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SOMMAIRE

BLUTE, LUCYSHYN-WRIGHT & O'NEILL - Derivations in codifferential categories	243
J. BOURKE, Note on the construction of globular weak ω -groupoids from types, topological spaces...	281
W. THOLEN, A mathematical tribute to Reinhard Börger	295
RESUMES des articles parus dans le Volume LVII	315

DERIVATIONS IN CODIFFERENTIAL CATEGORIES

by Richard BLUTE, Rory B. B. LUCYSHYN-WRIGHT
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Résumé. Nous étudions une notion générale de *dérivation* dans le contexte des *catégories codifférentielles* de Blute-Cockett-Seely, généralisant la notion de dérivation K -linéaire de l'algèbre commutative. Pour une catégorie codifférentielle (\mathcal{C}, T, d) , une T -dérivation $\partial : A \rightarrow M$ sur une algèbre A de la monade T est définie comme un morphisme de \mathcal{C} dans un A -module M vérifiant une forme du théorème de dérivation des fonctions composées par rapport à la *transformation dérivateur* d . Nous montrons que ces T -dérivations correspondent aux T -homomorphismes $A \rightarrow W(A, M)$ au-dessus de A dans une T -algèbre associée $W(A, M)$. Nous établissons l'existence de T -dérivations universelles $A \rightarrow \Omega_A^T$ dans un A -module associé Ω_A^T , le *module de différentiels* de type Kähler. Tandis que l'article précédent de Blute-Cockett-Porter-Seely sur les *catégories Kähleriennes* a utilisé une notion de dérivation exprimable sans référence à la monade T , nous montrons que l'usage de la notion de T -dérivation ci-dessus résout un problème ouvert concernant les catégories Kähleriennes, montrant que la *Propriété K* pour catégories codifférentielles n'est pas nécessaire. Le long du chemin, nous établissons une définition succincte et équivalente de la notion de catégorie codifférentielle en termes d'un morphisme de monades $S \rightarrow T$ sur la monade S de l'algèbre symétrique et d'une transformation d vérifiant le théorème de dérivation des fonctions composées.

Abstract. We define and study a novel, general notion of *derivation* in the setting of the *codifferential categories* of Blute-Cockett-Seely, generalizing the notion of K -linear derivation from commutative algebra. Given a codifferential category (\mathcal{C}, T, d) , a T -derivation $\partial : A \rightarrow M$ on an algebra A of the monad T is defined as a morphism in \mathcal{C} into an A -module M satisfying a form of the chain rule expressed in terms of the *deriving transformation* d . We show that such T -derivations correspond to T -homomorphisms $A \rightarrow W(A, M)$ over A valued in an associated T -algebra. We establish the existence of universal T -derivations $A \rightarrow \Omega_A^T$ valued in an associated A -module of Kähler-type differentials Ω_A^T . Whereas previous work of Blute-Cockett-Porter-Seely on *Kähler categories* employed a notion of derivation expressible without reference to the monad T , we show that the use of the above T -based notion of derivation resolves an open problem concerning Kähler categories, showing that *Property K* for codifferential categories is unnecessary. Along the way, we establish a succinct equivalent definition of codifferential categories in terms of a given monad morphism $S \rightarrow T$ on the symmetric algebra monad S and a compatible transformation d satisfying the chain rule.

Keywords. Derivation; Kähler differential; differential category; codifferential category; monoidal category; commutative algebra; module; monad.

Mathematics Subject Classification (2010). 13N05, 13N15, 18D10, 18D35, 18C15, 18C20, 14F10, 18E05.

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1. Synopsis

Derivations provide a way of transporting ideas from the calculus of manifolds to algebraic settings where there is no sensible notion of limit. In this paper, we consider derivations in certain monoidal categories, called *codifferential categories*. Differential categories were introduced as the categorical framework for modelling differential linear logic. The deriving transform of a differential category, which models the differentiation inference rule, is a derivation in the dual category. We here explore that derivation's universality.

One of the key structures associated to a codifferential category is an *algebra modality*. This is a monad T such that each object of the form TC is canonically an associative, commutative algebra. Consequently, every T -algebra has a canonical commutative algebra structure, and we show that universal derivations for these algebras can be constructed quite generally.

It is a standard result that there is a bijection between derivations from an associative algebra A to an A -module M and algebra homomorphisms over A from A to $A \oplus M$, with $A \oplus M$ being considered as an infinitesimal extension of A . We lift this correspondence to our setting by showing that in a codifferential category there is a canonical T -algebra structure on $A \oplus M$. We call T -algebra morphisms from TA to this T -algebra structure *Beck T -derivations*. This yields a novel, generalized notion of derivation.

The remainder of the paper is devoted to exploring consequences of that definition. Along the way, we prove that the symmetric algebra construction in any suitable symmetric monoidal category provides an example of codifferential structure, and using this, we give an alternative definition for differential and codifferential categories.

2. Introduction

The theory of Kähler differentials [15, 20] provides an analogue of the theory of differential forms and all of its various uses in settings other than the usual setting of smooth manifolds. They were originally introduced by Kähler as an abstract algebraic notion of differential form. One of their advantages is that they can be applied to varieties which are not also smooth manifolds, such as singular varieties in characteristic 0 or arbitrary varieties over a field of characteristic p . In a setting where one does not have access to limits, one can still talk about *derivations*. That is to say one passes from the variety to its coordinate ring, and then considers a module over that ring. A derivation is then a linear map from the algebra to the module satisfying the Leibniz rule. The module of *Kähler differentials* or *Kähler module* is then a module equipped with a universal derivation. As usual, such a module is unique up to isomorphism.

Since this initial work, the idea of extending differential forms to more and more abstract settings has advanced in a number of different directions. As one important example, we mention the noncommutative differential

forms that arise in noncommutative geometry [18].

Differential linear logic [11, 12] arose originally from semantic concerns. Ehrhard [9, 10] had constructed several models of linear logic [14] in which the hom-sets had a natural differentiation operator. Ehrhard and Regnier then described this operation as a sequent rule and represented it as a construction and a rewrite rule for both interaction nets and for λ -calculus. The corresponding categorical structures were introduced in [3, 4] and called *differential categories* and *cartesian differential categories*. Cartesian differential categories are an axiomatization of the coKleisli category of a differential category.

The notion of *Kähler category* [2] began with the observation that the *deriving transform*, the key feature of differential categories, is a derivation and, under certain assumptions, has a universal property discussed below. (Actually, we must work with the dual notion of codifferential category. If we worked with coalgebras and coderivations, we could work in differential categories and all of the following work, suitably *op*-ed, would still hold.) It thus seemed likely that an abstract monoidal setting in which Kähler differential modules could be defined would apply to differential categories. In fact, the original paper only partially resolved this issue. In the present paper, we provide a much more satisfying answer by generalizing the notion of derivation to take into account all of the codifferential structure, thereby establishing a suitable universal property in full generality.

A Kähler category is an additive, symmetric monoidal category with an *algebra modality*, i.e. a monad T for which each object of the form TC is equipped with a commutative, associative algebra structure and several coherence equations hold, such that each of these algebras has a universal derivation. In essence, we are requiring a Kähler module for each free T -algebra.

The present paper extends the work of [2] in several ways. It is not surprising that, given all the structure at hand, one can endow every T -algebra with the structure of a commutative, associative algebra. We show that in a Kähler category, one can use the existence of Kähler objects for free T -algebras to derive Kähler objects for all algebras¹ that arise as underlying

¹We realize that the unavoidable use of the word *algebra* in two different ways is confusing. The word algebra without a T – in front of it will always mean commutative, associative algebra.

algebras of T -algebras. Thus if the algebra category is monadic over the base, we can derive Kähler modules for all algebras by a single uniform procedure. These results follow from the M.Sc. thesis of the third author [21].

We also tackle the idea of what it means to be a derivation. It is well-known [6] that if A is a commutative algebra and M is an A -module, then there is a canonical algebra structure on $A \oplus M$ such that derivations from A to M are in bijective correspondence to algebra maps over A from A to $A \oplus M$. Essentially the algebra $A \oplus M$ is the extension of A by M -infinitesimals. This idea was used in a much more general setting by Beck [1].

While this is a straightforward calculation, it has far-reaching generalizations. First we show that in a codifferential category, given a T -algebra (A, a) and a module M over the algebra associated to A , there is a canonical T -algebra structure on $A \oplus M$ which under the passage from T -algebras to algebras yields the traditional associative algebra structure on $A \oplus M$ from [1]. We call this T -algebra $W(A, M)$. We then define a *Beck T -derivation on A valued in M* to be a map of T -algebras from (A, a) to $W(A, M)$ in the slice category over A . Beck T -derivations can be equivalently given by morphisms $\partial: A \rightarrow M$ satisfying a *chain rule* condition with respect to T .

We show that the symmetric algebra monad yields a codifferential category in a very general setting and in this case, our notion of Beck T -derivation is equivalent to the usual notion of derivation.

We define a *module of Kähler T -differentials* to be an A -module with a universal Beck T -derivation. We then show that the deriving transform in a codifferential category is always universal in this sense. In fact, every T -algebra has a universal T -derivation. Our analysis also yields an equivalent definition of differential category we believe will be valuable in generalizations of this abstract notion of differentiation. For example, it generalizes in a straightforward way to noncommutative settings.

We note that in [8], Dubuc and Kock define a notion of derivation on an algebra of a *Fermat theory*, the latter being a finitary set-based algebraic theory extending the theory of commutative rings and satisfying a certain axiom. It would be interesting to compare their notion with the notion of T -derivation defined here in the monoidal context of codifferential categories.

The extension of Kähler categories and codifferential categories to non-

commutative settings is an important project, and work of this sort has been initiated by R. Cockett [7]. In that paper, the author has also explored the relationship between T -algebras and derivations. In particular, he considers the implications of demanding for each T -algebra A and each A -bimodule M a *given* T -algebra structure on $A \oplus M$ satisfying certain axioms, whereas here we have shown that in the setting of a codifferential category, a T -algebra structure on $A \oplus M$ can be defined in terms of the given codifferential structure.

3. Derivations and categorical frameworks

This section covers the theory of derivations, both in its classical formulation with respect to algebras over a field and several of its more abstract categorical formulations.

3.1 Classical case

Derivations were originally considered for commutative algebras over a field and are employed in algebraic geometry and commutative algebra [13, 15].

Definition 3.1. Let k be a commutative ring, A a commutative k -algebra, and M an A -module. (All modules throughout the paper will be *left* modules.)

A *k*-derivation from A to M is a k -linear map $\partial : A \rightarrow M$ such that $\partial(aa') = a\partial(a') + a'\partial(a)$.

One can readily verify under this definition that $\partial(1) = 0$ and hence $\partial(r) = 0$ for any $r \in k$.

Definition 3.2. Let A be a k -algebra. A *module of A-differential forms* is an A -module Ω_A together with a k -derivation $\partial : A \rightarrow \Omega_A$ which is universal in the following sense: For any A -module M , and for any k -derivation $\partial' : A \rightarrow M$, there exists a unique A -module homomorphism $f : \Omega_A \rightarrow M$ such that $\partial' = \partial; f$.

Lemma 3.3. *For any commutative k-algebra A, a module of A-differential forms exists.*

There are several well-known constructions. The most straightforward, although the resulting description is not that useful, is obtained by constructing the free A -module generated by the symbols $\{\partial a \mid a \in A\}$ divided out by the evident relations, most significantly $\partial(aa') = a\partial(a') + a'\partial(a)$.

3.2 Derivations as algebra maps

We suppose we are working in the category of vector spaces over a field k , that A is a commutative k -algebra and M an A -module. Define a commutative algebra structure on $A \oplus M$ by

$$(a, m) \cdot (a', m') = (aa', am' + a'm)$$

It is evident that this is associative, commutative and unital. We will refer to this algebra structure as the *infinitesimal extension of A by M* . But its interest comes from the following observation.

Lemma 3.4. *There is a bijective correspondence between k -derivations from A to M and k -algebra homomorphisms from A to $A \oplus M$ which are the identity in the first component. Or more succinctly:*

$$\text{Der}_k(A, M) \cong \text{Alg}/A(A, A \oplus M)$$

Here, Alg/A is the slice category of objects over A in the category Alg of k -algebras.

We also note that it is straightforward to lift this result to the level of additive symmetric monoidal categories, see Section 3.3. The notions of commutative algebra and module are expressible in any symmetric monoidal category. Once one has additive structure then the notion of derivation is definable as well. The correspondence of Lemma 3.4 then extends to this more general setting. Lemma 3.4 also provided Jon Beck [1] a starting point for a far-reaching generalization of the notion of derivation for the purposes of cohomology theory. One of the primary contributions of this paper is to lift the correspondence of Lemma 3.4 to the level of codifferential categories. The fact that these ideas continue to hold at this level is testament to the importance of Beck's ideas about cohomology.

3.3 Categorical structure

It is a standard observation [19, 17] that the notions of algebra (monoid) and module over an algebra make sense in any monoidal category and the notion of commutative algebra makes sense in any symmetric monoidal category. But to discuss derivations for an algebra we also need additive structure.

- Definition 3.5.** 1. A symmetric monoidal category \mathcal{C} is *additive* if it is enriched over commutative monoids and the tensor functor is additive in both variables².
2. Let (A, m_A, e_A) be an algebra in an additive symmetric monoidal category³, and $M = \langle M, \bullet_M : A \otimes M \rightarrow M \rangle$ an A -module. Then a *derivation to M* is an arrow $\partial : A \rightarrow M$ such that (with m being the multiplication)

$$m; \partial = c; 1 \otimes \partial; \bullet_M + 1 \otimes \partial; \bullet_M \quad \text{and} \quad \partial(1) = 0$$

Remark 3.6. We note that Lemma 3.4 holds at this level of generality as well. Indeed, given a commutative algebra A in an additive symmetric monoidal category \mathcal{C} with finite coproducts (equivalently, finite biproducts) and an A -module M , we can equip $A \oplus M$ with the structure of a commutative algebra [2]. Derivations $A \rightarrow M$ then correspond to maps $A \rightarrow A \oplus M$ in the slice category Alg/A over A in the category Alg of commutative algebras in \mathcal{C} [2]. As noted in [2, §4.2], every map of A -modules $h : M \rightarrow N$ determines an algebra map $1 \oplus h : A \oplus M \rightarrow A \oplus N$, whence each derivation $\partial : A \rightarrow M$ determines a composite derivation $A \xrightarrow{\partial} M \xrightarrow{h} N$. Further, given a map of commutative algebras $g : A \rightarrow B$, each B -module N determines an A -module N_A , the *restriction of scalars of N along g* , consisting of the object N of \mathcal{C} equipped with the composite A -action

$$A \otimes N \xrightarrow{g \otimes 1} B \otimes N \xrightarrow{\bullet_N} B.$$

Moreover, given an algebra map $g : A \rightarrow B$ and a derivation $\partial : B \rightarrow N$, the composite $A \xrightarrow{g} B \xrightarrow{\partial} N$ is a derivation $A \rightarrow N_A$.

²In particular, we only need addition and unit on Hom-sets, rather than abelian group structure.

³We will use the notation m_A and e_A for the multiplication and unit for A .

As for most algebraic structures, when one adds in an appropriate notion of universality, the result is a very powerful mathematical object. For derivations, we obtain the module of *Kähler differentials* or *Kähler module*. We cite [15, 20] for calculations and examples.

Definition 3.7. Let \mathcal{C} be an additive symmetric monoidal category and let A be a commutative algebra in \mathcal{C} . A *module of Kähler differentials* is an A -module Ω_A together with a derivation $\partial : A \rightarrow \Omega_A$, such that for every A -module M , and for every derivation $\partial' : A \rightarrow M$, there exists a unique A -module map $h : \Omega_A \rightarrow M$ such that $\partial; h = \partial'$.

$$\begin{array}{ccc} A & \xrightarrow{\partial} & \Omega_A \\ & \searrow \partial' & \downarrow \text{dotted } h \\ & & M \end{array}$$

An axiomatization of a very different sort which attempted to capture the process of differentiation axiomatically is the theory of *differential categories* [3]. Since in this paper we wish to work with algebras and derivations as opposed to coalgebras and coderivations, we work in the dual theory of *codifferential categories*.

Definition 3.8. An *algebra modality* on a symmetric monoidal category \mathcal{C} consists of a monad (T, μ, η) on \mathcal{C} , and for each object C in \mathcal{C} , a pair of morphisms (note we are denoting the tensor unit by k)

$$m : T(C) \otimes T(C) \rightarrow T(C), \quad e : k \rightarrow T(C)$$

making $T(C)$ a commutative algebra such that this family of associative algebra structures satisfies evident naturality conditions [2].

Definition 3.9. An additive symmetric monoidal category with an algebra modality is a *codifferential category* if it is also equipped with a *deriving transform*⁴, i.e. a transformation natural in C

$$d_{T(C)} : T(C) \rightarrow T(C) \otimes C$$

satisfying the following four equations⁵:

⁴We use the terminology of a *deriving transform* in both differential and codifferential categories.

