending the use in classical algebraic geometry, where one could talk about e.g. a complex-valued point, or point defined over $\mathbb{C}$, for R an arbitrary algebraic variety.

R and S, respectively, the latter is naturally written $f(a)$:

$$f(a) := f \circ a.$$

Subobjects of an arbitrary object $R \in \mathcal{E}$ may be characterized by which of generalized elements of R they contain. Maps $R \to S$ may be described by what they do to generalized elements of R (by post-composition of maps). This is essentially Yoneda's Lemma.

Consider in particular the theory of neighbour maps and their affine combinations, as developed in the previous sections. It deals with the category $\mathcal{A}$ of commutative k-algebras. We shall translate some of the notions and constructions into the dual category $\mathcal{E}$, i.e. into the category of affine schemes over k, using the terminology of generalized elements, or generalized points.

Thus (assuming for simplicity that 2 is invertible in k), we can consider the criterion (3) for the neighbour relation of algebra maps $f, g : B \to C$; it translates as follows. Let $\overline{f}$ and $\overline{g}$ be points of $\overline{B}$ (defined at stage $\overline{C}$). Then $\overline{f} \sim \overline{g}$ iff for all $a : \overline{B} \to R$, we have $(a(\overline{f}) - a(\overline{g}))^2 = 0$, or, changing the names of the objects and maps/elements in question, e.g. $X = \overline{B}$, and refraining from mentioning the common stage of definition of the elements x and y:

Two points x and y in X are neighbours iff for any scalar valued function α on X, $(\alpha(x) = \alpha(y))^2 = 0$.

Thus, the basic (first order) neighbour relation $\sim$ on any object M is determined by the set of scalar valued functions on it, and by which points in the ring object R of scalars have square 0. This implies that the neighbour relation is preserved by any map $\overline{B} \to \overline{B'}$ between affine schemes. The naturality of the construction of affine combinations of mutual neighbour points in $\overline{B}$ implies that the construction is preserved by any map $\overline{B} \to \overline{B'}$ between affine schemes.

The Proposition 2.3 gets the formulation:

Proposition 5.1. *Given two points $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n) \in R^n$. Then they are neighbours iff*

$$(b_i - a_i) \cdot (b_j - a_j) = 0 \tag{18}$$

for all $i, j = 1, \ldots, n$.

Here the (common) parameter space $\overline{C}$ of the $a_i s$ and b_is is not mentioned explicitly; it could be any affine scheme. Note that (18) is typographically the same as (8); in (18), the a_is etc. are (parametrized) points of R (parametrized by $\overline{C}$), in (8), they are elements in the algebra C; but these data correspond, by (17), and this correspondence preserves algebraic structure. – Similarly, Proposition 2.4 gets the reformulation:

Proposition 5.2. *A $p+1$-tuple $\{a_{ij}\}$ of points in R^n form an infinitesimal p-simplex iff the equations (9) hold.*

This formulation, as the other formulations in "synthetic" terms, are the ones that are suited to axiomatic treatment, as in Synthetic Differential Geometry, which almost exclusively[4] assumes a given commutative ring object R in a category $\mathcal{E}$, preferably a topos, as a basic ingredient in the axiomatics. (The category $\mathcal{E}$ of affine schemes is not a topos, but the category of presheaves on $\mathcal{E}$ is, and it, and some of its subtoposes, are the basic categories considered in modern algebraic geometry, like in [3].)

Proposition 5.3. *Given an affine scheme $\overline{B}$, with the k-algebra B finitely presentable. Then for any finite presentation (with n generators, say) of the algebra, the corresponding embedding $e : \overline{B} \to R^n$ preserves and reflects the relation $\sim$, and it preserves affine combinations of neighbour points.*

For, any map between affine schemes preserves the neighbour relation, and affine combinations of mutual neighbours. The argument for reflection is as for (6) since the presentation amounts to a surjective map of k-algebras $k[X_1, \ldots, X_n] \to B$.

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ON THE CATEGORY OF STRATIFOLDS

by Toshiki AOKI and Katsuhico KURIBAYASHI

Résumé. Nous étudions les espaces stratifiés de Kreck (stratifolds)[a] d'un point de vue catégorique. Nous montrons entre autre que la catégorie des espaces stratifiés de Kreck admet un plongement pleinement fidèle dans la catégorie des $\mathbb{R}$-algèbres tout comme la catégorie des variétés lisses. Nous établissons une variante du théorème de Serre-Swan pour les espaces stratifiés de Kreck. En particulier, nous montrons que les fibrés vectoriels sur un espace stratifié de Kreck forment une catégorie équivalente à celle formée par les fibrés vectoriels sur un schéma affine qui est canoniquement associé à, mais en général plus grand que, l'espace stratifié lui-même.

Abstract. Stratifolds are considered from a categorical point of view. We show among others that the category of stratifolds fully faithfully embeds into the category of $\mathbb{R}$-algebras as does the category of smooth manifolds. We prove that a variant of the Serre-Swan theorem holds for stratifolds. In particular, the category of vector bundles over a stratifold is shown to be equivalent to the category of vector bundles over an associated affine scheme although the latter is in general larger than the stratifold itself.

Keywords. Stratifold, differential space, ringed space, vector bundle, the Serre-Swan theorem.
Mathematics Subject Classification (2010). 18F15, 58A35, 58A40, 55R99.

[a]The review article [7] of Kloeckner contains a nice historical review of different notions of stratified space.

1. Introduction

Stratifolds have been introduced by Kreck [8]. The new notion subsumes manifolds and algebraic varieties with isolated singularities as examples; see [3]. One of its advantages is that stratifolds give geometric counterparts of singular homology classes of a CW complex in much the same way as manifolds give geometric homology classes in the sense of Jakob [5]. More precisely, such a homology class is represented by an appropriate bordism class of stratifolds. One might therefore expect that stratifolds share some of the fascinating properties of manifolds and varieties. In this article, we focus on such properties for stratifolds and investigate them from categorical and sheaf-theoretical points of view.

Pursell [14] showed that the category of smooth manifolds fully faithfully embeds into the category of $\mathbb{R}$-algebras. We extend this result to the category of stratifolds.

Theorem 1.1. *The category of stratifolds fully faithfully embeds into the category of* $\mathbb{R}$*-algebras.*

A stratifold $(S, \mathcal{C})$ consists of a topological space S and a subalgebra $\mathcal{C}$ of the $\mathbb{R}$-algebra of continuous real-valued functions on the underlying space. Such a subalgebra defines a ringed space which is called the *structure sheaf* of the stratifold. Indeed, the subalgebra is nothing but the algebra of global sections of the sheaf. The assignment of the algebra to a stratifold, namely the forgetful functor F defined by $F(S, \mathcal{C}) = \mathcal{C}$, gives rise to the embedding in Theorem 1.1.

Let M be a smooth manifold. Then the prime spectrum of the ring $C^\infty(M)$ of real-valued smooth functions with the Zariski topology is larger than the underlying space M in general. However, the *real spectrum*, which is a subspace of the prime spectrum, is homeomorphic to M. This fact is shown to extend to stratifolds; see Propositions 2.6 and 3.3.

The results mentioned above lead us naturally to considering the affine scheme of the global sections of the structure sheaf of a stratifold. In consequence, we see that the restriction of the affine scheme to the real spectrum is isomorphic to a given stratifold as a ringed space; see Theorem 3.5. We are convinced that the result, a sheaf-theoretic description of a stratifold, enables one to consider stratifolds in the framework of derived differential geometry [6, 20] though this issue is not pursued in this manuscript; see Remark 3.6.

The category of vector bundles over a smooth manifold M is equivalent to the category of finitely generated projective modules over $C^\infty(M)$ by a classical result of Swan [21]. An analogous result for algebraic varieties has been obtained by Serre [15]. It is thus worthwhile to investigate a Serre-

Swan type theorem for stratifolds. To this end, we introduce the appropriate notion of vector bundle over stratifolds; see Definition 4.1 and Proposition 4.7. With our definition, we get the following result; see Theorem 4.9 for the precise statement.

Theorem 1.2. *The Serre-Swan theorem holds for stratifolds.*

As a consequence, we deduce that the category of vector bundles over a stratifold is equivalent to that of vector bundles over the affine scheme associated to the stratifold though the underlying prime spectrum is larger than the stratifold itself in general; see Remark 4.18.

The rest of this article is organized as follows. In Section 2, after recalling the definition of a stratifold and its important properties, we prove Theorem 1.1. We investigate stratifolds and their category from a sheaf-theoretical point of view in Section 3. In Section 4, the notion of *vector bundle* over a stratifold is introduced and the Serre-Swan theorem is shown to hold for any stratifold. In Section 5, we characterize morphisms of stratifolds by local data, and describe them inside the category of *diffeological spaces*; see [19, 4]. In Section 6, we study the product of stratifolds from a categorical perspective. This is used in Section 4 in the course of proving the Serre-Swan theorem.

We conclude this section with comments. An important device in the study of stratifolds is the existence of so-called *local retractions* near each point of the stratifolds; see [8]. These retractions are essential at several places in this article; see Sections 4, 5 and 6. Some of proofs in Sections 2 and 3 are straightforward. Yet, they are instructive for clarifying what properties of manifolds and stratifolds are responsible for the obtained results. These results are needed to set up a framework for describing the Serre-Swan theorem in our context. One of highlights in this manuscript is that a version of the Serre-Swan theorem for stratifolds is proved without using tautological bundles or the Whitney immersion theorem as is usually done for proving the theorem in the case of a manifold.

2. The real spectrum of a stratifold

This section contains a brief review of stratifolds. We begin with the definition of a differential space in the sense of Sikorski [18].

Definition 2.1. A *differential space* is a pair $(S, \mathcal{C})$ consisting of a topological space S and an $\mathbb{R}$-subalgebra $\mathcal{C}$ of the $\mathbb{R}$-algebra $C^0(S)$ of continuous real-valued functions on S, which is supposed to be *locally detectable* and C^∞*-closed*.

Local detectability means that $f \in \mathcal{C}$ if and only if for any $x \in S$, there exist an open neighborhood U of x and an element $g \in \mathcal{C}$ such that $f|_U = g|_U$.

C^∞-closedness means that for each $n \geq 1$, each n-tuple $(f_1, ..., f_n)$ of maps in $\mathcal{C}$ and each smooth map $g : \mathbb{R}^n \to \mathbb{R}$, the composite $h : S \to \mathbb{R}$ defined by $h(x) = g(f_1(x), ..., f_n(x))$ belongs to $\mathcal{C}$.

Let $(S, \mathcal{C})$ and $(S', \mathcal{C}')$ be differential spaces. We call a continuous map $f : S \to S'$ a *morphism of the differential spaces*, denoted $f : (S, \mathcal{C}) \to (S', \mathcal{C}')$, if f induces a map $f^* : \mathcal{C}' \to \mathcal{C}$; that is, $\varphi \circ f \in \mathcal{C}$ for each $\varphi \in \mathcal{C}'$. Thus we define a category Diff of differential spaces. Let Mfd denote the category of smooth manifolds. It is readily seen that the functor $i : \mathsf{Mfd} \to \mathsf{Diff}$ defined by $i(M) = (M, C^\infty(M))$ is a fully faithful embedding.

For any smooth paracompact manifold M, the defining subalgebra $C^\infty(M)$ of $C^0(M)$ has two additional properties:

(i) It extends to a sheaf of $\mathbb{R}$-algebras $U \mapsto C^\infty(U)$.

