

# cahiers de topologie et géométrie différentielle catégoriques

créés par CHARLES EHRESMANN en 1958  
dirigés par Andrée CHARLES EHRESMANN  
VOLUME LV-1, 1<sup>er</sup> trimestre 2014

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## A THEORY OF 2-PRO-OBJECTS

by *Emilia Descotte and Eduardo J. Dubuc*

**Résumé.** Dans [1] Grothendieck développe la théorie des pro-objets sur une catégorie  $\mathcal{C}$ . La propriété fondamentale de la catégorie  $\text{Pro}(\mathcal{C})$  des pro-objets est qu'il y a une immersion  $\mathcal{C} \xrightarrow{c} \text{Pro}(\mathcal{C})$ ,  $\text{Pro}(\mathcal{C})$  est fermée par petites limites cofiltrées, et ces limites sont libres dans le sens que pour une catégorie quelconque  $\mathbf{E}$  fermée par petites limites cofiltrées, la précomposition par  $c$  détermine une équivalence des catégories  $\text{Cat}(\text{Pro}(\mathcal{C}), \mathbf{E})_+ \simeq \text{Cat}(\mathcal{C}, \mathbf{E})$  (où "+" indique la sous-catégorie des foncteurs qui préservent les limites cofiltrées). Dans cet article nous développons une théorie des pro-objets en dimension 2. Étant donnée une 2-catégorie  $\mathcal{C}$ , nous construisons une 2-catégorie  $2\text{-Pro}(\mathcal{C})$ , dont nous appelons les objets 2-pro-objets. Nous montrons que  $2\text{-Pro}(\mathcal{C})$  a toutes les propriétés basiques attendues, correctement relativisées au contexte 2-catégorique, y compris la propriété universelle analogue à celle mentionnée ci-dessus. Bien que nous ayons à notre disposition les résultats de la théorie des catégories enrichies, notre théorie va au-delà du cas des catégories enrichies sur  $\text{Cat}$ , car nous considérons la notion non-strict de pseudo-limite, qui est usuellement celle d'intérêt pratique.

**Abstract.** In [1], Grothendieck develops the theory of pro-objects over a category  $\mathcal{C}$ . The fundamental property of the category  $\text{Pro}(\mathcal{C})$  is that there is an embedding  $\mathcal{C} \xrightarrow{c} \text{Pro}(\mathcal{C})$ , the category  $\text{Pro}(\mathcal{C})$  is closed under small cofiltered limits, and these limits are free in the sense that for any category  $\mathbf{E}$  closed under small cofiltered limits, pre-composition with  $c$  determines an equivalence of categories  $\text{Cat}(\text{Pro}(\mathcal{C}), \mathbf{E})_+ \simeq \text{Cat}(\mathcal{C}, \mathbf{E})$ , (where the "+" indicates the full subcategory of the functors preserving cofiltered limits). In this paper we develop a 2-dimensional theory of pro-objects. Given a 2-category  $\mathcal{C}$ , we define the 2-category  $2\text{-Pro}(\mathcal{C})$  whose

objects we call 2-pro-objects. We prove that  $2\text{-Pro}(\mathcal{C})$  has all the expected basic properties adequately relativized to the 2-categorical setting, including the universal property corresponding to the one described above. We have at hand the results of *Cat*-enriched category theory, but our theory goes beyond the *Cat*-enriched case since we consider the non strict notion of pseudo-limit, which is usually that of practical interest.

**Key words.** 2-pro-object, 2-filtered, pseudo-limit.

**MS classification.** Primary 18D05, Secondary 18A30.

**Introduction.** In this paper we develop a 2-dimensional theory of pro-objects. Our motivation are intended applications in homotopy, in particular strong shape theory. The Čech nerve before passing modulo homotopy determines a 2-pro-object which is not a pro-object, leaving outside the actual theory of pro-objects. Also, the theory of 2-pro-objects reveals itself a very interest subject in its own right.

Given a 2-category  $\mathcal{C}$ , we define the 2-category  $2\text{-Pro}(\mathcal{C})$ , whose objects we call 2-pro-objects. A 2-pro-object is a 2-functor (or diagram) indexed in a 2-cofiltered 2-category. Our theory goes beyond enriched category theory because in the definition of morphisms, instead of strict 2-limits, we use the non strict notion of pseudo-limit, which is usually that of practical interest. We prove that  $2\text{-Pro}(\mathcal{C})$  has all the expected basic properties of the category of pro-objects, adequately relativized to the 2-categorical setting.

Section 1 contains some background material on 2-categories. Most of this is standard, but some results (for which we provide proofs) do not appear to be in the literature. In particular we prove that pseudolimits are computed pointwise in the 2-functor 2-categories  $\mathcal{H}om(\mathcal{C}, \mathcal{D})$  and  $\mathcal{H}om_p(\mathcal{C}, \mathcal{D})$  (definition 1.1.11), with 2-natural or pseudonatural transformations as arrows. This result, although expected, needs nevertheless a proof. We recall from [8] the construction of 2-filtered pseudocolimits of categories which is essential for the computations in the 2-category of 2-pro-objects introduced in section 2. Finally, we consider the notion of flexible functors from [4] and state a useful characterization independent of the left adjoint to the inclusion  $\mathcal{H}om(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{H}om_p(\mathcal{C}, \mathcal{D})$  (Proposition 1.3.2). With this characterization the pseudo Yoneda lemma just says

that the representable 2-functors are flexible. It follows also that the 2-functor associated to any 2-pro-object is flexible, and this has important consequences for a Quillen model structure in the 2-category of 2-pro-objects currently being developed by the authors in ongoing research.

Section 2 contains the main results of this paper. In a first subsection we define the 2-category of 2-pro-objects of a 2-category  $\mathcal{C}$  and establish the basic formula for the morphisms and 2-cells between 2-pro-objects in terms of pseudo limits and pseudo colimits of the hom categories of  $\mathcal{C}$ . With this, inspired in the notion of an arrow representing a morphism of pro-objects found in [3], in the next subsection we introduce the notion of an arrow and a 2-cell in  $\mathcal{C}$  representing an arrow and a 2-cell in  $2\text{-}\mathcal{P}ro(\mathcal{C})$ , and develop computational properties of 2-pro-objects which are necessary in our proof that the 2-category  $2\text{-}\mathcal{P}ro(\mathcal{C})$  is closed under 2-cofiltered pseudo limits. In the third subsection we construct a 2-filtered category which serves as the index 2-category for the 2-filtered pseudolimit of 2-pro-objects (Definition 2.3.1 and Theorem 2.3.3). This is also inspired in a construction and proof for the same purpose found in [3], but which in our 2-dimensional case reveals itself very complex and difficult to manage effectively. We were forced to have recourse to this complicated construction because the conceptual treatment of this problem found in [1] does not apply in the 2-dimensional case. This is so because a 2-functor is not the pseudocolimit of 2-representables indexed by its 2-category of elements. Finally, in the last subsection we prove the universal properties of  $2\text{-}\mathcal{P}ro(\mathcal{C})$  (Theorem 2.4.6), in a way which is novel even if applied to the classical theory of pro-objects.

## 1 Preliminaries on 2-categories

We distinguish between *small* and *large* sets. For us *legitimate* categories are categories with small hom sets, also called *locally small*. We freely consider without previous warning illegitimate categories with large hom sets, for example the category of all (legitimate) categories, or functor categories with large (legitimate) exponent. They are legitimate as categories in some higher universe, or they can be considered as convenient notational abbreviations for large collections of data. In fact, questions of size play no overt role in this paper, except that

we elect for simplicity to consider only small 2-pro-objects. We will explicitly mention whether the categories are legitimate or small when necessary. We reserve the notation  $Cat$  for the legitimate 2-category of small categories, and we will denote  $\mathcal{CAT}$  the illegitimate category (or 2-category) of all legitimate categories in some arbitrary sufficiently high universe.

**Notation.** 2-Categories will be denoted with the “mathcal” font  $\mathcal{C}$ ,  $\mathcal{D}$ , ... , 2-functors with the capital “mathff” font,  $F$ ,  $G$ , ... and 2-natural transformations, pseudonatural transformations and modifications with the greek alphabet. For objects in a 2-category, we will use capital “mathff” font  $C$ ,  $D$ , ... , for arrows in a 2-category small case letters in “mathff” font  $f$ ,  $g$ , ... , and for the 2-cells the greek alphabet. However, when a 2-category is intended to be used as the index 2-category of a 2-diagram, we will use small case letters  $i$ ,  $j$ , ... to denote its objects, and small case letters  $u$ ,  $v$ , ... to denote its arrows. Categories will be denoted with capital “mathff” font.

We begin with some background material on 2-categories. Most of this is standard, but some results (for which we provide proofs) do not appear to be in the literature. We also set notation and terminology as we will explicitly use in this paper.

## 1.1 Basic theory

Let  $Cat$  be the category of small categories. By a 2-category, we mean a  $Cat$  enriched category. A 2-functor, a 2-fully-faithful 2-functor, a 2-natural transformation and a 2-equivalence of 2-categories, are a  $Cat$ -functor, a  $Cat$ -fully-faithful functor, a  $Cat$ -natural transformation and a  $Cat$ -equivalence respectively.

In the sequel we will call *2-category* an structure satisfying the following descriptive definition free of the size restrictions implicit above. Given a 2-category, as usual, we denote horizontal composition by juxtaposition, and vertical composition by a “ $\circ$ ”.

**1.1.1. 2-Category.** A 2-category  $\mathcal{C}$  consists on objects or 0-cells  $C$ ,  $D$  ... , arrows or 1-cells  $f$ ,  $g$  ... , and 2-cells  $\alpha$ ,  $\beta$ , ... .

$$\begin{array}{ccc} & f & \\ & \longrightarrow & \\ C & \xrightarrow{\alpha \Downarrow} & D \\ & g & \end{array}$$

The objects and the arrows form a category (called the underlying category of  $\mathcal{C}$ ), with composition (called "horizontal") denoted by juxtaposition. For a fixed  $C$  and  $D$ , the arrows between them and the 2-cells between these arrows form a category  $\mathcal{C}(C, D)$  under "vertical" composition, denoted by a " $\circ$ ". There is also an associative horizontal composition between 2-cells denoted by juxtaposition, with units  $id_{id_C}$ . The following is the basic 2-category diagram:

$$\begin{array}{ccccc}
 & \xrightarrow{f} & & \xrightarrow{f'} & \\
 & \downarrow \alpha & & \downarrow \alpha' & \\
 C & \xrightarrow{g} & D & \xrightarrow{g'} & E \\
 & \downarrow \beta_h & & \downarrow \beta'_h & \\
 & \longrightarrow & & \longrightarrow & 
 \end{array}$$

with the equations  $(\beta' \beta) \circ (\alpha' \alpha) = (\beta' \circ \alpha')(\beta \circ \alpha)$ ,  $id_{f'} id_f = id_{ff}$ .

We consider juxtaposition more binding than " $\circ$ ", thus  $\alpha \beta \circ \gamma$  means  $(\alpha \beta) \circ \gamma$ . We will abuse notation by writing  $f$  instead of  $id_f$  for morphisms  $f$  and  $C$  instead of  $id_C$  for objects  $C$ .

**1.1.2. Dual 2-Category.** If  $\mathcal{C}$  is a 2-category, we denote by  $\mathcal{C}^{op}$  the 2-category with the same objects as  $\mathcal{C}$  but with  $\mathcal{C}^{op}(C, D) = \mathcal{C}(D, C)$ , i.e. we reverse the 1-cells but not the 2-cells.

**1.1.3. 2-functor.** A 2-functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between 2-categories is an enriched functor over  $\mathcal{Cat}$ . As such, sends objects to objects, arrows to arrows and 2-cells to 2-cells, strictly preserving all the structure.

**1.1.4. 2-fully-faithful.** A 2-functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is said to be 2-fully-faithful if  $\forall C, D \in \mathcal{C}$ ,  $F_{C,D} : \mathcal{C}(C, D) \rightarrow \mathcal{D}(FC, FD)$  is an isomorphism of categories.

**1.1.5. Pseudonatural.** A pseudonatural transformation  $\mathcal{C} \xrightarrow{\theta} \mathcal{D}$

between 2-functors consists in a family of morphisms  $\{FC \xrightarrow{\theta_C} GC\}_{C \in \mathcal{C}}$  and a family of invertible 2-cells  $\{Gf \theta_C \xrightarrow{\theta_f} \theta_D Ff\}_{C \xrightarrow{f} D \in \mathcal{C}}$

$$\begin{array}{ccc}
FC & \xrightarrow{\theta_C} & GC \\
Ff \downarrow & \Downarrow \theta_f & \downarrow Gf \\
FD & \xrightarrow{\theta_D} & GD
\end{array}$$

satisfying the following conditions:

$$PN0: \forall C \in \mathcal{C}, \quad \theta_{id_C} = id_{\theta_C}$$

$$PN1: \forall C \xrightarrow{f} D \xrightarrow{g} E, \quad \theta_{gf} = \theta_g Ff \circ Gg \theta_f.$$

$$PN2: \forall C \xrightarrow[\alpha \Downarrow]{f} D, \quad \theta_g \circ G\alpha \theta_C = \theta_D F\alpha \circ \theta_f$$

**1.1.6. 2-Natural.** A 2-natural transformation  $\theta$  between 2-functors is a pseudonatural transformation such that  $\theta_f = id \forall f \in \mathcal{C}$ . Equivalently, it is a *Cat*-enriched natural transformation, that is, a natural transformation between the functors determined by  $F$  and  $G$ , such that for each

$$2\text{-cell } C \xrightarrow[\alpha \Downarrow]{f} D, \text{ the equation } G\alpha \theta_C = \theta_D F\alpha \text{ holds.} \quad \square$$

**1.1.7. Modification.** Given 2-functors  $F$  and  $G$  from  $\mathcal{C}$  to  $\mathcal{D}$ , a

$$\text{modification } F \xrightarrow[\rho \Downarrow]{\theta} G \text{ between pseudonatural transformations is a}$$

family  $\{\theta_C \xrightarrow{\rho_C} \eta_C\}_{C \in \mathcal{C}}$  of 2-cells of  $\mathcal{D}$  such that:

$$\forall C \xrightarrow{f} D \in \mathcal{C}, \quad \rho_D Ff \circ \theta_f = \eta_f \circ Gf \rho_C.$$

As a particular case, we have modifications between 2-natural transformations, which are families of 2-cells as above satisfying  $\rho_D Ff = Gf \rho_C$ .

**1.1.8. 2-Equivalence.** A 2-functor  $\mathcal{C} \xrightarrow{F} \mathcal{D}$  is said to be a 2-equivalence of 2-categories if there exists a 2-functor  $\mathcal{D} \xrightarrow{G} \mathcal{C}$  and invertible 2-natural transformations  $FG \xrightarrow{\alpha} id_{\mathcal{D}}$  and  $GF \xrightarrow{\beta} id_{\mathcal{C}}$ .  $G$  is said to be a quasi-inverse of  $F$ , and it is determined up to invertible 2-natural transformations.

**1.1.9 Proposition.** [11, 1.11] *A 2-functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a 2-equivalence of 2-categories if and only if it is 2-fully-faithful and essentially surjective on objects.*  $\square$

**1.1.10.** It is well known that 2-categories, 2-functors and 2-natural transformations form a 2-category (which actually underlies a 3-category) that we denote  $2\text{-CAT}$ . Horizontal composition of 2-functors and vertical composition of 2-natural transformations are the usual ones, and the horizontal composition of 2-natural transformations is defined by:

$$\text{Given } \mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \alpha \Downarrow \\ \xrightarrow{G} \end{array} \mathcal{D} \begin{array}{c} \xrightarrow{F'} \\ \alpha' \Downarrow \\ \xrightarrow{G'} \end{array} \mathcal{E}, \quad (\alpha' \alpha)_\mathcal{C} = \alpha'_{\mathcal{G}\mathcal{C}} \circ F'(\alpha_\mathcal{C}) \quad (= G'(\alpha_\mathcal{C}) \circ \alpha'_{\mathcal{F}\mathcal{C}}).$$

**1.1.11 Definition.** *Given two 2-categories  $\mathcal{C}$  and  $\mathcal{D}$ , we consider two 2-categories defined as follows:*

$\mathcal{H}om(\mathcal{C}, \mathcal{D})$  : 2-functors and 2-natural transformations.

$\mathcal{H}om_p(\mathcal{C}, \mathcal{D})$  : 2-functors and pseudonatural transformations.

*In both cases the 2-cells are the modifications. To define compositions we draw the basic 2-category diagram:*

$$\begin{array}{ccc} \xrightarrow{\theta} & \xrightarrow{\theta'} & \\ \Downarrow \rho & \Downarrow \rho' & \\ \mathcal{F} \xrightarrow{\eta} \mathcal{G} & \xrightarrow{\eta'} \mathcal{H} & \\ \Downarrow \varepsilon & \Downarrow \varepsilon' & \\ \xrightarrow{\mu} & \xrightarrow{\mu'} & \end{array} \quad \begin{array}{l} (\theta' \theta)_\mathcal{C} = \theta'_\mathcal{C} \theta_\mathcal{C} \\ (\rho' \rho)_\mathcal{C} = \rho'_\mathcal{C} \rho_\mathcal{C} \\ (\varepsilon \circ \rho)_\mathcal{C} = \varepsilon_\mathcal{C} \circ \rho_\mathcal{C} \end{array}$$

*It is straightforward to check that these definitions determine a 2-category structure.*  $\square$

**1.1.12 Remark.** [9, I,4.2.] Evaluation determines a *quasifunctor*  $\mathcal{H}om_p(\mathcal{C}, \mathcal{D}) \times \mathcal{C} \xrightarrow{ev} \mathcal{D}$  (in the sense of [9, I,4.1.], in particular, fixing a variable, it is a 2-functor in the other). In the strict case  $\mathcal{H}om$ , evaluation is actually a 2-bifunctor.  $\square$

**1.1.13 Remark.** [9, I,4.2.] Both constructions  $\mathcal{H}om$  and  $\mathcal{H}om_p$  determine a bifunctor  $2\text{-CAT}^{op} \times 2\text{-CAT} \rightarrow 2\text{-CAT}$ . Given 2-functors



$\mathcal{C}' \xrightarrow{H_0} \mathcal{C}$  and  $\mathcal{D} \xrightarrow{H_1} \mathcal{D}'$ , and  $F \xrightarrow[\eta]{\rho \Downarrow \theta} G$  in  $\mathcal{H}om_\epsilon(\mathcal{C}, \mathcal{D})(F, G)$ , the

definition  $\mathcal{H}om_\epsilon(H_0, H_1)(F \xrightarrow[\eta]{\rho \Downarrow \theta} G) = H_1 F H_0 \xrightarrow[\begin{smallmatrix} H_1 \rho H_0 \Downarrow \\ H_1 \eta H_0 \end{smallmatrix}]{H_1 \theta H_0} H_1 G H_0$  determines a functor  $\mathcal{H}om_\epsilon(\mathcal{C}, \mathcal{D})(F, G) \longrightarrow \mathcal{H}om_\epsilon(\mathcal{C}', \mathcal{D}')(H_1 F H_0, H_1 G H_0)$ , and this assignation is bifunctorial in the variable  $(\mathcal{C}, \mathcal{D})$  (here  $\mathcal{H}om_\epsilon$  denotes either  $\mathcal{H}om$  or  $\mathcal{H}om_p$ ).

If  $\mathcal{C}$  and  $\mathcal{D}$  are 2-categories, the product 2-category  $\mathcal{C} \times \mathcal{D}$  is constructed in the usual way, and this together with the 2-category  $\mathcal{H}om(\mathcal{C}, \mathcal{D})$  determine a symmetric cartesian closed structure as follows (see [11, chapter 2] or [9, I,2.3.]):

**1.1.14 Proposition.** *The usual definitions determine an isomorphism of 2-categories :*

$$\mathcal{H}om(\mathcal{C}, \mathcal{H}om(\mathcal{D}, \mathcal{A})) \xrightarrow{\cong} \mathcal{H}om(\mathcal{C} \times \mathcal{D}, \mathcal{A}).$$

Composing with the symmetry  $\mathcal{C} \times \mathcal{D} \xrightarrow{\cong} \mathcal{D} \times \mathcal{C}$  yields an isomorphism:

$$\mathcal{H}om(\mathcal{C}, \mathcal{H}om(\mathcal{D}, \mathcal{A})) \xrightarrow{\cong} \mathcal{H}om(\mathcal{D}, \mathcal{H}om(\mathcal{C}, \mathcal{A})). \quad \square$$

We use the following notation:

**Notation:** Let  $\mathcal{C}$  be a 2-category,  $C \in \mathcal{C}$  and  $D \xrightarrow[\mathbf{g}]{\mathbf{f}} E \in \mathcal{C}$ .

1.  $f_* : \mathcal{C}(C, D) \xrightarrow{f_*} \mathcal{C}(C, E)$ ,  $f_*(h \xrightarrow{\beta} h') = (fh \xrightarrow{f\beta} fh')$ .
2.  $f^* : \mathcal{C}(E, C) \xrightarrow{f^*} \mathcal{C}(D, C)$ ,  $f^*(h \xrightarrow{\beta} h') = (hf \xrightarrow{\beta f} hf')$ .
3.  $\alpha_* : f_* \xrightarrow{\alpha_*} g_*$ ,  $(\alpha_*)_h = \alpha h$ .
4.  $\alpha^* : f^* \xrightarrow{\alpha^*} g^*$ ,  $(\alpha^*)_h = h\alpha$ .
5.  $\mathcal{C} \xrightarrow{\mathcal{C}(C, -)} \mathcal{C}at : \mathcal{C}(C, -)(D \xrightarrow[\mathbf{g}]{\mathbf{f}} E) = (\mathcal{C}(C, D) \xrightarrow[\mathbf{g}^*]{\mathbf{f}_*} \mathcal{C}(C, E))$ .

6.  $\mathcal{C}^{op} \xrightarrow{\mathcal{C}(-, \mathcal{C})} \mathcal{C}at: \mathcal{C}(-, \mathcal{C})(D \xrightarrow[\underset{\mathbf{g}}{\alpha \Downarrow}}{\mathbf{f}}]{\mathbf{f}} E) = (\mathcal{C}(D, \mathcal{C}) \xrightarrow[\underset{\mathbf{g}^*}{\alpha^* \Downarrow}]{\mathbf{f}^*} \mathcal{C}(E, \mathcal{C}))$ .
7. We will also denote by  $\mathbf{f}^*$  the 2-natural transformation from  $\mathcal{C}(E, -)$  to  $\mathcal{C}(D, -)$  defined by  $(\mathbf{f}^*)_C = \mathbf{f}^*$ .
8. We will also denote by  $\mathbf{f}_*$  the 2-natural transformation from  $\mathcal{C}(-, D)$  to  $\mathcal{C}(-, E)$  defined by  $(\mathbf{f}_*)_C = \mathbf{f}_*$ .
9. We will also denote by  $\alpha^*$  the modification from  $\mathbf{f}^*$  to  $\mathbf{g}^*$  defined by  $(\alpha^*)_C = \alpha^*$ .
10. We will also denote by  $\alpha_*$  the modification from  $\mathbf{f}_*$  to  $\mathbf{g}_*$  defined by  $(\alpha_*)_C = \alpha_*$ .  $\square$

**1.1.15.** *Given a locally small 2-category  $\mathcal{C}$ , the Yoneda 2-functors are the following (note that each one is the other for the dual 2-category):*

- a.  $\mathcal{C} \xrightarrow{y^{(-)}} \mathcal{H}om(\mathcal{C}, \mathcal{C}at)^{op}$ ,  $y^C = \mathcal{C}(C, -)$ ,  $y^f = \mathbf{f}^*$ ,  $y^\alpha = \alpha^*$ .
- b.  $\mathcal{C} \xrightarrow{y^{(-)}} \mathcal{H}om(\mathcal{C}^{op}, \mathcal{C}at)$ ,  $y_C = \mathcal{C}(-, C)$ ,  $y_f = \mathbf{f}_*$ ,  $y_\alpha = \alpha_*$ .

Recall the Yoneda Lemma for enriched categories over  $\mathcal{C}at$ . We consider explicitly only the case a. in 1.1.15.

**1.1.16 Proposition (Yoneda lemma).** *Given a locally small 2-category  $\mathcal{C}$ , a 2-functor  $F: \mathcal{C} \rightarrow \mathcal{C}at$  and an object  $C \in \mathcal{C}$ , there is an isomorphism of categories, natural in  $F$ .*

$$\begin{array}{ccc} \mathcal{H}om(\mathcal{C}, \mathcal{C}at)(\mathcal{C}(C, -), F) & \xrightarrow{h} & FC \\ \theta \xrightarrow{\rho} \eta & \longmapsto & \theta_C(id_C) \xrightarrow{(\rho_C)id_C} \eta_C(id_C) \end{array}$$

*Proof.* The application  $h$  has an inverse

$$\begin{array}{ccc} FC & \xrightarrow{\ell} & \mathcal{H}om(\mathcal{C}, \mathcal{C}at)(\mathcal{C}(C, -), F) \\ C \xrightarrow{f} D & \longmapsto & \ell C \xrightarrow{\ell f} \ell D \end{array}$$

where  $(\ell C)_D(f \xrightarrow{\alpha} g) = \mathbf{F}f(C) \xrightarrow{(\mathbf{F}\alpha)_C} \mathbf{F}g(C)$  and  $((\ell f)_D)_f = \mathbf{F}f$ .  $\square$

**1.1.17 Corollary.** *The Yoneda 2-functors in 1.1.15 are 2-fully-faithful.*  $\square$

Beyond the theory of  $Cat$ -enriched categories, the lemma also holds for pseudonatural transformations in the following way:

**1.1.18 Proposition (Pseudo Yoneda lemma).** *Given a locally small 2-category  $\mathcal{C}$ , a 2-functor  $F : \mathcal{C} \rightarrow Cat$  and an object  $C \in \mathcal{C}$ , there is an equivalence of categories, natural in  $F$ .*

$$\begin{array}{ccc} \mathcal{H}om_p(\mathcal{C}, Cat)(\mathcal{C}(C, -), F) & \xrightarrow{\tilde{h}} & FC \\ \theta \xrightarrow{\rho} \eta & \longmapsto & \theta_C(id_C) \xrightarrow{(\rho_C)id_C} \eta_C(id_C) \end{array}$$

Furthermore, the quasi-inverse  $\tilde{\ell}$  is a section of  $\tilde{h}$ ,  $\tilde{h}\tilde{\ell} = id$ .

*Proof.*  $\tilde{h}$  and  $\tilde{\ell}$  are defined as in 1.1.16, but now  $\tilde{\ell}$  is only a section quasi-inverse of  $\tilde{h}$ . The details can be checked by the reader. One can find a guide in [13] for the case of lax functors and bicategories. We refer to the arguing and the notation there: In our case, the unit  $\eta$  is the equality because  $F$  is a 2-functor, and the counit  $\epsilon$  is an isomorphism because  $\alpha$  is pseudonatural and the unitor  $r$  is the equality.  $\square$

**1.1.19 Corollary.** *For any locally small 2-category  $\mathcal{C}$ , and  $C \in \mathcal{C}$ , the inclusion  $\mathcal{H}om(\mathcal{C}, Cat)(\mathcal{C}(C, -), F) \xrightarrow{i} \mathcal{H}om_p(\mathcal{C}, Cat)(\mathcal{C}(C, -), F)$  has a retraction  $\alpha$ , natural in  $F$ ,  $\alpha i = id$ ,  $i\alpha \cong id$ , which determines an equivalence of categories.*

*Proof.* Note that  $i = \tilde{\ell}h$ , then define  $\alpha = \ell\tilde{h}$ .  $\square$

**1.1.20 Corollary.** *The Yoneda 2-functors in 1.1.15 can be considered as 2-functors landing in the  $\mathcal{H}om_p$  2-functor categories. In this case, they are pseudo-fully faithful (meaning that they determine equivalences and not isomorphisms between the hom categories).*  $\square$

## 1.2 Weak limits and colimits

By *weak* we understand any of the several ways universal properties can be relaxed in 2-categories. Note that pseudolimits and pseudocolimits (already considered in [2]) require isomorphisms, and have many advantages over bilimits and bicolimits, which only require equivalences. Their universal properties are both stronger and more convenient to use, and they play the principal role in this paper. The defining universal properties characterize bilimits up to equivalence and pseudolimits up to isomorphism.

