cahiers de topologie et géométrie différentielle catégoriques

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ADJOINTS FOR MULTIPLE CATEGORIES
(ON WEAK AND LAX MULTIPLE CATEGORIES, III)
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Résumé. Suite aux deux premiers articles de cette série, on étudie ici les foncteurs adjoints entre catégories multiples de dimension infinie. Le cadre général est constitué par les *catégories multiples chirales* - une forme faible partiellement laxe ayant des interchangeurs dirigés entre les compositions faibles.

Abstract. Continuing our first two papers in this series, we study adjoints for infinite-dimensional multiple categories. The general setting is chiral multiple categories - a weak, partially lax form with directed interchanges between the weak composition laws.

Keywords. multiple category, double category, cubical set, adjoint functor.

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0. Introduction

This is the third paper in a series on weak and lax multiple categories, of finite or infinite dimension - an extension of weak double and weak cubical categories.

Our main framework, a *chiral multiple category*, was introduced in the first article [GP8], cited below as Part I; it is a partially lax multiple category with a strict composition $gf = f +_0 g$ in direction 0 (the *transversal direction*), weak compositions $x +_i y$ in all positive (or *geometric*) directions $i \in \mathbb{N} \setminus \{0\}$ and a directed *ij-interchanger* for the i- and j-compositions (for $0 < i < j$)

$$\chi_{ij}(x, y, z, u) \colon (x +_i y) +_j (z +_i u) \to (x +_j z) +_i (y +_j u). \qquad (1)$$

Part II [GP9] studies multiple limits in this setting. We now investigate multiple adjoints, extending the study of double adjunctions in [GP2] and cubical adjunctions in [G3].

Section 1 is an informal introduction to multiple adjunctions. After a synopsis of weak and chiral multiple categories, we describe a natural colax/lax adjunction $F \dashv G$ between the weak *double* categories of spans and cospans (already studied in [GP2]) and its extension to the corresponding infinite dimensional, weak *multiple* categories (of cubical type); the functor F is constructed with pushouts and is colax, while G is constructed with pullbacks and is lax. Then we derive from this adjunction other instances, between chiral multiple categories that are not of cubical type. Other examples are given in 1.7.

In Section 2 we introduce the strict *double* category $\mathbb{C}$mc of chiral multiple categories (or *cm-categories*), lax and colax cm-functors and suitable double cells. Comma cm-categories are also considered. Both topics extend notions of weak double categories developed in [GP2].

Section 3 reviews the notions of companions and adjoints *in a double category*, from [GP2]. Then Sections 4 and 5 introduce and study multiple colax/lax adjunctions, as adjoint arrows *in* the double category $\mathbb{C}$mc.

Finally, Section 6 deals with the preservation of limits by right adjoints, for cm-categories.

Literature. Strict double and multiple categories were introduced and studied by C. Ehresmann and A.C. Ehresmann [Eh, BE, EE1, EE2, EE3]. Strict cubical categories can be seen as a particular case of multiple categories (as shown in Part I); their links with strict ω-categories are made clear in [BM, ABS]. Weak double categories (or pseudo double categories) were introduced and studied in our series [GP1 - GP4]; adjunctions and monads in this setting are also studied in [FGK1, FGK2, Ni]; other aspects are developed in [DPR, Fi, Ga, P2, P3]. For weak cubical categories see [G1 - G3] and [GP5]. The three-dimensional case of lax triple categories covers and combines diverse structures like duoidal categories, Gray categories, Verity double bicategories and monoidal double categories; see [GP6, GP7]. Further information on literature for higher dimensional category theory can be found in the Introduction of Part I.

Notation. We follow the notation of Parts I and II [GP8, GP9]; the reference I.2.3 or II.2.3 points to Subsection 2.3 of Part I or Part II. The symbol $\subset$ denotes weak inclusion. Categories and 2-categories are generally denoted as $\mathbf{A}, \mathbf{B}$...; weak double categories as $\mathbb{A}, \mathbb{B}$...; weak or lax multiple categories

as $A, B...$ More specific points of notation are recalled below, in 1.1.

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1. Some basic examples of adjunctions

We begin by recalling examples of adjunctions for *weak double categories*, studied in [GP2], and their extension to (infinite-dimensional) *weak cubical categories*, studied in [G3]. Then we derive from the latter some instances of adjunctions between *chiral multiple* categories that are not of a cubical type.

This section is an informal introduction to such adjunctions, precise definition will be given later.

1.1 Notation

The definitions of weak and chiral multiple categories can be found in Part I, or - briefly reviewed - in Part II, Section 1. Here we only give a sketch of them, while recalling the notation we are using.

The two-valued index α (or β) varies in the set $2 = \{0, 1\}$, also written as $\{-, +\}$.

A *multi-index* $\mathbf{i}$ is a finite subset of $\mathbb{N}$, possibly empty. Writing $\mathbf{i} \subset \mathbb{N}$ it is understood that $\mathbf{i}$ is finite; writing $\mathbf{i} = \{i_1, ..., i_n\}$ it is understood that $\mathbf{i}$ has n *distinct* elements, written in the natural order $i_1 < i_2 < ... < i_n$; the integer $n \geqslant 0$ is called the *dimension* of $\mathbf{i}$. We write:

$$\begin{aligned} \mathbf{i}j = j\mathbf{i} = \mathbf{i} \cup \{j\} && (\text{for } j \in \mathbb{N} \setminus \mathbf{i}), \\ \mathbf{i}|j = \mathbf{i} \setminus \{j\} && (\text{for } j \in \mathbf{i}). \end{aligned} \tag{2}$$

For a weak multiple category A, the set of $\mathbf{i}$-cells $A_\mathbf{i}$ is written as A_*, A_i, A_{ij} when $\mathbf{i}$ is $\varnothing, \{i\}, \{i, j\}$ respectively. Faces and degeneracies, satisfying the *multiple relations*, are denoted as

$$\partial_j^\alpha \colon X_\mathbf{i} \to X_{\mathbf{i}|j}, \qquad e_j \colon X_{\mathbf{i}|j} \to X_\mathbf{i}. \tag{3}$$

The *transversal direction* $i = 0$ is set apart from the positive, or *geometric*, directions. For a *positive multi-index* $\mathbf{i} = \{i_1, ..., i_n\} \subset \mathbb{N} \setminus \{0\}$ the *augmented multi-index* $0\mathbf{i} = \{0, i_1, ..., i_n\}$ has dimension $n + 1$, but both $\mathbf{i}$

and 0i have *degree n*. An i-cell $x \in A_\mathbf{i}$ of A is also called an i-*cube*, while a 0i-cell $f \in A_{0\mathbf{i}}$ is viewed as an i-*map* $f\colon x \to_0 y$ (also written as $f\colon x \to y$), where $x = \partial_0^- f$ and $y = \partial_0^+ f$. Composition in direction 0 is categorical (and generally realised by ordinary composition of mappings); it is written as $gf = f +_0 g$, with identities $1_x = \mathrm{id}(x) = e_0(x)$. The *transversal category* $\mathrm{tv}_\mathbf{i}(\mathsf{A})$ consists of the i-cubes and i-maps of A, with transversal composition and identities.

On the other hand, composition of i-cubes and i-maps in a *positive* direction $i \in \mathbf{i}$ (often realised by pullbacks, pushouts, tensor products, etc.) is written in additive notation

$$x +_i y \qquad\qquad\qquad\qquad\qquad\qquad (\partial_i^+ x = \partial_i^- y),$$
$$f +_i g\colon x +_i y \to x' +_i y' \quad (f\colon x \to x',\ g\colon y \to y',\ \partial_i^+ f = \partial_i^- g). \qquad (4)$$

These operations are *categorical* and *interchangeable* up to transversally-invertible comparisons (for $0 < i < j$, see I.3.2)

$$\lambda_i x\colon (e_i \partial_i^- x) +_i x \to_0 x \qquad\qquad\qquad (\textit{left i-unitor}),$$
$$\rho_i x\colon x +_i (e_i \partial_i^+ x) \to_0 x \qquad\qquad\qquad (\textit{right i-unitor}),$$
$$\kappa_i(x,y,z)\colon x +_i (y +_i z) \to_0 (x +_i y) +_i z \qquad (\textit{i-associator}), \quad (5)$$
$$\chi_{ij}(x,y,z,u)\colon (x +_i y) +_j (z +_i u) \to_0 (x +_j z) +_i (y +_j u)$$
$$(\textit{ij-interchanger}).$$

The comparisons are natural with respect to transversal maps; λ_i, ρ_i and κ_i are *special in direction i* (i.e. their i-faces are transversal identities), while χ_{ij} is *special in both directions i, j*; all of them commute with ∂_k^α for $k \neq i$ (or $k \neq i, j$ in the last case). Finally the comparisons must satisfy various conditions of coherence, listed in I.3.3 and I.3.4.

More generally for a *chiral multiple category* A the ij-interchangers χ_{ij} are not assumed to be invertible (see I.3.7).

1.2 Cubical spans and cospans

Weak multiple categories generalise *weak cubical categories* and *weak symmetric cubical categories*; the latter were introduced in [G1] for studying higher cobordism, and give our main examples of weak multiple categories

of infinite dimension. We begin by recalling two instances in an informal, incomplete way.

The weak symmetric cubical category $\mathsf{Span}(\mathbf{C})$ *of cubical spans* (or $\omega\mathsf{Span}(\mathbf{C})$) was constructed in [G1] over a category $\mathbf{C}$ with (a fixed choice of) pullbacks. An *n-cube* is a functor $x\colon \vee^n \to \mathbf{C}$, where $\vee$ is the formal-span category

$$0 \longleftarrow u \longrightarrow 1 \qquad \vee$$

$$
\begin{array}{ccccc}
(0,0) & \longleftarrow & (u,0) & \longrightarrow & (1,0) \\
\uparrow & & \uparrow & & \uparrow \\
(0,u) & \longleftarrow & (u,u) & \longrightarrow & (1,u) \\
\downarrow & & \downarrow & & \downarrow \\
(0,1) & \longleftarrow & (u,1) & \longrightarrow & (1,1)
\end{array}
\qquad
\begin{array}{c}
\bullet \longrightarrow {}^1 \\
{\downarrow}_2
\end{array}
\qquad (6)
$$

$$\vee^2.$$

(Identities and composites are understood.) An *n-map*, or transversal map of n-cubes, is a natural transformation $f\colon x \to y\colon \vee^n \to \mathbf{C}$ of such functors; these maps form the category $\mathrm{Span}_n(\mathbf{C}) = \mathbf{Cat}(\vee^n, \mathbf{C})$, with composition written as gf and identities $1_x = \mathrm{id}(x)$. There are obvious *geometric faces and degeneracies* (satisfying the cubical relations)

$$
\begin{aligned}
&\partial_i^\alpha\colon \mathrm{Span}_n(\mathbf{C}) \to \mathrm{Span}_{n-1}(\mathbf{C}), \\
&e_i\colon \mathrm{Span}_{n-1}(\mathbf{C}) \to \mathrm{Span}_n(\mathbf{C}) \qquad (i = 1, ..., n;\ \alpha = \pm).
\end{aligned}
\qquad (7)
$$

Moreover there are *geometric* composition laws: the *i-concatenation* $x +_i y$ is defined for *i-consecutive* n-cubes ($i = 1, ..., n;\ \partial_i^+ x = \partial_i^- y$), and constructed with pullbacks; it is categorical up to invertible n-maps (*unitors* and *associators*); similarly we have the *i*-concatenation $f +_i g$ of *i*-consecutive n-maps. All pairs of composition laws have a strict interchange.

Viewing $\mathsf{Span}(\mathbf{C})$ as a weak multiple category (of cubical type), an n-cube $x\colon \vee^n \to \mathbf{C}$ is viewed as an **i**-cube, for every positive multi-index $\mathbf{i} = \{i_1, ..., i_n\}$ of dimension $n \geqslant 0$; an n-map is viewed as an **i**-map.

The 2-dimensional and 3-dimensional truncations of $\mathsf{Span}(\mathbf{C})$ are written as:

$$\mathbb{S}\mathrm{pan}(\mathbf{C}) = \mathbf{2}\mathsf{Span}(\mathbf{C}), \qquad \mathbf{3}\mathsf{Span}(\mathbf{C}). \qquad (8)$$

The weak double category $\mathbb{S}\mathrm{pan}(\mathbf{C})$ was studied in our series [GP1] - [GP4]: its horizontal and vertical arrows are ordinary arrows and spans of $\mathbf{C}$,

respectively, while a double cell is a morphism of spans. The 3-dimensional truncation $\mathbf{3}\mathsf{Span}(\mathbf{C})$ consists of $\mathbf{i}$-cells for $\mathbf{i} \subset \mathbf{3} = \{0, 1, 2\}$ (or i-cubes and i-maps for $i < 3$, in the cubical framework).

Similarly one can find in [G1] the construction of the weak symmetric cubical category $\mathsf{Cosp}(\mathbf{C})$ *of cubical cospans* over a category $\mathbf{C}$ with (a fixed choice of) pushouts. An n-cube is now a functor $x\colon \wedge^n \to \mathbf{C}$, where $\wedge = \vee^{\mathrm{op}}$ is the formal-cospan category $0 \to u \leftarrow 1$; again, a transversal map of n-cubes is a natural transformation of such functors. $\mathsf{Cosp}(\mathbf{C})$ is *transversally dual* to $\mathsf{Span}(\mathbf{C}^{\mathrm{op}})$.

The 2-dimensional and 3-dimensional truncations are written as:

$$\mathbb{C}\mathrm{osp}(\mathbf{C}) = \mathbf{2}\mathsf{Cosp}(\mathbf{C}), \qquad \mathbf{3}\mathsf{Cosp}(\mathbf{C}). \tag{9}$$

1.3 The chiral case

Chiral multiple categories *of non-cubical type* are constructed in [GP7] and Part I, Section 4.

For instance, if the category $\mathbf{C}$ has pullbacks and pushouts, the weak double category $\mathbb{S}\mathrm{pan}(\mathbf{C})$, of arrows and spans of $\mathbf{C}$, can be 'amalgamated' with the weak double category $\mathbb{C}\mathrm{osp}(\mathbf{C})$, of arrows and cospans of $\mathbf{C}$, to form a 3-dimensional structure: the chiral triple category $\mathsf{SC}(\mathbf{C})$ whose arrows in direction 0, 1 and 2 are the arrows, spans and cospans of $\mathbf{C}$, *in this order* (as required by the 12-interchanger).

The highest cubes, of type $\{1, 2\}$, are functors $x\colon \vee \times \wedge \to \mathbf{C}$, the highest (3-dimensional) cells are the natural transformations of the latter

$$
\begin{array}{ccccc}
(0,0) & \leftarrow & (u,0) & \longrightarrow & (1,0) \\
\downarrow & & \downarrow & & \downarrow \\
(0,u) & \leftarrow & (u,u) & \longrightarrow & (1,u) \\
\uparrow & & \uparrow & & \uparrow \\
(0,1) & \leftarrow & (u,1) & \longrightarrow & (1,1)
\end{array}
\qquad \vee \times \wedge.
\qquad
\begin{array}{c}
\bullet \overset{1}{\longrightarrow} \\
\Big\downarrow {\scriptstyle 2} \\
{}
\end{array}
\tag{10}
$$

Here 0-composition works by ordinary composition in $\mathbf{C}$, 1-composition by composing spans (with pullbacks) and 2-composition by composing cospans (with pushouts).

Higher dimensional examples, like $\mathsf{S}_p\mathsf{C}_q(\mathbf{C})$, $\mathsf{S}_p\mathsf{C}_\infty(\mathbf{C})$ and $\mathsf{S}_{-\infty}\mathsf{C}_\infty(\mathbf{C})$ (and the corresponding *left-chiral* cases) can be found in I.4.4; note that

$\mathsf{S}_{-\infty}\mathsf{C}_{\infty}(\mathbf{C})$ is indexed *by all integers*, with spans in each negative direction, ordinary arrows in direction 0 and cospans in positive directions.

1.4 A double adjunction

Let $\mathbf{C}$ be a category with distinguished pullbacks and pushouts. For the sake of simplicity we assume that the distinguished pullback (resp. pushout) of an identity along any map is an identity.

The weak double categories $\mathbb{S}\mathrm{pan}(\mathbf{C})$ and $\mathbb{C}\mathrm{osp}(\mathbf{C})$ of spans and cospans of $\mathbf{C}$ are linked by an obvious colax/lax adjunction

$$F\colon \mathbb{S}\mathrm{pan}(\mathbf{C}) \rightleftarrows \mathbb{C}\mathrm{osp}(\mathbf{C}) \colon G,$$
$$\eta\colon 1 \dashrightarrow GF, \qquad \varepsilon\colon FG \dashrightarrow 1, \tag{11}$$

that we describe here *in an informal way*. (Writing $\eta\colon 1 \dashrightarrow GF$ and $\varepsilon\colon FG \dashrightarrow 1$ is an abuse of notation, since the comparisons of F and G have conflicting directions and cannot be composed. The precise definition of a colax/lax adjunction of weak double categories can be found in [GP2]; but the reader will find here its multiple extension, in Section 4, and can easily recover the truncated notion.)

At the basic level of $\mathrm{tv}_*(\mathbb{S}\mathrm{pan}(\mathbf{C})) = \mathrm{tv}_*(\mathbb{C}\mathrm{osp}(\mathbf{C})) = \mathbf{C}$ everything is an identity. At the level 1 (of 1-cubes and 1-maps) F operates by pushout and G by pullback; the special transversal 1-maps $\eta x\colon x \to GFx$ and $\varepsilon y\colon FGy \to y$ are obvious (for a span $x = (x', x'')$ and a cospan $y = (y', y'')$ with 1-faces A and B):

$$\tag{12}$$

The triangle identities are plainly satisfied:

$$\varepsilon(Fx).F(\eta x) = \mathrm{id}(Fx), \qquad\qquad G(\varepsilon y).\eta(Gy) = \mathrm{id}(Gy).$$

Finally it is easy to check that F is, in a natural way, a colax double functor (while G is lax). The comparison cell

$$\underline{F}(x, y)\colon F(x +_1 y) \to Fx +_1 Fy$$

for concatenation is given by the natural map from the pushout of $x +_1 y = (x'z', y''z'')$ to the cospan $Fx +_1 Fy$ (with vertex Z' in the diagram below)

$$\tag{13}$$

Since we agreed to follow the unitarity constraint for the choice of pullbacks and pushouts in $\mathbf{C}$, the adjunction is *unitary*, in the sense that this property holds for the weak double categories $\mathbb{S}\mathrm{pan}(\mathbf{C})$, $\mathbb{C}\mathrm{osp}(\mathbf{C})$ and the colax/lax double functors F, G. It is also interesting to note that the restricted adjunction at the $\star$-level

$$F_* \colon \mathbf{C} \rightleftarrows \mathbf{C} \colon G_* \qquad \eta_* \colon 1 \to G_* F_*, \qquad \varepsilon_* \colon F_* G_* \to 1, \quad (14)$$

is composed of identity functors and identity transformations.

We also remark that the natural transformations $F\eta$, εF, ηG, $G\varepsilon$ at level 1 are invertible (which means that the ordinary adjunction at level 1 is idempotent: see [AT, LS]).

1.5 A multiple adjunction

Following [G3], the unitary colax double functor $F \colon \mathbb{S}\mathrm{pan}(\mathbf{C}) \to \mathbb{C}\mathrm{osp}(\mathbf{C})$ can be extended to a unitary colax multiple functor of cubical type

$$F \colon \mathsf{Span}(\mathbf{C}) \to \mathsf{Cosp}(\mathbf{C}).$$

For instance, let us take a 2-dimensional span $x \in \mathrm{Span}_{\mathbf{i}}(\mathbf{C})$ indexed by $\mathbf{i} = \{i, j\}$, as in the left diagram below

$$\begin{array}{ccc}
x_{00} \leftarrow x_{u0} \to x_{10} & \quad & x_{00} \longrightarrow F(\partial_j^- x)(u) \longleftarrow x_{10} \\
\uparrow \quad \uparrow \quad \uparrow & & \downarrow \qquad \downarrow \qquad \downarrow \\
x_{0u} \leftarrow x_{uu} \to x_{1u} & & F(\partial_i^- x)(u) \longrightarrow \mathrm{colim}(x) \longleftarrow F(\partial_i^+ x)(u) \quad (15) \\
\downarrow \quad \downarrow \quad \downarrow & & \uparrow \qquad \uparrow \qquad \uparrow \\
x_{01} \leftarrow x_{u1} \to x_{11} & & x_{01} \longrightarrow F(\partial_j^+ x)(u) \longleftarrow x_{11}
\end{array}$$

The 2-dimensional cospan $F(x) = F_{\mathbf{i}}(x)$ is constructed at the right hand, with the pushouts $F(\partial_i^\alpha x)$, $F(\partial_j^\alpha x)$ of the four faces and, in the central vertex, the colimit of the whole diagram $x\colon \vee^2 \to \mathbf{C}$. (The latter can be constructed in $\mathbf{C}$ as a pushout of pushouts; a general characterisation of the dual topic, limits 'generated' by pullbacks, can be found in [P1].)

One proceeds in a similar way, defining $F_{\mathbf{i}}$ for a positive multi-index $\mathbf{i} = \{i_1, ..., i_n\}$ of degree n, after all instances of degree $n - 1$

$$
\begin{aligned}
\partial_i^\alpha(F_{\mathbf{i}}(x)) &= F_{\mathbf{i}|i}(\partial_i^\alpha x) &&\text{for } \alpha = \pm \text{ and } i \in \mathbf{i}, \\
F_{\mathbf{i}}(x)(\underline{u}) &= \operatorname{colim}(x), &&\text{where } \underline{u} = (u, ..., u) \in \vee^n.
\end{aligned}
\tag{16}
$$

The definition of F on transversal $\mathbf{i}$-maps is obvious, as well as the comparison cells for the i-directed concatenation $\underline{F}_i(x, y)\colon F(x +_i y) \to Fx +_i Fy$.

The unitary lax double functor $G\colon \mathbb{C}\mathrm{osp}(\mathbf{C}) \to \mathbb{S}\mathrm{pan}(\mathbf{C})$ is similarly extended, using distinguished limits instead of colimits, and gives a unitary lax multiple functor $G\colon \mathsf{Cosp}(\mathbf{C}) \to \mathsf{Span}(\mathbf{C})$ of cubical type.

One extends the unit $\eta\colon 1 \dashrightarrow GF$ by a similar inductive procedure:

$$
\begin{aligned}
\partial_i^\alpha(\eta_{\mathbf{i}}(x)) &= \eta_{\mathbf{i}|i}(\partial_i^\alpha x) &&(\alpha = \pm, \ i \in \mathbf{i}), \\
(\eta_{\mathbf{i}}x)(\underline{u})&\colon x(\underline{u}) \to (G_{\mathbf{i}}F_{\mathbf{i}}x)(\underline{u}) = \lim(F_{\mathbf{i}}x),
\end{aligned}
\tag{17}
$$

where the map $(\eta_{\mathbf{i}}x)(\underline{u})$ is given by the universal property of $\lim(F_{\mathbf{i}}x)$ as the limit of the cubical cospan $F_{\mathbf{i}}x\colon \wedge^n \to \mathbf{C}$.