⁵For simplicity, we write as if the monoidal structure is strict.

- (d1) $e; d = 0$ (Derivative of a constant is 0.)
- (d2) $m; d = (1 \otimes d); (m \otimes 1) + (d \otimes 1); c; (m \otimes 1)$ (where c is the appropriate symmetry) (Leibniz Rule)
- (d3) $\eta; d = e \otimes 1$ (Derivative of a linear function is constant.)
- (d4) $\mu; d = d; \mu \otimes d; m \otimes 1$ (Chain Rule)

We make the following evident observation, noting that the morphism $u_C^{TC} := e \otimes 1: C = k \otimes C \rightarrow T(C) \otimes C$ exhibits $T(C) \otimes C$ as the free $T(C)$ -module on C .

Lemma 3.10. *When $T(C) \otimes C$ is considered as the free $T(C)$ -module generated by C , then the above deriving transform is a derivation.*

This leaves the question of its universality. We know there is a universal property for the object $T(C) \otimes C$ as the free $T(C)$ -module generated by C . Is this sufficient to guarantee the universality necessary to be a Kähler module? With this question in mind, the paper [2] introduced the notion of a *Kähler category* but only partially answered this question.

Definition 3.11. A *Kähler category* is an additive symmetric monoidal category with

- a monad T ,
- a (commutative) algebra modality for T ,
- for all objects C , a $T(C)$ -module of Kähler differential forms, satisfying the universal property of a Kähler module.

Thus the previous question can be formulated as whether every codifferential category is a Kähler category. The original paper [2] had a partial answer to this question. In the present paper, we give a much more satisfying answer to this question. The key is to generalize even further the notion of derivation. We use ideas from Jon Beck's remarkable thesis [1]. This will be covered in Section 5. In particular, see Definition 5.7 and Theorem 5.11.

3.4 Universal derivations for T -algebras

In a category with an algebra modality we may endow each T -algebra with the structure of a commutative algebra, in such a way that the structure map of the T -algebra is a morphism of associative algebras. Since universal derivations are a priori only defined for the algebras arising by virtue of the algebra modality in a Kähler category, it is natural to ask if universal derivations from these new algebras exist and, if so, how they are constructed. We examine this issue now and demonstrate that there is a very pleasing answer. The construction of such Kähler modules is from the third author's M.Sc. thesis [21]. We first note the following procedure for assigning algebra structure to T -algebras.

Theorem 3.12. *Let \mathcal{C} be a symmetric monoidal category equipped with an algebra modality T . The following construction determines a functor from the category of T -algebras to the category of commutative algebras in \mathcal{C} . Let (A, a) be a T -algebra in such a category. Define the multiplication for an algebra structure on A by the formula*

$$A \otimes A \xrightarrow{\eta \otimes \eta} TA \otimes TA \xrightarrow{m} TA \xrightarrow{a} A$$

with unit given by

$$k \xrightarrow{e} TA \xrightarrow{a} A$$

In particular, every map of T -algebras becomes an associative algebra map.

Also note that if we apply this construction to the free T -algebra (TA, μ) , we get back the original associative algebra (TA, m, e) .

Definition 3.13. Let \mathcal{C} be an additive symmetric monoidal category. Let A and B be algebras with universal derivations as in the diagram below. Let $f: A \rightarrow B$ be an algebra homomorphism. Define $\Omega_f: \Omega_A \rightarrow \Omega_B$ to be the unique morphism of A -modules making

$$\begin{array}{ccc}
 \Omega_A & \xrightarrow{\Omega_f} & \Omega_B \\
 d_A \uparrow & & \uparrow d_B \\
 A & \xrightarrow{f} & B
 \end{array}$$

commute, which exists by universality of d_A . One can verify that $\Omega_{(-)}$ is functorial.

The existence of Kähler modules for free T -algebras entails that Kähler modules for arbitrary T -algebras can be obtained by taking a quotient, as is seen in the following theorem.

Theorem 3.14. *Defining $\Omega_{A,a}$ as the following coequalizer*

$$\Omega_{T^2A} \begin{array}{c} \xrightarrow{\Omega_\mu} \\ \xrightarrow{\Omega_{Ta}} \end{array} \Omega_{TA} \xrightarrow{\Omega_a} \Omega_{A,a}$$

gives us the module of Kähler differentials for T -algebra (A, a) .

This result was in the M.Sc. thesis of the third author [21]. We do not give a proof of this result here as it can be obtained in a method similar to Theorem 5.23. We also note that, under suitable hypotheses, the existence of Kähler modules for arbitrary commutative algebras follows from Theorem 5.23.

4. The symmetric algebra monad

The most canonical example of an algebra modality is the symmetric algebra construction. This construction as applied to the category of vector spaces gives one of the most basic examples of a codifferential category. In this case, elements of the symmetric algebra are essentially polynomials, which are differentiated in the evident way. A similar construction works on the

category of sets and relations [23]. What we observe here is that the symmetric algebra construction provides examples of codifferential categories in a much more general setting.

First, we need to explore a theme which will be the centrepiece of the last sections of the paper. This is the idea of viewing derivations as algebra homomorphisms.

Remark 4.1. *For the remainder of this section, we assume \mathcal{C} is an additive symmetric monoidal category with finite coproducts and reflexive coequalizers, the latter of which are preserved by the tensor product in each variable. Let Alg be the category of commutative algebras in \mathcal{C} , and suppose that the forgetful functor $\text{Alg} \rightarrow \mathcal{C}$ has a left adjoint. The resulting adjunction is then monadic; denote its induced monad by S , so that $\text{Alg} \cong \mathcal{C}^S$, and we henceforth identify these categories. See [19] for details.*

4.1 Structure related to the symmetric algebra

We will also need the following straightforward observation:

Proposition 4.2. *The (commutative) algebra modalities on \mathcal{C} are in bijective correspondence to pairs (T, ψ) , where T is a monad and ψ is a monad morphism $\psi: S \rightarrow T$. Such a morphism induces a functor*

$$F_\psi: T\text{-Alg} \rightarrow S\text{-Alg}$$

Furthermore, the map $\psi_C: SC \rightarrow TC$ is a map of algebras.

4.2 Codifferential structure

Definition 4.3. Given an object C in \mathcal{C} , recall that $SC \otimes C$ is the free SC -module on C . Hence by Remark 3.6, the direct sum $SC \oplus (SC \otimes C)$ carries the structure of an algebra, and derivations $SC \rightarrow (SC \otimes C)$ correspond to algebra homomorphisms $SC \rightarrow SC \oplus (SC \otimes C)$ whose first coordinate is the identity. But since SC is the free algebra on C , the latter correspond to morphisms $C \rightarrow SC \oplus (SC \otimes C)$ whose first coordinate is $\eta: C \rightarrow SC$.

So let $d_{SC}: SC \rightarrow SC \otimes C$ be the derivation corresponding to the algebra homomorphism $SC \rightarrow SC \oplus (SC \otimes C)$ given on generators as

$\begin{pmatrix} \eta_C \\ u_C \end{pmatrix} : C \longrightarrow SC \oplus (SC \otimes C)$, where u_C is the map $u_C : C \cong k \otimes C \xrightarrow{e \otimes 1} SC \otimes C$.

Theorem 4.4. (\mathcal{C}, S, d) is a codifferential category.

Proof. S is a commutative algebra modality on \mathcal{C} . Since each d_{SC} is by definition a derivation, the Leibniz rule holds and precomposing d_{SC} by e_{SC} is the zero map. By the definition of d_{SC} ,

$$\eta_C; \begin{pmatrix} 1_{SC} \\ d_{SC} \end{pmatrix} = \begin{pmatrix} \eta_C \\ u_C \end{pmatrix} : C \longrightarrow SC \oplus (SC \otimes C)$$

so that

$$\eta_C; d_{SC} = \eta_C; \begin{pmatrix} 1_{SC} \\ d_{SC} \end{pmatrix}; \pi_2 = \begin{pmatrix} \eta_C \\ u_C \end{pmatrix}; \pi_2 = u_C$$

and consequently (d3) holds.

It remains only to demonstrate naturality of d and adherence to the chain rule condition. For naturality, consider a map $f : C \longrightarrow D$ in \mathcal{C} ; naturality of d is equivalent to the commutativity of the following square:

$$\begin{array}{ccc} SC & \xrightarrow{\begin{pmatrix} 1 \\ d_{SC} \end{pmatrix}} & SC \oplus (SC \otimes C) \\ sf \downarrow & & \downarrow sf \oplus (sf \otimes f) \\ SD & \xrightarrow{\begin{pmatrix} 1 \\ d_{SD} \end{pmatrix}} & SD \oplus (SD \otimes D) \end{array}$$

Since each morphism in the square is an algebra morphism, commutativity of this square may be demonstrated by showing that the square is commutative when preceded by $\eta_C : C \longrightarrow SC$. By naturality of η and definition of d_D we have on the left:

$$\eta_C; Sf; \begin{pmatrix} 1 \\ d_{SD} \end{pmatrix} = f; \eta_D; \begin{pmatrix} 1 \\ d_{SD} \end{pmatrix} = f; \begin{pmatrix} \eta_D \\ u_D \end{pmatrix}$$

By naturality of η and u and by definition of d we have on the right:

$$\begin{aligned} \eta_C; \begin{pmatrix} 1 \\ d_{SC} \end{pmatrix}; Sf \oplus (Sf \otimes f) &= \begin{pmatrix} \eta_C \\ u_{SC} \end{pmatrix}; Sf \oplus (Sf \otimes f) \\ &= \begin{pmatrix} \eta_C; Sf \\ u_{SC}; Sf \otimes f \end{pmatrix} = f; \begin{pmatrix} \eta_D \\ u_D \end{pmatrix} \end{aligned}$$

and so naturality of d is established.

To show that d adheres to the chain rule, it is necessary and sufficient to show that the following square commutes

$$\begin{array}{ccc} S^2C & \xrightarrow{\mu_C} & SC \\ \downarrow d_{S^2C} & & \downarrow d_{SC} \\ S^2C \otimes SC & \xrightarrow{\mu_C \otimes d_{SC}} SC \otimes SC \otimes C \xrightarrow{m_{SC} \otimes 1} & SC \otimes C \end{array}$$

When preceded by η_{SC} , commutativity of the resultant diagram is established by a routine verification. In order to show that this verification suffices, it must be shown that both paths in the above diagram yield derivations when preceded by η_{SC} ; the correspondence between derivations and morphisms of algebras then enables the utilization of the universal property of η to deduce that the associated morphisms of algebras are equal.

Since μ_C is an associative algebra homomorphism, $\mu_C; d_{SC}$ is a derivation with respect to the S^2C -module structure that $SC \otimes C$ acquires by restriction of scalars along μ_C . As for the counterclockwise composite, the following computation demonstrates that it adheres to the Leibniz rule

$$\begin{aligned} & m_{S^2C}; d_{S^2C}; \mu_C \otimes d_{SC}; m_{SC} \otimes 1 \\ &= (1 \otimes d_{S^2C} + c; 1 \otimes d_{S^2C}); m_{S^2C} \otimes 1; \mu_C \otimes d_{SC}; m_{SC} \otimes 1 \\ &= (1 \otimes d_{S^2C} + c; 1 \otimes d_{S^2C}); \mu_C \otimes \mu_C \otimes 1; m_{SC} \otimes 1; 1 \otimes d_{SC}; m_{SC} \otimes 1 \\ &= (1 \otimes (d_{S^2C}; \mu_C \otimes d_{SC}) + c; 1 \otimes (d_{S^2C}; \mu_C \otimes d_{SC})); \mu_C \otimes 1 \otimes 1 \otimes 1; \\ & \quad m_{SC} \otimes 1 \otimes 1; m_{SC} \otimes 1 \\ &= (1 \otimes (d_{S^2C}; \mu_C \otimes d_{SC}; m_{SC} \otimes 1) + c; 1 \otimes (d_{S^2C}; \mu_C \otimes d_{SC}; m_{SC} \otimes 1)); \\ & \quad \mu_C \otimes 1 \otimes 1; m_{SC} \otimes 1 \end{aligned}$$

That the counterclockwise composite is 0 when preceded by e_{S^2C} is immediate, and the proof is complete. \square

5. Beck T -derivations

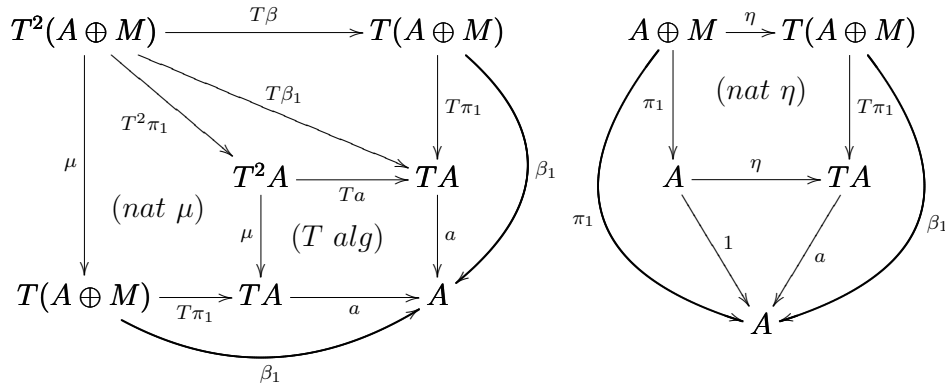
We now explore what we consider to be the main contribution of this paper. The first step in this project is the following theorem, due to the second author. It lifts the correspondence between derivations and algebra homomorphisms to the level of T -algebras. Throughout this section, we assume that \mathcal{C} has finite coproducts.

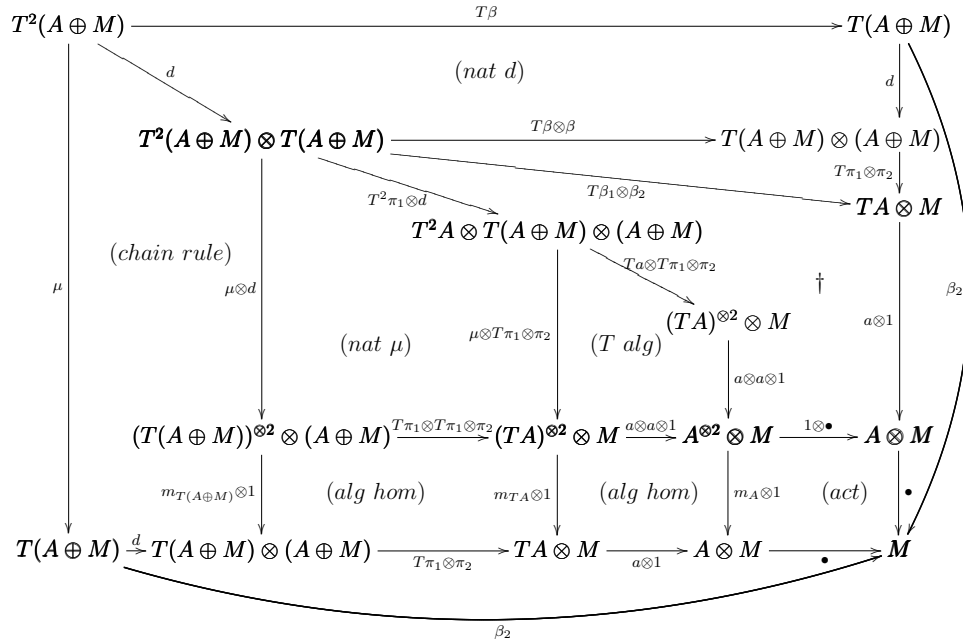
Theorem 5.1. *Let \mathcal{C} be a codifferential category with finite coproducts. Let (A, a) be a T -algebra and M a module over its associated algebra. Then $(A \oplus M, \beta)$ is a T -algebra with $\beta: T(A \oplus M) \rightarrow A \oplus M$ induced by the following maps.*

$$\beta_1: T(A \oplus M) \xrightarrow{T\pi_1} TA \xrightarrow{a} A$$

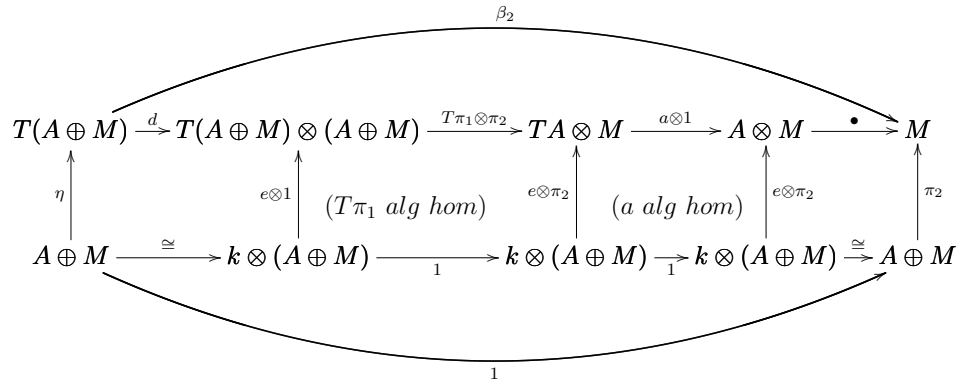
$$\beta_2: T(A \oplus M) \xrightarrow{d} T(A \oplus M) \otimes (A \oplus M) \xrightarrow{T(\pi_1) \otimes \pi_2} T(A) \otimes M \xrightarrow{a \otimes 1} A \otimes M \xrightarrow{\bullet} M$$

Proof. The following four diagrams capture all of the necessary equations.