**Notation** We consider pseudocolimits  $\varinjlim_{i \in \mathcal{I}} Fi$ , and bicolimits  $\mathbf{bi}\varinjlim_{i \in \mathcal{I}} Fi$ , of covariant 2-functors, and its dual concepts, pseudolimits  $\varprojlim_{i \in \mathcal{I}} Fi$ , and bilimits  $\mathbf{bi}\varprojlim_{i \in \mathcal{I}} Fi$ , of contravariant 2-functors.

**1.2.1 Definition.** Let  $F : \mathcal{I} \rightarrow \mathcal{A}$  be a 2-functor and  $A$  an object of  $\mathcal{A}$ . A pseudocone for  $F$  with vertex  $A$  is a pseudonatural transformation from  $F$  to the 2-functor which is constant at  $A$ , i.e. it consists in a family of morphisms of  $\mathcal{A}$   $\{Fi \xrightarrow{\theta_i} A\}_{i \in \mathcal{I}}$  and a family of invertible 2-cells of  $\mathcal{A}$   $\{\theta_i \xrightarrow{\theta_u} \theta_j Fu\}_{i \xrightarrow{u} j \in \mathcal{I}}$  satisfying the following equations:

$$\begin{aligned} PC0: & \quad \theta_{id_i} = id_{\theta_i} . \\ PC1: & \quad \forall i \xrightarrow{u} j \xrightarrow{v} k \in \mathcal{I}, \quad \theta_v Fu \circ \theta_u = \theta_{vu} \\ PC2: & \quad \forall i \xrightarrow[\alpha \Downarrow]{u} j \in \mathcal{I}, \quad \theta_i = \theta_j F\alpha \circ \theta_u \end{aligned}$$

A morphism of pseudocones between  $\theta$  and  $\eta$  with the same vertex is a modification, i.e. a family of 2-cells of  $\mathcal{A}$   $\{\theta_i \xrightarrow{\rho_i} \eta_i\}_{i \in \mathcal{I}}$  satisfying the following equation:

$$PCM: \quad \eta_u \circ \rho_i = \rho_j Fu \circ \theta_u .$$

Pseudocones form a category  $PC_{\mathcal{A}}(F, A) = \mathcal{H}om_p(\mathcal{I}, \mathcal{A})(F, A)$  furnished with a pseudocone  $PC_{\mathcal{A}}(F, A) \rightarrow \mathcal{A}(F, A)$ ,  $\{\theta_i\}_{i \in \mathcal{I}} \mapsto \theta_i$ , for the 2-functor  $\mathcal{I}^{op} \xrightarrow{\mathcal{A}(F(-), A)} \mathcal{CAT}$ .

### 1.2.2 Remark.

Since  $\mathcal{H}om_p(\mathcal{I}, \mathcal{A})$  is a 2-category, it follows:

a. Pseudocones determine a 2-bifunctor  $\mathcal{H}om(\mathcal{I}, \mathcal{A})^{op} \times \mathcal{A} \xrightarrow{\text{PC}_{\mathcal{A}}} \mathcal{CAT}$ .

From Remark 1.1.13 it follows in particular:

b. A 2-functor  $\mathcal{A} \xrightarrow{\text{H}} \mathcal{B}$  induces a functor between the categories of pseudocones  $\text{PC}_{\mathcal{A}}(\text{F}, \text{A}) \xrightarrow{\text{PC}_{\mathcal{H}}} \text{PC}_{\mathcal{B}}(\text{HF}, \text{HA})$ .  $\square$

**1.2.3 Definition.** *The pseudocolimit in  $\mathcal{A}$  of the 2-functor  $\text{F}$  is the universal pseudocone, denoted  $\{\text{Fi} \xrightarrow{\lambda_i} \varinjlim_{i \in \mathcal{I}} \text{Fi}\}_{i \in \mathcal{I}}$ , in the sense that  $\forall \text{A} \in \mathcal{A}$ , pre-composition with the  $\lambda_i$  is an isomorphism of categories  $\mathcal{A}(\varinjlim_{i \in \mathcal{I}} \text{Fi}, \text{A}) \xrightarrow{\lambda^*} \text{PC}_{\mathcal{A}}(\text{F}, \text{A})$ . Equivalently, there is an isomorphism of categories  $\mathcal{A}(\varinjlim_{i \in \mathcal{I}} \text{Fi}, \text{A}) \xrightarrow{\cong} \varprojlim_{i \in \mathcal{I}^{op}} \mathcal{A}(\text{Fi}, \text{A})$  commuting with the pseudocones. Remark that there is also an isomorphism of categories  $\text{PC}_{\mathcal{A}}(\text{F}, \text{A}) \xrightarrow{\cong} \varprojlim_{i \in \mathcal{I}^{op}} \mathcal{A}(\text{Fi}, \text{A})$*

*Requiring  $\lambda^*$  to be an equivalence (which implies that also the other two isomorphisms above are equivalences) defines the notion of bicolimit. Clearly, pseudocolimits are bicolimits.*

*We omit the explicit consideration of the dual concepts.*  $\square$

It is well known that in the strict 2-functor 2-categories the strict limits and colimits are performed pointwise (if they exists in the codomain category). Here we establish this fact for the pseudo limits and pseudocolimits in both the strict and the pseudo 2-functor 2-categories. Abusing notation we can say that the formula  $(\varinjlim_{i \in \mathcal{I}} \text{Fi})(\text{C}) = \varinjlim_{i \in \mathcal{I}} \text{Fi}(\text{C})$  holds in both 2-categories. The verification of this is straightforward but requires some care.

**1.2.4 Proposition.** *Let  $\mathcal{I} \xrightarrow{\text{F}} \mathcal{A}$ ,  $i \mapsto \text{Fi}$  be a 2-functor where  $\mathcal{A}$  is either  $\mathcal{H}om(\mathcal{C}, \mathcal{D})$  or  $\mathcal{H}om_p(\mathcal{C}, \mathcal{D})$ . For each  $\text{C} \in \mathcal{C}$  let  $\text{FiC} \xrightarrow{\lambda_i^{\text{C}}} \text{LC}$  be a pseudocolimit pseudocone in  $\mathcal{D}$  for the 2 functor  $\mathcal{I} \xrightarrow{\text{F}} \mathcal{A} \xrightarrow{\text{ev}(-, \text{C})} \mathcal{D}$  (where  $\text{ev}$  is evaluation, see 1.1.12). Then  $\text{LC}$  is 2-functorial in  $\mathcal{C}$  in such a way that  $\lambda_i^{\text{C}}$  becomes 2-natural and  $\text{Fi} \xrightarrow{\lambda_i} \text{L}$  is a pseudocolimit pseudocone in  $\mathcal{A}$  in both cases. By duality the same assertion holds for pseudolimits.*

*Proof.* Given  $C \xrightarrow[\alpha \Downarrow]{f} D$  in  $\mathcal{C}$ , evaluation determines a 2-cell in  $\mathcal{H}om(\mathcal{I}, \mathcal{D})$   $FC \xrightarrow[\text{Fg}]{\text{Ff}} FD = ev(F(-), C \xrightarrow[\alpha \Downarrow]{f} D)$ . (note that  $(FC)_i = F_i C$ , and similarly for  $f$ ,  $g$  and  $\alpha$ ). Then, for each  $X \in \mathcal{D}$ , it follows (from Remark 1.2.2 a.) that precomposing with this 2-cell determines a 2-cell (clearly 2-natural in the variable  $X$ ) in the right leg of the diagram below. Since the rows are isomorphisms, there is a unique 2-cell (also natural in the variable  $X$ ) in the left leg which makes the diagram commutative.

$$\begin{array}{ccc} \mathcal{D}(\text{LD}, X) & \xrightarrow[\cong]{(\lambda^D)^*} & \text{PC}_{\mathcal{D}}(\text{FD}, X) \\ \downarrow \Rightarrow \downarrow & & \downarrow \Rightarrow \downarrow \\ \mathcal{D}(\text{LC}, X) & \xrightarrow[\cong]{(\lambda^C)^*} & \text{PC}_{\mathcal{D}}(\text{FC}, X) \end{array}$$

Then, by the Yoneda lemma 1.1.17, the left leg is given by precomposing with a unique 2-cell in  $\mathcal{D}$ , that we denote  $\text{LC} \xrightarrow[\text{Lg}]{\text{Lf}} \text{LD}$ . It is clear by uniqueness that this determines a 2-functor  $\mathcal{C} \xrightarrow{\text{L}} \mathcal{D}$ .

Putting  $X = \text{LD}$  in the upper left corner and tracing the identity down the diagram yields the following commutative diagram of pseudocones in  $\mathcal{D}$ :

$$\begin{array}{ccc} F_i C & \xrightarrow{\lambda_i^C} & \text{LC} \\ F_i f \downarrow \begin{array}{c} F_i \alpha \\ \Rightarrow \\ F_i g \end{array} & & \downarrow \begin{array}{c} \text{L} \alpha \\ \Rightarrow \\ \text{L} g \end{array} \\ F_i D & \xrightarrow{\lambda_i^D} & \text{LD} \end{array}$$

This shows that  $\text{L}$  is furnished with a pseudocone for  $F$  and that the  $\lambda_i$  are 2-natural. It only remains to check the universal property:

Let  $\mathcal{C} \xrightarrow{\text{G}} \mathcal{D}$  be a 2-functor, consider the 2-functor  $\mathcal{A} \xrightarrow{ev(-, \mathcal{C})} \mathcal{D}$ . We have the following diagram, where the right leg is given by

Remark 1.2.2 b.:

$$\begin{array}{ccc} \mathcal{A}(\mathbf{L}, \mathbf{G}) & \xrightarrow{\lambda^*} & \mathrm{PC}_{\mathcal{A}}(\mathbf{F}, \mathbf{G}) \\ \downarrow \mathrm{ev}(-, \mathbf{C}) & & \downarrow \mathrm{PC}_{\mathrm{ev}(-, \mathbf{C})} \\ \mathcal{D}(\mathbf{LC}, \mathbf{GC}) & \xrightarrow[\cong]{(\lambda^{\mathbf{C}})^*} & \mathrm{PC}_{\mathcal{D}}(\mathbf{FC}, \mathbf{GC}) \end{array}$$

We prove now that the upper row is an isomorphism. Given  $\mathbf{F}_i \xrightarrow[\eta_i]{\rho_i \downarrow} \mathbf{G}$  in  $\mathrm{PC}_{\mathcal{A}}(\mathbf{F}, \mathbf{G})$ , it follows there exists a unique  $\mathbf{LC} \xrightarrow[\tilde{\eta}\mathbf{C}]{\tilde{\rho}\mathbf{C} \downarrow} \mathbf{GC}$  in  $\mathcal{D}(\mathbf{LC}, \mathbf{GC})$  such that  $\tilde{\rho}\mathbf{C} \lambda_i^{\mathbf{C}} = \rho_i \mathbf{C}$ . It is necessary to show that this 2-cell actually lives in  $\mathcal{A}$ . This has to be checked for any  $\mathbf{C} \xrightarrow[\mathbf{g}]{\alpha \downarrow} \mathbf{D}$  in  $\mathcal{C}$ . In both cases it can be done considering the isomorphism  $\mathcal{D}(\mathbf{LC}, \mathbf{GD}) \xrightarrow[\cong]{(\lambda^{\mathbf{C}})^*} \mathrm{PC}_{\mathcal{D}}(\mathbf{FC}, \mathbf{GD})$ .  $\square$

We precise now what we do consider as *preservation* properties of a 2-functor. We do it in the case of pseudolimits and bilimits, but the same clearly applies to pseudocolimits and bicolimits. Let  $\mathcal{I}^{op} \xrightarrow{\mathbf{X}} \mathcal{C} \xrightarrow{\mathbf{H}} \mathcal{A}$  be any 2-functors.

**1.2.5 Definition.** *We say that  $\mathbf{H}$  preserves a pseudolimit (resp. bilimit) pseudocone  $\mathbf{L} \xrightarrow{\pi_i} \mathbf{X}_i$  in  $\mathcal{C}$ , if  $\mathbf{HL} \xrightarrow{\mathbf{H}\pi_i} \mathbf{HX}_i$  is a pseudolimit (resp. bilimit) pseudocone in  $\mathcal{A}$ . Equivalently, if the (usual) comparison arrow is an isomorphism (resp. an equivalence) in  $\mathcal{A}$ .*

Note that by the very definition, the 2-representable 2-functors preserve pseudolimits and bilimits. Also, from proposition 1.2.4 it follows:

**1.2.6 Proposition.** *The Yoneda 2-functors in 1.1.15 preserve pseudolimits.*  $\square$

Recall that small pseudolimits and pseudocolimits of locally small categories exist and are locally small, as well that the 2-category  $\mathcal{C}at$  of small categories has all small pseudolimits and pseudocolimits (see for example [4], [12]).

**1.2.7.** We refer to the explicit construction of pseudolimits of category valued 2-functors, which is similar to the construction of pseudolimits of category-valued functors in [2, Exposé VI 6.], see full details in [5].

It is also key to our work the explicit construction of 2-filtered pseudocolimits of category valued 2-functors developed in [8]. We recall this now.

**1.2.8 Definition** (Kennison, [10]). *Let  $\mathcal{C}$  be a 2-category.  $\mathcal{C}$  is said to be 2-filtered if the following axioms are satisfied:*

*F0. Given two objects  $C, D \in \mathcal{C}$ , there exists an object  $E \in \mathcal{C}$  and arrows  $C \rightarrow E, D \rightarrow E$ .*

*F1. Given two arrows  $C \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} D$ , there exists an arrow  $D \xrightarrow{h} E$  and an invertible 2-cell  $\alpha : hf \cong hg$ .*

*F2. Given two 2-cells  $C \begin{array}{c} \xrightarrow{\alpha \Downarrow \beta \Downarrow} \\ \xrightarrow{\alpha \Downarrow \beta \Downarrow} \end{array} D$  there exists an arrow  $D \xrightarrow{h} E$  such that  $h\alpha = h\beta$ .*

*The dual notion of 2-cofiltered 2-category is given by the duals of axioms F0, F1 and F2.*

**1.2.9. Construction LL** (Dubuc-Street [8]) Let  $\mathcal{I}$  be a 2-filtered 2-category and  $F : \mathcal{I} \rightarrow \mathcal{Cat}$  a 2-functor. We define a category  $\mathcal{L}(F)$  in two steps as follows:

**First step** ([8, Definition 1.5]):

Objects:  $(C, i)$  with  $C \in Fi$

Premorphisms: A premorphism between  $(C, i)$  and  $(D, j)$  is a triple  $(u, f, v)$  where  $i \xrightarrow{u} k, j \xrightarrow{v} k$  in  $\mathcal{I}$  and  $F(u)(C) \xrightarrow{f} F(v)(D)$  in  $Fk$ .

Homotopies: An homotopy between two premorphisms  $(u_1, f_1, v_1)$  and  $(u_2, f_2, v_2)$  is a quadruple  $(w_1, w_2, \alpha, \beta)$  where  $k_1 \xrightarrow{w_1} k, k_2 \xrightarrow{w_2} k$  are 1-cells of  $\mathcal{I}$  and  $w_1v_1 \xrightarrow{\alpha} w_2v_2, w_1u_1 \xrightarrow{\beta} w_2u_2$  are invertible 2-cells of  $\mathcal{I}$  such that the following diagram commutes in  $Fk$ :

$$\begin{array}{ccc} F(w_1)F(u_1)(C) = F(w_1u_1)(C) & \xrightarrow{F(\beta)C} & F(w_2u_2)(C) = F(w_2)F(u_2)(C) \\ \downarrow F(w_1)(f_1) & & \downarrow F(w_2)(f_2) \\ F(w_1)F(v_1)(D) = F(w_1v_1)(D) & \xrightarrow{F(\alpha)D} & F(w_2v_2)(D) = F(w_2)F(v_2)(D) \end{array}$$



We say that two premorphisms  $f_1, f_2$  are equivalent if there is an homotopy between them. In that case, we write  $f_1 \sim f_2$ .

Equivalence is indeed an equivalence relation, and premorphisms can be (non uniquely) composed. Up to equivalence, composition is independent of the choice of representatives and of the choice of the composition between them. Since associativity holds and identities exist, the following actually does define a category:

**Second step** ([8, Definition 1.13]):

Objects:  $(C, i)$  with  $C \in Fi$ .

Morphisms: equivalence classes of premorphisms.

Composition: defined by composing representative premorphisms.

**1.2.10 Proposition.** [8, Theorem 1.19] *Let  $\mathcal{I}$  be a 2-filtered 2-category,  $F : \mathcal{I} \rightarrow \mathcal{C}at$  a 2-functor,  $i \xrightarrow{u} j$  in  $\mathcal{I}$  and  $C \xrightarrow{f} D \in Fi$ . The following formulas define a pseudocone  $F \xrightarrow{\lambda} \mathcal{L}(F)$ :*

$$\lambda_i(C) = (C, i) \quad \lambda_i(f) = [i, f, i] \quad (\lambda_u)_C = [u, Fu(C), j]$$

which is a pseudocolimit for the 2-functor  $F$ . □

### 1.3 Further results.

A. Joyal pointed to us the notion of *flexible* functors, related with some of our results on pseudo colimits of representable 2-functors. We recall now this notion since it bears some significance for the concept of 2-pro-object developed in this paper. Any 2-pro-object determines a 2-functor which is flexible, and some of our results find their right place stated in the context of flexible 2-functors.

**Warning:** *In this subsection 2-categories are assumed to be locally small, except the illegitimate constructions  $\mathcal{H}om$  and  $\mathcal{H}om_p$ .*

The inclusion  $\mathcal{H}om(\mathcal{C}, \mathcal{C}at) \xrightarrow{i} \mathcal{H}om_p(\mathcal{C}, \mathcal{C}at)$  has a left adjoint  $(-)' \dashv i$ , we refer the reader to [4]. The 2-natural counit of this adjunction  $F' \xrightarrow{\varepsilon_F} F$  is an equivalence in  $\mathcal{H}om_p(\mathcal{C}, \mathcal{C}at)$ , with a section given by the pseudonatural unit  $F \xrightarrow{\eta_F} F'$ ,  $\varepsilon_F \eta_F = id_F$ ,  $\eta_F \varepsilon_F \cong id_{F'}$ , [4, Proposition 4.1.]

**1.3.1 Definition.** [4, Proposition 4.2] A 2-functor  $\mathcal{C} \xrightarrow{F} \mathcal{C}at$  is flexible if the counit  $F' \xrightarrow{\varepsilon_F} F$  has a 2-natural section  $F \xrightarrow{\lambda} F'$ ,  $\varepsilon_F \lambda = id_F$ ,  $\lambda \varepsilon_F \cong id_{F'}$ , which determines an equivalence in  $\mathcal{H}om(\mathcal{C}, \mathcal{C}at)$ .

We state now a useful characterization of flexible 2-functors  $F$  independent of the left adjoint  $(-)'$ , the proof will appear elsewhere [6].

**1.3.2 Proposition.** A 2-functor  $\mathcal{C} \xrightarrow{F} \mathcal{C}at$  is flexible  $\iff$  for all 2-functors  $G$ , the inclusion  $\mathcal{H}om(\mathcal{C}, \mathcal{C}at)(F, G) \xrightarrow{i_G} \mathcal{H}om_p(\mathcal{C}, \mathcal{C}at)(F, G)$  has a retraction  $\alpha_G$  natural in  $G$ ,  $\alpha_G i_G = id$ ,  $i_G \alpha_G \cong id$ , which determines an equivalence of categories.  $\square$

Let  $\mathcal{H}om(\mathcal{C}, \mathcal{C}at)_f$  and  $\mathcal{H}om_p(\mathcal{C}, \mathcal{C}at)_f$  be the subcategories whose objects are the flexible 2-functors. We have the following corollaries:

**1.3.3 Corollary.** The 2-categories  $\mathcal{H}om(\mathcal{C}, \mathcal{C}at)_f$  and  $\mathcal{H}om_p(\mathcal{C}, \mathcal{C}at)_f$  are pseudoequivalent in the sense they have the same objects and retract equivalent hom categories.  $\square$

We mention that following the usual lines (based in the axiom of choice) in the proof of 1.1.9, it can be seen that the inclusion 2-functor  $\mathcal{H}om(\mathcal{C}, \mathcal{C}at)_f \rightarrow \mathcal{H}om_p(\mathcal{C}, \mathcal{C}at)_f$  has the identity (on objects) as a retraction quasi-inverse *pseudofunctor*, with the equality as the invertible pseudonatural transformation  $F \xrightarrow{=} F$  in  $\mathcal{H}om_p(\mathcal{C}, \mathcal{C}at)_f$ .

An important property of flexible 2-functors, false in general, is the following:

**1.3.4 Corollary.** Let  $\theta : G \Rightarrow F \in \mathcal{H}om(\mathcal{C}, \mathcal{C}at)_f$  be such that  $\theta_C : GC \rightarrow FC$  is an equivalence of categories for each  $C \in \mathcal{C}$ . Then,  $\theta$  is an equivalence in  $\mathcal{H}om(\mathcal{C}, \mathcal{C}at)_f$ .

*Proof.* It is easy to check that there is a pseudonatural transformation  $\eta' : F \Rightarrow G$  such that  $\theta \eta' \cong F$  and  $\eta' \theta \cong G$  in  $\mathcal{H}om_p(F, F)$  and  $\mathcal{H}om_p(G, G)$  respectively. Now, by 1.3.2, there is a 2-natural transformation  $\eta : F \Rightarrow G$  such that  $\eta \cong \eta'$  in  $\mathcal{H}om_p(F, G)$ . Then,  $\theta \eta \cong F$  and  $\eta \theta \cong G$  in  $\mathcal{H}om(F, F)$  and  $\mathcal{H}om(G, G)$  respectively and so  $\theta$  is an equivalence in  $\mathcal{H}om(\mathcal{C}, \mathcal{C}at)$ .  $\square$

**1.3.5 Proposition.** Small pseudocolimits of flexible 2-functors are flexible.

*Proof.* Let  $F = \varinjlim_{j \in \mathcal{I}} F_j$ , where each  $F_j$  is flexible, and let  $G$  be any other 2-functor. Set  $\mathcal{A} = \mathcal{H}om(\mathcal{C}, \mathcal{C}at)$  and  $\mathcal{A}_p = \mathcal{H}om_p(\mathcal{C}, \mathcal{C}at)$ . Then:

$$\mathcal{A}(F, G) \cong \varprojlim_{j \in \mathcal{I}} \mathcal{A}(F_j, G) \xrightarrow{i} \varprojlim_{j \in \mathcal{I}} \mathcal{A}_p(F_j, G) \cong \mathcal{A}_p(F, G).$$

The two isomorphisms are given by definition 1.2.3. The arrow  $i$  is the pseudolimit of the equivalences with retraction quasi-inverses corresponding to each  $F_j$ . It is not difficult to check that  $i$  is also such an equivalence.  $\square$

It follows also from 1.3.2 that the pseudo-Yoneda lemma (1.1.18, 1.1.19) says that the representable 2-functors are flexible, so we have:

**1.3.6 Corollary.** *Small pseudocolimits of representable 2-functors are flexible.*  $\square$

Note that 1.3.5 and 1.3.6 hold for any pseudocolimit that may exist.

## 2 2-Pro-objects

**Warning:** *In this section 2-categories are assumed to be locally small, except illegitimate constructions as  $\mathcal{H}om$ ,  $\mathcal{H}om_p$  or  $2\text{-CAT}$ .*

The main results of this paper are in this section. In the first subsection we define the 2-category of 2-pro-objects of a 2-category  $\mathcal{C}$  and establish the basic formula for the morphisms and 2-cells of this 2-category. Then in the next subsection we develop the notion of a 2-cell in  $\mathcal{C}$  representing a 2-cell in  $2\text{-Pro}(\mathcal{C})$ , inspired in the 1-dimensional notion of an arrow representing a morphism of pro-objects found in [3]. We use this in the third subsection to construct the 2-filtered category which serves as the index 2-category for the 2-filtered pseudolimit of 2-pro-objects. This is also inspired in a construction for the same purpose found in [3]. We were forced to have recourse to this complicated construction because the conceptual treatment of this problem found in [1] does not apply in the 2-category case. This is so because a 2-functor is not the pseudocolimit of 2-representables indexed by its 2-category of elements. Finally, in the last subsection we prove the universal properties of  $2\text{-Pro}(\mathcal{C})$ .

## 2.1 Definition of the 2-category of 2-pro-objects

In this subsection we define the 2-category of 2-pro-objects of a fixed 2-category and prove its basic properties. A 2-pro-object over a 2-category  $\mathcal{C}$  will be a small 2-cofiltered diagram in  $\mathcal{C}$  and it will be the pseudolimit of it's own diagram in the 2-category  $2\text{-Pro}(\mathcal{C})$ .

**2.1.1 Definition.** *Let  $\mathcal{C}$  be a 2-category. We define the 2-category of 2-pro-objects of  $\mathcal{C}$ , which we denote by  $2\text{-Pro}(\mathcal{C})$ , as follows:*

1. *Its objects are the 2-functors  $\mathcal{I}^{op} \xrightarrow{\mathbf{X}} \mathcal{C}$ ,  $\mathbf{X} = (\mathbf{X}_i, \mathbf{X}_u, \mathbf{X}_\alpha)_{i, u, \alpha \in \mathcal{I}}$ , with  $\mathcal{I}$  a small 2-filtered 2-category. Often we are going to abuse the notation by saying  $\mathbf{X} = (\mathbf{X}_i)_{i \in \mathcal{I}}$ .*
2. *If  $\mathbf{X} = (\mathbf{X}_i)_{i \in \mathcal{I}}$  and  $\mathbf{Y} = (\mathbf{Y}_j)_{j \in \mathcal{J}}$  are two 2-pro-objects,*

$$\begin{aligned} 2\text{-Pro}(\mathcal{C})(\mathbf{X}, \mathbf{Y}) &= \mathcal{H}om(\mathcal{C}, \mathcal{C}at)^{op}(\varprojlim_{i \in \mathcal{I}^{op}} \mathcal{C}(\mathbf{X}_i, -), \varprojlim_{j \in \mathcal{J}^{op}} \mathcal{C}(\mathbf{Y}_j, -)) \\ &= \mathcal{H}om(\mathcal{C}, \mathcal{C}at)(\varinjlim_{j \in \mathcal{J}} \mathcal{C}(\mathbf{Y}_j, -), \varinjlim_{i \in \mathcal{I}} \mathcal{C}(\mathbf{X}_i, -)) \end{aligned}$$

*The compositions are given by the corresponding compositions in the 2-category  $\mathcal{H}om(\mathcal{C}, \mathcal{C}at)^{op}$  so it is easy to check that  $2\text{-Pro}(\mathcal{C})$  is indeed a 2-category.*

**2.1.2 Proposition.** *By definition there is a 2-fully-faithful 2-functor  $2\text{-Pro}(\mathcal{C}) \xrightarrow{\mathbf{L}} \mathcal{H}om(\mathcal{C}, \mathcal{C}at)^{op}$ . Thus, there is a contravariant 2-equivalence of 2-categories  $2\text{-Pro}(\mathcal{C}) \xrightarrow{\mathbf{L}} \mathcal{H}om(\mathcal{C}, \mathcal{C}at)_{fc}^{op}$ , where  $\mathcal{H}om(\mathcal{C}, \mathcal{C}at)_{fc}$  stands for the full subcategory of  $\mathcal{H}om(\mathcal{C}, \mathcal{C}at)$  whose objects are those 2-functors which are small 2-filtered pseudocolimits of representable 2-functors. However, it is important to note that this equivalence is not injective on objects.  $\square$*

From Corollary 1.3.6 it follows:

**2.1.3 Proposition.** *For any 2-pro-object  $\mathbf{X}$ , the corresponding 2-functor  $\mathbf{L}\mathbf{X}$  is flexible.  $\square$*

**2.1.4 Remark.** If we use pseudonatural transformations to define morphisms of 2-pro-objects we obtain a 2-category  $2\text{-}\mathcal{P}ro_p(\mathcal{C})$ , which anyway, by 2.1.3, results pseudoequivalent (see 1.3.3) to  $2\text{-}\mathcal{P}ro(\mathcal{C})$ , with the same objects and retract equivalent hom categories. We think our choice of morphisms, which is much more convenient to use, will prove to be the good one for the applications.