Analogously for the counit $\varepsilon\colon FG \dashrightarrow 1$. The triangular identities hold.

1.6 Chiral examples

The colax/lax adjunction of weak triple categories of cubical type

$$
F\colon \mathbf{3Span}(\mathbf{C}) \ \rightleftarrows\ \mathbf{3Cosp}(\mathbf{C}) : G,
\tag{18}
$$

can be factorised through the chiral triple category $\mathsf{SC}(\mathbf{C})$ *of spans and cospans of* $\mathbf{C}$, obtaining two colax/lax adjunctions of chiral triple categories (no longer of cubical type)

$$
\begin{aligned}
F'&\colon \mathbf{3Span}(\mathbf{C}) \ \rightleftarrows\ \mathsf{SC}(\mathbf{C}) : G', \\
F''&\colon \mathsf{SC}(\mathbf{C}) \ \rightleftarrows\ \mathbf{3Cosp}(\mathbf{C}) : G''.
\end{aligned}
\tag{19}
$$

Here the functor $F' \colon \mathbf{3Span}(\mathbf{C}) \to \mathsf{SC}(\mathbf{C})$ acts on a 12-cube x

- by pushout on the three 2-directed spans of x,
- as the identity on the two 1-directed boundary spans $\partial_2^\alpha(x)$,
- by induced morphisms on the middle 1-directed span.

On the other hand the functor $G' \colon \mathsf{SC}(\mathbf{C}) \to \mathbf{3Span}(\mathbf{C})$ acts by pullback on the three (2-directed) cospans of x, as the identity on the (1-directed) boundary spans $\partial_2^\alpha(x)$ and by induced morphisms on the middle span. Similarly for F'' and G''.

One can also factorise the adjunction (18) through the *left* chiral triple category $\mathsf{CS}(\mathbf{C})$ *of cospans and spans*, obtaining two colax/lax adjunctions of left chiral triple categories.

Similarly, the multiple adjunction constructed in 1.5 can be factorised through any right chiral multiple category $\mathsf{S}_p\mathsf{C}_\infty(\mathbf{C})$, *or* through any left chiral multiple category $\mathsf{C}_p\mathsf{S}_\infty(\mathbf{C})$.

However, in infinite dimension, one may prefer to consider a more symmetric situation, starting from a colax/lax adjunction of weak multiple categories *indexed by the ordered set* $\mathbb{Z}$ of integers (pointed at 0)

$$F \colon \mathbb{Z}\mathsf{Span}(\mathbf{C}) \rightleftarrows \mathbb{Z}\mathsf{Cosp} \colon G. \tag{20}$$

This can be factorised through the chiral multiple category $\mathsf{S}_{-\infty}\mathsf{C}_\infty(\mathbf{C})$, obtaining two colax/lax adjunctions of 'unbounded' chiral multiple categories

$$\begin{aligned} F' &\colon \mathbb{Z}\mathsf{Span}(\mathbf{C}) \rightleftarrows \mathsf{S}_{-\infty}\mathsf{C}_\infty(\mathbf{C}) \colon G', \\ F'' &\colon \mathsf{S}_{-\infty}\mathsf{C}_\infty(\mathbf{C}) \rightleftarrows \mathbb{Z}\mathsf{Cosp}(\mathbf{C}) \colon G''. \end{aligned} \tag{21}$$

1.7 Other examples

Now we start from an ordinary adjunction $F \dashv G$

$$F \colon \mathbf{X} \rightleftarrows \mathbf{A} \colon G, \qquad \eta \colon 1 \to GF, \qquad \varepsilon \colon FG \to 1, \tag{22}$$

between categories with (a choice of) pullbacks. This can be extended in a natural way to a unitary *colax/pseudo* adjunction between the weak multiple categories of higher spans (of cubical type)

$$\mathsf{Span}(F) \colon \mathsf{Span}(\mathbf{X}) \rightleftarrows \mathsf{Span}(\mathbf{A}) \colon \mathsf{Span}(G). \tag{23}$$

In fact there is an obvious 2-functor

$$\mathsf{Span}\colon \mathbf{Cat}_{\mathrm{pb}} \to \mathrm{Cx}\mathbf{Cmc}, \tag{24}$$

defined on the full sub-2-category of $\mathbf{Cat}$ containing all categories with (a choice of) pullbacks, with values in the 2-category of chiral multiple categories, colax functors and their transversal transformations (see 2.1).

It sends a category $\mathbf{C}$ with pullbacks to the chiral multiple category $\mathsf{Span}(\mathbf{C})$ (actually a weak multiple category of symmetric cubical type).

For a functor $F\colon \mathbf{X} \to \mathbf{A}$ (between categories with pullbacks), $\mathsf{Span}(F)$, also written as F for brevity, simply acts by computing F over the diagrams of $\mathbf{X}$ that form i-cubes and i-maps; formally, over an i-map $f\colon x \to y\colon \vee^n \to \mathbf{X}$, $F(f)\colon F(x) \to F(y)$ is the composite

$$F.f\colon F.x \to F.y\colon \vee^n \to \mathbf{A}.$$

This extension is, in a natural way, a unitary *colax* functor, since identities of $\mathbf{X}$ go to identities of $\mathbf{A}$ and a composition $x +_i y$ of two spans $x, y\colon \vee \to \mathbf{X}$ (in any direction $i > 0$) gives rise to a diagram in $\mathbf{X}$ and a diagram in $\mathbf{A}$

$$\tag{25}$$

where the comparison $\underline{F}_i(x, y)\colon F(x +_i y) \to F(x) +_i F(y)$ is an i-special transversal map given by the $\mathbf{A}$-morphism $a\colon FP \to Q$ determined by the universal property of the pullback Q. Similarly we define $\underline{F}_i(x, y)$ for every i-composition of i-cubes. Note that $\mathsf{Span}(F)$ is pseudo if (and only if) the functor $F\colon \mathbf{X} \to \mathbf{A}$ preserves pullbacks.

Finally, a natural transformation $\varphi\colon F \to F'\colon \mathbf{X} \to \mathbf{A}$ yields a transversal transformation

$$\mathsf{Span}(\varphi)\colon \mathsf{Span}(F) \to \mathsf{Span}(F')\colon \mathsf{Span}(\mathbf{X}) \to \mathsf{Span}(\mathbf{A}), \tag{26}$$

that again will often be written as φ. On an $\mathbf{i}$-cube $x \colon \vee^n \to \mathbf{X}$, $\varphi x \colon F(x) \to_0 F'(x)$ is the composite of the functor $x \colon \vee^n \to \mathbf{X}$ with the natural transformation $\varphi \colon F \to F' \colon \mathbf{X} \to \mathbf{A}$. Concretely, the transversal $\mathbf{i}$-map $\varphi x \colon F(x) \to_0 F'(x)$ has components $\varphi(x(t))$, for every vertex t of $\vee^n$.

Now, letting the 2-functor $\mathsf{Span} \colon \mathbf{Cat}_{\mathrm{pb}} \to \mathrm{Cx}\mathbf{Cmc}$ act on the adjunction (22), we get an adjunction of weak multiple categories with the properties stated above: $\mathsf{Span}(F)$ is colax, $\mathsf{Span}(G)$ is pseudo and both are unitary.

On the other hand, if $\mathbf{X}$ and $\mathbf{A}$ have pushouts, the adjunction (22) yields a *pseudo/lax* adjunction of weak multiple categories

$$F \colon \mathsf{Cosp}(\mathbf{X}) \; \rightleftarrows \; \mathsf{Cosp}(\mathbf{A}) \colon G. \tag{27}$$

Finally, if $\mathbf{X}$ and $\mathbf{A}$ have pullbacks and pushouts, we can extend (22) to a colax/lax adjunction of chiral triple categories

$$F \colon \mathsf{SC}(\mathbf{X}) \; \rightleftarrows \; \mathsf{SC}(\mathbf{A}) \colon G, \tag{28}$$

or consider higher-dimensional extensions of 'type' $\mathsf{S}_p\mathsf{C}_q$, $\mathsf{S}_p\mathsf{C}_\infty$, $\mathsf{S}_{-\infty}\mathsf{C}_\infty$, etc. (see 1.3).

Note that, according to the analysis of [GP6], Section 5, F is a *colax-pseudo* morphism of chiral triple categories (i.e. colax for the 1-directed composition, realised by pullbacks, and pseudo for the 2-directed composition, realised by pushouts) while G is pseudo-lax.

2. The double category of lax and colax multiple functors

In the 2-dimensional case, weak double categories with lax and colax double functors and suitable double cells form a *strict* double category $\mathbb{D}$bl, a crucial structure introduced in [GP2] to define colax/lax double adjunctions - also recalled in Part I.

This construction was extended in [G3], Section 2, to form the strict *double* category $\mathbb{W}$sc of weak symmetric cubical categories, lax and colax symmetric cubical functors (and suitable double cells), in order to define colax/lax adjunctions between weak symmetric cubical categories.

We now give a further extension, building the strict *double* category $\mathbb{C}$mc of chiral multiple categories, lax and colax multiple functors, that will be

used below to define colax/lax adjunctions between chiral multiple categories.

Comma chiral multiple categories are also considered, extending again the cases of double and symmetric cubical categories, dealt with in [GP2, G3].

For $\mathbb{C}$mc we follow the notation for double categories used in [GP1] - [GP4]: the horizontal and vertical compositions of cells are written as $(\pi \,|\, \rho)$ and $\left(\frac{\pi}{\sigma}\right)$, or more simply as $\pi|\rho$ and $\pi \otimes \sigma$. Horizontal identities, of an object or a vertical arrow, are written as 1_A and 1_u; vertical identities, of an object or a horizontal arrow, as $1_A^{\bullet}$ and $1_f^{\bullet}$.

2.1 Lax cm-functors

A chiral multiple category is also called a *cm-category*, for short.

As defined in I.3.9, a *lax multiple functor* $F\colon \mathsf{A} \to \mathsf{B}$ between chiral multiple categories, or *lax cm-functor*, has components $F_{\mathbf{i}}\colon A_{\mathbf{i}} \to B_{\mathbf{i}}$ for all multi-indices $\mathbf{i}$ (often written as F) that agree with all faces, 0-degeneracies and 0-composition. Moreover, for every positive multi-index $\mathbf{i}$ and $i \in \mathbf{i}$, F is equipped with i-special comparison $\mathbf{i}$-maps $\underline{F}_i$ that agree with faces

$$
\begin{aligned}
&\underline{F}_i(x)\colon e_i F(x) \to_0 F(e_i x) &&\text{(for } x \text{ in } A_{\mathbf{i}|i}), \\
&\underline{F}_i(x,y)\colon F(x) +_i F(y) \to_0 F(z) &&\text{(for } z = x +_i y \text{ in } A_{\mathbf{i}}), \\
&\partial_j^{\alpha} \underline{F}_i(x) = \underline{F}_i(\partial_j^{\alpha} x) &&\text{(for } j \neq i), \\
&\partial_j^{\alpha} \underline{F}_i(x,y) = \underline{F}_i(\partial_j^{\alpha} x, \partial_j^{\alpha} y) &&\text{(for } j \neq i).
\end{aligned}
\tag{29}
$$

and satisfy some axioms. We write down the naturality conditions (lmf.1-2), frequently used below, while the coherence conditions (lmf.3-5) can be found in I.3.9.

(lmf.1) (*naturality of unit comparisons*) For every $\mathbf{i}|j$-map $f\colon x \to_0 y$ in A we have:

$$
F(e_j f).\underline{F}_j(x) = \underline{F}_j(y).e_j(Ff)\colon e_j F(x) \to_0 F(e_j y). \tag{30}
$$

(lmf.2) (*naturality of composition comparisons*) For two j-consecutive $\mathbf{i}$-maps $f\colon x \to_0 x'$ and $g\colon y \to_0 y'$ in A we have:

$$
F(f +_j g).\underline{F}_j(x,y) = \underline{F}_j(x',y').(F(f) +_j F(g)). \tag{31}
$$

A *transversal transformation* $h\colon F \to G\colon$ A $\to$ B between lax multiple functors of chiral multiple categories consists of a face-consistent family of **i**-maps in B (its components), one for every positive multi-index **i** and every **i**-cube x in A

$$hx\colon F(x) \to_0 G(x), \qquad h(\partial_j^\alpha x) = \partial_j^\alpha(hx), \qquad (32)$$

under the axioms (trt.1) and (trt.2L), see I.3.9.

We have thus the 2-category $\mathrm{Lx}\mathbf{Cmc}$ of cm-categories, lax cm-functors and their transversal transformations. Similarly one defines the 2-category $\mathrm{Cx}\mathbf{Cmc}$, for the colax case where the comparisons of *colax cm-functors* have the opposite direction. A *pseudo cm-functor* is a lax cm-functor whose comparisons are invertible; it is made colax by the inverse comparisons.

2.2 The double category $\mathbb{C}\mathrm{mc}$

Lax and colax cm-functors do not compose well, since we cannot compose their comparisons. But they give the horizontal and vertical arrows of a *strict* double category $\mathbb{C}\mathrm{mc}$, crucial for our study, where 'internal' orthogonal adjunctions (recalled below, in Section 3) will provide our general notion of multiple adjunction (Section 4) while companion pairs amount to pseudo cm-functors (Section 5).

The objects of $\mathbb{C}\mathrm{mc}$ are the cm-categories A, B, C, ...; its horizontal arrows are the *lax* cm-functors $R, S...$; its vertical arrows are the *colax* cm-functors $F, G...$

A double cell $\pi\colon \left(F \, {}^{R}_{S} \, G \right)$

$$
\begin{array}{ccc}
\mathsf{A} & \xrightarrow{R} & \mathsf{B} \\
{\scriptstyle F}\downarrow & \pi & \downarrow{\scriptstyle G} \\
\mathsf{C} & \xrightarrow{S} & \mathsf{D}
\end{array}
\qquad (33)
$$

is - roughly speaking - a 'transformation' $\pi\colon GR \dashrightarrow SF$. But the composites GR and SF are neither lax nor colax: the coherence conditions of π require the individual knowledge of the four 'functors', including the comparison cells of each of them.

Precisely, the double cell π consists of the following data:

(a) two lax cm-functors $R\colon \mathsf{A} \to \mathsf{B}$ and $S\colon \mathsf{C} \to \mathsf{D}$, with comparisons as follows:

$$\underline{R}_i(x)\colon e_i(Rx) \to_0 R(e_i x), \qquad \underline{R}_i(x,y)\colon Rx +_i Ry \to_0 R(x +_i y),$$
$$\underline{S}_i(x)\colon e_i(Sx) \to_0 S(e_i x), \qquad \underline{S}_i(x,y)\colon Sx +_i Sy \to_0 S(x +_i y),$$

(b) two colax cm-functors $F\colon \mathsf{A} \to \mathsf{C}$ and $G\colon \mathsf{B} \to \mathsf{D}$, with comparisons as follows:

$$\underline{F}_i(x)\colon F(e_i x) \to_0 e_i(Fx), \qquad \underline{F}_i(x,y)\colon F(x +_i y) \to_0 Fx +_i Fy,$$
$$\underline{G}_i(x)\colon G(e_i x) \to_0 e_i(Gx), \qquad \underline{G}_i(x,y)\colon G(x +_i y) \to_0 Gx +_i Gy,$$

(c) a family of $\mathbf{i}$-maps $\pi x\colon GR(x) \to_0 SF(x)$ of D (for every $\mathbf{i}$-cube x in A), consistent with faces

$$\pi(\partial_i^\alpha x) = \partial_i^\alpha(\pi x). \tag{34}$$

These data have to satisfy the naturality condition (dc.1) and the coherence conditions (dc.2), (dc.3) (with respect to i-directed degeneracies and composition, respectively)

(dc.1) $SFf.\pi x = \pi y.GRf\colon GR(x) \to SF(y) \quad$ (for $f\colon x \to y$ in $\mathrm{tv}_{\mathbf{i}}(\mathsf{A})$),

(dc.2) $S\underline{F}_i(x).\pi e_i(x).G\underline{R}_i(x) = \underline{S}_i F(x).e_i(\pi x).\underline{G}_i R(x) \qquad$ (for x in $A_{\mathbf{i}|i}$),

$$
\begin{array}{ccccc}
Ge_i(Rx) & \xrightarrow{G\underline{R}_i(x)} & GR(e_i x) & \xrightarrow{\pi e_i(x)} & SF(e_i x) \\
{\scriptstyle \underline{G}_i R(x)}\downarrow & & & & \downarrow{\scriptstyle S\underline{F}_i(x)} \\
e_i GR(x) & \xrightarrow[e_i \pi(x)]{} & e_i SF(x) & \xrightarrow[\underline{S}_i F(x)]{} & Se_i F(x)
\end{array}
$$

(dc.3) $S\underline{F}_i(x,y).\pi z.G\underline{R}_i(x,y) = \underline{S}_i(Fx, Fy).(\pi x +_i \pi y).\underline{G}_i(Rx, Ry)$

$$(\text{for } z = x +_i y \text{ in } A_{\mathbf{i}}),$$

$$
\begin{array}{ccccc}
G(Rx +_i Ry) & \xrightarrow{G\underline{R}_i(x,y)} & GR(z) & \xrightarrow{\pi z} & SF(z) \\
{\scriptstyle \underline{G}_i(Rx,Ry)}\downarrow & & & & \downarrow{\scriptstyle S\underline{F}_i(x,y)} \\
GRx +_i GRy & \xrightarrow[\pi x +_i \pi y]{} & SFx +_i SFy & \xrightarrow[\underline{S}_i(Fx,Fy)]{} & S(Fx +_i Fy)
\end{array}
$$

The horizontal composition $(\pi \mid \rho)$ and the vertical composition $\pi \otimes \sigma = \left(\frac{\pi}{\sigma}\right)$ of double cells are both defined *via the composition of transversal maps* (in a cm-category)

$$
\begin{array}{ccccc}
\mathsf{A} & \xrightarrow{\ R\ } & \bullet & \xrightarrow{\ R'\ } & \bullet \\
\scriptstyle F\ \downarrow & \pi & \scriptstyle G\ \downarrow \ \rho & & \downarrow\ \scriptstyle H \\
\bullet & -S\to & \bullet & -S'\to & \bullet \\
\scriptstyle F'\ \downarrow & \sigma & \scriptstyle G'\ \downarrow\ \tau & & \downarrow\ \scriptstyle H' \\
\bullet & \xrightarrow{\ T\ } & \bullet & \xrightarrow{\ T'\ } & \bullet
\end{array}
\tag{35}
$$

$$
\begin{aligned}
(\pi \mid \rho)(x) &= S'\pi x.\rho Rx \colon HR'R(x) \to_0 S'GR(x) \to_0 S'SF(x), \\
\left(\tfrac{\pi}{\sigma}\right)(x) &= \sigma Fx.G'\pi x \colon G'GR(x) \to_0 G'SF(x) \to_0 TF'F(x),
\end{aligned}
\tag{36}
$$

(for x in A). We verify below, in Theorem 2.3, that these compositions are well-defined and satisfy the axioms of a double category.

Within $\mathbb{C}$mc, we have the strict 2-category Lx**Cmc** of *cm-categories, lax cm-functors and transversal transformations*: namely, Lx**Cmc** is the restriction of $\mathbb{C}$mc to trivial vertical arrows. Symmetrically the strict 2-category Cx**Cmc**, whose arrows are the colax cm-functors, also lies in $\mathbb{C}$mc.

As in I.1.2 (for weak double categories), we can also note that a double cell $\pi \colon (F \ {}^R_1 \ 1)$ gives a notion of *transversal transformation* $\pi \colon R \dashrightarrow F \colon \mathsf{A} \to \mathsf{B}$ *from a lax to a colax functor*, while a double cell $\pi \colon (1 \ {}^1_S \ G)$ gives a notion of *transversal transformation* $\pi \colon G \dashrightarrow S \colon \mathsf{A} \to \mathsf{B}$ *from a colax to a lax functor*. Moreover, for a fixed pair A, B of chiral multiple categories, all the transversal transformations between lax and colax functors (of the four possible kinds) compose, forming a category $\{\mathsf{A}, \mathsf{B}\}$ whose objects are the lax *and* the colax functors $\mathsf{A} \to \mathsf{B}$.

2.3 Theorem

The structure $\mathbb{C}$mc, as defined above, is indeed a strict double category.

Proof. The argument is much the same as for $\mathbb{D}$bl in [GP2] and for $\mathbb{W}$sc in [G3].

First, to show that the double cells defined in (36) are indeed coherent, we verify the condition (dc.3) for $(\pi \mid \rho)$, with respect to a concatenation

$z = x +_i y$ in A. Our property amounts to the commutativity of the outer diagram below, formed of transversal maps (omitting the index i in $+_i$ and all comparisons $\underline{R}_i$ etc.)

$$
\begin{array}{ccccc}
HR'Rz & \xrightarrow{\rho Rz} & S'GRz & \xrightarrow{S'\pi z} & S'SFz \\
{\scriptstyle HR'\underline{R}}\uparrow & & {\scriptstyle S'G\underline{R}}\uparrow & & \downarrow{\scriptstyle S'S\underline{F}} \\
HR'(Rx + Ry) & \xrightarrow{\rho(Rx+Ry)} & S'G(Rx + Ry) & & S'S(Fx + Fy) \\
{\scriptstyle H\underline{R}'R}\uparrow & & \downarrow{\scriptstyle S'\underline{G}R} & & \uparrow{\scriptstyle S'\underline{S}F} \\
H(R'Rx + R'Ry) & & S'(GRx + GRy) & \xrightarrow{S'(\pi x+\pi y)} & S'(SFx + SFy) \\
{\scriptstyle \underline{H}R'R}\downarrow & & \uparrow{\scriptstyle \underline{S}'GR} & & \uparrow{\scriptstyle \underline{S}'SF} \\
HR'Rx + HR'Ry & \xrightarrow[\rho Rx+\rho Ry]{} & S'GRx + S'GRy & \xrightarrow[S'\pi x+S'\pi y]{} & S'SFx + S'SFy
\end{array}
$$

Indeed, the two hexagons commute applying (dc.3) to π and ρ, respectively. The upper square commutes by naturality of ρ on $\underline{R}_i(x, y)$; the lower one by axiom (lmf.2) (see 2.1) on the lax functor S', with respect to the **i**-maps $\pi x\colon GR(x) \to_0 SF(x)$ and $\pi y\colon GR(y) \to_0 SF(y)$

$$S'(\pi x +_i \pi y).\underline{S}'_i(GR(x), GR(y)) = \underline{S}'_i(SF(x), SF(y)).(S'(\pi x) +_i S'(\pi y)).$$

Now, both composition laws of double cells have been defined, in (36), *via the composition of transversal maps* (in a cm-category), and therefore are strictly unitary and associative.

