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# TANGENT SPACES AND TANGENT BUNDLES FOR DIFFEOLOGICAL SPACES

*by J. Daniel CHRISTENSEN and ENXIN WU*

**Résumé.** Nous étudions comment la notion d'espace tangent peut être étendue aux espaces difféologiques, qui généralisent les variétés régulières et comprennent les espaces singuliers et de dimension infinie. Nous nous intéressons à l'espace tangent interne, défini en utilisant les courbes régulières, et à l'espace tangent externe, défini en utilisant les dérivations régulières. Nous prouvons des résultats fondamentaux sur ces espaces tangents, les calculons sur des exemples, et observons qu'en général ils sont différents, même s'ils concident sur de nombreux exemples. Rappelant la définition du fibré tangent donnée par Hector, nous montrons que la multiplication scalaire et l'addition peuvent ne pas y être régulières. Une définition du fibré tangent résolvant ce problème est donnée, à l'aide de ce que nous appelons la difféologie dvs. Ces fibrés tangents sont étudiés, calculés sur des exemples, et nous étudions si leurs fibres sont des espaces vectoriels munis de la difféologie fine. Parmi les exemples: espaces singuliers, tores irrationels, espaces vectoriels de dimension infinie, groupes difféologiques, espaces d'applications régulières entre variétés régulières.

**Abstract.** We study how the notion of tangent space can be extended to diffeological spaces, which are generalizations of smooth manifolds that include singular spaces and infinite-dimensional spaces. We focus on the internal tangent space, defined using smooth curves, and the external tangent space, defined using smooth derivations. We prove fundamental results about these tangent spaces, compute them in many examples, and observe that while they agree for many of the examples, they do not agree in general. Next, we recall Hector's definition of the tangent bundle, and show that both scalar multiplication and addition can fail to be smooth. We then give an improved definition of the tangent bundle, using what we call the dvs diffeology, which fixes these deficiencies. We establish basic facts about these tangent bundles, compute them in many examples, and study the question of whether the fibres of tangent bundles are fine diffeological vector spaces. Our examples include singular spaces, irrational tori, infinite-dimensional vector spaces and diffeological groups, and spaces of smooth maps between smooth manifolds.

**Keywords.** diffeological space, tangent space, tangent bundle.

**Mathematics Subject Classification (2010).** 57P99, 58A05.

# 1. Introduction

The notion of smooth manifold has been generalized in many ways, some of which are summarized and compared in [St1]. Diffeological spaces were introduced by Souriau [So1, So2] and are one such generalization, which includes as special cases manifolds with corners, infinite-dimensional manifolds, and a wide variety of spaces with complicated local behaviour. In fact, the collection of diffeological spaces is closed under taking subsets, quotients and function spaces, and thus gives rise to a very well-behaved category. Moreover, the definition of diffeological space is extremely simple, and we encourage the reader not familiar with the definition to read Definition 2.1 now. The standard textbook for diffeological spaces is [I3], and we briefly summarize the basic theory in Section 2.

Tangent spaces and tangent bundles are important tools for studying smooth manifolds. There are many equivalent ways to define the tangent space of a smooth manifold at a point. These approaches have been generalized to diffeological spaces by many authors, including the following. In the papers that introduced diffeological groups and spaces, Souriau [So1, So2] defined tangent spaces for diffeological groups by identifying smooth curves using certain states. Hector [He] defined tangent spaces and tangent bundles for all diffeological spaces using smooth curves and a more intrinsic identification, and these were developed further in [HM, La]. (We point out some errors in all of [He, HM, La], and give corrected proofs when possible or counterexamples in other cases.) Vincent [V] defined tangent spaces for diffeological spaces by looking at smooth derivations represented by smooth curves, and built associated tangent bundles using the same construction we use in this paper. Iglesias-Zemmour [I3, 6.53] defined the tangent space to a diffeological space at a point as a subspace of the dual of the space of 1-forms at that point, and used these to define tangent bundles.

In this paper, we begin by studying two approaches to defining the tangent space of a general diffeological space at a point. The first is the approach introduced by Hector, which uses smooth curves, and which we call the internal tangent space (see Subsection 3.1). The second is a new approach that uses smooth derivations on germs of smooth real-valued functions, which we call the external tangent space (see Subsection 3.2). In these subsections, we prove basic facts about these tangent spaces, such as locality, and give tools that allow for the computations we do later. We also show that the internal tangent space respects finite products, and prove the non-trivial result that the internal tangent space depends only on the plots of dimension at most $2$, while the external tangent space depends only on the plots of dimension at most $1$.

In Subsections 3.3, 4.1 and 4.4, we compute these two tangent spaces for a diverse set of examples. We summarize some of the computations in the following table. If the basepoint is not specified, the result is true for any base point.

| **Diffeological space and base point** | **Internal** | **External** |
|---|---|---|
| discrete diffeological space | $\mathbb{R}^0$ | $\mathbb{R}^0$ |
| indiscrete diffeological space | $\mathbb{R}^0$ | $\mathbb{R}^0$ |
| topological space with continuous diffeology | $\mathbb{R}^0$ | $\mathbb{R}^0$ |
| smooth manifold of dimension $n$ | $\mathbb{R}^n$ | $\mathbb{R}^n$ |
| axes in $\mathbb{R}^2$ with the pushout diffeology at $0$ | $\mathbb{R}^2$ | $\mathbb{R}^2$ |
| axes in $\mathbb{R}^2$ with the sub-diffeology at $0$ | $\mathbb{R}^2$ | $\mathbb{R}^2$ |
| three lines intersecting at $0$ in $\mathbb{R}^2$ with the sub-diffeology at $0$ | $\mathbb{R}^3$ | $\mathbb{R}^3$ |
| $\mathbb{R}^n$ with wire diffeology ($n \geq 2$) | uncountable dimension | $\mathbb{R}^n$ |
| 1-dimensional irrational torus | $\mathbb{R}$ | $\mathbb{R}^0$ |
| quotient space $\mathbb{R}^n/O(n)$ at $[0]$ | $\mathbb{R}^0$ | $\mathbb{R}$ |
| $[0, \infty)$ with the sub-diffeology of $\mathbb{R}$ at $0$ | $\mathbb{R}^0$ | $\mathbb{R}$ |
| vector space $V$ with fine diffeology | $V$ | |
| diffeomorphism group of a compact smooth manifold $M$ at $1_M$ | $C^\infty$ vector fields on $M$ | |

We see that these two tangent spaces coincide in many cases, including of course for smooth manifolds, but that they are different in general. In Subsection 3.4, we briefly describe some variants of our definitions that one could also consider.

In Section 4, we study tangent bundles. Since the internal tangent space has better formal properties, and we are able to compute it in more examples, we define our tangent bundle (as a set) to be the disjoint union of the internal tangent spaces. Subsection 4.1 begins by describing the diffeology that Hector put on this internal tangent bundle [He], and then shows that it is not well-behaved in general. For example, we show in Example 4.3 that the fibrewise addition and scalar multiplication maps are not smooth in general, revealing errors in [He, HM, La]. We then introduce a refinement of Hector's diffeology, which we call the dvs diffeology, that avoids these problems. We also reprove the fact that the internal tangent bundle of a diffeological group with Hector's diffeology is trivial, since the original proof was partially based on a false result, and as a result we conclude that Hector's diffeology and the dvs diffeology coincide in this case. In Subsection 4.2, we give a conceptual explanation of the relationship between Hector's diffeology and the dvs diffeology: they are colimits, taken in different categories, of the same diagram.

The two diffeologies on the tangent bundle give rise to diffeologies on each internal tangent space. In Subsection 4.3, we study the question of when internal tangent spaces, equipped with either of these diffeologies, are fine diffeological vector spaces. Here the **fine diffeology** on a vector space is the smallest diffeology making the addition and scalar multiplication maps smooth. We show that for many infinite-dimensional spaces, both Hector's diffeology and the dvs diffeology on the internal tangent spaces are not fine. On the other hand, we also show that the internal tangent space of any fine diffeological vector space $V$ at any point is isomorphic to $V$ as a diffeological vector space. As a by-product, we show that the inverse function theorem does not hold for general diffeological spaces.

Finally, in Subsection 4.4, we study the internal tangent bundles of function spaces, and generalize a result in [He, HM] that says that the internal tangent space of the diffeomorphism group of a compact smooth manifold at the identity is isomorphic to the vector space of all smooth vector fields on the manifold. Again we find that in these cases, Hector's diffeology coincides with the dvs diffeology.

The paper [CW2] is a sequel to the present paper. It proves that a diffeological bundle gives rise to an exact sequence of tangent spaces, and gives conditions under which Hector's diffeology and the dvs diffeology on the tangent bundle agree.

All smooth manifolds in this paper are assumed to be finite-dimensional, Hausdorff, second countable and without boundary, all vector spaces are assumed to be over the field $\mathbb{R}$, and all linear maps are assumed to be $\mathbb{R}$-linear.

## 2. Background on diffeological spaces

We provide a brief overview of diffeological spaces. All the material in this section can be found in the standard textbook [I3]. For a concise introduction to diffeological spaces, we recommend [CSW], particularly Section 2 and the introduction to Section 3.

**Definition 2.1** ([So2])**.** *A* ***diffeological space*** *is a set $X$ together with a specified set $\mathcal{D}_X$ of functions $U \to X$ (called* ***plots****) for each open set $U$ in $\mathbb{R}^n$ and for each $n \in \mathbb{N}$, such that for all open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$:*

1. *(Covering) Every constant function $U \to X$ is a plot;*
2. *(Smooth Compatibility) If $U \to X$ is a plot and $V \to U$ is smooth, then the composite $V \to U \to X$ is also a plot;*
3. *(Sheaf Condition) If $U = \cup_i U_i$ is an open cover and $U \to X$ is a function such that each restriction $U_i \to X$ is a plot, then $U \to X$ is a plot.*

*We usually denote a diffeological space by its underlying set.*

*A function $f : X \to Y$ between diffeological spaces is* ***smooth*** *if for every plot $p : U \to X$ of $X$, the composite $f \circ p$ is a plot of $Y$.*

Write $\mathfrak{Diff}$ for the category of diffeological spaces and smooth maps. Given two diffeological spaces $X$ and $Y$, we write $C^\infty(X, Y)$ for the set of all smooth maps from $X$ to $Y$. An isomorphism in $\mathfrak{Diff}$ will be called a **diffeomorphism**.

Every smooth manifold $M$ is canonically a diffeological space with the same underlying set and plots taken to be all smooth maps $U \to M$ in the usual sense. We call this the **standard diffeology** on $M$, and, unless we say otherwise, we always equip a smooth manifold with this diffeology. It is easy to see that smooth maps in the usual sense between smooth manifolds coincide with smooth maps between them with the standard diffeology.

The set $\mathcal{D}$ of diffeologies on a fixed set $X$ is ordered by inclusion, and is a complete lattice. The largest element in $\mathcal{D}$ is called the **indiscrete diffeology** on $X$, and consists of all functions $U \to X$. The smallest element in $\mathcal{D}$ is called the **discrete diffeology** on $X$, and consists of all locally constant functions $U \to X$. The smallest diffeology on $X$ containing a set of functions $\mathcal{A} = \{U_i \to X\}_{i \in I}$ is called the diffeology **generated** by $\mathcal{A}$. It consists of all functions $f : U \to X$ that locally either factor through the given functions via smooth maps, or are constant. The standard diffeology on a smooth manifold is generated by any smooth atlas on the manifold, and for every diffeological space $X$, $\mathcal{D}_X$ is generated by $\cup_{n \in \mathbb{N}} C^\infty(\mathbb{R}^n, X)$.

For a diffeological space $X$ with an equivalence relation $\sim$, the smallest diffeology on $X/{\sim}$ making the quotient map $X \twoheadrightarrow X/{\sim}$ smooth is called the **quotient diffeology**. It consists of all functions $U \to X/{\sim}$ that locally factor through the quotient map. Using this, we call a smooth map $f : X \to Y$ a **subduction** if it induces a diffeomorphism $X/{\sim} \to Y$, where $x \sim x'$ if and only if $f(x) = f(x')$, and $X/{\sim}$ has the quotient diffeology.

For a diffeological space $Y$ and a subset $A$ of $Y$, the largest diffeology on $A$ making the inclusion map $A \hookrightarrow Y$ smooth is called the **sub-diffeology**. It consists of all functions $U \to A$ such that $U \to A \hookrightarrow Y$ is a plot of $Y$. Using this, we call a smooth map $f : X \to Y$ an **induction** if it induces a diffeomorphism $X \to \mathrm{Im}(f)$, where $\mathrm{Im}(f)$ has the sub-diffeology of $Y$.

The category of diffeological spaces is very well-behaved:

**Theorem 2.2.** *The category $\mathfrak{Diff}$ is complete, cocomplete and cartesian closed.*

The descriptions of limits, colimits and function spaces are quite simple, and are concisely described in [CSW, Section 2]. We will make use of these concrete descriptions. Unless we say otherwise, every function space is equipped with the functional diffeology.

We can associate to every diffeological space the following interesting topology:

**Definition 2.3** ([I1]). *Given a diffeological space $X$, the final topology induced by its plots, where each domain is equipped with the standard topology, is called the **D-topology** on $X$.*

In more detail, if $(X, \mathcal{D})$ is a diffeological space, then a subset $A$ of $X$ is open in the $D$-topology of $X$ if and only if $p^{-1}(A)$ is open for each $p \in \mathcal{D}$. We call such subsets ***D*-open**.

A smooth map $X \to X'$ is continuous when $X$ and $X'$ are equiped with the $D$-topology, and so this defines a functor $D : \mathfrak{Diff} \to \mathfrak{Top}$ to the category of topological spaces.

*Example* 2.4. (1) The $D$-topology on a smooth manifold coincides with the usual topology.

(2) The $D$-topology on a discrete diffeological space is discrete, and the $D$-topology on an indiscrete diffeological space is indiscrete.

For more discussion of the $D$-topology, see [CSW].

We will make use of the concept of diffeological group at several points, so we present it here.

**Definition 2.5.** *A **diffeological group** is a group object in* $\mathfrak{Diff}$. *That is, a diffeological group is both a diffeological space and a group such that the group operations are smooth maps.*

A smooth manifold of dimension $n$ is formed by gluing together open subsets of $\mathbb{R}^n$ via diffeomorphisms. A diffeological space is also formed by gluing together open subsets of $\mathbb{R}^n$ via smooth maps, possibly for all $n \in \mathbb{N}$. To make this precise, let $\mathcal{DS}$ be the category with objects all open subsets of $\mathbb{R}^n$ for all $n \in \mathbb{N}$ and morphisms smooth maps between them. Given a diffeological space $X$, we define $\mathcal{DS}/X$ to be the category with objects all plots of $X$ and morphisms the commutative triangles

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X, & \end{array}$$

with $p, q$ plots of $X$ and $f$ a smooth map. We call $\mathcal{DS}/X$ the **category of plots of *X***. Then we have:

**Proposition 2.6** ([CSW, Proposition 2.7])**.** *The colimit of the functor* $F : \mathcal{DS}/X \to \mathfrak{Diff}$ *sending the above commutative triangle to* $f : U \to V$ *is* $X$.

Given a diffeological space $X$, the category $\mathcal{DS}/X$ can be used to define geometric structures on $X$. For example, see [I3] for a discussion of differential forms and the de Rham cohomology of a diffeological space. In this paper, we will use a pointed version of $\mathcal{DS}/X$ to define the internal tangent space of $X$.

## 3. Tangent spaces

We discuss two approaches to defining the tangent space of a diffeological space at a point: the internal tangent space introduced by Hector using plots, and the external tangent space defined using smooth derivations of the algebra of germs of smooth functions. We prove basic facts about these tangent spaces in Subsections 3.1 and 3.2, and then compute them for a wide variety of examples in Subsection 3.3. Although they are isomorphic for smooth manifolds, we find that the two approaches are different for a general diffeological space; see Examples 3.22, 3.23, 3.24 and 3.25. In Subsection 3.4, we mention some other approaches to defining tangent spaces.

### 3.1 Internal tangent spaces

The internal tangent space of a pointed diffeological space is defined using plots. It was first introduced in [He], and is closely related to the kinematic tangent space of [KM] (see Subsection 3.4).

To start, we will define a pointed analog of the category $\mathcal{DS}/X$ of plots of $X$, introduced just before Proposition 2.6. Let $\mathcal{DS}_0$ be the category with objects all connected open neighbourhoods of $0$ in $\mathbb{R}^n$ for all $n \in \mathbb{N}$ and morphisms the smooth maps between them sending $0$ to $0$. Given a pointed diffeological space $(X, x)$, we define $\mathcal{DS}_0/(X, x)$ to be the category with objects the plots $p : U \to X$ such that $U$ is connected, $0 \in U$ and $p(0) = x$, and morphisms the commutative triangles

$$\begin{array}{ccc} U & \xrightarrow{\;f\;} & V \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X, & \end{array}$$

where $p, q \in \mathrm{Obj}(\mathcal{DS}_0/(X, x))$ and $f$ is a smooth map with $f(0) = 0$. We call $\mathcal{DS}_0/(X,x)$ the **category of plots of $X$ centered at $x$**. It is the comma category of the natural functor from $\mathcal{DS}_0$ to $\mathfrak{Diff}_*$, the category of pointed diffeological spaces.

**Definition 3.1** ([He]). *Let $(X,x)$ be a pointed diffeological space. The **internal tangent space** $T_x(X)$ of $X$ at $x$ is the colimit of the composite of functors $\mathcal{DS}_0/(X,x) \to \mathcal{DS}_0 \to \mathrm{Vect}$, where $\mathrm{Vect}$ denotes the category of vector spaces and linear maps, the first functor is the forgetful functor and the second functor is given by $(f : U \to V) \mapsto (f_* : T_0(U) \to T_0(V))$. Given a plot $p : U \to X$ sending $0$ to $x$ and an element $u \in T_0(U)$, we write $p_*(u)$ for the element these represent in the colimit.*

Let $f : (X,x) \to (Y,y)$ be a smooth map between pointed diffeological spaces. Then $f$ induces a functor $\mathcal{DS}_0/(X,x) \to \mathcal{DS}_0/(Y,y)$. Therefore, we have a functor $T : \mathfrak{Diff}_* \to \mathrm{Vect}$. Indeed, the functor $T$ is the left Kan extension along the inclusion functor $\mathcal{DS}_0 \to \mathfrak{Diff}_*$ of the functor $\mathcal{DS}_0 \to \mathrm{Vect}$ sending $f : U \to V$ to $f_* : T_0(U) \to T_0(V)$.

Note that the category of plots of a diffeological space centered at a point is usually complicated. In order to calculate the internal tangent space at a point efficiently, we need a simpler indexing category.

Let $(X,x)$ be a pointed diffeological space. We define a category $\mathcal{G}(X,x)$ whose objects are the objects in $\mathcal{DS}_0/(X,x)$, and whose morphisms are germs at 0 of morphisms in $\mathcal{DS}_0/(X,x)$. In more detail, morphisms from $p : (U,0) \to (X,x)$ to $q : (V,0) \to (X,x)$ in $\mathcal{G}(X,x)$ consist of equivalence classes of smooth maps $f : W \to V$, where $W$ is an open neighborhood of 0 in $U$ and $p|_W = q \circ f$. Two such maps are equivalent if they agree on an open neighborhood of $0$ in $U$. Then there is a functor $\mathcal{G}(X,x) \to \mathrm{Vect}$ sending the morphism

$$\begin{array}{ccc} U & \xrightarrow{[f]} & V \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

in $\mathcal{G}(X,x)$ to $f_* : T_0(U) \to T_0(V)$, and its colimit is $T_x(X)$.

A **local generating set of $X$ at $x$** is a subset $G$ of $\mathrm{Obj}(\mathcal{G}(X,x))$, such that for each object $p : (U,0) \to (X,x)$ in $\mathcal{G}(X,x)$, there exist an element $q : (W,0) \to (X,x)$ in $G$ and a morphism $p \to q$ in $\mathcal{G}(X,x)$. A **local**

**generating category of $X$ at $x$** is a subcategory $\mathcal{G}$ of $\mathcal{G}(X, x)$ such that $\mathrm{Obj}(\mathcal{G})$ is a local generating set of $X$ at $x$.

**Proposition 3.2.** *If $\mathcal{G}$ is a local generating category of $X$ at $x$, then there is a natural epimorphism from* $\mathrm{colim}(\mathcal{G} \hookrightarrow \mathcal{G}(X, x) \to \mathrm{Vect})$ *to* $T_x(X)$. *Moreover, if $\mathcal{G}$ is a final subcategory of $\mathcal{G}(X, x)$, then this is an isomorphism.*

*Proof.* The first statement follows from the definition of a local generating category of $X$ at $x$, and the second statement follows from [Mac, Theorem IX.3.1]. □

A **local generating set of curves of $X$ at $x$** is a subset $C$ of the intersection $\mathrm{Obj}(\mathcal{G}(X, x)) \cap C^\infty(\mathbb{R}, X)$ such that for each object $p : \mathbb{R} \to X$ in $\mathcal{G}(X, x)$, there exist an element $q : \mathbb{R} \to X$ in $C$ and a morphism $p \to q$ in $\mathcal{G}(X, x)$. A **local generating category of curves of $X$ at $x$** is a subcategory $\mathcal{C}$ of $\mathcal{G}(X, x)$ such that $\mathrm{Obj}(\mathcal{C})$ is a local generating set of curves of $X$ at $x$.

**Proposition 3.3.** *If $\mathcal{C}$ is a local generating category of curves of $X$ at $x$, then the natural map* $\mathrm{colim}(\mathcal{C} \hookrightarrow \mathcal{G}(X, x) \to \mathrm{Vect}) \to T_x(X)$ *is an epimorphism.*

In particular, every internal tangent vector in $T_x(X)$ is a linear combination of internal tangent vectors of the form $p_*(\frac{d}{dt})$, where $p : \mathbb{R} \to X$ is a smooth curve with $p(0) = x$, and $\frac{d}{dt}$ is the standard unit vector in $T_0(\mathbb{R})$.

*Proof.* This is because for each $u \in T_0(U)$ there exists a pointed smooth map $f : (\mathbb{R}, 0) \to (U, 0)$ such that $f_*(\frac{d}{dt}) = u$. □

Moreover, the relations between internal tangent vectors are determined by the two-dimensional plots:

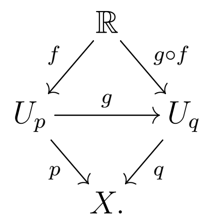
**Proposition 3.4.** *Let $(X, x)$ be a pointed diffeological space. Let $X'$ be the diffeological space with the same underlying set as $X$, with diffeology generated by all plots $\mathbb{R}^2 \to X$. Then the identity map $X' \to X$ is smooth and induces an isomorphism $T_x(X') \to T_x(X)$.*

*Proof.* Since $\mathbb{R}$ is a retract of $\mathbb{R}^2$, every plot $\mathbb{R} \to X$ factors through a plot $\mathbb{R}^2 \to X$, and so $X'$ contains the same 1-dimensional plots as $X$. Therefore, by Proposition 3.3, the linear map $T_x(X') \to T_x(X)$ is surjective.

To prove injectivity, we need a more concrete description of the internal tangent spaces. From the description of $T_x(X)$ as a colimit indexed by the category $\mathcal{G}(X,x)$, we can describe $T_x(X)$ as a quotient vector space $F/R$. Here $F = \oplus_p T_0(U_p)$, where the sum is indexed over plots $p : U_p \to X$ sending 0 to $x$, and $R$ is the span of the vectors of the form $(p,v) - (q, g_*(v))$, where $p : (U_p, 0) \to (X,x)$ and $q : (U_q, 0) \to (X, x)$ are pointed plots, $g : (U_p, 0) \to (U_q, 0)$ is a germ of smooth maps with $p = q \circ g$ as germs at 0, $v$ is in $T_0(U_p)$, $(p,v)$ denotes $v$ in the summand of $F$ indexed by $p$, and $(q, g_*(v))$ denotes $g_*(v)$ in the summand of $F$ indexed by $q$. Unless needed for clarity, we will write such a formal difference as simply $v - g_*(v)$, and not repeat the conditions on $p$, $q$, $g$ and $v$. We call $v - g_*(v)$ a **basic** relation, and call $U_p$ the **domain** of the relation.

Similarly, $T_x(X') = F'/R'$, where $F'$ and $R'$ are defined as above, but restricting to plots that locally factor through plots of dimension 2. In this notation, the natural map $T_x(X') \to T_x(X)$ is induced by the inclusion $F' \subseteq F$, and the surjectivity of this map says that every element of $F$ is equal modulo $R$ to an element of $F'$.

