cahiers de topologie et géométrie différentielle catégoriques

créés par CHARLES EHRESMANN en 1958
dirigés par Andrée CHARLES EHRESMANN
VOLUME LVII-3, 3ème trimestre 2016

SOMMAIRE

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES Vol. LVII-3 (2016)

LIMITS IN MULTIPLE CATEGORIES (ON WEAK AND LAX MULTIPLE CATEGORIES, II)

by Marco GRANDIS and Robert PARE

Résumé. Suite au premier article de cette série, on étudie ici les limites multiples dans les *catégories multiples chirales* (de dimension infinie) - une forme faible partiellement laxe ayant des interchangeurs dirigés.
Après avoir défini les limites multiples, nous prouvons qu'elles sont engendrées par les *produits*, *égalisateurs* et *tabulateurs* multiples - tous étant supposés être respectés par les oprations de faces et dégénérescence. Les tabulateurs sont donc les limites supérieures de base, comme dans le cas des catégories doubles.
On considère aussi les *intercatégories*, une forme plus laxe de catégorie multiple étudiée dans deux articles précédents. Dans ce cadre plus général les limites de base ci-dessus peuvent encore être définies, mais une théorie générale des limites multiples n'est pas développée ici.
Abstract. Continuing our first paper in this series, we study multiple limits in infinite-dimensional *chiral multiple categories* - a weak, partially lax form with directed interchangers.
After defining multiple limits, we prove that all of them can be constructed from (multiple) *products*, *equalisers* and *tabulators* - all of them assumed to be respected by faces and degeneracies. Tabulators appear thus to be the basic higher limits, as was already the case for double categories.
Intercategories, a laxer form of multiple category already studied in two previous papers, are also considered. In this more general setting the basic limits mentioned above can still be defined, but a general theory of multiple limits is not developed here.
Keywords. multiple category, double category, cubical set, limit.
Mathematics Subject Classification (2010). 18D05, 55U10, 18A30.

0. Introduction

Strict double and multiple categories were introduced and studied by C. Ehresmann and A.C. Ehresmann [Eh, BE, EE1, EE2, EE3]. Strict cubical categories can be seen as a particular case of multiple categories; their links with strict ω-categories are made clear in the article [ABS].

The present series studies various 'forms' of weak or lax multiple categories, of finite or infinite dimension. They extend weak double categories [GP1 - GP4] and weak cubical categories [G1, G2, GP5]. More information on literature on higher dimensional category theory can be found in the Introduction of the first paper [GP8], here referred to as Part I.

Our main framework, a *chiral multiple category*, is briefly reviewed here, in Section 1; it is a partially lax multiple category with a strict composition $gf = f +_0 g$ in direction 0 (the *transversal* direction), weak compositions $x +_i y$ in all positive (or *geometric*) directions i and directed interchanges for the i- and j-compositions (for $0 < i < j$)

$$\chi_{ij} \colon (x +_i y) +_j (z +_i u) \to_0 (x +_j z) +_i (y +_j u) \qquad \textit{(ij-interchanger)}. \quad (1)$$

Part I also considers a laxer form already studied in two previous papers [GP6, GP7] for the 3-dimensional case, under the name of 'intercategory', that is particularly powerful: it covers duoidal categories, Gray categories, Verity double bicategories, monoidal double categories, etc. In this framework, extended in Part I to infinite dimension and recalled here in 1.9, there are also *lower interchangers* $(\tau_{ij}, \mu_{ij}, \delta_{ij})$ where positive degeneracies (i.e. weak identities) intervene; in particular degeneracies are *no longer assumed to commute*, but have a directed interchange for $0 < i < j$

$$\tau_{ij} \colon e_j e_i(x) \to_0 e_i e_j(x) \qquad (\textit{ij-interchanger for identities}). \quad (2)$$

Here we study multiple limits in the setting of *chiral multiple categories*. Part of the theory is briefly extended to intercategories, *with the problems discussed below*.

Our general definition of multiple limits (in 4.4) requires their preservation by faces and degeneracies (as in the cubical case [G2]). We prove that all of them can be constructed from (multiple) *products*, *equalisers* and *tabulators*. The latter appear thus to be the basic higher form of a limit, as was already the case for double and cubical categories. In particular this holds in a 2-category, where tabulators (of vertical identities) reduce to cotensors by the ordinal **2**; the previous result agrees thus with Theorem 10 of R. Street [St1], according to which all weighted limits in a 2-category can be constructed from such cotensors and ordinary limits.

More analytically, Section 1 contains a review of the basic notions of strict, weak and chiral multiple categories. We also introduce the 'lift functors' that will play a relevant role below.

Then, in Section 2, we begin our study of limits with the simple case of $\mathbf{i}$-*level limits*, for a *positive* multi-index $\mathbf{i} = \{i_1, ..., i_n\}$. In a chiral multiple category A, $\mathbf{i}$-level limits are ordinary limits in the transversal category $\mathrm{tv}_{\mathbf{i}}(\mathsf{A})$. When all these exist, *and are preserved by faces and degeneracies* between transversal categories, we say that A has *level multiple limits*. Of course, multiple products and multiple equalisers generate all of them.

Non-level limits, where the diagram and the limit object are not confined to a transversal category, are studied in the next two sections. The main theorems on the construction and preservation of multiple limits are stated in 3.6 and 4.5, and proved in Section 5.

The main example treated here is the chiral triple category $\mathsf{SC}(\mathbf{C})$ of *spans and cospans* over a category $\mathbf{C}$ with pushouts and pullbacks (see 1.8, 2.1, 2.2, 3.7 and 4.6). One can similarly study multiple limits (and colimits) in other weak or chiral multiple categories of finite or infinite dimension, listed at the beginning of Section 2.

The relationship with the double limits of [GP1] are discussed in Sections 2 and 4. In the case of level limits (see 2.6) there are only some variations in terminology; for non-level limits there is a difference (see 4.7).

The general theory of multiple colimits is dual to that of multiple limits and is not written down explicitly. Showing this requires some technical expedient because - as we have seen in Part I - transversal duality turns a (right) chiral multiple category into a *left-hand version* where all interchangers have the opposite direction. Thus, a multiple colimit in the chiral multiple category A is a multiple limit in a *left* chiral multiple category A^{tv}; but it can also be viewed as a *multiple limit in a right chiral multiple category* $(\mathsf{A}^{\mathrm{tv}})^{-}$ indexed by the integers $\leqslant 0$ (reversing indices).

An extension of the general theory of multiple limits from the *chiral* case to *intercategories* presents serious problems, linked to the crucial fact that *degeneracies no longer commute*. Yet, the basic limits can be easily extended.

To begin with, *level limits* can be defined as here, in 2.2; one should nevertheless be aware that they do not behave so well as in the chiral case: see the end of Proposition 2.3. Tabulators can also be extended *and even acquire* *richer forms*: for instance, the total tabulator of a 12-cube gives now rise to two distinct notions, the e_1e_2-*tabulator* and the e_2e_1-*tabulator*, as already shown in Part I, Section 6. However it is not clear what a general definition of limit should be: in a situation where degeneracies do not commute, even defining the diagonal functor becomes complicated (see 3.1).

Notation. We follow the notation of Part I; the reference I.2.3 points to its Subsection 2.3. The two-valued index α (or β) varies in the set $2 = \{0, 1\}$, often written as $\{-, +\}$. The symbol $\subset$ denotes weak inclusion. Categories and 2- categories are generally denoted as $\mathbf{A}, \mathbf{B}$...; weak double categories as $\mathbb{A}, \mathbb{B}$...; weak or lax multiple categories as A, B...

Acknowledgments. The authors would like to thank the anonymous referee for a very careful reading of the paper and detailed comments. This work was partially supported by a research grant of Università di Genova.

1. Multiple categories

After a review of the basic notions of strict multiple categories, taken from Part I, we introduce the 'lift functors' that will play a relevant role in the study of multiple limits. As it will be made clear later (in 4.8) they are a surrogate for the path endofunctor of symmetric cubical categories. These notions are then extended to *chiral* multiple categories, a weak and partially lax version introduced in Part I.

1.1 Multiple sets

A *multi-index* $\mathbf{i}$ is a finite set of natural numbers, possibly empty. Writing $\mathbf{i} \subset \mathbb{N}$ it will be understood that $\mathbf{i}$ is finite; writing $\mathbf{i} = \{i_1, ..., i_n\}$ we always mean that $\mathbf{i}$ has n *distinct* elements, written in the natural order $i_1 < i_2 < ... < i_n$; the integer n is called the *dimension* of $\mathbf{i}$.

We use the following symbols

$$\mathbf{i}j = j\mathbf{i} = \mathbf{i} \cup \{j\} \quad (\text{for } j \in \mathbb{N} \setminus \mathbf{i}), \qquad \mathbf{i}|j = \mathbf{i} \setminus \{j\} \quad (\text{for } j \in \mathbf{i}). \qquad (3)$$

A *multiple set* is a system of sets and mappings $X = ((X_\mathbf{i}), (\partial_i^\alpha), (e_i))$ under the following two assumptions.

(mls.1) For every multi-index $\mathbf{i} = \{i_1, ..., i_n\}$, $X_\mathbf{i}$ is a set whose elements are called $\mathbf{i}$-*cells* of X and said to be of *dimension* n. We write X_*, X_i, X_{ij},... instead of $X_\varnothing$, $X_{\{i\}}$, $X_{\{i,j\}}$,...; thus X_* is of dimension 0 while X_0, X_1,... are of dimension 1.

(mls.2) For $j \in \mathbf{i}$ and $\alpha = \pm$ we have mappings, called *faces* and *degeneracies* of $X_\mathbf{i}$

$$\partial_j^\alpha \colon X_\mathbf{i} \to X_{\mathbf{i}|j}, \qquad\qquad e_j \colon X_{\mathbf{i}|j} \to X_\mathbf{i}, \tag{4}$$

satisfying the *multiple relations*

$$\begin{array}{ll} \partial_i^\alpha.\partial_j^\beta = \partial_j^\beta.\partial_i^\alpha \quad (i \neq j), & \qquad e_i.e_j = e_j.e_i \quad (i \neq j), \\ \partial_i^\alpha.e_j = e_j.\partial_i^\alpha \quad (i \neq j), & \qquad \partial_i^\alpha.e_i = \mathrm{id}. \end{array} \tag{5}$$

Faces commute and degeneracies commute, but ∂_i^α and e_i do not. These relations look similar to the cubical ones but much simpler, because here an index i stands for a particular sort, instead of a mere position, and is never 'renamed'. Note also that ∂_i^α acts on $X_\mathbf{i}$ if i belongs to the multi-index $\mathbf{i}$ (and cancels it), while e_i acts on $X_\mathbf{i}$ if i does not belong to $\mathbf{i}$ (and inserts it); therefore $\partial_i^\alpha.\partial_i^\beta$ and $e_i.e_i$ make no sense, here: one cannot cancel or insert twice the same index.

If $\mathbf{i} = \mathbf{j} \cup \mathbf{k}$ is a disjoint union and $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_r)$ is a mapping $\mathbf{k} = \{k_1, ..., k_r\} \to 2$, we have an *iterated face* and an *iterated degeneracy* (independent of the order of composition)

$$\partial_\mathbf{k}^{\boldsymbol{\alpha}} = \partial_{k_1}^{\alpha_1} ... \partial_{k_r}^{\alpha_r} \colon X_\mathbf{i} \to X_\mathbf{j}, \qquad e_\mathbf{k} = e_{k_1} ... e_{k_r} \colon X_\mathbf{j} \to X_\mathbf{i}. \tag{6}$$

In particular, the *total* $\mathbf{i}$-*degeneracy* is the mapping

$$e_\mathbf{i} = e_{i_1} ... e_{i_n} \colon X_* \to X_\mathbf{i}. \tag{7}$$

1.2 Multiple categories

We recall the definition, from Part I.

(mlc.1) A *multiple category* A is, first of all, a multiple set of components $A_\mathbf{i}$, whose elements are called $\mathbf{i}$-*cells*. As above, $\mathbf{i}$ is any multi-index, i.e. any finite subset of $\mathbb{N}$, and we write A_*, A_i, A_{ij}... for $A_\varnothing$, $A_{\{i\}}$, $A_{\{i,j\}}$,...

(mlc.2) Given two $\mathbf{i}$-cells x, y which are *i-consecutive* (i.e. $\partial_i^+(x) = \partial_i^-(y)$, with $i \in \mathbf{i}$), the *i-composition* $x +_i y$ is defined and satisfies the following interactions with faces and degeneracies, for $j \neq i$

$$\begin{array}{ll} \partial_i^-(x +_i y) = \partial_i^-(x), & \partial_i^+(x +_i y) = \partial_i^+(y), \\ \partial_j^\alpha(x +_i y) = \partial_j^\alpha(x) +_i \partial_j^\alpha(y), & e_j(x +_i y) = e_j(x) +_i e_j(y). \end{array} \tag{8}$$

(mlc.3) For every multi-index $\mathbf{i}$ containing j we have a category $\mathrm{cat}_{\mathbf{i},j}(\mathsf{A})$ with objects in $A_\mathbf{i}$, arrows in $A_{\mathbf{i}j}$, faces ∂_j^α, identities e_j and composition $+_j$.

(mlc.4) For $i < j$ we have

$$(x +_i y) +_j (z +_i u) = (x +_j z) +_i (y +_j u) \qquad (\textit{binary ij-interchange}), \tag{9}$$

whenever these composites make sense. (Note that the lower interchanges are already expressed above.)

More generally, for an ordered pointed set $N = (N, 0)$, an *N-indexed multiple category* A has components $A_\mathbf{i}$ *indexed by* (finite) *multi-indices* $\mathbf{i} \subset N$. If N is the ordinal set $\mathbf{n} = \{0, ..., n-1\}$ we obtain the n-dimensional version of a multiple category, called an *n-tuple category*. The 0-, 1- and 2-dimensional versions amount - respectively - to a set, a category or a double category.

1.3 Transversal categories

The *transversal* direction, corresponding to the index $i = 0$, is treated differently in the theory: we think of it as the ‘dynamic’ direction, along which ‘transformation occurs’, while the positive directions $i > 0$ are viewed as the ‘static’ or ‘geometric’ ones.

A *positive* multi-index $\mathbf{i} = \{i_1, ..., i_n\}$ (with $n \geqslant 0$ positive elements) has an ‘augmented’ multi-index $0\mathbf{i} = \{0, i_1, ..., i_n\}$. The *transversal category of* $\mathbf{i}$*-cubes of* A

$$\mathrm{tv}_\mathbf{i}(\mathsf{A}) = \mathrm{cat}_{\mathbf{i},0}(\mathsf{A}), \tag{10}$$

- has objects in $A_\mathbf{i}$, called $\mathbf{i}$-*cubes* and viewed as n-dimensional objects,

- has arrows $f\colon x^- \to_0 x^+$ in $A_{0\mathbf{i}}$, called $\mathbf{i}$-*maps*, with domain and codomain $\partial_0^\alpha(f) = x^\alpha$,

- has identities $1_x = \mathrm{id}(x) = e_0(x)\colon x \to_0 x$ and composition $gf = f +_0 g$.

All these items are said to be *of degree* n (though their dimension may be n or $n+1$): the degree always refers to the number of positive indices. In all of our examples, 0-composition is realised by the usual composition of mappings, while the 'positive' compositions (also called *concatenations*) are often obtained by operations (products, sums, tensor products, pullbacks, pushouts...) where reversing the order of the operands would only be confusing.

Faces and degeneracies give (ordinary) functors

$$\partial_j^\alpha\colon \mathrm{tv}_{\mathbf{i}j}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A}), \quad e_j\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}j}(\mathsf{A}) \quad (j \notin \mathbf{i},\, \alpha = \pm). \tag{11}$$

In particular, the unique positive multi-index of degree 0, namely $\varnothing$, gives the category $\mathrm{tv}_*(\mathsf{A})$ of *objects* of A (i.e. $\star$-cells) and their transversal maps (i.e. 0-cells).

An $\mathbf{i}$-map $f\colon x \to_0 y$ is said to be i-*special*, or *special in direction* $i \in \mathbf{i}$, if its i-faces are transversal identities (of $\mathbf{i}|i$-cubes)

$$\partial_i^\alpha f = e_0\partial_i^\alpha x = e_0\partial_i^\alpha y. \tag{12}$$

This, of course, implies that the $\mathbf{i}$-cubes x, y have the same i-faces. We say that f is *ij-special* if it is special in both directions i, j.

1.4 Multiple functors and transversal transformations

A *multiple functor* $F\colon \mathsf{A} \to \mathsf{B}$ between multiple categories is a morphism of multiple sets $F = (F_{\mathbf{i}})$ that preserves all the composition laws. For an $\mathbf{i}$-map $f\colon x \to_0 y$, we use one of the following forms

$$F(f)\colon F(x) \to_0 F(y), \qquad\qquad F_{0\mathbf{i}}(f)\colon F_{\mathbf{i}}(x) \to_0 F_{\mathbf{i}}(y),$$

as may be convenient.

A *transversal transformation* $h\colon F \to G\colon \mathsf{A} \to \mathsf{B}$ between multiple functors consists of a face-consistent family of $\mathbf{i}$-maps in B (its components), for every *positive* multi-index $\mathbf{i}$ and every $\mathbf{i}$-cube x in A

$$\begin{aligned} &hx\colon F(x) \to_0 G(x) && (h_{\mathbf{i}}x\colon F_{\mathbf{i}}(x) \to_0 G_{\mathbf{i}}(x)),\\ &h(\partial_j^\alpha x) = \partial_j^\alpha(hx) && (j \in \mathbf{i}). \end{aligned} \tag{13}$$

The following axioms of naturality and coherence are required:

(trt.1) $Gf.hx = hy.Ff \qquad$ (for $f\colon x \to_0 y$ in A),

(trt.2) h commutes with positive degeneracies and compositions:

$$h(e_j z) = e_j(hz), \qquad\qquad h(x +_j y) = hx +_j hy.$$

where $\mathbf{i}$ is a positive multi-index, $j \in \mathbf{i}$, x and y are j-consecutive $\mathbf{i}$-cubes, z is an $\mathbf{i}|j$-cube.

Given two multiple categories A, B we have thus the category $\mathbf{Mlc}(\mathsf{A}, \mathsf{B})$ of their multiple functors and transversal transformations. All these form the 2-category $\mathbf{Mlc}$, in an obvious way.

More generally for any ordered pointed set $N = (N, 0)$ we have the 2-category $\mathbf{Mlc}_N$ of N-indexed multiple categories, formed of ordinary categories $\mathbf{Mlc}_N(\mathsf{A}, \mathsf{B})$.

1.5 Lift functors

For a *positive* integer j there is a *j-directed lift functor* with values in the 2-category of multiple categories indexed by the pointed set $\mathbb{N}|j = \mathbb{N} \setminus \{j\}$

$$Q_j\colon \mathbf{Mlc} \to \mathbf{Mlc}_{\mathbb{N}|j}. \tag{14}$$

For a multiple category A, the multiple category $Q_j\mathsf{A}$ is - loosely speaking - that part of A that contains the index j, reindexed without it:

$$\begin{aligned}
&(Q_j\mathsf{A})_{\mathbf{i}} = A_{\mathbf{i}j},\\
&(\partial_i^\alpha\colon (Q_j\mathsf{A})_{\mathbf{i}} \to (Q_j\mathsf{A})_{\mathbf{i}|i}) = (\partial_i^\alpha\colon A_{\mathbf{i}j} \to A_{\mathbf{i}j|i}),\\
&(e_i\colon (Q_j\mathsf{A})_{\mathbf{i}|i} \to (Q_j\mathsf{A})_{\mathbf{i}}) = (e_i\colon A_{\mathbf{i}j|i} \to A_{\mathbf{i}j}) \qquad (i \in \mathbf{i} \subset \mathbb{N}|j),
\end{aligned} \tag{15}$$

and similarly for compositions. In the same way for multiple functors $F, G\colon \mathsf{A} \to \mathsf{B}$ and a transversal transformation $h\colon F \to G\colon \mathsf{A} \to \mathsf{B}$ we let

$$(Q_jF)_{\mathbf{i}} = F_{\mathbf{i}j}, \qquad (Q_jh)_{\mathbf{i}} = h_{\mathbf{i}j} \qquad\qquad (\mathbf{i} \subset \mathbb{N}|j). \tag{16}$$

There is also an obvious restriction 2-functor $R_j\colon \mathbf{Mlc} \to \mathbf{Mlc}_{\mathbb{N}|j}$, where the multiple category $R_j\mathsf{A}$ is that part of A that does not contain the index j. The j-directed faces and degeneracies of A are not used in $Q_j\mathsf{A}$, but yield three natural transformations, also called *faces* and *degeneracy*, with the following components for $\mathbf{i} \subset \mathbb{N}|j$

$$\begin{aligned}
&D_j^\alpha\colon Q_j \to R_j\colon \mathbf{Mlc} \to \mathbf{Mlc}_{\mathbb{N}|j}, \qquad (D_j^\alpha)_{\mathbf{i}} = \partial_j^\alpha\colon A_{\mathbf{i}j} \to A_{\mathbf{i}},\\
&E_j\colon R_j \to Q_j\colon \mathbf{Mlc} \to \mathbf{Mlc}_{\mathbb{N}|j}, \qquad (E_j)_{\mathbf{i}} = e_j\colon A_{\mathbf{i}} \to A_{\mathbf{i}j},\\
&D_j^\alpha E_j = \mathrm{id}.
\end{aligned} \tag{17}$$

In particular, the objects and $\star$-maps of $Q_j(\mathsf{A})$ are the j-cubes and j-maps of A, so that

$$\begin{gathered}
\mathrm{tv}_*(Q_j(\mathsf{A})) = \mathrm{tv}_j\mathsf{A}, \qquad \mathrm{tv}_*(R_j(\mathsf{A})) = \mathrm{tv}_*\mathsf{A},\\
\mathrm{tv}_*(D_j^\alpha) = \partial_j^\alpha\colon \mathrm{tv}_j\mathsf{A} \to \mathrm{tv}_*\mathsf{A}, \qquad \mathrm{tv}_*(E_j) = e_j\colon \mathrm{tv}_*\mathsf{A} \to \mathrm{tv}_j\mathsf{A}.
\end{gathered} \tag{18}$$

Plainly all the functors Q_j commute. By composing n of them in any order we get an *iterated lift functor* of degree n, in a *positive* direction $\mathbf{i} = \{i_1, ..., i_n\}$

$$\begin{gathered}
Q_{\mathbf{i}}\colon \mathbf{Mlc} \to \mathbf{Mlc}_{\mathbb{N}|\mathbf{i}}, \qquad Q_{\mathbf{i}}(\mathsf{A}) = Q_{i_n}...Q_{i_1}(\mathsf{A}),\\
\mathrm{tv}_*(Q_{\mathbf{i}}(\mathsf{A})) = \mathrm{tv}_{\mathbf{i}}(\mathsf{A}).
\end{gathered} \tag{19}$$

Again, there are faces and degeneracies (where $j \notin \mathbf{i}$, $\mathbf{h} \subset \mathbb{N}|\mathbf{i}j$ and $\mathbf{hi} = \mathbf{h} \cup \mathbf{i}$)

$$\begin{aligned}
&D_j^\alpha\colon Q_{\mathbf{i}j} \to R_jQ_{\mathbf{i}}\colon \mathbf{Mlc} \to \mathbf{Mlc}_{\mathbb{N}|\mathbf{i}j}, \quad (D_j^\alpha)_{\mathbf{h}} = \partial_j^\alpha\colon A_{\mathbf{hi}j} \to A_{\mathbf{hi}},\\
&E_j\colon R_jQ_{\mathbf{i}} \to Q_{\mathbf{i}j}\colon \mathbf{Mlc} \to \mathbf{Mlc}_{\mathbb{N}|\mathbf{i}j}, \quad (E_j)_{\mathbf{h}} = e_j\colon A_{\mathbf{hi}} \to A_{\mathbf{hi}j},
\end{aligned} \tag{20}$$

$$\mathrm{tv}_*(D_j^\alpha) = \partial_j^\alpha\colon \mathrm{tv}_{\mathbf{i}j}\mathsf{A} \to \mathrm{tv}_{\mathbf{i}}\mathsf{A}, \qquad \mathrm{tv}_*(E_j) = e_j\colon \mathrm{tv}_{\mathbf{i}}\mathsf{A} \to \mathrm{tv}_{\mathbf{i}j}\mathsf{A}. \tag{21}$$

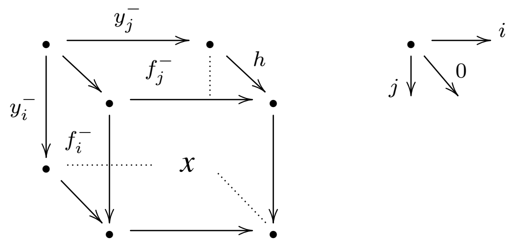
1.6 Transversal invariance

We now extend the notion of 'horizontal invariance' of double categories [GP1], obtaining a property that will be of use for multiple limits and should be expected of every 'well formed' multiple category.