Finally, to verify the middle-four interchange law on the four double cells of diagram (35), we compute the compositions $(\pi \,|\, \rho) \otimes (\sigma \,|\, \tau)$ and $(\pi \otimes \sigma) \,|\, (\rho \otimes \tau)$ on an **i**-cube x, and we obtain the two transversal maps $H'HR'Rx \to_0 T'TF'Fx$ of the upper or lower path in the following diagram

$$
\begin{array}{ccccc}
H'HR'Rx & \xrightarrow{H'\rho Rx} & H'S'GRx & \xrightarrow{H'S'\pi x} & H'S'SFx \\
& & \downarrow{\scriptstyle \tau GRx} & & \downarrow{\scriptstyle \tau SFx} \\
& & T'G'GRx & \xrightarrow[T'G'\pi x]{} T'G'SFx & \xrightarrow[T'\sigma Fx]{} T'TF'Fx
\end{array}
$$

But these two composites coincide because the square commutes: a consequence of the naturality of τ on the transversal map $\pi x\colon GR(x) \to SF(x)$, by axiom (dc.1) for the double cell τ. $\qquad\square$

2.4 Comma cm-categories

Comma double categories [GP2] also have a natural extension to the multiple case.

Given a *colax* cm-functor $F\colon \mathsf{A} \to \mathsf{C}$ and a *lax* cm-functor $R\colon \mathsf{X} \to \mathsf{C}$ with the same codomain, we can construct the *comma cm-category* $F \downarrow R$, where the projections P and Q are strict cm-functors and π is a double cell of $\mathbb{C}\mathrm{mc}$

$$
\begin{array}{ccc}
F \downarrow R & \xrightarrow{\ P\ } & \mathsf{A} \\
Q\Big\downarrow & \pi & \Big\downarrow F \\
\mathsf{X} & \xrightarrow{\ R\ } & \mathsf{C}
\end{array}
\qquad (37)
$$

An **i**-cube of $F \downarrow R$ is a triple $(a, x; c\colon Fa \to_0 Rx)$ where a is an **i**-cube of A, x is an **i**-cube of X and c is an **i**-map of C. An **i**-map $(h, f)\colon (a, x; c) \to_0 (a', x'; c')$ comes from a pair of **i**-maps $h\colon a \to_0 a'$ (in A) and $f\colon x \to_0 x'$ (in X) that give in C a commutative square of transversal maps

$$
\begin{array}{ccc}
Fa & \xrightarrow{\ c\ } & Rx \\
Fh\Big\downarrow & & \Big\downarrow Rf \\
Fa' & \xrightarrow[\ c'\ ]{} & Rx'
\end{array}
\qquad\qquad Rf.c = c'.Fh.
\qquad (38)
$$

Faces and transversal composition are obvious. The degeneracies are defined using the fact that F is *colax* and R is *lax*:

$$
e_i(a, x; c\colon Fa \to_0 Rx) = (e_i a, e_i x; \underline{R}_i(x).e_i c.\underline{F}_i(a)).
\qquad (39)
$$

Similarly the i-concatenation is defined as follows

$$
\begin{aligned}
(a, x; c) &+_i (b, y; d) \\
&= (a +_i b,\ x +_i y;\ u\colon F(a +_i b) \to_0 R(x +_i y)), \\
u = \underline{R}_i&(x, y).(c +_i d).\underline{F}_i(a, b)\colon \\
&F(a +_i b) \to_0 Fa +_i Fb \to_0 Rx +_i Ry \to_0 R(x +_i y).
\end{aligned}
\qquad (40)
$$

The associativity comparison for the i-composition of three i-consecutive **i**-cubes of $F \downarrow R$

$$
(a, x; c), \qquad\qquad (a', x'; c'), \qquad\qquad (a'', x''; c''),
$$

is given by the pair $(\kappa_i(\mathbf{a}), \kappa_i(\mathbf{x}))$ of associativity $\mathbf{i}$-isomorphisms for our two triples of $\mathbf{i}$-cubes, namely $\mathbf{a} = (a, a', a'')$ in A and $\mathbf{x} = (x, x', x'')$ in X

$$(\kappa_i(\mathbf{a}), \kappa_i(\mathbf{x})) \colon (a, x; c) +_i ((a', x'; c') +_i (a'', x''; c''))$$
$$\to_0 ((a, x; c) +_i (a', x'; c')) +_i (a'', x''; c''). \tag{41}$$

The coherence of this $\mathbf{i}$-map, as in diagram (38) above, is proved in the lemma below.

Similarly one constructs the unitors λ_i, ρ_i and the interchangers χ_{ij} of $F \downarrow R$, using those of A and X.

Finally, the strict cm-functors P and Q are projections, while the component of the 'transformation' π on the $\mathbf{i}$-cube $(a, x; c)$ of $F \downarrow R$ is the transversal map:

$$\pi(a, x; c) = c \colon Fa \to Rx. \tag{42}$$

2.5 Lemma

The pair $(\kappa(\mathbf{a}), \kappa(\mathbf{x}))$ is indeed an $\mathbf{i}$-map of $F \downarrow R$, with domain and codomain as specified in (41).

Proof. First, these two $\mathbf{i}$-cubes of $F \downarrow R$ can be written as:

$$(a, x; c) + ((a', x'; c') + (a'', x''; c'')) = (a_1, x_1; c_1),$$
$$((a, x; c) + (a', x'; c')) + (a'', x''; c'') = (a_2, x_2; c_2), \tag{43}$$

where $a_1, x_1, a_2, x_2, c_1, c_2$ are defined as follows (by commutative diagrams in the last two cases, and always leaving the index i understood for $+, \underline{F}, \underline{R}$)

$$a_1 = a + (a' + a''), \qquad x_1 = x + (x' + x''),$$
$$a_2 = (a + a') + a'', \qquad x_2 = (x + x') + x'', \tag{44}$$

$$\begin{array}{ccc}
Fa_1 & \xrightarrow{\quad c_1 \quad} & Rx_1 \\
{\scriptstyle \underline{F}(a, a'+a'')} \downarrow & & \uparrow {\scriptstyle \underline{R}(x, x'+x'')} \\
Fa + F(a' + a'') & & Rx + R(x' + x'') \\
{\scriptstyle 1+\underline{F}(a', a'')} \downarrow & & \uparrow {\scriptstyle 1+\underline{R}(x', x'')} \\
Fa + (Fa' + Fa'') & \xrightarrow[\; c + (c'+c'') \;]{} & Rx + (Rx' + Rx'')
\end{array}$$

$$\begin{array}{ccc}
Fa_2 & \xrightarrow{\;\;c_2\;\;} & Rx_2 \\[2pt]
{\scriptstyle \underline{F}(a+a',\,a'')}\Big\downarrow & & \Big\uparrow{\scriptstyle \underline{R}(x+x',\,x'')} \\[2pt]
F(a+a')+Fa'' & & R(x+x')+Rx'' \\[2pt]
{\scriptstyle \underline{F}(a,a')+1}\Big\downarrow & & \Big\uparrow{\scriptstyle \underline{R}(x,x')+1} \\[2pt]
(Fa+Fa')+Fa'' & \xrightarrow[(c+c')+c'']{} & (Rx+Rx')+Rx''
\end{array}$$

Now our claim, i.e. the condition for $(\kappa(\mathbf{a}),\kappa(\mathbf{x}))$ expressed in diagram (38), amounts to

$$R\kappa(\mathbf{x}).c_1 = c_2.F\kappa(\mathbf{a})\colon\; Fa_1 \to Rx_2. \qquad (45)$$

First, the coherence of the lax functor R with the associator κ gives (applying axiom (lmf.4) of I.3.9):

$$R\kappa(\mathbf{x}).c_1$$
$$= R\kappa(\mathbf{x}).\underline{R}(x,x'+x'').(1+\underline{R}(x',x'')).(c+(c'+c'')).$$
$$(1+\underline{F}(a',a'')).\underline{F}(a,a'+a'')$$
$$= \underline{R}(x+x',x'').(\underline{R}(x,x')+1).\kappa R(\mathbf{x}).(c+(c'+c'')).$$
$$(1+\underline{F}(a',a'')).\underline{F}(a,a'+a'').$$

Second, the coherence of the colax functor F with κ gives (applying the corresponding axiom, with reversed comparisons $\underline{F}$):

$$c_2.F\kappa(\mathbf{a})$$
$$= \underline{R}(x+x',x'').(\underline{R}(x,x')+1).((c+c')+c'').$$
$$(\underline{F}(a,a')+1).\underline{F}(a+a',a'').F\kappa(\mathbf{a})$$
$$= \underline{R}(x+x',x'').(\underline{R}(x,x')+1).((c+c')+c'').$$
$$\kappa(F\mathbf{a}).(1+\underline{F}(a',a'')).\underline{F}(a,a'+a'').$$

Finally, condition (45) follows from the naturality of κ on the triple of transversal maps $(c,c',c'')\colon F\mathbf{a} \to R\mathbf{x}$, which gives the commutative diagram

$$\begin{array}{ccc}
Fa+(Fa'+Fa'') & \xrightarrow{\;\kappa(F\mathbf{a})\;} & (Fa+Fa')+Fa'' \\[4pt]
{\scriptstyle c+(c'+c'')}\Big\downarrow & & \Big\downarrow{\scriptstyle (c+c')+c''} \\[4pt]
Rx+(Rx'+Rx'') & \xrightarrow[\;\kappa(R\mathbf{x})\;]{} & (Rx+Rx')+Rx''
\end{array}$$

$\square$

2.6 Theorem (Universal properties of commas)

(a) For a pair of lax *cm-functors S, T and a cell φ as below (in $\mathbb{C}$mc) there is a unique lax cm-functor $L\colon \mathsf{Z} \to F \downarrow R$ such that $S = PL$, $T = QL$ and $\varphi = (\psi \,|\, \pi)$ where the cell ψ is defined by the identity $1\colon QL \to T$ (a horizontal transformation of lax cm-functors)*

$$
\begin{array}{ccc}
\mathsf{Z} \xrightarrow{\ S\ } \mathsf{A} & & \mathsf{Z} \xrightarrow{\ L\ } F{\downarrow}R \xrightarrow{\ P\ } \mathsf{A} \\
1\downarrow \quad \varphi \quad \downarrow F & = & 1\downarrow \quad \psi \quad \downarrow Q \quad \pi \quad \downarrow F \\
\mathsf{Z} \xrightarrow[T]{} \mathsf{X} \xrightarrow[R]{} \mathsf{C} & & \mathsf{Z} \xrightarrow[T]{} \mathsf{X} \xrightarrow[R]{} \mathsf{C}
\end{array}
\tag{46}
$$

Moreover L is pseudo or strict if and only if both S and T are.

(b) A similar property holds for a pair of colax *cm-functors G, H and a double cell $\varphi'\colon (G\ {}^{1}_{R}\ FH)$.*

Proof. (a) L is defined as follows on an **i**-cube z and an **i**-map $f\colon z \to z'$ of Z

$$
\begin{aligned}
L(z) &= (Sz, Tz;\ \varphi z\colon FSz \to RTz), \\
L(f) &= (Sf, Tf).
\end{aligned}
\tag{47}
$$

The comparison transversal maps $\underline{L}_i$ are constructed with the laxity transversal maps $\underline{S}$ and $\underline{T}$ (and are invertible or degenerate if and only if the latter are)

$$
\begin{aligned}
\underline{L}_i x &= (\underline{S}_i x, \underline{T}_i x)\colon e_i(Lx) \to Le_i(x) & \text{(for } x \text{ in } Z_{\mathbf{i}|i}), \\
\underline{L}_i(x, y) &= (\underline{S}_i(x, y), \underline{T}_i(x, y))\colon Lx +_i Ly \to L(z) & \\
& & \text{(for } z = x +_i y \text{ in } Z_{\mathbf{i}}).
\end{aligned}
\tag{48}
$$

Here $Lx +_i Ly$ is the **i**-cube defined as below (applying (40) and writing $+_i$ as $+$)

$$
\begin{aligned}
Lx + Ly &= (Sx, Tx;\ \varphi x\colon FSx \to RTx) + (Sy, Ty;\ \varphi y\colon FSy \to RTy) \\
&= (Sx + Sy, Tx + Ty;\ u), \\
u &= \underline{R}_i(Tx, Ty).(\varphi x + \varphi y).\underline{F}_i Sx, Sy)\colon \\
& \quad F(Sx + Sy) \to FSx +_i FSy \to RTx + RTy \to R(Tx + Ty).
\end{aligned}
$$

Letting $z = x +_i y$, the coherence condition (38) on the transversal map $\underline{L}_i(x,y) = (\underline{S}_i(x,y), \underline{T}_i(x,y))$ of $F \downarrow R$ (with $z = x +_i y$)

$$RT_i(x,y).u = \varphi z.F\underline{S}_i(x,y), \qquad (49)$$

$$\begin{array}{ccc} F(Sx +_i Sy) & \xrightarrow{\ u\ } & R(Tx +_i Ty) \\ {\scriptstyle F\underline{S}_i(x,y)} \downarrow & & \downarrow {\scriptstyle R\underline{T}_i(x,y)} \\ FS(z) & \xrightarrow[\varphi z]{} & RT(z) \end{array}$$

follows from the coherence condition (dc.3) of φ as a double cell in $\mathbb{C}$mc (where $\underline{RT}_i(x,y) = R\underline{T}_i(x,y).\underline{R}_i(Tx,Ty)$)

$$\underline{RT}_i(x,y).(\varphi x +_i \varphi y).\underline{F}_i(Sx,Sy) = \varphi z.F\underline{S}_i(x,y), \qquad (50)$$

$$\begin{array}{ccccc} F(Sx +_i Sy) & \xrightarrow{F\underline{S}_i(x,y)} & FS(z) & \xrightarrow{\varphi z} & RT(z) \\ {\scriptstyle \underline{F}_i(Sx,Sy)} \downarrow & & & & \downarrow {\scriptstyle 1} \\ FSx +_i FSy & \xrightarrow[\varphi x +_i \varphi y]{} & RTx +_i RTy & \xrightarrow[\underline{RT}_i(x,y)]{} & RT(z) \end{array}$$

The uniqueness of L is obvious. □

3. Companions and adjoints in double categories

This brief section, taken from [GP2], Section 1, studies the connections between horizontal and vertical morphisms in a double category: horizontal morphisms can have vertical *companions* and vertical *adjoints* (the latter were called 'conjoints' in [DPR]). Such phenomena are interesting in themselves and typical of double categories.

$\mathbb{D}$ is always a weak double category, that we assume to be *unitary* for the sake of simplicity. We shall apply these notions to $\mathbb{C}$mc, which is strict.

3.1 Orthogonal companions

In the weak double category $\mathbb{D}$, the horizontal morphism $f\colon A \to B$ and the vertical morphism $u\colon A \longrightarrow B$ are made (orthogonal) *companions* by

assigning a pair (η, ε) of cells as below, called the *unit* and *counit*, that satisfy the identities $\eta|\varepsilon = 1_f^\bullet$ and $\eta \otimes \varepsilon = 1_u$

$$
\begin{array}{ccc}
A &=\!=\!=& A \\
1\downarrow & \eta & \downarrow u \\
A &\xrightarrow{\ f\ }& B
\end{array}
\qquad\qquad
\begin{array}{ccc}
A &\xrightarrow{\ f\ }& B \\
u\downarrow & \varepsilon & \downarrow 1 \\
B &=\!=\!=& B
\end{array}
\qquad\qquad (51)
$$

Given f, this is equivalent (*by unitarity*) to saying that the pair (u, ε) satisfies the following universal property:

(a) for every cell $\varepsilon'\colon (u' \ {}^{f}_{g} \ B)$ there is a unique cell $\lambda\colon (u' \ {}^{A}_{g} \ u)$ such that $\varepsilon' = \lambda|\varepsilon$

$$
\begin{array}{ccc}
A &\xrightarrow{\ f\ }& B \\
u'\downarrow & \varepsilon' & \downarrow 1 \\
A' &\xrightarrow{\ g\ }& B
\end{array}
\qquad\qquad
\begin{array}{ccccc}
A &=\!=\!=& A &\xrightarrow{\ f\ }& B \\
u'\downarrow & \lambda & \downarrow u & \varepsilon & \downarrow 1 \\
A' &\xrightarrow{\ g\ }& B &=\!=\!=& B
\end{array}
\qquad (52)
$$

In fact, given (η, ε), we can (and must) take $\lambda = \eta \otimes \varepsilon'$; on the other hand, given ε' we define $\eta\colon (A \ {}^{A}_{f} \ u)$ by the equation $\eta|\varepsilon = 1_f^\bullet$ and deduce that $\eta \otimes \varepsilon = 1_u$ because $(\eta \otimes \varepsilon)\,|\,\varepsilon = (\eta|\varepsilon) \otimes \varepsilon = \varepsilon = (1_u|\varepsilon)$.

Similarly, also the pair (u, η) is characterised by a universal property

(b) for every cell $\eta'\colon (A \ {}^{g}_{f} \ u')$ there is a unique cell $\mu\colon (u \ {}^{g}_{B} \ u')$ such that $\eta' = \eta|\mu$.

Therefore, if f has a vertical companion, this is determined up to a unique special isocell, *and will often be written as f_*.* Companions compose in the obvious (covariant) way: if $g\colon B \to C$ also has a companion g_*, then the vertical arrow $g_* f_*\colon A \dashrightarrow C$ is companion to $gf\colon A \to C$, with unit

$$
\left(\frac{\eta \mid 1}{1^\bullet \mid \eta'} \right)\colon \ (A \ {}^{A}_{gf} \ g_* f_*).
$$

Companionship is preserved by *unitary* lax or colax double functors.

We say that $\mathbb{D}$ *has vertical companions* if every horizontal arrow has a vertical companion. The weak double categories recalled in Section 1 have

vertical companions, given by the obvious embedding of horizontal arrows into the vertical ones.

Companionship is simpler for horizontal *iso*morphisms. If f is one and has a companion u, then its unit and counit are also horizontally invertible and determine each other:

$$(\varepsilon \,|\, 1_g^\bullet \,|\, \eta) = \eta \otimes \varepsilon = 1_u \qquad\qquad (g = f^{-1}), \qquad\qquad (53)$$

as one can see rewriting $(\varepsilon \,|\, 1_g^\bullet \,|\, \eta)$ as follows, and then applying middle-four interchange

$$
\begin{array}{ccccccc}
A & \xrightarrow{\;f\;} & B & \xrightarrow{\;g\;} & A & =\!=\!= & A \\
{\scriptstyle 1}\big\downarrow & 1_f^\bullet & \big\downarrow{\scriptstyle 1} & 1_g^\bullet & \big\downarrow{\scriptstyle 1} & \eta & \big\downarrow{\scriptstyle u} \\
A & -f\!\rightarrow & B & -g\!\rightarrow & A & -f\!\rightarrow & B \\
{\scriptstyle u}\big\downarrow & \varepsilon & \big\downarrow{\scriptstyle 1} & 1_g^\bullet & \big\downarrow{\scriptstyle 1} & 1_f^\bullet & \big\downarrow{\scriptstyle 1} \\
B & =\!=\!= & B & \xrightarrow{\;g\;} & A & \xrightarrow{\;f\;} & B
\end{array}
$$

Conversely, the existence of a horizontally invertible cell $\eta\colon (A \;{}^A_f\; u)$ implies that f is horizontally invertible, with companion u and counit as above.

3.2 Orthogonal adjoints

Transforming companionship by horizontal (or vertical) duality, the arrows $f\colon A \to B$ and $v\colon B \rightarrowtail A$ are made *orthogonal adjoints* by a pair (α, β) of cells as below

$$
\begin{array}{ccc}
A \xrightarrow{\;f\;} B & \qquad & B =\!=\!= B \\
{\scriptstyle 1}\big\downarrow \;\; \alpha \;\; \big\downarrow{\scriptstyle v} & & {\scriptstyle v}\big\downarrow \;\; \beta \;\; \big\downarrow{\scriptstyle 1} \\
A =\!=\!= A & & A \xrightarrow[\;f\;]{} B
\end{array}
\qquad (54)
$$

with $\alpha|\beta = 1_f^\bullet$ and $\beta \otimes \alpha = 1_v$. Then, f is the *horizontal adjoint* and v the *vertical* one. (In the general case there is no reason of distinguishing 'left' and 'right', unit and counit; see the examples of [GP2], Section 1.3).

Again, given f, these relations can be described by universal properties for (v, β) or (v, α)

(a) for every cell $\beta'\colon (v' \ {}^{g}_{f} \ B)$ there is a unique cell $\lambda\colon (v' \ {}^{g}_{A} \ v)$ such that $\beta' = \lambda|\beta$,

(b) for every cell $\alpha'\colon (A \ {}^{f}_{g} \ v')$ there is a unique cell $\mu\colon (v \ {}^{B}_{g} \ v')$ such that $\alpha' = \alpha|\mu$.

The vertical adjoint of f is determined up to a special isocell and will often be written as f^*; vertical adjoints compose, contravariantly: $(gf)^*$ can be constructed as f^*g^*.

We say that $\mathbb{D}$ *has vertical adjoints* if every horizontal arrow has a vertical adjoint. Plainly, this is the case for the weak double categories recalled in Section 1.

3.3 Proposition

Let $f\colon A \to B$ have a vertical companion $u\colon A \ \to\!\!\!\!\to \ B$. Then the arrow $v\colon B \to\!\!\!\!\to A$ is vertical adjoint to f if and only if $u \dashv v$ in the bicategory $\mathbf{V}\mathbb{D}$ (of objects, vertical arrows and special cells).

Proof. Given four cells $\eta, \varepsilon, \alpha, \beta$ as above (in 3.1, 3.2), we have two special cells

$$\eta \otimes \alpha\colon 1^\bullet \to u \otimes v, \qquad\qquad \beta \otimes \varepsilon\colon u \otimes v \to 1^\bullet,$$

that are easily seen to satisfy the triangle identities in $\mathbf{V}\mathbb{D}$. The converse is similarly obvious. $\qquad\square$

4. Multiple adjunctions

We now define a colax/lax adjunction between chiral multiple categories, a notion that occurs naturally in various situations, as already seen in Section 1.

4.1 Colax/lax adjunctions

A *colax/lax cm-adjunction* $(\eta, \varepsilon)\colon F \dashv G$, or a colax/lax adjunction between chiral multiple categories, will be an orthogonal adjunction *in* the double category $\mathbb{C}\mathrm{mc}$ (as defined in 3.2).

The data amount thus to:

- a *colax* cm-functor $F\colon \mathsf{X} \to \mathsf{A}$, the left adjoint,

- a *lax* cm-functor $G\colon \mathsf{A} \to \mathsf{X}$, the right adjoint,

- two $\mathbb{C}$mc-cells $\eta\colon 1_\mathsf{X} \dashrightarrow GF$ and $\varepsilon\colon FG \dashrightarrow 1_\mathsf{A}$ (unit and counit) that satisfy the triangle equalities:

$$
\begin{array}{ccc}
\mathsf{X} =\!\!=\!\!= \mathsf{X} & \mathsf{A} \xrightarrow{\ G\ } \mathsf{X} & \eta \otimes \varepsilon = 1_F, \\
{\scriptstyle F}\big\downarrow\ \ \eta\ \ \big\| & \big\|\ \ \varepsilon\ \ \big\downarrow{\scriptstyle F} & \\
\mathsf{A} \xrightarrow[\ G\ ]{} \mathsf{X} & \mathsf{A} =\!\!=\!\!= \mathsf{A} & \varepsilon \mid \eta = 1_G^\bullet.
\end{array}
\tag{55}
$$

We speak of a *pseudo/lax* (resp. a *colax/pseudo*) adjunction when the left (resp. right) adjoint is pseudo, and of a *pseudo* (or *strict*) adjunction when both adjoints are pseudo (or strict).

