

cahiers de topologie et géométrie différentielle catégoriques

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**CATEGORIES VS. GROUPOIDS
VIA GENERALISED MAL'TSEV PROPERTIES**

by *Nelson MARTINS-FERREIRA and Tim VAN DER LINDEN*

Résumé. On étudie la différence entre les catégories internes et les groupoïdes internes en termes de propriétés de Malcev généralisées—la propriété de Malcev faible d'un côté, et l' n -permutabilité de l'autre. Dans la première partie de l'article on donne des conditions sur les structures catégoriques internes qui détectent si la catégorie ambiante est naturellement de Malcev, de Malcev ou faiblement de Malcev. On démontre que celles-ci ne dépendent pas de l'existence de produits binaires. Dans la seconde partie on se concentre sur les variétés d'algèbres universelles.

Abstract. We study the difference between internal categories and internal groupoids in terms of generalised Mal'tsev properties—the weak Mal'tsev property on the one hand, and n -permutability on the other. In the first part of the article we give conditions on internal categorical structures which detect whether the surrounding category is naturally Mal'tsev, Mal'tsev or weakly Mal'tsev. We show that these do not depend on the existence of binary products. In the second part we focus on varieties of algebras.

Keywords. Mal'tsev condition, n -permutable variety, internal category

Mathematics Subject Classification (2010). 17D10, 18B99, 18D35

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Introduction

In this article we study the difference between internal categories and internal groupoids through the generalised Mal'tsev properties their surrounding category may have—the weak Mal'tsev property on the one hand, and n -permutability on the other. Conversely, or equivalently, we try to better understand these Mal'tsev conditions by providing new characterisations and new examples for them, singling out distinctive properties of a given type of category via properties of its internal categorical structures: internal categories, (pre)groupoids, relations.

The first part of the text gives a conceptual unification of three levels of Mal'tsev properties: *naturally Mal'tsev* categories [10] using groupoids, categories, pregroupoids, etc. (Theorem 2.2), *Mal'tsev* categories [2, 3] using equivalence relations, preorders, difunctional relations (Theorem 2.5), and *weakly Mal'tsev* categories [14] via strong equivalence relations, strong preorders, difunctional strong relations (Theorem 2.8). Each of the resulting collections of equivalent conditions is completely parallel to the others, and such that a weaker collection of conditions is characterised by a smaller class of internal structures.

Some of these characterisations are well established, whereas some others are less familiar; what is new in all cases is the context in which we prove them: we never use binary products, but restrict ourselves to categories in which kernel pairs and split pullbacks exist.

The notion of weakly Mal'tsev category is probably not as well known as the others. It was introduced in [14] as a setting where any internal reflexive graph admits at most one structure of internal category. It turned out that this new notion is weaker than the concept of Mal'tsev category. But, unlike in Mal'tsev categories, in this setting not every internal category is automatically an internal groupoid. This gave rise to the following problem: to characterise those weakly Mal'tsev categories in which internal categories and internal groupoids coincide.

In Section 3 we observe that, in a weakly Mal'tsev category with kernel pairs and equalisers, the following hold: (1) the forgetful functor from internal categories to multiplicative graphs is an isomorphism; (2) the forgetful functor from internal groupoids to internal categories is an isomorphism if and only if every internal preorder is an equivalence relation (Theorem 3.1).

We study some varietal implications of this result in Section 4. In finitary quasivarieties of universal algebra, the latter condition—that reflexivity and transitivity together imply symmetry—is known to be equivalent to the variety being *n-permutable*, for some *n* (Proposition 4.4). On the way we recall Proposition 4.3, a result due to Hagemann [6]—see also the monograph [4], and the article [9] where it is proved in the context of regular categories. We furthermore explain how to construct a weakly Mal'tsev quasivariety starting from a Goursat (= 3-permutable) quasivariety (Proposition 4.8), and use this procedure to show that categories which are both weakly Mal'tsev and Goursat still need not be Mal'tsev (Example 4.9).

Of course, via part (2) of Theorem 3.1, our Proposition 4.4 implies that, in an *n*-permutable weakly Mal'tsev variety, every internal category is an internal groupoid—but surprisingly, here in fact the weak Mal'tsev property is not needed: *n*-permutability suffices, as was recently proved by Rodelo [17] and further explored in the paper [15]. This indicates that there may still be hidden connections between these two (a priori independent) weakenings of the Mal'tsev axiom.

1 Preliminaries

We recall the definitions and basic properties of some internal categorical structures which we shall use throughout this article.

1.1 Split pullbacks

Let \mathcal{C} be any category. A diagram in \mathcal{C} of the form

$$\begin{array}{ccc}
 E & \begin{array}{c} \xrightarrow{p_2} \\ \xleftarrow{e_2} \end{array} & C \\
 \begin{array}{c} \uparrow e_1 \\ \downarrow p_1 \end{array} & & \begin{array}{c} \uparrow g \\ \downarrow s \end{array} \\
 A & \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{r} \end{array} & B
 \end{array} \tag{A}$$

such that

$$gp_2 = fp_1, \quad p_1e_2 = rg, \quad e_1r = e_2s, \quad p_2e_1 = sf$$

and

$$p_1e_1 = 1_A, \quad fr = 1_B, \quad gs = 1_B, \quad p_2e_2 = 1_C$$

is called a **double split epimorphism**. When we call a double split epimorphism a **pullback** we refer to the commutative square of split epimorphisms $fp_1 = gp_2$. Any pullback of a split epimorphism along a split epimorphism gives rise to a double split epimorphism; we say that \mathcal{C} **has split pullbacks** when the pullback of a split epimorphism along a split epimorphism always exists.

In a category with split pullbacks \mathcal{C} , any diagram such as

$$\begin{array}{ccccc}
 & & f & & g \\
 & & \rightarrow & & \rightarrow \\
 A & \xleftarrow{r} & B & \xleftarrow{s} & C \\
 & \searrow \alpha & \downarrow \beta & \swarrow \gamma & \\
 & & D & &
 \end{array} \tag{B}$$

where $fr = 1_B = gs$ and $\alpha r = \beta = \gamma s$ induces a diagram

$$\begin{array}{ccccc}
 & & C & & \\
 & \nearrow \pi_2 & \swarrow s & \searrow \gamma & \\
 & & B & \xrightarrow{\beta} & D \\
 A \times_B C & \xleftarrow{e_2} & C & \xleftarrow{g} & B \\
 & \searrow e_1 & \swarrow f & \swarrow r & \\
 & & A & \xrightarrow{\alpha} & D \\
 & \nearrow \pi_1 & & &
 \end{array} \tag{C}$$

in which the square is a double split epimorphism. This kind of diagram will appear in the statements of Theorem 2.2, 2.5 and 2.8 as part of a universal property: under certain conditions we expect it to induce a (unique) morphism $\varphi: A \times_B C \rightarrow D$ such that $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$.

1.2 Internal groupoids

A **reflexive graph** in \mathcal{C} is a diagram of the form

$$C_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} C_0 \tag{D}$$

such that $de = 1_{C_0} = ce$.

A **multiplicative graph** in \mathcal{C} is a diagram of the form

$$\begin{array}{ccccc}
 & \xrightarrow{\pi_2} & & & \\
 C_2 & \xleftarrow{e_2} & C_1 & \xrightarrow{d} & C_0 \\
 & \xleftarrow{m} & & \xleftarrow{e} & \\
 & \xleftarrow{e_1} & & \xrightarrow{c} & \\
 & \xrightarrow{\pi_1} & & &
 \end{array} \quad (\mathbf{E})$$

where

$$me_1 = 1_{C_1} = me_2, \quad dm = d\pi_2 \quad \text{and} \quad cm = c\pi_1$$

and the double split epimorphism

$$\begin{array}{ccc}
 C_2 & \xrightarrow{\pi_2} & C_1 \\
 \uparrow e_1 & & \downarrow e \\
 C_1 & \xrightarrow{d} & C_0 \\
 \downarrow \pi_1 & & \uparrow c
 \end{array}$$

is a pullback. Observe that a multiplicative graph is in particular a reflexive graph ($de = 1_{C_0} = ce$) and that the morphisms e_1 and e_2 are universally induced by the pullback:

$$e_1 = \langle 1_{C_1}, ed \rangle \quad \text{and} \quad e_2 = \langle ec, 1_{C_1} \rangle.$$

When the category \mathcal{C} admits split pullbacks we shall refer to a multiplicative graph simply as

$$C_2 \xrightarrow{m} C_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} C_0.$$

An **internal category** is a multiplicative graph which satisfies the associativity condition $m(1 \times m) = m(m \times 1)$.

An **internal groupoid** is an internal category where both squares $dm = d\pi_2$ and $cm = c\pi_1$ are pullbacks (see for instance [1, Proposition A.3.7]). Equivalently, there should be a morphism $t: C_1 \rightarrow C_1$ with $ct = d$, $dt = c$ and $m\langle 1_{C_1}, t \rangle = ec$, $m\langle t, 1_{C_1} \rangle = ed$.

In the following sections we shall consider the obvious forgetful functors

$$\text{Grpd}(\mathcal{C}) \xrightarrow{U_3} \text{Cat}(\mathcal{C}) \xrightarrow{U_2} \text{MG}(\mathcal{C}) \xrightarrow{U_1} \text{RG}(\mathcal{C})$$

from groupoids in \mathcal{C} to internal categories, to multiplicative graphs, to reflexive graphs. We write U_{12} and U_{123} for the induced composites $U_1 U_2$ and $U_1 U_2 U_3$, respectively.

1.3 Internal pregroupoids

A **pregroupoid** [12, 11, 7] in \mathcal{C} is a span

$$(d, c) = \begin{array}{ccc} & D & \\ d \swarrow & & \searrow c \\ D_0 & & D'_0 \end{array}$$

together with a structure of the form

$$\begin{array}{ccccccc} D \times_{D_0} D & \times_{D'_0} D & \xleftarrow{p_2} & D \times_{D'_0} D & \xrightarrow{c_2} & D & \\ \downarrow p_1 & \uparrow i_1 & & \downarrow c_1 & & \downarrow c & \\ D \times_{D_0} D & \xrightarrow{d_2} & D & \xrightarrow{c} & D'_0 & & \\ \downarrow d_1 & & \downarrow d & & & & \\ D & \xrightarrow{d} & D_0 & & & & \end{array} \quad \text{(F)}$$

where (1), (2) and (3) are pullback squares, the morphisms i_1, i_2 are determined by

$$p_1 i_1 = 1_{D \times_{D_0} D}, \quad p_2 i_1 = \langle d_2, d_2 \rangle$$

and

$$p_2 i_2 = 1_{D \times_{D'_0} D}, \quad p_1 i_2 = \langle c_1, c_1 \rangle$$

and there is a further morphism $p: D \times_{D_0} D \times_{D'_0} D \rightarrow D$ which satisfies the conditions

$$p i_1 = d_1 \quad \text{and} \quad p i_2 = c_2, \quad \text{(G)}$$

$$d p = d c_2 p_2 \quad \text{and} \quad c p = c d_1 p_1. \quad \text{(H)}$$

When \mathcal{C} admits split pullbacks and kernel pairs, we shall refer to a pregroupoid structure simply as a structure

$$\begin{array}{ccc}
 D \times_{D_0} D \times_{D'_0} D & \xrightarrow{p} & D \\
 & & \swarrow c \\
 & & D'_0 \\
 & & \searrow d \\
 & & D_0.
 \end{array} \tag{I}$$

In order to have a visual picture, we may think of the object D as having elements of the form

$$c(x) \leftarrow^x d(x) \quad \text{or} \quad \cdot \leftarrow^x \cdot$$

and hence the “elements” of $D \times_{D_0} D$, $D \times_{D'_0} D$ and $D \times_{D_0} D \times_{D'_0} D$ are, respectively, of the form

$$\cdot \leftarrow^x \cdot \xrightarrow{y} \cdot, \quad \cdot \xrightarrow{x} \cdot \leftarrow^y \cdot$$

and

$$\cdot \leftarrow^x \cdot \xrightarrow{y} \cdot \leftarrow^z \cdot$$

Observe that the morphism p is a kind of Mal'tsev operation in the sense that $p(x, y, y) = x$ and $p(x, x, y) = y$ (the conditions **(G)**). Furthermore, $dp(x, y, z) = dz$ and $cp(x, y, z) = cx$ by **(H)**.

In the following sections we shall also consider the forgetful functor

$$V: \text{PreGrpd}(\mathcal{C}) \rightarrow \text{Span}(\mathcal{C})$$

from the category of pregroupoids to the category of spans in \mathcal{C} .

The definition of pregroupoid also contains the associativity axiom, asking that $p(p(x, y, z), u, v) = p(x, y, p(z, u, v))$ whenever both sides of the equation make sense. We shall not assume this, but rather deduce the property in the naturally Mal'tsev and (weakly) Mal'tsev contexts.

1.4 Relations

The notions of reflexive relation, preorder (or reflexive and transitive relation), equivalence relation, and difunctional relation, may all be obtained, respectively, from the notions of reflexive graph, internal category (or multiplicative graph), internal groupoid, and pregroupoid, simply by imposing the extra condition that the pair of morphisms (d, c) is jointly monomorphic. We will also consider **strong** relations: here the pair of morphisms (d, c) is jointly strongly monomorphic.

2 Mal'tsev conditions

In this section we study some established and some less known characterisations of *Mal'tsev* and *naturally Mal'tsev* categories in terms of internal categorical structures. We extend these characterisations, which are usually considered in a context with finite limits, to a more general setting: categories with kernel pairs and split pullbacks. In particular we shall never assume that binary products exist. This allows for a treatment of *weakly Mal'tsev* categories in a manner completely parallel to the treatment of the two stronger notions.

2.1 Naturally Mal'tsev categories

We first consider the notion of naturally Mal'tsev category [10] in a context where binary products are not assumed to exist. This may seem strange, as the original definition takes place in a category with binary products (and no other limits). We can do this because the main characterisation of naturally Mal'tsev categories—as those categories for which the forgetful functor from internal groupoids to reflexive graphs is an isomorphism—is generally stated in a finitely complete context. This context may be even further reduced: we shall show that the existence of kernel pairs and split pullbacks is sufficient.

Theorem 2.2. *Let \mathcal{C} be a category with kernel pairs and split pullbacks. The following are equivalent:*

- (i) *the functor $U_{123} : \text{Grpd}(\mathcal{C}) \rightarrow \text{RG}(\mathcal{C})$ is an isomorphism;*

- (ii) the functor $U_{12}: \text{Cat}(\mathcal{C}) \rightarrow \text{RG}(\mathcal{C})$ has a section;
- (iii) the functor $U_1: \text{MG}(\mathcal{C}) \rightarrow \text{RG}(\mathcal{C})$ has a section;
- (iv) the functor $V: \text{PreGrpd}(\mathcal{C}) \rightarrow \text{Span}(\mathcal{C})$ has a section;
- (v) for every diagram such as **(B)** in \mathcal{C} , given any span

$$\begin{array}{ccc}
 & D & \\
 d \swarrow & & \searrow c \\
 D_0 & & D'_0
 \end{array}$$

such that $d\alpha = d\beta f$ and $c\gamma = c\beta g$, there is a unique $\varphi: A \times_B C \rightarrow D$ such that

$$\varphi e_1 = \alpha, \quad \varphi e_2 = \gamma \quad \text{and} \quad d\varphi = d\gamma\pi_2, \quad c\varphi = c\alpha\pi_1. \quad (\mathbf{J})$$

If the above equivalent conditions hold, then the functors U_{12} , U_1 and V are also isomorphisms. Furthermore, any pregroupoid is associative.

Proof. (i) \Rightarrow (ii) follows by composing the inverse of U_{123} from (i) with the functor $U_3: \text{Grpd}(\mathcal{C}) \rightarrow \text{Cat}(\mathcal{C})$. For (ii) \Rightarrow (iii) we compose with $U_2: \text{Cat}(\mathcal{C}) \rightarrow \text{MG}(\mathcal{C})$. Let us prove (iii) \Rightarrow (iv).

Suppose that the functor U_1 has a section. Then any reflexive graph admits a canonical morphism m

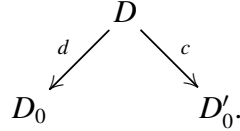
$$C_2 \xrightarrow{m} C_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} C_0$$

such that $me_1 = 1_{C_1} = me_2$, $dm = d\pi_2$ and $cm = c\pi_1$ as in the definition of a multiplicative graph. Furthermore, this morphism is natural, in the sense that, for any morphism $f = (f_1, f_0)$ of reflexive graphs, the diagram

$$\begin{array}{ccccc}
 C_2 & \xrightarrow{m} & C_1 & \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} & C_0 \\
 f_2 \downarrow & & f_1 \downarrow & & \downarrow f_0 \\
 C'_2 & \xrightarrow{m'} & C'_1 & \begin{array}{c} \xrightarrow{d'} \\ \xleftarrow{e'} \\ \xrightarrow{c'} \end{array} & C'_0
 \end{array} \quad (\mathbf{K})$$

with $f_2 = f_1 \times_{f_0} f_1$ commutes.

To prove that the functor V has a section, we have to construct a pregroupoid structure for any given span



Let us consider the reflexive graph

$$D \times_{D_0} D \times_{D'_0} D \begin{array}{c} \xrightarrow{c_2 p_2} \\ \xleftarrow{\Delta} \\ \xrightarrow{d_1 p_1} \end{array} D \quad (\mathbf{L})$$

(see diagram **(F)**) where an “element” of $D \times_{D_0} D \times_{D'_0} D$

$$\cdot \xleftarrow{x} \cdot \xrightarrow{y} \cdot \xleftarrow{z} \cdot$$

is viewed as an arrow y having domain x and codomain z . It is clearly reflexive, with $\Delta(x) = (x, x, x)$ being the identity on x . It is a multiplicative graph because the functor U_1 has a section. The desired pregroupoid structure p for (D, d, c) is obtained by the following procedure: given

$$\cdot \xleftarrow{x} \cdot \xrightarrow{y} \cdot \xleftarrow{z} \cdot$$

in $D \times_{D_0} D \times_{D'_0} D$, consider the pair of composable arrows

$$(\cdot \xrightarrow{x} \cdot \xleftarrow{x} \cdot \xrightarrow{y} \cdot, \cdot \xrightarrow{y} \cdot \xleftarrow{z} \cdot \xrightarrow{z} \cdot)$$

in the reflexive graph **(L)**. Since this reflexive graph is multiplicative, multiply in order to obtain

$$\cdot \xrightarrow{x} \cdot \xleftarrow{p(x,y,z)} \cdot \xrightarrow{z} \cdot$$

and project to the middle component.

The equalities $p(x, y, y) = x$ and $p(x, x, y) = y$ simply follow from the multiplicative identities $me_1 = 1_{C_1} = me_2$ of the multiplicative graph. Likewise, $dp(x, y, z) = dz$ and $cp(x, y, z) = cx$. This construction is functorial because the multiplication is natural.

Next we prove that, if V has a section, then the category \mathcal{C} satisfies Condition (v). Consider a diagram such as (C) above and a suitable span (d, c) . We have to construct a morphism $\varphi: A \times_B C \rightarrow D$ which satisfies the needed conditions, and prove that this φ is unique. To do so, we use the natural pregroupoid structure $p: D \times_{D_0} D \times_{D'_0} D \rightarrow D$. Since $d\alpha = d\beta f$, $c\gamma = c\beta g$ and $\alpha r = \beta = \gamma s$, there is an induced morphism

$$\langle \alpha\pi_1, \beta f\pi_1, \gamma\pi_2 \rangle: A \times_B C \rightarrow D \times_{D_0} D \times_{D'_0} D.$$

It assigns to any (a, c) with $f(a) = b = g(c)$ in $A \times_B C$ a triple

$$\cdot \xleftarrow{\alpha(a)} \cdot \xrightarrow{\beta(b)} \cdot \xleftarrow{\gamma(c)} \cdot$$

in $D \times_{D_0} D \times_{D'_0} D$. The desired morphism $\varphi: A \times_B C \rightarrow D$ is then obtained by taking its composition in the pregroupoid, i.e., $\varphi(a, c) = p(\alpha(a), \beta(b), \gamma(c))$ or

$$\varphi = p\langle \alpha\pi_1, \beta f\pi_1, \gamma\pi_2 \rangle.$$

This proves existence; the equalities $\varphi(a, b, s(b)) = \alpha(a)$ and $\varphi(r(b), b, c) = \gamma(c)$ follow from the properties of p , as do $d\varphi = d\gamma\pi_2$ and $c\varphi = c\alpha\pi_1$.

Now we show that the equalities (J) determine φ uniquely. Let us consider the span

$$\begin{array}{ccc} & A \times_B C & \\ \pi_2 \swarrow & & \searrow \pi_1 \\ C & & A \end{array}$$

with its induced pregroupoid structure

$$p: (A \times_B C) \times_C (A \times_B C) \times_A (A \times_B C) \rightarrow A \times_B C;$$

if the morphisms in this pregroupoid are viewed as arrows

$$a \xleftarrow{(a,c)} c$$

then the operation p takes a composable triple

$$a \xleftarrow{(a,c)} c \xrightarrow{(a',c)} a' \xleftarrow{(a',c')} c'$$

and sends it to

$$a \xleftarrow{(a,c')} c'$$

in $A \times_B C$. Note that this pregroupoid structure is unique, because the given span is a relation; in fact, its existence expresses the relation's difunctionality. Further note that it is a strong relation (cf. Theorem 2.8 below).

