cahiers de topologie et géométrie différentielle catégoriques

créés par CHARLES EHRESMANN en 1958
dirigés par Andrée CHARLES EHRESMANN
VOLUME LIV-4, 4e trimestre 2013

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3ème Volume dédié à René GUITART

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES
Vol. LIV-4 (2013)

NORMALITY, COMMUTATION AND SUPREMA IN THE REGULAR MAL'TSEV AND PROTOMODULAR SETTINGS

by Dominique BOURN

dedicated to René Guitart on the occasion of his sixty-fifth birthday

Résumé. Dans le contexte des catégories régulières de Mal'tsev et protomodulaires, nous développons les conséquences d'une caractérisation, acquise dans le simple cadre des catégories unitales sans condition de colimites, du fait que le sup de deux sous-objets qui commutent est leur commun codomaine. Nous retrouvons ainsi, mais avec des preuves conceptuelles, quelques résultats bien connus de la catégorie Gp des groupes.

Abstract. We develop, in the contexts of regular Mal'tsev and protomodular categories, the consequences of a characterization, obtained in a mere unital category without any cocompleteness assumption, of the fact that the supremum of two commuting subobjects is their common codomain. In this way we recover, with conceptual proofs, some well-known results in the category Gp of groups.

Keywords. Fibration of points, Mal'tsev and protomodular category, commutation of subobjects, centralisation of equivalence relations

Mathematics Subject Classification (2010). 08B05, 18C99, 18D30, 18D35, 20F12, 54H11

Introduction

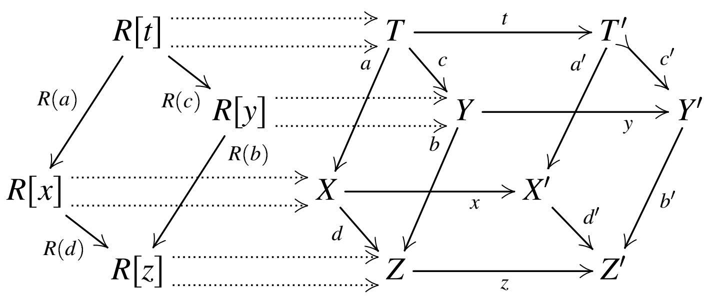
This work is devoted to unfold as far as possible the consequences of the mere observation according to which, in the unital setting, the construction of the supremum of two subobjects does not need on the ground category more cocompleteness than the regular [1] assumption provided that these two subobjects commute (Lemma 2.1). From that, in a pointed regular Mal'tsev setting, such a pair of subobjects is necessarily a pair of normal subobjects, and, in the homological setting, the supremum of two commuting normal subobjects is necessarily normal. We are then able to set in the homological setting a result already noticed in the much stricter context of semi-abelian categories by T. Everaert and M. Gran (and published in [14]), namely that, given two equivalence relations (R, S) on an object X such that the supremum of their normalizations is 1_X, they centralize each other if and only if their normalizations commute.

From that, we can derive two non-pointed applications:
1) in a regular Mal'tsev category, the centralizers of equivalence relations are stable under product provided that their base objects have global supports
2) in a regular protomodular category, the change of base functors f^* : $Pt_Y\mathbb{C} \to Pt_X\mathbb{C}$ with respect to the fibration of points, reflect the commutation of normal subobjects if and only if they reflect the mutual centralization of equivalence relations.
This last point extends to the non-pointed context some aspects of results obtained in different pointed situations in [9] and [19].

1 Direct image of a normal monomorphism

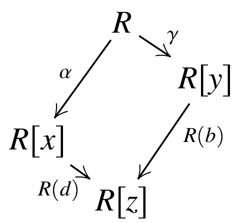
In this article any category will be assumed to be finitely complete. The aim of this section is to show that, in a regular Mal'tsev category, the direct image of a normal monomorphism along a regular epimorphism is still a normal monomorphism. Let us begin by the following observation:

Lemma 1.1. *Given any commutative right hand side cube in $\mathbb{E}$:*



where the face containing a and b is a pullback and the map c' is a monomorphism, the left hand side square given by the extensions to the kernel equivalence relations of the central quadrangles is a pullback.

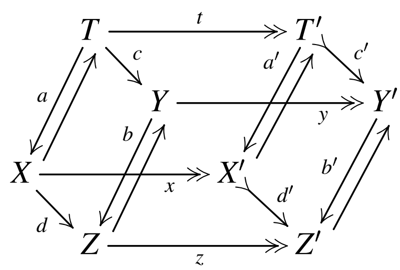
Proof. We denote by $R[f]$ the kernel equivalence relation of the morphism f. Now consider the following pullback of equivalence relations:



It produces an equivalence relation $R \rightrightarrows T$ on the object T and consequently a monomorphism of equivalence relations $j : R[t] \rightarrowtail R$. On the other hand $R \rightrightarrows T$ is coequalized by the map t since it is coequalized by $y.c = c'.t$ where c' is a monomorphism. Accordingly we have an inclusion $j' : R \rightarrowtail R[t]$ in the other direction. □

Recall that a Mal'tsev category is a category in which any reflexive relation is an equivalence relation, see [10], [11].

Proposition 1.1. *Suppose $\mathbb{C}$ is a regular Mal'tsev category. Consider any cube satisfying the previous conditions:*



and which, in addition, is such that x, y, z, t are regular epimorphisms and the maps a, b, a', b' are split epimorphisms such that all the squares are morphisms of split epimorphisms. Then the square with (d', c') is a pullback of split epimorphisms.

Proof. Since c' is a monomorphism and (d', c') is a morphism of split epimorphisms, the map d' is a monomorphism as well. Now let Θ be the vertex of the pullback of b' along d' and $\tau : T \to \Theta$ the induced factorization. According to the previous lemma we have $R[t] \simeq R[\tau]$:

$$\begin{array}{ccccc} R[t] & \overset{p_0}{\underset{p_1}{\rightleftarrows}} & T & \overset{t}{\twoheadrightarrow} & T' \\ \simeq \downarrow & & \| & & \downarrow j \\ R[\tau] & \overset{p_0}{\underset{p_1}{\rightleftarrows}} & T & \underset{\tau}{\rightarrow} & \Theta \end{array}$$

and the induced factorization j is a monomorphism. Since $\mathbb{C}$ is a regular Mal'tsev category and both T and Θ are the vertices of pullbacks of split epimorphisms, the factorization τ is a regular epimorphism (see Lemma 2.5.7 in [2]), since so are x, y and z; consequently the map j is also a regular epimorphism, and thus an isomorphism. □

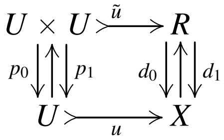
Given a finitely complete category $\mathbb{E}$, recall that $Pt(\mathbb{E})$ denotes the category whose objects are the split epimorphisms in $\mathbb{E}$ and whose arrows are the commuting squares between such split epimorphisms, and that $\P_{\mathbb{E}} : Pt(\mathbb{E}) \to \mathbb{E}$ denotes the functor associating its codomain with any split epimorphism: it is *the fibration of points*. The $\P$-cartesian maps are nothing but the pullbacks of split epimorphisms. Recall that a category $\mathbb{C}$ is protomodular [3] when any change of base functor with respect to this fibration reflects isomorphisms.

Corollary 1.1. *Let $\mathbb{C}$ be a regular Mal'tsev category. Then, in the category $Pt\mathbb{C}$, the images along a regular epimorphism of a $\P$-cartesian monomorphism is a $\P$-cartesian monomorphism. In the category $Grd\mathbb{C}$, the images along a regular epimorphic functor of a monomorphic discrete fibration is a monomorphic discrete fibration.*

Proof. This comess from the fact that in $Pt\mathbb{C}$ the regular epimorphisms are levelwise, and in $Grd\mathbb{C}$ as well, see [13]. □

Recall from [3] the following:

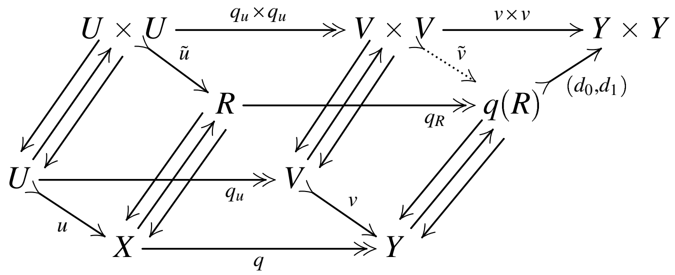
Definition 1.1. A monomorphism u in $\mathbb{E}$ is said to be normal to an equivalence relation R when: i) we have: $u^{-1}(R) = \nabla_U$ the indiscrete relation on U ii) the induced internal functor is a discrete fibration:



In the set theoretical context, when U is not empty, it is equivalent to saying that U is an equivalence class of R. Clearly, in this context, a monomorphism can be normal to several equivalence relations. It is also the case in a Mal'tsev setting. In a protomodular category a monomorphism is normal to at most one equivalence relation, see [2]; any kernel monomorphism is normal, but a normal monomorphism in this algebraic sense is not necessarily a kernel one.

Corollary 1.2. *Let $\mathbb{C}$ be a regular Mal'tsev category. Let $q : X \twoheadrightarrow Y$ be a regular epimorphism, R an equivalence relation on X and $u : U \rightarrowtail X$ a monomorphism which is normal to the equivalence relation R. Then the direct image $v : V \rightarrowtail Y$ of u along q is normal to the direct image $q(R)$ of R along q.*

Proof. Consider the following diagram:

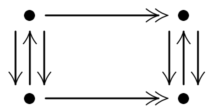


Since $q_u \times q_u$ is a regular epimorphism and (d_0, d_1) a monomorphism, there is a factorization $\tilde{v}$ which shows that $v^{-1}(q(R)) = \nabla_V$. The previous proposition shows that $(v, \tilde{v})$ is underlying a discrete fibration since so is $(u, \tilde{u})$. $\square$

This corollary shows that a weaker version of the Hofmann axiom [15] following which the kernel monomorphisms are stable under direct image is already valid in the non-pointed context and under the mild assumption that the base category is a regular Mal'tsev one. Finally we get:

Proposition 1.2. *Let $\mathbb{C}$ be a regular Mal'tsev category. Then, in the category $Pt\mathbb{C}$, the direct image along a regular epimorphism of a ¶-cartesian equivalence relation is a ¶-cartesian equivalence relation.*

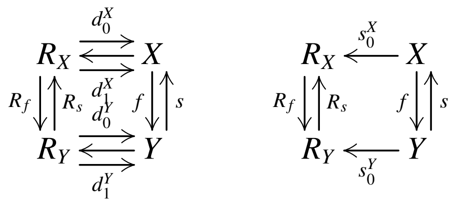
Proof. The regular epimorphisms in $Pt\mathbb{C}$ are levelwise. Consider the following diagram in $Pt\mathbb{C}$:



where the vertical parts are equivalence relations. Suppose the left hand side one is ¶-cartesian which means that any of its maps is ¶-cartesian. This is the case in particular for the vertical monomorphism. According to Corollary 1.1 the vertical monomorphism on the right hand side is ¶-cartesian as well. The fact that the whole relation is ¶-cartesian is a consequence of the following lemma. □

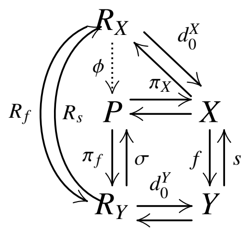
Lemma 1.2. *Suppose $\mathbb{C}$ is a Mal'tsev category. Then an equivalence relation in $Pt\mathbb{C}$ is ¶-cartesian if and only if its subdiagonal is ¶-cartesian. Any equivalence relation contained in ¶-cartesian one is itself ¶-cartesian.*

Proof. Let be given any equivalence relation in $Pt\mathbb{C}$ such that the right hand side square is a pullback:



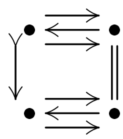
The pullback of R_f along s_0^Y is nothing but the equivalence relation $R_X \cap R[f]$. Accordingly, saying that the right hand side square is a pullback is equivalent to saying that the intersection $R_X \cap R[f]$ is the discrete equivalence relation

Δ_X. Now consider the following diagram where the lower square is a pullback and ϕ is the natural factorization:



Thanks to the Yoneda embedding, it is easy to check that, in any kind of category, $R_X \cap R[f] = \Delta_X$ implies that ϕ is a monomorphism. When, in addition, the category $\mathbb{C}$ is a Mal'tsev category, the factorization ϕ, being involved in a pullback of split epimorphisms, is necessarily a strong epimorphism. Accordingly ϕ is an isomorphism and the leg d_0 of the relation R in $Pt\mathbb{C}$ is ¶-cartesian. Clearly the same holds for the leg d_1.

