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ON REFLECTIVE AND COREFLECTIVE HULLS*

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RÉSUMÉ. Cet article étudie l'enveloppe réflective engendrée par une sous-catégorie pleine d'une catégorie complète. Il s'agit de la plus petite sous-catégorie qui est pleine et réflective. Comme application nous obtenons l'enveloppe coréflective de la sous-catégorie pleine de cubes pointés dans la catégorie des espaces topologiques pointés. Par la suite nous déterminons l'enveloppe réflective de la catégorie des espaces metriques dans la catégorie des espaces uniformes, ainsi que certaines autres sous-catégories.

ABSTRACT. This paper explores the reflective hull (smallest full reflective subcategory) generated by a full subcategory of a complete category. We apply this to obtain the coreflective hull of the full subcategory of pointed cubes inside the category of pointed topological spaces. We also find the reflective hull of the category of metric spaces inside the category of uniform spaces as well as certain subcategories.

Keywords: reflective subcategory, (Isbell)limit closure, adjoint functor, pointed cubes, uniform spaces.
AMS classification numbers (2010): 18A30, 18A32, 18B30, 54E15

1. Introduction

If $\mathcal{A} \subseteq \mathcal{C}$ is a full subcategory, then the **reflective hull** of $\mathcal{A}$ in $\mathcal{C}$ is, if it exists, the smallest reflective subcategory of $\mathcal{C}$ which contains $\mathcal{A}$. This paper concerns the existence and nature of the reflective hull and, dually, the coreflective hull. We particularly want to **describe** the reflective hull because this often reveals subtle ways a subcategory relates to the category in which it is embedded. Therefore, the purpose of this paper is partly to study the general question, but mainly to consider some specific instances of reflective hulls and coreflective hulls that seem especially interesting.

Reflective subcategories and reflective hulls have been studied in many papers, including [2, 3, 12, 13, 14, 19, 21, 25]. A useful summary of work on this subject up through 1987 can be found in [25]. The existence of a reflective hull is closely related to the question of whether the intersection

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of an arbitrary collection of full reflective subcategories is reflective. (We assume that arbitrary collections of classes always have an intersection.) The existence question for reflective subcategories of the category of topological spaces was raised in [13]. A counter-example was found in [2].

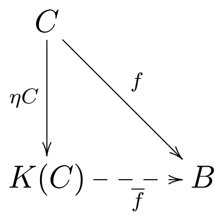
In Section 2 we discuss in detail the notion of the limit closure of a subcategory as well as a stronger version originally introduced by Isbell. In addition we give what appears to be a new adjoint functor theorem. In Section 3 we describe a construction of the reflective hull, largely due to [21, 22], based on a factorization system.

In Section 4 we look at the coreflective subcategory of pointed spaces generated by the cubes. In Section 5 we characterize those uniform spaces that are limits of metric spaces and show that the full subcategory of such spaces is the reflective hull.

In [4] we have studied in much greater detail the limit closures of certain full subcategories of integral domains in the category of commutative rings.

CONVENTION. *A subcategory will always be assumed to be full as well as* **replete***, that is, closed under isomorphic copies of its objects.*

1.1. DEFINITION. A subcategory $\mathcal{K} \subseteq \mathcal{C}$ is **reflective** if for each object $C \in \mathcal{C}$, there is a **reflection map** $\eta C : C \longrightarrow K(C)$ with $K(C) \in \mathcal{K}$ and such that whenever $f : C \longrightarrow B$ with $B \in \mathcal{K}$, then



there is a unique **extension** $\overline{f}$ with $\overline{f}.\eta C = f$. This property of ηC is called the **unique extension property**, see 3.8 for a full definition.

Under the above circumstances, it is well known (and easy to prove) that K *can be made into a functor and* η *into a natural transformation such that* K *is left adjoint to the inclusion of* $\mathcal{K} \longrightarrow \mathcal{C}$.

We will say that $\mathcal{K} \subseteq \mathcal{C}$ is **epireflective** if it is reflective in such a way that for every $C \in \mathcal{C}$, the reflection map $\eta C : C \longrightarrow K(C)$ is epic in $\mathcal{C}$.

2. Limit closure and Isbell limit closure

2.1. LIMIT CLOSURES OF SUBCATEGORIES OF COMPLETE CATEGORIES. We emphasize that when we talk of a **limit closed subcategory** $\mathcal{B} \subseteq \mathcal{C}$, we mean not only that $\mathcal{B}$ is complete, but also that the limits in $\mathcal{B}$ are the same as in those of $\mathcal{C}$, that is, that the inclusion $\mathcal{B} \subseteq \mathcal{C}$ preserves limits. The limit closure of $\mathcal{B}$ is the meet of all limit closed subcategories of $\mathcal{C}$ that contain $\mathcal{B}$.

2.2. ISBELL LIMITS. It has long been observed that if a category with small homsets is not a poset, it cannot have limits of arbitrary sized diagrams. If A, B are objects of the category $\mathcal{C}$ such that $\mathrm{Hom}(A, B)$ has more than one element and ∞ is the cardinality of the universe, then $\mathrm{Hom}(A, B^{\infty}) = \mathrm{Hom}(A, B)^{\infty}$ is not small. However, Isbell observed that there is no problem in supposing that a category have, in addition to all small limits, meets of arbitrary families of subobjects of some object, since if $B' \subseteq B$ is an arbitrary subobject, $\mathrm{Hom}(A, B')$ is a subset of $\mathrm{Hom}(A, B)$.

Following Isbell, we call a monic m **extremal** if $m = fe$ with e epic implies that e is an isomorphism. Isbell considered the case of a category with all small limits and having meets of arbitrary families of extremal subobjects. We will call such categories **Isbell complete**.

A subcategory $\mathcal{B}$ of an Isbell complete category $\mathcal{C}$ is **Isbell limit closed** (or "Left closed" in Isbell's terminology, [17]) if it is Isbell complete and the inclusion $\mathcal{B} \subseteq \mathcal{C}$ preserves small limits and arbitrary meets of $\mathcal{B}$-extremal subobjects (that is extremal in the category $\mathcal{B}$). The Isbell limit closure of a subcategory is the smallest Isbell limit closed subcategory containing it.

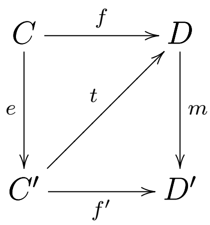
We also consider categories that have arbitrary meets of some other class $\mathcal{M}$ of subobjects of any object. Such categories will be called **$\mathcal{M}$-complete**.

Before relating Isbell limit closures to reflective hulls, we need to discuss Factorization Systems.

2.3. FACTORIZATION SYSTEMS. Factorization systems go back to the notion of a bicategory, see [23, 15]. With some modification, we will be using Isbell's terminology.

2.4. DEFINITION. Let $m : D \longrightarrow D'$ and $e : C \longrightarrow C'$ be two arrows in a category $\mathcal{C}$. We will say that m is **right orthogonal to** e and that e is **left orthogonal to** m if, for any maps $f : C \longrightarrow D$ and $f' : C' \longrightarrow D'$ with

$mf = f'e$ there is a unique $t : C' \longrightarrow D$ such that



commutes. We note that if either e is epic or m is monic, then we need assume only the existence of t; uniqueness follows.

Let $\mathcal{E}$ be a class of morphisms of a category $\mathcal{C}$. We denote by $\mathcal{E}^{\perp}$, the class of morphisms that are right orthogonal to $\mathcal{E}$. Dually if $\mathcal{M}$ is a class of morphisms, we denote by ${}^{\perp}\mathcal{M}$, the class of morphisms that are left orthogonal to $\mathcal{M}$.

2.5. DEFINITION. A **factorization system** in $\mathcal{C}$ is a pair $(\mathcal{E}, \mathcal{M})$ of classes of maps in $\mathcal{C}$ such that

FS-1. $\mathcal{M}$ and $\mathcal{E}$ contain all isomorphisms and are closed under composition.

FS-2. Every morphism f factors as $f = me$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$.

FS-3. If $m \in \mathcal{M}$ and $e \in \mathcal{E}$, then m is right orthogonal to e (and therefore e is left orthogonal to m).

We will say that a factorization system $(\mathcal{E}, \mathcal{M})$ is a **left factorization system** if every $m \in \mathcal{M}$ is monic, a **right factorization system** if every $e \in \mathcal{E}$ is epic and a **strict factorization system** if it is both a left factorization system and a right factorization system.

If $(\mathcal{E}, \mathcal{M})$ is a factorization system and $m : C \longrightarrow C'$ is a morphism in $\mathcal{M}$, we will sometimes say that C is an $\mathcal{M}$-subobject of C' even though m need not be monic. Dually, when $e : C \longrightarrow C'$ is a morphism in $\mathcal{E}$, we will sometimes say that C' is an $\mathcal{E}$-quotient of C, even though e need not be epic.

Finally, we say that the category is **$\mathcal{E}$-cowell-powered** if, up to isomorphism, each object has only a set of $\mathcal{E}$-quotients and dually for **$\mathcal{M}$-well-powered**.

2.6. (EPIC, EXTREMAL MONIC) FACTORIZATION IN ISBELL COMPLETE CATEGORIES.

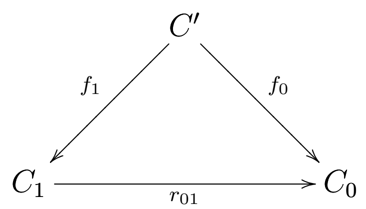
2.7. DEFINITION. By an **ordinal indexed family of subobjects (respectively, regular subobjects, extremal subobjects)** of an object C we mean a family of subobjects $\{u_\alpha : C_\alpha \longrightarrow C\}$, indexed so that α varies in some small (or possibly large) ordinal such that

1. For $\alpha > \beta$, $C_\alpha \subseteq C_\beta$ with inclusion $u_{\beta\alpha}$;
2. $u_{\alpha,\alpha+1}$ is monic (respectively, regular monic, extremal monic);
3. when α is a limit ordinal, $C_\alpha = \bigcap_{\beta<\alpha} C_\beta$.

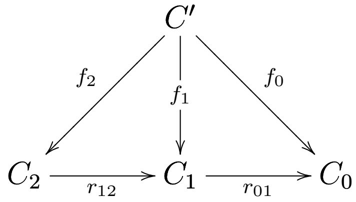
These definitions are found in [17]. We follow Isbell and assume we have small sets, large sets, and extraordinary sets. We will sometimes use the term "class" to describe a set that is no bigger than large. A model of this situation uses a strongly inaccessible cardinal we will call ∞. Then small sets have cardinality less than ∞, large sets have cardinality equal to ∞ and extraordinary sets have cardinality greater than ∞.

2.8. **Lemma.** *Let $\mathcal{C}$ be an Isbell complete category. Then any morphism can be factored as an epic followed by the limit of an ordinal string of regular monics.*

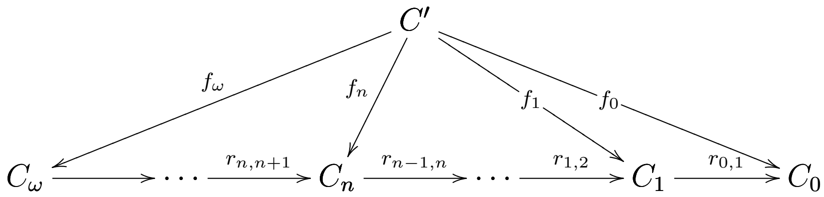
PROOF. Let $f : C' \longrightarrow C$ be a morphism. If f is epic, there is nothing to prove. If not, there are two maps out of C whose composite with f is the same and whose equalizer, therefore, factors f, so that with $C_0 = C$ and $f_0 = f$ we have a diagram:



with r_{01} regular epic. If f_1 is epic, we can stop here. If not repeat to get the diagram



with r_{12} regular monic. Let $r_{02} = r_{01}r_{12}$. Continue in this way, if possible, to get a diagram



in which $C_\omega = \lim C_n$ and $r_{n\omega} : C_\omega \longrightarrow C_n$ is the element of the limit cone. For $m < n$, let $r_{mn} = r_{n-1,n}r_{m,n-1}$. Then the commutativity in the limit diagram implies that $r_{m\omega} = r_{nm}r_{m.\omega}$. So long as we do not get an epic first factor, continue to define $C_{\omega+1}$, $C_{\omega+2}$ and all the relevant maps. This might continue through all small ordinals and we can let $C_\infty = \lim C_\alpha$. The map $f_\infty : C' \longrightarrow C_\infty$ might not be an isomorphism. So continue to define $C_{\infty+1}$, etc. through all large ordinals. An important observation is that if for $\beta < \alpha$, $r_\beta = r_\alpha$ as subobjects of C_0, then $r_\beta = r_\alpha = r_\beta r_{\beta\alpha}$ from which the monic r_β can be cancelled to conclude that $r_{\beta\alpha}$ is an isomorphism. But one easily sees that this implies that $r_{\beta,\beta+1}$ is an isomorphism, contradicting the construction. Since the class of large ordinals is an extra-large class, while C_0 can have only a large class of subobjects, this construction must stop.

2.9. **Corollary.** *Every extremal epic in an Isbell complete category factors as a limit of an ordinal string of regular monics.* ∎

2.10. **Corollary.** *Every morphism in an Isbell complete category factors as an epic followed by an extremal monic.* ∎

2.11. ISBELL LIMIT CLOSURES. If $\mathcal{A}$ is a subcategory of the Isbell complete category $\mathcal{C}$, its Isbell limit closure is the meet of all Isbell limit closed subcategories of $\mathcal{C}$ that contain $\mathcal{A}$.

2.12. **Proposition.** *A reflective subcategory of an Isbell complete category is Isbell limit closed.*

PROOF. Let $\mathcal{C}$ be Isbell complete and let $\mathcal{B} \subseteq \mathcal{C}$ be a reflective subcategory. Then $\mathcal{B}$ is limit closed in $\mathcal{C}$, which implies that $\mathcal{B}$ is complete.

We claim that every $\mathcal{B}$-extremal monic is a $\mathcal{C}$-extremal monic. Suppose $m : B_1 \longrightarrow B_2$ be a $\mathcal{B}$-extremal mono. To prove that m is a $\mathcal{C}$-extremal mono, assume that $m = ge$ where $e : B_1 \longrightarrow C$ is an epic in $\mathcal{C}$. It suffices to prove that e is invertible. Let $\eta : C \longrightarrow B$ be the reflection of C into $\mathcal{B}$. Then there exists a map $\overline{g} : B \longrightarrow B_2$ such that $\overline{g}\eta = g$. It is easily proven that ηe is an epic in $\mathcal{B}$ and, since $m = \overline{g}(\eta e)$ we see that ηe is a right factor of m in the subcategory $\mathcal{B}$. This implies that ηe is invertible, and that $(\eta e)^{-1}\eta$ is a left inverse for e. But then e is an epic with a left inverse which implies that e is invertible.

It follows that every family of $\mathcal{B}$-extremal subobjects of an object of $\mathcal{B}$ is a family of $\mathcal{C}$-extremal subobjects and thus has a greatest lower bound. That greatest lower bound is an intersection and is in $\mathcal{B}$ as $\mathcal{B}$ is reflective and is closed under all limits, including large intersections. Therefore $\mathcal{B}$ is Isbell complete and, in view of the proof, the inclusion $\mathcal{B} \subseteq \mathcal{C}$ preserves limits and meets of extremal subobjects. ∎

2.13. **Theorem.** *Let $\mathcal{A} \subseteq \mathcal{C}$ be a subcategory, where $\mathcal{C}$ is Isbell complete. If the Isbell limit closure of $\mathcal{A}$ is reflective, then it is the reflective hull of $\mathcal{A}$.* ∎

2.14. NOTATION. If $\mathcal{M}$ is a class of monics of $\mathcal{C}$, then $\mathrm{Sub}_{\mathcal{M}}\mathrm{Prod}(\mathcal{A})$ denotes the full subcategory of $\mathcal{M}$-subobjects of products of objects of $\mathcal{A}$.

2.15. AN ADJOINT FUNCTOR THEOREM. Let $\mathcal{C}$ be a category with a strict factorization system $(\mathcal{E},\mathcal{M})$. We will say that $\mathcal{C}$ is $\mathcal{M}$**-complete** if it is complete and every class of $\mathcal{M}$-subobjects of an object of $\mathcal{C}$ has a meet. Note that if $\mathcal{M}$ is the class of extremal monics then $\mathcal{M}$-complete is the same as Isbell complete.

A subcategory $\mathcal{A} \subseteq \mathcal{C}$ will be called $\mathcal{M}$**-dense** if $\mathcal{C} = \mathrm{Sub}_{\mathcal{M}}\mathrm{Prod}(\mathcal{A})$.

We thank the referee for simplifying the next development.

2.16. **Lemma.** *Suppose $\mathcal{C}$ is $\mathcal{M}$-complete and the $\mathcal{M}$-dense subcategory $\mathcal{B} \subseteq \mathcal{C}$ has a small weakly initial set. Then $\mathcal{C}$ has an initial object.*

PROOF. The product of all the objects in a weak initial set is a weak initial object. If I is a weak initial object of $\mathcal{B}$, then we claim the meet of all the $\mathcal{M}$-subobjects of I in $\mathcal{C}$ is initial. In fact, if $C \in \mathcal{C}$, there is an $\mathcal{M}$-embedding $m : C \hookrightarrow \prod B_i$, with all $B_i \in \mathcal{B}$. Now form the pullback

$$\begin{array}{ccc} I_C & \xrightarrow{m_C} & I \\ \downarrow & & \downarrow \\ C & \xrightarrow[m]{} & \prod B_i \end{array}$$

Then $m_C \in \mathcal{M}$ since $m \in \mathcal{M}$ and so the $\mathcal{M}$ subobjects of I are a weak initial family in $\mathcal{C}$. The meet I_0 of all the $\mathcal{M}$ subobjects of I is at least weakly initial. But if there were two maps $I_0 \rightrightarrows C$ for some object $C \in \mathcal{C}$ their equalizer would be a smaller $\mathcal{M}$ subobject of I, a contradiction. ∎

2.17. **Theorem.** *Suppose $\mathcal{C}$ is $\mathcal{M}$-complete, $\mathcal{A} \subseteq \mathcal{C}$ is $\mathcal{M}$-dense, and the functor $U : \mathcal{C} \longrightarrow \mathcal{B}$ preserves limits as well as arbitrary meets of $\mathcal{M}$-subobjects. If for each object $B \in \mathcal{B}$, the comma category $(B, U|\mathcal{A})$ has a small weakly initial set, then U has a left adjoint.* ∎

HOW THE GENERAL AND SPECIAL ADJOINT FUNCTOR THEOREMS DIFFER. The proof of the GAFT basically boils down to the fact that if I is weakly initial (the solution set condition), then the equalizer of all the endomorphisms of I is initial. Crucial to the argument is that there is some morphism from I to that equalizer. In the SAFT, I is weakly initial in a dense subcategory and you need the meet of all the subobjects of I to get an initial object and that requires some control over the class of subobjects.

3. A two-step construction of the reflective hull

3.1. CONVENTION *Throughout this paper we assume we are given a category $\mathcal{C}$ with a strict factorization system $(\mathcal{E}, \mathcal{M})$ such that $\mathcal{C}$ is $\mathcal{M}$-complete, cocomplete and $\mathcal{E}$-cowell-powered. We also assume that $\mathcal{A} \subseteq \mathcal{C}$ is a full subcategory and we are trying to describe the reflective hull of $\mathcal{A}$ in $\mathcal{C}$ or show that it does not exist.*

To explain the two-step approach, we need some definitions and a proposition:

3.2. DEFINITION. Let $\mathcal{A} \subseteq \mathcal{C}$ be as above. We define $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A}) \subseteq \mathcal{C}$ as $\mathrm{Sub}_{\mathcal{M}}\mathrm{Prod}(\mathcal{A})$.

3.3. DEFINITION. Let $(\mathcal{P}, I)$ be a right factorization system in $\mathcal{C}$ (see 2.5). A reflective subcategory $\mathcal{K} \subseteq \mathcal{C}$ is **$\mathcal{P}$-reflective** if every reflection map $\eta C : C \longrightarrow K(C)$ is in $\mathcal{P}$. So if $\mathcal{E}$ is the class of all epis, then an $\mathcal{E}$-reflective subcategory is the same thing as an epireflective subcategory.

The next result is well-known, see Proposition 1.2 of [21].

3.4. **Proposition.** *Let $(\mathcal{P}, I)$ be a right factorization system in a category $\mathcal{B}$ that has arbitrary products and is $\mathcal{P}$-cowell-powered. Then $\mathcal{K} \hookrightarrow \mathcal{B}$ is $\mathcal{P}$-reflective if and only if $\mathcal{K}$ is closed under the formation of products and I-subobjects.*

It easily follows that $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is an $\mathcal{E}$-reflective subcategory of $\mathcal{C}$ and the two-step approach is based on the inclusions $\mathcal{A} \subseteq \mathrm{Ref}_{\mathcal{E}}(\mathcal{A}) \subseteq \mathcal{C}$. It is well-known that $\mathcal{A}$ is a reflective subcategory of $\mathcal{C}$ if and only if $\mathcal{A}$ is reflective in $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ which is in turn reflective in $\mathcal{C}$. Similarly, the reflective hull of $\mathcal{A}$ in $\mathcal{C}$ is related to the reflective hull of $\mathcal{A}$ in $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ since these reflective hulls coincide if they exist.

We will call $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$**the $\mathcal{E}$-reflective hull of** $\mathcal{A}$ since it is the smallest $\mathcal{E}$-reflective subcategory of $\mathcal{C}$ that contains $\mathcal{A}$. We will define a class $\mathcal{P}$ of morphisms on $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ such that the smallest $\mathcal{P}$-reflective subcategory of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ which contains $\mathcal{A}$ would be, if it exists, the reflective hull of $\mathcal{A}$ in $\mathcal{C}$ and would coincide with the limit closure of $\mathcal{A}$ in $\mathcal{C}$. But if $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is not $\mathcal{P}$-cowell-powered, then the reflective hull of $\mathcal{A}$ may need to be bigger than the limit closure or might fail to exist. The Isbell limit closure, the UEP-closure (see 3.15) and the tame closure (see 3.31) of $\mathcal{A}$ all contain the limit closure and would be contained in the reflective hull. If the tame closure is not reflective, then the reflective hull of $\mathcal{A}$ in $\mathcal{C}$ fails to exist.

The next result follows from Proposition 1.3 of [21].

3.5. **Proposition.** *Let $\mathcal{C}$ be a cocomplete category and let $\mathcal{P}$ be a class of epimorphisms of $\mathcal{C}$. Assume that $\mathcal{C}$ is $\mathcal{P}$-cowell-powered. Then there exists an I such that $(\mathcal{P}, I)$ is a right factorization system if and only if the following two conditions are satisfied:*

1. *$\mathcal{P}$ contains all isomorphisms and is closed under compositions.*

2. *$\mathcal{P}$ is closed under cointersections and pushouts.*

3.6. **Proposition.** *Let $\mathcal{B}$ be a cocomplete, cowell-powered category. Let $\mathcal{E}$ denote the class of all epimorphisms of $\mathcal{B}$ and $\mathcal{M}$ denote the class of all extremal monomorphisms of $\mathcal{B}$. Then $(\mathcal{E}, \mathcal{M})$ is a strict factorization system on $\mathcal{B}$.* ∎

3.7. **Corollary.** *Let $\mathcal{B}$ be complete, cocomplete and cowell-powered. Then $\mathcal{K} \subseteq \mathcal{B}$ is epireflective if and only if $\mathcal{K}$ is closed under products and extremal subobjects.* ∎

3.8. DEFINITION. The map $p : B \longrightarrow B'$ of $\mathcal{K}$ is **epic with respect to** $A \in \mathcal{K}$ if the induced map $\mathrm{Hom}(B', A) \longrightarrow \mathrm{Hom}(B, A)$ is an injection. Furthermore, p has the **unique extension property with respect to** $A \in \mathcal{K}$ if the induced map $\mathrm{Hom}(B', A) \longrightarrow \mathrm{Hom}(B, A)$ is a bijection. Finally, p has the **unique extension with respect to** $\mathcal{A}$ if it has this property with respect to every $A \in \mathcal{A}$.