In the third diagram, the cell marked \dagger commutes by the definitions of β_1 and β_2 .



□

Definition 5.2. We denote this T -algebra by $W(A, M) = \langle A \oplus M, \beta^{AM} \rangle$.

The following result is straightforward.

Lemma 5.3. *Let (A, a) be a T -algebra, and let M be an A -module. Then $\pi_1 : A \oplus M \rightarrow A$ is a map of T -algebras, where $A \oplus M$ is given the T -algebra structure just defined.*

We also note that the algebra associated to this T -algebra under the process of Theorem 3.12 coincides with the algebra structure associated to $A \oplus M$ in Remark 3.6.

Proposition 5.4. *Let (A, a) be a T -algebra in \mathcal{C} and let M be an A -module. Then the commutative algebra structure carried by the T -algebra $A \oplus M$ coincides with the commutative algebra structure on $A \oplus M$ described in Remark 3.6.*

Proof. Since β^{AM} is an algebra homomorphism the multiplication associated to $W(A, M)$ is

$$m_{W(A, M)} = \eta_{A \oplus M} \otimes \eta_{A \oplus M}; m_{T(A \oplus M)}; \beta^{AM}$$

Since $\pi_1 : W(A, M) \rightarrow A$ is a T -homomorphism and hence an algebra homomorphism, $m_{W(A, M)}; \pi_1 = \pi_1 \otimes \pi_1; m_A$ and so the first component of $m_{W(A, M)}$ is given as in Remark 3.6.

The second component is the composite

$$\eta \otimes \eta; m_{T(A \oplus M)}; d_{T(A \oplus M)}; T\pi_1 \otimes \pi_2; a \otimes 1; \bullet$$

Calculate as follows:

$$\begin{aligned} & \eta_{A \oplus M} \otimes \eta_{A \oplus M}; m_{T(A \oplus M)}; d_{T(A \oplus M)}; T\pi_1 \otimes \pi_2; a \otimes 1; \bullet \\ &= \eta_{A \oplus M} \otimes \eta_{A \oplus M}; (1 \otimes d_{T(A \oplus M)} + c; 1 \otimes d_{T(A \oplus M)}); \\ & \quad m_{T(A \oplus M)} \otimes 1; S\pi_1 \otimes \pi_2; a \otimes 1; \bullet \\ &= (\eta_{A \oplus M} \otimes (\eta_{A \oplus M}; d_{T(A \oplus M)}) + c; \eta_{A \oplus M} \otimes (\eta_{A \oplus M}; d_{A \oplus M})); \\ & \quad (T\pi_1; a) \otimes (T\pi_1; a) \otimes \pi_2; m_A \otimes 1; \bullet \\ &= (1 \otimes (\eta_{A \oplus M}; d_{A \oplus M}) + c; 1 \otimes (\eta_{A \oplus M}; d_{A \oplus M})); \\ & \quad (\pi_1; \eta_A; a) \otimes (T\pi_1; a) \otimes \pi_2; m_A \otimes 1; \bullet \\ &= (1 + c); 1 \otimes e_{A \oplus M} \otimes 1; \pi_1 \otimes (T\pi_1; a) \otimes \pi_2; m_A \otimes 1; \bullet \\ &= (1 + c); \pi_1 \otimes \pi_2; 1 \otimes e_A \otimes 1; m_A \otimes 1; \bullet \\ &= (1 + c); \pi_1 \otimes \pi_2; \bullet \end{aligned}$$

□

We will need the following technical lemmas concerning the T -algebra $W(A, M)$.

Lemma 5.5. *Let (A, a) be a T -algebra, and let M and N be A -modules. Suppose $h: M \rightarrow N$ is an A -module map. Then $A \oplus h: A \oplus M \rightarrow A \oplus N$ is a T -algebra map $W(A, M) \rightarrow W(A, N)$.*

Proof. The result follows from the commutativity of the following two diagrams.

$$\begin{array}{ccc}
 T(A \oplus M) & \xrightarrow{T(1_A \oplus h)} & T(A \oplus N) \\
 \downarrow T\pi_1 & & \downarrow T\pi_1 \\
 TA & \xrightarrow{1_{TA}} & TA \\
 \downarrow a & & \downarrow a \\
 A & \xrightarrow{1_A} & A
 \end{array}
 \begin{array}{l}
 \beta_1^{AM} \curvearrowright \\
 \beta_1^{AN} \curvearrowleft
 \end{array}$$

$$\begin{array}{ccc}
 T(A \oplus M) & \xrightarrow{T(1_A \oplus h)} & T(A \oplus N) \\
 \downarrow d & & \downarrow d \\
 T(A \oplus M) \otimes (A \oplus M) & \xrightarrow{T(1_A \oplus h) \otimes (1_A \oplus h)} & T(A \oplus N) \otimes (A \oplus N) \\
 \downarrow T\pi_1 \otimes \pi_2 & & \downarrow T\pi_1 \otimes \pi_2 \\
 TA \otimes M & \xrightarrow{1_{TA} \otimes h} & TA \otimes N \\
 \downarrow a \otimes 1_M & & \downarrow a \otimes 1_N \\
 A \otimes M & \xrightarrow{1_A \otimes h} & A \otimes N \\
 \downarrow \bullet_M & & \downarrow \bullet_N \\
 M & \xrightarrow{h} & N
 \end{array}
 \begin{array}{l}
 \beta_2^{AM} \curvearrowright \\
 \beta_2^{AN} \curvearrowleft
 \end{array}$$

□

The above calculations allow us to conclude:

Proposition 5.6. *Given a T -algebra A , the above construction defines a functor:*

$$W(A, -): A\text{-Mod} \longrightarrow \mathcal{C}^T/A$$

Here, \mathcal{C}^T is the category of T -algebras and \mathcal{C}^T/A is the slice category over A .

It is the above series of observations that allows us to define a generalized notion of derivation depending on the given codifferential structure of \mathcal{C} .

Definition 5.7.

- Let (A, a) be a T -algebra. Let M be an A -module. A *Beck T -derivation* for A valued in M is a T -algebra map

$$A \longrightarrow W(A, M) \quad \text{in } \mathcal{C}^T/A$$

in the slice category \mathcal{C}^T/A .

- A *T -derivation* is a morphism $\partial: A \rightarrow M$ such that

$$\langle 1, \partial \rangle: A \longrightarrow A \oplus M$$

is a T -algebra homomorphism $A \rightarrow W(A, M)$.

Remark 5.8. Under the assumptions of Remark 3.1, suppose we are given $A \in \mathcal{C}^S$ where S is the symmetric algebra monad and $M \in A\text{-Mod}$. Then a morphism $\partial: A \rightarrow M$ in \mathcal{C} is an S -derivation if and only if ∂ is a derivation.

Remark 5.9. Evidently, the two notions of *Beck T -derivation* and *T -derivation* are in bijective correspondence and we will use the two interchangeably.

We now give several equations for a map $\partial: A \rightarrow M$ which are equivalent to ∂ being a T -derivation.

Proposition 5.10. *Let (A, a) be a T -algebra, and let M be an A -module. A morphism $\partial: A \rightarrow M$ is a T -derivation if and only if the following diagram*

commutes.

$$\begin{array}{ccc}
 TA & \xrightarrow{T\begin{pmatrix} 1_A \\ \partial \end{pmatrix}} & T(A \oplus M) \\
 a \downarrow & & \downarrow \beta_2 \\
 A & \xrightarrow{\partial} & M
 \end{array}$$

Proof. Since $A \oplus M$ is a product, the requirement that $\langle 1_A, \partial \rangle: A \rightarrow A \oplus M$ be a T -algebra homomorphism amounts to two equations, the second of which is expressed by the above diagram whereas the first commutes by the following calculation

$$\begin{array}{ccc}
 TA & \xrightarrow{T\begin{pmatrix} 1_A \\ \partial \end{pmatrix}} & T(A \oplus M) \\
 a \downarrow & \searrow 1_{TA} & \downarrow T\pi_1 \\
 & & TA \\
 & & \downarrow a \\
 A & \xrightarrow{1_A} & A
 \end{array}$$

□

Theorem 5.11. *Let (A, a) be a T -algebra, and let M be an A -module. A morphism $\partial: A \rightarrow M$ is a T -derivation if and only if*

$$\begin{array}{ccccc}
 TA & \xrightarrow{a} & & & A \\
 d \downarrow & & & & \downarrow \partial \\
 TA \otimes A & \xrightarrow{a \otimes \partial} & A \otimes M & \xrightarrow{\bullet_M} & M
 \end{array}$$

commutes.

Proof. Calculate as follows:

$$\begin{array}{ccccc}
 TA & \xrightarrow{d} & TA \otimes A & & \\
 \downarrow T\left(\begin{smallmatrix} 1_A \\ \partial \end{smallmatrix}\right) & & \downarrow T\left(\begin{smallmatrix} 1_A \\ \partial \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1_A \\ \partial \end{smallmatrix}\right) & & \\
 T(A \oplus M) & \xrightarrow{d} & T(A \oplus M) \otimes (A \oplus M) & \xrightarrow{T\pi_1 \otimes \pi_2} & TA \otimes M \xrightarrow{a \otimes 1_M} A \otimes M \xrightarrow{\bullet_M} M \\
 & & & & \nearrow \beta_2
 \end{array}$$

$\begin{array}{l} \searrow a \otimes \partial \\ \searrow 1_{TA} \otimes \partial \end{array}$

Thus the result follows from the previous proposition. \square

Whereas we have defined the notion of T -derivation in the setting of a given codifferential category, Theorem 5.11 furnishes an equivalent definition that is applicable more generally, as follows.

Definition 5.12. Let \mathcal{C} be a symmetric monoidal category equipped with an algebra modality T and arbitrary morphisms $d_{TC} : TC \rightarrow TC \otimes C$ ($C \in \mathcal{C}$). Given a T -algebra A and an A -module M , a T -derivation is a morphism $\partial : A \rightarrow M$ such that the diagram of Theorem 5.11 commutes.

The new understanding of derivations captured by the above propositions allows us, among other things, to reexamine the definition of (co)differential categories, as seen by the following:

Theorem 5.13. Let \mathcal{C} be a symmetric monoidal category equipped with an algebra modality T and arbitrary morphisms $d_{TC} : TC \rightarrow TC \otimes C$ ($C \in \mathcal{C}$). The Chain Rule equation for d in the definition of codifferential category is equivalent to the statement that each component d_{TC} is a T -derivation, where $TC \otimes C$ is viewed as the free TC -module generated by C .

Proof.

$$\begin{array}{ccccc}
 T^2C & \xrightarrow{\mu} & TC & & \\
 d_{T^2C} \downarrow & & \downarrow d_{TC} & & \\
 T^2C \otimes TC & \xrightarrow{\mu \otimes d_{TC}} & TC \otimes TC \otimes C & \xrightarrow{m_{TC} \otimes 1_C} & TC \otimes C
 \end{array}$$

This equation is both the chain rule and the statement that d_{TC} is a derivation.

□

5.1 Universal Beck T -derivations

Definition 5.14. Given a T -algebra A , a *module of Kähler T -differentials* is an A -module, denoted Ω_A^T , equipped with a universal T -derivation on A . This can be expressed in either of the following two equivalent ways:

- A T -derivation $d: A \rightarrow \Omega_A^T$ such that for all T -derivations $\partial: A \rightarrow M$, there is a unique A -linear map $\hat{\partial}: \Omega_A^T \rightarrow M$ such that $d; \hat{\partial} = \partial$.
- A morphism $g: A \rightarrow W(A, \Omega_A^T)$ in \mathcal{C}^T/A such that for each map $\partial: A \rightarrow W(A, M)$ in \mathcal{C}^T/A , there is a unique A -linear homomorphism $\hat{\partial}: \Omega_A^T \rightarrow M$ such that $g; W(A, \hat{\partial}) = \partial$.

We now explore the existence of universal derivations from this new T -perspective.

Theorem 5.15. *Let \mathcal{C} be a codifferential category, and let C be an object of \mathcal{C} . Then $d_{TC}: TC \rightarrow T(C) \otimes C$ is a universal T -derivation.*

Proof. Since d_{TC} satisfies the chain rule, it is a T -derivation. Since $T(C) \otimes C$ is the free $T(C)$ -module on C , given any T -derivation $\partial: T(C) \rightarrow M$ there exists a unique $T(C)$ -linear morphism $\partial^\#: TC \otimes C \rightarrow M$ such that $u_C^{T(C)}; \partial^\# = \eta_C; \partial$. Hence by axiom (d3), the two morphisms from C to M in the following diagram are equal:

$$\begin{array}{ccccc}
 C & \xrightarrow{\eta_C} & T(C) & \xrightarrow{d_{TC}} & T(C) \otimes C \\
 & & & \searrow \partial & \downarrow \partial^\# \\
 & & & & M
 \end{array}$$

Equivalently,

$$\begin{array}{ccc}
 TC & \xrightarrow{\begin{pmatrix} 1 \\ d_{TC} \end{pmatrix}} & W(TC, TC \otimes C) \\
 & \searrow \begin{pmatrix} 1 \\ \partial \end{pmatrix} & \downarrow W(TC, \partial^\#) \\
 & & W(TC, M)
 \end{array}$$

commutes when preceded by η_C . Since this is a diagram of T -algebra homomorphisms, it commutes if and only if it commutes when preceded by η_C

□

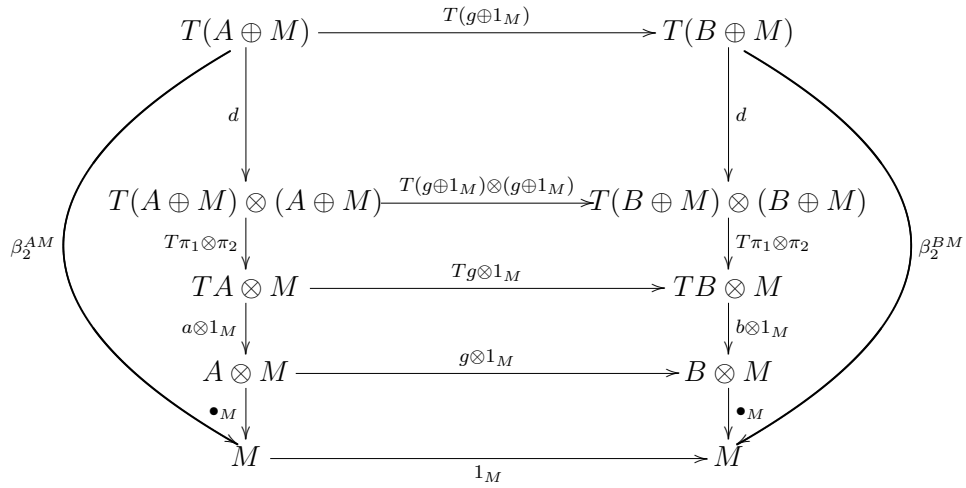
We now address the issue of extending the existence of universal T -derivations to arbitrary T -algebras.

Proposition 5.16. *Let (A, a) and (B, b) be T -algebras and M a B -module. Let $g: A \rightarrow B$ be a T -algebra homomorphism. Then $g \oplus M: A \oplus M \rightarrow B \oplus M$ is a map of T -algebras $W(A, M_A) \rightarrow W(B, M)$, where M_A is M with evident induced action of A .*

Proof. The result follows from the commutativity of the following two diagrams.

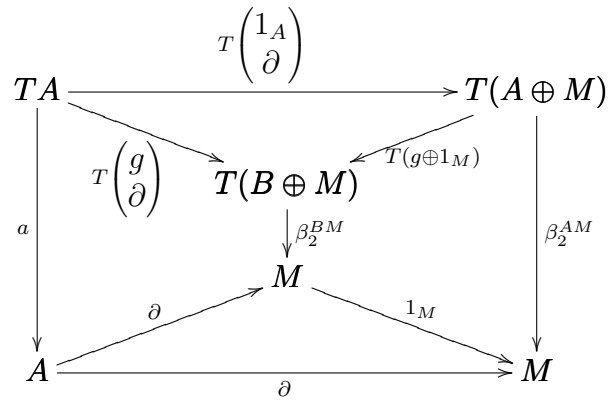
$$\begin{array}{ccc}
 T(A \oplus M) & \xrightarrow{T(g \oplus 1_M)} & T(B \oplus M) \\
 \downarrow T\pi_1 & & \downarrow T\pi_1 \\
 TA & \xrightarrow{Tg} & TB \\
 \downarrow a & & \downarrow b \\
 A & \xrightarrow{g} & B
 \end{array}$$

β_1^{AM} (left curved arrow) β_1^{BM} (right curved arrow)



Proposition 5.17. *With assumptions as in previous proposition, let $\partial: A \rightarrow M$ be such that $\langle g, \partial \rangle: A \rightarrow W(B, M)$ is a map of T -algebras. Then $\partial: A \rightarrow M_A$ is a T -derivation.*

Proof. This follows from the following calculation, which uses that $g \oplus 1_M$ is a T -algebra homomorphism by the previous proposition.