Consider the following inclusion of equivalence relations in $Pt\mathbb{C}$:



Suppose the lower one is ¶-cartesian. Its subdiagonal is ¶-cartesian. Since the vertical left hand side map is a monomorphism, the leftward square is a pullback, so that the upper subdiagonal is ¶-cartesian, and so is the whole upper equivalence relation. □

2 Commutation and supremum

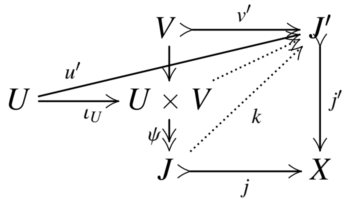
In this section, we shall show that, in a homological category, given two equivalence relations R and S on an object X, if the supremum of their normalizations is 1_X, then R and S centralize each other as soon as u and v commute in the sense of [16] and [4]. Let us begin by the following:

Lemma 2.1. *Let $\mathbb{C}$ be a unital category, and $(u : U \rightarrowtail X, v : U \rightarrowtail X)$ a pair of commuting subobjects with cooperator $\phi : U \times V \to X$. Then 1_X is the supremum of the pair (u, v) if and only if ϕ is an extremal epimorphism.*

Suppose, in addition, that $\mathbb{C}$ is regular; the supremum of a commuting pair (u, v) of monomorphisms is the image of their cooperator ϕ.

Proof. Suppose we have $1_X = u \vee v$ and a factorization $\phi = j.\psi$ with $j : J \rightarrowtail X$ a monomorphism. Then we have factorizations $\psi.\iota_U : U \rightarrowtail J$ and $\psi.\iota_V : V \rightarrowtail J$; and j is an isomorphism. Accordingly ϕ is an extremal epimorphism. Conversely suppose ϕ is an extremal epimorphism. and $j : J \rightarrowtail X$ a monomorphism with factorizations $u' : U \to J$, $v' : V \to J$. Since u and v commute and j is a monomorphism, so do u' and v'. Whence a map $\psi : U \times V \to J$ such that $\phi = j.\psi$. Since ϕ is an extremal epimorphism, the monomorphism j is an isomorphism.

Suppose in addition that $\mathbb{C}$ is regular. Let (u, v) be a pair of commuting subobjects with cooperator $\phi : U \times V \to X$, and $\phi = j.\psi$ its canonical regular decomposition. Now consider the following diagram:



where J' is a subobject containing U and V. Since j' is a monomorphism, and u and v commute, so do u' and v'; whence a cooperator $\psi' : U \times V \to J'$ such that $j'.\psi' = \phi = \psi.j$. Now, since ψ is a regular epimorphism and j' a monomorphism, we get the desired factorization k. □

From that, we can extend, rather unexpectedly, to any pointed regular Mal'tsev setting, a well known result of the category Gp of groups:

Proposition 2.1. *Let $\mathbb{C}$ be a pointed regular Mal'tsev category, and $(u : U \rightarrowtail X, v : V \rightarrowtail X)$ a pair of commuting subobjects such that their supremum is 1_X. Then u and v are normal to two equivalence relations on X which centralize each other.*

Proof. A pointed Mal'tsev category is unital. According to the previous lemma the cooperator $\phi : U \times Y \to X$ is a regular epimorphism. The inclusion $\iota_U : U \rightarrowtail U \times V$ is normal to $\nabla_U \times V$, while $\iota_V : V \rightarrowtail U \times V$ is normal to $U \times \nabla_V$. According to Corollary 1.2, the monomorphism u is normal to the direct image $\phi(\nabla_U \times V)$ while the monomorphism v is normal to the direct image $\phi(U \times \nabla_V)$. Now since we have $(\nabla_U \times V) \cap (U \times \nabla_V) = \Delta_{U\times V}$, the equivalence relations $\nabla_U \times V$ and $U \times \nabla_V$ centralize each other, and since ϕ is a regular epimorphism, it is the case for their direct images along ϕ, see [6]. □

In a pointed category, any equivalence relation R on X produces a normal subobject called its *normalization*, just take: $Kerd_0^R \overset{k}{\rightarrowtail} R \overset{d_1^R}{\to} X$.

Corollary 2.1. *Let $\mathbb{C}$ be a homological (i.e. pointed, regular and protomodular) category. Let R and S be two equivalence relations on X. Suppose that their normalizations $u : U \rightarrowtail X$ and $v : V \rightarrowtail X$ are such that 1_X is their supremum. Then R and S centralize each other if and only if u and v commute.*

Proof. We know that the normalizations of two equivalence relations which centralize each other do commute. Conversely suppose that the normalizations u and v of R and S do commute [7]. If, moreover, 1_X is their supremum, then, according to the previous proposition and the fact that any protomodular category is a Mal'tsev one, the unique R and S of which they are the normalizations centralize each other. □

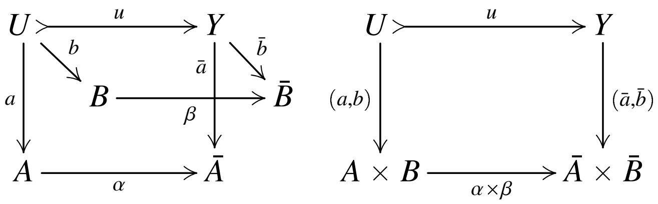
This last point was already observed in the stricter context of semi-abelian categories, see Proposition 4.6 in [14].

3 Supremum of two normal monomorphisms

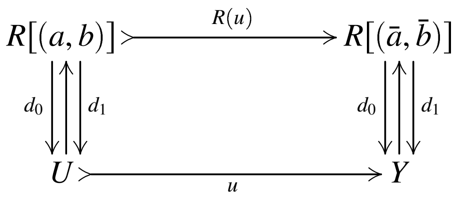
In this section, we shall show that, in a homological category, the supremum of two normal subobjects which commute is necessarily normal. Let us start with the following:

Lemma 3.1. *Let us consider, in a category $\mathbb{E}$, any left hand side diagram where any commutative square is a pullback and the map u is a monomorph-*

ism; then the right hand side square is a pullback:



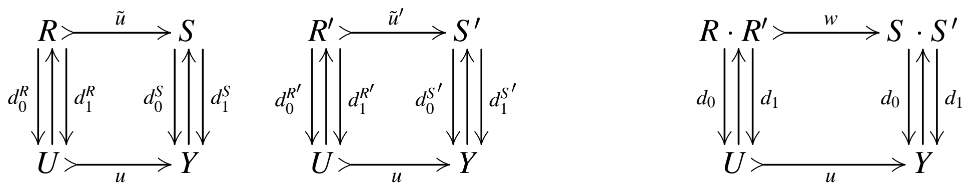
Accordingly the following diagram is a discrete fibration between equivalence relations:



Proof. The first assertion can be easily checked in Set. The second one is a consequence of the fact that the right hand side square is a pullback. □

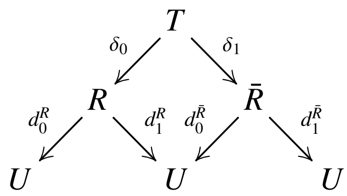
Recall that in a regular Mal'tsev category $\mathbb{C}$, the supremum of a pair (R, S) of equivalence relations on an object X is nothing but their composition $R \cdot S$.

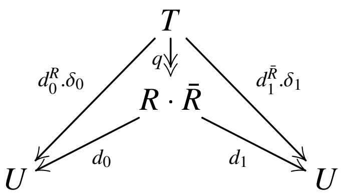
Proposition 3.1. *Let $\mathbb{C}$ be a regular Malt'sev category. Given any pair of monomorphic discrete fibrations between equivalence relations above a monomorphism u as on the left hand side below:*



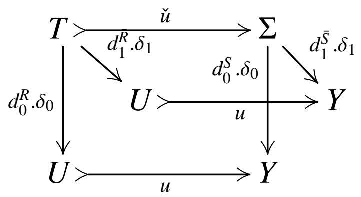
the induced monomorphism between the associated suprema, on the right hand side, is still a discrete fibration.

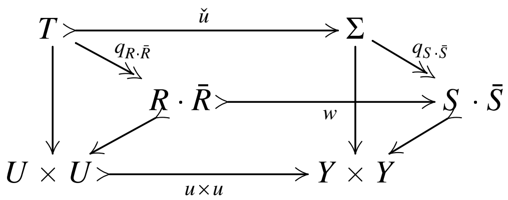
Proof. The category $\mathbb{C}$ being a regular Mal'tsev one, the supremum of the pair $(R, \bar{R})$ is $R \cdot \bar{R}$ which is given by the following construction where the upper quadrangle on the left hand side is a pullback and the pair (d_0, d_1) on the right hand side is jointly monic:



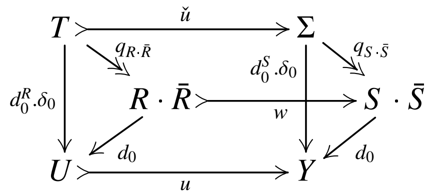


Accordingly, when we have discrete fibrations between equivalence relations, the following left hand side diagram is such that any of the commutative squares is a pullback, where Σ is defined in the same way as T with respect to $S \cdot S'$:





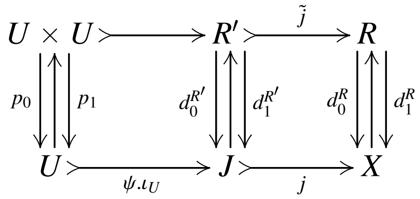
So, according to the previous lemma, the induced vertical rectangle on the right hand side is a pullback. Now, the category being regular, the right hand side upper quadrangle is a pullback, and the factorization w is necessarily a monomorphism. Finally, since the following rectangle is also a pullback, the lower quadrangle is a pullback as well:



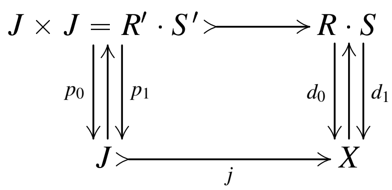
which means that the morphism of equivalence relations $R \cdot \bar{R} \rightarrowtail S \cdot \bar{S}$ above u is a discrete fibration. $\square$

Theorem 3.1. *Let $\mathbb{C}$ be a homological category. Suppose that two normal subobjects u and v commute. Then their surpremum (which exists by Lemma 2.1) is a normal monomorphism.*

Proof. Let $\phi : U \times V \to X$ be the cooperator of the commuting pair; their supremum is given by the image $j : J \rightarrowtail X$ of ϕ. Let R and S be the equivalence relations to which u and v are normal. The inverse image of the normal monomorphism u along j is $\psi.\iota_U$; it is normal to $R' = j^{-1}(R)$ which is equal to the direct image $\psi(\nabla_U \times V)$ by Lemma 2.1, since $\mathbb{C}$ is protomodular. Similarly $S' = j^{-1}(S)$ is $\psi(U \times \nabla_V)$. On the other hand the supremum of $\nabla_U \times V$ and $U \times \nabla_V$ is the indiscrete equivalence relation $\nabla_{U\times V}$, and the image of the equivalence relations along the regular epimorphism ψ preserves the supremum. Then the supremum of R' and S' is the indiscrete equivalence relation ∇_J. Now consider the diagram:



The category $\mathbb{C}$ being protomodular, the right hand side part of the diagram is a discrete fibration, since so are the left hand part and the whole diagram. The same holds for S and $S' = j^{-1}(S)$. Now, according to Proposition 3.1 the following diagram is a discrete fibration:



which means that the supremum j of u and v is normal to $R \cdot S$. □

4 Applications

4.1 Action distinctive categories

Here we shall investigate the product of centralizers of equivalence relations in the regular Mal'tsev setting. Recall, from [5], the following:

Definition 4.1. A Mal'tsev category $\mathbb{C}$ is said to be action distinctive when, in the category $Pt\mathbb{C}$, any object (f, s) admits a largest ¶-cartesian equivalence relation on it, called its *action distinctive* equivalence relation.

Given any split epimorphism $(f, s) : X \rightleftarrows Y$ in $\mathbb{C}$, we shall denote its associated action distinctive equivalence relation $\mathbb{D}[f, s]$ in $Pt\mathbb{C}$ in the following way in $\mathbb{C}$:

$$\begin{array}{ccc} D_X[f,s] & \overset{\delta_0^X}{\underset{\delta_1^X}{\overrightarrow{\overleftarrow{\longrightarrow}}}} & X \\ D_f \downarrow \uparrow D_s & & f \downarrow \uparrow s \\ D_Y[f,s] & \overset{\delta_0^Y}{\underset{\delta_1^Y}{\overrightarrow{\overleftarrow{\longrightarrow}}}} & Y \end{array}$$

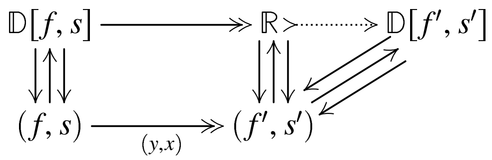
A Mal'tsev category $\mathbb{C}$ is action distinctive if and only if any equivalence relation $(d_0, d_1) : R \rightrightarrows X$ admits a centralizer [5] which is nothing but the ground equivalence relation of the action distinctive equivalence relation of the split epimorphism $(d_0, s_0) : R \rightleftarrows X$:

$$\begin{array}{cccc} D_R[d_0,s_0] & \overset{\delta_0^R}{\underset{\delta_1^R}{\overrightarrow{\overleftarrow{\longrightarrow}}}} & R & \\ D_{d_0} \downarrow \uparrow D_{s_0} & & d_0 \downarrow \uparrow s_0 & \vdots\, d_1 \\ D_X[d_0,s_0] & \overset{\delta_0^X}{\underset{\delta_1^X}{\overrightarrow{\overleftarrow{\longrightarrow}}}} & X & \end{array}$$

Examples. From the previous observation, it is clear that the categories Gp of groups is action distinctive; starting with any split epimorphism $(f, s) : X \rightleftarrows Y$, the equivalence relation $D_Y[f, s]$ is nothing but the kernel equivalence relation of its associated canonical action $\phi : Y \to AutK$, where K is the kernel of the homomorphism f. The categories R-Lie of Lie R-algebras, Rg of non-commutative rings, Rg^* of non-commutative rings with unit and the category $TopGp$ of topological groups are action distinctive as well. It is clear also that this notion is stable under coslicing and easy to show that it is stable under slicing (given an equivalence relation R on the object h of $\mathbb{C}/T$, its centralizer in $\mathbb{C}/T$ is $Z[R] \cap R[h]$). Accordingly the notion of action distinctiveness is stable under the passage to the fibres $Pt_Y\mathbb{C}$.