3.9. PARTIAL SOLUTION TO THE REFLECTIVE HULL PROBLEM. Our next result gives a useful description of the reflective hull, assuming reasonable conditions for $\mathcal{C}$, the ambient category. We first need the following definition:

3.10. **Proposition.** *Recall that we are given $\mathcal{A} \subseteq \mathcal{C}$ satisfying the conditions in 3.1. Let $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ be the $\mathcal{E}$-reflective hull of $\mathcal{A}$. Let $\mathcal{P}$ denote the class of maps of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ which have the unique extension property with respect to $\mathcal{A}$. Then every map in $\mathcal{P}$ is in $\mathcal{M}$ and is epic in $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$.*

Note that not every epic in $\mathcal{K}$ need be an epic in $\mathcal{C}$.

PROOF. Assume that $p : C \longrightarrow D$ is in $\mathcal{P}$. Let $m : C \longrightarrow P$ be in $\mathcal{M}$ where $P = \prod\{A_i\}$ is the product of the objects $A_i \in \mathcal{A}$. Since p has the UEP with respect to $\mathcal{A}$, there is, for each i, a map $g_i : D \longrightarrow A_i$ such that $g_i p = \pi_i m$ where $\pi_i : P \longrightarrow A_i$ is the projection. Clearly there exists $g : D \longrightarrow P$ for which $\pi_i g = g_i$ for all i. It is easily verified that $\pi_i gp = \pi_i m$ for all i so $gp = m$. This implies that $p \in \mathcal{M}$ as it is the right factor of a member of $\mathcal{M}$.

To show that $p : C \longrightarrow D$ is an epic in $\mathcal{K}$, let $g, h : D \longrightarrow E$ be such that $gp = hp$. Let $m : E \longrightarrow P$ be in $\mathcal{M}$ where $P = \prod A_i$ is the product of the objects $A_i \in \mathcal{A}$ and $\pi_i : P \longrightarrow A_i$ the projection. Since p has unique extensions to objects of $\mathcal{A}$, and since $\pi_i mgp = \pi_i mhp$ we see that $\pi_i mg = \pi_i mh$ for all i. But this implies that $mg = mh$ so $g = h$ as m is monic. ■

3.11. **Theorem.** *Assume that $\mathcal{A} \subseteq \mathcal{C}$ and ($\mathcal{E}$,$\mathcal{M}$) satisfy the conditions in the convention 3.1. Let $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ and $\mathcal{P}$ be as above. Let I denote the class $\mathcal{P}^{\perp}$, in the category $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ and let $\widehat{\mathcal{A}}$ be the full subcategory of all objects $B \in \mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ such that every $p \in \mathcal{P}$ has the unique extension property with respect to B.*

Assume that $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is $\mathcal{P}$-cowell-powered. Then:

1. *$\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is the smallest $\mathcal{E}$-reflective subcategory of $\mathcal{C}$ that contains $\mathcal{A}$.*

2. *$(\mathcal{P}, I)$ is a right factorization system on $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$. Moreover $g \in I$ if and only if $g = he$ with $e \in \mathcal{P}$ implies that e is an isomorphism.*

3. *$\widehat{\mathcal{A}}$ is a $\mathcal{P}$-reflective subcategory of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$.*

4. *$B \in \widehat{\mathcal{A}}$ if and only if $B \in \mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ and B has no proper $\mathcal{P}$-quotient.*

5. *$B \in \widehat{\mathcal{A}}$ if and only if every map of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ with domain B lies in I.*

6. *$\widehat{\mathcal{A}}$ is the reflective hull of $\mathcal{A}$ in $\mathcal{C}$.*

PROOF. 1 follows from Proposition 3.4 and 2 follows from 3.5. 3 and 6 follow from Theorem 3.1 of [21], while 4 and 5 follow from Proposition 3.2 of K'. ■

3.12. **Proposition.** *Let $\mathcal{A} \subseteq \mathcal{C}$ be as in 3.1. Assume that $\mathcal{C}$ is well-powered and that $\mathcal{A}$ has a small cogenerating family. Then $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is cowell-powered and the reflective hull of $\mathcal{A}$ is its limit closure.*

PROOF. The small family that cogenerates $\mathcal{A}$ clearly cogenerates its limit closure, which is then reflective by the special adjoint functor theorem. That $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is then cowell-powered follows from Theorem 2.2.2 of [21]. ■

3.13. SOME THEORETICAL OBSERVATIONS ON THE REFLECTIVE HULL. In what follows we investigate when the reflective hull of $\mathcal{A}$ is its limit closure or is its Isbell limit closure or, perhaps the still larger $\mathrm{UEP}^{\perp}$ closure, or the tame closure.

3.14. NOTATION. If $\mathcal{A} \subseteq \mathcal{C}$ then we let $\mathrm{UEP}_{\mathcal{A}}$ denote the class of all morphisms of $\mathcal{C}$ which have the unique extension property with respect to $\mathcal{A}$.

3.15. DEFINITION. Let $\mathcal{A} \subseteq \mathcal{C}$ be given. We say that $C \in \mathcal{C}$ is **attached to** $\mathcal{A}$ if every morphism with domain C is in $(\mathrm{UEP}_{\mathcal{A}})^{\perp}$.

We say that $\mathcal{A} \subseteq \mathcal{C}$ is $\mathrm{UEP}^{\perp}$ **closed in** C if $C \in \mathcal{A}$ whenever $C \in \mathcal{C}$ is attached to $\mathcal{A}$.

The $\mathrm{UEP}^{\perp}$ **closure of** $\mathcal{A}$, denoted by $\widehat{\mathcal{A}}$, is the class of all objects which are in every $\mathrm{UEP}^{\perp}$ closed subcategory of $\mathcal{C}$ that contains $\mathcal{A}$.

3.16. **Lemma.** *Assume $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ and let $C \in \mathcal{C}$ be attached to $\mathcal{A}$. Then C is also attached to $\mathcal{B}$.*

PROOF. Since $\mathcal{A} \subseteq \mathcal{B}$ it is clear that $\mathrm{UEP}_{\mathcal{B}} \subseteq \mathrm{UEP}_{\mathcal{A}}$. It follows that $(\mathrm{UEP}_{\mathcal{A}})^{\perp} \subseteq (\mathrm{UEP}_{\mathcal{B}})^{\perp}$ and the result is then obvious. ∎

3.17. **Lemma.** *Let $\mathcal{A} \subseteq \mathcal{C}$ be given and let $\widehat{\mathcal{A}}$ be the $\mathrm{UEP}^{\perp}$ closure of $\mathcal{A}$. Then $\widehat{\mathcal{A}}$ is itself $\mathrm{UEP}^{\perp}$ closed.*

PROOF. Let C be attached to $\widehat{\mathcal{A}}$. Then whenever $\mathcal{A} \subseteq \mathcal{B}$ where $\mathcal{B}$ is $\mathrm{UEP}^{\perp}$ closed, it follows by definition that $\widehat{\mathcal{A}} \subseteq \mathcal{B}$. So, by the above lemma, C is attached to $\mathcal{B}$ and so $C \in \mathcal{B}$. Since this is true for all such $\mathcal{B}$, it follows that $C \in \widehat{\mathcal{A}}$. ∎

3.18. **Corollary.** *$\widehat{\mathcal{A}}$ is the smallest $\mathrm{UEP}^{\perp}$-closed subcategory of $\mathcal{C}$ which contains $\mathcal{A}$.* ∎

3.19. **Proposition.** *Every reflective subcategory is $\mathrm{UEP}^{\perp}$-closed.*

PROOF. Suppose that $\mathcal{B} \subseteq \mathcal{C}$ is a reflective subcategory and that $C \in \mathcal{C}$ is attached to $\mathcal{B}$. Let $\eta : C \longrightarrow B(C)$ be the associated reflection map. Then, obviously, η is in $\mathrm{UEP}_{\mathcal{B}}$. Since C is attached to $\mathcal{B}$, we see that $\eta \in (\mathrm{UEP}_{\mathcal{B}})^{\perp}$ which implies that η is invertible as it is in $\mathrm{UEP}_{\mathcal{B}} \cap (\mathrm{UEP}_{\mathcal{B}})^{\perp}$. It follows that $C \in \mathcal{B}$. ∎

3.20. **Corollary.** *Every reflective subcategory of $\mathcal{C}$ that contains $\mathcal{A}$ also contains its* $\mathrm{UEP}^{\perp}$ *closure,* $\widehat{\mathcal{A}}$*. So if* $\mathcal{A}_{\mathrm{Hull}}$*, the reflective hull of* $\mathcal{A}$ *exists, then* $\widehat{\mathcal{A}} \subseteq \mathcal{A}_{\mathrm{Hull}}$*. Moreover, if* $\widehat{\mathcal{A}}$ *is reflective, then it is* $\mathcal{A}_{\mathrm{Hull}}$*.* ■

3.21. EXAMPLE. Let $\mathcal{A}$ be the class of all small ordinals, with the opposite of their usual ordering. We extend this ordered class by including two additional elements, B, C which are both lower bounds for $\mathcal{A}$ with B and C non-comparable. Let $\mathcal{C}$ denote the category corresponding to the ordered class $\mathcal{A} \cup \{B, C\}$. It is easily shown that the only member of $\mathrm{UEP}_{\mathcal{A}}$ with domain B is the identity map, 1_B. So B is attached to $\mathcal{A}$. Similarly C is attached to $\mathcal{A}$. Even though $\mathcal{A}$ is limit closed in $\mathcal{C}$, the larger $\mathrm{UEP}^{\perp}$ closure of $\mathcal{A}$ is all of $\mathcal{C}$, and is therefore the reflective hull of $\mathcal{A}$.

3.22. **Proposition.** *Let $\mathcal{A} \subseteq \mathcal{C}$ where $\mathcal{C}$ has pushouts. Let $\mathcal{T}$ be the class of all objects that are attached to $\mathcal{A}$. Then*

1. $\widehat{\mathcal{A}} = \mathcal{A} \cup \mathcal{T}$;
2. $\mathrm{UEP}_{\mathcal{A}} = \mathrm{UEP}_{\widehat{\mathcal{A}}}$.

PROOF. Let $\mathcal{A}' = \mathcal{A} \cup \mathcal{T}$. We claim that $\mathrm{UEP}_{\mathcal{A}'} = \mathrm{UEP}_{\mathcal{A}}$ (which will prove 2 once we prove 1). Obviously $\mathrm{UEP}_{\mathcal{A}'} \subseteq \mathrm{UEP}_{\mathcal{A}}$. To show the opposite inclusion, assume that $p : D \longrightarrow E$ is in $\mathrm{UEP}_{\mathcal{A}}$. Let $f : D \longrightarrow C$ be given where $C \in \mathcal{T}$. Consider the following pushout diagram:

$$\begin{array}{ccc} D & \xrightarrow{\;p\;} & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[\;q\;]{} & F \end{array}$$

It is readily shown that $\mathrm{UEP}_{\mathcal{A}}$ is closed under pushouts, so $q \in \mathrm{UEP}_{\mathcal{A}}$. Since $C \in \mathcal{T}$, we see that q is in $(\mathrm{UEP}_{\mathcal{A}})^{\perp}$ which implies that q is invertible. It follows that $\overline{f} = q^{-1}h$ is an extension of f in the sense that $\overline{f}p = f$. Now suppose $d : E \longrightarrow C$ also extends f, meaning that $dp = f$. Observe that $dp = 1_C f$ so, by the pushout property, there exists $r : F \longrightarrow C$ such that $rq = 1_C$ and $rh = d$. Since $rq = 1_C$, and since q^{-1} exists, we see that $r = q^{-1}$ and so $d = rh = q^{-1}h = \overline{f}$. This proves the claim that

$\mathrm{UEP}_{\mathcal{A}} = \mathrm{UEP}_{\mathcal{A}'}$. But this immediately implies that C is attached to $\mathcal{A}$ if and only if C is attached to $\mathcal{A}'$ if and only if $C \in \mathcal{T}$, which shows that $\mathcal{A}'$ is $\mathrm{UEP}^{\perp}$ closed. So $\widehat{\mathcal{A}} = \mathcal{A} \cup \mathcal{T}$, which proves 1. ∎

3.23. REMARK. In our next set of results, we examine the reflective hull of $\mathcal{A}$ in $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$. This will coincide with its reflective hull in $\mathcal{C}$ if both reflective hulls exist. However, we cannot rule out the possibility that $\mathcal{A}$ has no reflective hull in $\mathcal{C}$ but does have such a hull in $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$.

3.24. NOTATION. If $\mathcal{P}$ is a class of epimorphisms of a category $\mathcal{K}$, we say that $C \in \mathcal{K}$ has "no proper $\mathcal{P}$-quotients" if every quotient $p : C \longrightarrow D$ with $p \in \mathcal{P}$ is such that p is invertible.

3.25. **Proposition.** *Assume that $\mathcal{A} \subseteq \mathcal{C}$ satisfies the conditions in 3.1. Further assume that the $\mathcal{E}$-reflective hull, $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$, is $\mathcal{M}$-well-powered. In the category $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$, let $\mathcal{P} = \mathrm{UEP}_{\mathcal{A}}$.*

The following conditions on an object $C \in \mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ are equivalent:

1. *C is attached to $\mathcal{A}$.*
2. *$C \in \widehat{\mathcal{A}}$.*
3. *Every $p \in \mathcal{P}$ has the UEP with respect to C.*
4. *C has no proper $\mathcal{P}$-quotients.*

PROOF. $1 \Rightarrow 2$: Obvious.
$2 \Rightarrow 3$: This is proven in Proposition 3.22.2.
$3 \Rightarrow 4$: Let $p : C \longrightarrow D$ be a $\mathcal{P}$-quotient of C. Since $\mathcal{P}$ has the UEP with respect to C, there exists $r : D \longrightarrow C$ such that $rp = 1_C$. So p is invertible as it is epic in $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$.

$4 \Rightarrow 1$: Suppose that $f : C \longrightarrow D$ be given. We must show that $f \in \mathcal{P}^{\perp}$. For $p \in \mathcal{P}$, let $hp = fg$. Since p has the UEP with respect to C, there exists d such that $dp = g$. It follows that $fdp = hp$ and so $fd = h$ as p is epic. The uniqueness of d also follows as p is epic. ∎

3.26. **Proposition.** *Let $\mathcal{A} \subseteq \mathrm{Ref}_{\mathcal{E}}(\mathcal{A}) \subseteq \mathcal{C}$ and $\mathcal{P}$ be as above. Assume that $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is $\mathcal{M}$-well-powered. Let $\mathcal{L}_0$ be the limit closure and $\mathcal{L}$ the Isbell limit closure of $\mathcal{A}$. Let $\mathcal{B}$ be any reflective subcategory of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ which contains $\mathcal{A}$. Then:*

$$\mathcal{A} \subseteq \mathcal{L}_0 \subseteq \mathcal{L} \subseteq \widehat{\mathcal{A}} \subseteq \mathcal{B}$$

PROOF. It is readily shown that objects of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$, with respect to which $\mathcal{P}$ has the UEP form a subcategory that contains $\mathcal{A}$ and is closed under all limits (even large limits) and therefore closed under Isbell limits. By the above proposition, $\widehat{\mathcal{A}}$ is precisely this class of objects, so we easily see that $\mathcal{A} \subseteq \mathcal{L}_0 \subseteq \mathcal{L} \subseteq \widehat{\mathcal{A}}$. Finally, if $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ where $\mathcal{B}$ is reflective in $\mathcal{K}$, then, by Corollary 3.20, we see that $\widehat{\mathcal{A}} \subseteq \mathcal{B}$. ∎

For example, Theorem 3.11 gave conditions under which $\widehat{\mathcal{A}}$ is the reflective hull of $\mathcal{A}$. Next we give sufficient conditions for $\widehat{\mathcal{A}}$ to be the limit closure $\mathcal{L}_0$, the Isbell limit closure $\mathcal{L}$ or the $\mathrm{UEP}^{\perp}$ closure.

3.27. **Theorem.** *Let $\mathcal{A} \subseteq \mathcal{C}$ be as in our conventions, 3.1. Let $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ be the $\mathcal{E}$-reflective hull of $\mathcal{A}$, and let $\mathcal{P}$ and $\widehat{\mathcal{A}}$ be as in theorem 3.11. Then the reflective hull of $\mathcal{A}$ is:*

1. *The Isbell limit closure of $\mathcal{A}$ if $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is cowell-powered,*
2. *The limit closure of $\mathcal{A}$ if $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is cowell-powered and extremal-well-powered,*
3. *The $\mathrm{UEP}^{\perp}$ closure of $\mathcal{A}$ if $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is $\mathcal{P}$-cowell-powered.*

PROOF.

1. We note that under our assumptions in 3.1, the category $\mathcal{C}$ is $\mathcal{M}$-complete. Since it easily shown that every extremal mono of $\mathcal{C}$ is in $\mathcal{M}$, we see that $\mathcal{C}$ is Isbell complete. We claim that $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is also Isbell complete because every extremal mono of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is in $\mathcal{M}$. To prove this claim, let $m : B \longrightarrow C$ be an extremal mono of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$. Factor $m = gh$ with $h : B \longrightarrow D$ in $\mathcal{E}$ and $g : D \longrightarrow C$ in $\mathcal{M}$. Since $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is closed under $\mathcal{M}$-subobjects, we see that $D \in \mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$. Clearly h is an epi of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ and since m is

an extremal mono of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$, we see that h is invertible, which implies that $m \in \mathcal{M}$.

We further note that $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is $\mathcal{P}$-cowell-powered (as it is cowell-powered) so, by theorem 3.11, $\widehat{\mathcal{A}}$ is the reflective hull of $\mathcal{A}$. We note that $\widehat{\mathcal{A}}$ is epireflective in $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ (as every map in $\mathcal{P}$ is epi in $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$). Therefore, $\widehat{\mathcal{A}}$ is closed under subobjects which are extremal in $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$.

Since $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is cowell-powered, the subcategory $\mathcal{A}_1 \subseteq \mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ of all extremal subobjects of products of objects of $\mathcal{A}$ is reflective in $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ (where the extremal subobjects are extremal in $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$). Since $\widehat{\mathcal{A}}$ is the reflective hull of $\mathcal{A}$, we see that $\widehat{\mathcal{A}} \subseteq \mathcal{A}_1$. So if $B \in \widehat{\mathcal{A}}$, there exists $m : B \longrightarrow \prod A_\alpha$ where m is an extremal mono in $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ and each $A_\alpha \in \mathcal{A}$ We claim that B is in the Isbell limit closure of $\mathcal{A}$ because $\prod A_\alpha$ clearly is, and because m is an ordinal-indexed limit of regular maps which are determined by being equalizers. We observe that if $r : S \longrightarrow T$ is an equalizer of maps $v, w : T \longrightarrow W$ and if T is in the Isbell limit closure of $\mathcal{A}$, then so is S. To prove this, we need to replace W by an object in the Isbell limit closure of $\mathcal{A}$. But since $W \in \mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$, there exists $m : W \longrightarrow W'$ where $m \in \mathcal{M}$ and W' is a product of objects from $\mathcal{A}$. Then W' is in the Isbell limit closure of $\mathcal{A}$ and r is the equalizer of mv, mw. So B is in the Isbell limit closure of $\mathcal{A}$.

2. If $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is extremal-well-powered, then the limit closure of $\mathcal{A}$ coincides with the Isbell limit closure, so this case follows from the one above.

3. If $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is $\mathcal{P}$-cowell-powered, then theorem 3.11 applies which shows that $\widehat{\mathcal{A}}$ is the reflective hull and also the $\mathrm{UEP}^{\perp}$ closure of $\mathcal{A}$ as $\widehat{\mathcal{A}}$ consists of the objects with no proper $\mathcal{P}$ quotients and $\mathcal{P} = \mathrm{UEP}_{\mathcal{A}}$.

3.28. REMARK. We note that even when the reflective hull of $\mathcal{A} \subseteq \mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ exists and coincides with $\mathcal{L}$, the characterization of the reflective hull in terms of $\mathcal{P}^{\perp}$ (which may be denoted I when it yields a right factorization system) will prove to be very useful in the example in the next section.

3.29. REMARK. The above proposition does not address the case when $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ fails to be $\mathcal{P}$-cowell-powered. In this situation, the reflective hull might be bigger than $\widehat{\mathcal{A}}$ or might fail to exist. To analyze what then happens, we define the tame closure of $\mathcal{A}$

3.30. **Lemma.** *Let $\mathcal{A} \subseteq \mathrm{Ref}_{\mathcal{E}}(\mathcal{A}) \subseteq \mathcal{C}$ and $\mathcal{P}$ be as above. Assume that $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is $\mathcal{M}$-well-powered.*

Let $\mathcal{B}$ be a reflective subcategory of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ with $\mathcal{A} \subseteq \mathcal{B}$. Let $\mathcal{P}_1 = \mathrm{UEP}_{\mathcal{B}}$. Then $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is $\mathcal{P}_1$-cowell-powered.

PROOF. Let $C \in \mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ be given, and let $\eta : C \longrightarrow R(C)$ be its reflection into $\mathcal{B}$. We first claim that $\eta \in \mathcal{M}$. By assumption, there exists $m : C \longrightarrow \prod A_\alpha$ where $m \in \mathcal{M}$ and $A_\alpha \in \mathcal{A}$ for all α. Since $\prod A_\alpha$ is clearly in $\mathcal{B}$, there exists a map $f : R(C) \longrightarrow \prod A_\alpha$ such that $f\eta = m$. But this implies that $\eta \in \mathcal{M}$ as it is a right factor of $m \in \mathcal{M}$

Now let $g : C \longrightarrow D$ be any map in $\mathcal{P}_1$, Since $R(C) \in \mathcal{B}$, there exists a map $h : D \longrightarrow R(C)$ such that $hg = \eta$. And this implies that $g \in \mathcal{M}$ as it is a right factor of $\eta \in \mathcal{M}$. It follows that every $\mathcal{P}_1$ quotient of C is an $\mathcal{M}$-subobject of $R(C)$ so C has only a small set of $\mathcal{P}_1$-quotients as $R(C)$ has only a small set of $\mathcal{M}$-subobjects. ∎

3.31. DEFINITION. Assume that $\mathcal{A} \subseteq \mathcal{C}$ satisfies the conditions in 3.1. Further assume that the $\mathcal{E}$-reflective hull, $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$, is $\mathcal{M}$-well-powered. In the category $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$, let $\mathcal{P} = \mathrm{UEP}_{\mathcal{A}}$. For each epimorphism e of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$, let P_e denote the smallest class of epimorphisms of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ which contains e and all isomorphisms and is closed under compositions, cointersections, and pushouts as in Proposition 3.5. We then say that e is a **tame epimorphism** if $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is P_e-cowell-powered.

We let $\mathcal{P}_T$ denote the class of all tame epics in $\mathcal{P}$. We define the **tame closure** of $\mathcal{A}$, denoted by $\overline{\mathcal{A}}$, as the class of all objects $B \in \mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ which have no proper $\mathcal{P}_T$ quotients.