As a start to proving injectivity, we first show that the basic relations $v - g_*(v)$ are generated by those that have 1-dimensional domain $U_p$. Choose a germ $f : (\mathbb{R}, 0) \to (U_p, 0)$ so that $f_*(\frac{d}{dt}) = v$, giving

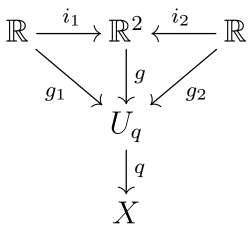



Then

$$\begin{aligned}
&\big[(p \circ f, -\tfrac{d}{dt}) - (p, f_*(-\tfrac{d}{dt}))\big] + \big[(q \circ g \circ f, \tfrac{d}{dt}) - (q, (g \circ f)_*(\tfrac{d}{dt}))\big] \\
&= \big[(p \circ f, -\tfrac{d}{dt}) + (p, v)\big] + \big[(p \circ f, \tfrac{d}{dt}) - (q, g_*(v))\big] \\
&= (p, v) - (q, g_*(v)),
\end{aligned}$$

which shows that our given relation is a sum of relations with domain $\mathbb{R}$. (This argument is similar to that of Proposition 3.3, and shows how that argument could be made more formal.)

We now know that a general element of $R$ can be written in the form $r = \sum_i v_i - (g_i)_*(v_i)$, where $g_i : \mathbb{R} \to U_{q_i}$. Next we show that $r$ can be written as a sum of basic relations such that any plot $q : U_q \to X$ (sending $0$ to $x$) with $\dim(U_q) > 2$ appears in at most one term of the sum. Suppose $q$ is such a plot that appears in more than one term of $r$. Without loss of generality, suppose $g_1$ and $g_2$ are the germs $(\mathbb{R}, 0) \to (U_q, 0)$ and $v_1$ and $v_2$ are the vectors in $T_0(\mathbb{R})$ which give two such terms in $r$. Let $i_1$ and $i_2$ be the inclusions of $\mathbb{R}$ into $\mathbb{R}^2$ as the $x$- and $y$-axes, and define a germ $g : (\mathbb{R}^2, 0) \to (U_q, 0)$ by $g(x, y) = g_1(x) + g_2(y)$. Then we have a commutative diagram



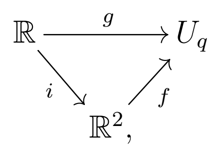

from which it follows that

$$[v_1 - (i_1)_*(v_1)] + [v_2 - (i_2)_*(v_2)] + [v - g_*(v)] = \\ [v_1 - (g_1)_*(v_1)] + [v_2 - (g_2)_*(v_2)],$$

where $v := (i_1)_*(v_1) + (i_2)_*(v_2) \in T_0(\mathbb{R}^2)$. The first two basic relations on the left-hand-side involve maps $\mathbb{R} \to \mathbb{R}^2$, while the third involves a map $\mathbb{R}^2 \to U_q$. Next, as in the previous paragraph, we replace this third basic relation by two, one using a map $\mathbb{R} \to \mathbb{R}^2$ and the other a map $\mathbb{R} \to U_q$. The result is that our new set of terms still consists of basic relations with domain $\mathbb{R}$, but the number of occurrances of the plot $q$ has been reduced by $1$. Proceeding in this way, one can ensure that each such $q$ appears at most once.

Finally, to prove injectivity of the map $F'/R' \to F/R$, we need to show that every element $r$ of $F' \cap R$ is in $R'$. By the above, we can write $r$ as a sum $r = \sum_i v_i - (g_i)_*(v_i)$, where $g_i : \mathbb{R} \to U_{q_i}$ and each $q_i$ with domain of dimension bigger than 2 appears in at most one term. Since $r$ is in $F'$, its component in any summand $T_0(U_q)$ of $F$ must be zero if $q$ does not locally factor through a plot of dimension 2. Since such a plot can appear in at most one term $v - g_*(v)$, no cancellation can occur, so we must have that

$g_*(v) = 0$. So it suffices to show that we can eliminate such terms. If $v = 0$, then $v - g_*(v) = 0$, so this term can be dropped. Otherwise, we must have $g(0) = g'(0) = 0$, and so we can write $g(x) = x^2 h(x)$ for a smooth map $h : \mathbb{R} \to U_q$. Thus we can factor $g$ as



where $i(x) = (x, x^2)$ and $f(s,t) = t \cdot h(s)$. Then $v - g_*(v) = [v - i_*(v)] + [w - f_*(w)]$, where $w = i_*(v) = c\frac{d}{dx}$. The map $i$ has codomain $\mathbb{R}^2$, so the first basic relation is of the required form, but we must still deal with the second relation. Consider the map $i_1 : \mathbb{R} \to \mathbb{R}^2$ sending $x$ to $(x, 0)$. Then $(i_1)_*(c\frac{d}{dt}) = w$, so $w - f_*(w) = [-c\frac{d}{dt} - (i_1)_*(-c\frac{d}{dt})] + [c\frac{d}{dt} - (f \circ i_1)_*(c\frac{d}{dt})]$. Again, the first basic relation is of the required form. For the second, note that the map $f \circ i_1$ is the zero map, and so $c\frac{d}{dt} - (f \circ i_1)_*(c\frac{d}{dt}) = c\frac{d}{dt} = c\frac{d}{dt} - k_*(c\frac{d}{dt})$, where $k : \mathbb{R} \to \mathbb{R}^0$ is the zero map in the category $\mathcal{G}(X, x)$.

This completes the proof. □

*Remark* 3.5. While the internal tangent space depends only on the two-dimensional plots, the diffeology on the internal tangent bundle $TX$ that we define in Section 4 contains all of the information about the diffeology on $X$, since $X$ is a retract of $TX$.

The internal tangent space is local, in the following sense:

**Proposition 3.6.** *Let $(X, x)$ be a pointed diffeological space, and let $A$ be a $D$-open neighborhood of $x$ in $X$. Equip $A$ with the sub-diffeology of $X$. Then the natural inclusion map induces an isomorphism $T_x(A) \cong T_x(X)$.*

*Proof.* This is clear. □

We now investigate the tangent space of a product of diffeological spaces, beginning with binary products.

**Proposition 3.7.** *Let $(X_1, x_1)$ and $(X_2, x_2)$ be two pointed diffeological spaces. Then there is a natural isomorphism of vector spaces*

$$T_{(x_1,x_2)}(X_1 \times X_2) \cong T_{x_1}(X_1) \times T_{x_2}(X_2).$$

*Proof.* The projections $\mathrm{pr}_j : X_1 \times X_2 \to X_j$, $j = 1, 2$, induce a natural map

$$\alpha = ((\mathrm{pr}_1)_*, (\mathrm{pr}_2)_*) : T_{(x_1,x_2)}(X_1 \times X_2) \to T_{x_1}(X_1) \times T_{x_2}(X_2).$$

Define inclusion maps $i_j : X_j \to X_1 \times X_2$ for $j = 1, 2$ by $i_1(y_1) = (y_1, x_2)$ and $i_2(y_2) = (x_1, y_2)$, and consider the map

$$\beta = (i_1)_* + (i_2)_* : T_{x_1}(X_1) \times T_{x_2}(X_2) \to T_{(x_1,x_2)}(X_1 \times X_2)$$

sending $(v_1, v_2)$ to $(i_1)_*(v_1)+(i_2)_*(v_2)$. We claim that these maps are inverse to each other.

To check one of the composites, we first compute

$$\begin{aligned}(\mathrm{pr}_1)_*((i_1)_*(v_1) + (i_2)_*(v_2)) &= (\mathrm{pr}_1)_*(i_1)_*(v_1) + (\mathrm{pr}_1)_*(i_2)_*(v_2)\\ &= (\mathrm{id}_{X_1})_*(v_1) + (c_{x_1})_*(v_2) = v_1,\end{aligned}$$

where $c_{x_1} : X_2 \to X_1$ is the constant map at $x_1$ and hence $(c_{x_1})_*(v_2) = 0$. Similarly, $(\mathrm{pr}_2)_*((i_1)_*(v_1) + (i_2)_*(v_2)) = v_2$, and so $\alpha \circ \beta$ is the identity.

By Proposition 3.3, it is enough to check the other composite on internal tangent vectors of the form $p_*(\frac{d}{dt})$, where $p = (p_1, p_2) : \mathbb{R} \to X_1 \times X_2$ and $p(0) = (x_1, x_2)$. We have

$$\begin{aligned}\beta(\alpha(p_*(\tfrac{d}{dt}))) &= (i_1)_*(\mathrm{pr}_1)_*p_*(\tfrac{d}{dt}) + (i_2)_*(\mathrm{pr}_2)_*p_*(\tfrac{d}{dt})\\ &= (i_1)_*(p_1)_*(\tfrac{d}{dt}) + (i_2)_*(p_2)_*(\tfrac{d}{dt}) = p_*(\tfrac{d}{dt}),\end{aligned}$$

where the last equality follows from the diagram

$$\begin{array}{ccccc} & & \mathbb{R} & & \\ & & \big\downarrow{\scriptstyle \Delta=(1,1)} & & \\ \mathbb{R} & \xrightarrow{i_x} & \mathbb{R}^2 & \xleftarrow{i_y} & \mathbb{R} \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_1\times p_2} & & \big\downarrow{\scriptstyle p_2} \\ X_1 & \xrightarrow[i_1]{} & X_1 \times X_2 & \xleftarrow[i_2]{} & X_2, \end{array}$$

using that $(p_1 \times p_2) \circ \Delta = p$ and $(i_x)_*(\frac{d}{dt}) + (i_y)_*(\frac{d}{dt}) = \frac{d}{dx} + \frac{d}{dy} = \Delta_*(\frac{d}{dt})$. □

*Remark* 3.8. For an arbitrary product $X = \prod_{j \in J} X_j$ of diffeological spaces and a point $x = (x_j)$, there is also a natural linear map $\alpha : T_x(X) \to \prod_{j \in J} T_{x_j}(X_j)$ induced by the projections. We will characterize when $\alpha$ is surjective, and show that this is not always the case.

To do so, we introduce some terminology. An internal tangent vector of the form $p_*(\frac{d}{dt})$ is said to be **representable**, and an internal tangent vector that can be expressed as a sum of $m$ or fewer representables is said to be ***m*-representable**. Recall that by Proposition 3.3, every internal tangent vector in the domain of $\alpha$ is $m$-representable for some $m$. Thus a family of internal tangent vectors $(v_j)$ in the image of $\alpha$ must have the property that each component $v_j$ is $m$-representable for $m$ independent of $j$. In fact, one can show that the image consists of exactly such families, and therefore that the map $\alpha$ is surjective if and only if there is an $m$ in $\mathbb{N}$ such that for all but finitely many $j$ in $J$, every internal tangent vector in $T_{x_j}(X_j)$ is $m$-representable. Manifolds and many other diffeological spaces have the property that every internal tangent vector is 1-representable (see the following remark), so $\alpha$ is often surjective.

Here is an example for which $\alpha$ is not surjective. For each $j \in \mathbb{N}$, consider the diffeological space $X_j$ which is the quotient of $j$ copies of $\mathbb{R}$ where the origins have been identified to one point $[0]$. One can show that $T_{[0]}(X_j) \cong \mathbb{R}^j$ and contains internal tangent vectors which are $j$-representable but not $(j-1)$-representable; see Example 3.17. So, with this family of diffeological spaces, $\alpha$ is not surjective.

We suspect that the map $\alpha$ can fail to be injective as well. The injectivity is related to the existence of a global bound on the number of basic relations (see the proof of Proposition 3.4) needed to show that two representable internal tangent vectors are equal.

See Example 4.24(1) and Proposition 4.27 (with $X$ discrete) for non-trivial cases in which $\alpha$ is an isomorphism.

*Remark* 3.9. Let $(X, x)$ be a pointed diffeological space. If every internal tangent vector in $T_x(X)$ is $m$-representable, then we say that $T_x(X)$ is ***m*-representable**. This is the case, for example, when the vector space $T_x(X)$ is $m$-dimensional. We will show that in fact many diffeological spaces have 1-representable internal tangent spaces.

Consider the case where $X$ is a smooth manifold. Then the internal tangent space agrees with the usual tangent space defined using curves, so

$T_x(X)$ is 1-representable for any $x$ in $X$.

Remark 4.14 shows that the internal tangent space of a diffeological group at any point is also 1-representable. In particular, this holds for a diffeological vector space. It is easy to see that if $A \subseteq X$ is either $D$-open in $X$ or a retract of $X$, $x$ is in $A$ and $T_x(X)$ is $m$-representable, then so is $T_x(A)$. In particular, the proof of Proposition 4.28 shows that if $Y$ is a diffeological space with compact $D$-topology and $N$ is a smooth manifold, then the internal tangent space of $C^\infty(Y, N)$ at any point is 1-representable.

Finally, one can also show that the internal tangent space of a homogeneous diffeological space (see [CW1, Definition 4.33]) at any point is 1-representable.

### 3.2 External tangent spaces

In contrast to the internal tangent space, which is defined using plots, the external tangent space of a pointed diffeological space $(X, x)$ is defined using germs of real-valued smooth functions on $X$. This is analogous to the operational tangent space defined in [KM].

Let $G_x(X) = \operatorname{colim}_B C^\infty(B, \mathbb{R})$ be the diffeological space of **germs of smooth functions of $X$ at $x$**, where the colimit is taken in $\mathfrak{D}$iff, $B$ runs over all $D$-open subsets of $X$ containing $x$ together with the sub-diffeology, $C^\infty(B, \mathbb{R})$ has the functional diffeology, and the maps in the colimit are restrictions along inclusions. $G_x(X)$ is a diffeological $\mathbb{R}$-algebra under pointwise addition, pointwise multiplication and pointwise scalar multiplication, i.e., all these operations are smooth, and the evaluation map $G_x(X) \to \mathbb{R}$ sending $[f]$ to $f(x)$ is a well-defined smooth $\mathbb{R}$-algebra map.

**Definition 3.10.** *An **external tangent vector** on $X$ at $x$ is a smooth derivation on $G_x(X)$. That is, it is a smooth linear map $F : G_x(X) \to \mathbb{R}$ such that the **Leibniz rule** holds: $F([f][g]) = F([f])g(x) + f(x)F([g])$. The **external tangent space** $\hat{T}_x X$ is the set of all external tangent vectors of $X$ at $x$.*

Clearly $\hat{T}_x X$ is a vector space under pointwise addition and pointwise scalar multiplication. The Leibniz rule implies that $F([c]) = 0$ for every external tangent vector $F$ on $X$ at $x$ and every constant function $c : X \to \mathbb{R}$.

Let $f : (X, x) \to (Y, y)$ be a pointed smooth map between two pointed diffeological spaces. Then $f$ induces a linear map $f_* : \hat{T}_x(X) \to \hat{T}_y(Y)$ with

$f_*(F)([g]) = F([g \circ f])$ for $F \in \hat{T}_x(X)$ and $[g] \in G_y(Y)$, where $g \circ f$ really means $g \circ f|_{f^{-1}(\mathrm{Dom}(g))}$. This gives a functor $\hat{T} : \mathfrak{D}\mathrm{iff}_* \to \mathrm{Vect}$.

We next give an equivalent characterization of the external tangent space of a diffeological space. Recall that $G_x(X)$ is a diffeological $\mathbb{R}$-algebra, and the evaluation map $G_x(X) \to \mathbb{R}$ is a smooth $\mathbb{R}$-algebra homomorphism. Let $I_x(X)$ be the kernel of the evaluation map, equipped with the sub-diffeology, and let $I_x(X)/I_x^2(X)$ be the quotient vector space with the quotient diffeology. We define $T'_x(X) = L^\infty(I_x(X)/I_x^2(X), \mathbb{R})$, the set of all smooth linear maps $I_x(X)/I_x^2(X) \to \mathbb{R}$. It is a vector space.

**Proposition 3.11.** *The map $\alpha : \hat{T}_x(X) \to T'_x(X)$ defined by the equation $\alpha(F)([f] + I_x^2(X)) = F([f])$ is an isomorphism.*

*Proof.* Define $\beta : T'_x(X) \to \hat{T}_x(X)$ by $\beta(G)([g]) = G([g]-[g(0)]+I_x^2(X))$, where $G \in T'_x(X)$ and $[g] \in G_x(X)$. It is straightforward to check that both $\alpha(F) : I_x(X)/I_x^2(X) \to \mathbb{R}$ and $\beta(G) : G_x(X) \to \mathbb{R}$ are smooth, and that $\alpha$ and $\beta$ are well-defined inverses to each other. □

The external tangent space is also local:

**Proposition 3.12.** *Let $(X, x)$ be a pointed diffeological space, and let $A$ be a $D$-open subset of $X$ containing $x$. Equip $A$ with the sub-diffeology of $X$. Then the natural inclusion map induces an isomorphism $\hat{T}_x(A) \cong \hat{T}_x(X)$.*

*Proof.* This is clear. □

Moreover, the external tangent space is determined by the one-dimensional plots:

**Proposition 3.13.** *Let $(X, x)$ be a pointed diffeological space, and write $X'$ for the set $X$ with the diffeology generated by $C^\infty(\mathbb{R}, X)$. Then the natural smooth map $X' \to X$ induces an isomorphism $\hat{T}_x(X') \to \hat{T}_x(X)$.*

*Proof.* By [CSW, Theorem 3.7], we know that the $D$-topology on $X'$ coincides with the $D$-topology on $X$, as they have the same smooth curves. Let $B$ be a $D$-open subset of $X$, equipped with the sub-diffeology of $X$, and write $B'$ for the same set equipped with the sub-diffeology of $X'$. The natural smooth map $1 : B' \to B$ induces a smooth map $1^* : C^\infty(B, \mathbb{R}) \to C^\infty(B', \mathbb{R})$ of $\mathbb{R}$-algebras. We will show that this is a diffeomorphism, which then implies that $G_x(X) \cong G_x(X')$ and therefore that $\hat{T}_x(X') \cong \hat{T}_x(X)$. The map $1^*$ is clearly injective. Surjectivity follows from Boman's theorem [KM, Corollary 3.14], which says that a map $U \to \mathbb{R}$ is smooth if and only if it sends smooth curves to smooth curves, where $U$ is open in some $\mathbb{R}^n$. To see that the inverse of $1^*$ is smooth, note that the plots $U \to C^\infty(B, \mathbb{R})$ correspond to smooth maps $U \times B \to \mathbb{R}$. Applying Boman's theorem again, we see that these are the same as the plots of $C^\infty(B', \mathbb{R})$. □

### 3.3 Examples and comparisons

We now calculate the internal and external tangent spaces of some pointed diffeological spaces. A table summarizing the results is in the Introduction.

*Example* 3.14. (1) Let $X$ be a discrete diffeological space. Then for each $x \in X$, $T_x(X) = 0$, since the local generating category of $X$ at $x$ with one object $x : \mathbb{R}^0 \to X$ is final in $\mathcal{G}(X, x)$. Also, $\hat{T}_x(X) = 0$, since $I_x(X) = 0$.

(2) Let $X$ be an indiscrete diffeological space. Then for each $x \in X$, $T_x(X) = 0$, since for any $p : \mathbb{R} \to X \in \mathrm{Obj}(\mathcal{DS}_0/(X, x))$, there exists $q : \mathbb{R} \to X \in \mathrm{Obj}(\mathcal{DS}_0/(X, x))$ such that $q \circ f = p$, where $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^3$. Also, one can show easily that $\hat{T}_x(X) = 0$.

*Example* 3.15. Let $(X, x)$ be a pointed topological space. Write $C(X)$ for the diffeological space with underlying set $X$ whose plots are the continuous maps. Then [He, Proposition 4.3] says that $T_x(C(X)) = 0$.

We now show that $\hat{T}_x(C(X)) = 0$ as well. Let $A$ be a $D$-open subset of $C(X)$, equipped with the sub-diffeology, and fix a smooth map $g : A \to \mathbb{R}$. It suffices to show that $g$ is locally constant. Let $p : \mathbb{R} \to A$ be a plot of $A$; that is, $p$ is a continuous map $\mathbb{R} \to X$ whose image is in $A$. Since $g$ is smooth, so is $g \circ p : \mathbb{R} \to \mathbb{R}$. Moreover, for any continuous $h : \mathbb{R} \to \mathbb{R}$, $p \circ h$ is a plot of $A$ and so $g \circ p \circ h$ is smooth as well. Taking $h(t) = |t| + a$ for each $a \in \mathbb{R}$, for example, one sees that $g \circ p$ is constant and so $g$ is locally constant.

*Example* 3.16. It is a classical result that for every pointed smooth manifold $(X, x)$, $T_x(X) \cong \mathbb{R}^n \cong \hat{T}_x(X)$, with $n = \dim(X)$. In fact, any derivation $F : G_x(X) \to \mathbb{R}$ is smooth. This follows from Proposition 3.12 and [CSW, Lemma 4.3].

*Example* 3.17. Let $X$ be the pushout of $\mathbb{R} \xleftarrow{0} \mathbb{R}^0 \xrightarrow{0} \mathbb{R}$ in $\mathfrak{Diff}$. The commutative diagram

$$\begin{CD} \mathbb{R}^0 @>{0}>> \mathbb{R} \\ @V{0}VV @VV{i_2}V \\ \mathbb{R} @>>{i_1}> \mathbb{R}^2, \end{CD}$$

where $i_1(s) = (s, 0)$ and $i_2(t) = (0, t)$, induces a smooth injective map $i : X \to \mathbb{R}^2$, and we identify points in $X$ with points in $\mathbb{R}^2$ under the map $i$. Note that the diffeology on $X$ is different from the sub-diffeology of $\mathbb{R}^2$ (see Example 3.19), but the $D$-topology on $X$ is the same as the sub-topology of $\mathbb{R}^2$. It is not hard to check that

$$T_x(X) = \begin{cases} \mathbb{R}, & \text{if } x \neq (0,0) \\ \mathbb{R}^2, & \text{if } x = (0,0). \end{cases}$$

We claim that the same is true for the external tangent spaces $\hat{T}_x(X)$:

$$\hat{T}_x(X) = \begin{cases} \mathbb{R}, & \text{if } x \neq (0,0) \\ \mathbb{R}^2, & \text{if } x = (0,0). \end{cases}$$

The first equality follows from Proposition 3.12 and Example 3.16. For the second equality, we define maps $a : \hat{T}_{(0,0)}(X) \to \hat{T}_0(\mathbb{R}) \oplus \hat{T}_0(\mathbb{R})$ sending $F$ to $(F_1, F_2)$ with $F_k([f]) = F([\tilde{f}_k])$, where $\tilde{f}_k(x_1, x_2) = f(x_k)$ for $k = 1, 2$, and $b : \hat{T}_0(\mathbb{R}) \oplus \hat{T}_0(\mathbb{R}) \to \hat{T}_{(0,0)}(X)$ sending $(G_1, G_2)$ to $G$ with $G([g]) = G_1([g \circ j_1]) + G_2([g \circ j_2])$, where $j_1, j_2$ are the structural maps from the pushout diagram of $X$. It is clear that both $a$ and $b$ are well-defined linear maps and that they are inverses, so the second equality follows.

By the same method, one can show that if $X_j$ is the quotient of $j$ copies of $\mathbb{R}$ with the origins identified to one point $[0]$, then

$$T_x(X_j) = \hat{T}_x(X_j) = \begin{cases} \mathbb{R}, & \text{if } x \neq [0] \\ \mathbb{R}^j, & \text{if } x = [0]. \end{cases}$$

*Remark* 3.18. Note that in the above example $\dim_{(0,0)}(X) = 1 < 2 = \dim(T_{(0,0)}(X))$; see [I2] for the definition of the dimension of a diffeological space at a point. In general, unlike smooth manifolds, there is no relationship between the dimension of a diffeological space at a point and the dimension of its tangent space at that point.

*Example* 3.19. Let $Y = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$ with the sub-diffeology of $\mathbb{R}^2$. The map $i$ introduced in Example 3.17 gives a smooth bijection $X \to Y$. However, this map is not a diffeomorphism. To see this, let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0. \end{cases}$$

Then $f$ is a smooth function, and $\mathbb{R} \to \mathbb{R}^2$ defined by $x \mapsto (f(x), f(-x))$ induces a plot of $Y$, but not a plot of $X$.