The morphism φ now gives rise to a morphism of pregroupoids, determined by the morphism of spans

$$\begin{array}{ccccc} C & \xleftarrow{\pi_2} & A \times_B C & \xrightarrow{\pi_1} & A \\ d\gamma \downarrow & & \downarrow \varphi & & \downarrow c\alpha \\ D_0 & \xleftarrow{d} & D & \xrightarrow{c} & D'_0 \end{array}$$

We write

$$\varphi' : (A \times_B C) \times_C (A \times_B C) \times_A (A \times_B C) \rightarrow D \times_{D_0} D \times_{D'_0} D$$

for the induced morphism to see that

$$\begin{aligned} \varphi(a, c) &= \varphi p(a \longleftarrow sf(a) \longrightarrow rg(c) \longleftarrow c) \\ &= p\varphi'(a \longleftarrow sf(a) \longrightarrow rg(c) \longleftarrow c) \\ &= p(\cdot \xleftarrow{\varphi e_1(a)} \cdot \xrightarrow{\varphi e_1 r(b)} \cdot \xleftarrow{\varphi e_2(c)} \cdot) \\ &= p(\cdot \xleftarrow{\alpha(a)} \cdot \xrightarrow{\beta(b)} \cdot \xleftarrow{\gamma(c)} \cdot) \\ &= p(\alpha(a), \beta(b), \gamma(c)) \end{aligned}$$

and φ is uniquely determined.

Next we prove that (v) implies Condition (i) in our theorem. Given a reflexive graph (\mathbf{D}) , a unique multiplication m satisfying (\mathbf{J}) , so

$$me_1 = 1_{C_1}, \quad me_2 = 1_{C_1} \quad \text{and} \quad dm = d\pi_2, \quad cm = c\pi_1,$$

is induced by the diagram

$$\begin{array}{ccccc} C_1 & \xleftarrow{d} & C_0 & \xrightarrow{c} & C_1 \\ & \searrow e & \downarrow e & \swarrow e & \\ & & C_1 & & \end{array}$$

together with the span (d, c) .

The naturality of m (see diagram **(K)**) follows from the uniqueness of the morphism induced by the diagram

$$\begin{array}{ccccc}
 C_1 & \xrightleftharpoons[d]{e} & C_0 & \xrightleftharpoons[c]{e} & C_1 \\
 & \searrow f_1 & \downarrow e' f_0 & \swarrow f_1 & \\
 & & C'_1 & &
 \end{array}$$

and the span (d', c') : indeed, both $f_1 m$ and $m' f_2$ qualify. This already gives us Condition (iii) in its strong form where U_1 is an isomorphism.

The associativity condition (needed for (ii)) follows from the uniqueness of the morphism induced by the diagram

$$\begin{array}{ccccc}
 C_2 & \xrightleftharpoons[\pi_2]{e_2} & C_1 & \xrightleftharpoons[\pi_1]{e_1} & C_2 : \\
 & \searrow m & \parallel & \swarrow m & \\
 & & C_1 & &
 \end{array}$$

indeed, both $m(1_{C_1} \times m)$ and $m(m \times 1_{C_1})$ satisfy the required conditions **(J)**, so they coincide.

The existence of inverses (needed for (i)) follows from the diagram

$$\begin{array}{ccccc}
 C_2 & \xrightleftharpoons[m]{e_2} & C_1 & \xrightleftharpoons[m]{e_1} & C_2 \\
 & \searrow \pi_2 & \parallel & \swarrow \pi_1 & \\
 & & C_1 & &
 \end{array}$$

as explained in [14].

To show that the functor V is an isomorphism, given a span (d, c) , we use the diagram

$$\begin{array}{ccccc}
 D_{d,c} & \xrightleftharpoons[c_2 p_2]{\Delta} & D & \xrightleftharpoons[d_1 p_1]{\Delta} & D_{d,c} \\
 & \searrow & \downarrow & \swarrow & \\
 & & D_{d,c} & &
 \end{array}$$

where $D_{d,c} = D \times_{D_0} D \times_{D'_0} D$ and $\Delta = \langle 1_D, 1_D, 1_D \rangle$ to prove uniqueness of its pregroupoid structure.

Finally, given a pregroupoid (\mathbf{I}) , its associativity follows by using (v) on the diagram

$$\begin{array}{ccccc}
 D_{d,c} & \xrightleftharpoons[c_2 p_2]{c_1 p_1} & D & \xrightleftharpoons[d_1 p_1]{d_2 p_2} & D_{d,c} \\
 & \searrow \Delta & \parallel & \swarrow \Delta & \\
 & p & & p & \\
 & & D & &
 \end{array}$$

because the morphisms

$$D_{d,c} \times_D D_{d,c} \rightarrow D$$

defined by sending (x, y, z, u, v) to $p(p(x, y, z), u, v)$ or to $p(x, y, p(z, u, v))$ both meet the requirements, so they must agree by the uniqueness in (v). \square

Observe that, in the case of finite limits, any one of the equivalent conditions of Theorem 2.2 is a characterisation for the notion of naturally Mal'tsev category introduced in [10]. Indeed, the Mal'tsev operation on an object X is determined by the diagram

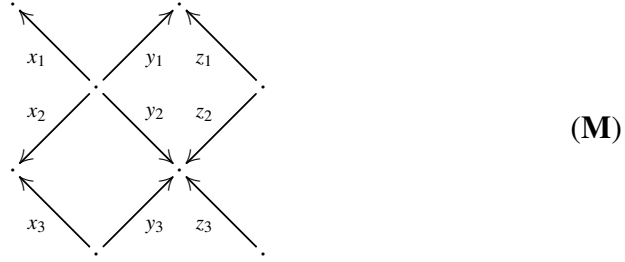
$$\begin{array}{ccccc}
 X \times X & \xrightleftharpoons[\langle 1_X, 1_X \rangle]{\pi_2} & X & \xrightleftharpoons[\langle 1_X, 1_X \rangle]{\pi_1} & X \times X \\
 & \searrow \pi_1 & \parallel & \swarrow \pi_2 & \\
 & & X & &
 \end{array}$$

together with the span $1 \leftarrow X \rightarrow 1$.

In the presence of coequalisers, when every span in \mathcal{C} is naturally endowed with a unique pregroupoid structure, there is an interchange law for composable strings valid in any pregroupoid in \mathcal{C} .

Proposition 2.3. *Let \mathcal{C} be a category with kernel pairs, split pullbacks and coequalisers satisfying the conditions (i)–(v). Consider a pregroupoid (\mathbf{I})*

in \mathcal{C} . Then for any configuration of the shape

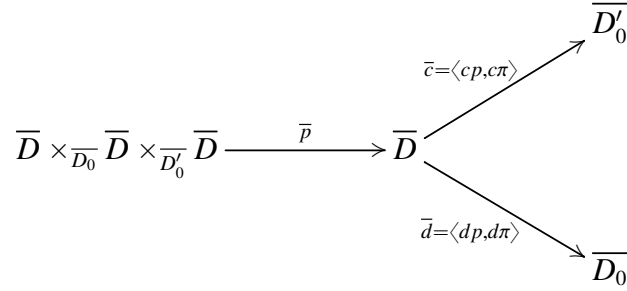


in this pregroupoid, the equality

$$\begin{aligned} p(p(x_1, x_2, x_3), p(y_1, y_2, y_3), p(z_1, z_2, z_3)) \\ = p(p(x_1, y_1, z_1), p(x_2, y_2, z_2), p(x_3, y_3, z_3)) \end{aligned} \quad (\mathbf{N})$$

holds.

Proof. It suffices to consider the pregroupoid in \mathcal{C} in which the configurations (M) are the composable triples, and then the equality will follow by naturality of the pregroupoid structures. This pregroupoid



is determined by the span $(\langle dp, d\pi \rangle, \langle cp, c\pi \rangle)$ where $\bar{D} = D \times_{D_0} D \times_{D'_0} D$,

$$\begin{array}{ccc} \bar{D}_0 = D_0 \times_Q D_0 & D_0 \xrightarrow{\text{Coeq}(dp, d\pi)} Q \\ \bar{D}'_0 = D'_0 \times_{Q'} D'_0 & D'_0 \xrightarrow{\text{Coeq}(cp, c\pi)} Q' \end{array}$$

and the middle projection $\pi = d_2 p_1 = c_1 p_2: D \times_{D_0} D \times_{D'_0} D \rightarrow D$ (diagram (F)) maps a composable triple (x_1, x_2, x_3) to x_2 . It is easily checked

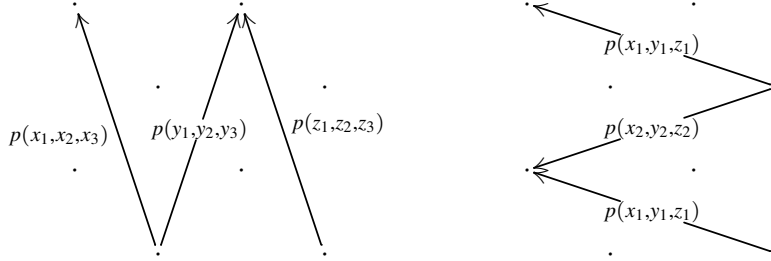


Figure 1: Vertical and horizontal composition

that the morphism \bar{p} which sends (\mathbf{M}) to its horizontal composite—the composable triple

$$(p(x_1, y_1, z_1), p(x_2, y_2, z_2), p(x_3, y_3, z_3))$$

in D , see Figure 1—determines a pregroupoid structure (hence, the unique one) on this span.

Furthermore, by naturality of pregroupoid structures, the morphism of spans

$$\begin{array}{ccccc} \bar{D}_0 & \xleftarrow{\langle dp, d\pi \rangle} & \bar{D} & \xrightarrow{\langle cp, c\pi \rangle} & \bar{D}'_0 \\ \downarrow & & \downarrow p & & \downarrow \\ D_0 & \xleftarrow{d} & D & \xrightarrow{c} & D'_0 \end{array}$$

induces a morphism $p' : \bar{D} \times_{\bar{D}_0} \bar{D} \times_{\bar{D}'_0} \bar{D} \rightarrow \bar{D}$ such that $pp' = p\bar{p}$, which gives us the required equality (\mathbf{N}) . Indeed, the induced morphism p' takes (\mathbf{M}) and sends it to its vertical composite—the composable triple

$$(p(x_1, x_2, x_3), p(y_1, y_2, y_3), p(z_1, z_2, z_3))$$

in D , see again Figure 1. □

Note that the equality (\mathbf{N}) is a partial version of the Mal'tsev operation p being *autonomous*, see [10].

2.4 Mal'tsev categories

Restricting Theorem 2.2 to the case where the morphisms d and c are jointly monomorphic we obtain the well known characterisation [3] for Mal'tsev

categories.

Theorem 2.5. *Let \mathcal{C} be a category with kernel pairs and split pullbacks. The following are equivalent:*

- (i') every reflexive relation is an equivalence relation;
- (ii') every reflexive relation is a preorder;
- (iii') every reflexive relation is transitive;
- (iv') every relation is difunctional;
- (v') for every diagram such as **(B)** in \mathcal{C} , given any relation

$$\begin{array}{ccc} & D & \\ d \swarrow & & \searrow c \\ D_0 & & D'_0 \end{array}$$

such that $d\alpha = d\beta f$ and $c\gamma = c\beta g$, there is a unique $\varphi: A \times_B C \rightarrow D$ such that

$$\varphi e_1 = \alpha, \quad \varphi e_2 = \gamma \quad \text{and} \quad d\varphi = d\gamma\pi_2, \quad c\varphi = c\alpha\pi_1.$$

Proof. By restricting to relations one easily adapts the proof of Theorem 2.2 to the present situation. \square

An important result on Mal'tsev categories is the following one, usually stated for finite limits [3]; it follows, for instance, from Theorem 3.1.

Theorem 2.6. *Let \mathcal{C} be a category with kernel pairs, split pullbacks and equalisers, satisfying the equivalent conditions of Theorem 2.5. Then the forgetful functor*

$$U_3: \text{Grpd}(\mathcal{C}) \rightarrow \text{Cat}(\mathcal{C})$$

is an isomorphism. \square

2.7 Weakly Mal'tsev categories

A category is said to be **weakly Mal'tsev** when it has split pullbacks and every induced pair of morphisms into the pullback (e_1, e_2) as in Diagram (A) above is jointly epimorphic [14].

Further restricting the conditions of Theorem 2.2 to the case where the morphisms d and c are jointly strongly monomorphic—and calling such a span a **strong relation** [8]—we obtain a characterisation of weakly Mal'tsev categories.

Theorem 2.8. *Let \mathcal{C} be a category with kernel pairs and split pullbacks. The following are equivalent:*

- (i'') every reflexive strong relation is an equivalence relation;
- (ii'') every reflexive strong relation is a preorder;
- (iii'') every reflexive strong relation is transitive;
- (iv'') every strong relation is difunctional;
- (v'') for every diagram such as (B) in \mathcal{C} , given any strong relation

$$\begin{array}{ccc} & D & \\ d \swarrow & & \searrow c \\ D_0 & & D'_0 \end{array}$$

such that $d\alpha = d\beta f$ and $c\gamma = c\beta g$, there is a unique $\varphi: A \times_B C \rightarrow D$ such that

$$\varphi e_1 = \alpha, \quad \varphi e_2 = \gamma \quad \text{and} \quad d\varphi = d\gamma\pi_2, \quad c\varphi = c\alpha\pi_1.$$

Proof. By restricting to strong relations one easily adapts the proof of Theorem 2.2 to the present situation. \square

Theorem 2.9. *Let \mathcal{C} be a category with kernel pairs, split pullbacks and equalisers. The following are equivalent:*

1. \mathcal{C} is a weakly Mal'tsev category;

2. \mathcal{C} satisfies the equivalent conditions of Theorem 2.8.

Proof. In the presence of equalisers, the weak Mal'tsev axiom is equivalent to Condition (iv'')—see [8]. \square

Mimicking the argument at the end of the proof of Theorem 2.2, it is easily seen that in a weakly Mal'tsev category, any internal pregroupoid is associative. The corresponding result for internal multiplicative graphs is treated in the following section.

3 Internal categories vs. internal groupoids

We prove that, in a weakly Mal'tsev category with kernel pairs and equalisers, internal categories are internal groupoids if and only if every preorder is an equivalence relation.

Theorem 3.1. *Let \mathcal{C} be a weakly Mal'tsev category with kernel pairs and equalisers. Then:*

1. *the forgetful functor*

$$U_2: \text{Cat}(\mathcal{C}) \rightarrow \text{MG}(\mathcal{C})$$

is an isomorphism;

2. *the forgetful functor*

$$U_3: \text{Grpd}(\mathcal{C}) \rightarrow \text{Cat}(\mathcal{C})$$

is an isomorphism if and only if every internal preorder in \mathcal{C} is an equivalence relation.

Part (1) of this result was already obtained in [14] where the definition of multiplicative graph does not include the conditions $dm = d\pi_2$ and $cm = c\pi_1$. Indeed, in this context they automatically hold. The proof of Part (2) depends on the following lemma.

Lemma 3.2. *Let \mathcal{C} be a weakly Mal'tsev category with equalisers. Given a category (\mathbf{E}) in \mathcal{C} , the morphisms*

$$\langle \pi_1, m \rangle: C_2 \rightarrow C_1 \times_c C_1 \quad \text{and} \quad \langle m, \pi_2 \rangle: C_2 \rightarrow C_1 \times_d C_1$$

are monomorphisms; this means that the multiplication is cancellable on both sides.

Proof. We shall prove $\langle \pi_1, m \rangle$ is a monomorphism. A similar argument shows the same for $\langle m, \pi_2 \rangle$.

First observe that the kernel pairs $C_1 \times_c C_1, C_1 \times_d C_1, C_2 \times_m C_2, C_2 \times_{\pi_1} C_2$ and $C_2 \times_{\pi_2} C_2$ exist because c, d, m, π_1 and π_2 are split epimorphisms. To prove that $\langle \pi_1, m \rangle$ is a monomorphism is the same as proving for every $x, y: Z \rightarrow C_2$ that

$$\left. \begin{array}{l} \pi_1 x = \pi_1 y \\ mx = my \end{array} \right\} \Rightarrow \pi_2 x = \pi_2 y.$$

Assuming that $\pi_1 x = \pi_1 y$ we have induced morphisms

$$\langle x, y \rangle \quad \text{and} \quad \langle e_2 \pi_2 x, e_2 \pi_2 y \rangle: Z \rightarrow C_2 \times_{\pi_1} C_2.$$

Indeed, $\pi_1 e_2 \pi_2 x = \pi_1 e_2 \pi_2 y$ as $\pi_1 e_2 \pi_2 = ec\pi_2 = ed\pi_1$. Considering the equaliser $(S, \langle s_1, s_2 \rangle)$ of the pair of morphisms

$$C_2 \times_{\pi_1} C_2 \rightrightarrows C_2 \xrightarrow{m} C_1,$$

and identifying $C_2 \times_{\pi_1} C_2$ with $C_1 \times_{c_0} (C_1 \times_c C_1)$ we obtain a strong relation

$$\begin{array}{ccc} & S & \\ s_1 \swarrow & & \searrow s_2 \\ C_1 & & C_1 \times_c C_1 \end{array}$$

which may be pictured as

$$\begin{array}{ccc} \cdot & \xleftarrow{x_1} & \cdot \\ \parallel & & \parallel \\ \cdot & \xleftarrow{x_2} & \cdot \\ \parallel & & \parallel \\ \cdot & \xleftarrow{y_1} & \cdot \\ & \xleftarrow{y_2} & \cdot \end{array}$$

with $x_1 = y_1$ and $(x_1 = y_1)S(x_2, y_2)$ if and only if $x_1 x_2 = y_1 y_2$.

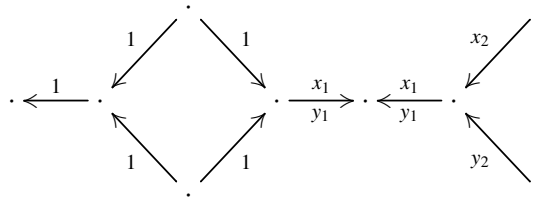
By Theorem 2.8, this relation, being a strong relation, is also difunctional and the argument used on page 103 of [3] also applies here to show that

$$\langle e_2\pi_2x, e_2\pi_2y \rangle = \langle s_1, s_2 \rangle \overline{pi\langle x, y \rangle},$$

where $p: SS^{-1}S \rightarrow S$ is obtained by difunctionality, $\overline{\langle x, y \rangle}: Z \rightarrow S$ is the factorisation of $\langle x, y \rangle$ through the equaliser (we are assuming that $mx = my$), and the morphism $i: S \rightarrow SS^{-1}S$, which sends $(x_1 = y_1)S(x_2, y_2)$ to

$$(1 = 1)S(1, 1)S^{-1}(x_1 = y_1)S(x_2, y_2),$$

may be pictured as follows.



This proves that $\langle e_2\pi_2x, e_2\pi_2y \rangle$ factors through the equaliser S , so we may conclude that

$$me_2\pi_2x = me_2\pi_2y,$$

or $\pi_2x = \pi_2y$ as desired. □

Proof of Theorem 3.1. If the functor U_3 is an isomorphism then in particular any preorder is an equivalence relation. For the converse, assume that every preorder is an equivalence relation (and every strong relation is difunctional). Given any category (\mathbf{E}) we shall prove that it is a groupoid. For this to happen it suffices that there is a morphism $t: C_1 \rightarrow C_1$ with $ct = d$ and $m\langle 1_{C_1}, t \rangle = ec$ (see, for instance, [14]).

By Lemma 3.2 we already know that the morphisms $\langle m, \pi_2 \rangle$ and $\langle \pi_1, m \rangle$ are monomorphisms. This means that the reflexive graph

$$C_2 \begin{array}{c} \xrightarrow{m} \\ \xleftarrow{e_1} \\ \xrightarrow{\pi_1} \end{array} C_1$$

is a reflexive relation, and since it is transitive—by assumption it is a multiplicative graph—it is an equivalence relation. Hence there is a morphism

$$\tau = \langle m, q \rangle: C_2 \rightarrow C_2$$

such that $m\tau = \pi_1$. Now $t = qe_2$ is the needed morphism $C_1 \rightarrow C_1$. Indeed $dm = cq$, because $\langle m, q \rangle$ is a morphism into the pullback C_2 , so that

$$ct = cq e_2 = dme_2 = d;$$

furthermore,

$$m\langle 1_{C_1}, t \rangle = m\langle me_2, qe_2 \rangle = m\langle m, q \rangle e_2 = \pi_1 e_2 = ec,$$

which completes the proof. \square

Remark 3.3. In general, a category can be weakly Mal'tsev without Condition (2) of Theorem 3.1 holding. For instance, in the category of commutative monoids with cancellation, the relation \leq on the monoid of natural numbers \mathbb{N} is a preorder which is not an equivalence relation.

Remark 3.4. It is possible for a category to satisfy both Condition (1) and Condition (2) of Theorem 3.1 without being Mal'tsev: see the following section.

4 The varietal case

When we restrict to varieties, the condition “every internal preorder is an equivalence relation” singled out in part (2) of Theorem 3.1 is known to be equivalent to the variety being n -permutable for some n . We explain how to prove this when passing via a characterisation of n -permutability due to Hagemann.

4.1 Finitary quasivarieties

Just like a variety of algebras is determined by certain identities between terms, a quasivariety also admits quasi-identities in its definition, i.e., ex-

pressions of the form

$$\left. \begin{array}{l} v_1(x_1, \dots, x_k) = w_1(x_1, \dots, x_k) \\ \vdots \\ v_n(x_1, \dots, x_k) = w_n(x_1, \dots, x_k) \end{array} \right\} \Rightarrow v_{n+1}(x_1, \dots, x_k) = w_{n+1}(x_1, \dots, x_k)$$

—see, for instance, [13] for more details. It is well known that any quasivariety may be obtained as a regular epi-reflective subcategory of a variety, and more generally the sub-quasivarieties of a quasivariety correspond to its regular epi-reflective subcategories. In particular, sub-quasivarieties are closed under subobjects.