From general properties (see 3.2), we already know that the left adjoint of a lax cm-functor G is determined up to transversal isomorphism (which amounts to a special invertible cell between vertical arrows in $\mathbb{C}$mc) and that left adjoints compose, contravariantly. Similarly for right adjoints.

As in 2.2, the arrow of a colax cm-functor is marked with a dot *when displayed vertically*, in a double cell of $\mathbb{C}$mc. Again, we may write unit and counit as $\eta\colon 1 \dashrightarrow GF$ and $\varepsilon\colon FG \dashrightarrow 1$, *but we should recall that the coherence conditions of such 'transformations' involve the comparison cells of F and G. Therefore (as with double categories, in [GP2]), a general* colax/lax *adjunction cannot be seen as an adjunction in some bicategory*; although this is possible for a *pseudo/lax* or a *colax/pseudo* adjunction, as we shall prove in the next section.

4.2 A description

To make the previous definition explicit, a *colax/lax adjunction* $(\eta, \varepsilon)\colon F \dashv G$ between the cm-categories X and A consists of the following items.

(a) A *colax* cm-functor $F\colon \mathsf{X} \to \mathsf{A}$, with comparison transversal maps

$$\underline{F}_i(x)\colon F(e_i x) \to_0 e_i(Fx), \qquad \underline{F}_i(x, y)\colon F(x +_i y) \to_0 Fx +_i Fy.$$

(b) A *lax* cm-functor $G\colon \mathsf{A} \to \mathsf{X}$, with comparison transversal maps

$$\underline{G}_i(a)\colon e_i(Ga) \to_0 G(e_ia), \qquad \underline{G}_i(a,b)\colon Ga +_i Gb \to_0 G(a +_i b).$$

(c) An ordinary adjunction $F_{\mathbf{i}} \dashv G_{\mathbf{i}}$ for every positive multi-index $\mathbf{i}$

$$\eta_{\mathbf{i}}\colon 1 \to G_{\mathbf{i}}F_{\mathbf{i}}\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{X}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{X}), \qquad \varepsilon_{\mathbf{i}}\colon F_{\mathbf{i}}G_{\mathbf{i}} \to 1\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A}),$$

$$\varepsilon_{\mathbf{i}}F_{\mathbf{i}}.F_{\mathbf{i}}\eta_{\mathbf{i}} = 1_{F_{\mathbf{i}}}, \qquad G_{\mathbf{i}}\varepsilon_{\mathbf{i}}.\eta_{\mathbf{i}}G_{\mathbf{i}} = 1_{G_{\mathbf{i}}}.$$

(Note that *there is an abuse of notation*: for the sake of simplicity we write as $F_{\mathbf{i}}$ the functor $\mathrm{tv}_{\mathbf{i}}(F)\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{X}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$, which involves the components $F_{\mathbf{i}}\colon X_{\mathbf{i}} \to A_{\mathbf{i}}$ and $F_{0\mathbf{i}}\colon X_{0\mathbf{i}} \to A_{0\mathbf{i}}$ of F.)

Explicitly this means that we are assigning:

- transversal maps $\eta_{\mathbf{i}}x\colon x \to_0 G_{\mathbf{i}}F_{\mathbf{i}}x$ in X (for x in $X_{\mathbf{i}}$), also written as $\eta x\colon x \to_0 GFx$,

- transversal maps $\varepsilon_{\mathbf{i}}a\colon F_{\mathbf{i}}G_{\mathbf{i}}a \to_0 a$ in A (for a in $A_{\mathbf{i}}$), also written as $\varepsilon a\colon FGa \to_0 a$,

satisfying the naturality conditions (c1) (for transversal maps $f\colon x \to_0 y$ in X and $h\colon a \to_0 b$ in A) and the triangle identities (c2),

(c1)　$\eta y.f = GFf.\eta x,$ 　　　　　$\varepsilon b.FGh = h.\varepsilon a,$

(c2)　$\varepsilon Fx.F\eta x = 1_{Fx},$ 　　　　$G\varepsilon a.\eta Ga = 1_{Ga}.$

(d) These families $\eta = (\eta_{\mathbf{i}})$ and $\varepsilon = (\varepsilon_{\mathbf{i}})$ must respect faces

$$\eta(\partial_i^\alpha x) = \partial_i^\alpha(\eta x), \qquad\qquad \varepsilon(\partial_i^\alpha a) = \partial_i^\alpha(\varepsilon a), \qquad\qquad (56)$$

and be coherent with the comparison cells of F and G:

(d1) (*coherence of η and ε with i-identities*) for x in X and a in A:

$$G\underline{F}_i(x).\eta(e_ix) = \underline{G}_i(Fx).e_i(\eta x),$$
$$(\eta(e_i(x) = e_i(\eta x),\ \text{if } F \text{ and } G \text{ are unitary}), \qquad (57)$$

$$\varepsilon(e_ia).F\underline{G}_i(a) = e_i(\varepsilon a).\underline{F}_i(Ga),$$
$$(\varepsilon(e_ia) = e_i(\varepsilon a),\ \text{if } F \text{ and } G \text{ are unitary}), \qquad (58)$$

(d2) (*coherence of η and ε with i-composition*) for $z = x +_i y$ in X and $c = a +_i b$ in A:

$$G\underline{F}_i(x, y).\eta z = \underline{G}_i(Fx, Fy).(\eta x +_i \eta y), \tag{59}$$

$$\varepsilon c.F\underline{G}_i(a, b) = (\varepsilon a +_i \varepsilon b).\underline{F}_i(Ga, Gb). \tag{60}$$

$$
\begin{array}{ccc}
z \xrightarrow{\ \eta x +_i \eta y\ } GFx +_i GFy & \qquad & F(Ga + Gb) \xrightarrow{\ F\underline{G}_i(a,b)\ } FGc \\
\eta z \downarrow \qquad \underline{G}_i(Fx,Fy) \downarrow & & \underline{F}_i(Ga,Gb) \downarrow \qquad \downarrow \varepsilon c \\
GFz \xrightarrow{\ G\underline{F}_i(x,y)\ } G(Fx +_i Fy) & & FGa + FGb \xrightarrow{\ \varepsilon a +_i \varepsilon b\ } c
\end{array}
$$

4.3 Lemma

(a) In a colax/lax cm-adjunction $(\eta, \varepsilon) \colon F \dashv G$, the comparison maps of G determine the comparison maps of F, through the ordinary adjunctions $F_{\mathbf{i}} \dashv G_{\mathbf{i}}$, as

$$
\begin{aligned}
\underline{F}_i(x) &= \varepsilon e_i(Fx).F\underline{G}_i(Fx).Fe_i(\eta x) \colon \\
& \quad Fe_i(x) \to Fe_i(GFx) \to FG(e_i Fx) \to e_i Fx,
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
\underline{F}_i(x, y) &= \varepsilon(Fx +_i Fy).F\underline{G}_i(Fx, Fy).F(\eta x +_i \eta y) \colon \\
F(x +_i y) &\to F(GFx + GFy) \to FG(Fx + Fy) \to Fx +_i Fy.
\end{aligned} \tag{62}
$$

Dually, the comparison maps of F determine the comparison maps of G, through the ordinary adjunctions.

(b) If all the components of η, ε are invertible, then G is pseudo if and only if F is.

Note. Loosely speaking, point (a) says that a lax multiple functor *can only have* a colax left adjoint (if any), and symmetrically. This fact will be completed in Theorem 5.3, showing that if a lax functor has a *lax* adjoint, the latter is necessarily pseudo.

Proof. (a) The first equation of (d1) says that the adjoint map of $\underline{F}_i(x)$, i.e. $(\underline{F}_i(x))' = G\underline{F}_i(x).\eta(e_i x)$, must be equal to $f = \underline{G}_i(Fx).e_i(\eta x)$. The adjoint map of the latter gives $\underline{F}_i(x) = f' = \varepsilon e_i(Fx).F(f)$.

In the same way the first equation of (d2) determines $\underline{F}_i(x, y)$. Point (b) is a straightforward consequence. □

4.4 Theorem (Characterisation by transversal hom-sets)

A multiple adjunction $(\eta, \varepsilon)\colon\ F \dashv G$ can equivalently be given by a colax cm-functor $F\colon \mathsf{X} \to \mathsf{A}$, a lax cm-functor $G\colon \mathsf{A} \to \mathsf{X}$ and a family $(L_{\mathbf{i}})$ of functorial isomorphisms indexed by the positive multi-indices $\mathbf{i} \subset \mathbb{N}$

$$L_{\mathbf{i}}\colon\ \mathrm{tv}_{\mathbf{i}}(\mathsf{A})(F_{\mathbf{i}}(-), =)\ \to\ \mathrm{tv}_{\mathbf{i}}(\mathsf{X})(-, G_{\mathbf{i}}(=))\colon$$
$$\mathrm{tv}_{\mathbf{i}}(\mathsf{X})^{\mathrm{op}} \times \mathrm{tv}_{\mathbf{i}}(\mathsf{A}) \to \mathbf{Set}, \qquad (63)$$
$$L_{\mathbf{i}}(x, a)\colon\ \mathrm{tv}_{\mathbf{i}}(\mathsf{A})(Fx, a) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{X})(x, Ga).$$

The components $L_{\mathbf{i}}(x, a)$, also written as $L(x, a)$ or just L, have to commute with faces and be coherent with the positive operations (through the comparison maps of F and G), i.e. must satisfy the following conditions (ad.1-3):

(ad.1) $L_{\mathbf{i}}(\partial_i^\alpha x, \partial_i^\alpha a) = \partial_i^\alpha(L_{\mathbf{i}}(x, a)),$

(ad.2) $L(e_i x, e_i a)(e_i(h).\underline{F}(x)) = \underline{G}(a).e_i(Lh)$ *(for $h\colon Fx \to a$ in A),*

$$Fe_i(x) \xrightarrow{\ F(x)\ } e_i(Fx) \xrightarrow{\ e_i(h)\ } e_i(a)$$

$$e_i(x) \xrightarrow{\ e_i(Lh)\ } e_i(Ga) \xrightarrow{\ G(a)\ } Ge_i(a)$$

(ad.3) $L((h +_i k).\underline{F}_i(x, y)) = \underline{G}_i(a, b).(Lh +_i Lk)$

$$\text{(for } h\colon Fx \to a,\ k\colon Fy \to b\ \text{ in } \mathrm{tv}_{\mathbf{i}}(\mathsf{A})),$$

$$F(x +_i y) \xrightarrow{\ \underline{F}_i(x,y)\ } Fx +_i Fy \xrightarrow{\ h+ik\ } a +_i b$$

$$x +_i y \xrightarrow{\ Lh+_i Lk\ } Ga +_i Gb \xrightarrow{\ \underline{G}_i(a,b)\ } G(a +_i b)$$

In this equivalence, $L_{\mathbf{i}}(x, a)$ is defined by the unit η as

$$L_{\mathbf{i}}(x, a)(h) = Gh.\eta_{\mathbf{i}} x\colon x \to GFx \to Ga \quad (h\colon Fx \to a \text{ in } \mathrm{tv}_{\mathbf{i}}(\mathsf{A})). \quad (64)$$

The other way round, the component $\eta_{\mathbf{i}}\colon 1 \to G_{\mathbf{i}} F_{\mathbf{i}}\colon X_{\mathbf{i}} \to X_{\mathbf{i}}$ of the unit is defined by L as

$$\eta_{\mathbf{i}}(x) = L_{\mathbf{i}}(x, Fx)(\mathrm{id} Fx)\colon x \to GF(x) \qquad (\textit{for } x \textit{ in } X_{\mathbf{i}}). \qquad (65)$$

Proof. We have only to verify the equivalence of the conditions (56)-(60) with the conditions above.

This is straightforward. For instance, to show that (59) implies (ad.3), let $h\colon Fx \to a$ and $k\colon Fy \to b$ be i-consecutive **i**-maps in A, and apply $L = L(x +_i y, a +_i a)$ as defined above, in (64):

$$
\begin{aligned}
L((h +_i k).\underline{F}_i(x,y)) &= G(h +_i k).G\underline{F}_i(x,y).\eta(x +_i y) \\
&= G(h +_i k).\underline{G}_i(Fx, Fy).(\eta x +_i \eta y) && \text{(by (59))} \\
&= \underline{G}_i(a,b).(Gh +_i Gk).(\eta x +_i \eta y) && \text{(by (lmf.2))} \\
&= \underline{G}_i(a,b).(Lh +_i Lk).
\end{aligned}
$$

$\square$

4.5 Corollary (Characterisation by commas)

With the previous notation, a multiple adjunction amounts to an isomorphism of chiral multiple categories $L\colon F\downarrow A \to X\downarrow G$ over the cartesian product $X\times A$

$$
\begin{array}{ccc}
F\downarrow A & \xrightarrow{\ \ L\ \ } & X\downarrow G \\
& \searrow \quad = \quad \swarrow & \\
& X\times A &
\end{array}
\tag{66}
$$

Proof. A straightforward consequence of the previous theorem. $\square$

4.6 Theorem (Right adjoint by universal properties)

Let a colax cm-functor $F\colon X \to A$ be given.

The existence (and choice) of a right adjoint lax cm-functor G amounts to a family (rad.**i**) *of conditions and choices, indexed by the positive multi-indices* **i***:*

(rad.**i**) for every **i**-cube a in A there is a universal arrow $(Ga, \varepsilon_\mathbf{i}a\colon F(Ga) \to a)$ from the functor $F_\mathbf{i}\colon \mathrm{tv}_\mathbf{i}(X) \to \mathrm{tv}_\mathbf{i}(A)$ to the object a, and we choose one,

provided that these choices commute with faces.

Explicitly, the universal property means that, for each **i***-cube x in X and* **i***-map $h\colon Fx \to_0 a$ in A there is a unique $f\colon x \to_0 Ga$ such that $h = \varepsilon a.Ff\colon Fx \to_0 F(Ga) \to_0 a$.*

The comparison $\mathbf{i}$-maps of G

$$\underline{G}_i(a)\colon e_i(Ga) \to_0 G(e_i a), \quad \underline{G}_i(a,b)\colon Ga +_i Gb \to_0 G(a +_i b), \quad (67)$$

are then given by the universal property of ε, as the unique solution of the equations (58)*,* (60)*, respectively.*

Proof. The conditions (rad.$\mathbf{i}$) are plainly necessary, including consistency with faces.

Conversely, each (rad.$\mathbf{i}$) provides an ordinary adjunction $(\eta_{\mathbf{i}}, \varepsilon_{\mathbf{i}})\colon F_{\mathbf{i}} \dashv G_{\mathbf{i}}$ for the categories $\mathrm{tv}_{\mathbf{i}}(\mathsf{X})$, $\mathrm{tv}_{\mathbf{i}}(\mathsf{A})$, so that G, η and ε are correctly defined - as far as cubes, transversal maps, faces, transversal composition and transversal identities are concerned.

Now we define the comparison maps $\underline{G}_i$ as specified in the statement, so that the coherence properties of ε are satisfied (see (58), (60)). One verifies easily, for such transversal maps, the axioms of naturality and coherence (see 2.1).

Finally, we have to prove that $\eta\colon 1 \dashrightarrow GF$ satisfies the coherence property (59)

$$G\underline{F}_i(x,y).\eta z = \underline{G}_i(Fx, Fy).(\eta x +_i \eta y), \quad (68)$$

with respect to a composition $z = x +_i y$ of $\mathbf{i}$-cubes in X (similarly one proves (57)). By the universal property of ε, it will suffice to show that the composite $\varepsilon(Fx +_i Fy).F(-)$ takes the same value on both terms of (68). In fact, on the left-hand term we get $\underline{F}_i(x,y)$

$$\varepsilon(Fx +_i Fy).FG\underline{F}_i(x,y).F\eta z = \underline{F}_i(x,y).\varepsilon Fz.F\eta z = \underline{F}_i(x,y).$$

We get the same on the right-hand term of (68), using (60), the naturality of $\underline{F}_i$, the $0i$-interchange in A and a triangle identity

$$\varepsilon(Fx +_i Fy).F\underline{G}_i(Fx, Fy).F(\eta x +_i \eta y)$$
$$= (\varepsilon Fx +_i \varepsilon Fy).\underline{F}_i(GFx, GFy).F(\eta x +_i \eta y)$$
$$= (\varepsilon Fx +_i \varepsilon Fy).(F\eta x +_i F\eta y).\underline{F}_i(x,y)$$
$$= (\varepsilon Fx.F\eta x +_i \varepsilon Fy.F\eta y).\underline{F}_i(x,y)$$
$$= (1_{Fx} +_i 1_{Fy}).\underline{F}_i(x,y)$$
$$= \underline{F}_i(x,y).$$

$\square$

4.7 Theorem (Factorisation of adjunctions)

Let $F \dashv G$ be a colax/lax adjunction between X and A. Then, using the isomorphism of cm-categories $L \colon F \downarrow \mathsf{A} \to \mathsf{X} \downarrow G$ (Corollary 4.5), we can factorise the adjunction

$$\mathsf{X} \xrightleftharpoons[P]{F'} F \downarrow \mathsf{A} \xrightleftharpoons[L^{-1}]{L} \mathsf{X} \downarrow G \xrightleftharpoons[G']{Q} \mathsf{A} \tag{69}$$

$$F = QLF', \qquad G = PL^{-1}G',$$

as a composite of:

- a coreflective *colax/strict adjunction $F' \dashv P$ (with unit $PF' = 1$),*
- an isomorphism $L \dashv L^{-1}$,
- a reflective *strict/lax adjunction $Q \dashv G'$ (with counit $QG' = 1$),*

where the comma projections P and Q are strict cm-functors.

Proof. We define the lax cm-functor $G' \colon \mathsf{A} \to \mathsf{X} \downarrow G$ by the universal property of commas 2.6(a), applied to $G \colon \mathsf{A} \to \mathsf{X}$, $1 \colon \mathsf{A} \to \mathsf{A}$ and $\varphi = 1_G^{\bullet}$ as in the diagram below

$$\tag{70}$$

$$G'(a) = (Ga, a; 1 \colon Ga \to Ga),$$

$$\underline{G}'_i(a) = (\underline{G}_i(a), 1) \colon (e_i Ga, e_i a; \underline{G}_i(a)) \to (G(e_i a), e_i a; 1),$$

$$\underline{G}'_i(a, b) = (\underline{G}_i(a, b), 1) \colon$$
$$\qquad (Ga +_i Gb, a +_i b; \underline{G}_i(a, b)) \to (G(a +_i b), a +_i b; 1).$$

Similarly, we define the colax cm-functor $F' \colon \mathsf{X} \to F \downarrow \mathsf{A}$ by the dual result 2.6(b)

$$F'(x) = (x, Fx; 1 \colon Fx \to Fx),$$

$$\underline{F}'_i(x) = (1, \underline{F}_i(x)) \colon (e_i x, F(e_i x); 1) \to (e_i x, Fe_i x; \underline{F}_i(x)),$$

$$\underline{F}'_i(x, y) = (1, \underline{F}_i(x, y)) \colon$$
$$\qquad (x +_i y, F(x +_i y); 1) \to (x +_i y, Fx +_i Fy; \underline{F}_i(x, y)).$$

$$\tag{71}$$

The coreflective adjunction $F' \dashv P$ is obvious

$$
\begin{aligned}
\eta' x &= 1_x \colon x \to PF'x, \\
\varepsilon'(x, a; f \colon Fx \to a) &= (1_x, f) \colon \\
&\quad (x, Fx; 1 \colon Fx \to Fx) \to (x, a; f \colon Fx \to a),
\end{aligned}
\tag{72}
$$

as well as the reflective adjunction $Q \dashv G'$ and the factorisation above. $\quad\square$

5. Multiple adjunctions and pseudo cm-functors

We consider now cm-adjunctions where the left or right adjoint is a pseudo cm-functor. Then we introduce adjoint equivalences of chiral multiple categories.

5.1 Comments

Let us recall, from 4.1, that a *pseudo/lax* cm-adjunction $F \dashv G$ is a colax/lax adjunction between cm-categories where the left adjoint F is pseudo.

Then the comparison cells of F are horizontally invertible and the composites GF and FG are lax cm-functors; it follows (from definition 2.2) that the unit and counit are horizontal transformations of such functors. Therefore *a pseudo/lax cm-adjunction gives an adjunction in the 2-category* $\mathrm{Lx}\mathbf{Cmc}$ of cm-categories, lax cm-functors and transversal transformations (see 2.2); and we shall prove that these two facts are actually equivalent (Theorem 5.3).

Dually a *colax/pseudo* cm-adjunction, where the right adjoint G is pseudo, will amount to an adjunction in the 2-category $\mathrm{Cx}\mathbf{Cmc}$ of cm-categories, colax cm-functors and transversal transformations. Finally a *pseudo* cm-adjunction, where both F and G are pseudo, will be the same as an adjunction in the 2-category $\mathrm{Ps}\mathbf{Cmc}$ whose arrows are the pseudo cm-functors.

5.2 Theorem (Companions in $\mathbb{C}$mc)

A lax cm-functor G has an orthogonal companion F in the double category $\mathbb{C}$mc if and only if it is pseudo; then one can define $F = G_*$ as the colax

cm-functor which coincides with G except for comparison maps, that are transversally inverse to those of G.

Proof. We restrict to unitary cm-categories, for simplicity. If G is pseudo, it is obvious that G_*, as defined above, is an orthogonal companion.

Conversely, suppose that $G\colon \mathsf{A} \to \mathsf{X}$ (lax) has an orthogonal companion F (colax). There are thus two double cells η, ε in $\mathbb{C}$mc)

$$
\begin{array}{ccc}
\mathsf{A} \mathrel{=\!=\!=} \mathsf{A} & & \mathsf{A} \xrightarrow{\ G\ } \mathsf{X} \\
\| \quad \eta \quad \Big\downarrow {\scriptstyle F} & & {\scriptstyle F}\Big\downarrow \quad \varepsilon \quad \| \\
\mathsf{A} \xrightarrow[\ G\ ]{} \mathsf{X} & & \mathsf{X} \mathrel{=\!=\!=} \mathsf{X}
\end{array}
\tag{73}
$$

which satisfy the identities $\eta|\varepsilon = 1_G^\bullet,\ \ \eta \otimes \varepsilon = 1_F$.

This means two 'transformations' $\eta\colon F \dashrightarrow G,\ \varepsilon\colon G \dashrightarrow F$, as defined in 2.2; for every **i**-cube a in A, we have two transversal maps ηa and εa in X

$$
\eta a\colon Fa \to Ga, \qquad\qquad \varepsilon a\colon Ga \to Fa, \tag{74}
$$

consistently with faces. These maps are transversally inverse, because of the previous identities (see (36))

$$
\eta a.\varepsilon a = (\eta \,|\, \varepsilon)(a) = 1_{Ga}, \qquad \varepsilon a.\eta a = (\eta \otimes \varepsilon)(a) = 1_{Fa}. \tag{75}
$$

Applying now the coherence condition (dc.3) (of 2.2) for the transformations η, ε and a concatenation $c = a +_i b$ in A we find

$$
\begin{aligned}
\eta c &= \underline{G}_i(a,b).(\eta a +_i \eta b).\underline{F}_i(a,b)\colon Fc \to Gc, \\
\varepsilon a +_i \varepsilon b &= \underline{F}_i(a,b).\varepsilon c.\underline{G}_i(a,b)\colon Ga +_i Gb \to Fa +_i Fb.
\end{aligned}
\tag{76}
$$

Since all the components of η and ε are transversally invertible, this proves that $\underline{G}_i(a,b)$ has a left inverse and a right inverse transversal map, whence it is invertible. Similarly for degeneracies.