Definition 5.18. Let Alg be the category of commutative algebras in a codifferential category \mathcal{C} and let $(-)-Mod: Alg^{op} \rightarrow Cat$ be the usual functor associating to an algebra its category of representations. The functor acts on morphisms by the usual restriction of scalars.

Composing with the functor $F^{op}: (\mathcal{C}^T)^{op} \rightarrow Alg^{op}$ we obtain a functor $H: \mathcal{C}^{T^{op}} \rightarrow Cat$. When we apply the usual Grothendieck construction to this functor, we obtain a category fibred over \mathcal{C}^T which we call Mod_T . Objects are pairs (A, M) with A a T -algebra and M an A -module. Arrows are pairs $(g, h): (A, M) \rightarrow (B, N)$ with $g: A \rightarrow B$ a T -algebra map and $h: M \rightarrow N_A$ a map of A -modules. Here N_A is the restriction of scalars of N along g (Remark 3.6).

Theorem 5.19. *There is a functor $W: Mod_T \rightarrow (\mathcal{C}^T)^\rightarrow$ that makes the following diagram commute:*

$$\begin{array}{ccc} Mod_T & \xrightarrow{W} & (\mathcal{C}^T)^\rightarrow \\ & \searrow & \swarrow \text{cod} \\ & \mathcal{C}^T & \end{array}$$

The functor is defined by:

On objects: $(A, M) \mapsto [W(A, M) \xrightarrow{\pi_1} A]$

On arrows: $(A, M) \xrightarrow{(g,h)} (B, N) \mapsto$ the following:

$$\begin{array}{ccc} W(A, M) & \xrightarrow{W(h,g):=g \oplus h} & W(B, N) \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ A & \xrightarrow{g} & B \end{array}$$

This functor is fibred over the base category \mathcal{C}^T .

Proof. We evidently have that $(1 \oplus h); (g \oplus 1) = g \oplus h$ is a map of T -algebras by Lemma 5.5 and Proposition 5.16, and so we have a functor making the triangle commute.

Now given a T -algebra homomorphism $g: A \rightarrow B$ and a B -module N , we get a cartesian arrow over g in Mod_T as $(g, 1_N): (A, N_A) \rightarrow (B, N)$. It suffices to show that

$$\begin{array}{ccc} W(A, N_A) & \xrightarrow{W(g, 1_N)} & W(B, N) \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ A & \xrightarrow{g} & B \end{array}$$

is a pullback. Given $f: Q \rightarrow A$ and $q: Q \rightarrow W(B, N)$ in \mathcal{C}^T such that $f; g = q; \pi_1$, we find that $q = \langle f; g, \partial \rangle$ for some $\partial: Q \rightarrow N$. By Lemma 5.17, we conclude $\partial: Q \rightarrow N_Q$ is a T -derivation. So $\langle 1_Q, \partial \rangle$ is a T -algebra map and thus $\langle 1_Q, \partial \rangle; f \oplus 1 = \langle f, \partial \rangle: Q \rightarrow W(A, N_A)$ is a T -algebra map. The result now follows. \square

Definition 5.20. Let A be a T -algebra and (B, M) in Mod_T . Let $Der(A, (B, M))$ be the set of all pairs (g, ∂) with $g: A \rightarrow B$ a T -algebra map and $\partial: A \rightarrow M_A$ a T -derivation.

We now record two related results which are straightforward.

Proposition 5.21. *The operation Der of the previous definition is functorial in both variables and forms part of a natural isomorphism:*

$$\mathcal{C}^T(A, W(B, M)) \cong Der(A, (B, M))$$

This result extends to the slice category in a straightforward way.

Proposition 5.22. *Given a T -algebra map $g: A \rightarrow B$, we have the following natural isomorphism:*

$$\mathcal{C}^T/B(A, W(B, M)) \cong Der(A, M_A)$$

We now present the main result of the section, demonstrating that the construction of Kähler modules for T -algebras lifts to the setting of T -derivations.

Theorem 5.23. *Suppose \mathcal{C} has reflexive coequalizers, and that these are preserved by \otimes in each variable. Then every T -algebra (A, a) has a universal T -derivation.*

Proof. Let $g : A \rightarrow B$ be a morphism of T -algebras, and suppose that universal T -derivations $d_A : A \rightarrow \Omega_A^T, d_B : B \rightarrow \Omega_B^T$ exist. Then there is a unique A -linear morphism Ω_g^T such that

$$\begin{array}{ccc} \Omega_A^T & \xrightarrow{\Omega_g^T} & \Omega_B^T \\ d_A \uparrow & & \uparrow d_B \\ A & \xrightarrow{g} & B \end{array}$$

commutes, where Ω_B^T is considered as an A -module by restriction of scalars along g . This follows from the observation that $g; d_B : A \rightarrow \Omega_B^T$ is a T -derivation.

Lemma 5.24. *Suppose we are given morphisms in the category Alg as follows which constitute a reflexive coequalizer in \mathcal{C}*

$$A_1 \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} \rightrightarrows A_2 \xrightarrow{k} A_3$$

Let M_i be an A_i -module for $i = 1, 2$, and let $\phi : M_1 \rightarrow f^(M_2)$ and $\gamma : M_1 \rightarrow g^*(M_2)$ be A_1 -linear, where $f^*(M_2)$ and $g^*(M_2)$ denote M_2 equipped with the A_1 -module structures induced by f and g , respectively. Suppose*

$$M_1 \begin{array}{c} \xrightarrow{\phi} \\ \xrightarrow{\gamma} \end{array} \rightrightarrows M_2 \xrightarrow{\kappa} M_3$$

is a reflexive coequalizer in \mathcal{C} . Then there is a unique A_3 -module structure on M_3 such that $\kappa : M_2 \rightarrow k^(M_3)$ is A_2 -linear.*

Proof. Since \otimes preserves reflexive coequalizers, the rows and columns of

the following diagram are reflexive coequalizers:

$$\begin{array}{ccccc}
 A_1 \otimes M_1 & \begin{array}{c} \xrightarrow{1 \otimes \phi} \\ \xrightarrow{1 \otimes \gamma} \end{array} & A_1 \otimes M_2 & \xrightarrow{1 \otimes \kappa} & A_1 \otimes M_3 \\
 \begin{array}{c} \Downarrow \\ g \otimes 1 \quad f \otimes 1 \\ \Downarrow \end{array} & & \begin{array}{c} \Downarrow \\ g \otimes 1 \quad f \otimes 1 \\ \Downarrow \end{array} & & \begin{array}{c} \Downarrow \\ g \otimes 1 \quad f \otimes 1 \\ \Downarrow \end{array} \\
 A_2 \otimes M_1 & \begin{array}{c} \xrightarrow{1 \otimes \phi} \\ \xrightarrow{1 \otimes \gamma} \end{array} & A_2 \otimes M_2 & \xrightarrow{1 \otimes \kappa} & A_2 \otimes M_3 \\
 \downarrow k \otimes 1 & & \downarrow k \otimes 1 & & \downarrow k \otimes 1 \\
 A_3 \otimes M_1 & \begin{array}{c} \xrightarrow{1 \otimes \phi} \\ \xrightarrow{1 \otimes \gamma} \end{array} & A_3 \otimes M_2 & \xrightarrow{1 \otimes \kappa} & A_3 \otimes M_3
 \end{array}$$

By Johnstone's lemma, Lemma 0.17, p. 4 [16], it follows that the top row of

$$\begin{array}{ccccc}
 A_1 \otimes M_1 & \begin{array}{c} \xrightarrow{f \otimes \phi} \\ \xrightarrow{g \otimes \gamma} \end{array} & A_2 \otimes M_2 & \xrightarrow{k \otimes \kappa} & A_3 \otimes M_3 \\
 \bullet_1 \downarrow & & \bullet_2 \downarrow & & \bullet_3 \downarrow \\
 M_1 & \begin{array}{c} \xrightarrow{\phi} \\ \xrightarrow{\gamma} \end{array} & M_2 & \xrightarrow{\kappa} & M_3
 \end{array}$$

is also a reflexive coequalizer. We have that

$$\begin{aligned}
 f \otimes \phi; \bullet_2; \kappa &= 1_{A_1} \otimes \phi; f \otimes 1_{M_2}; \bullet_2; \kappa \\
 &= \bullet_1; \phi; \kappa \\
 &= \bullet_1; \gamma; \kappa \\
 &= 1_{A_1} \otimes \gamma; g \otimes 1_{M_2}; \bullet_2; \kappa \\
 &= g \otimes \gamma; \bullet_2; \kappa
 \end{aligned}$$

It follows that $\bullet_3 : A_3 \otimes M_3 \rightarrow M_3$ is constructed as the unique map making the right-hand square in the above diagram commute. Hence it suffices to

show that \bullet_3 is an A_3 -module structure map on M_3 . Again using Johnstone's Lemma, the top row of the following diagram is a reflexive coequalizer

$$\begin{array}{ccccc}
 A_1 \otimes A_1 \otimes M_1 & \xrightarrow[\text{\scriptsize } g \otimes g \otimes \gamma]{\text{\scriptsize } f \otimes f \otimes \phi} & A_2 \otimes A_2 \otimes M_2 & \xrightarrow{\text{\scriptsize } k \otimes k \otimes \kappa} & A_3 \otimes A_3 \otimes M_3 \\
 \downarrow \text{\scriptsize } 1 \otimes \bullet_1; \bullet_1 = m_{A_1} \otimes 1; \bullet_1 & & \downarrow \text{\scriptsize } 1 \otimes \bullet_2; \bullet_2 = m_{A_2} \otimes 1; \bullet_2 & & \downarrow \text{\scriptsize } \bullet_3 \\
 M_1 & \xrightarrow[\text{\scriptsize } \gamma]{\text{\scriptsize } \phi} & M_2 & \xrightarrow{\text{\scriptsize } \kappa} & M_3
 \end{array}$$

It follows that there is a unique map $A_3 \otimes A_3 \otimes M_3 \rightarrow M_3$ making the right-hand square commute. Since both $1_{A_3} \otimes \bullet_3; \bullet_3$ and $m_{A_3} \otimes 1_{M_3}; \bullet_3$ satisfy this, the result follows. \square

Continuing with the proof of our theorem, since μ_A and Ta are T -algebra morphisms, they induce maps Ω_μ^T and Ω_{Ta}^T from $\Omega_{T^2A}^T$ to Ω_{TA}^T , which exist by Theorem 4.14. Furthermore, there exists a map $\Omega_{T\eta}^T$ induced by $T\eta$, which splits both of these maps. Consider the following diagram. We define d_A as the unique morphism in \mathcal{C} such that $a; d_A = d_{TA}; \Omega_a^T$, which exists since a is the coequalizer of μ and Ta . Here we take $\Omega_a^T : \Omega_{TA}^T \rightarrow \Omega_A^T$ to be the coequalizer.

$$\begin{array}{ccccc}
 \Omega_{T^2A}^T & \xrightarrow[\text{\scriptsize } \Omega_{Ta}^T]{\text{\scriptsize } \Omega_\mu^T} & \Omega_{TA}^T & \xrightarrow{\text{\scriptsize } \Omega_a^T} & \Omega_A^T \\
 \uparrow \text{\scriptsize } d_{T^2A} & & \uparrow \text{\scriptsize } d_{TA} & & \uparrow \text{\scriptsize } d_A \\
 T^2A & \xrightarrow[\text{\scriptsize } Ta]{\text{\scriptsize } \mu} & TA & \xrightarrow{\text{\scriptsize } a} & A
 \end{array}$$

One readily verifies that the preceding lemma applies so that Ω_A^T is equipped with an A -module structure, which makes Ω_a^T TA -linear. We find that $d_A = \eta_A; d_{TA}; \Omega_a^T$ since $a; \eta_A; d_{TA}; \Omega_a^T = d_{TA}; \Omega_a^T = a; d_A$, where the first equation is established through a short computation using the fact that $a; \eta_A = \eta_{TA}; Ta$.

Since

$$\begin{array}{ccc}
 TA & \xrightarrow{a} & A \\
 \left(\begin{array}{c} 1_{TA} \\ d_{TA} \end{array} \right) \downarrow & & \downarrow \left(\begin{array}{c} 1_A \\ d_A \end{array} \right) \\
 W(TA, \Omega_{TA}^T) & \xrightarrow{W(a, \Omega_a^T)} & W(A, \Omega_A^T)
 \end{array}$$

commutes, it follows that the right-hand map is a T -algebra homomorphism and therefore that d_A is a T -derivation. Indeed, the counterclockwise composite is evidently a T -algebra homomorphism, and since a is a T -algebra homomorphism that is split epi in \mathcal{C} , the fact that the right-hand map is a T -algebra homomorphism follows readily.

Now suppose that $\partial : A \rightarrow M$ is a T -derivation. Then $a; \partial$ is a T -derivation, which must factor through d_{TA} via a morphism of TA -modules ∂' . Since

$$\begin{aligned}
 d_{T^2A}; \Omega_{Ta}^T; \partial' &= Ta; d_{TA}; \partial' \\
 &= Ta; a; \partial \\
 &= \mu; a; \partial \\
 &= \mu; d_{TA}; \partial' \\
 &= d_{T^2A}; \Omega_\mu^T; \partial'
 \end{aligned}$$

it follows from the universal property of d_{T^2A} that $\Omega_{Ta}^T; \partial' = \Omega_\mu^T; \partial'$, so that ∂' factors uniquely through Ω_a^T via a map $\partial^\# : \Omega_a^T \rightarrow M$. Since $a \otimes \Omega_a^T$ is a coequalizer, the following computation shows that this map is A -linear:

$$\begin{aligned}
 a \otimes \Omega_a^T; \bullet_A; \partial^\# &= \bullet_{TA}; \Omega_a^T; \partial^\# \\
 &= \bullet_{TA}; \partial' \\
 &= 1_{TA} \otimes \partial'; a \otimes 1_A; \bullet_A \\
 &= a \otimes \Omega_a^T; 1_A \otimes \partial^\#; \bullet_A
 \end{aligned}$$

Finally, we show that $\partial^\#$ is the unique A -linear morphism, which makes

$$\begin{array}{ccc}
 A & \xrightarrow{d_A} & \Omega_A \\
 & \searrow \partial & \downarrow \partial^\# \\
 & & M
 \end{array}$$

commute. First, observe that $a; d_A; \partial^\# = d_{TA}; \Omega_a^T; \partial^\# = d_{TA}; \partial' = a; \partial$ so that this does indeed commute after cancellation of a . Now suppose that there exists another A -linear map $k : \Omega_A^T \rightarrow M$ such that $d_A; k = \partial$. Then

$$\begin{aligned} d_{TA}; \Omega_a^T; k &= a; d_A; k \\ &= a; \partial \\ &= d_{TA}; \partial' \\ &= d_{TA}; \Omega_a^T; \partial^\# \end{aligned}$$

The universal property of d_{TA} dictates that $\Omega_a^T; k = \Omega_a^T; \partial^\#$ and therefore $k = \partial^\#$ and the proof is complete. \square

6. An alternative definition of (co)differential category

Realization of the importance of the symmetric algebra in the analysis of Kähler categories also has the benefit that it leads to a succinct alternative definition of codifferential category as follows.

Theorem 6.1. *Let \mathcal{C} be an additive symmetric monoidal category for which the symmetric algebra monad S on \mathcal{C} exists. Assume that \mathcal{C} has reflexive coequalizers and that these are preserved by the tensor product in each variable. Then to equip \mathcal{C} with the structure of a codifferential category is, equivalently, to equip \mathcal{C} with*

1. a monad T ,
2. a monad morphism $\lambda : S \rightarrow T$, and
3. a transformation $d_{TC} : TC \rightarrow TC \otimes C$ natural in $C \in \mathcal{C}$

such that

(a) the diagram

$$\begin{array}{ccc} SC & \xrightarrow{\lambda_C} & TC \\ d_{SC} \downarrow & & \downarrow d_{TC} \\ SC \otimes C & \xrightarrow{\lambda_C \otimes 1_C} & TC \otimes C \end{array}$$

commutes for each $C \in \mathcal{C}$, where d_{SC} is the deriving transformation carried by S , and

(b) the Chain Rule axiom of Definition 3.9 holds, i.e. each d_{TC} is a T -derivation.

Proof. By Remark 4.1, the category of commutative algebras in \mathcal{C} is monadic over \mathcal{C} and so can be identified with the category of S -algebras. By Theorem 4.2, we know that algebra modalities on \mathcal{C} are in bijective correspondence with pairs (T, λ) consisting of a monad T on \mathcal{C} and a monad morphism $\lambda: S \rightarrow T$. Suppose we are given such a pair (T, λ) , together with a natural transformation $d_{T(-)}$ satisfying (a) and (b).