4.2 n -Permutable varieties

The following equivalent conditions due to Hagemann [6] describe what it means for a variety to be **n -permutable**. (Recall that 2-permutability is just the Mal'tsev property and a regular category which is 3-permutable is called **Goursat** [2].)

Proposition 4.3. *For a finitary quasivariety \mathcal{V} and a natural number $n \geq 2$, the following are equivalent:*

1. *for any two equivalence relations R and S on an object A , we have $(R, S)_n = (S, R)_n$;*
2. *there exist $n - 1$ terms w_1, \dots, w_{n-1} in \mathcal{V} such that*

$$\begin{cases} w_1(x, z, z) = x \\ w_i(x, x, z) = w_{i+1}(x, z, z) \\ w_{n-1}(x, x, z) = z; \end{cases}$$

3. *for any reflexive relation R , we have $R^{-1} \subset R^{n-1}$.*

In fact, this result is valid in regular categories, as shown in [9]. Also the following result is known [5]:

Proposition 4.4. *For a finitary quasivariety \mathcal{V} , the following are equivalent:*

1. in \mathcal{V} , every internal preorder is an equivalence relation;
2. \mathcal{V} is n -permutable for some n .

Proof. By Proposition 4.3, if Condition (2) holds then for every reflexive relation R in \mathcal{V} we have that $R^{-1} \subset R^{n-1}$. Now if R is transitive then $R^{n-1} \subset R$, so that $R^{-1} \subset R$, which means that R is symmetric.

To prove the converse, suppose that every internal preorder in \mathcal{V} is an equivalence relation. Let A be the free algebra on the set $\{x, z\}$ and let R be the reflexive relation on A consisting of all pairs

$$(w(x, x, z), w(x, z, z))$$

for w a ternary term. Then the pair (x, z) is in R . By assumption, the transitive closure \bar{R} of R is also symmetric, hence contains the pair (z, x) . This means that (z, x) may be expressed through a chain of finite length in R . More precisely, there exists a natural number n and ternary terms w_1, \dots, w_{n-1} such that

$$\begin{aligned} z &= w_{n-1}(x, x, z)Rw_{n-1}(x, z, z) = w_{n-2}(x, x, z)Rw_{n-2}(z, z, x) = \\ &\dots = w_1(x, x, z)Rw_1(x, z, z) = x. \end{aligned}$$

By Proposition 4.3 this means that \mathcal{V} is n -permutable. □

Remark 4.5. This of course raises the question whether a similar result would hold in a purely categorical context. It seems difficult to obtain the number n which occurs in Condition (2) of Proposition 4.4 without using free algebra structures, which are not available in general. And indeed, a counterexample exists [15]. On the other hand, the implication (2) \Rightarrow (1) admits a proof which is *almost* categorical—but depends on a characterisation of n -permutability for regular categories as in Condition (3) of Proposition 4.3. This is the subject of the articles [18] and [9].

Remark 4.6. Through Theorem 3.1, this result implies that in an n -permutable weakly Mal'tsev variety, every internal category is an internal groupoid. On the other hand, using different techniques, and without assuming the weak Mal'tsev condition, Rodelo recently proved that in any n -permutable variety, internal categories and internal groupoids coincide [17]. Whence

the question: how different are n -permutable varieties from weakly Mal'tsev ones? The only thing we know about this so far is that the two conditions together are not strong enough to imply that the variety is Mal'tsev (see Example 4.9). Further note that the conditions (IC1) and (IC2) considered in the paper [17], that is, $dm = d\pi_2$ and $cm = c\pi_1$ in **(E)**, come for free in a weakly Mal'tsev category. Outside this context, however, it is no longer clear whether or not they will always hold.

4.7 Constructing weakly Mal'tsev quasivarieties

A 3-permutable (quasi)variety always contains a canonical subvariety which is also weakly Mal'tsev. This allows us to construct examples of weakly Mal'tsev categories which are 3-permutable but not 2-permutable—thus we see, in particular, that in a weakly Mal'tsev category \mathcal{C} , categories and groupoids may coincide, even without \mathcal{C} being Mal'tsev.

Proposition 4.8. *Let \mathcal{V} be a Goursat finitary quasivariety with w_1, w_2 the terms obtained using Proposition 4.3. Then the sub-quasivariety \mathcal{W} of \mathcal{V} defined by the quasi-identity*

$$\left. \begin{aligned} w_1(x, a, b) = w_2(a, b, c) = w_1(x', a, b) \\ w_2(b, c, x) = w_1(a, b, c) = w_2(b, c, x') \end{aligned} \right\} \Rightarrow x = x'$$

is weakly Mal'tsev.

Proof. For any split pullback

$$\begin{array}{ccc} A \times_B C & \begin{array}{c} \xrightarrow{p_2} \\ \xleftarrow{e_2} \end{array} & C \\ \begin{array}{c} \uparrow e_1 \\ \downarrow p_1 \end{array} & & \begin{array}{c} \uparrow s \\ \downarrow g \end{array} \\ A & \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{r} \end{array} & B \end{array}$$

we have to show that e_1 and e_2 are jointly epic: any two $\varphi, \varphi': A \times_B C \rightarrow D$ such that

$$\varphi e_1 = \alpha = \varphi' e_1 \quad \text{and} \quad \varphi e_2 = \gamma = \varphi' e_2$$

must coincide. We use the notations from Diagram (C) and consider $a \in A$ and $c \in C$ with $f(a) = b = g(c)$. Then

$$\begin{aligned}
w_1(\varphi(a, c), \alpha(a), \beta(b)) &= w_1(\varphi(a, c), \varphi(a, s(b)), \varphi(r(b), s(b))) \\
&= \varphi(w_1(a, a, r(b)), w_1(c, s(b), s(b))) \\
&= \varphi(w_2(a, r(b), r(b)), c) \\
&= \varphi(w_2(a, r(b), r(b)), w_2(s(b), s(b), c)) \\
&= w_2(\varphi(a, s(b)), \varphi(r(b), s(b)), \varphi(r(b), c)) \\
&= w_2(\alpha(a), \beta(b), \gamma(c))
\end{aligned}$$

and

$$\begin{aligned}
w_2(\beta(b), \gamma(c), \varphi(a, c)) &= w_2(\varphi(r(b), s(b)), \varphi(r(b), c), \varphi(a, c)) \\
&= \varphi(w_2(r(b), r(b), a), w_2(s(b), c, c)) \\
&= \varphi(a, w_1(s(b), s(b), c)) \\
&= \varphi(w_1(a, r(b), r(b)), w_1(s(b), s(b), c)) \\
&= w_1(\varphi(a, s(b)), \varphi(r(b), s(b)), \varphi(r(b), c)) \\
&= w_1(\alpha(a), \beta(b), \gamma(c)),
\end{aligned}$$

which proves that

$$w_1(\varphi(a, c), \alpha(a), \beta(b)) = w_2(\alpha(a), \beta(b), \gamma(c)) = w_1(\varphi'(a, c), \alpha(a), \beta(b))$$

and

$$w_2(\beta(b), \gamma(c), \varphi(a, c)) = w_1(\alpha(a), \beta(b), \gamma(c)) = w_2(\beta(b), \gamma(c), \varphi'(a, c)),$$

since both expressions only depend on $\alpha(a)$, $\beta(b)$ and $\gamma(c)$. Hence by definition of \mathscr{W} we have that $\varphi(a, c) = \varphi'(a, c)$ for all $(a, c) \in A \times_B C$. \square

We could actually leave out the middle equalities (the ones not involving x and x') in the quasi-identity and still obtain a weakly Mal'tsev quasivariety, but the result of this procedure would be too small to include the following example, so we are not sure that it wouldn't force the quasivariety to become Mal'tsev.

a	1	2	1	2	1	2	1	2
b	1	1	2	2	1	1	2	2
c	1	1	1	1	2	2	2	2
x	1	2	2	-	1	1	1	2

Table 1: x is uniquely determined by a, b and c in A

a	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3			
b	1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3
c	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3
x	1	2	3	2	-	-	3	-	-	1	1	3	1	2	3	3	3	-	1	2	1	2	-	2	1	2	3

Table 2: x is uniquely determined by a, b and c in B

Example 4.9. The example due to Mitschke [16] of a category which is Goursat but not Mal'tsev may be modified using Proposition 4.8 to yield an example of a category which is Goursat and weakly Mal'tsev but not Mal'tsev. In fact, Proposition 4.8 makes it possible to construct such examples ad libitum.

Let the variety \mathcal{V} consist of **implication algebras**, i.e., (I, \cdot) which satisfy

$$\begin{cases} (xy)x = x \\ (xy)y = (yx)x \\ x(yz) = y(xz) \end{cases}$$

where we write $x \cdot y = xy$. It is proved in [6, 16] that \mathcal{V} is Goursat, and this is easily checked using Proposition 4.3 as witnessed by the terms $w_1(x, y, z) = (zy)x$ and $w_2(x, y, z) = (xy)z$. The further quasi-identity

$$\left. \begin{aligned} (ba)x = (ab)c = (ba)x' \\ (bc)x = (cb)a = (bc)x' \end{aligned} \right\} \Rightarrow x = x'$$

determines a weakly Mal'tsev sub-quasivariety \mathcal{W} of \mathcal{V} by Proposition 4.8. This quasivariety certainly stays Goursat, and the counterexample given in the paper [16] still works to prove that \mathcal{W} is not Mal'tsev.

Indeed, the implication algebras $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ with re-

spective multiplication tables

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

also belong to the quasivariety \mathscr{W} : given any choice of a , b and c , the system of equations

$$\begin{cases} (ba)x = (ab)c \\ (bc)x = (cb)a \end{cases}$$

either has no solution or just one, as pictured in Table 1 for the algebra A and in Table 2 for B .

To see that the quasivariety \mathscr{W} is not Mal'tsev, it now suffices to consider the homomorphisms $f, g: B \rightarrow A$ defined respectively by

$$f(1) = f(2) = 1, \quad f(3) = 2$$

and

$$g(1) = g(3) = 1, \quad g(2) = 2.$$

It is easy to check that the respective kernel relations R and S of f and g do not commute: RS contains the element $(3, 2)$, but not $(2, 3)$, which is in SR .

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Nelson Martins Ferreira

Departamento de Matemática, Escola Superior de Tecnologia e Gestão
Centro para o Desenvolvimento Rápido e Sustentado do Produto
Instituto Politécnico de Leiria, 2411–901 Leiria, Portugal
martins.ferreira@ipleiria.pt

Tim Van der Linden

CMUC, Universidade de Coimbra, 3001–454 Coimbra, Portugal
and
Institut de Recherche en Mathématique et Physique
Université catholique de Louvain
chemin du cyclotron 2 bte L7.01.02, 1348 Louvain-la-Neuve, Belgium
tim.vanderlinden@uclouvain.be

SUFFICIENT COHESION OVER ATOMIC TOPOSES

by *Matias MENNI*

Résumé. Soit (\mathcal{D}, J_{at}) un site atomique et $j : \mathbf{Sh}(\mathcal{D}, J_{at}) \rightarrow \widehat{\mathcal{D}}$ le topos des faisceaux associé. Tout foncteur $\phi : \mathcal{C} \rightarrow \mathcal{D}$ induit un morphisme géométrique $\widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$ et, en prenant le produit fibré le long de j , un morphisme géométrique $q : \mathcal{F} \rightarrow \mathbf{Sh}(\mathcal{D}, J_{at})$. Nous donnons une condition suffisante sur ϕ pour que q satisfasse le Nullstellensatz et la Cohésion Suffisante au sens de la Cohésion Axiomatique. Ceci est motivé par les exemples où \mathcal{D}^{op} est une catégorie d’extensions finies d’un corps.

Abstract. Let (\mathcal{D}, J_{at}) be an atomic site and $j : \mathbf{Sh}(\mathcal{D}, J_{at}) \rightarrow \widehat{\mathcal{D}}$ be the associated sheaf topos. Any functor $\phi : \mathcal{C} \rightarrow \mathcal{D}$ induces a geometric morphism $\widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$ and, by pulling-back along j , a geometric morphism $q : \mathcal{F} \rightarrow \mathbf{Sh}(\mathcal{D}, J_{at})$. We give a sufficient condition on ϕ for q to satisfy the Nullstellensatz and Sufficient Cohesion in the sense of Axiomatic Cohesion. This is motivated by the examples where \mathcal{D}^{op} is a category of finite field extensions.

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1. Introduction and outline

The first paragraph of Section II in [13] explains that the contrast of cohesion with non-cohesion (expressed by a geometric morphism $p : \mathcal{E} \rightarrow \mathcal{S}$ with certain special properties) can be made relative, so that \mathcal{S} may be an ‘arbitrary’ topos. The inverted commas should be taken seriously because reasonable hypotheses on the geometric morphism p imply strong restrictions on the base \mathcal{S} . Having said this, the base is not forced to be the category **Set** of sets and functions. As an example, it is proposed loc. cit. that in the case

of algebraic geometry the base topos \mathcal{S} may be usefully taken as the Galois topos of Barr-atomic sheaves on finite extensions of the ground field. What does ‘usefully’ mean here? To give a concrete idea let \mathcal{E} be the (Gros) Zariski topos of a field k . If k is algebraically closed, the canonical geometric morphism $\mathcal{E} \rightarrow \mathbf{Set}$ satisfies certain simple intuitive axioms (formalized in Definitions 1.1 and 1.3 below). These axioms do not hold if k is not algebraically closed, but may be restored by changing the base as suggested.

The purpose of the present paper is to give a detailed construction of sufficiently cohesive pre-cohesive toposes over Galois bases. We recall some of the basic definitions and results but the reader is assumed to be familiar with [13]. (See also [12, 9].) For general background on topos theory see [16, 7] and for atomic toposes in particular see also [3].

Let \mathcal{E} and \mathcal{S} be cartesian closed extensive categories.

Definition 1.1. The category \mathcal{E} is called *pre-cohesive (relative to \mathcal{S})* if it is equipped with a string of adjoint functors

$$\begin{array}{c} \mathcal{E} \\ \downarrow p_! \quad \uparrow p^* \quad \downarrow p_* \quad \uparrow p^! \\ \mathcal{S} \end{array}$$

with $p_! \dashv p^* \dashv p_* \dashv p^!$ and such that:

1. $p^* : \mathcal{S} \rightarrow \mathcal{E}$ is full and faithful.
2. $p_! : \mathcal{E} \rightarrow \mathcal{S}$ preserves finite products.
3. (Nullstellensatz) The canonical natural transformation $\theta : p_* \rightarrow p_!$ is (pointwise) epi.

For brevity we will say that $p : \mathcal{E} \rightarrow \mathcal{S}$ is pre-cohesive. The notation is devised to be consistent with that for geometric morphisms. Indeed, if \mathcal{E} and \mathcal{S} are toposes then the functors above determine a geometric morphism $p : \mathcal{E} \rightarrow \mathcal{S}$ with direct image p_* . On the other hand, if $p : \mathcal{E} \rightarrow \mathcal{S}$ is a geometric morphism between toposes then we call p *pre-cohesive* if the adjunction $p^* \dashv p_*$ extends to one $p_! \dashv p^* \dashv p_* \dashv p^!$ making \mathcal{E} pre-cohesive over \mathcal{S} .

Definition 1.2. A pre-cohesive $p : \mathcal{E} \rightarrow \mathcal{S}$ is called *cohesive* if the canonical natural $p_!(X^{p^*W}) \rightarrow (p_!X)^W$ is an iso for all X in \mathcal{E} and W in \mathcal{S} . (This is the ‘continuity’ property in Definition 2 in [13].)

We still do not fully understand the Continuity property defining cohesive categories and for this reason we introduce and concentrate on pre-cohesive ones. It is relevant to stress that most of the results in [13] hold for pre-cohesive p ; Theorem 1 loc. cit. being the most important exception.

Let $p : \mathcal{E} \rightarrow \mathcal{S}$ be pre-cohesive. An object X in \mathcal{E} is called *connected* if $p_! X = 1$. An object Y in \mathcal{E} is called *contractible* if Y^A is connected for all A .

Definition 1.3. The pre-cohesive $p : \mathcal{E} \rightarrow \mathcal{S}$ is called *sufficiently cohesive* if for every X in \mathcal{E} there exists a monic $X \rightarrow Y$ with Y contractible. (We may also say that p satisfies *Sufficient Cohesion*.)

Useful intuition about sufficiently cohesive categories is gained by contrasting them with an opposing class of pre-cohesive categories.

Definition 1.4. The pre-cohesive p is a *quality type* if $\theta : p_* \rightarrow p_!$ is an iso. (See Definition 1 in [13].)

In other words, p is a quality type if the (full) reflective subcategory $p^* : \mathcal{S} \rightarrow \mathcal{E}$ is a quintessential localization in the sense of [6]. Quality types and sufficiently cohesive categories are contrasting in the precise sense given by Proposition 3 in [13]: if $p : \mathcal{E} \rightarrow \mathcal{S}$ is both sufficiently cohesive and a quality type, then \mathcal{S} is inconsistent. (Although stated for cohesive categories, it is clear from the proof that it also holds for pre-cohesive ones.) Loosely speaking, Sufficient Cohesion positively ensures that \mathcal{E} and \mathcal{S} are decidedly different. In particular, assuming that $0 \rightarrow 1$ is not an iso in \mathcal{S} , Sufficient Cohesion implies that $p_* : \mathcal{E} \rightarrow \mathcal{S}$ cannot be an equivalence.

There are many examples of sufficiently cohesive pre-cohesive toposes over \mathbf{Set} , including the topos of simplicial sets and the Zariski toposes determined by algebraically closed fields. As already mentioned in the first paragraph, the main contribution of the present paper is the detailed construction of a class of sufficiently cohesive pre-cohesive $p : \mathcal{E} \rightarrow \mathcal{S}$ over toposes \mathcal{S} different from \mathbf{Set} , namely the Galois toposes of (non algebraically closed) perfect fields. The construction will make evident what is the connection between the Nullstellensatz condition in Definition 1.1 and Hilbert's classical result. The reader will see that each of these geometric morphisms $p : \mathcal{E} \rightarrow \mathcal{S}$ is induced by the inclusion of the category of finite extensions of a given field into a category of finitely presented algebras over the same

field. It is then reasonable to expect that the same examples can be more directly constructed using a characterization of the morphisms of sites that induce sufficiently cohesive pre-cohesive geometric morphisms; but since we do not have such a characterization at present, we take a more indirect route using some results from [8] which studies the Nullstellensatz in the context of connected and locally connected geometric morphisms.

Notice that any string of adjoint functors $p_! \dashv p^* \dashv p_* : \mathcal{E} \rightarrow \mathcal{S}$ with fully faithful $p^* : \mathcal{E} \rightarrow \mathcal{S}$ determines a canonical natural $\theta : p_* \rightarrow p_!$ and then it is fair to say that the string of adjoints satisfies the Nullstellensatz if θ is epi. We will need to use this generality for such a string of adjoints given by a connected essential geometric morphism $p : \mathcal{E} \rightarrow \mathcal{S}$. (Recall that p is *connected* if p^* is full and faithful and it is *essential* if p^* has a left adjoint, typically denoted by $p_! : \mathcal{E} \rightarrow \mathcal{S}$.)

It is also relevant to briefly explain the relation with local connectedness. Recall that a geometric morphism $p : \mathcal{E} \rightarrow \mathcal{S}$ is *locally connected* if p^* has an \mathcal{S} -indexed left adjoint $p_! : \mathcal{E} \rightarrow \mathcal{S}$. Such geometric morphisms are, of course, essential. Theorem 3.4 and Proposition 3.5 in [8] imply that if \mathcal{S} has a natural number object (nno) and $p : \mathcal{E} \rightarrow \mathcal{S}$ is bounded, connected, locally connected and satisfies the Nullstellensatz then p is pre-cohesive. (Connected locally connected geometric morphisms satisfying the Nullstellensatz are called ‘punctually locally connected’ in [8] but we will stick to the terminology of [13].) Reorganizing the hypotheses of these results we obtain the following fact.

Corollary 1.5. *If \mathcal{S} has a nno and $p : \mathcal{E} \rightarrow \mathcal{S}$ is bounded, connected and locally connected, then p is pre-cohesive if and only if p satisfies the Nullstellensatz.*

In the case that $\mathcal{S} = \mathbf{Set}$ there is of course a stronger result because $p_!$ is automatically indexed. Recall that a site (\mathcal{C}, J) is *locally connected* if every covering sieve is connected (as a subcategory of the corresponding slice). Such a site is called *connected* if \mathcal{C} has a terminal object.

Proposition 1.6. *If $p : \mathcal{E} \rightarrow \mathbf{Set}$ is bounded then the following are equivalent:*

1. p is pre-cohesive,

2. p is connected, essential and satisfies the Nullstellensatz,
3. \mathcal{E} has a connected and locally connected site of definition (\mathcal{C}, J) such that every object of \mathcal{C} has a point.

Proof. By Corollary 1.5 above and Proposition 1.4 in [8]. □

We now outline the main results in the paper. In Section 2.2 we prove the following characterization of sufficiently cohesive pre-cohesive toposes over **Set**.

Corollary 1.7. *Let (\mathcal{C}, J) be a connected and locally connected site such that every object has a point and let $p : \mathbf{Sh}(\mathcal{C}, J) \rightarrow \mathbf{Set}$ be the induced pre-cohesive geometric morphism. Then p is sufficiently cohesive if and only if there is an object in \mathcal{C} with (at least) two distinct points.*

There is a precedent to both results above. In the last paragraph of p. 421 of [11] Lawvere states that it follows from a remark in Grothendieck’s 1983 Pursuing Stacks that product preservation of $p_!$ and Sufficient Cohesion “will be satisfied by a topos of M -actions if the generic individual I ($= M$ acting on itself) has at least two distinct points”.

