cahiers de topologie et géométrie différentielle catégoriques

créés par CHARLES EHRESMANN en 1958
dirigés par Andrée CHARLES EHRESMANN
VOLUME LVII-2, 2e trimestre 2016

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES
Vol. LVII-2 (2016)

CALCULUS OF E-RELATIONS IN INCOMPLETE RELATIVELY REGULAR CATEGORIES

by Tamar JANELIDZE-GRAY

Résumé. Nous définissons une catégorie régulière relative incomplète comme une paire $(\mathbf{C}, \mathbf{E})$, où $\mathbf{C}$ est une catégorie arbitraire et $\mathbf{E}$ est une classe d'épimorphismes réguliers dans $\mathbf{C}$ satisfaisant certaines conditions. Nous développons ce que nous appelons un calcul relatif des relations dans ces catégories; on peut l'appliquer aux relations $(R, r_1, r_2) : A \to A$ dans $\mathbf{C}$ telles que les morphismes r_1 et r_2 sont dans $\mathbf{E}$. Cela généralise plusieurs résultats connus, y compris le travail récent avec J. Goedecke sur les catégories relatives de Goursat. Nous définissons les catégories régulières relatives incomplètes de Goursat et : (a) nous prouvons les versions *relatives incomplètes* des conditions équivalentes définissant les catégories régulières relatives de Goursat, (b): nous montrons que dans ce contexte l'axiome $\mathbf{E}$ -Goursat est équivalent à la version relative du Lemme 3×3.

Abstract. We define an incomplete relative regular category as a pair $(\mathbf{C}, \mathbf{E})$, where $\mathbf{C}$ is an arbitrary category and $\mathbf{E}$ is a class of regular epimorphisms in $\mathbf{C}$ satisfying certain conditions. We then develop what we call a relative calculus of relations in such categories; it applies to relations $(R, r_1, r_2) : A \to B$ in $\mathbf{C}$ having the morphisms r_1 and r_2 in $\mathbf{E}$. This generalizes previous results, including the recent work with J. Goedecke on relative Goursat categories. We define incomplete relative regular Goursat categories, and: (a) prove the *incomplete relative* versions of the equivalent conditions defining relative regular Goursat categories, (b): show that in this setting the $\mathbf{E}$-Goursat axiom is equivalent to the relative 3×3-Lemma.

Keywords. Normal epimorphism, incomplete relative regular category, $\mathbf{E}$-relations, incomplete relative Goursat cagetory.
Mathematics Subject Classification (2010). 18A20, 18B10, 18D99.
Acknowledgement. Partially supported by South African National Research Foundation and Shota Rustaveli National Science Foundation Grant DI/18/5-113/13.

1. Introduction

An incomplete relative regular category is defined as a pair $(\mathbf{C}, \mathbf{E})$ where $\mathbf{C}$ is a category and $\mathbf{E}$ is a class of regular epimorphisms in $\mathbf{C}$ satisfying suitable conditions. These conditions are such that:

(a) *Finitely complete relative case*: If $\mathbf{C}$ is a finitely complete category and $\mathbf{E}$ is a class of pullback stable regular epimorphisms in $\mathbf{C}$, then $(\mathbf{C}, \mathbf{E})$ is an incomplete relative regular category if and only if $(\mathbf{C}, \mathbf{E})$ is a relative regular category [4];

(b) *Absolute Case*: If $\mathbf{C}$ is finitely complete category with coequalizers of kernel pairs, and $\mathbf{E}$ is a class of all regular epimorphisms in $\mathbf{C}$, and pullback stable, then $(\mathbf{C}, \mathbf{E})$ is an incomplete relative regular category if and only if $\mathbf{C}$ is a regular category;

(c) *Trivial Case*: If $\mathbf{E}$ is the class of all isomorphisms in any category $\mathbf{C}$, then $(\mathbf{C}, \mathbf{E})$ always is an incomplete relative regular category.

Assuming that $(\mathbf{C}, \mathbf{E})$ is an incomplete relative regular category, we define an $\mathbf{E}$-relation $(R, r_1, r_2) : A \to B$ in $\mathbf{C}$ as a relation R from A to B with r_1 and r_2 jointly monic morphisms in $\mathbf{E}$. The $\mathbf{E}$-relations have already been studied in the context of relative regular categories in [7] and [4], and also in a more general "*incomplete relative*" context in [8] and [9]. However, that incomplete relative context still assumed the existence of certain limits, as well as the pullbacks of morphisms in $\mathbf{E}$. In this paper we consider a more general setting, namely, we do not require the existence of those "special" limits, we only require the existence of pullbacks of morphisms in $\mathbf{E}$. It turns out that most of the results we had for $\mathbf{E}$-relations in [8] and [9] can be extended to this incomplete relative regular category setting.

Relative Mal'tsev and relative Goursat categories were introduced in [3] (see also [2]), and [4] respectively, and now we introduce the incomplete relative Mal'tsev and incomplete relative Goursat categories. Substantial part of this paper is devoted to incomplete relative regular Goursat categories, we show that the results about Goursat categories (see [1] and [5]), which have been extended to relative Goursat categories in [4], can also be extended to these incomplete relative regular Goursat categories.

The paper is organised as follows: In Section 2 we define incomplete relative regular categories and extend the notion of **E**-relations (see [9] and references therein) to this setting. In Section 3 we give some of the properties of **E**-relations, omitting most of the proofs since they are essentially the same as in the finitely complete relative case ([9], [7], and [4]). In Section 4 we define equivalence **E**-relations and state some of their properties, and then we define incomplete relative regular Mal'tsev categories. In Section 5 we define incomplete relative regular Goursat categories and we prove that the **E**-Goursat axiom, just like in the absolute and in the finitely complete relative cases ([1], [5], and [4]), is equivalent to several other equivalent conditions. Finally, in Section 6, we show that also in this incomplete relative context, the **E**-Goursat axiom is equivalent to the 3×3-Lemma (see [5] for the absolute case).

2. Incomplete relative regular categories and E-relations

Throughout the paper we assume that **C** is a category and **E** is a class of morphisms in **C** containing all isomorphisms. Consider the following conditions on $(\mathbf{C}, \mathbf{E})$:

Condition 2.1. (a) Every morphism in **E** is a regular epimorphism;

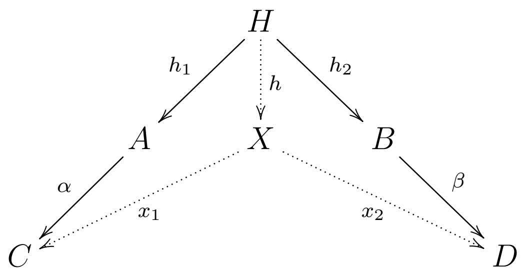
(b) The class **E** is closed under composition;

(c) If $f \in \mathbf{E}$ and $gf \in \mathbf{E}$, then $g \in \mathbf{E}$;

(d) If $f : A \to B$ and $f' : A' \to B$ are in **E**, then the pullback $(A \times_B A', \pi_1, \pi_2)$ of f and f' exists in **C** and the pullback projections π_1 and π_2 are in **E**;

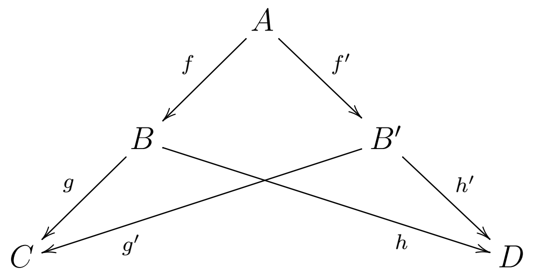
(e) If $h_1 : H \to A$ and $h_2 : H \to B$ are jointly monic morphisms in **C** and if $\alpha : A \to C$ and $\beta : B \to D$ are morphisms in **E**, then there exists a morphism $h : H \to X$ in **E** and jointly monic morphisms

$x_1 : X \to C$ and $x_2 : X \to D$ in $\mathbf{C}$ making the diagram



commutative.

Proposition 2.2. *Suppose* $(\mathbf{C}, \mathbf{E})$ *satisfies Conditions 2.1(a), 2.1(d) and 2.1(e), and let*



(2.1)

be a commutative diagram in $\mathbf{C}$*. If* f *and* f' *are in* $\mathbf{E}$ *and* (g, h) *and* (g', h') *are jointly monic pairs, then there exists a unique isomorphism* $\beta : B \to B'$ *with* $g'\beta = g$*,* $\beta f = f'$*, and* $h'\beta = h$*.*

Proof. Since f and f' are in $\mathbf{E}$, the kernel pairs of f and f' exist by Condition 2.1(d); moreover, they coincide since (g, h) and (g', h') are jointly monic pairs and the diagram (2.1) is commutative. Since every regular epimorphism is the coequalizer of its kernel pair (when the kernel pair exists), we conclude that there exists a unique isomorphism $\beta : B \to B'$ with $\beta f = f'$, and since f and f' are epimorphisms we obtain $g'\beta = g$ and $h'\beta = h$. □

Remark 2.3. As follows from Proposition 2.2, under the assumptions of Conditions 2.1(a) and 2.1(d), the factorization in Condition 2.1(e) is unique up to an isomorphism.

Proposition 2.4. *Suppose* $(\mathbf{C}, \mathbf{E})$ *satisfies Conditions 2.1(a), 2.1(d), and 2.1(e). If a morphism* f *in* $\mathbf{C}$ *factors as* $f = em$ *in which* e *is in* $\mathbf{E}$ *and* m *is a monomorphism, then it also factors (essentially uniquely) as* $f = m'e'$ *in which* m' *is a monomorphism and* e' *is in* $\mathbf{E}$*.*

Proof. Under the assumptions of Condition 2.1(e), take $h_1 = h_2 = m$ and $\alpha = \beta = e$. Then there exists a morphism $\bar{e}$ in $\mathbf{E}$ and jointly monic morphisms $\bar{m_1}$ and $\bar{m_2}$ in $\mathbf{C}$ such that $\bar{m_1}\bar{e} = \bar{m_2}\bar{e}$, and such factorization is unique by Remark 2.3. Since $\bar{e}$ is an epimorhism it follows that $\bar{m_1} = \bar{m_2}$, and therefore $em = \bar{m_1}\bar{e}$ is the desired factorization. □

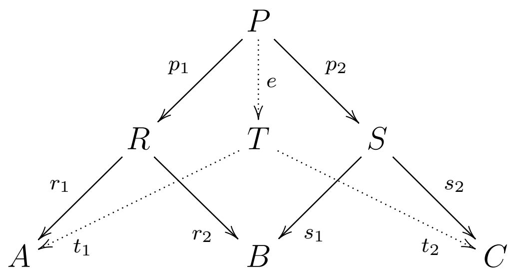
Definition 2.5. A pair $(\mathbf{C}, \mathbf{E})$ is said to be an incomplete relative regular category if it satisfies Condition 2.1.

As follows from Proposition 2.4 and Definition 2.5, if $\mathbf{C}$ is a finitely complete category and $\mathbf{E}$ is pullback stable class of regular epimorphisms in $\mathbf{C}$, then $(\mathbf{C}, \mathbf{E})$ is an incomplete relative regular category if and only if $(\mathbf{C}, \mathbf{E})$ is a relative regular category [4] (see also [9]) (note that, obviously, every relative regular category is incomplete relative regular). In the *"absolute case"*, that is, when $\mathbf{E}$ is the class of all regular epimorphisms in $\mathbf{C}$, if $\mathbf{C}$ has all finite limits and coequalizers of kernel pairs, and $\mathbf{E}$ is pullback stable, then the pair $(\mathbf{C}, \mathbf{E})$ is an incomplete relative regular category if and only if $\mathbf{C}$ is a regular category. On the other hand, if we take $\mathbf{E}$ to be the class of all isomorphisms in $\mathbf{C}$, which we call the *"trivial case"*, then any category $\mathbf{C}$ will satisfy Condition 2.1.

Throughout the rest of the paper we assume that $(\mathbf{C}, \mathbf{E})$ is an incomplete relative regular category. We now extend the calculus of $\mathbf{E}$-relations [9] (see also [7], [8], [4]) to this *incomplete relative* context.

Definition 2.6. An $\mathbf{E}$-relation R from an object A to an object B in $\mathbf{C}$, written as $R : A \to B$, is a triple $R = (R, r_1, r_2)$ in which $r_1 : R \to A$ and $r_2 : R \to B$ are jointly monic morphisms in $\mathbf{E}$.

Let $(R, r_1, r_2) = R : A \to B$ and $(S, s_1, s_2) = S : B \to C$ be $\mathbf{E}$-relations in $\mathbf{C}$ and let (P, p_1, p_2) be the pullback of s_1 and r_2; by Condition 2.1(d) this pullback does exist and p_1 and p_2 are in $\mathbf{E}$. Since p_1 and p_2 are jointly monic and r_1 and s_2 are in $\mathbf{E}$, using Condition 2.1(e) we obtain the commutative diagram



(2.2)

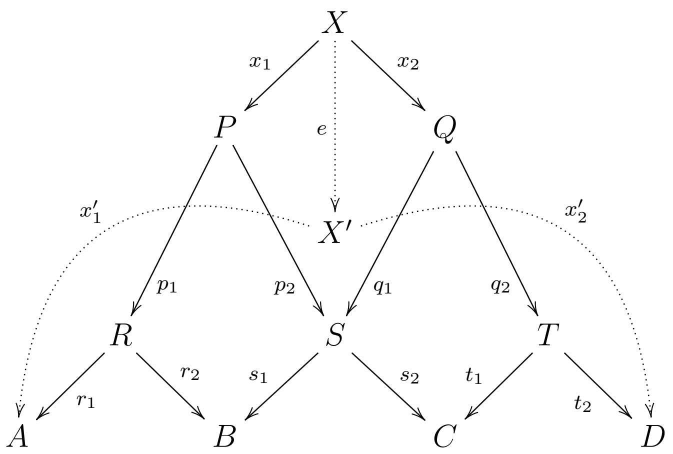
in which e is in $\mathbf{E}$, t_1 and t_2 are jointly monic, and such factorization ($t_1 e = r_1 p_1$ and $t_2 e = s_2 p_2$) is unique up to an isomorphism by Remark 2.3. Moreover, since r_1, p_1, s_2, and p_2 are in $\mathbf{E}$, the morphisms t_1 and t_2 are also in $\mathbf{E}$ by Conditions 2.1(b) and 2.1(c). Accordingly, we introduce:

Definition 2.7. If $R : A \to B$ and $S : B \to C$ are $\mathbf{E}$-relations in $\mathbf{C}$, then their composite $SR : A \to C$ is the $\mathbf{E}$-relation (T, t_1, t_2) in which T, t_1, and t_2 are defined as in the diagram (2.2) above.

It is well known that the composition of relations is associative in a regular category. The same is true for $\mathbf{E}$-relations in relative regular categories, and more generally in incomplete relative regular categories (the proof is essentially the same as in the finitely complete relative context, see Proposition 2.1.9 of [9]):

Proposition 2.8. *The composition of* $\mathbf{E}$*-relations in* $\mathbf{C}$ *is associative (if we identify isomorphic relations).* □

As follows from the proof of Proposition 2.8 (see Proposition 2.1.9 of [9]), to construct the composite of $\mathbf{E}$-relations $(R, r_1, r_2) : A \to B$, $(S, s_1, s_2) : B \to C$, and $(T, t_1, t_2) : C \to D$, we first take the pullbacks (P, p_1, p_2) and (Q, q_1, q_2), of r_2 and s_1, and of s_2 and t_1 respectively, (which exist by Condition 2.1(d), and moreover, p_1, p_2, q_1, q_2 are in $\mathbf{E}$), then take the pullback (X, x_1, x_2) of p_2 and q_1 (which again exists by Condition 2.1), and then their composite $(X', x_1', x_2') : A \to D$ will be the $\mathbf{E}$-relation obtained from the following factorization:



(2.3)

In a similar way we can compose any finite number of $\mathbf{E}$-relations accordingly.

From now on, in the rest of the paper, we will identify the isomorphic relations. For each $\mathbf{E}$-relation $R : A \to B$ in $\mathbf{C}$ there is an opposite $\mathbf{E}$-relation $R^\circ : B \to A$ given by the triple (R, r_2, r_1), and, just as in the absolute case, we have:

Proposition 2.9. *If* $(R, r_1, r_2) : A \to B$ *and* $(S, s_1, s_2) : B \to C$ *are* $\mathbf{E}$-*relations in* $\mathbf{C}$*, then:*

(i) $(R^\circ)^\circ = R$.

(ii) $(SR)^\circ = R^\circ S^\circ$. □

3. Properties of the E-relations

Most of the properties known for relations in a regular category have been extended to relative regular categories (see [7], [9], and [4]). In [8] we have proved that these properties also hold true when only some limits, namely the limits of some *special* diagrams (special case of which are pullbacks) existed. It turns out that the results can actually be proved in even more general setting, namely, when only the pullbacks of morphisms in $\mathbf{E}$ exist, i.e. in incomplete relative regular categories. We state some of these properties below, omitting the proofs since they are essentially the same as the proofs given in [9]:

Proposition 3.1. *Let* $(R, r_1, r_2) : A \to B$, $(R', r'_1, r'_2) : A \to B$, $(S, s_1, s_2) : B \to C$, *and* $(S', s'_1, s'_2) : B \to C$ *be* $\mathbf{E}$*-relations in* $\mathbf{C}$. *We have:*

(i) If $R \leq R'$ *then* $R^\circ \leq R'^\circ$.

(ii) If $R \leq R'$ *then* $SR \leq SR'$.

(iii) If $R \leq R'$ *and* $S \leq S'$ *then* $SR \leq S'R'$.

□

Recall that, $R \leq R'$ means that there exists a morphism $t : R \to R'$ such that $r'_1 t = r_1$ and $r'_2 t = r_2$.

Remark 3.2. Any morphism $f : A \to B$ in $\mathbf{E}$ can be considered as an $\mathbf{E}$-relation $(A, 1_A, f)$ from A to B. The opposite $\mathbf{E}$-relation f° from B to A will then be the triple $(A, f, 1_A)$.

Proposition 3.3. *Let* $(R, r_1, r_2) : A \to B$ *be an* $\mathbf{E}$*-relation in* $\mathbf{C}$. *If* $RR^\circ \leq 1_B$ *then* $r_1 : R \to A$ *is an isomorphism.* □

Proposition 3.4. *If* $(R, r_1, r_2) : A \to B$ *is an* $\mathbf{E}$*-relation in* $\mathbf{C}$ *then* $R = r_2 r_1{}^\circ$. □

Proposition 3.5. *If* $f : A \to B$ *and* $g : C \to B$ *are the morphisms in* $\mathbf{E}$, *then the* $\mathbf{E}$*-relation* $g^\circ f$ *from* A *to* C *in* $\mathbf{C}$ *is given by the pullback* $(A \times_B C, p_1, p_2)$ *of* f *along* g.

□

Remark 3.6. As follows from Proposition 3.5, if $f : A \to B$ is a morphism in $\mathbf{E}$, then the $\mathbf{E}$-relation $f^\circ f : A \to A$ is given by the pullback $(A \times_B A, f_1, f_2)$ of f with itself. That is, $f^\circ f = (A \times_B A, f_1, f_2)$ is the kernel pair of f, and therefore $1_A \leq f^\circ f$.

Proposition 3.7. *If a morphism* $f : A \to B$ *is in* $\mathbf{E}$, *then* $ff^\circ = 1_B$.

□

Remark 3.8. It follows from Proposition 3.7 that for every morphism $f : A \to B$ in $\mathbf{E}$ the following equalities

$$ff^\circ f = f,$$

$$f^\circ f f^\circ = f^\circ$$

hold.

Theorem 3.9. *Let*

$$\begin{array}{ccc} D & \xrightarrow{k} & C \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array} \tag{3.1}$$

be a diagram in $\mathbf{C}$*. If the morphisms f, g, h, and k are in* $\mathbf{E}$*, then:*

(i) $kh^\circ \leq g^\circ f$ *if and only if the diagram (3.1) commutes.*

(ii) $kh^\circ = g^\circ f$ *if and only if the diagram (3.1) commutes and the canonical morphism* $\langle h, k \rangle : D \to A \times_B C$ *is in* $\mathbf{E}$*.*

□

4. Equivalence E-relations

Just as in the absolute case, we can define equivalence $\mathbf{E}$-relations in an incomplete relative regular category $(\mathbf{C}, \mathbf{E})$ as follows:

Definition 4.1. An $\mathbf{E}$-relation $R : A \to A$ in $\mathbf{C}$ is said to be

(a) a reflexive $\mathbf{E}$-relation if $1_A \leq R$;

(b) a symmetric $\mathbf{E}$-relation if $R^\circ \leq R$ (so that $R^\circ = R$);

(c) a transitive $\mathbf{E}$-relation if $RR \leq R$;

(d) an equivalence $\mathbf{E}$-relation if it is reflexive, symmetric, and transitive.

As follows from Definition 4.1, if R is a reflexive and a transitive $\mathbf{E}$-relation then $RR = R$; indeed, since R is reflexive we have $R \leq RR$, which together with transitivity gives $RR = R$.

We now state some properties of equivalence $\mathbf{E}$-relations in incomplete relative regular categories, omitting the proofs again, since they are essentially the same as the proofs given in [9].

Proposition 4.2. *The composite of reflexive* $\mathbf{E}$*-relations in* $\mathbf{C}$ *is a reflexive* $\mathbf{E}$*-relation.* □

Proposition 4.3. *Let* $R : A \to A$ *and* $S : A \to A$ *be equivalence* $\mathbf{E}$*-relations in* $\mathbf{C}$*. If the composite* SR *is an equivalence* $\mathbf{E}$*-relation, then* $SR = S \vee R$ *(i.e.* SR *is the smallest equivalence* $\mathbf{E}$*-relation containing both* S *and* R*).* □

Proposition 4.4. *If a morphism* $f : A \to B$ *is in* $\mathbf{E}$*, then the kernel pair* $(A \times_B A, f_1, f_2)$ *of* f *is an equivalence* $\mathbf{E}$*-relation in* $\mathbf{C}$*.* □

Definition 4.5. An $\mathbf{E}$-relation $R : A \to B$ in $\mathbf{C}$ is said to be difunctional if $RR^{\circ}R = R$.

Theorem 4.6. *If* $(R, r_1, r_2) : A \to A$ *and* $(S, s_1, s_2) : A \to A$ *are equivalence* $\mathbf{E}$*-relations in* $\mathbf{C}$ *then the following conditions are equivalent:*

(a) $SR : A \to A$ *is an equivalence* $\mathbf{E}$*-relation.*

(b) $SR = RS$*.*

(c) Every $\mathbf{E}$*-relation is difunctional.*

(d) Every reflexive $\mathbf{E}$*-relation is an equivalence* $\mathbf{E}$*-relation.*

(e) Every reflexive $\mathbf{E}$*-relation is symmetric.*

(f) Every reflexive $\mathbf{E}$*-relation is transitive.*

□

Recall that a relative regular Mal'tsev category was defined in [3] as a relative regular category which satisfies any one of the conditions of Theorem 4.6 above (see also [2] and [6]). We now extend that definition to the "*incomplete relative*" context.

Definition 4.7. A pair $(\mathbf{C}, \mathbf{E})$ is said to be an incomplete relative regular Mal'tsev category, if it is an incomplete relative regular category and satisfies any one of the conditions of Theorem 4.6 above.