Although $X$ and $Y$ are not diffeomorphic, their internal and external tangent spaces at any point are isomorphic. This is clear away from the origin, so consider $y = (0, 0) \in Y$. Both [He, Example 4.4(i)] and [HM, Example 6.2] claim without proof that $T_y(Y) = \mathbb{R}^2$. We sketch a proof here. The inclusion map $i : Y \to \mathbb{R}^2$ induces a linear map $i_* : T_y(Y) \to T_y(\mathbb{R}^2) = \mathbb{R}^2$. There is also a linear map $f : \mathbb{R}^2 \to T_y(Y)$ sending $(1, 0)$ to $p_*(\frac{d}{dt})$ and sending $(0, 1)$ to $q_*(\frac{d}{dt})$, where $p : \mathbb{R} \to Y$ is defined by $x \mapsto (x, 0)$ and $q : \mathbb{R} \to Y$ is defined by $x \mapsto (0, x)$. It is clear that $i_* \circ f = 1_{\mathbb{R}^2}$, so $f$ is injective. To prove that $f$ is surjective, it is enough to show that if $r : \mathbb{R} \to Y$ is a plot sending 0 to $y$ and $r$ does not locally factor through $p$ or $q$ near 0, then $r_*(\frac{d}{dt}) = 0$. It is easy to observe that if we write the composite $i \circ r : \mathbb{R} \to \mathbb{R}^2$ as $(f_1, f_2)$, then $f_1(0) = f_2(0) = 0 = f_1'(0) = f_2'(0)$, and hence the conclusion follows from an argument similar to that used in Example 3.25 below.

Now we claim that $G_y(Y) = G_y(X)$ as diffeological spaces, from which it follows that $\hat{T}_y(Y) = \hat{T}_y(X)$, which is $\mathbb{R}^2$ by Example 3.17. To see that $G_y(Y) = G_y(X)$, we first note that the $D$-topologies on $X$ and $Y$ are equal, both being the sub-topology of $\mathbb{R}^2$. There is a cofinal system of $D$-open neighbourhoods of $y$ each of which is diffeomorphic to $Y$ (resp. $X$) when given the sub-diffeology of $Y$ (resp. $X$). Thus it is enough to show that $C^\infty(Y, \mathbb{R}) = C^\infty(X, \mathbb{R})$ as diffeological spaces. There is a canonical smooth injection $i : C^\infty(Y, \mathbb{R}) \to C^\infty(X, \mathbb{R})$ induced by the smooth bijection $X \to Y$. To show that this is a diffeomorphism, it suffices to show that every plot $V \to C^\infty(X, \mathbb{R})$ factors through $i$. By adjointness, this is equivalent to showing that every smooth map $V \times X \to \mathbb{R}$ factors through the bijection $V \times X \to V \times Y$. A smooth map $V \times X \to \mathbb{R}$ is the same as a pair of smooth maps $g, h : V \times \mathbb{R} \to \mathbb{R}$ such that $g(v, 0) = h(v, 0)$ for all $v \in V$. Such a map extends to a smooth map $V \times \mathbb{R}^2 \to \mathbb{R}$ which sends $(v, (x, y))$ to $g(v, x) + h(v, y) - g(v, 0)$, and therefore the restriction to $V \times Y$ is smooth. (See [V, Example 12(c)] for a similar argument.)

By the same method, one can show that if $Y_j$ is the union of all coordinate axes in $\mathbb{R}^j$ with the sub-diffeology, then

$$T_y(Y_j) = \hat{T}_y(Y_j) = \begin{cases} \mathbb{R}, & \text{if } y \neq 0 \\ \mathbb{R}^j, & \text{if } y = 0. \end{cases}$$

*Example* 3.20. Let $A = Y_3$ from the previous example, and let

$$B = \{(x, y) \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0 \text{ or } x = y\},$$

a union of three lines through the origin, with the sub-diffeology of $\mathbb{R}^2$. We prove below that $A$ and $B$ are diffeomorphic. This is also proved in [Wa, Example 2.72], using a more complicated argument. It follows that $T_0(B) = \hat{T}_0(B) = \mathbb{R}^3$, which shows that the induction $B \hookrightarrow \mathbb{R}^2$ does not induce a monomorphism under $T_0$ or $\hat{T}_0$.

Consider the smooth function $\mathbb{R}^3 \to \mathbb{R}^2$ sending $(x, y, z)$ to the pair $(x + z/\sqrt{2},\ y + z/\sqrt{2})$. This restricts to a smooth bijection $A \to B$ sending $(x, 0, 0)$ to $(x, 0)$, $(0, y, 0)$ to $(0, y)$, and $(0, 0, z)$ to $(z, z)/\sqrt{2}$. Write $f : B \to A$ for its inverse. We will show that $f$ is smooth. By Boman's theorem [KM, Corollary 3.14], it is enough to check that for any plot $p : \mathbb{R} \to B$, $f \circ p$ is smooth. And for this, it is enough to check that if we regard $f \circ p$ as a map $\mathbb{R} \to \mathbb{R}^3$, all derivatives exist at all points. This follows from the following claim, where we also regard the derivatives of $p$ as maps $\mathbb{R} \to \mathbb{R}^2$.

**Claim.** *For each $t$ in $\mathbb{R}$, $p^{(k)}(t)$ is in $B$ and $(f \circ p)^{(k)}(t) = f(p^{(k)}(t))$.*

*Proof.* We first prove this for $k = 1$. If $p(t) \neq (0, 0)$ or $p'(t) \neq (0, 0)$, then this is clear, since, near $t$, $p$ must stay within one of the three lines.

Now suppose $p(t) = (0, 0) = p'(t)$. Then

$$(0, 0) = p'(t) = \lim_{h \to 0} \frac{p(t + h) - p(t)}{h} = \lim_{h \to 0} \frac{p(t + h)}{h}.$$

Therefore, if the limits exist, we have

$$(f \circ p)'(t) = \lim_{h \to 0} \frac{f(p(t+h)) - f(p(t))}{h} = \lim_{h \to 0} \frac{f(p(t+h))}{h}.$$

But $\|f(p(t+h))\| = \|p(t+h)\|$, and so the last limit exists and equals $(0,0,0)$, which is $f(p'(t))$.

For general $k$, we have $(f \circ p)^{(k)}(t) = (f \circ p')^{(k-1)}(t) = \cdots = (f \circ p^{(k)})(t)$, as required. □

*Remark* 3.21. Given a pointed diffeological space $(X, x)$, there is a natural morphism $\beta : T_x(X) \to \hat{T}_x(X)$ which is defined on the generators by $\beta(p_*(u))([f]) = u([f \circ p])$, where $u$ is in $T_0(U) \cong \hat{T}_0(U)$, $p : U \to X$ is a plot of $X$ with $U$ connected, $0 \in U$ and $p(0) = x$, and $[f]$ is in $G_x(X)$. It is straightforward to show that $\beta(p_*(u)) : G_x(X) \to \mathbb{R}$ is smooth. By the definition of push-forwards of tangent vectors on smooth manifolds, it is clear that $\beta$ is well-defined. Then we can linearly extend it to be defined on $T_x(X)$. Clearly $\beta$ is linear, and $\beta$ induces a natural transformation $T \to \hat{T} : \mathfrak{Diff}_* \to \text{Vect}$. However in general, $\beta$ is neither injective nor surjective; see the following examples.

*Example* 3.22. (1) Let $X$ be $\mathbb{R}^n$ equipped with the diffeology generated by the set $C^\infty(\mathbb{R}, \mathbb{R}^n)$ of smooth curves. This is sometimes called the "wire" or "spaghetti" diffeology. It is clear that $X$ is not diffeomorphic to $\mathbb{R}^n$ with the standard diffeology, if $n \geq 2$. By Proposition 3.13, we know that $\hat{T}_x(X) \cong \hat{T}_x(\mathbb{R}^n) \cong \mathbb{R}^n$ for any $x \in X$.

On the other hand, one can show that in this case the relations between generating internal tangent vectors are determined by smooth curves in $X$, and hence that $T_x(X)$ has uncountable dimension when $n \geq 2$. For example, the internal tangent vectors $(p_\alpha)_*(\frac{d}{dt})$ for $\alpha \in \mathbb{R}$ are all linearly independent, where $p_\alpha : \mathbb{R} \to X$ sends $x$ to $(x, \alpha x)$.

This example shows that the internal tangent space of a diffeological space at a point is not determined by the plots of curves.

(2) Let $Y$ be $\mathbb{R}^n$ equipped with the diffeology generated by the set $C^\infty(\mathbb{R}^2, \mathbb{R}^n)$ of smooth planes. (One might call this the "lasagna" diffeology.) Then $Y$ is neither diffeomorphic to $X$ in (1) if $n \geq 2$, nor diffeomorphic to $\mathbb{R}^n$ with the standard diffeology if $n \geq 3$. But by Proposition 3.4 and (1) above, $T_y(Y) \cong \mathbb{R}^n \cong \hat{T}_y(Y)$ for any $y \in Y$.

*Example* 3.23. Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the usual 2-torus, and let $\mathbb{R}_\theta$ be the image of the line $\{y = \theta x\}$ under the quotient map $\mathbb{R}^2 \to T^2$, with $\theta$ a fixed irrational number. Note that $T^2$ is an abelian Lie group, and $\mathbb{R}_\theta$ is a subgroup. The quotient group $T^2/\mathbb{R}_\theta$ with the quotient diffeology is called the **1-dimensional irrational torus of slope $\boldsymbol{\theta}$**. One can show that $T^2/\mathbb{R}_\theta$ is diffeomorphic to $T^2_\theta := \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z})$ with the quotient diffeology; see [I3, Exercise 31 on page 31].

Let $x$ be the identity element in $T^2_\theta$. Since the $D$-topology on $T^2_\theta$ is indiscrete, the only smooth maps $T^2_\theta \to \mathbb{R}$ are the constant maps, which implies that $\hat{T}_x(T^2_\theta) = 0$. This was also observed in [He, Example 4.2].

On the other hand, we claim that $T_x(T^2_\theta) = \mathbb{R}$. Given a commutative solid square

$$\begin{array}{ccccc} & & U & & \\ & {}^{f}\swarrow & & \searrow^{g} & \\ \mathbb{R} & & & & \mathbb{R} \\ & {}_{\pi}\searrow & & \swarrow_{\pi} & \\ & & T^2_\theta & & \end{array}$$

with $U$ a connected open subset of $\mathbb{R}^n$, $f$ and $g$ smooth maps, and $\pi$ the quotient map, the difference $f-g$ is a continuous function landing in $\mathbb{Z}+\theta\mathbb{Z}$, and thus is constant. In particular, if $0 \in U$ and $f(0) = 0 = g(0)$, then $f = g$. Moreover, since plots $U \to T^2_\theta$ locally lift to $\mathbb{R}$, they lift as germs. (In fact, the uniqueness implies that they lift globally.) This shows that $\pi$ is a terminal object of $\mathcal{G}(T^2_\theta, x)$ and therefore that $T_x(T^2_\theta) \cong T_0(\mathbb{R}) \cong \mathbb{R}$. See [He, Example 4.2] for a geometric explanation of this result.

*Example* 3.24. Let $O(n)$ act on $\mathbb{R}^n$ in the usual way. Then the orbit space $H_n$ with the quotient diffeology bijects naturally with the half line $[0, \infty)$, but $H_n$ and $[0, \infty)$ with the sub-diffeology of $\mathbb{R}$ are not diffeomorphic, nor are $H_n$ and $H_m$ for $n \neq m$ (see [I2]). By Proposition 3.3, it is easy to see that $T_{[0]}(H_n) = 0$.

We claim that $\hat{T}_{[0]}(H_n) = \mathbb{R}$. To prove this, observe that the natural map

$$C^\infty(H_n, \mathbb{R}) \longrightarrow \{f \in C^\infty(\mathbb{R}^n, \mathbb{R}) \mid f \text{ is } O(n)\text{-invariant}\}$$

is a diffeomorphism, where the right-hand-side is equipped with the sub-diffeology of $C^\infty(\mathbb{R}^n, \mathbb{R})$. This follows from the fact that the functor $U \times -$

is a left adjoint for any open subset $U$ of a Euclidean space, and therefore commutes with quotients. Next, consider the natural smooth map

$$\phi\colon \{f \in C^\infty(\mathbb{R}^n, \mathbb{R}) \mid f \text{ is } O(n)\text{-invariant}\} \to \{g \in C^\infty(\mathbb{R}, \mathbb{R}) \mid g \text{ is even}\},$$

which sends $f$ to the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = f(t, 0, \ldots, 0)$, where the right-hand-side is equipped with the sub-diffeology of $C^\infty(\mathbb{R}, \mathbb{R})$. The map $\phi$ is injective, by rotational invariance. By [Wh, Theorem 1], any smooth even function $g : \mathbb{R} \to \mathbb{R}$ can be written as $h(t^2)$ for a (non-unique) smooth function $h : \mathbb{R} \to \mathbb{R}$, which implies that $\phi$ is surjective: given $g$, define $f$ by $f(x) = h(\|x\|^2)$. To see that $\phi$ is a diffeomorphism, use the Remark at the end of [Wh], which says that Theorem 1 holds for functions of several variables which are even in one of them. Note that under the identifications provided by the above maps, the diffeological algebra $G_{[0]}(H_n)$ of germs coincides with the diffeological algebra $G$ of germs of even smooth functions from $\mathbb{R}$ to $\mathbb{R}$ at $0$. Thus it suffices to compute the smooth derivations on $G$.

It is easy to check that the map $D : G \to \mathbb{R}$ which sends $g$ to $g''(0)$ is a derivation. It is smooth since the second derivative operator [CSW, Lemma 4.3] and evaluation at $0$ are. And it is non-zero, since $D(j) = 2$, where $j : \mathbb{R} \to \mathbb{R}$ is defined by $j(x) = x^2$. Now suppose that $F$ is any derivation on $G$. If $g$ is a germ of an even smooth function $\mathbb{R} \to \mathbb{R}$ near $0$, then $g(x) = g(0) + x^2 h(x)$ for some even smooth germ $h$. Thus $F(g) = F(j)h(0)$ while $D(g) = 2h(0)$, and so $F = (F(j)/2)D$. Therefore, $\hat{T}_{[0]}(H_n)$ is 1-dimensional.

We note in passing that $G$ can also be described as the algebra of germs of smooth functions on $H_1$, so the argument above can be viewed as a reduction to the case where $n = 1$.

The technique used in the following example to show that an internal tangent vector is zero is used in many other examples in this paper.

*Example* 3.25. Let $H_{sub} = [0, \infty)$, as a diffeological subspace of $\mathbb{R}$. Let us calculate $T_0(H_{sub})$ and $\hat{T}_0(H_{sub})$.

We will first show that $T_0(H_{sub}) = 0$. By Proposition 3.3, it suffices to show that every tangent vector of the form $p_*(\frac{d}{dt})$ is zero, where $p : \mathbb{R} \to H_{sub}$ is a smooth curve with $p(0) = 0$ and $\frac{d}{dt}$ is the standard unit vector in $T_0(\mathbb{R})$. It follows from Taylor's formula that $p'(0) = 0$, and hence that

$p(x) = x^2 r(x)$ for some plot $r : \mathbb{R} \to H_{sub}$. We define $q : \mathbb{R}^2 \to H_{sub}$ by $q(x, y) = y^2 r(x)$, which is clearly a plot of $H_{sub}$. This restricts to $p$ on the diagonal $y = x$, and so $p_*(\frac{d}{dt}) = q_*(\frac{d}{dx} + \frac{d}{dy}) = q_*(\frac{d}{dx}) + q_*(\frac{d}{dy})$. Thus our tangent vector is the sum of the tangent vectors obtained by restricting $q$ to the axes. The restriction of $q$ to the $x$-axis gives the zero function, and the restriction of $q$ to the $y$-axis gives the function $h : \mathbb{R} \to H_{sub}$ sending $y$ to $y^2 r(0)$. The former clearly gives the zero tangent vector. For the latter, since $h(y) = h(-y)$, we have that $h_*(\frac{d}{dy}) = h_*(-\frac{d}{dy}) = -h_*(\frac{d}{dy})$, which implies that $h_*(\frac{d}{dy}) = 0$. Hence our original vector is 0, and so $T_0(H_{sub}) = 0$.

To calculate $\hat{T}_0(H_{sub})$, consider the squaring map $\alpha : H_1 \to H_{sub}$, where $H_1 = \mathbb{R}/O(1)$ is described in Example 3.24. The map $\alpha$ is smooth, since it fits into a diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{j} & \mathbb{R} \\ \downarrow & & \uparrow \\ H_1 & \xrightarrow[\alpha]{} & H_{sub} . \end{array}$$

We therefore have a smooth map $\alpha^* : C^\infty(H_{sub}, \mathbb{R}) \to C^\infty(H_1, \mathbb{R})$. Regarding the latter as consisting of even smooth functions $\mathbb{R} \to \mathbb{R}$, $\alpha^*$ sends a smooth function $h : H_{sub} \to \mathbb{R}$ to the smooth function mapping $t$ to $h(t^2)$. This is clearly injective, and by [Wh] it is surjective and a diffeomorphism. Also note that by [CSW, Lemma 3.17], the $D$-topology on $H_{sub}$ is the same as the sub-topology of $\mathbb{R}$. This implies that the algebras $G_0(H_{sub})$ and $G_{[0]}(H_1)$ of germs are isomorphic as diffeological algebras, and therefore that $\hat{T}_0(H_{sub}) \cong \hat{T}_{[0]}(H_1) \cong \mathbb{R}$. A non-zero element of $\hat{T}_0(H_{sub})$ is given by the smooth derivation sending $f$ to $f'(0)$. Under the isomorphism, this corresponds to $D/2 \in \hat{T}_{[0]}(H_1)$ from Example 3.24.

### 3.4 Other approaches to tangent spaces

We have studied the internal and external tangent spaces in some detail because we find them to be the most natural definitions. Rather than arguing that one of them is the "right" definition, we simply point out that it depends on the application. The internal tangent space fits most closely with the definition of a diffeological space, as it works directly with the plots, so it will form the basis for the next section, on tangent bundles.

There are many other possible approaches to defining tangent spaces, and again, these will be useful for different applications. We briefly summarize a few variants of our approaches here.

### 3.4.1 Variants of the internal tangent space

We have seen in Proposition 3.3 that the internal tangent space $T_x(X)$ is spanned by vectors of the form $p_*(\frac{d}{dt})$, where $p : \mathbb{R} \to X$ is a smooth curve sending $0$ to $x$. One could instead use smooth maps from $[0, \infty)$ to $X$ sending $0$ to $x$, with relations coming from higher-dimensional quadrants. With such a definition, $[0, \infty)$ as a diffeological subspace of $\mathbb{R}$ would have a non-trivial tangent space at $0$.

Our internal tangent spaces were defined as colimits in the category of vector spaces. One could instead take the colimit in the category of sets, which would mean that every tangent vector is representable by a smooth curve.

These two choices are independent; one could form a tangent space using either one or both of them.

### 3.4.2 Variants of the external tangent space

An external tangent vector is defined to be a smooth derivation $G_x(X) \to \mathbb{R}$. One might instead consider all derivations. For all examples in Subsection 3.3 we would get the same external tangent spaces, but in general we don't know if every derivation is automatically smooth.

Independently, one could also change the definition of the space $G_x(X)$ of germs of smooth functions at $x$, for a general diffeological space $X$. It was defined at the beginning of Subsection 3.2 using the $D$-topology on $X$. However, when studying diffeological bundles [I1], the correct notion of "locality" uses the plots, not the $D$-open subsets. Moreover, for global smooth functions, we have that $C^\infty(X, \mathbb{R}) = \lim C^\infty(U, \mathbb{R})$, where the limit is taken in the category of sets and the indexing category is the category $\mathcal{DS}/X$ of plots $U \to X$. Thus, by analogy, one might define the germs of smooth functions at $x$ to be $\lim G_0(U)$, where the indexing category is the category $\mathcal{DS}_0/(X, x)$ of pointed plots $(U, 0) \to (X, x)$. This has the advantage of being closer to the definition of the internal tangent space, but we have not investigated it in detail.

There is an established definition of differential forms on a diffeological space [I3], and one could instead define the tangent space to be the dual of the space of 1-forms at a point. This is closely related to the external tangent space we study, since a germ $f$ of a smooth function gives rise to a 1-form $df$.

### 3.4.3 A mixed variant

Finally, recall that in Remark 3.21 we defined a natural transformation $\beta : T \to \hat{T}$. One could consider the image of this natural transformation. That is, one identifies internal tangent vectors if they give rise to the same directional derivative operators. Something very close to this is done in [V], except that the derivative operators are only compared on global smooth functions rather than germs of smooth functions.

In [I3], a mixed variant of the definition involving 1-forms is proposed, and worked out in some detail. Each 1-dimensional plot centered at a point gives rise to a smooth linear functional on 1-forms, and the tangent space is defined to be the span of such linear functionals. This definition agrees with the internal tangent space in all examples where we know both.

One can also consider mixed variants involving sets rather than vector spaces. For example, the set of external tangent vectors which are represented by smooth curves is called the kinematic tangent space in [St2]. A very similar approach was taken in [So1], which discussed the case in which $X$ is a diffeological group, and used certain "states" $X \to \mathbb{C}$ to determine when two internal tangent vectors are equal.

## 4. Internal tangent bundles

We discussed the internal tangent space of a diffeological space at a point in Subsection 3.1. As usual, if we gather all of the internal tangent spaces together, we can form the internal tangent bundle. In Subsection 4.1, we begin by recalling the diffeology Hector defined on this bundle [He]. We show that it is not well-behaved in general and also point out some errors in [He, HM, La]. We then introduce a refinement of Hector's diffeology, called the dvs diffeology, give several examples and counterexamples, and describe the internal tangent bundle of a diffeological group. In Subsection 4.2, we give a conceptual explanation of the relationship between Hector's diffeology and the dvs diffeology: they are colimits, taken in different categories, of the same diagram. In Subsection 4.3, we study the question of when internal tangent spaces are fine diffeological vector spaces. Finally, in Subsection 4.4, we study the internal tangent bundles of function spaces, and generalize a result in [He, HM] that says that the internal tangent space of the diffeomorphism group of a compact smooth manifold at the identity is isomorphic to the vector space of all smooth vector fields on the manifold.

### 4.1 Definitions and examples

In this subsection, we recall Hector's diffeology on the internal tangent bundle of a diffeological space. Then we observe in Example 4.3 that when the internal tangent bundle is equipped with Hector's diffeology, neither addition nor scalar multiplication are smooth in general. This provides a counterexample to some claims in [He, HM, La]. To overcome this problem, we introduce a refinement of Hector's diffeology, called the dvs diffeology, on the tangent bundle. Proposition 4.13 extends some results about internal tangent spaces to internal tangent bundles. Then we prove in Theorem 4.15 that the internal tangent bundle of a diffeological group is always trivial, as the original proof in [HM] was partially based on a false result. We also show that, in this case, Hector's diffeology and the dvs diffeology agree.

**Definition 4.1.** *The* ***internal tangent bundle*** $TX$ *of a diffeological space* $X$ *is defined to be the set* $\coprod_{x\in X} T_x(X)$. ***Hector's diffeology*** *[He] on* $TX$ *is generated by the maps* $Tf : TU \to TX$, *where* $f : U \to X$ *is a plot of* $X$ *with* $U$ *connected,* $TU$ *has the standard diffeology, and for each* $u \in U$, $T_u f : T_u(U) \to T_{f(u)}(X)$ *is defined to be the composite* $T_u(U) \to T_0(U - u) \to T_{f(u)}(X)$, *with* $U - u$ *the translation of* $U$ *by* $u$. *The internal tangent bundle of* $X$ *with Hector's diffeology is denoted* $T^H(X)$, *and* $T_x(X)$ *with the sub-diffeology of* $T^H(X)$ *is denoted* $T^H_x(X)$. *We write elements of* $TX$ *as* $(x, v)$, *where* $v \in T_x(X)$.

Recall that by the universal property of colimits, for any smooth map $f : X \to Y$ and any $x \in X$, we have a linear map $f_* : T_x(X) \to T_{f(x)}(Y)$. It is straightforward to check that $T^H : \mathfrak{Diff} \to \mathfrak{Diff}$ is a functor, and hence that $f_* : T_x^H(X) \to T_{f(x)}^H(Y)$ is smooth. Moreover, the natural map $\pi_X : T^H(X) \to X$ is smooth (indeed, it is a subduction), and therefore $\pi : T^H \to 1$ is a natural transformation. Also the zero section $X \to T^H(X)$ is smooth.

**Definition 4.2.** *A **diffeological vector space** is a vector space object in* $\mathfrak{D}\text{iff}$. *More precisely, it is both a diffeological space and a vector space such that the addition and scalar multiplication maps are smooth.*

The following example shows that $T_x^H(X)$ is not a diffeological vector space in general. In fact, both the addition map $T_x^H(X) \times T_x^H(X) \to T_x^H(X)$ and the scalar multiplication map $\mathbb{R} \times T_x^H(X) \to T_x^H(X)$ can fail to be smooth. Therefore, both [HM, Proposition 6.6] and [La, Lemma 5.7] are false.