A little trick will allow us to apply Corollary 1.7 to prove Sufficient Cohesion over other bases; so it remains to explain how to build pre-cohesive toposes over bases that are not **Set**. In order to sketch the main ideas fix a geometric morphism $p : \mathcal{E} \rightarrow \mathcal{S}$, a Lawvere-Tierney topology j in \mathcal{S} and consider the following pullback

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i} & \mathcal{E} \\ q \downarrow & & \downarrow p \\ \mathcal{S}_j & \xrightarrow{j} & \mathcal{S} \end{array}$$

of toposes. We are interested in conditions on p and j implying that q is pre-cohesive. Assume for simplicity that all toposes involved are Grothendieck and that p is connected and locally connected. Then q is also connected and locally connected by Theorem C3.3.15 in [7]. Corollary 1.5 leads us to consider conditions on p and j implying that q satisfies the Nullstellensatz.

The pullback stability result for locally connected geometric morphisms also shows that the (Beck-Chevalley) natural transformation $p^*j_* \rightarrow i_*q^*$ is

an iso. Taking left adjoints we obtain an iso $j^*p_! \rightarrow q_!i^*$; pre-composing with i_* we get another iso $j^*p_!i_* \rightarrow q_!i^*i_*$ and we can use the counit of $i^* \dashv i_*$ to get the canonical iso $j^*p_!i_* \rightarrow q_!i^*i_* \rightarrow q_!$ that appears in the next result.

Lemma 1.8. *Given the pullback diagram above, the following diagram*

$$\begin{array}{ccccc}
 j^*p_*i_* & \xrightarrow{=} & j^*j_*q_* & \xrightarrow{\text{counit}} & q_* \\
 \downarrow j^*\theta_{i_*} & & & & \downarrow \theta' \\
 j^*p_!i_* & \xrightarrow{\cong} & q_!i^*i_* & \xrightarrow{\cong} & q_!
 \end{array}$$

commutes, where $\theta' : q_ \rightarrow q_!$ is the natural transformation associated to the connected essential q .*

This result is probably folklore but we give a detailed proof in Section 3.1.

As suggested in [15], we denote the image of the map $\theta_X : p_*X \rightarrow p_!X$ by $HX \rightarrow p_!X$. This is an “invariant of objects in the bigger category, recorded in the smaller”.

Definition 1.9. Let us say that p satisfies the Nullstellensatz *relative to j* if for every X in \mathcal{E} , the mono $HX \rightarrow p_!X$ is j -dense.

Combining the above we obtain the following fact.

Lemma 1.10. *If, in the pullback diagram above, $p : \mathcal{E} \rightarrow \mathcal{S}$ satisfies the Nullstellensatz relative to j then $q : \mathcal{F} \rightarrow \mathcal{S}_j$ is pre-cohesive.*

Proof. By hypothesis, the image $H(i_*X) \rightarrow p_!(i_*X)$ of the canonical map $\theta_{i_*X} : p_*(i_*X) \rightarrow p_!(i_*X)$ is j -dense; so the canonical $\theta' : q_*X \rightarrow q_!X$ is epi by Lemma 1.8. \square

In the examples that motivate this work, \mathcal{S} is the topos $\widehat{\mathcal{D}}$ of presheaves on a category \mathcal{D} that can be equipped with the atomic topology (inducing a Lawvere-Tierney topology j in $\widehat{\mathcal{D}}$) and p is induced by a functor $\phi : \mathcal{C} \rightarrow \mathcal{D}$ that has a fully faithful right adjoint ι . Also, the fact that p satisfies the Nullstellensatz relative to j naturally follows from a more concrete related condition that holds for the adjunction $\phi \dashv \iota$.

Definition 1.11. A full reflective subcategory $\iota : \mathcal{D} \rightarrow \mathcal{C}$ is said to satisfy the *primitive Nullstellensatz* if for every C in \mathcal{C} there exists a map $\iota D \rightarrow C$ for some D in \mathcal{D} .

For example, if \mathcal{C} has a terminal object then the inclusion $\iota : 1 \rightarrow \mathcal{C}$ of the terminal object is reflective and it satisfies the primitive Nullstellensatz if and only if every object of \mathcal{C} has a point. In contrast notice that if \mathcal{C} has initial object then the inclusion $\iota : 1 \rightarrow \mathcal{C}$ of the initial object trivially satisfies that for every C in \mathcal{C} there exists a map $\iota D \rightarrow C$ for some D in \mathcal{D} , but the subcategory is not reflexive (unless \mathcal{D} is trivial). In other words, the requirement of a left adjoint to ι excludes the situation just described from the examples of the primitive Nullstellensatz.

We now discuss how the primitive Nullstellensatz relates to Hilbert’s classical result. Lawvere suggests that the relation is better explained by the conjunction of two facts: “traditionally, the heart of Hilbert’s result is the existence of points, and that is merely a consequence of Zorn’s Lemma”; the other fact is that that fields k have, as rings, the property that finitely-generated k -algebras that happen to be fields are in fact finitely-generated k -modules. (See also Tholen’s analysis in [17], which is particularly well suited for our purposes.)

Fix a field k . A classical commutative algebra textbook may formulate the two facts above as follows.

Lemma 1.12. *Let A be a k -algebra.*

1. *If A is not trivial then it has at least one maximal ideal.*
2. *If A is finitely generated as a k -algebra and $M \subseteq A$ is a maximal ideal then $k \rightarrow A \rightarrow A/M$ is a finite algebraic extension.*

Proof. The first item is proved in Theorem 1.3 in [2] as a “standard application of Zorn’s lemma”. The second item is Corollary 7.10 in [2] and it is referred to as the ‘weak’ version of Hilbert’s Nullstellensatz. \square

A k -algebra is called *connected* if it has exactly two idempotents. Let \mathbf{Con} be the category of finitely presented and connected k -algebras. Denote the full subcategory of separable extensions of k by $\mathbf{Ext} \rightarrow \mathbf{Con}$.

Lemma 1.13. *The full inclusion $\mathbf{Ext} \rightarrow \mathbf{Con}$ has a right adjoint.*

Proof. This does not seem to be very well-known so we recall the proof taken from Proposition I, §4, 6.5 in [5]. Let A in \mathbf{Con} and choose a maximal ideal $M \subseteq A$. Since A is connected every separable sub(k -)algebra $K \rightarrow A$ is a field and $[K : k] \leq [A/M : k]$. That is, the degrees of all possible such K are bounded; so the filtered system of such $K \subseteq A$ must have a maximum. \square

We recall this, of course, because the primitive Nullstellensatz holds as explained below.

Example 1.14. Assume that k is perfect to avoid complications with separable extensions. Lemma 1.12 implies that for any A in \mathbf{Con} there exists a map $A \rightarrow A/M$ with A/M in the subcategory $\mathbf{Ext} \rightarrow \mathbf{Con}$. This means that the full reflective $\mathbf{Ext}^{\text{op}} \rightarrow \mathbf{Con}^{\text{op}}$ satisfies the primitive Nullstellensatz. If k is algebraically closed then this says that every object of \mathbf{Con}^{op} has a point.

Fix a small category \mathcal{C} and a full reflective subcategory $\iota : \mathcal{D} \rightarrow \mathcal{C}$ with reflector $\phi : \mathcal{C} \rightarrow \mathcal{D}$. The geometric morphism $\phi : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$ induced by the reflector is essential, connected and local and so induces a string of functors

$$\begin{array}{c} \widehat{\mathcal{C}} \\ \downarrow \phi_! \quad \uparrow \phi^* \quad \downarrow \phi_* \quad \uparrow \phi^! \\ \widehat{\mathcal{D}} \end{array}$$

with $\phi_! \dashv \phi^* \dashv \phi_* \dashv \phi^!$ and $\phi^* : \widehat{\mathcal{D}} \rightarrow \widehat{\mathcal{C}}$ fully faithful. That is, a structure analogous to that in Definition 1.1 except that $\phi_!$ need not preserve products and the Nullstellensatz may not hold.

Assume now that \mathcal{D} satisfies the (right) Ore condition so that it can be equipped with the atomic topology J_{at} . Denote the resulting Lawvere-Tierney topology on $\widehat{\mathcal{D}}$ by j_{at} . In Section 3.2 we prove the following.

Lemma 1.15. *If $\phi \dashv \iota : \mathcal{D} \rightarrow \mathcal{C}$ satisfies the primitive Nullstellensatz then the geometric morphism $\phi : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$ satisfies the Nullstellensatz relative to j_{at} .*

Lemmas 1.10 and 1.15 imply the first part of the next result. The second part will be proved in Section 3.2.

Proposition 1.16. *Let the following diagram be a pullback*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i} & \widehat{\mathcal{C}} \\ q \downarrow & & \downarrow \phi \\ \mathbf{Sh}(\mathcal{D}, J_{at}) & \xrightarrow{j} & \widehat{\mathcal{D}} \end{array}$$

of toposes. If $\phi : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$ is locally connected and $\phi \dashv \iota : \mathcal{D} \rightarrow \mathcal{C}$ satisfies the primitive Nullstellensatz then $q : \mathcal{F} \rightarrow \mathbf{Sh}(\mathcal{D}, J_{at})$ is pre-cohesive. If, moreover, \mathcal{C} has a terminal object and some object with two distinct points then q is sufficiently cohesive.

In Section 4 we discuss how to apply Proposition 1.16 to Example 1.14 and we also give a presentation of the theory classified by \mathcal{F} in the case of $k = \mathbb{R}$.

2. Sufficient Cohesion

Here we characterize sufficiently cohesive pre-cohesive toposes $\mathcal{E} \rightarrow \mathbf{Set}$. The strategy to analyse Sufficient Cohesion is suggested by the following result.

Proposition 2.1. *Let $p : \mathcal{E} \rightarrow \mathcal{S}$ be a pre-cohesive topos. Then p is sufficiently cohesive if and only if the subobject classifier of \mathcal{E} is connected (i.e. $p_! \Omega = 1$).*

Proof. Simply observe that the proof of Proposition 4 in [13] does not need the Continuity condition. \square

Let $p : \mathcal{E} \rightarrow \mathcal{S}$ be an essential geometric morphism. As usual we denote the left adjoint to p^* by $p_! : \mathcal{E} \rightarrow \mathcal{S}$, the subobject classifier of \mathcal{E} by Ω and the top and bottom elements of its canonical lattice structure by $\top, \perp : 1 \rightarrow \Omega$.

Lemma 2.2. *If $p_! : \mathcal{E} \rightarrow \mathcal{S}$ preserves finite products then $p_! \Omega = 1$ if and only if the maps $p_! \top, p_! \perp : p_! 1 \rightarrow p_! \Omega$ are equal.*

Proof. One direction is trivial (and does not require that $p_!$ preserves finite products). On the other hand, if $p_!$ preserves products then $p_! \Omega$ is equipped with a lattice structure with $p_! \top$ and $p_! \perp$ as top and bottom elements respectively. Since they are equal, $p_! \Omega = 1$. \square

So the consideration of Sufficient Cohesion naturally leads to essential geometric morphisms whose leftmost adjoint preserves finite products. For example, recall that a small category \mathcal{D} is *sifted* if and only if the colimit functor $\mathbf{Set}^{\mathcal{D}} \rightarrow \mathbf{Set}$ preserves finite products [1] and that this holds if and only if \mathcal{D} is nonempty and the diagonal $\mathcal{D} \rightarrow \mathcal{D} \times \mathcal{D}$ is final. So, if we let $p : \widehat{\mathcal{C}} \rightarrow \mathbf{Set}$ be the (essential) canonical geometric morphism then $p_!$ preserves finite products if and only if \mathcal{C} is cosifted. To characterize those such p that satisfy $p_! \Omega = 1$ the following terminology will be useful.

Definition 2.3. A cospan $A \rightarrow B \leftarrow C$ in a category is said to be *disjoint* if it cannot be completed to a commutative square.

The next source of examples will also be relevant. (See the proof of Proposition 1.6(iii) in [8] for details.)

Lemma 2.4. *If \mathcal{C} has a terminal object and every object of \mathcal{C} has a point then \mathcal{C} is cosifted.*

2.1 The case of presheaf toposes

Let \mathcal{C} be a small category and $p : \widehat{\mathcal{C}} \rightarrow \mathbf{Set}$ the canonical (essential) geometric morphism. Let us recall a description of $p_! : \mathcal{E} \rightarrow \mathbf{Set}$.

Fix a presheaf P in $\widehat{\mathcal{C}}$. A cospan $C \xrightarrow{\sigma_l} U \xleftarrow{\sigma_r} C'$ is said to *connect* the elements $x \in PC$ and $x' \in PC'$ if there is a $y \in PU$ such that $x = y \cdot \sigma_l$ and $x' = y \cdot \sigma_r$. In this case we may denote the situation by the following diagram

$$x \longleftarrow y \longrightarrow x'$$

$$C \xrightarrow{\sigma_l} U \xleftarrow{\sigma_r} C'$$

or simply write $x \sigma x'$.

A *path* from C to C' is a sequence of cospans $\sigma_1, \sigma_2, \dots, \sigma_n$ as below

$$C_0 \xrightarrow{\sigma_{1,l}} U_1 \xleftarrow{\sigma_{1,r}} C_1 \xrightarrow{\sigma_{2,l}} U_2 \xleftarrow{\sigma_{2,r}} C_2 \cdots C_{n-1} \xrightarrow{\sigma_{n,l}} U_n \xleftarrow{\sigma_{n,r}} C_n$$

with $C_0 = C$ and $C_n = C'$. Such a path *connects* elements $x \in PC$ and $x' \in PC'$ if there exists a sequence $(x_i \in PC_i \mid 0 \leq i \leq n)$ of elements such

that $x_0 = x \in PC$, $x_n = x' \in PC'$ and for every $1 \leq i \leq n$, $x_{i-1}\sigma_i x_i$. We say that $x \in PC$ and $x' \in PC'$ are *connectable* if there is a path from C to C' that connects x and x' . An element in p_1P is given by a ‘tensor’ $x \otimes C$ with $x \in PC$. Two such tensors $x \otimes C$ and $x' \otimes C'$ are equal if and only if they are connectable.

We now concentrate on the set $p_1\Omega$ whose elements are of the form $S \otimes C$ with S a sieve on C . Let M_C be the maximal sieve on C and Z_C be the empty sieve on C . We will sometimes write M instead of M_C and similarly for Z_C .

Lemma 2.5. *A cospan $C \xrightarrow{\sigma_l} U \xleftarrow{\sigma_r} C'$ is disjoint if and only if it connects $M \in \Omega C$ and $Z \in \Omega C'$.*

Proof. If the cospan is disjoint, the sieve on U generated by σ_l witnesses the fact that the cospan connects M_C and $Z_{C'}$. Conversely, if S is a sieve on U such that $S \cdot \sigma_l = M_C$ and $S \cdot \sigma_r = Z_{C'}$ then $\sigma_l \in S$ and there is no map $h : D \rightarrow C'$ such that $\sigma_r h$ is in S . In particular, there is no h such that $\sigma_r h$ factors through σ_l . So the cospan in the statement is disjoint. \square

A path $\sigma_1, \dots, \sigma_n$ as above is called *singular at i* (for some $1 \leq i \leq n$) if the cospan

$$C_{i-1} \xrightarrow{\sigma_{i,l}} U_i \xleftarrow{\sigma_{i,r}} C_i$$

is disjoint. We say that the path is *singular* if it is singular at some i .

Lemma 2.6. *If the cospan $C \xrightarrow{\sigma_l} U \xleftarrow{\sigma_r} C'$ connects a non-empty sieve $S \in \Omega C$ and the empty sieve $Z_{C'} \in \Omega C'$ then there exists a singular path from C to C' .*

Proof. By hypothesis there is a sieve T on U as in the diagram below

$$S \longleftarrow T \longrightarrow Z$$

$$C \xrightarrow{\sigma_l} U \xleftarrow{\sigma_r} C'$$

and, since S is non-empty, T is also non-empty. Let $\tau : D \rightarrow U$ a map in T . Since, $T \cdot \sigma_r = Z$, the cospan (τ, σ_r) is disjoint and so, the path below

$$C \xrightarrow{\sigma_l} U \xleftarrow{\tau} D \xrightarrow{\sigma_r} U \xleftarrow{\sigma_r} C'$$

from C to C' is singular. \square

The main technical fact of the section is the following.

Lemma 2.7. *For any C and C' , M_C is connectable with $Z_{C'}$ if and only if there exists a singular path from C to C' .*

Proof. Consider a path $\sigma_1, \dots, \sigma_n$ from C to C' . Assume first that this path is singular at i . By Lemma 2.5, the cospan σ_i connects the maximal sieve on C_{i-1} with the empty sieve on C_i . Now observe that any path connects the maximal sieves on its ‘extremes’, and it also connects the empty sieves on its extremes. In particular, the path $\sigma_1, \dots, \sigma_{i-1}$ connects M_C with $M_{C_{i-1}}$ and the path $\sigma_{i+1}, \dots, \sigma_n$ connects Z_{C_i} with $Z_{C'}$. So the whole path $\sigma_1, \dots, \sigma_n$ connects M_C and $Z_{C'}$.

For the converse assume that the path $\sigma_1, \dots, \sigma_n$ connects M_C and $Z_{C'}$. Then there exist sieves S_0, \dots, S_n such that $S_0 = M$, $S_n = Z$ and for every $1 \leq i \leq n$, $S_{i-1} \sigma_i S_i$. So there exists a k such $S_k = Z$ and S_{k-1} is non-empty. By Lemma 2.6 there exists a singular path from C_{k-1} to C_k . Of course, this path can be extended to (a singular) one from C to C' . \square

If C is an object of \mathcal{C} and we let the terminal object 1 in $\widehat{\mathcal{C}}$ be such that $1C = \{*\}$ then the morphisms $p_1 \top, p_1 \perp: p_1 1 \rightarrow p_1 \Omega$ map $* \otimes C$ to $M_C \otimes C$ and $Z_C \otimes C$ respectively.

Proposition 2.8. *If \mathcal{C} is connected then the maps $p_1 \top, p_1 \perp: p_1 1 \rightarrow p_1 \Omega$ are equal if and only if \mathcal{C} contains a disjoint cospan.*

Proof. As \mathcal{C} is connected, there is an object C in \mathcal{C} and also: \mathcal{C} has a disjoint cospan if and only if there is a singular path from C to C . Now, the maps $p_1 \top, p_1 \perp: 1 \rightarrow p_1 \Omega$ are equal if and only if $M_C \otimes C = Z_C \otimes C$. By Lemma 2.7, this holds if and only if there exists a singular path from C to C . \square

Since cosifted categories are connected the next result follows.

Corollary 2.9. *Let \mathcal{C} be cosifted and $p: \widehat{\mathcal{C}} \rightarrow \mathbf{Set}$ the canonical geometric morphism. Then $p_1 \Omega = 1$ if and only if \mathcal{C} contains a disjoint cospan.*

We can now characterize the sufficiently cohesive pre-cohesive presheaf toposes. For this it is convenient to state the presheaf version of Proposition 1.6 and, in fact, it is worth sketching a direct proof.

Proposition 2.10. *Let \mathcal{C} be a small category whose idempotents split. The canonical $p : \widehat{\mathcal{C}} \rightarrow \mathbf{Set}$ is pre-cohesive if and only if \mathcal{C} has a terminal object and every object of \mathcal{C} has a point.*

Proof. The canonical $p : \widehat{\mathcal{C}} \rightarrow \mathbf{Set}$ is essential and $p_!C = 1$ for every representable C in $\widehat{\mathcal{C}}$. Example C3.6.3(b) in [7] shows that p is local if and only if \mathcal{C} has a terminal object. In this case, of course, p is connected. So we can assume that \mathcal{C} has a point and then $p_*X = \widehat{\mathcal{C}}(1, X) = X1$ for every X in $\widehat{\mathcal{C}}$. If the Nullstellensatz holds then $\mathcal{C}(1, C) = p_*C \rightarrow p_!C = 1$ is epi and so every object of \mathcal{C} has a point. For the converse assume that every object of \mathcal{C} has a point and let P in $\widehat{\mathcal{C}}$. Recall that an element of $p_!P$ may be described as a ‘tensor’ $x \otimes C$ with $x \in PC$. The natural transformation $\theta : p_*P \rightarrow p_!P$ sends each $y \in P1$ to the tensor $y \otimes 1$. Since every C in \mathcal{C} has a point, any tensor $x \otimes C$ is equal to one of the form $y \otimes 1$. Finally, $p_!$ preserves finite products by Lemma 2.4. \square

If \mathcal{C} has a terminal object and every object has a point then the existence of a disjoint cospan is equivalent to the existence of an object with two distinct points, so the next result follows from Corollary 2.9 and Proposition 2.10.

Corollary 2.11. *Let \mathcal{C} be a small category whose idempotents split and let $p : \widehat{\mathcal{C}} \rightarrow \mathbf{Set}$ be pre-cohesive. Then p is sufficiently cohesive if and only if there is an object in \mathcal{C} with two distinct points.*

2.2 The case of sheaves

Proposition 1.3 in [8] proves a characterization of the bounded locally connected $p : \mathcal{E} \rightarrow \mathbf{Set}$ such that $p_!$ preserves finite products. In this section we characterize, among these, those which satisfy $p_!\Omega = 1$. Some key ingredients may be isolated as basic facts about dense subtoposes and we treat them first.