We say that the multiple category A is *transversally invariant* if its cubes are 'transportable' along transversally invertible maps. Precisely:

(i) given an $\mathbf{i}$-cube x of degree n and a family of $2n$ *invertible* transversal maps $f_i^\alpha\colon y_i^\alpha \to_0 \partial_i^\alpha x$ ($i \in \mathbf{i}$, $\alpha = \pm$) *with consistent positive faces* (and otherwise arbitrary domains y_i^α)

$$\partial_i^\alpha(f_j^\beta) = \partial_j^\beta(f_i^\alpha) \qquad\qquad (\text{for } i \neq j \text{ in } \mathbf{i}), \tag{22}$$



$$(h = \partial_i^+(f_j^-) = \partial_j^-(f_i^+)),$$

there exists an invertible $\mathbf{i}$-map $f\colon y \to_0 x$ (a 'filler', as in the Kan extension property) with positive faces $\partial_i^\alpha f = f_i^\alpha$ (and therefore $\partial_i^\alpha y = y_i^\alpha$).

Of course this property can be equivalently stated for a family of invertible maps $g_i^\alpha\colon \partial_i^\alpha x \to_0 y_i^\alpha$.

1.7 Weak multiple categories

Weak multiple categories have been introduced in Part I, Section 3.

Extending weak double categories [GP1 - GP4] and weak triple categories [GP6, GP7], the basic structure of a weak multiple category A is a multiple set with compositions in all directions. The composition laws in direction 0 are categorical and have a strict interchange with the other compositions.

On the other hand, the 'positive' compositions have transversally invertible comparisons called *left i-unitor, right i-unitor, i-associator* and *ij-interchanger*, for $0 < i < j$

$$\begin{aligned}
&\lambda_i x\colon (e_i\partial_i^- x) +_i x \to_0 x,\\
&\rho_i x\colon x +_i (e_i\partial_i^+ x) \to_0 x,\\
&\kappa_i(x,y,z)\colon x +_i (y +_i z) \to_0 (x +_i y) +_i z,\\
&\chi_{ij}(x,y,z,u)\colon (x +_i y) +_j (z +_i u) \to_0 (x +_j z) +_i (y +_j u),
\end{aligned} \tag{23}$$

under coherence conditions listed in I.3.3 and I.3.4.

Our main infinite-dimensional examples are *of cubical type* (see I.3.5). Essentially, this means that components, faces and degeneracies are *invariant under renaming positive indices*, in the same order. An $\mathbf{i}$-cube can thus be indexed by $[n] = \{1, ..., n\}$ and called an *n-cube*; an $\mathbf{i}$-map can be indexed by $0[n] = \{0, 1, ..., n\}$ and called an *n-map*; again, such items are of *order* n and *dimension* n or $n + 1$, respectively. (The examples below are also *symmetric*, by a natural action of each symmetric group S_n on the sets of n-cubes and n-maps, permuting the *positive* directions; see Part I.)

(a) The strict symmetric cubical category $\omega\mathsf{Cub}(\mathbf{C})$ *of commutative cubes* over a category $\mathbf{C}$. An n-cube is a functor $x\colon \mathbf{2}^n \to \mathbf{C}$ $(n \geqslant 0)$, where $\mathbf{2}$ is the ordinal category $\bullet \to \bullet$; an n-map is a natural transformation of such functors. Applications of this multiple category (and its truncations) to algebraic K-theory can be found in [Sh].

(b) The weak symmetric cubical category $\omega\mathsf{Cosp}(\mathbf{C})$ *of cubical cospans* over a category $\mathbf{C}$ with (a fixed choice) of pushouts has been constructed in [G1], to deal with higher-dimensional cobordism. An n-cube is a functor $x\colon \wedge^n \to \mathbf{C}$, where $\wedge$ is the formal-cospan category $\bullet \to \bullet \leftarrow \bullet$; again, an n-map is a natural transformation of such functors.

(c) The weak symmetric cubical category $\omega\mathsf{Span}(\mathbf{C})$ *of cubical span*, over a category $\mathbf{C}$ with pullbacks, is similarly constructed over $\vee = \wedge^{\mathrm{op}}$, the formal-span category $\bullet \leftarrow \bullet \to \bullet$ (see [G1]). It is *transversally dual* to $\omega\mathsf{Cosp}(\mathbf{C}^{\mathrm{op}})$.

(d) The weak symmetric cubical category of *cubical bispans*, or *cubical diamonds* $\omega\mathsf{Bisp}(\mathbf{C})$, over a category $\mathbf{C}$ with pullbacks and pushouts, is similarly constructed over a formal diamond category [G1].

1.8 Chiral multiple categories

Our main framework here is more general and partially lax.

A *chiral*, or *χ-lax*, *multiple category* A (see I.3.7) has the same data and axioms of a weak multiple category, except for the fact that the interchange comparisons χ_{ij} $(0 < i < j)$ recalled above (in 1.7) are not supposed to be invertible.

Various examples are given in [GP7] and Part I, Section 4. For instance, if the category $\mathbf{C}$ has pullbacks and pushouts, the weak double category $\mathbb{S}\mathrm{pan}(\mathbf{C})$, of arrows and spans of $\mathbf{C}$, can be 'amalgamated' with the weak double category $\mathbb{C}\mathrm{osp}(\mathbf{C})$, of arrows and cospans of $\mathbf{C}$, to form a 3-dimensional structure: the chiral triple category $\mathsf{SC}(\mathbf{C})$ whose 0-, 1- *and* 2-directed arrows are the arrows, spans and cospans of $\mathbf{C}$, *in this order* (as required by the 12-interchanger). For higher dimensional examples, like $\mathsf{S}_p\mathsf{C}_q(\mathbf{C})$, $\mathsf{S}_p\mathsf{C}_\infty(\mathbf{C})$ and $\mathsf{S}_{-\infty}\mathsf{C}_\infty(\mathbf{C})$ (and the corresponding *left-chiral* cases) see I.4.4; the latter structure is indexed by all integers, with spans in each negative direction, ordinary arrows in direction 0 and cospans in positive directions.

Chiral multiple categories, with their strict multiple functors and transversal transformations, form the 2-category $\mathrm{St}\mathbf{Cmc}$.

As defined in I.3.9, a *lax multiple functor* $F\colon \mathsf{A} \to \mathsf{B}$ between chiral multiple categories, or *lax functor* for short, has components $F_\mathbf{i}\colon A_\mathbf{i} \to B_\mathbf{i}$ for all multi-indices $\mathbf{i}$ (often written as F) that agree with all faces, 0-degeneracies and 0-composition. Moreover, for every positive multi-index $\mathbf{i}$ and $i \in \mathbf{i}$, F is equipped with i-special comparison $\mathbf{i}$-maps $\underline{F}_i$ that agree with faces

$$\begin{array}{ll}
\underline{F}_i(x)\colon e_iF(x) \to_0 F(e_ix) & (x \text{ in } A_{\mathbf{i}|i}),\\
\underline{F}_i(x,y)\colon F(x) +_i F(y) \to_0 F(z) & (z = x +_i y \text{ in } A_\mathbf{i}),\\
\partial^\alpha_j\underline{F}_i(x) = \underline{F}_i(\partial^\alpha_j x) & (j \neq i),\\
\partial^\alpha_j\underline{F}_i(x,y) = \underline{F}_i(\partial^\alpha_j x, \partial^\alpha_j y) & (j \neq i).
\end{array} \tag{24}$$

These comparisons have to satisfy some axioms. We write down the naturality conditions (lmf.1-2), frequently used below, while the coherence conditions (lmf.3-5) can be found in I.3.9

(lmf.1) (*Naturality of unit comparisons*) For an $\mathbf{i}|i$-map $f\colon x \to_0 y$ in A we have:

$$F(e_if).\underline{F}_i(x) = \underline{F}_i(y).e_i(Ff)\colon\ e_iF(x) \to_0 F(e_iy).$$

(lmf.2) (*Naturality of composition comparisons*) For two i-consecutive $\mathbf{i}$-maps $f\colon x \to_0 x'$ and $g\colon y \to_0 y'$ in A we have:

$$F(f +_i g).\underline{F}_i(x,y) = \underline{F}_i(x',y').(Ff +_i Fg)\colon\ Fx +_i Fy \to_0 F(x' +_i y').$$

A *transversal transformation* $h\colon F \to G\colon \mathsf{A} \to \mathsf{B}$ between lax functors consists of a face-consistent family of $\mathbf{i}$-maps in B (its components), one for every positive multi-index $\mathbf{i}$ and every $\mathbf{i}$-cube x in A

$$hx\colon F(x) \to_0 G(x), \qquad\qquad h(\partial_i^\alpha x) = \partial_i^\alpha(hx), \tag{25}$$

under the axioms (trt.1) and (trt.2L) of I.3.9

(trt.1) $Gf.hx = hy.Ff$ (for $f\colon x \to_0 y$ in A),

(trt.2L) for every positive multi-index $\mathbf{i}$ and $i \in \mathbf{i}$:

$$h(e_i x)\underline{F}_i(x) = \underline{G}_i(x).e_i(hx)\colon e_iF(x) \to_0 G(e_ix),$$
$$h(x +_i y).\underline{F}_i(x,y) = \underline{G}_i(x,y).(hx +_i hy)\colon F(x) +_i F(y) \to_0 G(z).$$

We have thus the 2-category $\mathrm{Lx}\mathbf{Cmc}$ of chiral multiple categories, lax functors and their transversal transformations. The lax multiple functor F is said to be *unitary* if all its unit comparisons $\underline{F}_i(x)$ are transversal identities, so that $F(e_ix) = e_iF(x)$ and F is a morphism of multiple sets.

The *lift functor* and *restriction functor* in direction j (see 1.5) are extended in the same form, for $j > 0$, $j \notin \mathbf{i}$:

$$\begin{aligned} &Q_j\colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}, &&(Q_j\mathsf{A})_{\mathbf{i}} = A_{\mathbf{i}j},\\ &R_j\colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}, &&(R_j\mathsf{A})_{\mathbf{i}} = A_{\mathbf{i}}. \end{aligned} \tag{26}$$

Similarly one defines the 2-category $\mathrm{Cx}\mathbf{Cmc}$ for the colax case, where the comparisons of *colax (multiple) functors* have the opposite direction. A *pseudo (multiple) functor* is a lax functor whose comparisons are invertible (and is made colax by inverting its comparisons); such functors are the arrows of the 2-category $\mathrm{Ps}\mathbf{Cmc}$.

1.9 Intercategories

The more general case of *intercategories*, studied in [GP6, GP7] and Part I (Sections 5 and 6), will only be considered here in a marginal way.

Let us recall that an intercategory A has, besides χ_{ij}, other three kinds of directed ij-interchangers (for $0 < i < j$), where identities intervene:

(a) $\tau_{ij}(x)\colon e_je_i(x) \to_0 e_ie_j(x)$,

(b) $\mu_{ij}(x,y)\colon e_i(x) +_j e_i(y) \to_0 e_i(x +_j y)$,

(c) $\delta_{ij}(x,y)\colon e_j(x +_i y) \to_0 e_j(x) +_i e_j(y)$.

As proved in [GP7], three-dimensional intercategories comprise under a common form various structures previously studied, like duoidal categories, Gray categories, Verity double bicategories and monoidal double categories. Literature on the these structures can be found in [GP7]; the inspiring case of duoidal (or 2-monoidal) categories can be found in [AM, BS, St2].

As already noted in Part I, various 'anomalies' appear with respect to the chiral case, that make problems for a theory of multiple limits in this setting. These will be briefly considered below (see 2.3 and 3.1), without further investigating a situation for which we do not yet have examples sufficiently rich to have good limits.

Some anomalies can already be remarked here. First, an intercategory A *is no longer a multiple set* (unless each τ_{ij} is the identity). Second, a degeneracy e_i $(i > 0)$ is now *lax* with respect to every higher j-composition (for $j > i$, via τ_{ij} and μ_{ij}) but *colax* with respect to every lower j-composition (for $0 < j < i$, via τ_{ji} and δ_{ji}). Therefore, in the truncated n-dimensional case e_1 is lax with respect to all other compositions and e_n is colax, but the other positive degeneracies (if any, i.e. for $n > 3$) are neither lax nor colax.

2. Multiple level limits

We begin our study of limits with the simple case of $\mathbf{i}$-*level limits*, for a positive multi-index $\mathbf{i}$.

In a chiral multiple category A, $\mathbf{i}$-level limits are ordinary limits in the transversal category $\mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ (as in the cubical case, see [G2]). When all these exist, and are preserved by faces and degeneracies, we say that A has *level multiple limits*; of course they are 'generated' by multiple products and multiple equalisers.

Examples are given in the chiral triple category $\mathsf{SC}(\mathbf{C})$ recalled in 1.8; they can be easily adapted to the weak multiple categories

$$\omega\mathsf{Cub}(\mathbf{C}), \qquad \omega\mathsf{Cosp}(\mathbf{C}), \qquad \omega\mathsf{Span}(\mathbf{C}), \qquad \omega\mathsf{Bisp}(\mathbf{C})$$

of 1.7, and to the chiral multiple categories

$$\mathsf{S}_p\mathsf{C}_q(\mathbf{C}), \qquad \mathsf{S}_p\mathsf{C}_\infty(\mathbf{C}), \qquad \mathsf{S}_{-\infty}\mathsf{C}_\infty(\mathbf{C})$$

recalled in 1.8.

Note that all of these are transversally invariant, a property of interest for limits as we show below, in 2.3 and 2.4.

Level limits can be extended to intercategories with the same definitions (see 1.9). But Proposition 2.3 and its consequences in 2.4 would partially fail.

Non-level limits, where the diagram and the limit cube are not confined to a transversal category, will be studied in the next two sections.

2.1 Products

Let us begin by examining various kinds of products in the chiral triple category $\mathsf{A} = \mathsf{SC}(\mathbf{C})$.

Supposing that $\mathbf{C}$ has products, the same is true of its categories of diagrams, and (using the formal-span category $\vee$ and the formal cospan $\wedge$ recalled in 1.7) we have four types of products in A (indexed by a small set Λ):

- of objects (in $\mathbf{C}$), with projections in A_0:

$$C = \Pi_{\lambda}\, C_\lambda, \qquad\qquad p_\lambda \colon C \to_0 C_\lambda,$$

- of 1-arrows (in $\mathbf{C}^{\vee}$), with projections in A_{01}:

$$f = \Pi_{\lambda}\, f_\lambda, \qquad\qquad p_\lambda \colon f \to_0 f_\lambda,$$

- of 2-arrows (in $\mathbf{C}^{\wedge}$), with projections in A_{02}:

$$u = \Pi_{\lambda}\, u_\lambda, \qquad\qquad p_\lambda \colon u \to_0 u_\lambda,$$

- of 12-cells (in $\mathbf{C}^{\vee\times\wedge}$), with projections in A_{012}:

$$\pi = \Pi_{\lambda}\, \pi_\lambda, \qquad\qquad p_\lambda \colon \pi \to_0 \pi_\lambda,$$

Faces and degeneracies preserve these products. Saying that the triple category $\mathsf{SC}(\mathbf{C})$ *has triple products* we mean all this. It is important to note that *this is a global condition*: we shall *not* define when, in a chiral triple category, a *single* product of objects $\Pi_{\lambda}\, C_\lambda$ should be called 'a triple product'.

It is now simpler and clearer to work in a chiral *multiple* category A, rather than in a truncated case, as above.

Let $n \geqslant 0$ and let $\mathbf{i}$ be a positive multi-index (possibly empty). An $\mathbf{i}$-*product* $a = \Pi_{\lambda\in\Lambda}\, a_\lambda$ will be an ordinary product in the transversal category $\mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ of $\mathbf{i}$-cubes of A (recalled in Section 1). It comes with a family

$p_\lambda \colon a \to_0 a_\lambda$ of $\mathbf{i}$-maps (i.e. cells of $A_{0\mathbf{i}}$) that satisfies the obvious universal property.

We say that A:

(i) *has* $\mathbf{i}$-*products*, or products of type $\mathbf{i}$, if all these products (indexed by an arbitrary small set Λ) exist,

(ii) *has products* if it has $\mathbf{i}$-products for all positive multi-indices $\mathbf{i}$,

(iii) *has multiple products* if it has all products, and these are preserved by faces and degeneracies, viewed as (ordinary) functors (see (11))

$$\partial_j^\alpha \colon \mathrm{tv}_\mathbf{i}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}|j}(\mathsf{A}), \quad e_j \colon \mathrm{tv}_{\mathbf{i}|j}(\mathsf{A}) \to \mathrm{tv}_\mathbf{i}(\mathsf{A}) \qquad (j \in \mathbf{i},\ \alpha = \pm). \tag{27}$$

Of course this preservation is meant in the usual sense, up to isomorphism (i.e. invertible transversal maps); however, if this holds and A is transversally invariant (see 1.6), one can construct a choice of products that is *strictly* preserved by faces and degeneracies, starting from $\star$-products and going up. This will be proved, more generally, in Proposition 2.3.

A $\star$-product is also called a *product of degree* 0.

2.2 Level limits

We now let Λ be a small category.

There is an obvious chiral multiple category A^Λ whose $\mathbf{i}$-cubes are the functors $F \colon \Lambda \to \mathrm{tv}_\mathbf{i}(\mathsf{A})$ and whose $\mathbf{i}$-maps are their natural transformations, composed as such. The positive faces, degeneracies and compositions are pointwise (as well as their comparisons):

$$(\partial_i^\alpha F)(\lambda) = \partial_i^\alpha(F(\lambda)), \qquad (e_i F)(\lambda) = e_i(F(\lambda)),$$

$$(F +_i G)(\lambda) = F(\lambda) +_i G(\lambda).$$

The diagonal functor $D \colon \mathsf{A} \to \mathsf{A}^\Lambda$ takes each $\mathbf{i}$-cube a to the constant a-valued functor $Da \colon \Lambda \to \mathrm{tv}_\mathbf{i}(\mathsf{A})$, and each $\mathbf{i}$-map $h \colon a \to_0 b$ to the constant h-valued natural transformation $Dh \colon Da \to Db \colon \Lambda \to \mathrm{tv}_\mathbf{i}(\mathsf{A})$.

The limit of the functor F, called an $\mathbf{i}$-*level limit* in A, is an $\mathbf{i}$-cube $L \in A_\mathbf{i}$ equipped with a universal natural transformation $t \colon DL \to F \colon \Lambda \to \mathrm{tv}_\mathbf{i}(\mathsf{A})$, where $DL \colon \Lambda \to \mathrm{tv}_\mathbf{i}(\mathsf{A})$ is the constant functor at L. It is an $\mathbf{i}$-product if Λ is discrete and an $\mathbf{i}$-*equaliser* if Λ is the category $0 \rightrightarrows 1$.

We say that A:

(i) *has* $\mathbf{i}$*-level limits on* Λ if all the functors $\Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ have a limit,

(ii) *has level limits on* Λ if it has such limits for all positive multi-indices $\mathbf{i}$,

(iii) *has level multiple limits on* Λ if it has such level limits, and these are preserved by faces and degeneracies (as specified in (27)),

(iv) *has level multiple limits* if the previous property holds for every small category Λ.

Obviously, the multiple category A has level multiple limits if and only if it has multiple products and multiple equalisers. *Finite level limits* work in the same way, with finite multiple products.

In particular, a $\star$*-level limit* is a limit in the transversal category $\mathrm{tv}_*(\mathsf{A})$, associated to the multi-index $\varnothing$, of degree 0; it will also be called a *level limit of degree 0*.

Extending the case of multiple products considered in 2.1, if the category $\mathbf{C}$ is complete (or finitely complete) so are its categories of diagrams, and the chiral triple category $\mathsf{SC}(\mathbf{C})$ has level triple limits (or the finite ones).

2.3 Proposition (Level limits and invariance)

Let Λ *be a category and* A *a* transversally invariant *chiral multiple category (see 1.6). If* A *has level multiple limits on* Λ*, one can find a* consistent choice of such limits*. More precisely, one can fix for every positive multi-index* $\mathbf{i}$ *and every functor* $F\colon \Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ *a limit of* F

$$L(F) \in A_{\mathbf{i}}, \qquad t(F)\colon DL(F) \to F\colon \Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A}), \tag{28}$$

so that these choices are strictly preserved by faces and degeneracies:

$$\begin{array}{lll} \partial_i^\alpha(L(F)) = L(\partial_i^\alpha F), & \partial_i^\alpha(t(F)) = t(\partial_i^\alpha F) & (i \in \mathbf{i}), \\ e_i(L(F)) = L(e_i F), & e_i(t(F)) = t(e_i F) & (i \notin \mathbf{i}). \end{array} \tag{29}$$

Proof. We proceed by induction on the degree n of positive multi-indices. For $n = 0$ we just fix a choice $(L(F), t(F))$ of $\star$-level limits on Λ, for all $F\colon \Lambda \to \mathrm{tv}_*(\mathsf{A})$. Then, for $n \geqslant 1$, we suppose to have a consistent choice for all positive multi-indices of degree up to $n-1$ and extend this choice to degree n, as follows.

For a functor $F\colon \Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ of degree n, we already have a choice $(L(\partial_i^\alpha F), t(\partial_i^\alpha F))$ of the limit of each of its faces. Let (L, t) be an arbitrary limit of F; since faces preserve limits (in the usual, non-strict sense), there is a unique family of transversal isomorphisms h_i^α coherent with the limit cones (of degree $n-1$)

$$h_i^\alpha\colon L(\partial_i^\alpha F) \to_0 \partial_i^\alpha L, \quad t(\partial_i^\alpha F) = (\partial_i^\alpha t).h_i^\alpha \quad (i \in \mathbf{i},\ \alpha = \pm), \qquad (30)$$

and this family has consistent faces (see (22)), as follows easily from their coherence with the limit cones of a lower degree (when $n \geqslant 2$, otherwise the consistency condition is void).

Now, because of the hypothesis of transversal invariance, this family can be filled with a transversal isomorphism h, *yielding a choice* for $L(F)$ and $t(F)$

$$h\colon L(F) \to_0 L, \qquad t(F) = t.Dh\colon DL(F) \to F. \qquad (31)$$

By construction this extension of L is strictly preserved by all faces. To make it also consistent with degeneracies, we assume that - in the previous construction - the following constraint has been followed: for an i-degenerate functor $F = e_iG\colon \Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ we always choose the pair $(e_iL(G), e_it(G))$ as its limit (L, t). This allows us to take $h_i^\alpha = \mathrm{id}(L(G))$ (for all $i \in \mathbf{i}$ and $\alpha = \pm$), and finally $h = \mathrm{id}(L)$, that is

$$L(F) = e_iL(G), \qquad t(F) = e_it(G)\colon DL(F) \to F. \qquad (32)$$

If F is also j-degenerate, then $F = e_ie_jH = e_je_iH$; therefore, by the inductive assumption, both procedures give the same result: $e_iL(G) = e_ie_jL(H) = e_je_iL(H) = e_jL(e_iH)$.

Note that this point would fail in an intercategory with $e_ie_j \neq e_je_i$. $\square$

2.4 Level limits as unitary lax functors

The previous proposition shows that, if the chiral multiple category A is transversally invariant and has level multiple limits on the small category Λ, we can form a *unitary* lax functor L and a transversal transformation t

$$L\colon \mathsf{A}^\Lambda \to \mathsf{A}, \qquad t\colon DL \to 1\colon \mathsf{A}^\Lambda \to \mathsf{A}^\Lambda, \qquad (33)$$

such that, on every $\mathbf{i}$-cube F, the pair $(L(F), t(F))$ is the level limit of the functor F, as in (28).