*Example* 4.3. Let $X$ be the diffeological space introduced in Example 3.17, two copies of $\mathbb{R}$ glued at the origin thought of as a subset of $\mathbb{R}^2$. We first show that the addition map $T_{(0,0)}^H(X) \times T_{(0,0)}^H(X) \to T_{(0,0)}^H(X)$ is not smooth. To see this, let $f, g : \mathbb{R} \to T_{(0,0)}(X)$ be given by $f(x) = (x, 0)$ and $g(x) = (0, x)$, where we use the natural identification $T_{(0,0)}(X) \cong \mathbb{R}^2$. Clearly $f$ and $g$ are smooth as maps $\mathbb{R} \to T_{(0,0)}^H(X)$. We will show that the sum $h = f + g : \mathbb{R} \to T_{(0,0)}^H(X)$ given by $x \mapsto (x, x)$ is not smooth. Any plot $p : U \to X$ must locally factor through one of the axes, and so the diffeology on $T_{(0,0)}^H(X)$ is generated by plots factoring through one of the axes. Clearly $h$ is not locally constant and does not locally factor through a generating plot, so it is not smooth. Now we show that scalar multiplication $\mathbb{R} \times T_{(0,0)}^H(X) \to T_{(0,0)}^H(X)$ is not smooth. This is because the (non-smooth) map $h$ can be described as the composite $\mathbb{R} \to \mathbb{R} \times T_{(0,0)}^H(X) \to T_{(0,0)}^H(X)$, where the first map is given by $t \mapsto (t, (1, 1))$, which is clearly smooth. Note that, away from the axes, the diffeology on $T_{(0,0)}^H(X)$ is discrete.

*Remark* 4.4. It follows that the fibrewise scalar multiplication map $\mathbb{R} \times T^H(X) \to T^H(X)$ is also not smooth in general. This is particularly surprising given the following observation. A generating plot of $T^H(X)$ is of the form $Tf : TU \to T^H(X)$, where $f : U \to X$ is a plot of $X$ with $U$ connected. Thus one might expect the plots $1_{\mathbb{R}} \times Tf : \mathbb{R} \times TU \to \mathbb{R} \times T^H(X)$ to generate the product diffeology on $\mathbb{R} \times T^H(X)$. If so, the commutative square

$$\begin{array}{ccc} \mathbb{R} \times TU & \xrightarrow{1_{\mathbb{R}} \times Tf} & \mathbb{R} \times T^H(X) \\ \downarrow & & \downarrow \\ TU & \xrightarrow[Tf]{} & T^H(X) \end{array}$$

would imply that scalar multiplication on $T^H(X)$ is smooth. The problem is that there may be tangent vectors in $T^H(X)$ which are not in the image of any generating plot $TU \to T^H(X)$, in which case one needs to consider constant plots as well. However, this argument has shown that if every tangent vector in $X$ is 1-representable (Remarks 3.8 and 3.9), then scalar multiplication is smooth. This will be the case when $X$ is a diffeological group (Remark 4.14). It is not hard to see that the converse is true as well: when scalar multiplication is smooth, every tangent vector in $X$ is 1-representable.

We will introduce a new diffeology on the internal tangent bundle of a diffeological space that makes the addition and scalar multiplication maps smooth. We first introduce the following concept:

**Definition 4.5.** *Let $X$ be a diffeological space. A **diffeological vector space over $X$** is a diffeological space $V$, a smooth map $p : V \to X$ and a vector space structure on each of the fibres $p^{-1}(x)$ such that the addition map $V \times_X V \to V$, the scalar multiplication map $\mathbb{R} \times V \to V$ and the zero section $X \to V$ are smooth. Here $\mathbb{R} \times V$ has the product diffeology and $V \times_X V$ has the sub-diffeology of the product diffeology on $V \times V$. In other words, a diffeological vector space over $X$ is a vector space object in the overcategory $\mathfrak{D}\mathrm{iff}/X$ over the field object $\mathrm{pr}_2 : \mathbb{R} \times X \to X$.*

*In the case when $X$ is a point, we recover the concept of diffeological vector space.*

Note that if $V$ is a diffeological vector space over $X$, then it is automatically the case that each fibre of $p$, with the sub-diffeology, is a diffeological vector space. It also follows that $p$ is a subduction.

Now we give a construction that we will use to enlarge the diffeology on $T^H(X)$ in order to make it a diffeological vector space over $X$. A similar construction can be found in [V, Theorem 5.1.6 and Definition 6.2.1], using a different definition of the tangent spaces. One can show that the notion of *regular vector bundle* in [V] exactly matches our notion of a diffeological vector space over a diffeological space.

**Proposition 4.6.** *Let $p : V \to X$ be a smooth map between diffeological spaces, and suppose that each fibre of $p$ has a vector space structure. Then there is a smallest diffeology $\mathcal{D}$ on $V$ which contains the given diffeology and which makes $V$ into a diffeological vector space over $X$.*

*Proof.* We first take the largest diffeology on $V$ making $p : V \to X$ smooth. It is easy to see that this diffeology contains the original diffeology on $V$ and makes $V$ into a diffeological vector space over $X$. Now consider the intersection $\mathcal{D}$ of all diffeologies $\mathcal{D}_i$ on $V$ which have these two properties. We claim that $\mathcal{D}$ also has these two properties. It is clear that $\mathcal{D}$ contains the original diffeology on $V$ and that $p : (V, \mathcal{D}) \to X$ and the zero section $X \to (V, \mathcal{D})$ are smooth. A plot in $V \times_X V$ consists of plots $q_1, q_2 : U \to V$ in $\mathcal{D}$ such that $p \circ q_1 = p \circ q_2$. Since the pointwise sum $q_1 + q_2 : U \to V$ is in each of the diffeologies $\mathcal{D}_i$, it is in $\mathcal{D}$ as well. Thus $V \times_X V \to V$ is smooth with respect to $\mathcal{D}$. Similarly, $\mathbb{R} \times V \to V$ is smooth. Therefore, $(V, \mathcal{D})$ is a diffeological vector space over $X$. □

We write $\tilde{V}$ for $V$ equipped with the diffeology $\mathcal{D}$.

In the special case where $X$ is a point and the vector space $V$ starts with the discrete diffeology, the above proposition proves that there is a smallest diffeology on $V$ making it into a diffeological vector space. This diffeology is called the fine diffeology; see Subsection 4.3 and [I3, Chapter 3].

Note that the largest diffeology on $V$ making $p : V \to X$ smooth, which was used in the proof of the above proposition, is not generally interesting, since it induces the indiscrete diffeology on each fibre.

*Remark* 4.7. One can give a more concrete description of the diffeology $\mathcal{D}$ on $V$ described in Proposition 4.6: it is generated by the linear combinations of the original plots of $V$ and the composite of the zero section with plots of $X$. More precisely, given a plot $q : U \to X$, plots $q_1, q_2, \ldots, q_k : U \to V$ such that $p \circ q_i = q$ for all $i$, and plots $r_1, r_2, \ldots, r_k : U \to \mathbb{R}$ in the standard diffeology on $\mathbb{R}$, the linear combination $U \to V$ sending $u$ to $r_1(u)q_1(u) + \cdots + r_k(u)q_k(u)$ in $p^{-1}(q(u))$ is a plot in $\mathcal{D}$, and every plot in $\mathcal{D}$ is locally of this form. Note that when $k = 0$, this is the composite of the plot $q$ of $X$ with the zero section of $V$.

One consequence is the following description of the fibres of $\tilde{V}$. For $x \in X$, write $\tilde{V}_x$ for $p^{-1}(x)$ with the sub-diffeology of $\tilde{V}$ and $\widetilde{V_x}$ for the same set with the diffeology obtained by starting with the sub-diffeology of $V$ and completing it to a vector space diffeology using Proposition 4.6. It is not hard to see that these diffeologies agree. (See also [V, Proposition 6.2.2(iiii)].) More generally, this construction commutes with pullbacks.

The following two results also follow immediately from Remark 4.7.

**Proposition 4.8.** *Let*

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[g]{} & Y \end{array}$$

*be a commutative diagram in* $\mathfrak{D}\mathrm{iff}$*, such that each fibre of* $p$ *and* $q$ *has a vector space structure and* $f|_{p^{-1}(x)} : p^{-1}(x) \to q^{-1}(g(x))$ *is linear for each* $x \in X$*. Then the map* $f : \widetilde{V} \to \widetilde{W}$ *is smooth. Furthermore, if both* $f$ *and* $g$ *in the original square are inductions, then so is* $f : \widetilde{V} \to \widetilde{W}$*.*

**Proposition 4.9.** *Let* $p : V \to X$ *and* $q : W \to Y$ *be smooth maps between diffeological spaces. Assume that each fibre of* $p$ *and* $q$ *has a vector space structure. Then* $\widetilde{V} \times \widetilde{W}$ *is isomorphic to* $\widetilde{V \times W}$ *as diffeological vector spaces over* $X \times Y$*.*

Here is the new diffeology on the internal tangent bundle of a diffeological space:

**Definition 4.10.** *We define the* ***dvs diffeology*** *on* $TX$ *to be the smallest diffeology containing Hector's diffeology which makes* $TX$ *into a diffeological vector space over* $X$*. The internal tangent bundle of* $X$ *with the dvs diffeology is denoted* $T^{dvs}(X)$*, and* $T_x(X)$ *with the sub-diffeology of* $T^{dvs}(X)$ *is denoted* $T_x^{dvs}(X)$*.*

As a convention, whenever Hector's diffeology coincides with the dvs diffeology for a tangent space or a tangent bundle, we omit the superscript.

By Proposition 4.8 and the corresponding results for Hector's diffeology, it is clear that $T^{dvs} : \mathfrak{D}\mathrm{iff} \to \mathfrak{D}\mathrm{iff}$ is a functor, and we have a natural transformation $\pi : T^{dvs} \to 1$. Hence, for any smooth map $f : X \to Y$ and any $x$

in $X$, the induced map $f_* : T_x^{dvs}(X) \to T_{f(x)}^{dvs}(Y)$ is a smooth linear map between diffeological vector spaces. Also, the natural map $\pi_X : T^{dvs}(X) \to X$ is a subduction.

*Example* 4.11. When $X$ is a smooth manifold, it follows from Example 3.16 that $TX$ agrees with the usual tangent bundle as a set. In fact, it is not hard to check that Hector's diffeology on $TX$ coincides with the standard diffeology, regarding $TX$ as a smooth manifold. Since $TX$ is a diffeological vector space over $X$, the dvs diffeology on $TX$ also coincides with the standard diffeology.

*Remark* 4.12. Example 3.17 shows that $TX \to X$ is not a diffeological bundle [I1] in general, whether $TX$ is equipped with Hector's diffeology or the dvs diffeology, since the pullback along a non-constant plot passing through the origin would have fibres of different dimensions. The same example also shows that $TX \to X$ is not a fibration in the sense of [CW1], with either diffeology. Indeed, there is no dashed arrow in $\mathfrak{Diff}$ making the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & TX \\ \big\downarrow & \nearrow & \big\downarrow \\ \mathbb{R} & \longrightarrow & X \end{array}$$

commute, where the top map sends $0$ to $(1,1) \in T_{(0,0)}(X)$ and the bottom map is the inclusion of the $x$-axis.

Now we extend Propositions 3.6 and 3.7 from internal tangent spaces to internal tangent bundles:

**Proposition 4.13.**

1. *If $A$ is a $D$-open subset of a diffeological space $X$, equipped with the sub-diffeology of $X$, then $T^H(A) \to T^H(X)$ is an induction such that $T_a^H(A) \to T_a^H(X)$ is a diffeomorphism for each $a \in A$. The same is true for the dvs diffeology.*

2. *Let $X$ and $Y$ be diffeological spaces. Then there is a natural diffeomorphism $T^H(X \times Y) \to T^H(X) \times T^H(Y)$ which commutes with the projections to $X \times Y$, and the same is true for the dvs diffeology.*

*Proof.* (1) To see that $T^H(A) \to T^H(X)$ is an induction, note that for any plot $p : U \to X$, $p^{-1}(A)$ is an open subset of $U$ since $A$ is $D$-open in $X$, and we have a commutative square

$$\begin{array}{ccc} T(p^{-1}(A)) & \hookrightarrow & TU \\ \downarrow & & \downarrow{\scriptstyle Tp} \\ T^H(A) & \longrightarrow & T^H(X). \end{array}$$

It follows that, for each $a$ in $A$, the map $T_a^H(A) \to T_a^H(X)$ is an induction as well. By Proposition 3.6, this map is an isomorphism of vector spaces, and therefore it is a diffeomorphism. The corresponding result for the dvs diffeology then essentially follows from Proposition 4.8.

(2) The first statement follows directly from Proposition 3.7 and the fact that for any plots $p : U \to X$ and $q : V \to Y$, we have the following commutative square

$$\begin{array}{ccc} T(U \times V) & \xrightarrow{\cong} & TU \times TV \\ {\scriptstyle T(p\times q)}\downarrow & & \downarrow{\scriptstyle Tp\times Tq} \\ T^H(X \times Y) & \longrightarrow & T^H(X) \times T^H(Y), \end{array}$$

and the second statement follows then from Proposition 4.9. □

We end this subsection with a discussion of the internal tangent bundles of diffeological groups (see Definition 2.5).

*Remark* 4.14.

1. Let $e$ be the identity in a diffeological group $G$. It is shown in [HM, Proposition 6.4] that the multiplication map $G \times G \to G$ induces the addition map $T_e(G) \times T_e(G) \to T_e(G)$. It follows that the addition map $T_e^H(G) \times T_e^H(G) \to T_e^H(G)$ is smooth. Moreover, [HM, Corollary 6.5] implies that every element in $T_e(G)$ can be written as $p_*(\frac{d}{dt})$, where $p : \mathbb{R} \to G$ is a single smooth curve with $p(0) = e$, and so it follows from Remark 4.4 that scalar multiplication is smooth as well. Therefore, $T_e^H(G)$ is a diffeological vector space. Since left-multiplication by any $g \in G$ is a diffeomorphism, we also see that $T_g^H(G)$ is a diffeological vector space and that $T_e^H(G)$ and $T_g^H(G)$

are isomorphic as diffeological vector spaces. Similarly, $T_e^{dvs}(G)$ and $T_g^{dvs}(G)$ are isomorphic as diffeological vector spaces.

2. If $V$ is a diffeological vector space, one might hope that the scalar multiplication map $\mathbb{R} \times V \to V$ induces the scalar multiplication map $\mathbb{R} \times T_0(V) \to T_0(V)$, where we use that $T_r(\mathbb{R}) \cong \mathbb{R}$ for any $r \in \mathbb{R}$ and that $T$ respects finite products. (We omit superscripts here because we are not using the diffeology in this remark.) But since induced maps are always linear, and scalar multiplication is bilinear, this will only happen when $T_0(V) = 0$. Instead, for fixed $r \in \mathbb{R}$, we can consider the map $V \to V$ sending $v$ to $rv$. It is not hard to show that for $r \in \mathbb{Q}$, this induces the map $T_0(V) \to T_0(V)$ sending $u$ to $ru$. We conjecture that this is true for any $r \in \mathbb{R}$. It is true if for any $u \in T_0(V)$, there exist plots $p : \mathbb{R} \to V$ and $q : \mathbb{R} \to V$ such that $p(0) = 0$, $p_*(\frac{d}{dt}) = u$ and $p(t) = tq(t)$ for $t$ in a neighborhood of $0 \in \mathbb{R}$. This condition is satisfied when $V = C^\infty(X, \mathbb{R}^n)$ with the functional diffeology, for any diffeological space $X$, or when $V$ is a retract of such a space in the category of diffeological vector spaces.

Although the proof of [HM, Proposition 6.8] was partly based on a false result, the proposition is still correct:

**Theorem 4.15.** *Let $G$ be a diffeological group. Then $T^H(G)$ is a diffeological vector space over $G$ and all of $T^H(G)$, $G \times T_e^H(G)$, $T^{dvs}(G)$ and $G \times T_e^{dvs}(G)$ are isomorphic as diffeological vector spaces over $G$. Therefore, $T_g^H(G) = T_g^{dvs}(G)$ for any $g \in G$.*

*Proof.* In the proof of [HM, Proposition 6.8], Hector and Macias-Virgos defined the map $F : G \times T_e^H(G) \to T^H(G)$ sending $(g, v)$ to $(g, (L_g)_*(v))$, where $L_g : G \to G$ is left multiplication by $g$, and they argued that $F$ is a diffeomorphism. But in the proof that $F$ is smooth, they used that $T_e^H(G)$ is a fine diffeological vector space, which is not true in general; see Example 4.24 for counterexamples. It is easy to fix this. Let $a$ be the composite $G \times T^H(G) \to T^H(G) \times T^H(G) \cong T^H(G \times G) \to T^H(G)$, where the first map is given by $\sigma \times 1_{T^H(G)}$ with $\sigma : G \to T^H(G)$ the zero section, the second map is the isomorphism from Proposition 4.13(2), and the third map is induced from the multiplication $G \times G \to G$. Clearly $a$ is smooth, and it is straightforward to check that $F$ equals $a|_{G \times T_e^H(G)}$, and is therefore smooth. One can also see that $F^{-1} : T^H(G) \to G \times T^H_e(G)$ is given by $F^{-1}(g, v) = (g, a(g^{-1}, (g, v)))$, which is smooth. So $T^H(G)$ is diffeomorphic to $G \times T^H_e(G)$, and the diffeomorphism respects the projections to $G$ and the linear structures on the fibres. In particular, $T^H(G)$ is a diffeological vector space over $G$. The rest then follows directly from Definition 4.10. (The last statement also follows from the second paragraph of Remark 4.7 and Remark 4.14(1).) $\square$

Because of the previous result, our convention allows us to write $TG$ and $T_g(G)$ (without superscripts) when $G$ is a diffeological group. This includes the case when $G$ is a diffeological vector space.

### 4.2 A conceptual description of the Hector and dvs diffeologies

In this subsection we give a categorical explanation of the difference between Hector's diffeology and the dvs diffeology on the internal tangent bundle of a diffeological space. To summarize briefly, they can both be described as the colimit of a natural diagram, but the colimits take place in different categories. This material is not needed in the rest of the paper.

For a diffeological space $X$, a **vector space with diffeology over $X$** is a diffeological space $V$, a smooth map $p : V \to X$ and a vector space structure on each of the fibres $V_x = p^{-1}(x)$, with no compatibility conditions. The category $\mathcal{VSD}$ has as objects the vector spaces with diffeology over diffeological spaces, and as morphisms the commutative squares

$$\begin{array}{ccc} V & \xrightarrow{g} & W \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

in $\mathfrak{Diff}$ such that for each $x \in X$, $g|_{V_x} : V_x \to W_{f(x)}$ is linear.

**Proposition 4.16.** *The category $\mathcal{VSD}$ is complete and cocomplete.*

*Proof.* Let $F : I \to \mathcal{VSD}$ be a functor from a small category, and write $V_i \to X_i$ for $F(i)$. There are two functors $t, b : \mathcal{VSD} \to \mathfrak{Diff}$, sending the above commutative square to $g : V \to W$ and $f : X \to Y$, respectively. Write $\lim V_i$ and $\lim X_i$ for the limits of the functors $t \circ F$ and $b \circ F$, respectively.

Then it is easy to check that there is a canonical smooth map $\lim V_i \to \lim X_i$ which is the limit of $F$ in $\mathcal{VSD}$.

Next, write $X = \operatorname{colim} X_i$ for the colimit of the functor $b \circ F$. Let $\mathcal{C}$ be the category of elements of $b \circ F$. Then $\mathcal{C}$ has as objects the pairs $(i, a)$ for $i \in \mathrm{Obj}(I)$ and $a \in X_i$, and as morphisms $(i, a) \to (j, b)$ the morphisms $f : i \to j$ in $I$ such that $b(F(f)) : X_i \to X_j$ sends $a$ to $b$. There is a natural bijection between $\pi_0(\mathcal{C})$ and the underlying set of $X$. For any $x \in X$, the connected full subcategory $\mathcal{C}_x$ of $\mathcal{C}$ corresponding to $x$ consists of the objects $(i, a)$ such that $a \in X_i$ is sent to $x \in X$ by the cocone map $X_i \to X$. There is a functor $\mathcal{C}_x \to \mathrm{Vect}$ sending $(i, a)$ to $V_{i,a}$, the fibre above $a \in X_i$ in $V_i$, and sending $f : (i, a) \to (j, b)$ to $t(F(f))|_{V_{i,a}} : V_{i,a} \to V_{j,b}$. Let $V_x$ be the colimit of this functor and let $V$ be the disjoint union $\coprod_{x \in X} V_x$. There is a canonical map $V_i \to V$ for each $i \in \mathrm{Obj}(I)$, and we equip $V$ with the smallest diffeology making these maps smooth. Then the canonical projection $V \to X$ is smooth, and one can check that this is the colimit of $F$. □

We write $\mathcal{DVS}$ for the full subcategory of $\mathcal{VSD}$ with objects diffeological vector spaces over diffeological spaces (see Definition 4.2). Propositions 4.6 and 4.8 imply that the forgetful functor $\mathcal{DVS} \to \mathcal{VSD}$ has a left adjoint, sending $p : V \to X$ to $p : \tilde{V} \to X$. It is clear that if $p : V \to X$ is a diffeological vector space over $X$, then $\tilde{V} = V$. Therefore, the category $\mathcal{DVS}$ is also complete and cocomplete, with the limit computed in $\mathcal{VSD}$, and the colimit obtained by applying the left adjoint functor to the corresponding colimit in $\mathcal{VSD}$. Also, Proposition 4.9 says that the left adjoint commutes with finite products.

Recall that $\mathcal{DS}$ is the category with objects all open subsets of $\mathbb{R}^n$ and morphisms smooth maps between them. The main result of this subsection is now straightforward to check.

**Theorem 4.17.** *Let $X$ be a fixed diffeological space. Consider the functor $\mathcal{DS}/X \to \mathcal{DVS}$ defined by*

$$\begin{array}{ccc} U \xrightarrow{\ f\ } V & & TU \xrightarrow{\ Tf\ } TV \\ {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \longmapsto & {\scriptstyle \pi_U}\downarrow \qquad\qquad \downarrow{\scriptstyle \pi_V} \\ X & & U \xrightarrow{\ f\ } V. \end{array}$$

*The colimit of this functor is $\pi_X : T^{dvs}(X) \to X$, and the colimit of the composite functor $\mathcal{DS}/X \to \mathcal{DVS} \to \mathcal{VSD}$ is $\pi_X : T^H(X) \to X$.*

This can also be phrased as saying that the functors $T^H : \mathfrak{D}\mathrm{iff} \to \mathcal{VSD}$ and $T^{dvs} : \mathfrak{D}\mathrm{iff} \to \mathcal{DVS}$ are left Kan extensions of the natural functors $\mathcal{DS} \to \mathcal{VSD}$ and $\mathcal{DS} \to \mathcal{DVS}$ along the inclusion functor $\mathcal{DS} \to \mathfrak{D}\mathrm{iff}$.

### 4.3 Fineness of internal tangent spaces

Recall that any vector space $V$ has a smallest diffeology making it into a diffeological vector space. This diffeology is called the **fine** diffeology, and its plots are exactly those maps which locally are of the form $U \to \mathbb{R}^n \to V$, where the first map is smooth and the second map is linear; see [I3, Chapter 3]. In this subsection, we study when $T_x^H(X)$ and $T_x^{dvs}(X)$ are fine diffeological vector spaces, giving examples where they are fine and where they are not.

The material in this subsection is independent of the material in the following subsection.

From Example 4.3, we know that $T_x^H(X)$ in general is not a fine diffeological vector space. We now give a condition implying that $T_x^{dvs}(X)$ is fine.

**Proposition 4.18.** *Let $(X, x)$ be a pointed diffeological space. Assume that there exists a local generating set $G$ of $X$ at $x$ with the property that for any $q : U \to X$ in $G$, if $f : V \to U$ is a smooth map from an open subset $V$ of $\mathbb{R}^n$ such that $q \circ f = c_x$ (the constant map), then $f$ is locally constant. Then $T_x^{dvs}(X)$ is a fine diffeological vector space.*

*Proof.* By the second half of Remark 4.7, the diffeology on $T_x^{dvs}(X)$ is the smallest diffeology containing Hector's diffeology which makes $T_x^{dvs}(X)$ into a diffeological vector space. Thus it suffices to show that Hector's diffeology is contained in the fine diffeology. That is, we need to show that every plot $V \to T_x^H(X)$ locally smoothly factors through a linear map $\mathbb{R}^n \to T_x^H(X)$. Let $p : V \to T_x^H(X)$ be a plot. Since Hector's diffeology on $TX$ is generated by maps $TU \to TX$ induced by plots $U \to X$, for each $v_0 \in V$ there is a connected open neighbourhood $V'$ of $v_0$ in $V$ such that $p|_{V'}$ is either constant or of the form $V' \xrightarrow{g} TU \to TX$, with image in $T_x(X)$, where $g$ is smooth. In the first case, every constant map factors smoothly through a linear map $\mathbb{R} \to T_x(X)$. In the second case, shrinking $V'$ further if necessary, we can assume that $U \to X$ is in the local generating set $G$.