In this paper we will emphasise on what we will define in the next section *incomplete relative regular Goursat category*. For, we will need the following

Proposition 4.8. *The following conditions are equivalent in* $(\mathbf{C}, \mathbf{E})$*:*

(a) for equivalence $\mathbf{E}$*-relations* R *and* S *on an object* A*, we have* $RSR = SRS$*;*

(b) this 3*-permutability* $RSR = SRS$ *holds when* R *and* S *are effective equivalence* $\mathbf{E}$*-relations;*

(c) every $\mathbf{E}$*-relation* P *satisfies* $PP^{\circ}PP^{\circ} = PP^{\circ}$*;*

(d) for every reflexive $\mathbf{E}$*-relation* E *on an object* A*, the* $\mathbf{E}$*-relation* EE° *is an equivalence* $\mathbf{E}$*-relation;*

(e) for every reflexive $\mathbf{E}$*-relation* E*, the* $\mathbf{E}$*-relation* EE° *is transitive;*

(f) for every reflexive $\mathbf{E}$*-relation* E *we have* $EE^{\circ} = E^{\circ}E$*.* □

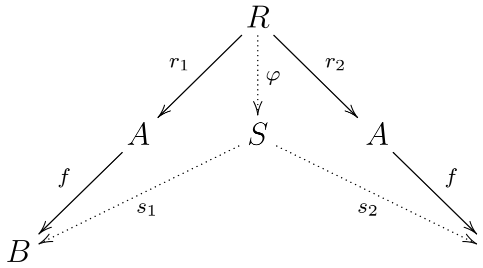
Again, we omit the proof since it follows the proof of Proposition 1.6 of [4].

5. Incomplete relative Goursat categories

Relative regular Goursat categories were introduce in [4], we now extend that definition to the "*incomplete relative*" context. First, let us define an $\mathbf{E}$-image of an endo-$\mathbf{E}$-relation in an incomplete relative regular category:

Definition 5.1. Let $(\mathbf{C}, \mathbf{E})$ be an incomplete relative regular category. Given an $\mathbf{E}$-relation (R, r_1, r_2) on an object A in $\mathbf{C}$ and a morphism $f : A \to B$ in

E, we define the **E**-image of R along f to be the relation S on B which is obtained from the factorization



(5.1)

which exists by Condition 2.1(e). We write $f(R) = S$, which again is an **E**-relation by Conditions 2.1(b) and 2.1(c).

Note that if **C** has products then this definition is the same as Definition 1.7 of [4].

Proposition 5.2. *Let $R = (R, r_1, r_2) : A \to A$ be an* **E**-*relation in* **C** *and let $f : A \to B$ be a morphism in* **E**. *We have:*

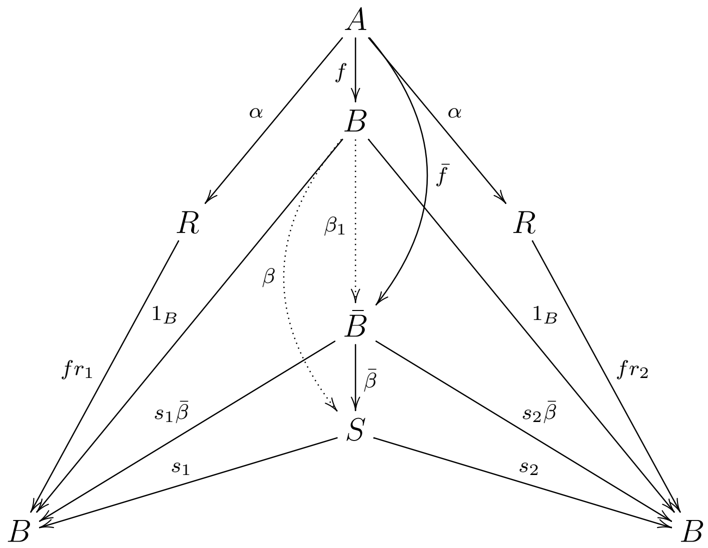
(i) If R is a reflexive **E**-*relation then $f(R)$ is also a reflexive* **E**-*relation.*

(ii) If R is a symmetric **E**-*relation then $f(R)$ is also a symmetric* **E**-*relation.*

Proof. (i): Suppose $R = (R, r_1, r_2) : A \to A$ is a reflexive **E**-relation in **C**. By the definition of a reflexive **E**-relation, there exists a morphism $\alpha : A \to R$ such that $r_1\alpha = 1_A = r_2\alpha$. Note here that α is a split monomorphism and therefore it is a monomorphism. Let $f : A \to B$ be a morphism in **E**; we have $(fr_1)\alpha = f = (fr_2)\alpha$, where fr_1 and fr_2 are in **E** since so are the morphisms f, r_1 and r_2. By Definition 5.1, the **E**-image of R along f is the **E**-relation (S, s_1, s_2) obtained from the factorization (5.1), therefore $fr_1 = s_1\varphi$ and $fr_2 = s_2\varphi$. Composing with α from the right on both sides of the last equality, we obtain $fr_1\alpha = s_1\varphi\alpha$ and $fr_2\alpha = s_2\varphi\alpha$.

On the other hand, since $\alpha : A \to R$ is a monomorphism and $\varphi : R \to S$ is in **E**, there exists a monomorphism $\bar{\beta} : \bar{B} \to S$ and a morphism $\bar{f} : A \to \bar{B}$ in **E** such that $\varphi\alpha = \bar{\beta}\bar{f}$.

We obtain the following diagram :



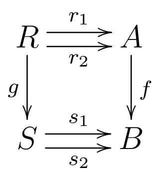
To prove that (S, s_1, s_2) is a reflexive $\mathbf{E}$-relation, we need to prove that there exists a morphism $\beta : B \to S$ such that $\beta s_1 = 1_B = \beta s_2$. Since $\bar{\beta}$ is a monomorphism, the morphisms $s_1\bar{\beta}$ and $s_2\bar{\beta}$ are jointly monic. Therefore, since f and $\bar{f}$ are in $\mathbf{E}$, and obvioulsy 1_B is jointly monic with itself, by Remark 2.3, the equalities $fr_1\alpha = f$, $fr_2\alpha = f$, $s_1\bar{\beta}\bar{f} = fr_1\alpha$, and $s_2\bar{\beta}\bar{f} = fr_2\alpha$ imply that there exists a unique morphism $\beta_1 : B \to \bar{B}$ such that $\beta_1 f = \bar{f}$. Now take $\beta = \bar{\beta}\beta_1$, then $s_1\beta = 1_B = s_2\beta$, as desired.

(ii): The proof easily follows from Remark 2.3. Indeed, if $R = (R, r_1, r_2) : A \to A$ is a symmetric $\mathbf{E}$-relation then there exists an isomorphism $r : R \to R$ such that $r_1 r = r_2$ and $r_2 r = r_1$. Letting $f(R) = (S, s_1, s_2)$, by Definition 5.1 we have that $s_1\varphi = fr_1$ and $s_2\varphi = fr_2$, yielding that $s_1\varphi r = fr_1 r = fr_2$ and $s_2\varphi r = fr_2 r = fr_1$. Therefore, by Remark 2.3 there exists a unique morphism $s : S \to S$ such that $s_2 s = s_1$ and $s_1 s = s_2$, i.e. $S^\circ \leq S$, proving that S is a symmetric $\mathbf{E}$-relation. □

The following Lemma and Corollary (Lemma 1.9 and Corollary 1.10 of [4]) also hold true in an incomplete relative regular category $(\mathbf{C}, \mathbf{E})$:

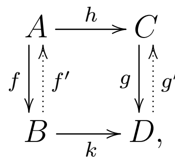
Lemma 5.3. *Given an* $\mathbf{E}$*-relation* (R, r_1, r_2) *on an object* A *in* $\mathbf{C}$ *and a morphism* $f : A \to B$ *in* $\mathbf{E}$*, the* $\mathbf{E}$*-image* $f(R)$ *can be formed as the composite* $f(R) = fRf^\circ = fr_2r_1^\circ f^\circ$. □

Corollary 5.4. *Given a commutative diagram*



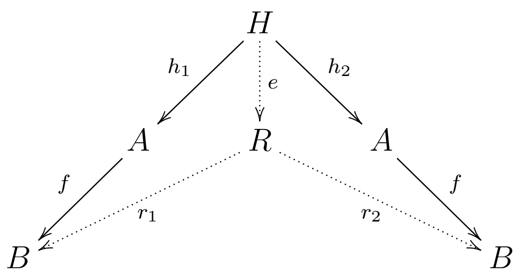
where R *and* S *are* $\mathbf{E}$*-relations in* $\mathbf{C}$ *and* f *is in* $\mathbf{E}$*, the morphism* g *is in* $\mathbf{E}$ *if and only if* $S = f(R)$*, or equivalently if and only if* $s_2 s_1^\circ = f r_2 r_1^\circ f^\circ$*. If* (R, r_1, r_2) *and* (S, s_1, s_2) *are kernel pairs with coequalizers* r *and* s *in* $\mathbf{E}$*, then the latter is also equivalent to* $s^\circ s = f r^\circ r f^\circ$*.* □

Lemma 5.5. *Let* $(\mathbf{C}, \mathbf{E})$ *be an incomplete relative regular category. Given a morphism of (downward) split epimorphisms*



that is, f *and* g *are split epimorphisms with splittings* f' *and* g' *respectively, and* $kf = gh$ *and* $g'k = hf'$*, if* f*,* g*,* h*, and* k *in are in* $\mathbf{E}$*, then the induced morphism between the kernel pairs of* h *and* k *is also in* $\mathbf{E}$*.*

Proof. We follow the proof of Lemma 1.11 of [4]. Let (H, h_1, h_2) and (K, k_1, k_2) be the kernel pairs of h and k (they do exist since h and k are in $\mathbf{E}$), clearly the induced morphism $H \to K$ is again a split epimorphism. Since h_1 and h_2 are jointly monic and f is in $\mathbf{E}$, using Condition 2.1(e) we obtain the factorization

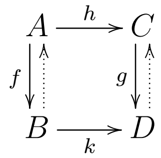


where e is in $\mathbf{E}$, and r_1 and r_2 are jointly monic morphisms in $\mathbf{E}$. Since e is in particular an epimorphism, the $\mathbf{E}$-relation R factors through the kernel pair K of k. But since $H \to K$ is a split epimorphism it follows that the induced morphism $R \to K$ is an isomorphism, therefore, $H \to K$ is in $\mathbf{E}$. □

We are now ready to prove the "*incomplete relative*" version of Theorem 2.1 of [4], which in the absolute case characterises regular Goursat categories (see [1] and [5]).

Theorem 5.6. *The following conditions are equivalent on* $(\mathbf{C}, \mathbf{E})$*:*

(a) the $\mathbf{E}$***-Goursat axiom*** *holds: given a morphism of (downward) split epimorphisms*



(5.2)

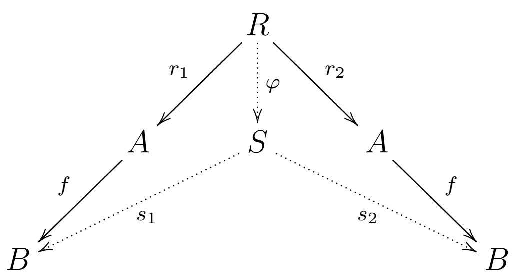
in $\mathbf{C}$ *with* f*,* g*,* h *and* k *in* $\mathbf{E}$*, the induced morphism between the kernel pairs of* f *and* g *is also in* $\mathbf{E}$*;*

(b) the $\mathbf{E}$*-image of an equivalence* $\mathbf{E}$*-relation is an equivalence* $\mathbf{E}$*-relation;*

(c) for every reflexive $\mathbf{E}$*-relation* E *on an object* A*, the* $\mathbf{E}$*-relation* EE° *is an equivalence* $\mathbf{E}$*-relation;*

(d) for equivalence $\mathbf{E}$*-relations* R *and* S *on an object* A*, we have* $RSR = SRS$*.*

Proof. Here again, we follow the proof of Theorem 2.1 from [4].
(a) ⇒ (b): Let (R, r_1, r_2) be an equivalence $\mathbf{E}$-relation on A and let $f : A \to B$ be in $\mathbf{E}$. We want to show that the $\mathbf{E}$-image $f(R) = (S, s_1, s_2)$ of R along f, obtained from the factorization



(5.3)

is again an equivalence $\mathbf{E}$-relation. Since S is a reflexive and a symmetric $\mathbf{E}$-relation, by Proposition 5.2 we only have to show that it is transitive, that is, $SS \leq S$. However, since S is a symmetric $\mathbf{E}$-relation, the transitivity of S will be proved if we show that $SS^{\circ} \leq S$. For, it is sufficient to show that there exists a morphism $t_S : S_1 \to S$, where (S_1, π_1, π_2) is the kernel pair of s_1, which makes the diagram

$$\begin{array}{ccc} S_1 & \overset{t_S}{\dashrightarrow} & S \\ {\scriptstyle \pi_1}\downarrow\downarrow{\scriptstyle \pi_2} & & {\scriptstyle s_1}\downarrow\downarrow{\scriptstyle s_2} \\ S & \underset{s_2}{\longrightarrow} & B \end{array} \tag{5.4}$$

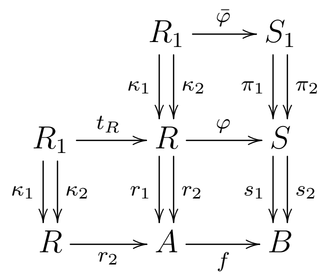
commutative. Since R is a (symmetric and) transitive $\mathbf{E}$-relation, there exists a morphism $t_R\colon R_1 \to R$, where $(R_1, \kappa_1, \kappa_2)$ is the kernel pair of r_1, making the corresponding diagram for R commutative:

$$\begin{array}{ccc} R_1 & \overset{t_R}{\longrightarrow} & R \\ {\scriptstyle \kappa_1}\downarrow\downarrow{\scriptstyle \kappa_1} & & {\scriptstyle r_1}\downarrow\downarrow{\scriptstyle r_2} \\ R & \underset{r_2}{\longrightarrow} & A \end{array}$$

Using the morphisms e_R and e_S which define the reflexivity of R and S respectively, we obtain a diagram

$$\begin{array}{ccc} R & \overset{\varphi}{\longrightarrow} & S \\ {\scriptstyle r_1}\downarrow\uparrow{\scriptstyle e_R} & & {\scriptstyle s_1}\downarrow\uparrow{\scriptstyle e_S} \\ A & \underset{f}{\longrightarrow} & B \end{array}$$

where φ is the $\mathbf{E}$-part of the factorization in (5.3). By assumptions, the morphism $\overline{\varphi}\colon R_1 \to S_1$ between the kernel pairs of r_1 and s_1 is in $\mathbf{E}$. Combining the above two diagrams and adding the morphism $\bar{\varphi}$ to it, we obtain the diagram

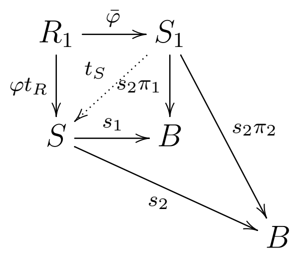


where, recall that, $(R_1, \kappa_1, \kappa_2)$ and (S_1, π_1, π_2) are the kernel pair of r_1 and s_1 respectively. We have:

$$s_1 \varphi t_R = f r_1 t_R = f r_2 \kappa_1 = s_2 \varphi \kappa_1 = s_2 \pi_1 \bar{\varphi}$$

$$s_2 \varphi t_R = f r_2 t_R = f r_2 \kappa_2 = s_2 \varphi \kappa_2 = s_2 \pi_2 \bar{\varphi}$$

Therefore, the following diagram



(5.5)

of solid arrows is commutative. We define the required morphism $t_S : S_1 \to S$ as follows. Since $\bar{\varphi}$ is in **E**, the kernel pair (X, x_1, x_2) of $\bar{\varphi}$ exists. Moreover, since the above diagram is commutative and s_1 and s_2 are jointly monic, it follows that $\varphi t_R x_1 = \varphi t_R x_2$. Furthermore, since $\bar{\varphi}$ is a regular epimorphism, $\bar{\varphi}$ is the coequalizer of its kernel pair, and therefore there exists a unique morphism $t_S : S_1 \to S$ with $t_S \bar{\varphi} = \varphi t_R$. Now since $\bar{\varphi}$ is an epimorphism, the commutativity of the diagram (5.5) implies that $s_1 t_S = s_2 \pi_1$ and $s_2 t_S = s_2 \pi_2$, which gives us the commutativity of the desired diagram (5.4). This proves $(a) \Rightarrow (b)$.

The proofs for the remaining implications are the same as the proofs of the corresponding implications of Theorem 2.1 in [4]. □

We are now ready to give the following

Definition 5.7. A pair $(\mathbf{C}, \mathbf{E})$ is said to be an incomplete relative regular Goursat category, if it is an incomplete relative regular category and satisfies any one of the conditions of Theorem 5.6 above.

6. The relative 3x3 Lemma

In this section we extend the results of Section 3 of [4] to the *"incomplete relative"* context. Just as in the absolute case, we have the following

Definition 6.1. Let $(\mathbf{C}, \mathbf{E})$ be an incomplete relative regular category. We will say that the diagram

$$F \underset{f_2}{\overset{f_1}{\rightrightarrows}} A \xrightarrow{f} B \tag{6.1}$$

is $\mathbf{E}$-exact when (f_1, f_2) is the kernel pair of f and f is in $\mathbf{E}$.

Notice that when (6.1) is $\mathbf{E}$-exact, the morphisms f_1 and f_2 are also in $\mathbf{E}$ by the pullback-stability if $\mathbf{E}$.

Since Theorems 3.9 and 5.6, Corollary 5.4, and Lemma 5.5, hold in incomplete relative regular categories, Theorem 3.3 and Theorem 3.4 of [4] also hold true in incomplete relative regular categories :

Theorem 6.2 (The relative 3×3-Lemma)**.** *Let $(\mathbf{C}, \mathbf{E})$ be a relative Goursat category. Given a commutative diagram*

$$\begin{array}{ccccc}
\overline{F} & \underset{\overline{h}_2}{\overset{\overline{h}_1}{\rightrightarrows}} & F & \xrightarrow{\overline{h}} & G \\
{\scriptstyle \overline{f}_2}\downarrow\downarrow{\scriptstyle \overline{f}_1} & & {\scriptstyle f_2}\downarrow\downarrow{\scriptstyle f_1} & & {\scriptstyle g_2}\downarrow\downarrow{\scriptstyle g_1} \\
H & \underset{h_2}{\overset{h_1}{\rightrightarrows}} & A & \xrightarrow{h} & C \\
{\scriptstyle \overline{f}}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
K & \underset{k_2}{\overset{k_1}{\rightrightarrows}} & B & \xrightarrow{k} & D
\end{array} \tag{6.2}$$

with $\mathbf{E}$-exact columns and middle row, the first row is $\mathbf{E}$-exact if and only if the third row is $\mathbf{E}$-exact.

Theorem 6.3. *Let* $(\mathbf{C}, \mathbf{E})$ *be an incomplete relative regular category. The following conditions are equivalent:*

(a) $(\mathbf{C}, \mathbf{E})$ *is an incomplete relative Goursat category;*

(b) the relative 3×3-Lemma holds in $(\mathbf{C}, \mathbf{E})$*;*

(c) in a diagram such as (6.2), if the first row is $\mathbf{E}$*-exact then the third row is also* $\mathbf{E}$*-exact;*

(d) in a diagram such as (6.2), if the third row is $\mathbf{E}$*-exact then the first row is also* $\mathbf{E}$*-exact.*

□

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CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Vol. LVII-2 (2016)

AN INTRODUCTION TO MULTIPLE CATEGORIES (ON WEAK AND LAX MULTIPLE CATEGORIES, I)

by Marco GRANDIS and Robert PARE

Résumé. Nous introduisons ici les *catégories multiples faibles* de dimension infinie, une extension des catégories doubles et triples. Nous considérons aussi une forme 'chirale' partiellement laxe, ayant des interchangeurs dirigés, et une forme plus laxe déjà étudiée dans deux articles précédents en dimension trois sous le nom de *intercatégorie*. Dans ce contexte nous entreprenons une étude des *tabulateurs*, les limites supérieures de base, qui sera conclue dans un article à suivre.

Abstract. We introduce here infinite dimensional *weak multiple categories*, an extension of double and triple categories. We also consider a partially lax, 'chiral' form with directed interchangers and a laxer form already studied in two previous papers for the 3-dimensional case, under the name of *intercategory*. In these settings we also begin a study of *tabulators*, the basic higher limits, that will be concluded in a sequel.

Keywords. multiple category, double category, duoidal category, cubical set.
Mathematics Subject Classification (2010). 18D05, 18D10, 55U10.

0. Introduction

Higher category theory takes various forms, based on different 'geometries'.

The best known is the *globular* form of 2-categories, n-categories and ω-categories (with their weak variations), based on a (possibly truncated) *globular set*; this is a system X of sets and mappings (faces and degeneracies)

$$X_0 \underset{e}{\overset{\partial^\alpha}{\leftleftarrows\!\!\rightarrow}} X_1 \underset{e}{\overset{\partial^\alpha}{\leftleftarrows\!\!\rightarrow}} X_2 \ \dots\ X_{n-1} \underset{e}{\overset{\partial^\alpha}{\leftleftarrows\!\!\rightarrow}} X_n \dots \quad (n \geqslant 0,\ \alpha = \pm), \tag{1}$$

that satisfies the globular relations. Without entering in problems of size, a 2-category can be formally defined as a category enriched over the cartesian closed category $\mathbf{Cat}$ of categories and functors; and so on for higher n-categories.

Here we are interested in a different, more general setting, that was introduced by C. Ehresmann prior to the previous one: the *multiple* form of double categories, n-tuple categories and multiple categories, based on a (possibly truncated) *multiple set*; this is a system X of sets $X_{i_1 i_2 ... i_n}$ and mappings

$$\partial^{\alpha}_{i_j} \colon X_{i_1 i_2 ... i_n} \to X_{i_1 ... \hat{i}_j ... i_n}, \quad e_{i_j} \colon X_{i_1 ... \hat{i}_j ... i_n} \to X_{i_1 i_2 ... i_n} \\ (n \geqslant 0, \;\; 0 \leqslant i_1 < ... < i_j < ... < i_n, \;\; \alpha = \pm), \tag{2}$$

that satisfies the *multiple relations* (see Subsection 2.2). Formally, a double category is a category object in $\mathbf{Cat}$, and a weak double category is a pseudo category object in $\mathbf{Cat}$, as a 2-category (cf. [Mr]); this structure, with its limits, adjoints and Kan extensions, has been introduced and studied in our series [GP1] - [GP4]. Weak and lax triple categories have been introduced in [GP6, GP7].

(*Cubical categories* can be viewed as a particular case of multiple categories, based on the geometry of *cubical sets* well known from Algebraic Topology; see 2.3 and 2.8. References are cited below.)

This series is devoted to the study of multiple categories. In the present introductory paper we give an explicit definition of the strict and weak cases (Sections 2 and 3), including the partially lax case of a *chiral*, or χ-lax, multiple category (see 3.7), where the weak composition laws in directions $i < j$ have a lax interchange χ_{ij}; an interesting 3- (or infinite-) dimensional example based on spans and cospans is presented in Section 4. Marginally, in Sections 5 and 6, we also consider the laxer notion of *intercategory* already studied in dimension three in [GP6, GP7], where we showed that it includes, besides weak and chiral triple categories, various 3-dimensional structures that have been previously established, like duoidal categories, Gray categories, Verity double bicategories and monoidal double categories.

Let us note that all these lax notions come in two forms, transversally dual to each other, according to the direction of interchangers; these forms are named ‘left’ and ‘right’, respectively, as explained in 3.7. We mainly work in the *right-hand* case, as in [GP6, GP7].