Proposition 4.1. *Let $\mathbb{C}$ be a regular Mal'tsev category which is action distinctive. Any regular epimorphism in $Pt\mathbb{C}$ has an extension up to the level of* *the action distinctive equivalence relations.*

Proof. Consider the following diagram in $Pt\mathbb{C}$ where (y, x) is a regular epimorphism and $\mathbb{R}$ the image of $\mathbb{D}[f, s]$ along it:



By Proposition 1.2 the equivalence relation $\mathbb{R}$ is ¶-cartesian; accordingly there is the dotted factorization. □

Theorem 4.1. *Let $\mathbb{C}$ be a regular Mal'tsev category which is action distinctive. The action distinctive equivalence relations are stable under finite products provided that the base objects of the split epimorphisms have global supports. If, in addition, the category is pointed, the action distinctive equivalence relations are stable under finite products without any restriction.*

Proof. Given any pair (f, s) and (f', s') of split epimorphisms, then the product $\mathbb{D}[f, s] \times \mathbb{D}[f', s']$ is a ¶-cartesian equivalence relation and produces an inclusion $\mathbb{D}[f, s] \times \mathbb{D}[f', s'] \subset \mathbb{D}[f \times f', s \times s']$. Suppose the base object Y of the split epimorphism (f, s) has a global support; its domain X has a global support as well. If Y' and X' denote respectively the codomain and the domain of (f', s'), then the projections $p_{Y'} : Y \times Y' \to Y'$ and $p_{X'} : X \times X' \to X'$ are regular epimorphisms and, according to the previous proposition, we get an extension: $\mathbb{D}[f \times f', s \times s'] \to \mathbb{D}[f', s']$. Similarly when Y' has a global support we get an extension: $\mathbb{D}[f \times f', s \times s'] \to \mathbb{D}[f, s]$. These two extensions led to an inverse factorization $\mathbb{D}[f \times f', s \times s'] \subset \mathbb{D}[f, s] \times \mathbb{D}[f', s']$. □

Corollary 4.1. *Let $\mathbb{C}$ be a regular Mal'tsev category which is action distinctive. The centralizers of equivalence relations are stable under finite products provided that the base objects of these relations have global supports. If, in addition, the category is pointed, the centralizers of equivalence relations are stable under finite products without any restriction.*

An action distinctive category $\mathbb{C}$ is said to be *functorially action distinctive* when, in addition, there is a (unique) functorial extension of any ¶-cartesian map up to the level of the ¶-distinctive equivalence relations. Any action accessible category [8] is functorially action distinctive. According to Proposition 5.10 in [5], any functorially action distinctive Mal'tsev category is such that any fibre $Pt_Y\mathbb{C}$ is functorially action distinctive and any change of base functor with respect to the fibration of points preserves the centralizers of equivalence relations. Let us end this section by a result we shall need as an illustration of the next section:

Corollary 4.2. *Let $\mathbb{C}$ be any functorially action distinctive protomodular category. Then any change of base functor with respect to the fibration of points reflects the centralization of equivalence relations.*

Proof. In a protomodular category the change of base functors with respect to the fibration of points reflects the inclusion of equivalence relations since it is the case for any left exact functor which reflects the isomorphisms. If moreover $\mathbb{C}$ is functorially action distinctive, the change of base functors preserve the centralizers; accordingly they reflect the centralization of equivalence relations. □

4.2 Reflection of commutation and centralization

In the pointed Mal'tsev setting, when two equivalence relations centralize each other [23] [22] [6], then their normalizations commute, see [7]. A pointed Mal'tsev category is said to satisfy the condition (SH) when the converse is true. The condition (SH) is satisfied in any pointed strongly protomodular category (see Theorem 6.1 in [7]), in any action accessible category [8], [12] and in any category of interest [20], [21]. See also [18] for other remarks. Let us recall the following conditions introduced in [9]:

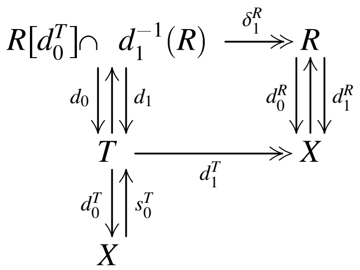
(C) any change of base functor with respect to the fibration of points reflects the commutation of normal subobjects;

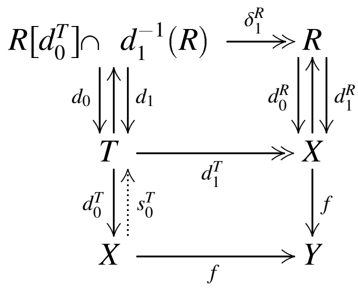
($\bar{\text{C}}$) any change of base functor with respect to the fibration of points reflects the centralization of equivalence relations.

We noticed at the end of the previous section that any functorially action distinctive protomodular category satisfies condition ($\bar{\text{C}}$). It is showed in [9] that, in a Mal'tsev category $\mathbb{C}$, (C) implies ($\bar{\text{C}}$) and that, when in addition the category $\mathbb{C}$ is pointed, the first condition is equivalent to the condition (SH).

It is also proved that the two conditions are stable under slicing and coslicing. This section will be devoted to prove that, in the regular protomodular setting, we have the converse, namely: ($\bar{C}$) implies (C). Let us begin with the two following lemmas:

Lemma 4.1. *Let $R \subset T$ be two equivalence relations on X in $\mathbb{E}$. Then the following left hand side upper diagram determines a regular epic discrete fibration in $\mathbb{E}$:*

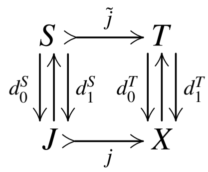




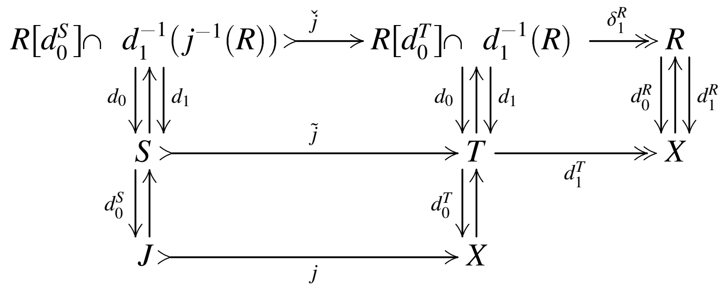
such that $R[d_0^T] \cap\ d_1^{-1}(R)$ is an equivalence relation on the point (d_0^T, s_0^T) in $Pt_X\mathbb{E}$. The equivalence relation $R[d_0^T] \cap\ d_1^{-1}(R)$ is the unique one to satisfy these two properties. Accordingly, when the equivalence relation T is effective, namely equal to some $R[f]$, the equivalence $R[d_0^T] \cap\ d_1^{-1}(R)$ is obtained by the right hand side iterated pullbacks above.

Proof. The fact that $R[d_0^T] \cap\ d_1^{-1}(R)$ is an equivalence relation on the point (d_0^T, s_0^T) in $Pt_X\mathbb{E}$ means that d_0^T coequalizes the pair (d_0, d_1). By the Yoneda Lemma, it is sufficient to check the assertion of the lemma in Set which is straightforward. □

Lemma 4.2. *Given a monomorphic discrete fibration between equivalence relations:*

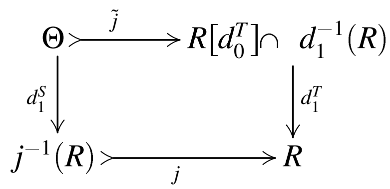


and an equivalence relation $R \subset T$, there is a unique map $\check{j}$:



making the left hand side upper part a monomorphic discrete fibration between equivalence relations; namely the left hand side vertical diagram in the fibre $Pt_J\mathbb{E}$ is the image by the change of base funtor $j^* : Pt_X\mathbb{E} \to Pt_J\mathbb{E}$ of the middle vertical diagram in the fibre $Pt_X\mathbb{E}$.

Proof. Consider the following pullback of equivalence relations in $\mathbb{E}$ where the morphisms of equivalence relations are only labelled by their underlying maps in $\mathbb{E}$:



The equivalence relation Θ is a relation on the object S which is coequalized by d_0^S since it is coequalized by $d_0^T.\tilde{j} = j.d_0^S$ and j is a monomorphism. On the other hand, the discrete fibrations being stable under pullback, the morphism labelled by d_1^S is a discrete fibration since so is the one labelled by d_1^T.Then, according to the previous lemma, we have $\Theta = R[d_0^S] \cap\ d_1^{-1}(j^{-1}(R))$. Moreover, since $j : S \rightarrowtail T$ is a discrete fibration, and $R \subset T$, the morphism $j^{-1}(R) \rightarrowtail R$ is a discrete fibration, and consequently so is:

$$\tilde{j} : \Theta = R[d_0^S] \cap\ d_1^{-1}(j^{-1}(R)) \rightarrowtail R[d_0^T] \cap\ d_1^{-1}(R)$$

□

Then we get:

Proposition 4.2. *Suppose $\mathbb{C}$ is a homological category. Then the condition* ($\bar{\mathrm{C}}$) *implies the condition (SH) (which is equivalent to the condition* (C)*). Accordingly, in the homological setting, the conditions* (C) *and* ($\bar{\mathrm{C}}$) *are equivalent.*

Proof. Suppose that (R, S) is a pair of equivalence relations on X such that their normalizations u and v commute. Let us denote by $j : J \rightarrowtail X$ their supremum. Then, according to the construction of Lemma 2.1, the factorizations $u' : U \rightarrowtail J$ of u and $v' : V \rightarrowtail J$ of v commute and their supremum is 1_J; according to Corollary 2.1 the equivalence relations $R' = j^{-1}(R)$ and $S' = j^{-1}(S)$ on J centralize each other. Moreover the supremum j is normal by Theorem 1.2 via the following discrete fibration :

$$\begin{array}{ccc} J \times J & \overset{\tilde{j}}{\rightarrowtail} & R \cdot S \\ p_0 \downarrow \uparrow \downarrow p_1 & & d_0 \downarrow \uparrow \downarrow d_1 \\ J & \underset{j}{\rightarrowtail} & X \end{array}$$

Now, R and S being contained in their supremum $R \cdot S$, we can apply them Lemma 4.2 with respect to this discrete fibration. So, considering the change of base functor $j^* : Pt_X\mathbb{C} \to Pt_J\mathbb{C}$, we have:

$$j^*(R[d_0^{R\cdot S}] \cap d_1^{-1}(R)) = R[p_0^J] \cap d_1^{-1}(R')$$

On the other hand, by Lemma 4.1, $R[p_0^J] \cap d_1^{-1}(R')$ is nothing but the pullback of R' along the terminal map $\tau_J : J \to 1$ since the indiscrete equivalence relation ∇_J is effective. Accordingly since R' and S' commute in $\mathbb{C}$, so do $R[p_0^J] \cap d_1^{-1}(R') = j^*(R[d_0^{R\cdot S}] \cap d_1^{-1}(R))$ and $R[p_0^J] \cap d_1^{-1}(S') = j^*(R[d_0^{R\cdot S}] \cap d_1^{-1}(S))$ in the fibre $Pt_J\mathbb{C}$. Now since the category $\mathbb{C}$ satisfies the condition (C), the equivalence relations $R[d_0^{R\cdot S}] \cap d_1^{-1}(R)$ and $R[d_0^{R\cdot S}] \cap d_1^{-1}(S)$ commute in the fibre $Pt_X\mathbb{C}$ and thus in $\mathbb{C}$. Now the direct images of these equivalence relations (in $\mathbb{C}$) along the regular epimorphism $d_1^{R\cdot S}$ are nothing but R and S (see Lemma 4.1), which consequently commute in $\mathbb{C}$. □

This equivalence is set in the stricter context of semi-abelian categories in [19].

Theorem 4.2. *Let $\mathbb{C}$ be any regular protomodular category. Then the condition* (C) *is equivalent to the condition* ($\bar{\mathrm{C}}$).