Therefore G is pseudo and F is transversally isomorphic to G_*. $\qquad\square$

5.3 Theorem

(a) (Pseudo/lax adjunctions) *For every adjunction $F \dashv G$ in the 2-category* $\mathrm{Lx}\mathbf{Cmc}$, *the functor F is pseudo and the adjunction is pseudo/lax, in the sense of 4.1 (or 5.1).*

(b) (Colax/pseudo adjunctions) *For every adjunction $F \dashv G$ in the 2-category* $\mathrm{Cx}\mathbf{Cmc}$, *the functor G is pseudo and the adjunction is colax/pseudo, in the sense of 4.1 (or 5.1).*

Note. More formally, (a) can be rewritten saying that, in the double category $\mathbb{C}\mathrm{mc}$, if the horizontal arrow G has a 'horizontal left adjoint' F (within the horizontal 2-category $\mathbf{H}\mathbb{C}\mathrm{mc} = \mathrm{Lx}\mathbf{Cmc}$), then it also has an orthogonal adjoint G^* (colax). (Then, applying Proposition 3.3, it would follow that F and G^* are companions, whence F is pseudo, by Theorem 5.2, and isomorphic to G^*.)

Proof. It suffices to prove (a); again, we only deal with the comparisons of a composition.

Let the lax structures of $F\colon \mathsf{X} \to \mathsf{A}$ and $G\colon \mathsf{A} \to \mathsf{X}$ be given by the following comparison maps, where $z = x +_i y$ and $c = a +_i b$

$$\lambda_i(x, y)\colon Fx +_i Fy \to F(x +_i y), \quad \underline{G}_i(a, b)\colon Ga +_i Gb \to G(a +_i b),$$

so that we have:

$$
\begin{aligned}
\eta z &= G\lambda_i(x, y).\underline{G}_i(Fx, Fy).(\eta x +_i \eta y)\colon \\
&\qquad z \to GFx +_i GFy \to G(Fx +_i Fy) \to GFz, \\
\varepsilon a +_i \varepsilon b &= \varepsilon c.F\underline{G}_i(a, b).\lambda_i(Ga, Gb)\colon \\
&\qquad FGa +_i FGb \to F(Ga +_i Gb) \to FG(a +_i b) \to c.
\end{aligned}
\tag{77}
$$

We construct a colax structure $\underline{F}$ for F, letting

$$
\begin{aligned}
\underline{F}_i(x, y) &= \varepsilon(Fx +_i Fy).F\underline{G}_i(Fx, Fy).F(\eta x +_i \eta y)\colon \\
&\qquad Fz \to F(GFx +_i GFy) \to FG(Fx +_i Fy) \to Fx +_i Fy.
\end{aligned}
$$

Now it is sufficient to verify that $\underline{F}_i(x, y)$ and $\lambda_i(x, y)$ are transversally

inverse, using the naturality of ε, λ and (77):

$$\lambda_i(x,y).\underline{F}_i(x,y)$$
$$= \lambda_i(x,y).\varepsilon(Fx +_i Fy).F\underline{G}_i(Fx, Fy).F(\eta x +_i \eta y)$$
$$= \varepsilon Fz.FG\lambda_i(x,y).F\underline{G}_i(Fx, Fy).F(\eta x +_i \eta y)$$
$$= \varepsilon F(z).F(\eta z) = 1_{Fz},$$

$$\underline{F}_i(x,y).\lambda_i(x,y)$$
$$= \varepsilon(Fx +_i Fy).F\underline{G}_i(Fx, Fy).F(\eta x +_i \eta y).\lambda_i(x,y)$$
$$= \varepsilon(Fx +_i Fy).F\underline{G}_i(Fx, Fy).\lambda_i(GFx, GFy).(F\eta x +_i F\eta y)$$
$$= (\varepsilon Fx +_i \varepsilon Fy).(F\eta x +_i F\eta y)$$
$$= \varepsilon Fx.F\eta x +_i \varepsilon Fy.F\eta y = 1_{Fx} +_i 1_{Fy} = 1_{Fx +_i Fy}.$$

$\square$

5.4 Equivalences of cm-categories

An *adjoint equivalence* between two cm-categories X and A will be a pseudo cm-adjunction $(\eta, \varepsilon)\colon F \dashv G$ where the transversal transformations $\eta\colon 1_\mathsf{X} \to GF$ and $\varepsilon\colon FG \to 1_\mathsf{A}$ are invertible.

The following properties of a cm-functor $F\colon \mathsf{X} \to \mathsf{A}$ will allow us (in the next theorem) to characterise this fact in the usual way, under the mild restriction of *transversal invariance* (see II.1.6):

(a) We say that F is *faithful* if all the ordinary functors $F_\mathbf{i}\colon \mathrm{tv}_\mathbf{i}(\mathsf{X}) \to \mathrm{tv}_\mathbf{i}(\mathsf{A})$ (between the categories of $\mathbf{i}$-cubes and their transversal maps) are faithful: given two $\mathbf{i}$-maps $f, g\colon x \to_0 y$ of X between the same $\mathbf{i}$-cubes, $F(f) = F(g)$ implies $f = g$.

(b) Similarly, we say that F is *full* if all the ordinary functors $F_\mathbf{i}\colon \mathrm{tv}_\mathbf{i}(\mathsf{X}) \to \mathrm{tv}_\mathbf{i}(\mathsf{A})$ are: for every $\mathbf{i}$-map $h\colon F(x) \to_0 F(y)$ in A there is an $\mathbf{i}$-map $f\colon x \to_0 y$ in X such that $F(f) = h$.

(c) Finally, we say that F is *essentially surjective on cubes* if every $F_\mathbf{i}$ is: for every $\mathbf{i}$-cube a in A there is some $\mathbf{i}$-cube x in X and some invertible $\mathbf{i}$-map $h\colon F(x) \to_0 a$ in A.

5.5 Theorem (Characterisations of equivalences)

Let $F\colon \mathsf{X} \to \mathsf{A}$ be a pseudo cm-functor between two transversally invariant *cm-categories (see II.1.6). The following conditions are equivalent:*

(i) $F\colon \mathsf{X} \to \mathsf{A}$ belongs to an adjoint equivalence of cm-categories,

(ii) F is faithful, full and essentially surjective on cubes (see 5.4),

(iii) every ordinary functor $F_{\mathbf{i}}\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{X}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ is an equivalence of categories.

Moreover, if F is unitary, one can make its 'quasi-inverse' unitary as well.

Remark. *The axiom of choice is assumed.*

Proof. By our previous definitions in 5.4, conditions (ii) and (iii) are about the family of ordinary functors $(F_{\mathbf{i}})$ and are well known to be equivalent (assuming (AC)). Moreover, if F belongs to an adjoint equivalence $(\eta, \varepsilon)\colon F \dashv G$, every $F_{\mathbf{i}}$ is obviously an equivalence of categories.

Conversely, let us assume that every $F_{\mathbf{i}}$ is an equivalence of ordinary categories and let us extend the pseudo cm-functor F to an adjoint equivalence, proceeding by induction on the degree $n \geqslant 0$ of the positive multi-index $\mathbf{i}$.

First, $F_*\colon \mathrm{tv}_*(\mathsf{X}) \to \mathrm{tv}_*(\mathsf{A})$ is an equivalence of categories and we begin by choosing an adjoint quasi-inverse $G_*\colon \mathrm{tv}_*(\mathsf{A}) \to \mathrm{tv}_*(\mathsf{X})$.

In other words, we choose for every $\star$-cube (or object) a some $G(a)$ in X and some isomorphism $\varepsilon a\colon FG(a) \to a$ in A; then a transversal map $h\colon a \to b$ in A is sent to the unique X-map $G(h)\colon G(a) \to G(b)$ coherent with the previous choices (since F_* is full and faithful). Finally the isomorphism $\eta x\colon x \to GF(x)$ is determined by the triangle equations (for every $\star$-cube x of X).

Assume now that the components of G, ε and η have been defined up to degree $n - 1 \geqslant 0$, and let us define them for a multi-index $\mathbf{i}$ of degree n, taking care that the new choices be consistent with the previous ones.

First, for every $\mathbf{i}$-cube a in A we want to choose some $\mathbf{i}$-cube $G(a)$ in X and some $\mathbf{i}$-isomorphism $\varepsilon a\colon FG(a) \to a$ in A, consistently with all faces ∂_i^α ($i \in \mathbf{i}$). In fact there exists (and we choose) some $\mathbf{i}$-cube x and some $\mathbf{i}$-isomorphism $u\colon F(x) \to a$. Then, by the inductive hypothesis, we have a

family of $2n$ transversal isomorphisms of A

$$v_i^\alpha = \partial_i^\alpha u^{-1}.\varepsilon(\partial_i^\alpha a)\colon FG(\partial_i^\alpha a) \to \partial_i^\alpha a \to F(\partial_i^\alpha x) \qquad (i \in \mathbf{i},\ \alpha = \pm),$$

which can be uniquely lifted as transversal isomorphisms t_i^α of X, since F is full and faithful

$$t_i^\alpha\colon G(\partial_i^\alpha a) \to \partial_i^\alpha x, \qquad\qquad v_i^\alpha = F(t_i^\alpha).$$

The family (v_i^α) has consistent positive faces (see II.1.6), because this is true of the family $(\partial_i^\alpha u^{-1})_{i,\alpha}$, by commuting faces, and of the family $(\varepsilon(\partial_i^\alpha a))_{i,\alpha}$ by inductive assumption. It follows that also the family (t_i^α) has consistent positive faces.

By transversal invariance in X we can fill this family (t_i^α) with a (chosen) transversal $\mathbf{i}$-isomorphism $t\colon y \to x$, and we define the $\mathbf{i}$-cube G(a) and the $\mathbf{i}$-isomorphism εa as follows:

$$G(a) = y, \qquad\qquad \varepsilon a = u.Ft\colon FG(a) \to F(x) \to a.$$

This choice is consistent with faces:

$$\partial_i^\alpha(\varepsilon a) = (\partial_i^\alpha u).Ft_i^\alpha = (\partial_i^\alpha u).v_i^\alpha = \varepsilon(\partial_i^\alpha a).$$

Now, since $F_{\mathbf{i}}$ is full and faithful, a transversal $\mathbf{i}$-map $h\colon a \to b$ in A is sent to the unique X-map $G(h)\colon G(a) \to G(b)$ satisfying the condition $\varepsilon b.F(Gh) = h.\varepsilon a$ (naturality of ε).

Again, the $\mathbf{i}$-isomorphism $\eta x\colon x \to GF(x)$ is determined by the triangle equations, for every $\mathbf{i}$-cube x of X.

The comparison $\mathbf{i}$-maps $\underline{G}_i$ are uniquely determined by their coherence conditions (see 4.2), for an $\mathbf{i}|i$-cube a and an i-composition of i-cubes $c = a +_i b$ in A

$$\varepsilon e_i a.F\underline{G}_i(a) = e_i(\varepsilon a).\underline{F}_i(Ga),$$

$$\varepsilon c.F\underline{G}_i(a,b) = (\varepsilon a +_i \varepsilon b).\underline{F}_i(Ga, Gb).$$

Moreover $\underline{G}_i(a)$ and $\underline{G}_i(a,b)$ are invertible, because so are their images by F, full and faithful.

The construction of G, ε and η is now achieved. One ends by proving that G is indeed a pseudo cm-functor, and that ε, η are coherent with the comparison cells of F and G.

Finally, let us assume that F is unitary: $\underline{F}_i(x)\colon F(e_i x) \to e_i(Fx)$ is always an identity. To make G unitary we assume that - in the previous inductive construction - the following constraint has been followed: for a j-degenerate $\mathbf{i}$-cube $a = e_j c$ we always choose the transversal isomorphism $u = e_j(\varepsilon c)\colon F(e_j(Gc)) \to e_j c$. It follows that each

$$v_i^\alpha\colon FG(\partial_i^\alpha e_j c) \to F(\partial_i^\alpha e_j Gc)$$

is the identity; then $t_i^\alpha\colon G(\partial_i^\alpha e_j c) \to \partial_i^\alpha e_j Gc$ is the identity as well. We (choose to) fill their family with the identity $t\colon e_j Gc \to e_j Gc$, which gives

$$G(e_j c) = e_j Gc, \qquad \varepsilon(e_j c) = u.Ft = e_j(\varepsilon c).$$

If a is also j'-degenerate, the commutativity of degeneracies ensures that both constructions give the same result. $\qquad\square$

6. Limits and adjoints for cm-categories

We briefly recall the definition of cones and limits from Part II, Section 3, and prove that unitary right adjoints preserve the limits of cm-functors.

6.1 Lift functors

First we recall a tool from II.1.5, II.1.8. For the *positive* integer j there is a *j-directed lift 2-functor* with values in the 2-category of chiral multiple categories indexed by the ordered set $\mathbb{N}|j = \mathbb{N} \setminus \{j\}$, pointed at 0

$$Q_j\colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}. \tag{78}$$

On a cm-category A the cm-category $Q_j\mathsf{A}$ is - loosely speaking - that part of A that contains the index j, reindexed without it:

$$(Q_j\mathsf{A})_\mathbf{i} = A_{\mathbf{i}j},$$
$$(\partial_i^\alpha\colon (Q_j\mathsf{A})_\mathbf{i} \to (Q_j\mathsf{A})_{\mathbf{i}|i}) = (\partial_i^\alpha\colon A_{\mathbf{i}j} \to A_{\mathbf{i}j|i}), \tag{79}$$
$$(e_i\colon (Q_j\mathsf{A})_{\mathbf{i}|i} \to (Q_j\mathsf{A})_\mathbf{i}) = (e_i\colon A_{\mathbf{i}j|i} \to A_{\mathbf{i}j}) \qquad (i \in \mathbf{i} \subset \mathbb{N}|j);$$

similarly for compositions and comparisons. In the same way for a lax cm-functor $F\colon \mathsf{A} \to \mathsf{B}$ and a transversal transformation $h\colon F \to G\colon \mathsf{A} \to \mathsf{B}$ we let

$$(Q_j F)_\mathbf{i} = F_{\mathbf{i}j}, \qquad (Q_j h)_\mathbf{i} = h_{\mathbf{i}j} \qquad\qquad (\mathbf{i} \subset \mathbb{N}|j). \tag{80}$$

There is also an obvious restriction 2-functor $R_j \colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}$ where the multiple category $R_j\mathsf{A}$ is that part of A that does not contain the index j. The j-directed faces and degeneracies of A are not used in $Q_j\mathsf{A}$, but yield three natural transformations, also called *faces* and *degeneracy*, that act as follows for $\mathbf{i} \subset \mathbb{N}|j$

$$D_j^\alpha \colon Q_j \to R_j \colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}, \quad (D_j^\alpha)_{\mathbf{i}} = \partial_j^\alpha \colon A_{\mathbf{i}j} \to A_{\mathbf{i}},$$

$$E_j \colon R_j \to Q_j \colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}, \quad (E_j)_{\mathbf{i}} = e_j \colon A_{\mathbf{i}} \to A_{\mathbf{i}j}, \quad (81)$$

$$D_j^\alpha E_j = \mathrm{id}.$$

All the functors Q_j commute. By composing n of them in any order we get an *iterated lift functor* of degree n, in a *positive* direction $\mathbf{i} = \{i_1, ..., i_n\}$

$$Q_{\mathbf{i}} \colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|\mathbf{i}}, \qquad Q_{\mathbf{i}}(\mathsf{A}) = Q_{i_n}...Q_{i_1}(\mathsf{A}),$$

$$\mathrm{tv}_*(Q_{\mathbf{i}}(\mathsf{A})) = \mathrm{tv}_{\mathbf{i}}(\mathsf{A}). \tag{82}$$

6.2 Cones

Let X and A be cm-categories, and let X be small. Consider the *diagonal* functor (of ordinary categories)

$$D \colon \mathrm{tv}_*\mathsf{A} \to \mathrm{Ps}\mathbf{Cmc}(\mathsf{X}, \mathsf{A}). \tag{83}$$

where $\mathrm{tv}_*\mathsf{A}$ is the ordinary category of $\star$-cubes (objects) of A and their transversal maps.

D takes each object A of A to a unitary pseudo functor $\mathsf{X} \to \mathsf{A}$, 'constant' at A via the family of the total $\mathbf{i}$-degeneracies $e_{\mathbf{i}} = e_{i_1}...e_{i_n} \colon A_* \to A_{\mathbf{i}}$

$$DA \colon \mathsf{X} \to \mathsf{A}$$

$$DA(x) = e_{\mathbf{i}}(A), \qquad DA(f) = \mathrm{id}(e_{\mathbf{i}}A) \qquad \text{(for } x, f \text{ in } \mathrm{tv}_{\mathbf{i}}\mathsf{X}\text{),}$$

$$\underline{DA}_i(x) = \mathrm{id}(e_{\mathbf{i}}A) \colon e_i(DA(x)) \to DA(e_ix) \qquad \text{(for } x \text{ in } X_{\mathbf{i}|i}\text{),} \tag{84}$$

$$\underline{DA}_i(x, y) = \lambda_i \colon e_{\mathbf{i}}(A) +_i e_{\mathbf{i}}(A) \to e_{\mathbf{i}}(A) \qquad \text{(for } x, y \text{ in } X_{\mathbf{i}}\text{),}$$

where $\lambda_i = \lambda_i(e_{\mathbf{i}}A) = \rho_i(e_{\mathbf{i}}A)$ is a left and right unitor of A.

Similarly, a $\star$-map $f \colon A \to B$ in A is sent to the constant transversal transformation

$$Df \colon DA \to DB \colon \mathsf{X} \to \mathsf{A},$$

$$(Df)(x) = e_{\mathbf{i}}(f) \colon e_{\mathbf{i}}(A) \to e_{\mathbf{i}}(B) \qquad (x \text{ in } \mathrm{tv}_{\mathbf{i}}\mathsf{X}). \tag{85}$$

Let $T\colon \mathsf{X} \to \mathsf{A}$ be a lax functor. A *(transversal) cone* of T will be a pair $(A, h\colon DA \to T)$ formed of an object A of A (the *vertex* of the cone) and a transversal transformation of lax functors $h\colon DA \to T\colon \mathsf{X} \to \mathsf{A}$; in other words, it is an object of the ordinary comma category $(D \downarrow T)$, where T is viewed as an object of the category $\mathrm{Lx}\mathbf{Cmc}(\mathsf{X}, \mathsf{A})$.

By definition (cf. II.1.8), the transversal transformation h amounts to assigning the following data:

- a transversal **i**-map $hx\colon e_{\mathbf{i}}(A) \to Tx$, for every **i**-cube x in X,

subject to the following axioms of naturality and coherence:

(tc.1) $\quad Tf.hx = hy$ $\hspace{3cm}$ (for every **i**-map $f\colon x \to_0 y$ in X),

(tc.2) h commutes with positive faces, and agrees with positive degeneracies and compositions:

$$h(\partial_i^\alpha x) = \partial_i^\alpha(hx), \hspace{3cm} (x \text{ in } X_{\mathbf{i}}),$$
$$h(e_i x) = \underline{T}_i(x).e_i(hx)\colon e_{\mathbf{i}}(A) \to_0 T(e_i x) \hspace{2cm} (x \text{ in } X_{\mathbf{i}|i}),$$
$$h(z) = \underline{T}_i(x, y).(hx +_i hy).\lambda_i^{-1}\colon e_{\mathbf{i}}(A) \to_0 T(z) \hspace{1cm} (z = x +_i y \text{ in } X_{\mathbf{i}}).$$

As remarked in II.3.2, a *unitary* lax functor $G\colon \mathsf{A} \to \mathsf{B}$ preserves diagonalisation, in the sense that $G.DA = D(GA)$; therefore G takes a cone $(A, h\colon DA \to T)$ of T to a cone (GA, Gh) of GT.

6.3 Limits of degree zero

As defined in II.3.3, the *(transversal) limit of degree zero* $\lim(T) = (L, t)$ of a lax functor $T\colon \mathsf{X} \to \mathsf{A}$ between chiral multiple categories is a universal cone $(L, t\colon DL \to T)$.

In other words:

(tl.0) L is an object of A and $t\colon DL \to T$ is a transversal transformation of lax functors,

(tl.1) for every cone $(A, h\colon DA \to T)$ there is precisely one $\star$-map $f\colon A \to L$ in A such that $t.Df = h$.

We say that A *has limits of degree zero* on X if all these exist.

Theorem II.3.6 proves that all limits of degree zero in A can be constructed from products, equalisers and *tabulators* - all of degree zero; it also

gives a corresponding result for the preservation of such limits by *unitary* lax multiple functors. (Tabulators, the basic form of higher limits, were sketched in Part I and studied in Part II, Section 3.)

6.4 Multiple limits

The general definition of multiple limits in a chiral multiple category A was given in II.4.4.

(a) For a positive multi-index $\mathbf{i} \subset \mathbb{N}$ and a chiral multiple category X we say that A has *limits of type* $\mathbf{i}$ on X if $Q_\mathbf{i}A$ has limits of degree zero on X.

(b) We say that A *has limits of type* $\mathbf{i}$ if this happens for all small chiral multiple categories X.

(c) We say that A *has limits of all degrees* (or *all types*) if this happens for all positive multi-indices $\mathbf{i}$.

(d) We say that A *has multiple limits of all degrees* if all the previous limits exist and are preserved by the multiple functors (see 6.1)

$$D_j^\alpha \colon Q_{\mathbf{i}j}(\mathsf{A}) \to R_j Q_\mathbf{i}(\mathsf{A}), \qquad E_j \colon R_j Q_\mathbf{i}(\mathsf{A}) \to Q_{\mathbf{i}j}(\mathsf{A}) \qquad (j \notin \mathbf{i}). \tag{86}$$

In this case, if A is transversally invariant one can always operate a choice of multiple limits such that this preservation is strict.

The Main Theorem of Part II (II.4.5) shows that all multiple limits in A can be constructed from multiple products, multiple equalisers and multiple tabulators; again, it also gives a corresponding result for the preservation of such limits by multiple functors.

We are now ready to prove the preservation properties of unitary adjoints.

6.5 Theorem (Adjoints and limits of degree zero)

Let $(\eta, \varepsilon) \colon F \dashv G$ be a colax/lax cm-adjunction, where both functors are unitary.

Then $G \colon \mathsf{A} \to \mathsf{B}$ preserves all (the existing) limits of degree zero of lax cm-functors $T \colon \mathsf{X} \to \mathsf{A}$.

Proof. The argument is the usual one. Let $(A, h \colon D_\mathsf{A}(A) \to T)$ be a limit of T in A. We want to prove that the pair $(GA, Gh \colon G.D_\mathsf{A}(A) \to GT)$ is a limit of GT in B.