Next we establish the basic formula which is essential in many computations in the 2-category  $2\text{-}\mathcal{P}ro(\mathcal{C})$ :

**2.1.5 Proposition.** *There is an isomorphism of categories:*

$$(2.1.5) \quad 2\text{-}\mathcal{P}ro(\mathcal{C})(\mathbf{X}, \mathbf{Y}) \cong \underset{j \in \mathcal{J}^{op}}{\leftarrow \text{Lim}} \underset{i \in \mathcal{I}}{\text{Lim}} \mathcal{C}(X_i, Y_j)$$

*Proof.*

$$\begin{aligned} 2\text{-}\mathcal{P}ro(\mathcal{C})(\mathbf{X}, \mathbf{Y}) &= \text{Hom}(\mathcal{C}, \text{Cat})(\underset{j \in \mathcal{J}}{\text{Lim}} \mathcal{C}(Y_j, -), \underset{i \in \mathcal{I}}{\text{Lim}} \mathcal{C}(X_i, -)) \cong \\ &\underset{j \in \mathcal{J}^{op}}{\leftarrow \text{Lim}} \text{Hom}(\mathcal{C}, \text{Cat})(\mathcal{C}(Y_j, -), \underset{i \in \mathcal{I}}{\text{Lim}} \mathcal{C}(X_i, -)) \cong \underset{j \in \mathcal{J}^{op}}{\leftarrow \text{Lim}} \underset{i \in \mathcal{I}}{\text{Lim}} \mathcal{C}(X_i, Y_j) \end{aligned}$$

The first isomorphism is due to 1.2.3 and the second one to 1.1.16.  $\square$

**2.1.6 Corollary.** *The 2-category  $2\text{-}\mathcal{P}ro(\mathcal{C})$  is locally small.*

**2.1.7 Corollary.** *There is a canonical 2-fully-faithful 2-functor  $\mathcal{C} \xrightarrow{c} 2\text{-}\mathcal{P}ro(\mathcal{C})$  which sends an object of  $\mathcal{C}$  into the corresponding 2-pro-object with index 2-category  $\{*\}$ . Since this 2-functor is also injective on objects, we can identify  $\mathcal{C}$  with a 2-full subcategory of  $2\text{-}\mathcal{P}ro(\mathcal{C})$ .  $\square$*

Where there is no risk of confusion, we will omit to indicate notationally this identification. By the very definition of  $2\text{-}\mathcal{P}ro(\mathcal{C})$  it follows:

**2.1.8 Proposition.** *If  $\mathbf{X} = (X_i)_{i \in \mathcal{I}}$  is any 2-pro-object of  $\mathcal{C}$ , then  $\mathbf{X} = \underset{i \in \mathcal{I}^{op}}{\leftarrow \text{Lim}} X_i$  in  $2\text{-}\mathcal{P}ro(\mathcal{C})$ .  $\mathbf{X}$  is equipped with projections, for each  $i \in \mathcal{I}$ ,  $\mathbf{X} \xrightarrow{\pi_i} X_i$ , and a pseudocone structure, for each  $i \xrightarrow{u} j \in \mathcal{I}$ , invertible 2-cells  $\pi_i \xrightarrow{\pi_u} X_u \pi_j$ .*

Under the isomorphism  $2\text{-Pro}(\mathcal{C})(X, X_i) \cong \varinjlim_{k \in \mathcal{I}} \mathcal{C}(X_k, X_i)$  (2.1.5),

the projections  $X \xrightarrow{\pi_i} X_i$  correspond to the object  $(id_{X_i}, i)$  in construction 1.2.9.  $\square$

Note that from this proposition it follows:

**2.1.9 Remark.** Given any two pro-objects  $X, Z \in 2\text{-Pro}(\mathcal{C})$ , there is an isomorphism of categories  $2\text{-Pro}(\mathcal{C})(Z, X) \xrightarrow{\cong} \text{PC}_{2\text{-Pro}(\mathcal{C})}(Z, cX)$ , where  $\text{PC}_{2\text{-Pro}(\mathcal{C})}$  is the category of pseudocones for the 2-functor  $cX$  with vertex  $Z$ .

It is important to note that when  $\varprojlim_{i \in \mathcal{I}^{op}} X_i$  exists in  $\mathcal{C}$ , this pseudolimit would not be isomorphic to  $X$  in  $2\text{-Pro}(\mathcal{C})$ . In general, the functor  $c$  does not preserve 2-cofiltered pseudolimits, in fact, it will preserve them only when  $\mathcal{C}$  is already a category of 2-pro-objects, in which case  $c$  is an equivalence.

## 2.2 Lemmas to compute with 2-pro-objects.

### 2.2.1 Definition.

1. Let  $X \xrightarrow{f} Y$  be an arrow in  $2\text{-Pro}(\mathcal{C})$ . We say that a pair  $(r, \varphi)$  represents  $f$ , if  $\varphi$  is an invertible 2-cell  $\pi_j f \xrightarrow{\varphi} r \pi_i$ . That is, if we have the following diagram in  $2\text{-Pro}(\mathcal{C})$ :

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi_i \downarrow & \Downarrow \cong \varphi & \downarrow \pi_j \\ X_i & \xrightarrow{r} & Y_j \end{array}$$

2. Let  $X \begin{array}{c} \xrightarrow{f} \\ \alpha \Downarrow \\ \xrightarrow{g} \end{array} Y$  and  $X_i \begin{array}{c} \xrightarrow{r} \\ \theta \Downarrow \\ \xrightarrow{s} \end{array} Y_j$  be 2-cells in  $2\text{-Pro}(\mathcal{C})$

and  $\mathcal{C}$  as in the following diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \alpha \Downarrow & \xrightarrow{g} & \\
 \downarrow \pi_i & & \downarrow \pi_j \\
 X_i & \xrightarrow{r} & Y_j \\
 \theta \Downarrow & \xrightarrow{s} & 
 \end{array}$$

We say that  $(\theta, r, \varphi, s, \psi)$  represents  $\alpha$  if  $(r, \varphi)$  represents  $f$ ,  $(s, \psi)$  represents  $g$ , and the following diagram commutes in  $2\text{-Pro}(\mathcal{C})$ :

$$\begin{array}{ccc}
 \pi_j f & \xrightarrow{\varphi} & r \pi_i \\
 \pi_j \alpha \Downarrow & \cong & \Downarrow \theta \pi_i \\
 \pi_j g & \xrightarrow{\psi} & s \pi_i
 \end{array}
 \quad i.e. \quad \theta \pi_i \circ \varphi = \psi \circ \pi_j \alpha$$

That is,  $\theta \pi_i = \pi_j \alpha$  "modulo" a pair of invertible 2-cells  $\varphi, \psi$ .

Clearly, if  $\alpha$  is invertible, then so is  $\theta$ .

**2.2.2 Proposition.** Let  $X = (X_i)_{i \in \mathcal{I}}$  and  $Y = (Y_j)_{j \in \mathcal{J}}$  be any two objects in  $2\text{-Pro}(\mathcal{C})$ :

1. Let  $X \xrightarrow{f} Y$ , then, for any  $j \in \mathcal{J}$  there is an  $i \in \mathcal{I}$  and  $X_i \xrightarrow{r} Y_j$  in  $\mathcal{C}$ , such that  $(r, id)$  represents  $f$ .

2. Let  $X \xrightarrow[\alpha \Downarrow]{f} Y$ , then, for any  $j \in \mathcal{J}$  there is an  $i \in \mathcal{I}$ ,

$X_i \xrightarrow[\theta \Downarrow]{r} Y_j$  in  $\mathcal{C}$ , and appropriate invertible 2-cells  $\varphi$  and  $\psi$  such that  $(\theta, r, \varphi, s, \psi)$  represents  $\alpha$ .

*Proof.* Consider  $X \xrightarrow[\pi_j \alpha \Downarrow]{\pi_j f} Y_j$  and use formula 2.1.5 plus the constructions of pseudolimits and 2-filtered pseudocolimits, 1.2.7, 1.2.9.  $\square$

**2.2.3 Lemma.** Let  $X = (X_i)_{i \in \mathcal{I}} \in 2\text{-Pro}(\mathcal{C})$ , let  $X_i \xrightarrow{r} C$ ,  $X_j \xrightarrow{s} C \in \mathcal{C}$ , and let  $X \xrightarrow[\alpha \Downarrow]{r\pi_i} C \in 2\text{-Pro}(\mathcal{C})$ . Then,  $\exists j \xrightarrow[u]{v} k$  and  $X_k \xrightarrow[\theta \Downarrow]{rX_u} C$  such that:

$$\begin{array}{ccc}
 X & \xrightarrow{\pi_i} & X_i \\
 \pi_k \downarrow & \searrow \pi_u \Downarrow X_u & \downarrow r \\
 X_k & & C \\
 X_v \downarrow & \Downarrow \theta & \\
 X_j & \xrightarrow{s} & C
 \end{array}
 =
 \begin{array}{ccc}
 X & \xrightarrow{\pi_i} & X_i \\
 \pi_k \swarrow & & \downarrow \alpha \\
 X_k & \xleftarrow{\pi_v} & X_j \\
 X_v \searrow & & \downarrow r \\
 & & C \\
 & & \xrightarrow{s}
 \end{array}$$

$$\begin{array}{ccc}
 r\pi_i & \xrightarrow[\cong]{r\pi_u} & rX_u\pi_k \\
 \alpha \downarrow & & \downarrow \theta\pi_k \\
 s\pi_j & \xrightarrow[\cong]{s\pi_v} & sX_v\pi_k
 \end{array}
 \quad i.e. \quad \theta\pi_k \circ r\pi_u = s\pi_v \circ \alpha$$

Clearly, if  $\alpha$  is invertible, then so is  $\theta$ .

*Proof.* By formula 2.1.5 and the construction of 2-filtered pseudocolimits (1.2.9),  $\alpha$  corresponds to a  $(r, i) \xrightarrow{[u, \theta, v]} (s, j) \in \varinjlim_{i \in \mathcal{I}} \mathcal{C}(X_i, C)$ . So,

$\exists j \xrightarrow[u]{v} k$  and  $X_k \xrightarrow[\theta \Downarrow]{rX_u} C$  such that  $\theta\pi_k \circ r\pi_u = s\pi_v \circ \alpha$ , as we wanted to prove.  $\square$

The following is an immediate consequence of [8, Lemma 2.2.]

**2.2.4 Remark.** If  $i = j$ , then one can choose  $u = v$ .  $\square$

**2.2.5 Lemma.** Let  $X = (X_i)_{i \in \mathcal{I}} \in 2\text{-Pro}(\mathcal{C})$  and  $X_i \xrightarrow[\theta \Downarrow \theta' \Downarrow]{f} C \in \mathcal{C}$

be such that  $\theta\pi_i = \theta'\pi_i$  in  $2\text{-Pro}(\mathcal{C})$ . Then  $\exists i \xrightarrow{u} i'$  such that  $\theta X_u = \theta' X_u$ .

*Proof.* It follows from 2.1.5 and [8, Lemma 1.20.]  $\square$



**2.2.6 Lemma.** Let  $X \xrightarrow[\alpha \Downarrow \mathbf{g}]{\mathbf{f}} Y$  in  $2\text{-Pro}(\mathcal{C})$  and  $X_i \xrightarrow[\mathbf{s}]{\mathbf{r}} \theta \Downarrow \theta' \Downarrow Y_j$  in  $\mathcal{C}$  such that  $(\theta, \mathbf{r}, \varphi, \mathbf{s}, \psi)$  and  $(\theta', \mathbf{r}, \varphi, \mathbf{s}, \psi)$  both represents  $\alpha$ . Then, there exists  $i \xrightarrow{u} i' \in \mathcal{I}$  such that  $\theta X_u = \theta' X_u$ .

*Proof.* Since both  $(\theta, \mathbf{r}, \varphi, \mathbf{s}, \psi)$  and  $(\theta', \mathbf{r}, \varphi, \mathbf{s}, \psi)$  represents  $\alpha$ , and  $\varphi, \psi$  are invertible, it follows that  $\theta \pi_i = \theta' \pi_i$ . Then, by 2.2.5, there exists  $i \xrightarrow{u} i' \in \mathcal{I}$  such that  $\theta X_u = \theta' X_u$ .  $\square$

**2.2.7 Lemma.** Let  $X \xrightarrow[\alpha \Downarrow \mathbf{g}]{\mathbf{f}} Y \in 2\text{-Pro}(\mathcal{C})$ ,  $(\mathbf{r}, \varphi)$  representing  $\mathbf{f}$ ,  $X_i \xrightarrow{\mathbf{r}} Y_j$  and  $(\mathbf{s}, \psi)$  representing  $\mathbf{g}$ ,  $X_{i'} \xrightarrow{\mathbf{s}} Y_j$ . Then,  $\exists \begin{matrix} i \\ \xrightarrow{u} \\ i' \\ \xrightarrow{v} \end{matrix} k$  and  $X_k \xrightarrow[\mathbf{s} X_v]{\mathbf{r} X_u} Y_j$  such that  $(\theta, \mathbf{r} X_u, \mathbf{r} \pi_u \circ \varphi, \mathbf{s} X_v, \mathbf{s} \pi_v \circ \psi)$  represents  $\alpha$ . Clearly, if  $\alpha$  is invertible, then so is  $\theta$ .

*Proof.* In lemma 2.2.3, take  $C = Y_j$ , and  $\alpha = \psi \circ \pi_j \alpha \circ \varphi^{-1}$ . Then,

$$\exists \begin{matrix} i \\ \xrightarrow{u} \\ i' \\ \xrightarrow{v} \end{matrix} k \text{ and } X_k \xrightarrow[\mathbf{s} X_v]{\mathbf{r} X_u} Y_j \text{ such that } \theta \pi_k \circ \mathbf{r} \pi_u \circ \varphi = \mathbf{s} \pi_v \circ \psi \circ \pi_j \alpha,$$

$$\begin{array}{ccccc} & & \mathbf{r} \pi_i & \xrightarrow{\mathbf{r} \pi_u} & \mathbf{r} X_u \pi_k & \xrightarrow{\theta \pi_k} & \mathbf{s} X_v \pi_k \\ & \nearrow \varphi & & & & & \\ \pi_j \mathbf{f} & & & & & & \\ & \searrow \pi_j \alpha & & & & & \\ & & \pi_j \mathbf{g} & \xrightarrow{\psi} & \mathbf{s} \pi_{i'} & & \end{array}$$

It is not difficult to check that  $(\theta, \mathbf{r} X_u, \mathbf{r} \pi_u \circ \varphi, \mathbf{s} X_v, \mathbf{s} \pi_v \circ \psi)$  represents  $\alpha$ .  $\square$

From remark 2.2.4 we have:

**2.2.8 Remark.** If  $i = i'$ , then one can choose  $u = v$ .  $\square$

### 2.3 2-cofiltered pseudolimits in $2\text{-Pro}(\mathcal{C})$ .

Let  $\mathcal{J}$  be a small 2-filtered 2-category and  $\mathcal{J}^{op} \xrightarrow{\mathbf{X}} 2\text{-Pro}(\mathcal{C})$  a 2-functor,  $\mathbf{X}^j = (X_i^j)_{i \in \mathcal{I}_j}$ ,  $\mathcal{I}_j^{op} \xrightarrow{\mathbf{X}^j} \mathcal{C}$ . Recall (2.1.8) that for each

$j$  in  $\mathcal{J}$ ,  $\mathbf{X}^j$  is equipped with a pseudolimit pseudocone  $\{\pi_i^j\}_{i \in \mathcal{I}_j}$ ,  $\{\pi_u^j\}_{i \xrightarrow{u} i' \in \mathcal{I}_j}$  for the 2-functor  $\mathbf{X}^j$ .

We are going to construct a 2-pro-object which is going to be the pseudolimit of  $\mathbf{X}$  in  $2\text{-Pro}(\mathcal{C})$ . First we construct its index category

**2.3.1 Definition.** Let  $\mathcal{K}_{\mathbf{X}}$  be the 2-category consisting on:

1. 0-cells of  $\mathcal{K}_{\mathbf{X}}$ :  $(i, j)$ , where  $j \in \mathcal{J}$ ,  $i \in \mathcal{I}_j$ .
2. 1-cells of  $\mathcal{K}_{\mathbf{X}}$ :  $(i, j) \xrightarrow{(a, r, \varphi)} (i', j')$ , where  $j \xrightarrow{a} j' \in \mathcal{J}$ ,  $\mathbf{X}_{i'}^{j'} \xrightarrow{r} \mathbf{X}_i^j$  are such that  $(r, \varphi)$  represents  $\mathbf{X}^a$ .
3. 2-cells of  $\mathcal{K}_{\mathbf{X}}$ :  $(a, r, \varphi) \xrightarrow{(\alpha, \theta)} (b, s, \psi)$ , where  $a \xrightarrow{\alpha} b \in \mathcal{J}$  and  $(\theta, r, \varphi, s, \psi)$  represents  $\mathbf{X}^\alpha$ .

The 2-category structure is given as follows:

$$\begin{array}{ccccc}
 & \xrightarrow{(a, r, \varphi)} & & \xrightarrow{(a', r', \varphi')} & \\
 & \Downarrow(\alpha, \theta) & & \Downarrow(\alpha', \theta') & \\
 (i, j) & \xrightarrow{(b, s, \psi)} & (i', j') & \xrightarrow{(b', s', \psi')} & (i'', j'') \\
 & \Downarrow(\beta, \eta) & & \Downarrow(\beta', \eta') & \\
 & \xrightarrow{(c, t, \phi)} & & \xrightarrow{(c', t', \phi')} & 
 \end{array}$$

1.  $(a', r', \varphi')(a, r, \varphi) = (a'a, rr', r\varphi' \circ \varphi \mathbf{X}^{a'})$
2.  $(\alpha', \theta')(\alpha, \theta) = (\alpha'\alpha, \theta\theta')$
3.  $(\beta, \eta) \circ (\alpha, \theta) = (\beta \circ \alpha, \eta \circ \theta)$

One can easily check that the structure so defined is indeed a 2-category, which is clearly small.

**2.3.2 Proposition.** The 2-category  $\mathcal{K}_{\mathbf{X}}$  is 2-filtered.

*Proof.* F0. Let  $(i, j), (i', j') \in \mathcal{K}_{\mathbf{X}}$ . Since  $\mathcal{J}$  is 2-filtered,  $\exists \begin{array}{c} j \\ \xrightarrow{a} \\ j' \xrightarrow{b} \end{array} j''$ .

By 2.2.2,  $\exists \mathbf{X}_{i_1}^{j''} \xrightarrow{r_1} \mathbf{X}_i^j$  and  $\mathbf{X}_{i_2}^{j''} \xrightarrow{r_2} \mathbf{X}_{i'}^{j'}$  such that  $(r_1, id)$  represents

$X^a$  and  $(r_2, id)$  represents  $X^b$ . Since  $\mathcal{I}_{j''}$  is 2-filtered,  $\exists \begin{matrix} i_1 \\ \xrightarrow{u} \\ i_2 \end{matrix} \xrightarrow{v} i''$ . Then, we have the following situation in  $\mathcal{K}_X$  which proves  $F0$  .:

$$\begin{array}{ccc} (i, j) & \xrightarrow{(a, r_1 X_u^{j''}, r_1 \pi_u^{j''})} & (i'', j'') \\ & \searrow & \nearrow \\ (i', j') & \xrightarrow{(b, r_2 X_v^{j''}, r_2 \pi_v^{j''})} & \end{array}$$

F1. Let  $(i, j) \xrightarrow[(b, s, \psi)]{(a, r, \varphi)} (i', j') \in \mathcal{K}_X$ . Since  $\mathcal{J}$  is 2-filtered,

$\exists j' \xrightarrow{c} j''$  and an invertible 2-cell  $ca \xrightarrow{\alpha} cb$ . By 2.2.2,  $\exists X_k^{j''} \xrightarrow{t} X_{i'}^{j'}$  such that  $(t, id)$  represents  $X^c$ . Then  $(rt, \varphi X^c)$  represents  $X^{ca}$  and  $(st, \psi X^c)$  represents  $X^{cb}$ , so, by 2.2.7, there exists  $k \xrightarrow{w} i'' \in \mathcal{I}_{j''}$  and an invertible 2-cell  $rtX_w^{j''} \xrightarrow{\theta} stX_w^{j''}$  such that  $(\theta, rtX_w^{j''}, rt\pi_w \circ \varphi X^c, stX_w^{j''}, st\pi_w \circ \psi X^c)$  represents  $X^\alpha$ . Then

$$\text{we have an invertible 2-cell in } \mathcal{K}_X \quad (i, j) \xrightarrow[(c, tX_w^{j''}, t\pi_w)(b, s, \psi)]{(c, tX_w^{j''}, t\pi_w)(a, r, \varphi)} \downarrow (\alpha, \theta) \downarrow (i'', j'')$$

which proves  $F1$ .

F2. Let  $(i, j) \xrightarrow[(b, s, \psi)]{(a, r, \varphi)} \downarrow (\alpha, \theta) \downarrow (\alpha', \theta') (i', j') \in \mathcal{K}_X$ . Since  $\mathcal{J}$  is

2-filtered,  $\exists j' \xrightarrow{c} j'' \in \mathcal{J}$  such that  $c\alpha = c\alpha'$ . Also, by 2.2.2,  $\exists X_k^{j''} \xrightarrow{t} X_{i'}^{j'}$  such that  $(t, id)$  represents  $X^c$ . Then, it is easy to check that  $(t, t, id, t, id)$  represents  $X^c$  and therefore we have that  $(\theta t, rt, \varphi X^c, st, \psi X^c)$  and  $(\theta' t, rt, \varphi X^c, st, \psi X^c)$  both represent  $X^{c\alpha}$ . Then, by 2.2.6,  $\exists k \xrightarrow{w} i'' \in \mathcal{I}_{j''}$  such that  $\theta tX_w^{j''} = \theta' tX_w^{j''}$ , so  $(c, tX_w^{j''}, t\pi_w)(\alpha, \theta) = (c, tX_w^{j''}, t\pi_w)(\alpha', \theta')$ , which proves  $F2$ .  $\square$

**2.3.3 Theorem.** Let  $\tilde{X}$  be the 2-pro-object  $\mathcal{K}_X^{op} \xrightarrow{\tilde{X}} \mathcal{C}$  defined by  $\tilde{X}_{(i,j)} = X_i^j$ ,  $\tilde{X}_{(a,r,\varphi)} = r$ , and  $\tilde{X}_{(\alpha,\theta)} = \theta$ . Then the following equation holds in 2-Pro( $\mathcal{C}$ ):

$$\tilde{X} = \underset{j \in \mathcal{J}^{op}}{\text{L}\lim} X^j$$

*Proof.* Let  $Z \in 2\text{-Pro}(\mathcal{C})$ , and  $\{Z \xrightarrow{h_j} X^j\}_{j \in \mathcal{J}}$ ,  $\{h_j \xrightarrow{h_a} X^a h_{j'}\}_{j \xrightarrow{a} j' \in \mathcal{J}}$  be a pseudocone for  $X$  with vertex  $Z$  (1.2.1). Given  $(i, j) \xrightarrow{(a, r, \varphi)} (i', j') \in \mathcal{K}_X$ , check that the definitions  $h_{(i, j)} = \pi_i^j h_j$  and  $h_{(a, r, \varphi)} = \varphi h_{j'} \circ \pi_i^j h_a$  determine a pseudocone for  $c\tilde{X}$  with vertex  $Z$ . It is straightforward to check that this extends to a functor, that we denote  $p$  (for the isomorphism below see 2.1.9):

$$\text{PC}_{2\text{-Pro}(\mathcal{C})}(Z, X) \xrightarrow{p} \text{PC}_{2\text{-Pro}(\mathcal{C})}(Z, c\tilde{X}) \cong 2\text{-Pro}(\mathcal{C})(Z, \tilde{X})$$

The theorem follows if  $p$  is an isomorphism. In the sequel we prove that, in fact,  $p$  is an isomorphism. Let  $Z \in 2\text{-Pro}(\mathcal{C})$ , and

$$\{h_{(i, j)} \xrightarrow{h_{(a, r, \varphi)}} \tilde{X}_{(a, r, \varphi)} h_{(i', j')} = r h_{(i, j)}\}_{(i, j) \xrightarrow{(a, r, \varphi)} (i', j') \in \mathcal{K}_X}, \quad \{Z \xrightarrow{h_{(i, j)}} X_i^j\}_{(i, j) \in \mathcal{K}_X}$$

be a pseudocone for  $c\tilde{X}$  with vertex  $Z$  (1.2.1).

1.  $p$  is bijective on objects:

Check that for each  $j \in \mathcal{J}$ ,  $\{Z \xrightarrow{h_{(i, j)}} X_i^j\}_{i \in \mathcal{I}_j}$  together with  $\{h_u = h_{(j, X_u^j, \pi_u^j)} : h_{(i, j)} \implies X_u^j h_{(i', j')}\}_{i \xrightarrow{u} i' \in \mathcal{I}_j}$  is a pseudocone for  $X^j$ . Then, since  $X^j \xrightarrow{\pi_i^j} X_i^j$  is a pseudolimit pseudocone, it follows that there exists a unique  $Z \xrightarrow{h_j} X^j$  such that

$$(2.3.4) \quad \forall i \in \mathcal{I}_j \quad \pi_i^j h_j = h_{(i, j)} \quad \text{and} \quad \forall i \xrightarrow{u} i' \in \mathcal{I}_j \quad \pi_u^j h_j = h_u.$$

It only remains to define the 2-cells of the pseudocone structure. That is, for each  $j \xrightarrow{a} j' \in \mathcal{J}$ , we need invertible 2-cells  $h_j \xrightarrow{h_a} h_{j'} X^a$ , such that  $\{h_j\}_{j \in \mathcal{J}}$  together with  $\{h_a\}_{j \xrightarrow{a} j' \in \mathcal{J}}$  form a pseudocone for  $X$  with vertex  $Z$ .