We also introduce here in an informal way the *tabulators* - the basic form of higher multiple limits, already studied in the 2-dimensional case of weak double categories [GP1] (where they extend the cotensors by $\mathbf{2}$ of 2-categories).

Part II, the next paper in this series, will study *multiple limits* for *chiral* multiple categories, proving that all of them can be constructed from (multiple) products, equalisers and tabulators. It should be noted that multiple limits are - by definition - preserved by faces and degeneracies, in a suitable form. While some particular limits can be extended to intercategories, an extension of the general theory seems to be problematic, as we shall discuss there.

We end by remarking that the weak and lax forms of multiple categories are *much simpler* than the globular ones, because here all the weak composition laws are associative, unitary and interchangeable up to cells in the *strict* 0-indexed direction; the latter are strictly coherent. This aspect has already been discussed in dimension three in [GP6], and for the cubical case in [GP5], where we showed how the 'simple' comparisons of a weak 3-cubical category produce - via some associated cells - the 'complicated' ones of a tricategory.

Literature. Higher category theory in the globular form has been studied in many papers and books; we only cite: Bénabou [Be] for bicategories; Gordon, Power and Street [GPS] for tricategories; Leinster [Le] for weak ω-categories.

Infinite dimensional weak and lax multiple categories are introduced here; but strict multiple categories and some of their weak or lax variations (possibly of a cubical type) have already been treated in the following papers (among others):

- strict double and multiple categories: [Eh, BE, EE],
- Gray categories: [Gr],
- weak double categories: [GP1] - [GP4],
- Verity double bicategories: [Ve],
- monoidal double categories: [Sh],
- strict cubical categories: [ABS],
- weak and lax cubical categories: [G1] - [G5],
- duoidal (or 2-monoidal) categories: [AM, BS, St],
- weak triple categories and 3-dimensional intercategories: [GP6, GP7],

- links between the cubical and the globular setting, in the strict case [ABS] or the weak one [GP5].

Conventions. The two-valued index α (or β) takes values in the cardinal $2 = \{0, 1\}$, generally written as $\{-, +\}$ in superscripts. We generally ignore set-theoretical problems, that can be fixed with a suitable hierarchy of universes. The symbol $\subset$ denotes weak inclusion.

Acknowledgments. The authors would like to thank the anonymous referee for a careful reading of the paper and detailed comments. This work was supported by a research grant of Università di Genova.

1. A triple category of weak double categories

Formally, a (strict) double category is a category object in $\mathbf{Cat}$, and a *triple category* is a category object in the category of double categories and double functors; an explicit definition of multiple categories of any dimension will be given in Section 2. This introductory section gives a first motivation for studying them.

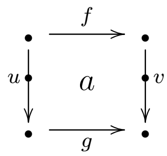
We start from the (strict) double category $\mathbb{D}\mathrm{bl}$ of weak double categories, lax and colax double functors (with suitable double cells), introduced in [GP2]. This structure plays a central role in the definition of adjunctions for weak double categories, where the left adjoint is generally colax while the right adjoint is lax: because of this, a general adjunction cannot live in a 2-category (or in a bicategory) but must be viewed *in* this double category. $\mathbb{D}\mathrm{bl}$ is also crucial for the study of Kan extensions in the same context [GP3, GP4]. It is also extensively used in [GP6, GP7].

We now embed $\mathbb{D}\mathrm{bl}$ in a triple category $\mathbb{S}\mathbb{D}\mathrm{bl}$, adding new arrows - the strict double functors - in an additional *transversal direction* $i = 0$. Then we briefly sketch some advantages of this embedding with respect to limits, in preparation for Part II.

1.1 Notation

For weak double categories we follow the notation of our series [GP1] - [GP4].

In particular, a vertical arrow $u\colon A \to\!\!\!\bullet\; B$ is often marked with a dot and the vertical composite of u and $v\colon B \to\!\!\!\bullet\; C$ is written as $v{\bullet}u$, or more often as $u \otimes v$; the vertical identity of an object A is written as $1^\bullet_A$. The boundary of a double cell a is presented as $a\colon (u\, {}^{f}_{g}\, v)$



(3)

or also as $a\colon u \to v$ (which is particularly convenient when we view a vertical arrow as a higher, 1-dimensional object). The horizontal composition of double cells is written as $(a \mid b)$; the vertical composition (or pasting, concatenation) as $\left(\frac{a}{c}\right)$ or $a \otimes c$. Horizontal composition of arrows and double cells is unitary and associative. The interchange law holds strictly:

$$\left(\frac{a \mid b}{c \mid d}\right) = \left(\frac{a}{c} \,\Big|\, \frac{b}{d}\right),$$

so that the pasting of a *consistent* matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of double cells is well defined - 'consistent' meaning that faces agree, so that the previous compositions make sense (as in diagram (6), below).

A cell $a\colon (u\ {}^{f}_{g}\ v)$ is said to be *special* if its horizontal arrows f, g are identities, and a *special isocell* if - moreover - it is horizontally invertible. The composition of vertical arrows is unitary and associative up to *special isocells* (for $u\colon A \to\!\!\!\bullet\, B$, $v\colon B \to\!\!\!\bullet\, C$, $w\colon C \to\!\!\!\bullet\, D$)

(a) $\lambda(u)\colon 1^\bullet_A \otimes u \to u$ (*left unitor*),
(b) $\rho(u)\colon u \otimes 1^\bullet_B \to u$ (*right unitor*),
(c) $\kappa(u, v, w)\colon u \otimes (v \otimes w) \to (u \otimes v) \otimes w$ (*associator*).

In a (strict) double category these comparison cells are trivial, i.e. horizontal identities.

A (strict) double functor between weak double categories preserves the whole structure; for the sake of brevity it will often be called a 'functor'. *Lax* and *colax* (double) functors are also used below; the definition can be found in [GP2], Section 2.1 (or deduced from their infinite-dimensional extension here, in 3.9).

1.2 The double category $\mathbb{D}\mathrm{bl}$

Let us recall the *strict* double category $\mathbb{D}\mathrm{bl}$, from [GP2], Section 2.2.

The objects of $\mathbb{D}\mathrm{bl}$ are the *weak* (or pseudo) double categories $\mathbb{A}, \mathbb{B}, ...$; its horizontal arrows are the *lax* (double) functors $F, G...$; its vertical arrows are the *colax* functors $U, V...$ A cell π

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{F} & \mathbb{B} \\ {\scriptstyle U}\big\downarrow\!\bullet & \pi & \bullet\!\big\downarrow{\scriptstyle V} \\ \mathbb{C} & \xrightarrow[G]{} & \mathbb{D} \end{array} \tag{4}$$

is - roughly speaking - a 'horizontal transformation' $\pi\colon VF \dashrightarrow GU$. But this is an abuse of notation, since the composites VF and GU are neither lax nor colax (just morphisms of double graphs, respecting the horizontal structure): the coherence conditions of π are based on the *four* 'functors' F, G, U, V and all their comparison cells.

Precisely, the cell π consists of the following data:

(a) a lax functor F with comparison special cells $\underline{F}$ (indexed by the objects A and pairs (u, v) of consecutive vertical arrows of $\mathbb{A}$) and a lax functor G with comparison special cells $\underline{G}$ (similarly indexed by $\mathbb{C}$)

$$F\colon \mathbb{A} \to \mathbb{B}, \quad \underline{F}(A)\colon 1^\bullet_{FA} \to F(1^\bullet_A), \quad \underline{F}(u, v)\colon Fu \otimes Fv \to F(u \otimes v),$$

$$G\colon \mathbb{C} \to \mathbb{D}, \quad \underline{G}(C)\colon 1^\bullet_{GC} \to G(1^\bullet_C), \quad \underline{G}(u, v)\colon Gu \otimes Gv \to G(u \otimes v),$$

(b) two colax functors U, V with comparison special cells $\underline{U}$, $\underline{V}$ (indexed by $\mathbb{A}$ and $\mathbb{B}$)

$$U\colon \mathbb{A} \to \mathbb{C}, \quad \underline{U}(A)\colon U(1^\bullet_A) \to 1^\bullet_{UA}, \quad \underline{U}(u, v)\colon U(u \otimes v) \to Uu \otimes Uv,$$

$$V\colon \mathbb{B} \to \mathbb{D}, \quad \underline{V}(B)\colon V(1^\bullet_B) \to 1^\bullet_{VB}, \quad \underline{V}(u, v)\colon V(u \otimes v) \to Vu \otimes Vv,$$

(c) horizontal maps $\pi A\colon VF(A) \to GU(A)$ and cells πu in $\mathbb{D}$ (for A and $u\colon A \to\!\!\bullet\, A'$ in $\mathbb{A}$)

$$\begin{array}{ccc} VFA & \xrightarrow{\pi A} & GUA \\ {\scriptstyle VFu}\big\downarrow\!\bullet & \pi u & \bullet\!\big\downarrow{\scriptstyle GUu} \\ VFA' & \xrightarrow[\pi A']{} & GUA' \end{array} \tag{5}$$

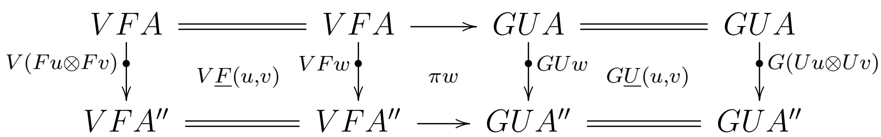
These data must satisfy the naturality conditions (c0), (c1) (the former is redundant, being implied by the latter) and the coherence conditions (c2), (c3)

$$(c0)\quad GUf.\pi A = \pi A'.VFf \qquad \text{(for } f\colon A \to A' \text{ in } \mathbb{A}\text{)},$$

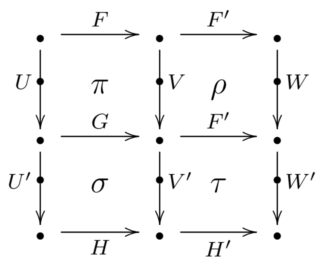
$$(c1)\quad (\pi u \mid GUa) = (VFa \mid \pi v) \qquad \text{(for } a\colon (u \,{}^{f}_{g}\, v) \text{ in } \mathbb{A}\text{)},$$

$$(c2)\quad (V\underline{F}(A) \mid \pi 1^{\bullet}_{A} \mid G\underline{U}(A)) = (\underline{V}(FA) \mid 1^{\bullet}_{\pi A} \mid \underline{G}(UA)) \quad \text{(for } A \text{ in } \mathbb{A}\text{)},$$

$$(c3)\quad (V\underline{F}(u,v) \mid \pi w \mid G\underline{U}(u,v))$$
$$= (\underline{V}(Fu, Fv) \mid (\pi u \otimes \pi v) \mid \underline{G}(Uu, Uv)) \quad \text{(for } w = u \otimes v \text{ in } \mathbb{A}\text{)},$$



The horizontal and vertical composition of double cells are both defined *using the horizontal composition* of the weak double category $\mathbb{D}$. Namely, for a consistent matrix of double cells



(6)

we have:

$$(\pi \mid \rho)(u) = (\rho Fu \mid G'\pi u), \qquad \left(\frac{\pi}{\sigma}\right)(u) = (V'\pi u \mid \sigma Uu). \qquad (7)$$

This 'explains' why these composition laws are strictly associative and unitary (like the horizontal composition in $\mathbb{D}$). One can find in [GP2] the proof of the coherence of the double cells defined in (7) and the middle-four interchange law on the matrix (6).

It will be relevant for our 3-dimensional extension to note that: if the horizontal (resp. vertical) arrows of π are strict (or just pseudo) functors, then our cell simply amounts to a horizontal transformation $\pi\colon VF \to GU$ of colax (resp. lax) functors (as defined in [GP2]).

(One can also note that a double cell $\pi\colon (U \; {}^{F}_{1} \; 1)$ gives a notion of *horizontal transformation* $\pi\colon F \to U\colon \mathbb{A} \to \mathbb{B}$ *from a lax to a colax functor*, while a double cell $\pi\colon (1 \; {}^{1}_{G} \; V)$ gives a notion of *horizontal transformation* $\pi\colon V \to G\colon \mathbb{A} \to \mathbb{B}$ *from a colax to a lax functor*. Moreover, for a fixed pair $\mathbb{A}, \mathbb{B}$ of weak double categories, all the four kinds of transformations compose, forming a category $\{\mathbb{A}, \mathbb{B}\}$ whose objects are the lax *and* the colax functors $\mathbb{A} \to \mathbb{B}$.)

1.3 The new triple category

The definition of a triple category will be made explicit in Section 2.

The triple category $\mathsf{S} = \mathsf{S}\mathbb{D}\mathrm{bl}$ that we introduce here (adding 'transversal arrows' and new cells to those considered above, in 1.2) is a clear instance of this structure and a good example for our study of limits.

(a) The set S_* of *objects* of S consists of all (small) weak double categories.

(b) The sets S_0, S_1, S_2 of the *0-arrows* (or *transversal arrows*), *1-arrows* and *2-arrows* of S, respectively, consist of:

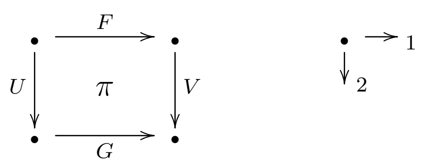
- (strict) functors between weak double categories,
- lax functors between weak double categories,
- colax functors between weak double categories,

Each set S_i (for $i = 0, 1, 2$) has a degeneracy and two faces

$$\begin{array}{ll} e_i\colon S_* \to S_i, & e_i(\mathbb{A}) = \mathrm{id}\mathbb{A}, \\ \partial_i^\alpha\colon S_i \to S_*, & \partial_i^- = \mathrm{Dom}, \quad \partial_i^+ = \mathrm{Codom}. \end{array} \tag{8}$$

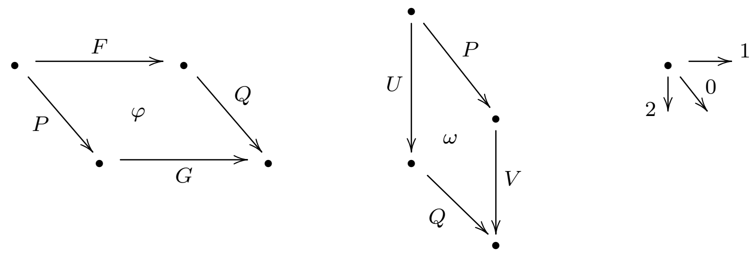
(c) The sets S_{12}, S_{01}, S_{02} of *double cells* of S consist of the following items:

- a 12-cell is an arbitrary double cell of $\mathbb{D}\mathrm{bl}$, with lax (resp. colax) functors in direction 1 (resp. 2) and components $\pi A\colon VF(A) \to GU(A)$, $\pi u\colon VF(u) \to GU(u)$ (cf. 1.2)



(9)

- a 01-cell, as shown in the *left* diagram below, is a double cell of $\mathbb{D}\mathrm{bl}$ with strict functors in direction 0, lax functors in direction 1 and a horizontal transformation $\varphi\colon QF \to GP$ (of lax functors)



(10)

- a 02-cell, as shown in the *right* diagram above, is a double cell of $\mathbb{D}\mathrm{bl}$ with strict functors in direction 0, colax functors in direction 2 and a horizontal transformation $\omega\colon VP \to QU$ (of colax functors).

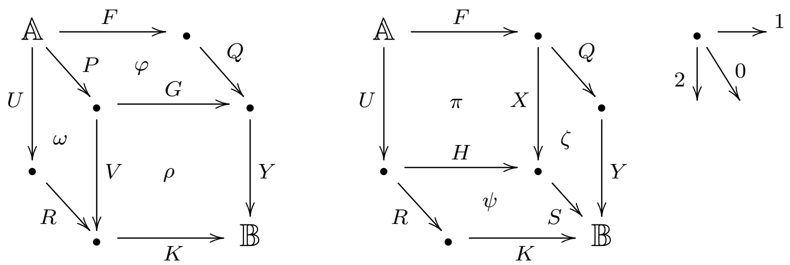
Each S_{ij} (for $0 \leqslant i < j \leqslant 2$) has two degeneracies and four faces, that are obvious

$$e_i\colon S_j \to S_{ij}, \qquad e_j\colon S_i \to S_{ij},$$
$$\partial_i^\alpha\colon S_{ij} \to S_j, \qquad \partial_j^\alpha\colon S_{ij} \to S_i. \tag{11}$$

Thus $e_1\colon S_2 \to S_{12}$ assigns to a 2-arrow U the identity cell $e_1(U)$ of the original double category for the 1-directed (i.e. horizontal) composition, while the 1-faces of the 12-cell π are the domain and codomain of the 1-directed composition (note that they are 2-*arrows*)

$$\partial_1^\alpha(\pi) = U \text{ or } V, \qquad \partial_2^\alpha(\pi) = F \text{ or } G. \tag{12}$$

(d) Finally S_{012} is the set of *triple cells* of $\mathsf{S}\mathbb{D}\mathrm{bl}$: such an item Π is a 'commutative cube' determined by its six faces; the latter are double cells of the previous three types



(13)

The commutativity condition means the following equality of pasted double cells in $\mathbb{D}\mathrm{bl}$ (the non-labelled ones being inhabited by natural transformations that are identities):

$$
\begin{array}{ccccc}
\mathbb{A} & \xrightarrow{1} & \mathbb{A} & \xrightarrow{F} & \bullet \\
{\scriptstyle 1}\downarrow & & {\scriptstyle P}\downarrow & \varphi & \downarrow{\scriptstyle Q} \\
\mathbb{A} & \xrightarrow{P} & \bullet & \xrightarrow{G} & \bullet \\
{\scriptstyle U}\downarrow & \omega & {\scriptstyle V}\downarrow & \rho & \downarrow{\scriptstyle Y} \\
\bullet & \xrightarrow[R]{} & \bullet & \xrightarrow[K]{} & \mathbb{B} \\
{\scriptstyle R}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\
\bullet & \xrightarrow[1]{} & \bullet & \xrightarrow[K]{} & \mathbb{B}
\end{array}
\qquad
\begin{array}{ccccc}
\mathbb{A} & \xrightarrow{F} & \bullet & \xrightarrow{1} & \bullet \\
{\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle Q} \\
\mathbb{A} & \xrightarrow{F} & \bullet & \xrightarrow{Q} & \bullet \\
{\scriptstyle U}\downarrow & \pi & {\scriptstyle X}\downarrow & \zeta & \downarrow{\scriptstyle Y} \\
\bullet & \xrightarrow[H]{} & \bullet & \xrightarrow[S]{} & \mathbb{B} \\
{\scriptstyle R}\downarrow & \psi & {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle 1} \\
\bullet & \xrightarrow[K]{} & \mathbb{B} & \xrightarrow[1]{} & \mathbb{B}
\end{array}
\qquad (14)
$$

More explicitly, the commutativity condition amounts to the following equality of components (horizontal composites of double cells in the weak double category $\mathbb{B}$):

$$
\begin{array}{l}
(YQFu \xrightarrow{Y\varphi u} YGPu \xrightarrow{\rho Pu} KVPu \xrightarrow{K\omega u} KRUu) \\
= (\,YQFu \xrightarrow{\zeta Fu} SXFu \xrightarrow{S\pi u} SHUu \xrightarrow{\psi Uu} KRUu),
\end{array}
\qquad (15)
$$

where u is any vertical arrow in the weak double category $\mathbb{A}$.

(e) All composition laws are derived from those of $\mathbb{D}\mathrm{bl}$, using the fact that the additional 0-directed structure is a particular case of the 1- and 2-directed ones. Therefore all these laws are strictly associative, unitary and interchangeable

The fact that any triple cell of $\mathrm{S}\mathbb{D}\mathrm{bl}$ is determined by its boundary (i.e. its six faces) can be expressed saying that the triple category $\mathrm{S}\mathbb{D}\mathrm{bl}$ is *box-like*.

1.4 Comments

Inserting the double category $\mathbb{D}\mathrm{bl}$ into the triple category $\mathrm{S}\mathbb{D}\mathrm{bl}$ can be motivated by the fact that:

(a) the horizontal and vertical limits in $\mathbb{D}\mathrm{bl}$ remain as *transversal limits* in $\mathrm{S}\mathbb{D}\mathrm{bl}$, where their projections are duly recognised as *strict* double functors,

(b) (more interestingly) new transversal limits appear in $\mathrm{S}\mathbb{D}\mathrm{bl}$, for which there is ‘no sufficient room’ in the original double category.

These aspects will be studied in Part II, but we anticipate now a sketch of tabulators, showing point (a) in 1.5, 1.6 and point (b) in 1.7, 1.8.

1.5 Horizontal tabulators in $\mathbb{D}\mathrm{bl}$

In the double category $\mathbb{D}\mathrm{bl}$ every vertical arrow $U\colon \mathbb{A} \to\!\!\!\bullet\; \mathbb{B}$ has a *horizontal tabulator* $(\mathbb{T}, P, Q, \tau)$, providing a horizontally universal cell τ as in the left diagram below (see [GP1])

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{P} & \mathbb{A} \\ {\scriptstyle 1}\bullet\big\downarrow & \tau & \big\downarrow\bullet{\scriptstyle U} \\ \mathbb{T} & \xrightarrow[Q]{} & \mathbb{B} \end{array} \qquad\qquad \begin{array}{ccccc} \mathbb{S} & \xrightarrow{F} & \mathbb{T} & \xrightarrow{P} & \mathbb{A} \\ {\scriptstyle 1}\bullet\big\downarrow & 1^\bullet_F & {\scriptstyle 1}\bullet\big\downarrow & \tau & \big\downarrow\bullet{\scriptstyle U} \\ \mathbb{S} & \xrightarrow[F]{} & \mathbb{T} & \xrightarrow[Q]{} & \mathbb{B} \end{array} \qquad (16)$$

The universal property says that every similar double cell $\tau'\colon (1^\bullet_{\mathbb{S}}\ {}^{P'}_{Q'}\ U)$ factorises as $\tau' = (1^\bullet_F \mid \tau)$, by a unique horizontal arrow $F\colon \mathbb{S} \to \mathbb{T}$, as in the right diagram above. (In [GP1] we also considered a two-dimensional universal property for the tabulator, which is not used here and will be discussed in Part II.)

The weak double category $\mathbb{T}$ has objects

$$(A, B, b\colon UA \to B),$$

with A in $\mathbb{A}$ and b horizontal in $\mathbb{B}$. A horizontal arrow of $\mathbb{T}$

$$(a, b)\colon (A_1, B_1, b_1) \to (A_2, B_2, b_2),$$

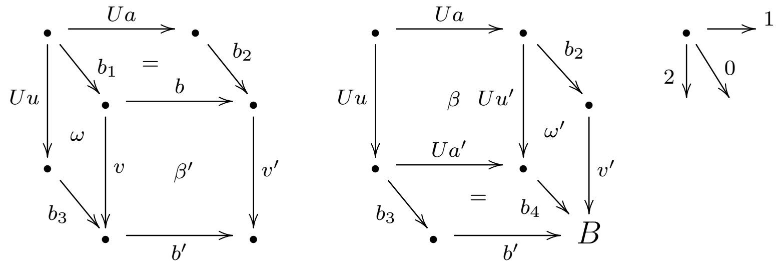
‘is’ a commutative square in $\mathbb{B}$, as in the upper square of diagram (17), below. A vertical arrow of $\mathbb{T}$

$$(u, v, \omega)\colon (A_1, B_1, b_1) \to (A_3, B_3, b_3),$$

‘is’ a double cell in $\mathbb{B}$, as in the left square of diagram (17). A double cell (β, β') of $\mathbb{T}$

$$(\beta, \beta')\colon \left((u, v, \omega)\ \begin{matrix}(a, b)\\(a', b')\end{matrix}\ (u', v', \omega')\right), \qquad\quad (\omega \mid \beta') = (\beta \mid \omega),$$

forms a commutative diagram of double cells of $\mathbb{B}$, as below (where the slanting direction must be viewed as horizontal)



(17)

The composition laws of $\mathbb{T}$ are obvious, as well as the (strict) double functors P, Q. The double cell τ has components

$$\tau(A, B, b) = b\colon UA \to B, \qquad \tau(u, v, \omega) = \omega\colon Uu \to v. \tag{18}$$

Now the lax functor $F\colon \mathbb{S} \to \mathbb{T}$ is defined on the objects as

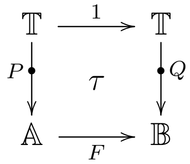
$$F(S) = (P'(S), Q'(S), \tau'S : UP'(S) \to Q'(S))$$

and is strict whenever P' and Q' are.