(ii) For any open cover $\mathcal{U}$ of M, there exists a *smooth* partition of unity subordinate to $\mathcal{U}$. In particular, the sheaf C^∞ is generated by global sections in the sense that the canonical map $C^\infty(M) \to (C^\infty)_x$ is surjective for any $x \in M$, where $(C^\infty)_x$ denotes the $\mathbb{R}$-algebra of the germs at x.

This in turn implies that $C^\infty(U)$ can be recovered from $C^\infty(M)$ as the set of *locally extendable functions* on U. With this in mind, we introduce such functions in the context of differential spaces.

For a differential space $(S, \mathcal{C})$ and a subspace Y of S, we call an element $g \in C^0(Y)$ a *locally extendable function* if for any $x \in Y$, there exists an open neighborhood V of x in Y and $h \in \mathcal{C}$ such that $g|_V = h|_V$. Let $\mathcal{C}_Y$ be the subalgebra of $C^0(Y)$ consisting of locally extendable functions on Y. Then it follows that $(Y, \mathcal{C}_Y)$ is a differential space; see [8, page 8]. Thus any subspace of a differential space inherits the structure of a differential space.

Let $(S, \mathcal{C})$ be a differential space and $x \in S$. The vector space consisting of derivations on the $\mathbb{R}$-algebra $\mathcal{C}_x$ of the germs at x is denoted by T_xS, which is called the *tangent space* of the differential space at x; see [8, Chapter 1, section 3].

Definition 2.2. A *stratifold* is a differential space $(S, \mathcal{C})$ such that the following four conditions hold:

(1) S is a locally compact Hausdorff space with countable basis;

(2) the *skeleta* $sk_k(S) := \{x \in S \mid \dim T_xS \leq k\}$ are closed in S;

(3) for each $x \in S$ and open neighborhood U of x in S, there exists a *bump function* at x subordinate to U; that is, a non-negative function $\rho \in \mathcal{C}$ such that $\rho(x) \neq 0$ and such that the support $\mathrm{supp}\,\rho := \overline{\{p \in S \mid \rho(p) \neq 0\}}$ is contained in U;

(4) the *strata* $S^k := sk_k(S) - sk_{k-1}(S)$ are k-dimensional smooth manifolds such that restriction along $i : S^k \hookrightarrow S$ induces an isomorphism of stalks

$$i^* : \mathcal{C}_x \xrightarrow{\cong} C^\infty(S^k)_x.$$

for each $x \in S^k$.

A stratifold is *finite-dimensional* if there is a non-negative integer n such that $S = sk_n(S)$. In particular, the tangent spaces of a finite-dimensional stratifold are finite-dimensional.

In what follows, we assume that all stratifolds are finite-dimensional. We may simply write S for a stratifold or differential space $(S, \mathcal{C})$ if no confusion arises. A smooth manifold $(M, C^\infty(M))$ is a typical example of a stratifold. We define the category Stfd of stratifolds as the full subcategory of Diff spanned by the stratifolds. Observe that the embedding $\mathsf{Mfd} \to \mathsf{Diff}$ mentioned above factors through Stfd.

We here recall important properties of a stratifold.

Remark 2.3. Let $(S, \mathcal{C})$ be a stratifold with strata $\{S^i\}$.
(i) Let U be an open subset of S and $\mathcal{C}_U$ the subalgebra of $\mathcal{C}$ consisting of locally extendable functions of $C^0(U)$ in $\mathcal{C}$. Then $(U, \mathcal{C}_U)$ is a stratifold with strata $\{S^i \cap U\}$; see [8, Example 5, page 22].

(ii) For any $x \in S^i$, there exist an open neighborhood U of x in S and a morphism

$$r_x : (U, \mathcal{C}_U) \to (U \cap S^i, \mathcal{C}_{U \cap S^i})$$

such that $r_x|_{U \cap S^i} = id$. Such a map is called a *local retraction* near x; see [8, Proposition 2.1]

(iii) Any locally compact Hausdorff space with countable basis is paracompact and in particular countable at infinity. This together with the other properties of a stratifold $(S, \mathcal{C})$ shows that for any open cover $\mathcal{U}$ of S, there exists a partition of unity subordinate to $\mathcal{U}$ *consisting of functions in* $\mathcal{C}$, i.e. the structure sheaf $\mathcal{O}_S$ of the stratifold $(S, \mathcal{C})$ is *fine*; see [8, Proposition 2.3] and Sections 3 and 4.

We refer the reader to the book [8] of Kreck for other fundamental properties, examples of stratifolds and fascinating results on the stratifold homology.

For an $\mathbb{R}$-algebra $\mathcal{F}$, we define $|\mathcal{F}|$ to be the set of all morphisms of $\mathbb{R}$-algebras from $\mathcal{F}$ to $\mathbb{R}$ which preserve the unit. Moreover, we define a map $\widetilde{f} : |\mathcal{F}| \to \mathbb{R}$ by $\widetilde{f}(x) = x(f)$ for any $f \in \mathcal{F}$. Let $\widetilde{\mathcal{F}}$ be the $\mathbb{R}$-algebra of maps from $|\mathcal{F}|$ to $\mathbb{R}$ of the form $\widetilde{f}$ for $f \in \mathcal{F}$. Then we consider the Gelfand topology on $|\mathcal{F}|$; that is, $|\mathcal{F}|$ is regarded as the topological space with the open basis

$$\{\widetilde{f}^{-1}(U) \mid U : \text{open in } \mathbb{R}, \widetilde{f} \in \widetilde{\mathcal{F}}\};$$

see [11, 2.1] and [12, 3.12]. Thus the assignment of a topological space to an $\mathbb{R}$-algebra gives rise to a contravariant functor

$$| \ | : \mathbb{R}\text{-}\mathsf{Alg} \to \mathsf{Top}$$

which is called the *realization functor*, where $\mathbb{R}$-Alg denotes the category of $\mathbb{R}$-algebras.

By definition, the map $\tau : \mathcal{F} \to \widetilde{\mathcal{F}}$ defined by $\tau(f) = \widetilde{f}$ is surjective. It follows that τ is an isomorphism if $\mathcal{F}$ is a subalgebra of the $\mathbb{R}$-algebra of continuous functions on a space; see [12, 3.14].

Lemma 2.4. *Let $(S, \mathcal{C})$ be a stratifold. Then the map $\theta : S \to |\mathcal{C}|$ defined by $\theta(p)(f) = f(p)$ is a homeomorphism.*

Proof. In virtue of Remark 2.3, the same argument as in the proof of [12, Theorem 7.2] shows that θ is a bijection.

For any open set U in $\mathbb{R}$ and $f \in \mathcal{C}$, we see that $\theta^{-1}(\widetilde{f}^{-1}(U)) = f^{-1}(U)$ since $\widetilde{f} \circ \theta = f$. This implies that θ is continuous.

Let W be an open set of S. By definition, a stratifold has a bump function for each $x \in W$; that is, there exists a non-negative function $f_x \in \mathcal{C}$ such that $\mathrm{supp} f_x \subset W$ and $f_x(x) \neq 0$. Then we see that $x \in f_x^{-1}(\mathbb{R}^+) \subset W$ for any $x \in W$ and hence $W = \bigcup_{x \in W} f_x^{-1}(\mathbb{R}^+)$. Therefore, it follows that

$$\theta(W) = \theta(\bigcup_{x \in W} f_x^{-1}(\mathbb{R}^+)) = \bigcup_{x \in W} (\theta^{-1})^{-1} f_x^{-1}(\mathbb{R}^+) = \bigcup_{x \in W} \widetilde{f_x}^{-1}(\mathbb{R}^+).$$

Observe that $\widetilde{f_x} \circ \theta = f_x$ as mentioned above. This shows that θ is open. $\square$

Let $\mathcal{F}$ be a subalgebra of $C^0(X)$ the $\mathbb{R}$-algebra of continuous maps from a space X to $\mathbb{R}$. We call the pair $(X, \mathcal{F})$ a *continuous space*. Let Csp be the category of continuous spaces. Observe that a morphism $\varphi : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is a continuous map $\varphi : X \to Y$ which satisfies the condition that $f \circ \varphi \in \mathcal{F}_X$ for any $f \in \mathcal{F}_Y$. By definition, the category Diff of differential spaces is a full subcategory of Csp; therefore, the categories Mfd and Stfd are full subcategories of Csp as well.

Proposition 2.5. *The map $\theta : S \to |\mathcal{C}|$ gives rise to an isomorphism $\theta : (S, \mathcal{C}) \to (|\mathcal{C}|, \widetilde{\mathcal{C}})$ of continuous spaces.*

Proof. Recall the isomorphism $\tau : \mathcal{C} \to \widetilde{\mathcal{C}}$. We consider the composite $\theta^* \circ \tau : \mathcal{C} \to \widetilde{\mathcal{C}} \to \mathcal{C}$. Then it is readily seen that $(\theta^* \circ \tau)(f) = f$ for any $f \in \mathcal{C}$. This implies that $\theta^* : \widetilde{\mathcal{C}} \to \mathcal{C}$ is a well-defined isomorphism. Since $(\theta^{-1})^* \theta^*(\widetilde{f}) = \widetilde{f}$, it follows that $(\theta^{-1})^* : \mathcal{C} \to C^0(|\mathcal{C}|)$ factors through the subalgebra $\widetilde{\mathcal{C}}$ and that $(\theta^{-1})^* : \mathcal{C} \to \widetilde{\mathcal{C}}$ is an isomorphism. This completes the proof. $\square$

We call a maximal ideal m of $\mathcal{C}$ real if the quotient $\mathcal{C}/\mathsf{m}$ is isomorphic to $\mathbb{R}$ as an $\mathbb{R}$-algebra. Let $\mathrm{Spec}_r\, \mathcal{C}$ be the *real spectrum*, namely the subset of the prime spectrum $\mathrm{Spec}\, \mathcal{C}$ of $\mathcal{C}$ consisting of real ideals. We consider $\mathrm{Spec}_r\, \mathcal{C}$ the subspace of $\mathrm{Spec}\, \mathcal{C}$ with the Zariski topology. It is readily seen that a map $u : |\mathcal{C}| \to \mathrm{Spec}_r\, \mathcal{C}$ defined by $u(\varphi) = \mathrm{Ker}\, \varphi$ is bijective. Moreover, the map u is continuous. In fact, for an open base $D(f) = \{\mathsf{m} \in \mathrm{Spec}_r\, \mathcal{C} \mid f \notin \mathsf{m}\}$ for some $f \in \mathcal{C}$, we see that $u^{-1}(D(f)) = \widetilde{f}^{-1}(\mathbb{R} \backslash \{0\})$.

Proposition 2.6. (*cf. [11, Remark, page 23]*) *The bijection* $u : |\mathcal{C}| \xrightarrow{\cong} \operatorname{Spec}_r \mathcal{C}$ *is a homeomorphism.*

Proof. With the same notation as in Lemma 2.4, we see that $u(\widetilde{f_x}^{-1}(\mathbb{R}^+)) = D(f_x)$. We observe that $\widetilde{f_x}$ is non-negative since $\widetilde{f_x} \circ \theta = f_x$ with θ the bijection. □

In consequence, the space $\operatorname{Spec}_r \mathcal{C}$ is homeomorphic to $|\mathcal{C}|$ and hence the underlying space S:

$$S \cong |\mathcal{C}| \cong \operatorname{Spec}_r \mathcal{C} \subset \operatorname{Spec} \mathcal{C}.$$

Remark 2.7. In [6, 4.3] and [2], the spectrum for an $\mathbb{R}$-algebra corresponds to what we call the real spectrum of the $\mathbb{R}$-algebra, which in general is not the same as its prime spectrum.