3.32. **Lemma.** *With the above assumptions and notation, the $\mathrm{UEP}^{\perp}$ closure of $\mathcal{A}$ is contained in its tame closure. In symbols, $\widehat{\mathcal{A}} \subseteq \overline{\mathcal{A}}$.*

PROOF. By Proposition 3.25, an object is in $\widehat{\mathcal{A}}$ if and only if it has no proper $\mathcal{P}$-quotients and, by definition, it is in $\overline{\mathcal{A}}$ if and only if it has no proper $\mathcal{P}_T$ quotients. Since $\mathcal{P}_T \subseteq \mathcal{P}$, the result follows. ∎

3.33. **Proposition.** *With the above assumptions and notation, $\overline{\mathcal{A}}$, the tame closure of $\mathcal{A}$, is the intersection of all reflective subcategories of $\mathcal{K}$ which contain $\mathcal{A}$.*

PROOF. Assume that $B \in \mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is in the intersection of all reflective subcategories $\mathcal{A}_1$ with $\mathcal{A} \subseteq \mathcal{A}_1 \subseteq \mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ and let $e : B \longrightarrow Q$ be a $\mathcal{P}_T$-quotient of B. Since $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is P_e-cowell-powered, there is, by Proposition 3.5, a class I_e for which (I_e, P_e) is a right factorization system on $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$. Then $\mathcal{A}_e = \mathrm{Sub}_{I_e}\mathrm{Prod}(\mathcal{A})$ is reflective by Proposition 3.4. Since $\mathcal{A} \subseteq \mathcal{A}_e$, we see that $B \in \mathcal{A}_e$. Let $i : B \longrightarrow \prod A_\alpha$ be a map in I_e. Let $p_\alpha : \prod A_\alpha \longrightarrow A_\alpha$ be the projection. Since $e \in \mathcal{P}$ there is, for each α, a map $g_\alpha : Q \longrightarrow A_\alpha$ for which $g_\alpha e = p_\alpha i$. Let $g : Q \longrightarrow \prod A_\alpha$ be determined so that $p_\alpha g = g_\alpha$ for all α. Then, clearly, we have $ge = i$ and since $e \in P_e$ and $i \in I_e$ we get that e has a left inverse and, since e is epic in $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$, we see that e is invertible which implies that $B \in \overline{\mathcal{A}}$.

Conversely, if $B \in \overline{\mathcal{A}}$, then B has no proper $\mathcal{P}_T$-quotient. Let $\mathcal{A}_1$ be a reflective subcategory of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ with $\mathcal{A} \subseteq \mathcal{A}_1$. Let $\mathcal{P}_1 = \mathrm{UEP}_{\mathcal{A}_1}$. By the above lemma, $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is $\mathcal{P}_1$-cowell-powered. It readily follows that $\mathcal{P}_1 \subseteq \mathcal{P}_T$. But this implies that the reflection map $\eta : B \longrightarrow R(B)$ is in $\mathcal{P}_T$ and, by our assumption on B, we have that η is invertible so $B \in \mathcal{A}_1$. ∎

3.34. **Theorem.** *Assume that $\mathcal{A} \subseteq \mathcal{C}$ satisfies the conditions in 3.1. Further assume that the $\mathcal{E}$-reflective hull, $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$, is $\mathcal{M}$-well-powered. Let $\mathcal{P}_T$ be as defined in 3.31. The following statements are then equivalent:*

1. *The tame closure of $\mathcal{A}$ is the reflective hull of $\mathcal{A}$ in $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$.*
2. *The reflective hull of $\mathcal{A}$ in $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ exists.*
3. *The tame closure of $\mathcal{A}$ is a reflective subcategory of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$.*
4. *$\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is $\mathcal{P}_T$-cowell-powered and $\mathcal{P}_T$ satisfies the hypotheses of Proposition 3.5.*

PROOF. As before, let $\overline{\mathcal{A}}$ denote the tame closure of $\mathcal{A}$. The equivalence of 1, 2, and 3 follows from the fact that $\overline{\mathcal{A}}$ is, by Proposition 3.33, the intersection of all reflective subcategories of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ which contain $\mathcal{A}$. To prove that 3 implies 4, we let $\overline{P}$ denote the class of all epimorphisms of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ which have the unique extension property with respect to all objects of $\overline{\mathcal{A}}$. By Lemma 3.30, we see that 3 implies that $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is $\overline{P}$-cowell-powered and this shows that $\overline{P} \subseteq \mathcal{P}_T$. On the other hand, that $\mathcal{P}_T \subseteq \overline{P}$ is obvious given the definition of $\overline{\mathcal{A}}$. It now follows that 3 implies 4 by Theorem 3.11.2 applied to $\overline{\mathcal{A}}$ instead of $\mathcal{A}$. The proof that 4 implies 3 follows from Theorem 3.11.4. ∎

3.35. APPLICATION TO THE CONSTRUCTION OF $\mathcal{L}$. We now suppose that $\mathcal{C}$ is Isbell complete, that $\mathcal{A}$ is a full subcategory and that $\mathcal{L}$ is the Isbell limit closure of $\mathcal{A}$. The aim is to find a left adjoint to the inclusion $\mathcal{L} \hookrightarrow \mathcal{C}$.

As we have seen, one way of getting $\mathcal{L}$ is to first close $\mathcal{A}$ under products and then repeatedly under equalizers. It follows immediately that $\mathcal{A}$ cogenerates $\mathcal{L}$. Thus we can apply Theorem 2.17 above to conclude:

3.36. **Theorem.** *Suppose that $\mathcal{A}$ is a full subcategory of the Isbell complete category $\mathcal{C}$ and that $\mathcal{L}$ is the Isbell limit closure of $\mathcal{A}$. If for every object $C \in \mathcal{C}$, the comma category $(C, \mathcal{A})$ has a weak initial set, then $\mathcal{L}$ is a reflective subcategory of $\mathcal{C}$.* ∎

3.37. AN EXAMPLE: THE LIMIT CLOSURE OF $\mathbf{Z}$ IN THE CATEGORY OF COMMUTATIVE RINGS. Here we look at the single ring of integers and show, under the hypothesis that there are no measurable cardinals, that the limit closure consists exactly of all powers of $\mathbf{Z}$. Alternately, this can be thought of as showing that all rings in the limit closure of $\mathbf{Z}$ of cardinality below the first measurable cardinal, are powers of $\mathbf{Z}$.

For this example, we assume that there are no measurable cardinals.

We begin with Łoś's theorem, [8, Theorem 47.2], which implies that any group homomorphism $\mathbf{Z}^I \longrightarrow \mathbf{Z}$ is a linear combination of projections.

3.38. **Proposition.** *Any ring homomorphism $\mathbf{Z}^I \longrightarrow \mathbf{Z}$ is a projection.*

PROOF. Let $f : \mathbf{Z}^I \longrightarrow \mathbf{Z}$ be a ring homomorphism. Since it is also a group homomorphism, we can write $f = \sum n_i p_i$ where the $n_i \in \mathbf{Z}$ and p_i is projection on the ith coordinate. Let j be a coordinate for which $n_j \neq 0$. Suppose $k \neq j$ and let e_j, respectively e_k, denote the element that has a 1 in the jth, respectively kth coordinate, and 0 elsewhere. Then $e_j e_k = 0$, whence$0 = f(e_j e_k) = f(e_j)f(e_k) = n_j n_k$ and, since $n_j \neq 0$, it follows that $n_k = 0$. Since k was any coordinate other than j, we see that $f = n_j p_j$. Then $f(1) = 1$ implies that $n_j = 1$ and so $f = p_j$. ■

In view of Proposition 3.12. we have shown:

3.39. **Theorem.** *Assuming there are no measurable cardinals. Let $\mathcal{A}$ be the subcategory of the category of rings consisting of the ring $\mathbf{Z}$ of integers (and isomorphic copies). Then the reflective hull of $\mathcal{A}$ coincides with its limit closure and consists of the powers of $\mathbf{Z}$.*

PROOF. If R is an ring, there is a canonical injection $R \longrightarrow \mathrm{Hom}(R, \mathbf{Z})$. If $f : R \longrightarrow \mathbf{Z}$ is a homomorphism, we have a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & R^{\mathrm{Hom}(R,\mathbf{Z})} \\ {\scriptstyle f}\big\downarrow & \swarrow {\scriptstyle p_f} & \\ \mathbf{Z} & & \end{array}$$

and Proposition 3.38 implies the uniqueness. This shows that the reflection of R is $\mathbf{Z}^{\mathrm{Hom}(R,\mathbf{Z})}$ so that the powers of $\mathbf{Z}$ are the reflective subcategory generated by $\mathbf{Z}$. ■

4. The coreflective subcategory generated by the finite dimensional cubes

In this section we look at the coreflective hull of $\mathcal{C}ube$, the subcategory of finite powers of $([0,1], 0)$ in the category $\mathcal{C}$ of pointed topological spaces. We let $(\mathcal{E},\mathcal{M})$ be the strict factorization system for which $\mathcal{E}$ is the class of surjections in $\mathcal{C}$ and $\mathcal{M}$ is the class of embeddings. We apply the dual of Theorem 3.11. The dual of $\mathrm{Ref}_{\mathcal{E}}(\mathcal{A})$ is $\mathrm{Crfl}_{\mathcal{M}}(\mathcal{C}ube)$, the images of sums of cubes, is the subcategory of pointed path-connected spaces.

We let I be the dual of the class $\mathcal{P}$ in Theorem 3.11 so I is the class of all maps with the unique lifting property (ULP), the dual of the unique extension property, with respect to the cubes. We will prove that I is precisely the class of all Serre fibrations with totally path-disconnected fibers (cf. [21]). Such maps, in the category of pointed path-connected spaces will be called **Serre coverings**. It follows that the coreflective hull consists of those spaces which have no non-trivial Serre coverings. Thus they are like simply connected spaces. In fact for nice spaces, the coreflection is the universal connected covering. For spaces which are not locally simply connected, the coreflection is the universal Serre covering. The topology of the fiber over a point reflects the local situation near that point. The points in the fiber correspond to elements of the deck-translation group.

4.1. NOTATION.

1. Let $\overline{\mathcal{Cube}}$ be the colimit closure of the subcategory $\mathcal{Cube}$.
2. A **Serre fibration** is a map which has the covering homotopy property for homotopies between maps from cubes.
3. A **Serre covering** is a Serre fibration with pathwise totally disconnected fibers in $\mathrm{Crfl}_{\mathcal{M}}(\mathcal{Cube})$.
4. If $f : [0,1] \longrightarrow X$ is a path on X, we let $f^{\leftarrow}$ denote the path for which $f^{\leftarrow}(t) = f(1-t)$. (Note that $f^{\leftarrow}$ need not preserve the base point.)
5. If $f, g : [0,1] \longrightarrow X$, with $f(1) = g(0)$, let $f \bullet g : [0,1] \longrightarrow X$ be defined by

$$(f \bullet g)(t) = \begin{cases} f(2t) & \text{for } 0 \le t \le \frac{1}{2} \\ g(2t-1) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

4.2. **Lemma.** *$\overline{\mathcal{Cube}}$ is the coreflective hull of $\mathcal{Cube}$.*

PROOF. This follows from the dual of Proposition 3.12. ∎

4.3. NOTATION. We let $\epsilon X : K(X) \longrightarrow X$ denote the coreflection of $X \in \mathcal{C}$ into $\overline{\mathcal{Cube}}$. By the dual of Theorem 3.11, there is a class $\mathcal{P}$ of maps for which $(\mathcal{P}, I)$ is a left factorization system on path-connected pointed spaces. Recall that the objects of $\overline{\mathcal{Cube}}$ are those which have no proper I-subobjects which implies that every map into an object of $\overline{\mathcal{Cube}}$ is in $\mathcal{P}$. We proceed to characterize the class of maps I.

We need some technical lemmas for which the following notation is convenient:

4.4. NOTATION.

1. If $x \in [0,1]^n$ then we write $x = (x_1, x_2, \ldots, x_n)$.
2. If $r \in [0,1]$ then $\hat{r} \in [0,1]^n$ is defined so that $\hat{r}_i = r$ for all i.
3. If $x \in [0,1]^n$ then $\overline{x} = \hat{a}$ where $a = (\sum x_i)/n$ is the average value of the entries of x.
4. In what follows, we assume that n is given. Define $D = \{\hat{r} \mid 0 \leq r \leq \frac{1}{2}\}$, $C = [\frac{1}{2}, 1]^n$, and $CD = (C \cup D, \widehat{0})$. (We think of CD as the cube C together with a "tail" D.)

4.5. **Lemma.** *For each n, the subset CD is a retract of $[0,1]^n$.*

PROOF. For all $x \in [0,1]^n$ and all $t \in [0,1]$ let $f(x,t) = (1-t)x + t\overline{x}$, a convex combination of x with its "average value" $\overline{x}$. Clearly f is continuous in (x,t). For each x, we define $t_0(x)$ as the smallest value of t for which $f(x,t) \in CD$. Define $r(x) = f(x, t_0(x))$. It is simple to show that r is a retraction in $\mathcal{C}$, that is, r is continuous, base-point preserving and $ri = 1_{CD}$, where $i : CD \longrightarrow [0,1]^n$ is the inclusion. ■

4.6. **Corollary.** $(CD, \widehat{0}) \in \overline{\mathcal{Cube}}$.

PROOF. The map $r : [0,1]^n \longrightarrow CD$ is the coequalizer of the maps $ir, \mathrm{id}_{[0,1]^n} : [0,1]^n \to [0,1]^n$. ■

4.7. REMARK. CD is not the sum (in pointed topological spaces) of $[0,1]$ and $[0,1]^n$ because its base point is not in the right place for the sum. Later we will show that we can move base points and stay in $\overline{\mathcal{Cube}}$.

4.8. **Lemma.** *Let $s \in I$ where $s : (E, e_0) \longrightarrow (X, x_0)$ and suppose $e_1 \in E$. Then $s : (E, e_1) \longrightarrow (X, x_1)$ is also in I, where $x_1 = s(e_1)$.*

PROOF. Since $([0,1]^n, \widehat{0})$ is isomorphic to $([\frac{1}{2}, 1]^n, \widehat{\frac{1}{2}})$ it suffices to show that every map $f : ([\frac{1}{2}, 1]^n, \widehat{\frac{1}{2}}) \longrightarrow (X, x_1)$ has a unique lift to (E, e_1). Since E is path-connected, there exists a path from e_0 to e_1. Since D is isomorphic to the unit interval, we may as well assume that this path is given by $g : D \longrightarrow E$ with $g(\widehat{0}) = e_0$ and $g(\widehat{\frac{1}{2}}) = e_1$. Clearly there exists $h : (CD, \widehat{0}) \longrightarrow (X, x_0)$ such that $h \mid D = sg$ and $h \mid C = f$. Since the class of objects for which s has the unique lifting property is closed under colimits, the map h has a unique lift to (E, e_0) and this easily gives us the desired unique lift of f. ■

The following is well-known, e.g. [26], Theorem 3.1. We note that Ungar assumes that all spaces are Hausdorff, but that is not used in this proof. We could also apply our construction to the category of pointed Hausdorff spaces, with strictly analogous results.

4.9. **Proposition.** *The maps in I are precisely the Serre coverings.*

PROOF. Assume $s : (E, e_0) \longrightarrow (X, x_0)$ is in I. We first need to show that s is a Serre fibration. Let $h : [0,1]^n \times [0,1] \longrightarrow X$ be a homotopy and let $g : [0,1]^n \times \{0\} \longrightarrow E$ be such that $sg = h \mid [0,1]^n \times \{0\}$. Then $[0,1]^n \times [0,1]$ and $[0,1]^n \times \{0\}$ are both (homeomorphic to) cubes, so by the dual of Theorem 3.11.5 the inclusion map $f : [0,1]^n \times \{0\} \longrightarrow [0,1]^n \times [0,1]$ must be in $\mathcal{P}$. Since $s \in I$ there is a diagonal fill-in d for which $sd = h$ and $df = g$. Clearly d is the required lifting of the homotopy h. Moreover, it is easily verified that the fibers of s must be pathwise totally disconnected in order for s to be in I (with the unique lifting property for all cubes).

Conversely, assume $s : E \longrightarrow X$ is a Serre covering (so that E, X are path-connected). We will show that $s \in I$. It follows by an easy induction that, once base points have been assigned, then s has the lifting property for cubes. To show that these lifts are unique, it suffices to show that s has the unique lifting property for $[0,1]$. But this is obvious since if a path had two lifts, they would have to be homotopic and this would lead to a non-trivial path component in one of the fibers of s. ∎

4.10. **Lemma.** *The path-connected, pointed space A is in $\overline{\mathcal{Cube}}$ if and only every $s \in I$ has the ULP (unique lifting property) with respect to A.*

PROOF. Assume $A \in \overline{\mathcal{Cube}}$. By definition, every $s \in I$ has the ULP with respect to every cube and, moreover, the class of objects with respect to which s has the ULP is easily seen to be limit closed. It follows that s has the ULP with respect to A.

Conversely, assume that every $s \in I$ has the ULP with respect to A. Let $\epsilon : K(A) \longrightarrow A$ be the coreflection map. Then, clearly, $\epsilon \in I$, so the identity map $1_A : A \longrightarrow A$ lifts to a map $r : A \longrightarrow K(A)$ with $\epsilon r = 1_A$. But ϵ is epic in the category of path-connected pointed spaces, and, since ϵ also has a right inverse r, it follows that $r = \epsilon^{-1}$ and so $A \in \overline{\mathcal{Cube}}$. ∎

4.11. **Proposition.** *If* $(A, a_0) \in \overline{\mathit{Cube}}$ *then, for each* $a_1 \in A$*, we have* $(A, a_1) \in \overline{\mathit{Cube}}$*.*

PROOF. By the above lemma, it suffices to show that every map in I has the ULP with respect to (A, a_1). Let $s : (E, e_1) \longrightarrow (X, x_1)$ be in I. Let $h : (A, a_1) \longrightarrow (X, x_1)$ be given. We need to show that h lifts to (E, e_1).

Since A is path-connected, there is a path $g : [0, 1] \longrightarrow A$ with $g(0) = a_0$ and $g(1) = a_1$. Then $g : ([0, 1], 1) \longrightarrow (X, x_1)$. Since $([0, 1], 1)$ is isomorphic to $([0, 1], 0)$, it is in $\overline{\mathit{Cube}}$ and therefore $hg : ([0, 1], 1) \longrightarrow (X, x_1)$ has a unique lift $\overline{hg} : ([0, 1], 1) \longrightarrow (E, e_1)$. Let $e_0 = \overline{hg}(0)$. Now let $x_0 = h(a_0)$. Then $s(e_0) = a_0$ as $s\overline{hg} = hg$. But by Lemma 4.8, we see that $s : (E, e_0) \longrightarrow (X, x_0)$ is in I. Since $(A, a_0) \in \overline{\mathit{Cube}}$, there exists $\overline{h} : (A, a_0) \longrightarrow (E, e_0)$ with $s\overline{h} = h$. We claim that $\overline{h}(a_1) = e_1$ and so $\overline{h}$ gives us the required lift (and $\overline{h}$ is unique as s is epic in path-connected, pointed spaces). It obviously suffices to show that $\overline{h}g = \overline{hg}$. But this follows as $s\overline{h}g = s\overline{hg}$ and both $\overline{h}g$ and $\overline{hg}$ are base point preserving maps from $([0, 1], 0)$ to (E, e_0). ∎

4.12. REMARK. In view of the above proposition, the condition that $A \in \overline{\mathit{Cube}}$ is independent of the choice of a base point. By Lemma 4.8, the condition that a morphism s is in I is also independent of the base point. It follows that we can safely omit any explicit reference to base points when saying $s : K(B) \longrightarrow B$ is the coreflection of B into $\overline{\mathit{Cube}}$. More precisely:

4.13. **Proposition.** *Assume* $s : (Y, y_0) \longrightarrow (X, x_0)$ *is the coreflection of* (X, x_0) *into* $\overline{\mathit{Cube}}$ *and assume* $y_1 \in Y$ *and* $x_1 = s(y_1)$*. Let* s' *be the function s, regarded as a map from* (Y, y_1) *to* (X, x_1)*. Then* $s' : (Y, y_1) \longrightarrow (X, x_1)$ *is the coreflection of* (X, x_1) *into* $\overline{\mathit{Cube}}$*.*

PROOF. To say that $s : (Y, y_0) \longrightarrow (X, x_0)$ is the coreflection of (X, x_0) into $\overline{\mathit{Cube}}$ means that $(Y, y_0) \in \overline{\mathit{Cube}}$ and that $s \in I$. But, by the previous results, both of these conditions are preserved by change of base point, so s_1 is the coreflection of (X, x_1) into $\overline{\mathit{Cube}}$. ∎

4.14. **Theorem.** *A path-connected, non-empty space* A *is in* $\overline{\mathit{Cube}}$ *if and only if* A *has no non-trivial Serre covering.*

PROOF. This follows from the dual of Theorem 3.11.5. ∎

4.15. DEFINITION. Let $s = \epsilon B : K(B) \longrightarrow B$ be the coreflection of B into $\overline{\mathcal{C}ube}$. By $\Delta(B)$, the **deck translation group** of B, we mean the group of all continuous, not-necessarily base point preserving, functions $f : K(B) \longrightarrow K(B)$ for which $sf = s$. (We will prove that all such maps f are necessarily homeomorphisms and that $\Delta(B)$ is really a group.)

4.16. **Proposition.** *Let $s : (K(B), e_0) \longrightarrow (B, b_0)$ be the coreflection of (B, b_0) into $\overline{\mathcal{C}ube}$. Let $e_1 \in s^{-1}(b_0)$ be given. Then there is a unique $\delta \in \Delta(B)$ for which $\delta(e_0) = e_1$.*

PROOF. Let s' be the same function as s, but regard it as a morphism $(K(B), e_1) \longrightarrow (B, b_0)$. Then, by Proposition 4.13, we see that s' is also a coreflection of (B, b_0) into $\overline{\mathcal{C}ube}$. It follows that there exists an isomorphism $\delta : (K(B), e_0) \longrightarrow (K(B), e_1)$ for which $s\delta = s$ so that δ is the required member of $\Delta(B)$. ∎

4.17. **Corollary.** *Every map in the deck translation group is a homeomorphism and the deck translation group is actually a group under composition of functions.*

PROOF. Let B be a given path-connected pointed space with base point b_0. If $f : K(B) \longrightarrow K(B)$ is a continuous, not necessarily base point preserving, map for which $sf = s$, let e_0 be the base point of $K(B)$ and let $e_1 = f(e_0)$. Then $f : (K(B), e_0) \longrightarrow (K(B), e_1)$ is a lift of $s : (K(B), e_1) \longrightarrow (B, b_0)$ to a map $(K(B), e_1) \longrightarrow (K(B), e_0)$ which factors through $s : (K(B), e_0) \longrightarrow (B, b_0)$. By the uniqueness of the lift, f must coincide with the homeomorphism δ constructed in the above proof.

The fact that the deck translation group is actually a group under composition of functions is now easily verified. ∎

4.18. DEFINITION. Let $g : [0, 1] \longrightarrow (B, b_0)$ be a loop, (meaning that $b_0 = g(0) = g(1)$) and let $s : (S, s_0) \longrightarrow (B, b_0)$ be a Serre covering. Let $\overline{b} : [0, 1] \longrightarrow (S, s_0)$ be the unique lifting of g. Then s **deloops** g if $\overline{b}$ is no longer a loop; that is if $\overline{b}(0) \neq \overline{b}(1)$.