Proof. In any Mal'tsev context, (C) implies ($\bar{C}$). Suppose now $\mathbb{C}$ is regular, protomodular and satisfies ($\bar{C}$). Then any fibre $Pt_X\mathbb{C}$ is homological. Since the condition ($\bar{C}$) is stable under slicing and coslicing, any fibre $Pt_X\mathbb{C}$ satisfies ($\bar{C}$). According to the previous proposition any fibre $Pt_X\mathbb{C}$ satisfies (SH). The fact that, for any change of base functor $f^* : Pt_Y\mathbb{C} \to Pt_X\mathbb{C}$, the two conditions are equivalent is now a consequence of the following lemma. □

Lemma 4.3. *Let $U : \mathbb{C} \to \mathbb{D}$ be a left exact functor between pointed Mal'tsev categories. When $\mathbb{C}$ satisfies (SH), then U reflects the centralizing equivalence relations as soon as it reflects the commutation of normal monomorphisms. When $\mathbb{D}$ satisfies (SH), then U reflects the commutation of normal monomorphisms as soon as it reflects the centralized equivalence relations. When both $\mathbb{C}$ and $\mathbb{D}$ satisfy (SH), the two conditions on U are equivalent.*

Proof. Suppose $\mathbb{C}$ satisfies (SH) and U reflects the commutation of normal monomorphisms. Start with a pair (R, S) of equivalence relations whose images $(U(R), U(S))$ centralize each other. Denote by u and v their normalizations. Then the normalizations $U(u)$ and $U(v)$ of $U(R)$ and $U(S)$ commute in $\mathbb{D}$. Since U reflects the commutation of normal monomorphisms, u and v commute in $\mathbb{C}$; since $\mathbb{C}$ satisfies (SH), (R, S) centralize each other.

Suppose $\mathbb{D}$ satisfies (SH) and U reflects the centralized equivalence relations. Denote by u and v the normalizations of two equivalence relations R and S in $\mathbb{C}$ and suppose that $U(u)$ and $U(v)$ commute. Then $U(R)$ and $U(S)$ centralize each other in $\mathbb{D}$. Since U reflects the centralized equivalence relations, R and S centralize each other in $\mathbb{C}$. So their normalizations u and v do commute. □

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CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES
Vol. LIV-4 (2013)

EQUALITY IN LAMBDA CALCULUS. WEAK UNIVERSALITY IN CATEGORY THEORY AND REVERSIBLE COMPUTATIONS

by Sergey BARANOV and Sergei SOLOVIEV

Résumé. On montre d'abord comment le type d'égalité ("extensionnelle" ou "intensionnelle") du lambda calcul avec types inductifs et récursion influence les constructions universelles dans certaines catégories basées sur ce calcul (l'universalité faible est reliée à l'égalité intensionnelle). Ensuite on établit un lien entre universalité faible et réversibilité conditionnelle dans la théorie du calcul réversible.

Abstract. First we consider how the type of equality (extensional or intensional) in lambda calculus with inductive types and recursion influences universal constructions in certain categories based on this calculus (weak universality is connected with intensional equality). Then we establish the link between weak universality and conditional reversibility in the theory of reversible computations.

Keywords. Weak Universality in Categories; Extensional and Intensional Equality in Type Theory; Conditionnally Reversible Computations.

Mathematics Subject Classification (2010). 03B15; 03D75; 03D80; 18A15; 18A40.

1. Introduction

Mathematical models based on category theory are often used in computer science [1], but the approaches to categories in category theory and in computer science are very different. Researchers in "mainstream" category theory usually seek higher levels of abstraction and universality, while in computer science categories are used (if at all) as a source of more or less concrete models and constructions with the main objective to provide a viable proof-of-concept for particular architectural solutions with respect to their consistency, completeness and other important properties. Even in case of a highly general and abstract categorical notion of monad, mostly concrete aspects of this notion are exploited, for example, in programming (the Haskell language), compilation (MLj), and development of (implementable) categorical abstract machines (cf. [2], [4]).

One point where this difference of approaches may be seen very clearly, is the role played by computational aspects of equality. In category theory the equality of objects and morphisms is included in definitions of categories, but seldom any attention is paid to the computational aspects of this equality. If computational aspects of equality are taken into account, it is done in connection with general questions of decidability/undecidability. In computer science their conceptual (and practical) importance is much greater.

There is, for example, an opposition between so called intensional and extensional equalities. Two functions are extensionally equal if they always produce equal outputs for equal inputs. Functions in computer science are usually represented by some syntactical expressions (programs). In difference from extensional equality, intensional equality is defined w.r.t. a certain system of conversions of these expressions (syntactic transformations corresponding to certain basic identities).

Intensional equality plays in computer science much greater role than in “mainstream” mathematics based on classical logic. One would be not mistaken to say that verification of equality via syntactic transformations of programs and syntactic expressions is the principal method used for equality check in computer science. An obvious reason is that in general verification of extensional equality on a (potentially) infinite domain is indecidable. Even on a finite domain the complexity of this check may be overwhelming. Another source of difficulties concerning extensional equality is that the “static” equality check is only a special case. Dynamic equality check is in practice more common, e.g. the elements of datatypes may be generated dynamically by some process. In categories used in computer science the datatypes often play the role of objects. The importance of equality on objects and morphisms for various categorical constructions needs no further argument.

One of our main observations is that many standard universal constructions of category theory become weakly universal when considered in categories with intensional equality used in computer science. We explore this fact in context of on-going research in computer science concerning, for example, the notion of canonical elements of inductive types [20] (one may speak also about “concrete” and “abstract” elements), the theme of reversible computation [23] (where we introduce the notion of conditional reversibility), etc.

As a main “illustration tool” we define several categorical structures on simply typed lambda-calculus with inductive types and recursion operators T_{ind} (cf. [8, 9]). All have types as objects but differ by the notion of equality. There is a well studied structure of free cartesian closed category on simply typed lambda-calculus with surjective pairing and terminal object without inductive types [16]. We considered the calculus with inductive types and recursion because it strengthens and makes more explicit computational aspects, and we think that categorical structures on T_{ind} are of interest in themselves when computational aspects of category theory are studied.

The simple opposition of the approaches of the “mainstream” category theory and computer science does not, of course, give a complete picture of contemporary research in the domain. This is why, to complete this introduction, we have to outline the place of this paper with respect to recent research in categorical logic and type theory.

One of the first works where the relationship of extensional and intensional equality in type theory (including important categorical models) has been studied in depth was the habilitation thesis of Thomas Streicher [29]. The work of Streicher contains many profound results, but its main motivation lies in semantics of type theory: “In this thesis we will give semantic proofs of inderivability for most of these propositions which are derivable in extensional type theory but have resisted any attempt to derive them formally in ICST” (Intuitionistic Constructive Set Theory), [29], p.5.

The book by Bart Jacobs [13] on categorical logic and type theory also is mostly devoted to categorical semantics: “The emphasis here lies on categorical semantics.” [13], p.7.

The approach of this paper differs in that our interest lies in categorical structures defined on various systems of logic and type theory considered as a tool to study the phenomena that are more or less external to logic and type theory as such. For example, in his previous work the second author applied categorical structures defined on the systems of propositional logic to study coherence in categories [26], isomorphism of types [25, 7, 27] etc. In this paper we are interested in conditionally reversible computations as a possible application.

From certain point of view, our approach may have some affinity with the approach of "Homotopic Type Theory", or HOTT [12]. This subject is "hot" nowadays (slight pun intended). More seriously, we think that one of the motivations for development of this theory (independent of philosophical arguments for the Univalence Axiom) lies in the fact that work with proof assistants based on intensional type theory contains many unpleasant surpises for the naive user, as a consequence of the difference between intensional and extensional equalities (cf. [29]).

To our opinion, the Univalence Axiom may force the collapse of many relevant mathematical structures (when isomorphic objects must be distinguished). It may be too strong, but it is not contradictory since there exist interesting models [12]. Due to its power, the efficient work with intensional equality in remaining structures may become possible, and the univalence foundations program will probably produce new and efficent tools for proof assistants.

2. Universal and Weakly Universal Constructions

In this section we kept, for the history's sake, the notation used by S. Mac Lane.

Mac Lane [19], p.55, defines the notion of a universal arrow as follows.

Definition 2.1. *If $S : D \to C$ is a functor and c is an object of C, then a universal arrow from c to S is a pair $\langle r, u\rangle$ consisting of an object r of D and an arrow $u : c \to Sr$ of C, such that to every pair $\langle d, f\rangle$ with d an object of D and $f : c \to Sd$ an arrow of C, there is a unique arrow $f' : r \to d$ of D with $Sf' \circ u = f$. In other words, every arrow f to S factors uniquely through the universal arrow u, as in the commutative diagram*

$$\begin{array}{ccccc} c & \xrightarrow{u} & Sr & \qquad & r \\ \Big\downarrow || & & \Big\downarrow Sf' & & \Big\downarrow f' \\ c & \xrightarrow{f} & Sd & & d \end{array}$$

Equivalently (Mac Lane continues), $u : c \to Sr$ is universal from c to S when the pair $\langle r, u\rangle$ is an initial object in the comma category $(c \downarrow S)$... As with any initial object, it follows that $\langle r, u\rangle$ is unique up to *isomorphism* in $(c \downarrow S)$; in particular, the object r of D is unique up to isomorphism in D.

The dual concept of universal arrows from a functor $S : D \to C$ to an object $c \in C$ can be defined as well. It is used, for example, to define a product in C ([19], p.58).

Let us elaborate this in slightly more details. Recall that *comma category* $(T \downarrow S)$ of two functors $T : E \to C$ and $S : D \to C$ is the category whose objects are triples $\langle e, d, f\rangle$ with $d \in Ob(D), e \in Ob(E)$ and $f : Te \to Sd \in Mor(C)$, and whose morphisms $\langle e, d, f\rangle \to \langle e', d', f'\rangle$ are pairs $\langle k, h\rangle$ of arrows, $k : e \to e' \in Mor(E)$, $h : d \to d' \in Mor(E)$ such that the diagram

$$
(*)\ \begin{array}{ccc}
Te) & \xrightarrow{Tk} & Te \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
Sd & \xrightarrow{Sh} & Sd'
\end{array}
$$

is commutative. The composite $\langle k', h'\rangle \circ \langle k, h\rangle$ is $\langle k' \circ k, h' \circ h\rangle$ when defined [19], p.46. All the cases considered above may be seen as the cases of this definition with a particular choice of functors. For example (as Mac Lane notices), in case of $(c \downarrow S)$ one may take the constant functor with value c as T.

Notice that the equality of objects in $(T \downarrow S)$ is "heterogenous":

- $\langle e, d, f\rangle = \langle e', d', f'\rangle$ iff $e = e'$ in E, $d = d'$ in D, and $f = f'$ in C.

The equality of morphisms comes from D and E: $\langle k, h\rangle = \langle k', h'\rangle$ iff $k = k'$ in E and $h = h'$ in D.

Remark 2.2. Still, other equality relations may be of use. Below we shall consider, for example:

$$\langle k, h\rangle =_w \langle k', h'\rangle : \langle e, d, f\rangle \to \langle e', d', f'\rangle \ \ iff \ \ Sh \circ f = Sh' \circ f.$$

(Because of the commutativity of the square above, this is equivalent also to $f' \circ Tk = f' \circ Tk'$.)

Obviously, if we take the relation $=_w$ instead of $=$ we obtain a factor category of $(T \downarrow S)$ that we will denote by $(T \downarrow S)^*$.

Initiality of $\langle r, u\rangle$ above means that for any other object $\langle r', u'\rangle$ there exists a unique arrow $f : r \to r'$ that makes (*) commutative. If we have another realization $\langle r', u' \rangle$ of the universal arrow, then there exist unique $f : r \to r'$ and $f' : r' \to r$ that must be mutually inverse isomorphisms.

In spite of its triviality, let us recall the proof of this fact, since we will need in the end of this section to show exactly what the difference is in case of weak universality. First, let us take $\langle r, u \rangle$ itself as $\langle r', u' \rangle$. The identity morphism $1_r : r \to r$ may be taken as f' in the definition, and because of unicity it is the only f' possible. Now, if we take a different initial pair $\langle r', u' \rangle$ then by definition we will have certain $f' : r \to r'$ and $f'' : r' \to r$, such that $u = Sf'' \circ (Sf' \circ u) = S(f'' \circ f') \circ u$. By unicity $f'' \circ f' = 1_r$. In a similar way, we derive that $f' \circ f'' = 1_{r'}$ and hence f' and f'' are mutually inverse isomorphisms in D.

The definition of a *weak universal arrow* ([19], p.235) differs from the definition of a universal arrow only in that f' in the diagram is *not* required to be unique. As Mac Lane remarks, it is possible to modify all the various types of universals, defining weak products, weak limits, weak coproducts (requiring just existence rather than uniqueness in each case). There is no more unicity up to isomorphism, but it does not mean that instead of isomorphisms we will have arbitrary arrows. Some *conditional* reversibility will be preserved.

In the proposition below we use the same notation as in the definition 2.1 above.

Proposition 2.3. *Let the pair $\langle r, u \rangle$ be a weak universal arrow. Then r is unique up to isomorphism in the factor category $(c \downarrow S)^*$.*

*Proof.*Without unicity condition, we still have the equalities $u = S(f'' \circ f') \circ u$ and $u' = S(f' \circ f'') \circ u'$, and they correspond exactly to the definition of isomorphism in the category $(c \downarrow S)^*$.

Remark 2.4. The property that defines an isomorphism in $(c \downarrow S)^*$ may be seen as *conditional reversibility* (in this case, the reversibility that has the composition with u as a precondition, and "modulo" application of S).

Of course, similar proposition will hold also for dual case.

3. The system T_{ind}

The system of λ-calculus considered below is a subsystem of the simply typed λ-calculus with inductive types, considered in detail in [8, 9, 7, 27]. It is more restricted: we excluded from the syntax the "canonical" terminal object and pairing. In [8, 9, 7, 27] the relationship of this "canonical" data with singletons and pairing defined using inductive type construction is studied in presence of additional reductions. Here we want to use it only to illustrate the general principles discussed above, and these extra data are not included.