Since P and Q are strict double functors, *this construction also gives the tabulator*, or e_2*-tabulator*, of the 2-arrow U of $\mathsf{S}\mathbb{D}\mathrm{bl}$: it will be defined in Part II as an object $\top_2 U$ with a universal 02-cell $\tau\colon e_2(\top_2 U) \to_0 U$; now the universal property says that every 02-cell $\tau'\colon e_2(\mathbb{S}) \to_0 U$ factorises as $\tau' = \tau.e_2(F)$, by a unique 0-arrow $F\colon \mathbb{S} \to_0 \mathbb{T}$. (Note that now $\tau'\colon (1^{\bullet}_{\mathbb{S}}\ {}^{P'}_{Q'}\ U)$ is a double cell whose horizontal arrows P', Q' are strict functors, so that F is strict as well.)

1.6 Vertical tabulators in $\mathbb{D}\mathrm{bl}$

Similarly, in the double category $\mathbb{D}\mathrm{bl}$ every horizontal arrow $F\colon \mathbb{A} \to \mathbb{B}$ has a *vertical tabulator* $(\mathbb{T}, P, Q, \tau)$, providing a vertically universal cell τ



(19)

Now, the weak double category $\mathbb{T}$ has objects $(A, B, b\colon B \to FA)$, with A in $\mathbb{A}$ and b a horizontal arrow of $\mathbb{B}$. The horizontal duality of weak double categories interchanges the horizontal and vertical tabulator, sparing us describing the whole structure.

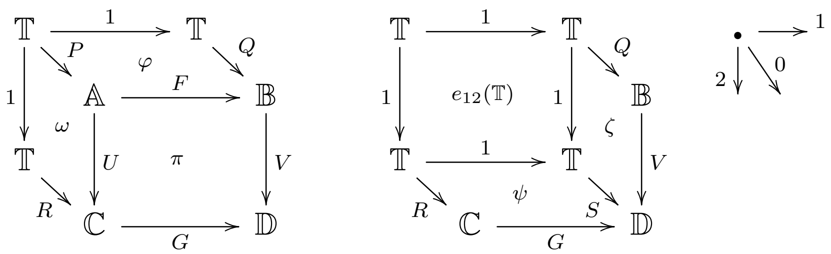
Again, P and Q are strict double functors, and *this construction also gives the tabulator*, or e_1*-tabulator*, of the 1-arrow F of $\mathsf{S}\mathbb{D}\mathrm{bl}$: it will be defined in Part II as an object $\top_1 F$ with a universal 01-cell $\tau\colon e_1(\top_1 F) \to_0 F$.

1.7 Higher tabulators, I

A double cell π of $\mathbb{D}\mathrm{bl}$

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{F} & \mathbb{B} \\ {\scriptstyle U}\big\downarrow\!\bullet & \pi & \bullet\!\big\downarrow{\scriptstyle V} \\ \mathbb{C} & \xrightarrow[G]{} & \mathbb{D} \end{array} \qquad (20)$$

is a 12-cell of the triple category $\mathsf{S}\mathbb{D}\mathrm{bl}$. *In the latter* we can define and construct the *total tabulator*, or e_{12}*-tabulator*, of π as an object $\mathbb{T} = \top\pi = \top_{12}\pi$ with a universal 012-cell $\Pi\colon e_{12}(\mathbb{T}) \to_0 \pi$, where $e_{12} = e_1e_2 = e_2e_1$

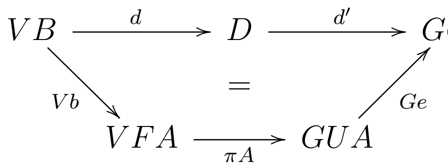


(21)

Now, an object X of the weak double category $\mathbb{T}$ consists of four objects, one in each of $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$, and four horizontal morphisms of $\mathbb{B}, \mathbb{C}, \mathbb{D}$ (two of them in $\mathbb{D}$)

$$\begin{gathered} X = (A, B, C, D;\ b, c, d, d'), \\ b\colon B \to FA, \quad c\colon UA \to C, \quad d\colon VB \to D, \quad d'\colon D \to GC, \end{gathered} \qquad (22)$$

so that the following pentagon of horizontal arrows commutes in $\mathbb{D}$



(23)

The arrows and double cells of $\mathbb{T}$ are essentially as in 1.5, if more complicated. The strict double functors P, Q, R, S are obvious projections and the double cells $\varphi, \psi, \omega, \zeta$ have the following components on the object X of (22) (and similar components on the vertical arrows of $\mathbb{T}$, which we have not described)

$$\begin{array}{ll} \varphi X = b\colon B \to FA, & \omega X = c\colon UA \to C, \\ \psi X = d'\colon D \to GC, & \zeta X = d\colon VB \to D. \end{array} \qquad (24)$$

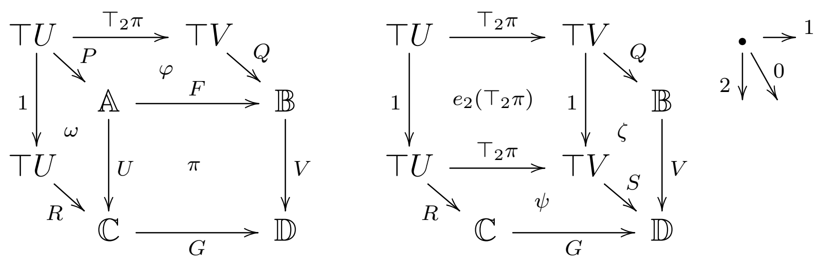
1.8 Higher tabulators, II

Finally, the 12-cell π also has two other higher tabulators $\top_i\pi$ ($i = 1, 2$), whose results are 1-*dimensional* cells (i.e. a lax or colax double functor), instead of an object as above:

- the e_1-*tabulator* is a 2-arrow $\top_1\pi$ with a universal 012-cell $e_1(\top_1\pi) \to_0 \pi$,
- the e_2-*tabulator* is a 1-arrow $\top_2\pi$ with a universal 012-cell $e_2(\top_2\pi) \to_0 \pi$.

(Note that the e_1-tabulator of π is 2-directed, like the 1-faces of π.)

For instance, $\top_2\pi$ is a lax double functor $\top U \to_1 \top V$ between the tabulators (computed in 1.5) of the two vertical arrows $\partial_1^\alpha\pi$, namely U and V



(25)

Thus $\top U$ has objects $(A, C, c\colon UA \to C)$, $\top V$ has objects $(B, D, d\colon VB \to D)$ and

$$\begin{aligned}
&(\top_2\pi)(A, C,\ c\colon UA \to C) = (FA, GC,\ Gc.\pi A\colon VFA \to GC),\\
&\varphi(A, C, c\colon UA \to C) = 1_{FA}, \qquad \psi(A, C, c\colon UA \to C) = 1_{GC},\\
&\omega(A, C, c\colon UA \to C) = c, \qquad\qquad \zeta(B, D, d\colon VB \to D) = d.
\end{aligned}$$

It will be important to note that these limits are *preserved by faces and degeneracies*, in a way that will be analysed in Part II

$$\partial_i^\alpha(\top_j\pi) = \top_j(\partial_i^\alpha\pi), \qquad \top_j(e_iX) = e_i(\top_jX) \qquad (i \neq j). \tag{26}$$

Moreover, by a composition of universal arrows, the total e_{12}-tabulator of π can be obtained as

$$\top_{12}\pi = \top_2\top_1\pi = \top_1\top_2\pi. \tag{27}$$

In fact, computing for instance $\top_1\top_2\pi$, we find that an object is a family

$$((A, C, c\colon UA \to C), (B, D, d\colon VB \to D), b\colon B \to FA, d'\colon D \to GC),$$

(with A in $\mathbb{A}$, etc.) such that the following square of horizontal arrows commutes in $\mathbb{D}$, as in the pentagon (23)

$$\begin{array}{ccc}
VB & \xrightarrow{\;d\;} & D\\
{\scriptstyle Vb}\downarrow & & \downarrow{\scriptstyle d'}\\
VFA & \xrightarrow[Ge.\pi A]{} & GC
\end{array}$$

2. Strict multiple categories

We give now an explicit definition of a (strict) multiple category. It is similar to that of [G1], Section 5 (where it was given as an extension of a *strict cubical category*) but is rewritten in a simplified, equivalent form.

2.1 The geometry

Loosely speaking, a (strict) multiple category A is a generalised (strict) cubical category where all the directions are of different *sorts*. An index $i \in \mathbb{N}$ will represent such a sort or direction, *including the transversal one* $i = 0$ (that will be treated differently, from 2.5 on).

We have thus

- a set A_* of objects,

- a set A_i of i-arrows, or i-*directed* arrows, for every index $i \geqslant 0$ (with faces in A_*),

- a set A_{ij} of 2-dimensional ij-cells, for indices $i < j$ (with faces in A_i and A_j),

- and generally, for every *multi-index* $\mathbf{i}$ of n indices

$$0 \leqslant i_1 < i_2 < ... < i_n \qquad\qquad (n \geqslant 0), \tag{28}$$

a set $A_\mathbf{i} = A_{i_1 ... i_n}$ of $\mathbf{i}$-cells of dimension n (with faces in the various $A_{i_1 ... \hat{i}_j ... i_n}$).

2.2 Multiple sets

A *multi-index* $\mathbf{i}$ is a finite subset of $\mathbb{N}$, possibly empty. Writing $\mathbf{i} \subset \mathbb{N}$ it will be understood that $\mathbf{i}$ is finite; writing $\mathbf{i} = \{i_1, ..., i_n\}$ it will be understood that $\mathbf{i}$ has n *distinct* elements, written in the natural order $i_1 < i_2 < ... < i_n$; the integer $n \geqslant 0$ is called the *dimension* of $\mathbf{i}$.

We shall use the following symbols

$$\mathbf{i}j = j\mathbf{i} = \mathbf{i} \cup \{j\} \;\; (\text{for } j \in \mathbb{N} \setminus \mathbf{i}), \qquad \mathbf{i}|j = \mathbf{i} \setminus \{j\} \;\; (\text{for } j \in \mathbf{i}). \tag{29}$$

A *multiple set* is a system of sets and mappings $X = ((X_\mathbf{i}), (\partial_i^\alpha), (e_i))$ under the following two assumptions.

(mls.1) For every multi-index $\mathbf{i} = \{i_1, ..., i_n\}$, $X_\mathbf{i}$ is a set whose elements are called $\mathbf{i}$-*cells* of X and said to be of *dimension* n. For the sake of simplicity, we write X_*, X_i, X_{ij}... instead of $X_\varnothing$, $X_{\{i\}}$, $X_{\{i,j\}}$, ... (To assume that the sets $X_\mathbf{i}$ are disjoint would often be inconvenient; when useful one can redefine $X'_\mathbf{i} = X_\mathbf{i} \times \{\mathbf{i}\}$.)

(mls.2) For $j \in \mathbf{i}$ and $\alpha = 0, 1$ we have mappings, called *faces* and *degeneracies* of $X_\mathbf{i}$

$$\partial_j^\alpha \colon X_\mathbf{i} \to X_{\mathbf{i}|j}, \qquad e_j \colon X_{\mathbf{i}|j} \to X_\mathbf{i}, \tag{30}$$

that satisfy the *multiple relations*

$$\begin{array}{ll} \partial_i^\alpha.\partial_j^\beta = \partial_j^\beta.\partial_i^\alpha \quad (i \neq j), & e_i.e_j = e_j.e_i \quad (i \neq j), \\ \partial_i^\alpha.e_j = e_j.\partial_i^\alpha \quad (i \neq j), & \partial_i^\alpha.e_i = \mathrm{id}. \end{array} \tag{31}$$

Faces commute and degeneracies commute, but ∂_i^α and e_i do not. These relations look much simpler than the cubical ones because here an index i stands for a particular sort, instead of a mere position, and is never 're-named'. Notice also that ∂_i^α acts on $X_\mathbf{i}$ if i belongs to the multi-index $\mathbf{i}$ (and cancels it), while e_i acts on $X_\mathbf{i}$ if i does not belong to $\mathbf{i}$ (and inserts it); therefore $\partial_i^\alpha.\partial_i^\alpha$ and $e_i.e_i$ make no sense, here: one cannot cancel or insert twice the same index.

If $\mathbf{i} = \mathbf{j} \cup \mathbf{k}$ is a disjoint union and $\boldsymbol{\alpha}$ is a mapping

$$\boldsymbol{\alpha} \colon \mathbf{k} = \{k_1, ..., k_r\} \to \{-, +\}, \qquad \boldsymbol{\alpha} = (\alpha_1, ..., \alpha_r),$$

we have an *iterated face* and an *iterated degeneracy* (independent of the order of composition)

$$\partial_\mathbf{k}^{\boldsymbol{\alpha}} = \partial_{\mathbf{k}_1}^{\alpha_1} ... \partial_{\mathbf{k}_r}^{\alpha_r} \colon X_\mathbf{i} \to X_\mathbf{j}, \qquad e_\mathbf{k} = e_{k_1} ... e_{k_r} \colon X_\mathbf{j} \to X_\mathbf{i}. \tag{32}$$

In particular the *total* $\mathbf{i}$*-degeneracy* is the mapping

$$e_\mathbf{i} = e_{i_1} ... e_{i_n} \colon X_* \to X_\mathbf{i}. \tag{33}$$

A *morphism of multiple sets* $F \colon X \to Y$ is a family of mappings $F_\mathbf{i} \colon X_\mathbf{i} \to Y_\mathbf{i}$ that commute with faces and degeneracies.

2.3 Multiple sets and cubical sets

Let us recall that the cubical sets form the presheaf category $\mathbf{Set}^{\mathbb{I}^{\mathrm{op}}}$, where the 'cubical site' $\mathbb{I}$ [GM, G1] has for objects the powers 2^n of the cardinal $2 = \{0, 1\}$ (with $n \in \mathbb{N}$) and a morphism $2^m \to 2^n$ takes out some coordinates and inserts some 0's or 1's (without modifying the order of the remaining coordinates). Such morphisms are generated by the following cofaces and codegeneracies (under the well-known cocubical relations):

$$\begin{array}{ll} \partial_j^\alpha\colon 2^{n-1} \to 2^n, & \partial_j^\alpha(t_1, ..., t_{n-1}) = (t_1, ..., t_{j-1}, \alpha, ..., t_{n-1}), \\ e_j\colon 2^n \to 2^{n-1}, & e_j(t_1, ..., t_n) = (t_1, ..., \hat{t}_j, ..., t_n) \\ & (\alpha = 0, 1;\ 1 \leqslant j \leqslant n). \end{array} \tag{34}$$

Modifying all this, the *multiple site* $\mathbb{M}$ has an object $2^{\mathbf{i}} = \mathbf{Set}(\mathbf{i}, 2)$ for every multi-index $\mathbf{i} \subset \mathbb{N}$. The category $\mathbb{M} \subset \mathbf{Set}$ is generated by the following mappings (with $i \neq j$ in $\mathbf{i}$ and $\alpha = 0, 1$)

$$\begin{array}{lll} \partial_j^\alpha\colon 2^{\mathbf{i}|j} \to 2^{\mathbf{i}}, & (\partial_j^\alpha \varphi)(i) = \varphi(i), & (\partial_j^\alpha \varphi)(j) = \alpha, \\ e_j\colon 2^{\mathbf{i}} \to 2^{\mathbf{i}|j}, & (e_j \varphi)(i) = \varphi(i), & \end{array} \tag{35}$$

under the *comultiple relations*, dual to the multiple relations of (31). (Since commutativity relations are invariant under duality, the only comultiple relation different from the previous ones is $e_i.\partial_i^\alpha = \mathrm{id}$.)

There is a canonical (covariant) functor (where $\mathbf{i} = \{i_1, ..., i_j, ..., i_n\}$)

$$\begin{array}{ll} F\colon \mathbb{M} \to \mathbb{I}, & F(2^{\mathbf{i}}) = 2^n, \\ F(\partial_{i_j}^\alpha\colon 2^{\mathbf{i}|i_j} \to 2^{\mathbf{i}}) = \partial_j^\alpha\colon 2^{n-1} \to 2^n, & \\ F(e_{i_j}\colon 2^{\mathbf{i}} \to 2^{\mathbf{i}|i_j}) = e_j\colon 2^n \to 2^{n-1}. & \end{array} \tag{36}$$

F transforms every cubical set $K\colon \mathbb{I}^{\mathrm{op}} \to \mathbf{Set}$ into $KF^{\mathrm{op}}\colon \mathbb{M}^{\mathrm{op}} \to \mathbf{Set}$, a *multiple set of cubical type*. Thus, the multiple set X is *of cubical type* if and only if it is 'invariant under renaming indices, in the same order'. Precisely, X has to satisfy the following relations, where $\mathbf{i} = \{i_1, ..., i_j, ..., i_n\} \subset \mathbb{N}$ is replaced with the 'normalised' multi-index $[n] = \{1, ..., j, ..., n\}$ (for $n \geqslant 0$)

$$\begin{array}{c} X_{\mathbf{i}} = X_{1...n}, \\ (\partial_{i_j}^\alpha\colon X_{\mathbf{i}} \to X_{\mathbf{i}|i_j}) = (\partial_j^\alpha\colon X_{1...n} \to X_{1...\hat{j}...n}), \\ (e_{i_j}\colon X_{\mathbf{i}|i_j} \to X_{\mathbf{i}}) = (e_j\colon X_{1...\hat{j}...n} \to X_{1...n}). \end{array} \tag{37}$$

This notion is equivalent to the classical notion of a cubical set, by a *rewriting of multi-indices*: in fact, the multi-index $\{1, ..., \hat{j}, ..., n\}$ has to be normalised with consecutive integers. This rewriting transforms the *multiple relations* that hold in a multiple set of cubical type into the *cubical relations* of the associated cubical set.

Here *we prefer to avoid such rewritings and stay within multiple sets*.

More generally, we have a *multiple site* $\mathbb{M}(N)$ *based on any ordered pointed set* $N = (N, 0)$; a multi-index is now a finite subset $\mathbf{i} \subset N$. We shall mostly use this extension for subsets $\mathbf{n} = \{0, 1, ..., n-1\}$ of the natural integers, but also for $N = \mathbb{Z}$. (The base point and the order will be used later.)

2.4 Multiple categories

We are now ready for a formal definition of our main strict structure.

(mlc.1) A *multiple category* A is, first of all, a multiple set of components $A_{\mathbf{i}}$, whose elements will be called $\mathbf{i}$-*cells*. As above, $\mathbf{i}$ is any multi-index, i.e. any finite subset of $\mathbb{N}$, and we write A_*, A_i, A_{ij}... for $A_\varnothing, A_{\{i\}}, A_{\{i,j\}}$,...

(mlc.2) Given two $\mathbf{i}$-cells x, y which are *i-consecutive* (i.e. $\partial_i^+(x) = \partial_i^-(y)$, with $i \in \mathbf{i}$), the *i-composition* $x +_i y$ is defined and satisfies the following interactions with faces and degeneracies (for $j \neq i$)

$$\begin{array}{ll} \partial_i^-(x +_i y) = \partial_i^-(x), & \partial_i^+(x +_i y) = \partial_i^+(y), \\ \partial_j^\alpha(x +_i y) = \partial_j^\alpha(x) +_i \partial_j^\alpha(y), & e_j(x +_i y) = e_j(x) +_i e_j(y). \end{array} \tag{38}$$

It will be important to remark that the last condition is a *strict interchange between i-composition and j-identities*, while the *strict interchange between i- and j-identities* (or zeroary compositions) is already written in the axioms of multiple sets: $e_j e_i = e_i e_j$ for $j \neq i$.

(mlc.3) For $j \notin \mathbf{i}$ we have a category $\mathrm{cat}_{\mathbf{i},j}(\mathsf{A})$ with objects in $A_{\mathbf{i}}$, arrows in $A_{\mathbf{i}j}$, faces ∂_j^α, identities e_j and composition $+_j$.

(mlc.4) For $i < j$ we have

$$(x +_i y) +_j (z +_i u) = (x +_j z) +_i (y +_j u) \qquad (\textit{binary } ij\textit{-interchange}), \tag{39}$$

whenever these composites make sense.

Again, we can more generally consider *N-indexed multiple categories*, where $N = (N, 0)$ is an ordered pointed set.

2.5 Transversal categories

The *transversal* direction, corresponding to the index $i = 0$, will play a special role. It will be used for the transformations of multiple functors and for the structural arrows of limits and colimits; its composition will stay strict, in all the weak or lax versions we shall consider. We think of it as the 'dynamic' direction, along which 'transformation occurs', while the positive directions are viewed as the 'static' or 'geometric' ones.

For a *positive* multi-index $\mathbf{i} = \{i_1, ..., i_n\} \subset \mathbb{N}\setminus\{0\}$ of dimension n, both $\mathbf{i}$ and the *augmented* multi-index $0\mathbf{i} = \{0, i_1, ..., i_n\}$ will be said to be of *degree* n, counting the number of positive indices that they contain.

We are interested in the $\mathbf{i}$-*transversal* category $\mathrm{tv}_\mathbf{i}(\mathsf{A}) = \mathrm{cat}_{\mathbf{i},0}(\mathsf{A})$ of A, where

- an object x, usually called an $\mathbf{i}$-*cube* of A, is an n-dimensional cell belonging to $A_\mathbf{i}$,

- a morphism $f\colon x^- \to_0 x^+$, usually called an $\mathbf{i}$-*map* of A, is an $(n+1)$-dimensional cell $f \in A_{0\mathbf{i}}$ with $\partial_0^\alpha f = x^\alpha$,

- their composition $gf = f +_0 g$ is the transversal one (in direction 0), with identities $1_x = \mathrm{id}(x) = e_0(x)$.

All these terms are said to be of *type* $\mathbf{i}$ and *degree* n in A; but let us recall that their *dimension* is either n or $n+1$. A *transversal isomorphism* is an isomorphism of a transversal category.

In all of our examples, 0-composition is realised by the *usual composition* of mappings. On the other hand, in the *non-strict* structures considered below, the 'positive' compositions are generally obtained by operations (products, sums, tensor products, pullbacks, pushouts...) where reversing the order of the operands would only be confusing.

2.6 Multiple functors and transversal transformations

A *multiple functor* $F\colon \mathsf{A} \to \mathsf{B}$ between multiple categories is a morphism of multiple sets $F = (F_\mathbf{i})$ (cf. 2.2) that preserves all the composition laws. For an $\mathbf{i}$-map $f\colon x \to_0 y$, we write $F(f)\colon F(x) \to_0 F(y)$ or $F_{0\mathbf{i}}(f)\colon F_\mathbf{i}(x) \to_0 F_\mathbf{i}(y)$, as may be convenient.