Indeed, after defining L and t on all $\mathbf{i}$-cubes F, *by a consistent choice* (which is possible by the proposition itself), we define $L(h)$ for every natural transformation $h\colon F \to G\colon \Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$. By the universal property of limits, there is precisely one $\mathbf{i}$-map $L(h)$ such that

$$L(h)\colon L(F) \to_0 L(G), \qquad\qquad h.t(F) = t(G).DL(h), \tag{34}$$

and this extension on $\mathbf{i}$-maps is obviously the only one that makes the family $t(F)\colon\ DL(F) \to F$ into a transversal transformation $DL \to 1$. The lax comparison for i-composition (with $i \in \mathbf{i}$)

$$\begin{gathered} \underline{L}_i(F,G)\colon L(F) +_i L(G) \to_0 L(F +_i G), \\ t(F +_i G).D\underline{L}_i(F,G) = t(F) +_i t(G), \end{gathered} \tag{35}$$

comes from the universal property of $L(F +_i G)$ as a limit.

In the hypotheses above we say that A *has lax functorial limits* on Λ. We say that A *has pseudo* (resp. *strict*) *functorial limits* on Λ if L is a pseudo functor (resp. can be chosen as a strict functor).

2.5 Level limits and liftings

Let us recall (from (19) and 1.8) that, for a positive multi-index $\mathbf{i}$, the chiral multiple category A has a lifting $Q_{\mathbf{i}}(\mathsf{A})$ such that

$$\mathrm{tv}_*(Q_{\mathbf{i}}(\mathsf{A})) = \mathrm{tv}_{\mathbf{i}}(\mathsf{A}). \tag{36}$$

Therefore, an $\mathbf{i}$-level limit in A is the same as a $\star$-level limit in $Q_{\mathbf{i}}(\mathsf{A})$. The chiral multiple category A

(i) *has* $\mathbf{i}$*-level limits on* Λ if and only if its lifting $Q_{\mathbf{i}}(\mathsf{A})$ has $\star$-level limits on Λ,

(ii) *has level limits on* Λ if and only if all its liftings $Q_{\mathbf{i}}(\mathsf{A})$ have $\star$-level limits,

(iii) *has level multiple limits on* Λ if and only if all its liftings $Q_{\mathbf{i}}(\mathsf{A})$ have $\star$-level limits and these are preserved by faces and degeneracies, namely the multiple functors $D_j^\alpha = D_j^\alpha(\mathsf{A})$ and $E_j = E_j(\mathsf{A})$ for $j \notin \mathbf{i}$ and $\alpha = \pm$ (see 1.5)

$$\begin{aligned} D_j^\alpha\colon Q_{\mathbf{i}j}(\mathsf{A}) \to R_jQ_{\mathbf{i}}(\mathsf{A}), \qquad & E_j\colon R_jQ_{\mathbf{i}}(\mathsf{A}) \to Q_{\mathbf{i}j}(\mathsf{A}), \\ \mathrm{tv}_*(D_j^\alpha) = \partial_j^\alpha\colon \mathrm{tv}_{\mathbf{i}j}\mathsf{A} \to \mathrm{tv}_{\mathbf{i}}\mathsf{A}, \qquad & \mathrm{tv}_*(E_j) = e_j\colon \mathrm{tv}_{\mathbf{i}}\mathsf{A} \to \mathrm{tv}_{\mathbf{i}j}\mathsf{A} \end{aligned} \tag{37}$$

(iv) *has level multiple limits* if the previous property holds for every small category Λ.

2.6 Level limits in weak double categories

Let $\mathbb{A}$ be a weak double category, viewed as the weak multiple category $\mathrm{sk}_2(\mathbb{A})$, by adding degenerate items of all the missing types (cf. I.2.7).

The present $\star$-level limits in $\mathbb{A}$, i.e. limits of ordinary functors $\Lambda \to \mathrm{tv}_*(\mathbb{A})$, correspond to the 'limits of horizontal functors' in [GP1]. There are slight differences in terminology, essentially because the '2-dimensional universal property' of double limits (see [GP1], 4.2) here is not required from the start but comes out of a condition of preservation by degeneracies.

As a particular case of the definitions in 2.2, we have the following cases.

(i) $\mathbb{A}$ *has* $\star$*-level limits on* a (small) category Λ if all the functors $\Lambda \to \mathrm{tv}_*(\mathbb{A})$ have a limit. By the usual theorem on ordinary limits, all of them can be constructed from:
- small products $\Pi\, A_\lambda$ of objects,
- equalisers of pairs $f, g\colon A \to B$ of parallel horizontal arrows.

(i$'$) $\mathbb{A}$ *has* 1*-level limits on* Λ if all the functors $\Lambda \to \mathrm{tv}_1(\mathsf{A})$ have a limit. All of them can be constructed from:
- small products $\Pi\, u_\lambda$ of vertical arrows,
- equalisers of pairs $a, b\colon u \to v$ of double cells (between the same vertical arrows).

(ii) $\mathbb{A}$ *has level limits on* Λ if it has $\star$- and 1-level limits on Λ.

(iii) $\mathbb{A}$ *has level double limits on* Λ if it has such level limits, preserved by faces and degeneracies.

(iv) $\mathbb{A}$ *has level double limits* if the previous property holds for every small category Λ; this is equivalent to the existence of small *double products* and *double equalisers*.

Let us note again, as in 2.1, that the existence of (say) double products is now a *global condition*: it means the existence of products of objects *and* vertical arrows, *consistently* with faces and degeneracies. Here we are *not* defining when a *single* product $\Pi\, A_\lambda$ should be called a 'double product' (while in [GP1] this meant a product of objects preserved by vertical identities).

In [GP1] case (i) would be expressed saying that $\mathbb{A}$ has 1-*dimensional* limits of horizontal functors on Λ. Case (iii) (resp. (iv)) would be expressed saying that $\mathbb{A}$ can be given a lax choice of double limits for all horizontal functors defined on Λ (resp. defined on some small category).

3. Multiple limits of degree zero

We now define 'multiple limits' of degree zero - *namely those limits that produce objects*. They extend the previous *level limits of degree zero* (or $\star$-level limits), and are generated by the latter together with *tabulators of degree zero* (Theorem 3.6). The general case - *limits that produce cubes of arbitrary dimension* - will be treated in the next section.

3.1 The diagonal functor

Let X and A be chiral multiple categories, and let X be small. Consider the *diagonal* functor (of ordinary categories)

$$D\colon \mathrm{tv}_*\mathsf{A} \to \mathrm{Ps}\mathbf{Cmc}(\mathsf{X},\mathsf{A}). \qquad (38)$$

D takes each object A of A to a unitary pseudo functor, 'constant' at A, via the family of the total $\mathbf{i}$-degeneracies (see (7))

$$\begin{array}{ll} DA\colon \mathsf{X}\to\mathsf{A}, & \\ DA(x) = e_{\mathbf{i}}(A) \qquad DA(f) = \mathrm{id}(e_{\mathbf{i}}A) & (\text{for } x \text{ and } f \text{ in } \mathrm{tv}_{\mathbf{i}}\mathsf{X}), \\ \underline{DA}_i(x) = 1_{e_{\mathbf{i}}A}\colon e_i(DA(x)) \to DA(e_ix) & (\text{for } x \text{ in } X_{\mathbf{i}|i}), \\ \underline{DA}_i(x,y) = \lambda_i\colon e_{\mathbf{i}}A +_i e_{\mathbf{i}}A \to e_{\mathbf{i}}A & (\text{for } z = x +_i y \text{ in } X_{\mathbf{i}}). \end{array} \qquad (39)$$

In fact, as required by axiom (lmf.3) of lax multiple functors (in I.3.9), the comparison $\underline{DA}_i(x,y)$ above must be the unitor $\lambda_i(e_{\mathbf{i}}A) = \rho_i(e_{\mathbf{i}}A)$ of A, equivalently left or right (see I.3.3), that will generally be written as λ_i for short.

Similarly, a $\star$-map $h\colon A\to B$ in A is sent to the constant transversal transformation

$$Dh\colon DA\to DB\colon \mathsf{X}\to\mathsf{A}, \;\; (Dh)(x) = e_{\mathbf{i}}h\colon e_{\mathbf{i}}A \to e_{\mathbf{i}}B \;\; (x\in X_{\mathbf{i}}). \qquad (40)$$

DA is a *strict* multiple functor whenever A is pre-unitary (cf. I.3.2).

Note also that the definition of the diagonal functor D depends on the commutativity of degeneracies in A, *which holds in the present chiral case*. For a general 3-dimensional *intercategory* A one could define two functors

$$D_{12}\colon \mathrm{tv}_*\mathsf{A} \to \mathrm{Lx}\mathbf{Cmc}(\mathsf{X}, \mathsf{A}), \qquad D_{21}\colon \mathrm{tv}_*\mathsf{A} \to \mathrm{Cx}\mathbf{Cmc}(\mathsf{X}, \mathsf{A}), \tag{41}$$

where $D_{ij}(A)$ sends a 12-cube x to $D_{ij}(A)(x) = e_i e_j(A)$ (and any lower $\mathbf{i}$-cube to $e_{\mathbf{i}}(A)$). In higher dimension the situation is even more complex.

Still, in an intercategory we have *level* limits, defined as in Section 2, and some simple non-level limits that can be defined ad hoc, like the e_1e_2-*tabulator* and the e_2e_1-*tabulator* of a 12-cube considered in Part I, Section 6.

3.2 Cones

Let $F\colon \mathsf{X} \to \mathsf{A}$ be a lax functor. A *(transversal) cone* of F will be a pair $(A, h\colon DA \to F)$ comprising an object A of A (the *vertex* of the cone) and a transversal transformation of lax functors $h\colon DA \to F\colon \mathsf{X} \to \mathsf{A}$; in other words, it is an object of the ordinary comma category $(D \downarrow F)$, where F is viewed as an object of the category $\mathrm{Lx}\mathbf{Cmc}(\mathsf{X}, \mathsf{A})$.

By definition (see 1.8), the transversal transformation h amounts to assigning the following data:

- a transversal $\mathbf{i}$-map $hx\colon e_{\mathbf{i}}(A) \to Fx$, for every $\mathbf{i}$-cube x in X, subject to the following axioms of naturality and coherence:

(tc.1) $Ff.hx = hy$ (for every $\mathbf{i}$-map $f\colon x \to_0 y$ in X),

(tc.2) h commutes with positive faces, and agrees with positive degeneracies and compositions:

$$h(\partial_i^\alpha x) = \partial_i^\alpha(hx), \qquad \text{(for } x \text{ in } X_{\mathbf{i}}),$$

$$h(e_i x) = \underline{F}_i(x).e_i(hx)\colon e_{\mathbf{i}}A \to_0 F(e_i x) \qquad \text{(for } x \text{ in } X_{\mathbf{i}|i}),$$

$$h(z) = \underline{F}_i(x, y).(hx +_i hy).\lambda_i^{-1}\colon e_{\mathbf{i}}A \to_0 F(z) \qquad \text{(for } z = x +_i y \text{ in } X_{\mathbf{i}}),$$

where $\lambda_i\colon e_{\mathbf{i}}(A) +_i e_{\mathbf{i}}(A) \to e_{\mathbf{i}}(A)$ is a unitor of A (see (39)).

It is easy to see that a *unitary* lax functor $S\colon \mathsf{A} \to \mathsf{B}$ *preserves diagonalisation*, in the sense that $S.DA = D(SA)$. Therefore S takes a cone

$(A, h\colon DA \to F)$ of $F\colon \mathsf{X} \to \mathsf{A}$ to a cone (SA, Sh) of $SF\colon \mathsf{X} \to \mathsf{B}$, and one can consider whether S preserves a limit. For a 'general' lax functor S one should transform cones using the comparison $\underline{S}(A)$, which will not be done here, for the sake of simplicity.

3.3 Definition (Limits of degree zero)

Given a lax functor $F\colon \mathsf{X} \to \mathsf{A}$ between chiral multiple categories, the *(transversal) limit of degree zero* $\lim(F) = (L, t\colon DL \to F)$ is a universal cone.

In other words:

(tl.0) L is an object of A and $t\colon DL \to F$ is a transversal transformation of lax functors,

(tl.1) for every cone $(A, h\colon DA \to F)$ there is precisely one $\star$-map $f\colon A \to L$ in A such that $t.Df = h$.

We say that A *has limits of degree zero* on X if all these exist.

In particular, if X is the multiple category freely generated by a category Λ, at $\star$-level, then A has 0-degree limits on X if and only if it has 0-degree *level* limits on Λ (see 2.2). Here *freely generated at $\star$-level* refers to a universal arrow from Λ to the functor $\mathrm{tv}_*\colon \mathbf{Mlc} \to \mathbf{Cat}$.

3.4 Tabulators of degree zero

A is always a chiral multiple category. Let us recall that every positive multi-index $\mathbf{i}$ gives a 'total' degeneracy

$$e_{\mathbf{i}} = e_{i_1}...e_{i_n}\colon \mathrm{tv}_*\mathsf{A} \to \mathrm{tv}_{\mathbf{i}}\mathsf{A}. \tag{42}$$

An $\mathbf{i}$-cube x of A can be viewed as a unitary pseudo functor $x\colon \mathsf{u}_{\mathbf{i}} \to \mathsf{A}$ where $\mathsf{u}_{\mathbf{i}}$ is the strict multiple category freely generated by one $\mathbf{i}$-cube $u_{\mathbf{i}}$. The pseudo functor x sends $u_{\mathbf{i}}$ to x, and has comparisons $\underline{x}_i$ for i-composites that derive from the unitors of A, as in the following cases

$$\underline{x}_i(e_i\partial_i^- u_{\mathbf{i}}, u_{\mathbf{i}}) = \lambda_i(x)\colon e_i\partial_i^- x +_i x \to x,$$

$$\underline{x}_i(u_{\mathbf{i}}, e_i\partial_i^+ u_{\mathbf{i}}) = \rho_i(x)\colon x +_i e_i\partial_i^+ x \to x.$$

Again, it is easy to see that this unitary pseudo functor $x\colon \mathsf{u}_{\mathbf{i}} \to \mathsf{A}$ is preserved by a unitary lax functor $S\colon \mathsf{A} \to \mathsf{B}$, in the sense that the composite $S.x$ coincides with $S(x)\colon \mathsf{u}_{\mathbf{i}} \to \mathsf{B}$. All the pseudo functors $x\colon \mathsf{u}_{\mathbf{i}} \to \mathsf{A}$ are strict precisely when A is unitary.

The *tabulator of degree zero* of x in A will be the limit of this pseudo functor $x\colon \mathsf{u}_{\mathbf{i}} \to \mathsf{A}$; we also speak of the *total tabulator*, or **i**-*tabulator*, of x.

The tabulator is thus an object $T = \top x \ (= \top_{\mathbf{i}} x)$ equipped with an **i**-map $t_x\colon e_{\mathbf{i}}T \to_0 x$ such that the pair $(T, t_x\colon e_{\mathbf{i}}T \to_0 x)$ is a universal arrow from the functor $e_{\mathbf{i}}\colon \mathrm{tv}_*\mathsf{A} \to \mathrm{tv}_{\mathbf{i}}\mathsf{A}$ to the object x of $\mathrm{tv}_{\mathbf{i}}\mathsf{A}$. Explicitly, this means that, for every object A and every **i**-map $h\colon e_{\mathbf{i}}A \to_0 x$ there is a unique $\star$-map f such that

$$
\begin{tikzcd}
e_{\mathbf{i}}(A) \arrow[r, "e_{\mathbf{i}}(f)"] \arrow[dr, "h"'] & e_{\mathbf{i}}(T) \arrow[d, "t_x"] \\
& x
\end{tikzcd}
\qquad\qquad
\begin{array}{c}
f\colon A \to_0 T, \\[2em]
t_x.e_{\mathbf{i}}(f) = h.
\end{array}
\tag{43}
$$

We say that A *has tabulators of degree zero* if all these exist, for every positive multi-index **i**. Obviously, the tabulator of an object always exists and is the object itself.

When such tabulators exist, we can form for every positive multi-index **i** a right adjoint functor

$$\top_{\mathbf{i}}\colon \mathrm{tv}_{\mathbf{i}}\mathsf{A} \to \mathrm{tv}_*\mathsf{A}, \qquad\qquad e_{\mathbf{i}} \dashv \top_{\mathbf{i}}, \tag{44}$$

which is just the identity for $\mathbf{i} = \varnothing$.

Assuming that the tabulators of $x \in A_{\mathbf{i}}$ and $z = \partial_j^\alpha x$ exist (for $j \in \mathbf{i}$), the *projection* $p_j^\alpha x$ of $\top x \ (= \top_{\mathbf{i}} x)$ will be the following $\star$-map of A

$$
\begin{tikzcd}
e_{\mathbf{i}|j}\top x \arrow[r, "e_{\mathbf{i}|j}(p_j^\alpha x)"] \arrow[dr, "\partial_j^\alpha(t_x)"'] & e_{\mathbf{i}|j}\top(\partial_j^\alpha x) \arrow[d, "t_z"] \\
& z = \partial_j^\alpha x
\end{tikzcd}
\qquad
\begin{array}{c}
p_j^\alpha x\colon \top x \to_0 \top(\partial_j^\alpha x), \\[2em]
t_z.e_{\mathbf{i}|j}(p_j^\alpha x) = \partial_j^\alpha(t_x).
\end{array}
\tag{45}
$$

3.5 Tabulators and concatenation

We now examine the relationship between tabulators of **i**-cubes and (zero-ary or binary) j-concatenation, for $j \in \mathbf{i}$.

(a) If the degenerate $\mathbf{i}$-cube $x = e_j z$ and the $\mathbf{i}|j$-cube z have total tabulators in A, they are linked by a *diagonal* transversal $\star$-map $d_j z$, defined as follows

$$\begin{tikzcd} e_{\mathbf{i}}(\top z) \arrow[r, "e_{\mathbf{i}}(d_j z)"] \arrow[dr, "e_j t_z"'] & e_{\mathbf{i}}(\top(e_j z)) \arrow[d, "t_x"] \\ & x = e_j z \end{tikzcd} \qquad \begin{array}{c} d_j z \colon \top z \to_0 \top(e_j z), \\ \\ t_x.e_{\mathbf{i}}(d_j z) = e_j t_z. \end{array} \tag{46}$$

This $\star$-map $d_j z$ is a section of both projections $p_j^\alpha x$ (defined above) because

$$t_z.e_{\mathbf{i}|j}(p_j^\alpha x.d_j z) = \partial_j^\alpha(t_x).e_{\mathbf{i}|j}(d_j z) = \partial_j^\alpha(t_x.e_{\mathbf{i}}(d_j z)) = \partial_j^\alpha(e_j t_z) = t_z.$$

(b) For a concatenation $z = x +_j y$ of $\mathbf{i}$-cubes, the three total tabulators of x, y, z are also related. The link goes through the ordinary pullback $\top_j(x, y)$ of the objects $\top x$ and $\top y$, over the tabulator $\top w$ of the $\mathbf{i}|j$-cube $w = \partial_j^+ x = \partial_j^- y$ (*provided* all these tabulators and such a pullback exist)

$$\begin{tikzcd} & \top x \arrow[dr, "p_j^+ x"] & \\ \top_j(x,y) \arrow[ur, "p_j(x,y)"] \arrow[dr, "q_j(x,y)"'] & & \top w \\ & \top y \arrow[ur, "p_j^- y"'] & \end{tikzcd} \qquad \begin{array}{c} t_w.e_{\mathbf{i}|j}(p_j^+ x) = \partial_j^+(t_x), \\ \\ t_w.e_{\mathbf{i}|j}(p_j^- y) = \partial_j^-(t_y). \end{array} \tag{47}$$

We now have a *diagonal* transversal $\star$-map $d_j(x, y)$ given by the universal property of $\top z$

$$\begin{gathered} d_j(x, y) \colon \top_j(x, y) \to_0 \top z, \\ t_z.e_{\mathbf{i}}(d_j(x, y)) = t_x.e_{\mathbf{i}} p_j(x, y) \ +_j \ t_y.e_{\mathbf{i}} q_j(x, y). \end{gathered} \tag{48}$$

The j-composition above is legitimate, by construction

$$\begin{aligned} \partial_j^+(t_x.e_{\mathbf{i}} p_j(x, y)) &= \partial_j^+(t_x).e_{\mathbf{i}|j}(p_j(x, y)) \\ &= t_w.e_{\mathbf{i}|j}(p_j^+ x).e_{\mathbf{i}|j}(p_j(x, y)) = t_w.e_{\mathbf{i}|j}(p_j^- y).e_{\mathbf{i}|j}(q_j(x, y)) \\ &= \partial_j^-(t_y).e_{\mathbf{i}|j}(q_j(x, y)) = \partial_j^-(t_y.e_{\mathbf{i}} q_j(x, y)). \end{aligned}$$

It is easy to show (and it also follows from the proof of the theorem below) that $\top_j(x, y)$ is the transversal limit of the diagram 'formed' by $z = x +_j y$ (based on the multiple category freely generated by two j-consecutive $\mathbf{i}$-cubes).

3.6 Theorem (Construction and preservation of 0-degree limits)

Let A *and* B *be chiral multiple categories.*

(a) All limits of degree zero in A *can be constructed from level limits of degree zero and tabulators of degree zero, or also from products, equalisers and tabulators - all of degree zero.*

(b) If A *has all limits of degree zero, a unitary lax multiple functor* $\mathsf{A} \to \mathsf{B}$ *preserves them if and only if it preserves products, equalisers and tabulators of degree zero.*

Proof. See Section 5. □

3.7 Examples

In the chiral triple category $\mathsf{SC}(\mathbf{C})$ (over a category $\mathbf{C}$ with pullbacks and pushouts) we have the following three kinds of tabulators of degree zero (apart from the trivial tabulator of an object), already described in I.4.3.

(a) The tabulator of a 1-arrow f (i.e. a span) is an object $\top_1 f$ with a universal 1-map $e_1(\top_1 f) \to_0 f$; the solution is the (trivial) limit of the span f, i.e. its middle object.

(b) The tabulator of a 2-arrow u (a cospan) is an object $\top_2 u$ with a universal 2-map $e_2(\top_2 u) \to_0 u$; the solution is the pullback of u.

(c) The *total tabulator* of a 12-cell π (a span of cospans) is an object $\top_{12}\pi$ with a universal 12-map $e_{12}(\top_{12}\pi) \to_0 \pi$; the solution is the limit of the diagram, i.e. the pullback of its middle cospan.

The two (non total) tabulators *of degree* 1 of the 12-cell π will be reviewed below, in 4.6.

4. Multiple limits of arbitrary degree

We now introduce general limits in a chiral multiple category A, taking advantage of the iterated lift functors $Q_{\mathbf{i}}$ (see 1.5), where $\mathbf{i}$ is a positive multi-index of degree $n \geqslant 0$. X is always a small chiral multiple category.

Let us recall that $\mathsf{u}_{\mathbf{i}}$ denotes the multiple category freely generated by one $\mathbf{i}$-cube $u_{\mathbf{i}}$ (as in 3.4).

4.1 A motivation

For a positive multi-index $\mathbf{i}$ of degree $n \geqslant 0$, the limits (of degree 0) of multiple functors with values in the lifted chiral multiple category $Q_{\mathbf{i}}\mathsf{A}$ will be called *limits of type* $\mathbf{i}$ (*and degree n*) in A; their results are thus $\mathbf{i}$-cubes of A. They extend the limits of degree zero considered above, for $\mathbf{i} = \varnothing$ and $Q_*\mathsf{A} = \mathsf{A}$.

Let us begin with some simple examples, based on a 2-dimensional cube $x \in A_{12}$, introducing definitions that will be made precise below.

(a) The cube $x \in A_{12}$ is the same as a unitary pseudo functor $x\colon \mathsf{u}_{12} \to \mathsf{A}$. We have already considered its *tabulator of degree zero*, namely an object $\top x = \top_{12}x$ with a universal 12-map $t\colon e_{12}(\top_{12}x) \to_0 x$ (where $e_{12} = e_1e_2 = e_2e_1\colon A_* \to A_{12}$ is the composed degeneracy).

(b) But x can also be viewed as a 1-arrow of $Q_2\mathsf{A}$, i.e. a unitary pseudo functor $x\colon \mathsf{u}_1 \to Q_2\mathsf{A}$. Its e_1-*tabulator* (of degree 1) will be the total tabulator of x as a 1-arrow of $Q_2\mathsf{A}$; this amounts to a 2-arrow $\top_1x$ of A with a universal 12-map $t\colon e_1(\top_1x) \to_0 x$ (where $e_1\colon A_2 \to A_{12}$ is the degeneracy $e_1\colon (Q_2\mathsf{A})_* \to (Q_2\mathsf{A})_1$).