Claim: Any T -derivation $\partial: A \rightarrow M$ is, in particular, an S -derivation, equivalently by Remark 5.8, a derivation in the ordinary sense (Definition 3.5).

To prove this claim, observe that the following diagram commutes, by (a) and Definition 5.12, where a is the given T -algebra structure on A .

$$\begin{array}{ccccc}
 SA & \xrightarrow{\lambda_A} & TA & \xrightarrow{a} & A \\
 d_{SA} \downarrow & & \downarrow d_{TA} & & \downarrow \partial \\
 SA \otimes A & \xrightarrow{\lambda_A \otimes 1} & TA \otimes A & \xrightarrow{a \otimes \partial} & A \otimes M \xrightarrow{\bullet} M
 \end{array}$$

But the upper row is the S -algebra structure acquired by A via Theorem 4.2, so by Definition 5.12 the Claim is proved.

We have assumed that d satisfies the Chain Rule axiom, equivalently that each component $d_{TC}: TC \rightarrow TC \otimes C$ is a T -derivation (Theorem 5.13), so by the Claim, d_{TC} is an S -derivation, equivalently, an ordinary derivation. Hence the axioms (d1) and (d2) of Definition 3.9 hold, since together they assert exactly that each component d_{TC} is an ordinary derivation. We also know that axiom (d4) (the Chain Rule) holds, by assumption (b), so it suffices to prove that (d3) holds. Indeed, (d3) asserts that the periphery of the

following diagram commutes

$$\begin{array}{ccccc}
 & & \eta_C^T & & \\
 & \curvearrowright & & \curvearrowleft & \\
 C & \xrightarrow{\eta_C^S} & SC & \xrightarrow{\lambda_C} & TC \\
 \downarrow \wr & & \downarrow d_{SC} & & \downarrow d_{TC} \\
 k \otimes C & \xrightarrow{e \otimes 1} & SC \otimes C & \xrightarrow{\lambda_C \otimes 1} & TC \otimes C \\
 & \curvearrowleft & & \curvearrowright & \\
 & & e \otimes 1 & &
 \end{array} \tag{1}$$

where η^T and η^S are the units of T and S , respectively. The upper cell commutes since λ is a monad morphism, and the lower cell commutes since λ_C is an S -homomorphism, i.e. a homomorphism of algebras. The leftmost cell commutes since \mathcal{C} is a codifferential category when equipped with S (4.4), and the rightmost cell commutes by (a).

Conversely, let us instead assume that (\mathcal{C}, T, d) is a codifferential category. Then since axiom (d3) holds, the periphery of the diagram (1) commutes, but we also know that the upper, lower, and leftmost cells in (1) commute. Hence, whereas our aim is to show that (a) holds, i.e., that the rightmost square in (1) commutes, we know that this square ‘commutes when preceded by η_C^S ’. But by axioms (d1) and (d2), d_{TC} is an ordinary derivation, equivalently, an S -derivation (3.5), so the composite $\lambda_C; d_{TC}$ is an S -derivation since λ_C is an algebra map. Also, d_{SC} is an S -derivation, and one readily checks that $\lambda_C \otimes 1 : SC \otimes C \rightarrow TC \otimes C$ is a morphism of SC -modules (where $TC \otimes C$ carries the SC -module structure that it acquires by restriction of scalars along the algebra homomorphism λ_C). Hence the composite $d_{SC}; \lambda_C \otimes 1$ is an S -derivation. Therefore both composites in the square in question are S -derivations and so are uniquely determined by their composites with $\eta_C^S : C \rightarrow SC$, which are equal. \square

An advantage of this definition is that it immediately paves the way for variations of the theory of differential categories and differential linear logic. For example, to obtain noncommutative variants, one can replace the symmetric algebra in the above construction with a different endofunctor.

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NOTE ON THE CONSTRUCTION OF GLOBULAR WEAK ω -GROUPOIDS FROM TYPES, TOPOLOGICAL SPACES...

by John BOURKE

Résumé. Nous donnons une concise introduction aux ω -groupeïdes faibles de Grothendieck. Notre but est de démontrer que, dans certains contextes, ce simple langage est utile à la construction de ω -groupeïdes faibles globulaires. Pour cela, nous reformulons brièvement la construction, due à van den Berg et Garner, d'un ω -groupeïde faible de Batanin à partir d'un type en utilisant le langage des ω -groupeïdes faibles de Grothendieck. Cette construction s'applique aussi aux espaces topologiques ainsi qu'aux complexes de Kan.

Abstract. A short introduction to Grothendieck weak ω -groupoids is given. Our aim is to give evidence that, in certain contexts, this simple language is a convenient one for constructing globular weak ω -groupoids. To this end, we give a short reworking of van den Berg and Garner's construction of a Batanin weak ω -groupoid from a type using the language of Grothendieck weak ω -groupoids. This construction also applies to topological spaces and Kan complexes.

Keywords. Higher groupoids. Globular sets.

Mathematics Subject Classification (2010). 18D05.

1 Introduction

Around 2009/2010 van den Berg and Garner [3] and Lumsdaine [11] independently showed that a type in intensional type theory gives rise to a weak ω -category in the sense of Batanin [2].¹ In [3] this weak ω -category was shown, moreover, to be a weak ω -groupoid.

Shortly after these papers appeared, Georges Maltsiniotis [12] brought to attention, and simplified, a further globular definition of weak ω -groupoid that first appeared at the beginning of Grothendieck's manuscript *Pursuing*

¹To be precise, both papers employed the mild reformulation of Batanin's definition given by Leinster in [10].

Stacks [9].

At the end of 2015 I read the papers of van den Berg–Garner and Maltsiniotis around the same time. Being struck by the low-tech and transparent nature of the Grothendieck definition, I figured that it should be significantly easier to communicate the main results of [3] by substituting Batanin’s weak ω -groupoids for Grothendieck’s.

The goal of this largely expository note is to explain precisely that. We give a self contained introduction to Grothendieck weak ω -groupoids and in that language give a direct reworking, attempting nothing original, of the central results and proofs of [3]. For the reader unfamiliar with type theory let us point out that the main construction applies to topological spaces and Kan complexes as well as to types. Our thesis is that Grothendieck weak ω -groupoids provide a transparent and workable notion of globular weak ω -groupoid, and our economical reworking of the main result of *loc.cit.* is intended as evidence to that effect.

On setting down to write the present note I became aware that a closely related connection between Grothendieck weak ω -groupoids and intensional type theory was already made by Brunerie [5] in 2013. He defined an intensional type theory whose models provide a notion of weak ω -groupoid, and has shown that each type naturally gives rise to a weak ω -groupoid of that kind. It is expected that these type theoretic weak ω -groupoids, after some minor modifications², are essentially the same as the Grothendieck weak ω -groupoids described here, although the precise details of this correspondence are not yet written down.

Let us now give a brief summary of what follows. In Section 2 we recall the notion of Grothendieck weak ω -groupoid. This material is from [12] up to insignificant notational distinctions. Section 3 closely follows [3] in introducing identity type categories and iterating the path object construction to build globular objects in such categories. We additionally point out that topological spaces and Kan complexes form identity type categories. Section 4 introduces endomorphism globular theories whilst Section 5 reinterprets the main result and proof of [3] using Grothendieck weak ω -groupoids.

The author thanks Clemens Berger, Guillaume Brunerie, Richard Garner

²One needed modification concerns the shapes of operations that are allowed: the contractible contexts of [5] encode globular sets such as the free span that are not encoded by the tables of dimensions described here.

and Mark Weber for useful discussions on this topic.

2 Globular theories and ω -groupoids

2.1 The globe category and ω -graphs

The *category of globes* \mathbb{G} is freely generated by the graph

$$0 \begin{array}{c} \xrightarrow{\sigma_1} \\ \xrightarrow{\tau_1} \end{array} 1 \begin{array}{c} \xrightarrow{\sigma_2} \\ \xrightarrow{\tau_2} \end{array} \cdots \begin{array}{c} \xrightarrow{\sigma_{n-1}} \\ \xrightarrow{\tau_{n-1}} \end{array} n-1 \begin{array}{c} \xrightarrow{\sigma_n} \\ \xrightarrow{\tau_n} \end{array} n \cdots$$

subject to the relations $\sigma_n \circ \sigma_{n-1} = \tau_n \circ \sigma_{n-1}$ and $\tau_n \circ \sigma_{n-1} = \sigma_n \circ \sigma_{n-1}$.

These relations ensure that $\mathbb{G}(n, m) = \{\sigma_{n,m}, \tau_{n,m}\}$ for $n < m$ where $\sigma_{n,m}$ and $\tau_{n,m}$ are obtained by composing sequences of σ_i 's and τ_i 's respectively. We typically abbreviate $\sigma_{n,m}$ and $\tau_{n,m}$ by σ and τ when the context is clear.

A functor $A : \mathbb{G}^{op} \rightarrow \mathcal{C}$ is called an ω -graph or *globular object* in \mathcal{C} and is specified by objects $A(n)$ together with morphisms

$$A(n) \begin{array}{c} \xrightarrow{s_n} \\ \xrightarrow{t_n} \end{array} A(n-1)$$

where we write $s_n = A(\tau_n)$ and $t_n = A(\sigma_n)$. Similarly we write $s_{n,m} = A(\tau_{n,m})$ and $t_{n,m} = A(\sigma_{n,m})$, or just s and t if the context is clear.

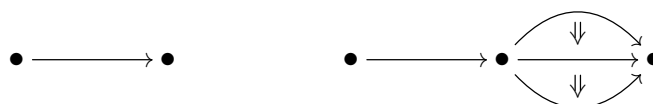
2.2 Globular sums and globular products

A *table of dimensions* is a sequence $\bar{n} = (n_1, \dots, n_k)$ of natural numbers with $n_{2i-1} > n_{2i} < n_{2i+1}$ and $k \in \{1, 3, 5, \dots\}$. Given \bar{n} , a functor $D : \mathbb{G} \rightarrow \mathcal{C}$ determines a diagram

$$\begin{array}{ccccccc} & & D(n_2) & & D(n_4) & & \cdots & & D(n_{k-1}) & & \\ & & \swarrow_{D\tau} & & \swarrow_{D\tau} & & & & \swarrow_{D\tau} & & \\ D(n_1) & & & D(n_3) & & D(n_5) & & \cdots & & D(n_{k-2}) & & D(n_k) \\ & & \searrow_{D\sigma} & & \searrow_{D\sigma} & & & & \searrow_{D\sigma} & & \end{array}$$

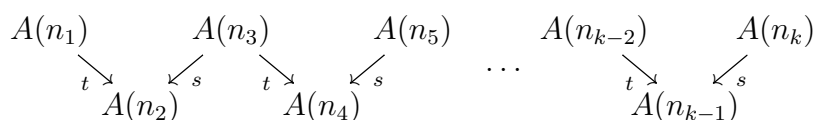
in \mathcal{C} whose colimit is called a *globular sum* and denoted by $D(\bar{n})$. If all such colimits exist then we say that \mathcal{C} admits D -globular sums (or just globular

sums). For $Y : \mathbb{G} \rightarrow [\mathbb{G}^{op}, \mathbf{Set}]$ the globular sums $Y(1)$ and $Y(1, 0, 2, 1, 2)$ are depicted below.



Though we have not labelled them differently, it is important to note that all of the cells depicted above are distinct.

Likewise an ω -graph $A : \mathbb{G}^{op} \rightarrow \mathcal{C}$ determines a diagram



whose limit, denoted $A(\bar{n})$, is called a *globular product*.

2.3 Globular theories

We now describe the category Θ_0 that plays the same role for globular theories as the skeletal category of finite sets plays for Lawvere theories.

To construct Θ_0 observe that the category of globular sets $[\mathbb{G}^{op}, \mathbf{Set}]$ is cocomplete and therefore admits Y -globular sums. Taking the full subcategory of $[\mathbb{G}^{op}, \mathbf{Set}]$ on the globular sums yields the initial, up to equivalence, category with globular sums. Θ_0 is a skeleton of this: we can view its objects as the tables of dimensions whilst $\Theta_0(\bar{n}, \bar{m}) = [\mathbb{G}^{op}, \mathbf{Set}](Y(\bar{n}), Y(\bar{m}))$. The functor

$$D : \mathbb{G} \rightarrow \Theta_0$$

factors the Yoneda embedding and is given by $Dn = (n)$ on objects. We record the universal property of its dual.

Lemma 2.1. *Let \mathcal{C} be a category admitting A -globular products. There exists an essentially unique extension*

$$\begin{array}{ccc}
 \Theta_0^{op} & & \\
 \uparrow D^{op} & \searrow A(-) & \\
 \mathbb{G}^{op} & \xrightarrow{A} & \mathcal{C}
 \end{array}$$

of A to a globular product preserving functor $A(-) : \Theta_0^{op} \rightarrow \mathcal{C}$. This sends \bar{n} to the globular product $A(\bar{n})$.

Definition 2.2. A globular theory consists of an identity on objects functor

$$J : \Theta_0^{op} \rightarrow \mathbb{T}$$

that preserves globular products.

The category $Mod(\mathbb{T}, \mathcal{C})$ of \mathbb{T} -algebras in \mathcal{C} is the full subcategory of $[\mathbb{T}, \mathcal{C}]$ containing the globular product preserving functors. Observe that there is a forgetful functor

$$U : Mod(\mathbb{T}, \mathcal{C}) \rightarrow [\mathbb{G}^{op}, \mathcal{C}]$$

given by restriction along $J \circ D^{op} : \mathbb{G}^{op} \rightarrow \mathbb{T}$. If $U(X) = A$ then we call X a \mathbb{T} -algebra structure on A .

Remark 2.3. The category Θ_0 was first described by Berger [4] using level trees. Globular theories were also first described in *ibid.*, in which the definition was formulated using a sheaf condition equivalent to J 's preserving globular products. The only difference with Definition 2.2 is that Definition 1.5 of *ibid.* required that J be faithful, as it typically is.

2.4 Contractibility and weak ω -groupoids

Let $A : \mathbb{G}^{op} \rightarrow \mathcal{C}$. By a parallel pair of n -cells in A is meant a pair

$$f, g : X \rightrightarrows A(n)$$

such that either $n = 0$ or $s_n \circ f = s_n \circ g$ and $t_n \circ f = t_n \circ g$. A lifting for such a pair is an arrow $h : X \rightarrow A(n+1)$ such that

$$\begin{array}{ccc} & & A(n+1) \\ & \nearrow h & \downarrow s \\ X & \xrightarrow{f} & A(n) \\ & \searrow g & \downarrow t \end{array}$$

commutes..

The ω -graph A is said to be *contractible* if each parallel pair of n -cells in A has a lifting, whilst a globular theory $J : \Theta_0^{op} \rightarrow \mathbb{T}$ is said to be contractible if its underlying ω -graph

$$J \circ D^{op} : \mathbb{G}^{op} \rightarrow \mathbb{T}$$

is contractible.

Definition 2.4. A *Grothendieck weak ω -groupoid* is an algebra for some contractible globular theory.

Let $J : \Theta_0^{op} \rightarrow \mathbb{T}$ be a contractible globular theory and let us agree not to write the action of J . Where are the operations for a weak ω -groupoid in \mathbb{T} ? The map representing composition of 1-cells should have domain the pullback below left.

$$\begin{array}{ccc}
 \begin{array}{ccc} (1, 0, 1) & \xrightarrow{q} & (1) \\ p \downarrow & & \downarrow s \\ (1) & \xrightarrow{t} & (0) \end{array} & \begin{array}{ccc} & & (1) \\ & \nearrow m & \downarrow s \\ (1, 0, 1) & \xrightarrow{s \circ p} & (0) \\ & \searrow t \circ q & \end{array} & \begin{array}{ccc} & & (2) \\ & \nearrow a & \downarrow s \\ (1, 0, 1, 0, 1) & \xrightarrow{m \circ (m, 1)} & (1) \\ & \searrow m \circ (1, m) & \end{array}
 \end{array}$$

Now the parallel 0-cells in the second diagram admit, by contractibility of \mathbb{T} , a lifting m and this encodes the sought for composition. Associativity of composition up to a 2-cell is encoded by the lifting a for the parallel 1-cells in the third diagram. Weak inverses are encoded by the lifting for the parallel pair

$$(1) \begin{array}{c} \xrightarrow{t} \\ \xrightarrow{s} \end{array} (0) \quad .$$

And so on. For further details see Section 1.7 of [12] or Section 3 of [1].

Remark 2.5. In [12] a weak ω -groupoid is defined to be an algebra for a *Gr-coherator* – a certain kind of contractible globular theory. By Theorem 3.14 of *loc.cit.* the *Gr-coherators* are precisely the *cellular* contractible globular theories and therefore are weakly initial amongst contractible globular theories. It follows that an ω -graph admits weak ω -groupoid in the present sense just when it admits an algebra structure for a *Gr-coherator*.