The following result yields Theorem 1.1.

Theorem 2.8. *The forgetful functor* $F : \mathsf{Stfd} \to \mathbb{R}\text{-}\mathsf{Alg}$ *defined by* $F(S, \mathcal{C}) = \mathcal{C}$ *is fully faithful; that is, the induced map* $F : \operatorname{Hom}_{\mathsf{Stfd}}((S, \mathcal{C}), (S', \mathcal{C}')) \to \operatorname{Hom}_{\mathbb{R}\text{-}\mathsf{Alg}}(\mathcal{C}', \mathcal{C})$ *is a bijection.*

Proof. For a morphism $\varphi : (S, \mathcal{C}) \to (S', \mathcal{C}')$ of stratifolds, namely a morphism of continuous spaces, we have a commutative diagram

(2.1)
$$\begin{array}{ccc} S & \xrightarrow[\cong]{\theta} & |\mathcal{C}| \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle|\varphi^*|} \\ S' & \xrightarrow[\cong]{\theta} & |\mathcal{C}'|. \end{array}$$

In fact, we see that for any $f' \in \mathcal{C}'$ and $p \in S$,

$$\begin{aligned} (|\varphi^*| \circ \theta)(p)(f') &= |\varphi^*|(\theta(p))(f') = \theta(p)(\varphi^*(f')) \\ &= \theta(p)(f' \circ \varphi) = (f' \circ \varphi)(p) \end{aligned}$$

and that $(\theta \circ \varphi)(p)(f') = \theta(\varphi(p))(f') = f'(\varphi(p))$. This yields that F is injective.

For any morphism $u : \mathcal{C}' \to \mathcal{C}$ of $\mathbb{R}$-algebras, we define $\varphi : S \to S'$ to be the composite

$$S \xrightarrow[\cong]{\theta} |\mathcal{C}| \xrightarrow{|u|} |\mathcal{C}'| \xrightarrow[\cong]{\theta^{-1}} S'.$$

Observe that $|u|$ is a continuous map defined by $|u|(p) = p \circ u$; see [12, 3.19]. For any $x \in |\mathcal{C}|$, we see that $|u|^*(\widetilde{f})(x) = (\widetilde{f} \circ |u|)(x) = \widetilde{f}(x \circ u) = (x \circ u)(f) = \widetilde{u(f)}(x)$. Thus it follows that $|u|^* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ is well defined. Moreover, we have a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{C}'} & \xrightarrow{|u|^*} & \widetilde{\mathcal{C}} \\ {\scriptstyle \theta^*}\downarrow & & \downarrow{\scriptstyle \theta^*} \\ \mathcal{C}' & \xrightarrow[u]{} & \mathcal{C}. \end{array} \tag{2.2}$$

This follows from the fact that for any $\widetilde{f'} \in \widetilde{\mathcal{C}}$ and $x \in S$,

$$\begin{aligned}(\theta^* \circ |u|^*)(\widetilde{f'})(x) &= (|u| \circ \theta)^*(\widetilde{f'})(x) = (\widetilde{f'} \circ (|u| \circ \theta))(x) \\ &= ((|u| \circ \theta)(x))(f') = (|u|(\theta(x)))(f') \\ &= \theta(x)(u(f')) = u(f')(x).\end{aligned}$$

Furthermore, we see that $(u \circ \theta^*)(\widetilde{f'})(x) = u(\theta(\widetilde{f'}))(x) = (u(\widetilde{f'} \circ \theta))(x) = u(f')(x)$. This enables us to deduce that $\varphi^* = u$. It turns out that F is a bijection. □

We conclude this section with comments concerning Theorems 1.1 and 2.8.

Remark 2.9. A stratifold $(S, \mathcal{C})$ is a differential space. Then it is readily seen that

$$\mathrm{Hom}_{\mathsf{Stfd}}((S, \mathcal{C}), (\mathbb{R}, C^\infty(\mathbb{R}))) = \mathcal{C}.$$

Remark 2.10. The result [12, 7.19] asserts that the category Mfd of manifolds is equivalent to the category of *smooth* $\mathbb{R}$-algebras, which is a full subcategory of the category $\mathbb{R}$-Alg. Moreover, we have the embedding j : Mfd $\to$ Stfd as mentioned above. However, Theorem 1.1 is not an immediate consequence of these results.

3. The structure sheaf of a stratifold

The goal of this section is to give a sheaf-theoretical extension of Theorem 1.1.

Let X be a space and $\mathcal{C}$ an $\mathbb{R}$-subalgebra of $C^0(X)$ the $\mathbb{R}$-algebra of real-valued continuous functions on X. Recall that for any open subset U of X, an element $f \in C^0(U)$ is called *locally extendable* in $\mathcal{C}$ if for any element x in U, there exist an open neighborhood V_x of x in U and a function $g \in \mathcal{C}$ such that $f|_{V_x} = g|_{V_x}$. It is readily seen that the pair $(X, \mathcal{O}_X)$ is a ringed subspace of (X, C^0) of real-valued continuous functions, where $\mathcal{O}_X(X) = \mathcal{C}$ and $\mathcal{O}_X(U)$ is the $\mathbb{R}$-subalgebra of $C^0(U)$ consisting of locally extendable elements in $\mathcal{C}$ for any open subset U of X. Such a ringed subspace $(X, \mathcal{O}_X)$ is called a *ringed continuous space*.

A map between the underlying spaces of ringed continuous spaces, which induces a well-defined map between global sections, gives rise to a morphism of ringed spaces. The proof is straightforward. More precisely, we have

Lemma 3.1. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed continuous spaces. Let $f : X \to Y$ be a continuous map. Suppose that $f^\sharp(g) := g \circ f$ is in $\mathcal{O}_X(X)$ for any $g \in \mathcal{O}_Y(Y)$. Then $f^\sharp$ induces a well-defined morphism of sheaves $f^\sharp| : \mathcal{O}_Y \to f_*\mathcal{O}_X$.*

Let $(\mathsf{RS})_{C^0}$ be the category of ringed continuous spaces on locally compact, Hausdorff spaces with countable basis whose morphisms are continuos maps between underlying spaces satisfying the assumption on Lemma 3.1. Observe that every stratifold $(S, \mathcal{C})$ gives rise to a ringed continuous space $(S, \mathcal{O}_S)$. Its sheaf of rings $\mathcal{O}_S$ will be called the *structure sheaf* of the stratifolds $(S, \mathcal{C})$, and is explicitly given by $\mathcal{O}_S(U) = \mathcal{C}_U$ for an open set U of S; see Remark 2.3.

Definition 3.2. A ringed continuous space $(X, \mathcal{O}_X)$ is called *fine* if the sheaf of rings $\mathcal{O}_X$ is fine; that is, if for every locally finite cover $\mathcal{U}$ of X, there exists a partition of unity into a sum of global sections $s_i \in \mathcal{O}_X(X)$ whose supports are subordinate to $\mathcal{U}$.

We work with the full subcategory $\mathsf{f}(\mathsf{RS})_{C^0}$ of $(\mathsf{RS})_{C^0}$ consisting of fine ringed continuous spaces. We have seen in Remark 2.3 that the category

Stfd fully faithfully embeds into $\mathsf{f(RS)}_{C^0}$. Moreover, we have the following extensions of results in Section 2.

Proposition 3.3. (i) *The functor F which assigns global sections gives rise to fully and faithful embedding form the category $\mathsf{f(RS)}_{C^0}$ into $\mathbb{R}$-Alg.*
(ii) *Let $(X, \mathcal{O}_X)$ be in $\mathsf{f(RS)}_{C^0}$. Then there exist functional homeomorphisms*

$$X \cong |\mathcal{O}_X(X)| \cong \mathrm{Spec}_r \mathcal{O}_X(X).$$

(iii) *The category $\mathsf{f(RS)}_{C^0}$ is a full subcategory of RS the category of ringed spaces.*

Proof. The proofs of Theorem 1.1 and Proposition 2.5 yield those of (i) and (ii).

Let $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphisms of ringed spaces. In order to prove (iii), it suffices to show that $\varphi = f^\sharp$. We define a continuous map $g : X \to Y$ by $g = \theta^{-1} \circ |\varphi_Y| \circ \theta$. The commutative diagram (2.2) allows us to deduce that $g^\sharp : \mathcal{O}_Y(Y) \to (f_*\mathcal{O}_X)(Y) = \mathcal{O}_X(X)$ is nothing but the map φ_Y. For any open set U of Y, we consider a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{\varphi_U, g^\sharp|} & (f_*\mathcal{O}_X)(U) \\ {\scriptstyle i^\sharp}\uparrow & & \uparrow{\scriptstyle i^\sharp} \\ \mathcal{O}_Y(Y) & \xrightarrow[\varphi_Y = g^\sharp]{} & (f_*\mathcal{O}_X)(Y), \end{array}$$

where $i : U \to Y$ denotes the inclusion. By applying the realization functor $|\ |$ to the diagram above, we see that $\varphi_U = g^\sharp|$. In fact, $|i^\sharp|$ is the inclusion i up to homeomorphism θ; see the commutative diagram (2.1). Suppose that $g(x) \neq f(x)$ for some $x \in X$. Then there exists an open neighborhood $V_{f(x)}$ of $f(x)$ such that $g(x)$ is not in $V_{f(x)}$. On the other hand, since the map $g^\sharp| = \varphi_{V_{f(x)}} : \mathcal{O}_Y(V_{f(x)}) \to (f_*\mathcal{O}_X)(V_{f(x)})$ is well defined, it follows that $V_{f(x)} \supset g(f^{-1}(V_{f(x)}))$ and hence $g(x)$ is in $V_{f(x)}$, which is a contradiction. We have $(f, f^\sharp) = (g, g^\sharp) = (f, \varphi)$. This completes the proof of (iii). □

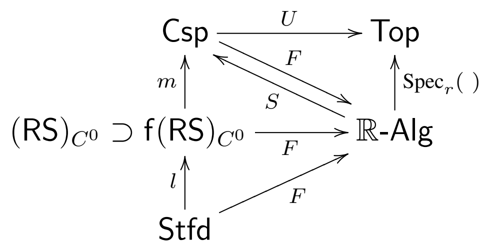
We recall the category Csp of continuous spaces; see Section 2. Let $S : \mathbb{R}\text{-}\mathsf{Alg} \to \mathsf{Csp}$ be the contravariant functor defined by $S\mathcal{F} = (|\mathcal{F}|, \widetilde{\mathcal{F}})$.