(c) Symmetrically, x can be viewed as a 2-arrow of $Q_1\mathsf{A}$, i.e. a unitary pseudo functor $x\colon \mathsf{u}_2 \to Q_1\mathsf{A}$. Its e_2-*tabulator* (of degree 1, again) will be the total tabulator of x as a 2-arrow of $Q_1\mathsf{A}$; this amounts to a 1-arrow $\top_2x$ of A with a universal 12-map $t\colon e_2(\top_2x) \to_0 x$ (where $e_2\colon A_1 \to A_{12}$ is the degeneracy $e_2\colon (Q_1\mathsf{A})_* \to (Q_1\mathsf{A})_2$).

(d) The 2-dimensional cube x is also an object of $Q_{12}\mathsf{A}$. Its *tabulator of degree two* is x itself. This is a (trivial) level limit, while the previous limits are not level, i.e. are not limits in some transversal category of A.

4.2 General tabulators

An $\mathbf{i}$-cube $x \in A_{\mathbf{i}}$ is a unitary pseudo functor $x\colon \mathsf{u}_{\mathbf{i}} \to \mathsf{A}$. For every $\mathbf{k} \subset \mathbf{i}$ we can also view x as a pseudo functor $\mathsf{u}_{\mathbf{j}} \to Q_{\mathbf{k}}\mathsf{A}$ where $\mathbf{j} = \mathbf{i} \setminus \mathbf{k}$, so that x can have an $e_{\mathbf{j}}$-*tabulator*, namely a $\mathbf{k}$-cube $T = \top_{\mathbf{j}}x \in A_{\mathbf{k}}$ with a universal $\mathbf{i}$-map $t_x\colon e_{\mathbf{j}}(\top_{\mathbf{j}}x) \to_0 x$. (Total tabulators correspond to $\mathbf{j} = \mathbf{i}$, while $\mathbf{j} = \varnothing$ gives the trivial case $\top_\varnothing x = x$.)

The universal property says now that, for every $\mathbf{k}$-cube A and every $\mathbf{i}$-map $h\colon e_{\mathbf{j}}(A) \to_0 x$ there is a unique $\mathbf{k}$-map u such that

$$\begin{tikzcd} e_{\mathbf{j}}(A) \arrow[r, "e_{\mathbf{j}}(u)"] \arrow[dr, "h"'] & e_{\mathbf{j}}(T) \arrow[d, "t_x"] \\ & x \end{tikzcd} \qquad \begin{array}{l} u\colon A \to_0 T, \\ \\ t_x.e_{\mathbf{j}}(u) = h. \end{array} \tag{49}$$

We say that the chiral multiple category A *has tabulators of all degrees* if every $\mathbf{i}$-cube $x \in A_{\mathbf{i}}$ has all $\mathbf{j}$-tabulators $\top_{\mathbf{j}} x \in A_{\mathbf{k}}$ (for $\mathbf{i} = \mathbf{j} \cup \mathbf{k}$, disjoint union). We say that A *has multiple tabulators* if it has *tabulators of all degrees*, preserved by faces and degeneracies.

In this case, if A is transversally invariant, one can always make a choice of multiple tabulators such that this preservation is strict (as we have already seen in various examples of Part I):

$$\partial_i^\alpha(\top_{\mathbf{j}} x) = \top_{\mathbf{j}}(\partial_i^\alpha x), \quad \top_{\mathbf{j}}(e_i y) = e_i(\top_{\mathbf{j}} y) \qquad (\mathbf{j} \subset \mathbf{i},\ i \in \mathbf{i} \setminus \mathbf{j}), \tag{50}$$

for $x \in A_{\mathbf{i}}$ and $y \in A_{\mathbf{i}|i}$.

Note that these conditions are trivial if $\mathbf{j} = \varnothing$ or $\mathbf{j} = \mathbf{i}$, *whence for all weak double categories* (where there is only one positive index). This remark will be important when reconsidering double limits, in 4.7.

4.3 Lemma (Basic properties of tabulators)

Let A be a chiral multiple category.

(a) For an $\mathbf{i}$-cube x and a disjoint union $\mathbf{i} = \mathbf{j} \cup \mathbf{k}$ we have

$$\top_{\mathbf{i}} x = \top_{\mathbf{k}} \top_{\mathbf{j}} x, \tag{51}$$

provided that $\top_{\mathbf{j}} x$ and $\top_{\mathbf{k}}(\top_{\mathbf{j}} x)$ exist.

(b) A has tabulators of all degrees if and only it has all elementary tabulators *$\top_j x$ (for every positive multi-index $\mathbf{i}$, every $j \in \mathbf{i}$ and every $\mathbf{i}$-cube x).*

(c) If all e_j-tabulators of $\mathbf{i}$-cubes exist in A there is an ordinary adjunction

$$e_j\colon \mathrm{tv}_{\mathbf{i}|j}(\mathsf{A}) \rightleftarrows \mathrm{tv}_{\mathbf{i}}(\mathsf{A}) : \top_j, \qquad e_j \dashv \top_j \qquad (j \in \mathbf{i}), \tag{52}$$

and $e_j\colon \mathrm{tv}_{\mathbf{i}|j}\mathsf{A} \to \mathrm{tv}_{\mathbf{i}}\mathsf{A}$ *preserves colimits.*

(d) If all e_j-cotabulators *of* $\mathbf{i}$*-cubes exist in* A*, then* $e_j\colon \mathrm{tv}_{\mathbf{i}|j}\mathsf{A} \to \mathrm{tv}_{\mathbf{i}}\mathsf{A}$ *is a right adjoint and preserves the existing limits (so that a condition on multiple* level *limits in 2.2(iii) is automatically satisfied).*

(e) In a weak double category $\mathbb{A}$ *the existence of cotabulators of vertical arrows implies that all ordinary limits in* $\mathrm{tv}_*(\mathbb{A})$ *are preserved by vertical identities. (This has already been used in I.5.5.)*

Proof. (a) Composing universal arrows for

$$e_{\mathbf{i}} = e_{\mathbf{j}}e_{\mathbf{k}}\colon\ \mathrm{tv}_*\mathsf{A} \to \mathrm{tv}_{\mathbf{k}}\mathsf{A} \to \mathrm{tv}_{\mathbf{i}}\mathsf{A},$$

one gets (a choice of) $\top_{\mathbf{i}}x$ from (a choice of) $\top_{\mathbf{j}}x$ and $\top_{\mathbf{k}}(\top_{\mathbf{j}}x)$. The rest is obvious. □

4.4 Definition (Multiple limits)

We are now ready for a general definition of multiple limits in a chiral multiple category A.

(a) For a positive multi-index $\mathbf{i} \subset \mathbb{N}$ and a chiral multiple category X we say that A has *limits of type* $\mathbf{i}$ on X if $Q_{\mathbf{i}}\mathsf{A}$ has limits of degree zero on X.

(b) We say that A *has limits of type* $\mathbf{i}$ if this happens for all small chiral multiple categories X.

(c) We say that A *has limits of all degrees* (or *all types*) if this happens for all positive multi-indices $\mathbf{i}$.

(d) We say that A *has multiple limits of all degrees* if all the previous limits exist and are preserved by the multiple functors (see 1.5)

$$D_j^\alpha\colon Q_{\mathbf{i}j}(\mathsf{A}) \to R_jQ_{\mathbf{i}}(\mathsf{A}),\quad E_j\colon R_jQ_{\mathbf{i}}(\mathsf{A}) \to Q_{\mathbf{i}j}(\mathsf{A})\qquad (j \notin \mathbf{i}). \tag{53}$$

In this case, if A is transversally invariant, one can always operate a choice of multiple limits such that this preservation is strict (working as in Proposition 2.3).

We do not speak here of *completeness*: this notion should also involve the existence of 'companions' and 'adjoints' for all transversal maps, as shown by our study of Kan extensions in the domain of weak double categories [GP3, GP4].

4.5 Main Theorem (Construction and preservation of multiple limits)

Let A *and* B *be chiral multiple categories.*

(a) All multiple limits in A *can be constructed from level multiple limits and multiple tabulators, or also from multiple products, multiple equalisers and multiple tabulators.*

(b) If A *has all multiple limits, a unitary lax multiple functor* $S\colon \mathsf{A} \to \mathsf{B}$ *preserves them if and only if it preserves multiple products, multiple equalisers and multiple tabulators.*
Similarly for finite limits and finite products.

Proof. Follows from Theorem 3.6, applied to the family of chiral multiple categories $Q_{\mathbf{i}}\mathsf{A}$, together with the multiple functors of faces and degeneracies (see (53)) and the lax multiple functors $Q_{\mathbf{i}}S\colon Q_{\mathbf{i}}\mathsf{A} \to Q_{\mathbf{i}}\mathsf{B}$. □

4.6 Examples

For a category $\mathbf{C}$ with pushouts and pullbacks we complete the discussion of tabulators in the chiral triple category $\mathsf{SC}(\mathbf{C})$, after the three types of tabulators of degree zero examined in 3.7. We start again from a 12-cube $\pi\colon \vee\times\wedge \to \mathbf{C}$ (a span of cospans in $\mathbf{C}$).

(a) The e_1-tabulator of π is a 2-arrow $\top_1\pi$ (a cospan) with a universal 12-map $e_1\top_1\pi \to_0 \pi$; the solution is the middle cospan of π.

(b) The e_2-tabulator of π is a 1-arrow $\top_2\pi$ (a span) with a universal 12-map $e_2\top_2\pi \to_0 \pi$; the solution is the obvious span whose objects are the pullbacks of the three cospans of π.

These limits are preserved by faces and degeneracies. For instance:

- $\partial_1^-(\top_2\pi) = \top_2(\partial_1^-\pi)$, which means that the domain of the span $\top_2\pi$ (described above) is the pullback of the cospan $\partial_1^-\pi$,

- $\top_2(e_1u) = e_1(\top_2u)$, i.e. the e_2-tabulator of the 1-degenerate cell e_1u (on the cospan u) is the degenerate span on the pullback of u.

Finally, putting together the previous results (in 2.2 and 3.7): if $\mathbf{C}$ is a complete (or finitely complete) category with pushouts, then the chiral triple category $\mathsf{SC}(\mathbf{C})$ has multiple limits (or the finite ones).

4.7 Limits in weak double categories

We now complete the discussion of limits in a weak double category $\mathbb{A}$, after the case of level limits examined in 2.6.

Here a consistent difference appears between the present analysis and that of [GP1]. In that paper all limits, including tabulators, were assumed to satisfy also a 'two-dimensional universal property' (namely condition (dl.2) in Definition 4.2). On the other hand multiple tabulators are here subject to preservation properties that only become non-trivial in dimension three or higher (as already remarked at the end of 4.2); the examples above (in 4.6) show that at least two positive indices are required to formulate non-trivial conditions of this type.

In other words, tabulators in a weak double category $\mathbb{A}$ are here double tabulators, and the only limits that must be preserved by faces and degeneracies are the level ones, generated by products and equalisers of objects or vertical arrows of $\mathbb{A}$.

The present terminology, a particular case of the definitions in 4.2 and 4.4, can thus be summarised as follows.

(a) $\mathbb{A}$ *has tabulators* if every vertical arrow u (a 1-cube) has an object $\top u = \top_1 u$ with a universal double cell $e_1(\top_1 u) \to u$.

(b) $\mathbb{A}$ *has limits of degree zero* (namely the limits that produce *objects*) if all the functors $\mathbb{X} \to \mathbb{A}$ (defined on a small weak double category) have a limit. Theorem 3.6 says that this condition amounts to the existence of:
- all products $\Pi\, A_\lambda$ of objects,
- all equalisers of pairs $f, g\colon A \to B$ of parallel horizontal arrows,
- all tabulators $\top u$ of vertical arrows.

(c) $\mathbb{A}$ *has limits of degree 1* (namely the limits that produce *vertical arrows*) if all the functors $\Lambda \to \mathrm{tv}_1(\mathbb{A}) = Q_1\mathbb{A}$ defined on a small category) have a limit. By the usual theorem on ordinary limits, this condition amounts to the existence of:
- products $\Pi\, u_\lambda$ of vertical arrows,
- equalisers of pairs $a, b\colon u \to v$ of double cells (between the same vertical arrows).

(d) $\mathbb{A}$ *has limits of all degrees* if both conditions (b) and (c) are satisfied.

(e) $\mathbb{A}$ *has double limits* if all the previous limits exist and are preserved by the ordinary functors

$$D_1^\alpha \colon \mathrm{tv}_1\mathbb{A} \to \mathrm{tv}_*\mathbb{A}, \qquad E_1 \colon \mathrm{tv}_*\mathbb{A} \to \mathrm{tv}_1\mathbb{A}, \tag{54}$$

inasmuch as this makes sense (i.e. for ordinary limits in $\mathrm{tv}_*\mathbb{A}$ and $\mathrm{tv}_1\mathbb{A}$, which amount to $\star$- and 1-level limits of $\mathbb{A}$).

Theorem 4.5 says that $\mathbb{A}$ has double limits if and only if it has: double products, double equalisers and tabulators. Concretely, this amounts to the existence of the limits listed in (b) and (c), together with the conditions:
- products are preserved by domain, codomain and vertical identities,
- equalisers are preserved by domain, codomain and vertical identities.

If this holds *and* $\mathbb{A}$ is transversally invariant ('horizontally invariant' in [GP1]), Proposition 2.3 says one can always choose double limits such that this preservation is strict. For products this means that:
- for a family of vertical arrows $u_\lambda \colon A_\lambda \to B_\lambda$ we have $\Pi\, u_\lambda \colon \Pi\, A_\lambda \to \Pi\, B_\lambda$,
- for a family of objects A_λ the product of their vertical identities is the vertical identity of $\Pi\, A_\lambda$.

4.8 The symmetric cubical case

As analysed in [G1], weak symmetric cubical categories (with lax cubical functors) have a path endofunctor

$$\begin{aligned} &P \colon \mathrm{Lx}\mathbf{Wsc} \to \mathrm{Lx}\mathbf{Wsc}, \\ &P((\mathrm{tv}_n\mathsf{A}), (\partial_i^\alpha), (e_i), (+_i), (s_i), ...) \\ &\quad = ((\mathrm{tv}_{n+1}\mathsf{A}), (\partial_{i+1}^\alpha), (e_{i+1}), (+_{i+1}), (s_{i+1}), ...), \end{aligned} \tag{55}$$

which lifts all components of one degree and discards 1-indexed faces, degeneracies, transpositions and comparisons (the latter are omitted above). The discarded faces and degeneracy yield three natural transformations

$$\partial_1^\alpha \colon P \rightleftarrows 1 : e_1, \qquad \partial_1^\alpha . e_1 = \mathrm{id}, \tag{56}$$

which make P into a *path endofunctor*, from a structural point of view. The role of symmetries is crucial (without them we would have two *non-isomorphic* path-functors, and a plethora of higher path functors, their composites, see [G1]).

This situation cannot be extended to chiral multiple categories: the path endofunctor was replaced by the lift functors $Q_j\colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}$ and the restriction functors $R_j\colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}$ of 1.8, with faces and degeneracy

$$D_j^\alpha\colon Q_j \rightleftarrows R_j : E_j, \qquad\qquad D_j^\alpha.E_j = \mathrm{id}. \tag{57}$$

The whole system is consistent, by means of commutative squares

$$\begin{tikzcd} \mathrm{Lx}\mathbf{Wsc} \arrow[r, "P"] \arrow[d, "U"'] & \mathrm{Lx}\mathbf{Wsc} \arrow[d, "U_j"] \\ \mathrm{Lx}\mathbf{Cmc} \arrow[r, "Q_j"'] & \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Lx}\mathbf{Wsc} \arrow[r, "1"] \arrow[d, "U"'] & \mathrm{Lx}\mathbf{Wsc} \arrow[d, "U_j"] \\ \mathrm{Lx}\mathbf{Cmc} \arrow[r, "R_j"'] & \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j} \end{tikzcd} \tag{58}$$

where $U\colon \mathrm{Lx}\mathbf{Wsc} \to \mathrm{Lx}\mathbf{Cmc}$ is the embedding described in I.2.8 (that gives rise to weak multiple categories of a symmetric cubical type) and $U_j = R_jU$.

In this way, cubical limits in weak symmetric cubical categories, dealt with in [G2], agree with multiple limits as presented here.

5. Proof of the theorem on multiple limits

We now prove Theorem 3.6. The argument is similar to the proof of the corresponding theorem for double limits [GP1], or its extension to cubical limits [G2].

5.1 Comments

Of course we only have to prove the 'sufficiency' part of the statement. We write down the argument for the construction of limits; the preservation property is proved in the same way.

The chiral multiple category A is supposed to have all level limits of degree zero and all tabulators of degree zero (or total tabulators). The proof works by transforming a lax functor $F\colon \mathsf{X} \to \mathsf{A}$ of chiral multiple categories into a graph-morphism $G\colon \mathbf{X} \to \mathrm{tv}_*\mathsf{A}$ and taking the limit of the latter. The (directed) graph $\mathbf{X}$ is a sort of 'transversal subdivision' of X, where every $\mathbf{i}$-cube of X is replaced with an object *simulating its total tabulator*.

The procedure is similar to computing the end of a functor $S\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ as the limit of the associated functor $S^{\S}\colon \mathbf{C}^{\S} \to \mathbf{D}$ based on Kan's *subdivision category* of $\mathbf{C}$ ([Ka], 1.10; [Ma], IX.5).

5.2 Transversal subdivision

The *transversal subdivision* $\mathbf{X}$ of X is a graph, formed by the following objects and arrows, for an arbitrary positive multi-index $\mathbf{i}$ of degree $n \geqslant 0$, with arbitrary $j \in \mathbf{i}$ and $\alpha = \pm$. (Note that this graph is finite whenever X is.)

(a) For every $\mathbf{i}$-cell x of X there is an object x in $\mathbf{X}$. For every $\mathbf{i}$-map $f\colon x \to y$ of X there is an arrow $f\colon x \to y$ in $\mathbf{X}$.

(b) For every $\mathbf{i}$-cell x of X, we also add $2n$ arrows $p_j^\alpha x\colon x \to \partial_j^\alpha x$ (that simulate the projections (45) of the total tabulator of x, for $j \in \mathbf{i}$ and $\alpha = \pm$).

(c) If $x = e_j z$ is degenerate (in direction j) we also add an arrow $d_j z\colon z \to e_j z$ (that simulates the diagonal map (46)).

(d) For every j-concatenation of $\mathbf{i}$-cells $z = x +_j y$ in X, we also add an object $(x, y)_j$ in $\mathbf{X}$ and three arrows

$$\begin{gathered} p_j = p_j(x,y)\colon (x,y)_j \to x, \qquad q_j = q_j(x,y)\colon (x,y)_j \to y, \\ d_j(x,y)\colon (x,y)_j \to z, \end{gathered} \tag{59}$$

that simulate the pullback-object $\top_j(x, y)$ of (47), with its projections and the diagonal map (48).

5.3 The associated morphism of graphs

We now construct a graph-morphism $G\colon \mathbf{X} \to \mathrm{tv}_*\mathsf{A}$ that naturally comes from F and the existence of level limits and tabulators (of degree zero) in A.

(a) For every $\mathbf{i}$-cell x of X, we define Gx as the following total tabulator (a $\star$-cube) of A

$$G(x) = \top(Fx) \qquad\qquad (t_{Fx}\colon e_{\mathbf{i}}G(x) \to_0 F(x)). \tag{60}$$

For every $\mathbf{i}$-map $f\colon x \to_0 y$ of X, we define Gf as the transversal map of A determined by the universal property of t_{Fy}, as follows

$$\begin{tikzcd} e_{\mathbf{i}}\top(Fx) \arrow[r, "e_{\mathbf{i}}(Gf)"] \arrow[d, "t_{Fx}"'] & e_{\mathbf{i}}\top(Fy) \arrow[d, "t_{Fy}"] \\ Fx \arrow[r, "Ff"'] & Fy \end{tikzcd} \qquad \begin{array}{l} Gf\colon \top(Fx) \to_0 \top(Fy), \\ \\ t_{Fy}.e_{\mathbf{i}}(Gf) = Ff.t_{Fx}. \end{array} \tag{61}$$

(b) For $z = \partial^\alpha_j x$ we define $G(p^\alpha_j x)\colon Gx \to_0 Gz$ as the following transversal map of A

$$\begin{tikzcd} e_{\mathbf{i}|j}\top(Fx) \arrow[r, "e_{\mathbf{i}|j}(Gp^\alpha_j x)"] \arrow[dr, "\partial^\alpha_j(t_{Fx})"'] & e_{\mathbf{i}|j}\top(Fz) \arrow[d, "t_{Fz}"] \\ & Fz \end{tikzcd} \qquad \begin{array}{l} G(p^\alpha_j x)\colon \top Fx \to_0 \top Fz, \\ \\ t_{Fz}.e_{\mathbf{i}|j}(G(p^\alpha_j x)) = \partial^\alpha_j(t_{Fx}). \end{array} \tag{62}$$

(c) For a degenerate $\mathbf{i}$-cube $x = e_j z$ (where z is an $\mathbf{i}|j$-cube) the map $G(d_j z)\colon Gz \to_0 G(e_j z)$ is defined as follows

$$\begin{tikzcd} e_{\mathbf{i}}(\top Fz) \arrow[r, "e_{\mathbf{i}}(Gd_j z)"] \arrow[d, "e_j t_{Fz}"'] & e_{\mathbf{i}}(\top Fe_j z) \arrow[d, "t_{Fx}"] \\ e_j Fz \arrow[r, "\underline{F}_j z"'] & Fe_j z = Fx \end{tikzcd} \qquad \begin{array}{l} G(d_j z)\colon \top Fz \to_0 \top(Fe_j z), \\ \\ t_{Fx}.e_{\mathbf{i}}(G(d_j z)) = \underline{F}_j z.e_j(t_{Fz}). \end{array} \tag{63}$$

(d) For a concatenation $z = x +_j y$ of $\mathbf{i}$-cubes, the object $G(x,y)_j = \top_j(Fx, Fy)$ is the pullback of the objects $\top Fx$ and $\top Fy$, over the tabulator $\top Fw$ associated to the $\mathbf{i}|j$-cube $w = \partial^+_j x = \partial^-_j y$ (see (47)).

The arrows $p_j(x,y)\colon (x,y)_j \to x$ and $q_j(x,y)\colon (x,y)_j \to y$ of $\mathbf{X}$ are taken by G to the projections (47) of $\top_j(Fx, Fy)$

$$G(p_j(x,y))\colon G(x,y)_j \to_0 Gx, \quad G(q_j(x,y))\colon G(x,y)_j \to_0 Gy, \tag{64}$$

so that $(G(x,y)_j; Gp_j(x,y), Gq_j(x,y))$ is the pullback of $(p^+_j(Fx), p^-_j(Fy))$ in $\mathrm{tv}_*\mathsf{A}$.

Finally, the arrow $d_j(x,y)\colon (x,y)_j \to z$ of $\mathbf{X}$ is sent by G to the diagonal (48) of $G(x,y)_i = \top_j(Fx, Fy)$, determined as follows

$$\begin{aligned}
&G(d_j(x,y))\colon \top_j(Fx,Fy) \to_0 \top F(z),\\
&t_{Fz}.e_{\mathbf{i}}(G(d_j(x,y))\\
&\quad = \underline{F}_j(x,y).(t_{Fx}.e_{\mathbf{i}}G(p_j(x,y)) +_j t_{Fy}.e_{\mathbf{i}}G(q_j(x,y))).\lambda_j^{-1},
\end{aligned} \tag{65}$$

$$\begin{tikzcd}
e_{\mathbf{i}}(G(x,y)_i) \arrow[r, "e_{\mathbf{i}}(G(d_j(x,y))"] \arrow[d, "\lambda_j^{-1}"'] & e_{\mathbf{i}}(\top(Fz)) \arrow[r, "t_{Fz}"] & Fz \\
e_{\mathbf{i}}(G(x,y)_i +_j e_{\mathbf{i}}(G(x,y)_i) \arrow[rr, "t_{Fx}.e_{\mathbf{i}}Gp_j +_j t_{Fy}.e_{\mathbf{i}}Gq_j"'] & & Fx +_j Fy \arrow[u, "\underline{F}_j(x,y)"']
\end{tikzcd}$$

The limit of this diagram $G\colon \mathbf{X} \to \mathrm{tv}_*\mathsf{A}$ exists, by hypothesis.

5.4 From multiple cones to cones

In order to prove that the limit of G gives the limit of degree 0 of F we construct an isomorphism

$$(D{\downarrow}F) \to (D'{\downarrow}G),$$

from the comma category of transversal cones of the lax functor F to the comma category of ordinary cones of the graph-morphism G. We proceed first in this direction, and then backwards.