Recall that a subtopos $i : \mathcal{F} \rightarrow \mathcal{E}$ is *dense* if $i_* : \mathcal{F} \rightarrow \mathcal{E}$ preserves the initial object 0 (see A4.5.20 in [7]). For any subtopos $i : \mathcal{F} \rightarrow \mathcal{E}$ consider the split mono $i_*\Omega_{\mathcal{F}} \rightarrow \Omega_{\mathcal{E}}$ presenting the subobject classifier of \mathcal{F} as a retract

of that of \mathcal{E} . The diagram on the left below

$$\begin{array}{ccc} i_*1 & \xrightarrow{i_*\top} & i_*\Omega_{\mathcal{F}} \\ \cong \downarrow & & \downarrow \\ 1 & \xrightarrow{\top} & \Omega_{\mathcal{E}} \end{array} \qquad \begin{array}{ccc} i_*1 & \xrightarrow{i_*\perp} & i_*\Omega_{\mathcal{F}} \\ \cong \downarrow & & \downarrow \\ 1 & \xrightarrow{\perp} & \Omega_{\mathcal{E}} \end{array}$$

always commutes. On the other hand, the square on the right above commutes if and only if the subtopos is dense.

Lemma 2.12. *Let $p : \mathcal{E} \rightarrow \mathcal{S}$ be an essential geometric morphism. If the geometric $i : \mathcal{F} \rightarrow \mathcal{E}$ is a dense subtopos then the maps on the left below*

$$p_!i_*1 \begin{array}{c} \xrightarrow{p_!(i_*\top)} \\ \xrightarrow{p_!(i_*\perp)} \end{array} p_!(i_*\Omega_{\mathcal{F}}) \qquad p_!1 \begin{array}{c} \xrightarrow{p_!\top} \\ \xrightarrow{p_!\perp} \end{array} p_!\Omega_{\mathcal{E}}$$

are equal if and only if the ones on the right above are.

Proof. Since $i : \mathcal{F} \rightarrow \mathcal{E}$ is dense, the map $\perp : 1 \rightarrow \Omega_{\mathcal{E}}$ factors through the retract $i_*\Omega_{\mathcal{F}} \rightarrow \Omega_{\mathcal{E}}$. Then the diagram below

$$\begin{array}{ccccc} & & p_!\top & & \\ & \searrow & \curvearrowright & \searrow & \\ p_!1 & \xrightarrow{p_!(i_*\top)} & & \xrightarrow{\quad} & p_!\Omega_{\mathcal{E}} \\ & \xrightarrow{p_!(i_*\perp)} & p_!(i_*\Omega_{\mathcal{F}}) & \longrightarrow & \\ & & p_!\perp & & \end{array}$$

commutes and the result follows because $p_!(i_*\Omega_{\mathcal{F}}) \rightarrow p_!\Omega_{\mathcal{E}}$ is (split) mono. \square

This is applied in the next result where the subtopos is dense as a result of a stronger condition.

Lemma 2.13. *Consider a diagram*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i} & \mathcal{E} \\ & \searrow q & \downarrow p \\ & & \mathcal{S} \end{array}$$

with i an inclusion. If p^* factors through i_* (in the sense that the canonical $p^* \rightarrow i_*i^*p^* = i_*q^*$ is an iso) then i is a dense subtopos and q is essential. If, moreover, $p_!$ preserves finite products then so does $q_!$. Also, in this case, $p_!\Omega_{\mathcal{E}} = 1$ if and only if $q_!\Omega_{\mathcal{F}} = 1$.

Proof. Start with the iso $p^* \rightarrow i_*q^*$. Since p^* and q^* preserve 0 then so does i_* . It is straightforward to check that the functor $p_!i_* : \mathcal{F} \rightarrow \mathcal{S}$ is left adjoint to $q^* : \mathcal{S} \rightarrow \mathcal{F}$ so q is essential and we can define $q_! = p_!i_* : \mathcal{F} \rightarrow \mathcal{S}$. Clearly, if $p_!$ preserves finite products then so does $q_!$. It remains to prove that $p_!\Omega_{\mathcal{E}} = 1$ if and only if $q_!\Omega_{\mathcal{F}}$. By Lemma 2.2 it is enough to prove that $p_!\top = p_!\perp : 1 \rightarrow p_!\Omega_{\mathcal{E}}$ if and only if $q_!\top = q_!\perp : 1 \rightarrow q_!\Omega_{\mathcal{F}}$. Since $q_! = p_!i_*$ the result follows from Lemma 2.12. \square

One of the equivalences in Proposition 1.3 of [8] states that if the canonical $p : \mathcal{E} \rightarrow \mathbf{Set}$ is bounded and locally connected then, $p_!$ preserves finite products if and only if \mathcal{E} has a locally connected site of definition (\mathcal{C}, J) with \mathcal{C} cosifted.

Proposition 2.14. *Let (\mathcal{C}, J) be a locally connected site with \mathcal{C} cosifted and $q : \mathbf{Sh}(\mathcal{C}, J) \rightarrow \mathbf{Set}$ be the induced geometric morphism. Then $q_!\Omega = 1$ if and only if \mathcal{C} contains a disjoint cospan.*

Proof. We have a diagram

$$\begin{array}{ccc} \mathcal{F} = \mathbf{Sh}(\mathcal{C}, J) & \xrightarrow{i} & \widehat{\mathcal{C}} = \mathcal{E} \\ & \searrow q & \downarrow p \\ & & \mathbf{Set} \end{array}$$

where p and q are locally connected, $p_!$ and $q_!$ preserve finite products and $i : \mathbf{Sh}(\mathcal{C}, J) \rightarrow \widehat{\mathcal{C}}$ is a subtopos. In the proof of Proposition 1.3 in [8] it is observed that if a site (\mathcal{C}, J) is locally connected then constant presheaves on \mathcal{C} are J -sheaves. That is, $p^* : \mathbf{Set} \rightarrow \widehat{\mathcal{C}}$ factors through the embedding $\mathbf{Sh}(\mathcal{C}, J) \rightarrow \widehat{\mathcal{C}}$, so Lemma 2.13 applies. Therefore $q_!\Omega_{\mathcal{F}} = 1$ if and only if $p_!\Omega_{\mathcal{E}} = 1$. The result follows from Corollary 2.9. \square

Corollary 1.7 follows from Proposition 2.14 and Lemma 2.4.

3. The Nullstellensatz

In Section 3.1 we prove Lemma 1.8 and then the proof of Lemma 1.10 will be complete. In Section 3.2 we show Lemma 1.15 and complete the proof of Proposition 1.16.

3.1 Proof of Lemma 1.8

As already mentioned in Section 1, this result is probably folklore. It should follow from 2-categorical generalities about morphisms of adjunctions, but I have failed to find the necessary machinery in the material I have access to, so I give here a simple minded proof. I try to keep the notation in Section 2 of [8].

Let $F \dashv R : \mathcal{E} \rightarrow \mathcal{S}$ and denote its unit and counit by η and ε respectively. In parallel, consider another adjunction $F' \dashv R' : \mathcal{E}' \rightarrow \mathcal{S}'$ with unit and counit denoted by η' and ε' . Fix also a commutative diagram

$$\begin{array}{ccc} \mathcal{E}' & \xrightarrow{i_*} & \mathcal{E} \\ R' \downarrow & & \downarrow R \\ \mathcal{S}' & \xrightarrow{j_*} & \mathcal{S} \end{array}$$

with i_* and j_* having left adjoints denoted by i^* and j^* respectively. We denote the unit and counit of $i^* \dashv i_*$ by u and c , and those of $j^* \dashv j_*$ by u' and c' .

Because left adjoints are essentially unique there exists a canonical isomorphism $\varphi : i^*F \rightarrow F'j^*$ such that the following diagram

$$\begin{array}{ccccc} Id & \xrightarrow{\eta} & RF & \xrightarrow{Ru_F} & Ri_*i^*F \\ u' \downarrow & & & & \downarrow Ri_*\varphi \\ j_*j^* & \xrightarrow{j_*\eta'_{j^*}} & j_*R'F'j^* & \xrightarrow{=} & Ri_*F'j^* \end{array}$$

commutes. (The top map is the unit of the composite adjunction $i^*F \dashv Ri_*$.)

The transposition of φ is the composite

$$Fj_* \xrightarrow{u_{Fj_*}} i_*i^*Fj_* \xrightarrow{i_*\varphi_{j_*}} i_*F'j^*j_* \xrightarrow{i_*F'c'} i_*F'$$

and will be denoted by $\zeta : Fj_* \rightarrow i_*F'$. We call it the *Beck-Chevalley natural transformation*. Trivial calculations show the following.

Lemma 3.1. *The diagrams*

$$\begin{array}{ccc}
 i^*F & \xrightarrow{i^*Fu'} & i^*Fj_*j^* \xrightarrow{i^*\zeta_{j^*}} i^*i_*F'j^* \\
 & \searrow \varphi & \downarrow c_{F'j^*} \\
 & & F'j^*
 \end{array}
 \qquad
 \begin{array}{ccc}
 j_* & \xrightarrow{\eta} & RFj_* \xrightarrow{R\zeta} Ri_*F \\
 & \searrow j_*\eta' & \downarrow = \\
 & & j_*RF
 \end{array}$$

commute.

Assume from now on that F has a left adjoint $L : \mathcal{E} \rightarrow \mathcal{S}$ and denote the unit of $L \dashv F$ by $\alpha : Id \rightarrow FL$.

Lemma 3.2. *If i_* is full and faithful then the following diagram*

$$\begin{array}{ccccccc}
 i_*i^*i_* & \xrightarrow{i^*i_*\alpha_{i_*}} & & & i_*i^*FLi_* & & \\
 i_*c \downarrow & & & & \downarrow i_*\varphi_{Li_*} & & \\
 i_* & \xrightarrow{\alpha_{i_*}} & FLi_* & \xrightarrow{Fu'_{Li_*}} & Fj_*j^*Li_* & \xrightarrow{\zeta_{j^*Li_*}} & i_*F'j^*Li_*
 \end{array}$$

commutes.

Proof. The transposition of the top-right map is

$$i^*i_*i^*i_* \xrightarrow{c_{i^*i_*}} i^*i_* \xrightarrow{i^*\alpha_{i_*}} i^*FLi_* \xrightarrow{\varphi_{Li_*}} F'j^*Li_*$$

while that of the left-bottom one is

$$i^*i_*i^*i_* \xrightarrow{i^*i_*c} i^*i_* \xrightarrow{i^*\alpha_{i_*}} i^*FLi_* \xrightarrow{\varphi_{Li_*}} F'j^*Li_*$$

by Lemma 3.1. But $c_{i^*i_*} = i^*i_*c : i^*i_*i^*i_* \rightarrow i^*i_*$ because $c : i^*i_* \rightarrow Id$ is an iso by hypothesis. \square

We say that the *Beck-Chevalley condition* holds if $\zeta : Fj_* \rightarrow i_*F'$ is an iso. (See A4.1.16 in [7].)

Lemma 3.3. *Assume the Beck-Chevalley condition holds and that i_* , j_* and F are full and faithful. Then F' is full and faithful and has a left adjoint defined by $L' = j^*Li_* : \mathcal{E} \rightarrow \mathcal{S}$.*

Proof. First calculate:

$$\mathcal{E}'(F'X, F'Y) \cong \mathcal{E}(i_*F'X, i_*F'Y) \cong \mathcal{E}(Fj_*X, Fj_*Y) \cong \mathcal{S}'(X, Y)$$

to show that F' is full and faithful. To prove that $L' \dashv F'$ notice that:

$$\mathcal{S}'(L'X, S) \cong \mathcal{E}(i_*X, Fj_*S) \cong \mathcal{E}(i_*X, i_*F'S)$$

by adjointness and Beck-Chevalley. So $\mathcal{S}'(LX, S) \cong \mathcal{E}'(X, F'S)$ because i_* is full and faithful. \square

Assume from now on that the hypotheses of Lemma 3.3 hold and that $L' : \mathcal{E}' \rightarrow \mathcal{S}$ is defined as in that statement. Moreover, let α' denote the unit of $L' \dashv F'$.

Lemma 3.4. *The composition*

$$Id \xrightarrow{c^{-1}} i^*i_* \xrightarrow{i^*\alpha_{i_*}} i^*FLi_* \xrightarrow{\varphi_{Li_*}} F'j^*Li_* = F'L'$$

equals the unit $\alpha' : Id \rightarrow F'L'$ of $L' \dashv F'$.

Proof. If we chase the identity $L' \rightarrow L'$ in the proof of Lemma 3.3 then we obtain that the unit of $L' \dashv F'$ is the top-right composition in the diagram below:

$$\begin{array}{c} Id \xrightarrow{c^{-1}} i^*i_* \xrightarrow{i^*\alpha_{i_*}} i^*FLi_* \xrightarrow{i^*Fu_{Li_*}'} i^*Fj_*j^*Li_* \xrightarrow{i^*\zeta_{j^*Li_*}} i^*i_*F'j^*Li_* \\ \searrow \varphi_{Li_*} \qquad \qquad \qquad \downarrow c \\ \qquad \qquad \qquad F'j^*Li_* \end{array}$$

and the triangle commutes by Lemma 3.1. \square

The units α and α' may be related as follows.

Lemma 3.5. *The following diagram*

$$\begin{array}{ccccc}
 i_* & \xrightarrow{i_*\alpha'} & i_*F'L' & \xrightarrow{\zeta_{L'}^{-1}} & Fj_*L' \\
 \alpha_{i_*} \downarrow & & & & \downarrow = \\
 FLi_* & \xrightarrow{Fu'_{Li_*}} & Fj_*j^*Li_* & &
 \end{array}$$

commutes.

Proof. Post-composing with $\zeta_{L'}$ and replacing α' with its expression given in Lemma 3.4 the statement is equivalent to the commutativity of the diagram

$$\begin{array}{ccccc}
 i_* & \xrightarrow{i_*c^{-1}} & i_*i^*i_* & \xrightarrow{i_*i^*\alpha_{i_*}} & i_*i^*FLi_* \\
 \alpha_{i_*} \downarrow & & & & \downarrow i_*\varphi_{Li_*} \\
 FLi_* & \xrightarrow{Fu'_{Li_*}} & Fj_*j^*Li_* & \xrightarrow{\zeta_{j^*Li_*}} & i_*F'j^*Li_* = i_*F'L'
 \end{array}$$

but pre-composing with $i_*c : i_*i^*i_* \rightarrow i_*$ this is equivalent to Lemma 3.2. \square

Following [8] define $\theta = (\eta_L)^{-1}(R\alpha) : R \rightarrow L$ and $\theta' : R' \rightarrow L'$ analogously.

Lemma 3.6. *The diagram*

$$\begin{array}{ccccc}
 j^*Ri_* & \xrightarrow{=} & j^*j_*R' & \xrightarrow{c'_{R'}} & R' \\
 j^*\theta_{i_*} \downarrow & & & & \downarrow \theta' \\
 j^*Li_* & \xrightarrow{=} & & & L'
 \end{array}$$

commutes.

Proof. Start from the top-right and calculate:

$$\begin{array}{ccccc}
 j^*Ri_* & \xrightarrow{=} & j^*j_*R' & \xrightarrow{c'_{R'}} & R' \\
 j^*Ri_*\alpha' \downarrow & & j^*j_*R'\alpha' \downarrow & & \downarrow R'\alpha' \\
 j^*Ri_*F'L' & \xrightarrow{=} & j^*j_*R'F'L' & \xrightarrow{c'} & R'F'L' \\
 j^*R(\zeta_{L'})^{-1} \downarrow & & \downarrow j^*j_*(\eta'_{L'})^{-1} & & \downarrow (\eta'_{L'})^{-1} \\
 j^*RFj_*L' & \xrightarrow{j^*(\eta_{j_*L'})^{-1}} & j^*j_*L' & \xrightarrow{c_{L'}} & L'
 \end{array}$$

where the bottom-left square commutes by Lemma 3.1. Now observe that, by Lemma 3.5, the left-edge equals the composition

$$j^* Ri_* \xrightarrow{j^* R\alpha_{i_*}} j^* RFLi_* \xrightarrow{j^* RFu'_{Li_*}} j^* RFj_*L'$$

which, followed by the bottom edge, equals $j^*\theta_{i_*}$. \square

To complete the proof of Lemma 1.8 just observe that the pullback diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i} & \mathcal{E} \\ q \downarrow & & \downarrow p \\ \mathcal{S}_j & \xrightarrow{j} & \mathcal{S} \end{array}$$

discussed there satisfies all the hypotheses used in this section: we have already mentioned that, by Theorem C3.3.15 in [7], q is connected and locally connected and the square is Beck-Chevalley; also, i is a subtopos by Example A4.15.14(e) loc. cit.

3.2 Proof of Proposition 1.16

Here we prove Lemma 1.15 and Proposition 1.16. Fix small categories \mathcal{C} and \mathcal{D} .

Definition 3.7. A functor $\iota : \mathcal{D} \rightarrow \mathcal{C}$ is said to satisfy the (right) Ore condition if for every C in \mathcal{C} and diagram as on the left below

$$\begin{array}{ccc} & C & \\ & \downarrow f & \\ \iota D_1 & \xrightarrow{\iota g} & \iota D_0 \end{array} \quad \begin{array}{ccc} \iota D_2 & \xrightarrow{h} & C \\ \iota f' \downarrow & & \downarrow f \\ \iota D_1 & \xrightarrow{\iota g} & \iota D_0 \end{array}$$

in \mathcal{C} , there exists a map $f' : D_2 \rightarrow D_1$ in \mathcal{D} and a map $h : \iota D_2 \rightarrow C$ in \mathcal{C} such that the diagram on the right above commutes.

Clearly, a category \mathcal{D} satisfies the right Ore condition in the usual sense if and only if the identity functor $\mathcal{D} \rightarrow \mathcal{D}$ does so in the sense of Definition 3.7. We now relate this condition to the one defining the primitive Nullstellensatz (Definition 1.11).

Lemma 3.8. *If $\iota : \mathcal{D} \rightarrow \mathcal{C}$ is full and satisfies that for every C in \mathcal{C} there is a map $\iota D \rightarrow C$ for some D in \mathcal{D} then the first item below:*

1. \mathcal{D} satisfies the Ore condition in the usual sense,
2. ι satisfies the Ore condition in the sense of Definition 3.7,

implies the second. If, moreover, ι is faithful then the converse holds.

Proof. Consider a diagram as on the left below

$$\begin{array}{ccc}
 & & C \\
 & & \downarrow f \\
 \iota D_1 & \xrightarrow{\iota g} & \iota D_0
 \end{array}
 \qquad
 \begin{array}{ccc}
 \iota D & \xrightarrow{h} & C \\
 & \searrow t & \downarrow f \\
 & & \iota D_0
 \end{array}$$

in \mathcal{D} . By hypothesis there is a map $h : \iota D \rightarrow C$ for some D and, because ι is full, there is a map $t : D \rightarrow D_0$ in \mathcal{D} such that the diagram on the right above commutes. As \mathcal{D} satisfies the Ore condition, there is a diagram as on the left below

$$\begin{array}{ccc}
 D_2 & \xrightarrow{w} & D \\
 t' \downarrow & & \downarrow t \\
 D_1 & \xrightarrow{g} & D_0
 \end{array}
 \qquad
 \begin{array}{ccc}
 \iota D_2 & \xrightarrow{\iota w} & \iota D & \xrightarrow{h} & C \\
 \iota t' \downarrow & & \searrow t & & \downarrow f \\
 \iota D_1 & \xrightarrow{\iota g} & \iota D_0 & &
 \end{array}$$

in \mathcal{D} . The diagram on the right above shows that ι satisfies the Ore condition.

For the converse consider a cospan $g : D \rightarrow E \leftarrow D' : g'$ in \mathcal{D} . As ι satisfies the Ore condition there is an $f' : D_2 \rightarrow D$ in \mathcal{D} and an $h : \iota D_2 \rightarrow \iota D'$ in \mathcal{C} such that the diagram on the left below

$$\begin{array}{ccc}
 \iota D_2 & \xrightarrow{h} & \iota D' \\
 \iota f' \downarrow & & \downarrow \iota g' \\
 \iota D & \xrightarrow{\iota g} & \iota E
 \end{array}
 \qquad
 \begin{array}{ccc}
 D_2 & \xrightarrow{h'} & D' \\
 f' \downarrow & & \downarrow f \\
 D & \xrightarrow{g} & E
 \end{array}$$

commutes. Because ι is full there is an $h' : D_2 \rightarrow D'$ such that $\iota h' = h$ and, since ι is faithful, the diagram on the right above commutes. \square

We can now prove Lemma 1.15. Let \mathcal{D} be a small category satisfying the right Ore condition and let (\mathcal{D}, J_{at}) be the resulting atomic site. Fix a full reflective subcategory $\phi \dashv \iota : \mathcal{D} \rightarrow \mathcal{C}$ satisfying the primitive Nullstellensatz. Lemma 1.15 states that the induced (essential connected) geometric morphism $\phi : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$ satisfies the Nullstellensatz relative to the Lawvere-Tierney topology in $\widehat{\mathcal{D}}$ induced by J_{at} . Concretely this means that for any P in $\widehat{\mathcal{C}}$, the image $HP \rightarrow \phi_! P$ of θ_P is J_{at} -dense. This holds if and only if the map $\theta_P : \phi_* P \rightarrow \phi_! P$ is locally surjective. (Recall that a morphism $\alpha : F \rightarrow G$ in $\widehat{\mathcal{D}}$ is *locally surjective* w.r.t. J_{at} if for each D in \mathcal{D} and each $y \in GD$, there is map $e : D' \rightarrow D$ such that $y \cdot e$ is in the image of $\alpha_{D'}$. See Corollary III.7.6 in [16].)