First, since G is unitary, $GD_{\mathsf{A}}(A) = D_{\mathsf{B}}(GA)$ and the pair (GA, Gh) is indeed a cone of the lax cm-functor $GT\colon \mathsf{X} \to \mathsf{B}$.

Moreover, given a cone $(B, k'\colon D_{\mathsf{B}}(a) \to GT)$ of GT, with transversal components $k'x\colon e_{\mathbf{i}}(B) \to GTx$ for every $\mathbf{i}$-cube x in X, the adjunction gives a family $h'x\colon Fe_{\mathbf{i}}(B) \to Tx$, that is a cone $(FB, h'\colon D_{\mathsf{A}}(FB) \to T)$ in A. Therefore there is precisely one transversal map $f\colon FB \to A$ in A such that $h.Df = h'$. This means precisely one transversal map $g\colon B \to GA$ in B such that $Gh.Dg = k'$. $\qquad\square$

6.6 Remark

Since the lift 2-functor

$$Q_{\mathbf{i}}\colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}$$

preserves cm-adjunctions, it follows that, if the cm-category A has *multiple limits on* X, these are preserved by a right adjoint cm-functor $G\colon \mathsf{A} \to \mathsf{B}$ (under the previous unitarity assumptions).

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ON CERTAIN TOPOLOGICAL *-AUTONOMOUS CATEGORIES

by Michael BARR

Résumé. Etant donné une catégorie additive et equationelle, munie d'une structure fermeé monoïdale symetrique ainsi que d'un objet dualisateur potentiel, on trouve des conditions suffisantes pour que la catégorie des objets topologiques sur cette catégorie admette une bonne notion des souscatégories pleines qui contiennent des objets fortement et faiblement topologisés. On montre que chacune des souscatégories est équivalente à la catégorie chu de la catégorie originale par rapport à l'objet dualisateur.

Abstract. Given an additive equational category with a closed symmetric monoidal structure and a potential dualizing object, we find sufficient conditions that the category of topological objects over that category admits a good notion of full subcategories of strong and weakly topologized objects and show that each is equivalent to the chu category of the original category with respect to the dualizing object.

Keywords. spherically complete fields, duality, Chu construction

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1. Introduction

This paper is a continuation of [5, 6, 7]. The first reference showed that the full subcategory of the category of (real or complex) topological vector spaces that consists of the Mackey spaces (defined in 2.5 below)is *-autonomous and equivalent to both the full subcategory of weakly topologized topological vector spaces and to the full subcategory of topological vector spaces topologized with the strong, or Mackey topology. This means, first, that those subcategories can, in principle at least, be studied without taking the topology into consideration. Second it implies that both of those categories are *-autonomous.

In [6], we showed that the category of topological abelian groups had

similar properties: that both the weakly and strongly topologized abelian groups formed a $*$-autonomous category.

Later, André Joyal raised the question whether the results of [5] remained valid for vector spaces over the field $\mathbf{Q}_p$ of p-adic rationals. This question was mentioned, but not answered, in [7]. Thinking about this question, I realized that there is a useful general theorem that answers this question for any locally compact field and also for locally compact abelian groups. The current paper provides a positive answer to Joyal's question.

All these results follow from a systematic use of the chu construction, see Section 3 below.

To state the main results, we need a definition. A normed field is **spherically complete** if any family of closed balls with the finite intersection property has non-empty intersection. It is known that every locally compact field is spherically complete (so this answers Joyal's question since $\mathbf{Q}_p$, as well as its finite extensions, is locally compact) and spherically complete is known to be strictly stronger than metrically complete.

Theorem 1.1. *Let K be a spherically complete field and $|K|$ its underlying discrete field. Then the following five categories are equivalent:*

1. $\mathrm{chu}(K\text{-Vect}, |K|)$ *(Section 3)*

2. *The category $\mathcal{V}_w(K)$ of topological K-spaces topologized with the weak topology for all their continuous linear functionals into K.*

3. *The category $\mathcal{V}_s(K)$ of topological K-spaces topologized with the strong topology (see Section 2) for all their continuous linear functionals into K.*

4. *The category $\mathcal{V}_w(|K|)$ of topological $|K|$-spaces topologized with the weak topology for all their continuous linear functionals into $|K|$.*

5. *The category $\mathcal{V}_s(|K|)$ of topological $|K|$-spaces topologized with the strong topology (see Section 2) for all their continuous linear functionals into $|K|$.*

and all are $*$-autonomous (see beginning of Section 3).

The methods also apply to give the results of [6].

1.1 Terminology

*We assume that all topological objects are Hausdorff. As we will see, each of the categories contains an object K with special properties. It will be convenient to call a morphism $V \longrightarrow K$ a **functional** on V. In the case of abelian groups, the word "character" would be more appropriate, but it is convenient to have one word. In a similar vein, we may refer to a mapping of topological abelian groups as "linear" to mean additive. We will be dealing with topological objects in categories of topological vector spaces and abelian groups. If V is such an object, we will denote by $|V|$ the underlying vector space or group.*

*If K is a topological field, we will say that a vector space is **linearly discrete** if it is a categorical sum of copies of the field.*

2. *The strong and weak topologies*

2.1 Blanket assumptions.

*In this section, we deal with a certain category $\mathcal{T}$ of topological algebras and a distinguished object K, usually called the **dualizing object**. Maps $V \longrightarrow K$ in $\mathcal{T}$ will be called **functionals**. A bijective map $V \longrightarrow V'$ will be called a **weak isomorphism** if it induces a bijection $\mathrm{Hom}(V', K) \longrightarrow \mathrm{Hom}(V, K)$. We show that for any V, there is a space τV with the finest possible topology for which $\tau V \longrightarrow V$ is a weak isomorphism and a space σV with the coarsest possible topology for which $V \longrightarrow \sigma V$ is a weak isomorphism. We show that σ and τ are functors for which the weak isomorphisms just mentioned are natural transformations.*

Throughout this section, we make the following assumptions.

1. *$\mathcal{A}$ is an additive equational closed symmetric monoidal category and $\mathcal{T}$ is the category of topological $\mathcal{A}$-algebras.*

2. *K is a uniformly complete object of $\mathcal{T}$.*

3. *there is a neighbourhood U of 0 in K such that*

   (a) U contains no non-zero subobject;

(b) *whenever $\varphi : T \longrightarrow K$ is such that $\varphi^{-1}(U)$ is open, then φ is continuous.*

In connection with point 2, in the application to spherically complete fields, K will be the ground field and we have already noted that sperically complete fields are metrically complete. In the application to topological groups, K will the compact circle group.

Point 3 says that, in some sense, the neighbourhood U is small. The existence of such a neighbourhood in the circle group is well known, although we provide an argument.

Lemma 2.1. *Suppose there is an embedding $T \hookrightarrow \prod_{i \in I} T_i$ and there is a morphism $\varphi : T \longrightarrow K$. Then there is a finite subset $J \subseteq I$ and a commutative diagram*

Moreover, we can take T_0 closed in $\prod_{j \in J} T_j$.

Proof. Since $\varphi^{-1}(U)$ is a neighbourhood of 0 in T, it must be the meet with T of a neighbourhood of 0 in $\prod_{i \in I} T_i$. From the definition of the product topology, we must have a finite subset $J \subseteq I$ and neighbourhoods U_j of 0 in T_j such that

$$\varphi^{-1}(U) \supseteq T \cap (\prod_{j \in J} U_j \times \prod_{i \in I-J} T_i)$$

It follows that

$$U \supseteq \varphi(T \cap (\prod_{j \in J} 0 \times \prod_{i \in I-J} T_i))$$

But the latter is a subobject of K contained in U and therefore must be 0. Now let

$$T_0 = \frac{T}{T \cap (\prod_{j \in J} 0 \times \prod_{i \in I-J} T_i)}$$

topologized as a subspace of $\prod_{j \in J} T_j$ and φ_0 be the induced map. It is immediate that $\varphi_0^{-1}(U) \supseteq \prod_{j \in J} U_j$ which is a neighbourhood of 0 in the induced topology and hence φ_0 is continuous. Finally, since K is complete, we can replace T_0 by its closure in $\prod_{j \in J} T_j$. $\square$

Theorem 2.2. *Suppose $\mathcal{S}$ is a full subcategory of $\mathcal{T}$ that is closed under finite products and closed subobjects and that $K \in \mathcal{S}$ satisfies the assumptions in 2.1. If $\mathcal{V}$ is the closure of $\mathcal{S}$ under all products and all subobjects and K is injective in $\mathcal{S}$, then it is also injective in $\mathcal{V}$.*

Proof. It is sufficient to show that if $V \subseteq \prod_{i \in I} S_i$ with each $S_i \in \mathcal{S}$, then every morphism $V \longrightarrow K$ extends to the product. But the object V_0 constructed in the preceding lemma is a closed subobject of $\prod_{j \in J} S_j$ so that $V_0 \in \mathcal{S}$ and the fact that K is injective in $\mathcal{S}$ completes the proof. $\square$

Recall that a weak isomorphism $V \longrightarrow V'$ is a bijective morphism that induces a bijection on the functionals.

Of course, a bijective morphism induces an injection so the only issue is whether the induced map is a surjection.

Proposition 2.3. *A finite product of weak isomorphisms is a weak isomorphism.*

Proof. Assume that J is a finite set and for each $j \in J$, $V_j \longrightarrow V_j'$ is a weak isomorphism. Then since finite products are the same as finite sums in an additive category, we have

$$\text{Hom}(\prod V_j', K) \cong \text{Hom}(\sum V_j', K) \cong \prod \text{Hom}(V_j', K)$$

$$\cong \prod \text{Hom}(V_j, K) \cong \text{Hom}(\sum V_j, K) \cong \text{Hom}(\prod V_j, k) \qquad \blacksquare$$

Theorem 2.4. *Assume the conditions of Theorem 2.2 and also suppose that for every object of $\mathcal{S}$, and therefore of $\mathcal{V}$, there are enough functionals to separate points. Then for every object V of $\mathcal{V}$, there are weak isomorphisms $\tau V \longrightarrow V \longrightarrow \sigma V$ with the property that σV has the coarsest topology that has the same functionals as V and τV has the finest topology that has same functionals as V.*

Proof. The argument for σ is standard. Simply retopologize V as a subspace of $K^{\text{Hom}(V,K)}$. This is the weakest topology for which all the functionals are continuous and obviously no weaker topology will admit all the functionals.

Let $\{V_i \longrightarrow V\}$ range over the isomorphism classes of weak isomorphisms to V. We define τV as the pullback in

$$
\begin{array}{ccc}
\tau V & \longrightarrow & \prod V_i \\
\downarrow & & \downarrow \\
V & \longrightarrow & V^I
\end{array}
$$

The bottom map is the diagonal and is a topological embedding so that the top map is also a topological embedding. We must show that every functional on τV is continuous on V. Let φ be a functional on τV. From injectivity, it extends to a functional ψ on $\prod V_i$. By Lemma 2.1, there is a finite subset $J \subseteq I$ and a functional ψ_0 on $\prod_{j \in J} V_j$ such that ψ is the composite

$\prod_{i \in I} V_i \longrightarrow \prod_{j \in J} V_j \xrightarrow{\psi_0} K$. Thus we have the commutative diagram

$$
\begin{array}{ccccc}
\tau V & \longrightarrow & \prod_{i \in I} V_i & \longrightarrow & \prod_{j \in J} V_j \\
\downarrow & & \downarrow & & \downarrow \quad \searrow^{\psi_0} \\
& & & & \qquad K \\
V & \longrightarrow & V^I & \longrightarrow & V^J \quad \nearrow
\end{array}
$$

The dashed arrow exists because of Proposition 2.3, which completes the proof. $\square$

Remark 2.5. We will call the topologies on σV and τV the **weak** and **strong** topologies, respectively. They are the coarsest and finest topology that have the same underlying $\mathcal{A}$ structure and the same functionals as V. The strong topology is also called the **Mackey topology**.

Proposition 2.6. *Weak isomorphisms are stable under pullback.*

Proof. Suppose that

$$
\begin{array}{ccc}
W' & \longrightarrow & W \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\
V' & \xrightarrow{\iota} & V
\end{array}
$$

and the bottom arrow is a weak isomorphism. Clearly, $W' \longrightarrow W$ is a bijection, so we need only show that $\mathrm{Hom}(W, K) \longrightarrow \mathrm{Hom}(W', K)$ is surjective.

I claim that $W' \subseteq W \times V'$ with the induced topology. Let us define W'' to be the subobject $W \times_V V'$ with the induced topology. Since $W' \longrightarrow W$ and $W' \longrightarrow V$ are continuous, the topology on W' is at least as fine as that of W''. On the other hand, we do have $W'' \longrightarrow W$ and $W'' \longrightarrow V'$ with the same map to V so that we have $W'' \longrightarrow W'$, so that the topology on W'' is at least as fine as that of W'. Then we have a commutative diagram

$$
\begin{array}{ccc}
W' & \longrightarrow & W \\
\uparrow & & \downarrow{\scriptstyle (\mathrm{id}, f)} \\
W \times V' & \longrightarrow & W \times V
\end{array}
$$

Apply $\mathrm{Hom}(-, K)$ and use the injectivity of K to get:

$$
\begin{array}{ccc}
\mathrm{Hom}(W', K) & \longleftarrow & \mathrm{Hom}(W, K) \\
\uparrow & & \uparrow \\
\mathrm{Hom}(W, K) \times \mathrm{Hom}(V', K) & \xleftarrow{\;\cong\;} & \mathrm{Hom}(W, K) \times \mathrm{Hom}(V, K)
\end{array}
$$

The bottom arrow is a bijection and the left hand arrow is a surjection, which implies that the top arrow is a surjection. □

Proposition 2.7. *σ and τ are functors on $\mathcal{V}$.*

Proof. For σ, this is easy. If $f : W \longrightarrow V$ is a morphism, the induced $\sigma f : \sigma W \longrightarrow \sigma V$ will be continuous if and only if its composite with every functional on V is a functional on W, which obviously holds.

To see that τ is a functor, suppose $f : W \longrightarrow V$ is a morphism. Form the pullback

$$
\begin{array}{ccc}
W' & \longrightarrow & W \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
\tau V & \longrightarrow & V
\end{array}
$$

Since $\tau V \longrightarrow V$ is a weak isomorphism, the preceding proposition implies that $W' \longrightarrow W$ is a weak isomorphism. But since τW has the finest topology with that property, it follows that the topology on τW is finer than that of W' and hence $\tau W \longrightarrow W$ factors through W' and the composite $\tau W \longrightarrow W' \longrightarrow \tau V$. $\qquad\square$

Proposition 2.8. *If $V \longrightarrow V'$ is a weak isomorphism, then $\sigma V \longrightarrow \sigma V'$ and $\tau V \longrightarrow \tau V'$ are isomorphisms.*

Proof. For σ, this is obvious. Clearly, $\tau V \longrightarrow V \longrightarrow \tau V'$ is also a weak isomorphism so that τV is one of the factors in the computation of $\tau V'$ and then $\tau V' \longrightarrow \tau V$ is a continuous bijection, while the other direction is evident. $\qquad\square$

Corollary 2.9. *Both σ and τ are idempotent, while $\sigma\tau \cong \sigma$ and $\tau\sigma \cong \tau$.* $\quad\square$

Proposition 2.10. *For any $V, V' \in \mathcal{V}$, we have $\mathrm{Hom}(\sigma V, \sigma V') \cong \mathrm{Hom}(\tau V, \tau V')$.*

Proof. It is easiest to assume that the underlying objects $|V| = |\sigma V| = |\tau V|$ and similarly for V'. Then for any $f : V \longrightarrow V'$, we also have that $|f| = |\sigma f| = |\tau f|$. Thus the two composition of the two maps below

$$
\mathrm{Hom}(\sigma V, \sigma V') \longrightarrow \mathrm{Hom}(\tau\sigma V, \tau\sigma V') = \mathrm{Hom}(\tau V, \tau V')
$$

and

$$
\mathrm{Hom}(\tau V, \tau V') \longrightarrow \mathrm{Hom}(\sigma\tau V, \sigma\tau V') \cong \mathrm{Hom}(\sigma V, \sigma V')
$$

give the identity in each direction. $\qquad\square$

Let $\mathcal{V}_w \subseteq \mathcal{V}$ and $\mathcal{V}_s \subseteq \mathcal{V}$ denote the full subcategories of weak and strong objects, respectively. Then as an immediate corollary to the preceding, we have:

Theorem 2.11. $\tau : \mathcal{V}_w \longrightarrow \mathcal{V}_s$ *and* $\sigma : \mathcal{V}_s \longrightarrow \mathcal{V}_w$ *determine inverse equivalences of categories.*

3. Chu and chu

A $*$-autonomous is a symmetric monoidal closed category equipped with a "dualizing object" $\perp$. We will denote the monoidal structure by $\otimes$ with tensor unit $\top$ and the closed structure by $\multimap$. The basic assumption is that for every object A the canonical map $A \longrightarrow (A \multimap \perp) \multimap \perp$ is an isomorphism. We let $A^* = A \multimap \perp$. Many things follow from this, e.g. $A \multimap B \cong B^* \multimap A^*$, $A \otimes B \cong (A \multimap B^*)^*$, and $A \multimap B \cong (A \otimes B^*)^*$. See [2] for all details.

Now we add to the assumptions on $\mathcal{A}$ that it be a symmetric monoidal closed category in which the underlying set of $A \multimap B$ is $\mathrm{Hom}(A, B)$. We denote by $\mathcal{E}$ and $\mathcal{M}$ the classes of surjections and injections, respectively.

We briefly review the categories $\mathrm{Chu}(\mathcal{A}, K)$ and $\mathrm{chu}(\mathcal{A}, K)$. See [4] for details. The first has a objects pairs (A, X) of objects of $\mathcal{A}$ equipped with a "pairing" $\langle -, - \rangle : A \otimes X \longrightarrow K$. A morphism $(f, g) : (A, X) \longrightarrow (B, Y)$ consists of a map $f : A \longrightarrow B$ and a map $g : Y \longrightarrow X$ such that

$$
\begin{array}{ccc}
A \otimes Y & \xrightarrow{\ f \otimes Y\ } & B \otimes Y \\
{\scriptstyle A \otimes g} \downarrow & & \downarrow {\scriptstyle \langle -, - \rangle} \\
A \otimes X & \xrightarrow[\ \langle -, - \rangle\ ]{} & K
\end{array}
$$

commutes. This says that $\langle fa, y \rangle = \langle a, gy \rangle$ for all $a \in A$ and $y \in Y$. This can be enriched over $\mathcal{A}$ by internalizing this definition as follows. Note first that the map $A \otimes X \longrightarrow K$ induces, by exponential transpose, a map $X \longrightarrow A \multimap K$. This gives a map $Y \multimap X \longrightarrow Y \multimap (A \multimap K) \cong A \otimes Y \multimap K$. There is a similarly defined arrow $A \multimap B \longrightarrow A \otimes Y \multimap K$. De-

fine $[(A, X), (B, Y)]$ so that

$$
\begin{array}{ccc}
[(A, X), (B, Y)] & \longrightarrow & A \multimap B \\
\downarrow & & \downarrow \\
Y \multimap X & \longrightarrow & A \otimes Y \multimap K
\end{array}
$$

is a pullback. Then define

$$(A, X) \multimap (B, Y) = ([[(A, X), (B, Y)]], A \otimes Y)$$

with $\langle (f, g), a \otimes y \rangle = \langle fa, y \rangle = \langle a, gy \rangle$ and

$$(A, X) \otimes (B, Y) = (A \otimes B, [(A, X), (Y, B)])$$

with pairing $\langle a \otimes b, (f, g) \rangle = \langle b, fa \rangle = \langle a, gb \rangle$. The duality is given by $(A, X)^* = (X, A) \cong (A, X) \multimap (K, \top)$ where $\top$ is the tensor unit of $\mathcal{A}$. Incidentally, the tensor unit of $\mathrm{Chu}(\mathcal{A}, K)$ is $(\top, K)$.

The category $\mathrm{Chu}(\mathcal{A}, K)$ is complete (and, of course, cocomplete). The limit of a diagram is calculated using the limit of the first coordinate and the colimit of the second. The full subcategory $\mathrm{chu}(\mathcal{A}, K) \subseteq \mathrm{Chu}(\mathcal{A}, K)$ consists of those objects (A, X) for which the two transposes of $A \otimes X \longrightarrow K$ are injective homomorphisms. When $A \rightarrowtail X \multimap K$, the pair is called separated and when $X \rightarrowtail A \multimap K$, it is called extensional. In the general case, one must choose a factorization system $(\mathcal{E}, \mathcal{M})$ and assume that the arrows in $\mathcal{E}$ are epic and that $\mathcal{M}$ is stable under $\multimap$, but here these conditions are clear. Let us denote by $\mathrm{Chu}_s(\mathcal{A}, K)$ the full subcategory of separated pairs and by $\mathrm{Chu}_e(\mathcal{A}, K)$ the full subcategory of extensional pairs.

The inclusion $\mathrm{Chu}_s(\mathcal{A}, K) \hookrightarrow \mathrm{Chu}(\mathcal{A}, K)$ has a left adjoint S and the inclusion $\mathrm{Chu}_e(\mathcal{A}, K) \hookrightarrow \mathrm{Chu}(\mathcal{A}, K)$ has a right adjoint E. Moreover, S takes an extensional pair into an extensional one and E does the dual. In addition, when (A, X) and (B, Y) are separated and extensional, $(A, X) \multimap (B, Y)$ is separated but not necessarily extensional and, dually, $(A, X) \otimes (B, Y)$ is extensional, but not necessarily separated. Thus we must apply the reflector to the internal hom and the coreflector to the tensor, but everything works out and $\mathrm{chu}(\mathcal{A}, K)$ is also $*$-autonomous. See [4] for details.

In the chu category it is evident that for any $(f, g) : (A, X) \longrightarrow (B, Y)$, f and g determine each other uniquely. So a map could just as well be described as an $f : A \longrightarrow B$ such that $x.\tilde{y} \in X$ for every $y \in Y$. Here $\tilde{y} : B \longrightarrow K$ is the evaluation at $y \in Y$ of the exponential transpose $Y \longrightarrow B \multimap K$.

Although the situation in the category of abelian groups is as described, in the case of vector spaces over a field, the hom and tensor of two separated extensional pairs turns out to be separated and extensional already ([3]).

4. The main theorem

Theorem 4.1. *Assume the hypotheses of Theorem 2.4 and also assume that the canonical map $\top \longrightarrow K \multimap K$ is an isomorphism. Then the categories of weak spaces and strong spaces are equivalent to each other and to $chu(\mathcal{A}, K)$ and are thus $*$-autonomous.*

Proof. The first claim is just Theorem 2.11. Now define $F : \mathcal{V} \longrightarrow$ chu by $F(V) = (|V|, \mathrm{Hom}(V, K))$ with evaluation as pairing. We first define the right adjoint R of F. Let $R(A, X)$ be the object A, topologized as a subobject of K^X. Since it is already inside a power of K, it has the weak topology. Let $f : |V| \longrightarrow A$ be a homomorphism such that for all $x \in X$, $\tilde{x}.f \in \mathrm{Hom}(V, K)$. This just means that the composite $V \longrightarrow R(A, X) \longrightarrow K^X \xrightarrow{\pi_x} K$ is continuous for all $x \in X$, exactly what is required for the map into $R(A, X)$ to be continuous. The uniqueness of f is clear and this establishes the right adjunction.