Let $(A, h\colon DA \to F)$ be a cone of F. For every $\mathbf{i}$-cube x of X, we define $k(x)\colon A \to_0 Gx = \top(Fx)$ as the $\star$-map of A determined by the $\mathbf{i}$-map hx, via the tabulator property

$$t_{Fx}.e_{\mathbf{i}}(kx) = hx. \tag{66}$$

Further, we define $k(x,y)_j\colon A \to_0 G(x,y)_j$ by means of the pullback-property of $G(x,y)_j$

$$\begin{aligned}
p_j(x,y).k(x,y)_j &= kx\colon A \to_0 Gx,\\
q_j(x,y).k(x,y)_j &= ky\colon A \to_0 Gy.
\end{aligned} \tag{67}$$

Let us verify that this family k is indeed a cone of $G\colon \mathbf{X} \to \mathrm{tv}_*\mathsf{A}$.

(a) Coherence with an $\mathbf{i}$-map $f\colon x \to_0 y$ (viewed as an arrow of $\mathbf{X}$) means that $Gf.kx = ky$, which follows from the cancellation property of t_{Fy}

$$t_{Fy}.e_{\mathbf{i}}(Gf.kx) = Ff.t_{Fx}.e_{\mathbf{i}}(kx) = Ff.hx = hy = t_{Fy}.e_{\mathbf{i}}(ky). \tag{68}$$

(b), (c) Coherence with the $\mathbf{X}$-arrows $p_j^\alpha(x)\colon x \to \partial_j^\alpha x$ and $d_j z\colon z \to e_j z = x$ follows from (62) and (63)

$$\begin{aligned} G(p_j^\alpha(x)).kx &= k(\partial_j^\alpha x), \\ t_{Fx}.e_{\mathbf{i}}(G(d_j z).kz) &= \underline{F}_j z.e_j(t_{Fz}).e_{\mathbf{i}}(kz) = \underline{F}_j z.e_j(t_{Fz}.e_{\mathbf{i}|j}(kz)) \\ &= \underline{F}_j z.e_j(hz) = h(e_j z) = h(x) = t_{Fx}.e_{\mathbf{i}}(kx). \end{aligned} \tag{69}$$

(d) Coherence with the $\mathbf{X}$-arrows $p_j = p_j(x,y)$ and $q_j = q_j(x,y)$ holds by construction (see (64)). For $d_j(x,y)$ and $z = x +_j y$ we have

$$\begin{aligned} & t_{Fz}.e_{\mathbf{i}}(G(d_j(x,y).k(x,y)_j) \\ &\quad = \underline{F}_j(x,y).(t_{Fx}.e_{\mathbf{i}}p_j +_j t_{Fy}.e_{\mathbf{i}}q_j).\lambda_j^{-1}.e_{\mathbf{i}}k(x,y)_j \\ &\quad = \underline{F}_j(x,y).(t_{Fx}.e_{\mathbf{i}}p_j +_j t_{Fy}.e_{\mathbf{i}}q_j).(e_{\mathbf{i}}k(x,y)_j +_j e_{\mathbf{i}}k(x,y)_j).\lambda_j^{-1} \\ &\quad = \underline{F}_j(x,y).(hx +_j hy).\lambda_j^{-1} = hz = t_{Fz}.e_{\mathbf{i}}(kx). \end{aligned} \tag{70}$$

Finally, a map of multiple cones

$$f\colon (A, h\colon DA \to F) \to (A', h'\colon DA' \to F)$$

determines a map of G-cones $f\colon (A,k) \to (A',k')$, since

$$t_{Fx}.e_{\mathbf{i}}(k'x.f) = h'x.e_{\mathbf{i}}(f) = hx = t_{Fx}.e_{\mathbf{i}}(kx). \tag{71}$$

5.5 From cones to multiple cones

In the reverse direction $(D'{\downarrow}G) \to (D{\downarrow}F)$ we just specify the procedure on cones. Given an ordinary cone $(A, k\colon D'A \to G)$ of G, one forms a multiple cone $(A, h\colon DA \to F)$ by letting

$$hx = t_{Fx}.e_{\mathbf{i}}(kx)\colon e_{\mathbf{i}}(A) \to x \qquad\qquad (x \in A_{\mathbf{i}}). \tag{72}$$

This satisfies (tc.1) (see 3.2) since, for $f\colon x \to_0 y$ in X

$$Ff.hx = Ff.t_{Fx}.e_{\mathbf{i}}(kx) = t_{Fy}.e_{\mathbf{i}}(Gf.kx) = t_{Fy}.e_{\mathbf{i}}(ky) = hy. \qquad (73)$$

Finally, to verify the condition (tc.2) for j-units and j-composition in X we operate much as above (with $x = e_j z$ in the first case and $z = x +_j y$ in the second)

$$\begin{aligned} &\underline{F}_j(z).e_j(hz) = \underline{F}_j(z).e_j(t_{Fz}.e_{\mathbf{i}|j}(kz)) = \underline{F}_j(z).e_j(t_{Fz}).e_{\mathbf{i}}(kz) \\ &\quad = t_{Fx}.e_{\mathbf{i}}(G(d_j z).kz) = t_{Fx}.e_{\mathbf{i}}(kx) = hx. \end{aligned} \qquad (74)$$

$$\begin{aligned} &hz = t_{Fz}.e_{\mathbf{i}}(kz) = t_{Fz}.e_{\mathbf{i}}(G(d_j(x,y)).k(x,y)_j) = \\ &\quad = \underline{F}_j(x,y).(t_{Fx}.e_{\mathbf{i}}p_j +_j t_{Fy}.e_{\mathbf{i}}q_j).\lambda_j^{-1}.e_{\mathbf{i}}k(x,y)_j \\ &\quad = \underline{F}_j(x,y).(t_{Fx}.e_{\mathbf{i}}p_j +_j t_{Fy}.e_{\mathbf{i}}q_j).(e_{\mathbf{i}}k(x,y)_j +_j e_{\mathbf{i}}k(x,y)_j).\lambda_j^{-1} \\ &\quad = \underline{F}_j(x,y).(hx +_j hy).\lambda_j^{-1}. \end{aligned} \qquad (75)$$

References

[ABS] F.A.A. Al-Agl - R. Brown - R. Steiner, Multiple categories: the equivalence of a globular and a cubical approach, Adv. Math. 170 (2002), 71-118.

[AM] M. Aguiar - S. Mahajan, Monoidal functors, species and Hopf algebras, CRM Monograph Series, 29. American Mathematical Society, Providence, RI, 2010.

[BE] A. Bastiani - C. Ehresmann, Multiple functors I. Limits relative to double categories, Cahiers Top. Géom. Diff. 15 (1974), 215-292.

[BS] T. Booker - R. Street, Tannaka duality and convolution for duoidal categories, Theory Appl. Categ. 28 (2013), No. 6, 166-205.

[EE1] A. Ehresmann - C. Ehresmann, Multiple functors II. The monoidal closed category of multiple categories, Cahiers Top. Géom. Diff. 19 (1978), 295-333.

[EE2] A. Ehresmann - C. Ehresmann, Multiple functors III. The Cartesian closed category Cat_n. Cahiers Top. Géom. Diff. 19 (1978), 387-443.

[EE3] A. Ehresmann - C. Ehresmann, Multiple functors IV. Monoidal closed structures on Cat_n, Cahiers Top. Géom. Diff. 20 (1979), 59-104.

[Eh] C. Ehresmann, Catégories structurées, Ann. Sci. Ecole Norm. Sup. 80 (1963), 349-425.

[G1] M. Grandis, Higher cospans and weak cubical categories (Cospans in Algebraic Topology, I), Theory Appl. Categ. 18 (2007), No. 12, 321-347.

[G2] M. Grandis, Limits in symmetric cubical categories (On weak cubical categories, II), Cah. Topol. Géom. Différ. Catég. 50 (2009), 242-272.

[GP1] M. Grandis - R. Paré, Limits in double categories, Cah. Topol. Géom. Différ. Catég. 40 (1999), 162-220.

[GP2] M. Grandis - R. Paré, Adjoint for double categories, Cah. Topol. Géom. Différ. Catég. 45 (2004), 193-240.

[GP3] M. Grandis - R. Paré, Kan extensions in double categories (On weak double categories, Part III), Theory Appl. Categ. 20 (2008), No. 8, 152-185.

[GP4] M. Grandis - R. Paré, Lax Kan extensions for double categories (On weak double categories, Part IV), Cah. Topol. Géom. Différ. Catég. 48 (2007), 163-199.

[GP5] M. Grandis - R. Paré, From cubical to globular higher categories, in: Liber Amicorum en l'honneur de Mme A.C. Ehresmann, Diagrammes 67-68 suppl. (2012), 117-148. Available at: http://www.dima.unige.it/~grandis/CGlb.pdf

[GP6] M. Grandis - R. Paré, Intercategories, Theory Appl. Categ. 30 (2015), No. 38, 1215-1255.

[GP7] M. Grandis - R. Paré, Intercategories: a framework for three dimensional category theory, submitted. Preprint available at: http://arxiv.org/abs/1412.0212

[GP8] M. Grandis - R. Paré, An introduction to multiple categories (On weak and lax multiple categories, I), Cah. Topol. Géom. Différ. Catég., to appear.

[Gr] J.W. Gray, Formal category theory: adjointness for 2-categories, Lecture Notes in Mathematics, Vol. 391, Springer, Berlin, 1974.

[Ka] D.M. Kan, Adjoint functors, Trans. Amer. Math. Soc. 87 (1958), 294-329.

[Ma] S. Mac Lane, Natural associativity and commutativity, Rice Univ. Studies 49 (1963), 28-46.

[Sh] K. Shimakawa, Multiple categories and algebraic K-theory, J. Pure Appl. Algebra 41 (1986), 285-304.

[St1] R.H. Street, Limits indexed by category valued 2-functors, J. Pure Appl. Algebra 8 (1976), 149-181.

[St2] R. Street, Monoidal categories in, and linking, geometry and algebra, Bull. Belg. Math. Soc. Simon Stevin 19 (2012), 769-821.

Marco Grandis
Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35
16146 - Genova, Italy
grandis@dima.unige.it

Robert Paré
Department of Mathematics and Statistics
Dalhousie University
Halifax NS
Canada B3H 4R2
pare@mathstat.dal.ca

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES Vol. LVII-3 (2016)

THE COMPLETION OF A QUANTUM B-ALGEBRA

by Wolfgang RUMP

Résumé. On montre que les quantales sont les objets injectifs dans la catégorie des quantum B-algèbres, et que chaque quantum B-algèbre a une enveloppe injective. Par une construction explicite, l'enveloppe injective se révèle comme une complétion, plus générale que la complétion de Dedekind-MacNeille. Un resultat récent de Lambek et al., où des structures résiduelles surviennent de manière surprenante, est expliqué à la lumière des quantum B-algèbres, fournissant un autre exemple de leur ubiquité. Des connexions aux structures promonoïdales et aux multi-catégories sont indiquées.

Abstract. It is shown that quantales are the injective objects in the category of quantum B-algebras, and that every quantum B-algebra has an injective envelope. By an explicit construction, the injective envelope is revealed as a completion, more general than the Dedekind-MacNeille completion. A recent result of Lambek et al., where residual structures unexpectedly arise, is explained in the light of quantum B-algebras, which gives another instance for their ubiquity. Connections to promonoidal structures and multi-categories are indicated.

Keywords. Quantum B-algebra, quantale, completion, injective envelope, partially ordered monoid, promonoidal category, Day convolution, multi-poset, ternary frame.

Mathematics Subject Classification (2010). 08B30, 68R01, 06F07, 06F05, 03B47, 20M50, 20M50, 03G27.

1. Introduction

Recently, J. Lambek et al. [20] proved that the injective hull of a partially ordered monoid, viewed as an object in a suitable category, is a quantale, and that quantales are injective in that category. The construction is natural, but not straightforward. For example, morphisms are submultiplicative rather than multiplicative, which appears to be natural in the presence of a partial order. Surprisingly, the construction depends on the left and right residuals of a quantale, which led to an unexpected solution, as Lambek remarks: "*mirabile dictu*, it worked!"

In this paper, we show that the reason behind this mystery is the covert presence of a quantum B-algebra. Recall that a *quantum B-algebra* [30, 31] is a partially ordered set X with two binary operations $\rightarrow$ and $\rightsquigarrow$ satisfying

$$x \leqslant y \rightarrow z \iff y \leqslant x \rightsquigarrow z \tag{1}$$

$$x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z) \tag{2}$$

$$y \leqslant z \implies x \rightarrow y \leqslant x \rightarrow z. \tag{3}$$

A certain ubiquity of quantum B-algebras was observed in [30] and [31]. To mention the two extreme cases: A group is equivalent to a quantum B-algebra with trivial partial order, while on the other hand, any partial order with a greatest element determines a quantum B-algebra. In terms of non-commutative logic, the operations $\rightarrow$ and $\rightsquigarrow$ stand for one-sided implications, while $\leqslant$ interprets the logical entailment relation. By [30], Theorem 2.3, quantum B-algebras can be characterized as systems with two operations $\rightarrow, \rightsquigarrow$ and a partial order which can be embedded into a quantale. Note that quantales can be viewed in several respects as non-commutative spaces [4, 3, 6, 5, 24, 25, 26].

We prove that quantales are the injective objects in the category of quantum B-algebras, and that every quantum B-algebra X has an injective envelope (Theorem 1). Moreover, we give an explicit construction of the injective envelope, generalizing various types of completions (Theorem 2). For example, the Dedekind-MacNeille completion of a poset, or of an archimedean lattice-ordered group, occurs as a special case.

A particular instance is Lambek's above mentioned construction [20] of the injective hull of a partial ordered monoid M. As a first step of this construction, M is embedded into the quantale $L(M)$ of lower sets in M. We consider the slightly more general issue where M is a partially ordered semigroup. The injective hull $Q(M)$ of M is then obtained as a quantalic quotient $q\colon L(M) \twoheadrightarrow Q(M)$ with a natural embedding $q|_M\colon M \hookrightarrow Q(M)$. We show that the map q is determined by the quantum B-algebra $X_M \subset$

$L(M)$ generated by M, and that $q|_{X_M}: X_M \to Q(M)$ is nothing else than the completion of X_M as a quantum B-algebra. This reveals the nature of $Q(M)$ and explains the occurrence of residuals in a context of semigroups. In particular, $q|_{X_M}$ is injective, i. e. the quantum B-algebra X_M remains unaffected by passing to the quotient $L(M) \twoheadrightarrow Q(M)$.

Following a referee's suggestion who pointed out that unital quantum B-algebras form a special instance of a promonoidal category, we explain in Section 6 how quantum B-algebras X and their enveloping quantales $U(X)$ fit into the much broader framework of enriched categories. In particular, we relate the multiplication in $U(X)$ to the Day convolution of $U(X)$ as a functor category. It turns out that promonoidal posets can be characterized as a special class of multicategories, enriched over the two-element quantale. Unital quantum B-algebras form a reflective full subcategory of the category of promonoidal posets (Proposition 11). In the context of multi-posets, the universal property of enveloping quantales is derived in Proposition 12.

Some examples are given in a final section. For instance, we exhibit a quantum B-algebra X for which the underlying partially ordered set is a complete lattice, but where X is not a quantale. This also provides an example where the completion of X is strictly larger than the Dedekind-MacNeille completion. On the other hand, we show that the completion of a quantum B-algebra does not coincide with the canonical extension [14]. Another example shows that the isomorphism class of a partially ordered monoid M need not be determined by the quantum B-algebra X_M, though M can be recovered from the quantale $L(M)$.

2. Quantum B-algebras and quantales

Quantum B-algebras form a category **qBAlg** [31], morphisms $f: X \to Y$ being monotone and satisfying the equivalent inequalities

$$f(x \to y) \leqslant f(x) \to f(y), \qquad f(x \rightsquigarrow y) \leqslant f(x) \rightsquigarrow f(y). \tag{4}$$

If these inequalities are equations, we call $f: X \to Y$ a *strict* morphism. For example, every *quantale* [23], that is, a complete lattice with an associative multiplication that distributes over arbitrary joins, is a quantum B-algebra. More generally, every *residuated poset*, that is, a partially ordered semigroup with binary operations $\rightarrow$ and $\rightsquigarrow$ satisfying

$$x \leqslant y \rightarrow z \iff xy \leqslant z \iff y \leqslant x \rightsquigarrow z, \tag{5}$$

is a quantum B-algebra.

Definition 1. Let X be a quantum B-algebra. We say that a product xy of elements $x, y \in X$ *exists* if the set $\{z \in X \mid x \leqslant y \rightarrow z\}$ has a smallest element. This element will be denoted by xy.

Thus, if xy exists, it is unique and satisfies (5).

Proposition 1. *Let X be a quantum B-algebra and $x, y, z \in X$. Assume that the products xy and yz exist in X. Then $(xy)z$ extsts if and only if $x(yz)$ exists, and in case they exist, these products are equal.*

Proof. Assume that $(xy)z$ exists. Then $(xy)z \leqslant t \Leftrightarrow xy \leqslant z \rightarrow t \Leftrightarrow y \leqslant x \rightsquigarrow (z \rightarrow t) \Leftrightarrow y \leqslant z \rightarrow (x \rightsquigarrow t) \Leftrightarrow yz \leqslant x \rightsquigarrow t \Leftrightarrow x \leqslant yz \rightarrow t$, which shows that $x(yz)$ exists and is equal to $(xy)z$. The converse follows by symmetry. $\square$

By Proposition 1, it makes sense to speak of a *submonoid* or a *sub-semigroup* of a quantum B-algebra. The latter means a subset M with existing products $xy \in M$ for each pair $x, y \in M$. As usual, we endow any subset of X with the induced partial order.

Definition 2. We define a *morphism* $f\colon M \rightarrow N$ of partially ordered semigroups to be a monotone map satisfying $f(a)f(b) \leqslant f(ab)$ for all $a, b \in M$.

The following result shows how the inequalities (4) have to be changed in terms of existing products.

Proposition 2. *Let X be a quantum B-algebra with a sub-semigroup M such that every $a \in X$ satisfies $a = \bigvee\{z \in M \mid z \leqslant a\}$. Let $f\colon X \rightarrow Q$ be a map into a quantale Q with $f(a) = \bigvee\{f(z) \mid a \geqslant z \in M\}$ for all $a \in X$. Then f is a morphism of quantum B-algebras if and only if $f|_M$ is a morphism of partially ordered semigroups.*

Proof. Clearly, f is monotone. Assume that f satisfies (4). For all $x, y \in M$, we have $x \leqslant y \to xy$. Hence $f(x) \leqslant f(y \to xy) \leqslant f(y) \to f(xy)$, which gives $f(x)f(y) \leqslant f(xy)$.

Conversely, assume that this inequality holds for $x, y \in M$. For given $a, b \in X$, assume that $x \leqslant a$ and $y \leqslant a \to b$. Then (1) implies that $x \leqslant a \leqslant (a \to b) \rightsquigarrow b$, hence $y \leqslant a \to b \leqslant x \to b$. Thus $yx \leqslant b$. So we obtain $f(y)f(x) \leqslant f(yx) \leqslant f(b)$, which yields $f(a \to b)f(a) \leqslant f(b)$. Whence $f(a \to b) \leqslant f(a) \to f(b)$. □

As an immediate consequence, we have

Corollary. *Let Q and Q' be quantales. A map $f\colon Q \to Q'$ is a morphism of quantum B-algebras if and only if f is a morphism of partially ordered semigroups.*

Note that such a morphism $f\colon Q \to Q'$ satisfies $\bigvee f(A) \leqslant f(\bigvee A)$ for subsets $A \subset Q$. Thus, a *quantale homomorphism* is obtained if the inequalities are replaced by equations: $f(\bigvee A) = \bigvee f(A)$ and $f(ab) = f(a)f(b)$ for $A \subset Q$ and $a, b \in Q$.

Let X be a quantum B-algebra, and $x, x_1, \ldots, x_n \in X$ with $n > 1$. Inductively, we define

$$x_1 \cdots x_n \leqslant x \;:\Longleftrightarrow\; x_1 \cdots x_{n-1} \leqslant x_n \to x. \tag{6}$$

The fundamental rôle of this relation will become apparent in Section 4. From (5) it follows that (6) becomes a true fact if the products on both sides exist. Note that every morphism $f\colon X \to Y$ of quantum B-algebras satisfies

$$x_1 \cdots x_n \leqslant x \implies f(x_1) \cdots f(x_n) \leqslant f(x) \tag{7}$$

for all $x, x_1, \ldots, x_n \in X$. Indeed, this is trivial for $n = 1$. Now $x_1 \cdots x_n \leqslant x$ gives $x_1 \cdots x_{n-1} \leqslant x_n \to x$. Hence, by induction, we can assume that $f(x_1) \cdots f(x_{n-1}) \leqslant f(x_n \to x) \leqslant f(x_n) \to f(x)$. Thus $f(x_1) \cdots f(x_n) \leqslant f(x)$.

Definition 3. We define an *embedding* $X \rightarrowtail Y$ of quantum B-algebras X, Y to be a morphism $e\colon X \to Y$ for which the implication (7) is an equivalence. If $X \hookrightarrow Y$ is a strict embedding, we call X a *quantum B-subalgebra* of Y.

In particular, an embedding $X \rightarrowtail Y$ is injective, and X can be regarded as a subposet of Y. The converse holds for strict morphisms:

Proposition 3. *Let $e\colon X \to Y$ be a strict morphism of quantum B-algebras such that $x \leqslant y \iff e(x) \leqslant e(y)$ holds for $x, y \in X$. Then e is an embedding.*

Proof. We have to show that $e(x_1)\cdots e(x_n) \leqslant e(x)$ implies $x_1 \cdots x_n \leqslant x$ for given $x, x_1, \ldots, x_n \in X$. For $n = 1$, this follows by the assumption. Otherwise, $e(x_1)\cdots e(x_n) \leqslant e(x)$ yields $e(x_1)\cdots e(x_{n-1}) \leqslant e(x_n) \to e(x) = e(x_n \to x)$. Thus, by induction, we can assume that $x_1 \cdots x_{n-1} \leqslant x_n \to x$. Whence $x_1 \cdots x_n \leqslant x$. □

3. The injective envelope

In this section, we construct an injective envelope for every quantum B-algebra. We say that an object Q in **qBAlg** is *injective* if every morphism $X \to Q$ factors through any embedding $X \rightarrowtail Y$ of quantum B-algebras.

Proposition 4. *With respect to embeddings, quantales are injective objects in the category* **qBAlg** *of quantum B-algebras.*

Proof. Let $X \hookrightarrow Y$ be an embedding, and let $f\colon X \to Q$ be a morphism into a quantale Q. By [30], Theorem 2.3, Y embeds into a quantale Q'. So it suffices to prove that f factors through $X \hookrightarrow Q'$. Define $f'\colon Q' \to Q$ by

$$f'(a) := \bigvee\{f(x_1)\cdots f(x_n) \mid x_1, \ldots, x_n \in X \text{ and } x_1 \cdots x_n \leqslant a\}.$$

For $a, b \in Q'$, this gives

$$\begin{aligned} &f'(a)f'(b) = \\ &\bigvee\{f(x_1)\cdots f(x_n)f(y_1)\cdots f(y_m) \mid x_i, y_j \in X, x_1 \cdots x_n \leqslant a, y_1 \cdots y_m \leqslant b\} \\ &\leqslant \bigvee\{f(x_1)\cdots f(x_n)f(y_1)\cdots f(y_m) \mid x_1 \cdots x_n y_1 \cdots y_m \leqslant ab\} = f'(ab). \end{aligned}$$

Since f' is monotone, the corollary of Proposition 2 shows that f' is a morphism of quantum B-algebras. Furthermore, $f'|_X = f$ follows by (7) since $X \hookrightarrow Q'$ is an embedding. □

Definition 4. We call an embedding $e\colon X \rightarrowtail Y$ of quantum B-algebras *essential* if every morphism $f\colon Y \to Z$ in **qBAlg** for which fe is an embedding is itself an embedding. If, in addition, Y is injective, we call e an *injective envelope* of X.

As usual, an injective envelope is unique, up to isomorphism.