The system considered in [8, 9, 7, 27] was itself obtained from the system UTT of Z. Luo [18] by a series of simplifications. UTT is a dependent type theory closely related to Martin-Löf type theory and Calculus of Constructions. Our system was obtained by a) retaining only non-dependent types, b) exclusion of kinds, in particular the kind $Type$, type universes, unpredicative type $Prop$ and all logical part of UTT. All machinery concering inductive types that *was* retained is well known. It is a particular case of more general definitions for dependent types that can be found in the book of Z. Luo [18]. This is why below we do not give, for example, a self-contained definition of recursion operators over inductive types in T_{ind}.

Definition 3.1. *Types are either atomic types or obtained by application of type constructors.*

Atomic types are elements of a finite or infinite set $\mathcal{S} = \{\alpha, \beta, \ldots\}$ *of* type variables.

Type constructors are:

- $\rightarrow$ *for functional types, which constructs* $A \rightarrow B$ *for any types* A *and* B
- Ind*, defined as follows: let* $\mathcal{C}$ *be an infinite set of* introduction operators *(constructors of* elements *of inductive types), with* $\mathcal{C} \cap \mathcal{S} = \emptyset$*. an inductive type with* n *constructors* $c_1, \ldots, c_n \in \mathcal{C}$*, each of them having the arity* k_i *(with* $1 \leq i \leq n$*), has the form:*

  $$Ind(\alpha)\{c_1 : A_1^1 \rightarrow \ldots \rightarrow A_1^{k_1} \rightarrow \alpha \mid \ldots \mid c_n : A_n^1 \rightarrow \ldots \rightarrow A_n^{k_n} \rightarrow \alpha\},$$

  Here, every $A \equiv A_i^1 \rightarrow \ldots \rightarrow A_i^{k_i} \rightarrow \alpha$ *is an inductive schema, i.e.,* A_i^j *is:*

  - *either a type not containing* α*; (we call this* A_i^j *a* non-recursive operator*);*

– *or a type of the form* $A_i^j \equiv C_1 \to \ldots \to C_m \to \alpha$, *where* α *does not appear in any* $C_{\ell\in 1..m}$ *(such* A_i^j *are called* strictly positive operators*).*

Here $\to$ *associates to the right, i.e.,* $C_1 \to C_2 \to ...\alpha$ *means* $(C_1 \to (C_2 \to ...\alpha)...)$*;* $Ind(\alpha)$ binds *the variable* α*.*

Example 3.2. (The types $Bool$, Nat, functional space, and T_ω, the type of ω-trees, as inductive types.)
$Bool =_{def} Ind(\alpha)\{T : \alpha \,|\, F : \alpha\}$
$Nat =_{def} Ind(\alpha)\{0 : \alpha \,|\, succ : \alpha \to \alpha\}$
$[A, B] =_{def} Ind(\alpha)\{fun : (A \to B) \to \alpha\}$
$T_\omega = Ind(\alpha)\{0_\omega : \alpha \,|\, succ_\omega : \alpha \to \alpha \,|\, lim_\omega : (Nat \to \alpha) \to \alpha\}$.

Definition 3.3. *Let* $\mathcal{V}$ *be an infinite set of variables* $\mathcal{V}$ *(with* $\mathcal{V} \cap \mathcal{S} = \mathcal{V} \cap \mathcal{C} = \emptyset$*). The set of* λ-terms *is generated by the following grammar rules:*

$$M \ ::= \ c \,|\, Rec^{B\to D} \,|\, x \,|\, (\lambda x : B \cdot M) \,|\, (M\ M)$$

where $x \in \mathcal{V}$*,* $c \in \mathcal{C}$*,* B *and* D *are arbitraty types, and* $Rec^{B\to D}$ *denotes the recursion operator from* B *to* D *(for details, see [8, 9], [18]).*

We write $M_0M_1...M_n$ instead of $(...(M_0M_1)...M_n)$ to reduce the number of parentheses (associativity to the left). All terms and types are considered up to α-conversion, *i.e.*, renaming of bound variables. Context Γ is a set of term variables with types $x_1 : A_1, ..., x_n : A_n$ ($x_1, ..., x_n$ should be distinct). Γ, Δ denotes union of the contexts Γ, Δ (we assume that Γ, Δ have no common term variables).

Definition 3.4. *There are the following typing axioms and rules for the terms defined above (*A, B, D *denote arbitrary types,* Γ *is an arbitrary context).*

Axioms:

- $\Gamma, x : A \vdash x : A$,*;*
- *For each inductive type* $C = Ind(\alpha)\{c_1 : A_1 \,|\, \ldots \,|\, c_n : A_n\}$ *and* $1 \le i \le n$

$$\Gamma \vdash c_i : A_i[C/\alpha]$$

(e.g., if $C = Nat$*, then* $\Gamma \vdash 0 : Nat$ *and* $\Gamma \vdash succ : Nat \to Nat$*);*

- *For C as above and any type D the axiom* [1]*:*

$$\Gamma \vdash Rec^{C\to D} : \Upsilon_C(A_1, D) \to ... \to \Upsilon_C(A_n, D) \to C \to D.$$

Typing rules.

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A \cdot M) : A \to B}(\lambda)$$

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash (M\ N) : B}(app)$$

The constant $Rec^{C\to D}$ is called the *recursor* from C to D. Notice that applying it (using the rule app) to the terms $M_1 : \Upsilon_C(A_1, D), ..., M_n : \Upsilon_C(A_n, D)$ we define the function $Rec^{C\to D} M_1...M_n : C \to D$. The following derived rule is often included:

$$\frac{\Gamma \vdash M_i : \Upsilon_C(A_i, D)\ (1 \le i \le n)}{\Gamma \vdash (Rec^{C\to D}\ M_1\ \dots\ M_n) : C \to D}(elim)$$

Normalization and intensional equality. The terms of the system T_{ind} are considered up to equality generated by conversion relation. The α-conversion (renaming of bound variables) was already mentioned. Other conversions are:[2] (i) β-conversion $(\lambda x : A.M)N = [N/x]M$; (ii) η-conversion $\lambda x : A.(Mx) = M$ (where x must not be free in M); (iii) and ι-conversion for recursion. The ι-conversion corresponds to one step in recursive computation. For example, in case of $Rec^{Nat\to Nat}$ it is

- $(Rec^{Nat\to Nat} ag)(0) \to_\iota a$,
- $(Rec^{Nat\to Nat} ag)(succ\, x) \to_\iota gx((Rec^{Nat\to Nat} ag)x)$.

[1] $\Upsilon_C(A, D)$ are certain auxilliary types used to define recursion from C to D. They correspond to the types of functions that appear in standard recursive equations over C. For example, if $C = Nat$, $A_1 = Nat$ (the type of constant 0), $A_2 = Nat \to Nat$ (the type of successor S) in the definition of Nat, then $\Upsilon_{Nat}(A_1, D) = D, \Upsilon_{Nat}(A_2, D) = Nat \to D \to D$. In more general dependent type case a detailed description of these auxilliary types may be found in [18], p.178.

[2] We omit the contexts and types of terms.

T is confluent and strongly normalizing with respect to $\beta\eta\iota$-reductions (directed conversions), i.e. every reduction sequence is finite and ends by normal form which is unique up to α-conversion. Detailed description and normalization theorems for T_{ind} can be found in [8, 9]. Thus, the equivalence relation on terms based on conversion (often called $\alpha\beta\eta\iota$-equality) is decidable.

Closed terms and canonical elements.

As usual, closed terms are terms that have no free variables. In T_{ind} they include such terms as $succ(succ\,0) : Nat$, $\lambda x : Bool.x : Bool \to Bool$ etc. The terms that do not include variables at all, like $succ(succ\,0)$ are called constant terms.

Lemma 3.5. *Let C be an inductive type, and $\vdash M : C$ in T_{ind} some closed term. If M is normal, then M has the form $c_i M'$ where c_i is one of the constructors (introduction operators) of C.*

Proof. We use standard properties of normal forms in typed lambda-calculus (cf [8, 9]) and proceed by induction on the length of M. Since M has type C and C, being unductive type, does not contain $\to$, M cannot begin with λ. In this case M is necessarily an application of the form $M_0 M_1 ... M_n$ where all terms $M_0, ..., M_n$ are normal closed terms, and M_0 is not an application. M_0 cannot be a variable (it would be free). It cannot begin with λ (the term M would be not normal). Two remaining possibilities are that M_0 is an inductive type constructor c_i (then we are done) or that M_0 is a recursor. If it is a recursor, then it is a recursor from some inductive type C' to C, and M_1 should be of type C'. M_1 is closed normal term, and by induction it begins by some constructor of C'. In this case ι-reduction is applicable to M and it is not normal.

Some inductive types, like ω-trees, have constructors that may take functions as arguments, and this makes the precise (and useful) definition of canonical elements difficult. When functional arguments are excluded (such types are sometimes called 0-recursive [8, 9]), the lemma above permits to identify the canonical elements with closed terms and to show that they are the same as constant terms obtained by application of type constructors.

Definition 3.6. *Let us call an inductive type 0-recursive, if it is defined according to the defintion 3.1 with additional restriction applied recursively:*

- *in each inductive schema $A \equiv A_i^1 \to \ldots \to A_i^{k_i} \to \alpha$, A_i^j is either a 0-recursive type (if it does not contain α freely) or α (without any premise C_m).*

Theorem 3.7. *Let an inductive type C be 0-recursive, and $\vdash M : C$ in T_{ind} be some closed term. If M is normal, then M is constant term built by application using only constructors of 0-recursive inductive types.*

Proof by induction on the length of M (using standard properties of normal terms). By lemma 3.5 M has the form $c_i M_1 ... M_n$ where $M_1, ..., M_n$ are closed terms whose number and types are defined by the inductive schema corresponding to c_i.

Let this schema be $A \equiv A_i^1 \to \ldots \to A_i^{k_i} \to \alpha$.

The type of c_i is $[C/\alpha]A$. Those A_j^i in A that do not contain α are not changed (they are 0-recursive) and those that are α are replaced by C. Thus, the terms $M_1, ..., M_n$ are all closed terms of 0-recursive types, and inductive hypothesis can be applied.

Definition 3.8. *Let C be a 0-recursive inductive type. We shall call its canonical elements the terms $\vdash M : C$ built by application using only constructors of 0-recursive inductive types (including constructors without arguments, like $0 : Nat$).*

Example 3.9. An inductive type usually called product is defined as

$$A \times B =_{def} Ind(\alpha)\{pair : A \to (B \to \alpha)\}$$

(cf. [8, 9], [18]). If we take $Nat \times Nat$, then all normal closed terms of this type will be of the form $pair(succ...(succ\, 0)...)(succ...(succ\, 0)...)$ (possibly not the same number of $succ$).

Remark 3.10. If we take some type that is not 0-recursive, say, T_ω, there are closed terms of the form

$$lim((Rec^{Nat \to T_\omega}(succ_\omega 0_\omega))(\lambda x : Nat.\lambda y : T_\omega.y)) : T_\omega$$

that contain not only constructors (introduction operators). Another example is function space $[A, B] = Ind(\alpha)\{fun : (A \to B) \to \alpha\}$. The type $A \to B$ at the right is not 0-recursive. Now, if we take $A = B$ there are closed terms like $fun(\lambda x : A.x)$ that are not canonical elements in the above-mentioned sense.

Categorical structures on T_{ind}. For all variants of categorical structure we shall consider, the objects of the category T_{ind} are types, described above. Equality of objects is syntactic identity[3]. The morphisms from A to B are closed terms (i.e., terms that do not contain free term variables) of type $A \to B$, i.e. $\vdash f : A \to B$ should be derivable in T_{ind}.

The categorical structures will differ only by equality (equivalence relation) on morphisms. Speaking about morphisms, we shall usually omit $\vdash$. The composition of $f : A \to B$ and $f' : B \to C$ is defined as the (equivalence class of) the term $\lambda x : A.(f'(fx))$, $f' \circ f =_{def} \lambda x : A.(f'(fx))$. The identity is defined as the (equivalence class of) $id_A =_{def} \lambda x : A.x : A \to A$.

Below we shall consider two main equivalence relations on morphisms. If closed terms $f : A \to B$ are considered up to $\alpha\beta\eta\iota$-equality, we shall speak about T_{ind} with intensional equality.

Another equivalence relation, that we shall call extensional equality, is defined by the following condition. Let $f, g : A \to B$. We shall call f and g extensionally equal iff $ft =_{\alpha\beta\eta\iota} gt$ for every closed term $t : A$ in T_{ind}.

The axioms of category are trivially satisfied for T_{ind} with both variants of equality.

Remark 3.11. The intensional equality on T_{ind} is decidable: indeed, to verify $f =_{\alpha\beta\eta\iota} g$ the terms are reduced to normal form (T_{ind} is strongly normalizing [8, 9]) and then α-convertibility is trivially verified. To the contrary, extensional equality is not: the use of recursors permits to represent, e.g., all primitive recursive functions $f : Nat \to Nat$. Their equality on canonical elements of Nat coincides with ordinary equality of functions, and for primitive recursive functions it is not decidable [14].

Remark 3.12. In general, an extensional equality on terms of functional types $f, g : A \to B$ is defined by some condition of the form:

- For all t of type A satisfying certain condition $ft =_{\alpha\beta\eta\iota} gt$.