3 Identity type categories and iterated path objects

3.1 Identity type categories

An *identity type category* [3] is a category \mathcal{C} equipped with a weak factorisation system (L, R) ³ satisfying the following properties:

- A terminal object 1 exists and for each $X \in \mathcal{C}$ the unique map $! : X \rightarrow 1$ is an R -map.
- Pullbacks of R -maps exist and the pullback of an L -map along an R -map is again an L -map.

As shown in [8, 3] the syntactic category of an intensional type theory admits the structure of an identity type category.

Further examples arise from Quillen model categories \mathcal{C} whose cofibrations are pullback stable along fibrations. Since weak equivalences between fibrant objects are always stable under pullback along fibrations (see Proposition 13.1.2 of [6]) the trivial cofibrations between fibrant objects in such model categories are also stable under pullback along fibrations. So for such \mathcal{C} it follows that the full subcategory of fibrant objects \mathcal{C}_f is an identity type category when equipped with the restricted (trivial cofibration/fibration)-weak factorisation system.

In the Strøm model structure on topological spaces [16] the cofibrations –*closed cofibrations*– are stable under pullback along the fibrations, the *Hurewicz fibrations*. This is Theorem 12 of [15]. Since all topological spaces are fibrant the category of topological spaces is therefore an identity type category. In the standard model structure on simplicial sets [13] the cofibrations are the monos and so are pullback stable along all maps; it follows that the full subcategory of fibrant objects – *the Kan complexes* – is an identity type category.

³The definition of [3] actually only requires certain factorisations to exist but is equivalent to the present formulation by the argument of Lemma 2.4 of [14]. See also Lemma 11 of [8] for the original type theoretic argument.

3.2 Iterating the path object construction

Starting with an object X of \mathcal{C} the goal now is to build an ω -graph X_\star with $X_\star(0) = X$. $X_\star(1)$ is to be the *path object* of X : that is, an (L, R) -factorisation

$$X_\star(0) \xrightarrow{i_{0,1}} X_\star(1) \xrightarrow{\langle s_1, t_1 \rangle} X_\star(0) \times X_\star(0)$$

of the diagonal map. Then $s_1, t_1 : X_\star(1) \rightrightarrows X_\star(0)$ will be the underlying 1-graph of X_\star .

The inductive construction of an $(n + 1)$ -graph from an n -graph makes use of the $(n + 1)$ -boundary $B_{n+1}X_\star$ of an n -graph. This has $B_1X_\star = X_\star(0) \times X_\star(0)$ whilst for higher n , it is given by the pullback below

$$\begin{array}{ccc} B_{n+1}X_\star & \xrightarrow{p_n} & X_\star(n) \\ q_n \downarrow & & \downarrow \langle s_n, t_n \rangle \\ X_\star(n) & \xrightarrow{\langle s_n, t_n \rangle} & B_nX_\star \end{array} \quad (3.1)$$

in which the map

$$\langle s_n, t_n \rangle : X_\star(n) \rightarrow B_nX_\star \quad (3.2)$$

is inductively constructed.

Let us note that by restriction one can speak of the $(n + 1)$ -boundary of an ω -graph, and it is not hard to see that this represents parallel pairs of n -cells in the ω -graph, as were defined in Section 2.4.

Now the pullback (3.1) exists in an identity type category because the inductively defined map (3.2) is an R -map at each stage. For the inductive step, we observe that the identity on $X_\star(n)$ induces a diagonal $\langle 1, 1 \rangle : X_\star(n) \rightarrow B_{n+1}X_\star$ whose (L, R) -factorisation

$$X_\star(n) \xrightarrow{i_{n,n+1}} X_\star(n+1) \xrightarrow{\langle s_{n+1}, t_{n+1} \rangle} B_{n+1}X_\star$$

is taken to define $X_\star(n + 1)$. The two maps $s_{n+1}, t_{n+1} : X_\star(n + 1) \rightrightarrows X_\star(n)$ then extend X_\star to an $(n + 1)$ -graph. Because the projections in (3.1) are pullbacks of R -maps, they are R -maps too. And since s_{n+1} and t_{n+1} are obtained by composing these projections with the R -map $\langle s_n, t_n \rangle$ it follows that both s_{n+1} and t_{n+1} are R -maps as well.

Induction now produces an ω -graph X_\star that we call the *iterated path object* and whose relevant properties we now record.

Lemma 3.1. *The iterated path object X_\star is a reflexive globular context [3], i.e.,*

1. *There exist L -maps $i_{n,n+1} : X_\star(n) \rightarrow X_\star(n+1)$ with $s_n \circ i_{n,n+1} = t_n \circ i_{n,n+1}$.*
2. *The maps $s_n, t_n : X_\star(n+1) \rightrightarrows X_\star(n)$ and $\langle s_n, t_n \rangle : X_\star(n+1) \rightarrow B_n X_\star$ are R -maps.*

Remark 3.2. A couple of points are perhaps worth noting. Firstly, the maps $i_{n,n+1}$ exhibit X_\star as a *reflexive* (globular object / ω -graph). Secondly, the above construction of X_\star from X can be understood in terms of the *Reedy structure* on the *reflexive globe category* \mathbb{R} . For \mathbb{J} a Reedy category (see [7] for instance) let $\mathbb{J}_{\leq n}$ denote the full subcategory on the objects of degree at most n . Then extensions of $A : \mathbb{J}_{\leq n} \rightarrow \mathcal{C}$ to $\mathbb{J}_{\leq n+1}$ correspond to factorisations of the map $L_n A \rightarrow M_n A$ from the n -th *latching object* of A to the n -th *matching object* of A , a colimit and limit respectively. It follows that for \mathcal{C} a sufficiently bicomplete category equipped with a weak factorisation system, there is a canonical method of inductively constructing an object $X_\star : \mathbb{J} \rightarrow \mathcal{C}$ from $X \in \mathcal{C}$. Specialised to the Reedy category \mathbb{R} and an identity type category (\mathcal{C}, L, R) this yields the iterated path object construction.

4 Endomorphism theories

Let \mathcal{C} be a category with A -globular products and consider the extension $A(-) : \Theta_0^{op} \rightarrow \mathcal{C}$ of A as below.

$$\begin{array}{ccc}
 \Theta_0^{op} & \xrightarrow{J_A} & \text{End}(A) \\
 \uparrow D^{op} & \searrow A(-) & \downarrow K_A \\
 \mathbb{G}^{op} & \xrightarrow{A} & \mathcal{C}
 \end{array}$$

Factoring $A(-)$ as identity on objects followed by fully faithful yields the *endomorphism theory*

$$J_A : \Theta_0^{op} \rightarrow \text{End}(A)$$

of A . This has the same objects as Θ_0 whilst $\text{End}(A)(\bar{n}, \bar{m}) = \mathcal{C}(A(\bar{n}), A(\bar{m}))$. Since $A(-)$ preserves globular products so do both $J_A : \Theta_0^{op} \rightarrow \text{End}(A)$ and $K_A : \text{End}(A) \rightarrow \mathcal{C}$. The first fact establishes that $\text{End}(A)$ is a globular theory whilst the second exhibits the canonical $\text{End}(A)$ -algebra structure on A .

We will use the following lemma, whose proof is a matter of tracing through the definitions, to construct weak ω -groupoids.

Lemma 4.1. *Let \mathcal{C} admit A -globular products. Then $\text{End}(A)$ is contractible if and only if each parallel pair $f, g : A(\bar{n}) \rightrightarrows A(\bar{m})$ of m -cells in A with domain a globular product has a lifting.*

5 The weak ω -groupoid structure

Theorem 5.1. *Let \mathcal{C} be an identity type category. Then for each $X \in \mathcal{C}$ the iterated path object X_\star admits the structure of a weak ω -groupoid.*

Proof. More generally we will show that each reflexive globular context $A : \mathbb{G}^{op} \rightarrow \mathcal{C}$ admits the structure of a weak ω -groupoid. Firstly we establish some notation. On composing the L -maps $i_{n,n+1} : A(n) \rightarrow A(n+1)$ we obtain further L -maps $i_{n,m} : A(n) \rightarrow A(m)$ for $n < m$ which will be abbreviated by i , excepting the case $n = 0$ where we write $i_n : A(0) \rightarrow A(n)$.

Now the L -maps $i_n : A(0) \rightarrow A(n)$ assemble into a cone $i : \Delta A(0) \rightarrow A \in [\mathbb{G}^{op}, \mathcal{C}]$ under $A(0)$. This induces a factorisation

$$\begin{array}{ccc} \mathbb{G}^{op} & \xrightarrow{i/A} & A(0)/\mathcal{C} \\ & \searrow A & \downarrow U \\ & & \mathcal{C} \end{array}$$

of A through $A(0)/\mathcal{C}$. Here the functor i/A sends n to $i_n : A(0) \rightarrow A(n)$ whilst U is the forgetful functor.

We will prove the theorem by showing:

1. The category $A(0)/\mathcal{C}$ has i/A -globular products preserved by U ;
2. The endomorphism theory $\text{End}(i/A)$ is contractible.

Then the composite

$$\text{End}(i/A) \xrightarrow{K_{i/A}} A(0)/\mathcal{C} \xrightarrow{U} \mathcal{C}$$

will exhibit the structure of a $\text{End}(i/A)$ -algebra – and hence weak ω -groupoid – on A .

For (1) we proceed by induction over the length k of a table of dimensions $\bar{n} = (n_1, \dots, n_k)$. As usual we write $A(\bar{n})$ for the globular product in \mathcal{C} with $p_j^{\bar{n}} : A(\bar{n}) \rightarrow A(n_j)$ the j 'th projection. We write $i_{\bar{n}} : A(0) \rightarrow A(\bar{n})$ for the globular product in $A(0)/\mathcal{C}$ which then satisfies

$$A(0) \xrightarrow{i_{\bar{n}}} A(\bar{n}) \xrightarrow{p_j^{\bar{n}}} A(n_j) = A(0) \xrightarrow{i_{n_j}} A(n_j) .$$

For the base case $\bar{n} = (n_1)$ we have $A(\bar{n}) = A(n_1)$ with the identity projection, and $i_{n_1} : A(0) \rightarrow A(n_1)$ as globular product in $A(0)/\mathcal{C}$. For $\bar{n}^+ = (n_1, \dots, n_k, n_{k+1}, n_{k+2})$ the globular product $A(\bar{n}^+)$ in \mathcal{C} can be constructed as the pullback in the rectangle below

$$\begin{array}{ccc} A(0) & \xrightarrow{i_{n_{k+2}}} & A(n_{k+2}) \\ \downarrow i_{\bar{n}^+} & \searrow & \downarrow s \\ A(\bar{n}^+) & \xrightarrow{p_{k+2}^{\bar{n}^+}} & A(n_{k+2}) \\ \downarrow q & & \downarrow s \\ A(\bar{n}) & \xrightarrow{p_k^{\bar{n}}} A(n_k) \xrightarrow{t} & A(n_{k+1}) \end{array} \quad (5.1)$$

which exists since $s : A(n_{k+2}) \rightarrow A(n_{k+1})$ is an R -map. By the universal property of the pullback there exists a unique map $i_{\bar{n}^+} : A(0) \rightarrow A(\bar{n}^+)$ rendering commutative the two triangles. Since U creates pullbacks this is the pullback, and hence globular product, in $A(0)/\mathcal{C}$.

By induction we have now proven (1). A further consequence of the inductive construction is that the final projection

$$p_k^{\bar{n}} : A(\bar{n}) \rightarrow A(n_k)$$

is an R -map. This is trivial in the base case, and clear in the inductive step since the final projection $p_{k+2}^{\bar{n}^+}$ is the pullback of a composite $t \circ p_k^{\bar{n}}$ of R -maps.

Now the main ingredient in proving (2) is, in fact, to show that each morphism

$$i_{\bar{n}} : A(0) \rightarrow A(\bar{n})$$

is an L -map and again this is done by induction. In the base case we have the L -map $i_{n_1} : A(0) \rightarrow A(n_1)$. For the inductive step we start by observing that the right vertical arrow $s : A(n_{k+2}) \rightarrow A(n_{k+1})$ of (5.1) has section the L -map $i : A(n_{k+1}) \rightarrow A(n_{k+2})$. It follows that its pullback q has a unique section i' satisfying the commutativity in the left square below.

$$\begin{array}{ccccc}
 & & 1 & & \\
 & \searrow & & \searrow & \\
 A(\bar{n}) & \xrightarrow{i'} & A(\bar{n}^+) & \xrightarrow{q} & A(\bar{n}) \\
 \downarrow t \circ p_k^{\bar{n}} & & \downarrow p_{k+2}^{\bar{n}^+} & & \downarrow t \circ p_k^{\bar{n}} \\
 A(n_{k+1}) & \xrightarrow{i} & A(n_{k+2}) & \xrightarrow{s} & A(n_{k+1}) \\
 & \swarrow & & \swarrow & \\
 & & 1 & &
 \end{array}$$

Since the right and outer rectangles above are pullbacks the left one is a pullback too and, since $p_{k+2}^{\bar{n}^+} \in R$ and $i \in L$, it follows that $i' \in L$. Therefore to prove that $i_{\bar{n}^+} \in L$ it suffices to show that $i_{\bar{n}^+} = i' \circ i_{\bar{n}}$. Both maps give $i_{\bar{n}}$ when postcomposed by q . Composing with the other pullback projection gives

$$p_{k+2}^{\bar{n}^+} \circ i' \circ i_{\bar{n}} = i \circ s \circ p_k^{\bar{n}} \circ i_{\bar{n}} = i \circ t \circ i_{n_k} = i \circ i_{n_{k+1}} = i_{n_{k+2}} = p_{k+2}^{\bar{n}^+} \circ i_{\bar{n}^+}$$

as required.

To complete the proof we must show that the endomorphism theory $\text{End}(i/A)$ is contractible. By Lemma 4.1 this is equally to show that each parallel pair of m -cells in $i/A : \mathbb{G}^{op} \rightarrow A(0)/\mathcal{C}$ with domain a globular product $A(\bar{n})$ has a lifting. Such a parallel pair are depicted below left.

$$\begin{array}{ccc}
 A(0) & & A(0) \xrightarrow{i_{m+1}} A(m+1) \\
 \downarrow i_{\bar{n}} & \searrow i_m & \downarrow \langle s, t \rangle \\
 A(\bar{n}) & \xrightarrow{f} & A(\bar{n}) \xrightarrow{\langle f, g \rangle} B_{m+1}A \\
 & \searrow g & \downarrow p_m \\
 & & A(m)
 \end{array}$$

$\begin{array}{ccc} & \searrow s & \\ & & \downarrow t \\ & & A(m) \end{array}$

These induce a unique map $\langle f, g \rangle : A(\bar{n}) \rightarrow B_{m+1}A$ to the boundary such that $p_n \circ \langle f, g \rangle = f$ and $q_n \circ \langle f, g \rangle = g$. In the diagram above right all paths from $A(0)$ to $A(m)$ coincide as $i_m : A(0) \rightarrow A(m)$. Since the pullback projections p_m and q_m are jointly monic it follows that the square commutes. Now $i_{\bar{n}}$ is an L -map and $\langle s, t \rangle$ an R -map. Therefore there exists a diagonal filler $h : A(\bar{n}) \rightarrow A(m+1)$ in the square and this gives the desired lifting. \square

Remark 5.2. The preceding construction of a Grothendieck weak ω -groupoid is simpler than that of a Batanin weak ω -groupoid for a couple of reasons. One is that Batanin's weak ω -groupoids are defined as a special case of his weak ω -categories and so another step is required. Another reason is that endomorphism theories seem easier to handle with globular theories rather than globular operads.

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A MATHEMATICAL TRIBUTE TO REINHARD BÖRGER

by Walter THOLEN

Résumé. Un résumé de la vie et du travail de Reinhard Börger (1954–2014) est présenté en mettant l’accent sur ses oeuvres premières ou non publiées.

Abstract. A synopsis of the life and work of Reinhard Börger (1954–2014) is presented, with an emphasis on his early or unpublished works.

Keywords. cogenerator, strong generator, semi-topological functor, total category, extensive category, sequentially convex space.

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1. A Brief Curriculum Vitae

On June 6, 2014, Reinhard Börger passed away, after persistent heart complications. He had taught at *Fernuniversität* in Hagen, Germany, for over three decades where he had received his *Dr. rer. nat.* (Ph.D.) in 1981, with a thesis [15] on notions of connectedness, written under the direction of Dieter Pumplün. He had continued to work on mathematical problems until just hours before his death.

Born on August 19, 1954, Reinhard went to school in Gevelsberg (near Hagen) before beginning his mathematics studies at *Westfälische Wilhelms-Universität* in Münster in 1972. A year later he won a runner-up prize at the highly competitive national *Jugend forscht* competition. Quite visibly, mathematics seemed to always be on his mind, and he often seemed to appear out of nowhere at lectures, seminars or informal gatherings. These sudden appearances quickly earned him his nickname *Geist* (ghost), a name that he willingly adopted for himself as well. His trademark ability to then launch pointed and often unexpected, but always polite, questions, be it on mathematics or any other issue, quickly won him the respect of all.