We next claim that $FR \cong \mathrm{Id}$. That is equivalent to showing that $\mathrm{Hom}(R(A, X), K) = X$. Suppose $\varphi : R(A, X) \longrightarrow K$ is a functional. By injectivity, it extends to a $\psi : K^X \longrightarrow K$. It follows from 2.1, there is a finite set of elements $x_1, \ldots, x_n \in X$ and morphisms $\theta_1, \ldots, \theta_n$ such that ψ factors as $K^X \longrightarrow K^n \xrightarrow{(\theta_1, \ldots, \theta_n)} K$. Applied to $R(A, X)$, this means that $\varphi(a) = \langle \theta_1 x_1, a \rangle + \cdots + \langle \theta_n x_n, a \rangle$. But the $\theta_i \in I$ and the tensor products are over I so that the pairing is a homomorphism $A \otimes_I X \longrightarrow K$. This means that $\varphi(a) = \langle \theta_1 x_1 + \cdots \theta_n x_n, a \rangle$ and $\theta_n x_1 + \cdots + \theta_n x_n \in X$.

Finally, we claim that $RF = S$, the left adjoint of the inclusion $\mathcal{V}_w \subseteq \mathcal{V}$. If $V \in \mathcal{V}$, then $RFV = R(|V|, \mathrm{Hom}(V, K))$ which is just V with the weak

topology it inherits from $K^{\mathrm{Hom}(V,K)}$, exactly the definition of SV. It follows that $F|\mathcal{V}_w$ is an equivalence.

Since $\mathcal{V}_w$ and $\mathcal{V}_s$ are equivalent to a $*$-autonomous category, they are $*$-autonomous. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The fact that the categories of weak and Mackey spaces are equivalent was shown, for the case of B (Banach) spaces in [8, Theorem 15, p. 422]. Presumably, the general case has also been long known, but I am not aware of a reference.

5. Examples.

Example 1. Vector spaces over a spherically complete field

Let K be a spherically complete field. Let $U = \{\times \in K \mid \|x\| < 1\}$. As a ranges over the non-zero elements of K, the sets of the form aU are a neighbourhood base at 0. If V is a topological K-vector space and $\varphi : |V| \longrightarrow K$ is a linear mapping such that $\varphi^{-1}(U)$ is open in V, then $\varphi^{-1}(aU) = a\varphi^{-1}(U)$ which is open by continuity of division and thus φ is actually continuous on V. That U contains no K-subspace of K and that $K \longrightarrow K \multimap K$ is an isomorphism are obvious.

This example includes all locally compact fields, see [15, Corollary 20.3(i)].

We take for $\mathcal{S}$ the category of normed linear K-spaces, except in the case that K is discrete, we require also that the spaces have the discrete norm. We know that K is injective in the discrete case. The injectivity of K in the real or complex case is just the Hahn-Banach theorem, which has been generalized to ultrametric fields according to the theorem following the definition:

An ultrametric is a metric for which the ultratriangle inequality, $\|x + y\| \leq \|x\| \vee \|y\|$, holds. This is obviously true for p-adic and t-adic norms. Spherically complete means that the meet of any descending sequence of closed balls is non-empty. This is known to be satisfied by locally compact ultrametric spaces.

Theorem 5.1 (Ingelton). *Let K be a spherically complete ultrametric field. E a K-normed space and v a subspace of E. For every bounded linear*

functional φ defined on V, there exists a bounded linear functional ψ defined on E whose restriction to V is φ and such that $\|\varphi\| = \|\psi\|$.

The proof is found in [14]

Notice that if K is non-discrete, then what we have established is that both $\mathcal{V}_s$ and $\mathcal{V}_w$ are equivalent to $\text{chu}(\text{Vect-}|K|, |K|)$. But exactly the same considerations show that the same is true if we ignore the topology on K and use the discrete norm. The category $\mathcal{S}$ will now be the category of discrete finite-dimensional $|K|$-vector spaces. Its product and subobject closure will consist of spaces that are mostly not discrete, but there are still full subcategories of weakly and strongly topologized spaces within this category and they are also equivalent to $\text{chu}(\text{Vect-}|K|, |K|)$.

Thus, these categories really do not depend on the topologies. Another interpretation is that this demonstrates that, for these spaces, the space of functionals replaces the topology, which was arguably Mackey's original intention.

Example 2. Locally compact abelian groups.

For the abelian groups, we take for $\mathcal{V}$ the category of those abelian that are subgroups (with the induced topology) of products of locally compact abelian groups. The object K in this case is the circle group $\mathbf{R}/\mathbf{Z}$. A simple representation of this group is as the closed interval $[-1/2, 1/2]$ with the endpoints identified and addition mod 1. The group is compact. Let U be the open interval $(-1/3, 1/3)$. It is easy to see that any non-zero point in that interval, added to itself sufficiently often, eventually escapes that neighborhood so that U contains no non-zero subgroup. It is well-known that the endomorphism group of the circle is $\mathbf{Z}$.

If $f : G \longrightarrow K$ is a homomorphism such that $T = f^{-1}(U)$ is open in G, let $T = T_1, T_2, \ldots, T_n, \ldots$ be a sequence of open sets in G such that $T_{i+1} + T_{i+1} \subseteq T_i$ for all i. Let $U_i = (-2^{-i}/3, 2^{-i}/3) \subseteq K$. Then the $\{U_i\}$ form a neighborhood base in K and one readily sees that $f^{-1}(U_i) \subseteq T_i$ which implies that f is continuous.

We take for $\mathcal{S}$ the category of locally compact abelian groups. The fact that K is an injective follows directly from the Pontrjagin duality theorem. A result [9, Theorem 1.1] says that every locally compact group is strongly topologized. Thus both categories of weakly topologized and strongly topol-

ogized groups that are subobjects of products of locally compact abelian groups are equivalent to $\text{chu}(\mathcal{A}b, |K|)$ and thus are *-autonomous.

We can ask if the same trick of replacing $K = \mathbf{R}/\mathbf{Z}$ by $|K|$, as in the first example, can work. It doesn't appear so. While $\text{Hom}(K, K) = \mathbf{Z}$, the endomorphism ring of $|K|$ has cardinality 2^c and is non-commutative, so we cannot draw no useful inference about maps from $|K|^n \longrightarrow |K|$, even for finite n.

Example 3. Modules over a self injective cogenerator.

If we examine the considerations that are used in vector spaces over a field, it is clear that what is used is that a field is both an injective module over itself and a cogenerator in the category of vector spaces. Then if K is a such a commutative ring, we can let $\mathcal{T}$ be the category of topological K-modules, $\mathcal{S}$ be the full subcategory of submodules of finite powers of K with the discrete topology and $\mathcal{V}$ the limit closure of $\mathcal{S}$. Then $\text{chu}(\text{Mod}_K, K)$ is equivalent to each of the categories $\mathcal{V}_s$ and $\mathcal{V}_w$ of topological K-modules that are strongly and weakly topologized, respectively, with respect to their continuous linear functionals into K.

We now show that there is a class of commutative rings with that property. Let k be a field and $K = k[x]/(x^n)$. When $n = 2$, this is called the ring of dual numbers over k.

Proposition 5.2. *K is self injective.*

We base this proof on the following well-known fact:

Lemma 5.3. *Let k be a commutative ring, K is a k-algebra, Q an injective k-module, and P a flat right K-module then $\text{Hom}_k(K, Q)$ is an injective K-module.*

The K-module structure on the Hom set is given by $(rf)(a) = f(ar)$ for $r \in K$ and $a \in P$.

Proof. Suppose $A \rightarrowtail B$ is an injective homomorphism of K-modules.

Then we have

$$\begin{array}{ccc} \mathrm{Hom}_R(B, \mathrm{Hom}_k(P, Q)) & \longrightarrow & \mathrm{Hom}_R(A, \mathrm{Hom}_k(P, Q)) \\ \cong \big\downarrow & & \cong \big\downarrow \\ \mathrm{Hom}_k(P \otimes_R B, Q) & \rightarrowtail & \mathrm{Hom}_k(P \otimes_R A, Q) \end{array}$$

and the flatness of P, combined with the injectivity of Q force the bottom arrow to be a surjection. $\qquad\square$

of 5.2. From the lemma it follows that $\mathrm{Hom}_k(K, k)$ is a K-injective. We claim that, as K-modules, $\mathrm{Hom}_k(K, k) \cong K$. To see this, we map $f : K \longrightarrow \mathrm{Hom}_k(K, k)$. Since these are vector spaces over k, we begin with a k-linear map and show it is K-linear. A k-basis for K is given by $1, x, \ldots, x^{n-1}$. We define $f(x^i) : K \longrightarrow k$ for $0 \leq\leq n - 1$ by $f(x^i)(x^j) = \delta_{i+j,n}$ (the Kronecker δ). For this to be K-linear, we must show that $f(xx^i) = xf(x^i)$. But

$$f(xx^i)(x^j) = f(x^{i+1}(x^j)) = \delta_{i+1+j,n} = f(x^i)(x^{j+1}) = (xf(x^i))(x^j)$$

Clearly, the $f(x^i)$, for $0 \leq i \leq n$ are linearly independent and so f is an isomorphism. $\qquad\square$

Proposition 5.4. *K is a cogenerator in the category of K-modules.*

Proof. Using the injectivity, it suffices to show that every cyclic module can be embedded into K. Suppose M is a cyclic module with generator m. Let i be the first power for which $x^i m = 0$. I claim that $m, xm, \ldots, x^{i-1}m$ are linearly independent over k. If not, suppose that $\lambda_0 m + \lambda_1 xm + \cdots \lambda_{i-1} x^{i-1}m = 0$ and not all coefficients 0. Let λ_j be the first non-zero coefficient, so that $\lambda_j x^j + \cdots + \lambda_{i-1} x^{i-1}m = 0$. Multiply this by x^{i-j-1} and use that $x^l m = 0$ for $l \geq i$ to get $\lambda_j x^{i-1}m = 0$. But by assumption, $x^{i-1}m \neq 0$ so that this would imply that $\lambda_j = 0$, contrary to hypothesis. Thus there is a k-linear map $f : M \longrightarrow K$ given by $f(x^j m) = x^{n-i+j}$. Since the x^j are linearly independent, this is k-linear and then it is clearly K-linear. $\qquad\square$

6. Interpretation of the dual of an internal hom

These remarks are especially relevant to the vector spaces, although they are appropriate to the other examples. The fact that $(U \multimap V)^* \cong U \multimap V^*$ can be interpreted that the dual of $U \multimap V$ is a subspace of $V \multimap U$, namely those linear transformations of finite rank. An element of the form $u \otimes v^*$ acts as a linear transformation by the formula $(u \otimes v^*)(v) = \langle v, v^* \rangle u$. This is a transformation of row rank 1. Sums of these elements is similarly an element of finite rank.

This observation generalizes the fact that in the category of finite dimensional vector spaces, we have that $(U \multimap V)^* \cong V \multimap U$ (such a category is called a compact $*$-autonomous category). In fact, Halmos avoids the complications of the definition of tensor products in that case by *defining $U \otimes V$* as the dual of the space of bilinear forms on $U \oplus V$, which is quite clearly equivalent to the dual of $U \multimap V^* \cong V \multimap U^*$ ([10, Page 40]). (Incidentally, it might be somewhat pedantic to point out that Halmos's definition makes no sense since $U \oplus V$ is a vector space in its own right and a bilinear form on a vector space is absurd. It would have been better to use the equivalent form above or to define $\mathrm{Bilin}(U, V)$.)

Since linear transformations of finite rank are probably not of much interest in the theory of topological vector spaces, this may explain why the internal hom was not pursued.

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AUTOCATEGORIES : III. REPRESENTATIONS, AND EXPANSIONS OF PREVIOUS EXAMPLES
by René GUITART

Abstract. An *autograph* is a set A with an action of the free monoïd with 2 generators, and an *autographic monad* is a monad on the topos of autographs. In previous papers we have shown that knots and *double*-categories are examples, and we proved that basic graphic algebras are *autographic algebras*. In this third paper we add three new results. We explain how to get concrete representations of autographs and conversely how to collect any representation into an autograph. We precise previous results and extend them, showing that knots and general links and grid diagrams are autographs, and that general graphic algebras are some *autographic algebras*.

Résumé. Un *autographe* est un ensemble A équipé d'une action du monoïde libre à deux générateurs, une *algèbre autographique* est une algèbre d'une monade sur le topos des autographes. Dans deux articles précédents nous avons vu que les diagrammes de nœuds et les 2-graphes sont des exemples, et que les algèbres graphiques basiques sont autographiques.

Dans ce troisième article, nous ajoutons trois résultats nouveaux. Nous montrons comment représenter concrètement les autographes, et réciproquement comment collecter une représentation en un autographe, nous expliquons précisément comment les nœuds, les entrelacs, les diagrammes de grilles, et aussi les catégories doubles, sont des exemples d'autographes, et nous identifions les algèbres graphiques générales avec des algèbres autographiques.

Keywords. graph, autograph, autographic algebra, autographic monad, knot, link, double category.

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1. Category $\text{Rep}(A, d, c)$ of representations of an autograph

Of course the construction in this section could work when Set is replaced by an arbitrary topos $\mathcal{E}$, providing $\text{Auto}[\text{Rel}(\mathcal{E})]$ and $\text{Auto}[\mathcal{E}]$, and consequently

with the topos $Agraph$ we would get Auto[Agraph], etc.

Definition 1.1. *1 — An* autograph *is the data $A = (\underline{A}, d_A, c_A)$ of a set $\underline{A}$ and two maps $d_A : \underline{A} \to \underline{A}$, $c_A : \underline{A} \to \underline{A}$. Abusively, often the set $\underline{A}$ will be denoted by A, and d_A and c_A are denoted by d and c. If we denote by $\mathbb{FM}(2) = \{d, c\}^*$ the free monoïd on two generators d and c (and with unit v) then an autograph is an action $A(-)$ of $\mathbb{FM}(2)$, with $A(v) = A$, $A(d) = d_A$, $A(c) = c_A$. We represent $a \in \underline{A}$ with $d_A a = v$ and $c_A a = w$, by: $a : v \to w$, or $v \xrightarrow{a} w$.*
2 — The category of autographs is Agraph $= \mathrm{Set}^{\mathbb{FM}(2)}$ *— a topos of course — a morphism in it from A to A' being a map $f : \underline{A} \to \underline{A}'$ satisfying*

$$d'fa = fda, \quad c'fa = fca.$$

3 — An autocategory *[3, Definition 6.1] is an autograph with identifier and a unitary and associative composition for consecutive arrows.*

The purpose of this section is to show how to represent concretely such autographs, and, starting from these representations, how to elaborate new "collected" autographs.

1.1 From Autorelations to autographs, and conversely

Proposition 1.2. *Considered as sets we have $\mathbb{FM}(2) = \mathbb{FA}(()) = \mathbb{FA}(\{f\})$ (the free autograph on one generator, see [3]), and they consist in words written with c and d, with maps $d(-)$ and $c(-)$ given by $m \mapsto dm$, $m \mapsto cm$. By a (binary)* Autorelation *we mean a family of sets $R = (R_m)_{m \in \mathbb{FM}(2)}$, with*

$$R_m \subseteq R_{dm} \times R_{cm},$$

or with the induced "projections" $c_m : R_m \to R_{cm}$ and $d_m : R_m \to R_{dm}$. The set of these R is denoted by Auto[Rel].
In such an R, each element ξ in $R_{()}$ or in any R_m generates an image R_ξ of $\mathbb{FM}(2)$ which is an autograph, and the set R_∞ disjoint union of the R_m

$$R_\infty = \dot{\cup}_m R_m = \cup_m (R_m \times \{m\}),$$

is itself an autograph, union of these R_ξ. Furthermore $\pi_R : (\xi, m) \mapsto m$ is a morphism of autographs

$$\pi_R : R_\infty \to \mathbb{FA}(()).$$

Conversely, given a morphism of autographs $\pi : S \to \mathbb{FA}(())$ *we can reconstruct an autorelation, with* $R_m = \pi^{-1}(m)$.

Example 1.3. Given a data $\underline{B}$ of 3 sets X, Y, Z and 6 maps $c_X : X \to Y$, $c_Y : Y \to Z$, $c_Z : Z \to X$, and $d_X : X \to Z$, $d_Y : Y \to X$, $d_Z : Z \to Y$, we get maps $X \overset{(c_X, d_X)}{\longrightarrow} Y \times Z$, $Y \overset{(c_Y, d_Y)}{\longrightarrow} Z \times X$, $Z \overset{(c_Z, d_Z)}{\longrightarrow} X \times Y$, and a finite generator of an autorelation

$$X \subset Y \times Z, \quad Y \subset Z \times X, \quad Z \subset X \times Y,$$

the associated autorelation B being given by $B_{()} = X$ and

$$B_c = Y, B_d = Z, B_{cc} = Z, B_{dc} = X, B_{cd} = X, B_{dd} = Y, B_{ccc} = X, \ldots$$

Example 1.4. With notations from [3, Proposition 3.1.], interpreting each R_m as $\mathbb{N}$, with "projections" $c(n) = t_1(n) = 3n + 1$, $d(n) = t_2(n) = 3n+2$, we get an autorelation "on" $\mathbb{N} = \mathbb{FA}(3\mathbb{N})$, of which the corresponding autograph is $\mathbb{FA}(3\mathbb{N}) \times \mathbb{FA}(\{f\})$, equipped with a morphism

$$\mathbb{FA}(3\mathbb{N}) \times \mathbb{FA}(\{f\}) \to \mathbb{FA}(\{f\}).$$

Proposition 1.5. *An autograph* $A = (\underline{A}, d_A, c_A)$ *determines* $\underline{A} \overset{(d_A, c_A)}{\longrightarrow} \underline{A} \times \underline{A}$, *and so we get an autorelation "on"* $\underline{A}$, *as in examples 1.3 and 1.4.*

1.2 Set $\mathcal{R}(A, d, c)$ of relational representations of an autograph

Definition 1.6. *A* relational representation *(or a* spanning representation*) of an autograph* (A, d, c) *is a data* $\varphi = (\Phi, \phi^d, \phi^c)$, *with for each* $f \in A$, *the data of a set* $\Phi(f)$ *and of a span of functions*

$$\Phi(df) \overset{\phi^d(f)}{\longleftarrow} \Phi(f) \overset{\phi^c(f)}{\longrightarrow} \Phi(cf),$$

which are the induced "projections" associated to a specified inclusion

$$\Phi(f) \subset \Phi(df) \times \Phi(cf).$$

We denote by $\mathcal{R}(A, d, c)$ *the set of these relational representations.*

Proposition 1.7. *Given a relational representation* $\varphi = (\Phi, \phi^d, \phi^c)$ *of an autograph* $A = (\underline{A}, d_A, c_A)$*, we* collect *it over A, constructing a map of autograph*

$$\pi_\varphi : \Sigma\varphi = (S_\varphi, d_\varphi, c_\varphi) \longrightarrow (\underline{A}, d_A, c_A) = A$$

with

$$S_\varphi = \{(f, u); f \in A, u \in \Phi(f)\},$$

$$d_\varphi(f, u) = (\phi^d(u), d_A f), \quad c_\varphi(f, u) = (\phi^c(u), c_A f), \quad q_\varphi(f, u) = f.$$

Example 1.8. A relational representation of $\mathbb{FA}(())$ is exactly an autorelation as in 1.2, so the set Autorel[Set] of autorelations is $\mathcal{R}(\mathbb{FA})(())$, and the π_R is a case of a π_φ.

1.3 From autorelations to automaps, and conversely

Definition 1.9. *We define* Auto[Set] *as the set of automaps, an* automap *being a sequence* $f = (f_m)_{m \in \mathbb{FM}(2)}$ *of maps* $f_m : G_{dm} \to G_{cm}$*, each G_n being the graphic of f_n,*

$$G_n = \{(x, y); x \in G_{dn}, y \in G_{cn}, y = f_n(x)\} \simeq G_{dn}.$$

Of course such an automap is an autorelation, and Auto[Set] $\subset$ Auto[Rel]*.*

Proposition 1.10. *An autorelation R determines an automap $\hat{R}$ given by maps* $\hat{R}_m : \mathcal{P}(R_{cm}) \to \mathcal{P}(R_{dm})$ *with $\mathcal{P}(E)$ the set of subsets of E, and*

$$\hat{R}_m(X) = \{y \in R_{dm}; \exists z \in R_m, (c_m(z) \in X \wedge d_m(z) = y)\}.$$

So we get an injection $\hat{-}$: Auto[Rel] $\longrightarrow$ Auto[Set]*.*

1.4 Set $\mathcal{F}(A, d, c)$ of functional representations of an autograph

Definition 1.11. *A* functional representation *of an autograph (A, d, c) is a data (Φ, ϕ), with for each $f \in A$, the data of sets $\Phi(df)$ and $\Phi(cf)$, and of a function*

$$\phi(f) : \Phi(df) \to \Phi(cf).$$

The set of these functional representations is denoted by $\mathcal{F}(A, d, c)$, and as a functional representation is a special case of a relational representation — with $\Phi(f) = \Phi(df)$ — we have $\mathcal{F}(A, d, c) \subset \mathcal{R}(A, d, c)$.

Proposition 1.12. *The definition in the construction of* Auto[Set] *in Proposition 1.9 determines each automap as a functional representation of the free autograph* $\mathbb{F}\mathbb{A}(())$, *and* Auto[Set] *as a subset of* $\mathcal{F}(\mathbb{F}\mathbb{A}(()))$. *A fortiori,* Auto[Rel] *being a subset of* Auto[Set], *it is also a subset of* $\mathcal{F}(\mathbb{F}\mathbb{A}(()))$.

Proposition 1.13. *As in the case of Proposition 1.10 we have an injection*

$$\hat{-} : \mathcal{R}(A, d, c) \longrightarrow \mathcal{F}(A, d, c).$$

Proposition 1.14. *We get a category* $\mathrm{Rep}(A, d, c)$ *of representations of an autograph, with objects the elements of* $\mathcal{F}(A, d, c)$, *a morphism from* (Φ, ϕ) *to* (Φ', ϕ') *being a double collection* $(t_f^d, t_f^c)_{f \in A}$ *of maps*

$$t_f^d : \Phi(df) \to \Phi'(df), \ t_f^c : \Phi(cf) \to \Phi'(cf),$$

such that

$$t_f^c \phi_f = \phi'_f t_f^d.$$

1.5 The regular representation, object of $\mathrm{Rep}(A, d, c)$

The natural representation for a category is given by Yoneda's lemma, with at first the following basic fact. For each category $\mathcal{C}$ we have a faithful representation by a functor $U_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathrm{Set}$, given by $U_{\mathcal{C}}(A) = \dot{\cup}_X \mathrm{Hom}(X, A)$, and $U_{\mathcal{C}}(f)(u) = f.u$, when $f : A \to B$. In the special case where $\mathcal{C}$ is the free category of paths in a graph G, this provides the representation of G by action on its paths. Similarly for an autograph (A, d, c) we have:

Proposition 1.15. *For each autograph A we have the following faithful regular representation $f \mapsto (\Gamma(f), \gamma(f))$ with:*
1 — The set $\Gamma(f)$ of (d, c)-paths (cf. [3, Definition 1.4]) with end f i.e.