Proposition 5. *Every essential embedding $e\colon X \rightarrowtail Y$ of quantum B-algebras is strict.*

Proof. By [30], Theorem 2.3, there is a strict embedding $i\colon X \rightarrowtail Q$ into a quantale Q. Therefore, Proposition 4 implies that $i = fe$ for some morphism $f\colon Y \to Q$. Since e is essential, f is an embedding. For $x, y \in X$, we have

$$fe(x \to y) \leqslant f(e(x) \to e(y)) \leqslant fe(x) \to fe(y) = fe(x \to y).$$

Hence $fe(x \to y) = f(e(x) \to e(y))$, and thus $e(x \to y) = e(x) \to e(y)$. $\square$

Recall that a *nucleus* [27, 28] of a quantale Q is defined to be an endomorphism $j\colon Q \to Q$ which satisfies $a \leqslant j(a) = j^2(a)$ for all $a \in Q$. There is a natural one-to-one correspondence between quantic nuclei $j\colon Q \to Q$ and congruence relations on Q: For any surjective quantale homomorphism $p\colon Q \twoheadrightarrow Q'$, every fiber $p^{-1}(p(a))$ of p has a greatest element $j(a)$, which gives a nucleus j, and every nucleus arises in this way.

A special type of nucleus is obtained as follows. For a subset X of a quantale Q, let X^* denote the sub-semigroup generated by X. So the subquantale generated by X is $\{\bigvee A \mid A \subset X^*\}$.

Proposition 6. *Let Q be a quantale, generated by a quantum B-subalgebra X. Then $j(a) := \bigwedge\{x \in X \mid a \leqslant x\}$ defines a nucleus $j\colon Q \to Q$.*

Proof. By definition, j is a *closure operator*, that is, j is monotone with $a \leqslant j(a) = j^2(a)$ for all $a \in Q$. For given $a, b \in Q$, assume that $ab \leqslant x$ for some $x \in X$. For all $y \in X^*$ with $y \leqslant b$, this gives $ay \leqslant x$, hence $a \leqslant y \to x$. Thus $j(a) \leqslant y \to x$, which gives $j(a)y \leqslant x$. Since $b = \bigvee\{y \in X^* \mid y \leqslant b\}$, we obtain $j(a)b \leqslant x$. Similarly, every $z \in X^*$ with

$z \leqslant j(a)$ satisfies $zb \leqslant x$, which gives $b \leqslant z \rightsquigarrow x$. Thus $j(b) \leqslant z \rightsquigarrow x$, which yields $zj(b) \leqslant x$. So we get $j(a)j(b) \leqslant x$ for all $x \in X$ with $ab \leqslant x$. Whence $j(a)j(b) \leqslant j(ab)$. $\square$

Definition 5. We say that an embedding $X \hookrightarrow Q$ of a quantum B-algebra X into a quantale Q is *dense* if X generates the quantale Q and every $a \in Q$ is of the form $a = \bigwedge A$ with $A \subset X$.

Proposition 7. *An embedding $X \hookrightarrow Q$ of a quantum B-algebra X into a quantale Q is essential if and only if it is dense.*

Proof. Assume that $X \hookrightarrow Q$ is dense, and let $f\colon Q \to Y$ be a morphism of quantum B-algebras such that $f|_X$ is an embedding. By [30], Theorem 2.3, there exists a strict embedding $Y \hookrightarrow Q'$ into a quantale Q'. Now assume that $a, a_1, \ldots, a_n \in Q$ and $f(a_1)\cdots f(a_n) \leqslant f(a)$. To verify that $X \hookrightarrow Q$ is essential, we have to prove that $a_1 \cdots a_n \leqslant a$. To this end, it is enough to show that $x_1^* \cdots x_n^* \leqslant x$ holds for all $x_1^*, \ldots, x_n^* \in X^*$ and $x \in X$ with $x_i^* \leqslant a_i$ and $a \leqslant x$. If $x_i^* = x_{i,1} \cdots x_{i,m_i}$ with $x_{i,j} \in X$, then $f(x_{i,1}) \cdots f(x_{i,m_i}) \leqslant f(x_{i,1} \cdots x_{i,m_i}) \leqslant f(a_i)$. Therefore, the inequality $f(x_{1,1}) \cdots f(x_{1,m_1}) \cdots f(x_{n,1}) \cdots f(x_{n,m_n}) \leqslant f(x)$ holds in Q'. Since $X \xrightarrow{f|_X} Y \hookrightarrow Q'$ is an embedding, this yields

$$x_1^* \cdots x_n^* = x_{1,1} \cdots x_{1,m_1} \cdots x_{n,1} \cdots x_{n,m_n} \leqslant x.$$

Conversely, assume that $X \hookrightarrow Q$ is essential. By Proposition 5, X is a quantum B-subalgebra of Q. Let Q_0 be the subquantale of Q generated by X. Then $Q_0 \hookrightarrow Q$ is an embedding. By Proposition 6,

$$j(a) := \bigwedge \{x \in X \mid a \leqslant x\}$$

defines a nucleus $j\colon Q_0 \to Q_0$. So the quantale homomorphism $Q_0 \twoheadrightarrow jQ_0$ factors through $Q_0 \hookrightarrow Q$, which gives a commutative diagram

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, hook, "i"'] & Q_0 \arrow[r, hook] \arrow[d, two heads] & Q \arrow[dl, "f"] \\
& jQ_0 &
\end{tikzcd}$$

with an embedding i. Hence f is an embedding, and thus j is the identity map. So f is an injective retraction, which shows that $Q_0 = Q$. Consequently, $X \hookrightarrow Q$ is dense. □

Now we are ready to prove

Theorem 1. *Every quantum B-algebra X has an injective envelope.*

Proof. By [30], Theorem 2.3, there is a strict embedding $e\colon X \hookrightarrow Q$ into a quantale Q. As in the preceding proof, let Q_0 be the subquantale generated by X. So the nucleus $j\colon Q_0 \to Q_0$ on Q_0 yields a dense embedding $X \hookrightarrow jQ_0$ into the quantale jQ_0. □

Corollary. *For a quantum B-algebra Q, the following are equivalent.*

(a) *Q is a quantale.*

(b) *Q is injective in* **qBAlg**.

(c) *Every essential embedding $Q \rightarrowtail X$ is an isomorphism.*

Proof. (a) ⇒ (b) follows by Proposition 4.

(b) ⇒ (c): Let $e\colon Q \rightarrowtail X$ be an essential embedding. Then there is a morphism $f\colon X \to Q$ with $fe = 1_Q$. Since e is essential, f is an embedding. Hence e is invertible.

(c) ⇒ (a): This follows immediately by the proof of Theorem 1. □

4. The completion

By [30], Theorem 2.3, every quantum B-algebra X admits a strict embedding

$$X \hookrightarrow U(U(X))$$

into a quantale, where $U(X)$ denotes the quantale of upper sets of X, with multiplication

$$A \cdot B := \{x \in X \mid \exists\, y \in B\colon y \to x \in A\} \tag{8}$$

for $A, B \in U(X)$. Together with the proof of Theorem 1, this leads to an explicit construction of the injective envelope. In this section, we give a direct approach, without using the embedding $X \hookrightarrow U(U(X))$. As a byproduct, this yields an independent proof of the strict embeddability of X into a quantale.

Let X^f denote the free semigroup generated by X. Then (6) defines a relation $a \leqslant x$ between $a \in X^f$ and $x \in X$. For subsets $A \subset X^f$ and $Y \subset X$, we write

$$A \leqslant Y \ :\Longleftrightarrow \ \forall\, a \in A,\, y \in Y : a \leqslant y. \tag{9}$$

If A or Y is a singleton, we simply write $a \leqslant Y$ or $A \leqslant y$ instead of $\{a\} \leqslant Y$ or $A \leqslant \{y\}$, respectively. The relation (9) induces a Galois connection between the power sets $\mathfrak{P}(X^f)$ and $\mathfrak{P}(X)$, given by

$$A^{\uparrow} := \{x \in X \mid A \leqslant x\}$$
$$Y^{\downarrow} := \{a \in X^f \mid a \leqslant Y\}$$

for $A \in \mathfrak{P}(X^f)$ and $Y \in \mathfrak{P}(X)$. Thus every $Y \subset X$ has a *closure* $Y^{\downarrow\uparrow}$. We call Y *closed* if $Y = Y^{\downarrow\uparrow}$ and denote the set of closed subsets of X by $\widehat{X}$. For $Y, Z \in \widehat{X}$, we define

$$Y \cdot Z := \{x \in X \mid \forall\, b \leqslant Y,\, c \leqslant Z : bc \leqslant x\}, \tag{10}$$

and for a family of $Y_i \in \widehat{X}$, we set

$$\bigvee Y_i := \bigcap Y_i. \tag{11}$$

Note that $\bigcap Y_i$ is again closed. Finally, there is a natural injection $X \rightarrowtail \widehat{X}$ which maps $x \in X$ to the upper set $\uparrow x := \{y \in X \mid x \leqslant y\}$. We endow $\widehat{X}$ with the partial order

$$Y \leqslant Z \ :\Longleftrightarrow \ Y \supset Z.$$

Theorem 2. *Let X be a quantum B-algebra. Then $\widehat{X}$ is a quantale, and $X \rightarrowtail \widehat{X}$ is an injective envelope of X.*

Proof. Eq. (11) makes $\widehat{X}$ into a complete lattice. For $B, C \subset X^f$, we set

$$BC := \{bc \mid b \in B,\, c \in C\}.$$

Then Eq. (10) can be written as

$$Y \cdot Z = (Y^{\downarrow} Z^{\downarrow})^{\uparrow}.$$

To prove the associativity of (10), we thus have to verify

$$((Y_1^{\downarrow} Y_2^{\downarrow})^{\uparrow\downarrow} Y_3^{\downarrow})^{\uparrow} = (Y_1^{\downarrow} (Y_2^{\downarrow} Y_3^{\downarrow})^{\uparrow\downarrow})^{\uparrow} \tag{12}$$

for $Y_1, Y_2, Y_3 \in \widehat{X}$. For $a \in X^f$ and $x \in X$, we define $a \to x$, $a \rightsquigarrow x \in X$ inductively by

$$ay \to x := a \to (x \to y), \qquad ya \rightsquigarrow x := a \rightsquigarrow (y \rightsquigarrow x),$$

for $y \in X$. Then

$$ab \leqslant x \iff a \leqslant b \to x \iff b \leqslant a \rightsquigarrow x$$

holds for $a, b \in X^f$ and $x \in X$. Now

$$\begin{aligned}
(Y_1^{\downarrow} Y_2^{\downarrow})^{\uparrow\downarrow} Y_3^{\downarrow} \leqslant x &\iff \forall\, c \in Y_3^{\downarrow} \colon (Y_1^{\downarrow} Y_2^{\downarrow})^{\uparrow\downarrow} \leqslant c \to x \\
&\iff \forall\, c \in Y_3^{\downarrow} \colon Y_1^{\downarrow} Y_2^{\downarrow} \leqslant c \to x \\
&\iff Y_1^{\downarrow} Y_2^{\downarrow} Y_3^{\downarrow} \leqslant x \iff \forall\, a \in Y_1^{\downarrow} \colon Y_2^{\downarrow} Y_3^{\downarrow} \leqslant a \rightsquigarrow x \\
&\iff \forall\, a \in Y_1^{\downarrow} \colon (Y_2^{\downarrow} Y_3^{\downarrow})^{\uparrow\downarrow} \leqslant a \rightsquigarrow x \\
&\iff Y_1^{\downarrow} (Y_2^{\downarrow} Y_3^{\downarrow})^{\uparrow\downarrow} \leqslant x
\end{aligned}$$

is valid for all $x \in X$. This proves Eq. (12).

Next assume that $Y, Y_i \in \widehat{X}$ for $i \in I \neq \varnothing$. Then $Y_j^{\downarrow} \subset \bigcup_{i \in I} Y_i^{\downarrow}$ implies $(\bigcup_{i \in I} Y_i^{\downarrow})^{\uparrow} \subset Y_j^{\downarrow\uparrow} = Y_j$ for all $j \in I$. Hence $(\bigcup_{i \in I} Y_i^{\downarrow})^{\uparrow} \subset \bigcap_{i \in I} Y_i = (\bigcap_{i \in I} Y_i)^{\downarrow\uparrow}$. For $b \in Y^{\downarrow}$ and $x \in X$, this gives

$$\forall\, i \in I \colon Y_i^{\downarrow} \leqslant b \rightsquigarrow x \implies (\bigcap_{i \in I} Y_i)^{\downarrow} \leqslant b \rightsquigarrow x. \tag{13}$$

The reverse implication is trivial. Thus

$$\forall\, i \in I \colon Y^{\downarrow} Y_i^{\downarrow} \leqslant x \iff Y^{\downarrow} (\bigcap_{i \in I} Y_i)^{\downarrow} \leqslant x$$

holds for all $x \in X$. So we obtain $\bigcap_{i\in I}(Y^{\downarrow}Y_i^{\downarrow})^{\uparrow} = (Y^{\downarrow}(\bigcap_{i\in I} Y_i)^{\downarrow})^{\uparrow}$, which proves that $\bigvee_{i\in I}(Y \cdot Y_i) = Y \cdot \bigvee_{i\in I} Y_i$. If we replace $b \rightsquigarrow x$ in (13) by $b \to x$, we obtain $\bigvee_{i\in I}(Y_i \cdot Y) = (\bigvee_{i\in I} Y_i) \cdot Y$. Thus $\widehat{X}$ is a quantale.

For $x, y \in X$ and $Y \in \widehat{X}$, we have

$$\begin{aligned} Y \leqslant {\uparrow}x \to {\uparrow}y \iff Y\cdot {\uparrow}x \leqslant {\uparrow}y \iff {\uparrow}y \subset (Y^{\downarrow}\{x\}^{\downarrow})^{\uparrow} \iff Y^{\downarrow}\{x\}^{\downarrow} \leqslant y \\ \iff Y^{\downarrow} \leqslant x \to y \iff {\uparrow}(x \to y) \subset Y \iff Y \leqslant {\uparrow}(x \to y), \end{aligned}$$

which shows that ${\uparrow}(x \to y) = {\uparrow}x \to {\uparrow}y$. Furthermore,

$$ {\uparrow}x \leqslant {\uparrow}y \iff {\uparrow}y \subset {\uparrow}x \iff x \leqslant y. $$

Hence $X \rightarrowtail \widehat{X}$ is a strict embedding. In particular,

$$ {\uparrow}x_1 \cdots {\uparrow}x_n \leqslant {\uparrow}x \iff x_1 \cdots x_n \leqslant x \tag{14} $$

holds for $x, x_1, \ldots, x_n \in X$. For $Y \in \widehat{X}$ and $x \in X$,

$$ Y \leqslant {\uparrow}x \iff {\uparrow}x \subset Y \iff x \in Y. \tag{15} $$

Hence $Y = \bigwedge_{x\in Y} {\uparrow}x$. Furthermore, with the abbreviation $a^{\uparrow} := \{a\}^{\uparrow}$,

$$\begin{aligned} \bigvee\{a^{\uparrow} \mid a \in X^f,\, a \leqslant Y\} \leqslant {\uparrow}x &\iff \forall\, a \in X^f\colon (a \leqslant Y \Rightarrow x \in a^{\uparrow}) \\ &\iff \forall\, a \in X^f\colon (a \leqslant Y \Rightarrow a \leqslant x) \\ &\iff x \in Y \iff Y \leqslant {\uparrow}x. \end{aligned}$$

Hence $Y = \bigvee\{a^{\uparrow} | a \in X^f,\, a \leqslant Y\}$. For $a := x_1 \cdots x_n$ and $x_1, \ldots, x_n \in X$, the equivalences (14) and (15) show that $a^{\uparrow} = {\uparrow}x_1 \cdots {\uparrow}x_n$. Therefore, X is dense in $\widehat{X}$. Thus Proposition 7 completes the proof. □

Note that the construction of $\widehat{X}$ exhibits a strong similarity to the Dedekind-MacNeille completion, with the main difference that the partial order is replaced by the fundamental relation (6). Therefore, we call $\widehat{X}$ the *completion* of the quantum B-algebra X. This improves the same-named concept in [30], which was shown to be closely related, but not equivalent to the canonical extension [14] of X. The correctness of our adjustment, which makes use of the nucleus in Proposition 6 to pass to a quotient quantale, is now apparent by its affinity to the Dedekind-MacNeille completion.

5. The case of partially ordered semigroups

Lambek et al. [20] constructed injective hulls in the category **PoM** of partially ordered monoids and showed that they coincide with unital quantales if morphisms f in **PoM** are declared to satisfy Definition 2 and $f(1) = 1$. We will show now that the construction in [20] makes implicit use of a quantum B-algebra.

Let M be a partially ordered semigroup. As in [20], we embed M into the quantale $L(M)$ of *lower sets* $A \subset M$, that is, $A = \downarrow A := \{x \in M \mid \exists\, y \in A\colon x \leqslant y\}$. Thus $a \in M$ is mapped to the lower set $\downarrow a := \downarrow\{a\} \in L(M)$. Let X_M be the quantum B-subalgebra of $L(M)$ generated by M. Thus X_M consists of all terms built from elements of M by using the residuals

$$A \to B := \{c \in M \mid cA \subset B\}, \qquad A \rightsquigarrow B := \{c \in M \mid Ac \subset B\}$$

in $L(M)$. For example, $\downarrow a \rightsquigarrow (\downarrow b \to \downarrow c) = \{d \in M \mid adb \leqslant c\}$ is an element of X_M. We identify M with the image of $M \rightarrowtail X_M$. Thus $X_M = M$ if and only if M is a residuated poset.

Following [20], we say that a morphism $f\colon M \to N$ of partially ordered semigroups is an *embedding* if the implication

$$f(x_1)\cdots f(x_n) \leqslant f(x) \implies x_1 \cdots x_n \leqslant x$$

holds for all $x, x_1, \dots, x_n \in M$.

Proposition 8. *A morphism $f\colon M \to N$ of partially ordered semigroups is an embedding if and only if every morphism $M \to Q$ into a quantale Q factors through f.*

Proof. The necessity follows by the same argument as in the proof of [20], Theorem 4.1. Conversely, let $f\colon M \to N$ be a morphism of partially ordered semigroups. Assume that the embedding $i\colon M \hookrightarrow L(M)$ factors through f. So there is a morphism $g\colon N \to L(M)$ with $gf = i$. Suppose that $f(x_1)\cdots f(x_n) \leqslant f(x)$ holds for some $x, x_1, \dots, x_n \in M$. Then $i(x_1 \cdots x_n) = gf(x_1)\cdots gf(x_n) \leqslant g(f(x_1)\cdots f(x_n)) \leqslant gf(x) = i(x)$. Hence $x_1 \cdots x_n \leqslant x$. $\square$

As in [20], we define injectivity with respect to embeddings. We call an embedding $e\colon M \to N$ *essential* if every morphism $f\colon N \to N'$ for which

fe is an embedding is itself an embedding. An essential embedding into an injective object will be called an *injective envelope*.

Theorem 3. *Let M be a partially ordered semigroup, and let X_M be the associated quantum B-algebra. The completion of X_M is an injective envelope in the category of partially ordered semigroups.*

Proof. By Proposition 8, The quantale $\widehat{X_M}$ is injective as a partially ordered semigroup. Thus, it remains to verify that $M \hookrightarrow X_M \hookrightarrow \widehat{X_M}$ is an essential embedding. By Proposition 2, every morphism $M \to Q$ into a quantale Q extends to a morphism $X_M \to Q$ of quantum B-algebras, which further extends to a morphism $f\colon \widehat{X_M} \to Q$ in **qBAlg**. By the corollary of Proposition 2, f is a morphism of partially ordered semigroups. So Proposition 8 implies that $M \hookrightarrow X_M \hookrightarrow \widehat{X_M}$ is an embedding. Now let $f\colon \widehat{X_M} \to Q$ be a morphism of partially ordered semigroups such that $f|_M$ is an embedding. If the composed map $\widehat{X_M} \xrightarrow{f} Q \hookrightarrow L(Q)$ is an embedding, f is an embedding, too. So we can assume, without loss of generality, that Q is a quantale.

Next we show that $f|_{X_M}$ is an embedding of quantum B-algebras. Since $X_M \hookrightarrow \widehat{X_M}$ is strict by Proposition 5, we have to verify

$$f(a_1)\cdots f(a_n) \leqslant f(a) \implies a_1\cdots a_n \leqslant a \tag{16}$$

for $a, a_1, \ldots, a_n \in X_M$, where the product $a_1 \cdots a_n$ can be taken in $\widehat{X_M}$. Thus $a_1\cdots a_n = \bigvee\{x_1\cdots x_n \mid a_i \geqslant x_i \in M\}$. Hence, without loss of generality, we can assume that $a_1, \ldots, a_n \in M$. So the implication (16) is valid for $a \in M$. If $a \notin M$, then either $a = b \to c$ or $a = b \rightsquigarrow c$, with terms $b, c \in X_M$ of smaller complexity than a. If $a = b \to c$, we have $f(a_1)\cdots f(a_n) \leqslant f(a) \leqslant f(b) \to f(c)$, which gives $f(a_1)\cdots f(a_n)f(b) \leqslant f(c)$. Thus, by induction, we can assume that $a_1 \cdots a_n b \leqslant c$. Whence $a_1\cdots a_n \leqslant b \to c = a$. The case $a = b \rightsquigarrow c$ is treated similarly.

So we have proved that $f|_{X_M}$ is an embedding. Since $X_M \hookrightarrow \widehat{X_M}$ is essential, this shows that f is an embedding of quantum B-algebras, hence an embedding of partially ordered semigroups. $\square$

Remark. The construction in [20] embeds M into $L(M)$ first and than passes to some quotient quantale $p\colon L(M) \twoheadrightarrow Q(M)$ with $p|_M$ invertible.

The preceding proof shows that $p|_{X_M}$ is invertible, too, which highlights the relevance of the quantum B-algebra X_M as an intermediate step toward the injective envelope of M. A minor point is that Lambek et al. [20] deal with monoids instead of semigroups. We briefly address this special case now.

Recall that a quantum B-algebra X is said to be *unital* if there is an element $u \in X$ with

$$u \to x = u \rightsquigarrow x = x$$

for all $x \in X$. Such a *unit element* u is unique [31].

Proposition 9. *If M is a partialy ordered monoid, then X_M is unital. If X is a unital quantum B-algebra, $\widehat{X}$ is a unital quantale.*

Proof. Let M be a partialy ordered monoid with unit element u. For $a \in X_M$ and $x \in M$, we have $x \leqslant u \to a \iff xu \leqslant a \iff x \leqslant a \iff ux \leqslant a \iff x \leqslant u \rightsquigarrow a$. Hence $u \to a = a = u \rightsquigarrow a$. Now let X be a unital quantum B-algebra. For $x, y \in X$, this gives $x \leqslant u \to y \iff x \leqslant y$. Thus xu exists, and $xu = x$. Similarly, $ux = x$. Since $X \hookrightarrow \widehat{X}$ is strict, $ux = xu = x$ holds in $\widehat{X}$. Now every element of $\widehat{X}$ is a join of elements from X. Whence $ua = au = a$ for all $a \in \widehat{X}$. □

6. A categorical perspective

The preceding theorems admit far-reaching generalizations in the framework of enriched categories. We follow a referee's suggestion to put the above results into that wider perspective. All of this section is based upon the referee's detailed remarks.

First, every preordered set A can be regarded as a category, enriched over the cartesian monoidal category **2** with two objects and a single non-identity morphism. Since any such category A is equivalent to its skeleton, we can restrict ourselves to partially ordered sets. Then a **2**-*distributor* $\Phi\colon A \to B$ between posets A and B in the sense of Bénabou [1] is given by a monotone map $\Phi\colon B^{\mathrm{op}} \times A \to \mathbf{2}$. In other words, $\Phi^{-1}(1)$ is an upper set of $B^{\mathrm{op}} \times A$, an *ideal relation* between A an B. By adjunction, Φ can be viewed as a functor $A \to \mathbf{2}^{B^{\mathrm{op}}}$ into the category of **2**-valued presheaves over B, that is, a monotone map $\widehat{\Phi}\colon A \to L(B)$ into the set of lower sets of B. If I denotes the inclusion $B \hookrightarrow L(B)$, the composition $\Psi \otimes \Phi$ of a second distributor $\Psi\colon B \multimap C$ with Φ corresponds to $(\mathrm{Lan}_I \widehat{\Psi})\widehat{\Phi}\colon A \to L(C)$, where the left Kan-extension $\mathrm{Lan}_I \widehat{\Psi}\colon L(B) \to L(C)$ is given by

$$\mathrm{Lan}_I \widehat{\Psi}(b) := \bigvee_{b \geqslant x \in B} \widehat{\Psi}(x)$$

for $b \in L(B)$. Equivalently, $\Psi \otimes \Phi$ can be computed as a coend

$$\Psi \otimes \Phi = \int^{b \in B} \Psi(-, b) \times \Phi(b, -), \tag{17}$$

corresponding to the product of ideal relations

$$(\Psi \otimes \Phi)^{-1}(1) = \Psi^{-1}(1) \circ \Phi^{-1}(1).$$

Let **Idl** denote the category of posets with ideal relations as morphisms. For a poset B, we regard $L(B)$ as an object of **Sup**, the category of *sup-lattices* [19], that is, complete lattices with set-indexed join-preserving morphisms. So the morphisms $\Phi\colon A \multimap B$ in **Idl** can be viewed as morphisms $L(A) \to L(B)$ in **Sup**, which exhibits **Idl** as a reflective full subcategory of **Sup**.