The equality $f =_{\alpha\beta\eta\iota} g$ implies $ft =_{\alpha\beta\eta\iota} gt$. The extensional equality we are considering is sometimes called extensional equality in closed term model. Thus the extensional equality always contains the intensional equality based on $\alpha\beta\eta\iota$.

[3]Technically it is more convenient to compare different kinds of equality only on morphisms.

Remark 3.13. If the type A in $f : A \to B$ is 0-recursive, then, according to theorem 3.7, all closed terms $t : A$ represent canonical elements, and extensional equality we introduced corresponds to ordinary set-theoretical equality of functions on the sets of canonical elements.

By T^0_{ind} we shall denote the full subcategory of T_{ind} whose objects are 0-recursive inductive types.

4. Case Studies

4.1 Intensional and Extensional Equality in T_{ind}

Let us consider two terms of T_{ind}:

- $f_1 = S : Nat \to Nat$ and
- $f_2 = Rec^{Nat \to Nat}(\lambda y : Nat.succ)(succ\, 0) : Nat \to Nat$.

Each term is a morphism of T_{ind}. Each term represents also a function on the terms of type Nat defined by $f_i(t) =_{def} f_i t$.

Canonical elements of Nat are 0 and terms $succ(...(succ\, 0))$, and on any canonical element $n : Nat$ both f_1 and f_2 have the value $succ\, n$. At the same time f_1 and f_2 are not intensionally equal: both are already in normal form and these normal forms are different.

We can define also a one side inverse to f_1 with respect to intensional equality, given by $f_1' =_{def} Rec^{Nat \to Nat}(\lambda x : Nat.\lambda y : Nat.x)0$ (the value of f_1' on $succ\, n$ will be n, the value on 0 will be 0). For the composition with f_1,

$$\lambda x : Nat.((Rec^{Nat \to Nat}(\lambda x : Nat.\lambda y : Nat.x)0)(succ\, x)) \to_\iota$$

$$\lambda x : Nat.(\lambda x : Nat.\lambda y : Nat.x)x(succ\, x)) =_\beta \lambda x : Nat.x =_{def} id_{Nat}.$$

If we compose f_1' with f_2, applying the composition to canonical elements we have

$$f_1'(f_2 0) =_{\beta\eta\iota} 0, \;\; f_1'(f_2(n)) =_{\beta\eta\iota} f'(succ\, n) =_{\beta\eta\iota} n,$$

but the composition itself (Rec means $Rec^{Nat \to Nat}$)

$$\lambda x : Nat.((Rec(\lambda x : Nat.\lambda y : Nat.x)0)(Rec(\lambda y : Nat.succ)(succ\, 0)x))$$

is normal (i.e.,does not admit any reduction) and so it is not equal to id_{Nat}.

- It can be shown that f_2 does not have left inverse w.r.t. intensional equality at all.

  Outline of a proof. Observe that the term $f_2 x = (Rec^{Nat \to Nat}(\lambda y : Nat.succ)(succ\, 0))x$ is normal and has type Nat. Consider any composition $f_2' \circ f_2 = \lambda x : Nat.(f_2'(f_2 x))$ where f_2' also is normal. If it would reduce to id, at least one reduction would be possible, and, taking into account the form of the terms, this reduction could be only β-reduction, in particular f_2' must be of the form $\lambda y : Nat.f_2''$. Simple case analysis shows that the term $[f_2 x/y]f_2''$ will not allow any further reductions (will be normal). This term contains Rec and so cannot represent id.

- Of course, both f_1 and f_2 have left inverses w.r.t. extensional equality. This behaviour can be seen as a case of conditional reversibility: f_2 is reversible at the left if the arguments are canonical elements.

- More categorical view at this conditional reversibility would be that some functor from the category T_{ind} to the category of sets such that the types become sets of their canonical elements is applied first (and equality of morphisms in this "target" category is the extensional equality of functions represented by λ-terms).

4.2 Weak Terminal Objects in T_{ind}

An inductive type with one element may be defined in T_{ind} as $Ind(\alpha)\{c : \alpha\}$. Allowing some abuse of notation, we shall denote this type by $\{c\}$. The constant c may be considered as (the name of) its unique element. The related typing axiom is $\Gamma \vdash c : \{c\}$. There are other such types, obtained by changing c.

The definition of a terminal object $\top \in T_{ind}$ as a universal construction (in the strong sense) is equivalent to the condition that for every object $A \in T_{ind}$ there exists the unique $f : A \to \top$. For a weak terminal object only the existence is required. If we take any of the types $\{c\}$, for any A there is $\lambda x : A.c : A \to \{c\}$, but other non-equivalent closed terms of the same type may exist (for example, defined using recursors).

The recursor $Rec^{\{c\} \to A}$ has the type $A \to \{c\} \to A$, i.e., the functions from $\{c\}$ to A are defined by application of $Rec^{\{c\} \to A}$ to $a : A$, an obvious interpretation is that they are defined by their value on the unique element $c : \{c\}$. Still, with respect to intensional equality $Rec^{\{c\} \to \{c'\}} c' : \{c\} \to \{c'\}$ is not equal to $\lambda x : \{c\}.c' : \{c\} \to \{c'\}$. Moreover, with respect to this equality the morphisms $\lambda x : \{c\}.c' : \{c\} \to \{c'\}$, $\lambda x : \{c'\}.c : \{c'\} \to \{c\}$, $Rec^{\{c\} \to \{c'\}} c' : \{c\} \to \{c'\}$, $Rec^{\{c'\} \to \{c\}} c : \{c'\} \to \{c\}$ are not mutually inverse isomorphisms. For example, the composition of first two gives

$$\lambda y : \{c\}.(\lambda x : \{c\}.c'((\lambda x : \{c'\}.c)y)) =_{\beta\eta\iota} \lambda y : \{c\}.c \neq \lambda y : \{c\}.y =_{def} id_{\{c\}}.$$

The composition of second two is a normal term and so also is not equal to $id_{\{c\}}$. It is possible to show that with respect to intensional equality they are not isomorphisms at all.

The same remark as in the end of the previous subsection can be added concerning conditional reversibility.

4.3 Product as a Weakly Universal Construction

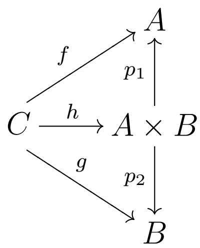
Let us take as an example the notion of product $A \times B$ of two objects A, B of a category K. It can be defined using the notion of universal arrow from diagonal functor $\Delta : K \to K \times K$ (in functor category) to the functor $F : \{1,2\} \to K$ from discrete category $\{1,2\}$ to K (with $F(1) = A, F(2) = B$). The details can be found in [19], p.69. We shall skip them (only the fact that this may be seen as a particular case of the notion of universal arrow is important) and pass directly to more common equivalent definition using projections.

The object $A \times B \in Ob(K)$ is called product of two objects $A, B \in Ob(K)$ iff

- there exist the unique arrows $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ (called projections) such that
- for every object $C \in Ob(K)$ and two arrows $f : C \to A, g : C \to B$ there exists a unique arrow $h : C \to A \times B$ that makes the following

diagram commute:

$(*)$



The arrow h is denoted $\langle f, g\rangle$. It is usually called product (or pair) of f, g, and f, g are called its components. Universality (in the strong sense) of this construction is reflected by the condition of unicity of projections and h. One of the consequences is that $A \times B$ is unique up to isomorphism.

Let us consider now, how all this will work in T_{ind}. Given two types A, B, an inductive type usually called product of A, B (cf. [18]) is defined as follows:

$$A \times B =_{def} Ind(\alpha)(pair : A \to (B \to \alpha)).$$

Its canonical elements are terms of the form $(pair\, s)t$ where $s : A$ and $t : B$. There may be other elements that do not have the constructor $pair$ at their head. For example, if we admit open terms as elements of objects, the variable $x : A \times B$ is a non-canonical element.

It turns out that in T_{ind} with intensional equality $A \times B$ can not be considered as product in the sense of strong universality.

As any inductive type, $A \times B$ in T_{ind} comes equipped with recursion operators. The recursion operator from $A \times B$ to D is a constant $R : (A \to (B \to D)) \to (A \times B \to D)$. The corresponding ι-conversion is $(Rf)((pair\, t_1)t_2) = (ft_1)t_2$ with $f : A \to (B \to D)$, $t_1 : A, t_2 : B$. Notice that if $s : A \times B$ is not of the form $(pair\, t_1)t_2$ then the conversion is not applicable. Let us denote R_1 and R_2 the recursion operators from $A \times B$ to A and B respectively. Projections are defined now as $p_1 = R_1(\lambda x : A.\lambda y : B.x) : A \times B \to A$ and $p_2 = R_2(\lambda x : A.\lambda y : B.y) : A \times B \to B$.

Given two terms $f : C \to A$ and $g : C \to B$, h of the diagram $(*)$ may be defined as $h = \langle f, g\rangle =_{def} \lambda z : C.(pair\, f(z))g(z)$. The diagram will be commutative, but what about the unicity of h?

Let $C = A \times B$ and $h : C \to A \times B$ be $id_{A\times B} = \lambda x : A \times B.x$. Let $f = p_1 \circ h = p_1, g = p_2 \circ h = p_2$. The diagram $(*)$ will be commutative.

Let us take $h' = \langle p_1, p_2 \rangle$. The diagram will be commutative, but h and h' are not equal w.r.t. the intensional equality in T_{ind}.

The product in T_{ind} with intensional equality is only weakly universal. It is possible to define another product as $A \times' B =_{def} Ind(\alpha)(pair' : A \to (B \to \alpha))$ (the only modification is the name of the constructor). The "products" $A \times B$ and $A \times' B$ will *not* be isomorphic in T_{ind} with intensional equality.

More precisely, let $\langle f, g \rangle' =_{def} \lambda z : C.((pair' f(z))g(z))$. The "candidates" to the role of isomorphisms are obvious:

$$\theta = \langle p_1, p_2 \rangle' : A \times B \to A \times' B, \;\; \theta' = \langle p_1', p_2' \rangle : A \times' B \to A \times B,$$

but they are not mutually inverse w.r.t. intensional equality. Using the technique similar to that we used in 4.1, it is possible to show that there is no isomorphism at all.

Remark 4.1. In fact, it is possible to consider the extensions of T_{ind} that include explicitly some product operator, and even add well-behaving reductions like $\langle p_1 \circ h, p_2 \circ h \rangle = h$ but this will not completely solve the problem, as the absence of unicity of product shows. To establish "equivalence" of different product operators, it will be necessary to introduce more reductions each time when one more product operator is added (cf. [8, 9]).

Remark 4.2. (Conditional inversibility.) Consider the following diagram:

$$C \xrightarrow{h} A \times B \underset{\theta'}{\overset{\theta}{\rightleftarrows}} A \times' B \, .$$

The morphisms θ and θ' are not mutually inverse w.r.t. intensional equality, but they are mutually inverse conditionally, in the following sense. If $h = \langle f, g \rangle$ for some $f : C \to A$ and $g : C \to B$ then $(\theta' \circ \theta) \circ h = h$. (Cf. with the defintion of equality in the categories $(S \downarrow T)^*$.)

4.4 Product and Extensional Equality in T^0_{ind}.

Obviously, T^0_{ind} is closed w.r.t. product defined as inductive type. Notice that it is not the same with functional space $[A, B]$. Below we consider T^0_{ind} with extensional equality.

Theorem 4.3. *In T^0_{ind} with extensional equality the product construction described above is universal in ordinary sense.*

Proof. We need to show the unicity of h in diagram $(*)$ above. Notice that the type $A \times B$ is 0-recursive. Consider another morphism $h' : C \to A \times B$ that makes the diagram commutative. Let us take any canonical element $c : C$. Since f, g, h, h' are represented by closed terms, the terms $fc, gc, hc, h'c$ are closed as well, and theorem 3.7 can be applied. It follows immediately that $h'c pair\, a\, b$ for some canonical elements $a : A, b : B$. Notice that $hc =_\beta pair\, (fc)\, (gc)$. Application of p_1 and p_2 gives $a = fc, b = gc$ and thus h and h' are extensionnally equal.

Corollary 4.4. *The category T^0_{ind} with extensional equality and product $\times$ defined as above is cartesian. Product is unique up to extensional isomorphism in T_{ind}.*

5. Discussion, Applications and Perspectives

The simple cases studied above may easily give an impression of "toy examples". To render them their due significance, we need to discuss them in a broader context, consider possible applications and perspectives of future research.

5.1 Discussion

The calculus T_{ind} has been chosen because of relative simplicty of its description, but it has considerable computational power: the inductive types of T_{ind} together with the associated recursion operators are sufficient to define all functionals of finite type [10, 15, 30].

The definition of inductive types and recursion in T_{ind} is a direct restriction to the simply typed case of the general definition used in powerful dependent type theories (we used, similarly to [8, 9, 27] the restricted form of the definitions from Z. Luo's system UTT [18]. Luo's UTT is not very much different in this respect from Martin-Löf type theory or the Calculus of Constructions used in Coq).

In the sense of metatheory, all inductive types of T_{ind} are also definable in UTT or in proof assistant Coq, but the category that contains these types *only* is not definable internally.

We did not include in T_{ind} the types $Prop$, Prf, identity types etc. In fact, we were not really interested in logical power of T_{ind}, but only in its computational properties.