A loop is **Serre trivial** if no Serre covering deloops it. Equivalently, if the coreflection $\epsilon B : K(B) \longrightarrow B$ fails to deloop g. We say that $\gamma \in \pi(B)$ is Serre trivial if γ is represented by a Serre trivial loop g.

It readily follows that:

4.19. **Theorem.** *For each path-connected pointed space* (B, b_0) *there is a natural surjective group homomorphism* $\pi(B) \longrightarrow \Delta(B)$ *whose kernel is the subgroup of Serre trivial members of* $\pi(B)$. ∎

4.20. REMARK. Covering maps are not closed under either composition [24] or limits in $\mathrm{Crfl}_{\mathcal{M}}(\mathcal{C}ube)$ of diagrams with a fixed codomain. But Serre coverings, characterized by having the ULP for cubes, are closed under both.

4.21. DEFINITION. For each path-connected, pointed topological space B, let $\hat{\pi}(B)$ denote the quotient of the fundamental group $\pi(B)$ by the normal subgroup of Serre trivial loops. As shown above, then $\hat{\pi}(B)$ is isomorphic to the deck translation group of B.

We say a pointed, path-connected space B is **Serre simply connected** if $\hat{\pi}(B)$ is the trivial group, equivalently, if all loops on B are Serre trivial.

4.22. DEFINITION. A subset U of a topological space X is **path-open** if for every path $f : [0,1] \longrightarrow X$, we have that $f^{-1}(U)$ open in $[0,1]$. A space X is $[0,1]$**-generated** if every path-open subset is open.

4.23. **Proposition.** $(X, x_0) \in \overline{\mathcal{C}ube}$ *if and only if* X *is* $[0,1]$*-generated, path-connected, and Serre simply connected.*

PROOF. Suppose (X, x_0) satisfies the given conditions. Since X is path-connected the coreflection map $\epsilon X : K(X, x_0) \longrightarrow (X, x_0)$ is surjective. Since every loop on X is Serre trivial, ϵX is injective. Since X is $[0,1]$-generated it follows that ϵX is a topological quotient map. To prove this, it suffices to show that every path on X lifts to $K(X, x_0)$. But since ϵX is a Serre fibration, this is immediate.

Conversely, assume that $(X, x_0) \in \overline{\mathcal{C}ube}$. Obviously every loop on it is trivial. Since $\overline{\mathcal{C}ube} \subseteq \mathrm{Crfl}_{\mathcal{M}}(\mathcal{C}ube)$, we see that it is path-connected. To prove that X is $[0,1]$-generated, it suffices to observe that the subcategory of all $[0,1]$-generated pointed spaces is coreflective and contains all the cubes, hence contains the coreflective hull $\overline{\mathcal{C}ube}$. ∎

4.24. DEFINITION. Recall that a space is locally path-connected if the path-connected open subsets form a base for the topology.

4.25. REMARK. It is well-known that the subcategories of sequential spaces and of locally path-connected spaces are both coreflective in $\mathcal{Top}$. It easily follows that both pointed sequential spaces and pointed locally connected spaces are coreflective in pointed spaces. So sequential spaces as well as locally path-connected spaces are both closed under the formation of coproducts and quotients. See [6, 10, 11].

It is also well known, and readily proven, that locally path-connected spaces are characterized by the property that path components of open subsets are open.

4.26. **Lemma.** *All* $[0,1]$*-generated spaces are locally path-connected and sequential.*

PROOF. It is obvious that the $[0,1]$-generated spaces form the coreflective hull of the space $[0,1]$. Since $[0,1]$ is contained in the coreflective subcategory of locally path-connected sequential spaces, it is clear that the coreflective hull of $[0,1]$ is also contained in locally path-connected sequential spaces. ∎

4.27. **Corollary.** *If* $(X, x_0) \in \overline{\mathcal{Cube}}$*, then* X *is locally path-connected and sequential.* ∎

4.28. **Lemma.** *An open subset of a* $[0,1]$*-generated space is* $[0,1]$*-generated.*

PROOF. Let X be [0,1]-generated, let $U \subseteq X$ be open and let $V \subseteq U$ be path-open in U. We need to show that V is open. Since X is [0,1]-generated, it suffices to show that V is path-open in X. So if $p : [0,1] \longrightarrow X$ is an arbitrary path on X, it suffices to show that $p^{-1}(V)$ is open in [0,1]. Let $r \in p^{-1}(V)$ be arbitrary. It suffices to show that $p^{-1}(V)$ is a neighbourhood of r. But U is open in X and p is continuous, so $p^{-1}(U)$ is open in [0,1] and it clearly contains r. Obviously, we can find a closed subinterval $[a, b] \subseteq [0,1]$ so that $[a, b]$ is a neighbourhood of r. Let p_1 be the restriction of p to [a,b]. Then $p_1 : [a, b] \longrightarrow U$ so, by recalibrating $[a, b]$, we can regard p_1 as a path on U. Since V is path-open in U, we see that ${p_1}^{-1}(V)$ is open in [0,1]. Note that $r \subseteq {p_1}^{-1}(V) \subseteq p^{-1}(V)$ which shows that $p^{-1}(V)$ is a neighbourhood of r. ∎

4.29. **Lemma.** *Let $U \subseteq X$ be an open subset of the $[0,1]$-generated, path-connected, pointed space X such that U is Serre simply connected. Let $\epsilon X : K(X) \longrightarrow X$ be the coreflection of X into $\overline{\mathcal{Cube}}$. Then $\epsilon X^{-1}(U)$ is a disjoint union of open sets each of which is mapped homeomorphically onto U by the restriction of ϵX.*

PROOF. Since X is $[0,1]$-generated, it is locally path-connected. Similarly, by 4.23, $K(X)$ is [0,1]-generated and locally path-connected. It suffices to prove the lemma in the case that U is path-connected (then apply that result to the path-components of U). Let V be any path-component of $(\epsilon X)^{-1}(U)$. Let $\epsilon_V : V \longrightarrow U$ be the restriction of ϵX to V. We note we can move the base point of X to a point in U and move the base point of $K(X)$ to a point in V lying over the base point in U. It is straightforward to prove that $\epsilon_V \in I$ and thus is a Serre Covering. By the above lemma, U is $[0,1]$-generated and since U is also Serre simply connected, it follows that U is in the colimit closure of $\mathcal{Cube}$. Therefore, the only Serre coverings into U are isomorphisms. Thus ϵ_V is a homeomorphism and the result follows. ∎

4.30. **Corollary.** *Suppose $\epsilon X : K(X) \longrightarrow X$ is a coreflection map where X is a $[0,1]$-generated, pointed, path-connected space. Let $U \subseteq X$ be an open subset which is Serre simply connected. Then the fibers $(\epsilon X)^{-1}(u)$ are discrete for every $u \in U$ and $(\epsilon X)^{-1}(U)$ covers U.* ∎

4.31. EXAMPLE. For each positive integer n, let C_n be the circle in $\mathbf{R} \times \mathbf{R}$ with center $(\frac{1}{n}, 0)$ and radius $\frac{1}{n}$. The "Hawaiian earring", HE is defined as $\bigcup\{C_n\}$. Note that all of these circles pass through $(0,0)$, which we take as the base point of E.

For each n, we let $\mathbf{a}_n$ be a loop in HE which starts at $(0,0)$ and travels once around C_n in a counterclockwise direction. There are additional loops of the form $\mathbf{a}_{n_1} \bullet \cdots \bullet \mathbf{a}_{n_k}$. Moreover, there are loops which wind around infinitely many C_n. For example, we can define a loop $\ell : [0,1] \longrightarrow E$ so that ℓ restricted to $[\frac{1}{2}, 1]$ goes once around C_1, then ℓ restricted to $[\frac{1}{3}, \frac{1}{2}]$ goes once around C_2 and so forth, with ℓ restricted to $[\frac{1}{n+1}, \frac{1}{n}]$ going around C_n (and $\ell(0) = (0,0)$). This map is continuous.

There are even stranger loops. Recall that the Cantor set is found by first removing $(\frac{1}{3}, \frac{2}{3})$, the "open middle third" of the interval $[0,1]$, then removing $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, the middle thirds of the two intervals that remain, then removing the middle thirds of four intervals that remain and so forth. Define a loop $\lambda : [0,1] \longrightarrow \mathrm{HE}$ so that on $[\frac{1}{3}, \frac{2}{3}]$ the loop λ goes once around C_1, and on each of the next two middle thirds, λ goes around C_2. In general, define λ so that it goes once around C_n on the closure of each open interval of length $(\frac{1}{3})^n$ in the complement of the Cantor set. Finally, define λ to be the base point, $(0,0)$, on each point in the Cantor set. Clearly HE has a rich set of loops and a complicated fundamental group.

We now construct the coreflection of the Hawaiian earring in $\overline{\mathcal{C}ube}$. For each positive integer n, let HE_n be $C_1 \cup C_2 \cup \cdots \cup C_n$. Let $q_n : \mathrm{HE} \longrightarrow \mathrm{HE}_n$ be the quotient map which collapses each circle C_m for $m > n$ to the base point. Note that HE_n is connected, locally path-connected, and locally simply connected so it has a universal covering space which pulls back along q_n the covering map $e_n : W_n \longrightarrow \mathrm{HE}$. The fiber of W_n over the base point is clearly the fundamental group of HE_n which is the free group on $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Note that whenever $m > n$ there is a map $e_{n,m} : W_m \longrightarrow W_n$ such that $e_n e_{n,m} = q_m$. Consider the filtered diagram formed by the family of maps $\{e_{n,m}\}$ We let $\hat{e} : W_\infty \longrightarrow \mathrm{HE}$ be the limit of this diagram taken in the category of path-connected, [0,1]-generated pointed spaces. This means starting with the filtered limit L_∞ in pointed spaces, then taking the path component of the base point of L_∞ and, finally, replacing the topology on this component with its [0,1]-generated coreflection. (Note that this final step does not change any of the paths or homotopies.)

We claim that $\hat{e} : W_\infty \longrightarrow \mathrm{HE}$ is the coreflection of HE. It is readily shown that $\hat{e}$ has the ULP with respect to all cubes as each covering map e_n does. Therefore $\hat{e} : W_\infty \longrightarrow \mathrm{HE}$ is a Serre covering. By Proposition 4.23, It suffices to show that $W_\infty \in \overline{\mathcal{C}ube}$ or that W_∞ is path connected, [0,1]-generated and Serre simply connected. It is, by construction, path-connected and [0,1]-generated. To show that W_∞ is simply connected, we note that by theorem 2.5 and the discussion in 3.3 of [5], a loop $p : [0,1] \longrightarrow E$ is homotopically trivial if and only if it is not delooped by any of the coverings $e_n : W_n \longrightarrow E$. Equivalently, the loop p is homotopically trivial if and only if it is not delooped by the Serre covering $\hat{e} : W_\infty \longrightarrow E$. It follows that the fundamental group of W_∞ is trivial, for if $P : [0,1] \longrightarrow W_\infty$ is a given loop, then P is a lifting of $p = \hat{e}P$. Observe that p is not delooped by $W_\infty \longrightarrow E$ because p lifts to the loop P, so p is homotopically trivial. It follows by the covering homotopy property that P is trivial too. So W_∞ is simply connected and a fortiori, Serre simply connected. [5] has a nice description of the fundamental group of HE as it is embedded in the inverse limit of the free groups on $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

4.32. REMARK. For the coreflective subcategory generated by the contractible pointed spaces, see [22, 21]. The results are similar except that the earlier papers do not have a nice characterization of the maps with the unique lifting property for contractible pointed spaces. Also in the earlier case, the fibers of the coreflection are homeomorphic to each other and therefore do not mirror local properties.

5. Limit closures in uniform spaces

5.1. UNIFORMITIES. In this section, we study limit closures in the category of uniform spaces. Uniform spaces were first studied in [27] to do for uniform continuity what topology does for continuity. The studies have been continued in many places including, for example [18, 16, 28]. There are three ways (at least) to present uniform spaces. The first way describes a uniformity on X as a family of subsets of $X \times X$, called entourages in [27] (sometimes translated as "surroundings"), subject to certain conditions. The second is in terms of a family of covers, called uniform covers, subject to certain conditions, and the third is in terms of a family of pseudometrics which we describe in detail below. For a development of the three ways and proofs that they are equivalent, see [28], Sections 35-39. The arguments in this section rely mainly on the pseudometrics, but the other approaches are important elsewhere. See also [7], Chapter IX, Sections 10, 11 (where pseudometrics are called *gauges*) for the equivalence of the three definitions, or see [18], Chapter 6 for the construction of a pseudometric from a sequence of entourages each a *-refinement of the next.

Basically, uniformities generalize metrics. In a metric space, a typical entourage is $\{(x, y) \mid d(x, y) < \epsilon\}$, a typical uniform cover is the set of all ϵ-balls and the single metric generates the family of pseudometrics. See [18, 16, 28] for further details.

5.2. A PRIMER ON PSEUDOMETRICS. Suppose X is a set and $d : X \times X \to \mathbf{R}$ is a function. We consider the following six conditions on d, for all $x, x' \in X$:

M-1. $d(x, x') \geq 0$;

M-2. $d(x, x) = 0$;

M-3. $d(x, x') = 0$ implies $x = x'$

M-4. $d(x, x') = d(x', x)$;

M-5. $d(x, x'') \leq d(x, x') + d(x', x'')$.

M-6. $d(x, x'') \leq d(x, x') \vee d(x', x'')$.

Obviously M-1 and M-6 imply M-5. If d satisfies M-1 to M-5, it is called a **metric** on X. If it satisfies M-1 to M-6, it is called an **ultrametric** on X. If it satisfies M-1, M-2, M-4, and M-5, it is called a **pseudometric** on X. If it satisfies M-1, M-2, M-4, and M-6, it is called an **ultra-pseudometric** on X.

The set of pseudometrics on X is clearly partially ordered by $d \leq e$ if for all $x, x' \in X$, $d(x, x') \leq e(x, x')$.

5.3. DEFINITION.

1. A **base for a uniform structure** (or a **base** for short) on a set X is a family $\mathcal{D}$ of pseudometrics which is directed by $\leq$. Note the parallel between this definition and that of base for a topology.

2. If $\mathcal{D}$ and $\mathcal{E}$ are bases for uniform structures on X and Y respectively, then a function $f : X \longrightarrow Y$ is **uniform** or a **unimorphism** if, for all $e \in \mathcal{E}$ and all $\epsilon > 0$, there are $d \in \mathcal{D}$ and $\delta > 0$ such that, for all $x, x' \in X$, $d(x, x') < \delta$ implies $e(fx, fx') < \epsilon$.

3. If $\mathcal{D}$ consists of a single pseudometric d, we will sometimes write (X, d) instead of $(X, \{d\})$.

4. A base $\mathcal{D}$ on a set X is **saturated** if whenever e is a pseudometric on X for which $\mathrm{id} : (X, \mathcal{D}) \longrightarrow (X, e)$ is a unimorphism, then $e \in \mathcal{D}$. This is equivalent to the assertion that adding any pseudometric to $\mathcal{D}$ changes the uniformity.

5. A **uniformity** (or a **uniform structure**) on a set X is a saturated base of pseudometrics.

We note that in [18] the term **gage** is used to denote what we have called a saturated base. We also note that if d, e are pseudometrics on a set X, then their sum $d + e$ and their sup $d \vee e$ are also pseudometrics.

A subset $\mathcal{D}_0 \subseteq \mathcal{D}$ **generates** $\mathcal{D}$ if $\mathcal{D}$ is the smallest (saturated) uniformity containing $\mathcal{D}_0$. We will sometimes let $(X, \mathcal{D}_0)$ denote the uniform space given by the saturated base generated by $\mathcal{D}_0$.

We say that $\mathcal{D}$ is a **pseudometric structure** if it is generated by a single pseudometric d. If d is a metric, we say that the uniformity is metrizable, or even that it is a metric space.

Let X be a uniform space with uniform structure given by $\mathcal{D}$. If $A \subseteq X$ is a subset then for any $d \in \mathcal{D}$ and any $x \in X$, we define $d(x, A) = d(A, x) = \inf_{a \in A} d(x, a)$. We say that x is **adherent to** A if $d(x, A) = 0$ for all $d \in \mathcal{D}$ and define $\mathrm{cl}(A)$ to consist of all points that are adherent to A. This is easily seen to be a Kuratowski closure operator and defines a topology on X called the **uniform topology**.

The following is quoted verbatim from [16], II.8.

5.4. **Proposition.** *The uniform topology of a sum, product, or subspace is the sum, product, subspace topology, respectively.* ∎

Two non-isomorphic uniform spaces can define homeomorphic uniform topologies. For example, the sequence $\{\frac{1}{n}\}$, n a positive integer, is a uniform space with the usual metric from **R**. The uniform topology is discrete. A second uniformity, called the **discrete uniformity** is generated by setting the distance between any two distinct elements to be 1. But the identity function from the second to the first is not uniform, although both uniform topologies are discrete. On the other hand, the uniformity that the space of integers inherits from the reals is isomorphic to the discrete metric generated by setting the distance between any two distinct elements to be 1.

A uniform space X is said to be **separated** if points are closed in the uniform topology. This turns out to be true if and only if the uniform topology is Hausdorff (in fact, completely regular Hausdorff). Obviously a pseudometric space is separated if and only if the generating pseudometric is a metric.

Important properties of compact Hausdorff spaces include that they have a unique uniformity, they are obviously separated, and every map from a compact Hausdorff space that is continuous in the uniform topologies of the spaces is also uniform.

For a uniform space X, the obvious definition of totally disconnected is that for every pair $x \neq x'$ of elements of X there is a uniform map to the discrete space 2 that separates them. This is equivalent to having an injective unimorphism into a power of 2. If X is also compact, such a map is clearly a uniform embedding, that is, it has the induced uniformity. In general, it will not be an embedding. However, we will say that X is **uniformly totally disconnected** if its uniform structure has a base of ultra-pseudometrics. As we will see, this is equivalent to having a uniform embedding into a product of discrete uniform spaces. If the discrete spaces are each given the uniformity in which the distance between any two distinct points is 1, then one easily sees that any product of such spaces has a uniformity given by a family of ultra-pseudometrics.

5.5. **Proposition.** *Every monic in the category of separated uniform spaces is injective. Hence the category is well-powered and Isbell complete.*

PROOF. If $f : X \longrightarrow Y$ is not injective, there are points $x, x' \in X$ such that $f(x) = f(x')$. These give two maps, evidently uniform, from a one point space to X with the same composite with f and thus f is not monic. ■

5.6. **Proposition.** *Suppose that $\mathcal{C}$ is a full subcategory of separated uniform spaces that is closed under products and closed subspaces, then $\mathcal{C}$ is reflective.*

PROOF. We let $(\mathcal{E},\mathcal{M})$ be the strict factorization system on the category of separated uniform spaces for which $\mathcal{M}$ is the class of embeddings of closed subspaces and $\mathcal{E}$ is the class of all maps $e : B \longrightarrow C$ for which $e(B)$, the image of e, is dense in C. The proof that this is a strict factorization system is straightforward. For example to prove orthogonality, suppose $fe = mg$ where $e : B \longrightarrow C$ is in $\mathcal{E}$ and $m : M \longrightarrow X$ is the embedding of the closed subspace M of X. By hypothesis, $e(B)$, is dense in C. Clearly, f maps $e(B)$ into M and, since M is closed and $e(B)$ is dense and f is continuous, we see that f maps all of C into M. The proof of the orthogonality condition is now obvious, and the proof of this Proposition follows from Proposition 3.4. ■

5.7. COLLAPSING A CLOSED SUBSPACE. If X is a separated uniform space and $C \subseteq X$ is a closed subspace, we denote by E_C the equivalence relation defined by xE_Cy if $x = y$ or $x, y \in C$. Then the space X/E_C is the quotient gotten by collapsing C to a point.

The sup of two pseudometrics is easily seen to be a pseudometric, but the inf is not in general. For example, let $X = \{x, y, z\}$ with distance functions d_1 and d_2 given by $d_1(x, z) = d_2(x, z) = 3$, $d_1(x, y) = d_2(y, z) = 1$, and $d_1(y, z) = d_2(x, y) = 2$. Then d_1 and d_2 satisfy the triangle inequality but $d = d_1 \wedge d_2$ does not.

However, we do have a simple criterion for the inf of two pseudometrics to be a pseudometric.

5.8. **Proposition.** *Let d_1 be a pseudometric (respectively, ultra-pseudometric) and d_2 satisfy M-1, M-4, and M-5 (respectively, M-1, M-4, and M-6) on the set X. Then a necessary and sufficient condition that $d = d_1 \wedge d_2$ be a pseudometric (respectively ultra-pseudometric) is that for all $x, y, z \in X$, we have $d(x, z) \le d_1(x, y) + d_2(y, z)$ (respectively, $d(x, z) \le d_1(x, y) \vee d_2(y, z)$).*

PROOF. We will use the sign $\forall$ to denote either $+$ or $\vee$ as appropriate in the argument below. Let us note that by exchanging x and z and using symmetry the inequality becomes $d(x, z) \le d_2(x, y) \forall d_1(y, z)$. It is obvious that d satisfies M-1, M-2, and M-4, so that the only issue is M-5 or M-6, as the case might be. The necessity of the condition is obvious since $d \le d_1$ and $d \le d_2$. To verify that

$$d(x, z) \le d(x, y) \forall d(y, z)$$

it is, in principle, necessary to consider 8 cases depending on which of the three terms in the inequality is given by d_1 or d_2. But if, for example, the two terms on the right are given by the same d_i, $i = 1, 2$, then we can argue that $d(x, z) \le d_i(x, z) \le d_i(x, y) \forall d_i(y, z) = d(x, y) \forall d(y, z)$. Thus we concentrate on the cases in which they differ, say $d(x, y) = d_1(x, y)$, while $d(y, z) = d_2(y, z)$. But then $d(x, z) \le d_1(x, y) \forall d_2(y, z)$, the latter being exactly our hypothesis. ∎

5.9. **Lemma.** *Suppose d is a pseudometric on X and $C \subseteq X$ is a subset. Then the function d^C defined by $d^C(x, y) = d(x, C) \forall d(y, C)$ satisfies M-1, M-4, and M-5, and M-6 if d is an ultra-pseudometric.*

PROOF. The only thing we need show is that $d^C(x,z) \le d^C(x,y) \forall d^C(y,z)$ or that $d(x,C) \forall d(z,C) \le d(x,C) \forall d(y,C) \forall d(y,C) \forall d(z,C)$ which is obvious since $\forall$ is monotone. ∎

5.10. **Proposition.** *Suppose d is a pseudometric on X and $C \subseteq X$ is a subset. Then the function d_C defined by $d_C(x,y) = d(x,y) \wedge d^C(x,y)$ is a pseudometric on X and is an ultra-pseudometric in the case that d is an ultra-pseudometric and $\forall = \vee$.*

PROOF. We must show that $d_C(x,z) \le d(x,y) \forall (d(y,C) \forall d(C,z))$. Since $d_C(x,z) \le d(x,C) \forall d(C,z)$, it suffices to show that $d(x,C) \forall d(C,z) \le d(x,y) \forall d(y,C) \forall d(C,z)$. Since both $+$ and $\vee$ are monotone, it suffices to show that $d(x,C) \le d(x,y) \forall d(y,C)$. For any $\epsilon > 0$, there is a $c \in C$ such that $d(C,y) > d(c,y) - \epsilon$. Then we have $d(x,C) \le d(x,c) \le d(x,y) \forall d(c,y) \le d(x,y) \forall (d(C,y) + \epsilon) \le (d(x,y) \forall d(C,y)) + \epsilon$. But for this to hold for all $\epsilon > 0$, we must have $d(x,C) \le d(x,y) \forall d(y,C)$ as required. ∎

We have not found the following in standard references although it would seem too obvious not to be known. The fact that it fails for completely regular spaces shows that 5.4 would not hold for quotients.