Proof of Lemma 1.15. For any P in $\widehat{\mathcal{C}}$ and D in \mathcal{D} , $(\phi_! P)D$ may be expressed as the following coequalizer:

$$\sum_{C, C'} PC \times \mathcal{C}(C', C) \times \mathcal{D}(D, \phi C') \xrightarrow[l]{l} \sum_C PC \times \mathcal{D}(D, \phi C) \longrightarrow (\phi_! P)D$$

where for $x \in PC$, $u : C' \rightarrow C$ and $a' \in \mathcal{D}(D, \phi C')$, $l(x, u, a') = (x \cdot u, a')$ and $r(x, u, a') = (x, (\phi u)a')$. The equivalence class determined by a pair (x, a) with $x \in PC$ and $a : D \rightarrow \phi C$ will be denoted by $x \otimes a \in (\phi_! P)D$. (Theorem VII.2.2 in [16].) Also, $(\phi_* P)D = P(\iota D)$ for any P in $\widehat{\mathcal{C}}$ and D in \mathcal{D} , and $\theta : \phi_* P \rightarrow \phi_! P$ assigns to each $x \in (\phi_* P)D = P(\iota D)$ the element $(x \otimes \varepsilon^{-1}) \in (\phi_! P)D$ where $\varepsilon : \phi(\iota D) \rightarrow D$ is the counit of $\phi \dashv \iota$.

As explained above we must prove that the map $\theta_P : \phi_* P \rightarrow \phi_! P$ is locally surjective. So let $x \otimes d \in (\phi_! P)D$ with $d : D \rightarrow \phi C$ and $x \in PC$. By Lemma 3.8 the functor $\iota : \mathcal{D} \rightarrow \mathcal{C}$ satisfies the right Ore condition. So there exists a diagram in \mathcal{C} as below

$$\begin{array}{ccc} \iota D' & \xrightarrow{h} & C \\ \downarrow \iota e & & \downarrow \eta \\ \iota D & \xrightarrow[\iota d]{} & \iota(\phi C) \end{array}$$

where η is the unit of $\phi \dashv \iota$. We claim that $(x \otimes d) \cdot e = x \otimes (de)$ in $(\phi_! P)D$ equals $\theta(x \cdot h) = (x \cdot h) \otimes \varepsilon^{-1} = x \otimes ((\phi h)\varepsilon^{-1})$. For this, it is enough to prove that $de = (\phi h)\varepsilon^{-1}$ in \mathcal{D} . Since the counit is an iso, it is enough to

prove that $de\varepsilon = \phi h$. So apply ϕ to the square above, post-compose with ε to obtain

$$\begin{array}{ccc} \phi(\iota D') & \xrightarrow{\phi h} & \phi C \\ \phi(\iota\varepsilon) \downarrow & & \downarrow \phi\eta \\ \phi(\iota D) & \xrightarrow{\phi(\iota d)} & \phi(\iota(\phi C)) \xrightarrow{\varepsilon} \phi C \end{array}$$

and observe that the left-bottom composition equals $de\varepsilon$. \square

To complete the proof of Proposition 1.16 assume that the connected geometric morphism $\phi : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$ is locally connected so that if we take the pullback

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i} & \widehat{\mathcal{C}} \\ q \downarrow & & \downarrow \phi \\ \mathbf{Sh}(\mathcal{D}, J_{at}) & \xrightarrow{j} & \widehat{\mathcal{D}} \end{array}$$

of toposes then $q : \mathcal{F} \rightarrow \mathbf{Sh}(\mathcal{D}, J_{at})$ is connected and locally connected. Lemmas 1.10 and 1.15 imply that q is pre-cohesive. So it remains to show that if \mathcal{C} has a terminal object and has an object with two distinct points then q is sufficiently cohesive. Denote $\widehat{\mathcal{C}}$ by \mathcal{E} and its subobject classifier by $\Omega_{\mathcal{E}}$.

Lemma 3.9. *If $\phi_! \top : 1 = \phi_! 1 \rightarrow \phi_! \Omega_{\mathcal{E}}$ is j -dense then $q_! \Omega_{\mathcal{F}} = 1$.*

Proof. By Lemma 3.3 we can assume that $q_! = j^* \phi_! i_* : \mathcal{F} \rightarrow \mathbf{Sh}(\mathcal{D}, J_{at})$. We know that $i_* \Omega_{\mathcal{F}}$ is a retract of $\Omega_{\mathcal{E}}$ so $j^*(\phi_!(i_* \Omega_{\mathcal{F}})) = q_! \Omega_{\mathcal{F}}$ is a retract of $j^*(\phi_! \Omega_{\mathcal{E}})$. Hence, $j^*(\phi_! \Omega_{\mathcal{E}}) = 1$ implies $q_! \Omega_{\mathcal{F}} = 1$. \square

Now recall that a mono in $\widehat{\mathcal{D}}$ is dense (for the atomic topology) in \mathcal{D} if and only if it is locally surjective.

Lemma 3.10. *Let $f : \widehat{\mathcal{D}} \rightarrow \mathbf{Set}$ be the canonical geometric morphism to \mathbf{Set} . For any $\alpha : X \rightarrow Y$ in $\widehat{\mathcal{D}}$, if $f_! \alpha : f_! X \rightarrow f_! Y$ is epi in \mathbf{Set} then α is locally surjective in $\widehat{\mathcal{D}}$.*

Proof. Let $y \in YD$. Then $(y \otimes D) \in f_! Y$ and, by hypothesis, there exists an $(x \otimes E) \in f_! X$ such that $(f_! \alpha)(x \otimes E) = (\alpha_E x) \otimes E = (y \otimes D) \in f_! Y$. Because of the Ore condition this is equivalent to the existence of a span

$$E \xleftarrow{l} A \xrightarrow{r} D$$

in \mathcal{D} such that $(\alpha_{Ex}) \cdot l = y \cdot r \in YA$. So $\alpha_A(x \cdot l) = y \cdot r$, showing that y is locally in the image of α . \square

Finally let $g : \widehat{\mathcal{C}} \rightarrow \mathbf{Set}$ be the canonical geometric morphism, so that $f_! \phi_! = g_! : \mathcal{E} = \widehat{\mathcal{C}} \rightarrow \mathbf{Set}$. If \mathcal{C} is cosifted and has a disjoint cospan then $g_! \top = f_!(\phi_! \top) : 1 \rightarrow f_!(\phi_! \Omega_{\mathcal{E}})$ is an iso by Corollary 2.9, $\phi_! \top : 1 \rightarrow \phi_! \Omega_{\mathcal{E}}$ is locally surjective by Lemma 3.10 and hence $q_! \Omega_{\mathcal{F}} = 1$ by Lemma 3.9. That is, q is sufficiently cohesive, as we needed to prove.

4. Sufficient Cohesion over Galois toposes

Let k be a field. Let \mathbf{Con} be the category of finitely presented connected k -algebras and $\ell : \mathbf{Ext} \rightarrow \mathbf{Con}$ the full subcategory of separable extensions of k . Lemma 1.13 shows that ℓ has a right adjoint $\rho : \mathbf{Con} \rightarrow \mathbf{Ext}$. It is now relevant to mention a related fact. Let \mathbf{Alg} be the category of finitely presented k -algebras and $\bar{\ell} : \mathbf{Sep} \rightarrow \mathbf{Alg}$ the full subcategory of separable k -algebras. It is clear that $\ell : \mathbf{Ext} \rightarrow \mathbf{Con}$ is the restriction of $\bar{\ell}$ along the inclusion $\mathbf{Ext} \rightarrow \mathbf{Sep}$ as displayed in the following diagram

$$\begin{array}{ccc} \mathbf{Con} & \longrightarrow & \mathbf{Alg} \\ \ell \uparrow & & \uparrow \bar{\ell} \\ \mathbf{Ext} & \longrightarrow & \mathbf{Sep} \end{array}$$

and that ρ extends to a right adjoint $\bar{\rho} : \mathbf{Alg} \rightarrow \mathbf{Sep}$ to $\bar{\ell}$.

Proposition 4.1. *For any A in \mathbf{Alg} and K in \mathbf{Ext} , the canonical map $(\bar{\rho}A) \otimes_k K \rightarrow \bar{\rho}(A \otimes_k K)$ is an iso. In other words if the square on the left below*

$$\begin{array}{ccc} A & \xrightarrow{in_0} & A \otimes_k K \\ j \uparrow & & \uparrow in_1 \\ k & \xrightarrow{b} & K \end{array} \qquad \begin{array}{ccc} \bar{\rho}A & \xrightarrow{\bar{\rho}(in_0)} & \bar{\rho}(A \otimes_k K) \\ \bar{\rho}(j) \uparrow & & \uparrow \bar{\rho}(in_1) \\ k & \xrightarrow{b} & K \end{array}$$

is a pushout in \mathbf{Alg} then the square on the right is a pushout in \mathbf{Sep} .

Proof. This is Proposition I, §4, 6.7 in [5]. \square

Assume for the moment that $\bar{\rho}(j) : k \rightarrow \bar{\rho}A$ is an iso in the right square above. In particular, the largest separable subalgebra of A does not have idempotents, so A is connected. Of course, $\bar{\rho}(in_1) : K \rightarrow \bar{\rho}(A \otimes_k K)$ is also an iso and, again, this implies that $A \otimes_k K$ is connected. Let us stress this fact, if $A \in \mathbf{Con}$, $\rho A = k$ and $K \in \mathbf{Ext}$ then the algebra $A \otimes_k K$ is also in \mathbf{Con} and $\rho(A \otimes_k K) = K$. Moreover, this is for every k .

Lemma 4.2. *The geometric morphism $[\mathbf{Con}, \mathbf{Set}] \rightarrow [\mathbf{Ext}, \mathbf{Set}]$ induced by $\rho : \mathbf{Con} \rightarrow \mathbf{Ext}$ is connected and locally connected.*

Proof. As we have already mentioned, connectedness follows from the fact that ρ has a full and faithful left adjoint. To prove local connectedness we use a sufficient condition proved in [7]. This condition involves a category $\mathcal{X}_\rho = \mathcal{X}$ of so called ρ -extracts. In general, its objects would be 4-tuples (U, V, r, i) with U in the domain of ρ , V in the codomain, $r : \rho U \rightarrow V$ a map and $i : V \rightarrow \rho U$ a section of r ; and maps $(U, V, r, i) \rightarrow (U', V', r', i')$ would be pairs $(a : U \rightarrow U', b : V \rightarrow V')$ such that $r'(a) = br$ and $i'b = (\rho a)i$. In our concrete case, every map in the codomain of $\rho : \mathbf{Con} \rightarrow \mathbf{Ext}$ is mono and ρ has a full and faithful left adjoint ℓ so each object (U, V, r, i) is completely determined by a map $j : \ell V \rightarrow U$ such that $\rho j : \rho(\ell V) \rightarrow \rho U$ is an iso. It is convenient to drop ℓ from the notation and denote objects in \mathbf{Ext} with decorated K 's. Then the category \mathcal{X} of ρ -extracts may be described as follows: its objects are triples $(U, K, j : K \rightarrow U)$ with U in \mathbf{Con} such that $\rho j : K \rightarrow \rho U$ is an iso; and a map $a : (U, K, j) \rightarrow (U', K', j')$ is just a map $a : U \rightarrow U'$ in \mathbf{Con} . There is an obvious functor $g : \mathcal{X} \rightarrow \mathbf{Ext}$ that sends (U, K, j) to K and $a : (U, K, j) \rightarrow (U', K', j')$ to the unique map $ga : K \rightarrow K'$ making the following square

$$\begin{array}{ccc} \rho U & \xrightarrow{\rho a} & \rho U' \\ \rho j \uparrow & & \uparrow \rho j' \\ K & \xrightarrow{ga} & K' \end{array}$$

commute. For any K in \mathbf{Ext} write $\mathcal{X}(K)$ for the fibre of g over K . Now, for each $b : K \rightarrow K'$ in \mathbf{Ext} and lifting of K to an object (U, K, j) in \mathcal{X} define the category $\mathcal{Y}_{U, K, j, b} = \mathcal{Y}$ whose objects are liftings of b to a morphism of \mathcal{X} with domain (U, K, j) and whose morphisms are morphisms of $\mathcal{X}(K')$

forming commutative triangles. Lemma C3.3.8 of [7] implies that: if for each b and (U, K, j) as above, the associated category \mathcal{Y} is connected then $[\mathbf{Con}, \mathbf{Set}] \rightarrow [\mathbf{Ext}, \mathbf{Set}]$ is locally connected. Let us first prove that \mathcal{Y} is nonempty. For this consider the pushout on the left below

$$\begin{array}{ccc} U & \xrightarrow{in_0} & U \otimes_K K' \\ j \uparrow & & \uparrow in_1 \\ K & \xrightarrow{b} & K' \end{array} \qquad \begin{array}{ccc} \rho U & \xrightarrow{\rho(in_0)} & \rho(U \otimes_K K') \\ \rho(j) \uparrow & & \uparrow \rho(in_1) \\ K & \xrightarrow{b} & K' \end{array}$$

calculated in the category of k -algebras. Since $\rho(j)$ is iso by hypothesis (recall that (U, K, j) is in \mathcal{X}) Proposition 4.1 implies that $U \otimes_K K'$ is connected and that $\rho(in_1) : K' \rightarrow \rho(U \otimes_K K')$ is an iso. Hence, the map $in_0 : (U, K, j) \rightarrow (U \otimes_K K', K', in_1)$ is an object in \mathcal{Y} . Finally, consider any object $a : (U, K, j) \rightarrow (U', K', j')$ in \mathcal{Y} as displayed on the left below

$$\begin{array}{ccc} U & \xrightarrow{a} & U' \\ j \uparrow & & \uparrow j' \\ K & \xrightarrow{b} & K' \end{array} \qquad \begin{array}{ccc} U & \xrightarrow{in_0} & U \otimes_K K' & \xleftarrow{in_1} & K' \\ & \searrow a & \downarrow h & \swarrow j' & \\ & & U' & & \end{array}$$

and notice that the pushout property determines a unique $h : U \otimes_K K' \rightarrow U'$ such that the triangles on the right above commute. So h is a map in \mathcal{Y} from $in_0 : (U, K, j) \rightarrow (U \otimes_K K', K', in_1)$ to $a : (U, K, j) \rightarrow (U', K', j')$. It follows that \mathcal{Y} is indeed connected. \square

After the proof of Lemma 4.2 we stress that we do not claim to have found the most efficient way to present the examples. It is to be expected that in a near future there will be simpler ways to explain how the inclusion $\mathbf{Ext} \rightarrow \mathbf{Con}$ determines a pre-cohesive topos. In any case, we have the following result.

Proposition 4.3. *If k is perfect, $p : [\mathbf{Con}, \mathbf{Set}] \rightarrow [\mathbf{Ext}, \mathbf{Set}]$ is the geometric morphism induced by the coreflector $\rho : \mathbf{Con} \rightarrow \mathbf{Ext}$ and the following diagram*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i} & [\mathbf{Con}, \mathbf{Set}] \\ q \downarrow & & \downarrow p \\ \mathbf{Sh}(\mathbf{Ext}^{\text{op}}, J_{at}) & \longrightarrow & [\mathbf{Ext}, \mathbf{Set}] \end{array}$$

is a pullback of toposes then $q : \mathcal{F} \rightarrow \mathbf{Sh}(\mathbf{Ext}^{\text{op}}, J_{\text{at}})$ is pre-cohesive and sufficiently cohesive.

Proof. The category $\mathcal{D} = \mathbf{Ext}^{\text{op}}$ satisfies the right Ore condition (see example 7 in [3]). Let $\mathcal{C} = \mathbf{Con}^{\text{op}}$, $\iota : \mathcal{D} \rightarrow \mathcal{C}$ the obvious full inclusion and $\phi = \rho^{\text{op}} : \mathcal{C} \rightarrow \mathcal{D}$ its left adjoint. Example 1.14 shows that the reflective subcategory $\iota : \mathcal{D} \rightarrow \mathcal{C}$ satisfies the primitive Nullstellensatz and $\phi : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$ is connected and locally connected by Lemma 4.2. Finally, the category \mathcal{C} has a terminal object and the two maps $k[x] \rightarrow k$ in \mathbf{Con} that send x to 0 and 1 in k respectively show that there is an object in \mathcal{C} with two distinct points. So we can apply Proposition 1.16. \square

The construction of examples in this section naturally leads to the following questions. Let \mathcal{C} be an extensive category with finite products and let $\mathcal{C}_s \rightarrow \mathcal{C}$ is its full subcategory of separable/decidable/unramified objects [10, 4]. When is this category reflective? Assuming that \mathcal{C} is small, when is it the case that the left adjoint $\phi : \mathcal{C} \rightarrow \mathcal{C}_s$ induces a locally connected $\widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{C}}_s$? To prove this for our examples we used Proposition 4.1 which highlights a special behaviour of tensor products in the category of k -algebras for a field k . So we are led to a more specific problem. Consider a coextensive algebraic category \mathcal{V} (such as those discussed in [14]) and let K be an object in \mathcal{V} . The category K/\mathcal{V} is also algebraic and coextensive. If we let \mathcal{C} be the opposite of the category of finitely presentable objects in K/\mathcal{V} then it would be interesting to understand those K that make $\mathcal{C}_s \rightarrow \mathcal{C}$ reflective etc.

If $k = \mathbb{C}$ then \mathbf{Ext} is terminal so the horizontal maps in the pullback in the statement of Proposition 4.3 are equivalences and (the canonical) $p : [\mathbf{Con}, \mathbf{Set}] \rightarrow [\mathbf{Ext}, \mathbf{Set}] = \mathbf{Set}$ is pre-cohesive. But we stress that, in general, the canonical geometric morphism $\mathcal{F} \rightarrow \mathbf{Set}$ is not pre-cohesive. This can be seen even in the simple case of $k = \mathbb{R}$ as we show in the next section.

4.1 The case of the real field

Of course, Galois groups need not be finite. Moreover, if Galois theory is to be done in an arbitrary ambient topos, then Galois groups are not internal groups of automorphisms in the naive sense [18]. Having said this, I believe that it is useful to illustrate the results in the previous sections in the simplest possible non trivial (although finite) case over sets.

Indeed, let us consider the case of $k = \mathbb{R}$ in \mathbf{Set} , so that \mathbf{Con} is the category of finitely presented connected \mathbb{R} -algebras and $\ell : \mathbf{Ext} \rightarrow \mathbf{Con}$ is the (finite) full subcategory determined by finite extensions of k . Of course, this full subcategory is equivalent to that determined by the (the initial object) \mathbb{R} and \mathbb{C} . The right adjoint $\rho : \mathbf{Con} \rightarrow \mathbf{Ext}$ may be described as follows. For A in \mathbf{Con} , ρA is the \mathbb{R} -subalgebra generated the square roots of -1 . Notice that $\rho A \cong \mathbb{R}$ if A does not have square roots of -1 and $\rho A \cong \mathbb{C}$ otherwise. To check that this is well-defined observe that if $i^2 = -1 = j^2$ then $j = i$ or $j = -i$. (This follows from connectedness and the fact that $\frac{ij+1}{2}$ is idempotent in A .)

The atomic topology J_{at} on $\mathcal{D} = \mathbf{Ext}^{\text{op}}$ has essentially one non-trivial sieve: that generated by the unique map $\mathbb{R} \rightarrow \mathbb{C}$ in \mathbf{Ext} . Also, since \mathcal{D} is essentially finite and all its idempotents are identities, J_{at} is rigid in the sense C2.2.8 in [7] and $\mathbf{Sh}(\mathcal{D}, J_{at})$ is equivalent to the topos of presheaves on the full subcategory of \mathcal{D} determined by those objects which only have trivial covers. That is, $\mathbf{Sh}(\mathcal{D}, J_{at}) \cong [\mathbf{C}_2, \mathbf{Set}]$ where $\mathbf{C}_2 \rightarrow \mathbf{Ext}$ is the full subcategory determined by those objects iso to \mathbb{C} . Of course, \mathbf{C}_2 is equivalent to the cyclic group C_2 of order two.

Let $\mathbf{Con}' \rightarrow \mathbf{Con}$ be the full subcategory determined by those connected \mathbb{R} -algebras A such that $\rho A \cong \mathbb{C}$ or, equivalently, there is an \mathbb{R} -algebra map $\mathbb{C} \rightarrow A$. The following diagram

$$\begin{array}{ccc} \mathbf{Con}' & \longrightarrow & \mathbf{Con} \\ \downarrow & & \downarrow \\ \mathbf{C}_2 & \longrightarrow & \mathbf{Ext} \end{array}$$

is a pullback of categories and the next result shows that it is preserved when passing to toposes of \mathbf{Set} -valued functors.

Lemma 4.4. *If we let $[\mathbf{Con}', \mathbf{Set}] \rightarrow [\mathbf{C}_2, \mathbf{Set}]$ be the geometric morphism induced by the full inclusion $\mathbf{C}_2 \rightarrow \mathbf{Con}'$ then the following diagram*

$$\begin{array}{ccc} [\mathbf{Con}', \mathbf{Set}] & \xrightarrow{i} & [\mathbf{Con}, \mathbf{Set}] \\ \downarrow & & \downarrow p \\ [\mathbf{C}_2, \mathbf{Set}] & \longrightarrow & [\mathbf{Ext}, \mathbf{Set}] \end{array}$$

is a pullback of toposes. (So $[\mathbf{Con}', \mathbf{Set}] \rightarrow [\mathbf{C}_2, \mathbf{Set}]$ is pre-cohesive and sufficiently cohesive.)