Let X be a quantum B-algebra. The ideal relation $P \subset (X \times X)^{\mathrm{op}} \times X$ with

$$(x, y, z) \in P \;:\Longleftrightarrow\; x \leqslant y \to z \tag{18}$$

gives a **2**-functor $X^{\mathrm{op}} \times X^{\mathrm{op}} \times X \to \mathbf{2}$. Recall that a *promonoidal category* $\mathscr{A}$ (over **2**) [9, 10, 11] is defined by a pair of **2**-functors

$$P\colon \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \mathbf{2} \quad \text{and} \quad J\colon \mathscr{A} \to \mathbf{2}$$

with natural isomorphisms

$$\alpha_{a,b,c,d}\colon \int^{x} P(a, b, x) \otimes P(x, c, d) \xrightarrow{\sim} \int^{x} P(b, c, x) \otimes P(a, x, d) \tag{19}$$

$$\lambda_{a,b}\colon \int^{x} J(x) \otimes P(x, a, b) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{A}}(a, b) \tag{20}$$

$$\rho_{a,b}\colon \int^{x} J(x) \otimes P(a, x, b) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{A}}(a, b) \tag{21}$$

satisfying two coherence conditions [9, 11]. Accordingly, a *promomoidal functor* [12] between promonoidal categories $\mathscr{A}, \mathscr{B}$ is a functor $\Phi\colon \mathscr{A} \to \mathscr{B}$ with two natural transformations $\varphi_{a,b,c}\colon P(a,b,c) \to P(\Phi a, \Phi b, \Phi c)$ and $\varphi_a\colon Ja \to J\Phi a$ satisfying certain relations [8, 11]. For the base category **2**, we speak of a *promonoidal poset*. Then a promomoidal **2**-functor between promonoidal posets A, B is just a monotone map $\Phi\colon A \to B$ which satisfies $\Phi(J) \subset J$ and

$$(x,y,z) \in P \implies (\Phi(x), \Phi(y), \Phi(z)) \in P. \tag{22}$$

Proposition 10. *With respect to* (18)*, every quantum B-algebra X satisfies the associativity condition* (19)*. If X is unital, X is a promonoidal poset.*

Proof. In terms of ideal relations, (19) states that for given $a,b,c,d \in X$, there exists an $x \leqslant c \to d$ with $a \leqslant b \to x$ if and only if there is an $x \in X$ with $b \leqslant c \to x$ and $a \leqslant x \to d$. The second condition is equivalent to the existence of an $x \leqslant a \rightsquigarrow d$ with $b \leqslant c \to x$. So we have to check the equivalence

$$a \leqslant b \to (c \to d) \iff b \leqslant c \to (a \rightsquigarrow d).$$

Indeed, $a \leqslant b \to (c \to d) \iff b \leqslant a \rightsquigarrow (c \to d) \iff b \leqslant c \to (a \rightsquigarrow d)$. If X is unital, the upper set $\uparrow u$ defines a morphism $J\colon X \to \mathbf{2}$. Then (20) and (21) are equivalent to

$$\exists\, x \geqslant u\colon x \leqslant a \to b \iff a \leqslant b \iff \exists\, x \geqslant u\colon a \leqslant x \to b.$$

This can be rewritten as

$$a \leqslant u \rightsquigarrow b \iff a \leqslant b \iff a \leqslant u \to b,$$

which is equivalent to $u \rightsquigarrow b = b = u \to b$. $\square$

Proposition 10 sheds some light upon the enveloping quantale $U(X) = \mathbf{2}^X$. Let X be a poset with a distributor $P\colon X \nrightarrow X \times X$. In **Sup** this gives a morphism $L(X) \to L(X \times X) = L(X) \otimes L(X)$, or dually, a morphism $U(X) \otimes U(X) \to U(X)$. Then (19) states that $U(X)$ is a semigroup object in **Sup**, a quantale. In terms of (18), the map $L(X) \to L(X \times X)$ is given by $z \mapsto \{(x, y) \mid x \leqslant y \to z\}$ for $z \in X$, or

$$C \mapsto \{(x, y) \mid \exists\, z \in C \colon x \leqslant y \to z\}$$

for $C \in L(X)$. The dual $f^\circ \colon U(Y) \to U(X)$ of a morphism $f \colon L(X) \to L(Y)$ is given by $f^\circ(A) := \uparrow f^{-1}(A)$. So the multiplication on $U(X)$ becomes

$$\begin{aligned} A \cdot B &= \{z \in X \mid \exists\, (x, y) \in A \times B \colon x \leqslant y \to z\} \\ &= \{z \in X \mid \exists\, y \in B \colon y \to z \in A\}, \end{aligned} \tag{23}$$

in conformity with formula (8). According to R. K. Meyer [22], the multiplication (8) is well known to logicians. Following L. Powers, he calls it *modus ponens product*. Fine [16] calls it *fusion*. If we regard $A, B \in U(X)$ as functors $X \to \mathbf{2}$, the first equation in (23) can be written as

$$A \cdot B = \int^{x,y} P(x, y, -) \otimes A(x) \otimes B(y), \tag{24}$$

which identifies the multiplication in $U(X)$ with the *Day convolution* in $\mathbf{2}^X$ [9].

It remains to clarify the difference between a promonoidal poset and a unital quantum B-algebra. By [31], Theorem 1, the category of quantum B-algebras is equivalent to the category of *logical quantales*, that is, quantales of the form $U(X)$ for some poset X with

$$x \cdot \big(\bigwedge_{i \in I} a_i\big) = \bigwedge_{i \in I} (x \cdot a_i), \qquad \big(\bigwedge_{i \in I} a_i\big) \cdot x = \bigwedge_{i \in I} (a_i \cdot x)$$

for all $x \in X$ and $a_i \in U(X)$. Thus, in comparison with a promonoidal poset, a unital quantum B-algebras satisfies this extra condition. The point is that the promonoidal structure does not guarantee that $X \subset U(X)$ is closed under $\to$ and $\rightsquigarrow$. In other words, a promonoidal poset is an implicational algebra without implicational operations.

To make this precise, let us interpret the relation (6) in the framework of multicategories [21]. Let X^f denote the free semigroup generated by a set X. We define a *multi-poset* to be a set X with a binary relation $a \leqslant x$ for $x \in X$ and $a \in X^f$, such that the following are satisfied for $x, y, x_i \in X$ and $a_i \in X^f$:

(a) $(a_1 \leqslant x_1, \ldots, a_n \leqslant x_n$ and $x_1 \cdots x_n \leqslant x) \implies a_1 \cdots a_n \leqslant x$.
(b) $x \leqslant y \leqslant x \iff x = y$.

Morphisms of multi-posets are multi-functors, that is, maps $f\colon X \to Y$ satisfying (7). For quantum B-algebras, the case $n = 2$ is equivalent to the first inequality of (4). Thus quantum B-algebras form a full subcategory **qBAlg** of the category **mPos** of multi-posets.

Just as in (18), we can define a distributor $X \multimap X \times X$ for any multi-poset X by the corresponding relation $P \subset X^{\mathrm{op}} \times X^{\mathrm{op}} \times X$ with

$$(x, y, z) \in P \;:\Longleftrightarrow\; xy \leqslant z. \tag{25}$$

The convolution formula (24) then makes $U(X)$ into a quantale with multiplication

$$A \cdot B = \{z \in X \mid \exists\,(x, y) \in A \times B\colon xy \leqslant z\}.$$

Define a *truth set* of a multi-poset X to be an upper set $U \subset X$ such that for all $x, y \in X$,

$$x \leqslant y \iff \exists\, t \in U\colon tx \leqslant y \iff \exists\, t \in U\colon xt \leqslant y.$$

If X admits a truth set, we call X *unital*. Let us call a multi-poset X *coherent* if the implication

$$axy \leqslant z \implies \exists\, t \in X\colon xy \leqslant t,\; at \leqslant z \tag{26}$$

holds for $x, y, z \in X$ and $a \in X^f$. Not every multi-poset is coherent. For example, let $\{x\}$ be a singleton with $x \cdots x \leqslant x$ if and only if the length of $x \cdots x$ is odd. Then X is not coherent.

Proposition 11. *The category of promonoidal posets is equivalent to the category of unital coherent multi-posets. The category of unital quantum B-algebras admits a full embedding into each of these categories.*

Proof. For a promonoidal poset X, note first that with (25), condition (19) turns into the equivalence

$$(ab)c \leqslant d \iff a(bc) \leqslant d \tag{27}$$

which defines a unique relation $abc \leqslant d$ for $a, b, c, d \in X$. As the reverse implication in (26) holds for multi-posets, we use (26) to define $x_1 \cdots x_n \leqslant x$ via induction. By (27), this gives a coherent multi-poset X. Moreover, (20) and (21) state that X is unital. Therefore, promonoidal posets are equivalent to coherent unital multi-posets. For a map $\Phi\colon X \to Y$ between multi-posets, the implication (22) states that $xy \leqslant z$ implies $\Phi(x)\Phi(y) \leqslant \Phi(z)$. By induction, this proves the first statement of the proposition. By (6), quantum B-algebras are coherent as multi-posets. This gives the second statement. □

To determine the full subcategory of unital quantum B-algebras, let us denote the one-element poset by **1**. Then a promonoidal poset X is given by a pair of distributors $\mathbf{1} \overset{J}{\leftarrow\!\!\circ\!-} X \overset{P}{-\!\!\circ\!\!\rightarrow} X \times X$, that is, monotone functions

$$J\colon X \to \mathbf{2}, \qquad P\colon (X \times X)^{\mathrm{op}} \times X \to \mathbf{2}, \tag{28}$$

satisfying (19)-(21). Let us call X *representable* if J and the presheaves $P(-, y, z)$ and $P(x, -, z)$ are representable for all $x, y, z \in X$. Then we have

Corollary. *A promonoidal poset* (28) *is representable if and only if it is a unital quantum B-algebra.*

Proof. Representability of J means that there is an element $u \in X$ with $J^{-1}(1) = {\uparrow}u$. Similarly, the presheaves $P(-, y, z)$ and $P(x, -, z)$ are representable if and only if there are binary operations $\to$ and $\rightsquigarrow$ on X with

$$P(-, y, z)^{-1}(1) = {\downarrow}(y \to z), \qquad P(x, -, z)^{-1}(1) = {\downarrow}(x \rightsquigarrow z)$$

for all $x, y, z \in X$. Thus, as a ternary relation, P is given by

$$(x, y, z) \in P \iff x \leqslant y \to z \iff y \leqslant x \rightsquigarrow z,$$

in accordance with (18). As shown in the proof of Proposition 10, condition (19) is equivalent to Eq. (2), while (20) and (21) state that $u \to x = u \rightsquigarrow x = x$ holds for all $x \in X$. As the monotonicity condition (3) holds for every promonoidal poset, the proof is complete. □

Remark. Note that with the above notation, a quantum B-algebra X admits a product xy for $x, y \in X$ (Definition 1) if and only if the presheaf $P(x, y, -)$ is representable.

For a complete and cocomplete symmetric monoidal closed base category $\mathscr{V}$, the main theorem of [18] gives a universal property for the category $\mathscr{V}^{\mathscr{A}^{\mathrm{op}}}$ of presheaves over a monoidal category $\mathscr{A}$. For $\mathscr{V} = \mathbf{2}$, this implies the obvious universal property of $L(X) \cong \mathbf{2}^{X^{\mathrm{op}}}$ for a partially ordered semigroup X via the Yoneda embedding $X \hookrightarrow L(X)$. The next result shows that a multiplication in X is not needed if $L(X)$ is replaced by $U(X)$.

Let **Qu** be the subcategory of **mPos** consisting of the quantales with set-indexed join-preserving morphisms, and let

$$M\colon \mathbf{Qu} \to \mathbf{mPos} \tag{29}$$

be the functor which associates the multi-poset $MQ := Q^{\mathrm{op}}$ to a quantale Q. So the defining relation in MQ is $x_1 \cdots x_n \geqslant x$.

Proposition 12. *The functor* (29) *makes* **Qu** *into a reflective subcategory of* **mPos** *with reflector* U.

Proof. For a multi-poset X, we show that the morphism $\eta_X\colon X \hookrightarrow MU(X)$ with $\eta_X(x) := \uparrow x$ is a unit of an adjunction $U \dashv M$. For $A \in U(X)$, we have $A = \bigcup_{x\in A} \uparrow x$. Hence, if $f\colon X \to MQ$ is a morphism in **mPos** and $f'\colon U(X) \to Q$ a morphism in **Qu** with $Mf' \circ \eta_X = f$, we necessarily have $f'(A) = \bigvee_{x\in A} f(x)$. For $A, B \in U(X)$, this gives

$$\begin{aligned} f'(A)f'(B) &= \bigvee\{f(x)f(y) \mid x \in A,\ y \in B\} \\ &\geqslant \bigvee\{f(z) \mid x \in A,\ y \in B,\ xy \leqslant z\} = f'(AB). \end{aligned} \qquad \square$$

At first glance, the replacement of $L(X)$ by $U(X)$ appears to be counterintuitive. However, it allows to embed arbitrary multi-posets X into a quantale $U(U(X))$. A similar switch led to the invention of quantum B-algebras [31].

Finally, we remark that a **2**-promonoidal structure gives rise to a *ternary frame* [29, 13, 22], that is, a ternary relation R on a poset X, with a compatibility condition which states that R is an upper set in $X^{\mathrm{op}} \times X^{\mathrm{op}} \times X$. The logical connectives can then be realized as operations on $U(X)$. For example, the *linear implication* is given by

$$A \to B := \{x \in X \mid \forall\, y \in A\, \forall\, z \in X\colon (x, y, z) \in R \Rightarrow z \in B\}.$$

Associativity of R is given by the relational analogue to (19). For details, we refer to [13, 15, 17, 22].

7. Further examples

In the introduction, partially ordered sets with a greatest element, and groups (with no partial order) were mentioned as two extreme types of quantum B-algebras. Let us discuss these two cases first.

Example 1. Every partially ordered set Ω with greatest element 1 is a quantum B-algebra with

$$x \to y = x \rightsquigarrow y := \begin{cases} 1 & \text{for } x \leqslant y \\ y & \text{for } x \nleqslant y \end{cases}$$

for $x, y \in \Omega$. The fundamental relation (6) is given by

$$x_1 \cdots x_n \leqslant x \iff x_i \leqslant x \text{ for some } i \in \{1, \ldots, n\}.$$

We introduce a topology on Ω by taking the sets

$$D(x) := \{y \in \Omega \mid x \nleqslant y\}$$

as a subbasis of open sets. Then $\widehat{\Omega}$ consists of the closed sets, with reverse inclusion as partial order. The natural embedding $\Omega \hookrightarrow \widehat{\Omega}$ is given by

$$x \mapsto \overline{\{x\}} = \uparrow x.$$

Thus $\widehat{\Omega}$ is a locale. If Ω is totally ordered, $\widehat{\Omega}$ coincides with the Dedekind-MacNeille completion of Ω.

As a special case, consider the poset

$$\Omega := \omega + \omega^* = \{0, 1, 2, 3, \ldots, 3^*, 2^*, 1^*, 0^*\}.$$

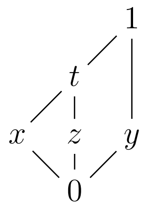
Here $\widehat{\Omega}$ has exactly one additional element, represented by the upper set ω^*. By contrast, the canonical extension [14] of Ω has two additional elements between ω and ω^*.

Example 2. A quantum B-algebra G with trivial partial order is equivalent to a group (see [30], Theorem 4.2). More generally, every semigroup M determines a quantum B-algebra X_M. For example, consider the commutative semigroup $M = \{x, y, z\}$ with multiplication table

$\cdot$	x	y	z
x	y	z	y
y	z	y	z
z	y	z	y

Then X_M has six elements, and its residuals coincide since M is commutative. Precisely, $X_M = \{0, x, y, z, t, 1\}$ with table for $\to$ and Hasse diagram

$\to$	0	x	y	z	t	1
0	1	1	1	1	1	1
x	0	0	t	y	y	1
y	0	0	y	t	t	1
z	0	0	t	y	y	1
t	0	0	t	y	y	1
1	0	0	0	0	0	1



Here $X_M = \widehat{X_M}$, but X_M is not a submonoid of $L(M)$.

Example 3. For a cancellative semigroup M with $|M| > 1$ which is not a group, the quantum B-algebra X_M is obtained from M by adjoining a greatest element 1 and a smallest element 0. For $x, y \in M$,

$$x \to y = \begin{cases} z & \text{if } zx = y \text{ for some } z \in M \\ 0 & \text{otherwise,} \end{cases}$$

and similarly for $x \rightsquigarrow y$. Furthermore,

$$0 \to x = 0 \rightsquigarrow x = x \to 1 = x \rightsquigarrow 1 = 1$$

for all $x \in X_M$, and $x \to 0 = x \rightsquigarrow 0 = 0$ for $x \neq 0$, and $1 \to x = 1 \rightsquigarrow x = 0$ for $x \neq 1$. Here X_M is the injective envelope of M. In particular, $N := X_M$ satisfies $X_N \cong X_M$, which shows that in general, a partially ordered semigroup M cannot be recovered from the quantum B-algebra X_M.

Example 4. Between the two extreme cases, every partially ordered group G is a unital quantum B-algebra with residuals $a \to b := ba^{-1}$ and $a \rightsquigarrow b := a^{-1}b$. As a partially ordered set, $\widehat{G}$ coincides with the Dedekind-MacNeille completion. If G is lattice-ordered and archimedean, $\widehat{G} \smallsetminus \{0,1\}$ is an ℓ-group. This group is usually called the *completion* of G (see [2, 7]).

Example 5. If a quantum B-algebra is a complete lattice, it need not be a quantale. For example, the quantum B-algebra $X := \{0, \ldots, \frac{1}{3}, \frac{1}{2}, 1\}$ with the natural order and

$$x \to y = x \rightsquigarrow y := \begin{cases} 0 & \text{for } x \neq 0 \text{ and } y = 0 \\ 1 & \text{otherwise} \end{cases}$$

is a complete lattice. However, $1 \leqslant 1 \to \frac{1}{n}$ for all positive integers n. Suppose that the product $1 \cdot 1$ exists. Then $1 \cdot 1 \leqslant \frac{1}{n}$ for all n. Hence $1 \cdot 1 = 0$, and thus $1 \leqslant 1 \to 0 = 0$, a contradiction. So the product $1 \cdot 1$ does not exist in X.

The completion of X is obtained by adjoining an element $\varepsilon > 0$ with $\varepsilon \leqslant \frac{1}{n}$ for all n. Indeed, the multiplication

$$ab := \begin{cases} \varepsilon & \text{for } a, b \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

makes $\widehat{X} := X \sqcup \{\varepsilon\}$ into a quantale. Moreover, X is dense in $\widehat{X}$ since $\varepsilon = \bigwedge \frac{1}{n}$, and it is easily checked that X is a quantum B-subalgebra of $\widehat{X}$. Note, however, that ε is not a join of elements from X.

Acknowledgement. We thank an anonymous referee for detailed remarks, especially for pointing out connections to promonoidal structures, multicategories, and ternary frames.

References

[1] J. Bénabou: Les distributeurs, Rapport No. 33 du Séminaire de Math. Pures, Univ. Catholoque de Louvain, 1973

[2] A. Bigard, K. Keimel, and S. Wolfenstein: Groupes et anneaux réticulés, Lecture Notes in Mathematics, Vol. 608, Springer-Verlag, Berlin-New York, 1977

[3] F. Borceux, G. van den Bossche: An essay on noncommutative topology, Topology Appl. 31 (1989), no. 3, 203-223

[4] F. Borceux, J. Rosický, G. Van den Bossche: Quantales and C^*-algebras, J. London Math. Soc. 40 (1989), no. 3, 398-404

[5] M. E. Coniglio, F. Miraglia: Non-commutative topology and quantales, Studia Logica 65 (2000), no. 2, 223-236

[6] A. Connes: Noncommutative geometry, Academic Press, San Diego, CA, 1994

[7] M. R. Darnel: Theory of lattice-ordered groups, Monographs and Textbooks in Pure and Applied Mathematics, 187, Marcel Dekker, Inc., New York, 1995

[8] B. J. Day: Construction of biclosed categories, PhD thesis, Univ. New South Wales, 1970

[9] B. J. Day: On closed categories of functors, Reports of the Midwest Category Seminar, IV, pp. 1-38, Springer LNM 137, Berlin, 1970

[10] B. J. Day: An embedding theorem for closed categories, Category Seminar, Proc. Sem., Sydney, 1972/1973, pp. 55-64

[11] B. J. Day: Note on monoidal monads, J. Austral. Math. Soc. 23 (1977), no. 3, 292-311

[12] B. J. Day, R. Street: Kan extensions along promonoidal functors, Theory Appl. Categ. 1 (1995), No. 4, 72-77

[13] K. Došen: A brief survey of frames for the Lambek calculus, Z. Math. Logik Grundlag. Math. 38 (1992), no. 2, 179-187

[14] M. Dunn, M. Gehrke, A. Palmigiano: Canonical extensions and relational completeness of some substructural logics, J. Symbolic Logic 70 (2005), no. 3, 713-740

[15] M. Emms: Models for polymorphic Lambek calculus, Logical aspects of computational linguistics (Nancy, 1996), 168-187, Lecture Notes in Comput. Sci. 1328, Springer, Berlin, 1997

[16] K. Fine: Models for entailment, J. Philos. Logic 3 (1974), 347- 372

[17] M. Finger: When is a substructural logic paraconsistent? Structural conditions for paraconsistency in ternary frames, Paraconsistency (São Sebastião, 2000), 353-367, Lecture Notes in Pure and Appl. Math. 228, Dekker, New York, 2002

[18] G. B. Im, G. M. Kelly: A universal property of the convolution monoidal structure, J. Pure Appl. Algebra 43 (1986), no. 1, 75-88

[19] A. Joyal, M. Tierney: An extension of the Galois theory of Grothendieck, Mem. Amer. Math. Soc. 51 (1984), no. 309, vii+71pp.

[20] J. Lambek, M. Barr, J. F. Kennison, R. Raphael: Injective hulls of partially ordered monoids, Theory Appl. Categ. 26 (2012), No. 13, 338-348

[21] T. Leinster: Higher operads, higher categories, London Mathematical Society Lecture Note Series, 298, Cambridge University Press, Cambridge, 2004

[22] R. K. Meyer: Ternary relations and relevant semantics, Provinces of logic determined, Ann. Pure Appl. Logic 127 (2004), no. 1-3, 195-217

[23] C. J. Mulvey: &, Second topology conference (Taormina, 1984), Rend. Circ. Mat. Palermo (2) Suppl. No. 12 (1986), 99-104

[24] C. J. Mulvey, J. W. Pelletier: On the quantisation of points, J. Pure Appl. Algebra 159 (2001), no,s. 2-3, 231-295

[25] C. J. Mulvey, J. W. Pelletier: On the quantisation of spaces, Special volume celebrating the 70th birthday of Professor Max Kelly, J. Pure Appl. Algebra 175 (2002), no. 1-3, 289-325

[26] C. J. Mulvey, P. Resende: A noncommutative theory of Penrose tilings, Internat. J. Theoret. Phys. 44 (2005), no. 6, 655-689

[27] S. Niefield, K. I. Rosenthal: Constructing locales from quantales, Math. Proc. Cambridge Philos. Soc. 104 (1988), no. 2, 215-234

[28] K. I. Rosenthal: Quantales and their applications, Pitman Research Notes in Mathematics Series 234, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1990

[29] R. Routley, R. K. Meyer: The semantics of entailment, I, Truth, syntax and modality (Proc. Conf. Alternative Semantics, Temple Univ., Philadelphia, Pa., 1970), pp. 199-243, Studies in Logic and the Foundations of Math., Vol. 68, North-Holland, Amsterdam, 1973

[30] W. Rump: Quantum B-algebras. Cent. Eur. J. Math. 11 (2013), no. 11, 1881-1899

[31] W. Rump, Y. Yang: Non-commutative logical algebras and algebraic quantales, Ann. Pure Appl. Logic 165 (2014), no. 2, 759-785

Wolfgang Rump
Institute for Algebra and Number Theory
University of Stuttgart
Pfaffenwaldring 57
D-70550 Stuttgart, Germany
rump@mathematik.uni-stuttgart.de

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Vol. LVII-3 (2016)

ON A CONJECTURE OF DEGOS

by Nick GILL

Résumé. Dans cette note nous prouvons une conjecture de Degos à propos des groupes engendrés par des matrices compagnons dans $\mathrm{GL}_n(q)$.