5.11. **Theorem.** *Suppose X is a separated uniform space and $C \subseteq X$ is a closed subspace. Then X/E_C has a separated uniform structure such that the projection $X \longrightarrow X/E_C$ is uniform. If X is uniformly totally disconnected, the same is true of X/E_C.*

PROOF. For each pseudometric d on X, let d_C be as above. For $x \in X$ and $y \in C$, it is clear that $d_C(x,y) = d(x,C)$ since $d(x,C) \le d(x,y)$. Also, if $x, y \in C$, then $d_C(x,y) = 0$ so that d_C is actually a pseudometric on X/E_C. If $x \notin C$, then the fact that C is closed implies that there is at least one pseudometric d on X such that $d(x,C) \neq 0$ so that the set of all d_C separates the points of X/E_C that are not in C from C. If $x \notin C$ and $y \notin C$, then choose a d_1 so that $d_1(x,y) \neq 0$ and, since C is closed, a d_2 so that $d_2(x,C) \neq 0$. Then $d = d_1 \forall d_2$, d_C separates x from y. Finally, the fact that $d_C(x,y) \le d(x,y)$ implies that the projection is uniform. Clearly d_C is an ultra-pseudometric when d is, which proves the last sentence. ∎

We note that the monotonicity of the operation $d \mapsto d_C$ trivially implies that $\{d_C\}$ is a base for a uniformity.

5.12. **Corollary.** *In the category of separated uniform spaces, an inclusion of a closed subspace is regular and every extremal monomorphism is regular.* ∎

5.13. **Corollary.** *Suppose $\mathcal{D} \subseteq \mathcal{C}$ are full subcategories of separated uniform spaces such that*

1. *$\mathcal{C}$ is limit closed in the category of separated uniform spaces.*
2. *Every object of $\mathcal{C}$ has a closed uniform embedding into a product of objects of $\mathcal{D}$;*
3. *Whenever $C \subseteq X = \prod D_i$ is isomorphic to a closed subspace of a product of objects of $\mathcal{D}$, then the object X/E_C has an injective unimorphism into such a product.*

Then $\mathcal{C}$ is the limit closure of $\mathcal{D}$ in the category of separated uniform spaces.

PROOF. Let $C \hookrightarrow X$ be a closed uniform embedding with X a product of objects of $\mathcal{D}$. Let $q : X \longrightarrow X/E_C$ be the canonical projection onto the quotient and let $u : X \longrightarrow X/E_C$ be the constant morphism $u(x) = C$ for all $x \in X$. It is clear that C is the equalizer of q and u. Now let $f : X/E_C \longrightarrow Y$ be an injective unimorphism into a product of objects of $\mathcal{D}$. Then the equalizer of fq and fu is still C and thus C belongs to the closure of $\mathcal{D}$ under products and equalizers. Conversely, it is clear that a regular subspace of a uniform space in the category of separated uniform spaces is closed and then Corollary 2.9 implies that so is every extremal subspace. The category of closed subspaces of objects of $\mathcal{D}$ is evidently complete. ∎

5.14. **Proposition.** *If X is a separated uniform space, $C \subseteq X$ a closed subspace, then $C \hookrightarrow X$ is an equalizer of a pair of maps to a power of* $[0, 1]$.

PROOF. Let $\mathcal{D}$ be a set of pseudometrics that define the uniformity. For each $d \in \mathcal{D}$, define $f_d : X \longrightarrow [0, 1]$ by $f_d(x) = d(x, C) \wedge 1$. Let $f : X \longrightarrow [0, 1]^{\mathcal{D}}$ be the map whose dth coordinate is f_d. Let $g : X \longrightarrow [0, 1]$ be the map whose every coordinate is 0. From the definition of closure in the uniform topology there is, for each $x \notin C$ some $d \in \mathcal{D}$ for which $d(x, C) \neq 0$. It is immediate that $C \hookrightarrow X$ is the equalizer of f and g. ■

5.15. **Proposition.** *Every separated uniform space can be embedded into a product of metric spaces.*

PROOF. For any pseudometric d on the separated uniform space X, let E_d be the equivalence relation defined by xE_dx' when $d(x, x') = 0$. Then one easily sees that the space $X_d = X/E_d$ is a metric space with metric given by d. Two uses of the triangle inequality easily shows that d is well defined mod E_d. If $\mathcal{D}$ is the set of all pseudometrics on X, then the uniformity induced on X by the injective function $X \longrightarrow \prod_{d \in \mathcal{D}} X_d$ is just the infimum of the (non-separated) uniformities given by all the pseudometrics, which is the given uniformity on X. ■

5.16. **Proposition.** *The full subcategory of separated uniform spaces whose objects are the closed subspaces of a product of metric spaces is limit closed and reflective in the category of separated uniform spaces.*

PROOF. It is reflective by Prop 5.6, and is therefore limit closed. ■

Putting the last three propositions together, we see:

5.17. **Theorem.** *The limit closure and reflective hull of metric spaces consists of those separated uniform spaces that have a closed embedding into a product of metric spaces.* ■

Along similar lines, we have:

5.18. **Theorem.** *The limit closure in separated uniform spaces of the unit interval* $[0, 1]$ *is the category of separated compact uniform spaces.*

PROOF. It is well known that every compact Hausdorff space X is embeddable in a cube, say $X \hookrightarrow Y = [0, 1]^I$. Since a compact Hausdorff space has a unique uniformity and every continuous map is uniform, this embedding is closed and uniform. Proposition 5.14 implies $X \hookrightarrow Y$ is an equalizer of two maps to a cube. The converse is trivial as compact Hausdorff spaces are closed under the formation of products and closed subspaces. ■

5.19. **Proposition.** *The category of separated uniform spaces is cowell-powered.*

PROOF. Let $e : X \longrightarrow Y$ be an epic of separated uniform spaces. Let $C \subseteq Y$ be the closure of the image of X. From 5.11, we know that X/E_C is separated. We now define two maps $g, h : Y \longrightarrow Y/E_C$ with $ge = he$. Let g be the canonical projection $Y \longrightarrow Y/E_C$ and h the map which sends all of Y to the point $C \in Y/E_C$. Since e is epic, it follows that C is all of Y and that e is epic if and only if its image is dense. So there can be no more points in Y than there are ultrafilters on X. ∎

As we were trying to work out which separated uniform spaces have a closed embedding into a product of metric spaces we posted the question on the web site MathOverflow (`http://mathoverflow.net/questions/`), we got a private answer from James Cooper who stated the following conjecture: "A separated uniform space satisfies your condition if and only if strong Cauchy nets (defined below) converge." We have proved Cooper's conjecture and the argument follows the definition below.

5.20. DEFINITION. Let $(X, \mathcal{D})$ be a separated uniform space. We will say that a net $\{x_i \mid i \in I\}$ is a **strong Cauchy net** if, for all $d \in \mathcal{D}$, there is an $i \in I$ such that $j \in I$ with $j \geq i$ implies $d(x_i, x_j) = 0$. This means that the image of the net in X/E_d is eventually constant.

We will say that a separated uniform space $(X, \mathcal{D})$ is **Cooper complete** if every strong Cauchy net converges.

5.21. **Theorem.** *A separated uniform space is uniformly isomorphic to a closed subspace (with the relative uniformity) of a product of metric spaces if and only if it is Cooper complete.*

PROOF. We begin by showing that a product of Cooper complete spaces is Cooper complete. Let $X = \prod X_\tau$ and $p_\tau : X \longrightarrow X_\tau$ be the projection on the product. We describe a base for the pseudometrics on X as follows. Let d_τ be a pseudometric on X_τ. Define $\hat{d}_\tau$ on X by $\hat{d}_\tau(x, x') = d_\tau(p_\tau x, p_\tau x')$. The set of finite sups of the set of all $\hat{d}_\tau$, taken over all the pseudometrics on X_τ and over all indices τ, is the canonical base of pseudometrics on X. It generates, as it must, the least uniformity of X for which the p_τ are all unimorphisms. Now if $\{x_i\}$ is a strongly Cauchy net on X, then given a τ, and a pseudometric d_τ on X_τ, there must exist an i such that $j \geq i$

implies $\hat{d}_\tau(x_i, x_j) = 0$. But this is the same as $d_\tau(p_\tau x_i, p_\tau x_j)$, so that $\{p_\tau x_i\}$ is a strongly Cauchy sequence in X_τ and so converges to $x_\tau \in X_\tau$. But then $\{x_i\}$ converges to the element $x = (x_\tau)$. Next we show that a closed subspace of a Cooper complete space is Cooper complete. Let X be Cooper complete and $C \subseteq X$ be a closed subspace. A pseudometric on C is simply the restriction to C of a pseudometric on X. It is immediate that a strongly Cauchy net in C is strongly Cauchy in X and thus converges to an element of X, which must lie in C since C is closed.

Conversely, suppose that X is Cooper complete. Let $\mathcal{D}$ denote a base for the pseudometrics on X. We use the notation introduced in the proof of 5.15: E_d is the equivalence relation on X defined by a $d \in \mathcal{D}$ and $X_d = X/E_d$. We denote by $q_d : X \longrightarrow X_d$ the quotient mapping. Recall that $\mathcal{D}$ is partially ordered by $d \le e$ if $d(x, x') \le e(x, x')$ for all $x, x' \in X$. It is also directed since if $d, e \in \mathcal{D}$ then $d \vee e \in \mathcal{D}$ and clearly $d \le d \vee e$ and $e \le d \vee e$. We saw in 5.15 that X is embedded in $\prod_{d \in \mathcal{D}} X_d$. We claim that if every strong Cauchy net in X converges, then X is closed in the product. For each $d \in \mathcal{D}$, denote by $p_d : \prod X_d \longrightarrow X_d$ the canonical projection. If $d \le e$, it is clear that there is a canonical quotient mapping $q_{d,e} : X_e \longrightarrow X_d$ such that $q_{d,e} q_e = q_d$. Since $q_d = p_d | X$, we also have that $d \le e$ implies that $q_{d,e} p_e | X = q_d | X$. But X_d is Hausdorff and hence we conclude that $q_{d,e} p_e | \operatorname{cl}(X) = q_d | \operatorname{cl}(X)$.

Now suppose that $y \in \operatorname{cl}(X)$. Since q_d is surjective we can choose, for each $d \in \mathcal{D}$, an element $x_d \in X$ such that $q_d(x_d) = p_d(y)$. If $d \le e$, then we have

$$q_d(x_e) = q_{d,e}(q_e(x_e)) = q_{d,e}(p_e(y)) = p_d(y) = q_d(x_d)$$

which is possible only if $d(x_d, x_e) = 0$. Since this holds whenever $d \le e$, it follows that the $\mathcal{D}$-indexed net $\{x_d\}$ is strong Cauchy and hence converges to some $x \in X$. Since $p_d(x) = p_d(y)$ for all $d \in \mathcal{D}$ and the p_d are jointly monic, we conclude that $y = x \in X$. ■

An examination of the proof of the converse above shows that:

5.22. **Proposition.** *If X is Cooper complete and $\mathcal{D}$ a base of pseudometrics on X, then the map $X \longrightarrow \prod_{d \in \mathcal{D}} X_d$ is a closed uniform embedding.* ■

EXAMPLE. This is an example of a separated uniform space in which there is a strong Cauchy net that does not converge. It was suggested by James Cooper. Let $X = \Omega$ denote the set of all countable ordinals. We give it the uniformity that it inherits from $\Omega + 1$ which is the set of all ordinals that are less than or equal to Ω. Since $\Omega + 1$ is compact in its order topology it has a unique uniformity. Every uniform function from $X \longrightarrow [0, 1]$ can be extended to $\Omega + 1$ since $[0, 1]$ is complete. In particular, every pseudometric d on X is the restriction to X of a pseudometric we will also call d on $\Omega + 1$. It is shown in [9], 5.12 (c) that any continuous function $\Omega + 1 \longrightarrow [0, 1]$ is eventually constant on X, . The same is true for any bounded uniform function $X \longrightarrow \mathbf{R}$. If d is one of these pseudometrics, then $d(x, \Omega)$ is a continuous function of x. Thus, by the above, it is eventually constant, so there is an ordinal $\alpha < \Omega$ such that $\beta > \alpha$ implies that $d(\beta, \Omega)$ is constant and, since $d(\Omega, \Omega) = 0$, that constant must be 0. Thus if d is one of these pseudometrics, there is an ordinal $\alpha < \Omega$ such that $\beta > \alpha$ implies that $d(\beta, \Omega)$ is constant and, since $d(\Omega, \Omega) = 0$ and pseudometrics are continuous, that constant must be 0. Then if $\beta, \gamma > \alpha$, we have that $d(\beta, \gamma) \leq d(\beta, \Omega) + d(\gamma, \Omega) = 0$. The result is that the identity function of X, which is an Ω-indexed net on X, is a strong Cauchy net that cannot have a limit.

5.23. EMBEDDINGS OF UNIFORMLY TOTALLY DISCONNECTED SPACES. We will now suppose that X is uniformly totally disconnected and that $\mathcal{D}$ is a base of ultra-pseudometrics for the uniform structure. To simplify the presentation, *from now on we may assume, without loss of generality, that $\mathcal{D}$ is closed under positive scalar multiplication*. Then we may simplify the definition of uniform map: $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is uniform if and only if for all $e \in \mathcal{E}$, there is a $d \in \mathcal{D}$ such that $d(x, x') < 1$ implies $e(fx, fx') < 1$. If $x \in X$, $d \in \mathcal{D}$ let

$$N(x, d) = \{y \in X \mid d(x, y) < 1\}$$

5.24. **Proposition.** *If $d \in \mathcal{D}$, the family $\{N(x, d) \mid x \in X\}$ covers X by a family of disjoint clopen sets.*

PROOF. From the ultra-pseudometric property, if $d(x, y) < 1$ and $d(y, z) < 1$, then $d(x, z) < 1$. Therefore the relation of having $d(x, y) < 1$ is transitive and thus the $N(x, d)$ partition X. Since each one is open, the union of all but that one is open and so each one is closed. ∎

5.25. **Corollary.** *Let X/d be the set of equivalence classes in the partition above, with the metric in which the distance between distinct elements is 1.*

We will call this the **unit discrete metric**. *Then the map $X \longrightarrow X/d$ that takes x to the equivalence class $N(x,d)$ is uniform.* ∎

5.26. **Theorem.** *A separated uniformly totally disconnected space can be embedded in a product of discrete uniform spaces.*

PROOF. Suppose $X \longrightarrow X/d$ is as in the preceding corollary and we let $X \longrightarrow \prod_{d\in\mathcal{D}} X/d$ be the resulting map into the product. We claim this is an embedding. First we show it is injective. For $x \neq y$, choose a $d \in \mathcal{D}$ so that $d(x,y) \geq 1$. Then x and y go to distinct elements in X/d and thus $X \longrightarrow \prod X/d$ is injective. If we give X/d the unit discrete metric, it is clear that the map into the product is an embedding. ∎

5.27. **Proposition.** *Suppose X and $\mathcal{D}$ are as above and $C \subseteq X$ is closed. Then for all $d \in \mathcal{D}$, the set $N(C,d) = \{x \in X \mid d(C,x) < 1\}$ is clopen.*

PROOF. Suppose $y \in N(x,d) \cap N(C,d)$. Then $d(C,y) < 1$ so there must exist $c \in C$ such that $d(c,y) < 1$. From $d(x,y) < 1$, we infer that $d(x,c) < 1$ and thus $d(C,x) < 1$. If $z \in N(x,d)$, then we also see that $d(z,C) < 1$ and therefore $N(x,d) \subseteq N(C,d)$. Clearly $N(C,d)$ is open and so is the union of all the $N(x,d)$ that do not meet $N(C,d)$ and hence $N(C,d)$ is also closed. ∎

5.28. **Corollary.** *The set consisting of $N(C,d)$ together with all the $N(x,d)$ for $x \notin N(C,d)$ is a clopen partition of X.* ∎

5.29. **Corollary.** *Let $X_{C,d}$ denote the set of partitions as just described, with the unit discrete metric. Then the map $f_{C,d} : X \longrightarrow X_{C,d}$ that takes each element to its equivalence class in the partition is uniform.* ∎

5.30. **Proposition.** *Suppose X, $\mathcal{D}$, and $C \subseteq X$ are as above. Let $f_C = (f_{C,d}) : X \longrightarrow \prod_{d\in\mathcal{D}} X_{C,d}$ and $g_C : X \longrightarrow \prod_{d\in\mathcal{D}} X_{C,d}$ be the map whose dth coordinate is the constant map at the element $\{N(C,d)\}$ of $X_{C,d}$. Then C is the equalizer of f_C and g_C.*

PROOF. Trivial. ∎

We now have all the elements needed to show:

5.31. **Theorem.** *The limit closure and reflective hull of the discrete uniform spaces in the category of separated uniform spaces is the category of separated, Cooper complete, uniformly totally disconnected spaces.*

PROOF. If X is a limit of discrete uniform spaces, then it is a closed subspace of a product of such spaces and each factor can be assumed to have the unit discrete metric. The induced metrics on the product will all be ultra-pseudometrics and the same is true of any subspace of the product.

For the converse, suppose X is Cooper complete and uniformly totally disconnected. Then there is a base of ultra-pseudometrics on X for which the corresponding X_d are all discrete and, by Proposition 5.22, the canonical map $X \longrightarrow \prod X_d$ is a uniform closed embedding. The previous proposition guarantees that X is the equalizer of two maps to a uniformly totally disconnected space. The latter can, in turn, be embedded in a product of uniformly discrete spaces, whence X is a limit of uniformly discrete spaces since Corollary 5.13 applies. ∎

5.32. **Corollary.** *The limit closure and reflective hull of the finite discrete uniform spaces in the category of separated uniform spaces is the category of uniformly totally disconnected compact Hausdorff spaces.*

PROOF. If X is compact in the theorem above, then all the X/d are quotients of X, hence are compact. They are also discrete, hence finite. The converse is trivial. ∎

5.33. **Corollary.** *The limit closure and reflective hull of the 2 element discrete spaces in the category of separated uniform spaces is the category of uniformly totally disconnected compact Hausdorff spaces.*

PROOF. Every finite discrete space is a limit of 2 element discrete spaces and hence the limit closure of a 2 element space includes all finite discrete spaces and hence has the same limit closure. ∎

5.34. A UNIFORMLY TOTALLY DISCONNECTED METRIC SPACE IS ULTRAMETRIC. Although this claim is not relevant to the rest of this section, it does answer an obvious question. We list a series of steps that will verify it. Let X denote a uniformly totally disconnected metric space with metric d.

1. X is embedded in a product of discrete spaces with metric bounded by 1. This is clear from the development above.

2. The uniformity on X has a base of bounded ultra-pseudometrics.

3. There is a countable base of bounded ultra-pseudometrics. To see this, use the fact that the map from $X \longrightarrow (X, d)$ is uniform. Then for each positive integer n there is a bounded ultra-pseudometric d_n such that for all $x, x' \in X$, $d_n(x, x') < 1$ implies $d(x, x') < \frac{1}{n}$. Then the set of finite sups of the $\{d_n\}$ are a base for the uniformity.

4. Assume that each d_n is bounded by 1. We may also suppose that $d_1 \le d_2 \le \cdots$. Then $\overline{d} = \bigvee \frac{1}{n} d_n$ is in the saturation of the $\{d_n\}$. To see this, we show that $X \longrightarrow (X, \overline{d})$ is uniform. If $x, x' \in X$ are such that $d_i(x, x') < \frac{i}{n}$ for $i < n$, then obviously $\overline{d}(x, x') < \frac{1}{n}$. We leave to the reader the easy proof that d is an ultrametric.

5. $\overline{d}$ generates the uniformity. This follows since $\frac{1}{n} d_n \le \overline{d}$ for all n.

5.35. THE NON-SEPARATED CASE. Until now, we have supposed that all uniform spaces were separated, which is well known to imply that the uniform topology is Hausdorff. We haven't explored the non-separated case deeply, but there is one result that seems to be interesting.

5.36. **Proposition.** *Let X be a uniform space (not necessarily separated) and $C \subseteq X$ be a subset. Let X/C denote the quotient in which C is collapsed to a point and equipped with the trivial pseudometric (the distance between any pair of points is 0). Let $f : X \longrightarrow X/C$ assign to each element its class and $g : X \longrightarrow X/C$ be constant at $\{C\}$. Then f and g are uniform and $C \hookrightarrow X$ is their equalizer.*

PROOF. Trivial. ∎

5.37. **Theorem.** *The limit closure and reflective hull of pseudometric spaces in the category of all (not necessarily separated) uniform spaces is the entire category.*

PROOF. Let $(X, \mathcal{D})$ be a uniform space. For $d \in \mathcal{D}$, we let (X, d) denote the same point set X, but with the uniformity generated by the sole pseudometric d. Then the diagonal $X \longrightarrow \prod(X, d)$ embeds X uniformly into the diagonal. Clearly X has the induced uniformity from the product. The previous proposition shows that $X \subseteq \prod(X, d)$ is an equalizer of two maps to a pseudometric space. ■

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LIMITS AND COLIMITS OF QUANTALOID-ENRICHED CATEGORIES AND THEIR DISTRIBUTORS

by Lili SHEN and Walter THOLEN

Résumé. Pour un petit quantaloïde $\mathcal{Q}$ les auteurs démontrent que la catégorie des petites $\mathcal{Q}$-catégories et leurs $\mathcal{Q}$-foncteurs est totale et cototale, et que la catégorie des $\mathcal{Q}$-distributeurs et leurs $\mathcal{Q}$-Chu-transformations est aussi totale et cototale.

Abstract. It is shown that, for a small quantaloid $\mathcal{Q}$, the category of small $\mathcal{Q}$-categories and $\mathcal{Q}$-functors is total and cototal, and so is the category of $\mathcal{Q}$-distributors and $\mathcal{Q}$-Chu transforms.

Keywords. Quantaloid, $\mathcal{Q}$-category, $\mathcal{Q}$-distributor, $\mathcal{Q}$-Chu transform, total category.

Mathematics Subject Classification (2010). 18D20, 18A30, 18A35.

1. Introduction

The importance of (small) categories enriched in a (unital) quantale rather than in an arbitrary monoidal category was discovered by Lawvere [18] who enabled us to look at individual mathematical objects, such as metric spaces, as small categories. Through the study of lax algebras [15], quantale-enriched categories have become the backbone of a larger array of objects that may be viewed as individual generalized categories. Prior to this development, Walters [31] had extended Lawvere's viewpoint in a different manner, replacing the quantale at work by a *quantaloid* (a term proposed later by Rosenthal [22]), thus by a bicategory with the particular property that its hom-objects are given by complete lattices such that composition from either side preserves suprema; quantales are thus simply one-object quantaloids.

Based on the theory of quantaloid-enriched categories developed by Stubbe [25, 26], recent works [16, 21, 27, 28] have considered in particular the case when the quantaloid in question arises from a given quantaloid by a "diagonal construction" whose roots go far beyond its use in this paper; see

[12, 13]. Specifically for the one-object quantaloids (i.e., quantales) whose enriched categories give (pre)ordered sets and (generalized) metric spaces, the corresponding small quantaloids of diagonals lead to truly partial structures, in the sense that the full structure is available only on a subset of the ambient underlying set of objects.