The composite $V' \xrightarrow{g} TU \to U \to X$ is constant, and so by hypothesis, $g$ must land in $T_u(U)$ for some $u \in U$. This shows that $p|_{V'}$ smoothly factors through the linear map $T_u(U) \to T_x(X)$. Since $T_u(U)$ is diffeomorphic to $\mathbb{R}^n$, for some $n$, we are done. □

Here are some results that follow from Proposition 4.18:

*Example* 4.19. $T_x^{dvs}(X)$ is a fine diffeological vector space when:

1. $X$ is a smooth manifold, and $x$ is any point of $X$;
2. $X$ is the axes in $\mathbb{R}^2$ with the pushout diffeology (Example 3.17), and $x$ is any point of $X$;
3. $X$ is a 1-dimensional irrational torus (Example 3.23), and $x$ is any point of $X$.

*Remark* 4.20. Since the irrational torus $T^2_\theta$ is a diffeological group, by Theorem 4.15 we know that $T^H_{[0]}(T^2_\theta) \cong T^{dvs}_{[0]}(T^2_\theta)$. So by (3) of the above example, both of them have the fine diffeology. Therefore, the map $T_0(\mathbb{R}) \to T_{[0]}(T^2_\theta)$ induced by the quotient map $\mathbb{R} \to T^2_\theta$ from Example 3.23 is a linear diffeomorphism. As a consequence, the inverse function theorem does not hold for general diffeological spaces when tangent spaces are equipped with either Hector's diffeology or the dvs diffeology, in the sense that, if $f : A \to B$ is a smooth map between diffeological spaces such that $f_* : T^H_a(A) \to T^H_{f(a)}(B)$ (or $f_* : T^{dvs}_a(A) \to T^{dvs}_{f(a)}(B)$) is a linear diffeomorphism for some $a \in A$, then it is not true that there exist $D$-open neighborhoods $A'$ and $B'$ of $a \in A$ and $f(a) \in B$, respectively, such that $f : A' \to B'$ is a diffeomorphism.

*Example* 4.21. Note that any smooth linear bijection from a diffeological vector space to a fine diffeological vector space is a linear diffeomorphism. As a consequence, we know that the tangent space $T^{dvs}_0(Y)$ of the axes $Y$ in $\mathbb{R}^2$ with the sub-diffeology (Example 3.19) has the fine diffeology.

**Proposition 4.22.** *Let $V$ be a fine diffeological vector space. Then $T_0(V) \cong V$ as diffeological vector spaces.*

*Proof.* Recall that the vector space $T_0(V)$ is the colimit of $T_0(U)$ taken over the category $\mathcal{G}(V, 0)$. Consider the full subcategory $\mathcal{G}$ of $\mathcal{G}(V, 0)$ consisting of inclusions $W \hookrightarrow V$ of finite-dimensional linear subspaces. It is not hard to see that this is a final subcategory, in the sense that the overcategory $p/\mathcal{G}$ is non-empty and connected for each object $p$ in $\mathcal{G}(V, 0)$. Therefore, we can restrict to the subcategory $\mathcal{G}$ and find that, as vector spaces,

$$T_0(V) \cong \operatorname*{colim}_{W \in \mathcal{G}} T_0(W) \cong \operatorname*{colim}_{W \in \mathcal{G}} W \cong V.$$

By the criteria in Proposition 4.18, we know that $T_0^{dvs}(V)$ is a fine diffeological vector space, and so $T_0^{dvs}(V) \cong V$ as diffeological vector spaces. By Theorem 4.15, $T_0^H(V) = T_0^{dvs}(V)$ as diffeological vector spaces. □

*Remark* 4.23. When $V$ is fine, it follows from Theorem 4.15 that $TV \cong V \times V$ as diffeological vector spaces over $V$. This is not true for an arbitrary diffeological vector space; see Example 3.14(2).

*Example* 4.24. The following examples show that when $G$ is a diffeological group, $T_e(G)$ is not necessarily fine, contradicting the argument given in [HM, Proposition 6.8]. As a consequence, although the vector space $T_e(G)$ is a colimit in the category of vector spaces of $T_0(U)$, the diffeological vector space $T_e(G)$ is not a colimit in the category of diffeological vector spaces of $T_0(U)$ with the fine diffeology. For properties of (fine) diffeological vector spaces, see [Wu].

1. Let $X = \prod_\omega \mathbb{R}$ be the countable product of copies of $\mathbb{R}$ with the product diffeology. Then there is a canonical smooth map $TX \to \prod_\omega T\mathbb{R}$ which induces a smooth linear map $\alpha : T_0(X) \to \prod_\omega \mathbb{R}$, where we identify $T_0(\mathbb{R})$ with $\mathbb{R}$ in the natural way. Consider the plot $p : \mathbb{R}^2 \to X$ sending $(s, t)$ to $(t, st, s^2t, s^3t, \ldots)$. It induces one of Hector's generating plots $T\mathbb{R}^2 \to TX$. The restriction of $T\mathbb{R}^2$ to the $s$-axis is a rank-2 bundle whose total space $U$ is diffeomorphic to $\mathbb{R}^3$. The composite $U \hookrightarrow T\mathbb{R}^2 \to TX$ gives a plot $q$ in $T_0(X)$, since the plot $p$ sends the $s$-axis to the point 0 in $X$. Now consider the composite $\alpha \circ q : U \to T_0(X) \to \prod_\omega \mathbb{R}$. At the point $(s, 0)$, it sends $\partial_s$ to 0 and $\partial_t$ to $(1, s, s^2, s^3, \ldots)$. As $s$ varies, these $(1, s, s^2, s^3, \ldots)$'s are all linearly independent in $\prod_\omega \mathbb{R}$. Since $\alpha$ is linear, $q$ doesn't locally factor through any finite-dimensional linear subspace of $T_0(X)$. Therefore, the diffeology on $T_0(X)$ is strictly larger than the fine diffeology. It follows from Theorem 4.15 that the same is true for $T_x(X)$ for any

$x \in X$, since $X$ is a diffeological group (in fact, a diffeological vector space).

While it isn't needed above, one can also show that $\alpha$ is an isomorphism, so $T_0(X) \cong X$ as vector spaces. The surjectivity follows from Remark 3.8, and the injectivity follows from the fact that $X$ is a diffeological group, using an argument similar to that used in Example 3.25. In fact, Proposition 4.27 shows that $\alpha$ is a diffeomorphism.

2. Write $Y$ for $C^\infty(\mathbb{R}^n, \mathbb{R})$, where $n \geq 1$, and write $\hat{0}$ for the zero function in $Y$. Define $\phi : Y \to X$ by $f \mapsto (f(0), \frac{\partial f}{\partial y_1}(0), \frac{\partial^2 f}{\partial y_1^2}(0), \ldots)$. Then $\phi$ is a smooth map, by [CSW, Lemma 4.3]. Let $p : \mathbb{R}^2 \to Y$ be defined by $p(s,t)(y) = te^{sy_1}$. Then $p$ is a plot of $Y$. The restriction of $T\mathbb{R}^2$ to the $s$-axis is a rank-2 bundle whose total space $U$ is diffeomorphic to $\mathbb{R}^3$. As in (1), one can check that the composite $U \hookrightarrow T\mathbb{R}^2 \to TY$ gives a plot $q$ in $T_{\hat{0}}(Y)$, but the composite $\alpha \circ T_{\hat{0}}\phi \circ q : U \to T_{\hat{0}}(Y) \to T_0(X) \to \prod_\omega \mathbb{R}$ does not locally factor through a finite-dimensional linear subspace of $\prod_\omega \mathbb{R}$. Therefore, $T_f(Y)$ is not a fine diffeological vector space for any $f \in Y$.

3. Let $M$ be a smooth manifold of positive dimension, and let $i : \mathbb{R}^n \to M$ be a smooth chart. Using a smooth partition of unity one can show that for any smooth map $f : \mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R}$, there exists a smooth map $g : \mathbb{R}^2 \times M \to \mathbb{R}$ such that $g \circ (1 \times i)|_{\mathbb{R}^2 \times B} = f|_{\mathbb{R}^2 \times B}$, where $B$ is an open neighborhood of $0$ in $\mathbb{R}^n$. By (2), we know that $T_f(C^\infty(M, \mathbb{R}))$ is not a fine diffeological vector space for any $f \in C^\infty(M, \mathbb{R})$.

As a corollary of (3), if $M$ is a compact smooth manifold of positive dimension, and $U$ is an open subset of $\mathbb{R}^n$ for some $n \in \mathbb{Z}^+$, then for any $f \in C^\infty(M, U) =: Z$, both $T_f^H(Z)$ and $T_f^{dvs}(Z)$ are not fine. This follows from Proposition 4.13(1) and the fact that $Z = C^\infty(M, U)$ is a $D$-open subset of $C^\infty(M, \mathbb{R}^n)$ [CSW, Proposition 4.2].

### 4.4 Internal tangent bundles of function spaces

In this subsection, we first observe in Proposition 4.25 that for any diffeological spaces $X$ and $Y$, there is always a natural smooth map

$$\gamma : T^H(C^\infty(X, Y)) \to C^\infty(X, T^H(Y)).$$

In Propositions 4.27 and 4.28, we give two special cases when the above map is actually a diffeomorphism, and it follows that in these cases we have $T^H(C^\infty(X,Y)) = T^{dvs}(C^\infty(X,Y))$. In particular, we recover in Corollary 4.29 the fact that the internal tangent space of the diffeomorphism group of a compact smooth manifold at the identity is isomorphic to the vector space of all smooth vector fields on the manifold.

Stacey [St2] also studies the map $\gamma$, but with a different focus.

**Proposition 4.25.** *Let $X$ and $Y$ be diffeological spaces. There is a smooth map*

$$\gamma : T^H(C^\infty(X,Y)) \longrightarrow C^\infty(X, T^H(Y)),$$

*which is natural in $X$ and $Y$, and which makes the following triangle commutative:*

$$\begin{array}{ccc} T^H(C^\infty(X,Y)) & \xrightarrow{\quad\gamma\quad} & C^\infty(X,T^H(Y)) \\ & {\scriptstyle \pi_{C^\infty(X,Y)}} \searrow \quad \swarrow {\scriptstyle (\pi_Y)_*} & \\ & C^\infty(X,Y). & \end{array}$$

*Proof.* Observe that the map $\tau : C^\infty(\mathbb{R},Y) \to T^H(Y)$ defined by $\alpha \mapsto (\alpha(0), \alpha_*(\frac{d}{dt}))$ is smooth, since for any plot $q : U \to C^\infty(\mathbb{R},Y)$, we have a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad q \quad} & C^\infty(\mathbb{R},Y) \\ \downarrow & & \downarrow{\scriptstyle \tau} \\ TU \times T\mathbb{R} & \xrightarrow[T\tilde{q}]{} & T^H(Y) \end{array}$$

in $\mathfrak{Diff}$, where the left vertical map is defined by $u \mapsto ((u,0),(0,\frac{d}{dt}))$, and $\tilde{q} : U \times \mathbb{R} \to Y$ is the adjoint of $q$.

Now we first partially define $\gamma : T^H(C^\infty(X,Y)) \to C^\infty(X,T^H(Y))$ by sending $(f, p_*(\frac{d}{dt}))$ to $\tau \circ \hat{p}$, where $p : \mathbb{R} \to C^\infty(X,Y)$ is a plot such that $p(0) = f$, and $\hat{p}$ is the double adjoint of $p$, using the cartesian closedness of $\mathfrak{Diff}$. One can show that the following triangle

$$\begin{array}{ccc} T^H(C^\infty(X,Y)) & \xrightarrow{\quad\gamma\quad} & C^\infty(X,T^H(Y)) \\ & {\scriptstyle \pi_{C^\infty(X,Y)}} \searrow \quad \swarrow {\scriptstyle (\pi_Y)_*} & \\ & C^\infty(X,Y), & \end{array}$$

commutes, so we can linearly extend this on each fibre to define $\gamma$. It is straightforward to check that the map $\gamma$ is well-defined. Moreover, $\gamma$ is smooth: given any plots $q : U \to C^\infty(X, Y)$ and $r : V \to X$, the composite

$$TU \xrightarrow{Tq} T^H(C^\infty(X, Y)) \xrightarrow{\gamma} C^\infty(X, T^H(Y))$$

is smooth since for the adjoint map $TU \times X \to T^H(Y)$ we have a commutative diagram

$$\begin{array}{ccc} TU \times V & \xrightarrow{1\times r} & TU \times X \\ {\scriptstyle 1\times\sigma}\downarrow & & \downarrow \\ TU \times TV & \xrightarrow[T\beta]{} & T^H(Y), \end{array}$$

where $\sigma$ is the zero section, and $\beta$ is the composite

$$U \times V \xrightarrow{1\times r} U \times X \xrightarrow{\tilde{q}} Y.$$

The naturality of $\gamma$ directly follows from its definition. $\square$

*Aside* 4.26. Note that when $X$ is discrete, the map $\gamma$ just defined coincides with the map $\alpha$ from Remark 3.8 in the case where each $X_j$ is equal to $Y$. In fact, there is a framework which encompasses both the map $\gamma$ in the previous proposition and the map $\alpha$ in Remark 3.8 as special cases. Let $f : Y \to X$ be a fixed smooth map between diffeological spaces. Write $\Gamma(f)$ for the set of all smooth sections of $f$ equipped with the sub-diffeology of $C^\infty(X, Y)$, and write $A$ for the set

$$\{g \in C^\infty(X, T^H(Y)) \mid \pi_Y \circ g \in \Gamma(f) \text{ and } f_*(g(x)) = (x, 0) \text{ for all } x \in X\}$$

of "vertical" sections of $T^H(Y) \to X$, equipped with the sub-diffeology of $C^\infty(X, T^H(Y))$. Then $(\pi_Y)_* : C^\infty(X, T^H(Y)) \to C^\infty(X, Y)$ restricts to a smooth map $(\pi_Y)_* : A \to \Gamma(f)$. Moreover, $\gamma : T^H(C^\infty(X, Y)) \to C^\infty(X, T^H(Y))$ from the previous proposition induces a smooth map $\gamma : T^H(\Gamma(f)) \to A$, and we have a commutative triangle

$$\begin{array}{ccc} T^H(\Gamma(f)) & \xrightarrow{\gamma} & A \\ & {\scriptstyle \pi_{\Gamma(f)}}\searrow \quad \swarrow{\scriptstyle (\pi_Y)_*} & \\ & \Gamma(f). & \end{array}$$

In particular, if the fixed map $f$ is the projection $\mathrm{pr}_1 : X \times Y \to X$, then we recover the previous proposition. And if $J$ is a discrete diffeological space, $\{X_j\}_{j\in J}$ is a set of diffeological spaces, and the fixed map $f : \coprod_{j\in J} X_j \to J$ sends each $X_j$ to $j$, then the map $\gamma : T^H(\Gamma(f)) \to A$ discussed above is the same as the map $\alpha$ discussed in Remark 3.8.

We will now focus on cases in which $T^H(Y) = T^{dvs}(Y)$, so we write $TY$ to simplify the notation. In this situation, it is straightforward to show that $(\pi_Y)_* : C^\infty(X, TY) \to C^\infty(X, Y)$ is a diffeological vector space over $C^\infty(X, Y)$. Given a smooth map $f : X \to Y$, we write $S_f(X, Y) = \{g \in C^\infty(X, TY) \mid \pi_Y \circ g = f\}$ for the fibre of $(\pi_Y)_*$ over $f$, a diffeological vector space. The map $\gamma$ restricts to a natural smooth linear map $\gamma_f : T^H_f(C^\infty(X, Y)) \to S_f(X, Y)$.

**Proposition 4.27.** *Let $X$ be a diffeological space. Then the natural map $\gamma : T(C^\infty(X, \mathbb{R}^n)) \to C^\infty(X, T\mathbb{R}^n)$ from Proposition 4.25 is an isomorphism of diffeological vector spaces over $C^\infty(X, \mathbb{R}^n)$.*

When $X$ is discrete, this shows that $T(\prod_{j\in X} \mathbb{R}^n) \cong \prod_{j\in X} T\mathbb{R}^n$ as diffeological vector spaces over $\prod_{j\in X} \mathbb{R}^n$; see Proposition 3.7, Remark 3.8 and Example 4.24(1).

*Proof.* We first prove that for each $f \in C^\infty(X, \mathbb{R}^n)$, the restriction $\gamma_f : T_f(C^\infty(X, \mathbb{R}^n)) \to S_f(X, \mathbb{R}^n)$ of $\gamma$ to the fibre over $f$ is an isomorphism of vector spaces.

Since in this case, $\gamma$ is a smooth map between diffeological vector spaces over $C^\infty(X, \mathbb{R}^n)$, it is enough to prove that $\gamma_f$ is a bijection for $n = 1$.

We first prove injectivity. Note that $C^\infty(X, \mathbb{R})$ is a diffeological group. Corollary 6.5 of [HM] says that for any diffeological group $G$, every element of $T_e(G)$ comes from a plot $U \to G$. (Compare with Example 4.3.) Since $T_e(G) \cong T_g(G)$, the same is true for $T_g(G)$. By the proof of Proposition 3.3, every internal tangent vector is in fact of the form $p_*(\frac{d}{dt})$, where $p : \mathbb{R} \to G$ is a plot with $p(0) = g$. Also note that:

(1) If $\alpha : U \times X \to \mathbb{R}$ is smooth for $U$ some open subset of a Euclidean space, then $\frac{\partial \alpha}{\partial u_i} : U \times X \to \mathbb{R}$ is also smooth for any coordinate $u_i$ in $U$, since for any plot $\beta : W \to X$, $\frac{\partial \alpha}{\partial u_i} \circ (1_U \times \beta) = \frac{\partial}{\partial u_i}(\alpha \circ (1_U \times \beta))$.

(2) If $K$ is a compact subset of $U$, $\phi : K \to \mathbb{R}$ is an integrable function, and $F : U \times X \to \mathbb{R}$ is smooth, then by [G, Theorem V.2.9.9], the function $X \to \mathbb{R}$ defined by $\int_K \phi(s) F(s, x)\, ds$ is smooth.

Now assume that a plot $p : \mathbb{R} \to C^\infty(X, \mathbb{R})$ with $p(0) = f$ induces $0 \in S_f(X, \mathbb{R})$; that is, the adjoint map $\tilde{p} : \mathbb{R} \times X \to \mathbb{R}$ is smooth, $\tilde{p}(0, x) = f(x)$ and $\frac{\partial \tilde{p}}{\partial t}(0, x) = 0$ for all $x \in X$. We can conclude that $\tilde{p}(t, x) = f(x) + tg(t, x)$ for $g(t, x) = \int_0^1 (D_1\tilde{p})(st, x)\, ds$, and this $g$ is in $C^\infty(\mathbb{R} \times X, \mathbb{R})$ by (1) and (2). Note that $g(0, x) = 0$ for all $x \in X$. Then we define $q : \mathbb{R}^2 \to C^\infty(X, \mathbb{R})$ by $q(t_1, t_2)(x) = f(x) + t_1 g(t_2, x)$. The restriction of $q$ to either axis is a constant map $\mathbb{R} \to C^\infty(X, \mathbb{R})$, so $q_*(\frac{d}{dt_1}) = 0 = q_*(\frac{d}{dt_2})$ in $T_f(C^\infty(X, \mathbb{R}))$. And the restriction of $q$ to the diagonal is $p$, so, as in the argument given in Example 3.25, $p_*(\frac{d}{dt}) = 0$ in $T_f(C^\infty(X, \mathbb{R}))$.

For surjectivity, take any $(f, g) \in S_f(X, \mathbb{R})$; that is $g : X \to \mathbb{R}$ is any smooth map. Define $p : \mathbb{R} \to C^\infty(X, \mathbb{R})$ by $t \mapsto (x \mapsto f(x) + tg(x))$. It is straightforward to check that $p$ is a plot with $p(0) = f$ and that the map $\gamma_f : T_f(C^\infty(X, \mathbb{R})) \to S_f(X, \mathbb{R})$ sends $p_*(\frac{d}{dt})$ to $g$.

Hence, together with Proposition 4.25, we have proved so far that the map $\gamma : T(C^\infty(X, \mathbb{R}^n)) \to C^\infty(X, T\mathbb{R}^n)$ is a smooth bijection and linear on each fibre. We claim that in this case, $\gamma^{-1}$ is also smooth, and hence $\gamma$ is an isomorphism of diffeological vector spaces over $C^\infty(X, \mathbb{R}^n)$. Here is the proof. Notice that it is enough to prove this for $n = 1$. For any plot $q = (q_1, q_2) : U \to C^\infty(X, T\mathbb{R})$, we define a smooth map $\rho : \mathbb{R} \times U \to C^\infty(X, \mathbb{R})$ by $\rho(t, u) = q_1(u) + tq_2(u)$. The smoothness of $\gamma^{-1}$ follows from the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad q \quad} & C^\infty(X, T\mathbb{R}) \\ \big\downarrow & & \big\downarrow \gamma^{-1} \\ T\mathbb{R} \times TU & \xrightarrow[T\rho]{} & T(C^\infty(X, \mathbb{R})), \end{array}$$

where the left vertical map is given by $u \mapsto ((0, \frac{d}{dt}), (u, 0))$. □

As direct consequences, the map $\gamma_f : T_f(C^\infty(X, \mathbb{R}^n)) \to S_f(X, \mathbb{R}^n)$ is an isomorphism between diffeological vector spaces, and

$$C^\infty(X, \mathbb{R}^n) \times C^\infty(X, \mathbb{R}^n) \cong C^\infty(X, T\mathbb{R}^n) \cong T(C^\infty(X, \mathbb{R}^n))$$

as diffeological vector spaces over $C^\infty(X, \mathbb{R}^n)$. Moreover, we have:

**Proposition 4.28.** *Let $X$ be a diffeological space such that its D-topology is compact, and let $N$ be a smooth manifold. Then the natural map*

$\gamma : T^H(C^\infty(X,N)) \to C^\infty(X,TN)$ *from Proposition 4.25 is an isomorphism of diffeological vector spaces over* $C^\infty(X,N)$.

*Proof.* Let $N \to \mathbb{R}^n$ be a smooth embedding, and let $U$ be an open tubular neighborhood of $N$ with inclusions $i : N \to U$ and $j : U \to \mathbb{R}^n$, and a smooth retraction $r : U \to N$ such that $r \circ i = 1_N$. Since the $D$-topology on $X$ is compact, [CSW, Proposition 4.2] implies that $C^\infty(X,U)$ is a $D$-open subset of $C^\infty(X,\mathbb{R}^n)$, and hence by Proposition 4.13(1), $j_* : T^H(C^\infty(X,U)) \to T(C^\infty(X,\mathbb{R}^n))$ is an induction, and for each $f$ in $C^\infty(X,U)$, the restriction $j_{*,f} : T^H_f(C^\infty(X,U)) \to T_f(C^\infty(X,\mathbb{R}^n))$ is an isomorphism. It is straightforward to see that $j_* : C^\infty(X,TU) \to C^\infty(X,T\mathbb{R}^n)$ is also an induction, and for each $f \in C^\infty(X,U)$, the restriction $j_{*,f} : S_f(X,U) \to S_f(X,\mathbb{R}^n)$ is an isomorphism, since $U$ is open in $\mathbb{R}^n$. So by Proposition 4.27 and the functoriality of $\gamma$, we know that $\gamma^U : T^H(C^\infty(X,U)) \to C^\infty(X,TU)$ is an isomorphism of diffeological vector spaces over $C^\infty(X,U)$. Since $\gamma : T^H(C^\infty(X,N)) \to C^\infty(X,TN)$ is a retract of $\gamma^U$, it is also an isomorphism of diffeological vector spaces over $C^\infty(X,N)$. □

In this case, since $T^H(C^\infty(X,N))$ is a diffeological vector space over $C^\infty(X,N)$, it follows that $T^H(C^\infty(X,N)) = T^{dvs}(C^\infty(X,N))$.

Note that, if $M$ is a smooth manifold, then $\mathfrak{D}\mathrm{iff}(M)$, the set of all diffeomorphisms from $M$ to itself, is a diffeological group, when equipped with the sub-diffeology of $C^\infty(M,M)$. So we recover Proposition 6.3 of [HM]:

**Corollary 4.29.** *Let* $M$ *be a compact smooth manifold. Then* $T_{1_M}(\mathfrak{D}\mathrm{iff}(M))$ *is isomorphic to the vector space of all smooth vector fields on* $M$.