In type theories with unductive types η-rules usually are understood in generalized sense. E.g., in [18] it is explained how in UTT a *logical η-rule* can be defined for an inductive type A defined by any finite sequence of inductive schemata $\Theta_1, ..., \Theta_n$ and logical validity of η-rules is proved (p.201)[4].

It is well known, that in the presence of η-rules and identity types the type-checking is indecidable (since conversion depends on inhabitance of types, cf. [13]). To our opinion, this makes the type theory with these rules and types useless as an underlying system for introduction of a categorical structure, because even the composability of morphisms will be undecidable [5].

Notice that if in the dependent type theories mentioned above the η-rules are not included at all, the situations similar to the situations considered in our examples will be easily reproduced.

Another reason why we did not include logical machinery in T_{ind} is that one of the main features of logical frameworks is that they permit to define application-oriented type theories. These theories may include some (not all) of the types that may be defined in the theory (e.g., some inductive types, some types such as $Prop$, Prf etc.), may contain, or not η-rules, and even contain some additional user-defined conversions. The question, what kind of categorical structure may exist on such a type theory (e.g. monoidal, cartesian, cartesian closed etc.) is of great interest, but there is no general theorem describing in advance all necessary properties of the underlying system, in particular with respect to extensional and intensional equality. To consider within this paper not only computational, but also logical properties of such intermediate systems would be a distraction.

To make this remark more clear, let us consider in more detail some ideas

[4]Another important class of rules is the class of *filling-up rules*. As Luo notices in [18], p.201, "The logical η-rules express that every object is equal to a canonical object and the filling-up rules express that the elimination operator covers all of the use of the inductive data type."

[5]The distinction has to be made between the existence of certain categorical models of a logical system that may be useful for its semantics, and introduction of an application-oriented categorical structure on the system itself that is mostly considered in this paper.

and results of [8, 9, 27] [6].

The principal idea explored in these works was that some reductions (interesting from computational point of view) may be added in such a way that strong normalization (SN) and Church-Rosser property (CR) will be preserved. The system considered there was an extended version of T_{ind} described above, including canonical surjective pairing and terminal object (below we shall call this system T_{ind} as well). One may note, that a dependent type system (Luo's UTT) extended with these reductions was considered in the thesis (in french) of another ph.d. student of S. Soloviev, Lionel Marie-Magdeleine [21] but there is no easily accessible publications of his work.

Among new reductions studied there was the reduction for isomorphism of "copy" between inductive types. For every inductive type A in T_{ind} and its copy A' that differs only by different choice of names of introduction operators in its definition there exist canonical closed terms $c : A \to A'$ and $c' : A' \to A$ (c and c' are defined by recursion in T_{ind} over A and A' respectively). The new reduction (called χ-reduction) was defined by rewriting rules $c'(ct) \to t$ and $c(c't') \to t'$. One may say, that χ-reduction makes copy an intensional isomorphism.

Other reductions included: η-reductions for products defined as inductive types; the reductions (similar to χ-reduction) that "make intensional" the isomorphisms between products defined as inductive types and canonical product defined by pairing; η-rules for finite types.

It was shown that T_{ind} with these reductions is SN and CR. Of course new examples that show the difference between extensional and intensional equality similar to elementary examples considered above may be constructed in the extended system. The fundamental difference between extensional and intensional equality cannot be cancelled by "local" extensions of the notion of intensional equality.

5.2 Applications

The importance of categorical models for computer science motivates also the study of behaviour of extensional and intensional equality in these mod-

[6] [9, 7, 27] are closely connected with ph.d. thesis of David Chemouil [8] (in French). Second author was the supervisor of his thesis.

els. We would like to attract attention to reversible computations as a possible domain of applications. In particular, the link between weak universality and conditional reversibility of computations may be exploited.

Reversible computations are actively studied since 1960es. Works on reversible computations link together such distant domains of science as logics, theory of algorithms, physics, thermodynamics and even biology, cf. [3, 17, 23, 31]. In practice, though, most of the computations considered as reversible are reversible only in more or less idealized models. Sometimes (but not always) the pre-conditions that make these models adequate may be clearly identified.

For example, the conditions may be purely mathematical. They may also concern the treatment of information (history of computations), the physical properties of a system (quantum state) etc.. Mathematical conditions may be concrete, e.g., expressed in terms of values of certain parameters, like non-zero determinant of a matrix, or more abstract (expressed in general terms characterizing the environment or the history of computations). Below we take into account only theoretical aspects of reversible computations related to mathematics and computer science.

From the categorical point of view, the reversible computations may be considered as morphisms of some category that are isomorphisms, or sometimes have only one-side inverse. If these morphisms are to be treated by computers, they should have some sort of termal representation, and this means that arise the problems concerning extensional and intensional equality. (Usually efficient treatment of isomorphisms by computers requires intensional equality.)

In the literature on reversible computations the history of computations is usually understood in the sense derived from Turing-machine protocols (commonly used are Turing machines with an additional *history tape*, cf. [3, 31]). So, pre-conditions of reversibility formulated in terms of history would require complete or partial preservation of history in this sense. From the point of view of category theory, natural are pre-conditions expressed in categorical terms, e.g. preliminary composition of a given morphism with some other morphism, application of a functor etc. This understanding of conditions of reversibility and that expressed in terms of 'history tape' are not mutually exclusive, but the categorical view accentuates other aspects of computation.

Let us consider several examples.

- If take the composition $\theta \circ h$ as in remark 4.2 of previous section, if $h =< f, g >$ then there exists θ' such that $\theta' \circ (\theta \circ h) = h$ (the part of computation represented by θ can be reversed).

- Let $f : A \to B$ be any morphism in T^0_{ind}, and assume that f is inversible on canonical elements of A and B, i.e., there exists $f' : B \to A$ such that for every canonical element $a : A$ $f'(fa)) = a$ and for every canonical element $b : B$ $f(f'(b)) = b$. Then by theorem 3.7 f is inversible in the sense of extensional equality. From the point of view of reversible computations, the condition of reversibility is that f is applied to a closed term.

- The same may be expressed more "diagrammatically". Let us consider a diagram of the form

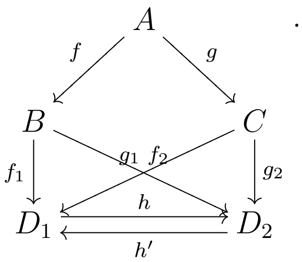
$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} ... \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A \xrightarrow{f} B$$

  in T_{ind}, where A_0, A and B are 0-recursive and f is inversible on canonical elements. Intermediate types $A_1, ..., A_n$ may be arbitrary types of T_{ind}. Then there exists f' such that $(f' \circ f) \circ (f_n \circ ... \circ f_0)) = f_n \circ ... \circ f_0$.

- The following example comes from our study of graph rewriting. In categorical graph rewriting [24] most often are used so called single pushout (SPO) and double pushout (DPO) approaches. Single (respectively, double) pushout construction[7] is used to define graph transformation rules. Working on generalization of SPO and DPO to the case of attributed graphs [5, 22] we arrived to the situations where only the existence of a weak pushout is guaranteed. Consider two weak

[7] We do not go into details here how the categories of graphs, equality of morphisms in these categories etc. are defined.

pushouts generated by the same "span" (f, g):



> Because of weak universality h and h' are not in general mutually inverse isomorphisms, but at the same time weak universality means commutativity of all diagrams above. For h and h', in particular, it means $(h' \circ h) \circ (f_1 \circ f) = f_1 \circ f$ and $(h \circ h') \circ (g_2 \circ g) = f_1 \circ g$. This can be seen as conditional reversibility.

These observations show the interest of application of diagrammatic methods and category theory to the theory of reversible computations. At the practical side, the use of graph rewriting techniques for treatment of diagrams may be advised.

5.3 Perspectives

In this paper we considered as main applications of our analysis of weak universality the applications to the study of reversible computations. It would be natural, if we would look at the perspectives in the same direction.

As we have outlined above, weak universality is related to conditional reversibility. Our motivating examples were coming mostly from categorical type theory. The same modelling language, categorical type theory, suggests that other forms of conditional reversibility would be interesting to study in future.

One of such forms could be context-dependent reversibility. Let us recall that the notion of retraction in λ-calculus, first defined in [6] (cf. also [28]) is context-dependent [8].

[8]The type ρ is a retract of type τ if there are terms $C : \rho \to \tau$ and $D : \tau \to \rho$ (witnesses) such that $D \circ C =_{\beta\eta} \lambda x^{\rho}.x$ [28].Witness terms in this definition may contain free variables. If a context (a list of typed free variables) is fixed, no witness may exist for some contexts.

More general question is the meaning of different kinds of equality when reversibility of computations is studied. It seems that there is little interaction between research communities studying reversible computations in connection with their physical (technical) realisation, computer architecture, thermodynamics etc., and more theoretical aspects such as meaning of reversibility itself. What does change when we consider the notion of reversibility with respect to different kinds of equality? When extensional and intensional equality are considered? When we modify the notion of equality between terms (programs), introducing new conversions? All this seems to be an important subject for future study.

At more theoretical side, it would be interesting to explore the behaviour of categorical universal constructions with respect to different types of equality in more powerful systems of λ-calculus (with or without inductive types).

6. Conclusion

In purely mathematical approach to category theory various types of equality are treated indifferently, as part of definition of categorical structure, and no special role is given to intensional equality.

We considered in this paper several examples of categories based on T_{ind}, a system of lambda-calculus with inductive types and recursion. The aim was to underline the connection between the "strength" of categorical universal constructions and equality of morphisms treated under the angle of computational efficiency (decidable intensional equality and extensional equality that is in general undecidable). The examples were rather elementary, but illustrated specific properties of categories with intensional equality (typically, used in computer science) with respect to basic universal constructions.

We paid special attention to another domain of research interesting for practical computing. So called "reversible computations" are actively studied nowadays. From categorical point of view, the reversible computations may be considered as morphisms of some category that are isomorphisms, or sometimes have one-side inverse. The equality of morphisms in this context is usually intensional.

The fact that in most cases there exist only weakly universal constructions may be connected with the notion of *conditional reversibility*. Category theory and categorical logic (type theory, lambda calculus) suggest new forms of reversibility conditions that were not considered before in the study of reversible computations and are much lighter (and may be more practical) than the conditions concerning the history of computation in the style of Turing machine protocols.

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES
Vol. LIV-4 (2013)

ESQUISSABILITE PROJECTIVE DES ESPACES DIFFEOLOGIQUES

par Jean-Pierre LAFFINEUR

À René Guitart, pour ton soixante cinquième anniversaire, en toute amitié.

Résumé. Dans cet article, nous présentons une esquisse projective des espaces difféologiques.
Abstract. In this article, we describe a projective sketch of diffeological spaces.
Keywords. Diffeological Spaces, Sketches, Locally Presentable Categories.
Mathematics Subject Classification (2010). Primary MSC 58A40, Secondary MSC 18C30 18C35 18F10 18F20.

1. Introduction

Lors du colloque Souriau's 90, à la mémoire de Jean-Marie Souriau, qui s'est tenu à Aix-en-Provence en juin 2012, Enxin Wu [10] a mentionné que la catégorie des espaces difféologiques était localement présentable. Les catégories localement présentables sont projectivement esquissables, ce qui nous a incité à en construire une esquisse projective. Nous présentons cette esquisse, après avoir rappelé la définition des espaces difféologiques.

2. Espaces Difféologiques

Pour les définitions et constructions relatives à la difféologie nous renvoyons à l'ouvrage de Patrick Iglesias-Zemmour [6].

Définition 2.1. *On appelle **domaine** la donnée d'un ouvert $\mathfrak{U}$ d'un $\mathbb{R}^n$.*

Définition 2.2. *Un **espace difféologique** est un ensemble X muni, pour tout domaine $\mathfrak{U}$, d'un ensemble, noté $X_\mathfrak{U}$, d'applications $\phi : \mathfrak{U} \to X$ appelées **plaques** et telles que :*

1. *(**Compatibilité**) si ϕ est une plaque de X de domaine $\mathfrak{U}$ et si $f : \mathfrak{U}' \to \mathfrak{U}$ est une application C^∞, alors $\phi \circ f : \mathfrak{U}' \to X$ est une plaque de X,*
2. *(**Localité**) si les $(\mathfrak{U}_j)_{j\in J}$ forment un recouvrement de $\mathfrak{U}$ d'inclusions $i_j : \mathfrak{U}_j \to \mathfrak{U}$ et si, pour tout j, $\phi \circ i_j : \mathfrak{U}_j \to X$ est une plaque, alors $\phi : \mathfrak{U} \to X$ est une plaque,*
3. *(**Recouvrement**) chaque application constante de $\mathfrak{U}$ dans X est une plaque.*

*L'ensemble des plaques est appelé **difféologie**.*

Définition 2.3. *Soient X et Y deux espaces difféologiques et $f : X \to Y$ une application. On dit que f est **lisse** si, pour toute plaque P de X, alors $f \circ P$ est une plaque de Y.*

La catégorie des espaces difféologiques, notée $\mathbb{D}$iff, est la catégorie dont les objets sont les espaces difféologiques et les flèches sont les applications lisses.

3. Une esquisse projective des difféologies

On désigne par $\mathbb{E}_{diff}$ l'esquisse projective construite comme suit.