Consider the pseudocone  $\{X^j \xrightarrow{\pi_i^j} X_i^j\}_{i \in \mathcal{I}_j}$ . Then the composites  $\pi_i^j h_j$ ,  $\pi_i^j X^a h_{j'}$ , determine two pseudocones  $\{Z \xrightarrow[\pi_i^j X^a h_{j'}]{\pi_i^j h_j} X_i^j\}_{i \in \mathcal{I}_j}$  for  $X^j$  with vertex  $Z$ .

**Claim 1** *Let  $(r, \varphi)$  and  $(s, \psi)$  be two pairs representing  $X^a$  as follows:*

$$\begin{array}{ccc} X^{j'} & \xrightarrow{X^a} & X^j \\ \pi_{i'}^{j'} \downarrow & \Downarrow \cong \varphi & \downarrow \pi_i^j \\ X_{i'}^{j'} & \xrightarrow{r} & X_i^j \end{array} \quad \begin{array}{ccc} X^{j'} & \xrightarrow{X^a} & X^j \\ \pi_{i''}^{j'} \downarrow & \Downarrow \cong \psi & \downarrow \pi_i^j \\ X_{i''}^{j'} & \xrightarrow{s} & X_i^j \end{array}$$

Then,  $\varphi^{-1}h_{j'} \circ h_{(a,r,\varphi)} = \psi^{-1}h_{j'} \circ h_{(a,s,\psi)}$  (proof below).

**Claim 2** *For each  $i \in \mathcal{I}_j$ , let  $(r, \varphi)$  be a pair representing  $X^a$ , and set  $\rho_i = \varphi^{-1}h_{j'} \circ h_{(a,r,\varphi)}$ . Then,  $\{\rho_i\}_{i \in \mathcal{I}_j}$  determines an isomorphism of*

*pseudocones  $\{Z \xrightarrow[\pi_i^j X^a h_{j'}]{\rho_i \Downarrow} X_i^j\}_{i \in \mathcal{I}_j}$*  (proof below).

Since  $X^j \xrightarrow{\pi_i^j} X_i^j$  is a pseudolimit pseudocone, the functor  $2\text{-Pro}(\mathcal{C})(Z, X^j) \xrightarrow{(\pi^j)^*} \text{PC}_{2\text{-Pro}(\mathcal{C})}(Z, X^j)$  is an isomorphism of categories. Then, from Claim 2 it follows that there are invertible 2-cells  $Z \xrightarrow[\pi_i^j X^a h_{j'}]{h_j} X^j \in 2\text{-Pro}(\mathcal{C})$  such that  $\rho_i = \pi_i^j h_a \forall i \in \mathcal{I}_j$ . It can be

checked that in fact  $\{Z \xrightarrow{h_j} X^j\}_{j \in \mathcal{J}}$  with  $\{h_j \xrightarrow{h_a} h_{j'} X^a\}_{j \rightarrow j' \in \mathcal{J}}$  is a pseudocone over  $X$ .

2. *p is full and faithful:*

Let  $\{Z \xrightarrow[\pi_i^j m_j]{\rho_{(i,j)} \Downarrow} X_i^j\}_{(i,j) \in \mathcal{K}_X}$  be a morphism of pseudocones for  $\tilde{X}$ . It

is easy to check that for each  $j \in \mathcal{J}$ ,  $\{Z \xrightarrow[\pi_i^j m_j]{\rho_{(i,j)} \Downarrow} X_i^j\}_{i \in \mathcal{I}_j}$  is a morphism

of pseudocones for  $X^j$ . Then arguing as above, there exists a unique morphism  $Z \xrightarrow[\pi_i^j m_j]{h_j} X^j \in 2\text{-Pro}(\mathcal{C})$  such that  $\forall i \in \mathcal{I}_j$ ,  $\pi_i^j \rho_j = \rho_{(i,j)}$ .

It can be checked that  $\{\rho_j\}_{j \in \mathcal{J}}$  is a morphism of pseudocones. This

proves the assertion.  $\square$

*Proof of Claim 1.* First assume that  $i' = i''$  and  $(r, \varphi)$ ,  $(s, \psi)$  are related by a 2-cell  $(i, j) \begin{array}{c} \xrightarrow{(a, r, \varphi)} \\ \xrightarrow{(a, \theta) \Downarrow} \\ \xrightarrow{(a, s, \psi)} \end{array} (i', j')$  in  $\mathcal{K}_X$ . Then:

$$(\psi^{-1} \mathbf{h}_{j'}) \circ (\mathbf{h}_{(a, s, \psi)}) = (\psi^{-1} \mathbf{h}_{j'}) \circ (\theta \mathbf{h}_{(i', j')}) \circ \mathbf{h}_{(a, r, \varphi)} = (\varphi^{-1} \mathbf{h}_{j'}) \circ (\mathbf{h}_{(a, r, \varphi)}),$$

the first equality by the pseudocone axiom PC2 (Definition 1.2.1), and the second because  $\theta$  represents *id* (the identity of  $X^a$ ).

The general case reduces to this one as follows:

We have  $(i, j) \begin{array}{c} \xrightarrow{(a, r, \varphi)} \\ \xrightarrow{(a, s, \psi)} \end{array} \begin{array}{c} (i', j') \\ (i'', j') \end{array}$ . Take  $\begin{array}{c} i' \\ i'' \end{array} \begin{array}{c} \xrightarrow{u} \\ \xrightarrow{v} \end{array} k$  in  $\mathcal{I}_j$ . This yields a particular instance of lemma 2.2.7:

$$\begin{array}{ccc} X^{j'} & \xrightarrow{X^a} & X^j \\ \text{id} \Downarrow & & \text{id} \Downarrow \\ X^{j'} & \xrightarrow{X^a} & X^j \\ \pi_k \downarrow & \text{rX}_u^{j'} & \downarrow \pi_i \\ X_k^{j'} & \xrightarrow{\quad} & X_i^j \\ & \text{sX}_v^{j'} & \end{array}$$

with  $(\text{rX}_u^{j'}, (\text{r}\pi_u^{j'}) \circ \varphi)$  and  $(\text{sX}_v^{j'}, (\text{s}\pi_v^{j'}) \circ \psi)$  both representing  $X^a$ .

It follows there exists  $k \xrightarrow{w} k'$  and  $X_{k'}^{j'} \xrightarrow[\text{sX}_v^{j'} X_w^{j'}]{\text{rX}_u^{j'} X_w^{j'}} X_i^j$  such that

$(\theta, \text{rX}_u^{j'} X_w^{j'}, \text{rX}_u^{j'} \pi_w^{j'} \circ \text{r}\pi_u^{j'} \circ \varphi, \text{sX}_v^{j'} X_w^{j'}, \text{sX}_v^{j'} \pi_w^{j'} \circ \text{s}\pi_v^{j'} \circ \psi)$  represents *id* (the identity of  $X^a$ ).

Considering  $(\text{rX}_u^{j'} X_w^{j'}, \text{rX}_u^{j'} \pi_w^{j'} \circ \text{r}\pi_u^{j'} \circ \varphi)$  and  $(\text{sX}_v^{j'} X_w^{j'}, \text{sX}_v^{j'} \pi_w^{j'} \circ \text{s}\pi_v^{j'} \circ \psi)$  both representing  $X^a$ , we have a situation that corresponds to the previous case. Thus:

$$\begin{aligned} & (\varphi^{-1} \mathbf{h}_{j'}^{-1} \circ \text{r}(\pi_u^{j'})^{-1} \circ \text{rX}_u^{j'} (\pi_w^{j'})^{-1}) \mathbf{h}_{j'} \circ \text{rh}_{(j', X_u^{j'} X_w^{j'}, X_u^{j'} \pi_w^{j'})} \circ \mathbf{h}_{(a, r, \varphi)} = \\ & = (\psi^{-1} \mathbf{h}_{j'}^{-1} \circ \text{s}(\pi_v^{j'})^{-1} \circ \text{sX}_v^{j'} (\pi_w^{j'})^{-1}) \mathbf{h}_{j'} \circ \text{sh}_{(j', X_v^{j'} X_w^{j'}, X_v^{j'} \pi_w^{j'})} \circ \mathbf{h}_{(a, s, \psi)}. \end{aligned}$$

From 2.3.4, it follows that  $(r(\pi_u^{j'})^{-1} \circ r\mathbf{X}_u^{j'}(\pi_w^{j'})^{-1})\mathbf{h}_{j'} \circ \mathbf{r}\mathbf{h}_{(j', \mathbf{X}_u^{j'}, \mathbf{X}_w^{j'}, \mathbf{X}_u^{j'} \pi_w^{j'})}$  and  $(s(\pi_v^{j'})^{-1} \circ s\mathbf{X}_v^{j'}(\pi_w^{j'})^{-1})\mathbf{h}_{j'} \circ \mathbf{s}\mathbf{h}_{(j', \mathbf{X}_v^{j'}, \mathbf{X}_w^{j'}, \mathbf{X}_v^{j'} \pi_w^{j'})}$  are identities. So  $\varphi^{-1}\mathbf{h}_{j'} \circ \mathbf{h}_{(a,r,\varphi)} = \psi^{-1}\mathbf{h}_{j'} \circ \mathbf{h}_{(a,s,\psi)}$  as we wanted to prove.  $\square$

*Proof of Claim 2.* Given any  $i \xrightarrow{u} k \in \mathcal{I}_j$ , we have to check the PCM equation in 1.2.1. Given the pair  $(s, \psi)$  used to define  $\rho_k$ , it is possible to choose a pair  $(r, \varphi)$  to define  $\rho_i$  in such a way that the equation holds. This arguing is justified by Claim 1.  $\square$

**2.3.5 Corollary.** *2-Pro( $\mathcal{C}$ ) is closed under small 2-cofiltered pseudolimits. Considering the equivalence in 2.1.2, it follows that the inclusion  $\mathcal{H}\text{om}(\mathcal{C}, \text{Cat})_{fc} \subset \mathcal{H}\text{om}(\mathcal{C}, \text{Cat})$  is closed under small 2-filtered pseudocolimits*  $\square$

## 2.4 Universal property of 2-Pro( $\mathcal{C}$ )

In this subsection we prove for 2-pro-objects the universal property established for pro-objects in [1, Ex. I, Prop. 8.7.3]. Consider the 2-functor  $\mathcal{C} \xrightarrow{c} 2\text{-Pro}(\mathcal{C})$  of Corollary 2.1.7 and a 2-pro-object  $\mathbf{X} = (\mathbf{X}_i)_{i \in \mathcal{I}}$ . Given a 2-functor  $\mathcal{C} \xrightarrow{F} \mathcal{E}$  into a 2-category closed under small 2-cofiltered pseudolimits, we can naively extend  $F$  into a 2-cofiltered pseudolimit preserving 2-functor  $2\text{-Pro}(\mathcal{C}) \xrightarrow{\widehat{F}} \mathcal{E}$  by defining  $\widehat{F}\mathbf{X} = \varprojlim_{i \in \mathcal{I}} F\mathbf{X}_i$ . This is just part of a 2-equivalence of 2-categories that we develop with the necessary precision in this subsection. First the universal property should be wholly established for  $\mathcal{E} = \text{Cat}$ , and only afterwards can be lifted to any 2-category  $\mathcal{E}$  closed under small 2-cofiltered pseudolimits.

**2.4.1 Lemma.** *Let  $\mathcal{C}$  be a 2-category and  $F : \mathcal{C} \rightarrow \text{Cat}$  a 2-functor. Then, there exists a 2-functor  $\widehat{F} : 2\text{-Pro}(\mathcal{C}) \rightarrow \text{Cat}$  that preserves small 2-cofiltered pseudolimits, and an isomorphism  $\widehat{F}c \xrightarrow{\cong} F$  in  $\mathcal{H}\text{om}(\mathcal{C}, \text{Cat})$ .*

*Proof.* Let  $\mathbf{X} = (\mathbf{X}_i)_{i \in \mathcal{I}} \in 2\text{-Pro}(\mathcal{C})$  be a 2-pro-object. Define:

$$\begin{aligned} \widehat{FX} &= (\mathcal{H}om(\mathcal{C}, \mathcal{C}at)(-, F) \circ L)X = \mathcal{H}om(\mathcal{C}, \mathcal{C}at)(\varinjlim_{i \in \mathcal{I}} \mathcal{C}(X_i, -), F) \xrightarrow{\cong} \\ &\xrightarrow{\cong} \varprojlim_{i \in \mathcal{I}} \mathcal{H}om(\mathcal{C}, \mathcal{C}at)(\mathcal{C}(X_i, -), F) \xrightarrow{\cong} \varprojlim_{i \in \mathcal{I}} FX_i. \end{aligned}$$

Where  $L$  is the 2-functor of 2.1.2, the first isomorphism is by definition of pseudocolimit 1.2.3, and the second is the Yoneda isomorphism 1.1.16. Since it is a 2-equivalence, the 2-functor  $L$  preserves any pseudocolimit. Then by Corollary 2.3.5 it follows that the composite  $\mathcal{H}om(\mathcal{C}, \mathcal{C}at)(-, F) \circ L$  preserves small 2-cofiltered pseudocolimits  $\square$

**2.4.2 Theorem.** *Let  $\mathcal{C}$  be any 2-category. Then, pre-composition with  $\mathcal{C} \xrightarrow{c} 2\text{-Pro}(\mathcal{C})$  is a 2-equivalence of 2-categories:*

$$\mathcal{H}om(2\text{-Pro}(\mathcal{C}), \mathcal{C}at)_+ \xrightarrow{c^*} \mathcal{H}om(\mathcal{C}, \mathcal{C}at)$$

(where  $\mathcal{H}om(2\text{-Pro}(\mathcal{C}), \mathcal{C}at)_+$  stands for the full subcategory whose objects are those 2-functors that preserve small 2-cofiltered pseudocolimits).

*Proof.* We check that the 2-functor  $c^*$  is essentially surjective on objects and 2-fully-faithful:

*Essentially surjective on objects:* It follows from lemma 2.4.1.

*2-fully-faithful:* We check that if  $F$  and  $G$  are 2-functors from  $2\text{-Pro}(\mathcal{C})$  to  $\mathcal{C}at$  that preserve small 2-cofiltered pseudocolimits, then

$$(2.4.3) \quad \mathcal{H}om(2\text{-Pro}(\mathcal{C}), \mathcal{C}at)_+(F, G) \xrightarrow{c^*} \mathcal{H}om(\mathcal{C}, \mathcal{C}at)(Fc, Gc)$$

is an isomorphism of categories.

Let  $Fc \xrightarrow[\eta_c]{\theta_c} Gc \in \mathcal{H}om(\mathcal{C}, \mathcal{C}at)(Fc, Gc)$ . It can be easily checked

that the composites  $\{FX \xrightarrow{F\pi_i} FX_i \xrightarrow[\eta_{X_i}]{\theta_{X_i}} GX_i\}_{i \in \mathcal{I}}$  determine two pseudocones for  $GX$  together with a morphism of pseudocones. Since  $G$  preserves small 2-cofiltered pseudocolimits, post-composing with  $GX \xrightarrow{G\pi_i} GX_i$  is an isomorphism of categories  $\mathcal{C}at(FX, GX) \xrightarrow{(G\pi)^*} \mathcal{P}C_{\mathcal{C}at}(FX, GX)$ . It follows there exists a unique 2-cell in  $\mathcal{C}at$ ,  $FX \xrightarrow[\eta'_X]{\theta'_X} GX$ , such that



$G\pi_i\theta'_X = \theta_{X_i}F\pi_i$ ,  $G\pi_i\eta'_X = \eta_{X_i}F\pi_i$ , and  $G\pi_i\mu'_X = \mu_{X_i}F\pi_i$ ,  $\forall i \in \mathcal{I}$ . It is not difficult to check that  $\theta'_X$ ,  $\eta'_X$  are in fact 2-natural on  $X$ , and that  $\mu'_X$  is a modification. Clearly  $\theta'c = \theta$ ,  $\eta'c = \eta$ , and  $\mu'c = \mu$ . Thus 2.4.3 is an isomorphism of categories.  $\square$

**2.4.4 Lemma.** *Let  $\mathcal{C}$  be a 2-category,  $\mathcal{E}$  a 2-category closed under small 2-cofiltered pseudolimits and  $F : \mathcal{C} \rightarrow \mathcal{E}$  a 2-functor. Then, there exists a 2-functor  $\widehat{F} : 2\text{-Pro}(\mathcal{C}) \rightarrow \mathcal{E}$  that preserves small 2-cofiltered pseudolimits, and an isomorphism  $\widehat{F}c \xrightarrow{\cong} F$  in  $\mathcal{H}om(\mathcal{C}, \mathcal{E})$ .*

*Proof.* If  $X = (X_i)_{i \in \mathcal{I}} \in 2\text{-Pro}(\mathcal{C})$ , define  $\widehat{F}X = \varprojlim_{i \in \mathcal{I}^{op}} FX_i$ . We will prove that this is the object function part of a 2-functor, and that this 2-functor has the rest of the properties asserted in the proposition.

Consider the composition  $y_{(-)}F : \mathcal{C} \xrightarrow{F} \mathcal{E} \xrightarrow{y_{(-)}} \mathcal{H}om(\mathcal{E}^{op}, \mathcal{C}at)$ , where  $y_{(-)}$  is the Yoneda 2-functor (1.1.15). Under the isomorphism 1.1.14 this corresponds to a 2-functor  $\widehat{(-)} : \mathcal{C} \rightarrow \mathcal{H}om(\mathcal{C}, \mathcal{C}at)$ . Composing this 2-functor with a quasi-inverse  $\widetilde{(-)}$  for the 2-equivalence in 2.4.2, we obtain a 2-functor  $\mathcal{E}^{op} \rightarrow \mathcal{H}om(2\text{-Pro}(\mathcal{C}), \mathcal{C}at)_+$ , which in turn corresponds to a 2-functor  $2\text{-Pro}(\mathcal{C}) \xrightarrow{\widetilde{F}} \mathcal{H}om(\mathcal{E}^{op}, \mathcal{C}at)$ . The 2-functor  $\widetilde{F}$  preserves small 2-cofiltered pseudolimits because they are computed pointwise in  $\mathcal{H}om(\mathcal{E}^{op}, \mathcal{C}at)$  (1.2.4). Chasing the isomorphisms shows that we have the following diagram:

$$(2.4.5) \quad \begin{array}{ccc} \mathcal{C} & \xrightarrow{c} & 2\text{-Pro}(\mathcal{C}) \\ \downarrow F & \Downarrow \cong & \downarrow \widetilde{F} \\ \mathcal{E} & \xrightarrow{y_{(-)}} & \mathcal{H}om(\mathcal{E}^{op}, \mathcal{C}at) \end{array}$$

$\widetilde{F}c \xrightarrow{\cong} y_{(-)}F$

Consider the following chain of isomorphisms (the first and the third because  $\widetilde{F}$  and  $y_{(-)}$  preserve pseudolimits (1.2.6), and the middle one given by 2.4.5):

$$\widetilde{F}X = \widetilde{F}\varprojlim_{i \in \mathcal{I}} X_i \xrightarrow{\cong} \varprojlim_{i \in \mathcal{I}} \widetilde{F}cX_i \xrightarrow{\cong} \varprojlim_{i \in \mathcal{I}} y_{(-)}FX_i \xleftarrow{\cong} y_{(-)}\varprojlim_{i \in \mathcal{I}} FX_i.$$

This shows that  $\widetilde{F}X$  is in the essential image of  $y_{(-)}$ . Since  $y_{(-)}$  is 2-fully faithful (1.1.17), it follows there is a factorization  $y_{(-)}\widehat{F} \xrightarrow{\cong} \widetilde{F}$ ,

given by a 2-functor  $2\text{-Pro}(\mathcal{C}) \xrightarrow{\widehat{F}} \mathbf{E}$ . Clearly  $\widehat{F}$  preserves small 2-cofiltered pseudolimits. We have  $y_{(-)}\widehat{F}c \xrightarrow{\cong} \widetilde{F}c \xrightarrow{\cong} y_{(-)}\widehat{F}$ . Finally, the fully faithfulness of  $y_{(-)}$  provides an isomorphism  $y_{(-)}\widehat{F} \xrightarrow{\cong} F$ . This finishes the proof.  $\square$

Exactly the same proof of theorem 2.4.2 applies with an arbitrary 2-category  $\mathcal{E}$  in place of  $\mathcal{C}at$ , and we have:

**2.4.6 Theorem.** *Let  $\mathcal{C}$  be any 2-category, and  $\mathcal{E}$  a 2-category closed under small 2-cofiltered pseudolimits. Then, pre-composition with  $\mathcal{C} \xrightarrow{c} 2\text{-Pro}(\mathcal{C})$  is a 2-equivalence of 2-categories:*

$$\mathcal{H}om(2\text{-Pro}(\mathcal{C}), \mathcal{E})_+ \xrightarrow{c^*} \mathcal{H}om(\mathcal{C}, \mathcal{E})$$

Where  $\mathcal{H}om(2\text{-Pro}(\mathcal{C}), \mathcal{E})_+$  stands for the full subcategory whose objects are those 2-functors that preserve small 2-cofiltered pseudolimits.  $\square$

From theorem 2.4.6 it follows automatically the pseudo-functoriality of the assignment of the 2-category  $2\text{-Pro}(\mathcal{C})$  to each 2-category  $\mathcal{C}$ , and in such a way that  $c$  becomes a pseudonatural transformation. But we can do better:

If we put  $\mathcal{E} = 2\text{-Pro}(\mathcal{D})$  in 2.4.6 it follows there is a 2-functor (post-composing with  $c$  followed by a quasi-inverse in 2.4.6)

$$(2.4.7) \quad \mathcal{H}om(\mathcal{C}, \mathcal{D}) \xrightarrow{\widehat{(-)}} \mathcal{H}om(2\text{-Pro}(\mathcal{C}), 2\text{-Pro}(\mathcal{D}))_+,$$

and for each 2-functor  $\mathcal{C} \xrightarrow{F} \mathcal{D}$ , a diagram:

$$(2.4.8) \quad \begin{array}{ccc} 2\text{-Pro}(\mathcal{C}) & \xrightarrow{\widehat{F}} & 2\text{-Pro}(\mathcal{D}) \\ \uparrow c & \Downarrow \cong & \uparrow c \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

Given any 2-pro-object  $X \in 2\text{-Pro}(\mathcal{C})$ , set  $2\text{-Pro}(F)(X) = \widehat{F}X$ . It is straightforward to check that this determines a 2-functor

$$2\text{-Pro}(\mathcal{C}) \xrightarrow{2\text{-Pro}(F)} 2\text{-Pro}(\mathcal{D})$$

making diagram 2.4.8 commutative. It follows we have an isomorphism  $\widehat{F}X \xrightarrow{\cong} 2\text{-Pro}(\mathbf{F})(X)$  2-natural in  $X$ . This shows that the 2-functor  $2\text{-Pro}(\mathbf{F})$  preserves small 2-cofiltered pseudolimits because  $\widehat{F}$  does. Also, it follows that  $2\text{-Pro}(\mathbf{F})$  determines a 2-functor as in 2.4.7. In conclusion, denoting now by  $2\text{-CAT}$  the 2-category of locally small 2-categories (see 1.1.10) we have:

**2.4.9 Theorem.** *The definition  $2\text{-Pro}(\mathbf{F})(X) = \widehat{F}X$  determines a 2-functor*

$$2\text{-Pro}(-) : 2\text{-CAT} \longrightarrow 2\text{-CAT}_+$$

*in such a way that  $c$  becomes a 2-natural transformation (where  $2\text{-CAT}_+$  is the full sub 2-category of locally small 2-categories closed under small 2-cofiltered pseudo limits and small pseudolimit preserving 2-functors).*  $\square$

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## STABLE COMPACTIFICATIONS OF FRAMES

by Guram Bezhanishvili and John Harding

**Résumé.** Nous proposons une nouvelle description des compactifications stables de Smyth des espaces  $T_0$  comme plongements dans des espaces compacts stables qui sont denses pour la “patch topology”, et nous relient ces compactifications stables au cas des espaces ordonnés. Dans ce cadre “sans point”, nous introduisons une notion de compactification stable d’un frame qui étend la compactification stable de Smyth d’un espace  $T_0$ , ainsi que la compactification de Banaschewski d’un frame. Nous caractérisons l’ensemble ordonné des compactifications stables d’un frame en termes de proximités sur le frame, et en termes de sous-frames stablement compacts du frame de ses idéaux. Ces résultats sont alors appliqués aux compactifications cohérentes de frames, et reliés à la compactification spectrale d’un espace  $T_0$  considérée par Smyth.

**Abstract.** In a classic paper, Smirnov [25] characterized the poset of compactifications of a completely regular space in terms of the proximities on the space. Banaschewski [1] formulated Smirnov’s results in the pointfree setting, defining a compactification of a completely regular frame, and characterizing these in terms of the strong inclusions on the frame. Smyth [26] generalized the concept of a compactification of a completely regular space to that of a stable compactification of a  $T_0$ -space and described them in terms of quasi-proximities on the space.

We provide an alternate description of stable compactifications of  $T_0$ -spaces as embeddings into stably compact spaces that are dense with respect to the patch topology, and relate such stable compactifications to ordered spaces. Each stable compactification of a  $T_0$ -space induces a companion topology on the space, and we show the companion topology induced by the largest stable compactification is the topology  $\tau^*$  studied by Salbani [21, 22].

In the pointfree setting, we introduce a notion of a stable compactification of a frame that extends Smyth’s stable compactification of a  $T_0$ -space, and Banaschewski’s compactification of a frame. We characterize the poset of stable compactifications of a frame in terms of proximities on the frame, and in terms of stably compact subframes of its ideal frame. These results are then specialized to coherent compactifications of frames, and related to Smyth’s spectral compactifications of a  $T_0$ -space.

**Keywords.** Pointfree topology, proximity, compactification, stable compactness.

**Mathematics Subject Classification (2010).** 06D22; 54D35; 54E05; 54D30; 54D45; 18B30.

## 1. Introduction

A classic result of Smirnov [25] shows that the poset of compactifications of a completely regular space  $X$  is isomorphic to the poset of proximities on  $X$  that are compatible with the topology on  $X$ . Banaschewski [1] generalized Smirnov's theorem to the pointfree setting by introducing the concept of a compactification of a frame. He also generalized the concept of a proximity on a space to that of a strong inclusion on a frame, and proved that the poset of compactifications of a frame  $L$  is isomorphic to the poset of strong inclusions on  $L$ . In particular, if  $L$  is the frame of open sets of a completely regular space, then Smirnov's theorem follows.

Smyth [26] generalized the theory of compactifications of completely regular spaces to that of stable compactifications of  $T_0$ -spaces. He also generalized the concept of proximity to that of quasi-proximity and proved that the poset of stable compactifications of a  $T_0$ -space  $X$  is isomorphic to the poset of quasi-proximities on  $X$  that are compatible with the topology on  $X$ . Restricting to completely regular spaces and proximities then yields Smirnov's theorem.

In this paper, we provide an alternate description of Smyth's stable compactification of a  $T_0$ -space  $X$  as an embedding of  $X$  into a stably compact space  $Y$  whose image is dense in the patch topology of  $Y$ . We then relate such stable compactifications to ordered spaces. Each stable compactification of a  $T_0$ -space  $X$  induces an ordered space structure on  $X$  whose open upsets are the given topology on  $X$ , and under which the stable compactification can be naturally viewed as an order-compactification. So each stable compactification of  $X$  yields a companion topology to the original, the topology of open downsets of the associated ordered space. We show the companion topology associated with the largest stable compactification of  $X$  is the topology  $\tau^*$  studied by Salbani [21, 22].