A *transversal transformation* $h\colon F \to G\colon \mathsf{A} \to \mathsf{B}$ between multiple functors consists of a family of **i**-maps in B (its components), for every positive multi-index **i** and every **i**-cube x in A, that agrees with positive faces

$$\begin{array}{ll} hx\colon F(x) \to_0 G(x) & (h_{\mathbf{i}}x\colon F_{\mathbf{i}}(x) \to_0 G_{\mathbf{i}}(x)), \\ h(\partial_i^\alpha x) = \partial_i^\alpha(hx) & (i \in \mathbf{i}). \end{array} \tag{40}$$

The following axioms of naturality and coherence are assumed:

(trt.1) $Gf.hx = hy.Ff$ (for $f\colon x \to_0 y$ in A),

(trt.2) h commutes with positive degeneracies and compositions:

$$h(e_i z) = e_i(hz), \qquad h(x +_i y) = hx +_i hy,$$

where **i** is a positive multi-index, $i \in \mathbf{i}$, x and y are i-consecutive **i**-cubes, z is an $\mathbf{i}|i$-cube.

We have thus the category $\mathbf{Mlc}(\mathsf{A}, \mathsf{B})$ of the multiple functors $\mathsf{A} \to \mathsf{B}$ and their transversal transformations. All these form the 2-category $\mathbf{Mlc}$, in an obvious way. More generally for any ordered pointed set $N = (N, 0)$, we have the 2-category $\mathbf{Mlc}_N$ of N-indexed multiple categories, formed of ordinary categories $\mathbf{Mlc}_N(\mathsf{A}, \mathsf{B})$.

Multiple categories have dualities, generated by reversing each direction i and permuting directions; they form an infinite-dimensional hyperoctahedral group. But we are mainly interested in the *transversal dual* A^{tv} that reverses all transversal faces ∂_0^α and all transversal compositions, so that $\mathrm{tv}_{\mathbf{i}}(\mathsf{A}^{\mathrm{tv}}) = (\mathrm{tv}_{\mathbf{i}}(\mathsf{A}))^{\mathrm{op}}$; for two consecutive **i**-maps f, g in A with $f\colon x^- \to_0 x^+$, we have corresponding maps f^*, g^* in A^{tv} with

$$f^*\colon x^+ \to_0 x^-, \qquad f^*.g^* = (gf)^*. \tag{41}$$

2.7 Truncation and triple categories

Restricting all indices to the subsets of the ordinal set $\mathbf{n} = \{0, ..., n-1\}$ we obtain the n-dimensional version of a multiple category, called an *n-tuple category*, where the highest cells have dimension n. The 0-, 1- and 2-dimensional versions amount - respectively - to a set, a category or a double category.

There is thus a truncation 2-functor with values in the 2-category $\mathbf{Mlc_n}$ of n-tuples categories, which has both adjoints

$$\mathrm{trc}_n\colon \mathbf{Mlc} \to \mathbf{Mlc_n}, \qquad\qquad \mathrm{sk}_n \dashv \mathrm{trc}_n \dashv \mathrm{cosk}_n. \qquad (42)$$

The left adjoint (*skeleton*) adds degenerate items of all missing types $\mathbf{i} \not\subset \mathbf{n}$. The right adjoint (*coskeleton*) is more compex: for instance, if $\mathbf{C}$ is a category and $\mathbf{i}$ a positive multi-index, an $\mathbf{i}$-cube of $\mathrm{cosk}_1(\mathbf{C})$ is a functor $x\colon 2^{\mathbf{i}} \to \mathbf{C}$ where $2 = \{0, 1\}$ is discrete (so that x is a family of objects of $\mathbf{C}$ indexed by the set $2^{\mathbf{i}}$); an $\mathbf{i}$-map is a natural transformation of such functors.

We are particularly interested in the 3-dimensional notion, called a *triple category*. Its cells, corresponding to multi-indices $\mathbf{i} \subset \{0, 1, 2\}$, are:

- objects, of one sort (for $\mathbf{i} = \varnothing$),

- arrows of three sorts, in direction 0 (*transversal*), 1 (*horizontal*) and 2 (*vertical*),

- 2-dimensional cells of three sorts, in direction 01 (*horizontal*), 02 (*lateral*), 12 (*basic*),

- 3-dimensional cells of one sort, in direction 012.

The terminology in parenthesis comes from [GP6], and is based on diagrams as drawn above (e.g. in 1.3): the horizontal and vertical direction are viewed in a vertical plane and the transversal direction as orthogonal to the plane. We have already studied in Section 1 the triple category $\mathrm{S}\mathbb{D}\mathrm{bl}$ of weak double categories, with arrows given by: strict functors, lax functors and colax functors.

2.8 Symmetric cubical categories

Let us first remark that the notion of a *cubical category* which we use here was defined in [G1, G3]: it includes transversal maps, that are crucial for the weak and lax extensions (as in the present case of multiple categories). It differs on this point from the notion of [ABS], that was called a 'reduced cubical category' in [G1, G3], even though in the strict case *the difference is just a formal reindexing*.

In the present setting we say that the multiple category A is *of cubical type* if its components, faces and degeneracies are *invariant under renaming* positive *indices*, in the same order.

With respect to multiple sets (in 2.3), we use now a different notion of *normalised multi-index*, that only operates on positive indices and preserves both dimension and degree. Precisely, a (general) multi-index $\mathbf{i} \subset \mathbb{N}$ has a *normalised multi-index* $\mathbf{h}$ defined as follows, according to the positivity of $\mathbf{i}$

$$\begin{aligned} &\mathbf{i} = \{i_1, ..., i_n\} \mapsto \mathbf{h} = [n] = \{1, ..., n\} &&\text{for } 0 < i_1 < ... < i_n, \\ &\mathbf{i} = \{0, i_1, ..., i_n\} \mapsto \mathbf{h} = 0[n] &&\text{for } 0 = i_0 < ... < i_n. \end{aligned} \tag{43}$$

The multiple category A is *of cubical type* if, with this notation:

$$\begin{gathered} A_{\mathbf{i}} = A_{\mathbf{h}}, \qquad (\partial^{\alpha}_{i_j} \colon A_{\mathbf{i}} \to A_{\mathbf{i}|i_j}) = (\partial^{\alpha}_{j} \colon A_{\mathbf{h}} \to A_{\mathbf{h}|j}), \\ (e_{i_j} \colon A_{\mathbf{i}|i_j} \to A_{\mathbf{i}}) = (e_j \colon A_{\mathbf{h}|j} \to A_{\mathbf{h}}). \end{gathered} \tag{44}$$

This notion is equivalent to that of a cubical category, as defined in [G1, G3]. In the truncated case the invariance condition is trivially satisfied up to dimension 2 (corresponding to sets, categories and double categories), since a subset $\mathbf{i} \subset \{0, 1\}$ is automatically normalised; on the other hand, a triple category can be of cubical type or not: the example $\mathbb{SD}\mathrm{bl}$ of Section 1 is not, as its 1- and 2-arrows are different.

In a multiple category of cubical type an $\mathbf{i}$-cube $x \in A_{\mathbf{i}} = A_{[n]}$ is called an n*-cube*, and an $\mathbf{i}$-map $f \colon x \to y$ (belonging to $A_{0\mathbf{i}} = A_{0[n]}$) is called an n*-map*. On the other hand, *the positive compositions* $x +_i y$ *need not be related*. Yet, all of our important examples of cubical categories (also in the weak case, see 3.6) are 'symmetric', with *positive faces, degeneracies and compositions related by symmetries* - so that composition in direction 1, for instance, determines all the positive ones. Again, symmetric cubical categories are studied in [G1, G3]; here they can be viewed as follows.

A *multiple category of symmetric cubical type* is a multiple category of cubical type A (as defined above) with an assigned action of the symmetric group S_n (non trivial for $n \geqslant 2$) on each set $A_{\mathbf{i}} = A_{\mathbf{h}}$ (where the multi-indices $\mathbf{i}, \mathbf{h}$ have *degree* n), generated by mappings called *transpositions*

$$s_i \colon A_{\mathbf{h}} \to A_{\mathbf{h}}, \qquad i = 1, ..., n-1 \qquad (n \geqslant 2). \tag{45}$$

These transpositions satisfy the well-known Moore relations of the symmetric group (listed for instance in [G1], 2.1.3). Moreover s_i exchanges the i-indexed structure with the $(i+1)$-indexed one, leaving the rest unchanged.

More precisely, the following axioms must be satisfied (for $i > 0$, $j \geqslant 0$ and $j \neq i, i+1$):

$$\begin{array}{ll} \partial_i^\alpha s_i = \partial_{i+1}^\alpha \;\; (\partial_{i+1}^\alpha . s_i = \partial_i^\alpha), & \partial_j^\alpha . s_i = s_i . \partial_j^\alpha, \\ s_i . e_i = e_{i+1} \;\; (s_i . e_{i+1} = e_i), & s_i . e_j = e_j . s_i, \\ s_i(x +_i y) = s_i(x) +_{i+1} s_i(y) & (s_i(x +_{i+1} y) = s_i(x) +_i s_i(y)), \\ s_i(x +_j y) = s_i(x) +_j s_i(y). & \end{array} \tag{46}$$

(Note that j need not be positive.) The relations in parentheses are redundant because of the involutive property of transpositions $s_i . s_i = \mathrm{id}$, which is part of the Moore relations.

The symmetric cubical category $\omega\mathsf{Cub}(\mathbf{C})$ *of commutative cubes* over a category $\mathbf{C}$ is recalled below, in 3.5.

In the truncated case the symmetric structure, that only works on *positive* indices, *is trivial up to dimension* 2 (for sets, categories and *double categories as well*); on the other hand, a triple category of cubical type A is made symmetric (*if this is possible*) by assigning two involutions $s_1 \colon A_{12} \to A_{12}$ and $s_1 \colon A_{012} \to A_{012}$ that satisfy the axioms above.

Infinite-dimensional globular categories, usually called ω-categories, can be analysed as cubical categories of a globular type: see [ABS] and [GP5], Section 2.

3. Weak and chiral multiple categories

We now extend multiple categories to the *weak* case. The basic structure of a weak multiple category A is a multiple set with compositions in all directions. The composition laws in direction 0 are categorical and have a strict interchange with the other compositions. On the other hand, the ‘positive’ compositions have invertible comparisons for unitarity, associativity and interchange (see 3.2), under coherence conditions listed in 3.3 and 3.4.

After some examples of a cubical type, we end with a more general notion, partially lax: a *chiral*, or χ-*lax*, *multiple category* (see 3.7); it has the same structure of a weak multiple category, except for the fact that the ‘positive’ interchange comparisons χ_{ij} (for $0 < i < j$) are not supposed to be invertible.

3.1 The basic structure

A *weak multiple category* A has a basic structure of multiple set (cf. 2.1) *with compositions*.

(wmc.1) A is, first of all, a multiple set of components $A_{\mathbf{i}}$, whose elements will be called **i**-*cells*; as above, **i** is any multi-index, i.e. a finite subset of $\mathbb{N}$. As in Section 2, the index 0 denotes the *transversal* direction and plays a special role, different from that of the positive indices.

(wmc.2) Given two **i**-cells x, y which are i-*consecutive* (i.e. $\partial_i^+(x) = \partial_i^-(y)$, with $i \in \mathbf{i}$), the i-*composition* $x +_i y$ is defined and satisfies the following 'geometric' interactions with faces and degeneracies (for $j \neq i$)

$$\begin{aligned} &\partial_i^-(x +_i y) = \partial_i^-(x), && \partial_i^+(x +_i y) = \partial_i^+(y), \\ &\partial_j^\alpha(x +_i y) = \partial_j^\alpha(x) +_i \partial_j^\alpha(y), && e_j(x +_i y) = e_j(x) +_i e_j(y). \end{aligned} \tag{47}$$

(Again, as in 2.4, the last condition is a *strict interchange between* i-*composition and* j-*identities*.)

(wmc.3) Transversal composition is categorical: for every positive multi-index **i** we have a *transversal category* $\mathrm{tv}_{\mathbf{i}}(\mathsf{A}) = \mathrm{cat}_{\mathbf{i},0}(\mathsf{A})$; its arrows are the 0**i**-cells $f\colon x \to_0 y$, also called **i**-maps between **i**-cubes (see 2.5); their composition is written as $gf = f +_0 g$.

(wmc.4) Transversal composition has a strict interchange with any positive i-composition

$$gf +_i kh = (g +_i k)(f +_i h) \qquad (0i\text{-}interchange), \tag{48}$$

for $i \in \mathbf{i}$ and four **i**-maps f, g, h, k such that these composites make sense. (We already remarked that the lower $0i$-interchanges are expressed above.)

For a positive multi-index **i**, an **i**-map $f\colon x \to_0 y$ is said to be i-*special*, or *special in direction* $i \in \mathbf{i}$, if its two i-faces are transversal identities

$$\partial_i^\alpha f = e_0\partial_i^\alpha x = e_0\partial_i^\alpha y \qquad (\alpha = \pm). \tag{49}$$

This, of course, implies that the **i**-cubes x, y have the same i-faces: $\partial_i^\alpha x = \partial_i^\alpha y$ (in $A_{\mathbf{i}|i}$).

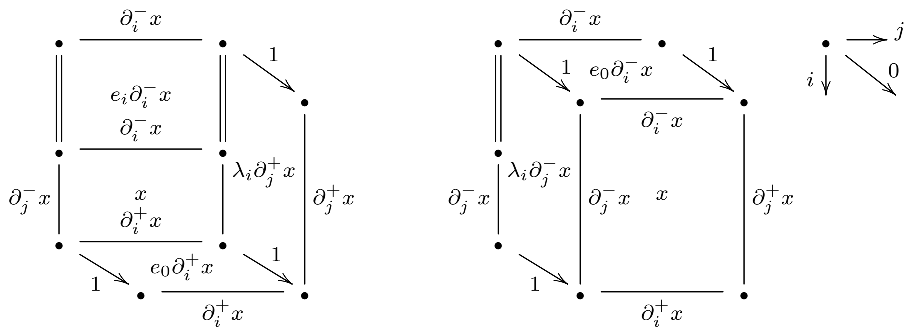
We say that f is ij-*special* if it is special in both directions i, j.

3.2 Comparisons

Now we require that the positive compositions are unitary, associative and interchangeable up to transversal isomorphisms: left unitors, right unitors, associators and interchangers. The letter **i** denotes a positive multi-index with $i \in \mathbf{i}$. (In the diagrams below a line in a positive direction represents a cell and a double line represents a cell degenerate in that direction.)

(wmc.5) For every **i**-cube x we have an invertible i-special **i**-map $\lambda_i x$, which is natural on **i**-maps and has the following faces (for $j \neq i$ in **i**)

$$\begin{array}{ll} \lambda_i x \colon (e_i\partial_i^- x) +_i x \to_0 x & \qquad (\textit{left } i\textit{-unitor}), \\ \partial_j^\alpha \lambda_i x = \lambda_i \partial_j^\alpha x \qquad (\partial_i^\alpha \lambda_i x = e_0 \partial_i^\alpha x), & \end{array} \tag{50}$$



The condition in parentheses says again that these maps are i-special, and will not be repeated below. The naturality condition means that for every **i**-map $f \colon x \to_0 x'$ the following square of **i**-maps commutes (in the category $\mathrm{tv}_{\mathbf{i}}(\mathsf{A})$)

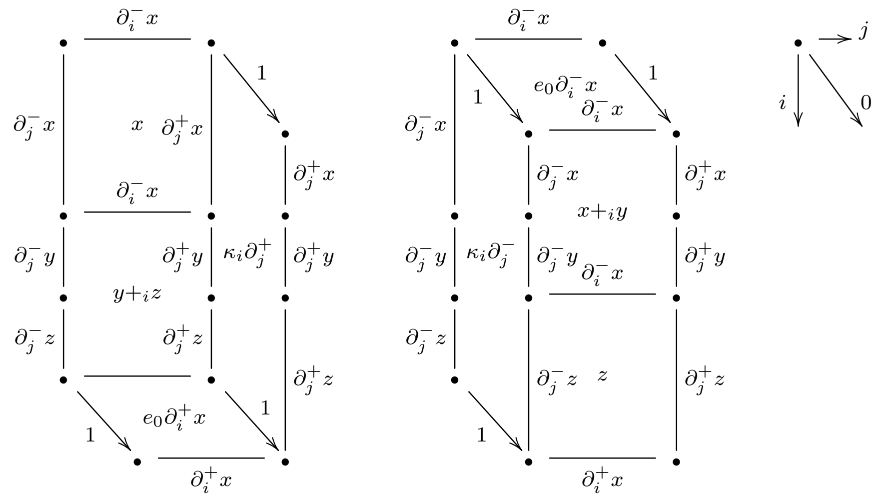
$$\begin{array}{ccc} (e_i\partial_i^- x) +_i x & \xrightarrow{\lambda_i x} & x \\ {\scriptstyle (e_i\partial_i^- f)+_i f}\downarrow & & \downarrow{\scriptstyle f} \\ (e_i\partial_i^- x') +_i x' & \xrightarrow{\lambda_i x'} & x' \end{array} \tag{51}$$

(wmc.6) For every **i**-cube x we have an invertible i-special **i**-map $\rho_i x$, which is natural on **i**-maps and has the following faces (for $j \neq i$ in **i**)

$$\begin{array}{ll} \rho_i x \colon x +_i (e_i\partial_i^+ x) \to_0 x & \qquad (\textit{right } i\textit{-unitor}), \\ \partial_j^\alpha \rho_i x = \rho_i \partial_j^\alpha x. & \end{array} \tag{52}$$

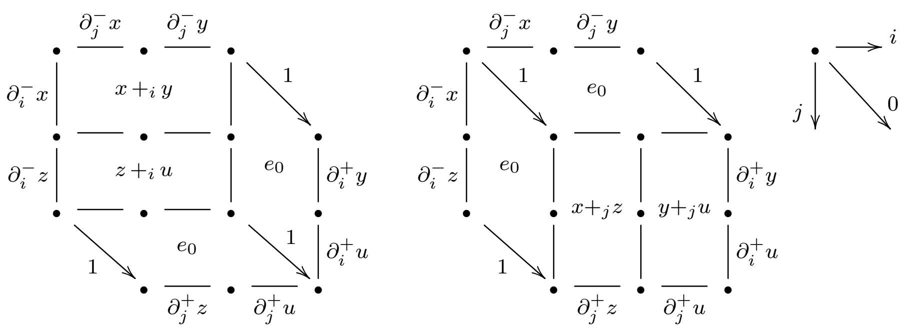
(wmc.7) For three i-consecutive **i**-cubes x, y, z we have an invertible i-special **i**-map $\kappa_i(x, y, z)$ which is natural on **i**-maps and has the following faces (for $j \neq i$ in **i**)

$$\begin{aligned} &\kappa_i(x, y, z)\colon x +_i (y +_i z) \to_0 (x +_i y) +_i z \qquad (i\text{-}associator), \\ &\partial_j^\alpha \kappa_i(x, y, z) = \kappa_i(\partial_j^\alpha x, \partial_j^\alpha y, \partial_j^\alpha z) \end{aligned} \tag{53}$$



(wmc.8) Given four **i**-cubes x, y, z, u which satisfy the boundary conditions displayed below, for $i < j$ in **i**, we have an invertible ij-special **i**-map $\chi_{ij}(x, y, z, u)$, the ij-*interchanger*, which is natural on **i**-maps and has the following k-faces (for $k \neq i, j$ in **i**)

$$\begin{aligned} \chi_{ij}(x, y, z, u)\colon (x +_i y) +_j (z +_i u) \to_0 (x +_j z) +_i (y +_j u), \\ \partial_k^\alpha \chi_{ij}(x, y, z, u) = \chi_{ij}(\partial_k^\alpha x, \partial_k^\alpha y, \partial_k^\alpha z, \partial_k^\alpha u), \end{aligned} \tag{54}$$



(wmc.9) Finally, these comparisons must satisfy some conditions of coherence, listed below in 3.3, 3.4.

We say that A is *unitary* if the comparisons λ, ρ are identities, and *pre-unitary* if every unitor of type $\lambda(e_iz) = \rho(e_iz)\colon e_iz +_i e_iz \to_0 e_iz$ is an identity (see (59)).

The *transversal dual* A^{tv} of a weak multiple category reverses the transversal faces and compositions (as in (41)), and has inverted comparisons $\lambda_i^*(x) = ((\lambda_i x)^{-1})^*$, etc.

3.3 Coherence conditions, I

As an extension of the coherence conditions for weak symmetric cubical categories [G1], the coherence axiom (wmc.9) means that various conditions on the comparisons are satisfied; for future reference it will be convenient to split them in two parts, deferring to the next point 3.4 all the conditions involving the interchangers χ_{ij}.

The following diagrams of transversal maps must commute (assuming that all the compositions in direction $i > 0$ make sense).

(wmc.9.i) *Coherence pentagon of the i-associator $\kappa = \kappa_i$*

$$\begin{array}{ccccc} & & (x+_iy)+_i(z+_iu) & & \\ & \overset{\kappa}{\nearrow} & & \overset{\kappa}{\searrow} & \\ x+_i(y+_i(z+_iu)) & & & & ((x+_iy)+_iz)+_iu \\ {\scriptstyle 1+_i\kappa}\downarrow & & & & \uparrow{\scriptstyle \kappa+_i1} \\ x+_i((y+_iz)+_iu) & & \xrightarrow{\quad\kappa\quad} & & (x+_i(y+_iz))+_iu \end{array} \qquad (55)$$

(wmc.9.ii) *Coherence conditions for $\kappa = \kappa_i$, $\lambda = \lambda_i$ and $\rho = \rho_i$*

$$\begin{array}{ccc} x +_i (e_i\partial_i^- y +_i y) & \xrightarrow{\kappa} & (x +_i e_i\partial_i^+ x) +_i y \\ & {\scriptstyle 1+_i\lambda}\searrow \quad \swarrow{\scriptstyle \rho+_i1} & \\ & x +_i y & \end{array} \qquad (56)$$

$$\begin{array}{ccc} e_i\partial_i^- x +_i (x +_i y) & \xrightarrow{\kappa} & (e_i\partial_i^- x +_i x) +_i y \\ & {\scriptstyle \lambda}\searrow \quad \swarrow{\scriptstyle \lambda+_i1} & \\ & x +_i y & \end{array} \qquad (57)$$

$$\begin{array}{ccc} x +_i (y +_i e_i\partial_i^+ y) & \xrightarrow{\kappa} & (x +_i y) +_i e_i\partial_i^+ y \\ & {\scriptstyle 1+_i\rho}\searrow \quad \swarrow{\scriptstyle \rho} & \\ & x +_i y & \end{array} \tag{58}$$

$$\lambda(e_i z) = \rho(e_i z)\colon e_i z +_i e_i z \to_0 e_i z, \tag{59}$$

(These conditions amount to asking that, for every positive multi-index $\mathbf{i}$ and $i \notin \mathbf{i}$, the $\mathbf{i}$-cubes of $\mathbb{A}$ form a weak double category with horizontal arrows in $A_{0\mathbf{i}}$, vertical arrows in $A_{\mathbf{i}i}$ and double cells in $A_{0\mathbf{i}i}$. We write (wmc.9.ii) in the form used by Mac Lane in his classical paper on coherence of monoidal categories [Ma]. As proved by Kelly [Ke] these axioms are redundant: properties (55) and (56) imply the other three; but we prefer to keep the latter, as they are useful in computation.)

3.4 Coherence conditions, II

Finally we list the conditions involving the interchangers χ_{ij} (for $0 < i < j$). Again, the following diagrams of transversal maps must commute (assuming that all the positive compositions make sense).