In the first instance then, this paper aims at exploring the categorical properties of the category $\mathcal{Q}$-$\mathbf{Cat}$ of small $\mathcal{Q}$-enriched categories and their $\mathcal{Q}$-functors for a small quantaloid $\mathcal{Q}$. By showing that $\mathcal{Q}$-$\mathbf{Cat}$ is topological [2] over the comma category $\mathbf{Set}/\operatorname{ob}\mathcal{Q}$ (Proposition 2.2) one easily describes small limits and colimits in this category, and beyond. In fact, one concludes that categories of this type are total [24] and cototal, hence possess even those limits and colimits of large diagrams whose existence is not made impossible by the size of the small hom-sets of $\mathcal{Q}$-$\mathbf{Cat}$ (see [9]).

Our greater interest, however, is in the category $\mathcal{Q}$-$\mathbf{Chu}$ whose objects are often called $\mathcal{Q}$-Chu spaces, the prototypes of which go back to [3, 20] and many others (see [4, 10]). Its objects are *$\mathcal{Q}$-distributors* of $\mathcal{Q}$-categories (also called $\mathcal{Q}$-(bi)modules or $\mathcal{Q}$-profunctors), hence they are compatible $\mathcal{Q}$-relations (or $\mathcal{Q}$-matrices) that have been investigated intensively ever since Bénabou [5] introduced them (see [6, 7]). While when taken as the morphisms of the category whose objects are $\mathcal{Q}$-categories, they make for a in many ways poorly performing category (as already the case $\mathcal{Q} = \mathbf{2}$ shows), when taken as objects of $\mathcal{Q}$-$\mathbf{Chu}$ with morphisms given by so-called *$\mathcal{Q}$-Chu transforms*, i.e., by pairs of $\mathcal{Q}$-functors that behave like adjoint operators, we obtain a category that in terms of the existence of limits and colimits behaves as strongly as $\mathcal{Q}$-$\mathbf{Cat}$ itself. In analogy to the property shown in [11] in a different categorical context, we first prove that the domain functor $\mathcal{Q}$-$\mathbf{Chu} \longrightarrow \mathcal{Q}$-$\mathbf{Cat}$ allows for initial liftings [2] of structured cones over small diagrams (Theorem 3.4), which then allows for an explicit description of all limits and colimits in $\mathcal{Q}$-$\mathbf{Chu}$ over small diagrams. But although the domain functor fails to be topological, just as for $\mathcal{Q}$-$\mathbf{Cat}$ we are able to show totality (and, consequently, cototality) of $\mathcal{Q}$-$\mathbf{Chu}$. A key ingredient for this result is the existence proof of a generating set in $\mathcal{Q}$-$\mathbf{Chu}$, and therefore also of a cogenerating set (Theorem 3.8).

2. Limits and colimits in $\mathcal{Q}$-Cat

Throughout, let $\mathcal{Q}$ be a small *quantaloid*, i.e., a small category enriched in the category $\mathbf{Sup}$ of complete lattices and sup-preserving maps. A small $\mathcal{Q}$-*category* is given by a set X (its set of objects) and a lax functor

$$a : X \longrightarrow \mathcal{Q},$$

where the set X is regarded as a quantaloid carrying the chaotic structure, so that for all $x, y \in X$ there is precisely one arrow $x \longrightarrow y$, called (x, y). Explicitly then, the $\mathcal{Q}$-category structure on X is given by

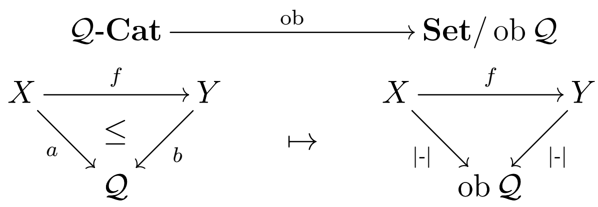
- a family of objects $|x|_X := ax$ in $\mathcal{Q}$ $(x \in X)$,
- a family of morphisms $a(x, y) : |x| \longrightarrow |y|$ in $\mathcal{Q}$ $(x, y \in X)$, subject to

$$1_{|x|} \leq a(x, x) \quad \text{and} \quad a(y, z) \circ a(x, y) \leq a(x, z)$$

$(x, y, z \in X)$. When one calls $|x| = |x|_X$ the *extent* (or *type*) of $x \in X$, a $\mathcal{Q}$-*functor* $f : (X, a) \longrightarrow (Y, b)$ of $\mathcal{Q}$-categories (X, a), (Y, b) is an extent-preserving map $f : X \longrightarrow Y$ such that there is a lax natural transformation $a \longrightarrow bf$ given by identity morphisms in $\mathcal{Q}$; explicitly,

$$|x|_X = |f(x)|_Y \quad \text{and} \quad a(x, y) \leq b(f(x), f(y))$$

for all $x, y \in X$. Denoting the resulting ordinary category by $\mathcal{Q}$-$\mathbf{Cat}$, we have a forgetful functor



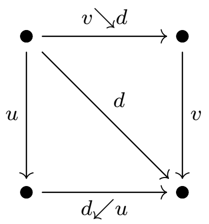
Example 2.1. (1) If $\mathcal{Q}$ is a *quantale*, i.e., a one-object quantaloid, then the extent functions of $\mathcal{Q}$-categories are trivial and $\mathcal{Q}$-$\mathbf{Cat}$ assumes its classical meaning (as in [17], where $\mathcal{Q}$ is considered as a monoidal (closed) category). Prominent examples are $\mathcal{Q} = \mathbf{2} = \{0 < 1\}$ and

$\mathcal{Q} = ([0,\infty], \geq, +)$, where then $\mathcal{Q}$-$\mathbf{Cat}$ is the category $\mathbf{Ord}$ (sets carrying a reflexive and transitive relation, called *order* here but commonly known as preorder, with monotone maps) and, respectively, the category $\mathbf{Met}$ (sets X carrying a distance function $a : X \times X \longrightarrow [0,\infty]$ required to satisfy only $a(x,x) = 0$ and $a(x,z) \leq a(x,y) + a(y,z)$ but, in accordance with the terminology introduced by Lawvere [18] and used in [15], nevertheless called *metric* here, with non-expanding maps $f : X \longrightarrow Y$, so that $b(f(x), f(y)) \leq a(x,y)$ for all $x, y \in X$).

(2) (Stubbe [27]) Every quantaloid $\mathcal{Q}$ gives rise to a new quantaloid $\mathsf{D}\mathcal{Q}$ whose objects are the morphisms of $\mathcal{Q}$, and for morphisms u, v in $\mathcal{Q}$, a morphism $(u, d, v) : u \rightsquigarrow v$ in $\mathsf{D}\mathcal{Q}$, normally written just as d, is a $\mathcal{Q}$-morphism $d : \operatorname{dom} u \longrightarrow \operatorname{cod} v$ satisfying

$$(d \swarrow u) \circ u = d = v \circ (v \searrow d),$$

also called a *diagonal* from u to v:



(Here $d \swarrow u$, $v \searrow d$ denote the *internal homs* of $\mathcal{Q}$, determined by

$$z \leq d \swarrow u \iff z \circ u \leq d, \quad t \leq v \searrow d \iff v \circ t \leq d$$

for all $z : \operatorname{cod} u \longrightarrow \operatorname{cod} d$, $t : \operatorname{dom} d \longrightarrow \operatorname{dom} v$.) With the composition of $d : u \rightsquigarrow v$ with $e : v \rightsquigarrow w$ in $\mathsf{D}\mathcal{Q}$ defined by

$$e \diamond d = (e \swarrow v) \circ d = e \circ (v \searrow d),$$

and with identity morphisms $u : u \rightsquigarrow u$, $\mathsf{D}\mathcal{Q}$ becomes a quantaloid whose local order is inherited from $\mathcal{Q}$. In fact, there is a full embedding

$$\mathcal{Q} \longrightarrow \mathsf{D}\mathcal{Q}, \quad (u : t \longrightarrow s) \mapsto (u : 1_t \rightsquigarrow 1_s)$$

of quantaloids.

We remark that the construction of D works for ordinary categories; indeed it is part of the proper factorization monad on $\mathbf{CAT}$ [13].

(2a) For $\mathcal{Q} = \mathbf{2}$, the quantaloid $\mathsf{D}\mathcal{Q}$ has object set $\{0, 1\}$. There are exactly two $\mathsf{D}\mathcal{Q}$-arrows $1 \rightsquigarrow 1$, given by $0, 1$, and 0 is the only arrow in every other hom-set of $\mathsf{D}\mathcal{Q}$; composition is given by infimum. A $\mathsf{D}\mathcal{Q}$-category is given by a set X, a distinguished subset $A \subseteq X$ (those elements of X with extent 1) and a (pre)order on A. Hence, a $\mathsf{D}\mathcal{Q}$-category structure on X is a (truly!) *partial order* on X. With morphisms $f : (X, A) \longrightarrow (Y, B)$ given by maps $f : X \longrightarrow Y$ monotone on $A = f^{-1}B$ we obtain the category

$$\mathbf{ParOrd} = \mathsf{D}\mathcal{Q}\text{-}\mathbf{Cat},$$

which contains $\mathbf{Ord}$ as a full coreflective subcategory.

(2b) For $\mathcal{Q} = ([0, \infty], \geq, +)$, the hom-sets of $\mathsf{D}\mathcal{Q}$ are easily described by

$$\mathsf{D}\mathcal{Q}(u, v) = \{s \in [0, \infty] \mid u, v \leq s\},$$

with composition given by $t \diamond s = s - v + t$ (for $t : v \rightsquigarrow w$). A $\mathsf{D}\mathcal{Q}$-category structure on a set X consists of functions $|\text{-}| : X \longrightarrow [0, \infty]$, $a : X \times X \longrightarrow [0, \infty]$ satisfying

$$|x|, |y| \leq a(x, y), \quad a(x, x) \leq |x|, \quad a(x, z) \leq a(x, y) - |y| + a(y, z)$$

$(x, y, z \in X)$. Obviously, since necessarily $|x| = a(x, x)$, these conditions simplify to

$$a(x, x) \leq a(x, y), \quad a(x, z) \leq a(x, y) - a(y, y) + a(y, z)$$

$(x, y, z \in X)$, describing a as a *partial metric* on X (see [16, 19, 21][1]). With non-expanding maps $f : (X, a) \longrightarrow (Y, b)$ satisfying $a(x, x) = b(f(x), f(x))$ for all $x \in X$ one obtains the category

$$\mathbf{ParMet} = \mathsf{D}\mathcal{Q}\text{-}\mathbf{Cat},$$

which contains $\mathbf{Met}$ as a full coreflective subcategory: the coreflector restricts the partial metric a on X to those elements $x \in X$ with $a(x, x) = 0$.

[1]Here our terminology naturally extends Lawvere's notion of metric and is synonymous with "generalized partial metric" as used by Pu-Zhang [21] who dropped finiteness ($a(x, y) < \infty$), symmetry ($a(x, y) = a(y, x)$) and separation ($a(x, x) = a(x, y) = a(y, y) \iff x = y$) from the requirement for the notion of "partial metric" as originally introduced by Matthews [19].

To see how limits and colimits in the (ordinary) category $\mathcal{Q}$-$\mathbf{Cat}$ are to be formed, it is best to first prove its topologicity over $\mathbf{Set}/\operatorname{ob}\mathcal{Q}$. Recall that, for any functor $U : \mathcal{A} \longrightarrow \mathcal{X}$, a *$U$-structured cone* over a diagram $D : \mathcal{J} \longrightarrow \mathcal{A}$ is given by an object $X \in \mathcal{X}$ and a natural transformation $\xi : \Delta X \longrightarrow UD$. A *lifting* of (X, ξ) is given by an object A in $\mathcal{A}$ and a cone $\alpha : \Delta A \longrightarrow D$ over D with $UA = X$, $U\alpha = \xi$. Such lifting (A, α) is *U-initial* if, for all cones $\beta : \Delta B \longrightarrow D$ over D and morphisms $t : UB \longrightarrow UA$ in $\mathcal{X}$, there is exactly one morphism $h : B \longrightarrow A$ in $\mathcal{A}$ with $Uh = t$ and $\alpha \cdot \Delta h = \beta$. We call U *small-topological* [11] if all U-structured cones over small diagrams admit U-initial liftings, and U is *topological* when this condition holds without the size restriction on diagrams. Recall also the following well-known facts:

- Topological functors are necessarily faithful [8], and for faithful functors it suffices to consider discrete cones to guarantee topologicity.
- $U : \mathcal{A} \longrightarrow \mathcal{X}$ is topological if, and only if, $U^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{X}^{\mathrm{op}}$ is topological.
- The two properties above generally fail to hold for small-topological functors. However, for any functor U, a U-initial lifting of a U-structured cone that is a limit cone in $\mathcal{X}$ gives also a limit cone in $\mathcal{A}$.
- Every small-topological functor is a fibration (consider singleton diagrams) and has a fully faithful right adjoint (consider the empty diagram).

Proposition 2.2. *For every (small) quantaloid $\mathcal{Q}$, the "object functor"*

$$\mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Set}/\operatorname{ob}\mathcal{Q}$$

is topological.

Proof. Given a (possibly large) family $f_i : (X, |\text{-}|) \longrightarrow (Y_i, |\text{-}|_i)$ $(i \in I)$ of maps over $\operatorname{ob}\mathcal{Q}$, where every Y_i carries a $\mathcal{Q}$-category structure b_i with extent function $|\text{-}|_i$, we must find a $\mathcal{Q}$-category structure a on X with extent function $|\text{-}|$ such that (1) every $f_i : (X, a) \longrightarrow (Y, b)$ is a $\mathcal{Q}$-functor, and (2) for every $\mathcal{Q}$-category (Z, c), any extent preserving map $g : Z \longrightarrow X$

becomes a $\mathcal{Q}$-functor $(Z,c)\longrightarrow(X,a)$ whenever all maps $f_i g$ are $\mathcal{Q}$-functors $(Z,c)\longrightarrow(Y_i,b_i)$ $(i\in I)$. But this is easy: simply define

$$a(x,y) := \bigwedge_{i\in I} b_i(f_i(x),f_i(y))$$

for all $x,y\in X$. Hence, a is the ob-initial structure on X with respect to the structured source $(f_i : X\longrightarrow(Y_i,b_i))_{i\in I}$. □

Corollary 2.3. $\mathcal{Q}$-$\mathbf{Cat}$ *is complete and cocomplete, and the object functor has both a fully faithful left adjoint and a fully faithful right adjoint.*

Remark 2.4. (1) The set of objects of the product (X,a) of a small family of $\mathcal{Q}$-categories (X_i,a_i) $(i\in I)$ is given by the fibred product of $(X_i,|\text{-}|_i)$ $(i\in I)$, i.e.,

$$X=\{((x_i)_{i\in I},q)\mid q\in\mathrm{ob}\,\mathcal{Q},\ \forall i\in I(x_i\in X_i,|x_i|=q)\},$$

and (when writing $(x_i)_{i\in I}$ instead of $((x_i)_{i\in I},q)$ and putting $|(x_i)_{i\in I}|=q$) we have

$$a((x_i)_{i\in I},(y_i)_{i\in I})=\bigwedge_{i\in I} a_i(x_i,y_i) : |(x_i)_{i\in I}|\longrightarrow|(y_i)_{i\in I}|$$

for its hom-arrows. In particular, $(\mathrm{ob}\,\mathcal{Q},\top)$ with

$$\top(q,r)=\top : q\longrightarrow r$$

the top element in $\mathcal{Q}(q,r)$ (for all $q,r\in\mathrm{ob}\,\mathcal{Q}$), is the terminal object in $\mathcal{Q}$-$\mathbf{Cat}$.

(2) The coproduct (X,a) of $\mathcal{Q}$-categories (X_i,a_i) $(i\in I)$ is simply formed by the coproduct in $\mathbf{Set}$, with all structure to be obtained by restriction:

$$X=\coprod_{i\in I} X_i,\quad |x|_X=|x|_{X_i} \text{ if } x\in X_i,$$

$$a(x,y)=\begin{cases} a_i(x,y) & \text{if } x,y\in X_i,\\ \bot : |x|\longrightarrow|y| & \text{else.}\end{cases}$$

In particular, $\varnothing$ with its unique $\mathcal{Q}$-category structure is an initial object in $\mathcal{Q}$-$\mathbf{Cat}$.

(3) The equalizer of $\mathcal{Q}$-functors $f, g : (X, a) \longrightarrow (Y, b)$ is formed as in $\mathbf{Set}$, by restriction of the structure of (X, a). The object set of their coequalizer (Z, c) in $\mathcal{Q}$-$\mathbf{Cat}$ is also formed as in $\mathbf{Set}$, so that $Z = Y/\sim$, with $\sim$ the least equivalence relation on Y with $f(x) \sim g(x)$, $x \in X$. With $\pi : Y \longrightarrow Z$ the projection, necessarily $|\pi(y)|_Z = |y|_Y$, and $c(\pi(y), \pi(y'))$ is the join of all

$$b(y_n, y'_n) \circ b(y_{n-1}, y'_{n-1}) \circ \cdots \circ b(y_2, y'_2) \circ b(y_1, y'_1),$$

where $|y| = |y_1|, |y'_1| = |y_2|, \ldots, |y'_{n-1}| = |y_n|, |y'_n| = |y'|$ $(y_i, y'_i \in Y, n \geq 1)$.

(4) The fully faithful left adjoint of $\mathcal{Q}$-$\mathbf{Cat} \longrightarrow \mathbf{Set}/\operatorname{ob}\mathcal{Q}$ provides a set $(X, |\text{-}|)$ over $\operatorname{ob}\mathcal{Q}$ with the discrete $\mathcal{Q}$-structure, given by

$$a(x, y) = \begin{cases} 1_{|x|} & \text{if } x = y, \\ \bot : |x| \longrightarrow |y| & \text{else;} \end{cases}$$

while the fully faithful right adjoint always takes $\top : |x| \longrightarrow |y|$ as the hom-arrow, i.e., it chooses the indiscrete $\mathcal{Q}$-structure.

Example 2.5. The product of partial metric spaces (X_i, a_i) $(i \in I)$ provides its carrier set

$$X = \{((x_i)_{i \in I}, s) \mid s \in [0, \infty],\ \forall i \in I(x_i \in X_i, |x_i| = s)\}$$

with the "sup metric":

$$a((x_i)_{i \in I}, (y_i)_{i \in I}) = \sup_{i \in I} a_i(x_i, y_i).$$

$[0, \infty]$ is terminal in $\mathbf{ParMet}$ when provided with the chaotic metric that makes all distances 0, and it is a generator when provided with the discrete metric d:

$$d(s, t) = \begin{cases} 0 & \text{if } s = t, \\ \infty & \text{else.} \end{cases}$$

Beyond small limits and colimits, $\mathcal{Q}$-$\mathbf{Cat}$ actually has all large-indexed limits and colimits that one can reasonably expect to exist. More precisely, recall that an ordinary category $\mathcal{C}$ with small hom-sets is (see [9])

- *hypercomplete* if a diagram $D : \mathcal{J} \longrightarrow \mathcal{C}$ has a limit in $\mathcal{C}$ whenever the limit of $\mathcal{C}(A, D-)$ exists in $\mathbf{Set}$ for all $A \in \mathrm{ob}\,\mathcal{C}$; equivalently: whenever, for every $A \in \mathrm{ob}\,\mathcal{C}$, the cones $\Delta A \longrightarrow D$ in $\mathcal{C}$ may be labeled by a set;
- *totally cocomplete* if a diagram $D : \mathcal{J} \longrightarrow \mathcal{C}$ has a colimit in $\mathcal{C}$ whenever the colimit of $\mathcal{C}(A, D-)$ exists in $\mathbf{Set}$ for all $A \in \mathrm{ob}\,\mathcal{C}$; equivalently: whenever, for every $A \in \mathrm{ob}\,\mathcal{C}$, the connected components of $(A \downarrow D)$ may be labelled by a set.

The dual notions are *hypercocomplete* and *totally complete*. It is well known (see [9]) that

- $\mathcal{C}$ is totally cocomplete if, and only if, $\mathcal{C}$ is *total*, i.e., if the Yoneda embedding $\mathcal{C} \longrightarrow \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ has a left adjoint;
- total cocompleteness implies hypercompleteness but not vice versa (with Adámek's monadic category over graphs [1] providing a counterexample);
- for a *solid* (=*semi-topological* [29]) functor $\mathcal{A} \longrightarrow \mathcal{X}$, if $\mathcal{X}$ is hypercomplete or totally cocomplete, $\mathcal{A}$ has the corresponding property [30];
- in particular, every topological functor, every monadic functor over $\mathbf{Set}$, and every full reflective embedding is solid.

It is also useful for us to recall [9, Corollary 3.5]:

Proposition 2.6. *A cocomplete and cowellpowered category with small hom-sets and a generating set of objects is total.*

Since $\mathcal{Q}$-$\mathbf{Cat}$ is topological over $\mathbf{Set}/\,\mathrm{ob}\,\mathcal{Q}$ which, as a complete, cocomplete, wellpowered and cowellpowered category with a generating and a cogenerating set, is totally complete and totally cocomplete, we conclude:

Theorem 2.7. $\mathcal{Q}$-$\mathbf{Cat}$ *is totally complete and totally cocomplete and, in particular, hypercocomplete and hypercomplete.*

Remark 2.8. Of course, we may also apply Proposition 2.6 directly to obtain Theorem 2.7 since the left adjoint of $\mathcal{Q}$-$\mathbf{Cat} \longrightarrow \mathbf{Set}/\operatorname{ob}\mathcal{Q}$ sends a generating set of $\mathbf{Set}/\operatorname{ob}\mathcal{Q}$ to a generating set of $\mathcal{Q}$-$\mathbf{Cat}$, and the right adjoint has the dual property. Explicitly then, denoting for every $s \in \operatorname{ob}\mathcal{Q}$ by $\{s\}$ the discrete $\mathcal{Q}$-category whose only object has extent s, we obtain the generating set $\{\{s\} \mid s \in \operatorname{ob}\mathcal{Q}\}$ for $\mathcal{Q}$-$\mathbf{Cat}$. Similarly, providing the disjoint unions $D_s = \{s\} + \operatorname{ob}\mathcal{Q}$ $(s \in \operatorname{ob}\mathcal{Q})$ with the identical extent functions and the indiscrete $\mathcal{Q}$-category structures, one obtains a cogenerating set in $\mathcal{Q}$-$\mathbf{Cat}$.