$(z_n)_{0 \le n \le k-1}$, *with* $cz_0 = dz_1, cz_1 = dz_2 \dots, cz_{k-2} = dz_{k-1}$ *and* $cz_{k-1} = f$.

2 — A map $\gamma(f) : \Gamma(df) \to \Gamma(cf)$, given by concatenation with f, by

$$\gamma(f)((z_n)_{0 \le n \le k-1}) = (z'_n)_{0 \le n \le k},$$

with $z'_n = z_n$ if $n < k$, and $z'_k = f$.
Shortly if $(z_n)_{0 \le n \le k-1} = z$, then $(z'_n)_{0 \le n \le k} = fz$, or $\gamma(f)(z) = fz$.
*So (A, d, c) can be identified with a special element of $\mathcal{F}(A, d, c)$ or object of* $\mathrm{Rep}(A, d, c)$.

2. Double categories, Knots, Links, Grid Diagrams

In the first paper of this series [3] we obtained that the topos Agraph of auto-graphs is a common setting for knots and 2-categories or double categories. Here this result is strengthened and extended, using 2-dimensional paths in double categories and grid diagrams.

2.1 Double categories and knots as well formed 2-dim words

Proposition 2.1. *A double category* $\mathbb{C}$ *is determined by an associated auto-category* $\mathrm{Ass}(\mathbb{C})$*, according to the following picture to represent a* 2*-block* b *as an autograph:*

A 2-block **b** as an autograph
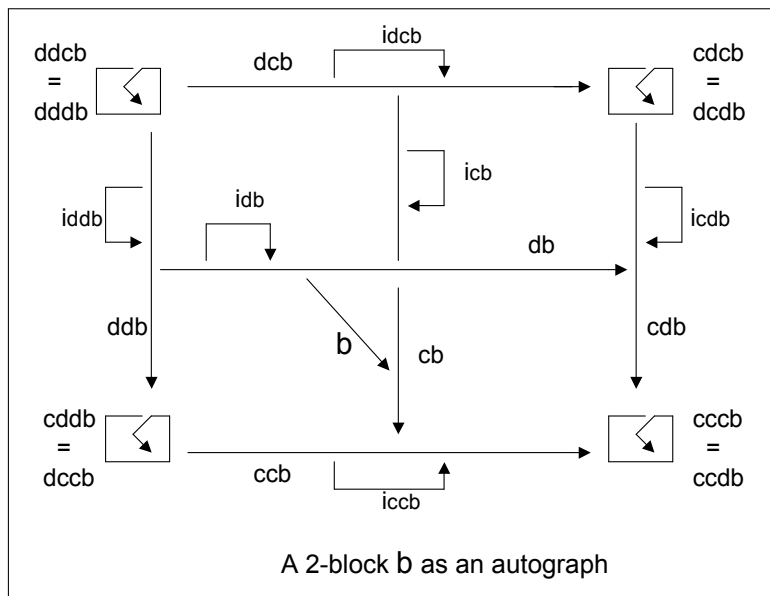
Proof. A 2-block b (fig.[1] below) is considered as an arrow from its two oriented versions, its vertical orientation b_v and its horizontal orientation b_h. Then b_v is an arrow from $d_v b$ to $c_v b$, etc. (fig.[2]). Hence (fig.[3]) a resulting autograph, which can be completed and redrawn as in Proposition 2.1 above. The full description of $\mathrm{Ass}(\mathbb{C})$ is explained in [3, Proposition 7.1], but we have to correct a typos there: in the picture an autoarrow $I_{b_v} = (b_v)^{\theta} : b_v \rightarrow b_v$ should be added.

$$[1] \qquad\qquad [2] \qquad\qquad [3]$$
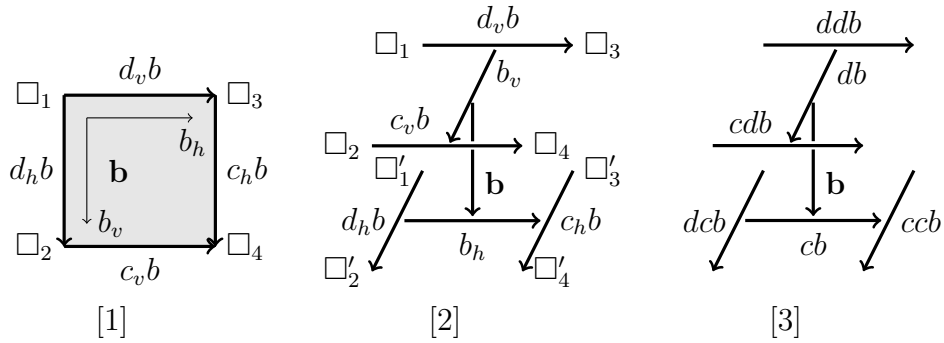
In particular we have shown how the two horizontal and vertical compositions, denoted by ∞ and 8
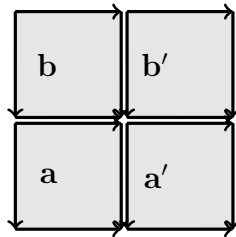


and compatible according to

$$(a'\infty a)8(b'\infty b) = (a'8b')\infty(a8b),$$

are replaced by a unique composition law. $\qquad\qquad\qquad\qquad\square$

Proposition 2.2. *Any knot or link can be presented as a 2-dim rectangular "well formed" word, on a rectangle $\mathcal{R}_{n,m}$ of dimension $n \times m$ made with the tiles from the set $\mathbb{T}$ of the 9 following tiles (a word is well formed if each line decoration into any tile arriving on a side of this tile is pursued in the next adjacent tile). Of course it is a map $L : \mathcal{R}_{n,m} \to \mathbb{T}$, and of course such a data is representable as an autograph.*
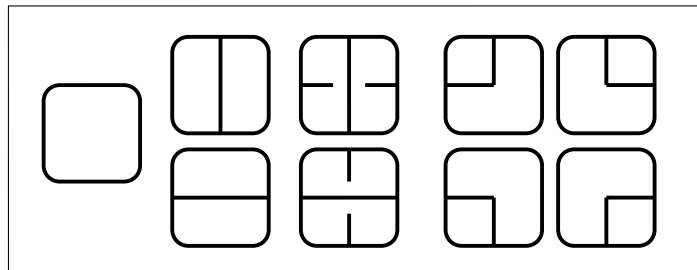
Consequently a link is a 2-*dim path in the double category generated by these tiles (or in the corresponding autograph according to Proposition 2.1).*

Remark 2.3. Hence the question of isotopy type of links becomes a question of 2-dim rewriting, as explained in [2] (This is also near from studies on mosaics [6]). There are vertical (or horizontal) dilatations: if a column consists only in empty tiles or horizontal line tiles, we can add a new similar column juxtaposed to the first one; and furthermore they are analogous to the three Reidemeister moves.

Example 2.4. The following 2-dim word is a borromean link.
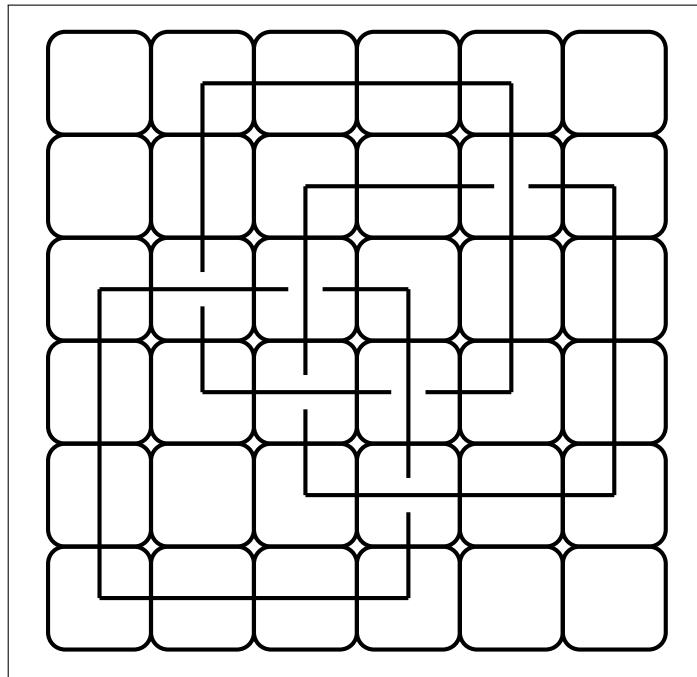

2.2 Knots, from their knot diagrams

In the paper [3, section 4] for a knot $\mathbb{K}$ we introduced an associated autograph $\mathrm{As}(\mathbb{K})$, used for trefoil or borromean knot and link. The following Proposition 2.5 strengthens this result.

Proposition 2.5. *If* $\mathbb{K}$ *is an alternating knot, then from* $\mathrm{As}(\mathbb{K})$ *we recover the Gauss' code of* $\mathbb{K}$*, and so this knot is determined by its associated autograph.*

For a general knot a modification of the construction is necessary, following Proposition 2.6

Proposition 2.6. *If $\mathbb{K}$ is an arbitrary knot, then from the autograph $\mathrm{As}(\mathbb{K}^{aa})$ — with $\mathbb{K}^{aa}$ defined in the proof — we recover the Gauss'code of $\mathbb{K}$, and so this knot is determined by this autograph.*

Proof. If the knot is not alternating, then we cannot recover the Gauss'code from $\mathrm{As}(\mathbb{K})$. For example the **K** in the next picture is a not-alternating knot, and we consider an arc which is not going in an alternative way, as e from b to h, passing over in two consecutive crossings; hence we have c and f arriving to e, but in $\mathrm{As}(\mathbf{K})$ we have no information on the order in which these arrows arrived on e: following e which one is the first met, c or f? So before considering $\mathrm{As}(\mathbf{K})$ we decide to modify **K** into $\mathbf{K}^{aa}$ as follows. In **K** we observe arcs which are alternating, as d, f, g, h, and the others, a, b, c, e are said to be not-alternating. Each of these not-alternating arcs (see in $\mathbf{K}^{aa}$) is now decomposed by introducing autoarrows, $2, 5, 8, 12$, and we have $a = 1.2.3$, $b = 4.5.6$, $c = 7.8.9$, $e = 11.12.13$. Now c or rather 9 arrives to 11, whereas 14 arrives to a different arc, namely 13, and we can recover the Gauss'code of **K** from $\mathrm{As}(\mathbf{K}^{aa})$. For the Gauss'code see [5, p.666].


$\square$
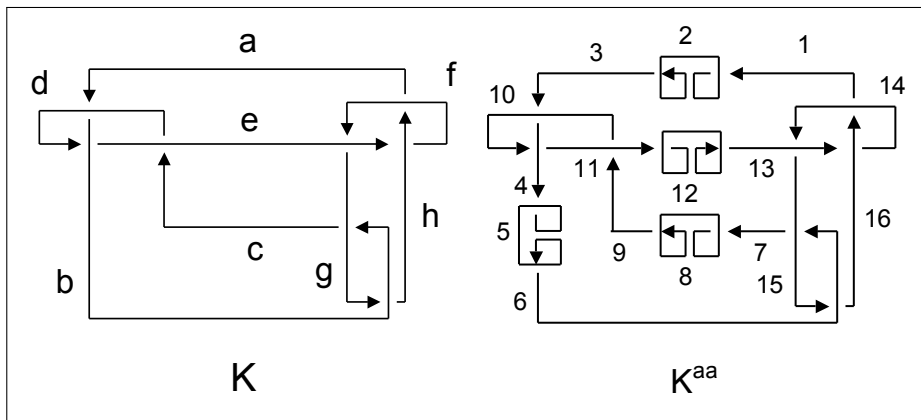
2.3 Grid diagrams and links isotopy types

Proposition 2.7. *Any link can be associated to an autograph determining its isotopy type.*

Proof. A *Grid diagram* [7], is an $n \times n$ square with n triangles and n circles placed in distinct places, such that each row and each column contains exactly one triangle and one circle. In the next picture the first left drawing is an example with $n = 8$. Given such a grid, we join the triangle and the circle in each column by continuous straight vertical lines (second step in the picture), and then in each row we join the triangle and the circle by a straight line passing under the previously vertical straight lines it meets (third step). And finally (fourth step) we look at a link. In this example it is a borromean link (but presented differently from the picture given in [3, Example 4.5]). Another borromean example is furnished by Example 2.4.

grid diagram   colums over   rows under   link
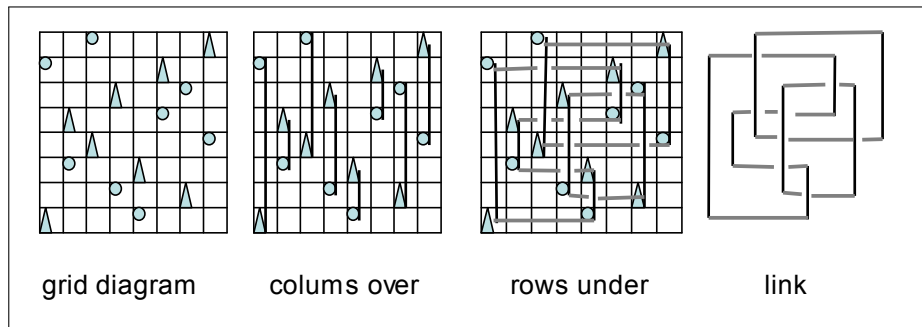
Now, as any isotopic type of link can be obtained in this way [1], we conclude if we can show that any grid can be determined by an autograph, and this is obvious since a grid is a graph. □

3. Graphic monads among autographic monads

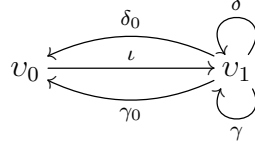**3.1 The topos** Agraph **of autographs, between** Graph **and** Set

3.1.1 Autographs and graphs

According to [3, def.1.1., p.66], [4, def.1.1.-1.2, p.152], we have:

Definition 3.1 (Graphs)**.** *Let* $\mathbb{G}(2)$ *be the category with objects* v_0 *and* v_1, *five non-identity arrows*

$$\gamma_0, \delta_0 : v_1 \to v_0, \quad \iota : v_0 \to v_1, \quad \delta, \gamma : v_1 \to v_1,$$

identities on v_1, v_0, equations: $\delta_0.\iota = 1_{v_0}$, $\gamma_0.\iota = 1_{v_0}$, $\gamma = \iota.\gamma_0$, $\delta = \iota.\delta_0$.

A presheaf G on $\mathbb{G}(2)$, i.e. an object of Graph $=$ Set$^{\mathbb{G}(2)}$ *is named a* graph. *Any $c \in G(v_0)$ is named a* vertex *or a* carfour, *and if $f \in G(v_1)$, f is named an* arrow; *the fact that $G(\delta_0)(f) = c$ and $G(\gamma_0)(f) = c'$ is written:* $f : c \to c'$.

3.1.2 The comparison V and its equivalent W

With [4, Prop.1.4 p.153, Prop.2.2. p.154] the comparison between autographs and graphs is given by a functor V : Graph $\to$ Agraph. :

Proposition 3.2. *The categories* Agraph *and* Graph *are toposes, inscribed in the sequence*

$$\text{Graph} \xrightarrow{\text{V}} \text{Agraph} \xrightarrow{\text{U}} \text{Set},$$

where U $= eva_v{}^{\mathbb{FM}(2)}$ *is the monadic forgetful functor given by evaluation at v, $(A, (d_A, c_A)) \mapsto A$, and* V $= \Phi = (-).\phi$ *is the monadic functor induced by the map $\phi : \mathbb{FM}(2) \to \mathbb{G}(2); v, c, d \mapsto v_1, \gamma, \delta$.*

Remark 3.3. In a graph G the $G(\iota) = \phi$ associates to each vertex an arrow. Hence we have the more general situation of *flexigraph* in [3, Defg. 5.4.]. An autograph A appears as a special case of a flexigraph, when $\phi = 1_A$. Let us recall also from [4] that with graphs we have 2 types of arities (vertices and arrows), wheras with autographs only 1 type (arrows) is considered. Next Proposition 3.4 clarifies thist point.

Proposition 3.4. *The category $\mathbb{G}(2)$ is the Karoubian envelope of the monoïd $\mathbb{M}(2) = \{1, c, d\}$, with equations $c^2 = c, d^2 = d, cd = d, dc = c$, and* Graph *is equivalent to* Set$^{\mathbb{M}(2)}$. *Up to this equivalence, the functor* V *is the functor* W $=$ Set$^{\bar{\phi}}$ *induced by the composition with the monoïd quotient homomorphism $\bar{\phi}$ given by $v \mapsto 1$, $c \mapsto c, d \mapsto d$:*

$$\bar{\phi} : \mathbb{FM}(2) \to \mathbb{M}(2),$$

$$\text{W} = \text{Set}^{\bar{\phi}} : \text{Set}^{\mathbb{M}(2)} \to \text{Set}^{\mathbb{FM}(2)}.$$

3.2 From graphic monads to autographic monads

In the second paper of this series [4], via graphic monoïds of Lawvere we have shown that basic Albert Burroni's graphic algebras are autographic algebras; and especially as autographic algebras we get categories as well as autocategories. Now we have to precise the study for graphic algebras which are not necessarily basic. Albert Burroni defined a *graphic algebra* as an algebra of a monad on Graph (a graphic monad); similarly we defined an *autographic algebra* as an algebra of a monad on Agraph (an autographic monad). So we have to complete the result in the general case, for algebras of arbitrary monads on Graph $= \mathrm{Set}^{\mathbb{G}(2)} \simeq \mathrm{Set}^{\mathbb{M}(2)}$ and their transport via the W in Proposition 3.4.

Generally speaking, for any homomorphism of monoïds $\varphi : F \to M$ the induced functor

$$\mathrm{Set}^{\varphi} : \mathrm{Set}^{M} \to \mathrm{Set}^{F}$$

is monadic, This works for our $\mathrm{W} = \mathrm{Set}^{\bar{\phi}} : \mathrm{Set}^{\mathbb{M}(2)} \to \mathrm{Set}^{\mathbb{FM}(2)}$ from Proposition 3.4, as well as for any quotient map of monoïds

$$q_M : \mathbb{FM}(2) \to M,$$

the corresponding $\mathrm{W}_{q_M} = \mathrm{Set}^{q_M}$, its left adjoint Lan_{q_M}, $T_M = (-)^{q_M} \mathrm{Lan}_{q_M}$ and $\mathbb{T}_M = (T_M, r)$ the associated idempotent monad on $\mathrm{Set}^{\mathbb{FM}(2)} = \mathrm{Agraph}$. So, for a given q_M — see [4, Proposition 2.6]) — the topos Set^{M} is a reflexive subcategory of Agraph, with for any $E = (A, d, c)$ a reflexion

$$r_E : E \to T_M(E),$$

given by a quotient set $T_M(E) = T_M(A, d, c) = A/[q_M]$, quotient of A by the smallest congruence $[q_M]$ on E compatible with q_M.

Proposition 3.5. *With the notations above for a given* $q_M : \mathbb{FM}(2) \to M$ *(and the associated monad* $\mathbb{T}_M$*) on Agraph, we consider another monad* $\mathbb{T} = (T, \eta, \mu)$ *on* Set^{M}*. Then the functor*

$$(\mathrm{Set}^{M})^{\mathbb{T}} \xrightarrow{U_{\mathbb{T}}} \mathrm{Set}^{M} \xrightarrow{\mathrm{W}_{q_M}} \mathrm{Set}^{\mathbb{FM}(2)} = \mathrm{Agraph}$$

determines a monad $\bar{\mathbb{T}} = (\bar{T}, \bar{\eta}, \bar{\mu})$ *on* Agraph*, of which an algebra* $\bar{\theta}$ *on* E *is a composition* $\bar{\theta} = \lambda\theta$ *of an algebra* $\theta = r_E\bar{\theta}$ *of* $\mathbb{T}$ *on* $T_M(E)$ *and of a*

special section $\lambda = \bar{\theta}\eta_{E/[q]}$ of the reflexion $r_E : E \to T_M(E)$.
Consequently we have

$$(\mathrm{Set}^M]^{\mathbb{T}} \simeq \mathrm{Agraph}^{\bar{\mathbb{T}}} \cap \mathrm{Agraph}^{\mathbb{T}_M}.$$

In particular this is true for $M = \mathbb{M}(2)$ and the corresponding W, and algebras of graphic monad are such special algebras of autographic monads.

Proof. With $(\bar{\mathbb{T}} = (\bar{T}, \bar{\eta}, \bar{\mu})$ the monad associated to $\mathrm{W}_{q_M} U_{\mathbb{T}}$ we have, with $T_M(E) = E/[q] = A/[q_M]$ and $r_E : E \to E/[q]$, the following formula: $\bar{T}(E) = T(E/[q]), \bar{\eta}_E = \eta_{E/[q]}r_E, \bar{\mu}_E = \mu_{E/[q]}, \bar{T}^2(E) = T^2(E/[q])$.

If $(E, \bar{\theta})$ is a $\bar{\mathbb{T}}$-algebra on E, then we introduce $\theta = r_E\bar{\theta}$, and so $T\theta = T(r_E\bar{\theta})$. The $\bar{\mathbb{T}}$-associativity $\bar{\theta}\bar{T}\bar{\theta} = \bar{\theta}\bar{\mu}_E$ implies, by composition on the left with r_E, $r_E\bar{\theta}T(r_E\bar{\theta}) = r_E\bar{\theta}\bar{\mu}_E$ i.e. the $\mathbb{T}$-associativity: $\theta.T\theta = \theta\mu_{E/[q]}$. Also from $\bar{\mathbb{T}}$-unitarity we obtain $\mathbb{T}$-unitarity, $\theta\eta_{E/[q]} = 1_{E/[q]}$, from $\bar{\theta}\bar{\eta}_E = 1_{E]}$ by composition on the left with r_E: $\theta\eta_{E/[q]}r_E = r_E$. So we obtain $(E/[q], \theta)$ a $\mathbb{T}$-algebra on $E/[q]$.

In fact introducing $\lambda = \bar{\theta}\eta_{E/[q]}$, we obtain $\lambda\theta = \bar{\theta}$, and $r_E\lambda = 1_{E/[q]}$. For the first we have $\bar{\theta}\eta_{E/[q]}r_E\bar{\theta} = \bar{\theta}$, i.e. $\bar{\theta}\bar{\eta}_E\bar{\theta} = \bar{\theta}$. For the second, by composition on the right with the epimorphism θ we get $r_E\lambda\theta = \theta$, or $r_E\bar{\theta}\eta_{E/[q]}\theta = \theta, \theta\eta_{E/[q]}\theta = \theta$.

So any $\bar{\theta}$, $\bar{\mathbb{T}}$-algebra on E, determines two things: θ, an algebra on $E/[q]$, and λ, a section of $r_E : E \to E/[q]$. Conversely, given θ and λ, we recover $\bar{\theta} = \lambda\theta$.

Especially, a $\mathbb{T}$-algebra is a $\bar{\mathbb{T}}$-algebra on a E such that $E \simeq E/[q]$, i.e. a E equipped with a $\mathbb{T}_M$-algebra structure ($\lambda = 1_E = r_E$). $\qquad\square$

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