*Proof.* This follows from Propositions 3.6 and 4.28, and the fact that $\mathfrak{D}\mathrm{iff}(M)$ is a $D$-open subset of $C^\infty(M,M)$ since $M$ is a compact smooth manifold [CSW, Corollary 4.15]. □

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# EQUALIZERS IN KLEISLI CATEGORIES

*by S. N. HOSSEINI and Y. QASEMI NEZHAD*

**Résumé.** Dans cet article, nous donnons des conditions nécessaires et suffisantes pour qu'une paire de morphismes d'une catégorie de Kleisli, associée à une monade générale, ait un égalisateur. Nous proposons aussi, dans différents cas de monades intéressantes, un meilleur critère pour l'existence d'un égalisateur et dans ces cas nous explicitons ce qu'est l'égalisateur (lorsqu'il existe).

**Abstract.** In this article, we give necessary and sufficient conditions for a pair of morphisms in a Kleisli category, corresponding to a general monad, to have an equalizer. We also present a better criterion for equalizers in a number of cases of interesting monads, and in all these cases we explain what an equalizer (if it exists) of a pair of morphisms is.

**Keywords.** equalizer, Kleisli category, (representation, add-point or exception, $M$-set, power object) monad.

**Mathematics Subject Classification (2010).** 18A30, 18C15, 18C20.

## 1. Introduction

The richness of a given category depends on to what degree the category is complete and/or cocomplete. In particular this is true about the kleisli categories, that are being widely used in different areas such as, the semantics of linear logic, [2], computing, [6], Maltsev varieties, [5], extension of functors, [11], factorization-related monads, [4], and information systems, [9], to mention a few.

In [10], the completeness/cocompleteness of Kleisli categories are tackled, however as the author mentions, the results are powerless in concrete instances. In this article, we attempt the problem of the existence of equalizers in Kleisli categories. It is known that the category of sets and relations, the Kleisli category for the power set monad, does not have equalizers, [3].

So we try to answer the question of when a given pair of morphisms has equalizers. First we give some equivalent conditions for the existence of equalizers of a given pair of maps in a general Kleisli category. Then we present a better criterion for the existence of equalizers in a number of cases of interesting monads.

## 2. Preliminaries

A monad on a category $\mathcal{E}$, [1], is a triple $\mathbb{T} = (T, \eta, \mu)$, where $T : \mathcal{E} \longrightarrow \mathcal{E}$ is a functor, $\eta : I \longrightarrow T$ and $\mu : T^2 \longrightarrow T$ are natural transformations rendering commutative the following square and triangles.

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array} \qquad\qquad \begin{array}{ccccc} T & \xrightarrow{T\eta} & T^2 & \xleftarrow{\eta T} & T \\ & {\scriptstyle 1_T}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1_T} & \\ & & T & & \end{array}$$

Some examples of monads that we use in this article, are:

**Example 2.1.** $a)$ [1], [7], [12]. Let $\mathcal{E}$ be a category in which partial morphisms are represented, such as a quasitopos. Let $\eta_X : X \longrightarrow \tilde{X}$ represent partial morphisms to $X$ (such a map is mono) and be universal, i.e., pullback of $\eta_X$ along all morphisms exists. The functor $\sim : \mathcal{E} \longrightarrow \mathcal{E}$, taking $X$ to $\tilde{X}$, the natural transformation $\eta : I \longrightarrow \sim$ and the natural transformation $\mu : \sim^2 \longrightarrow \sim$, where $\mu_X : \tilde{\tilde{X}} \longrightarrow \tilde{X}$ is defined by the following pullback square:

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle \eta_{\tilde{X}}\eta_X}\downarrow & p.b. & \downarrow{\scriptstyle \eta_X} \\ \tilde{\tilde{X}} & \xrightarrow[\exists!\mu_X]{} & \tilde{X} \end{array}$$

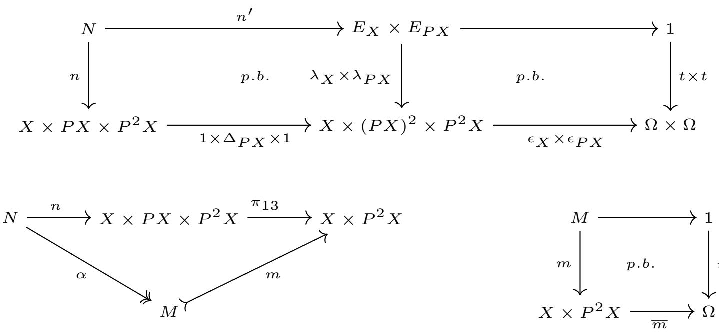
form a monad $\mathbb{R} = (\sim, \eta, \mu)$, called the representation monad.

$b)$ [3]. Let $\mathcal{E}$ be a category with a coproductive terminal object 1, i.e., a terminal object whose coproduct with any other object exists. The functor $A = - \amalg 1 : \mathcal{E} \longrightarrow \mathcal{E}$, the natural transformation $\eta : I \longrightarrow A$, with

$\eta_X = \nu_1 : X \longrightarrow X \amalg 1$, and the natural transformation $\mu : A^2 \longrightarrow A$ defined by $\mu_X = 1 \bigoplus \nu_2 : (X \amalg 1) \amalg 1 \longrightarrow X \amalg 1$, where $\nu_1$ and $\nu_2$ are respectively the first and the second injections of the coproduct, form a monad $\mathbb{A} = (A, \eta, \mu)$, called the add-point or exception monad.

$c)$ [3]. Let $(M, \star, 1)$ be a monoid. The functor $M : Set \longrightarrow Set$ defined on objects by $\hat{M}(X) = M \times X$, together with the natural transformations $\eta$ and $\mu$, with $\eta_X(x) = (1, x)$ and $\mu_X(m_1, (m_2, x)) = (m_1 \star m_2, x)$ form a monad $\mathbb{M} = (\hat{M}, \eta, \mu)$ called the $M$-Set monad. We denote $m \star n$ also by $mn$. Note that $\eta_X = \langle \tilde{1}, 1_X \rangle$, with $\tilde{1}$ the constant map with value $1 \in M$ and $1_X$ the identity function. Also $\mu_X \cong \star \times 1_X$.

$d)$ [1], [8], [12]. Let $\mathcal{E}$ be a topos. The covariant power object functor $P : \mathcal{E} \longrightarrow \mathcal{E}$, where $P( X \xrightarrow{f} Y ) = PX \xrightarrow{\exists_f} PY$ , the singleton natural transformation $I \xrightarrow{\eta} P$ and the monad multiplication $\mu : P^2 \longrightarrow P$, where $\mu_X = \tilde{\overline{m}} : P^2X \longrightarrow PX$ is the transpose of $\overline{m} : X \times P^2X \longrightarrow \Omega$ and $\overline{m}$ is defined as follows:




with $\epsilon$ denoting the evaluation, $\Delta$ the diagonal and $\pi_{13}$ the evident projection, form a monad $\mathbb{P} = (P, \eta, \mu)$, called the power object monad.

For naturality of $\mu$ in part $(c)$ of 2.1, see [12]. We show $\mu$ commutes with the internal join. But first, with:

$$U : Sub(X \times (PX)^2) \times Sub(X \times (PX)^2) \longrightarrow Sub(X \times (PX)^2)$$

the external join or union, $\vee_X : (PX)^2 \longrightarrow PX$ the internal join and the map $f^{-1}(g)$ denoting the pullback of $g$ along $f$, we have:

**Lemma 2.2.** *The map $v$ obtained by the pullback:*

$$\begin{array}{ccc} V & \xrightarrow{v'} & E_X \\ {\scriptstyle v}\downarrow & p.b. & \downarrow{\scriptstyle \lambda_X} \\ X \times (PX)^2 & \xrightarrow[1\times\vee_X]{} & X \times PX \end{array}$$

*is* $v = \pi_{12}^{-1}(\lambda_X) \cup \pi_{13}^{-1}(\lambda_X) = (\lambda_X \times 1) \cup \pi_{132}(\lambda_X \times 1)$, *where the maps* $X \times (PX)^2 \overset{\pi_{12}}{\underset{\pi_{13}}{\rightrightarrows}} X \times PX$ *and* $\pi_{132} : X \times (PX)^2 \longrightarrow X \times (PX)^2$ *are the evident projections.*

*Proof.* By the above pullback diagram, the classifying map of $v$ is $\hat{v} = \epsilon_X(1 \times \vee_X)$. Now form the following pullbacks.

$$\begin{array}{ccc} V_1 & \xrightarrow{v_1'} & E_x \\ {\scriptstyle v_1}\downarrow & p.b. & \downarrow{\scriptstyle \lambda_X} \\ X \times (PX)^2 & \xrightarrow[\pi_{12}]{} & X \times PX \end{array} \qquad \begin{array}{ccc} V_2 & \xrightarrow{v_2'} & E_x \\ {\scriptstyle v_2}\downarrow & p.b. & \downarrow{\scriptstyle \lambda_X} \\ X \times (PX)^2 & \xrightarrow[\pi_{13}]{} & X \times PX \end{array}$$

Take the epi-mono factorization of $v_1 \oplus v_2 : V_1 \amalg V_2 \longrightarrow X \times (PX)^2$ to get:

$$\begin{array}{ccc} V_1 \amalg V_2 & \xrightarrow{v_1 \oplus v_2} & X \times (PX)^2 \\ & \searrow \quad \nearrow{\scriptstyle v_1 \cup v_2} & \\ & V' & \end{array}$$

The following diagram shows $\widehat{v_1 \cup v_2} = \epsilon_X(1 \times \vee_X)$.

$$\begin{array}{ccc} (v_1, v_2) \in Sub(X \times (PX)^2) \times Sub(X \times (PX)^2) & \xrightarrow{\cup} & Sub(X \times (PX)^2) \ni v_1 \cup v_2 \\ \wr\wr & & \wr\wr \\ hom(X \times (PX)^2, \Omega) \times hom(X \times (PX)^2, \Omega) & \xrightarrow{\vee} & hom(X \times (PX)^2, \Omega) \ni \epsilon_X(1 \times \vee_X) \\ \wr\wr & & \wr\wr \\ hom((PX)^2, PX) \times hom((PX)^2, PX) & \xrightarrow{\vee} & hom((PX)^2, PX) \ni \vee_X \\ \wr\wr & & \| \\ 1 \in hom((PX)^2, (PX)^2) & \xrightarrow{\vee} & hom((PX)^2, PX) \ni \vee_X \end{array}$$

It follows that $\hat{v} = \widehat{v_1 \cup v_2}$ and so $v = v_1 \cup v_2$, proving the first equality. Since the squares:

$$\begin{array}{ccc} E_X \times PX & \xrightarrow{\pi_1} & E_X \\ {\scriptstyle \lambda_X \times 1}\downarrow & & \downarrow{\scriptstyle \lambda_X} \\ X \times (PX)^2 & \xrightarrow[\pi_{12}]{} & X \times PX \end{array} \quad \text{and} \quad \begin{array}{ccc} E_X \times PX & \xrightarrow{\pi_1} & E_X \\ {\scriptstyle \pi_{132}(\lambda_X \times 1)}\downarrow & & \downarrow{\scriptstyle \lambda_X} \\ X \times (PX)^2 & \xrightarrow[\pi_{13}]{} & X \times PX \end{array}$$

are pullbacks, we get the second equality.

□

**Theorem 2.3.** *The monad multiplication $\mu$ of $\mathbb{P}$ preserves the internal join, i.e., for each $X$ the following square commutes.*

$$\begin{array}{ccc} (P^2X)^2 & \xrightarrow{\vee_{PX}} & P^2X \\ {\scriptstyle \mu_X^2 = \mu_X \times \mu_X}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ (PX)^2 & \xrightarrow[\vee_X]{} & PX \end{array}$$

*Proof.* We have $\vee_X \mu_X^2 = \mu_X \vee_{PX}$ if and only if their transposes are equal, i.e., $\epsilon_X(1_X \times \vee_X \mu_X^2) = \epsilon_X(1_X \times (\mu_X \vee_{PX}))$ if and only if $\epsilon_X(1_X \times \vee_X)(1_X \times \mu_X^2) = \epsilon_X(1_X \times \mu_X)(1_X \times \vee_{PX})$ if and only if $w = k$, where $w$ and $k$ are obtained by the following pullbacks.

$$\begin{array}{ccc} W & \xrightarrow{w'} & V \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v} \\ X \times (P^2X)^2 & \xrightarrow[1 \times \mu^2]{} & X \times (PX)^2 \end{array} \quad \text{and} \quad \begin{array}{ccccc} K & \xrightarrow{k'} & M & \xrightarrow{m'} & E_X \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \lambda_X} \\ X \times (P^2X)^2 & \xrightarrow[1 \times \vee]{} & X \times P^2X & \xrightarrow[1 \times \mu]{} & X \times PX \end{array}$$

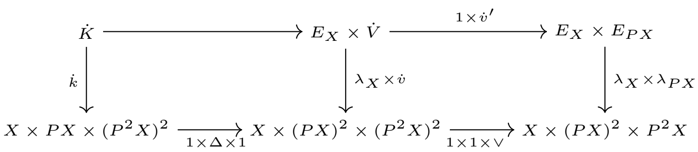
Form the pullback squares:

$$\begin{array}{ccc} \dot{K} & \xrightarrow{\dot{k}'} & N \\ {\scriptstyle \dot{k}}\downarrow & & \downarrow{\scriptstyle n} \\ X \times PX \times (P^2X)^2 & \xrightarrow[1 \times 1 \times \vee]{} & X \times PX \times P^2X \end{array} \quad \text{and} \quad \begin{array}{ccc} \dot{V} & \xrightarrow{\dot{v}'} & E_{PX} \\ {\scriptstyle \dot{v}}\downarrow & & \downarrow{\scriptstyle \lambda_{PX}} \\ PX \times (P^2X)^2 & \xrightarrow[1 \times \vee]{} & PX \times P^2X \end{array}$$

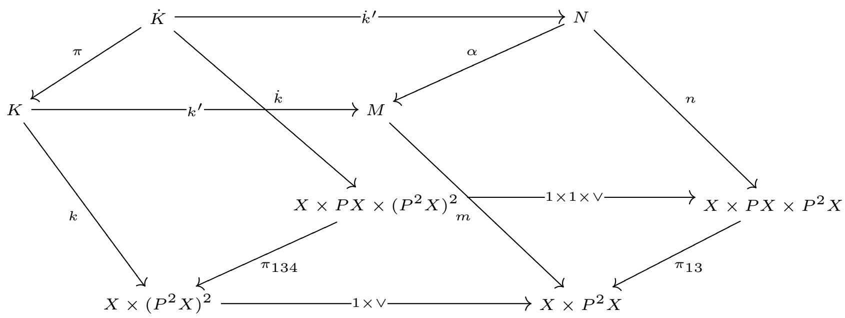
So we have the following diagram in which both squares are pullbacks.

$$\begin{array}{ccccc} \dot{K} & \xrightarrow{\dot{k}'} & N & \xrightarrow{n'} & E_X \times E_{PX} \\ {\scriptstyle \dot{k}}\downarrow & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle \lambda_X \times \lambda_{PX}} \\ X \times PX \times (P^2X)^2 & \xrightarrow[1 \times 1 \times \vee]{} & X \times PX \times P^2X & \xrightarrow[1 \times \Delta \times 1]{} & X \times (PX)^2 \times P^2X \end{array}$$

Since $(1 \times \Delta \times 1)(1 \times 1 \times \vee) = (1 \times 1 \times \vee)(1 \times \Delta \times 1)$, we get the following pullback squares.




Therefore in the following cube, the right and left faces are commutative and all the other faces are pullbacks. Also since $\alpha$ is epi, so is it's pullback $\pi$.



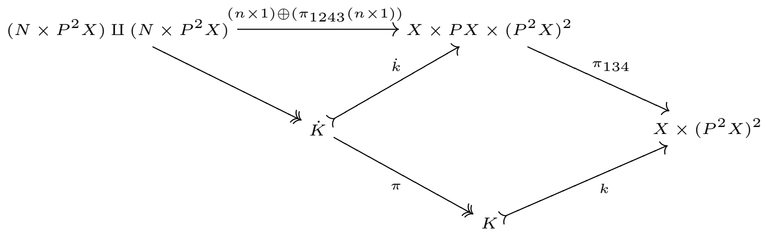

by 2.2, $\dot{v} = (\lambda_{PX} \times 1) \cup \pi_{132}(\lambda_{PX} \times 1)$ and so:

$$\lambda_X \times \dot{v} = \lambda_X \times [(\lambda_{PX} \times 1) \cup \pi_{132}(\lambda_{PX} \times 1)] =$$
$$[\lambda_X \times (\lambda_{PX} \times 1)] \cup [\lambda_X \times (\pi_{132}(\lambda_{PX} \times 1))]$$

Therefore

$$\dot{k} = (1 \times \triangle \times 1)^{-1}(\lambda_X \times \dot{v}) =$$
$$[(1 \times \triangle \times 1)^{-1}(\lambda_X \times (\lambda_{PX} \times 1))] \cup [(1 \times \triangle \times 1)^{-1}(\lambda_X \times (\pi_{132}(\lambda_{PX} \times 1)))] =$$
$$[(1 \times \triangle \times 1 \times 1)^{-1}((\lambda_X \times \lambda_{PX}) \times 1)] \cup [(1 \times \triangle \times 1 \times 1)^{-1}(\pi_{1243}((\lambda_X \times \lambda_{PX}) \times 1))] =$$
$$(n \times 1) \cup \pi_{1243}(n \times 1)$$

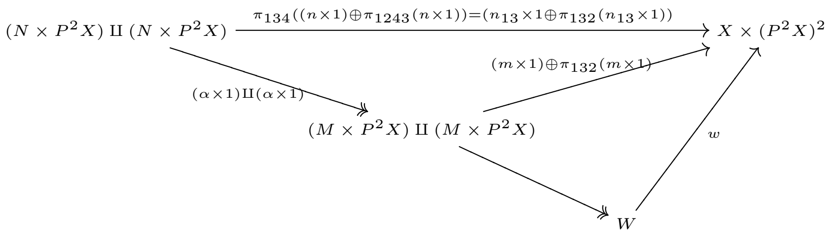
Hence taking the epi-mono factorization of $(n \times 1) \oplus (\pi_{1243}(n \times 1))$, we get the below commutative triangle; while the bottom square is the left face of the above cube.




On the other hand using 2.2, we have:

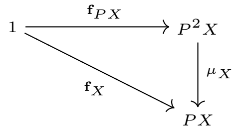
$$
\begin{gathered}
w = (1 \times \mu^2)^{-1}(v) = \\
(1 \times \mu^2)^{-1}[\pi_{12}^{-1}(\lambda_X) \cup \pi_{13}^{-1}(\lambda_X)] = \\
[(\pi_{12}(1 \times \mu^2))^{-1}(\lambda_X)] \cup [(\pi_{13}(1 \times \mu^2))^{-1}(\lambda_X)] = \\
[((1 \times \mu)\pi_{12})^{-1}(\lambda_X)] \cup [((1 \times \mu)\pi_{13})^{-1}(\lambda_X)] = \\
[\pi_{12}^{-1}(1 \times \mu)^{-1}(\lambda_X)] \cup [\pi_{13}^{-1}(1 \times \mu)^{-1}(\lambda_X)] = \\
[\pi_{12}^{-1}(m)] \cup [\pi_{13}^{-1}(m)] = \\
(m \times 1) \cup \pi_{132}(m \times 1)
\end{gathered}
$$

In the following diagram, the commutativity of the top triangle can be easily verified and that of the bottom triangle follows from the fact that $w = (m \times 1) \cup \pi_{132}(m \times 1)$.




Therefore $w$ and $k$ are the mono part of the morphism $\pi_{134}((n \times 1) \oplus \pi_{1243}(n \times 1))$ and so are equal. □

**Lemma 2.4.** *The monad multiplication $\mu$ of $\mathbb{P}$ preserves the false map, i.e., for each $X$ the following triangle commutes.*



*Proof.* Since $\mathbf{f}_{PX} \leq \eta_{PX}\mathbf{f}_X$, by 2.3, $\mu_X\mathbf{f}_{PX} \leq \mu_X\eta_{PX}\mathbf{f}_X$. So $\mu_X\mathbf{f}_{PX} \leq \mathbf{f}_X$, implying $\mu_X\mathbf{f}_{PX} = \mathbf{f}_X$.

□

## 3. Preservation of Equalizers

Denoting the Kleisli category of a monad $\mathbb{T}$ by $\mathcal{E}_{\mathbb{T}}$ and the morphism in $\mathcal{E}_{\mathbb{T}}$ associated to $f : X \longrightarrow TY$ in $\mathcal{E}$ by $\hat{f} : X \longrightarrow Y$, we have:

**Lemma 3.1.** *For a monad $\mathbb{T}$ in $\mathcal{E}$, the functor $U : \mathcal{E}_{\mathbb{T}} \longrightarrow \mathcal{E}$ defined by:*

$$U(\, X \xrightarrow{\hat{f}} Y \,) = \; TX \xrightarrow{\mu_Y T(f)} TY$$

*is right adjoint to the functor $I : \mathcal{E} \longrightarrow \mathcal{E}_{\mathbb{T}}$ defined by:*

$$I(\, X \xrightarrow{f} Y \,) = \; X \xrightarrow{\widehat{\eta_Y f}} Y$$

*Proof.* See [1], [3]. □

**Proposition 3.2.** *The functor $U : \mathcal{E}_{\mathbb{T}} \longrightarrow \mathcal{E}$ preserves and reflects equalizers.*

*Proof.* The preservation of equalizers follows from 3.1. To prove $U$ reflects equalizers, let $E \xrightarrow{\hat{e}} X \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} Y$ be a diagram in $\mathcal{E}_{\mathbb{T}}$ such that the diagram $TE \xrightarrow{\mu_X T(e)} TX \underset{\mu_Y T(g)}{\overset{\mu_Y T(f)}{\rightrightarrows}} TY$ is an equalizer in $\mathcal{E}$. It follows that $\mu_Y T(f)\mu_X T(e) = \mu_Y T(g)\mu_X T(e)$. Also by monad axioms $\mu_X\eta_{TX} = 1_{TX}$ and by naturality of $\eta$, $\eta_{TX}e = T(e)\eta_E$. So $\mu_Y T(f)e = \mu_Y T(f)\mu_X\eta_{TX}e = \mu_Y T(f)\mu_X T(e)\eta_E = \mu_Y T(g)\mu_X T(e)\eta_E = \mu_Y T(g)\mu_X\eta_{TX}e = \mu_Y T(g)e$ and thus $\hat{f}\hat{e} = \hat{g}\hat{e}$.

Now if there is $\hat{k} : Z \longrightarrow X$ such that $\hat{f}\hat{k} = \hat{g}\hat{k}$, then $\mu_Y T(f)k = \mu_Y T(g)k$ and so there is a unique $\bar{k} : Z \longrightarrow TE$ such that $\mu_X T(e)\bar{k} = k$. Therefore $\hat{\bar{k}} : Z \longrightarrow E$ is in $\mathcal{E}_{\mathbb{T}}$ and $\hat{e}\hat{\bar{k}} = \hat{k}$.

If there is $\hat{k'}$ such that $\hat{e}\hat{k'} = \hat{k}$, then $\widehat{\mu_X T(e) k'} = \hat{k}$. It follows that $\mu_X Tek' = k$ and so $k' = \bar{k}$. Therefore $\hat{\bar{k}} = \hat{k'}$ and the result follows.

□

**Theorem 3.3.** *Let $\mathcal{E}$ be a category and $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{E}$. For morphisms $\hat{f}, \hat{g} : X \longrightarrow Y$ in $\mathcal{E}_{\mathbb{T}}$, the following conditions are equivalent.*

*$a)$ In $\mathcal{E}_{\mathbb{T}}$, there is a morphism $E \xrightarrow{\hat{e}} X$ such that $E \xrightarrow{\hat{e}} X \overset{\hat{f}}{\underset{\hat{g}}{\rightrightarrows}} Y$ is an equalizer.*

*$b)$ In $\mathcal{E}$, there is a morphism $e : E \longrightarrow TX$ such that the diagram $TE \xrightarrow{\mu_X T(e)} TX \overset{\mu_Y T(f)}{\underset{\mu_Y T(g)}{\rightrightarrows}} TY$ is an equalizer.*

*$c)$ In $\mathcal{E}$, there is a morphism $m : M \longrightarrow TX$, an object $E$ and an isomorphism $\varphi : TE \longrightarrow M$ such that $M \xrightarrow{m} TX \overset{\mu_Y T(f)}{\underset{\mu_Y T(g)}{\rightrightarrows}} TY$ is an equalizer and the following diagram is commutative.*

$$\begin{array}{ccc} T^2E & \xrightarrow{\mu_E} & TE \\ {\scriptstyle T(m\varphi)}\downarrow & & \downarrow{\scriptstyle m\varphi} \\ T^2X & \xrightarrow[\mu_X]{} & TX \end{array}$$

*In this case the morphism $\hat{e}$ of part $(a)$ corresponds to the morphism $e$ of part $(b)$ which in turn corresponds to the morphism $m\varphi\eta_E$ of part $(c)$.*

*Proof.* The equivalence of $(a)$ and $(b)$ follows from 3.2.