3. Limits and colimits in $\mathcal{Q}$-$\mathbf{Chu}$

For $\mathcal{Q}$-categories $X = (X, a)$, $Y = (Y, b)$, a $\mathcal{Q}$-*distributor* [5] $\varphi : X \mathrel{\circ\!\!\!\longrightarrow} Y$ (also called $\mathcal{Q}$-*(bi)module* [18], $\mathcal{Q}$-*profunctor*) is a family of arrows $\varphi(x, y) : |x| \longrightarrow |y|$ $(x \in X, y \in Y)$ in $\mathcal{Q}$ such that

$$b(y, y') \circ \varphi(x, y) \circ a(x', x) \leq \varphi(x', y')$$

for all $x, x' \in X$, $y, y' \in Y$. Its composite with $\psi : Y \mathrel{\circ\!\!\!\longrightarrow} Z$ is given by

$$(\psi \circ \varphi)(x, z) = \bigvee_{y \in Y} \psi(y, z) \circ \varphi(x, y).$$

Since the structure a of a $\mathcal{Q}$-category (X, a) is neutral with respect to this composition, we obtain the category

$$\mathcal{Q}\text{-}\mathbf{Dis}$$

of $\mathcal{Q}$-categories and their $\mathcal{Q}$-distributors which, with the local pointwise order

$$\varphi \leq \varphi' \iff \forall x, y : \ \varphi(x, y) \leq \varphi'(x, y),$$

is actually a quantaloid. Every $\mathcal{Q}$-functor $f : X \longrightarrow Y$ gives rise to the $\mathcal{Q}$-distributors $f_\natural : X \mathrel{\circ\!\!\!\longrightarrow} Y$ and $f^\natural : Y \mathrel{\circ\!\!\!\longrightarrow} X$ with

$$f_\natural(x, y) = b(f(x), y) \quad \text{and} \quad f^\natural(y, x) = b(y, f(x))$$

$(x \in X, y \in Y)$. One has $f_\natural \dashv f^\natural$ in the 2-category $\mathcal{Q}$-$\mathbf{Dis}$, and if one lets $\mathcal{Q}$-$\mathbf{Cat}$ inherit the order of $\mathcal{Q}$-$\mathbf{Dis}$ via

$$f \leq g \iff f^\natural \leq g^\natural \iff g_\natural \leq f_\natural \iff 1_{|x|} \leq b(f(x), g(x)) \quad (x \in X),$$

then one obtains 2-functors

$$(-)_\natural : (\mathcal{Q}\text{-}\mathbf{Cat})^{\mathrm{co}} \longrightarrow \mathcal{Q}\text{-}\mathbf{Dis}, \quad (-)^\natural : (\mathcal{Q}\text{-}\mathbf{Cat})^{\mathrm{op}} \longrightarrow \mathcal{Q}\text{-}\mathbf{Dis}$$

which map objects identically; here "op" refers to the dualization of 1-cells and "co" to the dualization of 2-cells.

Example 3.1 (See Example 2.1)**.** (1) A $\mathbf{2}$-distributor is an *order ideal relation*; that is, a relation $\varphi : X \mathrel{-\!\!\circ\!\!\!\rightarrow} Y$ of ordered sets that behaves like a two-sided ideal w.r.t. the order:

$$x' \le x \;\&\; x\varphi y \;\&\; y \le y' \Longrightarrow x'\varphi y'.$$

A $[0,\infty]$-distributor $\varphi : X \mathrel{-\!\!\circ\!\!\!\rightarrow} Y$ introduces a distance function between metric spaces (X,a), (Y,b) that must satisfy

$$\varphi(x',y') \le a(x',x) + \varphi(x,y) + a(y,y')$$

for all $x,x' \in X$, $y,y' \in Y$.

(2) A $\mathsf{D}\mathbf{2}$-distributor $\varphi : X \mathrel{-\!\!\circ\!\!\!\rightarrow} Y$ is given by a $\mathbf{2}$-distributor $A \mathrel{-\!\!\circ\!\!\!\rightarrow} B$ where $A = \{x \in X \mid x \le x\}$, $B = \{y \in Y \mid y \le y\}$ are the coreflections of X, Y, respectively. Likewise, a $\mathsf{D}[0,\infty]$-distributor $\varphi : X \mathrel{-\!\!\circ\!\!\!\rightarrow} Y$ is given by a distributor of the metric coreflections of the partial metric spaces X and Y.

In our context $\mathcal{Q}\text{-}\mathbf{Dis}$ plays only an auxiliary role for us in setting up the category

$$\mathcal{Q}\text{-}\mathbf{Chu}$$

whose objects are $\mathcal{Q}$-distributors and whose morphisms $(f,g) : \varphi \longrightarrow \psi$ are given by $\mathcal{Q}$-functors $f : (X,a) \longrightarrow (Y,b)$, $g : (Z,c) \longrightarrow (W,d)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;f_\natural\;} & Y \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ W & \xrightarrow[\;g^\natural\;]{} & Z \end{array} \tag{3.i}$$

commutes in $\mathcal{Q}$-$\mathbf{Dis}$:

$$\psi(f(x), z) = \varphi(x, g(z)) \tag{3.ii}$$

for all $x \in X$, $z \in Z$. In particular, with $\varphi = a$, $\psi = b$ one obtains that the morphisms $(f, g) : 1_{(X,a)} \longrightarrow 1_{(Y,b)}$ in $\mathcal{Q}$-$\mathbf{Chu}$ are precisely the adjunctions $f \dashv g : (Y, b) \longrightarrow (X, a)$ in the 2-category $\mathcal{Q}$-$\mathbf{Cat}$. With the order inherited from $\mathcal{Q}$-$\mathbf{Cat}$, $\mathcal{Q}$-$\mathbf{Chu}$ is in fact a 2-category, and one has 2-functors

$$\begin{aligned} \mathrm{dom}: &\quad \mathcal{Q}\text{-}\mathbf{Chu} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}, &\quad (f, g) \mapsto f, \\ \mathrm{cod}: &\quad \mathcal{Q}\text{-}\mathbf{Chu} \longrightarrow (\mathcal{Q}\text{-}\mathbf{Cat})^{\mathrm{op}}, &\quad (f, g) \mapsto g. \end{aligned}$$

In order for us to exhibit properties of $\mathcal{Q}$-$\mathbf{Chu}$, it is convenient to describe *$\mathcal{Q}$-Chu transforms*, i.e., morphisms in $\mathcal{Q}$-$\mathbf{Chu}$, alternatively, with the help of *presheaves*, as follows. For every $s \in \mathrm{ob}\,\mathcal{Q}$, let $\{s\}$ denote the discrete $\mathcal{Q}$-category whose only object has extent s. For a $\mathcal{Q}$-category $X = (X, a)$, a *$\mathcal{Q}$-presheaf* φ *on* X of extent $|\varphi| = s$ is a $\mathcal{Q}$-distributor $\varphi : X \,{-\!\!\circ\!\!\rightarrow}\, \{s\}$. Hence, φ is given by a family of $\mathcal{Q}$-morphisms $\varphi_x : |x| \longrightarrow |\varphi|$ $(x \in X)$ with $\varphi_y \circ a(x, y) \leq \varphi_x$ $(x, y \in X)$. With

$$[\varphi, \psi] = \bigwedge_{x \in X} \psi_x \swarrow \varphi_x,$$

$\mathsf{P}X$ becomes a $\mathcal{Q}$-category, and one has the *Yoneda $\mathcal{Q}$-functor*

$$\mathsf{y}_X = \mathsf{y} : X \longrightarrow \mathsf{P}X, \quad x \mapsto (a(-, x) : X \,{-\!\!\circ\!\!\rightarrow}\, \{|x|\}).$$

y is fully faithful, i.e., $[\mathsf{y}(x), \mathsf{y}(y)] = a(x, y)$ $(x, y \in X)$. The point of the formation of $\mathsf{P}X$ for us is as follows (see [14, 23]):

Proposition 3.2. *The 2-functor* $(-)^{\natural} : (\mathcal{Q}\text{-}\mathbf{Cat})^{\mathrm{op}} \longrightarrow \mathcal{Q}\text{-}\mathbf{Dis}$ *has a left adjoint* P *which maps a $\mathcal{Q}$-distributor* $\varphi : X \,{-\!\!\circ\!\!\rightarrow}\, Y$ *to the $\mathcal{Q}$-functor*

$$\varphi^* : \mathsf{P}Y \longrightarrow \mathsf{P}X, \quad \psi \mapsto \psi \circ \varphi;$$

hence,

$$(\varphi^*(\psi))_x = \bigvee_{y \in Y} \psi_y \circ \varphi(x, y)$$

for all $\psi \in \mathsf{P}Y$, $x \in X$. *In particular, for a $\mathcal{Q}$-functor* $f : X \longrightarrow Y$ *one has*

$$f^* := (f_{\natural})^* : \mathsf{P}Y \longrightarrow \mathsf{P}X, \quad (f^*(\psi))_x = \psi_{f(x)}.$$

Denoting by $\widetilde{\varphi} : Y \longrightarrow \mathsf{P}X$ the transpose of $\varphi : X \looparrowright Y$ under the adjunction, determined by $\widetilde{\varphi}^\natural \circ (\mathsf{y}_X)_\natural = \varphi$, so that $(\widetilde{\varphi}(y))_x = \varphi(x,y)$ for all $x \in X$, $y \in Y$, we can now present $\mathcal{Q}$-Chu transforms, as follows:

Corollary 3.3. *A morphism* $(f,g) : \varphi \longrightarrow \psi$ *in* $\mathcal{Q}$-$\mathbf{Chu}$ *(as in* (3.i)*) may be equivalently presented as a commutative diagram*

$$\begin{array}{ccc} \mathsf{P}X & \xleftarrow{\ f^*\ } & \mathsf{P}Y \\ {\scriptstyle\widetilde{\varphi}}\uparrow & & \uparrow{\scriptstyle\widetilde{\psi}} \\ W & \xleftarrow[\ g\ ]{} & Z \end{array} \tag{3.iii}$$

in $\mathcal{Q}$-$\mathbf{Cat}$*. Condition* (3.ii) *then reads as*

$$(\widetilde{\varphi}(g(z)))_x = (\widetilde{\psi}(z))_{f(x)} \tag{3.iv}$$

for all $x \in X$*,* $z \in Z$*.*

Proof. For all $z \in Z$,

$$\begin{aligned} f^*(\widetilde{\psi}(z)) &= \widetilde{\psi}(z) \circ f_\natural \\ &= \mathsf{y}_Z(z) \circ \psi \circ f_\natural \\ &= \mathsf{y}_Z(z) \circ g^\natural \circ \varphi \\ &= \mathsf{y}_Y(g(z)) \circ \varphi \\ &= \widetilde{\varphi}(g(z)). \end{aligned}$$

□

Theorem 3.4. *Let* $D : \mathcal{J} \longrightarrow \mathcal{Q}$-$\mathbf{Chu}$ *be a diagram such that the colimit* $W = \operatorname{colim} \operatorname{cod} D$ *exists in* $\mathcal{Q}$-$\mathbf{Cat}$*. Then any cone* $\gamma : \Delta X \longrightarrow \operatorname{dom} D$ *in* $\mathcal{Q}$-$\mathbf{Cat}$ *has a* dom*-initial lifting* $\Gamma : \Delta\varphi \longrightarrow D$ *in* $\mathcal{Q}$-$\mathbf{Chu}$ *with* $\varphi : X \looparrowright W$*,* $\operatorname{dom}\Gamma = \gamma$*. In particular, if* γ *is a limit cone in* $\mathcal{Q}$-$\mathbf{Cat}$*,* Γ *is a limit cone in* $\mathcal{Q}$-$\mathbf{Chu}$*.*

Proof. Considering the functors

$$\mathcal{Q}\text{-}\mathbf{Chu} \xrightarrow{\ \operatorname{dom}\ } \mathcal{Q}\text{-}\mathbf{Cat} \xrightarrow{\ (-)_\natural\ } \mathcal{Q}\text{-}\mathbf{Dis},$$

$$\mathcal{Q}\text{-}\mathbf{Chu} \xrightarrow{\ \operatorname{cod}\ } (\mathcal{Q}\text{-}\mathbf{Cat})^{\mathrm{op}} \xrightarrow{\ (-)^\natural\ } \mathcal{Q}\text{-}\mathbf{Dis},$$

one has the natural transformation

$$\kappa : (\mathrm{dom}(-))_\natural \mathrel{-\!\!\circ\!\!\rightarrow} (\mathrm{cod}(-))^\natural, \quad \kappa_\varphi := \varphi \ (\varphi \in \mathrm{ob}\, \mathcal{Q}\text{-}\mathbf{Chu}).$$

By the adjunction of Proposition 3.2, $\kappa D : (\mathrm{dom}\, D)_\natural \mathrel{-\!\!\circ\!\!\rightarrow} (\mathrm{cod}\, D)^\natural$ corresponds to a natural transformation $\widetilde{\kappa D} : \mathrm{cod}\, D \longrightarrow \mathsf{P}(\mathrm{dom}\, D)_\natural$, and the given cone γ gives a cocone $\gamma^* : \mathsf{P}(\mathrm{dom}\, D)_\natural \longrightarrow \Delta \mathsf{P} X$. Forming the colimit cocone $\delta : \mathrm{cod}\, D \longrightarrow \Delta W$ one now obtains a unique $\mathcal{Q}$-functor $\widetilde{\varphi} : W \longrightarrow \mathsf{P} X$ making

$$\begin{array}{ccc} \Delta \mathsf{P} X & \xleftarrow{\ \gamma^*\ } & \mathsf{P}(\mathrm{dom}\, D)_\natural \\ {\scriptstyle \Delta\widetilde{\varphi}}\uparrow & & \uparrow{\scriptstyle \widetilde{\kappa D}} \\ \Delta W & \xleftarrow[\ \delta\ ]{} & \mathrm{cod}\, D \end{array}$$

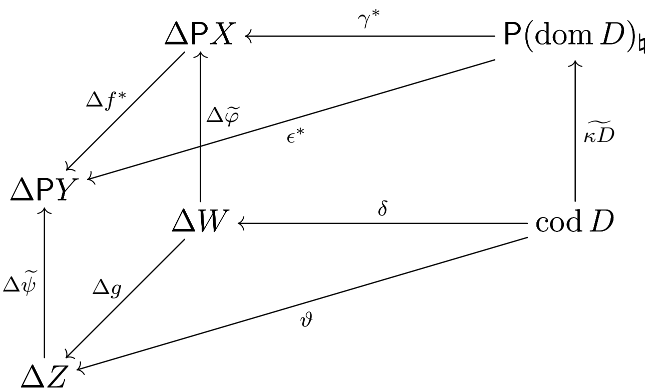
commute in $\mathcal{Q}$-$\mathbf{Cat}$ or, equivalently, making

$$\begin{array}{ccc} \Delta X & \xrightarrow{\ \gamma_\natural\ } & (\mathrm{dom}\, D)_\natural \\ {\scriptstyle \Delta\varphi}\downarrow & & \downarrow{\scriptstyle \kappa D} \\ \Delta W & \xrightarrow[\ \delta^\natural\ ]{} & (\mathrm{cod}\, D)^\natural \end{array}$$

commute in $\mathcal{Q}$-$\mathbf{Dis}$, with $\varphi : X \mathrel{-\!\!\circ\!\!\rightarrow} W$ corresponding to $\widetilde{\varphi}$. In other words, we have a cone $\Gamma : \Delta\varphi \longrightarrow D$ with $\mathrm{dom}\, \varphi = X$, $\mathrm{dom}\, \Gamma = \gamma$, namely $\Gamma = (\gamma, \delta)$.

Given a cone $\Theta : \Delta\psi \longrightarrow D$ with $\psi : Y \mathrel{-\!\!\circ\!\!\rightarrow} Z$ in $\mathcal{Q}$-$\mathbf{Dis}$ and a $\mathcal{Q}$-functor $f : Y \longrightarrow X$ with $\gamma \cdot \Delta f = \epsilon := \mathrm{dom}\, \Theta$, the cocone $\vartheta := \mathrm{cod}\, \Theta : \mathrm{cod}\, D \longrightarrow \Delta Z$ corresponds to a unique $\mathcal{Q}$-functor $g : W \longrightarrow Z$ with $\Delta g \cdot \delta = \vartheta$

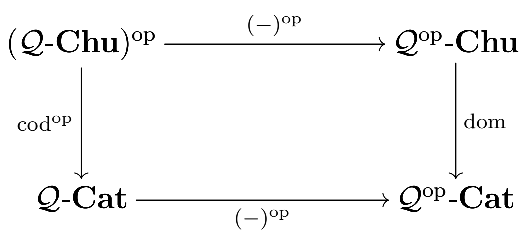
by the colimit property. As the diagram



shows, the colimit property of W also guarantees $f^*\widetilde{\varphi} = \widetilde{\psi} g$ (with $\widetilde{\psi}$ corresponding to ψ) which, by Corollary 3.3, means that $(f,g) : \psi \longrightarrow \varphi$ is the only morphism in $\mathcal{Q}$-**Chu** with $\mathrm{dom}(f,g) = f$ and $\Gamma \cdot \Delta(f,g) = \Theta$. $\square$

Corollary 3.5. $\mathrm{dom} : \mathcal{Q}\text{-}\mathbf{Chu} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$ *is small-topological; in particular,* dom *is a fibration with a fully faithful right adjoint which embeds* $\mathcal{Q}$-**Cat** *into* $\mathcal{Q}$-**Chu** *as a full reflective subcategory.* $\mathrm{cod} : \mathcal{Q}\text{-}\mathbf{Chu} \longrightarrow (\mathcal{Q}\text{-}\mathbf{Cat})^{\mathrm{op}}$ *has the dual properties.*

Proof. With the existence of small colimits guaranteed by Corollary 2.3, dom-initial liftings to small dom-structured cones exist by Theorem 3.4. For the assertion on cod, first observe that every $\mathcal{Q}$-category $X = (X, a)$ gives rise to the $\mathcal{Q}^{\mathrm{op}}$-category $X^{\mathrm{op}} = (X, a^{\circ})$, where $a^{\circ}(x,y) = a(y,x)$ $(x, y \in X)$. With the commutative diagram

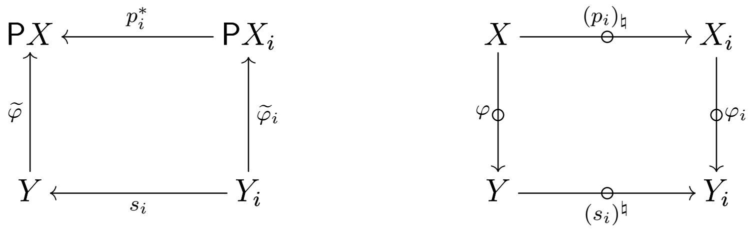


one sees that, up to functorial isomorphisms, $\mathrm{cod}^{\mathrm{op}} : (\mathcal{Q}\text{-}\mathbf{Chu})^{\mathrm{op}} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$ coincides with the small-topological functor $\mathrm{dom} : \mathcal{Q}^{\mathrm{op}}\text{-}\mathbf{Chu} \longrightarrow \mathcal{Q}^{\mathrm{op}}\text{-}\mathbf{Cat}$. $\square$

Corollary 3.6. $\mathcal{Q}$-$\mathbf{Chu}$ *is complete and cocomplete, all small limits and colimits in* $\mathcal{Q}$-$\mathbf{Chu}$ *are preserved by both* dom *and* cod.

Proof. The dom-initial lifting of a dom-structured limit cone in $\mathcal{Q}$-$\mathbf{Cat}$ is a limit cone in $\mathcal{Q}$-$\mathbf{Chu}$, which is trivially preserved. Having a right adjoint, dom also preserves all colimits. □

Remark 3.7. (1) Let us describe (small) products in $\mathcal{Q}$-$\mathbf{Chu}$ explicitly: Given a family of $\mathcal{Q}$-distributors $\varphi_i : X_i \mathrel{\circ\!\!\rightarrow} Y_i$ $(i \in I)$, one first forms the product X of the $\mathcal{Q}$-categories $X_i = (X_i, a_i)$ as in Remark 2.4(1) with projections p_i and the coproduct of the $Y_i = (Y_i, b_i)$ as in Remark 2.4(2) with injections s_i $(i \in I)$. The transposes $\widetilde{\varphi}_i$ then determines a $\mathcal{Q}$-functor $\widetilde{\varphi}$ making the left square of

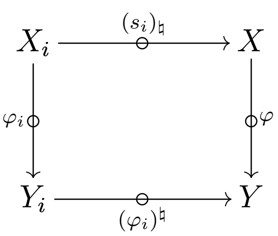


commutative, while the right square exhibits φ as a product of $(\varphi_i)_{i\in I}$ in $\mathcal{Q}$-$\mathbf{Chu}$ with projections (p_i, s_i), $(i \in I)$; explicitly,

$$\varphi(x,y) = (\widetilde{\varphi}(y))_x = (\widetilde{\varphi}_i(y) \circ (p_i)_\natural)_x = (\widetilde{\varphi}_i(y))_{p_i(x)} = \varphi_i(x_i, y)$$

for $x = ((x_i)_{i\in I}, q)$ in X and $y = s_i(y)$ in Y_i, $i \in I$.

(2) The coproduct of $\varphi_i : X_i \mathrel{\circ\!\!\rightarrow} Y_i$ $(i \in I)$ in $\mathcal{Q}$-$\mathbf{Chu}$ is formed like the product, except that the roles of domain and codomain need to be interchanged. Hence, one forms the coproduct X of $(X_i)_{i\in I}$ and the product Y of $(Y_i)_{i\in I}$ in $\mathcal{Q}$-$\mathbf{Cat}$ and obtains the coproduct $\varphi : X \mathrel{\circ\!\!\rightarrow} Y$ in $\mathcal{Q}$-$\mathbf{Chu}$ as in



so that $\varphi(x,y) = \varphi_i(x,y_i)$ for $y = ((y_i)_{i\in I}, q)$ in Y and $x = s_i(x)$ in X_i, $i \in I$.

- The equalizer of $(f,g), (\overline{f},\overline{g}) : \varphi \longrightarrow \psi$ in $\mathcal{Q}$-**Chu** is obtained by forming the equalizer and coequalizer

$$U \xrightarrow{\ i\ } X \underset{\overline{f}}{\overset{f}{\rightrightarrows}} Y \quad \text{and} \quad W \underset{\overline{g}}{\overset{g}{\rightrightarrows}} Z \xrightarrow{\ p\ } V$$

in $\mathcal{Q}$-$\mathbf{Cat}$, respectively. With $\widetilde{\chi} : V \longrightarrow \mathsf{P}U$ obtained from the coequalizer property making

$$\begin{array}{ccc} \mathsf{P}U & \xleftarrow{\quad i^* \quad} & \mathsf{P}X \\ {\scriptstyle \widetilde{\chi}}\uparrow & & \uparrow{\scriptstyle \widetilde{\varphi}} \\ V & \xleftarrow[\quad p \quad]{} & W \end{array}$$

commutative, Theorem 3.4 guarantees that

$$\chi \xrightarrow{(i,p)} \varphi \underset{(\overline{f},\overline{g})}{\overset{(f,g)}{\rightrightarrows}} \psi$$

is an equalizer diagram in $\mathcal{Q}$-**Chu**, where

$$\chi(x,p(w)) = (\widetilde{\chi}(p(w))_x = (i^*(\widetilde{\varphi}(w)))_x = (\widetilde{\varphi}(w) \circ i_\natural)_x = \varphi(i(x),w)$$

for all $x \in U$, $w \in W$.

- Coequalizers in $\mathcal{Q}$-**Chu** are formed like equalizers, except that the roles of domain and codomain need to be interchanged.