By the definition of a morphism in Csp, we see that the same maps Φ and Ψ as in Section 5 below give bijections

$$\mathrm{Hom}_{\mathsf{Csp}^{op}}(S\mathcal{F}, (X, \mathcal{C})) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathrm{Hom}_{\mathbb{R}\text{-}\mathsf{Alg}}(\mathcal{F}, F(X, \mathcal{C})),$$

where $F : \mathsf{Csp}^{op} \to \mathbb{R}\text{-}\mathsf{Alg}$ is the forgetful functor defined by $F(X, \mathcal{C}) = \mathcal{C}$. In fact, for g in Csp, the map $\Phi(g)$ factors through $\mathcal{C}$ and hence Φ is well defined. For $\varphi : \mathcal{F} \to \mathcal{C}$ and $\widetilde{f} \in \widetilde{\mathcal{F}}$, we have $(|\varphi| \circ \theta)^*(\widetilde{f}) = \varphi(f) \in \mathcal{C}$. This implies that Ψ is well defined. Thus S is the left adjoint of F. Let $U : \mathsf{Csp} \to \mathsf{Top}$ be the forgetful functor which assigns a continuous space the underlying space. We define a functor $m : \mathsf{f(RS)}_{C^0} \to \mathsf{Csp}$ by $m(X, \mathcal{O}) = (X, \mathcal{O}(X))$ for a fine ringed continuous space $(X, \mathcal{O})$. With these functors, we have a digram

(3.1)



in which the upper square and the triangles except for the upper right-hand side one are commutative up to isomorphism; see Propositions 2.5 and 2.6. Proposition 3.3 (i) yields that the functor $F : \mathsf{f(RS)}_{C^0} \to \mathbb{R}\text{-}\mathsf{Alg}$ gives rise to an equivalence of categories between $\mathsf{f(RS)}_{C^0}$ and its image, which is a full subcategory $\mathbb{R}$-Alg. One might remember the same result in algebraic geometry as the fact that the category of affine schemes is equivalent to the category of commutative rings with the global section functor.

Remark 3.4. An object in $\mathsf{f(RS)}_{C^0}$ which comes from Stfd is a locally ringed space; that is, the ring of germs at each point is local. This follows from the definition of a stratifold and [11, Theorem 1.8].

Let A be an $\mathbb{R}$-algebra and U an open set of $\mathrm{Spec}_r A$. We put $M_U := \bigcap_{\mathfrak{m} \in U} \mathfrak{m}^c$, where $\mathfrak{m}^c$ denotes the complement of $\mathfrak{m}$. Then M_U is a multiplicative set. We denote by $M_U^{-1}A$ the localization of A with respect to M_U. Define the *structure sheaf* $\widehat{A}$ on $\mathrm{Spec}_r A$ by the sheafification of the presheaf

$U \rightsquigarrow M_U^{-1}A$. Observe that the sheaf $\widehat{A}$ is the inverse image of the affine scheme $(\mathrm{Spec}A, \widetilde{A})$ of A along the inclusion $\mathrm{Spec}_r A \hookrightarrow \mathrm{Spec}A$.

The following proposition asserts that a stratifold is indeed a restriction of an affine scheme.

Theorem 3.5. *Let $(S, \mathcal{O}_S)$ be a fine ringed space which comes from a stratifold $(S, \mathcal{C})$ and $i : \mathrm{Spec}_r\mathcal{O}_S(S) \to \mathrm{Spec}\,\mathcal{O}_S(S)$ the inclusion. Then $(S, \mathcal{O}_S)$ is isomorphic to $i^*(\mathrm{Spec}\,\mathcal{O}_S(S), \widetilde{\mathcal{O}_S(S)})$ as a ringed space, where $(\mathrm{Spec}\,\mathcal{O}_S(S), \widetilde{\mathcal{O}_S(S)})$ is the affine scheme associated with the ring $\mathcal{O}_S(S)$.*

Proof. We recall the homeomorphism $\theta : S \xrightarrow{\cong} |S|$ and $u : |S| \xrightarrow{\cong} \mathrm{Spec}_r\mathcal{O}_S(S)$ in Section 2. Let m be the composite $u \circ \theta$. Then we have

$$m(p) = (u \circ \theta)(p) = \mathrm{Ker}\theta(p) = \{f \in \mathcal{C} \mid f(p) = 0\} =: \mathfrak{m}_p.$$

In order to prove the theorem, it suffices to show that $(S, \mathcal{O}_S)$ is isomorphic to the structure sheaf $(\mathrm{Spec}_r\mathcal{O}_S(S), \widehat{\mathcal{O}_S(S)})$. To this end, we construct an isomorphism from $\widehat{\mathcal{O}_S(S)}$ to $m_*\mathcal{O}_S$.

For an open set U of $\mathrm{Spec}_r\mathcal{O}_S(S)$, we define $\alpha_U : M_U^{-1}\mathcal{O}_S(S) \to (m_*\mathcal{O}_S)(U)$ by $\alpha([f/s]) = f \cdot \frac{1}{s}$. Observe that $s(p) \neq 0$ for each p in $m^{-1}(U)$. This implies that α_U is well defined. We see that α_U induces a morphism of presheaves. Moreover, the morphism of presheaves gives rise to a morphism $\widehat{\alpha} : \widehat{\mathcal{O}_S(S)} \to m_*\mathcal{O}_S$ of sheaves. The natural map

$$\alpha_p : \widehat{\mathcal{O}_S(S)}_{\mathfrak{m}_p} = \mathrm{colim}_{\mathfrak{m}_p \in V}\widehat{\mathcal{O}_S(S)}(V) = \mathcal{O}_S(S)_{\mathfrak{m}_p} \to \mathrm{colim}_{p \in U}\mathcal{O}_S(U) =: \mathcal{C}_p$$

defined by $\alpha([f/s]) = f_p \cdot (\frac{1}{s_p})$ is well defined. Here $\mathcal{O}_S(S)_{\mathfrak{m}_p}$ denotes the localization of the ring $\mathcal{O}_S(S)$ at $\mathfrak{m}_p$. In fact, if $s \in \mathfrak{m}_p^c = \mathcal{O}_S(S)\backslash\mathfrak{m}_p$, then $s(p) = \theta(p)(s) \neq 0$. Since $(S, \mathcal{O}_S(S))$ is a stratifold, it follows that $1/s \in \mathcal{O}_S(U)$ for some open set U of S. This follows from the condition (4) in Definition 2.2; see the proof of [8, Proposition 2.3]. Moreover, there exists a bump function at each $x \in S$. Thus the proof of [11, Corollary 1.6] enables us to conclude that α_p is an isomorphism. It turns out that $\widehat{\alpha} = \amalg_{p \in S}\alpha_p$ and hence $\widehat{\alpha}$ is an isomorphism. We have the result. □

Remark 3.6. For a stratifold $(S, \mathcal{C})$, we regard $\mathcal{C}$ as a C^∞-ring; see [6, 2, 9]. Theorem 3.5 asserts that the structure sheaf $(S, \mathcal{O}_S)$ of $(S, \mathcal{C})$ is a C^∞-ringed space in the sense of Joyce [6] and is isomorphic to the spectrum of the C^∞-ring $\mathcal{C}$; see [6, Definition 4.12] for example.

4. Vector bundles and the Serre-Swan theorem for stratifolds

Generalizing the notion of smooth vector bundle over a manifold, we define a vector bundle over a stratifold.

Definition 4.1. Let $(S, \mathcal{C}_S)$ be a stratifold and $(E, \mathcal{C}_E)$ a differential space. A morphism of differential spaces $\pi : (E, \mathcal{C}_E) \to (S, \mathcal{C}_S)$ is a *vector bundle* over $(S, \mathcal{C}_S)$ if the following conditions are satisfied.

(1) $E_x := \pi^{-1}(x)$ is a vector space over $\mathbb{R}$ for $x \in S$.

(2) There exist an open cover $\{U_\alpha\}_{\alpha \in J}$ of S and an isomorphism $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{n_\alpha}$ of differential spaces for each $\alpha \in J$. Here $\pi^{-1}(U_\alpha)$ is regarded as a differential subspace of $(E, \mathcal{C}_E)$; see Remark 2.3, and $U_\alpha \times \mathbb{R}^{n_\alpha}$ is considered the product of the substratifold $(U_\alpha, \mathcal{C}_{U_\alpha})$ of $(S, \mathcal{C}_S)$ and the manifold $(\mathbb{R}^{n_\alpha}, C^\infty(\mathbb{R}^{n_\alpha}))$; see Section 6.

(3) The diagram

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U_\alpha \times \mathbb{R}^{n_\alpha} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U_\alpha & \end{array}$$

is commutative, where pr_1 is the projection onto the first factor.

(4) The composite $pr_2 \circ \phi_\alpha|_{E_x} : E_x \to U_\alpha \times \mathbb{R}^{n_\alpha} \to \mathbb{R}^{n_\alpha}$ is a linear isomorphism, where $pr_2 : U_\alpha \times \mathbb{R}^{n_\alpha} \to \mathbb{R}^{n_\alpha}$ denotes the projection onto the second factor.

We call a vector bundle $\pi : (E, \mathcal{C}_E) \to (S, \mathcal{C}_S)$ *bounded* if for the index set J of the cover which gives the trivialization, the set of integer $\{n_\alpha\}_{\alpha \in J}$ is bounded. Observe that the integer n_α is constant on a connected component of S.

Let $\pi_E : E \to S$ and $\pi_F : F \to S$ be vector bundles over a stratifold S, We define a *morphism* $\varphi : E \to F$ *of bundles* to be a morphism of differential spaces from E to F such that $\pi_F \circ \varphi = \pi_E$ and the restrictions on each stalks $\varphi_x : E_x \to F_x$ are linear maps. We denote by $\mathsf{VBb}_{(S,\mathcal{C})}$ the category of vector bundles over $(S, \mathcal{C})$ of bounded rank.

Definition 4.2. Let $\pi : (E, \mathcal{C}_E) \to (S, \mathcal{C}_S)$ be a vector bundle over a stratifold $(S, \mathcal{C}_S)$. A morphism of differential spaces $s : (U, \mathcal{C}_U) \to (E, \mathcal{C}_E)$ is called a *section* on U if $\pi \circ s = id_{(U,\mathcal{C}_U)}$. We denote by $\Gamma(U, E)$ the set of all sections.

We observe that $\Gamma(U, E)$ is an $\mathcal{O}_S(U)$-module through the identification $\mathcal{C}_U = \mathcal{O}_S(U)$. Moreover, we have the following proposition.

Proposition 4.3. *The assignment* $\Gamma(\ , E) : U \rightsquigarrow \Gamma(U, E)$ *gives rise to an* $\mathcal{O}_S$*-module.*

Proof. We begin by showing that the assignment gives rise to a set-valued sheaf. Let $i : U \to V$ be an inclusion between open sets U and V of S. Since i is a morphism of stratifolds, restricting along i takes a section on V to a section on U.

Let $\{V_\gamma\}_\gamma$ be an open cover of an open set U of S. Suppose that $\{s_\gamma\}_\gamma$ in $\prod_\gamma \Gamma(V_\gamma, E)$ satisfies the condition that $res^{V_\gamma}_{V_\gamma \cap V_{\gamma'}}(s_\gamma) = res^{V_{\gamma'}}_{V_\gamma \cap V_{\gamma'}}(s_{\gamma'})$ for any γ and γ'. In the category Top of topological spaces, we have a section $s : U \to E$ with $res^U_{V_\gamma}(s) = s_\gamma$ for any γ. We need to verify that s is a morphism of differential spaces. For any $x \in U$, there exists an open set V_γ such that $x \in V_\gamma$. Since s_γ is a morphism of differential spaces, it follows that $(s^*(p))|_{V_\gamma} = s^*_\gamma(p) \in \mathcal{C}_{V_\gamma}$ for $p \in \mathcal{C}_E$. By the definition of $\mathcal{C}_{V_\gamma}$, we see that there exists an open neighborhood W_γ of x with $W_\gamma \subset V_\gamma$ such that $(s^*(p))|_{W_\gamma} = ((s^*(p))|_{V_\gamma})|_{W_\gamma} = p \circ s_\gamma|_{W_\gamma} = h|_{W_\gamma}$ for some $h \in \mathcal{C}_S$. This enables us to conclude that $s^*(p) \in \mathcal{C}_U$ and hence $\Gamma(\ , E)$ is a sheaf.