Proof. The subtopos $[\mathbf{C}_2, \mathbf{Set}] \rightarrow [\mathbf{Ext}, \mathbf{Set}]$ is open. Indeed, the sieve in \mathcal{D} generated by the unique morphism $\mathbb{R} \rightarrow \mathbb{C}$ in \mathbf{Ext} determines a subobject $U \rightarrow 1$ in the topos $[\mathbf{Ext}, \mathbf{Set}]$. More explicitly, $U\mathbb{R} = \emptyset$ and $U\mathbb{C} = 1$; and $[\mathbf{C}_2, \mathbf{Set}] \cong [\mathbf{Ext}, \mathbf{Set}]/U \rightarrow [\mathbf{Ext}, \mathbf{Set}]$. Since open subtoposes are closed under pullback it follows that the subtopos $\mathcal{F} \rightarrow [\mathbf{Con}, \mathbf{Set}]$ in Proposition 4.3 is equivalent to $[\mathbf{Con}, \mathbf{Set}]/p^*U \rightarrow [\mathbf{Con}, \mathbf{Set}]$ and hence \mathcal{F} must be a presheaf topos, say, of the form $[\mathbf{Con}', \mathbf{Set}]$ for some essentially small \mathbf{Con}' determined by $V = p^*U$ in $[\mathbf{Con}, \mathbf{Set}]$. In order to describe \mathbf{Con}' explicitly we first apply the general construction (see e.g. Proposition A1.1.7.). The objects of \mathbf{Con}' are pairs (x, C) with $x \in VC$ and $C \in \mathbf{Con}$. A map $f : (x, C) \rightarrow (x', C')$ in \mathbf{Con}' is a morphism $f : C \rightarrow C'$ in \mathbf{Con} such that $(Vf)x = x'$. But $VC = (p^*U)C = U(\rho C)$ for each connected \mathbb{R} -algebra C . In other words, $VC = (p^*U)C$ is terminal or initial depending on whether there is an \mathbb{R} -algebra map $\mathbb{C} \rightarrow C$ or not. \square

In order to give an explicit description of the Grothendieck topology on \mathbf{Con}^{op} inducing $\mathcal{F} = [\mathbf{Con}', \mathbf{Set}]$ we first isolate the following basic fact (clearly related to the far more general Proposition 4.1).

Lemma 4.5. *If the \mathbb{R} -algebra A is connected and without square roots of -1 then $A[i] = A \otimes_{\mathbb{R}} \mathbb{C}$ is connected.*

Proof. Let $a + bi$ in $A[i]$ be idempotent. Then $a^2 - b^2 = a$ and $2ab = b$ in A . Now calculate

$$b^2 = 4a^2b^2 = 4(a + b^2)b^2 = 4ab^2 + 4b^4 = 2b^2 + 4b^4$$

an record that $b^2 + 4b^4 = 0$. So $u = b^2$ satisfies $4u^2 = -u$ in A . Then $(4u)^2 = 16u^2 = -4u$ and so $c = 4u$ satisfies the equality $c^2 = -c$. But then $(c + 1)^2 = c^2 + 2c + 1 = -c + 2c + 1 = c + 1$. That is, $c + 1$ is idempotent in A which means, under our hypotheses, that either $c + 1 = 0$ or $c + 1 = 1$; so $c = -1$ or $c = 0$. If $-1 = c = 4u = 4b^2 = (2b)^2$ then we reach a contradiction (since we are assuming that A does not have a square root of -1). If $0 = c = 4b^2$ then $b^2 = 0$ so $a^2 = a$. Since A is connected $a = 0$ or $a = 1$. If $a = 0$ then $b = 2ab = 0$. If $a = 1$ then $b = 2b$ so $b = 0$. Altogether, $a + bi$ is either 0 or 1. \square

We can now define a basis K for a Grothendieck topology on \mathbf{Con}^{op} (in the sense of Exercise III.3 in [16]). We do this in terms of cocovers in \mathbf{Con} . First we state that the cocovering families consist of exactly one map, so it is enough to say what maps cocover. First all isos cocover. Also, if $\rho A \cong \mathbb{R}$ then a map $A \rightarrow A'$ also cocovers if it is iso over A to the canonical $A \rightarrow A[i]$. (This makes sense by Lemma 4.5.)

Lemma 4.6. *The function K that sends A in \mathbf{Con}^{op} to the collection of covering maps with codomain A is a basis and $\mathbf{Sh}(\mathbf{Con}^{\text{op}}, K) \cong [\mathbf{Con}', \mathbf{Set}]$ as subtoposes of $[\mathbf{Con}, \mathbf{Set}]$.*

Proof. It is easy to check that K is indeed a basis. The main ingredient is that if $A \in \mathbf{Con}$ is such that $\rho A \cong \mathbb{R}$ and $A \rightarrow A'$ is in \mathbf{Con} then there exists a cocovering map $A' \rightarrow B$ and a commutative square as below

$$\begin{array}{ccc} A[i] & \longrightarrow & B \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

in \mathbf{Con} . Indeed, if $\rho A' \cong \mathbb{C}$ then we can take $B = A'$ and $A' \rightarrow B$ to be the identity. On the other hand, if $\rho A' \cong \mathbb{R}$ then we can take $B = A'[i]$ and the canonical $A' \rightarrow A'[i] = B$.

To prove that $\mathbf{Sh}(\mathbf{Con}^{\text{op}}, K) = [\mathbf{Con}', \mathbf{Set}]$ we use the notation in the proof of Lemma 4.4. So the subobject $U \rightarrow 1$ is the image of the map $\mathbf{Ext}(\mathbb{C}, _) \rightarrow \mathbf{Ext}(\mathbb{R}, _) = 1$ in $[\mathbf{Ext}, \mathbf{Set}]$ and we denote the map $p^*U \rightarrow 1$ by $V \rightarrow 1$ in $[\mathbf{Con}, \mathbf{Set}]$. Recall that VC is terminal or initial depending on whether there is an \mathbb{R} -algebra map $\mathbb{C} \rightarrow C$ or not. For general reasons, the dense subobjects for the associated open topology in $[\mathbf{Con}, \mathbf{Set}]$ are those monos $X' \rightarrow X$ such that the projection $\pi_0 : X \times V \rightarrow X$ factors through $X' \rightarrow X$. In particular, for any \mathbb{R} -algebra A in \mathbf{Con} and cosieve $S \rightarrow \mathbf{Con}(A, _)$, S is dense if and only if for every A' in \mathbf{Con} such that $VA' = 1$ (that is, $\rho A' \cong \mathbb{C}$), every $A \rightarrow A'$ is in the cosieve S . Notice that if $VA = 1$ then the identity on A must be in S . In other words, if $VA = 1$ then the maximal cosieve is the only (co)covering one. On the other hand, if $VA = 0$ (i.e. $\rho A = \mathbb{R}$) then, S is cocovering if and only if the map $A \rightarrow A[i]$ is in S . Altogether, a sieve on A is dense with respect to the open topology determined by $V \rightarrow 1$ if and only if it contains a cocovering map. \square

Now let \mathbf{Alg} be the category of finitely presented \mathbb{R} -algebras. The extensive \mathbf{Alg}^{op} may be equipped with the Gaeta topology and it is well-known (see [14]) that the resulting topos of sheaves is equivalent to $[\mathbf{Con}, \mathbf{Set}]$. It is also well-known that the Gaeta topology is subcanonical and that the restricted Yoneda embedding $\mathbf{Alg}^{\text{op}} \rightarrow [\mathbf{Con}, \mathbf{Set}]$ into the Gaeta topos preserves finite coproducts.

Lemma 4.7. *The restricted Yoneda embedding $\mathbf{Alg}^{\text{op}} \rightarrow [\mathbf{Con}, \mathbf{Set}]$ factors through the subtopos inclusion $\mathcal{F} \rightarrow [\mathbf{Con}, \mathbf{Set}]$ and the factorization $\mathbf{Alg}^{\text{op}} \rightarrow \mathcal{F}$ preserves finite coproducts.*

Proof. Let A in \mathbf{Alg} . It is fair to write $\mathbf{Con}(A, -)$ for the non-representable associated object in the Gaeta topos $[\mathbf{Con}, \mathbf{Set}]$. It is enough to prove that every such $\mathbf{Con}(A, -)$ is a K -sheaf for the basis discussed in Lemma 4.6. We need only worry about objects that have non-trivial covers so let C in \mathbf{Con} be such that $\rho C = \mathbb{R}$ and consider the cocovering $C \rightarrow C[i]$. A compatible family consists of a map $f : A \rightarrow C[i]$ satisfying that for any pair of maps $g, h : C[i] \rightarrow D$ in \mathbf{Con} such that the diagram on the left below commutes

$$C \longrightarrow C[i] \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{h} \end{array} D \qquad A \xrightarrow{f} C[i] \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{h} \end{array} D$$

the diagram on the right above commutes too. But $C \rightarrow C[i]$ is the equalizer (in \mathbf{Alg}) of the identity on $C[i]$ and conjugation. Hence there exists a unique map $f' : A \rightarrow C$ factoring f through $C \rightarrow C[i]$. This implies that $\mathbf{Con}(A, -)$ is a sheaf. To confirm that the factorization $\mathbf{Alg}^{\text{op}} \rightarrow \mathcal{F}$ preserves finite coproducts just observe that since $1 + 1$ in the Gaeta topos $[\mathbf{Con}, \mathbf{Set}]$ is actually in the image of $\mathbf{Alg}^{\text{op}} \rightarrow [\mathbf{Con}, \mathbf{Set}]$ then it is also in the subtopos $\mathcal{F} \rightarrow [\mathbf{Con}, \mathbf{Set}]$. \square

In short, the geometric morphism $\mathcal{F} = [\mathbf{Con}', \mathbf{Set}] \rightarrow [\mathbf{C}_2, \mathbf{Set}]$ makes \mathcal{F} into a sufficiently cohesive pre-cohesive topos embedding the category of ‘affine \mathbb{R} -schemes’ \mathbf{Alg}^{op} in such a way that finite coproducts are preserved. In contrast, the canonical geometric morphism $f : \mathcal{F} \rightarrow \mathbf{Set}$ is not pre-cohesive. It is certainly locally connected because \mathcal{F} is a pre-sheaf topos but the leftmost adjoint $f_! : \mathcal{F} \rightarrow \mathbf{Set}$ does not preserve finite products (and hence the Nullstellensatz must fail). The simplest way to see this may be the following.

Example 4.8. The object $X = \mathbf{Con}'(\mathbb{C}, -)$ in \mathcal{F} is connected in the sense that $f_1 X = 1$ because it is representable but $f_1(X \times X) = 2$ as the next calculation shows. Since there are enough maps to \mathbb{C} , $f_1(X \times X)$ is a quotient of $(X \times X)\mathbb{C} = \mathbf{Con}'(\mathbb{C}, \mathbb{C}) \times \mathbf{Con}'(\mathbb{C}, \mathbb{C}) \cong C_2 \times C_2$. If $\kappa : \mathbb{C} \rightarrow \mathbb{C}$ denotes conjugation then the pairs (id, id) and (κ, κ) induce the same element in $f_1(X \times X)$. Similarly, (id, κ) and (κ, id) induce the same element; but (id, id) and (id, κ) cannot be equivalent.

It seems relevant at this point to compare \mathcal{F} with the Zariski topos. Let Z be the basis on \mathbf{Alg}^{op} determined by declaring that the cocovering families are (up to iso) those of the form $(A \rightarrow A[s^{-1}] \mid s \in S)$ with $S \subseteq A$ a finite subset not contained in any proper ideal of A in \mathbf{Alg} . (See III.3 in [16] or A2.1.11(f) in [7].) Denote the Zariski topos $\mathbf{Sh}(\mathbf{Alg}^{\text{op}}, Z)$ by \mathcal{Z} . Clearly the basis Z contains the Gaeta one so the inclusion $\mathcal{Z} \rightarrow [\mathbf{Alg}, \mathbf{Set}]$ factors through the Gaeta subtopos $[\mathbf{Con}, \mathbf{Set}] \rightarrow [\mathbf{Alg}, \mathbf{Set}]$. The basis Z is also subcanonical but we stress that the subtoposes $\mathcal{Z} \rightarrow [\mathbf{Con}, \mathbf{Set}]$ and $\mathcal{F} \rightarrow [\mathbf{Con}, \mathbf{Set}]$ are incomparable. This is clear if we contrast the basis K of Lemma 4.6 with the Zariski basis defined above. Certainly, the Grothendieck topology generated by K does not contain most of the sieves generated by the ‘open’ covers of Z . On the other hand, \mathbb{R} in \mathbf{Con}^{op} does not have a non-trivial Z -cocover. Hence, the composite

$$\mathcal{Z} \rightarrow [\mathbf{Con}, \mathbf{Set}] \rightarrow [\mathbf{Ext}, \mathbf{Set}]$$

does not factor through the subtopos $[\mathbf{C}_2, \mathbf{Set}] \rightarrow [\mathbf{Ext}, \mathbf{Set}]$.

The discussion above suggests considering the intersection of \mathcal{F} and \mathcal{Z} over $[\mathbf{Con}, \mathbf{Set}]$. Hopefully, the resulting topos would combine the benefits of a pre-cohesive topos with the colimit preservation properties of the embedding $\mathbf{Alg}^{\text{op}} \rightarrow \mathcal{Z}$. Alternatively, one can consider in \mathcal{F} the algebra object $R = \mathbf{Con}(\mathbb{R}[x], -)$ and the least Lawvere-Tierney topology that makes the subobject

$$\{a \in R \mid (\exists b \in R)(ab = 1) \vee (\exists b \in R)((1 - a)b = 1)\} \longrightarrow R$$

dense. The two subtoposes of \mathcal{F} suggested above may turn out to be the same but, in any case, this will have to be treated elsewhere.

Still in the case that $k = \mathbb{R}$; what does $\mathcal{F} = [\mathbf{Con}', \mathbf{Set}]$ classify? Assume a standard presentation of the theory of \mathbb{R} -algebras extending the usual

presentation of the theory of rings. The theory of connected \mathbb{R} -algebras may be presented by adding the axioms

$$0 = 1 \vdash \perp \quad \text{and} \quad x^2 = x \vdash_x (x = 0) \vee (x = 1)$$

and it is well-known (see [14]) that this induces the Gaeta topology on \mathbf{Alg}^{op} so the resulting topos of sheaves is equivalent to $[\mathbf{Con}, \mathbf{Set}]$.

Lemma 4.9. *The theory classified by \mathcal{F} can be presented by adding the axiom*

$$\vdash (\exists x)(x^2 = -1)$$

to the presentation of the theory of connected \mathbb{R} -algebras described above.

Proof. To prove this is convenient to use the presentation of \mathcal{F} given in Lemma 4.6 because the basis K on \mathbf{Con}^{op} is clearly generated by the map $\mathbb{R} \rightarrow \mathbb{C}$ in \mathbf{Con} ; and this sieve covers if and only if the theory classified by \mathcal{F} satisfies the evident axiom.

Alternatively, one can start with the theory presented as in the statement and regard it as a ‘quotient’ of the presentation of the theory of \mathbb{R} -algebras. It is well-known (see e.g. D3.1.10 in [7]) that one can construct the classifying topos as the topos of sheaves on a site whose underlying category is the opposite of the category of finitely presented algebras. Following this path (and factoring through the Gaeta site) one arrives at the site $(\mathbf{Con}^{\text{op}}, K)$. \square

For an arbitrary field the subtopos $\mathbf{Sh}(\mathcal{D}, J_{\text{at}}) \rightarrow [\mathbf{Ext}, \mathbf{Set}]$ will not be open but the description of the theory classified by \mathcal{F} can probably be modified by adding an appropriate sequent for each map in \mathbf{Ext} .

Lawvere suggested to discuss the classifying role of \mathcal{F} over its natural base. To do this recall (Theorem VIII.2.7 in [16]) that the base $[\mathbf{C}_2, \mathbf{Set}]$ classifies C_2 -torsors, where C_2 is cyclic group of order 2. For brevity let us define a (C_2) -torsored topos as a pair (\mathcal{T}, T) given by a topos \mathcal{T} and a C_2 -torsor T in it. A morphism $g : (\mathcal{T}, T) \rightarrow (\mathcal{T}', T')$ of torsored toposes is a geometric morphism $g : \mathcal{T} \rightarrow \mathcal{T}'$ such that $g^*T' \cong T$.

Definition 4.10. A torsored algebra in a torsored topos (\mathcal{T}, T) is an internal \mathbb{R} -algebra A in \mathcal{T} together with a map $T \rightarrow A$ such that the following

diagram

$$\begin{array}{ccccc}
 T & \xrightarrow{\quad} & & \xrightarrow{\quad} & 1 \\
 \downarrow & & & & \downarrow^{-1} \\
 A & \xrightarrow{\Delta} & A \times A & \xrightarrow{\cdot} & A
 \end{array}$$

is a pullback; where Δ is the diagonal and \cdot is the multiplication of the algebra A .

The pre-cohesive $\mathcal{F} \rightarrow [\mathbf{C}_2, \mathbf{Set}]$ makes \mathcal{F} into a torsored topos (\mathcal{F}, F) and the object $R = \mathbf{Con}(\mathbb{R}[x], -)$ is a \mathbb{R} -algebra in $\mathcal{F} = \mathbf{Sh}(\mathbf{Con}^{\text{op}}, K)$.

Proposition 4.11. *The \mathbb{R} -algebra R in \mathcal{F} may be equipped with a torsored algebra structure in (\mathcal{F}, F) and it is the generic one. That is, (\mathcal{F}, F, R) classifies torsored algebras among torsored toposes.*

Proof. The underlying object of the generic C_2 -torsor is the representable $\mathbf{C}_2(\mathbb{C}, -)$ in $[\mathbf{C}_2, \mathbf{Set}]$. The inverse image of the pre-cohesive $\mathcal{F} \rightarrow [\mathbf{C}_2, \mathbf{Set}]$ sends $\mathbf{C}_2(\mathbb{C}, -)$ to $\mathbf{C}_2(\mathbb{C}, \rho(-)) \cong \mathbf{Con}(\mathbb{C}, -) = F$. The unique \mathbb{R} -algebra map $\mathbb{R}[x] \rightarrow \mathbb{C}$ sending x to i determines a morphism $F \rightarrow R$ and since the diagram below

$$\begin{array}{ccccc}
 x & \xrightarrow{\quad} & & \xrightarrow{\quad} & x^2 \\
 \downarrow & & & & \downarrow \\
 & & \mathbb{R}[x] & \xrightarrow{\quad} & \mathbb{R}[x] \\
 & & \downarrow & & \downarrow \\
 & & \mathbb{R} & \xrightarrow{\quad} & \mathbb{C} \\
 \downarrow & & & & \downarrow \\
 -1 & & & & i
 \end{array}$$

is a pushout in \mathbf{Con} , the map $F \rightarrow R$ in \mathcal{F} makes R into a torsored algebra. To prove that it is the generic one let (\mathcal{T}, T) be a torsored topos and let A be a torsored algebra in \mathcal{T} . The unique map $T \rightarrow 1$ is epi because T is a torsor and so, the condition defining torsor algebras implies that $\vdash (\exists x)(x^2 = -1)$ holds in \mathcal{T} . By Lemma 4.9 there exists an essentially unique geometric morphism $g : \mathcal{T} \rightarrow \mathcal{F}$ such that $g^*R = A$. Since g^* preserves finite limits it must be the case that $g^*F \cong T$ so g is a morphism of torsored toposes. \square

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Matías Menni
Conicet and Universidad Nacional de La Plata
La Plata (Argentina)
matias.menni@gmail.com

ADDENDUM TO THE ARTICLE (in LIV-4)

by Sergey BARANOV and Sergei SOLOVIEV

The following footnote should be added to the article:

"Equality in Lambda calculus. Weak universality in
Category Theory and reversible computations"

by S. Baranov and S. Soloviev, published in Volume LIV-4 (pp. 264-291):

S. Soloviev was supported by the Climt project,
ANR-11-BS02-016".

AUTOCATEGORIES : II. AUTOGRAPHIC ALGEBRAS

by René GUITART

Abstract. An *autograph* is a set A with an action of the free monoid with 2 generators; it could be drawn as arrows between arrows. In [5] we have shown that knot diagrams as well as 2-graphs are examples. Of course the category of autographs is a topos, and an *autographic algebra* will be the algebra of a monad on this topos. In this paper we compare autographic algebras with *graphic algebras* of Burroni, via *graphic monoids* of Lawvere. For that we use monadicity criterions of Lair and of Coppey. The point is that when it is possible to replace graphic algebras by autographic algebras, we change a situation with 2 types of arities into a situation with only 1 type, the type “object” being avoided. So graphs, basic graphic algebras, autographs in a category of algebras of a Lawvere theory, elements of any 2-generated graphic topos, categories, autcategories, associative autographs, are autographic algebras.

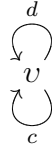
Résumé. Un *autographe* est un ensemble A équipé d’une action du monoïde libre à deux générateurs, et peut être représenté en dessinant des flèches entre des flèches. Dans [5] nous avons obtenu comme exemples les diagrammes de nœuds et les 2-graphes. Evidemment la catégorie de ces autographes est un topos, et une *algèbre autographique* sera une algèbre d’une monade sur ce topos. Ici nous comparons ces algèbres avec les algèbres graphiques de Burroni, via les monoïdes graphiques de Lawvere, en utilisant les critères de monadicité de Lair et de Coppey. Le point est que lorsque l’on remplace une situation graphique par une situation autographique, on transforme une situation à 2 types d’arités en une situation à 1 type, le type “objet” étant montré évitable. Ainsi les graphes, les algèbres graphiques basiques, les autographes dans une catégorie d’algèbres de Lawvere, les éléments de topos graphiques 2-engendrés, les catégories, les autocatégories, et les autographes associatifs sont des algèbres autographiques.

Keywords. graph, autograph, graphic algebras, graphic monoids.

Mathematics Subject Classification (2010). 18.