Abstract. In this note we prove a conjecture of Degos concerning groups generated by companion matrices in $\mathrm{GL}_n(q)$.

Keywords. Companion matrices; finite fields; general linear group; group generation.

Mathematics Subject Classification (2010). 20H30; 15A99.

1. Introduction

Let $\mathbb{F}$ be a field, and let $f \in \mathbb{F}[X]$ be a polynomial of degree n, i.e.

$$f(X) = a_n X^n + a_{n-1} X_{n-1} + \cdots + a_1 X + a_0$$

where $a_0, \ldots, a_n \in \mathbb{F}$. Recall that the *companion matrix* of f is the $n \times n$ matrix

$$C_f := \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 & -a_0 \\ 1 & 0 & & & 0 & -a_1 \\ 0 & 1 & 0 & & 0 & -a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & 0 & -a_{n-2} \\ 0 & \cdots & \cdots & 0 & 1 & -a_{n-1} \end{bmatrix}.$$

The matrix C_f has the property that its minimal polynomial and its characteristic polynomial are both equal to f. Conversely, if $g \in \mathrm{GL}_n(\mathbb{F})$ has minimal polynomial and characteristic polynomial both equal to some polynomial f, then g is conjugate in $\mathrm{GL}_n(\mathbb{F})$ to C_f.

Recall in addition that if $\mathbb{F}$ has order q and $f \in \mathbb{F}[X]$ has degree n, then f is called *primitive* if it is the minimal polynomial of a primitive element $x \in \mathbb{F}$. In [Deg13], J.-Y. Degos makes the following conjecture.

Conjecture 1. *Let $\mathbb{F}$ be a field of order p a prime, let $g = X^n - 1$ and let $f \in \mathbb{F}[X]$ be a primitive polynomial of degree n. Then $\langle C_f, C_g \rangle = \mathrm{GL}_n(p)$.*

We will prove a stronger version of this conjecture. Specifically, we prove the following.

Theorem 2. *Let $\mathbb{F}$ be a finite field of order q and let $f, g \in \mathbb{F}[X]$ be distinct polynomials of degree n such that f is primitive, and the constant term of g is non-zero. Then $\langle C_f, C_g \rangle = \mathrm{GL}_n(q)$.*

For the rest of this paper $\mathbb{F}$ is a finite field of order q.

2. Field-extension subgroups

Let $\mathbb{K} = \mathbb{F}(\alpha)$ be an algebraic extension of $\mathbb{F}$ of degree d. Let $W = \mathbb{K}^a$, and observe that W is both an a-dimensional vector space over $\mathbb{K}$ and an ad-dimensional space over $\mathbb{F}$.

A $\mathbb{K}/\mathbb{F}$-*semilinear automorphism* of W, ϕ, is an invertible map $\phi : W \to W$ for which there exists $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{F})$ such that, for all $v_1, v_2 \in W$ and $k_1, k_2 \in \mathbb{K}$,

$$\phi(k_1 v_1 + k_2 v_2) = k_1^\sigma \phi(v_1) + k_2^\sigma \phi(v_2).$$

We define a group

$$\Gamma\mathrm{L}_{\mathbb{K}/\mathbb{F}}(W) = \{\phi : W \to W \mid \phi \text{ is a } \mathbb{K}/\mathbb{F}\text{-semilinear automorphism of } W\}.$$

The group $\Gamma\mathrm{L}_{\mathbb{K}/\mathbb{F}}(W)$ can be written as a product $\mathrm{GL}_a(\mathbb{K}).F$ where F is a cyclic group of degree d generated by the automorphism

$$W \to W,\ (w_1, \dots, w_d) \mapsto (w_1^q, \dots, w_d^q).$$

We will refer to elements of F as *field-automorphisms* of W.

Now, for $\mathcal{B} = \{v_1, \dots, v_{ad}\}$ an ordered $\mathbb{F}$-basis of W and $\phi \in \Gamma\mathrm{L}_{\mathbb{K}/\mathbb{F}}(W)$, we define the following matrix

$$(\phi)_\mathcal{B} = \left[\ \phi(v_1) \mid \phi(v_2) \mid \cdots \mid \phi(v_{ad})\ \right].$$

It is a well-known fact that the map

$$\Phi_\mathcal{B} : \Gamma\mathrm{L}_{\mathbb{K}/\mathbb{F}}(W) \to \mathrm{GL}_{ad}(q), \phi \mapsto (\phi)_\mathcal{B}$$

is a well-defined injective group homomorphism, the image of which is a group E known as a *field-extension subgroup of degree* d in $\mathrm{GL}_{ad}(q)$. Indeed, more is true: if we define

$$\theta : W \to \mathbb{F}^{ad}, w \mapsto [w]_{\mathcal{B}},$$

and consider $\Phi_{\mathcal{B}}$ to be a map $\Gamma\mathrm{L}_{\mathbb{K}/\mathbb{F}}(W) \to E$, then the pair (Φ, θ) is a permutation group isomorphism. (Here, and throughout this note, we consider groups acting on the left.)

Note that the group $\Gamma\mathrm{L}_{\mathbb{K}/\mathbb{F}}(W)$ contains a unique normal subgroup N isomorphic to $\mathrm{GL}_a(\mathbb{K})$. Then $H = \Phi_{\mathcal{B}}(N)$ is a subgroup of $\mathrm{GL}_{ad}(q)$ isomorphic to $\mathrm{GL}_a(\mathbb{K})$ and, writing $G = \mathrm{GL}_{ad}(q)$, one can check that $N_G(H) = E$, the associated field-extension subgroup. (To see this, note, firstly, that $E \leq N_G(H) \leq N_G(Z(H))$; now [KL90, Proposition 4.3.3 (ii)] asserts that $N_G(Z(H))) = E$ and we are done.)

3. Singer cycles

Recall that a *Singer subgroup* of the group $\mathrm{GL}_n(q)$ is a cyclic subgroup of order $q^n - 1$. In this section we prove the following lemma.

Lemma 3. *Let $g \in \mathrm{GL}_n(q)$ and let f be its minimal polynomial. Then $\langle g \rangle$ is a Singer subgroup if and only if f is primitive of degree n.*

What is more, if $S = \langle g \rangle$ is a Singer subgroup, then $\langle g \rangle$ is conjugate to $\langle C_f \rangle$, and $S = \Phi_{\mathcal{B}}(GL_1(\mathbb{K}))$, where $\mathbb{K}$ is a degree n extension of $\mathbb{F}$, and $\mathcal{B}$ is an ordered $\mathbb{F}$-basis of $\mathbb{K}$.

Proof. Suppose that $S = \langle g \rangle$ is a Singer subgroup. Then g contains an eigenvalue α that lies in $\mathbb{K}$, a degree n extension of $\mathbb{F}$, and no smaller field. What is more, since g has order $q^n - 1$, so does α and so the minimal polynomial of g is primitive of degree n as required.

Suppose, on the other hand, that f is primitive of degree n. Then the eigenvalues of g are $\alpha, \alpha^q, \dots, \alpha^{q^{n-1}}$; in particular they are all distinct. Elementary linear algebra implies that g is conjugate to C_f, the companion matrix of f. It is enough, then, to prove that $\langle C_f \rangle$ is a Singer cycle.

Let α be a primitive element of degree n over $\mathbb{F}$ and a root of f; let $\mathbb{K} = \mathbb{F}(\alpha)$, an extension of $\mathbb{F}$ of degree n. We construct a field-extension subgroup

G of degree n in $\mathrm{GL}_n(q)$ as the image of the map $\Phi_{\mathcal{B}} : \Gamma\mathrm{L}_{\mathbb{K}/\mathbb{F}}(\mathbb{K}) \to \mathrm{GL}_n(q)$ where $\mathcal{B} = \{\alpha, \alpha^2, \dots, \alpha^{n-1}\}$.

By construction H is isomorphic to $\Gamma\mathrm{L}_{\mathbb{K}/\mathbb{F}}(\mathbb{K})$ and, in particular, contains a subgroup isomorphic to $\mathrm{GL}_1(\mathbb{K}) \cong \mathbb{K}^*$. This subgroup is cyclic of order $q^n - 1$ and is generated by the invertible linear transformation

$$L_\alpha : \mathbb{K} \to \mathbb{K}, x \mapsto \alpha \cdot x.$$

Now our construction guarantees that $\Phi_{\mathcal{B}}(L_\alpha) = C_f$ and we conclude, as required, that C_f generates a cyclic subgroup of $\mathrm{GL}_n(q)$ of order $q^n - 1$. In fact we have shown that $\langle C_f \rangle = \Phi_{\mathcal{B}}(GL_1(\mathbb{K}))$ and the final statement follows. $\square$

4. Two companion matrices

Lemma 4. *Let H be a field-extension subgroup of degree a in $\mathrm{GL}_{ad}(q)$. A non-trivial element of H fixes at most $(q^a)^{d-1}$ elements of $V = (\mathbb{F})^{ad}$.*

Proof. We observed in §2 that the action of H on V is isomorphic to the action of $\Gamma\mathrm{L}_{\mathbb{K}/\mathbb{F}}(W)$ on $W = \mathbb{K}^a$ where $\mathbb{K}$ is a degree d extension of $\mathbb{F}$. Thus we set ϕ to be a non-trivial element of $\Gamma\mathrm{L}_{\mathbb{K}/\mathbb{F}}(W)$.

If ϕ lies in $\mathrm{GL}_a(\mathbb{K})$ and is non-trivial, then basic linear algebra implies that the fixed-point set is a proper $\mathbb{K}$-subspace of W and so fixes at most $(q^a)^{d-1}$ elements of W.

Suppose that ϕ does not lie in $\mathrm{GL}_a(\mathbb{K})$. Thus we can write $\phi = h\sigma$ where h is linear and σ is a non-trivial field automorphism of W that fixes $(\mathbb{F})^a$.

Thus if $v \in \mathbb{K}^a$ and $v^\phi = v$ we obtain immediately that $v^h = v^{\sigma^{-1}}$. Now if c is a scalar that is not fixed by σ, then we obtain immediately that $(cv)^h \neq (cv)^{\sigma^{-1}}$. Since v and c were arbitary we conclude immediately that g fixes at most $(q^b)^d$ elements where b is some proper-divisor of a. The result follows. $\square$

Corollary 5. *If C_f and C_g are companion matrices of distinct monic polynomials $f, g \in \mathbb{F}[x]$ of degree n, then $\langle C_f, C_g \rangle$ does not lie in a field-extension subgroup of $\mathrm{GL}_n(q)$.*

Proof. We consider the action of $\mathrm{GL}_n(q)$ on $V = \mathbb{F}^n$. Observe that the images of the first $n-1$ elementary basis vectors are the same for both

C_f and C_g. In particular, then, the matrix $C_f^{-1}C_g$ fixes the $\mathbb{F}$-span of these $n-1$ vectors and so fixes at least q^{n-1} vectors. The previous lemma implies that, since $C_f \neq C_g$, we can conclude that $\langle C_f, C_g \rangle$ is not a subgroup of a field-extension subgroup of $\mathrm{GL}_n(q)$. □

5. A result about subgroups

To complete the proof of Theorem 2 we will need the result below, Theorem 7. In an earlier draft of this article, we attributed this result to Kantor [Kan80]. We are grateful to Peter Mueller who pointed out that Kantor's result relies on another paper – [CK79] – which has subsequently been found to contain a number of errors.

In fact it is clear that the errors in [CK79] are not fatal and that, with a little adjustment, the result still holds [Cam]. However, since no proof exists in the literature, we will sketch one below. Our approach uses a theorem of Hering [Her85], a proof of which can be found in [Lie87, Appendix 1]. The disadvantage of our proof is that it relies on the Classification of Finite Simple Groups (CFSG), which Kantor's original approach did not.

Lemma 6. *Suppose that S is a Singer cycle in $\mathrm{GL}_n(q)$. Then, for each integer d dividing n, there is a unique field-extension subgroup $\Phi_{\mathcal{B}}(\Gamma\mathrm{L}_{\mathbb{K}/\mathbb{F}}(W))$ (where $\mathbb{K}$ is a field extension of $\mathbb{F}$ of degree d) that contains S.*

Proof. Let H be a subgroup of $\mathrm{GL}_n(q)$ that contains S and suppose that $H \cong \mathrm{GL}_{n/d}(q^d)$ for some divisor d of n. Now S is a Singer cycle in H and so $S = \Phi_{\mathcal{C}}(\mathrm{GL}_1(\mathbb{L}))$ where $\mathbb{L}$ is a degree n/d extension of $\mathbb{F}_{q^d}$.

Write Z for the unique subgroup of S of order $q^d - 1$. Direct calculation confirms that Z coincides with the center of H. Thus $H \leq C_{\mathrm{GL}_n(q)}(Z)$. But Z is precisely the $\mathbb{F}_{q^d}$-scalar maps on $\mathbb{L}$, and so (as we saw earlier, using [KL90, Proposition 4.3.3(ii)]) $N_{\mathrm{GL}_n(q)}(Z)$ is a field-extension subgroup $\Phi_{\mathcal{B}}(\Gamma\mathrm{L}_{\mathbb{K}/\mathbb{F}}(\mathbb{L}))$ where $\mathbb{K}$ is a field extension of $\mathbb{F}$ of degree d. But now H must be the unique normal subgroup of this field-extension subgroup that is isomorphic to $\mathrm{GL}_{n/d}(q^d)$ and we are done. □

In the proof above we refer to two ordered $\mathbb{F}$-bases of $\mathbb{L}$, namely $\mathcal{B}$ and $\mathcal{C}$. It is an easy exercise to see that we can take $\mathcal{B}$ to be equal to $\mathcal{C}$.

Theorem 7. *Let L be a proper subgroup of $G = \mathrm{GL}_n(q)$ that contains a Singer cycle. Then L contains a normal subgroup H isomorphic to $\mathrm{GL}_a(q^c)$ with $n = ac$ and $c > 1$. What is more H is equal to $\Phi_{\mathcal{B}}(\mathrm{GL}_a(\mathbb{K}))$ for $\mathbb{K}$ some field extension of $\mathbb{F}$ of degree c, and $\mathcal{B}$ some ordered $\mathbb{F}$-basis of $\mathbb{K}^a$.*

Proof. It is convenient, first, to deal with the case when $n = 2$. If L lies inside the normalizer of a non-split torus, then L contains a normal subgroup $H \cong \mathrm{GL}_1(q^2)$, as required. Furthermore, order considerations imply that L is a subgroup of neither the normalizer of a split torus, nor a Borel subgroup of $\mathrm{GL}_2(q)$.

The remaining subgroups of $\mathrm{GL}_2(q)$ can be deduced from a classical theorem of [Dic58]. In particular, $L \cap \mathrm{SL}_2(q)$ is isomorphic to either A_4, S_4, A_5 or a double cover of one of these. In particular the maximal order of an element of $L \cap \mathrm{SL}_2(q)$ is 10. Since $L \cap \mathrm{SL}_2(q)$ must contain an element of order $q + 1$, we conclude that $q \leq 9$. Now computation in the remaining groups (using, for example, [GAP15]) rules out the remaining possibilities.

Assume, then that $n \geq 3$, and we refer to Hering's Theorem, as presented in [Lie87, Appendix 1]. This result lists those subgroups of $\mathrm{GL}_\ell(p)$ (for $\ell \in \mathbb{Z}^+$) that act transitively on the set of non-zero vectors of $(\mathbb{F}_p)^\ell$. Since G embeds naturally (inside a field extension subgroup) in $\mathrm{GL}_\ell(p)$ for $\ell = n \log_p q$ and, since a Singer cycle acts transitively (via this embedding) on the set of non-zero vectors in $(\mathbb{F}_p)^\ell$, this list contains all the possible groups L. In what follows we fix a field-extension embedding

$$\Phi_{\mathcal{D}} : G \hookrightarrow \mathrm{GL}_\ell(p)$$

for $\ell = n \log_p q$, and $\mathcal{D}$ an ordered $\mathbb{F}_p$-basis of $(\mathbb{F})^n$. We obtain an associated action on the vector space $V = (\mathbb{F}_p)^\ell$, and apply the theorem.

According to Hering's Theorem, the group L lies in one of three class (A), (B) and (C). Given that $\ell \geq n \geq 3$, the classes (B) and (C) reduce to the following possibilities:

1. $L = A_6, A_7$ or $\mathrm{SL}_2(13)$; $G = \mathrm{GL}_4(2), \mathrm{GL}_6(3)$ or $\mathrm{GL}_3(9)$.

2. L has a normal subgroup $R \cong D_8 \circ Q_8$, $L/R \leq S_5$ and $G = \mathrm{GL}_4(3)$.

In the first case, we note that all elements of L have order less than or equal to 14, and this case is immediately excluded. Similarly, in the second case, all elements of L have order less than or equal to 48, and this case is immediately excluded.

We are left with groups in Liebeck's class A. These come in four families; we examine them one at a time. For family (1), L is a subgroup of the normalizer of a Singer cycle. The result follows immediately in this case. For the remaining families, L has a normal subgroup N isomorphic to $\mathrm{SL}_a(q_0)$, $\mathrm{Sp}_a(q_0)$ or $G_2(q_0)$ with $q_0 = p^d$ and $\ell = ad$.

By examining the proof in [Lie87], we find that, in all cases, L lies in a field-extension subgroup $\Phi_{\mathcal{C}}(\Gamma\mathrm{L}_{\mathbb{K}_0/\mathbb{F}_p}(W))$ of $\mathrm{GL}_\ell(p)$, for $\mathbb{K}_0$ some field extension of $\mathbb{F}_p$ of degree $d \in \mathbb{Z}^+$ and $\mathcal{C}$ some ordered $\mathbb{F}_p$-basis of $W = (\mathbb{K}_0)^a$. What is more $q_0 = q^d$ and $N \leq \Phi_{\mathcal{C}}(\mathrm{GL}_a(\mathbb{K}_0))$.

In the symplectic case, this means that the action of N on $(\mathbb{K}_0)^a$ yields the natural module for $\mathrm{Sp}_a(\mathbb{K}_0)$ (see, for instance, [KL90, Proposition 5.4.13]). Now one can check that an irreducible cyclic subgroup of $\mathrm{Sp}_a(q_0)$ in the natural module has size dividing $q_0^{a/2} + 1$ (see, for instance, [Ber00]). Now Schur's Lemma implies that an irreducible cyclic subgroup of L has order dividing $(q_0^{a/2} + 1)2(q_0 - 1)\log_p(q_0)$. Since this must be at least $q_0^a - 1$, one immediately obtains that $a/2 = 1$ and, since $\mathrm{Sp}_2(\mathbb{K}_0) \cong \mathrm{SL}_2(\mathbb{K}_0)$ we are in one of the remaining cases.

If $G = G_2(q_0)$, then the proof in [Lie87] implies that, in fact, N is a subgroup of a symplectic group $\mathrm{Sp}_6(q_0)$ that acts on $(\mathbb{K}_0)^6$ via its natural module. Thus this situation can be excluded via the calculation of the previous paragraph.

We are left with the case where

$$N \cong \mathrm{SL}_a(q_0) \lhd L \leq \Phi_{\mathcal{C}}(\Gamma\mathrm{L}_{\mathbb{K}_0/\mathbb{F}_p}(W)) \leq \mathrm{GL}_\ell(p).$$

Direct computation inside $\Gamma\mathrm{L}_{\mathbb{K}_0/\mathbb{F}_p}(W)$ confirms that, since L contains a cyclic group of order $p^\ell - 1$, L must contain $M = \Phi_{\mathcal{C}}(\mathrm{GL}(W)) \cong GL_a(q_0)$ as a normal subgroup.

Observe, then, that the Singer cycle S lies in two field extension subgroups of $\mathrm{GL}_d(p)$, namely $N_{\mathrm{GL}_d(p)}(G)$ and $N_{\mathrm{GL}_d(p)}(M)$. Notice, though, that by Lemma 3, $S = \Phi_{\mathcal{B}}(GL_1(\mathbb{L}))$ for some ordered $\mathbb{F}_p$-basis $\mathcal{B}$ of $\mathbb{L}$, a degree n extension of $\mathbb{F}_p$. Clearly the groups $\Phi_{\mathcal{B}}(\Gamma\mathrm{L}_{\mathbb{F}/\mathbb{F}_p}(\mathbb{L}))$ and $\Phi_{\mathcal{B}}(\Gamma\mathrm{L}_{\mathbb{K}_0/\mathbb{F}_p}(\mathbb{L}))$ are also field extension subgroups that contain S.

Now Lemma 6 implies that $M = \Phi_{\mathcal{B}}(\mathrm{GL}_a(\mathbb{K}_0))$ and $G = \Phi_{\mathcal{B}}(\mathrm{GL}_n(\mathbb{F}))$. The second occurrence of the monomorphism $\Phi_{\mathcal{B}}$ here is simply a restriction of the first; it is an easy exercise to check that, in this situation, M is a field-extension subgroup of G as required. □

6. Proving Theorem 2

Observe that if f and g are as in Theorem 2, then they both have non-zero constant term and hence are invertible and so lie in $\mathrm{GL}_n(q)$. Now Lemma 3, Corollary 5 and Theorem 7 imply that $\langle C_f, C_g\rangle$ does not lie in a proper subgroup of $\mathrm{GL}_n(q)$. In other words $\langle C_f, C_g\rangle = \mathrm{GL}_n(q)$, as required.

References

[Ber00] Á. Bereczky. Maximal overgroups of Singer elements in classical groups. *J. Algebra*, 234(1):187–206, 2000.

[Cam] P. J. Cameron. Antiflag-transitive groups. 2015. Blogpost at: `https://cameroncounts.wordpress.com/2015/05/31/antiflag-transitive-groups/`.

[CK79] P. J. Cameron and W. M. Kantor. 2-transitive and antiflag transitive collineation groups of finite projective spaces. *J. Algebra*, 60:384–422, 1979.

[Deg13] J.-Y. Degos. Linear groups and primitive polynomials over $\mathbf{F}_p$. *Cah. Topol. Géom. Différ. Catég.*, 54(1):56–74, 2013.

[Dic58] L. E. Dickson. Linear groups. With an exposition of the Galois field theory, 1958.

[GAP15] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.7.8*, 2015.

[Her85] C. Hering. Transitive linear groups and linear groups which contain irreducible subgroups of prime order. II. *J. Algebra*, 93:151–164, 1985.

[Kan80] W. M. Kantor. Linear groups containing a Singer cycle. *J. Algebra*, 62:232–234, 1980.

[KL90] P. Kleidman and M. Liebeck. *The subgroup structure of the finite classical groups*, volume 129 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1990.

[Lie87] M. W. Liebeck. The affine permutation groups of rank three. *Proc. Lond. Math. Soc. (3)*, 54:477–516, 1987.

Nick Gill
Department of Mathematics
University of South Wales
Treforest, CF37 1DL
U.K.
nicholas.gill@southwales.ac.uk

DOWNLOAD OF CAHIERS' FORMER ISSUES

We are pleased to announce that the issues of the

Cahiers de Topologie et Géométrie Différentielle Catégoriques,

from Volume L (2009) to Volume LV (2014) included, are now freely downloadable from the internet site of the *Cahiers*

http://ehres.pagesperso-orange.fr/Cahiers/Ctgdc.htm

through hyperlinks posted on "Recent Volumes". We recall that papers published up to Volume XLIX are accessible on the NUMDAM site.
In the future the issues of the "Cahiers" will become freely downloadable on the site of the *Cahiers* two years after their paper publication.

Moreover, the 7 volumes of

Charles Ehresmann: Œuvres complètes et commentées

(edited by A. Ehresmann from 1980-83 as *Supplements* to the *Cahiers*) are now freely downloadable from the site

http://ehres.pagesperso-orange.fr/C.E.WORKS_fichiers/C.E_Works.htm

The 2 above sites are included in the site of Andree Ehresmann

http://ehres.pagesperso-orange.fr/

and they can also be accessed through hyperlinks on its first page.

Andrée EHRESMANN, Marino GRAN and René GUITART
Chief-Editors of the *Cahiers*