(wmc.9.iii) *Coherence hexagon of* $\chi = \chi_{ij}$ *and* κ_i $(0 < i < j)$

$$\begin{array}{ccc} (x{+_i}(y{+_i}z)){+_j}(x'{+_i}(y'{+_i}z')) & \xrightarrow{\kappa_i+_j\kappa_i} & ((x{+_i}y){+_i}z){+_j}((x'{+_i}y'){+_i}z') \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi} \\ (x{+_j}x'){+_i}((y{+_i}z){+_j}(y'{+_i}z')) & & ((x{+_i}y){+_j}(x'{+_i}y')){+_i}(z{+_j}z') \\ {\scriptstyle 1+_i\chi}\downarrow & & \downarrow{\scriptstyle\chi+_i1} \\ (x{+_j}x'){+_i}((y{+_j}y'){+_i}(z{+_j}z')) & \xrightarrow{\kappa_i} & ((x{+_j}x'){+_i}(y{+_j}y')){+_i}(z{+_j}z') \end{array} \tag{60}$$

(wmc.9.iv) *Coherence hexagon of* $\chi = \chi_{ij}$ *and* κ_j $(0 < i < j)$

$$\begin{array}{ccc} (x{+_i}x'){+_j}((y{+_i}y'){+_j}(z{+_i}z')) & \xrightarrow{\kappa_j} & ((x{+_i}x'){+_j}(y{+_i}y')){+_j}(z{+_i}z') \\ {\scriptstyle 1+_j\chi}\downarrow & & \downarrow{\scriptstyle\chi+_j1} \\ (x{+_j}x'){+_j}((y{+_j}z){+_i}(y'{+_j}z')) & & ((x{+_j}y){+_i}(x'{+_j}y')){+_j}(z{+_i}z') \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi} \\ (x{+_j}(y{+_j}z)){+_i}(x'{+_j}(y'{+_j}z')) & \xrightarrow{\kappa_j+_i\kappa_j} & ((x{+_j}y){+_j}z){+_i}((x'{+_j}y'){+_j}z') \end{array} \tag{61}$$

(wmc.9.v) *Coherence conditions for* $\chi = \chi_{ij}$, λ_i *and* ρ_i $(0 < i < j)$

$$\begin{array}{ccccc}
(e_i\partial_i^- x +_i x) +_j (e_i\partial_i^- y +_i y) & \xrightarrow{\lambda_i +_j \lambda_i} & x +_j y & \xleftarrow{\rho_i +_j \rho_i} & (x +_i e_i\partial_i^+ x) +_j (y +_i e_i\partial_i^+ y) \\
\downarrow{\scriptstyle \chi} & & \| & & \downarrow{\scriptstyle \chi} \\
(e_i\partial_i^- x +_j e_i\partial_i^- y) +_i (x +_j y) & & \| & & (x +_j y) +_i (e_i\partial_i^+ x +_j e_i\partial_i^+ y) \\
\| & & \| & & \| \\
e_i\partial_i^-(x +_j y) +_i (x +_j y) & \xrightarrow[\lambda_i]{} & x +_j y & \xleftarrow[\rho_i]{} & (x +_j y) +_i e_i\partial_i^+(x +_j y)
\end{array} \qquad (62)$$

(wmc.9.vi) *Coherence conditions for* $\chi = \chi_{ij}$, λ_j *and* ρ_j $(0 < i < j)$

$$\begin{array}{ccccc}
e_j\partial_j^-(x +_i y) +_j (x +_i y) & \xrightarrow{\lambda_j} & x +_j y & \xleftarrow{\rho_j} & (x +_i y) +_j e_j\partial_j^+(x +_i y) \\
\| & & \| & & \| \\
(e_j\partial_j^- x +_i e_j\partial_j^- y) +_j (x +_i y) & & \| & & (x +_i y) +_j (e_j\partial_j^+ x +_i e_j\partial_j^+ y) \\
\downarrow{\scriptstyle \chi} & & \| & & \downarrow{\scriptstyle \chi} \\
(e_j\partial_j^- x +_j x) +_i (e_j\partial_j^- y +_j y) & \xrightarrow[\lambda_j +_i \lambda_j]{} & x +_j y & \xleftarrow[\rho_j +_i \rho_j]{} & (x +_j e_j\partial_j^+ x) +_i (y +_j e_j\partial_j^+ y)
\end{array} \qquad (63)$$

(wmc.9.vii) *Coherence hexagon of the interchangers* χ_{ij}, χ_{jk} *and* χ_{ik} $(0 < i < j < k)$

$$\begin{array}{ccc}
 & ((x +_i y) +_j (z +_i u)) +_k ((x' +_i y') +_j (z' +_i u')) & \\
\swarrow{\scriptstyle \chi_{jk}} & & \searrow{\scriptstyle \chi_{ij} +_k \chi_{ij}} \\
((x +_i y) +_k (x' +_i y')) +_j ((z +_i u) +_k (z' +_i u')) & & ((x +_j z) +_i (y +_j u)) +_k ((x' +_j z') +_i (y' +_j u')) \\
\downarrow{\scriptstyle \chi_{ik} +_j \chi_{ik}} & & \downarrow{\scriptstyle \chi_{ik}} \\
((x +_k x') +_i (y +_k y')) +_j ((z +_k z') +_i (u +_k u')) & & ((x +_j z) +_k (x' +_j z')) +_i ((y +_j u) +_k (y' +_j u')) \\
\searrow{\scriptstyle \chi_{ij}} & & \swarrow{\scriptstyle \chi_{jk} +_i \chi_{jk}} \\
 & ((x +_k x') +_j (z +_k z')) +_i ((y +_k y') +_j (u +_k u')) &
\end{array} \qquad (64)$$

3.5 Weak multiple categories of symmetric cubical type

We have seen in 2.8 that multiple categories generalise cubical categories and symmetric cubical categories. In the same way, weak multiple categories generalise *weak cubical categories* and *weak symmetric cubical categories*; the latter were introduced in [G1] for higher cobordisms, and give here our main examples of weak multiple categories of infinite dimension.

Here we only recall the following examples that will be of use for studying multiple limits.

(a) The strict symmetric cubical category $\omega\mathsf{Cub}(\mathbf{C})$, or $\mathsf{Cub}(\mathbf{C})$, *of commutative cubes* over a category $\mathbf{C}$. An n-cube (see 2.8) is a functor $x\colon \mathbf{2}^n \to \mathbf{C}$ $(n \geqslant 0)$, where $\mathbf{2}$ is the ordinal category $\bullet \to \bullet$; a transversal map of n-cubes is a natural transformation of such functors.

(b) The weak symmetric cubical category $\omega\mathsf{Cosp}(\mathbf{C})$ *of cubical cospans* has been constructed in [G1] over a category $\mathbf{C}$ with (a fixed choice) of pushouts, in order to deal with higher-dimensional cobordism. An n-cube is a functor $x\colon \wedge^n \to \mathbf{C}$, where $\wedge$ is the formal-cospan category $\bullet \to \bullet \leftarrow \bullet$; again, a transversal map of n-cubes is a natural transformation of such functors.

(c) The weak symmetric cubical category $\omega\mathsf{Span}(\mathbf{C})$ *of cubical spans*, over a category $\mathbf{C}$ with pullbacks, is similarly constructed over the formal-span category $\vee = \wedge^{\mathrm{op}}$, namely $\bullet \leftarrow \bullet \to \bullet$. It is *transversally dual* to $\omega\mathsf{Cosp}(\mathbf{C}^{\mathrm{op}})$.

(d) The weak symmetric cubical category of *cubical bispans*, or *cubical diamonds* $\omega\mathsf{Bisp}(\mathbf{C})$, over a category $\mathbf{C}$ with pullbacks and pushouts, is similarly constructed over a 'formal bispan category' (which is just a 'formal commutative square', but becomes a *formal bispan* when equipped with the obvious structure of a formal interval; see [G1], Section 4.7).

The truncated case is considered below. Other examples treated in [G3] will be investigated later.

3.6 Weak n-tuple categories

As in 2.7, the n-dimensional structure of a *weak n-multiple category*, or *weak n-tuple category*, is obtained by restricting all multi-indices to the subsets of the ordinal $\mathbf{n} = \{0, 1, ..., n-1\}$.

As in 2.7, the 0- and 1-dimensional versions just amount to a set or a category, but the 2-dimensional notion is now a *weak* double category, or *pseudo* double category (as defined in [GP1]). Again we are particularly interested in the 3-dimensional case, a *weak triple category*.

Starting from a weak multiple category A, its (n-dimensional) truncation with multi-indices $\mathbf{i} \subset \mathbf{n}$ gives a weak n-tuple category $\mathrm{trc}_n A = \mathbf{n}\mathsf{A}$.

Thus $\mathbf{3}\mathsf{Cosp}(\mathbf{C})$ is the weak *triple* category of 2-cubical cospans (over a category with pushouts), where the highest-dimensional 'objects' are 2-cubes $x\colon \wedge^2 \to \mathbf{C}$, that is cospans of cospans, but the whole structure - including transversal maps - is 3-dimensional.

Similarly we have the weak n-tuple categories $\mathbf{n}\mathsf{Cosp}(\mathbf{C})$, $\mathbf{n}\mathsf{Span}(\mathbf{C})$ and $\mathbf{n}\mathsf{Bisp}(\mathbf{C})$ of $(n-1)$-dimensional cubical cospans, spans and bispans.

3.7 Chiral multiple categories

A *chiral multiple category*, or χ*-lax multiple category*, is a partially lax extension of a weak multiple category. ('Chiral' refers to something that cannot be superposed to its mirror image.)

This notion is no longer transversally selfdual and has two instances. A *right chiral multiple category* has the same structure and satisfies the same axioms considered above in the weak multiple case, except for the fact that the ij*-interchanger*, for $0 < i < j$

$$\chi_{ij}(x, y, z, u)\colon (x +_i y) +_j (z +_i u) \to_0 (x +_j z) +_i (y +_j u), \qquad (65)$$

is *no longer supposed to be invertible*.

By transversal duality, a *left chiral multiple category* has an interchanger directed the other way round, for $0 < i < j$

$$\chi_{ij}(x, y, z, u)\colon (x +_j z) +_i (y +_j u) \to_0 (x +_i y) +_j (z +_i u), \qquad (66)$$

with the obvious modification of the coherence axioms.

Both structures still have the three kinds of *strict* degenerate interchanges mentioned in 2.4 and 3.1, for $0 < i < j$:

$$e_i e_j = e_j e_i, \quad e_j x +_i e_j y = e_j(x +_i y), \quad e_i(x +_j y) = e_i x +_j e_i y. \qquad (67)$$

As in [GP6, GP7] we generally work in the *right* chiral case, that is just called 'chiral'.

Note that in the truncated n-dimensional case every left chiral n-tuple category can be turned into a right chiral one just by reversing the positive indices, $i \mapsto n - i$; *in this way we avoid resorting to transversal duality, which would turn transversal limits into colimits*. In the infinite dimensional case this only works if we are willing to replace the natural indices with the integral ones, or with any *self-dual* ordered pointed set $N = (N, 0)$.

A *chiral triple category* is the 3-dimensional truncated notion, with multi-indices $\mathbf{i} \subset \{0, 1, 2\}$. Our main example of this kind is the (right) chiral triple category $\mathsf{SC}(\mathbf{C}) = \mathsf{S}_1\mathsf{C}_1(\mathbf{C})$ of *spans and cospans* over a category $\mathbf{C}$ (with pushouts and pullbacks), that will be recalled in the next section, together with other structures of higher dimension. These examples motivate our terminology for the alternative right/left: in the right-hand case limits (i.e. right adjoints) are used in the lower composition laws, *before* colimits, that are used in the upper ones; for instance, pullbacks *before* pushouts in $\mathsf{SC}(\mathbf{C})$.

In Section 5 we shall briefly sketch *interchange categories*, a further generalisation of chiral multiple categories introduced in [GP6, GP7] (in dimension three), where not only χ but also the three strict interchanges listed above are laxified.

3.8 Extending multiple functors

Extending the definitions of the strict case (cf. 2.6), a *multiple functor* $F\colon \mathsf{A} \to \mathsf{B}$ between (say right) *chiral* multiple categories is a morphism of multiple sets which preserves all the composition laws and *also* all comparisons (listed in 3.2):

$$\begin{gathered} F(\lambda_i x) = \lambda_i(Fx), \qquad F(\rho_i x) = \rho_i(Fx), \\ F(\kappa_i(x, y, z)) = \kappa_i(Fx, Fy, Fz), \\ F(\chi_{ij}(x, y, z, u)) = \chi_{ij}(Fx, Fy, Fz, Fu). \end{gathered} \tag{68}$$

A *transversal transformation* $h\colon F \to G\colon \mathsf{A} \to \mathsf{B}$ between multiple functors of chiral multiple categories is a face-consistent family of transversal maps in B (its components), for every positive multi-index $\mathbf{i}$ and every $\mathbf{i}$-cube x in A

$$hx\colon F(x) \to_0 G(x), \qquad\qquad h(\partial_i^\alpha x) = \partial_i^\alpha (hx), \tag{69}$$

subject to the same axioms of naturality and coherence (trt.1, 2) of the strict case (cf. 2.6).

Given two chiral multiple categories A and B we have thus the category $\mathrm{St}\mathbf{Cmc}(\mathsf{A}, \mathsf{B})$ of their (strict) multiple functors and transversal transformations. All these form the 2-category $\mathrm{St}\mathbf{Cmc}$.

3.9 Lax multiple functors and transversal transformations

More generally, a *lax multiple functor* $F\colon \mathsf{A} \to \mathsf{B}$ between (right) chiral multiple categories has components $F_{\mathbf{i}}\colon A_{\mathbf{i}} \to B_{\mathbf{i}}$ (for *all* multi-indices $\mathbf{i}$) that agree with all faces, 0-degeneracies and 0-composition.

Moreover, for every positive multi-index $\mathbf{i}$ and $i \in \mathbf{i}$, F is equipped with i-special comparison $\mathbf{i}$-maps $\underline{F}_i$ that agree with faces

$$\begin{aligned} &\underline{F}_i(x)\colon e_iF(x) \to_0 F(e_ix) && (\text{for } x \text{ in } A_{\mathbf{i}|i}),\\ &\underline{F}_i(x, y)\colon F(x) +_i F(y) \to_0 F(z) && (\text{for } z = x +_i y \text{ in } A_{\mathbf{i}}),\\ &\partial_j^\alpha \underline{F}_i(x) = \underline{F}_i(\partial_j^\alpha x),\ \ \partial_j^\alpha \underline{F}_i(x, y) = \underline{F}_i(\partial_j^\alpha x, \partial_j^\alpha y) && (j \neq i \text{ in } \mathbf{i}). \end{aligned} \tag{70}$$

These comparisons must satisfy the following axioms of naturality and coherence, again for every positive multi-index $\mathbf{i}$ and $i \in \mathbf{i}$.

(lmf.1) (*Naturality of unit comparisons*) For every $\mathbf{i}|i$-map $f\colon x \to_0 y$ in A we have:

$$Fe_i(f).\underline{F}_i(x) = \underline{F}_i(y).e_i(Ff)\colon e_iF(x) \to_0 F(e_iy). \tag{71}$$

(lmf.2) (*Naturality of composition comparisons*) For two i-consecutive $\mathbf{i}$-maps $f\colon x \to_0 x'$ and $g\colon y \to_0 y'$ in A we have:

$$F(f{+_i}g).\underline{F}_i(x, y) = \underline{F}_i(x', y').(Ff{+_i}Fg)\colon Fx{+_i}Fy \to_0 F(x'{+_i}y'). \tag{72}$$

(lmf.3) (*Coherence with unitors*) For an $\mathbf{i}$-cube x with i-faces $\partial_i^- x = z$ and $\partial_i^+ x = w$ (preserved by F) we have two commutative diagrams of $\mathbf{i}$-maps:

$$\begin{array}{ccc} e_i(Fz){+_i}Fx & \xrightarrow{\lambda_i(Fx)} & Fx \\ {\scriptstyle \underline{F}_i(z)+_i1}\downarrow & & \uparrow{\scriptstyle F(\lambda_i x)} \\ F(e_iz){+_i}Fx & \xrightarrow[\underline{F}_i]{} & F(e_iz{+_i}x) \end{array} \qquad \begin{array}{ccc} Fx{+_i}e_i(Fw) & \xrightarrow{\rho_i(Fx)} & Fx \\ {\scriptstyle 1+_i\underline{F}_i(w)}\downarrow & & \uparrow{\scriptstyle F(\rho_i x)} \\ Fx{+_i}F(e_iw) & \xrightarrow[\underline{F}_i]{} & F(x{+_i}e_iw) \end{array} \tag{73}$$

(lmf.4) (*Coherence with associators*) For three i-consecutive **i**-cubes x, y, z in A we have a commutative diagram of **i**-maps:

$$\begin{array}{ccc} Fx +_i (Fy +_i Fz) & \xrightarrow{\kappa_i F} & (Fx +_i Fy) +_i Fz \\ {\scriptstyle 1+_i \underline{F}_i}\downarrow & & \downarrow{\scriptstyle \underline{F}_i +_i 1} \\ Fx +_i F(y +_i z) & & F(x +_i y) +_i Fz \\ {\scriptstyle \underline{F}_i}\downarrow & & \downarrow{\scriptstyle \underline{F}_i} \\ F(x +_i (y +_i z)) & \xrightarrow[F\kappa_i]{} & F((x +_i y) +_i z) \end{array} \qquad (74)$$

(lmf.5) (*Coherence with interchangers*) For $i < j$ in **i** and a consistent matrix of **i**-cubes $\left(\begin{smallmatrix} x & y \\ z & u \end{smallmatrix}\right)$ (with i-consecutive rows and j-consecutive columns) we have a commutative diagram of **i**-maps:

$$\begin{array}{ccc} (Fx +_i Fy) +_j (Fz +_i Fu) & \xrightarrow{\chi_{ij} F} & (Fx +_j Fz) +_i (Fy +_j Fu) \\ {\scriptstyle \underline{F}_i +_j \underline{F}_i}\downarrow & & \downarrow{\scriptstyle \underline{F}_j +_i \underline{F}_j} \\ F(x +_i y) +_j F(z +_i u) & & F(x +_j y) +_i F(z +_j u) \\ {\scriptstyle \underline{F}_j}\downarrow & & \downarrow{\scriptstyle \underline{F}_j} \\ F((x +_i y) +_j (z +_i u)) & \xrightarrow[F\chi_{ij}]{} & F((x +_j z) +_i (y +_j u)) \end{array} \qquad (75)$$

The lax multiple functor F is said to be *unitary* if all its unit comparisons $\underline{F}_i(x)$ are 0-directed identities; only *in this case* F commutes with all degeneracies and is a morphism of multiple sets. The importance of unitarity for lax or colax double functors is discussed in [GP3, GP4].

Lax multiple functors compose, in a categorical way.

A *transversal transformation* $h\colon F \to G\colon \mathsf{A} \to \mathsf{B}$ between lax multiple functors of chiral multiple categories consists of a face-consistent family of **i**-maps in B (its components), one for every positive multi-index **i** and every **i**-cube x in A

$$hx\colon F(x) \to_0 G(x), \qquad\qquad h(\partial_i^\alpha x) = \partial_i^\alpha (hx), \qquad (76)$$

subject to the same naturality axiom (trt.1) of the strict case (cf. 2.6) and an extended *coherence axiom* (trt.2L) that involves the comparison maps of F and G

(trt.1) $Gf.hx = hy.Ff$ (for $f\colon x \to_0 y$ in A),

(trt.2L) for a positive multi-index $\mathbf{i}$, $i \in \mathbf{i}$, $x \in A_{\mathbf{i}|i}$ and $z = x +_i y \in A_{\mathbf{i}}$:

$$h(e_i x)\underline{F}_i(x) = \underline{G}_i(x).e_i(hx)\colon e_i F(x) \to_0 G(e_i x),$$
$$h(z).\underline{F}_i(x, y) = \underline{G}_i(x, y).(hx +_i hy)\colon F(x) +_i F(y) \to_0 G(z).$$

We have now the 2-category $\mathrm{Lx}\mathbf{Cmc}$ of chiral multiple categories, lax multiple functors and their transversal transformations. Similarly one defines the 2-category $\mathrm{Cx}\mathbf{Cmc}$, for the colax case (where the comparisons of 'functors' have the opposite direction). A *pseudo (multiple) functor* is a lax functor whose comparisons are invertible (and is made colax by the inverse comparisons); they are the arrows of the 2-category $\mathrm{Ps}\mathbf{Cmc}$.

In [GP6], Section 6, one can find a complete analysis of the 'functors' that can occur in the 3-dimensional case (in the more general setting of intercategories). Namely:

- a lax triple functor, called a *lax-lax morphism* (because it is lax in directions 1 and 2),
- a colax triple functor, called a *colax-colax morphism*,
- a *colax-lax morphism*, which is colax in direction 1 and lax in direction 2,

while the lax-colax case makes no sense.

4. A chiral triple category of spans and cospans

In this section $\mathbf{C}$ is a category equipped with a choice of pullbacks and pushouts.

The weak double category $\mathbb{S}\mathrm{pan}(\mathbf{C})$, *of arrows and spans* of $\mathbf{C}$, can be 'amalgamated' with the weak double category $\mathbb{C}\mathrm{osp}(\mathbf{C})$, *of arrows and cospans* of $\mathbf{C}$, to form a 3-dimensional structure: the chiral triple category $\mathsf{SC}(\mathbf{C})$ whose 0-, 1- and 2-arrows are the arrows, spans *and* cospans of $\mathbf{C}$, *in this order*. It has been studied in [GP7], Subsection 6.4, with notation $\mathsf{SpanCosp}(\mathbf{C})$.

Interchanging the positive directions one gets the *left* chiral triple category $\mathsf{CS}(\mathbf{C})$ of *cospans and span* of $\mathbf{C}$. Higher dimensional examples are considered in 4.4.

For the sake of simplicity we assume that, in our choices, *the pullback or pushout of an identity along any map is an identity*. Omitting this convention would simply give non-trivial invertible unitors λ and ρ for 1- and 2-composition.

4.1 A triple set with compositions.

We begin by constructing a triple set $\mathsf{A} = \mathsf{SC}(\mathbf{C})$ enriched with composition laws.

(a) The objects of A are those of $\mathbf{C}$; they form the set A_*.

(b) The set A_0 is formed of the maps of $\mathbf{C}$, written as $p\colon X \to_0 Y$ or $p\colon X \to Y$; their composition, written as qp or $q.p$, is that of $\mathbf{C}$ and gives a category.

(b′) The set A_1 consists of the spans of $\mathbf{C}$, written as $f\colon X \to_1 Y$ or

$$(f', f'')\colon (X \leftarrow \bullet \rightarrow Y);$$

their composition, by our fixed choice of pullbacks, will be written as $f +_1 g$. Formally, f is functor $\vee \to \mathbf{C}$ defined on the formal-span category $\vee$ (as in 3.5).

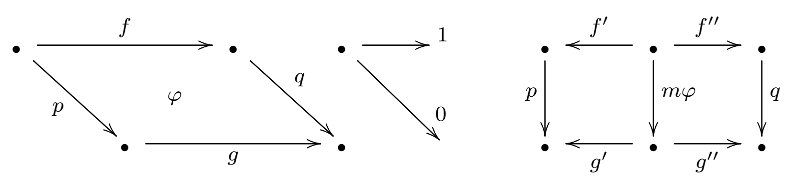
(b″) The set A_2 consists of the cospans of $\mathbf{C}$, written as $u\colon X \to_2 Y$ or

$$(u', u'')\colon (X \rightarrow \bullet \leftarrow Y);$$

their composition, by our fixed choice of pushouts, will be written as $u +_2 v$. Formally, u is functor $\wedge \to \mathbf{C}$ defined on the formal-cospan category $\wedge = \vee^{\mathrm{op}}$.