For s and t in $\Gamma(U, E)$, we define a section $(s+t)$ in Top by $(s+t)(x) = s(x) + t(x)$ for any $x \in U$. We show that $(s + t)$ is in $\Gamma(U, E)$. Let s_γ, $t_\gamma : V_\gamma \to \pi^{-1}(V_\gamma)$ be the restrictions of s and t to V_γ, respectively. We define $\widetilde{s_\gamma} : V_\gamma \to V_\gamma \times \mathbb{R}^n$ and $\widetilde{t_\gamma} : V_\gamma \to V_\gamma \times \mathbb{R}^n$ by $\phi_\gamma \circ s_\gamma$ and $\phi_\gamma \circ t_\gamma$, respectively. Here $\phi_\gamma : \pi^{-1}(V_\gamma) \xrightarrow{\cong} V_\gamma \times \mathbb{R}^n$ denotes a local trivialization.

Assertion 4.4. Let s_γ be in $\Gamma(V_\gamma, E)$. Then $s_\gamma : V_\gamma \to \pi^{-1}(V_\gamma)$ is a morphism of differential spaces.

Thus $\widetilde{s_\gamma}$ and $\widetilde{s_\gamma}$ are morphisms of differential spaces. The projection $pr_2 : V_\gamma \times \mathbb{R}^n \to \mathbb{R}^n$ into the second factor is a morphism of stratifolds and so are $pr_2 \circ \widetilde{s_\gamma}$ and $pr_2 \circ \widetilde{t_\gamma}$. We see that $pr_2 \circ \widetilde{s_\gamma} + pr_2 \circ \widetilde{t_\gamma}$ is a morphism of stratifolds. Proposition 6.1 below yields a morphism $(s_\gamma + t_\gamma)' : V_\gamma \to V_\gamma \times \mathbb{R}^n$ of stratifolds which fits into a commutative diagram

$$\begin{array}{ccccc} & & V_\gamma & & \\ & \swarrow^{pr_2 \circ \widetilde{s_\gamma} + pr_2 \circ \widetilde{t_\gamma}} & \downarrow{\scriptstyle (s_\gamma + t_\gamma)'} & \searrow^{1_{V_\gamma}} & \\ \mathbb{R}^n & \xleftarrow[pr_2]{} & V_\gamma \times \mathbb{R}^n & \xrightarrow[pr_1]{} & V_\gamma. \end{array}$$

Define $s_\gamma + t_\gamma : V_\gamma \to \pi^{-1}(V_\gamma)$ to be the composite $\phi_\gamma^{-1} \circ (s_\gamma + t_\gamma)'$. Observe that $(s_\gamma + t_\gamma)(x) = (s+t)(x) = (s_{\gamma'} + t_{\gamma'})(x)$ for $x \in U_\gamma \cap U_{\gamma'}$. Since $\Gamma(, E)$ is a sheaf, it follows that there exists a unique extension $\widetilde{s+t} \in \Gamma(U, E)$ of $\{s_\gamma + t_\gamma\}_\gamma$. It is readily seen that $\widetilde{s+t} = s + t$.

The same argument as above does work well to show that sk defined by $sk(x) = k(x)s(x)$ is in $\Gamma(U, E)$ for $k \in \mathcal{C}_U$ and $s \in \Gamma(U, E)$. This completes the proof. $\square$

Proof of Assertion 4.4. Let x be an element in V_γ. For any $\rho \in \mathcal{C}_{\pi^{-1}(V_\gamma)}$, by definition, there exists an open neighborhood $W_{s_\gamma(x)}$ of $s_\gamma(x)$ such that $\rho|_{W_{s_\gamma(x)}} = \overline{\rho}|_{W_{s_\gamma(x)}}$ for some $\overline{\rho} \in \mathcal{C}_E$. Thus we see that

$$\begin{aligned} \rho \circ s_\gamma|_{s_\gamma^{-1}(W_{s_\gamma(x)})} = \rho|_{W_{s_\gamma(x)}} \circ s_\gamma|_{s_\gamma^{-1}(W_{s_\gamma(x)})} &= \overline{\rho}|_{W_{s_\gamma(x))}} \circ s_\gamma|_{s_\gamma^{-1}(W_{s_\gamma(x)})} \\ &= \overline{\rho} \circ s_\gamma|_{s_\gamma^{-1}(W_{s_\gamma(x)})}. \end{aligned}$$

Since $s_\gamma : V_\gamma \to E$ is a morphism of differential spaces, it follows that $\overline{\rho} \circ s_\gamma$ is in $\mathcal{C}_{V_\gamma}$. Then $\overline{\rho} \circ s_\gamma$ is a restriction of a map in $\mathcal{C}_S$ to an appropriate open neighborhood of x and hence so is $\rho \circ s_\gamma$. We have the result. $\square$

We denote by $\mathcal{L}_E$ the $\mathcal{O}_S$-module of Proposition 4.3.

Lemma 4.5. *Let $pr_1 : S \times \mathbb{R}^n \to S$ be the product bundle over a stratifold $(S, \mathcal{C})$. The map $e_i : S \to S \times \mathbb{R}^n$ defined by $e_i(x) = (x, \mathsf{e}_i)$ is a section of this bundle for $i = 1, ..., n$, where $\{\mathsf{e}_1, ..., \mathsf{e}_n\}$ is a canonical basis for $\mathbb{R}^n$.*

Proof. We prove that e_i is a morphism of differential spaces. Suppose that f is in $\mathcal{C}_{S \times \mathbb{R}^n}$; see Section 6. Then there are local retractions $r_x : U_x \to U_x \cap S^j$ and $r_y = id : U_y \to U_y$ such that $f|_{U_x \times U_y} = f(r_x \times r_y)$ for $x \in S^j$ and $y = \mathsf{e}_i \in \mathbb{R}^n$. This yields that $f \circ e_i | U_x = f \circ e_i \circ r_x$. Since the restriction map $e_i|_{S^j} : S^j \to S^j \times \mathbb{R}^n$ is smooth, it follows that the composite $f \circ e_i|_{S^j} : S^j \to S^j \times \mathbb{R}^n \to \mathbb{R}$ is also smooth. In consequence, we have $f \circ e_i \in \mathcal{C}$. $\square$

Proposition 4.6. *The transition functions* $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$ *are morphisms of stratifolds.*

Proof. By the definition of the transition function, we see that $\phi_\beta\phi_\alpha^{-1}(x, v) = (x, g_{\alpha\beta}(x)v)$. It follows from Lemma 4.5 that the composite

$$\psi_j : U_\alpha \cap U_\beta \xrightarrow{e_j} U_\alpha \cap U_\beta \times \mathbb{R}^n \xrightarrow{\phi_\beta\phi_\alpha^{-1}} U_\alpha \cap U_\beta \times \mathbb{R}^n \xrightarrow{pr_2} \mathbb{R}^n$$

is a morphism of differential spaces. Therefore, for the well-defined map $\psi_j^* : C^\infty(\mathbb{R}^n) \to \mathcal{C}_{U_\alpha \cap U_\beta}$, we see that $u_{ij} := \psi_j^*(p_i) = p_i \circ \psi_j \in \mathcal{C}_{U_\alpha \cap U_\beta}$ and $g_{\alpha\beta}(x) = (u_{ij}(x))$, where $p_i : \mathbb{R}^n \to \mathbb{R}$ is the projection onto the ith factor. It turns out that $g_{\alpha\beta}$ is a morphism of stratifolds. In fact, for any $f \in \mathcal{C}_{GL_n(\mathbb{R})}$, there exists a smooth map $\overline{f} : M_{nn}(\mathbb{R}) = \mathbb{R}^{n^2} \to \mathbb{R}$ whose restriction coincides with f. Then we have $g_{\alpha\beta}^*(f)(x) = fg_{\alpha\beta}(x) = f(u_{11}(x), u_{12}(x), ..., u_{nn}(x)) = \overline{f}(u_{11}(x), u_{12}(x), ..., u_{nn}(x))$. This yields that $g_{\alpha\beta}^*(f)$ is in $\mathcal{C}_{U_\alpha \cap U_\beta}$. □

Proposition 4.7. *Let* $\pi : (E, \mathcal{C}_E) \to (S, \mathcal{C}_S)$ *be a vector bundle in the sense of Definition 4.1. Then the differential space* $(E, \mathcal{C}_E)$ *admits a stratifold structure for which* π *is a morphism of stratifolds.*

Proof. Without loss of generality, we assume that there exists a countable trivialization. Indeed S has a countable basis. Thus the existence of a countable basis of E follows from the local triviality. Moreover, the local triviality allows us to deduce that E is a Hausdorff space.

Let S^i be a stratum of S. Observe that S^i is a manifold for each i. By virtue of Proposition 4.6, we see that $\pi^{-1}(S^i)$ is a manifolds and $\pi : \pi^{-1}(S^i) \to S^i$ is a smooth vector bundle. It remains to prove that for any $x \in S^i$, the inclusion $i : \pi^{-1}(S^i) \to E$ induces an isomorphism $i^* : C(E)_x \to C^\infty(\pi^{-1}(S^i))_x$. Suppose that x is in U_α with $\phi_\alpha : \pi^{-1}(U) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^n$ a trivialization. Then we have a commutative diagram

$$\begin{array}{ccc}
(\mathcal{C}_E)_x & \xrightarrow{i^*} & C^\infty(\pi^{-1}(S^i))_x \\
{\scriptstyle res^*}\downarrow\cong & & \cong\downarrow{\scriptstyle res^*} \\
(\mathcal{C}_{\pi^{-1}(U_\alpha)})_x & \xrightarrow{i^*} & C^\infty(\pi^{-1}(S^i \cap U_\alpha))_x \\
{\scriptstyle \phi_\alpha^*}\uparrow\cong & & \cong\uparrow{\scriptstyle \phi_\alpha^*} \\
(\mathcal{C}_{U_\alpha \times \mathbb{R}^n})_{\phi_\alpha(x)} & \xrightarrow{(i \times 1_{\mathbb{R}^n})^*} & \mathcal{C}(S^i \cap U_\alpha \times \mathbb{R}^n)_{\phi_\alpha(x)}
\end{array}$$

The stratifold structure on $U_\alpha \times \mathbb{R}^n$ allows us to deduce that $(i \times 1_{\mathbb{R}^n})^*$ is an isomorphism. Then we see that the upper horizontal arrow i^* is an isomorphism.