$(b) \Rightarrow (c)$ : Setting $M = TE$, $m = \mu_X T(e)$ and $\varphi = id_{TE}$, it is enough to show the square in $(c)$ commutes. By naturality of $\mu$ and monad definition we have, $\mu_X T(e)\mu_E = \mu_X \mu_{TX} T^2 e = \mu_X T\mu_X T^2 e = \mu_X T(\mu_X T(e))$, as desired.

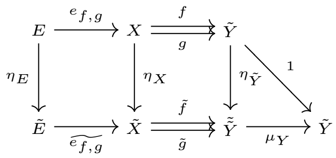
$(c) \Rightarrow (b)$ : Setting $e = m\varphi\eta_E : E \longrightarrow TX$, we have, $\mu_X T(e) = \mu_X T(m\varphi\eta_E) = \mu_X T(m\varphi)T\eta_E = m\varphi\mu_E T\eta_E = m\varphi$. The result now follows from the fact that $\varphi$ is an isomorphism and $m$ is an equalizer of $\mu_Y T(f)$ and $\mu_Y T(g)$.

The last assertion holds obviously. $\square$

**Lemma 3.4.** *Let partial morphisms in a category $\mathcal{E}$ be representable by universal arrows, and let $\mathbb{R}$ be the representation monad. The morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} \tilde{Y}$ have an equalizer in $\mathcal{E}$ if and only if the morphisms $\tilde{X} \overset{\mu_Y \tilde{f}}{\underset{\mu_Y \tilde{g}}{\rightrightarrows}} \tilde{Y}$ do.*

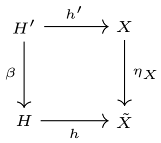
*In this case an equalizer of* $\mu_Y \tilde{f}$ *and* $\mu_Y \tilde{g}$ *is* $\widetilde{e_{f,g}}$, *where* $e_{f,g}$ *is an equalizer of* $f$ *and* $g$.

*Proof.* Suppose an equalizer $e_{f,g} : E \longrightarrow X$ of $f$ and $g$ exists in $\mathcal{E}$. Consider the following diagram in which all the squares are pullbacks and the triangle commutes.

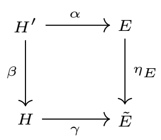


We show $\widetilde{e_{f,g}}$ is an equalizer of $\mu\tilde{f}$ and $\mu\tilde{g}$. Pulling back $\eta_Y$ along $\mu_Y \tilde{f} \widetilde{e_{f,g}}$ and $\mu_Y \tilde{g} \widetilde{e_{f,g}}$, and using the fact that $f e_{f,g} = g e_{f,g}$, we get the same pullback square. Since $\eta_Y$ represents partial morphisms, by uniqueness we get $\mu_Y \tilde{f} \widetilde{e_{f,g}} = \mu_Y \tilde{g} \widetilde{e_{f,g}}$.

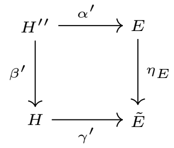
Now suppose $h : H \longrightarrow \tilde{X}$ is given such that $\mu_Y \tilde{f} h = \mu_Y \tilde{g} h$. Form the following pullback.



We have $fh' = \mu_Y \eta_{\tilde{Y}} f h' = \mu_Y \tilde{f} \eta_X h' = \mu_Y \tilde{f} h \beta = \mu_Y \tilde{g} h \beta = \mu_Y \tilde{g} \eta_X h' = \mu_Y \eta_{\tilde{Y}} g h' = g h'$. So there is a unique morphism $\alpha : H' \longrightarrow E$ such that $e_{f,g}\alpha = h'$. Now there is a unique morphism $\gamma$ rendering pullback the following square.

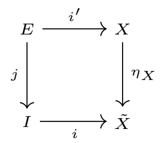


Now pullback of $\eta_X$ along $\widetilde{e_{f,g}}\gamma$ and along $h$ yields the same 2-source. So by uniqueness, $\widetilde{e_{f,g}}\gamma = h$. Now if $\gamma'$ satisfies $\widetilde{e_{f,g}}\gamma' = h$, then $\widetilde{e_{f,g}}\gamma' = \widetilde{e_{f,g}}\gamma$. Form the following pullback.



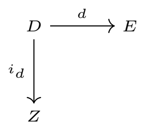

The equality $\widetilde{e_{f,g}}\gamma' = \widetilde{e_{f,g}}\gamma$, yields $H'' = H'$, $\beta' = \beta$ and $e_{f,g}\alpha = e_{f,g}\alpha'$. Since $e_{f,g}$ is mono, $\alpha = \alpha'$. Since $\eta_E$ represents partial morphisms, uniqueness yields $\gamma = \gamma'$, as desired.

Conversely suppose an equalizer $i : I \longrightarrow \tilde{X}$ of $\mu_Y \tilde{f}$ and $\mu_Y \tilde{g}$ exists in $\mathcal{E}$. Form the following pullback.

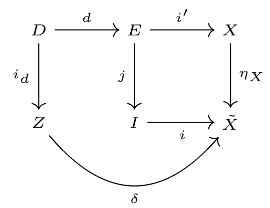



One can easily verify that $fi' = gi'$. Now suppose $h : H \longrightarrow X$ is given such that $fh = gh$. We have $\mu_Y \tilde{f}\eta_X h = \mu_Y \eta_{\tilde{Y}} fh = \mu_Y \eta_{\tilde{Y}} gh = \mu_Y \tilde{g}\eta_X h$. So there is a unique $\alpha : H \longrightarrow I$ such that $i\alpha = \eta_X h$. The above pullback square now yields a unique $k : H \longrightarrow E$ such that $i'k = h$ and $jk = \alpha$. Now if there is $k'$ such that $i'k' = h$, then $ijk' = \eta_X i'k' = \eta_X h = i\alpha$. Since $i$ is mono, $jk' = \alpha$. uniqueness implies $k' = k$. This proves an equalizer of $f$ and $g$ exists and is $i'$.

The last assertion holds for the direct implication. To prove it for the converse, first we show the morphism $j$ in the above pullback square is a partial morphism classifier. Let the partial morphism:




be given. There is a unique morphism $\delta$ making the big square below a pullback.



Since $fi'd = gi'd$, the pullback of $\eta_Y$ along both $\mu_Y \tilde{f}\delta$ and $\mu_Y \tilde{g}\delta$ yields the same 2-source. Uniqueness of representation gives $\mu_Y \tilde{f}\delta = \mu_Y \tilde{g}\delta$. Therefore there is a unique $\lambda : Z \longrightarrow I$ such that $i\lambda = \delta$. It can be easily shown that the square:

$$\begin{array}{ccc} D & \xrightarrow{d} & E \\ {\scriptstyle i_d}\downarrow & & \downarrow{\scriptstyle j} \\ Z & \xrightarrow[\lambda]{} & I \end{array}$$

commutes and so is a pullback.

To show uniqueness, suppose there is $\lambda' : Z \longrightarrow I$ making the above square a pullback. Then $i\lambda'$ makes the above big square a pullback and so $i\lambda' = i\lambda$. Therefore $\lambda' = \lambda$, as desired. Hence $j$ is a partial morphism classifier.

Since $\eta_E$ is also a partial morphism classifier, there is an isomorphism $\varphi : \tilde{E} \longrightarrow I$ such that $\varphi\eta_E = j$. It follows that the square:

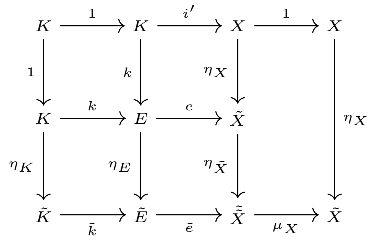
$$\begin{array}{ccc} E & \xrightarrow{i'} & X \\ {\scriptstyle \eta_E}\downarrow & & \downarrow{\scriptstyle \eta_X} \\ \tilde{E} & \xrightarrow[i\varphi]{} & \tilde{X} \end{array}$$

is a pullback and that $i\varphi = \widetilde{i'}$ is an equalizer of $\mu_Y \tilde{f}$ and $\mu_Y \tilde{g}$. The result follows. □

**Theorem 3.5.** *Let partial morphisms in a category $\mathcal{E}$ be representable by universal arrows, and let $\mathbb{R}$ be the representation monad. The morphisms $X \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} Y$ in the Kleisli category $\mathcal{E}_{\mathbb{R}}$ have an equalizer if and only if the morphisms $X \underset{g}{\overset{f}{\rightrightarrows}} \tilde{Y}$ in $\mathcal{E}$ have an equalizer.*

*In this case an equalizer $E \xrightarrow{\hat{e}} X$ of $\hat{f}$ and $\hat{g}$ corresponds to the map $E \xrightarrow{e=\eta_X e_{f,g}} \tilde{X}$, where $E \xrightarrow{e_{f,g}} X$ is an equalizer of $f$ and $g$.*

*Proof.* The first assertion follows from 3.3 and 3.4. To prove the last assertion, by 3.3 we have $\hat{e} : E \longrightarrow X$ is an equalizer of $\hat{f}$ and $\hat{g}$ in $\mathcal{E}_{\mathbb{R}}$ if and only if $\mu_X \tilde{e} : E \longrightarrow \tilde{X}$ is an equalizer of $\mu_Y \tilde{f}$ and $\mu_Y \tilde{g}$ and by 3.4, if and only if $\mu_X \tilde{e} = \widetilde{e_{f,g}}$, with $e_{f,g}$ an equalizer of $f$ and $g$. Now consider the following diagram in which pullback of $\eta_X$ along $e$ is formed to get the mono $k$; and all the other squares are known to be pullbacks.

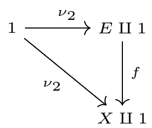


It follows from the above big pullback square that $\mu_X \tilde{e}\tilde{k} = \widetilde{i'}$. Now by 3.4, on the one hand $i'$ is an equalizer of $f$ and $g$ and on the other hand $\widetilde{i'}$ is an equalizer of $\mu_Y \tilde{f}$ and $\mu_Y \tilde{g}$. Therefore $\mu_X \tilde{e}\tilde{k}$ and $\mu_X \tilde{e}$ are both equalizers of $\mu_Y \tilde{f}$ and $\mu_Y \tilde{g}$. Thus $\tilde{k}$ and so $k$ are isomorphisms. Hence $e_{f,g} = i'k^{-1}$ is an equalizer of $f$ and $g$; and $e = \eta_X i'k^{-1} = \eta_X e_{f,g}$, concluding the proof. □

**Corollary 3.6.** *Let $\mathbb{R}$ be the representation monad on a topos $\mathcal{E}$. The Kleisli category $\mathcal{E}_{\mathbb{R}}$ has equalizers. Furthermore an equalizer $\hat{e} : E \longrightarrow X$ of a pair $X \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} Y$ in $\mathcal{E}_{\mathbb{R}}$ corresponds to $e = \eta_X e_{f,g} : E \longrightarrow \tilde{X}$, where the map $e_{f,g} : E \longrightarrow X$ is an equalizer of $f$ and $g$.*

*Proof.* Follows from 3.5 and the fact a topos has equalizers. □

Saying $f : A \amalg 1 \longrightarrow B \amalg 1$ is point preserving if it renders commutative the following triangle,



we have:

**Theorem 3.7.** *Let $\mathbb{A}$ be the add-point monad on a category $\mathcal{E}$ with a coproductive terminal object 1. The morphisms $X \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} Y$ in the Kleisli category*

*$\mathcal{E}_{\mathbb{A}}$ have an equalizer if and only if the morphisms* $X \amalg 1 \underset{g \oplus \nu_2}{\overset{f \oplus \nu_2}{\rightrightarrows}} Y \amalg 1$ *in $\mathcal{E}$ have an equalizer* $i : I \longrightarrow X \amalg 1$ *and there exists an object $E$ and an isomorphism* $\varphi : E \amalg 1 \longrightarrow I$ *such that $i\varphi$ is point preserving.*

*In this case an equalizer* $E \xrightarrow{\hat{e}} X$ *of $\hat{f}$ and $\hat{g}$ corresponds to the map* $E \xrightarrow{e = i\varphi\nu_1} X \amalg 1$.

*Proof.* The square:

$$\begin{array}{ccc} (E \amalg 1) \amalg 1 & \xrightarrow{\mu_E = 1 \oplus \nu_2} & E \amalg 1 \\ {\scriptstyle (i\varphi)\amalg 1}\downarrow & & \downarrow{\scriptstyle i\varphi} \\ (X \amalg 1) \amalg 1 & \xrightarrow[\mu_X = 1 \oplus \nu_2]{} & X \amalg 1 \end{array}$$

commutes if and only if $i\varphi(1 \oplus \nu_2) = (1 \oplus \nu_2)((i\varphi) \amalg 1)$ if and only if $(i\varphi) \oplus (i\varphi\nu_2) = (i\varphi) \oplus \nu_2$ if and only if $i\varphi\nu_2 = \nu_2$ if and only if $i\varphi$ is point preserving. The result follows by 3.3. □

**Definition 3.8.** *A category is called well add-pointed, provided that it has a coproductive terminal object 1, in which coproducts with 1 are disjoint and universal and squares of the form:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \nu_1}\downarrow & & \downarrow{\scriptstyle \nu_1} \\ A \amalg 1 & \xrightarrow{f \amalg 1} & B \amalg 1 \\ {\scriptstyle \nu_2}\uparrow & & \uparrow{\scriptstyle \nu_2} \\ 1 & \longrightarrow & 1 \end{array}$$

*are pullbacks.*

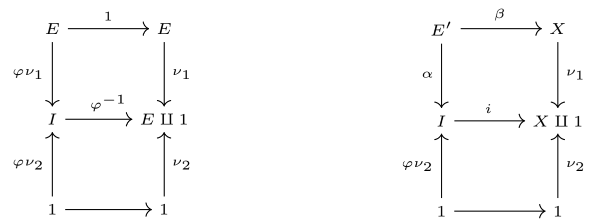
Denoting an equalizer of the pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y \amalg 1$ by $E_{f,g} \overset{e_{f,g}}{\hookrightarrow} X$, we have:

**Corollary 3.9.** *Let $\mathbb{A}$ be the add-point monad on a well add-pointed category $\mathcal{E}$. The morphisms* $X \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} Y$ *in the Kleisli category $\mathcal{E}_{\mathbb{A}}$ have an equalizer if and only if the morphisms* $X \underset{g}{\overset{f}{\rightrightarrows}} Y \amalg 1$ *in $\mathcal{E}$ have an equalizer.*

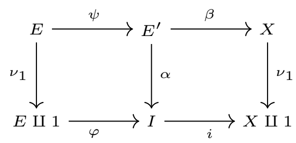
*In this case an equalizer* $E_{f,g} \xrightarrow{\hat{e}} X$ *of* $\hat{f}$ *and* $\hat{g}$ *corresponds to the map* $E_{f,g} \xrightarrow{e=\nu_1 e_{f,g}} X \amalg 1$.

*Proof.* Suppose an equalizer of $X \overset{\hat{f}}{\underset{\hat{g}}{\rightrightarrows}} Y$ exists in $\mathcal{E}_{\mathbb{A}}$. By 3.7, an object $E$ and an isomorphism $\varphi : E \amalg 1 \longrightarrow I$ exist such that $i\varphi$ is point preserving, where $i : I \longrightarrow X \amalg 1$ is an equalizer of $f \oplus \nu_2$ and $g \oplus \nu_2$.

In the below diagram, the left squares are easily seen to be pullbacks, the right top square is formed to be a pullback and the right bottom square is a pullback because $i\varphi$ is point preserving and $i$ is a mono.

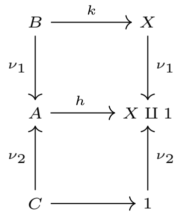


Since $\mathcal{E}$ is well add-pointed, the left vertical arrows in the above left and right diagrams are coproducts and so there is an isomorphism $\psi : E \longrightarrow E'$, such that $\alpha\psi = \varphi\nu_1$. Now we have the following pullback squares, the left of which can be verified easily.



One can now directly verify that $\beta\psi$ is an equalizer of $f$ and $g$, as desired.

Conversely, suppose $E_{f,g} \xhookrightarrow{e_{f,g}} X$ is an equalizer of $X \overset{f}{\underset{g}{\rightrightarrows}} Y \amalg 1$ in $\mathcal{E}$. We show $e \amalg 1 : E \amalg 1 \longrightarrow X \amalg 1$ is an equalizer of $f \oplus \nu_2$ and $g \oplus \nu_2$. Obviously $(f \oplus \nu_2)(e \amalg 1) = (g \oplus \nu_2)(e \amalg 1)$. Now given $h : A \longrightarrow X \amalg 1$ such that $(f \oplus \nu_2)h = (g \oplus \nu_2)h$, form the following pullbacks.




Now the morphism $k$ equalizes $f$ and $g$ and so there is a unique morphism $\bar{k} : B \longrightarrow E_{f,g}$ such that $e_{f,g}\bar{k} = k$. Since the left vertical arrows in the above diagram form a coproduct, $\bar{h} = (\nu_1\bar{k}) \oplus (\nu_2 !) : A \longrightarrow E \amalg 1$. We have $(e \amalg 1)\bar{h} = h$. Uniqueness follows from the fact that $e \amalg 1$ is mono, as $\mathcal{E}$ is well add-pointed. The result now follows from 3.7 by taking $\varphi = 1$.

The last assertion can be verified easily. □

**Lemma 3.10.** *Let* $(M, \star, 1)$ *be a monoid.*

$a)$ *The relation on* $M$*, defined by* $m \leq n$ *if there is* $a \in M$ *such that* $n = am$*, is reflexive and transitive, i.e., a preorder.*

$b)$ *The relation on* $M$*, defined by* $m \sim n$ *if there are* $a, b \in M$ *such that* $am = bn$*, is reflexive and symmetric.*

$c)$ *The equivalence relations* $R_{\leq}$ *induced by* " $\leq$ " *and* $R_{\sim}$ *induced by* " $\sim$ " *are equal.*

*Proof.* $(a)$ and $(b)$ are Obvious.

$c)$ Letting $k \leq\geq k'$ to mean $k \leq k'$ or $k' \leq k$, one has $nR_{\leq}n'$ if and only if there are $k_1, k_2, \cdots, k_i \in M$ such that $n \leq\geq k_1 \leq\geq k_2 \cdots \leq\geq k_i \leq\geq n'$. On the other hand we have $nR_{\sim}n'$ if and only if there are $k_1, k_2, \cdots, k_i \in M$ such that $n \sim k_1 \sim k_2 \cdots \sim k_i \sim n'$. The result follows from the facts that $k \leq\geq k'$ implies $k \sim k'$; and $k \sim k'$ implies there are $a, b \in M$ such that $ak = bk'$ so that $k \leq ak = bk' \geq k'$, and therefore $kR_{\leq}k'$. The result then follows. □

Setting $R = R_{\leq} = R_{\sim}$, we have:

**Lemma 3.11.** *Let* $(M, \star, 1)$ *be a monoid and* $X$ *be a set. On* $M \times X$*,*

$a)$ *the relation* $(m, x) \leq (n, y)$ *if* $m \leq n$ *and* $x = y$*, is a preorder.*

$b)$ *the relation* $(m, x) \sim (n, y)$ *if* $m \sim n$ *and* $x = y$*, is reflexive and symmetric.*

$c)$ *the relation* $(m, x)R(n, y)$ *if* $mRn$ *and* $x = y$*, is an equivalence relation.*

*Proof.* Follows from 3.10. □

**Definition 3.12.** *Let* $\mathbb{M} = (\hat{M}, \eta, \mu)$ *be the* $M$*-Set monad and consider the functions* $X \overset{f}{\underset{g}{\rightrightarrows}} M \times Y$. *We set:*

*a)* $I \overset{i}{\hookrightarrow} M \times X$ *to be the equalizer of* $M \times X \overset{f^*=(\star\times 1)(1\times f)}{\underset{g^*=(\star\times 1)(1\times g)}{\rightrightarrows}} M \times Y$.

*b)* $\dot{I}$ *to be those* $(m, x) \in I$ *for which* $m$ *is right cancelable, i.e.* $am = bm$ *implies* $a = b$.

The relations on $M \times X$ given in 3.11, induce relations on $I$ and $\dot{I}$. Denoting the quotient map to $\dot{I}/R$ by $q : \dot{I} \longrightarrow \dot{I}/R$, for sets $A \subseteq B \subseteq M \times X$, the up segment of $A$ in $B$ by $B \uparrow A = \{b \in B :$ there is $a \in A$ such that $b \geq a\}$ and the image of a function $s$ by $Im(s)$, we have:

**Definition 3.13.** *Let* $\mathbb{M} = (\hat{M}, \eta, \mu)$ *be the* $M$*-Set monad and consider the functions* $X \overset{f}{\underset{g}{\rightrightarrows}} M \times Y$. *We say* $E \subseteq I$ *is invariantly* $(f, g)$*-compatible if* $I \uparrow E = I$.

**Theorem 3.14.** *Let* $\mathbb{M} = (\hat{M}, \eta, \mu)$ *be the* $M$*-Set monad. The morphisms* $X \overset{\hat{f}}{\underset{\hat{g}}{\rightrightarrows}} Y$ *in the Kleisli category* $Set_{\mathbb{M}}$ *have an equalizer if and only if there is a section* $s : \dot{I}/R \longrightarrow \dot{I}$ *of* $q : \dot{I} \longrightarrow \dot{I}/R$ *such that* $Im(s)$ *is invariantly* $(f, g)$*-compatible.*

*In this case an equalizer* $Im(s) \overset{\hat{e}}{\longrightarrow} X$ *of* $\hat{f}$ *and* $\hat{g}$ *corresponds to the map* $Im(s) \overset{e}{\hookrightarrow} M \times X$.

*Proof.* Suppose an equalizer of the pair $X \overset{\hat{f}}{\underset{\hat{g}}{\rightrightarrows}} Y$ in $Set_{\mathbb{M}}$ is the map $\hat{e} : E \longrightarrow X$. Without loss of generality we assume the corresponding map $E \overset{e}{\hookrightarrow} M \times X$ is the inclusion. By 3.3, $M \times E \xrightarrow{e^*=(\star\times 1)(1\times e)} M \times X$ is an equalizer of $M \times X \overset{f^*}{\underset{g^*}{\rightrightarrows}} M \times Y$. Since $I \overset{i}{\hookrightarrow} M \times X$ is also an equalizer, there is a bijection $M \times E \xrightarrow{\psi} I$ such that $i\psi = e^*$. It follows that $\psi(m, n, x) = (mn, x)$. Let $(n, x) \in E$. On the one hand $(n, x) = \psi(1, n, x) \in I$, thus $E \subseteq I$. On the other hand if $a, b \in M$ and $an = bn$, then since $e^*$ is mono, $a = b$, thus $E \subseteq \dot{I}$.