Each set A_i (for $i = 0, 1, 2$) has thus two faces $\partial_i^\alpha\colon A_i \to A_*$ (implicitly used in the previous composition laws) and a degeneracy $e_i\colon A_* \to A_i$.

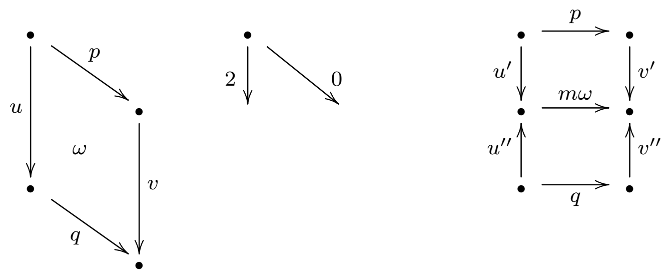
(c) A 01-cell $\varphi \in A_{01}$, as in the left diagram below, is a commutative diagram of $\mathbf{C}$ as in the right diagram below; formally, it is a natural transformation $\varphi\colon f \to g\colon \vee \to \mathbf{C}$



(77)

Their 0-composition, written as $\psi\varphi$, is obvious (that of natural transformations) and gives a category. Their 1-composition, written as $\varphi +_1 \psi$, is computed by two pullbacks in $\mathbf{C}$.

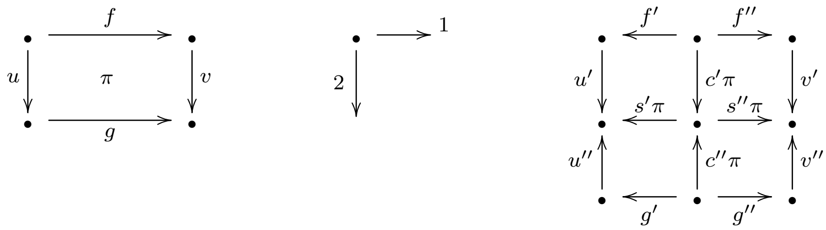
(c′) A 02-cell $\omega \in A_{02}$, as in the left diagram below, is a commutative diagram of $\mathbf{C}$ as in the right diagram below; formally, it is a natural transformation $\omega\colon u \to v\colon \wedge \to \mathbf{C}$



(78)

Their 0-composition, written as $\zeta\omega$, is obvious again and gives a category. Their 2-composition, written as $\omega +_2 \zeta$, is computed by two pushouts in $\mathbf{C}$.

(c″) A 12-cell $\pi \in A_{12}$ is a commutative diagram of $\mathbf{C}$, as at the right hand below, with three spans in direction 1 and three cospans in direction 2; formally, it is a functor $\pi\colon \vee\times\wedge \to \mathbf{C}$



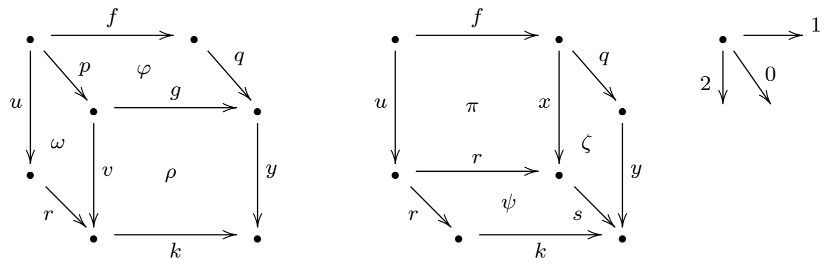
(79)

Their 1-composition, written as $\pi +_1 \rho$, is computed by three pullbacks in $\mathbf{C}$; their 2-composition, written as $\pi +_2 \rho$, by three pushouts in $\mathbf{C}$.

Each set A_{ij} (for $0 \leqslant i < j \leqslant 2$) has two obvious degeneracies and four obvious faces (implicitly used in the composition laws described above)

$$\begin{aligned} &e_i\colon A_j \to A_{ij}, \qquad && e_j\colon A_i \to A_{ij}, \\ &\partial_i^\alpha\colon A_{ij} \to A_j, && \partial_j^\alpha\colon A_{ij} \to A_i. \end{aligned} \tag{80}$$

(d) Finally A_{012} is the set of *triple cells* of $\mathsf{A} = \mathsf{SC}(\mathbf{C})$. Such an item Π is a commutative diagram of $\mathbf{C}$ forming a natural transformation $\Pi\colon \pi \to \rho\colon \vee \times \wedge \to \mathbf{C}$; its boundary consists of two 12-cells π, ρ (its 0-faces), two 01-cells φ, ψ and two 02-cells ω, ζ with consistent boundaries (but Π also has an additional transversal arrow $m\Pi$ between the central objects of π, ρ)



(81)

The set A_{012} has three obvious degeneracies and the six faces described above

$$\begin{array}{lll} e_0\colon A_{12} \to A_{012}, & e_1\colon A_{02} \to A_{012}, & e_2\colon A_{01} \to A_{012}, \\ \partial_0^\alpha\colon A_{012} \to A_{12}, & \partial_1^\alpha\colon A_{012} \to A_{02}, & \partial_2^\alpha\colon A_{012} \to A_{01}. \end{array} \qquad (82)$$

The 0-composition of such cells, written as $\Pi'\Pi$, is obvious (that of natural transformations) and gives a category. Their 1-composition, written as $\Pi +_1 \Pi'$, is computed by six pullbacks in $\mathbf{C}$; their 2-composition, written as $\Pi +_2 \Pi'$, by six pushouts in $\mathbf{C}$.

(e) The sets $A_*, A_0, ..., A_{01}, ..., A_{012}$, with the faces and degeneracies considered above, form a triple set (a 3-dimensional truncated multiple set) with composition laws; the *multiple relations* satisfied by faces and degeneracies are written down in 2.2.

4.2 Comparisons

We have already remarked that 0-directed composition is categorical (on each type). It is also easy to see that it has a strict interchange with the other compositions. Because of our assumption on the choice of pushouts and pullbacks, all 1- or 2-directed composition laws are strictly unitary.

On the other hand, there are *invertible* comparisons for the associativity of 1- and 2-directed composition, and a *directed* comparison for their interchange. The latter is defined for a consistent matrix $\binom{\pi\ \ \pi'}{\rho\ \ \rho'}$ of four 12-cells, and is a 12-special map natural under 0-composition

$$\chi(\pi, \pi', \rho, \rho')\colon (\pi +_1 \pi') +_2 (\rho +_1 \rho') \to_0 (\pi +_2 \rho) +_1 (\pi' +_2 \rho'). \quad (83)$$

All these comparisons are constructed in [GP7], where their coherence is proved.

4.3 Tabulators

As in Section 1, the chiral triple category $\mathsf{SC}(\mathbf{C})$ has all five kinds of tabulators (and cotabulators as well, of course).

The first two, namely the tabulators of 1-arrows and 2-arrows, are already known from the theory of weak double categories.

(a) The tabulator of a 1-arrow f (i.e. a span) is an object $\top_1 f$ with a universal 1-map $e_1(\top_1 f) \to_0 f$; the solution is the (trivial) limit of the span f, i.e. its middle object.

(b) The tabulator of a 2-arrow u (i.e. a cospan) is an object $\top_2 u$ with a universal 2-map $e_2(\top_2 u) \to_0 u$; the solution is the pullback of u.

Then we have three tabulators of a 12-cell π.

(c) The *total tabulator* $\top_{12}\pi$ is an object with a universal 12-map $e_{12}(\top_{12}\pi) \to_0 \pi$; the solution is the limit of the diagram, i.e. the pullback of its middle cospan.

(d) The e_1-tabulator of π is a 2-arrow $\top_1\pi$ (a cospan) with a universal 12-map $e_1(\top_1\pi) \to_0 \pi$; the solution is the middle cospan of π.

(e) The e_2-tabulator of π is a 1-arrow $\top_2\pi$ with a universal 12-map $e_2(\top_2\pi) \to_0 \pi$; the solution is the obvious span whose objects are the pullbacks of the three cospans of π.

Again, these limits are preserved by faces and degeneracies, in a way that will be made precise in Part II; here we just remark that:

- $\partial_1^-(\top_2\pi) = \top_2(\partial_1^-\pi)$, which means that the domain of the span $\top_2\pi$ (described above) is the pullback of the cospan $\partial_1^-\pi$,

- $\top_2(e_1 u) = e_1(\top_2 u)$, i.e. the e_2-tabulator of the 1-degenerate cell $e_1 u$ (on the cospan u) is the degenerate span on the pullback of u.

4.4 Higher dimensional examples

More generally one can form a chiral n-tuple category $\mathsf{S}_p\mathsf{C}_q(\mathbf{C})$ for $p, q > 0$ and $n = p + q + 1$: its non-transversal i-directed arrows are spans of $\mathbf{C}$ for $0 < i \leqslant p$ and cospans of $\mathbf{C}$ for $p < i \leqslant p + q$. Then we have the infinite-dimensional structure $\mathsf{S}_p\mathsf{C}_\infty(\mathbf{C})$.

Similarly we have a *left* chiral n-tuple category $\mathsf{C}_p\mathsf{S}_q(\mathbf{C})$ whose non-transversal i-arrows are cospans of $\mathbf{C}$ for $0 < i \leqslant p$ and spans of $\mathbf{C}$ for $p < i \leqslant p + q$. We also have the left chiral multiple category $\mathsf{C}_p\mathsf{S}_\infty(\mathbf{C})$.

In all these cases the ij-interchanger χ_{ij} is not invertible for $i \leqslant p < j$.

Here $\mathsf{C}_p\mathsf{S}_q(\mathbf{C})$ is transversally dual to $\mathsf{S}_q\mathsf{C}_p(\mathbf{C}^{\mathrm{op}})$, but there can be no relationship of this kind between any $\mathsf{C}_p\mathsf{S}_\infty(\mathbf{C})$ and any $\mathsf{S}_q\mathsf{C}_\infty(\mathbf{C}^{\mathrm{op}})$. To restore symmetry we can consider an 'unbounded' chiral multiple category $\mathsf{S}_{-\infty}\mathsf{C}_\infty(\mathbf{C})$ indexed by the ordered set of *integers*, where i-arrows are spans for $i < 0$, ordinary arrows for $i = 0$ and cospans for $i > 0$. This is transversally dual to the unbounded left chiral multiple category $\mathsf{C}_{-\infty}\mathsf{S}_\infty(\mathbf{C}^{\mathrm{op}})$.

5. A sketch of infinite-dimensional intercategories

Three-dimensional *intercategories*, introduced and studied in [GP6, GP7], generalise the notion of chiral triple category by replacing all *strict* or *weak* interchangers with *lax interchangers* of four types $(\tau, \mu, \delta, \chi)$, which deal with the four possible cases of zero-ary or binary composition in the *positive* directions 1, 2.

The difference can be better appreciated noting that a 3-dimensional intercategory is a pseudo category in the 2-category of weak double categories, lax double functors and horizontal transformations (see [GP6], Section 3), while a chiral triple category is a *unitary* pseudo category in the 2-category of weak double categories, *unitary* lax double functors and horizontal transformations.

Here we extend the definition of intercategories to the infinite dimensional case.

5.1 Intercategories.

An (infinite-dimensional, right) *intercategory* A is a kind of lax multiple category where, with respect to the notion of a chiral multiple category, we replace the three strict interchanges listed in (67) with lax interchangers.

Now, for any two *positive* directions $i < j$, we have the following families of ij-special maps (including χ_{ij}, already present in the chiral case):

(a) $\tau_{ij}(x)\colon e_je_i(x) \to_0 e_ie_j(x)$ (ij*-interchanger for identities*),

(b) $\mu_{ij}(x, y)\colon e_i(x) +_j e_i(y) \to_0 e_i(x +_j y)$
(ij*-interchanger for* i*-identities and* j*-composition* on j-consecutive cubes),

(c) $\delta_{ij}(x, y)\colon e_j(x +_i y) \to_0 e_j(x) +_i e_j(y)$
(ij*-interchanger for* i*-composition and* j*-identities* on i-consecutive cubes),

(d) $\chi_{ij}(x, y, z, u)\colon (x +_i y) +_j (z +_i u) \to_0 (x +_j z) +_i (y +_j u)$
(ij*-interchanger for binary compositions* on a consistent matrix of cubes).

All these maps must be natural for transversal maps. The coherence axioms stated in 3.3 are required. Furthermore there are coherence conditions for the interchangers, stated below in 5.2 and 5.3.

The transversally dual notion of a *left intercategory* has interchangers in the opposite direction.

Various 'anomalies' appear, with respect to the chiral case, that make problems for a theory of multiple limits in this setting. First, A *is no longer a multiple set* (unless each τ_{ij} is the identity). Second, a degeneracy e_i $(i > 0)$ is now *lax* with respect to every higher j-composition (for $j > i$, via τ_{ij} and μ_{ij}) but *colax* with respect to every lower j-composition (for $0 < j < i$, via τ_{ji} and δ_{ji}). Therefore, in the truncated n-dimensional case e_1 is lax with respect to all other compositions and e_n is colax, but the other positive degeneracies (if any, i.e. for $n > 3$) are neither lax nor colax.

5.2 Lower coherence axioms for the interchangers

We now list, here and in 5.3, the conditions involving the interchangers.

The following diagrams of transversal maps must commute (under 0-composition), assuming that $0 < i < j$ (and that the cubes $x, y...$ are consistent with the operations acting on them). We write $\tau = \tau_{ij}$, $\mu = \mu_{ij}$, $\delta = \delta_{ij}$, $\chi = \chi_{ij}$.

(i) *Coherence hexagon of* $\chi = \chi_{ij}$ *and* κ_i: see (wmc.9.iii) in 3.4.

(ii) *Coherence hexagon of* $\chi = \chi_{ij}$ *and* κ_j: see (wmc.9.iv).

(iii) *Coherence hexagon of* $\delta = \delta_{ij}$ *and* κ_i

$$\begin{array}{ccc}
e_j(x +_i (y +_i z)) & \xrightarrow{\;e_j\kappa_i\;} & e_j((x +_i y) +_i z) \\
{\scriptstyle\delta}\Big\downarrow & & \Big\downarrow{\scriptstyle\delta} \\
e_j(x) +_i e_j(y +_i z) & & e_j(x +_i y) +_i e_j(z) \\
{\scriptstyle 1+_i\delta}\Big\downarrow & & \Big\downarrow{\scriptstyle \delta+_i 1} \\
e_j(x) +_i (e_j(y) +_i e_j(z)) & \xrightarrow{\;\kappa_i e_j\;} & (e_j(x) +_i e_j(y)) +_i e_j(z))
\end{array} \tag{84}$$

(iv) *Coherence hexagon of* $\mu = \mu_{ij}$ *and* κ_j

$$\begin{array}{ccc}
e_i(x) +_j (e_i(y) +_j e_i(z)) & \xrightarrow{\;\kappa_j e_i\;} & (e_i(x) +_j e_i(y)) +_j e_i(z) \\
{\scriptstyle 1+_j\mu}\Big\downarrow & & \Big\downarrow{\scriptstyle \mu+_j 1} \\
e_i(x) +_j e_i(y +_j z) & & e_i(x +_j y) +_j e_i(z) \\
{\scriptstyle\mu}\Big\downarrow & & \Big\downarrow{\scriptstyle\mu} \\
e_i(x +_j (y +_j z)) & \xrightarrow{\;e_i\kappa_j\;} & e_i((x +_j y) +_j z)
\end{array} \tag{85}$$

(v) *Coherence laws for* χ, μ, λ_i *and* χ, μ, ρ_i

$$\begin{array}{ccccc}
(e_i\partial_i^- x +_i x) +_j (e_i\partial_i^- y +_i y) & \xrightarrow{\;\lambda_i +_j \lambda_i\;} & x +_j y & \xleftarrow{\;\rho_i +_j \rho_i\;} & (x +_i e_i\partial_i^+ x) +_j (y +_i e_i\partial_i^+ y) \\
{\scriptstyle\chi}\Big\downarrow & & \Big\Vert & & \Big\downarrow{\scriptstyle\chi} \\
(e_i\partial_i^- x +_j e_i\partial_i^- y) +_i (x +_j y) & & \Big\Vert & & (x +_j y) +_i (e_i\partial_i^+ x +_j e_i\partial_i^+ y) \\
{\scriptstyle\mu +_i 1}\Big\downarrow & & \Big\Vert & & \Big\downarrow{\scriptstyle 1 +_i \mu} \\
e_i\partial_i^-(x +_j y) +_i (x +_j y) & \xrightarrow[\;\lambda_i\;]{} & x +_j y & \xleftarrow[\;\rho_i\;]{} & (x +_j y) +_i e_i\partial_i^+(x +_j y)
\end{array} \tag{86}$$

(vi) *Coherence laws for* χ, δ, λ_j *and* χ, δ, ρ_j

$$\begin{array}{ccccc}
e_j\partial_j^-(x +_i y) +_j (x +_i y) & \xrightarrow{\;\lambda_j\;} & x +_i y & \xleftarrow{\;\rho_j\;} & (x +_i y) +_j e_j\partial_j^+(x +_i y) \\
{\scriptstyle\delta +_j 1}\Big\downarrow & & \Big\Vert & & \Big\downarrow{\scriptstyle 1 +_j \delta} \\
(e_j\partial_j^- x +_i e_j\partial_j^- y) +_j (x +_i y) & & \Big\Vert & & (x +_i y) +_j (e_j\partial_j^+ x +_i e_j\partial_j^+ y) \\
{\scriptstyle\chi}\Big\downarrow & & \Big\Vert & & \Big\downarrow{\scriptstyle\chi} \\
(e_j\partial_j^- x +_j x) +_i (e_j\partial_j^- y +_j y) & \xrightarrow[\;\lambda_j +_i \lambda_j\;]{} & x +_i y & \xleftarrow[\;\rho_j +_i \rho_j\;]{} & (x +_j e_j\partial_j^+ x) +_i (y +_j e_j\partial_j^+ y)
\end{array} \tag{87}$$

(vii) *Coherence laws for* δ, τ, λ_i *and* δ, τ, ρ_i

$$
\begin{CD}
e_j((e_i\partial_i^- x) +_i x) @>{e_j\lambda_i}>> e_j(x) @<{e_j\rho_i}<< e_j(x +_i (e_i\partial_i^- x)) \\
@V{\delta}VV @| @VV{\delta}V \\
e_je_i\partial_i^-(x) +_i e_j(x) @. @. e_j(x) +_i e_je_i\partial_i^-(x) \\
@V{\tau +_i 1}VV @| @VV{1 +_i \tau}V \\
e_i\partial_i^- e_j(x) +_i e_j(x) @>>{\lambda_i e_j}> e_j(x) @<<{\rho_i e_j}< e_j(x) +_i e_i\partial_i^- e_j(x)
\end{CD}
\tag{88}
$$

(viii) *Coherence laws for* μ, τ, λ_j *and* μ, τ, ρ_j

$$
\begin{CD}
e_j\partial_j^- e_i(x) +_j e_i(x) @>{\lambda_j e_i}>> e_i(x) @<{\rho_j e_i}<< e_i(x) +_j e_j\partial_j^- e_i(x) \\
@V{\tau +_j 1}VV @| @VV{1 +_j \tau}V \\
e_ie_j\partial_j^-(x) +_j e_i(x) @. @. e_i(x) +_j e_ie_j\partial_j^-(x) \\
@V{\mu}VV @| @VV{\mu}V \\
e_i((e_j\partial_j^- x) +_j x) @>>{e_i\lambda_j}> e_i(x) @<<{e_i\rho_j}< e_i(x +_j (e_j\partial_j^- x))
\end{CD}
\tag{89}
$$

5.3 Higher coherence axioms

Finally we list the coherence conditions involving three interchangers and three (positive) directions at a time. These axioms vanish in the three dimensional case, where we only have two positive indices. The first condition has already been considered as axiom (wmc.9.vii) of weak (and chiral) multiple categories, in 3.4, but we rewrite it here as a guideline for the others.

Again, the following diagrams of transversal maps must commute, for $0 < i < j < k$, assuming that all the positive compositions make sense.