The local triviality of the bundle implies the existence of a bump function. In fact, the existence is a local property. This completes the proof. □

Proposition 4.8. *Let $(S, \mathcal{C})$ be a stratifold and $(E, \pi) \in \mathsf{VBb}_{(S,\mathcal{C})}$. Then the $\mathcal{O}_S$-module $\mathcal{L}_E$ is a locally free module.*

Proof. Let $(\{U_\alpha\}, \{\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{n_\alpha}\})$ be a trivialization. Define sections $s_i \in \mathcal{L}_E(U_\alpha)$ by $s_i = \phi_\alpha^{-1} \circ e_i|_{U_\alpha}$ for $i = 1, ..., n_\alpha$. Then these sections are bases of $\mathcal{L}_E(U_\alpha)$, so there exists an isomorphism between $\mathcal{L}_E(U_\alpha)$ and $\mathcal{O}_S(U_\alpha)^{n_\alpha}$. This induces an isomorphism between $\mathcal{L}_E|_{U_\alpha}$ and $\mathcal{O}_S^{n_\alpha}|_{U_\alpha}$. □

For $f \in \mathrm{Hom}_{\mathsf{VBb}_{(S,\mathcal{C})}}(E, F)$, we define a map $f_* : \Gamma(U, E) \to \Gamma(U, F)$ by $f_*(s) = f \circ s$. Since $f_x : E_x \to F_x$ are linear maps, it follows that f_* is a morphism of $\mathcal{O}_S(U)$-modules. Thus f_* gives rise to a morphism $\mathcal{L}_f : \mathcal{L}_E \to \mathcal{L}_F$. Let $\mathsf{Lfb}(S)$ be the full subcategory of $\mathcal{O}_S$-Mod consisting of locally free $\mathcal{O}_S$-modules of bounded rank. Proposition 4.8 enables us to define a functor $\mathcal{L} : \mathsf{VBb}_{(S,\mathcal{C})} \to \mathsf{Lfb}(S)$. Our goal of this section is to verify that the global section functor is an equivalence of categories as well as the usual result in case of smooth manifolds.

Theorem 4.9. *Let $(S, \mathcal{C})$ be a stratifold. Then the global section functor*

$$\Gamma(S, -) : \mathsf{VBb}_{(S,\mathcal{C})} \to \mathsf{Fgp}(\mathcal{C})$$

gives rise to an equivalence of categories, where $\mathsf{Fgp}(\mathcal{C})$ denotes the category of finitely generated projective modules over $\mathcal{C}$.

We shall prove Theorem 4.9 by using the result due to Morye [10], Proposition 4.3 and an equivalence between categories $\mathsf{VBb}_{(S,\mathcal{C})}$ and $\mathsf{Lfb}(S)$ for a stratifold $(S, \mathcal{C})$, which is proved below.

Lemma 4.10. *Let X be a topological space and $\{X_\alpha\}$ an open cover of X. Suppose $(X_\alpha, \mathcal{C}_\alpha)$ is a differential space for each α. Define $\mathcal{C}$ to be the subalgebra of $C^0(X)$ consisting of $f : X \to \mathbb{R}$ such that $f|_{X_\alpha} \in \mathcal{C}_\alpha$ for all α. Then the pair $(X, \mathcal{C})$ is a differential space.*

Proof. The proof is straightforward. We check that $\mathcal{C}$ is a locally detectable $\mathbb{R}$-algebra. Let f be in $C^0(X)$. Assume further that, for each $x \in X$, there are an open neighborhood U_x of x and $h_x \in \mathcal{C}$ such that $f|_{U_x} = h_x|_{U_x}$. Since $h_x|_{X_\alpha} \in \mathcal{C}_\alpha$, $f|_{U_x \cap X_\alpha} = h_x|_{U_x \cap X_\alpha}$ and $\mathcal{C}_\alpha$ is locally detectable, it follows that $f|_{X_\alpha} \in \mathcal{C}_\alpha$ and hence $f \in \mathcal{C}$ by definition. □

Proposition 4.11. *The functor* $\mathcal{L} : \mathsf{VBb}_{(S,\mathcal{C})} \to \mathsf{Lfb}(S)$ *is essentially surjective.*

Proof. If $\mathcal{F} \in \mathsf{Lfb}(S)$, then there is an open cover $\{U_\alpha\}$ and isomorphisms $\varphi_\alpha : \mathcal{F}|_{U_\alpha} \to \mathcal{O}_S^{n_\alpha}|_{U_\alpha}$. Put $U_{\alpha,\beta} := U_\alpha \cap U_\beta$. We have a transition function $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}_n(\mathbb{R})$ induced by the isomorphisms φ_α and φ_β. More precisely, consider the sequence of morphisms of $\mathcal{O}_{U_{\alpha\beta}}$-modules

$$\mathcal{O}_S|_{U_{\alpha\beta}} \xrightarrow{in_j} \mathcal{O}_S^n|_{U_{\alpha\beta}} \xrightarrow[\cong]{\varphi_\beta^{-1}} \mathcal{F}|_{U_{\alpha\beta}} \xrightarrow[\cong]{\varphi_\alpha} \mathcal{O}_S^n|_{U_{\alpha\beta}} \xrightarrow{p_i} \mathcal{O}_S|_{U_{\alpha\beta}},$$

where in_j and p_i denote the inclusion into the jth factor and the projection onto the ith factor, respectively. We define $u_{ij} \in \mathcal{O}_S|_{U_{\alpha\beta}}(U_{\alpha\beta})$ by $u_{ij} = p_i\varphi_\alpha\varphi_\beta^{-1}in_j(\mathbf{1})$ with unit $\mathbf{1}$ in $\mathcal{O}_S|_{U_{\alpha\beta}}(U_{\alpha\beta}) = \mathcal{O}(U_{\alpha\beta})$. Then $g_{\alpha\beta}$ is defined by $g_{\alpha\beta}(x) = (u_{ij}(x))$ for $x \in U_{\alpha\beta}$.

For each $x \in U_\alpha \cap U_\beta \cap U_\gamma$, these transition functions satisfy the relation $g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$. This enables us to define a space E by the quotient space $(\bigsqcup_\alpha U_\alpha \times \mathbb{R}^{n_\alpha})/\sim$, where the equivalence relation $\sim$ is defined by $(x, \mathsf{v}) \sim (y, \mathsf{w})$ if $x = y \in U_{\alpha\beta}$ and $\mathsf{v} = g_{\alpha\beta}(x)\mathsf{w}$. Let $\rho : \bigsqcup_\alpha U_\alpha \times \mathbb{R}^{n_\alpha} \to E$ be the canonical projection. Then we define a continuous map $\pi : E \to S$ by $\pi(\rho(x, \mathsf{v})) = x$. Since the restriction $\rho_\alpha : U_\alpha \times \mathbb{R}^{n_\alpha} \to \pi^{-1}(U_\alpha)$ is a homeomorphism, it gives a subalgebra $\mathcal{C}_\alpha$ of $C^0(\pi^{-1}(U_\alpha))$ which is naturally isomorphic to $\mathcal{C}_{U_\alpha\times\mathbb{R}^{n_\alpha}}$. By Lemma 4.10, we have a differential space $(E, \mathcal{C}_E)$.

If $f \in \mathcal{C}_S$, then $(f \circ \pi)|_{\rho_\alpha(U_\alpha\times\mathbb{R}^{n_\alpha})} \in \mathcal{C}_\alpha$ since the projection $U_\alpha \times \mathbb{R}^{n_\alpha} \to U_\alpha$ is a morphism of differential spaces; see Section 6. This implies that $f \circ \pi \in \mathcal{C}_E$ and hence the map $\pi : (E, \mathcal{C}_E) \to (S, \mathcal{C}_S)$ is a morphism of differential spaces. Moreover, we can see that the morphism π is a vector bundle with trivializations $(\{U_\alpha\}, \{\rho_\alpha\})$.

We shall show that $\mathcal{L}_E$ is isomorphic to $\mathcal{F}$. For $s \in \mathcal{L}_E(U_\alpha)$, we define $\hat{s} \in \mathcal{O}_S(U_\alpha)^{n_\alpha}$ by the composite $pr_2 \circ \rho_\alpha^{-1} \circ s : U_\alpha \to \pi^{-1}(U_\alpha) \to U_\alpha \times$

$\mathbb{R}^n \to \mathbb{R}^n$. Since $\psi_\alpha : \mathcal{L}_E(U_\alpha) \to \mathcal{O}_S(U_\alpha)^{n_\alpha}$ defined by $\psi_\alpha(s) = \hat{s}$ is an isomorphism, it gives rise to an isomorphism $\psi_\alpha : \mathcal{L}_E|_{U_\alpha} \to \mathcal{O}_S^{n_\alpha}|_{U_\alpha}$. The definitions of ψ_α and E allow us to deduce that $\varphi_\alpha^{-1} \circ \psi_\alpha = \varphi_\beta^{-1} \circ \psi_\beta$. Therefore, we have a morphisms of equalizers

$$\begin{array}{ccccc} \mathcal{L}_F(U) & \longrightarrow & \prod_\alpha \mathcal{L}_F(U_\alpha \cap U) & \overset{res^\alpha_{\alpha\beta}}{\underset{res^\beta_{\alpha\beta}}{\rightrightarrows}} & \prod_{\alpha\beta} \mathcal{L}_F(U_\alpha \cap U_\beta \cap U) \\ & & \cong \big\downarrow \varphi_\alpha^{-1}\psi_\alpha & & \cong \big\downarrow \varphi_\alpha^{-1}\psi_\alpha = \varphi_\beta^{-1}\psi_\beta \\ \mathcal{F}(U) & \longrightarrow & \prod_\alpha \mathcal{F}(U_\alpha \cap U) & \overset{res^\alpha_{\alpha\beta}}{\underset{res^\beta_{\alpha\beta}}{\rightrightarrows}} & \prod_{\alpha\beta} \mathcal{F}(U_\alpha \cap U_\beta \cap U). \end{array}$$

This yields that $\mathcal{L}_E \cong \mathcal{F}$ as an $\mathcal{O}_S$-module. Hence the functor $\mathcal{L}$ is essentially surjective. □

Proposition 4.12. *The functor $\mathcal{L}$ is fully faithful.*

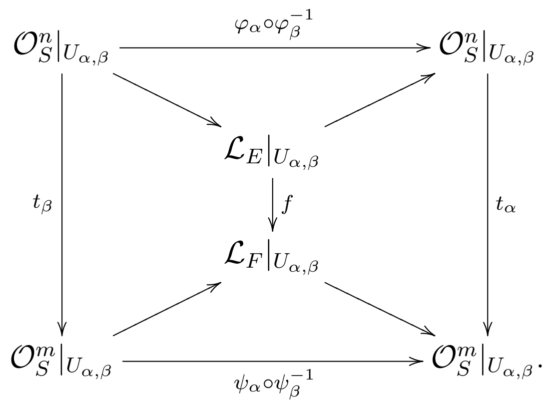
Proof. Let f and g be morphisms from (E, π_E) to (F, π_F). Assume that $\mathcal{L}_f = \mathcal{L}_g$. Then for all sections $s \in \Gamma(U, E)$, we see that $f \circ s = g \circ s$. This implies that $f = g$.