We will now strengthen Corollary 3.6 and show total completeness and total cocompleteness of $\mathcal{Q}$-**Chu** with the help of Proposition 2.6. To that end, let us observe that, since the limit and colimit preserving functors dom and cod must in particular preserve both monomorphisms and epimorphisms, a monomorphism $(f,g) : \varphi \longrightarrow \psi$ in $\mathcal{Q}$-**Chu** must be given by a monomorphism f and an epimorphism g in $\mathcal{Q}$-$\mathbf{Cat}$, i.e., by an injective $\mathcal{Q}$-functor f and a surjective $\mathcal{Q}$-functor g. Consequently, $\mathcal{Q}$-**Chu** is wellpowered, and so is its dual $(\mathcal{Q}\text{-}\mathbf{Chu})^{\mathrm{op}} \cong \mathcal{Q}^{\mathrm{op}}$-**Chu**. The main point is therefore for us to prove:

Theorem 3.8. $\mathcal{Q}$-**Chu** *contains a generating set of objects and, consequently, also a cogenerating set.*

Proof. With the notations explained below, we show that

$$\{\eta_s : \varnothing \longrightarrow\!\!\!\!\!\!\!\circ\;\; D_s \mid s \in \mathrm{ob}\,\mathcal{Q}\} \cup \{\lambda_t : \{t\} \longrightarrow\!\!\!\!\!\!\!\circ\;\; \widehat{C} \mid t \in \mathrm{ob}\,\mathcal{Q}\}$$

is generating in $\mathcal{Q}$-**Chu**. Here D_s belongs to a generating set of $\mathcal{Q}$-**Cat** (see Remark 2.8), and

$$C = \coprod_{t \in \mathrm{ob}\,\mathcal{Q}} \mathsf{P}\{t\}$$

is a coproduct in $\mathcal{Q}$-**Cat** (see Remark 2.4(2)) of the presheaf $\mathcal{Q}$-categories of the singleton $\mathcal{Q}$-categories $\{t\}$ (see Proposition 3.2). From C one obtains $\widehat{C}$ by adding an isomorphic copy of each object in C, which may be easily explained for a $\mathcal{Q}$-category (X, a): simply provide the set $\widehat{X} := X \times \{1, 2\}$ with the structure

$$|(x, i)|_{\widehat{X}} = |x|_X \quad \text{and} \quad \widehat{a}((x, i), (y, j)) = a(x, y)$$

for all $x, y \in X$, $i, j \in \{1, 2\}$. Noting that the objects of $\mathsf{P}\{t\}$ are simply $\mathcal{Q}$-arrows with domain t, we now define $\lambda_t : \{t\} \longrightarrow\!\!\!\!\!\!\!\circ\;\; \widehat{C}$ by

$$\lambda_t(u, i) = \begin{cases} u & \text{if } \mathrm{dom}\, u = t, \\ \bot & \text{else} \end{cases}$$

for $i \in \{1, 2\}$ and every object t and arrow u in $\mathcal{Q}$. For another element (v, j) in $\widehat{C}$, if $\mathrm{dom}\, v = \mathrm{dom}\, u = t$ one then has

$$[(u, i), (v, j)] \circ \lambda_t(u, i) = (v \swarrow u) \circ u \leq v = \lambda_t(v, j),$$

and in other cases this inequality holds trivially. Hence, λ_t is indeed a $\mathcal{Q}$-distributor.

Let us now consider $\mathcal{Q}$-Chu transforms $(f, g) \neq (\overline{f}, \overline{g}) : \varphi \longrightarrow \psi$ as in

$$\begin{array}{ccc} (X, a) & \overset{f_\natural}{\underset{\overline{f}_\natural}{\rightrightarrows}} & (Y, b) \\ \varphi \big\downarrow & & \big\downarrow \psi \\ (W, d) & \overset{g^\natural}{\underset{\overline{g}^\natural}{\rightrightarrows}} & (Z, c) \end{array}$$

Case 1: $X = \varnothing$ is the initial object of $\mathcal{Q}$-$\mathbf{Cat}$ (and $\mathcal{Q}$-$\mathbf{Dis}$). Then $g \neq \overline{g}$, and we find $s \in \mathrm{ob}\,\mathcal{Q}$ and $h : W \longrightarrow D_s$ with $hg \neq h\overline{g}$ in $\mathcal{Q}$-$\mathbf{Cat}$. Consequently, $(1_\varnothing, h) : \eta_s \longrightarrow \varphi$ satisfies $(f, g)(1_\varnothing, h) \neq (\overline{f}, \overline{g})(1_\varnothing, h)$.

Case 2: $f \neq \overline{f}$, so that $f(x_0) \neq \overline{f}(x_0)$ for some $x_0 \in X$. Then, for $t := |x_0|$, $e : \{t\} \longrightarrow X$, $|x_0| \mapsto x_0$, is a $\mathcal{Q}$-functor with $fe \neq \overline{f}e$, and it suffices to show that

$$h : W \longrightarrow \widehat{C}, \quad w \mapsto (\varphi(x_0, w), 1)$$

is a $\mathcal{Q}$-functor making $(e, h) : \lambda_t \longrightarrow \varphi$ a $\mathcal{Q}$-Chu transform. Indeed,

$$\begin{aligned} &d(w, w') \leq \varphi(x_0, w') \swarrow \varphi(x_0, w) = [h(w), h(w')], \\ &\lambda_t(h(w)) = \varphi(x_0, w) = \varphi(e(t), w) \end{aligned}$$

for all $w, w' \in W$.

Case 3: $X \neq \varnothing$ and $g \neq \overline{g}$. Then $g(z_0) \neq \overline{g}(z_0)$ for some $z_0 \in Z$, and with any fixed $x_0 \in X$ we may alter the previous definition of $h : W \longrightarrow \widehat{C}$ by

$$h(w) := \begin{cases} (\varphi(x_0, w), 2) & \text{if } w = \overline{g}(z_0), \\ (\varphi(x_0, w), 1) & \text{else.} \end{cases}$$

The verification for h to be a $\mathcal{Q}$-functor and $(e, h) : \lambda_t \longrightarrow \varphi$ a $\mathcal{Q}$-Chu transform remain intact, and since $hg \neq h\overline{g}$, the proof is complete. □

Remark 3.9. A generating set in $\mathcal{Q}$-$\mathbf{Chu}$ may be alternatively given by

$$\{\lambda_\varnothing : \varnothing \,{-\!\!\circ\!\!\rightarrow}\, \widehat{C}\} \cup \{\lambda_t : \{t\} \,{-\!\!\circ\!\!\rightarrow}\, \widehat{C} \mid t \in \mathrm{ob}\,\mathcal{Q}\},$$

so that in Case 1 one may proceed exactly as in Case 3 only by replacing $\varphi(x_0, w)$ with $\top : q \longrightarrow |w|$ for any fixed $q \in \mathrm{ob}\,\mathcal{Q}$.

With Theorem 3.8 we obtain:

Corollary 3.10. $\mathcal{Q}$-$\mathbf{Chu}$ *is totally complete and totally cocomplete and, in particular, hypercocomplete and hypercomplete.*

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GENERALISED PUSHOUTS, CONNECTED COLIMITS AND CODISCRETE GROUPOIDS

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Résumé. Nous étudions brièvement une espèce de colimites, appellée ici 'pushout généralisé'. On prouve que, dans une catégorie quelconque, l'existence de ces colimites correspond à celle des colimites connexes; dans le cas fini on sait que ceci se reduit à l'existence de pushouts ordinaires et coégalisateurs (R. Paré, 1993). Cette étude a été motivée par la remarque que tout groupoïde est, *à équivalence près*, un pushout généralisé de groupoïdes codiscrets. Pour les groupoïdes fondamentaux d'espaces convenables nous donnons des résultats plus fins concernant des pushouts généralisés *finis*.
Abstract. This is a brief study of a particular kind of colimit, called a 'generalised pushout'. We prove that, in any category, generalised pushouts amount to connected colimits; in the finite case the latter are known to amount to ordinary pushouts and coequalisers (R. Paré, 1993). This study was motivated by remarking that every groupoid is, *up to equivalence*, a generalised pushout of codiscrete subgroupoids. For the fundamental groupoids of suitable spaces we get finer results concerning *finite* generalised pushouts.
Keywords. generalised pushout, connected colimit, fundamental groupoid.
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1. Introduction

We are interested in colimits of a particular form, that will be called 'generalised pushouts' (see Section 2), following a line well represented in category theory: to study particular classes of (co)limits, like filtered colimits, flexible (co)limits, connected (co)limits, etc.

Generalised pushouts are *connected, non-simply-connected colimits* and therefore cannot be reduced to ordinary pushouts (see R. Paré [P1, P2]). We show in Sections 2 and 3 that generalised pushouts give all connected colimits and - in the finite case - amount to finite connected colimits; the latter can be reduced to pushouts and coequalisers, as proved in [P2].

The second part is about colimits of groupoids. In Section 4 we show that - up to categorical equivalence - *every* groupoid is a generalised pushout of codiscrete subgroupoids.

In the last Section 5 we consider cases where the fundamental groupoid $\pi_1 X$ of a space can be obtained as a *finite* colimit of this kind.

For instance, an obvious cover of the circle $\mathbf{S}^1$ with three open arcs X_1, X_2, X_3 shows that the fundamental groupoid $\pi_1(\mathbf{S}^1)$ is a *3-generalised pushout* of the codiscrete subgroupoids $\pi_1 X_i$ over codiscrete subgroupoids. Moreover if we form a subset A by choosing a point in each of the three intersections $X_i \cap X_j$, the (equivalent) restricted groupoid $\pi_1(\mathbf{S}^1)|A$ (with objects in A) is a 3-generalised pushout of *finite* codiscrete subgroupoids.

There are similar results for a compact differentiable manifold (Corollary 5.3), while a sphere with countably many handles would require a countable generalised pushout. More generally these facts hold for spaces having 'sufficiently good' covers, as we show in Theorem 5.2.

Let us stress the point that this article is not about the *concrete computation* of fundamental groupoids, which is already well covered in the literature (see [Bw2, BwS] and references therein): our goal is to isolate a kind of connected colimit motivated by the previous arguments and to study its categorical aspects.

As a related question, suggested by F.W. Lawvere, one might investigate 'codiscretely generated' toposes, like symmetric simplicial sets [Gr]. The van Kampen theorem for lextensive categories of [BwJ] might also be examined in the present line.

Here a connected category is assumed to be non-empty, as usual; but we do not follow this convention for spaces.

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2. Generalised pushouts

In an arbitrary category $\mathbf{C}$ we are interested in colimits of a particular form, that will be called 'generalised pushouts'.

We start from a non-empty set I and form the set $\hat{I}$ of its subsets $\{i, j\}$ having one or two elements, ordered by the relation $\{i, j\} \leq \{i\}$ and viewed as a connected category.

A functor $X\colon \hat{I} \to \mathbf{C}$ will be (partially) represented as the left diagram below

$$\begin{array}{ccc} & \overset{u_{ij}}{\nearrow} & X_i \\ X_{ij} & & \\ & \underset{u_{ji}}{\searrow} & X_j \end{array} \qquad (i, j \in I), \qquad \begin{array}{ccccc} & \nearrow & X_i & \overset{f_i}{\searrow} & \\ X_{ij} & & & & Y \\ & \searrow & X_j & \underset{f_j}{\nearrow} & \end{array} \qquad (1)$$

A cocone of vertex Y amounts to family $f_i\colon X_i \to Y$ of morphisms of $\mathbf{C}$ such that all the right-hand squares above commute. The colimit will be called the *generalised pushout* of the objects X_i *over* the objects X_{ij}. We speak of a *finite* generalised pushout, or n-*generalised pushout* when the set I is finite, with n elements.

Plainly, 1-generalised pushouts are trivial and 2-generalised pushouts are ordinary pushouts. A 3-generalised pushout is the colimit of a diagram $X\colon \hat{I} \to \mathbf{C}$ with $I = \{1, 2, 3\}$

$$\begin{array}{ccccc} & & 3 & & \\ & \nearrow & & \nwarrow & \\ \{1,3\} & & & & \{2,3\} \\ \swarrow & & & & \searrow \\ 1 & \longleftarrow & \{1,2\} & \longrightarrow & 2 \end{array} \quad \hat{I} \qquad \begin{array}{ccccc} & & X_3 & & \\ & \nearrow & & \nwarrow & \\ X_{13} & & & & X_{23} \\ \swarrow & & & & \searrow \\ X_1 & \longleftarrow & X_{12} & \longrightarrow & X_2 \end{array} \quad X \qquad (2)$$

as in the example of the Introduction for $\pi_1(\mathbf{S}^1)$. We prove below that they give all finite generalised pushouts and that their existence amounts to that of pushouts and coequalisers (Section 3).

The fact that pushouts are not sufficient to construct all finite generalised pushouts is already known from a paper of R. Paré [P1] (see Theorem 2) that characterises the limits that can be constructed with pullbacks. In fact the category $\hat{I}$ associated to $I = \{1, 2, 3\}$ (or any larger set) is not *simply connected*: the left figure above shows a non-trivial loop, that gives a non-trivial endomorphism in $\pi_1(\hat{I}) = \pi_1(\hat{I}^{op})$ (cf. [P1]).

3. Generalised pushouts and connected colimits

We shall now make use of a second paper of R. Paré on connected limits [P2], recalling a result from its Section 4 (written here in dual form).

Theorem 3.1 (R. Paré)**.** *The category* $\mathbf{C}$ *has arbitrary (resp. finite) connected colimits if and only if it has arbitrary (resp. finite) fibred coproducts and coequalisers.*

Here a *fibred coproduct* is the colimit of a family $h_i \colon X \to X_i$ of morphisms indexed by a non-empty set I. Of course the existence of finite fibred coproducts is equivalent to the existence of pushouts.

Lemma 3.2. *If the category* $\mathbf{C}$ *has arbitrary (resp. finite) generalised pushouts then it has arbitrary (resp. finite) fibred coproducts.*

Proof. Starting from a family $(h_i \colon X \to X_i)_{i \in I}$ $(I \neq \emptyset)$, we transform it into a diagram $X \colon \hat{I} \to \mathbf{C}$. After the given objects X_i, we let $X_{ij} = X$ for $i \neq j$ and

$$u_{ij} = h_i \colon X \to X_i, \text{ for } i \neq j, \qquad\qquad u_{ii} = \mathrm{id}X_i.$$

Now the given diagram $(h_i \colon X \to X_i)$ and the new one have the same cocones, namely the families of morphisms $f_i \colon X_i \to Y$ such that $f_i h_i = f_j h_j$ for all indices i, j. □

Lemma 3.3. *If the category* $\mathbf{C}$ *has 3-generalised pushouts then it has ordinary pushouts and coequalisers.*

Proof. Suppose that $\mathbf{C}$ has 3-generalised pushouts. To prove the existence of pushouts, starting from the left diagram below

$$X_{12} \begin{array}{l} \xrightarrow{u_{12}} X_1 \\ \xrightarrow[u_{21}]{} X_2 \end{array} \qquad X_{13} \begin{array}{l} \xrightarrow{u_{13}} X_1 \\ \xrightarrow[u_{31}]{} X_3 \end{array} \qquad X_{23} \begin{array}{l} \xrightarrow{u_{23}} X_2 \\ \xrightarrow[u_{32}]{} X_3 \end{array} \qquad (3)$$

we extend it to a diagram on $I = \{1, 2, 3\}$, where $X_3 = X_2$, the second span above coincides with the first and the third consists of identities $u_{23} = u_{32} = \mathrm{id}X_2$. Plainly the extended diagram has the same cocones as the original one and its colimit is the pushout of the latter.

To prove that $\mathbf{C}$ has coequalisers, we start from two arrows $u, v \colon X \to Y$ and construct a new diagram on $I = \{1, 2, 3\}$, where $X_1 = X_2 = X_3 = Y$ and the three spans are specified below

$$X \begin{array}{l} \xrightarrow{u} X_1 \\ \xrightarrow[v]{} X_2 \end{array} \qquad Y \begin{array}{l} \xrightarrow{\mathrm{id}} X_1 \\ \xrightarrow[\mathrm{id}]{} X_3 \end{array} \qquad Y \begin{array}{l} \xrightarrow{\mathrm{id}} X_2 \\ \xrightarrow[\mathrm{id}]{} X_3 \end{array} \qquad (4)$$

For the new diagram, a cocone $(f_i\colon X_i \to Z)$ is a map $f = f_1 = f_2 = f_3\colon Y \to Z$ such that $fu = fv$; the 3-generalised pushout is thus the coequaliser of u, v. □

Corollary 3.4. *The following conditions on a category* $\mathbf{C}$ *are equivalent:*

(a) $\mathbf{C}$ *has generalised pushouts,*

(b) $\mathbf{C}$ *has fibred coproducts and coequalisers,*

(c) $\mathbf{C}$ *has connected colimits.*

Proof. (a) ⇒ (b) From the previous lemmas. (b) ⇒ (c) From Theorem 3.1. (c) ⇒ (a) Obvious. □

Corollary 3.5. *The following conditions on a category* $\mathbf{C}$ *are equivalent:*

(a) $\mathbf{C}$ *has finite generalised pushouts,*

(b) $\mathbf{C}$ *has 3-generalised pushouts,*

(c) $\mathbf{C}$ *has pushouts and coequalisers,*

(d) $\mathbf{C}$ *has finite connected colimits.*

Proof. (d) ⇒ (a) ⇒ (b) Obvious. (b) ⇒ (c) From 3.3. (c) ⇒ (d) From 3.1. □

We end by remarking that the 'weight' of generalised pushouts within finite colimits can be measured noting that each of the conditions below implies the following one

(a) $\mathbf{C}$ has finite colimits,

(b) $\mathbf{C}$ has finite generalised pushouts, or (equivalently) finite connected colimits, or pushouts and coequalisers, or 3-generalised pushouts,

(c) $\mathbf{C}$ has pushouts.

These implications cannot be reversed. In fact the category of non-empty sets has all colimits of non-empty diagrams but lacks an initial object. Secondly, every groupoid has (trivial) pushouts, but it has coequalisers if and only if it is an equivalence relation.

4. Generalised pushouts of groupoids

We prove now that all groupoids can be obtained as generalised pushouts of codiscrete groupoids. Of course a groupoid G is said to be *codiscrete*, or chaotic, if it has precisely one arrow $x \to y$ for any two vertices x, y; this means that G is either empty or equivalent to the singleton groupoid.

Proposition 4.1. *Every groupoid is categorically equivalent to a groupoid G that is a generalised pushout of finite codiscrete groupoids and inclusions. These groupoids can be assumed to be subgroupoids of G, and non-empty if G is connected.*

Proof. As a motivation of making appeal to categorical equivalence, note that a group has only one codiscrete subgroupoid, the trivial one, and cannot be a generalised pushout of codiscrete *sub*groupoids - unless it is trivial. (But one can prove, by an argument similar to the following one, that every connected groupoid on at least three vertices *is a generalised pushout of codiscrete subgroupoids and inclusions*.)

We can suppose that our groupoid is connected (non-empty); then it is equivalent to its skeleton, which is a group G_0, and we replace the latter with an equivalent groupoid G on three vertices, say 1, 2, 3. Let I be the set of commutative diagrams in G of the following form

$$\begin{array}{ccccc} & & 2 & & \\ & {}^{x}\nearrow & & \searrow^{y} & \\ 1 & & \xrightarrow[z]{} & & 3 \end{array} \qquad\qquad z = yx. \tag{5}$$

It will be convenient to denote this diagram by the triple (x, y, z), even though each pair of these arrows determines the third; we write as $F(x, y, z)$ the subgroupoid of G generated by these arrows: it is formed by all the objects, their identities, the three given maps and their inverses. I can be identified with the set of all the wide codiscrete subgroupoids of G. It is also easy to see that the set I is in bijective (non-canonical) correspondence with $G_0{\times}G_0$: after fixing a diagram (5) and identifying $G(1,1)$ with G_0, each pair $(g, h) \in G_0 \times G_0$ determines a triple (xg, y', zh), with $y' = zhg^{-1}x^{-1}$.

We now form a diagram $F\colon \hat{I} \to \mathbf{Gpd}$ of finite codiscrete groupoids and inclusions. F is already defined on the triples $(x, y, z) \in I$. For two *distinct* triples (x, y, z), (x', y', z') we distinguish two cases:

(i) if $x = x'$ or $y = y'$ or $z = z'$, we let

$$F(x, y, z; x', y', z') = (F(x, y, z) \cap F(x', y', z'))\hat{},$$

where $(-)\hat{}$ means taking out the isolated vertex (since the intersection itself is not codiscrete),

(ii) otherwise we let $F(x, y, z; x', y', z')$ be the (co)discrete groupoid on the vertex 1 (since we do not want to use the empty groupoid).

It is now evident that every cocone $f_{xyz}\colon F(x, y, z) \to H$ (for $(x, y, z) \in I$) of F has precisely one extension to a 'mapping' $f\colon G \to H$, which necessarily preserves identities and composition. The only point that is not completely trivial is showing that two arbitrary morphisms f_{xyz} and $f_{x'y'z'}$ of this cocone coincide on each vertex. Working for instance on the vertex 3, it is sufficient to consider the objects

$$F(x, y, z), \qquad F(x'', y', z), \qquad F(x', y', z') \qquad\qquad (x'' = z^{-1}y'),$$

so that the cocone condition gives: $f_{xyz}(3) = f_{x''y'z}(3) = f_{x'y'z'}(3)$. □

5. Good covers of spaces and manifolds

We end by investigating cases where the fundamental groupoid $\pi_1(X)$ of a space can be obtained as a finite generalised pushout of codiscrete groupoids.

First we need an extension of the van Kampen theorem for fundamental groupoids, as formulated by R. Brown in [Bw1, Bw2].

Theorem 5.1. *Let X be a space equipped with a family of subspaces $(X_i)_{i\in I}$ such that X is covered by their interior parts. Then the fundamental groupoid $\pi_1 X$ is (strictly) the generalised pushout of the groupoids $\pi_1 X_i$ over the groupoids $\pi_1(X_i \cap X_j)$.*

Proof. The binary case, concerning a pushout, is proved in [Bw2], Section 6.7.2. This extension can be proved by the same argument. (See also Exercise 6 of [Bw2], 6.7.) □

Secondly we recall (from [BT], Theorem 5.1) that every differentiable n-manifold X has a *good cover*, i.e. an open cover $(X_i)_{i\in I}$ such that all non-empty intersections $X_{i_1} \cap ... \cap X_{i_k}$ are diffeomorphic to $\mathbb{R}^n$. (Up to dimension 3, but not beyond, every topological manifold has a differentiable structure, so that this result can be extended [Sc, Mo].)

With this motivation we consider a topological space X having an open cover (X_i) such that all subspaces $X_i \cap X_j$ (including the original X_i, of course) are 1-connected: in other words we assume that the following fundamental groupoids are codiscrete

$$\pi_1(X_i) = C(X_i), \quad \pi_1(X_i \cap X_j) = C(X_i \cap X_j) = C(X_i) \cap C(X_j), \tag{6}$$

where $C(S)$ denotes the codiscrete groupoid on a set of objects S, possibly empty.

Theorem 5.2. *(a) In this hypothesis the fundamental groupoid $\pi_1 X$ is (in a strict sense) the generalised pushout of* a diagram $C\colon \hat{I} \to \mathbf{Gpd}$ of codiscrete groupoids and inclusions

$$\begin{array}{ccc} & \overset{u_{ij}}{\nearrow} & C_i \\ C_{ij} & & \\ & \underset{u_{ji}}{\searrow} & C_j \end{array} \qquad C_i = C(X_i), \qquad C_{ij} = C(X_i \cap X_j) = C_i \cap C_j. \tag{7}$$

All these objects are subgroupoids of $\pi_1 X$, the colimit.

(b) If X has a finite *cover (X_i) of the previous type (which is a consequence of the previous assumption, for a compact X), then $\pi_1 X$ is a* finite *generalised pushout of codiscrete groupoids and inclusions.*

(c) If, moreover, each subspace $X_{ijk} = X_i \cap X_j \cap X_k$ has a finite number of path components, we can choose a finite subset A of X which meets every such component so that the (equivalent) restricted groupoid $\pi_1 X|A$ is a finite *generalised pushout of* finite *codiscrete subgroupoids and inclusions.*

Proof. The first two points follow from Theorem 5.1. Point (c) follows from the Main Theorem of Brown - Salleh [BwS]. This article also shows that - here - it is not sufficient to consider the binary intersections $X_i \cap X_j$; the authors are indebted to R. Brown for helpful comments on this aspect. □

Corollary 5.3. *These results are automatically true for every differentiable manifold X. In the compact case, also the additional hypotheses of 5.2 (b), (c) automatically hold.*

Proof. Follows from the existence of good covers recalled above. □

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