We then extend Smyth's theory of stable compactifications to the point-free setting. We introduce the concept of a stable compactification of a frame, and prove a generalization of Banaschewski's theorem, showing that the poset of stable compactifications of a frame  $L$  is isomorphic to the poset of proximities on the frame in the sense of [5], and to the poset of certain stably compact subframes of the ideal frame of  $L$ . The spatial case of this result yields Smyth's theorem.

The paper is organized as follows. The second section provides preliminaries. In the third, we discuss stable compactifications of  $T_0$ -spaces, giving a characterization of such compactifications in terms of the patch topology, and relating such compactifications to ordered spaces. In the fourth section we define stable compactifications of frames, and provide characterizations of such stable compactifications in terms of proximities, and in terms of certain subframes of the ideal frame. The fifth section specializes the results of the fourth to coherent and spectral compactifications.

## 2. Preliminaries

Recall the classical notion of a compactification of a topological space  $X$  is an embedding  $e : X \rightarrow Y$  into a compact Hausdorff space  $Y$  whose image is dense in  $Y$ . Here, embedding is used to mean that  $e$  is a homeomorphism from  $X$  to its image considered with the subspace topology from  $Y$ . Classical results characterize those spaces  $X$  having a compactification as the completely regular ones. It is standard to form a poset from the compactifications of a completely regular space, as in the following definition.

**Definition 2.1.** *For compactifications  $e : X \rightarrow Y$  and  $e' : X \rightarrow Y'$  of  $X$  write  $e' \sqsubseteq e$  if there is a continuous map  $f : Y \rightarrow Y'$  with  $e' = f \circ e$ .*

It is well known that  $\sqsubseteq$  is a quasi-order. This induces an equivalence relation on the class of compactifications, and the associated partially ordered set of equivalence classes is called the poset of compactifications of  $X$ . Smirnov described this poset in terms of proximities on  $X$ . Standard results show that the Stone-Ćech compactification of  $X$  is the largest member of this poset, and that this poset has a least element iff  $X$  is locally compact, and in this case the least element is the one-point compactification of  $X$  (see, e.g., [10, Sec. 3.5 and 3.6]). Also standard to the theory of compactifications is the following result.

**Theorem 2.2.** *A compact Hausdorff space  $X$  has up to homeomorphism only itself as a compactification.*

The notion of a compactification  $e : X \rightarrow Y$  can be extended in an obvious way simply by dropping the requirement that the compact space  $Y$  be

Hausdorff. However, this is a very poorly behaved notion, with a space  $X$  having such general compactifications of arbitrary cardinality. Smyth [26] introduced a notion of a stable compactification of a  $T_0$ -space, that although still pathological in some ways, is much better behaved. We describe these stable compactifications in detail in the following section, but remark they are certain dense embeddings into the stably compact spaces we describe next. A few basic definitions are required first.

A topological space  $X$  is *locally compact* if for each  $x \in X$  and open neighborhood  $U$  of  $x$ , there is an open neighborhood  $V$  of  $x$  and a compact set  $K$  with  $V \subseteq K \subseteq U$ . A subset  $A$  of  $X$  is *irreducible* if  $A \subseteq B \cup C$  with  $B, C$  closed implies  $A \subseteq B$  or  $A \subseteq C$ ; and  $X$  is *sober* if each closed irreducible set is the closure of a unique singleton. Finally, a subset of  $X$  is *saturated* if it is an intersection of open sets.

**Definition 2.3.** *A space  $X$  is stably compact if it is compact, locally compact, sober, and the intersection of two compact saturated sets is compact.*

The theory of stably compact spaces is developed in detail in [12], where it is shown that there is a close connection between stably compact spaces and certain ordered topological spaces. To recall this connection, we need to describe two additional topologies associated to any stably compact space.

**Definition 2.4.** *For a stably compact space  $X$  with topology  $\tau$ , the compact saturated sets are the closed sets of a topology  $\tau^k$  on  $X$  called the co-compact topology. The join of the topologies  $\tau$  and  $\tau^k$  is called the patch topology  $\pi$ .*

An *ordered topological space* is a triple  $(X, \leq, \pi)$  consisting of a set  $X$  with partial ordering  $\leq$  and topology  $\pi$ . A subset  $U$  of  $X$  is an *upset* if  $x \in U$  and  $x \leq y$  imply  $y \in U$ , and it is a *downset* if  $x \in U$  and  $y \leq x$  imply  $y \in U$ . An ordered topological space  $(X, \leq, \pi)$  is *order-Hausdorff* if  $x \not\leq y$  implies that there exist an upset neighborhood  $U$  of  $x$  and a downset neighborhood  $V$  of  $y$  such that  $U \cap V = \emptyset$ . It is well known (see, e.g., [14]) that  $(X, \leq, \pi)$  is order-Hausdorff iff  $\leq$  is a closed subset of  $X^2$ . Ordered topological spaces were introduced by Nachbin, who showed that compact order-Hausdorff spaces provide a natural generalization of compact Hausdorff spaces [15]. In honor of Nachbin, we make the following definition.

**Definition 2.5.** *A Nachbin space is a compact order-Hausdorff space.*



We recall that in a topological space, the specialization order  $\leq$  is defined by  $x \leq y$  iff the closure of  $y$  contains  $x$ . The following results are well known [12, Sec. VI-6].

**Theorem 2.6.** *If  $(X, \tau)$  is a stably compact space with specialization order  $\leq$  and patch topology  $\pi$ , then  $(X, \leq, \pi)$  is a Nachbin space whose open upsets are the  $\tau$ -open sets, and whose open downsets are the  $\tau^k$ -open sets. We call this the Nachbin space associated to  $(X, \tau)$ . Conversely, if  $(X, \leq, \pi)$  is a Nachbin space, then the open upsets form a topology  $\tau$  on  $X$ . The space  $(X, \tau)$  is stably compact, and its associated Nachbin space is  $(X, \leq, \pi)$ .*

Following [12], we call a continuous map  $f$  between stably compact spaces *proper* if the inverse image of each compact saturated set is compact. This is equivalent to  $f$  being continuous with respect to both the given and co-compact topologies. Let  $\text{StKSp}$  be the category of stably compact spaces and proper maps. Let  $\text{Nach}$  be the category of Nachbin spaces and the continuous order-preserving maps between them. The above result extends as follows [12, Sec. VI-6].

**Theorem 2.7.** *There is an isomorphism between the categories  $\text{StKSp}$  and  $\text{Nach}$  taking a stably compact space to its associated Nachbin space.*

We next turn our attention to frames.

**Definition 2.8.** *A frame is a complete lattice  $L$  that satisfies  $a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$ . A frame homomorphism is a map  $f : L \rightarrow M$  that preserves finite meets (including 1) and arbitrary joins (including 0).*

For a topological space  $X$ , its open sets  $\Omega(X)$  form a frame, and for any continuous map  $f : X \rightarrow Y$ , the map  $\Omega(f) = f^{-1} : \Omega(Y) \rightarrow \Omega(X)$  is a frame homomorphism. This gives a contravariant functor  $\Omega : \text{Top} \rightarrow \text{Frm}$  from the category of topological spaces and continuous maps to the category of frames and frame homomorphisms. A *point* of a frame  $L$  is a frame homomorphism  $p : L \rightarrow 2$  into the two-element frame. The points  $\text{pt}(L)$  of  $L$  are topologized by taking for all  $a \in L$  the sets  $\varphi(a) = \{p : p(a) = 1\}$  as open sets. For a frame homomorphism  $f : L \rightarrow M$ , the map  $\text{pt}(f) : \text{pt}(M) \rightarrow \text{pt}(L)$  defined by  $\text{pt}(f)(p) = p \circ f$  is continuous. This gives a contravariant functor  $\text{pt} : \text{Frm} \rightarrow \text{Top}$ . The following results are well known [13, 18].

**Theorem 2.9.** *The functors  $\Omega$  and  $\text{pt}$  give a dual adjunction between  $\text{Top}$  and  $\text{Frm}$ . For each frame  $L$ , the dual adjunction provides a frame homomorphism  $h : L \rightarrow \Omega(\text{pt}(L))$ , which is always onto. A frame is called *spatial* if  $h$  is an isomorphism. For each space  $X$ , the dual adjunction provides a continuous map  $s : X \rightarrow \text{pt}(\Omega(X))$  called the *sobrification* of the space, which is a topological embedding iff the space is  $T_0$ . A space is *sober* iff  $s$  is a homeomorphism. The functors  $\Omega$  and  $\text{pt}$  restrict to give a dual equivalence between the categories of spatial frames and sober spaces.*

For the convenience of the reader, we isolate two consequences of this result used later.

**Corollary 2.10.** *If  $X, Y$  are spaces with  $Y$  sober, then  $\Omega$  gives a bijection between the homsets  $\text{Top}(X, Y)$  and  $\text{Frm}(\Omega(Y), \Omega(X))$ . If  $X$  is  $T_0$ , then a continuous map  $e : X \rightarrow Y$  is an embedding iff the frame homomorphism  $\Omega(e) : \Omega(Y) \rightarrow \Omega(X)$  is onto.*

We turn now to finer properties of frames, see [13, 18] for further details.

**Definition 2.11.** *For  $a, b$  elements of a frame  $L$ , we say  $a$  is way below  $b$ , and write  $a \ll b$ , if for any  $T$  with  $b \leq \bigvee T$ , there is a finite subset  $S \subseteq T$  with  $a \leq \bigvee S$ . We say  $a$  is well inside  $b$ , and write  $a < b$ , if  $\neg a \vee b = 1$ , where  $\neg a$  is the pseudocomplement of  $a$  in  $L$ .*

An element  $a$  of a frame  $L$  is *compact* if  $a \ll a$ , and a frame  $L$  is *compact* if its top element  $1$  is compact. We next use the way below and well inside relations to define the particular classes of frames of primary interest here.

**Definition 2.12.** *We say a frame  $L$  is*

1. *locally compact* if  $a = \bigvee \{x : x \ll a\}$  for each  $a \in L$ .
2. *regular* if  $a = \bigvee \{x : x < a\}$  for each  $a \in L$ .
3. *stable* if  $a \ll b, c$  implies  $a \ll b \wedge c$  for all  $a, b, c \in L$ .

*We say  $L$  is compact regular if it is compact and regular, and stably compact if it is locally compact, compact, and stable.*

Let  $\text{KRFrm}$  be the category of compact regular frames and frame homomorphisms between them. A frame homomorphism  $f$  is called *proper* if  $a \ll b$  implies  $fa \ll fb$ . Let  $\text{StKFrm}$  be the category of stably compact frames and proper frame homomorphisms between them. Let  $\text{KHaus}$  be the category of compact Hausdorff spaces. The following are well known [12, 13].

**Theorem 2.13.** *A space  $X$  is stably compact iff the frame  $\Omega(X)$  is stably compact, and a frame  $L$  is stably compact iff it is isomorphic to  $\Omega(X)$  for some stably compact space  $X$ . Further, a continuous map  $f$  between stably compact spaces  $X$  and  $Y$  is proper iff the corresponding frame homomorphism between  $\Omega(Y)$  and  $\Omega(X)$  is proper. Thus, the functors  $\Omega$  and  $\text{pt}$  restrict to give a dual equivalence between  $\text{StKSp}$  and  $\text{StKFrm}$ .*

*Each compact Hausdorff space is stably compact, and every continuous map between compact Hausdorff spaces is proper. So  $\text{KHaus}$  is a full subcategory of  $\text{StKSp}$ ,  $\text{KRFrm}$  is a full subcategory of  $\text{StKFrm}$ , and  $\Omega$  and  $\text{pt}$  restrict to give a dual equivalence between  $\text{KHaus}$  and  $\text{KRFrm}$ .*

A frame is *coherent* if each element is the join of compact elements, and the meet of two compact elements is compact. A frame homomorphism  $h$  between two coherent frames  $L$  and  $M$  is *coherent* if  $a$  compact in  $L$  implies that  $h(a)$  is compact in  $M$ . Let  $\text{CohFrm}$  be the category of coherent frames and the coherent frame homomorphisms between them. A space  $X$  is a *spectral space* if it is sober, compact, and the compact open sets are closed under finite intersections and form a basis. A continuous map between spectral spaces is a *spectral map* if the inverse image of each compact open set is compact open. Let  $\text{Spec}$  be the category of spectral spaces and the spectral maps between them. We conclude the preliminaries with the following well known result [13].

**Theorem 2.14.** *The category  $\text{Spec}$  is a full subcategory of  $\text{StKSp}$ , the category  $\text{CohFrm}$  is a full subcategory of  $\text{StKFrm}$ , and the functors  $\Omega$ ,  $\text{pt}$  restrict to a dual equivalence between  $\text{Spec}$  and  $\text{CohFrm}$ .*

### 3. Stable compactifications of spaces

In this section we recall Smyth's definition of a stable compactification of a  $T_0$ -space  $X$ , and Smyth's ordering of the stable compactifications of  $X$ . We

provide an alternate description of such stable compactifications in terms of the patch topology of a stably compact space, and remark on the connections between stable compactifications and ordered spaces.

**Definition 3.1.** [26] *Let  $X$  be a  $T_0$ -space,  $Y$  be a stably compact space, and  $e : X \rightarrow Y$  be a homeomorphism from  $X$  to a subspace of  $Y$ . For  $U \in \Omega(Y)$ , let  $\overline{U}$  be the largest open set of  $Y$  whose intersection with the image of  $X$  is contained in  $U$ . We say the pair  $(Y, e)$  is a stable compactification of  $X$  if  $U \ll V \Rightarrow \overline{U} \ll V$  for all  $U, V \in \Omega(Y)$ , where  $U \ll V$  means  $U$  is way below  $V$  in the frame  $\Omega(Y)$ .*

If  $(Y, e)$  is a stable compactification of  $X$ , using  $\emptyset \ll \emptyset$  it follows that  $\overline{\emptyset} = \emptyset$ , hence the image  $e[X]$  is dense in  $Y$ . We next recall Smyth's ordering of the stable compactifications of a  $T_0$ -space  $X$ .

**Definition 3.2.** *For two stable compactifications  $e : X \rightarrow Y$  and  $e' : X \rightarrow Y'$ , define  $e' \sqsubseteq e$  if there is a proper map  $f : Y \rightarrow Y'$  with  $e' = f \circ e$ . We let  $\text{COMP } X$  be the poset of equivalence classes of stable compactifications of  $X$  under the partial order associated with the quasi-order  $\sqsubseteq$  and denote the equivalence class of a compactification  $e : X \rightarrow Y$  by  $[e]$ .*

Smyth characterized the poset  $\text{COMP } X$  in terms of his “quasi-proximities” on  $X$ , and showed it has a largest element given by the space of prime filters of the frame  $\Omega(X)$  of open sets of  $X$ .

**Remark 3.3.** Stable compactifications of  $T_0$ -spaces lack some of the familiar properties of classical compactifications of completely regular spaces. Smyth's result [26, Prop. 16] that the space of prime filters of  $\Omega(X)$  gives the largest stable compactification of  $X$  yields an example showing that a compact Hausdorff space can have a stable compactification that is not Hausdorff. One can further show that for stable compactifications  $e : X \rightarrow Y$  and  $k : Y \rightarrow Z$ , the composite  $k \circ e : X \rightarrow Z$  need not be a stable compactification. For an example of this let  $X$  be the negative integers with the obvious order and the upset topology. Let  $e : X \rightarrow Y$  be the largest stable compactification of  $X$  and let  $k : Y \rightarrow Z$  be the largest stable compactification of  $Y$ . Then  $k \circ e : X \rightarrow Z$  is not a stable compactification of  $X$ .

The condition  $U \ll V \Rightarrow \overline{U} \ll V$  in Smyth's definition of a stable compactification has a strongly frame-theoretic nature. We provide a description

of stable compactifications in more purely topological terms, namely as embeddings into stably compact spaces that are dense in the patch topology. We note that this is somewhat the inverse of the usual sequence of things in pointfree topology, when standard topological notions are given pointfree meaning. We begin with several standard facts from the theory of ordered spaces whose proofs can be found in [12, 15].

**Proposition 3.4.** *Let  $(Y, \leq, \pi)$  be a Nachbin space, and let  $\text{cl}_\pi$  be the closure operator with respect to the topology  $\pi$ .*

1.  $A \ll B$  in the frame of open upsets of  $Y$  iff  $\text{cl}_\pi(A) \subseteq B$ .
2. If  $B$  is an open downset,  $B = \bigcup\{A : A \text{ is an open downset and } \text{cl}_\pi(A) \subseteq B\}$ .
3. If  $A$  is closed, then its downset  $\downarrow A$  is closed.

**Theorem 3.5.** *For a  $T_0$ -space  $X$ , an embedding  $e : X \rightarrow Y$  into a stably compact space  $Y$  is a stable compactification of  $X$  iff the image of  $X$  is dense in the patch topology of  $Y$ .*

*Proof.* By identifying  $X$  with its image  $e[X]$  in  $Y$ , we assume that  $X$  is a subspace of  $Y$  and  $Y$  is a stably compact space with topology  $\tau$  and patch topology  $\pi$ . Let  $(Y, \leq, \pi)$  be the Nachbin space associated to  $Y$ .

“ $\Leftarrow$ ” Assume  $X$  is patch-dense in  $Y$ . To show the identical embedding of  $X$  into  $Y$  is a stable compactification, we must show that for  $U, V$   $\tau$ -open subsets of  $Y$ , that  $U \ll V$  implies  $\overline{U} \ll V$ . If  $U \ll V$ , then by Proposition 3.4.1,  $\text{cl}_\pi(U) \subseteq V$ . As  $X$  is patch-dense in  $Y$ , for each patch-open subset  $W$  of  $Y$ , we have  $\text{cl}_\pi(W) = \text{cl}_\pi(W \cap X)$ . Therefore, from the definition of  $\overline{U}$  as the largest  $\tau$ -open set whose intersection with  $X$  is contained in  $U$ , we have  $\text{cl}_\pi(\overline{U}) = \text{cl}_\pi(\overline{U} \cap X) = \text{cl}_\pi(U \cap X) = \text{cl}_\pi(U)$ . Thus,  $\text{cl}_\pi(\overline{U}) \subseteq V$ , so  $\overline{U} \ll V$ .

“ $\Rightarrow$ ” Assume  $X$  is not patch-dense in  $Y$ . We show that there exist  $\tau$ -open sets  $U, V$  such that  $U \ll V$  and  $\overline{U} \not\ll V$ . We recall (see Theorem 2.6) that the open upsets of the Nachbin space  $(Y, \leq, \pi)$  are the topology  $\tau$ , the open downsets are the co-compact topology  $\tau^k$ , and the join of these topologies is the patch topology  $\pi$ . We also use  $-T$  for the set-theoretic complement of  $T$ .

**Claim 3.6.**

1. *There exist an open upset  $S$  and an open downset  $T$  with  $S \cap T \neq \emptyset$  and  $X \cap S \cap T = \emptyset$ .*
2. *For each open upset  $A$ , if  $-T \subseteq A$ , then  $S \subseteq \overline{A}$ .*

*Proof of Claim:* (1) This is a consequence of the fact that the patch topology is the join of the topologies of the open upsets and open downsets and that  $X$  is not patch-dense. (2) As  $X \cap S \cap T = \emptyset$ , we have  $X \cap S \subseteq -T \subseteq A$ . So  $S \subseteq \overline{A}$ .  $\square$

**Claim 3.7.** *Let  $z \in S \cap T$ . There are open downsets  $P, Q$  with*

1.  *$z \in Q$  and  $\text{cl}_\pi(Q) \subseteq T$ .*
2.  *$z \in P$  and  $\text{cl}_\pi(P) \subseteq Q$ .*

*Further, both  $U = -\downarrow\text{cl}_\pi(Q)$  and  $V = -\downarrow\text{cl}_\pi(P)$  are open upsets.*

*Proof of Claim:* (1) As  $z \in T$  and  $T$  is an open downset, Proposition 3.4.2 gives an open downset  $Q$  with  $z \in Q$  and  $\text{cl}_\pi(Q) \subseteq T$ . (2) As  $z \in Q$  and  $Q$  is an open downset, another application of Proposition 3.4.2 gives an open downset  $P$  with  $z \in P$  and  $\text{cl}_\pi(P) \subseteq Q$ . For the further comment, by Proposition 3.4.3, both  $\downarrow\text{cl}_\pi(Q)$  and  $\downarrow\text{cl}_\pi(P)$  are closed, and are clearly downsets. Thus, their complements are open upsets.  $\square$

We show the open upsets  $U, V$  satisfy  $U \ll V$  and  $\overline{U} \not\ll V$ . As  $Q \subseteq \downarrow\text{cl}_\pi(Q)$ , we have  $U = -\downarrow\text{cl}_\pi(Q) \subseteq -Q$ , and as  $\text{cl}_\pi(P) \subseteq Q$  and  $Q$  is a downset,  $\downarrow\text{cl}_\pi(P) \subseteq Q$ , giving  $-Q \subseteq -\downarrow\text{cl}_\pi(P) = V$ . Thus,  $U \subseteq -Q \subseteq V$ , and as  $Q$  is open,  $\text{cl}_\pi(U) \subseteq V$ . So by Proposition 3.4.1,  $U \ll V$ . To see that  $\overline{U} \not\ll V$ , note  $z \in P \subseteq \downarrow\text{cl}_\pi(P)$ , so  $z \notin -\downarrow\text{cl}_\pi(P) = V$ . As  $\text{cl}_\pi(Q) \subseteq T$  and  $T$  is a downset, we have  $\downarrow\text{cl}_\pi(Q) \subseteq T$ , hence  $-T \subseteq -\downarrow\text{cl}_\pi(Q) = U$ . Since  $U$  is an open upset and  $-T \subseteq U$ , by Claim 3.6.2,  $S \subseteq \overline{U}$ . But  $z \in S$ , hence  $z \in \overline{U}$ , and  $z \notin V$ , so  $\overline{U} \not\subseteq V$ . Thus,  $\overline{U} \not\ll V$ .  $\square$

**Corollary 3.8.** *The stable compactifications of a  $T_0$ -space  $X$  determine, and are determined by, mappings of  $X$  into Nachbin spaces  $(Y, \leq, \pi)$  that are embeddings with respect to the topology of open upsets of  $Y$ , and are dense with respect to  $\pi$ .*

We next use this result to relate the poset of stable compactifications of a completely regular space  $X$  to its poset of classical compactifications. Since a compact Hausdorff space is stably compact and its patch topology coincides with the original topology, Theorem 3.5 shows that any compactification of  $X$  is a stable compactification. It also follows from Theorem 3.5 that any stable compactification into a compact Hausdorff space is a compactification. We use  $k : X \rightarrow \beta X$  for the Stone-Ćech compactification of  $X$ , and recall this is the largest compactification of  $X$ .

**Proposition 3.9.** *The poset of classical compactifications of a completely regular space  $X$  is a retract of the downset of  $\text{COMP } X$  generated by  $k : X \rightarrow \beta X$ . This retraction is realized by sending a stable compactification  $e : X \rightarrow (Y, \tau)$  that lies beneath  $k : X \rightarrow \beta X$  to the classical compactification  $e : X \rightarrow (Y, \pi)$ , where  $\pi$  is the patch topology of  $\tau$ .*

*Proof.* Suppose  $e : X \rightarrow Y$  is a stable compactification of  $X$  that lies beneath  $k : X \rightarrow \beta X$  in the poset of stable compactifications. Then there is a proper continuous map  $f : \beta X \rightarrow Y$  with  $e = f \circ k$ . Let  $\sigma$  be the topology on  $\beta X$  and  $\tau$  be the topology on  $Y$ . As  $\beta X$  is compact Hausdorff, its patch topology is  $\sigma$ . Let  $\pi$  be the patch topology on  $Y$ . Since  $f$  is proper with respect to  $\sigma$  and  $\tau$ , it is continuous with respect to the patch topologies  $\sigma$  and  $\pi$ . Let  $U \in \pi$ . Then  $f^{-1}(U) \in \sigma$ , so  $k^{-1}f^{-1}(U)$  is open in  $X$ , hence  $e^{-1}(U)$  is open in  $X$ . Thus,  $e : X \rightarrow (Y, \pi)$  is continuous. By Theorem 3.5,  $e[X]$  is dense in  $(Y, \pi)$ , and as  $(Y, \pi)$  is a compact Hausdorff space, this is a compactification of  $X$ . It is then routine to show that the map sending such a stable compactification  $e : X \rightarrow (Y, \tau)$  to the compactification  $e : X \rightarrow (Y, \pi)$  is the required retraction.  $\square$

We recall the classical result that a completely regular space has a least compactification iff it is locally compact, and in this case, its least compactification is the one-point compactification. As the construction of the one-point compactification of a locally compact Hausdorff space generalizes to any  $T_0$ -space (see, e.g., [6, Sec. 3]), every  $T_0$ -space  $X$  has a (possibly non-Hausdorff) one-point compactification. This one-point compactification does not have to be a stable compactification of  $X$ . In fact, as the next corollary shows, not every  $T_0$ -space has a least stable compactification.

**Corollary 3.10.** *The space of rationals  $\mathbb{Q}$  with the usual topology has no least stable compactification.*

*Proof.* If there were a least element in the poset of stable compactifications of  $\mathbb{Q}$ , then by Proposition 3.9, there would be a least element in the poset of classical compactifications of  $\mathbb{Q}$ . This is not the case since  $\mathbb{Q}$  is not locally compact.  $\square$

There are further connections between stable compactifications and ordered spaces. We describe some of these connections below. We start by recalling Nachbin's generalization of the concept of compactification to that of order-compactification.

**Definition 3.11.** *An order-compactification of an ordered space  $X$  is a pair  $(Y, e)$  such that  $Y$  is a Nachbin space,  $e : X \rightarrow Y$  is both a topological embedding and an order-embedding, and the image  $e[X]$  is topologically dense in  $Y$ .*

**Proposition 3.12.** *Let  $e : X \rightarrow Y$  be a stable compactification. Then the associated Nachbin space  $(Y, \leq, \pi)$  induces an ordered space structure on  $X$  whose open upsets are the original topology of  $X$ , and whose partial ordering is the specialization order of this topology. Further, the embedding  $e$  of this ordered structure into  $(Y, \leq, \pi)$  is an order-compactification.*

*Proof.* Let  $(X, \tau)$  and  $(Y, \delta)$  be our original  $T_0$  and stably compact spaces. By Theorem 3.5, the image  $e[X]$  is dense in the patch topology  $\pi$  of  $Y$ , so is a topologically dense subspace of the Nachbin space  $(Y, \leq, \pi)$ . The restriction of this Nachbin space to  $e[X]$  makes  $e[X]$  into an ordered space having  $(Y, \leq, \pi)$  as an order-compactification. So this induces an ordered space structure on  $X$  having  $(Y, \leq, \pi)$  as an order-compactification. It remains only to show the open upsets of this ordered space structure on  $X$  are the original topology  $\tau$ , and that the partial ordering on  $X$  is the specialization order.

The open upsets of  $e[X]$  are the restrictions of the open upsets of  $(Y, \leq, \pi)$ , hence are the restrictions of members of  $\delta$  to  $e[X]$ . So the open upsets of the induced structure on  $X$  are the inverse images under  $e$  of members of  $\delta$ , and as  $e$  is an embedding with respect to  $\tau$  and  $\delta$ , these are exactly the members of  $\tau$ . As the partial ordering of  $(Y, \leq, \pi)$  is the specialization order of  $\delta$ , the partial ordering of  $e[X]$  is the specialization order of the open upsets of  $e[X]$ , hence the partial ordering on  $X$  is the specialization order of  $\tau$ .  $\square$



Proposition 3.12 shows that every stable compactification can be viewed as an order-compactification. The following example shows the converse of this does not hold. The difficulty, roughly speaking, is in the fact that an order-compactification must be an embedding with respect to the patch topology, while a stable compactification must be an embedding with respect to the topology of open upsets.