Suppose that $f : \mathcal{L}_E \to \mathcal{L}_F$ is a morphism in $\mathsf{Lfb}(S)$ and $\varphi_\alpha : \mathcal{L}_E|_{U_\alpha} \to \mathcal{O}_S^n|_{U_\alpha}$ and $\psi_\alpha : \mathcal{L}_F|_{U_\alpha} \to \mathcal{O}_S^m|_{U_\alpha}$ are trivializations which is induced by the given trivializations of E and F; see Proposition 4.8. Then we obtain the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_S^n|_{U_\alpha} & \xrightarrow[\cong]{\varphi_\alpha^{-1}} & \mathcal{L}_E|_{U_\alpha} \\ t_\alpha \big\downarrow & & \big\downarrow f \\ \mathcal{O}_S^m|_{U_\alpha} & \xleftarrow[\psi_\alpha]{\cong} & \mathcal{L}_F|_{U_\alpha}. \end{array}$$

The morphism t_α induces a morphism $t_\alpha : U_\alpha \to \mathrm{Mat}_{m,n}(\mathbb{R})$ of stratifolds with such way of defining $g_{\alpha\beta}$ in the proof of Proposition 4.11. We define a map $\eta_\alpha : E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^n \to U_\alpha \times \mathbb{R}^m \cong F|_{U_\alpha}$ by $\eta_\alpha(x, \mathsf{v}) = (x, t_\alpha(x)\mathsf{v})$. This map is a morphism of stratifolds since the restriction on each manifold $(U_\alpha \cap S^i) \times \mathbb{R}^n$ is smooth and for $l \in \mathcal{C}_{S\times\mathbb{R}^m}$, there are local retraction $r_x : U_x \to U_x \cap S^i$ and open set V of $\mathbb{R}^n$ such that $l \circ \eta_\alpha|_{U_x \times V} = l \circ \eta_\alpha(r_x \times id_V)$. Then the maps η_α induce a morphism $\eta : E \to F$ with $\eta_\alpha = \eta|_{\pi^{-1}(U_\alpha)}$:

$\pi_E^{-1}(U_\alpha) \to \pi_F^{-1}(U_\alpha)$. In fact, we have a commutative diagram



By the construction of $\mathcal{L}_\eta$, it is readily seen that $\mathcal{L}_\eta = f$ and hence the functor $\mathcal{L}$ is full. □

Thanks to Propositions 4.11 and 4.12, we see that the functor $\mathcal{L}$ is an equivalence of categories. We shall prove that the category $\mathsf{Lfb}(S)$ is equivalent to the full subcategory of the category $\Gamma(S, \mathcal{O}_S)$-Mod consisting of finitely generated projective modules, which is denoted by $\mathsf{Fgp}(\Gamma(S, \mathcal{O}_S))$.

Following Morye, we say that the *Serre-Swan theorem* holds for a locally ringed space $(X, \mathcal{O}_X)$ if the global section functor induces an equivalence of categories between $\mathsf{Lfb}(X)$ and $\mathsf{Fgp}(\Gamma(X, \mathcal{O}_X))$. The following theorem then completes the proof of Theorem 4.9.

Theorem 4.13. (Morye, [10, Corollary 3.2]) *Let $(X, \mathcal{O}_X)$ be a locally ringed space such that X is a paracompact Hausdorff space of finite covering dimension, and $\mathcal{O}_X$ is a fine sheaf of rings (cf. Definition 3.2). Then the Serre-Swan theorem holds for $(X, \mathcal{O}_X)$.*

The structure sheaf $\mathcal{O}_S$ of a stratifold $(S, \mathcal{C})$ is fine and the underlying space S is paracompact; see Remark 2.3(iii). In order to prove Theorem 4.9, it is thus sufficient to show that the covering dimension $\dim S$ of S is finite.

Theorem 4.14. *[13, Proposition 5.1 in chapter 3] Any n-dimensional paracompact manifold M (without boundary) has covering dimension $\dim M = n$.*

Theorem 4.15. *[13, Proposition 5.11 in chapter 3] Let X be a normal space and A and B be subspaces of X such that $X = A \cup B$. Then, dim $X \leq$ dim $A +$ dim $B + 1$.*

Corollary 4.16. *Any finite-dimensional stratifold has finite covering dimension.*

Proof. By the definition of a stratifold, we see that $S = S^1 \sqcup S^2 \sqcup ... \sqcup S^n$, where S^i is a manifold of dimension i. Theorems 4.14 and 4.15 imply that $\dim S < \infty$. □

By definition, it follows that $\Gamma(S, \mathcal{O}_S) = \mathcal{O}_S(S) = \mathcal{C}$. Thus Proposition 4.13 and the results above enable us to deduce the following corollary.

Corollary 4.17. *Let $(S, \mathcal{C})$ be a stratifold and $\mathcal{O}_S$ the structure sheaf. Then the global sections functor $\Gamma(S, -) : \mathsf{Lfb}(S) \to \mathsf{Fgp}(\mathcal{C})$ is an equivalence.*

We are now ready to prove the main theorem in this section.

Proof of Theorem 4.9. Corollary 4.17, Proposition 4.11 and 4.12 yield Theorem 4.9 □

Remark 4.18. Theorem 3.5 states that a stratifold $(S, \mathcal{C})$ can be regarded as a subsheaf of an affine scheme of the form Spec $\mathcal{O}_S(S)$. Since the space Spec $\mathcal{O}_S(S)$ is compact, it follows that the real spectrum Spec$_r \mathcal{O}_S(S)$ is a proper subspace of the prime spectrum if S is non-compact; see Proposition 3.3. Moreover, in general, there exists a point in Spec $\mathcal{O}_S(S)$ which is a maximal ideal but not in the real spectrum. Such a point is called a *ghost*; see [12, 8.22]. However, Theorem 4.9 and the original Serre-Swan theorem yield that the category $\mathsf{VBb}_{(S,\mathcal{C})}$ is equivalent to $\mathsf{VBb}_{\mathrm{Spec}\,\mathcal{O}_S(S)}$ the category of vector bundles over the affine scheme Spec $\mathcal{O}_S(S)$ via the category $\mathsf{Fgp}(\Gamma(S, \mathcal{O}_S))$; see [10, Corollary 3.1] and [16, Theorem 6.2] for example.

5. A local characterization of morphisms of stratifolds

In this section, we describe morphisms of stratifolds inside the category of diffeological spaces. On the way we obtain a characterization of them by local data. We use the terminology of the book [4] for diffeology.

Let Diffeology be the category of diffeological spaces; see [4]. We define a functor k : Stfd $\to$ Diffeology by $k(S, \mathcal{C}) = (S, \mathcal{D}_{\mathcal{C}})$ and $k(\phi) = \phi$ for a morphism $\phi : S \to S'$ of stratifolds, where

$$\mathcal{D}_{\mathcal{C}} = \{u : U \to S \mid U : \text{open in } \mathbb{R}^q, q \geq 0, \phi \circ u \in C^\infty(U) \text{ for any } \phi \in \mathcal{C}\}.$$

Observe that a plot in $\mathcal{D}_{\mathcal{C}}$ is a set map. The functor k is faithful, but not full; that is, for a continuous map $f : S \to S'$, it is more restrictive to be a morphism of stratifolds $(S, \mathcal{C}) \to (S', \mathcal{C}')$ than to be a morphism of diffeological spaces $(S, \mathcal{D}_{\mathcal{C}}) \to (S', \mathcal{D}_{\mathcal{C}'})$.

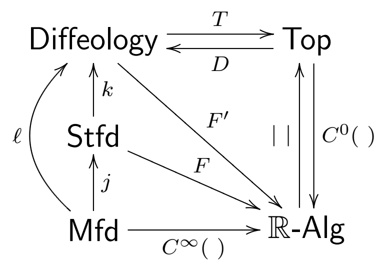
We recall the fully faithful functor ℓ : Mfd $\to$ Diffeology defined in [4, 4.3]; see also [1, Theorem 2.3]. For a diffeological space $(X, \mathcal{D})$, the set X admits a topology which is referred as the D-topology. More precisely, a subset A of X is open if and only if $p^{-1}(A)$ is open for any plot $p \in \mathcal{D}$. We denote by $T(X, \mathcal{D})$ the topological space. It is readily seen that the assignment of a topological space to a diffeological space induces a functor T : Diffeology $\to$ Top.

For a topological space Y, we define a diffeological space $D(Y) = (Y, \mathcal{D}_Y)$ in which the set of plots $\mathcal{D}_Y$ consists of all continuous maps $U \to Y$ for any open subset U of $\mathbb{R}^q$ and for $q \geq 0$.

Let $\mathcal{D}_{\mathbb{R}}$ be the standard diffeology on $\mathbb{R}$. For each diffeological space $(X, \mathcal{D})$, we have an $\mathbb{R}$-algebra $F'((X, \mathcal{D})) := \mathrm{Hom}_{\mathsf{Diffeology}}((X, \mathcal{D}), (\mathbb{R}, \mathcal{D}_{\mathbb{R}}))$ with the algebra structure defined pointwise. A usual argument enables us to conclude that F' gives rise to a contravariant functor F' : Diffeology $\to$ $\mathbb{R}$-Alg.

Summarizing the functors mentioned above, we have a diagram

(5.1)



in which the lower triangle and the left-hand side diagram are commutative. We observe that the functor T is a left adjoint to D; see [17, Proposition 3.1].

Moreover, it follows that the functor C^0 and $|\ |$ are adjoints. In fact, we have bijections

$$\mathrm{Hom}_{\mathsf{Top}}(X, |\mathcal{F}|) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathrm{Hom}_{\mathbb{R}\text{-}\mathsf{Alg}}(\mathcal{F}, C^0(X))$$

which are defined by the composites $\Phi(g) : \mathcal{F} \xrightarrow{\tau} \widetilde{\mathcal{F}} \xrightarrow{l} C^0(|\mathcal{F}|) \xrightarrow{g^*} C^0(X)$ with the inclusion l and $\Psi(\varphi) : X \xrightarrow{\theta} |C^0(X)| \xrightarrow{|\varphi|} |\mathcal{F}|$, respectively. The bijectivity follows from a straightforward computation.

We give here a characterization of morphisms of stratifolds in Diffeology with local data.

Proposition 5.1. *A morphism of diffeological spaces $f : (S, \mathcal{D}_{\mathcal{C}}) \to (S', \mathcal{D}_{\mathcal{C}'})$ stems from a morphism of stratifolds $f : (S, \mathcal{C}) \to (S', \mathcal{C}')$ if and only if for any $x \in S$, there exist local retractions $r_x : U_x \to U_x \cap S^i$ and $r_{f(x)} : V_{f(x)} \to V_{f(x)} \cap S'^j$ such that $r_{f(x)} \circ f \circ r_x = r_{f(x)} \circ f$ on some neighborhood of x.*

Proof. By definition, for any $x \in S$, there exists uniquely an integer i such that x is in S^i. Let $r_x : U_x \to U_x \cap S^i$ be a local retraction. The stratum S^i is an i-dimensional manifold. Therefore, we have a local diffeomorphism $\varphi_i : V_x \to U_x \cap S^i$ for some open subset V_x of $\mathbb{R}^i$. Let $u : V_x \to S$ be the composite $l \circ \varphi_i$, where $l : U_x \cap S^i \to S$ is the inclusion.

Suppose that $f : (S, \mathcal{D}_{\mathcal{C}}) \to (S', \mathcal{D}_{\mathcal{C}'})$ is a morphism of diffeological spaces. In order to prove the "if" part, it suffices to show that for any $x \in S$, the induced morphism $f^* : \mathcal{C}'_{f(x)} \to \mathcal{S}et(S, \mathbb{R})_x$ factors through the $\mathbb{R}$-algebra $\mathcal{C}_x$ of germs, where $\mathcal{S}et(S, \mathbb{R})_x$ denotes the germ at x of set maps $S \to \mathbb{R}$ associated with open neighborhoods of x. In fact, it follows that for any $\alpha \in \mathcal{C}'$, $f^*([\alpha]_{f(x)}) = [\alpha \circ f]_x \in \mathcal{C}_x$. Then there exists $\beta \in \mathcal{C}$ such that $(\alpha \circ f)|_{W_x} = \beta|_{W_x}$ for some open subset W_x of S. Since the $\mathbb{R}$-algebra $\mathcal{C}$ is locally detectable, we see that $f : S \to S'$ is a morphism of stratifolds and hence $k(f) = f$.