1. From graphs to autographs

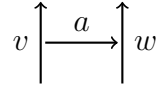
Definition 1.1. (see [5, def. 1.1., p.66]) We denote by $\mathbb{FM}(2) = \{d, c\}^*$ the free monoid on two generators d and c . As a category with one object v , this monoid $\mathbb{FM}(2)$ is the category of paths in the graph



Epecially the identity is the empty path “()”, also denoted by 1_v . An autograph $(A, (d_A, c_A))$ is a set A of arrows, equipped with two maps domain $d_A : A \rightarrow A$ and codomain $c_A : A \rightarrow A$; that is to say an action of $\mathbb{FM}(2)$ on A ; if necessary this action is again denoted by A , with

$$A(v) = A, A(d) = d_A, A(c) = c_A.$$

We represent $a \in A$ with $d_A a = v$ and $c_A a = w$, by: $a : v \rightarrow w$, or $v \xrightarrow{a} w$, or by:



The category of autographs is $\text{Agraph} = \text{Set}^{\mathbb{FM}(2)}$; in this category a morphism is a map $f : A \rightarrow A'$ satisfying $d'fa = fda$, $c'fa = fca$. We have a forgetful functor:

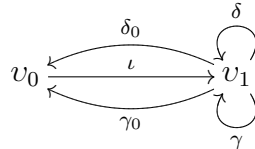
$$U : \text{Agraph} \rightarrow \text{Set} : (A, (d_A, c_A)) \mapsto A.$$

Definition 1.2. We denote by $\mathbb{G}(2)$ the category with two objects v_0 and v_1 , and five non-identity arrows

$$\gamma_0, \delta_0 : v_1 \rightarrow v_0, \quad \iota : v_0 \rightarrow v_1, \quad \delta, \gamma : v_1 \rightarrow v_1,$$

with identities on v_1 and v_0 , and with equations :

$$\delta_0 \cdot \iota = 1_{v_0}, \quad \gamma_0 \cdot \iota = 1_{v_0}, \quad \gamma = \iota \cdot \gamma_0, \quad \delta = \iota \cdot \delta_0.$$



Presheaves G on $\mathbb{G}(2)$, i.e. objects of $\text{Graph} = \text{Set}^{\mathbb{G}(2)}$ are named *graphs*. Any $V \in G(v_0)$ is named a *vertex*, and if $f \in G(v_1)$, f is named an *arrow*; then the fact that $G(\delta_0)(f) = V$ and $G(\gamma_0)(f) = V'$ is represented by: $f : V \rightarrow V'$.

Remark 1.3. When we work “over graphs” we have to consider 2 types of arities (vertices and arrows), whereas working “over autographs” introduces only 1 type (arrows). So our question here is to understand precisely when the reduction of a 2 types situation to a 1 type situation is possible.

Proposition 1.4. *The comparison between graphs and autographs is induced by pre-composition with the functor*

$$\mathbb{FM}(2) \xrightarrow{\phi} \mathbb{G}(2)$$

given by

$$\phi(v) = v_1, \phi(d) = \delta, \phi(c) = \gamma.$$

Up to an isomorphism, any graph $G : \mathbb{G}(2) \rightarrow \text{Set}$ is determined by its associated autograph $G\phi : \mathbb{FM}(2) \rightarrow \text{Set}$.

Proof. In a graph G for each vertex $V \in G(v_0)$, the arrow $G(\iota)(V) \in G(v_1)$ is exactly a fixed point of δ , i.e. an $x \in G(v_1)$ such that $\delta(x) = x$, as well as exactly a fixed point of γ . So in fact we recover the set of vertices of G as the splitting of the idempotent δ , or also the splitting of γ (these two splittings are isomorphic). \square

Proposition 1.5. *The comparison ϕ between graphs and autographs in Proposition 1.4 admits a factorisation through the image $\mathbb{M}(2)$ of ϕ , in such a way that $\mathbb{G}(2)$ is the strict karoubian envelope of $\mathbb{M}(2)$.*

$$\mathbb{G}(2) = \text{Kar}_0(\mathbb{M}(2)) \xleftarrow{j} \mathbb{M}(2) = \text{Im}\phi \xleftarrow{\bar{\phi}} \mathbb{FM}(2).$$

This monoid $\mathbb{M}(2)$ is introduced by Lawvere as a graphic monoid.

Proof. In $\mathbb{G}(2)$ we get $\delta^2 = \delta, \gamma^2 = \gamma, \gamma\delta = \delta, \delta\gamma = \gamma$, and the full subcategory $\text{End}_{\mathbb{G}}(v_1)$ of $\mathbb{G}(2)$ generated by v_1 has one unit and two idempotents, γ and δ , which are splitted in $\mathbb{G}(2)$ as v_0 . If we denote by $\mathbb{M}(2) = \{1, c, d\}$ the monoid with $c^2 = c, d^2 = d, cd = d, dc = c$, this monoid is included in $\mathbb{G}(2)$

as $\text{End}_{\mathbb{G}}(v_1)$ via $j : 1 \mapsto 1_{v_1}, c \mapsto \gamma, d \mapsto \delta$ and is a quotient of $\mathbb{F}\mathbb{M}(2)$, the free monoïd on two generators c and d , via $\bar{\phi}$, with $\bar{\phi}(1_v) = 1, \bar{\phi}(c) = c, \bar{\phi}(d) = d$. We have $j.\bar{\phi} = \phi$.

Our $\mathbb{M}(2)$ is denoted Δ_1 and $\mathbb{M}(2)$ by Lawvere (see [9] and [10]), and a diagram of shape Δ_1 is named a *cylinder*. Lawvere observed that we recover $\mathbb{G}(2) = \text{Kar}_0(\Delta_1)$ from the Cauchy completion $\text{Kar}(\Delta_1) = \bar{\Delta}_1$ of the monoïd $\Delta_1 = \mathbb{M}(2)$ (the category obtained by splitting idempotents in Δ_1 , also named “karoubian envelope” of Δ_1). As $\delta\gamma = \gamma$ and $\gamma\delta = \delta$, $\delta : (v_1, \delta) \rightarrow (v_1, \gamma)$ and $\gamma : (v_1, \gamma) \rightarrow (v_1, \delta)$ are morphisms between idempotents, and furthermore, for the same reason, they are inverse one of the other. We obtain $\text{Kar}_0(\Delta_1)$ from $\text{Kar}(\Delta_1)$ by reduction of these inverse isomorphisms to identities on one object v_0 . \square

2. Autographic algebras

Burroni [3] defines a *graphic algebra* as an algebra of a monad on Graph . Similarly we define:

Definition 2.1. *An autographic algebra is an algebra of a monad on Agraph .*

2.1 Graphs are autographic algebras

Proposition 2.2. *In the following diagram, all the functors are monadic:*

$$\begin{array}{ccccc}
 & & \Phi=(-).\phi & & \\
 & & \curvearrowright & & \\
 \text{Graph} = \text{Ens}^{\mathbb{G}(2)} & \xrightarrow{J=(-).j} & \text{Ens}^{\mathbb{M}(2)} & \xrightarrow{\bar{\Phi}=(-).\bar{\phi}} & \text{Ens}^{\mathbb{F}\mathbb{M}(2)} = \text{Agraph} \\
 & \searrow \text{eva}_v^{\mathbb{G}(2)} = & \downarrow \text{eva}_v^{\mathbb{M}(2)} = & \swarrow \text{eva}_v^{\mathbb{F}\mathbb{M}(2)} & \\
 & & \text{Ens} & &
 \end{array}$$

Epecially, graphs are autographic algebras.

Proof. It is easy to show that J is an equivalence of categories [1, ex. 3.4, p.107]. This comes from the fact that splitting idempotents is an absolute limit construction. Then we use the known fact that for any monoïd M the forgetful functor $\text{Ens}^M \rightarrow \text{Ens}$ is monadic [1, ex. 3.5, p.109], and we

get that the three evaluations are monadic. These facts could be proved by the Linton characterization of monadicity over \mathbf{Ens} , and the property for $\bar{\Phi}$ by its extension by Borceux and Day [2] for monadicity over a category of presheaves. \square

In the next sections we will need two criterions of monadicity which could have been used here.

Proposition 2.3. *The proposition 2.2 could also be proved using Coppey or Lair criterions of monadicity.*

Proof. 1 — According to the *Coppey's criterion* [4, Prop. 2, p.17], the monadicity of a functor $\mathbf{Ens}^{\mathbb{D}} \xrightarrow{\mathbf{Ens}^K} \mathbf{Ens}^{\mathbb{C}}$ for $K : \mathbb{C} \rightarrow \mathbb{D}$ a functor bijective on objects, is equivalent to the existence of a left adjoint.

This works especially for any morphism of monoïds $f : M' \rightarrow M$, and so here for $\bar{\phi} : \mathbb{FM}(2) \rightarrow \mathbb{M}(2)$.

2 — The *Lair's criterion* [7, thm.2, p.278] [8, Corollaire, p.8]. says that the VTT condition of Beck [11, Th. 1, p147, ex. 6, p. 151] for tripleability is satisfied for a projectively sketched functor $U = \mathbf{Ens}^K : \mathbf{Ens}^{S'} \rightarrow \mathbf{Ens}^S$, sketched by a morphism of projective sketches $K : S \rightarrow S'$ if K is *basic* (or of '*Kleisli*'), i.e. if any distinguished cone in S' is based in S , and any new object in S' is the top of a distinguished cone in S' .

Here this criterion can be applied to $\bar{\phi} : \mathbb{FM}(2) \rightarrow \mathbb{M}(2)$ considered as a morphism of projective sketches, with no cones, and with no new object in $\mathbb{M}(2)$, or it could be applied to $\phi : \mathbb{FM}(2) \rightarrow \mathbb{G}(2)$, with, as a distinguished cone, the one specifying the new object v_0 as a kernel, based in $\mathbb{M}(2)$. \square

2.2 Basic graphic algebras are autographic algebras

Proposition 2.4. *If $W : \mathcal{X} \rightarrow \mathbf{Graph}$ is algebraic, i.e. if, via W , \mathcal{X} is a category of graphic algebras in the sense of Burroni [3], and if, more strictly, W is sketched by a basic morphism of small projective sketches (in the sense of Lair) $K : \mathbb{G}(2) \rightarrow S'$, with $\mathcal{X} = \mathbf{Set}^{S'}$ and $W = \mathbf{Set}^K$, then the functor $\mathbf{Set}^{K\phi} : \mathcal{X} \rightarrow \mathbf{Agraph}$ is algebraic, i.e., via $\mathbf{Set}^{K\phi}$, \mathcal{X} is a category of autographic algebras. For example this works for $\mathcal{X} = \mathbf{Cat}$: categories are autographic algebras.*

Proof. As j induces an equivalence, the question is reduced to the transfer of monadicity from $\text{Set}^{\mathbb{M}(2)}$ to $\text{Set}^{\mathbb{FM}(2)}$, via $\text{Set}^{\bar{\phi}}$. At first let us recall that this functor is monadic, i.e. that the proposition is valid if $W = \text{Id}_{\text{Graph}}$. But as $K : \mathbb{G}(2) \rightarrow S'$ is basic (see definition in proposition 2.3), also $Kj : \mathbb{M}(2) \rightarrow S'$ is basic, and then $Kj\bar{\phi} = K\phi : \mathbb{M}(2) \rightarrow S'$ is basic. So \mathcal{X} is a category of (basic) autographic algebras. \square

2.3 Autographs in Lawvere algebras are autographic algebras

Proposition 2.5. *Let T be the sketch of a Lawvere theory. Then the category $\text{Agraph}(\text{Set}^T)$ of autographs in Set^T is monadic over Agraph .*

Proof. Let u be the object in T such that each object is specified as a u^n , and let $K : \mathbb{FM}(2) \rightarrow \mathbb{FM}(2) \times T : m \mapsto (m, u)$. This functor is basic, and so the functor $\text{Set}^K : \text{Set}^{\mathbb{FM}(2) \times T} \rightarrow \text{Set}^{\mathbb{FM}(2)} = \text{Agraph}$ is monadic. And $\text{Agraph}(\text{Set}^T) = (\text{Set}^T)^{\mathbb{FM}(2)} = \text{Set}^{T \times \mathbb{FM}(2)}$. \square

2.4 Quotients of $\mathbb{FM}(2)$ and reflexive subcategories of Agraph

Proposition 2.6. *Any presentation of a 2-generated monoid M , i.e. any quotient map of monoids $q_M : \mathbb{FM}(2) \rightarrow M$ determines by composition on the right a functor $\text{Set}^{q_M} : \text{Set}^M \rightarrow \text{Set}^{\mathbb{FM}(2)}$ with left and right adjoints $\text{Lan}_{q_M} \dashv (-)^{q_M} \dashv \text{Ran}_{q_M}$, and the adjunction $\text{Lan}_{q_M} \dashv (-)^{q_M}$ determines the topos Set^M as a reflexive subcategory of Agraph which is a category of autographic algebras. Especially this works for M the monoids $\mathbb{FIM}(2)$, $\mathbb{FGM}(2)$, $\mathbb{FSM}(2) = \mathbb{M}(2)$ given in Prop. 2.8.*

Proof. As in Prop. 2.3 it is a consequence of [4, Prop. 2, p.17]. Let us precise that the corresponding idempotent monad $T_M = (-)^{q_M} \text{Lan}_{q_M}$ is clear; for (A, d, c) in $\text{Set}^{\mathbb{FM}(2)}$ we have $T_M(A, d, c) = A/[q_M]$, with $[q_M]$ the smallest congruence on (A, d, c) such that

$$\forall m, m' \in \mathbb{FM}(2) \forall u \in A \quad (q_M(m) = q_M(m') \Rightarrow mu = m'u \pmod{[q_M]});$$

so, for any $x, y \in A$, we have $x = y \pmod{[q_M]}$ if and only if

$$\exists k \geq 1, \forall j \leq k, \exists m_j, m'_j \in \mathbb{FM}(2), q(m_j) = q(m'_j), \exists u_j \in E, \\ x = m_1 u_1, m_1 u_1 = m_2 u_2, \dots, m_{k-1} u_{k-1} = m_k u_k, m_k u_k = y.$$

\square

An example of Prop. 2.6 is:

Proposition 2.7. *With $\mathbb{FM}(1) = \mathbb{N}$ (equipped with $+$) the free monoid on one generator, $\text{Set}^{\mathbb{FM}(2)}$ is the pullback of $\text{Set}^{\mathbb{FM}(1)} \rightarrow \text{Set}$ with itself. The topos $\text{Set}^{\mathbb{FM}(1)}$ of “primary structures” is an algebraic category over the topos $\text{Set}^{\mathbb{FM}(2)}$ of autographs: primary structures are autographic algebras.*

Proof. The objects of $\text{Set}^{\mathbb{FM}(1)}$ are named *primary structures* by M. Lazard. Here we have a quotient map of monoids

$$q_1 : \mathbb{FM}(2) \rightarrow \mathbb{FM}(2)/(c = d) = \mathbb{FM}(1),$$

and so (Prop. 2.6) $\text{Set}^{q_1} : \text{Set}^{\mathbb{FM}(1)} \rightarrow \text{Set}^{\mathbb{FM}(2)}$ is monadic. \square

2.5 2-generated graphic topos

One example of Prop.2.6 is associated to the free 2-generated graphic monoid.

Proposition 2.8. *1 — The monoid $\mathbb{M}(2)$ is the free right singular monoid (in which $xy = y$) on 2 generators c and d , and so is denoted by $\mathbb{FSM}(2)$.*

2 — The monoid $\mathbb{M}(2)$ is a right graphic monoid (in which $xyx = yx$), but it is not a free one.

3 — The free right graphic monoid on 2 generators c and d is denoted by $\mathbb{FGM}(2)$. It has 5 elements.

4 — $\mathbb{G}(2) = \text{Kar}_0(\mathbb{M}(2))$ is a quotient of $\bar{\mathbb{G}}(2) = \text{Kar}_0(\mathbb{FGM}(2))$.

5 — The monoid $\mathbb{FGM}(2)$ is an idempotent monoid (in which $x^2 = x$), but it is not a free one.

6 — The free idempotent monoid on 2 generators c and d is denoted by $\mathbb{FIM}(2)$. It has 7 elements. A quotient of $\mathbb{FIM}(2)$ is $\mathbb{FGM}(2)$.

Proof. $\mathbb{FGM}(2) = \{1, c, d, cd, dc\}$ is obtained by adjunction of a unit to the free right regular semigroup on two generators c and d . Then $\mathbb{M}(2)$ is a quotient of $\mathbb{FGM}(2)$, with $cdc = dc$ and $dcd = cd$. The notion of a *graphic monoid* M is used by Lawvere to introduce *graphic toposes* $\text{Ens}^{M^{\text{op}}}$. A *band* is a semigroup where every element is idempotent, a *left regular band* is a band with $xyx = xy$, and so a *graphic monoid* according to Lawvere is exactly a left regular band with unit. Then an explicit construction of the free left regular band on n generators is given in [6], which for $n = 2$ gives

a semigroup with 4 elements, and with one more element as unit we get $\text{FGM}(2)$.

We have $\text{FIM}(2) = \{1, c, d, cd, dc, cdc, dcd\}$. The description of $\text{FIM}(2)$ is given in [12]. \square

Proposition 2.9. *Any graphic topos $\text{Set}^{\text{Kar}_0(M)} \simeq \text{Set}^M$ with M a 2-generated graphic monoid is a category of autographic algebras.*

Proof. Any such M is a quotient of $\text{FGM}(2)$, and so of $\text{FIM}(2)$. Of course as $\text{FIM}(2)$ is finite, there is only a finite number of such M . \square

3. Autocategories, associative autographs, flexicategories

The following definitions precise the situation of autocategories between associative autographs and flexicategories. We see that associative autographs and autocategories are examples of autographic algebras.

Definition 3.1. *An associative autograph \mathcal{A} is an autograph (A, d, c) equipped with composition $gf : p \rightarrow r$ for any $f : p \rightarrow q, g : q \rightarrow r$, such that the two compositions of three consecutive arrows are equal:*

$$\text{Associativity: } h(gf) = (hg)f, \text{ if } f : p \rightarrow q, g : q \rightarrow r, h : r \rightarrow s.$$

An associative autograph \mathcal{A} is unitary if the underlying autograph (A, d, c) is with identifiers — i.e. with specified arrows $i_{df} : df \rightarrow df, i_{cf} : cf \rightarrow cf$ for every $f \in A$ — and if these identifiers are identities i.e. units for composition:

$$\text{Unitarity: } fi_{df} = f = i_{cf}f.$$

An unitary associative autograph is shortly named an autocategory ([5, p.76]).

The category of associative autographs is denoted by AAGraph (with morphisms the maps $F : A \rightarrow A'$ with $d'Fa = Fda, c'Fa = Fca, F(ba) = F(b)F(a)$ if $db = ca$).

The forgetful functor $W : \text{AAGraph} \rightarrow \text{Agraph}$ is given by $W(\mathcal{A}) = (A, d, c)$. The category of autocategories is denoted by Acat : it is the full subcategory of AAGraph with objects the autocategories. We have a forgetful functor $W' = WI : \text{Acat} \rightarrow \text{Agraph}$, with $I : \text{Acat} \rightarrow \text{AAGraph}$ the inclusion functor.

Remark 3.2. In the definition of an autcategory, identities are unique, and so “unitary” is a property of an associative autograph, and not a supplementary data on it. So autcategories are to associative autographs as monoïds are to semigroups.

Definition 3.3. A flexicategory is a category \mathcal{C} equipped with a flex i.e. a map $\varphi : \text{Obj}(\mathcal{C}) \rightarrow \text{Arrow}(\mathcal{C})$. The category of flexicategories is Fcat , with morphisms functors $F : \mathcal{C} \rightarrow \mathcal{C}'$ such that $\varphi'(F(X)) = F(\varphi(X))$.

We have a forgetful functor $W'' : \text{Fcat} \rightarrow \text{Cat}$ given by $W''((\mathcal{C}, \varphi)) = \mathcal{C}$.

Proposition 3.4. An autcategory \mathcal{A} “is” a mono-flexicategory, i.e. a category $U\mathcal{A} = \mathcal{C}$ equipped with a flex $\varphi : \text{Obj}(\mathcal{C}) \rightarrow \text{Arrow}(\mathcal{C})$, with the condition that φ is injective. So is defined an inclusion $J : \text{Acat} \rightarrow \text{Fcat}$, and we have the forgetful functor $W''' = W''J : \text{Acat} \rightarrow \text{Cat}$.

Proof. In [5, Prop.6.2., p.76] we saw that any mono-flexicategory determines an autcategory. But conversely, given an autcategory \mathcal{A} we can introduce an underlying category \mathcal{C} with objects the O_u , with u any identifier $u = i_{df}$ or $u = i_{cf}$ in \mathcal{A} , and then sources and targets are given by $s(f) = O_{df}$, $t(f) = O_{cf}$, and the identities on objects are $\text{Id}_{O_u} = u$. \square

Proposition 3.5. 1 — The category of autcategories is a category of autographic algebras, the associated monad is the construction of “paths with identity” given in [5, Prop. 6.3, p. 77]

$$\mathbf{P}^t(A, (d, c)) = (\text{Path}^t(A, (d, c)), D, C).$$

2 — The category AAgraph of associative autographs is a category of autographic algebras, the associated monad is the construction of “paths without adding identities” given by

$$\mathbf{P}(A, (d, c)) = (\text{Path}(A, (d, c)), D, C),$$

analogous to $\mathbf{P}^t(A, (d, c))$ excepted that we do not add identities.

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René Guitart
 Université Paris Diderot Paris 7. IMJ-PRG. UMR 7586
 Bâtiment Sophie Germain. Case 7012
 75205 Paris Cedex 13
 rene.guitart@orange.fr