**Example 3.13.** *Let  $(X, \leq, \pi)$  be the natural numbers  $\mathbb{N}$  with discrete topology, ordered as an antichain, and let  $(Y, \leq, \pi)$  be the one-point compactification  $\mathbb{N} \cup \{\infty\}$  ordered as an antichain on  $\mathbb{N}$  and with  $n \leq \infty$  for each  $n \in \mathbb{N}$ . The identical embedding is an order-compactification of  $(X, \leq, \pi)$  into  $(Y, \leq, \pi)$ . But the open upsets of  $Y$  are the cofinite ones containing  $\infty$ , while all subsets of  $X$  are open upsets. So the identical embedding is not a stable compactification with respect to the topologies of open upsets.*

Proposition 3.12 can be viewed in another light. Each stable compactification of a  $T_0$ -space  $(X, \tau)$  induces an ordered space structure on  $X$  having  $\tau$  as its open upsets, and giving a companion topology  $\tau'$  of open downsets, so that the join of the topologies  $\pi = \tau \vee \tau'$  is a completely regular topology. Based on the work of [4, 24, 17], Salbani [21, 22] has considered a method to associate with any  $T_0$  topology  $\tau$  on  $X$  a companion topology he calls  $\tau^*$ , where  $\tau^*$  has the members of  $\tau$  as a basis for the closed sets.

**Proposition 3.14.** *For a  $T_0$ -space  $(X, \tau)$ , Salbani's topology  $\tau^*$  is the companion topology to  $\tau$  arising from the largest stable compactification of  $X$ .*

*Proof.* Smyth [26, Prop. 16] showed that the largest stable compactification of  $(X, \tau)$  is the space  $Y$  of prime filters of  $\Omega(X)$ , i.e. the spectral space of the distributive lattice  $\Omega(X)$ . Here  $Y$  has as a basis for its topology all sets  $\varphi(U) = \{F \in Y : U \in F\}$ , where  $U \in \Omega(X)$ , and the embedding  $e : X \rightarrow Y$  is given by  $e(x) = \{U : x \in U\}$ . Note  $e^{-1}\varphi(U) = U$  for each  $U \in \Omega(X)$ . The Nachbin space associated to  $Y$  is the Priestley space [20] of  $\Omega(X)$ . The closed sets of the topology of open downsets of  $Y$  are the closed upsets of the Priestley space. These are the intersections of the clopen upsets, hence of the sets  $\varphi(U)$ , where  $U \in \Omega(X)$ . So the companion topology on  $X$  induced by this stable compactification has the sets  $U = e^{-1}\varphi(U)$  for  $U \in \Omega(X)$  as a basis for its closed sets. Thus, this companion topology is Salbani's topology  $\tau^*$ .  $\square$

**Remark 3.15.** We may consider these results in one further context. We recall that a *bitopological space* is a set  $X$  equipped with two topologies  $\tau_1$  and  $\tau_2$ . For a bispace  $(X, \tau_1, \tau_2)$ , let  $\pi = \tau_1 \vee \tau_2$  be the patch topology. Following [21], we call a bispace  $(X, \tau_1, \tau_2)$  *compact* if  $(X, \pi)$  is compact,  $T_0$  if  $(X, \pi)$  is  $T_0$ , and *regular* if it is  $T_0$  and for each  $U \in \tau_i$ , we have  $U = \bigcup \{V \in \tau_i : \text{cl}_k(V) \subseteq U\}$  ( $i \neq k$ ,  $i, k = 1, 2$ ). The correspondence of Theorem 2.6 between stably compact spaces and Nachbin spaces extends to also include compact regular bispaces. Indeed, if  $(X, \leq, \pi)$  is a Nachbin space, then the open upsets and open downsets form a compact regular bispace, and each compact regular bispace arises this way (see, e.g., [12]).

Salbany [21] generalized the notion of compactification to that of bi-compactification. A *bicompactification* of a bispace  $(X, \tau_1, \tau_2)$  is a bispace embedding  $e : (X, \tau_1, \tau_2) \rightarrow (Y, \delta_1, \delta_2)$  into a compact regular bispace  $(Y, \delta_1, \delta_2)$  such that  $e[X]$  is dense in the patch topology  $\pi = \delta_1 \vee \delta_2$ . For any stable compactification  $e : (X, \tau_1) \rightarrow (Y, \delta_1)$  of a  $T_0$ -space, letting  $\delta_2$  be the co-compact topology of  $\delta_1$ , produces a compact regular bispace  $(Y, \delta_1, \delta_2)$ . This induces a completely regular bispace structure  $(X, \tau_1, \tau_2)$  on  $X$  with  $e : (X, \tau_1, \tau_2) \rightarrow (Y, \delta_1, \delta_2)$  a bispace compactification. (Note that  $\tau_2$  is not determined by  $\tau_1$ , but is dependent on the specific stable compactification of  $(X, \tau_1)$ .) This is the bispace analogue of Proposition 3.12, and indicates that every stable compactification can be viewed as a bicompactification. Conversely, it is easily seen from Theorem 3.5 that if  $e : (X, \tau_1, \tau_2) \rightarrow (Y, \delta_1, \delta_2)$  is a bicompactification, then  $e : (X, \tau_1) \rightarrow (Y, \delta_1)$  is a stable compactification.

## 4. Stable compactifications of frames

In this section we extend the notion of stable compactifications to the setting of frames, and describe the poset of stable compactifications of a frame in several ways. To begin, we recall Banaschewski's definition of a compactification of a frame [1].

**Definition 4.1.** A *compactification of a frame*  $L$  is a dense frame homomorphism  $f : M \rightarrow L$  from a compact regular frame  $M$  onto  $L$ . Here, a frame homomorphism is dense if for all  $x \in M$  we have  $f(x) = 0$  implies  $x = 0$ .

Banaschewski showed that a frame  $L$  has a compactification iff it is a

completely regular frame, and that for a completely regular space  $X$ , the compactifications of the frame  $\Omega(X)$  correspond to the compactifications of the space  $X$ . So compactifications of spatial frames amount to a translation of the notion of compactification to the frame language. However, while every compact regular frame  $M$  is spatial, there are completely regular frames  $L$  that are not spatial, and for these the notion of compactification is new. We now extend these ideas to stable compactifications of frames.

**Definition 4.2.** For a frame homomorphism  $f : M \rightarrow L$  we let  $r_f : L \rightarrow M$  be the right adjoint of  $f$ , namely the map  $r_f(a) = \bigvee \{x : f(x) \leq a\}$ .

The map  $r_f$  preserves finite meets, but need not preserve finite joins. When the map  $f$  is clear from the context, we often use  $r$  in place of  $r_f$ .

**Definition 4.3.** A stable compactification of a frame  $L$  is a pair  $(M, f)$  where  $M$  is a stably compact frame and  $f : M \rightarrow L$  is an onto frame homomorphism that satisfies

$$x \ll y \Rightarrow r(f(x)) \ll y. \quad (*)$$

The reader may notice that as with stable compactifications of spaces, density is not specifically required in the definition. As with spaces, it is a consequence of the definition.

**Lemma 4.4.** If  $f : M \rightarrow L$  is a stable compactification of  $L$ , then  $f$  is dense.

*Proof.* As  $0 \ll 0$ , we have  $r(0) \ll 0$ , giving  $r(0) = 0$ . □

**Proposition 4.5.** Every compactification of  $L$  is a stable compactification of  $L$ .

*Proof.* Suppose  $f : M \rightarrow L$  is a compactification. Then  $f$  is a dense onto frame homomorphism. Since  $M$  is compact regular,  $M$  is stably compact. So to show  $f : M \rightarrow L$  is a stable compactification it remains to verify condition  $(*)$  of Definition 4.3. The way below relation  $\ll$  and well inside relation  $<$  agree in any compact regular frame, so it is sufficient to show that if  $x, y, z \in M$  with  $x < y$  and  $f(z) \leq f(x)$ , then  $z < y$ . From  $f(z) \leq f(x)$  it follows that  $f(z) \wedge \neg f(x) = 0$ , so  $f(z) \wedge f(\neg x) = 0$ , and the density of  $f$  yields  $z \wedge \neg x = 0$ , hence  $\neg x \leq \neg z$ . But  $x < y$  means  $\neg x \vee y = 1$ , hence  $\neg z \vee y = 1$ , giving  $z < y$ . □

We next show that the stable compactifications of the frame of open sets of a  $T_0$ -space  $X$  correspond to Smyth's stable compactifications of the space  $X$ . We recall (see Theorem 2.13) that the stably compact frames are, up to isomorphism, exactly the frames  $\Omega(Y)$  for a stably compact space  $Y$ . Also, by Corollary 2.10, if  $Y$  is a stably compact space, hence a sober space, then  $\Omega$  provides a bijection between the homsets  $\text{Top}(X, Y)$  and  $\text{Frm}(\Omega(Y), \Omega(X))$ .

**Proposition 4.6.** *For  $X$  a  $T_0$ -space,  $Y$  a stably compact space, and  $e : X \rightarrow Y$  continuous,  $e$  is a stable compactification of  $X$  iff  $\Omega(e)$  is a stable compactification of  $\Omega(X)$ .*

*Proof.* Corollary 2.10 states that  $e$  is an embedding iff  $\Omega(e)$  is onto. For  $U$  and  $V$  open subsets of  $Y$  we have  $\Omega(e)(V) \subseteq \Omega(e)(U)$  iff  $e^{-1}(V) \subseteq e^{-1}(U)$  iff  $V \cap e[X] \subseteq U \cap e[X]$ . It follows that  $\overline{U} = r(\Omega(e))(U)$  for all open  $U \subseteq Y$ . Thus,  $e$  is a stable compactification iff  $\Omega(e)$  is a stable compactification.  $\square$

We turn next to providing internal ways to describe stable compactifications of a frame. This is similar in spirit to Smirnov's result [25] providing an internal characterization of the compactifications of a completely regular space, Banaschewski's result [1] characterizing compactifications of a frame, and generalizes Smyth's result [26] to the pointfree setting.

Our key notion is that of proximities. The idea of a proximity has a long history, see [16] for details, and occurs in the literature with related but different meanings. Proximities were originally considered as various types of relations on the powerset of a set [9, 25, 8, 16]. They were later extended to the pointfree setting [7, 1, 11, 26, 19]. Here we follow the path we began in [5] that views a proximity as a relation on a frame that generalizes Banaschewski's notion of a strong inclusion and is closely related to Smyth's approximating auxiliary relation. As in [5] we use  $<$  for a proximity on a frame. The symbol  $<$  is also used to denote the well inside relation on a frame; when it is used with this meaning in the sequel, we will specifically say so.

**Definition 4.7.** [5] *Let  $L$  be a frame. A proximity on  $L$  is a binary relation  $<$  on  $L$  satisfying:*

1.  $0 < 0$  and  $1 < 1$ .

2.  $a < b$  implies  $a \leq b$ .
3.  $a \leq b < c \leq d$  implies  $a < d$ .
4.  $a, b < c$  implies  $a \vee b < c$ .
5.  $a < b, c$  implies  $a < b \wedge c$ .
6.  $a < b$  implies there exists  $c \in L$  with  $a < c < b$ .
7.  $a = \bigvee \{b \in L : b < a\}$ .

**Example 4.8.** Some examples of proximity frames are the following. (1) The partial ordering of any frame is a proximity. (2) A strong inclusion on a frame [1] is a proximity that is contained in the well inside relation and satisfies  $a < b$  implies  $\neg b < \neg a$ . (3) The way below relation on a stably compact frame is a proximity. (4) The well inside relation on a compact regular frame is a proximity. (5) The really inside relation on any completely regular frame [13, Sec. IV.1] is a proximity. See [5] for further details.

**Definition 4.9.** For  $f : M \rightarrow L$  a stable compactification of  $L$ , define a relation  $<_f$  on  $L$  by setting  $a <_f b \Leftrightarrow r_f(a) \ll r_f(b)$ .

To make notation nicer, we often use  $<$  in place of  $<_f$  and  $r$  in place of  $r_f$ .

**Lemma 4.10.**  $a < b$  iff  $x \ll y$  for some  $x, y$  with  $f(x) = a$  and  $f(y) = b$ .

*Proof.*  $\Rightarrow$ : This is trivial as  $f(r(a)) = a$  and  $f(r(b)) = b$  since  $f$  is onto.

$\Leftarrow$ : Suppose  $x \ll y$ , where  $f(x) = a$  and  $f(y) = b$ . Then as  $y \leq r(b)$ , we have  $x \ll r(b)$ . But part of the definition of a stable compactification says  $p \ll q \Rightarrow r(f(p)) \ll q$ . Thus, as  $r(f(x)) = r(a)$ , we have  $r(a) \ll r(b)$ , so  $a < b$ .  $\square$

**Proposition 4.11.** If  $f : M \rightarrow L$  is a stable compactification, then  $<$  is a proximity on  $L$ .

*Proof.* For (1) note  $0 \ll 0$  always holds, and as  $M$  is compact  $1 \ll 1$ . By Lemma 4.10,  $f(0) < f(0)$  and  $f(1) < f(1)$ , giving  $0 < 0$  and  $1 < 1$ . For (2) suppose  $a < b$ . Then  $r(a) \ll r(b)$ , hence  $r(a) \leq r(b)$ , giving  $a = fr(a) \leq$

$fr(b) = b$ . For (3) suppose  $a \leq b < c \leq d$ . Then  $r(a) \leq r(b) \ll r(c) \leq r(d)$ , so  $r(a) \ll r(d)$ , hence  $a < d$ . For (4) suppose  $a, b < c$ . Then  $r(a), r(b) \ll r(c)$ , hence by general properties of the way below relation,  $r(a) \vee r(b) \ll r(c)$ . Then as  $f(r(a) \vee r(b)) = a \vee b$  and  $f(r(c)) = c$ , Lemma 4.10 gives  $a \vee b < c$ . For (5) suppose  $a < b, c$ . Then  $r(a) \ll r(b), r(c)$  and as  $M$  is stable,  $r(a) < r(b) \wedge r(c)$ , giving  $r(a) \ll r(b \wedge c)$ , hence  $a < b \wedge c$ . For (6) suppose  $a < b$ . Then  $r(a) \ll r(b)$ . As  $M$  is stably compact, we may interpolate to find  $z$  with  $r(a) \ll z \ll r(b)$ . Then letting  $f(z) = c$ , Lemma 4.10 shows  $a < c < b$ . For (7) as  $M$  is stably compact,  $r(a) = \bigvee \{x : x \ll r(a)\}$ , so  $f(r(a)) = \bigvee \{f(x) : x \ll r(a)\}$ . By Lemma 4.10, if  $x \ll r(a)$  then  $f(x) < a$ . It follows that  $a = \bigvee \{b : b < a\}$ .  $\square$

**Definition 4.12.** For a proximity  $\prec$  on  $L$ , we say an ideal  $I$  of  $L$  is  $\prec$ -round if for each  $a \in I$  there is  $b \in I$  with  $a < b$ . We let  $\mathfrak{I}_{\prec}L$  be the collection of all  $\prec$ -round ideals of  $L$ .

**Definition 4.13.** For a stably compact frame  $M$ , we say  $N \subseteq M$  is a stably compact subframe of  $M$  if

1.  $N$  is a subframe of  $M$ .
2.  $N$  is a stably compact frame.
3. The identical embedding of  $N$  in  $M$  is proper, so  $a \ll_N b \Rightarrow a \ll_M b$ .

A stably compact subframe  $M$  of the ideal frame  $\mathfrak{I}L$  is called dense if  $\bigvee \cdot : M \rightarrow L$  is onto.

We note that if  $N$  is a stably compact subframe of  $M$ , then  $a \ll b$  in  $N$  iff  $a \ll b$  in  $M$ . One direction is provided by the definition of stably compact subframe, the other as  $a \ll b$  in a frame implies  $a \ll b$  in any subframe containing  $a, b$ . We also point the reader to [5, Sec. 4], where a number of results were established for the frame of round ideals of a proximity frame. In [5], this frame was called  $\mathfrak{R}\mathfrak{I}L$  rather than  $\mathfrak{I}_{\prec}L$  as above because there was no need to consider more than one proximity on a given frame, as there will be here.

**Proposition 4.14.** For a proximity  $\prec$  on  $L$ , the set  $\mathfrak{I}_{\prec}L$  of  $\prec$ -round ideals is a dense stably compact subframe of the frame  $\mathfrak{I}L$  of ideals of  $L$ . Further, the join map  $\bigvee \cdot : \mathfrak{I}_{\prec}L \rightarrow L$  is a stable compactification of  $L$ .

*Proof.* For  $a \in L$  let  $\downarrow a = \{b \in L : b < a\}$ . In [5, Prop. 4.6] it is shown that  $\mathfrak{I}_{<}L$  is a subframe of  $\mathfrak{I}L$ , and  $I \ll J$  in  $\mathfrak{I}_{<}L$  iff  $I \subseteq \downarrow a$  for some  $a \in J$ . This second condition shows  $I \ll J$  in  $\mathfrak{I}_{<}L$  implies  $I \ll J$  in  $\mathfrak{I}L$ . That  $\mathfrak{I}_{<}L$  is a stably compact frame is given in [5, Prop. 4.8]. Together, these show  $\mathfrak{I}_{<}L$  is a stably compact subframe of  $\mathfrak{I}L$ .

The map  $\bigvee \cdot : \mathfrak{I}L \rightarrow L$  is known to be a frame homomorphism. So its restriction to  $\mathfrak{I}_{<}L$  is a frame homomorphism. To see it is onto, if  $a \in L$ , then  $\downarrow a$  is a  $<$ -round ideal and by properties of a proximity,  $\bigvee \downarrow a = a$ . Thus,  $\mathfrak{I}_{<}L$  is dense. Finally, we show  $\bigvee \cdot$  satisfies condition  $(*)$ . Suppose  $I, J$  are  $<$ -round ideals with  $I \ll J$ . Then there is  $a \in J$  with  $I \subseteq \downarrow a$ . Suppose  $\bigvee I = b$ . Then the largest  $<$ -round ideal mapped by  $\bigvee \cdot$  to  $b$  is  $\downarrow b$ , so  $r(\bigvee I) = \downarrow b$ . As  $I \subseteq \downarrow a$  and  $b = \bigvee I$ , we have  $b \leq a$ . Then  $\downarrow b \subseteq \downarrow a$  and  $a \in J$  imply  $\downarrow b \ll J$ . This shows  $\bigvee \cdot$  satisfies condition  $(*)$ , so is a stable compactification.  $\square$

**Proposition 4.15.** *If  $f : M \rightarrow L$  is a stable compactification, then there are mutually inverse frame isomorphisms  $g : M \rightarrow \mathfrak{I}_{<_f}L$  and  $h : \mathfrak{I}_{<_f}L \rightarrow M$  defined by  $g(m) = \{f(n) : n \ll m\}$  and  $h(I) = \bigvee r_f[I]$ . Further,  $(\bigvee \cdot) \circ g = f$  and  $f \circ h = \bigvee \cdot$ .*

*Proof.* For  $m \in M$  we first show  $I = g(m)$  is a  $<_f$ -round ideal of  $L$ . Suppose  $n \ll m$  and  $a \leq f(n)$ . As before, using  $r$  for  $r_f$ , we have  $r(a) \leq r f(n)$ . By condition  $(*)$  on  $f$  we have  $n \ll m \Rightarrow r f(n) \ll m$ , so  $r(a) \ll m$ , thus  $a = f r(a) \in I$ . So  $I$  is a downset. If  $n_1 \ll m$  and  $n_2 \ll m$ , then  $n_1 \vee n_2 \ll m$ , and as  $f$  is a frame homomorphism, it follows that  $I$  is closed under finite joins, so  $I$  is an ideal of  $L$ . Say  $n \ll m$ . Then there is  $p$  with  $n \ll p \ll m$ . Therefore, by Lemma 4.10,  $f(n) <_f f(p)$  and  $f(p) \in I$ . So  $I$  is  $<_f$ -round.

We have shown  $g$  is well-defined. Clearly  $h$  is also well-defined, and it is obvious that both  $g$  and  $h$  are order-preserving. For  $m \in M$  we have  $hg(m) = \bigvee \{r f(n) : n \ll m\}$ . Condition  $(*)$  on  $f$  shows  $n \ll m \Rightarrow r f(n) < m$ , hence  $n \ll m \Rightarrow n \leq r f(n) \leq m$ . As  $M$  is stably compact,  $m = \bigvee \{n : n \ll m\}$ , and it follows that  $m = \bigvee \{r f(n) : n \ll m\}$ . Thus,  $h \circ g$  is the identity map on  $M$ .

Suppose  $I$  is a  $<_f$ -round ideal of  $L$ . If  $a \in I$ , then there is  $b \in I$  with  $a <_f b$ . By the definition of  $<_f$  we have  $r(a) \ll r(b)$ , hence  $r(a) \ll r(b) \leq \bigvee r[I] = h(I)$ . As  $gh(I) = \{f(n) : n \ll h(I)\}$  we have  $a = f r(a) \in gh(I)$ . Thus,  $I \subseteq gh(I)$ . Conversely, suppose  $a \in gh(I)$ . Then  $a = f(n)$  for some  $n \ll h(I)$ . As  $h(I) = \bigvee r[I]$ , the definition of way below and the fact that

$r[I]$  is up-directed gives  $n \leq r(b)$  for some  $b \in I$ . Therefore,  $a = f(n) \leq fr(b) = b$ , and as  $I$  is an ideal, we have  $a \in I$ . Thus,  $I = gh(I)$ , showing  $g \circ h$  is the identity map on  $\mathfrak{J}_{<_f}L$ . So we have shown  $g$  and  $h$  are mutually inverse frame isomorphisms between  $M$  and  $\mathfrak{J}_{<_f}L$ .

For the further comment, suppose  $m \in M$ . As  $M$  is stably compact,  $m = \bigvee \{n : n \ll m\}$ , and as  $f$  is a frame homomorphism,  $f(m) = \bigvee \{f(n) : n \ll m\}$ . Thus,  $(\bigvee \cdot) \circ g = f$ . Then  $f \circ h = (\bigvee \cdot) \circ g \circ h = \bigvee \cdot$  as  $g$  and  $h$  are mutually inverse isomorphisms.  $\square$

**Proposition 4.16.** *If  $L$  is a frame and  $M$  is a dense stably compact subframe of  $\mathfrak{J}L$ , then  $\bigvee \cdot : M \rightarrow L$  is a stable compactification of  $L$ , and  $M$  is equal to  $\mathfrak{J}_{<_{\bigvee}}L$ .*

*Proof.* By the definition of a stably compact subframe, we have  $M$  is a stably compact frame. Also, this definition implies  $M$  is a subframe of  $\mathfrak{J}L$ , and as the join map from  $\mathfrak{J}L$  to  $L$  is a frame homomorphism, its restriction to  $M$  is also a frame homomorphism. We have assumed the join map from  $M$  to  $L$  is an onto mapping, so to show  $\bigvee \cdot : M \rightarrow L$  is a stable compactification we need only show this map satisfies condition (\*). Suppose  $I, J \in M$  with  $I \ll J$  in  $M$ , hence by the definition of a stably compact subframe,  $I \ll J$  in  $\mathfrak{J}L$ . Let  $r(I)$  be the largest element of  $M$  mapped by  $\bigvee \cdot$  to  $\bigvee I$ , and let  $\hat{r}(I)$  be the largest element of  $\mathfrak{J}L$  mapped by  $\bigvee \cdot$  to  $\bigvee I$ . As  $\bigvee \cdot : \mathfrak{J}L \rightarrow L$  is a stable compactification, we have  $\hat{r}(I) \ll J$  in  $\mathfrak{J}L$ , so  $r(I) \leq \hat{r}(I) \ll J$  in  $\mathfrak{J}L$ , giving  $r(I) \ll J$  in  $\mathfrak{J}L$ , hence  $r(I) \ll J$  in  $M$ . Thus,  $\bigvee \cdot : M \rightarrow L$  is a stable compactification.

We now show  $M = \mathfrak{J}_{<_{\bigvee}}L$ . Suppose  $I$  is an element of  $M$ . Surely  $I$  is an ideal of  $L$ , we must show it is  $<_{\bigvee}$ -round. Let  $a \in I$ . Then as  $\bigvee \cdot : M \rightarrow L$  is assumed to be onto, there is some  $J \in M$  with  $\bigvee J = a$ . So  $J \subseteq \downarrow a$  and  $a \in I$  give  $J \ll I$ . As  $M$  is stably compact,  $\ll$  is interpolating, so we can find  $K$  in  $M$  with  $J \ll K \ll I$ . Setting  $b = \bigvee K$ , the definition of  $<_{\bigvee}$  gives  $a <_{\bigvee} b$  since  $J \ll K$  and both  $a = \bigvee J$  and  $b = \bigvee K$ . Now  $K \ll I$  gives  $K \subseteq \downarrow c$  for some  $c \in I$ , so  $b \leq c$ , giving  $b \in I$ . So  $I$  is indeed  $<_{\bigvee}$ -round. Conversely, suppose  $I$  is a  $<_{\bigvee}$ -round ideal of  $L$ . As  $\bigvee \cdot : M \rightarrow L$  is an onto frame homomorphism, for each  $a \in I$  there is a largest ideal  $J_a$  in  $M$  with  $a = \bigvee J_a$ . Let  $J$  be the join in the ideal frame of  $\{J_a : a \in I\}$ . Then as  $M$  is a subframe of  $\mathfrak{J}L$ , we have  $J \in M$ . For each  $a \in I$  we have  $J_a \subseteq \downarrow a$ , so each  $J_a$  is contained in  $I$ , hence  $J \subseteq I$ . Suppose  $a \in I$ . As  $I$  is  $<_{\bigvee}$ -round, there is



$b \in I$  with  $a <_{\vee} b$ . This means there are ideals  $P \ll Q$  in  $M$  with  $a = \vee P$  and  $b = \vee Q$ . As  $P \ll Q$ , there is  $c \in Q$  with  $P \subseteq \downarrow c$ . Clearly  $a \leq c$ , and  $c \in Q \subseteq J_b$ . Thus,  $a \leq c \in J$ , so  $a \in J$ . This yields  $J = I$ , showing  $I$  belongs to  $M$ .  $\square$

**Definition 4.17.** For stable compactifications  $f : M \rightarrow L$  and  $f' : M' \rightarrow L$  of a frame  $L$ , define  $f \sqsubseteq f'$  if there is a proper frame homomorphism  $g : M \rightarrow M'$  with  $f = f' \circ g$ . Then  $\sqsubseteq$  is reflexive and transitive, so is a quasi-order on the class of stable compactifications of  $L$ . Let  $\text{COMP } L$  be the poset of equivalence classes of stable compactifications under the partial order associated with  $\sqsubseteq$ , and denote the equivalence class of  $f : M \rightarrow L$  by  $[f]$ .

**Remark 4.18.** Proposition 4.15 shows every equivalence class of  $\text{COMP } L$  contains a member of the form  $\vee \cdot : M \rightarrow L$  for some stably compact subframe  $M$  of  $\mathfrak{J}L$ . So  $\text{COMP } L$  is a set with a partial ordering even though there is a proper class of compactifications.

**Definition 4.19.** For a frame  $L$ , let  $\text{PROX } L$  be the poset of proximities on  $L$ , partially ordered by set inclusion, and  $\text{SUB } \mathfrak{J}L$  be the poset of dense stably compact subframes  $M$  of the ideal frame  $\mathfrak{J}L$ , partially ordered by set inclusion.