Consider the following diagram

$$\begin{array}{ccccccc} & & & \mathcal{C}'_{f(x)} & & & \\ & \swarrow^{(f\circ u)^*} & & & & \searrow^{f^*} & \\ C^\infty(V_x)_{\varphi_i^{-1}(x)} & \underset{\varphi_i^*}{\overset{\psi_i^*}{\rightleftarrows}}\cong & C^\infty(S^i)_x & \underset{l^*}{\overset{r_x^*}{\rightleftarrows}}\cong & \mathcal{C}_x & \xrightarrow{s} & \mathcal{S}et(S, \mathbb{R})_x, \end{array}$$

where ψ_i is the local inverse of φ_i and s denotes the inclusion. Observe that $(f \circ u)^* : \mathcal{C}'_{f(x)} \to C^\infty(V_x)_{\varphi_i^{-1}(x)}$ *is well defined since* f *is a morphism of diffeological spaces*. For any $\alpha \in \mathcal{C}'$, we see that $\alpha = r^*_{f(x)}(\alpha)$ in $\mathcal{C}'_{f(x)}$; see [8, page 19]. Thus it follows that

$$\begin{aligned} r_x^*\psi_i^*(f \circ u)^*(r^*_{f(x)}(\alpha)) &= \alpha \circ r_{f(x)} \circ f \circ l \circ \varphi_i \circ \psi_i \circ r_x \\ &= \alpha \circ r_{f(x)} \circ f \circ r_x = \alpha \circ r_{f(x)} \circ f = \alpha \circ f. \end{aligned}$$

The third equality follows from the assumption. This implies that f^* factors through the algebra $\mathcal{C}_x$ since $r_x^*\psi_i^*(f \circ u)^*(r^*_{f(x)}(\alpha))$ is in $\mathcal{C}_x$.

We prove the "only if" part. Let $f : (S, \mathcal{C}) \to (S', \mathcal{C}')$ be a morphism of stratifolds and $r_x : U_x \to U_x \cap S^i$ and $r_{f(x)} : V_{f(x)} \to V_{f(x)} \cap S'^j$ are appropriate local retractions. Without loss of generalities, we may assume that the image of $r_{f(x)}$ is contained in a local coordinate $V'_{f(x)}$ of the manifold $V_{f(x)} \cap S'^j$. We have $r_x^* \circ \psi_i^* \circ (f \circ u)^* = f^*$. Observe that $(f \circ u)^* = \varphi_i^* \circ l^* \circ f^*$ and that the target of f^* is the algebra $\mathcal{C}_x$. Let π_k be an element in $\mathcal{C}^\infty(V_{f(x)} \cap S'^j)_{f(x)}$ obtained by extending the composite $V'_{f(x)} \xrightarrow{\cong} V' \xrightarrow{t} \mathbb{R}^j \xrightarrow{pr_k} \mathbb{R}$ by a bump function at $f(x)$, where $V'_{f(x)} \xrightarrow{\cong} V'$ is the homeomorphism of the local coordinate, t is the inclusion and pr_k denotes the projection onto the kth factor. Then for the element $r^*_{f(x)}(\pi_k) \in \mathcal{C}'_{f(x)}$, we have $r_x^* \circ \psi_i^* \circ (f \circ u)^*(r^*_{f(x)}(\pi_k)) = f^* \circ r^*_{f(x)}(\pi_k)$. The same argument as above enables us to deduce that

$$\pi_k \circ r_{f(x)} \circ f \circ r_x = \pi_k \circ r_{f(x)} \circ f$$

on some neighborhood W_x of x and hence $r_{f(x)} \circ f \circ r_x = r_{f(x)} \circ f$ on W_x. This completes the proof. □

Corollary 5.2. *Let* M *be a manifold and* $(S, \mathcal{C})$ *a stratifold. Then the functor* $k : \mathsf{Stfd} \to \mathsf{Diffeology}$ *induces a bijection*

$$k_* : \mathrm{Hom}_{\mathsf{Stfd}}((M, C^\infty(M)), (S, \mathcal{C})) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{Diffeology}}((M, \mathcal{D}_{C^\infty(M)}), (S, \mathcal{D}_{\mathcal{C}})).$$

In the rest of this section, we give a subcategory of Diffeology which is equivalent to Stfd as a category.

Definition 5.3. Let $(S, \mathcal{C})$ and $(S', \mathcal{C}')$ be stratifolds. A continuous map $f : S \to S'$ is $(\mathcal{C}, \mathcal{C}')$-*admissible* if for any $x \in S$, there exist local retractions

r_x and $r_{f(x)}$ near x and $f(x)$, respectively such that $r_{f(x)} \circ f \circ r_x = r_{f(x)} \circ f$ and for each $\phi \in \mathcal{C}'$, the restriction of $\phi \circ f : S \to \mathbb{R}$ to any stratum S^i of $(S, \mathcal{C})$ is smooth.

The proof of Proposition 5.1 yields the following result, which recovers as special case [8, Exercise 2.6(11)].

Proposition 5.4. *A continuos map $f : S \to S'$ induces a morphism of stratifolds $(S, \mathcal{C}) \to (S', \mathcal{C}')$ if and only if f is $(\mathcal{C}, \mathcal{C}')$-admissible.*

Let $(S, \mathcal{C})$, $(S', \mathcal{C}')$ and $(S'', \mathcal{C}'')$ be stratifolds. Proposition 5.4 immediately implies that $(\mathcal{C}, \mathcal{C}')$-admissible continuous maps compose with $(\mathcal{C}', \mathcal{C}'')$-admissible continuous maps $S' \to S''$.

Let $k : \mathsf{Stfd} \to \mathsf{Diffeology}$ be the functor in (5.1) and $\langle \mathsf{Im} k \rangle$ the full subcategory of Diffeology consisting of objects which come from Stfd by k. By the argument above, we have a wide subcategory $\langle \mathsf{Im} k \rangle_W$ of $\langle \mathsf{Im} k \rangle$ consisting of admissible maps and the same class of objects as in $\langle \mathsf{Im} k \rangle$. Then Proposition 5.1 establishes the following theorem.

Theorem 5.5. *The functor $k : \mathsf{Stfd} \to \mathsf{Diffeology}$ induces an equivalence $k : \mathsf{Stfd} \to \langle \mathsf{Im} k \rangle_W$ of categories. In particular, one has a natural bijection*

$$k_* : \mathrm{Hom}_{\mathsf{Stfd}}((S, \mathcal{C}), (S', \mathcal{C}')) \xrightarrow{\cong} \mathrm{Hom}_{\langle \mathsf{Im} k \rangle_W}((S, \mathcal{D}_{\mathcal{C}}), (S', \mathcal{D}_{\mathcal{C}'})).$$

6. Cartesian product of stratifolds

We recall the product of stratifolds defined in [8]. Let $(S, \mathcal{C}_S)$ and $(S', \mathcal{C}_{S'})$ be stratifolds. We define a stratifold with the underlying topological space $S \times S'$. Let $\mathcal{C}_{S \times S'}$ be the $\mathbb{R}$-algebra consisting of functions $f : S \times S' \to \mathbb{R}$ which are smooth on every products $S^i \times (S')^j$ and for each $(x, y) \in S^i \times (S')^j$, there are local retractions $r_x : U_x \to S^i \cap U_x$ and $r_y : V_y \to (S')^j \cap V_y$ for which $f|_{U_x \times V_y} = f(r_x \times r_y)$. Then $(S \times S', \mathcal{C}_{S \times S'})$ is a stratifold and the projections into first and second factors are morphisms of stratifolds; see [8, Appendix A].

Proposition 6.1. *The product of stratifolds mentioned above is the cartesian product in the category* Stfd.

We use lemmas to prove Proposition 6.1.

Lemma 6.2. *Let* $f_1 : (S_1, \mathcal{C}_1) \to (S'_1, \mathcal{C}'_1)$ *and* $f_2 : (S_2, \mathcal{C}_2) \to (S'_2, \mathcal{C}'_2)$ *be morphisms of stratifolds. Then the product of maps*

$$f_1 \times f_2 : (S_1 \times S_2, \mathcal{C}_{S_1 \times S_2}) \to (S'_1 \times S'_2, \mathcal{C}_{S'_1 \times S'_2})$$

is a morphism of stratifolds.

Proof. For $x \in S_i$, assume that $x \in S_i^{k_i}$ and $f_i(x) \in S_i'^{j_i}$. Then we have a diagram

$$\begin{array}{ccc} \mathcal{C}'_{f_i(x)} & \xrightarrow{\cong} & C^\infty(S_i'^{j_i})_{f_i(x)} \\ {\scriptstyle f_i^*}\downarrow & & \\ \mathcal{C}_x & \xrightarrow{\cong} & C^\infty({S_i}^{k_i})_x \end{array}$$

in which horizontal maps induced by the inclusions are isomorphisms. Therefore, for a smooth map φ defined on an appropriate neighborhood of $f_i(x)$ in $S_i'^{j_i}$, we see that $\varphi \circ r_{f_i(x)} \circ f_i$ is a smooth map on some neighborhood of x in $S_i^{k_i}$, where $r_{f_i(x)}$ denotes a local retraction near $f_i(x)$. Thus, we infer that for any h in $\mathcal{C}_{S'_1 \times S'_2}$,

$$\begin{aligned} h \circ (f_1 \times f_2)|_{S_1^{k_1} \times S_2^{k_2}} &= h \circ (r_{f_1(x_1)} \times r_{f_2(x_2)}) \circ (f_1 \times f_2) \\ &= (h \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ (r_{f_1(x_1)} \times r_{f_2(x_2)}) \circ (f_1 \times f_2) \end{aligned}$$

on some neighborhood of (x_1, x_2) in $S_1^{k_1} \times S_2^{k_2}$, where φ_α is a local coordinate around $(f_1(x_1), f_2(x_2))$ of the manifold $S_1'^{j_1} \times S_2'^{j_2}$. This implies that $h \circ (f_1 \times f_2)|_{S_1^{k_1} \times S_2^{k_2}}$ is smooth. Since f_1 and f_2 are admissible, it follows that for $h \in \mathcal{C}_{S'_1 \times S'_2}$,

$$\begin{aligned} h \circ (f \times f_2) \circ (r_{x_1} \times r_{x_2}) &= h \circ (r_{f_1(x_1)} \times r_{f_2(x_2)}) \circ (f \times f_2) \circ (r_{x_1} \times r_{x_2}) \\ &= h \circ (r_{f_1(x_1)} \times r_{f_2(x_2)}) \circ (f \times f_2) \\ &= h \circ (f \times f_2) \end{aligned}$$

on an appropriate neighborhood of (x_1, x_2) in $S_1 \times S_2$. This completes the proof. □

By the same argument as in the proof of Lemma 6.2, we have the following lemma.

Lemma 6.3. *The diagonal map $\Delta : S \to S \times S$ is a morphism of stratifolds.*

Proof of Proposition 6.1. Let $(S, \mathcal{C})$, $(S', \mathcal{C}')$ and $(Z, \mathcal{C}_1)$ be stratifolds. Let $f_1 : (Z, \mathcal{C}_1) \to (S, \mathcal{C})$ and $f_2 : (Z, \mathcal{C}_1) \to (S', \mathcal{C}')$ be morphisms of stratifolds. It suffices to show that $(f_1 \times f_2) \circ \Delta$ is a morphism of stratifolds. This follows from Lemmas 6.2 and 6.3 □

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