Reinhard’s interest in category theory started early during his studies in

Münster when, supported by a scholarship of the prestigious *Studienstiftung des Deutschen Volkes*, he took Pumplün's course on the subject that eventually led him to write his 1977 *Diplomarbeit* (M.Sc. thesis) about congruence relations on categories [3]. For his doctoral studies he accepted a scholarship from the *Cusanuswerk* and followed Pumplün from Münster to Hagen where Pumplün had accepted an inaugural chair at the newly founded Fernuniversität in 1975. After the completion of his doctoral degree in 1981 with an award-winning thesis, he assumed a number of research assistantships, at the Universities of Karlsruhe (Germany) and of Toledo (Ohio, USA), and back at Fernuniversität. For his *Habilitationsschrift* [31], which earned him the *venia legendi* in 1989, he developed a categorical approach to integration theory. Beginning from 1990 he worked as a *Hochschuldozent* at Fernuniversität, interrupted by a visiting professorship at York University in Toronto (Canada) in 1993, and in 1995 he was appointed *Außerplanmäßiger Professor* at Fernuniversität, a position that he kept until his premature death in 2014.

In what follows I give a synopsis of Reinhard's mathematical work, emphasizing early, incomplete or not easily accessible contributions. After a brief account in Section 2 of his work up to the completion of his M.Sc. thesis, I recall some of his early contributions to the development of categorical topology (Section 3), before describing in Section 4 some aspects of his Ph.D. thesis and the work that emanated from it. Section 5 sketches the work on integration theory in his *Habilitationsschrift*, and Section 6 highlights some of his more isolated mathematical contributions. For Reinhard's substantial contributions in the area of convexity theory, inspired by the Pumplün–Röhrl works on convexity (such as [86, 87]), we refer to the article [81] by his frequent coauthor on the subject, Ralf Kemper.

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2. First Steps

The earliest written mathematical work of Reinhard that I am aware of and that may still be of interest today, is the three-page mimeographed note [1] giving a sufficient condition for the non-existence of a cogenerating (also called coseparating) set of objects in a category \mathcal{K} . While the existence of such a set in the category of R -modules and, in particular, of abelian groups, is standard, none of the following categories can possess one: fields; skew fields; (commutative; unital) rings; groups; semigroups; monoids; small categories. Reinhard's theorem, found when he was still an undergraduate student, gives a unified reason for this, as follows.

Theorem 2.1. *Let \mathcal{K} have (strong epi, mono)-factorizations and admit a functor U to \mathbf{Set} that preserves monomorphisms. If, for every cardinal number κ , there is a simple object A in \mathcal{K} with the cardinality of UA at least κ , then there is no cogenerating set in \mathcal{K} .*

(He defined an object A to be *simple* if the identity morphism on A is not constant while every strong epimorphism with domain A must be constant or an isomorphism; a morphism f is *constant* if for all parallel morphisms x, y composable with f one has $fx = fy$.) Reinhard returned to the theme of the existence of cogenerators repeatedly throughout his career, see [16, 34, 44, 45, 49, 50].

In 1975 Reinhard and I discussed various generalizations of the notion of right adjoint functor that had appeared in the literature at the time, in particular Kaput's [80] locally adjunctionable functors. We tightened that notion to *strongly locally right adjoint* and proved, among other things, preservation of connected limits by such functors. Our paper [2] was presented at the "Categories" conference in Oberwolfach in 1976, and we discussed it with Yves Diers who was working on a slightly stricter notion for his thesis [71] that today is known under the name *multi-right adjoint* functor. Diers' only further requirement to our strong local right adjointness was that the local adjunction units of an object, known as its *spectrum*, must form a set. Without this size restriction, Reinhard and I had already given in [2] a complete characterization of the spectrum of an object, as follows.

Theorem 2.2. *For a strongly locally right adjoint functor $U : \mathcal{A} \rightarrow \mathcal{X}$ and an object $X \in \mathcal{X}$, its spectrum is the only full subcategory of the comma cat-*

egory $(X \downarrow U)$ that is a groupoid, coreflective, and closed under monomorphisms.

These works actually precede Reinhard's M.Sc. thesis [3] (summarized in [5]) whose starting point was a notion presented in Pumplün's categories course that went beyond the classical notion (as given in [72]) which confines equivalent morphism to the same hom set. Without that restriction, for any equivalence relation \sim on the class of morphisms of a category \mathcal{K} to be (*uniquely*) *normal* Pumplün required the existence of a (uniquely determined) composition law for the equivalence classes that makes \mathcal{K}/\sim a category and the projection $P : \mathcal{K} \rightarrow \mathcal{K}/\sim$ a functor. Reinhard showed that the behaviour of a *compatible* equivalence relation \sim on the morphism class of a category \mathcal{K} (so that $u \sim u'$ and $v \sim v'$ implies $uv \sim u'v'$ whenever the composites are defined) requires great caution:

Theorem 2.3. *Each of the following statements on an equivalence relation \sim on the class of morphisms of a category \mathcal{K} implies the next, but none of these implications is reversible:*

- \sim is compatible, and $1_A \sim 1_B$ only if $A = B$, for all objects $A, B \in \mathcal{K}$;
- \sim is compatible, and for all $u : A \rightarrow B, v : C \rightarrow D$ with $1_B \sim 1_C$, there are $u' : A' \rightarrow B', v' : C' \rightarrow D'$ with $u \sim u', v \sim v'$ and $B' = C'$;
- \sim is uniquely normal;
- \sim is normal;
- there is a functor F with domain \mathcal{K} inducing \sim (so that $u \sim u' \iff Fu = Fu'$);
- \sim is compatible.

3. Semi-topological functors and total cocompleteness

Brümmer's [68], Shukla's [90], Hoffmann's [77] and Wischnewsky's [98] theses and Wyler's [100, 99], Manes' [84] and Herrlich's [75, 76] semi-

nal papers triggered the development of what became known as *Categorical Topology*, with various groups in Germany, South Africa, the United States and other countries working intensively throughout the 1970s on axiomatizations of “topologically behaved” functors and their generalizations and properties; see [69] for a survey. Reinhard and I, long before he started working on his doctoral dissertation, were very much part of this effort. Here are some examples of results that he has influenced the most.

Topologicity of a functor $P : \mathcal{A} \rightarrow \mathcal{X}$ may be defined by the sole requirement that initial liftings of (arbitrarily large) so-called P -structured sources exist, without the a-priori assumption of faithfulness of P . (This is Brümmer’s [68] definition, although he did not use the name *topological* for such functors in his thesis.) Herrlich realized that faithfulness is a consequence of the definition, with a proof that made essential use of the smallness of hom-sets for the categories in question. Reinhard’s spontaneous idea then was to use a Cantor-type diagonal argument instead that works also for not necessarily locally small categories. In [8] we came up with a general theorem that not only proves the faithfulness of topological and, more generally, *semi-topological* functors [94, 78, 95], but that also entails Freyd’s theorem that a small category with (co)products must be, up to categorical equivalence, a complete lattice, and that in fact reproduces Cantor’s original theorem about the cardinality of a set being always exceeded by that of its power set, as follows:

Theorem 3.1. *Consider a (possibly large) family $(t_i : A_i \rightarrow C)_{i \in I}$ of morphisms and an object B in a category \mathcal{K} , such that any family $(h_i : A_i \rightarrow B)_{i \in I}$ factors as $h_i = ht_i$ ($i \in I$) for some $h : C \rightarrow B$. If there is a surjection $I \rightarrow \mathcal{K}(C, B)$, then for any morphisms $f, g : C \rightarrow B$ one has $ft_i = gt_i$ for some $i \in I$.*

Semi-topological functors in their various incarnations (also called *solid* functors, at Herrlich’s urging) were a topic of Reinhard’s and my joint interest for considerable time, in particular in conjunction with strong completeness properties of the participating categories, as witnessed by our papers [9, 16, 34, 35, 38]. In [96] I had shown that the fundamental property of *totality* (or *total cocompleteness*) introduced by Street and Walters [92] lifts from \mathcal{X} to \mathcal{A} along a semi-topological $P : \mathcal{A} \rightarrow \mathcal{X}$, and in [66] total categories with a (strong) generating set of objects were characterized as the cat-

egories admitting a semi-topological (and conservative) functor into some small discrete power of **Set**. For our paper [34] Reinhard constructed an incredible example:

Theorem 3.2. *There is a total category \mathcal{A} with a (single-object) strong generator but no regularly generating set of objects. \mathcal{A} is cowell-powered with respect to regular epimorphisms but not with respect to strong epimorphisms; \mathcal{A} does not admit co-intersections of arbitrarily large families of strong epimorphisms. The colimit closure \mathcal{B} of the strong generator in \mathcal{A} fails to be complete since it doesn't even possess a terminal object.*

Since totality entails a very strong completeness property, called *hyper-completeness* by Reinhard (see [16]), the colimit closure \mathcal{B} in the example above fails badly to inherit totality from its ambient category \mathcal{A} . A comparison with the following affirmative result on totality of colimit closures obtained in [38] demonstrates how “tight” this example is:

Theorem 3.3. *Let the cocomplete category \mathcal{B} be the colimit closure of a small full subcategory \mathcal{G} , and assume that every extremal epimorphism in \mathcal{B} is the colimit of a chain of regular epimorphisms of length at most α , for some fixed ordinal α . Then \mathcal{B} is total and admits large co-intersections of strong epimorphisms, and \mathcal{G} is strongly generating in \mathcal{A} .*

4. Connectedness, coproducts, and extensive categories

Reinhard's doctoral dissertation [15] relates various categorical notions of connectedness studied throughout the 1970s with each other, adds new concepts and gives some surprising applications. Starting points for him were the notions of *component subcategory* (initiated by Herrlich [74] and developed further by Preuß [85], Strecker [91] and Tiller [97]), of *left-constant subcategory* (also initiated by Herrlich [74] in generalization of the correspondence between torsion and torsion-free classes and fully characterized within the category of topological spaces by Arhangel'skii and Wiegandt [67]), and the notion of strongly locally coreflective [2] or multi-coreflective [71] subcategory (already mentioned in Section 2 in the dual situation and applied in topology by Salicrup [88]).

Let us concentrate here on a more category-intrinsic approach to connectedness to which Reinhard greatly contributed and which led him to make

significant contributions to preservation properties of coproducts in abstract and concrete categories. The starting point is the easy observation that a topological space X is (not empty and) connected if, and only if, every continuous map $X \rightarrow \coprod_{i \in I} Y_i$ into a topological sum factors uniquely through exactly one coproduct injection; in other words, if the covariant hom-functor $\mathbf{Top} \rightarrow \mathbf{Set}$ represented by X preserves coproducts. Trading \mathbf{Top} for any category \mathcal{K} with coproducts Hoffmann [77] called such objects X *Z-objects*, Reinhard preferred the name *coprime*, while most people will nowadays use the term *connected* in \mathcal{K} . More specifically, for a cardinal number α , let us call X α -*connected* in \mathcal{K} if the hom-functor of X preserves coproducts indexed by a set of cardinality $\leq \alpha$.

In his thesis Reinhard was the first to explore this concept deeply in the dual category of the category \mathbf{Rng} of unital (but not necessarily commutative) rings. α -connectedness of a ring R now means that every unital homomorphism $f : \prod_{\beta < \alpha} S_\beta \rightarrow R$ depends only on exactly one coordinate (so that it factors uniquely through precisely one projection of the direct product). While it is easy to see that, without loss of generality, one may assume here that every ring S_β is the ring \mathbb{Z} of integers, and that the finitely-connected (i.e., α -connected, for every finite α) rings are precisely those that traditionally are called connected (i.e., those rings that have no idempotent elements other than 0 and 1), Reinhard unravelled several surprises in the infinite case. Calling a ring *ultraconnected* when it is \aleph_0 -connected, he proved in [15] (see also [21]) that the countable case governs the arbitrary infinite case precisely when there are no uncountable measurable cardinals:

Theorem 4.1. *If there are no uncountable measurable cardinals, then the connected objects in \mathbf{Rng}^{op} are precisely the ultraconnected rings. If there are uncountable measurable cardinals, then there are no ultraconnected objects in \mathbf{Rng}^{op} .*

The fields \mathbb{R} and \mathbb{C} of real and of complex numbers are ultraconnected, and so is every subring of an ultraconnected ring. But none of the following connected rings is ultraconnected: the cyclic rings of cardinality p^m (p prime, $m \geq 1$), the ring \mathbb{Z}_p of p -adic integers and its field of fractions \mathbb{Q}_p

The Theorem remains valid if \mathbf{Rng} is traded for the category of commutative unital rings. Its proof makes essential use of a general categorical result that Reinhard had first presented at a meeting on “Categorical Algebra

and Its Applications” held in Arnsberg (Germany) in 1979 (see [13]):

Theorem 4.2. *For a category \mathcal{K} with an initial object and α -indexed coproducts (α an infinite cardinal), a functor $F : \mathcal{K} \rightarrow \mathbf{Set}$ preserves such coproducts if, and only if, F preserves β -indexed coproducts for every measurable $\beta \leq \alpha$.*

He only subsequently learned that Trnková [93] had proved this theorem earlier in the special case that also the domain of F is \mathbf{Set} . In [25], keeping the general domain \mathcal{K} , he went on to expand it further to functors with target categories other than \mathbf{Set} .

The themes touched upon in, or emerging from, Reinhard’s thesis very much reverberate in today’s research. I can mention here only one example in this regard. It concerns the important notion of extensive category, a term introduced by Carboni, Lack and Walters in [70]: a category \mathcal{K} with (finite) coproducts and pullbacks is (finitely) extensive if (finite) coproducts are universal (i.e., stable under pullback) and disjoint (i.e., the pullback of any two coproduct injections with distinct labels is the initial object). This is a typically geometric property shared by \mathbf{Set} and \mathbf{Top} , while a pointed extensive category must be trivial. Every elementary topos is finitely extensive, and Grothendieck topoi (i.e., the localizations of presheaf categories) may be characterized as those Barr-exact categories with a generating set of objects that are extensive. In a (finitely) extensive category the (finitely) connected objects are characterized as a topologist would expect: they are precisely the coproduct-indecomposable objects, i.e., those non-initial objects X with the property that whenever X is presented as a coproduct of Y and Z , one of Y, Z must be initial.

Reinhard started his studies of the universality and disjointness properties of coproducts years before the appearance of [70]. His initial account [26] went through a multi-year period of refinement, extension and correction before it finally got published in [46]. But his first account already contains all the ingredients to the proof of a refined analysis of the notion of (finite) extensivity that is missing from [70]; it shows that universality almost implies disjointness, as follows:

Theorem 4.3. *A category with (finite) coproducts and pullbacks is (finitely) extensive if, and only if, non-empty (binary) coproducts are universal and pre-initial objects are initial.*

(A *pre-initial* object admits at most one morphism into any other object, while an initial object admits exactly one. A streamlined proof of the Theorem is contained in [79].) The dual of the category of commutative unital rings is finitely extensive, and Reinhard gave an example showing that commutativity is essential here, although \mathbf{Rng}^{op} still has the disjointness property.

5. Measure and Integration

Given the wide range of his mathematical interests, it is hardly surprising that a large part of Reinhard's work addresses analytic themes, which are also at the core of his *Habilitationschrift* [31], titled "A categorical approach to integration theory" (written in German, with the preprint [28] giving a compressed English version of it). Before Reinhard started his work in this area, there had been only few attempts to present measure and integration theory in a categorically satisfactory fashion, with limited follow-up work; among others, see [82, 83, 73]. Of these, Reinhard's approach may be seen as a further development of Linton's early work.

The starting point in his approach is the elementary, but crucial, observation that integration of simple functions is given by a *universal property*. Specifically, for a Boolean algebra B (with top and bottom elements 1 and 0) and a real vector space A , the space $M(B, A)$ of *charges* $\mu : B \rightarrow A$ (i.e., of maps μ with $\mu(u \vee v) = \mu(u) + \mu(v)$ for all $u, v \in B$ with $u \wedge v = 0$) is representable when considered as a functor in A , so that for the fixed Boolean algebra B there is a real vector space EB with $M(B, -) \cong \text{Hom}_{\mathbb{R}}(EB, -) : \mathbf{Vec}_{\mathbb{R}} \rightarrow \mathbf{Set}$. Hence, there is a charge $\chi_B : B \rightarrow EB$ such that any charge $\mu : B \rightarrow A$ factors as $\mu = l \cdot \chi_B$, for a uniquely determined \mathbb{R} -linear map $l : EB \rightarrow A$. For a set algebra B of a set Ω , EB is the space of simple functions, and χ_B assigns to a subset of Ω in B its characteristic function. In particular then, for $A = \mathbb{R}$ and a charge μ , the corresponding map l assigns to a simple function its integral with respect to μ .