We next show that the posets  $\text{COMP } L$ ,  $\text{PROX } L$ , and  $\text{SUB } \mathfrak{J}L$  are isomorphic.

**Theorem 4.20.** For a frame  $L$  there are isomorphisms

$$\Phi : \text{COMP } L \rightarrow \text{PROX } L \text{ where } \Phi([f]) = <_f$$

$$\Psi : \text{PROX } L \rightarrow \text{SUB } \mathfrak{J}L \text{ where } \Psi(<) = \mathfrak{J}_{<}L$$

$$\Pi : \text{SUB } \mathfrak{J}L \rightarrow \text{COMP } L \text{ where } \Pi(M) \text{ is the equivalence class of } \vee \cdot : M \rightarrow L.$$

Further,  $\Phi^{-1} = \Pi \circ \Psi$ ,  $\Psi^{-1} = \Phi \circ \Pi$ , and  $\Pi^{-1} = \Psi \circ \Phi$ .

*Proof.* To see  $\Phi$  is well-defined, suppose  $f : M \rightarrow L$  and  $f' : M' \rightarrow L$  are equivalent stable compactifications, so there are proper frame homomorphisms  $g : M \rightarrow M'$  and  $g' : M' \rightarrow M$  with  $f' \circ g = f$  and  $f \circ g' = f'$ . If  $a <_f b$ , then by Lemma 4.10, there are  $x \ll y$  in  $M$  with  $f(x) = a$  and

$f(y) = b$ . As  $g$  is proper, we have  $g(x) \ll g(y)$ , and as  $f' \circ g = f$ , we have  $a = f'g(x) \prec_{f'} f'g(y) = b$ . So  $\prec_f \subseteq \prec_{f'}$ , and by symmetry  $\prec_{f'} \subseteq \prec_f$ , hence equality. So the definition of  $\Phi$  does not depend on the member  $f$  of the equivalence class  $[f]$  chosen. That  $\Phi([f])$  is indeed a member of  $\text{PROX } L$  is given by Proposition 4.11. That  $\Psi$  is a map into  $\text{SUB } \mathfrak{J}L$  is given by Proposition 4.14, and that  $\Pi$  is a map into  $\text{COMP } L$  is given by Proposition 4.16.

To see  $\Phi$  is order-preserving, suppose  $[f] \leq [f']$  where  $f : M \rightarrow L$  and  $f' : M' \rightarrow L$  are stable compactifications. Then there is a proper frame homomorphism  $g : M \rightarrow M'$  with  $f' \circ g = f$ . We have just seen that this implies  $\prec_f \subseteq \prec_{f'}$ , so  $\Phi([f]) \leq \Phi([f'])$ . To see  $\Psi$  is order-preserving, suppose  $\prec \subseteq \prec'$ . Then  $\mathfrak{J}_{\prec}L$  is a subset of  $\mathfrak{J}_{\prec'}L$ , so  $\Psi(\prec) \subseteq \Psi(\prec')$ . Finally, to show  $\Pi$  is order-preserving, suppose  $M$  and  $M'$  are dense stably compact subframes of  $\mathfrak{J}L$  with  $M \subseteq M'$ . Let  $g : M \rightarrow M'$  be the identical embedding. As both  $M$  and  $M'$  are subframes of  $\mathfrak{J}L$ , we have finite meets and arbitrary joins in  $M$  and  $M'$  agree with those in  $\mathfrak{J}L$ , so  $g$  is a frame homomorphism. To see  $g$  is proper, we note that the definition of a stably compact subframe implies that the way below relations in  $M$  and  $M'$  are the restrictions of the way below relation in  $\mathfrak{J}L$ . Finally, for  $I \in M$  we have  $(\bigvee \cdot) \circ g(I)$  is simply the join of  $I$  in  $L$ , which is equal to  $(\bigvee \cdot)I$ . This shows  $\bigvee \cdot : M \rightarrow L$  is  $\sqsubseteq$  related to  $\bigvee \cdot : M' \rightarrow L$ , hence the equivalence class of the first compactification is beneath that of the second in the partial ordering of  $\text{COMP } L$ , showing  $\Pi(M) \leq \Pi(M')$ .

To show that  $\Phi, \Psi, \Pi$  are isomorphisms and the further remarks describing their inverses, it is enough to show  $\Pi\Psi\Phi, \Psi\Phi\Pi$ , and  $\Phi\Pi\Psi$  are the identity maps on  $\text{COMP } L, \text{SUB } \mathfrak{J}L$ , and  $\text{PROX } L$ , respectively.

To see  $\Pi\Psi\Phi$  is the identity on  $\text{COMP } L$ , let  $f : M \rightarrow L$  be a stable compactification. Then  $\Pi\Psi\Phi([f]) = \Pi\Psi(\prec_f) = \Pi(\mathfrak{J}_{\prec_f}L)$ , and this final item is the equivalence class of the compactification  $\bigvee \cdot : \mathfrak{J}_{\prec_f}L \rightarrow L$ . Proposition 4.15 shows  $f : M \rightarrow L$  and  $\bigvee \cdot : \mathfrak{J}_{\prec_f}L \rightarrow L$  are equivalent, so  $\Pi\Psi\Phi([f]) = [f]$ .

To see  $\Psi\Phi\Pi$  is the identity map on  $\text{SUB } \mathfrak{J}L$ , suppose  $M$  belongs to  $\text{SUB } \mathfrak{J}L$ . Proposition 4.16 shows  $M$  is equal to  $\mathfrak{J}_{\prec_{\bigvee}}L$ , hence  $\Psi\Phi\Pi(M) = M$ .

Finally, we show  $\Phi\Pi\Psi$  is the identity on  $\text{PROX } L$ . Suppose  $\prec$  is a proximity on  $L$  and let  $\prec'$  be the proximity  $\Phi\Pi\Psi(\prec)$ . Suppose  $a \prec b$ . Then there is  $c$  with  $a \prec c \prec b$ . The ideals  $\downarrow a$  and  $\downarrow b$  are  $\prec$ -round and as  $\prec$  is a proximity,

we have  $(\bigvee \cdot) \downarrow a = a$  and  $(\bigvee \cdot) \downarrow b = b$ . As  $\downarrow a \subseteq \downarrow c$  and  $c \in \downarrow b$ , we have  $\downarrow a \ll \downarrow b$ , and it follows from Lemma 4.10 and the definition of  $\prec' = \prec_{\bigvee}$  that  $a \prec' b$ . Conversely, suppose  $a \prec' b$ . Then the definition of  $\prec_{\bigvee}$  gives  $r(a) \ll r(b)$ , where  $r$  is the right adjoint of  $\bigvee \cdot : \mathfrak{J}_{\prec} L \rightarrow L$ . Clearly the largest  $\prec$ -round ideal of  $L$  mapped by  $\bigvee \cdot$  to  $a$  is  $\downarrow a$ , so  $r(a) = \downarrow a$ , and  $r(b) = \downarrow b$ . So  $\downarrow a \ll \downarrow b$ . This means there is  $c \in \downarrow b$  with  $\downarrow a \subseteq \downarrow c$ . As  $\prec$  is a proximity,  $a = \bigvee \downarrow a$ , so  $a \leq c$ , and as  $c \in \downarrow b$ , we have  $c < b$ , hence  $a < b$ . So  $\prec = \prec'$ , thus  $\prec = \Phi \Pi \Psi(\prec)$ .  $\square$

We conclude this section with a discussion of matters related to the poset of stable compactifications of a frame. We begin with a comparison to Smyth's poset  $\text{COMP } X$  of stable compactifications of a  $T_0$ -space described in Definition 3.2.

**Proposition 4.21.** *For a  $T_0$ -space  $X$ , the poset  $\text{COMP } X$  of stable compactifications of  $X$  is isomorphic to the poset  $\text{COMP } \Omega(X)$  of stable compactifications of the frame  $\Omega(X)$ .*

*Proof.* Proposition 4.6, and the discussion before it, show that each equivalence class of stable compactifications of the frame  $\Omega(X)$  contains an element of the form  $\Omega(e) : \Omega(Y) \rightarrow \Omega(X)$  for some stable compactification  $e : X \rightarrow Y$  of the space  $X$ . The result then follows as the proper frame homomorphisms from the frame  $\Omega(Y)$  to the frame  $\Omega(Z)$  of open sets of stably compact spaces  $Y$  and  $Z$  are exactly the  $\Omega(f)$  where  $f : Z \rightarrow Y$  is proper.  $\square$

**Corollary 4.22.** *For a  $T_0$ -space  $X$  and its sobrification  $s(X)$ , the poset  $\text{COMP } X$  is isomorphic to the poset  $\text{COMP } s(X)$ .*

*Proof.* This follows from Proposition 4.21 as the frames  $\Omega(X)$  and  $\Omega(sX)$  are isomorphic.  $\square$

**Remark 4.23.** The poset of stable compactifications of  $L$  always has a largest element. In terms of the poset of proximities on  $L$ , this corresponds to the largest proximity, namely the partial ordering on  $L$ , and in terms of the dense stably compact subframes of the ideal frame, this corresponds to the largest such subframe, namely the ideal frame  $\mathfrak{J}L$  itself. As we discuss in the next section, this largest stable compactification is coherent. We also point to Smyth's results on the largest stable compactification of a  $T_0$ -space and its

connection to Salbani's companion topology discussed in Proposition 3.14. We further note that as shown in Corollary 3.10, the poset of stable compactifications of  $L$  need not have a least element, even in the case when  $L$  is a spatial frame.

**Remark 4.24.** In [1] Banaschewski showed that for a completely regular frame  $L$ , there is an isomorphism between the poset of compactifications of  $L$  and the poset of strong inclusions on  $L$ . He also showed that each compactification of  $L$  is equivalent to one of the form  $\bigvee \cdot : M \rightarrow L$ , where  $M$  is a compact regular subframe of the regular coreflection  $\mathfrak{R}L$ . It follows that the poset of compactifications of  $L$  is isomorphic to the poset of dense compact regular subframes of  $\mathfrak{R}L$ . In particular,  $\mathfrak{R}L$  gives the largest element of the poset of compactifications of  $L$ . The above results form extensions of these to the setting of stable compactifications. Note, the largest stable compactification of  $L$  given by  $\mathfrak{J}L$  need not be a compactification of  $L$ .

**Remark 4.25.** In [3] Banaschewski, Brümmer, and Hardie introduced biframes as a pointfree version of bitopological spaces, much as frames are a pointfree version of topological spaces. A *biframe* is a triple  $M = (M_0, M_1, M_2)$ , where  $M_1, M_2$  are subframes of the frame  $M_0$  and  $M_0$  is generated as a frame by  $M_1 \cup M_2$ , and a *biframe homomorphism*  $h : M \rightarrow L$  is a frame homomorphism  $h : M_0 \rightarrow L_0$ , where  $h(M_i) \subseteq L_i$  for  $i = 1, 2$ .

The notions of compactness and regularity for biframes were introduced in [3], and in [2] Banaschewski and Brümmer constructed for any stably compact frame  $M_1$ , a compact regular biframe  $(M_0, M_1, M_2)$ . Their technique involved representing  $M_1$ , and the stably compact frame  $M_2$  of Scott open filters of  $M_1$ , in the congruence frame  $\text{CON}(M_1)$  of  $M_1$ , and then constructing  $M_0$  from the subframe of this congruence frame generated by the images of  $M_1$  and  $M_2$ . It follows that the category of compact regular biframes is equivalent to the category of stably compact frames, hence dually equivalent to the category of stably compact spaces, and also to the category of Nachbin spaces.

In [23] Schauerte studied bicompatifications of biframes. She defined a bicompatification of a biframe  $L$  to be a pair  $(M, f)$ , where  $M$  is a compact regular biframe and  $f : M \rightarrow L$  is a dense onto biframe homomorphism. Here density is used in the usual sense with respect to  $M_0$  and  $L_0$ , while onto means that the restrictions to  $M_i$  are onto  $L_i$  for  $i = 1, 2$ . Schauerte [23]

generalized Banaschewski's theorem by proving that the poset of bicompa-  
ctifications of a biframe is isomorphic to the poset of "strong inclusions" on  
 $L$ .

Our results on stable compactifications and ordered spaces can be placed  
in the context of biframes. Suppose  $f : M_1 \rightarrow L_1$  is a stable compactifica-  
tion of a frame  $L_1$ . As  $f$  is an onto frame homomorphism, there is a frame  
homomorphism  $\bar{f} : \text{CON}(M_1) \rightarrow \text{CON}(L_1)$  taking a congruence  $\theta$  on  $M_1$   
to the congruence on  $L$  associated with  $\theta \vee \ker f$ . For the compact regular  
biframe  $M = (M_0, M_1, M_2)$  constructed in [2], the frames  $M_0, M_1, M_2$  were  
realized inside the congruence frame  $\text{CON}(M_1)$ , and this yields a biframe  
 $L = (L_0, L_1, L_2)$  with  $L_i$  determined by the image of  $\bar{f}(M_i)$  for  $i = 0, 1, 2$ .  
This gives a biframe compactification  $\bar{f} : M \rightarrow L$ . So every stable com-  
pactification naturally yields a biframe compactification. Conversely, it fol-  
lows from Schauerte's characterization of biframe compactifications that  
if  $f : (M_0, M_1, M_2) \rightarrow (L_0, L_1, L_2)$  is a biframe compactification, then  
 $f|_{M_1} : M_1 \rightarrow L_1$  is a stable compactification. So the correspondence between  
stable compactifications of frames and bicompaactifications of biframes is  
similar to that between stable compactifications of  $T_0$ -spaces and bicom-  
pactifications of bispaces discussed in Remark 3.15.

## 5. Coherent and spectral compactifications

Recall that a frame is coherent if its compact elements are a bounded sublattice,  
and each element is a join of compact elements. A space is spectral if it  
is the space of prime filters of a bounded distributive lattice. Every coherent  
frame is stably compact, and every spectral space is stably compact. Here we  
consider stable compactifications in the context of coherent frames and spec-  
tral spaces. This is closely related to Smyth's characterization [26, Prop. 20]  
of spectral compactifications of a  $T_0$ -space  $X$  in terms of lattice bases of the  
frame of open sets  $\Omega(X)$ , where we call a stable compactification  $(Y, e)$  of  
a  $T_0$ -space  $X$  a *spectral compactification* if  $Y$  is a spectral space.

**Definition 5.1.** *Let  $L$  be a frame and  $f : M \rightarrow L$  be a stable compactification  
of  $L$ . We call  $f$  a coherent compactification of  $L$  if  $M$  is a coherent frame.  
Let  $\text{COH } L$  be the sub-poset of  $\text{COMP } L$  whose equivalence classes consist  
of coherent compactifications of  $L$ . A proximity  $<$  on  $L$  is called coherent if*

$a < b$  implies there is  $c$  with  $c < c$  and  $a < c < b$ .

**Proposition 5.2.** *COH  $L$  is isomorphic to the sub-poset of PROX  $L$  consisting of coherent proximities on  $L$ , and to the sub-poset of SUB  $\mathfrak{J}L$  consisting of dense stably compact subframes that are additionally coherent.*

*Proof.* Consider the isomorphism  $\Phi : \text{COMP } L \rightarrow \text{PROX } L$  of Theorem 4.20 and suppose that  $f : M \rightarrow L$  is a stable compactification of  $L$ . By Lemma 4.10,  $a <_f b$  iff  $f(x) = a$  and  $f(y) = b$  for some  $x \ll y$  in  $M$ . If  $M$  is coherent, the proximity  $\ll$  on  $M$  is coherent, and it follows that  $<_f$  is coherent as well. Next, consider the isomorphism  $\Psi : \text{PROX } L \rightarrow \text{SUB } \mathfrak{J}L$  and suppose that  $<$  is a coherent proximity. Then the frame  $\mathfrak{J}_<L$  of  $<$ -round ideals of  $L$  is coherent. Indeed, if  $I, J$  are  $<$ -round ideals with  $I \ll J$ , then there is  $a \in J$  with  $I \subseteq \downarrow a$ . As  $J$  is round, there is  $b \in J$  with  $a < b$ . Then as  $<$  is coherent, there is  $c < c$  with  $a < c < b$ . Therefore,  $I \ll \downarrow c \ll \downarrow c \ll J$ . Finally, consider the isomorphism  $\Pi : \text{SUB } \mathfrak{J}L \rightarrow \text{COMP } L$ . Clearly if  $M$  is a dense stably compact subframe of  $\mathfrak{J}L$  that is coherent, then the stable compactification  $\bigvee \cdot : M \rightarrow L$  is by definition coherent.  $\square$

In the coherent setting, there is an alternate path to a description of compactifications that is convenient. We call a bounded sublattice  $S$  of a frame  $L$  a *lattice basis* if  $S$  is join-dense in  $L$ , meaning each element of  $L$  is a join of elements of  $S$ . Let  $\text{LAT } L$  be the poset of lattice bases of  $L$ , where the ordering is set inclusion.

**Proposition 5.3.** *COH  $L$  is isomorphic to  $\text{LAT } L$ .*

*Proof.* By Proposition 5.2,  $\text{COH } L$  is isomorphic to the poset  $\text{CSUB } \mathfrak{J}L$  of dense stably compact subframes of  $\mathfrak{J}L$  that are themselves coherent.

If  $M$  belongs to  $\text{CSUB } \mathfrak{J}L$ , then as  $\ll$  in  $M$  is the restriction of  $\ll$  in  $\mathfrak{J}L$ , the compact elements of  $M$  are those principal ideals  $\downarrow a$  belonging to  $M$ . As  $M$  is coherent, we have  $S = \{a \in L : \downarrow a \in M\}$  is a sublattice of  $L$ , and as each element of  $M$  is the join of compact elements and the join map  $\bigvee \cdot : M \rightarrow L$  is onto,  $S$  is a join-dense sublattice of  $L$ , hence a lattice basis. Setting  $\Gamma(M) = \{a : \downarrow a \in M\}$  gives an order-preserving map from  $\text{CSUB } \mathfrak{J}L$  to  $\text{LAT } L$ .

If  $S$  is a lattice basis of  $L$ , set  $\mathfrak{J}_S L$  to be the set of ideals of  $L$  generated by  $S$ , and note that this is the subframe of  $\mathfrak{J}L$  generated by  $\{\downarrow a : a \in S\}$ .

The compact elements of  $\mathfrak{J}_S L$  are exactly the  $\downarrow a$  where  $a \in S$ , and it follows that  $\mathfrak{J}_S L$  is a coherent frame. If  $I \ll J$  in  $\mathfrak{J}_S L$ , then  $I \subseteq \downarrow a$  for some  $a \in J$  with  $a \in S$ , hence  $I \ll J$  in  $\mathfrak{J}L$ . So  $\mathfrak{J}_S L$  is a stably compact subframe of  $\mathfrak{J}L$  that is coherent, and it is dense as  $S$  is a join-dense sublattice of  $L$ . Setting  $\Lambda(S) = \mathfrak{J}_S L$  then gives an order-preserving map from  $\text{LAT } L$  to  $\text{CSUB } \mathfrak{J}L$ .

Our constructions show that  $\Lambda\Gamma(M) = M$  for each  $M \in \text{CSUB } \mathfrak{J}L$  and  $\Gamma\Lambda(S) = S$  for each  $S \in \text{LAT } L$ , so  $\Gamma$  and  $\Lambda$  establish an isomorphism of  $\text{CSUB } \mathfrak{J}L$  and  $\text{LAT } L$ .  $\square$

**Remark 5.4.** Smyth [26, Prop. 20] showed that the poset of spectral compactifications of  $X$  is isomorphic to the poset of lattice bases of  $\Omega(X)$ . The above result is an obvious extension of this to the setting of coherent compactifications of frames.

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## AUTOCATEGORIES : I. A COMMON SETTING FOR KNOTS AND 2-CATEGORIES

by René GUITART

**Abstract.** An *autograph* is an action of the free monoid with 2 generators; it could be drawn with no use of objects, by arrows drawn between arrows. As examples we get knot diagrams as well as 2-graphs. The notion of an *autocategory* is analogous to the notion of category, by replacing the underlying graph by an autograph. Examples are knots or links diagrams (unstratified case), categories, 2-categories, double categories (stratified case), which so live in the same context, the category of autocategories.

**Résumé.** Un *autographe* est une action du monoïde libre à deux générateurs  $d$  et  $c$ , et peut être représenté en dessinant des flèches entre des flèches, sans utiliser d'objets. Par exemple nous avons les graphes et les 2-graphes. La notion d'*autocatégorie* est semblable à celle de catégorie, en remplaçant le graphe sous-jacent par un autographe. Les exemples sont les diagrammes de nœuds ou d'entrelacs (cas non-stratifiés), les catégories, 2-catégories et catégories doubles (cas stratifiés), qui ainsi résident dans la même catégorie des autocatégories.

**Keywords.** knot, graph, 2-category

**Mathematics Subject Classification (2010).** 18

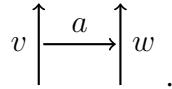
### 1. Autograph

**Definition 1.1.** An autograph  $(A, (d, c))$  is a set  $A$  of elements named arrows, equipped with two maps, domain  $d : A \rightarrow A$  and codomain  $c : A \rightarrow A$ , i.e. a map  $\partial = (d, c) : A \rightarrow A \times A$ . Of course it is the same thing that an action on  $A$  of  $\mathbb{FM}(2) = \{d, c\}^*$ , the free monoid on two generators  $d$  and  $c$ .

We denote by  $\text{Agraph}$  the category of autographs, with morphism maps  $f : A \rightarrow A'$  with  $d'fa = fda$ ,  $c'fa = fca$ , and  $U : \text{Agraph} \rightarrow \text{Set}$  the forgetful functor given by  $U((A, (d, c))) = A$ .

**Example 1.2.** For  $(G, (\delta, \gamma))$  a 2-generated group  $G$  (e.g. any finite simple group), with generators  $\delta$  and  $\gamma$ , any  $G$ -set  $E$  is an autograph, with  $d$  and  $c$  given by the actions of  $\delta$  and  $\gamma$  on  $E$ . In such a case  $d$  and  $c$  are invertible.

**Remark 1.3.** We represent the fact that the domain of an arrow  $a \in A$  is the arrow  $v$  and its codomain is the arrow  $w$ , i.e.  $da = v$  and  $ca = w$ , by:  $a : v \rightarrow w$ , or  $v \xrightarrow{a} w$ , or by the picture



**Definition 1.4.** 1.[path] — A  $(d, c)$ -path or a path of length  $k$  in an autograph  $(A, (d, c))$  is a finite sequence of consecutive arrows  $(z_n)_{0 \leq n \leq k-1}$  with

$$cz_0 = dz_1, \quad cz_1 = dz_2, \quad \dots \quad cz_{k-2} = dz_{k-1}.$$

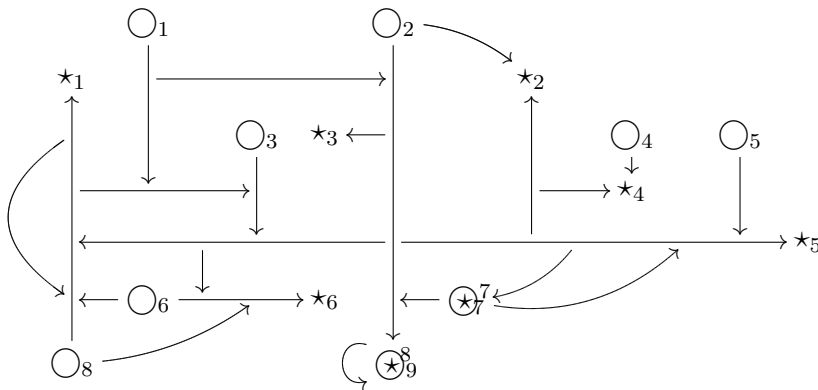
If there is no path of length  $> 1$ , then the autograph is  $U$ -free. The set of paths in  $A$  is denoted  $\text{Path}(A, (d, c))$  or shortly  $\text{Path}(A)$ .

2.[descent] — Given an autograph  $(A, (d, c))$ , a  $(d, c)$ -sequence or a downward sequence or a descent in  $(A, (d, c))$  is a sequence  $(x_n)_{n \geq 0}$  — finite or not — of elements of  $A$  with:

$$\forall n \geq 0 \quad x_{n+1} \in \{dx_n, cx_n\}.$$

If there is no cyclic (resp. infinite) descent, then the autograph is stratifiable (resp. foundable). The set of  $(d, c)$ -sequences or descents in  $A$  is denoted  $\text{Desc}(A, (d, c))$  or shortly  $\text{Desc}(A)$ .

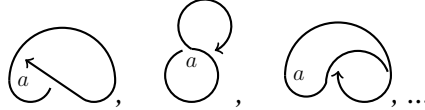
**Example 1.5.** A random example of a fragment of an autograph is:



where the  $\bigcirc_j$  are sources, and the  $\star_i$  are targets, with  $\bigcirc_9 = \star_8$ , in such a way that the picture looks like a kind of super-arrow from  $(\bigcirc_j)_{j=1,\dots,9}$  toward  $(\star_i)_{i=1,\dots,8}$ ; the sources and targets are ‘open’ or ‘empty’ places, in the sense that they must be filled by new arrows, for example by auto-arrows (cf. 2.1 and 2.2).

## 2. Auto-arrows and terminal autograph, self-reference

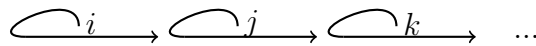
**Definition 2.1.** In an autograph, an ‘auto’-arrow is an auto-mapping from itself to itself, i.e. an arrow  $a$  with domain  $da = a$  and codomain  $ca = a$ , i.e. a data which could be written as “ $a : a \rightarrow a$ ”, and be drawn as



**Remark 2.2.** Of course very often — by way of *abbreviation* — we denote an ‘auto’-arrow by a simple closed curve, as a circle  $\bigcirc_a$ , or a square  $\square_a$ , or even by a bullet  $\bullet_a$ , or a star  $\star_a$ , or a crossing  $\times_a$ , etc., and so we get a ‘point’ in our picture; but such a ‘point’ is not at all considered as a static or stable ‘object’, rather it is an auto-modification.

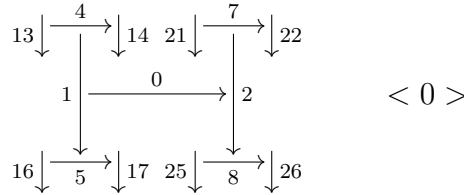
**Definition 2.3.** The terminal autograph, i.e. the terminal object in  $\text{Agraph}$ , consists of one letter named  $*$ ; it will be denoted by  $S_* = \{ * : * \rightarrow * \}$  (referencing to the shape ‘S’ of the second picture for an auto-arrow in 2.1). An auto-arrow  $a$  in  $A$  is equivalent to a morphism of autographs  $a^\bullet : S_* \rightarrow (A, (d, c))$ , the constant map on  $a$ .

**Remark 2.4.** The visualization of *self-reference* as an auto-arrow, or a ‘partial auto’-arrow was introduced by Jean Schneider [7, 8], in order to modelize graphically the structure of time : an instant  $i$  is an operation  $i$  applied to itself  $i$  and producing a new thing  $j$ , which itself produces  $k$ , etc. and so the time is constructed:





equipped with  $d = t_1$  and  $c = t_2$  given by  $(\star)$ . We recover  $\text{FA}(\{f\})$  as the sub-structure  $\langle 0 \rangle$  generated by 0 :



3 — We get  $\mathbb{N} = \sum_{n \geq 0} \langle 3n \rangle$ : for any  $N \in \mathbb{N}$ , successive divisions by 3 provide  $N = 3q_1 + x_1, q_1 = 3q_2 + x_2, \dots, q_{k-1} = 3q_k + x_k, q_k = 3q_{k+1}$ , with  $x_1, \dots, x_k \in \{1, 2\}, q_1, \dots, q_{k+1} \in \mathbb{N}$ , and then  $N \in \langle 3n \rangle$  for the unique value  $n = q_{k+1}$ . So  $(\mathbb{N}, (t_1, t_2))$ , with  $t_1$  and  $t_2$  given by  $(\star)$ , appears as  $\text{FA}(3\mathbb{N})$ , the free autograph on a denumerable set of generators  $\{g_n\}$  with  $g_n = 3n, 3\mathbb{N} = \{3n; n \in \mathbb{N}\}$ .