(i) (Case $2 \times 2 \times 2$) *Coherence hexagon of the interchangers* χ_{ij}, χ_{jk} *and* χ_{ik}

for a consistent $2\times 2\times 2$ matrix of $\mathbf{i}$-cubes

$$\begin{array}{l}
((x+_iy)+_j(z+_iu))+_k((x'+_iy')+_j(z'+_iu')) \\
\quad \downarrow \chi_{jk} \\
((x+_iy)+_k(x'+_iy'))+_j((z+_iu)+_k(z'+_iu')) \\
\quad \downarrow \chi_{ik}+_j\chi_{ik} \\
((x+_kx')+_i(y+_ky'))+_j((z+_kz')+_i(u+_ku')) \\
\quad \downarrow \chi_{ij} \\
((x+_kx')+_j(z+_kz'))+_i((y+_ky')+_j(u+_ku'))
\end{array}
\qquad
\begin{array}{l}
((x+_iy)+_j(z+_iu))+_k((x'+_iy')+_j(z'+_iu')) \\
\quad \downarrow \chi_{ij}+_k\chi_{ij} \\
((x+_jz)+_i(y+_ju))+_k((x'+_jz')+_i(y'+_ju')) \\
\quad \downarrow \chi_{ik} \\
((x+_jz)+_k(x'+_jz'))+_i((y+_ju)+_k(y'+_ju')) \\
\quad \downarrow \chi_{jk}+_i\chi_{jk} \\
((x+_kx')+_j(z+_kz'))+_i((y+_ky')+_j(u+_ku'))
\end{array}
\qquad (90)$$

(ii) (Case $0\times 2\times 2$) *Coherence hexagon of the interchangers χ_{jk}, μ_{ij} and μ_{ik}* for a 2×2 matrix of $\mathbf{i}|i$-cubes

$$\begin{array}{ccc}
(e_ix+_je_iy)+_k(e_iz+_je_iu) & \xrightarrow{\chi_{jk}e_i} & (e_ix+_ke_iz)+_j(e_iy+_ke_iu) \\
\downarrow{\scriptstyle \mu_{ij}+_k\mu_{ij}} & & \downarrow{\scriptstyle \mu_{ik}+_j\mu_{ik}} \\
e_i(x+_jy)+_ke_i(z+_ju) & & e_i(x+_kz)+_je_i(y+_ku) \\
\downarrow{\scriptstyle \mu_{ik}} & & \downarrow{\scriptstyle \mu_{ij}} \\
e_i((x+_jy)+_k(z+_ju)) & \xrightarrow[e_i\chi_{jk}]{} & e_i((x+_kz)+_j(y+_ku))
\end{array} \qquad (91)$$

(iii) (Case $2\times 0\times 2$) *Coherence hexagon of the interchangers χ_{ik}, μ_{jk} and δ_{ij}* for a 2×2 matrix of $\mathbf{i}|j$-cubes

$$\begin{array}{ccc}
e_j(x+_iy)+_ke_j(z+_iu) & \xrightarrow{\delta_{ij}+_k\delta_{ij}} & (e_jx+_ie_jy)+_k(e_jz+_ie_ju) \\
\downarrow{\scriptstyle \mu_{jk}} & & \downarrow{\scriptstyle \chi_{ik}e_j} \\
e_j((x+_iy)+_k(z+_iu) & & (e_jx+_ke_jz)+_i(e_jy+_ke_ju) \\
\downarrow{\scriptstyle e_j\chi_{ik}} & & \downarrow{\scriptstyle \mu_{jk}+_i\mu_{jk}} \\
e_j((x+_kz)+_i(y+_ju)) & \xrightarrow[\delta_{ij}]{} & e_j(x+_kz)+_ie_j(y+_ju))
\end{array} \qquad (92)$$

(iv) (Case $2\times2\times0$) *Coherence hexagon of the interchangers* χ_{ij}, δ_{ik} *and* δ_{jk} for a 2×2 matrix of $\mathbf{i}|k$-cubes

$$\begin{array}{ccc} e_k((x+_i y)+_j(z+_i u)) & \xrightarrow{e_k\chi_{ij}} & e_k((x+_j z)+_i(y+_j u)) \\ \downarrow{\scriptstyle \delta_{jk}} & & \downarrow{\scriptstyle \delta_{ik}} \\ e_k(x+_i y)+_j e_k(z+_i u) & & e_k(x+_j z)+_i e_k(y+_j u) \\ \downarrow{\scriptstyle \delta_{ik}+_j\delta_{ik}} & & \downarrow{\scriptstyle \delta_{jk}+_i\delta_{jk}} \\ (e_kx+_i e_ky)+_j(e_kz+_i e_ku) & \xrightarrow[\chi_{ij}e_k]{} & (e_kx+_j e_kz)+_i(e_ky+_j e_ku) \end{array} \tag{93}$$

(v) (Case $0\times0\times2$) *Coherence hexagon of the interchangers* τ_{ij}, μ_{ij} *and* μ_{ik} for a pair of cubes indexed by $\mathbf{i}\setminus\{i,j\}$

$$\begin{array}{ccc} e_je_ix+_k e_je_iy & \xrightarrow{\tau_{ij}+_k\tau_{ij}} & e_ie_jx+_k e_ie_jy \\ \downarrow{\scriptstyle \mu_{jk}e_i} & & \downarrow{\scriptstyle \mu_{ik}e_j} \\ e_j(e_ix+_k e_iy) & & e_i(e_jx+_k e_jy) \\ \downarrow{\scriptstyle e_j\mu_{ik}} & & \downarrow{\scriptstyle e_i\mu_{jk}} \\ e_je_i(x+_k y) & \xrightarrow[\tau_{ij}]{} & e_ie_j(x+_k y) \end{array} \tag{94}$$

(vi) (Case $0\times2\times0$) *Coherence hexagon of the interchangers* τ_{ik}, δ_{jk} *and* μ_{ij}

$$\begin{array}{ccc} e_k(e_ix+_j e_iy) & \xrightarrow{\delta_{jk}e_i} & e_ke_ix+_j e_ke_iy \\ \downarrow{\scriptstyle e_k\mu_{ij}} & & \downarrow{\scriptstyle \tau_{ik}+_j\tau_{ik}} \\ e_ke_i(x+_j y) & & e_ie_kx+_j e_ie_ky \\ \downarrow{\scriptstyle \tau_{ik}} & & \downarrow{\scriptstyle \mu_{ij}e_k} \\ e_ie_k(x+_j y) & \xrightarrow[e_i\delta_{jk}]{} & e_i(e_kx+_j e_ky) \end{array} \tag{95}$$

(vii) (Case $2{\times}0{\times}0$) *Coherence hexagon of the interchangers* τ_{jk}, δ_{ik} *and* δ_{jk}

$$\begin{array}{ccc} e_ke_j(x+_i y) & \xrightarrow{\tau_{jk}} & e_je_k(x+_i y) \\ {\scriptstyle e_k\delta_{ij}}\downarrow & & \downarrow{\scriptstyle e_j\delta_{ik}} \\ e_k(e_jx+_i e_jy) & & e_j(e_kx+_i e_ky) \\ {\scriptstyle \delta_{ik}e_j}\downarrow & & \downarrow{\scriptstyle \delta_{ij}e_k} \\ e_ke_jx+_i e_ke_jy & \xrightarrow[\tau_{jk}+_i\tau_{jk}]{} & e_je_kx+_i e_je_ky \end{array} \qquad (96)$$

(viii) (Case $0{\times}0{\times}0$) *Coherence hexagon of the interchangers* τ_{ij}, τ_{ik} *and* τ_{jk}

$$\begin{array}{ccc} e_ke_je_ix & \xrightarrow{\tau_{jk}e_i} & e_je_ke_ix \\ {\scriptstyle e_k\tau_{ij}}\downarrow & & \downarrow{\scriptstyle e_j\tau_{ik}} \\ e_ke_ie_jx & & e_je_ie_kx \\ {\scriptstyle \tau_{ik}e_j}\downarrow & & \downarrow{\scriptstyle \tau_{ij}e_k} \\ e_ie_ke_jx & \xrightarrow[e_i\tau_{jk}]{} & e_ie_je_kx \end{array} \qquad (97)$$

(This is a Moore relation for transpositions, in the symmetric group S_3.)

5.4 Duoidal categories

As proved in [GP7] Section 2.1, a triple intercategory on a single object, with trivial arrows in all directions and trivial 01- and 02-cells is the same as a duoidal category, with objects and morphisms given by 12-cubes and 12-maps, respectively.

A duoidal category A is thus a category equipped with two monoidal structures $(+_i, e_i, \kappa_i, \lambda_i, \rho_i)$ that are linked by four 12-interchangers

(a) $\tau\colon e_2 \to_0 e_1$ (*interchanger for identities*),

(b) $\mu\colon e_1 +_2 e_1 \to_0 e_1$ (*interchanger for 1-identities and 2-product*),

(c) $\delta\colon e_2 \to_0 e_2 +_1 e_2$ (*interchanger for 1-product and 2-identities*),

(d) $\chi(x, y, z, u)\colon (x +_1 y) +_2 (z +_1 u) \to_0 (x +_2 z) +_1 (y +_2 u)$ (*interchanger for products*).

As a basic example, we have seen in [GP7] that any category $\mathbf{C}$ with finite products and sums has a structure of duoidal category

$$\mathsf{Fps}(\mathbf{C}) = (\mathbf{C}, \times, \top, +, \bot)$$

given by (a choice of) these operations, in this order. The comparison are obvious.

In particular the canonical morphism $\tau\colon \bot \to \top$ is invertible if and only if $\mathbf{C}$ has a zero object. In this case we can adopt the same choice $\top = 0 = \bot$ for the terminal and initial object, so that $\mathsf{Fps}(\mathbf{C})$ becomes a substructure of the chiral triple category $\mathsf{SC}(\mathbf{C})$ on the object 0 (Section 4). Therefore $\mathsf{Fps}(\mathbf{C})$ is chiral itself: also μ and δ are identities (an obvious fact, actually).

Finally, if $\mathbf{C}$ is semiadditive finite products and sums coincide and we simply have a monoidal structure on $\mathbf{C}$.

It is interesting to note that in the (non-chiral) duoidal category $\mathsf{Fps}(\mathbf{Set})$ the interchanger $\delta\colon \varnothing \to \varnothing + \varnothing$ is trivial while the other interchangers are not invertible:

$$\begin{gathered} \tau\colon \varnothing \to 1, \qquad\qquad \mu\colon 1+1 \to 1, \\ \chi\colon (X \times Y) + (Z \times U) \to (X+Z) \times (Y+U). \end{gathered} \tag{98}$$

In $\mathsf{Fps}(\mathbf{Set} \times \mathbf{Set}^{\mathrm{op}})$ all the four interchangers are not invertible.

5.5 Other examples of intercategories

Many examples are considered in [GP7]. Here we only recall, from Section 6 therein, that starting from a weak double category $\mathbb{D}$ *with a lax choice of 1-dimensional pullbacks* (as defined in [GP1]), one can construct a 3-dimensional intercategory $\mathsf{Span}(\mathbb{D})$ with:

- objects and vertical arrows as in $\mathbb{D}$,
- the horizontal maps of $\mathbb{D}$ as transversal arrows,
- spans of horizontal maps of $\mathbb{D}$ as horizontal arrows.

As analysed in [GP7], this intercategory is chiral as soon as we have in $\mathbb{D}$ *a lax choice of double pullbacks* (including the two-dimensional universal property). In fact, τ and μ are always degenerate while δ is degenerate whenever the choice of pullbacks in $\mathbb{D}$ is preserved by vertical identities, as in all the examples of [GP1] based on profunctors, spans or cospans. (Note that if $\mathbb{D}$ has cotabulators, then its vertical degeneracy has a left adjoint and must preserve the existing limits.)

If $\mathbb{D} = \mathbb{S}\mathrm{pan}(\mathbf{C})$ is the weak double category of spans over a category $\mathbf{C}$ with (a choice of) pullbacks, then the intercategory $\mathsf{Span}\mathbb{S}\mathrm{pan}(\mathbf{C})$ is the weak symmetric 3-cubical category *of spans of spans of* $\mathbf{C}$, already recalled as $\mathbf{3}\mathsf{Span}(\mathbf{C})$ in 3.6.

On the other hand, if $\mathbb{D} = \mathbb{C}\mathrm{osp}(\mathbf{C})$ is the weak double category of cospans over a category $\mathbf{C}$ with pullbacks and pushouts, then $\mathsf{Span}\mathbb{C}\mathrm{osp}(\mathbf{C})$ $(= \mathsf{Cosp}\mathbb{S}\mathrm{pan}(\mathbf{C}))$ is the chiral triple category $\mathsf{SC}(\mathbf{C})$ of spans and cospans of $\mathbf{C}$ studied here in Section 4.

5.6 Comments on lax multiple categories

We end this section by remarking that the term 'lax multiple category' can cover various 'kinds' of laxity, where - with respect to a weak multiple category - some comparisons are still invertible while others (even some strict ones!) acquire a particular direction depending on the kind we are considering.

Thus, an intercategory is a particular type of 'interchange-lax' multiple category, which is not even a multiple set: the positive degeneracies need not commute.

Other kinds have already appeared in [G2], where $\omega\mathsf{COSP}(\mathbf{Top})$ denotes the 'symmetric quasi cubical category' of higher cospans of topological spaces composed with *homotopy pushouts* (which is of interest for higher cobordism, because homotopy pushouts are homotopy invariant while ordinary pushouts are not).

For the sake of simplicity, let us replace $\mathbf{Top}$ with a more regular 2-dimensional structure: let $\mathbf{C}$ be a 2-category with (a fixed choice of) *pseudo-pushouts*. Then we can modify $\omega\mathsf{Cosp}(\mathbf{C})$, the weak symmetric cubical category recalled in 3.5, *by composing cubical cospans with pseudo-pushouts*. We obtain a kind of lax symmetric cubical category $\omega\mathsf{COSP}(\mathbf{C})$ where all comparisons are invertible except the unitors, directed as

$$\lambda_i(x)\colon e_i(\partial_i^- x) +_i x \to x, \qquad \rho_i(x)\colon x +_i e_i(\partial_i^+ x) \to x. \qquad (99)$$

Dually, if the 2-category $\mathbf{C}$ has (a fixed choice of) pseudo-pullbacks, we can form a structure $\omega\mathsf{SPAN}(\mathbf{C})$ by composing cubical spans with pseudo-pullbacks; we get a different kind of laxity, where unitors are directed the other way round with respect to (99).

6. Tabulators in a 3-dimensional intercategory

We end by constructing an example of a 3-dimensional intercategory with $e_1e_2 \neq e_2e_1$, where a 12-cube π can have two different total tabulators $\top_{12}\pi$ and $\top_{21}\pi$. This example is rather artificial, but *non*-degenerate intercategories seem to be difficult to build, while there are important examples of *degenerate* intercategories, like duoidal categories; of course the latter (having a single object) lack tabulators.

6.1 An intercategory

Let us start from the chiral triple category of spans and cospans $\mathsf{A} = \mathsf{SC}(\mathbf{C})$, studied in Section 4.

We recall that the category $\mathbf{C}$ is equipped with a choice of pullbacks and pushouts that preserves identities. We also assume that $\mathbf{C}$ has a (chosen) initial object 0, and therefore all finite colimits; furthermore, we assume that every morphism $u\colon 0 \to X$ is mono (which fails in $\mathbf{Set}^{\mathrm{op}}$, for instance) and the chosen pullback of (u, u) is precisely 0.

We now restrict the items of A, so that the only remaining 1-arrows are the *null spans* $X \leftarrow 0 \to Y$. We can thus form an intercategory B that is not a substructure of A and is no longer chiral: it has a different e_1 and its interchangers τ, μ are directed - while δ stays degenerate.

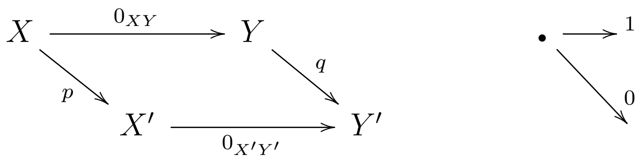
(a) B_*, B_0, B_2 and B_{02} coincide, respectively, with A_*, A_0, A_2, A_{02}, and have the same composition laws in direction 0 and 2.

(b) The subset $B_1 \subset A_1$ of the new 1-cells consists of the null spans $(X \leftarrow 0 \to Y)$ of $\mathbf{C}$, also written as $0_{XY}\colon X \to_1 Y$; they compose as in A (by our assumptions on the initial object) but have different identities (as the old ones do not belong to B_1)

$$0_{XY} +_1 0_{YZ} = 0_{XZ}, \qquad\qquad e_1(X) = 0_{XX} = (X \leftarrow 0 \to X).$$

This forms a category, isomorphic to the codiscrete category on the objects of $\mathbf{C}$.

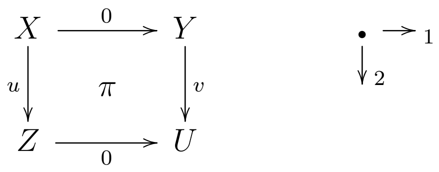
(c) A 01-cell in A_{01} amounts to an arbitrary pair (p, q) of morphisms of $\mathbf{C}$



(100)

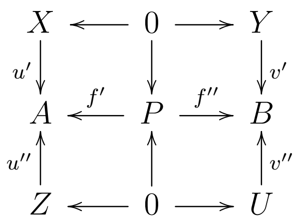
Their 0-composition is obvious and gives a category isomorphic to $\mathbf{C}\times\mathbf{C}$. Their 1-composition is that of the codiscrete category on the morphisms of $\mathbf{C}$, namely $(p, q) +_1 (q, r) = (p, r)$.

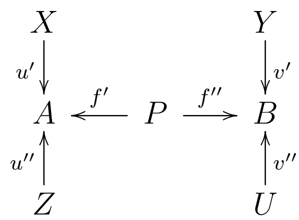
(c′) A 12-cell π of $B_{12} \subset A_{12}$ is a cell of A_{12} whose 1-arrows are null spans and 2-arrows are arbitrary cospans



(101)

as in the left diagram below (automatically commutative in $\mathbf{C}$)





It amounts to a triple (u, f, v) containing two cospans (u and v) and a span (f) as in the right diagram above. The 1- and 2-composition of these 12-cells are as in A (computed by pushouts and pullbacks, respectively), with the same associators. In particular

$$(u, f, v) +_1 (v, g, w) = (u, f +_1 g, w). \qquad (102)$$

The degeneracy $e_2 \colon B_1 \to B_{12}$ is the restriction of that of A, and sends a null span 0_{XY} to the obvious 12-cell with three degenerate cospans (on X, 0 and Y). The other degeneracy is different from that of A

$$\begin{gathered} e_2\colon B_1 \to B_{12}, \qquad e_1\colon B_2 \to B_{12}, \\ e_2(0_{XY}) = e_2(X \leftarrow 0 \to Y) = ((1_X, 1_X), 0_{XY}, (1_Y, 1_Y)), \\ e_1(u) = e_1(X \to A \leftarrow Y) = (u, (1_A, 1_A), u), \end{gathered} \tag{103}$$

$$\begin{array}{ccccc} X & & & & Y \\ {\scriptstyle 1}\downarrow & & & & \downarrow{\scriptstyle 1} \\ X & \longleftarrow & 0 & \longrightarrow & Y \\ {\scriptstyle 1}\uparrow & & & & \uparrow{\scriptstyle 1} \\ X & & & & Y \end{array} \qquad\qquad \begin{array}{ccccc} X & & & & Y \\ {\scriptstyle u'}\downarrow & & & & \downarrow{\scriptstyle u'} \\ A & \xleftarrow{1} & A & \xrightarrow{1} & A \\ {\scriptstyle u''}\uparrow & & & & \uparrow{\scriptstyle u''} \\ Y & & & & Y \end{array}$$

The interchangers are defined as follows. Firstly, δ is trivial

$$\delta(0_{XY}, 0_{YZ})\colon e_2(0_{XY} +_1 0_{YZ}) \to e_2(0_{XY}) +_1 e_2(0_{YZ}), \tag{104}$$

namely the identity of the 12-cell $((1_X, 1_X), 0_{XZ}, (1_X, 1_X))$.

Secondly, μ amounts to the canonical morphism $h\colon A + B \to C$, where A, B and $C = A +_Y B$ are the central objects of the 2-arrows u, v and $u +_2 v$, respectively:

$$\begin{gathered} \mu(u, v)\colon e_1(u) +_2 e_1(v) \to e_1(u +_2 v), \\ (u +_2 v, (h, h), u +_2 v) \to (u +_2 v, (1_C, 1_C), u +_2 v). \end{gathered} \tag{105}$$

Thirdly, χ is the (non-invertible) restriction of the binary interchanger of A; also τ is not invertible (in general)

$$\begin{gathered} \tau(X)\colon e_2 e_1(X) \to e_1 e_2(X), \\ ((1_X, 1_X), 0_{XX}, (1_X, 1_X)) \to ((1_X, 1_X), (1_X, 1_X), (1_X, 1_X)), \end{gathered} \tag{106}$$

$$\begin{array}{ccccc} X & & & & X \\ {\scriptstyle 1}\downarrow & & & & \downarrow{\scriptstyle 1} \\ X & \longleftarrow & 0 & \longrightarrow & X \\ {\scriptstyle 1}\uparrow & & & & \uparrow{\scriptstyle 1} \\ X & & & & X \end{array} \qquad\qquad \begin{array}{ccccc} X & & & & X \\ {\scriptstyle 1}\downarrow & & & & \downarrow{\scriptstyle 1} \\ X & \xleftarrow{1} & X & \xrightarrow{1} & X \\ {\scriptstyle 1}\uparrow & & & & \uparrow{\scriptstyle 1} \\ X & & & & X \end{array}$$

(d) Finally B_{012} is the set of triple cells of A whose 1-arrows are null spans. They compose as in A.

6.2 Tabulators

Let us suppose now that the category $\mathbf{C}$ also has a terminal object (and therefore all finite limits). Then our example has all kinds of tabulators; here there are *six* forms instead of five, because e_1 and e_2 do not commute.

(a) The tabulator of a 1-arrow $f = 0_{XY}$ (i.e. a null span) is an object $T = \top_1 f$ with a universal 1-map $e_1(T) = 0_{TT} \to_0 f$; the solution is the product $X \times Y$ in $\mathbf{C}$.

(b) The tabulator of a 2-arrow u (a cospan) is an object $\top_2 u$ with a universal 2-map $e_2(\top_2 u) \to_0 u$; the solution is the pullback of u in $\mathbf{C}$.

After these, we have four tabulators for the 12-cube $\pi = (u, f, v)$ of (101).

(c) The *total* e_1e_2*-tabulator* $T = \top_{12}\pi$ is an object with a universal 12-map $e_1e_2(T) \to_0 \pi$, where $e_1e_2(T) = ((1_T, 1_T), (1_T, 1_T), (1_T, 1_T))$ (see (106)). The solution is the limit in $\mathbf{C}$ of π, viewed as the left diagram below

$$\begin{array}{ccccc} X & & & & Y \\ {\scriptstyle u'}\downarrow & & & & \downarrow{\scriptstyle v'} \\ A & \xleftarrow{f'} & P & \xrightarrow{f''} & B \\ {\scriptstyle u''}\uparrow & & & & \uparrow{\scriptstyle v''} \\ Z & & & & U \end{array} \qquad\qquad \begin{array}{ccc} X & & Y \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle v'} \\ A & & B \\ {\scriptstyle u''}\uparrow & & \uparrow{\scriptstyle v''} \\ Z & & U \end{array}$$

(c′) The *total* e_2e_1*-tabulator* $T = \top_{21}\pi$ is an object with a universal 12-map $e_2e_1(T) \to_0 \pi$, where $e_2e_1(T) = ((1_T, 1_T), 0_{TT}, (1_T, 1_T))$ (see (106)). The solution is the limit in $\mathbf{C}$ of the right diagram above, namely the product $\top u \times \top v$ of two pullbacks.

(d) The e_1*-tabulator* is a 2-arrow (a cospan) $\top_1\pi$ with a universal 12-map $e_1(\top_1\pi) \to_0 \pi$; the solution, as in the left diagram below, is the cospan $z = (L' \to P \leftarrow L'')$, where $L' = \lim(u', f', f'', v')$ is the limit of the upper part of π and $L'' = \lim(u'', f', f'', v'')$ is the limit of its lower part.

$$\begin{array}{ccccc} L' & & & & L' \\ {\scriptstyle z'}\downarrow & & & & \downarrow{\scriptstyle z'} \\ P & \xleftarrow{\;1\;} & P & \xrightarrow{\;1\;} & P \\ {\scriptstyle z''}\uparrow & & & & \uparrow{\scriptstyle z''} \\ L'' & & & & L'' \end{array} \qquad\qquad \begin{array}{ccccc} S & & & & T \\ {\scriptstyle 1}\downarrow & & & & \downarrow{\scriptstyle 1} \\ S & \longleftarrow & 0 & \longrightarrow & T \\ {\scriptstyle 1}\uparrow & & & & \uparrow{\scriptstyle 1} \\ S & & & & T \end{array} \qquad (107)$$

(e) The e_2-*tabulator* of π is a 1-arrow $\top_2\pi = 0_{ST}$ (a null span) with a universal 12-map $e_2(\top_2\pi) \to_0 \pi$; the solution, as in the right diagram above, is the null span $(S \leftarrow 0 \to T)$ on the pullbacks of the two cospans $\partial_1^\alpha\pi$, namely $S = \top_2 u$ and $T = \top_2 v$.

These limits are *only partially* preserved by faces and degeneracies, in the following sense.

- $\partial_2^\alpha(\top_1\pi)$ *need not coincide with* $\top_1(\partial_2^\alpha\pi)$. For instance, for $\alpha = 0$, the domain L' of the cospan $\top_1\pi$ (described above) need not be the product $X\times Z$ of the 1-faces of $\partial_2^-\pi = 0_{XZ}$.

- $\top_1(e_2 f) = e_2(\top_1 f)$, i.e. the cell $e_2 f = ((1_X, 1_X), 0_{XY}, (1_Y, 1_Y))$ on the null span $f = 0_{XY}$ has an e_1-tabulator

$$(1_X, 1_X)\times(1_Y, 1_Y) = (X\times Y \leftarrow X\times Y \to X\times Y),$$

that coincides with $e_2(X\times Y)$.

- $\partial_1^\alpha(\top_2\pi) = \top_2(\partial_1^\alpha\pi)$, which means, for $\alpha = 0$, that the domain of the null span $\top_2\pi$ (described above) is the pullback of the cospan $\partial_1^-\pi$.

- $\top_2(e_1 u) = e_1(\top_2 u)$, i.e. the 1-degenerate cell $e_1 u = (u, 0, u)$ on the cospan u has an e_2-tabulator that coincides with the degenerate span on the pullback of u.

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