Next we define $s : \dot{I}/R \longrightarrow \dot{I}$. Let $[(m, x)] \in \dot{I}/R$, with $(m, x) \in \dot{I}$. Then $\psi^{-1}(m, x) = (m', \dot{m}, x)$, with $(\dot{m}, x) \in E$ and $m = m'\dot{m}$. Set $s([(m, x)]) = (\dot{m}, x)$. If $[(m, x)] = [(n, x)]$, then $mRn$, i.e., there are $k_1, k_2, \cdots, k_i$ such that $m \sim k_1$, $k_1 \sim k_2$, $\cdots$, $k_i \sim n$. Also $s([(m, x)]) = (\dot{m}, x)$, with $(\dot{m}, x) \in E$, $m = m'\dot{m}$ and $s([(n, x)]) = (\dot{n}, x)$, with $(\dot{n}, x) \in E$ and $n = n'\dot{n}$. Since each $(k_j, x) \in \dot{I} \subseteq I$, $k_j = k'_j\dot{k}_i$ with $(\dot{k}_j, x) \in E$; and since $m \sim k_1$, there are $a, b \in M$ such that $am = bk_1$. Then $am'\dot{m} = bk'_1\dot{k}_1$. Monotonicity of $e^*$ implies $\dot{m} = \dot{k}_1$. Continuing in this manner, we get $\dot{k}_1 = \dot{k}_2 = \cdots = \dot{k}_i = \dot{n}$. So $(\dot{m}, x) = (\dot{n}, x)$ and therefore $s$ is well defined. Now $m = m'\dot{m}$ implies $m \sim \dot{m}$ and so $[(\dot{m}, x)] = [(m, x)]$. It follows that $qs([(m, x)]) = q(\dot{m}, x) = [(\dot{m}, x)] = [(m, x)]$. Hence $qs = 1$, i.e., $s$ is a section of $q$. To show $E = Im(s)$, we know $Im(s) \subseteq E$. Now if $(m, x) \in E$, then $(m, x) \in I$ and so $m = m'\dot{m}$ with $(\dot{m}, x) \in E$. Since $(m, x)$ and $(\dot{m}, x)$ are in $E$ and $1m = m'\dot{m}$, monotonicity of $e^*$ yields $m = \dot{m}$. Hence $(m, x) = (\dot{m}, x) = s([(m, x)]) \in Im(s)$. Thus $E \subseteq Im(s)$. Therefore $E = Im(s)$. Finally we show $I \uparrow Im(s) = I$. Let $(n, x) \in I$. We know $n = n'\dot{n}$ with $(\dot{n}, x) \in E$. It follows that $(n, x) \geq (\dot{n}, x)$ and $(\dot{n}, x) \in E = Im(s)$. Thus $(n, x) \in I \uparrow Im(s)$. Hence $I \uparrow Im(s) = I$, i.e., $Im(s)$ is invariantly $(f, g)$-compatible.

Conversely suppose there is a section $s : \dot{I}/R \longrightarrow \dot{I}$ of the morphism $q : \dot{I} \longrightarrow \dot{I}/R$ such that $Im(s)$ is invariantly $(f, g)$-compatible. Denote by $Im(s) \overset{e}{\hookrightarrow} M \times X$ the inclusion. For $(m, n, x) \in M \times Im(s)$, we have $f^*e^*(m, n, x) = (mnf_1(x), f_2(x))$, where $f_1$ and $f_2$ denote the first and the second projection of $f$, respectively. Since $(n, x) \in Im(s) \subseteq I$, we have $(nf_1(x), f_2(x)) = (ng_1(x), g_2(x))$. It follows that $f^*e^*(m, n, x) = g^*e^*(m, n, x)$. Hence $f^*e^* = g^*e^*$.

Now suppose $h : A \longrightarrow M \times X$ is given such that $f^*h = g^*h$. Let $a \in A$ and $h(a) = (m, x)$. Then $f^*(m, x) = g^*(m, x)$ and so $(m, x) \in I = I \uparrow Im(s)$. So $(m, x) \geq (\dot{m}, x)$ with $(\dot{m}, x) \in Im(s)$. It follows that $m = m'\dot{m}$ and $(\dot{m}, x) \in Im(s)$. Define $\bar{h} : A \longrightarrow M \times Im(s)$ by $\bar{h}(a) =$

$(m', \dot{m}, x)$. If $a = b$, then $h(a) = h(b) = (m, x)$. $m = m'\dot{m} = n'\dot{n}$ with $(\dot{m}, x)$ and $(\dot{n}, x)$ in $Im(s)$. So $(\dot{m}, x) \sim (\dot{n}, x)$, $(\dot{m}, x) = s([(k, x)])$ and $(\dot{n}, x) = s([(l, x)])$, for some $k$ and $l$. It follows that $[(k, x)] = qs([(k, x)]) = [(\dot{m}, x)] = [(\dot{n}, x)] = qs([(l, x)]) = [(l, x)]$. Therefore $(\dot{m}, x) = (\dot{n}, x)$ and so $\dot{m} = \dot{n}$. Now $m'\dot{m} = n'\dot{n}$ implies $m'\dot{m} = n'\dot{m}$ and since $\dot{m}$ is right cancelable, we get $m' = n'$. So $\bar{h}(a) = (m', \dot{m}, x) = (n', \dot{n}, x) = \bar{h}(b)$. Hence $\bar{h}$ is well defined. One can easily prove $e^*\bar{h} = h$ and $e^*$ is mono. Uniqueness of $\bar{h}$ with $e^*\bar{h} = h$ therefore follows. Hence $e^*$ is an equalizer of $f^*$ and $g^*$ and by 3.3 the result follows.

Finally the last assertion obviously holds. □

Denoting an equalizer of the pair $X \overset{f}{\underset{g}{\rightrightarrows}} M \times Y$ by $E_{f,g} \overset{e_{f,g}}{\hookrightarrow} X$, we have:

**Corollary 3.15.** *Let $\mathbb{M} = (\hat{M}, \eta, \mu)$ be the $M$-Set monad with $(M, \star, 1)$ an abelian monoid. The morphisms $X \overset{\hat{f}}{\underset{\hat{g}}{\rightrightarrows}} Y$ in the Kleisli category $Set_{\mathbb{M}}$ have an equalizer if and only if for $(m, x) \in M \times X$, $mf_1(x) = mg_1(x)$ and $f_2(x) = g_2(x)$ implies $x \in E_{f,g}$.*

*In this case an equalizer $E_{f,g} \xrightarrow{\hat{e}} X$ of $\hat{f}$ and $\hat{g}$ corresponds to the map $E_{f,g} \xrightarrow{e = \langle \tilde{1}, e_{f,g} \rangle} M \times X$.*

*Proof.* Suppose the equalizer of $\hat{f}$ and $\hat{g}$ exists. Then by 3.14, we have the existence of a section $s : \dot{I}/R \longrightarrow \dot{I}$ of $q : \dot{I} \longrightarrow \dot{I}/R$ such that $Im(s)$ is invariantly $(f, g)$-compatible. Let $(m, x) \in Im(s)$. Then $(m, x) \in \dot{I} \subseteq I$ and so $mf_1(x) = mg_1(x)$, $f_2(x) = g_2(x)$ and $m$ is right cancelable. Since the monoid is abelian, $m$ is also left cancelable. Therefore $f_1(x) = g_1(x)$. It follows that $f(x) = g(x)$, i.e., $x \in E_{f,g}$. Now suppose $x \in E_{f,g}$. Then $(1, x) \in I$ and so there is $(u_x, x) \in Im(s)$ such that $1 \geq u_x$. Therefore $1 = au_x = u_x a$, i.e., $u_x$ is invertible. If $(m, x)$ and $(n, x)$ are in $Im(s)$, then since $mn = nm$, $e^*(m, n, x) = e^*(n, m, x)$. Therefore $m = n$ and so $(m, x) = (n, x)$. Putting these together we conclude that $Im(s) = \{(u_x, x) : x \in E_{f,g}\}$, where each $u_x$ is invertible.

Now if $(m, x) \in M \times X$, $mf_1(x) = mg_1(x)$ and $f_2(x) = g_2(x)$, then $(m, x) \in I$ and therefore $(m, x) \geq (n, x)$ with $(n, x) \in Im(s)$. This implies $x \in E_{f,g}$.

Conversely suppose for $(m, x) \in M \times X$, $mf_1(x) = mg_1(x)$ and $f_2(x) = g_2(x)$ implies $x \in E_{f,g}$. It is easy to see for$E_{f,g} \xrightarrow{e=\langle \tilde{1}, e_{f,g}\rangle} M \times X$, $M \times E_{f,g} \xrightarrow{e^*} M \times X$ is the inclusion. Now direct computation shows that $e^*$ is an equalizer of $f^*$ and $g^*$. The result then follows by 3.3.

The last assertion follows easily. □

**Definition 3.16.** *Let $\mathbb{P}$ be the power object monad on a topos $\mathcal{E}$ and let* $\mathbf{f} : 1 \longrightarrow PX$ *denote the false map. Given morphisms* $X \underset{g}{\overset{f}{\rightrightarrows}} PY$*,*

$a)$ *a morphism* $A \xrightarrow{a} PX$ *is said to be $(f, g)$-invariant relative to $\mu$, if it factors through the equalizer* $i : I \longrightarrow PX$ *of* $PX \underset{\mu P(g)}{\overset{\mu P(f)}{\rightrightarrows}} PY$*, i.e., if* $\mu P(f)a = \mu P(g)a$.

$b)$ *a morphism* $1 \xrightarrow{a} PX$ *is said to be $(f, g)$-simple if it is a minimal element of $\mathcal{E}(1, PX)$ that is not equal to* **f** *and that is $(f, g)$-invariant relative to $\mu$, where $\mathcal{E}(A, PX)$ is partially ordered by $a \leq b$ if $b = a \vee b$.*

**Lemma 3.17.** *Let $\mathbb{P}$ be the power object monad on a topos $\mathcal{E}$ and consider the maps* $X \underset{g}{\overset{f}{\rightrightarrows}} PY$ *in $\mathcal{E}$. A morphism* $e : E \longrightarrow PX$ *is $(f, g)$-invariant if and only if the morphism* $\mu_X P(e) : PE \longrightarrow PX$ *is.*

*Proof.* Suppose $e$ is $(f, g)$-invariant. We have:

$$\mu_Y P(f)\mu_X P(e) = \mu_Y \mu_{PY} P^2(f)P(e) =$$
$$\mu_Y P(\mu_Y)P^2(f)P(e) = \mu_Y P(\mu_Y P(f)e)$$

Similarly $\mu_Y P(g)\mu_X P(e) = \mu_Y P(\mu_Y P(g)e)$. The result then follows.

Now suppose $\mu_X P(e)$ is $(f, g)$-invariant. We have:

$$\mu_Y P(f)e = \mu_Y P(f)\mu_X \eta_{PX} e = \mu_Y P(f)\mu_X P(e)\eta_E$$

Similarly

$\mu_Y P(g)e = \mu_Y P(g)\mu_X P(e)\eta_E$. The result then follows. □

Recall that, [8], a well-pointed topos is one in which parallel morphisms are equal if they are equal when composed on the right with all the global elements; also that a well-pointed topos is two-valued.

**Lemma 3.18.** *A topos is well-pointed if and only if for all $X$, the sink $(\ 1 \xrightarrow{x} X\ )_{x \in hom(1,X)}$ is a coproduct.*

*Proof.* For an object $X$ of a well-pointed topos, set $G_X = \amalg_{hom(1,X)} 1$. There is a unique $\varphi$ making the following triangle commute.

$$\begin{array}{ccc} 1 & \xrightarrow{\nu_x} & G_X \\ & \searrow^{x} & \downarrow{\scriptstyle \varphi} \\ & & X \end{array}$$

If $\alpha\varphi = \beta\varphi$, then for all $x \in hom(1,X)$, $\alpha\varphi\nu_x = \beta\varphi\nu_x$ and so for all $x \in hom(1,X)$, $\alpha x = \beta x$. Therefore $\alpha = \beta$, implying $\varphi$ is an epimorphism. (Note if $hom(1,X) = \emptyset$, then the assertion for all $x \in hom(1,X)$, $\alpha x = \beta x$ is vacuously true, implying $\alpha = \beta$, so that there is at most one morphism with domain $X$, implying $\varphi$ is an epimorphism).

Now given $f : 1 \longrightarrow G_X$, form the pullback:

$$\begin{array}{ccc} A_x & \longrightarrow & 1 \\ {\scriptstyle f^{-1}(\nu_x)}\downarrow & p.b. & \downarrow{\scriptstyle \nu_x} \\ 1 & \xrightarrow[f]{} & G_X \end{array}$$

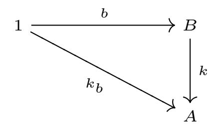
Since the topos is two-valued and $A_x$ is a subobject of $1$, it is either $0$ or $1$. If for all $x$, $A_x = 0$, then $f^{-1}(\nu_x)$ is $!: 0 \longrightarrow 1$. So $1 = f^{-1}(\oplus_x \nu_x) = \oplus_x f^{-1}(\nu_x) = \oplus_x ! = !$, which is a contradiction. Therefore there exists $x$ such that $A_x = 1$. It follows that there is $x$ such that $f = \nu_x$.

Now if $1 \overset{f}{\underset{g}{\rightrightarrows}} G_X$ are morphisms with $\varphi f = \varphi g$, then there are $x$ and $y$ such that $f = \nu_x$ and $g = \nu_y$. So $\varphi\nu_x = \varphi\nu_y$ implying $x = y$. It follows that $f = \nu_x = \nu_y = g$. So $\varphi$ is a monomorphism, hence an isomorphism.

The converse follows from the fact that a coproduct is an epi sink. □

**Lemma 3.19.** *In a well-pointed topos a morphism that has a unique right inverse is an isomorphism.*

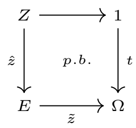
*Proof.* Suppose $f : A \longrightarrow B$ has a unique right inverse $g : B \longrightarrow A$. Let $a : 1 \longrightarrow A$ be a morphism. For $b \in hom(1, B)$, define $k_b \in hom(1, A)$ by $k_b = \begin{cases} gb \text{ if } b \neq fa \\ a \text{ if } b = fa \end{cases}$. By 3.18, there is a unique morphism $k : B \longrightarrow A$ rendering commutative the following triangle.




Now for all $b \in hom(1, B)$, $fkb = fk_b = \begin{cases} fgb \text{ if } b \neq fa \\ fa \text{ if } b = fa \end{cases} = b$. It follows that $fk = 1$. Uniqueness of $g$ yields $k = g$. So $a = k_{fa} = kfa = gfa$. Therefore for all $a \in hom(1, A)$, $gfa = a$. It follows that $gf = 1$. Hence $g$ is the inverse of $f$.

□

**Lemma 3.20.** *Let $\mathbb{P}$ be the power object monad on a topos $\mathcal{E}$. If the arrows $e : E \longrightarrow PX$ and $z : 1 \longrightarrow PE$ are given, then for all arrows $b : 1 \longrightarrow Z$, $e\hat{z}b \leq \mu_X P(e)z$, where $\hat{z}$ is the corresponding subobject of $z$ obtained by:*




*Proof.* Let $b : 1 \longrightarrow Z$ be given. $\hat{z}$ is classified by $\tilde{z}$ and one can easily verify that $\hat{z}b$ is classified by $\widetilde{\eta \hat{z} b} = \epsilon_E(1 \times \eta_E)(1 \times \hat{z})\langle 1, b! \rangle$. Now since $\hat{z}b \leq \hat{z}$, $\widetilde{\eta \hat{z} b} \leq \tilde{z}$. It follows that $\eta \hat{z} b \leq z$ and so by 2.3, $\mu_X P(e) \eta \hat{z} b \leq \mu_X P(e) z$ which in turn implies $e\hat{z}b \leq \mu_X P(e)z$. □

**Lemma 3.21.** *Let $\mathbb{P}$ be the power object monad on a topos $\mathcal{E}$. If the map $e : E \longrightarrow PX$ in $\mathcal{E}$ is such that $\mu_X P(e)$ is a monomorphism, then the map $\mathbf{f}_X : 1 \longrightarrow PX$ does not factor through $e$.*

*Proof.* Suppose there is $f' : 1 \longrightarrow E$ such that $\mathbf{f}_X = ef'$. Then we have $\mu_X P(e) \eta_E f' = \mu_X \eta_{PX} e f' = \mathbf{f}_X$ and by 2.4 and the fact that $P(e) = \exists_e$

preserves the false map, we have $\mu_X P(e)\mathbf{f}_E = \mathbf{f}_X$. So $\mu_X P(e)\eta_E f' = \mu_X P(e)\mathbf{f}_E$. Since $\mu_X P(e)$ is a monomorphism, we get $\eta_E f' = \mathbf{f}_E$. This implies the subobjects $f' : 1 \longrightarrow E$ and $! : 0 \longrightarrow E$, corresponding to $\eta_E f'$ and $\mathbf{f}_E$ respectively, are equal; which is a contradiction. □

**Lemma 3.22.** *$a)$ In a topos, if $!_E : 0 \longrightarrow E$ is the unique morphism, then $P(!_E) = \mathbf{f}_E$.*

*$b)$ In a well-pointed topos, if $b : 1 \longrightarrow E$ and $z : 1 \longrightarrow PE$ are morphisms such that $\mathbf{f}_E \neq z \leq \eta_E b$, then $z = \eta_E b$.*

*Proof.* $a)$ Since $\mathbf{f}_0 : 1 \longrightarrow P(0) = 1$ is the identity morphism and $P(!_A)\mathbf{f}_0 = \exists_{!_A}\mathbf{f}_0 = \mathbf{f}_A$, we get $P(!_A) = \mathbf{f}_A$.

$b)$ Since the corresponding subobjects of $z$ and $\eta_E b$ are respectively the maps $\hat{z} : Z \longrightarrow E$ and $b : 1 \longrightarrow E$, we get $\hat{z} \leq b$, i.e., there is a morphism $\alpha : 1 \longrightarrow Z$ such that $b\alpha = \hat{z}$. But then $\alpha$ is a monomorphism and since the topos is well-pointed, $Z = 0$ or $Z = 1$. If $Z = 0$, then $z = \mathbf{f}_E$, a contradiction. So $Z = 1$, implying $\hat{z} = b$. It then follows that $z = \eta_E b$. □

For a pair of morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} PY$, denoting an equalizer of the pair $PX \overset{\mu P(f)}{\underset{\mu P(g)}{\rightrightarrows}} PY$ by $i : I \longrightarrow PX$, and setting:

$$\mathcal{S} = \{\, a : 1 \longrightarrow PX \mid a \text{ is } (f,g)\text{-simple}\}$$

we have:

**Theorem 3.23.** *Let $\mathbb{P}$ be the power object monad on a well-pointed topos $\mathcal{E}$. The morphisms $X \overset{\hat{f}}{\underset{\hat{g}}{\rightrightarrows}} Y$ in the Kleisli category $\mathcal{E}_{\mathbb{P}}$ have an equalizer if and only if there is a unique morphism $I \xrightarrow{s} P(\amalg_{\mathcal{S}} 1)$ such that $\mu_X P(\oplus_{a\in\mathcal{S}} a)s = i$.*

*In this case an equalizer$\amalg_{\mathcal{S}} 1 \xrightarrow{\hat{e}} X$ of $\hat{f}$ and $\hat{g}$ corresponds to the map $\amalg_{\mathcal{S}} 1 \xrightarrow{e = \oplus_{a\in\mathcal{S}} a} PX$.*

*Proof.* Let$E \xrightarrow{\hat{e}} X$ be an equalizer of $\hat{f}$ and $\hat{g}$. By 3.3,$PE \xrightarrow{\mu_X P(e)} PX$ is an equalizer of $\mu P(f)$ and $\mu P(g)$. Since $i$ is also an equalizer of this pair, there is a unique isomorphism $s : I \longrightarrow PE$ such that $\mu P(e)s = i$. We are done as soon as we show $E = \amalg_{\mathcal{S}} 1$ and $e = \oplus_{a\in\mathcal{S}} a$.

To prove $E = \amalg_{\mathcal{S}} 1$, we show $\mathcal{S} \cong \mathcal{E}(1, E)$. The result then follows by 3.18.

In case $E \cong 0$, then $PE \cong 1$ and so there is a unique morphism from 1 to $PE$. It follows that the only morphism from 1 to $PE$ is $\mathbf{f}_E$. Now if $a \in \mathcal{S}$, then there is a unique $a' : 1 \longrightarrow PE$ such that $a = \mu_X P(e)a'$. But then $a' = \mathbf{f}_E$ and so by 2.4, $a = \mathbf{f}_X$, a contradiction. Hence $\mathcal{S} = \emptyset$ and so $\mathcal{S} = \mathcal{E}(1, E) = \emptyset$.

Now assume $E \ncong 0$. Let $a \in \mathcal{S}$. Since $a$ is $(f, g)$-invariant, there is a unique $1 \xrightarrow{z} PE$ such that $\mu_X P(e)z = a$. If $Z = 0$, then $\tilde{z} = \mathbf{f}!$ and $z = \mathbf{f}_E$. But then by 2.4, $a = \mathbf{f}_X$, a contradiction. So $Z \neq 0$ and therefore by 3.18, $hom(1, Z) \neq \emptyset$. Now let $b : 1 \longrightarrow Z$ be any morphism. By 3.20, $e\hat{z}b \leq a$. Now since $\mu P(e)$ is $(f, g)$-invariant, so is $e$ by 3.17. It follows that $e\hat{z}b$ is $(f, g)$-invariant and by 3.21, $e\hat{z}b \neq \mathbf{f}_X$. Since $a$ is $(f, g)$-simple, $e\hat{z}b = a$. Define $\kappa : \mathcal{S} \longrightarrow \mathcal{E}(1, E)$ to take $a$ to $\hat{z}b$.

Now let $b \in \mathcal{E}(1, E)$. We show $eb$ is $(f, g)$-simple. Since $e$ is $(f, g)$-invariant, so is $eb$. Also by 3.21, $eb \neq \mathbf{f}_X$. Now if $a : 1 \longrightarrow PX$ is $(f, g)$-invariant, $a \neq \mathbf{f}_X$ and $a \leq eb$, then there is $z : 1 \longrightarrow PE$ such that $a = \mu_X P(e)z$. By 2.3, we have $\mu_X Pe(z \vee \eta_E b) = (\mu_X P(e)z) \vee (\mu_X P(e)\eta_E b) = a \vee eb = eb = \mu_X P(e)\eta_E b$. So $z \vee \eta_E b = \eta_E b$, i.e., $z \leq \eta_E b$. Since $a \neq \mathbf{f}_X$, by 2.4, $z \neq \mathbf{f}_E$. Now by 3.22, $z = \eta_E b$ and so $a = \mu_X P(e)z = \mu_X P(e)\eta_E b = eb$ as desired. Define $\kappa' : \mathcal{E}(1, E) \longrightarrow \mathcal{S}$ to take $b$ to $eb$.

One can easily verify that $\mathcal{S} \underset{\kappa'}{\overset{\kappa}{\rightleftarrows}} \mathcal{E}(1, E)$ establishes an isomorphism.

Using the isomorphisms $\kappa$ and $\kappa'$, one could get the isomorphisms between $\amalg_{\mathcal{S}} 1$ and $\amalg_{\mathcal{E}(1,E)} 1 \cong E$. Some computations then show $e = \oplus_{a\in\mathcal{S}} a$.

Now suppose there is a unique morphism $I \xrightarrow{s} P(\amalg_{\mathcal{S}} 1)$ such that $\mu_X P(\oplus_{a\in\mathcal{S}} a)s = i$. Set $E = \amalg_{\mathcal{S}} 1$ and $e = \oplus_{a\in\mathcal{S}} a$. Since each $a \in \mathcal{S}$ is $(f, g)$-invariant, so is $e$ and by 3.17, so is $\mu_X P(e)$. Therefore there exists a unique $r : PE \longrightarrow I$ such that $ir = \mu_X P(e)$. Now $irs = \mu_X P(e)s = i$, implying $rs = 1$. Now if there is $s'$ such that $rs' = 1$, then $\mu_X P(e)s' = irs' = i$. Uniqueness implies $s' = s$. Therefore $r$ has a unique right inverse

$s$. By 3.19, $r$ is an isomorphism. Thus $PE \xrightarrow{\mu_X P(e)} PX \overset{\mu_Y P(f)}{\underset{\mu_Y P(g)}{\rightrightarrows}} PY$ is an equalizer and by 3.3, we are done.

The last assertion is easily seen to be valid.

□

Let $\mathbb{P}$ be the power set monad and $X \overset{f}{\underset{g}{\rightrightarrows}} PY$ be functions in the category $Set$. Let's call a subset $A$ of $X$, $(f,g)$-invariant if $\bigcup_{a\in A} f(a) = \bigcup_{a\in A} g(a)$ and $(f,g)$-simple if it is minimal non-empty $(f,g)$-invariant. Setting $S = \{A \subseteq X : A \text{ is } (f,g)\text{-simple}\}$, we have:

**Corollary 3.24.** *Let $\mathbb{P}$ be the power set monad. The morphisms $X \overset{\hat{f}}{\underset{\hat{g}}{\rightrightarrows}} Y$ in the Kleisli category $Set_{\mathbb{P}}$ have an equalizer if and only if every $(f,g)$-invariant subset of $X$ can be uniquely written as a union of $(f,g)$-simple subsets.*

*In this case an equalizer $S \xrightarrow{\hat{e}} X$ of $\hat{f}$ and $\hat{g}$ corresponds to the map $S \overset{e}{\hookrightarrow} PX$.*

*Proof.* Follows from 3.23 and the existence of a unique $s$ satisfying the equality $\mu_X P(\oplus_{a\in\mathcal{S}} a)s = i$. □

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