4 — Each  $x \in \mathbb{R}$  is representable as  $x = x_0 + \sum_{i \geq 1} \frac{x_i}{2^i}$ , where, for every  $i \geq 1, x_i \in \{0, 1\}$ , and  $x_0 \in \mathbb{N}$ ; if for one  $s \geq 0, x_{s+j} = 1$  for all  $j \geq 1$ , then we replace  $x_s$  by  $x_s + 1$  and all the  $x_{s+j}, j \geq 1$ , by 0: this new code determines the same  $x$ . After that, every  $x$  has a unique representation, with no infinite sequence of 1; then we associate to  $x$  the infinite sequence of elements of  $\langle 3x_0 \rangle$  with codes in  $d$  and  $c$  associated to  $[x]_n = .x_1x'_2x'_3 \dots x'_n$  with, for all  $i, x'_i = x_i + 1$ . For example to the real  $\frac{5}{3}$  is associated the sequence  $\dots dcdcdc(3)$ , or to the real  $\pi$ , of which the binary code is  $11.001001000011111101 \dots$  is associated the sequence  $\dots cdccccdddddcdcd(3)$ . So  $\mathbb{R}$  appears as a completion of the autograph  $\text{FA}(3\mathbb{N})$ , in terms of descents (definition 1.4):  $\mathbb{R} \subset \text{Desc}(\text{FA}(3\mathbb{N}))$ .  $\square$

#### 4. Knots and links, surgery

**Proposition 4.1.** Any oriented knot diagram  $\mathbb{K}$  determines an associated autograph denoted by  $\text{As}(\mathbb{K}) = (\text{Arc}(\mathbb{K}), \alpha, \omega)$ .

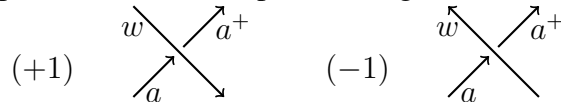
*Proof.* Given an oriented ‘knot diagram’  $\mathbb{K}$  [1, 5], i.e. a regular plan projection of a knot, with only isolated regular double points (the crossings of the diagram), following the orientation, from any crossing toward the next one, we get an oriented arc  $a$ , which is seen as an arrow from  $v = \alpha(a)$  to

$w = \omega(a)$ , if the first crossing is a crossing of  $v$  and  $a$ , and the second one is a crossing of  $w$  and  $a$ ; so, on the set  $\text{Arc}(\mathbb{K})$  of arcs with these two maps  $d = \alpha$  and  $c = \omega$  we get a structure of autograph, denoted by  $\text{As}(\mathbb{K})$ , in which for each  $a$  we have:

$$\begin{array}{c} a^- \longrightarrow \alpha(a) \quad \Bigg| \quad a \longrightarrow \Bigg| \quad \omega(a) \longrightarrow a^+ \\ \hline \hline \hline \end{array}$$

□

**Remark 4.2.** Elements of  $\text{Arc}(\mathbb{K})$  are not completely ‘abstract’, they are real arcs in the plan, and so we can precise a sign for each crossing :

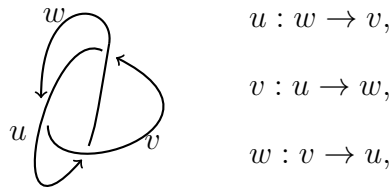


So each arc  $a$  of the oriented knot diagram is equipped with a double sign  $(\epsilon, \eta)$ , where  $\epsilon$  is the sign of its initial crossing and  $\eta$  is the sign of its final crossing; the data  $(a, (\epsilon, \eta))$  is named a *doubly signed arc*,  $(a, \epsilon\eta = \sigma)$  is named a *signed arc*, and  $a$  itself is an *unsigned arc* (unsigned, but oriented), with a predecessor  $a^-$  (crossing with  $v$ , sign  $\epsilon$ ) and a successor  $a^+$  (crossing with  $w$ , sign  $\eta$ ). Of course the values of  $\epsilon$  and  $\eta$  determine the orientations on  $d$  and  $c$ .

Now, as the set of unsigned arcs, the set of signed arcs and the set of doubly signed arcs — in the given knot diagram  $\mathbb{K}$  — are two other autographs associated to  $\mathbb{K}$ , denoted by  $\text{As}_\sigma(\mathbb{K})$  and  $\text{As}_{\epsilon,\eta}(\mathbb{K})$ .

Of course the same constructions work for any oriented link diagram  $\mathbb{L}$ .

**Example 4.3.** The simplest example is the *trefoil knot*, with an oriented diagram  $\mathbb{T}$ , in which the double sign of each arc is  $(+, +)$ ; the associated autograph  $\text{As}(\mathbb{T})$  is pictured and listed as follows:



or described by :  $\alpha u = v, \omega u = w, \alpha v = w, \omega v = u, \alpha w = u, \omega w = v$ .

**Remark 4.4.** In [8, p. 168-169], this diagram is commented almost as follows. With  $R = v_w, S = w_u, I = u_v$ , we write, as for fractions (or matrices composition or tensors calculus),  $S \star I = w_u \star u_v = w_v = R^{-1}$ ; hence Schneider think of the trefoil as representing relations like :

$$S \star I = R^{-1}, \quad I \star R = S^{-1}, \quad R \star S = I^{-1},$$

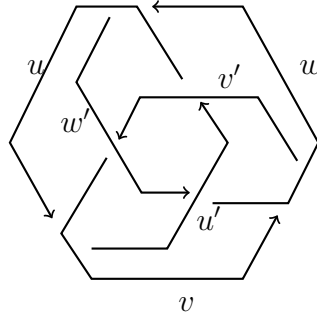
and the trefoil becomes a symbol of a borromean situation between  $R, S$  and  $I$ . Then the borromean schema is presented as an enrichment of the trefoil with three new crossing points  $r, s$  and  $i$ , with

$$R = I \star i, S = R \star r, I = S \star s, r = i \star I, s = r \star R, i = s \star S.$$

Here we proceed in a different way, trying to stay always at the level of autographs (arrows), i.e. with arcs rather than with crossing points (objects).

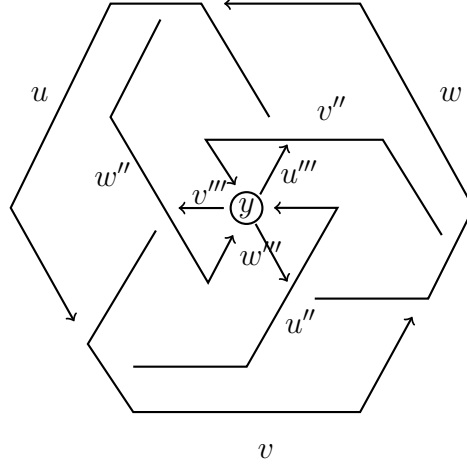
**Example 4.5.** An example is the *borromean* link, with an oriented diagram  $\mathbb{B}$ , and the associated autograph  $\text{As}(\mathbb{B})$  is listed and pictured as follows:

$$\begin{aligned} u : v' \rightarrow v, & & u' : v \rightarrow v', \\ v : w' \rightarrow w, & & v' : w \rightarrow w', \\ w : u' \rightarrow u, & & w' : u \rightarrow u'. \end{aligned}$$



**Example 4.6.** Starting with  $\mathbb{B}$ , in its ‘hexagonal center’ we can do a surgical procedure first consisting in the introduction of a Y-cut, cutting  $u'$  near the end in a point  $\alpha$ , now separated into  $\alpha^-$  and  $\alpha^+$ , and similarly for  $v'$  and  $w'$ , and then continued by the junctions of  $\alpha^+$  and  $\beta^-$ ,  $\beta^+$  and  $\gamma^-$ ,  $\gamma^+$  and  $\alpha^-$ . Finally, after three Reidemeister moves of type I (twist)[1] to eliminate the three new loops, we get the picture of the trefoil  $\mathbb{T}$ . The data of this construction determine an autograph  $\text{As}(\mathbb{B})_Y$ , pictured and listed as:





$$\begin{aligned}
 u &: v'' \rightarrow v, & v &: w'' \rightarrow w, & w &: u'' \rightarrow u, \\
 u'' &: v \rightarrow y, & v'' &: w \rightarrow y, & c'' &: a \rightarrow y, & y &: y \rightarrow y. \\
 u''' &: y \rightarrow v'', & v''' &: y \rightarrow w'', & w''' &: y \rightarrow u'',
 \end{aligned}$$

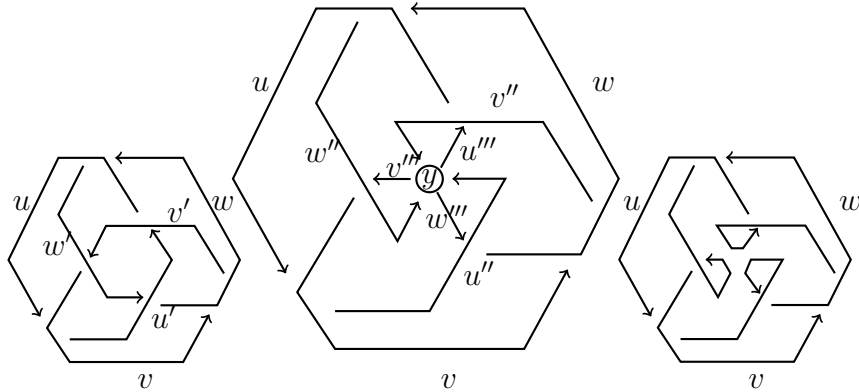
In  $\text{As}(\mathbb{B})_Y$ , the Y-cut is simulated by the introduction of an auto-arrow  $y$  and three arrows making the convenient junction:  $u''', v''', w'''$ .

**Proposition 4.7.** *At the level of autographs, the surgery procedure in 4.5 — explaining how to get  $\mathbb{T}$  from  $\mathbb{B}$  — could be translated by the construction of two maps between three free autcategories of paths on an autograph (see definition 6.1, 6.3):*

$$\text{Path}(\text{As}(\mathbb{B})) \xrightarrow{\bar{\beta}} \text{Path}(\text{As}(\mathbb{B})_Y) \xleftarrow{\bar{\tau}} \text{Path}(\text{As}(\mathbb{T})).$$

*Proof.* The reader is invited to follow the paths on the picture of  $A(\mathbb{B})_Y$ , in order to understand the transformations. The left mapping is determined by the map  $\beta : \mathbb{B} \rightarrow \text{Path}(\text{As}(\mathbb{B})_Y)$ , and the right one is determined by  $\tau : \mathbb{T} \rightarrow \text{Path}(\text{As}(\mathbb{B})_Y)$ , with

$$\begin{aligned}
 \beta(u) &= u, \beta(v) = v, \beta(w) = w, \\
 \beta(u') &= u'''u'', \beta(v') = v'''v'', \beta(w') = w'''w'', \\
 \tau(u) &= uu'''v'', \tau(v) = vv'''w'', \tau(w) = ww'''u''.
 \end{aligned}$$

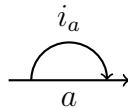


□

### 5. Identifiers, autographs and flexigraphs with identifiers

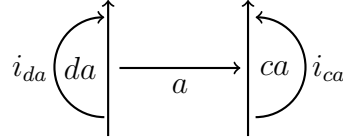
In the context of graphs, diagrams and categories, the notions of vertex, object, identity and unit, are almost identified. With the notion of autograph we clarify a distinction between them, at first by the elimination of the notion of object, and the introduction of the notion of auto-arrow. Secondly now we precise the notion of an *identifier*. In the next section we will introduce units and identities.

**Remark 5.1.** In presence of an object or a vertex  $X$ , in a category or a graph for example, we have to be careful and not to confuse the identity mapping  $1_X : X \rightarrow X$  of the object  $X$  with an auto-arrow on  $X$ ; in such a situation, it is  $X$  itself which determines an auto-arrow  $X : X \rightarrow X$ , or better we have to consider that  $1_X$  is an auto-arrow :  $1_X : 1_X \rightarrow 1_X$  (see 6.2). But, in the general situation an identity mapping could exist on an arrow different from any object, even for an arrow which is not an auto-arrow, i.e. for such an arrow  $\xrightarrow{a}$ . In fact we could have a data  $i_a : a \rightarrow a$  representing a so called 'identifier' on  $a$ , i.e. a selected endo-arrow of  $a$ ,  $i_a$ , such that  $di_a = a = ci_a$ :



Such a data  $i_a$  determines  $a$  (by  $di_a = a$ ), and so is really an 'identifier' of  $a$  (a 'name' of  $a$ ), but is distinct from  $a$ , and is not an auto-arrow, and is not necessarily unique. Later we will see if a given identifier has to be an identity.

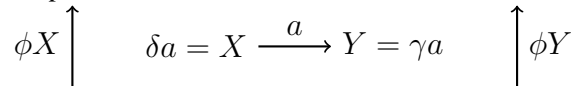
**Definition 5.2.** An autograph  $A$  in which each arrow  $a$  is equipped with the data of a selected endo-arrow  $i_{da} : da \rightarrow da$  named the identifier of  $da$ , and a selected endo-arrow  $i_{ca} : ca \rightarrow ca$  named the identifier of  $ca$ , is called an autograph with identifiers. Then, with each arrow  $a$ , we have the picture



**Remark 5.3.** Clearly  $i_{da}$  (or  $i_{ca}$ ) does not depend really of  $a$ , but only of  $v = da$  (or  $w = ca$ ). Of course in an autograph with identifiers, if  $a$  is the domain (or the codomain) of something, i.e.  $a = db$  or  $a = cb$ , then we have an identifier  $i_a$  on  $a$ . And if  $i_a$  itself is the domain (or codomain) of something, then we have an identifier  $i_{i_a}$  (or  $j_{i_a}$ ), etc. But if  $a$  is not a domain or a codomain, then no identity on  $a$  is assumed to be specified.

**Definition 5.4.** A flexigraph is the data of two sets  $G_0$  and  $G_1$ , and three maps  $\delta, \gamma : G_1 \rightarrow G_0$ ,  $\phi : G_0 \rightarrow G_1$ .

The map  $\phi$  is the flex, and, as in the case of an autograph for  $d$  and  $c$ ,  $\delta$  and  $\gamma$  are thought as ‘domain’ and ‘codomain’. The difference with an autograph is that there are two types of elements [0 (vertex), 1 (arrow)], and then domain and codomain look like ‘objects’ (or ‘vertices’). For any  $a \in G_1$  we get the picture :



There are also ‘isolated’  $\phi Z$ , for any  $Z$  which is not a  $\delta a$  or a  $\gamma a$ .

**Example 5.5.** An autograph  $(A, d, c)$  is a special case of flexigraph, with  $G_1 = A = G_0$ ,  $\delta = d$ ,  $\gamma = c$ , and  $\phi = 1_A$ . Conversely a flexigraph  $(G_1, G_0, \delta, \gamma, \phi)$  determines an autograph with  $A = G_1$ ,  $d = \phi\delta$ ,  $c = \phi\gamma$ . If the flexigraph is a flexigraph with ‘identities’, i.e. is equipped with a map  $\iota : G_0 \rightarrow G_1$  such that  $\delta\iota = 1_{G_0} = \gamma\iota$  — that is to say that  $(G_1, G_0, \delta, \gamma, \iota)$  is an oriented graph (usually named today a *graph*), and if  $\phi$  is injective — then the associated autograph is with identifiers, with  $i_{da} = \iota\delta a$ ,  $i_{ca} = \iota\gamma a$ . Of course this works in the special case of an oriented graph just seen as a flexigraph with  $\phi := \iota$ .

## 6. Autocategories

**Definition 6.1.** An autocategory  $\mathcal{A}$  is the data of an autograph  $(A, d, c)$  with identifiers (definitions 1.1 and 5.2), equipped with the data of a composition law for consecutive arrows, i.e. for any consecutive arrows

$$f : p \rightarrow q, \quad g : q \rightarrow r \quad (\text{where } q = dg = cf),$$

we have a composed arrow denoted  $gf$ , with  $d(gf) = df = p$  and  $c(gf) = cg = r$ , i.e.

$$\text{Position: } gf : p \rightarrow r,$$

such that identities are units for composition:

$$\text{Unitarity: } fi_{df} = f = i_{cf}f;$$

and such that the two compositions of three consecutive arrows are equal:

$$\text{Associativity: } h(gf) = (hg)f, \quad \text{if } dh = cg \quad \text{and} \quad dg = cf.$$

We denote by  $\text{Acat}$  the category of autocategories, with morphism the maps  $F : A \rightarrow A'$  with  $d'Fa = Fda$ ,  $c'Fa = Fca$ , and  $F(ba) = F(b)F(a)$  if  $db = ca$ .

The forgetful functor  $\mathbb{V} : \text{Acat} \rightarrow \text{Agraph}$  is given by  $\mathbb{V}(\mathcal{A}) = (A, d, c)$ .

**Proposition 6.2.** A category determines an autocategory  $\text{Ass}(\mathcal{C})$ . Furthermore, any structure of flexigraph on its underlying graph, such that the flex  $\phi$  is injective, determines another structure of autocategory  $\text{Ass}(\mathcal{C}, \phi)$ , with the “same” arrows and the same composition law (but a very different underlying autograph); such a structure is named a flexicategory or a category with a flexion.

*Proof.* This definition is almost the same as the definition of a category [4], [6], excepted that now there are no objects. As in a category, the identifier  $i_a$  on  $a$ , if it exists, is unique, because it has to be a unit: in this case it is named an *identity*. This identity does exist when  $a$  is a domain or a codomain, but not only in the case where  $a$  is the auto-arrow associated to an object.

So, starting with a category  $\mathcal{C}$ , we get an autocategory  $\text{Ass}(\mathcal{C})$  by replacing each object  $X \in \text{Obj}(\mathcal{C})$  by an auto-arrow  $i_X : i_X \rightarrow i_X$ , where  $i_X$  ‘is’

the arrow  $1_X : X \rightarrow X$ , identity on  $X$  in the category. If in  $\mathcal{C}$  we have  $f : X \rightarrow Y$ , i.e.  $X = \text{dom}(f)$  and  $X = \text{cod}(f)$ , then we consider that  $i_X = d_{\text{Ass}(\mathcal{C})}f$ ,  $i_X = c_{\text{Ass}(\mathcal{C})}(f)$ . Composition is the same as in  $\mathcal{C}$ . Of course in this autcategory we recover the objects as being the auto-arrows which are units (as already it works in categories [4]).

If on  $\mathcal{C}$  we have a flex  $\phi : \text{Obj}(\mathcal{C}) \rightarrow \text{Arrow}(\mathcal{C})$  (see definition 5.4), we define  $d_\phi$  and  $c_\phi$  by  $d_\phi(f) = \phi(d_{\mathcal{C}}(f))$  and  $c_\phi(f) = \phi(c_{\mathcal{C}}(f))$ , then for every object  $X$ ,  $i_X$  become an arrow from  $\phi(X)$  to  $\phi(X)$ , which is the identity on  $\phi(X)$ .  $\square$

**Proposition 6.3.** *To any autograph  $(A, (d, c))$  there is associated a  $\mathbb{V}$ -free autcategory  $\mathbb{P}(A, (d, c)) = (\text{Path}^t(A, (d, c)), D, C)$ , which is the free autcategory on  $A$ .*

*Proof.* We consider paths  $(z_n)_{0 \leq n \leq k-1}$  in  $A$ , shortly denoted by  $(z_n)_k$  (definition 1.4), with  $D((z_n)_k) := (dz_0)_0$  and  $C((z_n)_k) = (cz_k)_0$ , and so these paths are between paths of length 1 consisting in a domain or a codomain in  $A$ , i.e. an  $a$  of the form  $dx$  or  $cy$ ; for any of these  $a$  we have to add an identity element  $I_a$  to  $\text{Path}(A)$ . So we get the set  $\text{Path}^t(A)$ . If  $a$  is an identifier  $i_b$  in  $A$ , and possibly an identity when  $A$  is an autcategory, then we should not confuse  $i_b$ ,  $(i_b) = (i_b)_0$ , and  $I_{i_b}$ . In fact  $I_a$  plays the part of the empty sequence in the usual calculus of words; but here we need several empty words, one by domain or codomain  $a$ . Then the composition is given by concatenation of paths, and by the equations  $(z_n)_k I_{z_0} = (z_n)_K$  and  $I_{z_k} (z_n)_k = (z_n)_k$ .  $\square$

## 7. Double categories and 2-categories as autcategories

**Proposition 7.1.** *Any double category  $\mathbb{C}$  (and especially any 2-category) is determined by an associated autcategory  $\text{Ass}(\mathbb{C})$ .*

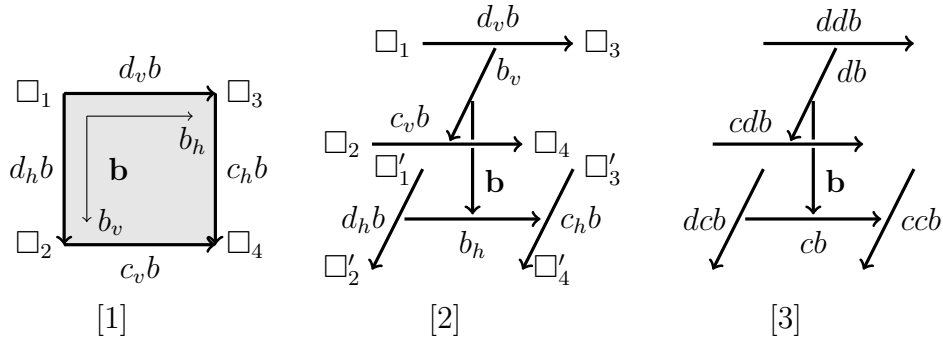
*Proof.* 1 — Let  $\mathbb{C}$  be a double category [3], where  $d_h, d_v, c_h, c_v$  are horizontal and vertical domains and codomains, where  $\infty$  and  $\delta$  are horizontal and vertical compositions. Let  $\underline{\mathbb{C}}$  be the set of all elements in  $\mathbb{C}$  (2-block, horizontal arrow, vertical arrow, or object).

The underlying set of  $\text{Ass}(\mathbb{C})$  will be  $\underline{\mathbb{C}}_v + \underline{\mathbb{C}}_u + \underline{\mathbb{C}}_h$ , a sum of three copies of  $\underline{\mathbb{C}}$ , and an arbitrary element of  $\mathbb{C}$  will have three avatars: if  $x$  is a designation

of any of these avatars, we get designations for all the avatars:  $x_u$  is the un-oriented version,  $x_h$  is the horizontally oriented version,  $x_v$  is the vertically oriented version. So  $(x_h)_h = x_h$ ,  $(x_h)_v = x_v$ ,  $(x_h)_u = x_u$ , etc. Each avatar determines the others. In  $\text{Ass}(\mathbb{C})$  we consider that  $x_u$  is an arrow from  $x_v$  to  $x_h$ , i.e.  $x_u : x_v \rightarrow x_h$ , and so we define:

$$d(x_u) = x_v, \quad c(x_u) = x_h.$$

Let  $\mathbf{b}$  be a not degenerated element of the double category, an unoriented 2-block, a 2-dim data as in picture [1]:



To get a determined operation (of type  $\infty$  or 8) with  $\mathbf{b} = b_u$  we need an orientation: to operate horizontally we use of  $b_h$ , and to operate vertically we use of  $b_v$ . We introduce, successively

$$d(b) = b_v, \quad c(b) = b_h$$

$$ddb = d(b_v) = (d_v b)_h, \quad cdb = c(b_v) = ((c_v b))_h,$$

$$dcb = d(b_h) = (d_h b)_v, \quad cdb = c(b_h) = (c_h b)_v,$$

$$\square_1 = dddb = (d_h d_v b)_v, \quad \square'_1 = ddc b = (d_v d_h b)_h.$$

We have

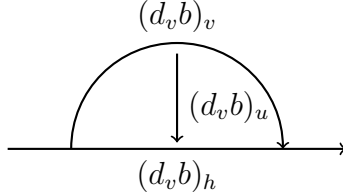
$$(\square_1)_h = \square'_1, \quad (\square'_1)_v = \square_1,$$

And the same facts for the three other corners.

Let us remark that, for clarity and simplicity, in picture [2] and [3] not all the existing arrows are drawn ; for example in fact  $d_v b$  is for  $(d_v b)_h$ , but there are also  $(d_v b)_v$ , which has to be the vertical unity on  $(d_v b)_h$  in  $\mathbb{C}$ , and here in  $\text{Ass}(\mathbb{C})$  its identity

$$i_{(d_v b)_h} = (d_v b)_v,$$

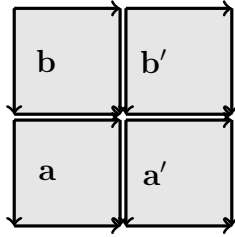
and then we have also  $(d_v b)_u : (d_v b)_v \rightarrow (d_v b)_h$ , etc.



In  $\text{Ass}(\mathbb{C})$ , if  $dy = cx$ , then  $x$  and  $y$  are both horizontal or both vertical. So a unique composition law “.” in  $\text{Ass}(\mathbb{C})$  is defined as follows, if and only if  $dy = cx$ :

$$y.x = \begin{cases} y \infty x & : \text{if } y \text{ and } x \text{ are horizontal,} \\ y \delta x & : \text{if } y \text{ and } x \text{ are vertical.} \end{cases}$$

Now, to conclude, we have to consider the axiom which relates the two composition laws  $\infty$  and  $\delta$  in  $\mathbb{C}$ , for a ‘square of squares’:



and to translate it in  $\text{Ass}(\mathbb{C})$ .

This compatibility (distributivity) in  $\mathbb{C}$  is

$$(a' \infty a) \delta (b' \infty b) = (a' \delta b') \infty (a \delta b),$$

when

$$d_v a = c_v b, d_v a' = c_v b', d_h a' = c_h a, d_h b' = c_h b.$$

The translation in  $\text{Ass}(\mathbb{C})$  is:

$$[(a'_h . a_h)_v . (b'_h . b_h)_v]_h = (a'_v . b'_v)_h . (a_v . b_v)_h,$$

as well as

$$(a'_h . a_h)_v . (b'_h . b_h)_v = [(a'_v . b'_v)_h . (a_v . b_v)_h]_v.$$

If we introduce on  $\text{Ass}(\mathbb{C})$  the involutive *transversal map*  $(-)^{\theta}$  by

$$(x_h)^{\theta} = x_v, \quad (x_u)^{\theta} = x_u, \quad (x_v)^{\theta} = x_h,$$

such that

$$(x^\theta)^\theta = x, \quad (\text{Inv})$$

then we could recover the identities  $i_d$  and  $i_c$  on domains and codomains:  $i_{dx} = (dx)^\theta$ ,  $i_{cx} = (cx)^\theta$ , in such a way that for any  $x$  we have

$$(cx)^\theta x = x = x(dx)^\theta. \quad (\text{Id})$$

And with  $\theta$  the compatibility becomes:

$$(a'.a)^\theta.(b'.b)^\theta = [(a'.b')^\theta.(a.b)^\theta]^\theta. \quad (\text{Comp})$$

□

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