Since every bounded measurable function is the uniform limit of simple functions, it is clear that one must provide for a "good" convergence setting to arrive at a satisfactory integration theory, and Reinhard formulates the following necessary steps to this end: 1. express the integration of simple functions categorically in sufficient generality; 2. provide for a "convenient

convergence environment”, by replacing the category of sets by a suitable category of topological spaces; 3. test the categorical theory obtained against classical approaches to, and results in, integration theory. Unfortunately, as Reinhard explains in the 18-page introduction to his *Habilitationsschrift*, this obvious roadmap is loaded with specific obstacles.

The “simple integration theory” sketched above relies crucially on the fact that the symmetric monoidal-closed category $\mathbf{Vec}_{\mathbb{R}}$ lives over the Cartesian closed category \mathbf{Set} , with the left adjoint L to the forgetful functor $V : \mathbf{Vec}_{\mathbb{R}} \rightarrow \mathbf{Set}$ preserving the monoidal structure: $L(X \times Y) \cong L(X) \otimes L(Y)$ for all sets X, Y . Since the category \mathbf{Top} fails to be Cartesian closed and can therefore not replace \mathbf{Set} , the first question then is which subtype of topological or analytic structure one should add on both sides of the adjunction without losing its “monoidal well-behavedness”. A good replacement candidate for \mathbf{Set} is the Cartesian closed category $\mathbf{SeqHaus}$ of *sequential Hausdorff spaces* (in which every sequentially closed subset is actually closed). However, since even its finite (categorical) products generally carry a finer topology than the product topology, vector space objects in $\mathbf{SeqHaus}$ may fail to be topological vector spaces. To overcome this and other “technical” obstacles, Reinhard restricts himself to considering only vector spaces in which convergence to 0 may be tested with *convex* neighbourhoods of 0, thus replacing the functor V above by the forgetful functor $\mathbf{SCS} \rightarrow \mathbf{SeqHaus}$ of *sequentially convex spaces*. Reassuringly, \mathbf{SCS} is still big enough to contain all Banach spaces (real or complex), even all locally convex Fréchet spaces.

His general categorical setting and theory is centred around a right-adjoint functor $V : \mathcal{A} \rightarrow \mathcal{X}$ with a (semi-)additive category \mathcal{A} where, for simplicity, I assume here that both \mathcal{A} and \mathcal{X} be finitely complete and cocomplete. For every Boolean algebra object B in \mathcal{X} and every A in \mathcal{A} he gives a categorical construction of the set $M(B, A)$ of A -valued measures on B . As described in the elementary case of set-based charges, a representation of $M(B, -) : \mathcal{A} \rightarrow \mathbf{Set}$ defines a *universal* measure $\chi_B : B \rightarrow EB$, where EB plays the role of $L_{(\infty)}(B)$ in concrete situations, and the factorization of an arbitrary measure μ through χ_B defines the integral with respect to μ . *Multiplicativity* of measures, a property that Reinhard defines in this abstract setting, requires a symmetric monoidal structure on \mathcal{A} and the well-behavedness of the left adjoint L of V with respect to that structure on \mathcal{A} and the Cartesian structure of \mathcal{X} . Under mild hypotheses he then shows that

the universal measure is automatically multiplicative and that E , considered as a functor $\mathcal{B} \rightarrow \mathcal{R}$ to the category \mathcal{R} of commutative monoid objects in the additive category \mathcal{A} , is left adjoint. As a particular consequence then, E preserves binary coproducts, a fact that may be interpreted as *Fubini's Theorem*, as one may explain for the specific categories considered earlier.

Indeed, for $\mathcal{A} = \mathbf{SCS}$, $\mathcal{X} = \mathbf{SeqHaus}$, a Boolean algebra object B in \mathcal{X} is now called a *sequential Hausdorff Boolean algebra*, and a commutative monoid object R in \mathcal{A} gives a *commutative sequentially convex algebra*. The fact that the functor $E : \mathbf{SHBool} \rightarrow \mathbf{SCA}$ preserves binary coproducts implies that, for B_0, B_1 in \mathbf{SHBool} , an element in $E(B_0 \otimes B_1)$, i.e., an *integrable functionoid* on the coproduct $B_0 \otimes B_1$ in \mathbf{SHBool} , may be considered a “functionoid in two variables”, and its “iterated integral” with respect to measures μ_0, μ_1 on B_0, B_1 respectively, coincides with its integral with respect to the (real-valued) “product measure” on the coproduct $B_0 \otimes B_1$ in \mathbf{SHBool} determined by μ_0, μ_1 .

This is only a coarse and partial sketch of the work presented in his *Habilitationschrift*. Reinhard kept working on refining and extending his integration theory till the end of his life. Beyond his published article [61] there are preliminary versions of a planned monograph on categorical integration theory of 2006 (see [57]) and 2010 (see [62]) which await some editorial work before they will hopefully be made available to a wider audience.

6. Across Mathematics

In the previous sections I have tried to give an impression of Reinhard's contributions to category theory and its applications to algebra, topology and analysis. But I haven't touched upon many of his other contributions (as listed in the References) that have no apparent connection to the type of work mentioned so far, for example in number theory (algebraic or analytic) and topology (general or algebraic), of which I can mention here only very few examples. They should underline his fascination with “concrete” objects and problems, his mastery of which was as strong as that of “abstract” mathematical theories. Take, for example, the intricate proof of his solution [39] to the problem of “*How to make a path injective*” that cleverly utilizes the order of the unit interval $I = [0, 1]$:

Theorem 6.1. *Let $\varphi : I \rightarrow X$ be a continuous path from a to b in a Hausdorff space X , $a \neq b$. Then there exist an injective continuous path $\psi : I \rightarrow X$ from a to b , a closed subset $A \subseteq I$ and a continuous order-preserving map $p : I \rightarrow I$ with $p(A) = I$ and $\psi \cdot p|_A = \varphi$.*

In [53] he constructs “A non-Jordan measurable regularly open subset of the unit interval”, and in [33] he exploits the role of rational numbers in \mathbb{R} to give a surprisingly easy example of a “reasonable” connected Hausdorff space in which every point has a hereditarily disconnected neighbourhood. In fact, he proves the following theorem.

Theorem 6.2. *There is a topology on the set of real numbers finer than the Euclidean topology, making it a connected Hausdorff space that is the union of two hereditarily disconnected open subspaces.*

His proof takes less than a page and “adds” just a little elementary number theory to everybody’s knowledge of the topology of the real line. Quite a different side of number theory is displayed in Reinhard’s informal discussion note [30] that was sparked by the observation $6! \cdot 7! = 10!$ and the quest for other integer solutions x, y, z of $x! \cdot y! = z!$ with $1 \leq x \leq y$. Hence, after discarding the “trivial” solutions $1, y, y$ with $y \geq 1$ and $x, x! - 1, x!$ with $x \geq 3$ he asked whether the set S of *non-trivial* solutions is finite or, in fact, contains any triple other than $6, 7, 10$. His note, which asks for input from specialist number theorists, does not settle this question, but it does provide the following constraint on members of S that he obtained with analytic methods:

Theorem 6.3. *Any non-trivial integer solution to $x! \cdot y! = z!$ with $1 \leq x \leq y$ must satisfy $2\sqrt{\frac{x}{2}} - x < y$. As a consequence, there is no non-trivial integer solution to that equation with $x = y$.*

7. Farewell

As a former colleague and frequent coauthor I belong to the many privileged people with whom Reinhard generously shared the depth and breadth of his mathematical knowledge and ideas. They include his teachers as much as his students and the accidental acquaintance at a conference, all of whom

may have experienced his initial shyness that, however, could quickly give way to a spark in his eyes when confronted with an interesting mathematical question, usually followed by a rapid flow of pointed remarks that were often difficult to comprehend at first. Reinhard's premature death is surely a great loss to all of us.

Despite his superior talents Reinhard was a fundamentally modest person, with firm beliefs in Christian values. He saw no conflict between science and his religion, the principles of which he consistently upheld as a letter writer to papers and author of non-mathematical articles. His life-long dedicated engagement in local parish work as well as his contributions to national organizations addressing social and environmental issues, especially regarding the impact of individual car traffic, may not have been as visible to the people around him as they deserved to be. For example, in spite of having known him since his early university student times, it took me years to understand that his passion for railways and especially the use of local trains and public transport were rooted in much more than just a hobby.

Reinhard hardly ever talked much about himself, neither about his accomplishments nor his problems. His mathematical coworkers would rarely hear from him about his engagements outside mathematics, even when these were professionally related to his mathematical activities, such as his ambition to learn the Czech language. And only when asked directly would one hear the proud father speak about his three sons Lukas, Simon and Jonas. He fought hard to overcome the consequences of a devastating stroke some seven years before his death, especially as he was looking forward to celebrate later in 2014 his sixtieth birthday and the thirtieth anniversary of his wedding to Andrea Börger. Sadly, he lost that battle.

In what follows I first list, in approximate chronological order and typeset in *italics*, Reinhard Börger's written mathematical contributions, including unpublished or incomplete works, to the extent I was able to trace them, followed by an alphabetical list of references to other works cited in this article.

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**RESUMES DES ARTICLES PUBLIES
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CHRISTENSEN & ENXIN WU, Tangent spaces and tangent bundles for diffeological spaces, 3-50.

Les auteurs étudient comment la notion d'espace tangent peut être étendue aux espaces difféologiques, qui généralisent les variétés régulières et comprennent les espaces singuliers et de dimension infinie. Ils s'intéressent à l'espace tangent interne, défini en utilisant les courbes régulières, et à l'espace tangent externe, défini en utilisant les dérivations régulières. Ils prouvent des résultats fondamentaux sur ces espaces tangents, les calculent sur des exemples, et observent qu'en général ils sont différents même s'ils coïncident sur de nombreux exemples. Rappelant la définition du fibré tangent donnée par Hector ils montrent que la multiplication scalaire et l'addition peuvent ne pas y être régulières. Une définition du fibré tangent résolvant ce problème est donnée, à l'aide de ce qu'ils appellent la difféologie dvs. Ces fibrés tangents sont étudiés, calculés sur des exemples, et on étudie si leurs fibres sont des espaces vectoriels munis de la difféologie fine. Parmi les exemples : espaces singuliers, tores irrationnels, espaces vectoriels de dimension infinie, groupes difféologiques, espaces d'applications régulières entre variétés régulières.

HOSSEINI & QASEMI-NEZHAD, Equalizers in Kleisli categories, 51-76.

Cet article donne des conditions nécessaires et suffisantes pour qu'une paire de morphismes d'une catégorie de Kleisli associée à une monade générale ait un égalisateur. On propose aussi, dans différents cas de monades intéressantes, un meilleur critère pour l'existence d'un égalisateur et dans ces cas, on explicite ce qu'est l'égalisateur (lorsqu'il existe).

T. JANELIDZE-GRAY, Calculus of E-Relations in incomplete relatively regular categories, 83-102.

L'auteur définit une catégorie régulière relative incomplète comme une paire $(C;E)$, où C est une catégorie arbitraire et E une classe

d'épimorphismes réguliers dans C satisfaisant certaines conditions. Elle développe un 'calcul relatif des relations' dans ces catégories ; on peut l'appliquer aux relations $(R; r_1; r_2) : A \rightarrow A$ dans C telles que les morphismes r_1 et r_2 sont dans E . Ceci généralise plusieurs résultats connus, y compris le travail récent avec J. Goedecke sur les catégories relatives de Goursat. On définit les catégories régulières relatives incomplètes de Goursat et : (a) on prouve les versions relatives incomplètes des conditions équivalentes définissant les catégories régulières relatives de Goursat ; (b) on montre que dans ce contexte l'axiome E-Goursat est équivalent à la version relative du Lemme 3 x 3.

GRANDIS & PARE, An introduction to multiple categories (on weak and lax multiple categories, I), 103-159.

Les auteurs introduisent les catégories multiples faibles de dimension infinie, une extension des catégories doubles et triples. Ils considèrent aussi une forme 'chirale' partiellement lax, ayant des inter-changeurs dirigés, et une forme plus lax déjà étudiée dans deux articles précédents en dimension trois sous le nom de « inter-catégorie ». Dans ce contexte ils entreprennent une étude des tabulateurs, des limites supérieures de base, qui sera poursuivie dans l'article suivant.

GRANDIS & PARE, Limits in multiple categories (on weak and lax multiple categories, II), 163-202.

Cette suite de l'article précédent étudie les limites multiples dans les *catégories multiples chirales* (de dimension infinie) -- une forme faible partiellement lax ayant des inter-changeurs dirigés. Après avoir défini les limites multiples, on prouve qu'elles sont engendrées par les produits, égalisateurs et tabulateurs multiples, tous étant supposés être respectés par les opérations de face et dégénérescence. Les tabulateurs sont donc les limites supérieures de base, comme dans le cas des catégories doubles. On considère aussi les *inter-catégories*, une forme plus lax de catégorie multiple étudiée dans deux articles précédents. Dans ce cadre plus général les limites de base ci-dessus peuvent encore être définies, mais une théorie générale des limites multiples n'est pas développée ici.

W. RUMP, The completion of a quantum B-algebra, 203-228.

On montre que les quantales sont les objets injectifs dans la catégorie des quantum B-algèbres, et que chaque quantum B-algèbre a une enveloppe injective. Par une construction explicite, l'enveloppe injective se révèle comme une complétion, plus générale que la complétion de Dedekind-MacNeille. Un résultat récent de Lambek et al., où des structures résiduelles surviennent de manière surprenante, est expliqué à la lumière des quantum B-algèbres, fournissant un autre exemple de leur ubiquité. Des connexions aux structures promonoïdales et aux multi-catégories sont indiquées.

N. GILL, On a conjecture of Degos, 229-238.

Cette note donne une preuve d'une conjecture de Degos à propos des groupes engendrés par des matrices compagnons dans $GL_n(q)$.

BLUTE, LUCYSHYN-WRIGHT & O'NEILL - Derivations in codifferential categories, 243-280.

Etude d'une notion générale de dérivation dans le contexte des catégories codifférentielles de Blute-Cockett-Seely, généralisant la notion de dérivation K-linéaire de l'algèbre commutative. Pour une catégorie codifférentielle $(C; T; d)$, une T-dérivation $\partial: A \rightarrow M$ sur une algèbre A de la monade T est définie comme un morphisme de C dans un A-module M vérifiant une forme du théorème de dérivation des fonctions composées par rapport à la transformation dérivateur d . On montre que ces T-dérivations correspondent aux T-homomorphismes de A dans une T-algèbre associée $W(A;M)$. Les auteurs montrent l'existence de T-dérivations universelles de A dans un A-module associé Ω_A^T , le module de différentiels de type Kähler. Tandis que l'article précédent de Blute-Cockett-Porter-Seely sur les catégories Kählériennes utilisait une notion de dérivation exprimable sans référence à la monade T, on montre que l'usage de la notion de T-dérivation ci-dessus résout un problème ouvert concernant les catégories Kählériennes, montrant que la Propriété K pour catégories codifférentielles n'est pas nécessaire. Au cours du chemin, on établit une définition succincte et équivalente de la notion de catégorie codifférentielle en termes d'un morphisme de monades $S \rightarrow T$ sur la monade S de l'algèbre symétrique et d'une transformation d vérifiant le théorème de dérivation des fonctions composées.

J. BOURKE, Note on the construction of globular weak ω -groupoids from types, topological spaces..., 281-294.

Courte introduction aux ω -groupoïdes faibles de Grothendieck. Le but est de démontrer que, dans certains contextes, ce simple langage est utile à la construction de ω -groupoïdes faibles globulaires. Pour cela, l'auteur reformule brièvement la construction, due à van den Berg et Garner, d'un ω -groupoïde faible de Batanin à partir d'un type en utilisant le langage des ω -groupoïdes faibles de Grothendieck. Cette construction s'applique aussi aux espaces topologiques ainsi qu'aux complexes de Kan.

W. THOLEN, A mathematical tribute to Reinhard Börger, 295-314.

Un bref résumé de la vie et du travail de Reinhard Börger (1954–2014) est présenté, en mettant l'accent sur ses premiers travaux et sur ses travaux non publiés.

TABLE DES MATIERES DU VOLUME LVII (2016)*Fascicule 1*

CHRISTENSEN & ENXIN WU, Tangent spaces and tangent bundles for diffeological spaces	3
HOSSEINI & QASEMI-NEZHAD, Equalizers in Kleisli categories	51

Fascicule 2

T. JANELIDZE-GRAY, Calculus of E-Relations in incomplete relatively regular categories	83
GRANDIS & PARE, An introduction to multiple categories (on weak and lax multiple categories, I)	103

Fascicule 3

GRANDIS & PARE, Limits in multiple categories (on weak and lax multiple categories, II)	163
W. RUMP, The completion of a quantum B-algebra	203
N. GILL, On a conjecture of Degos	229

Fascicule 4

BLUTE, LUCYSHYN-WRIGHT & O'NEILL - Derivations in codifferential categories	243
J. BOURKE, Note on the construction of globular weak ω-groupoids from types, topological spaces...,	281
W. THOLEN, A mathematical tribute to Reinhard Börger	295
RESUMES des articles parus dans le Volume LVII (2016)	315