Définition 3.1 (Les objets)**.** *On se donne :*

- *un objet X, l'objet de base,*
- *pour tout domaine $\mathfrak{U}$, un objet $X^\mathfrak{U}$,*
- *pour tout domaine $\mathfrak{U}$, un objet $X_\mathfrak{U}$.*

On a donc un objet de base plus deux familles d'objets.

Définition 3.2 (Les flèches)**.** *Outre les identités on se donne :*

- *pour toute application C^∞ $f : \mathfrak{U} \to \mathfrak{U}'$, une flèche $X^f : X^{\mathfrak{U}'} \to X^{\mathfrak{U}}$,*
- *pour toute application C^∞ $f : \mathfrak{U} \to \mathfrak{U}'$, une flèche $X_f : X_{\mathfrak{U}'} \to X_{\mathfrak{U}}$,*
- *pour toute application C^∞ $f : \mathfrak{U} \to \mathfrak{U}'$, une flèche $d_f : X_{\mathfrak{U}'} \to X^{\mathfrak{U}}$,*
- *pour tout domaine $\mathfrak{U}$, une flèche $h_{\mathfrak{u}} : X_{\mathfrak{U}} \to X^{\mathfrak{U}}$.*

On pose désormais $\mathbb{R}^0 = 1$ et on identifie chaque élément $e \in \mathfrak{U}$ avec la flèche e : $1 \to \mathfrak{U}$.

Définition 3.3 (Composition de flèches)**.** *Pour tout domaine $\mathfrak{U}$, pour tout e $\in \mathfrak{U}$ et pour toute application C^∞ $f : \mathfrak{U} \to \mathfrak{U}'$, on déclare que le diagramme suivant est commutatif :*

$$\begin{array}{ccc} X^{\mathfrak{U}'} & \xrightarrow{X^f} & X^{\mathfrak{U}} \\ {\scriptstyle X^{f(e)}}\downarrow & & \downarrow{\scriptstyle X^e} \\ X^1 & = & X^1 \end{array}$$

Définition 3.4 (Composition de flèches bis)**.** *Pour toute application C^∞ $f : \mathfrak{U} \to \mathfrak{U}'$, on déclare que le diagramme suivant est commutatif :*

$$\begin{array}{ccc} X^{\mathfrak{U}'} & \xrightarrow{X^f} & X^{\mathfrak{U}} \\ {\scriptstyle h_{\mathfrak{U}'}}\uparrow & \nearrow{\scriptstyle d_{\mathfrak{f}}} & \uparrow{\scriptstyle h_{\mathfrak{U}}} \\ X_{\mathfrak{U}'} & \xrightarrow{X_f} & X_{\mathfrak{U}} \end{array}$$

de sorte que pour tout domaine $\mathfrak{U}$ on a $h_{\mathfrak{U}} = d_{Id_{\mathfrak{U}}}$.

Définition 3.5 (Première série de cônes projectifs distingués)**.** *Pour tout domaine* $X^{\mathfrak{U}}$ *on distingue le cône projectif de sommet* $X^{\mathfrak{U}}$ *:*

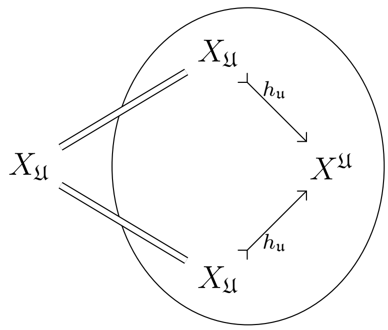
$$\{X^e\}_{e\in\mathfrak{U}} : X^{\mathfrak{U}} \xrightarrow{X^e} \{X\}_{e\in\mathfrak{U}}$$

Ainsi, pour tout modèle M de $\mathbb{E}_{diff}$ dans les ensembles :

$$\mathbb{E}_{diff} \xrightarrow{\quad M \quad} \mathbb{E}_{ns}$$

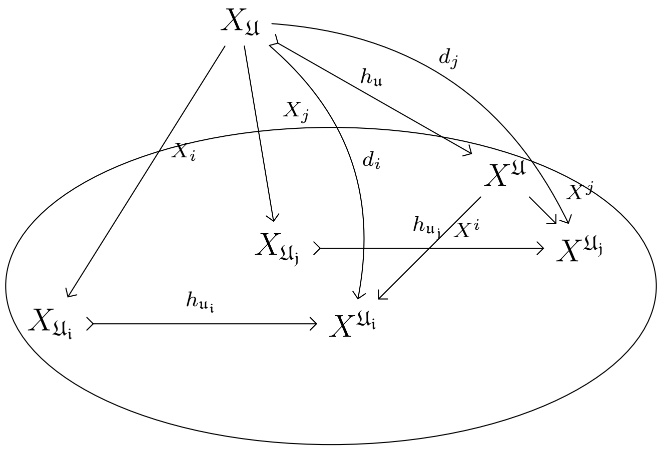
on aura $M(X^{\mathfrak{U}}) = M(\prod_{e\in\mathfrak{U}} X) = \prod_{e\in\mathfrak{U}} M(X)$ en bijection avec $M(X)^{\mathfrak{U}}$ l'ensemble de toutes les applications $f : \mathfrak{U} \to M(X)$.
Désormais on peut poser $X^1 = X$.

Définition 3.6 (Deuxième série de cônes projectifs distingués)**.** *Pour chaque* $\mathfrak{U}$*, on fait de* $h_{\mathfrak{u}}$ *un monomorphisme potentiel en distinguant le cône projectif suivant :*



Ainsi, pour tout modèle M de $\mathbb{E}_{diff}$ dans les ensembles, $M(h_{\mathfrak{u}})$ sera injective. Elle identifie $M(X_{\mathfrak{U}})$ avec un sous-ensemble de $M(X)^{\mathfrak{U}}$ et en fait un ensemble d'applications.

Définition 3.7 (Troisième série de cônes projectifs distingués)**.** *Pour tout recouvrement d'un domaine $\mathfrak{U}$ par des domaines $\mathfrak{U}_i$, on distingue le cône projectif suivant :*



Définition 3.8 (Une dernière équation)**.** *À tout cela il nous faut rajouter l'équation $h_{\mathfrak{1}} = Id_X$, pour identifier X^1 avec X_1 et ainsi ajouter les flèches constantes :*

$$X_{\mathfrak{1}} \xrightarrow[h_{\mathfrak{1}} = Id_X]{} X^{\mathfrak{1}} = X$$

Théorème 3.9. *La catégorie des modèles de l'esquisse $\mathbb{E}_{diff}$ ainsi construite est équivalente à $\mathbb{D}iff$ la catégorie des espaces difféologiques et des applications lisses.*

Preuve. En effet, par construction :

- les plaques sont des applications, ce que nous avons forcé en définissant $X^{\mathfrak{U}}$ comme produit, par la première série de cônes, et en y injectant $X_{\mathfrak{U}}$ par la deuxième série de cônes,
- la validité de l'axiome de compatibilité vient de la fonctorialité et de la prise en compte dans l'esquisse de toutes les applications C^∞ entre domaines de $\mathbb{R}^n$,
- l'axiome de localité vient de la troisième série de cônes,
- l'axiome de recouvrement vient de l'identité $X_{\mathfrak{1}}$=$X^{\mathfrak{1}}$ que nous avons ajoutée à la fin de la construction,
- les applications lisses sont les transformations naturelles entre espaces difféologiques.

□

Pour les questions relatives aux esquisses et aux catégories localement présentables, le lecteur pourra consulter [4] réédité dans [5], [9], [1], [2] , [7] et [8].

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PARCOURS DE RENE GUITART, UN MATHEMATICIEN AUX MULTIPLES FACETTES

par Andrée EHRESMANN

Résumé. Bref rappel des travaux de René Guitart, extrait de la présentation faite au cours de la "Journée en l'honneur de René GUITART" (Université Paris-Diderot, Novembre 2012) à l'occasion de son 65^e anniversaire.

Abstract. Brief recall of René Guitart's works, extracted from the presentation done during the "Journée en l'honneur de René GUITART" (Université Paris-Diderot, Novembre 2012) in honor of his 65^th birthday.

Keywords. Relations, Univers Algébriques, Carrés exacts, Diagrammes localement libres, Borroméens.

MS Classification. 01A70, 18-xx

1. Courte Biographie

René Guitart est né en 1947. Il a obtenu un Doctorat de 3^e cycle à l'Université Paris 7 en Juin 1970 sous la direction de C. Ehresmann ; puis un Doctorat d'Etat à l'Université de Picardie en Juin 1979 (Jury : A. Ehresmann, C. Ehresmann, G. Choquet, H. Kleisli, J. Bénabou, M. Tierney, L. Boasson) ; enfin son Habilitation à diriger des recherches à l'Université Paris 7 en 1988.

Nommé Assistant à l'Université de Picardie en 1968, il a été ensuite Maître Assistant puis Maître de Conférences à l'Université Paris 7 de 1970 à 2012. Il a dirigé les thèses de Luc Van den Bril (Amiens, 1978), Matthias Gerner (Paris 7, 1994), Monique Sassier (Paris 7, 2002) et l'Habilitation de Pierre Damphousse (Tours, 2003).

De 1982 à 1992 il a aussi été Ingénieur-mathématicien chez Essilor, et de 1992 à 1998, Directeur de Programmes au Collège International de Philosophie et il y a dirigé les Diplômes du CIPh de Bahia Monti, George Monti et Catherine Cisinski.

Il a organisé de nombreuses rencontres mathématiques de toute nature, en particulier, avec Pierre Damphousse, le "European Colloquium of Category Theory" (1994, Tours), placé sous la présidence de S. Eilenberg et S. Mac Lane. Et il est l'un des fondateurs du Séminaire Itinérant de Catégories (tri-annuel) qu'il continue régulièrement à organiser.

La liste de ses publications (cf. "Cahiers" LIV-2, pp. 85-90) contient une centaine d'articles de recherche, la plupart téléchargeables sur son site personnel :

http://rene.guitart.pagesperso-orange.fr

ainsi que 3 livres de nature mathématico-épistémologique.

Le graphique à la fin de cet article montre le "Système Evolutif" de ses travaux, dont il n'est pas possible de donner une idée complète ici, où je mentionnerai seulement certains de ses résultats qui me semblent les plus marquants.

2. Sur la théorie des relations

Dans ses premiers travaux, Guitart développe une théorie des relations dans une catégorie.

Dans sa thèse de 3^e cycle [1970], il étudie la catégorie des relations entre ensembles. Dans ce cas, une relation $A \to B$ peut être vue comme une application de A dans l'ensemble des parties P(B), et de cette façon la *catégorie des relations* dans *Ens* s'identifie à la catégorie de Kleisli KlP de la monade des parties.

2.1. *Relations associées à une monade. Univers algébriques* [1970, 1975a, 1977a, 1979a, 1982a]

Partant de ce résultat, il généralise le cadre en partant d'une catégorie C munie d'une monade, et il prend pour catégorie des relations la catégorie de Kleisli associée à la monade. Le problème est alors de déterminer quelles conditions mettre sur la monade sur C pour y interpréter la logique du 1[er] ordre. Ceci le conduit à traduire diverses propriétés ensemblistes de manière purement équationnelle :

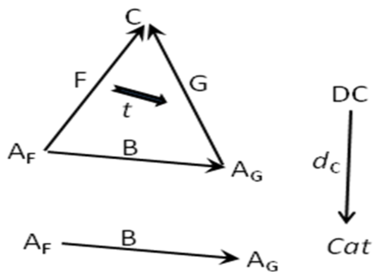
- Catégories avec égalité et appartenance ;
- Catégories avec monade involutive et transposition permettant un bon calcul des intersections et des relations inverses, valable éventuellement même en l'absence de limites ;
- *univers algébriques* dans lesquels on peut développer une logique du 1[er] ordre plus souple que celle donnée par les topos.

En plus des topos apparaissant avec leur monade des parties (logique intuitionniste), il donne divers exemples de tels univers, en particulier *Ens* avec la monade associée à un sup-monoïde abélien (logique floue), les relations associées étant les relations floues.

Ces univers se prêtent aussi à la description de propriétés topologiques, par l'introduction de *topogenèses* [1975b, 1976, 1979a] (exemples : topologies, espaces à fermeture, topologies de Grothendieck,...) et notamment permettent la construction des espaces de sous-espaces.

2.2. *Relations dans Cat. Monade des diagrammes* [1973, 1977c, 1979a]

Par contre les univers algébriques ne sont pas adaptés à l'étude des relations dans *Cat.* Pour étudier ces relations, l'idée est de prendre pour monade la "monade des diagrammes".



Etant donné une catégorie C, la catégorie DC des diagrammes de *C* a pour objets les foncteurs vers C, et les morphismes de F: $A_F \rightarrow C$ vers G: $A_G \rightarrow C$ sont les couples (B, *t*) où B est un foncteur de A_F vers A_G et où *t*: F => GB est une transformation naturelle. Cette catégorie est munie d'un foncteur 'base' d_C vers Cat qui associe B à (B, *t*).

Théorème *(Guitart-Van den Bril, 1977). Soit* DC *la catégorie des dia-*

grammes de C et d_C: DC → *Cat le foncteur 'base'.*

1. Le foncteur d: *Cat* → *CAT/Cat associant* d_C *à* C a pour *adjoint le foncteur* K *associant à un foncteur p:* M → *Cat la fibration kp:* K*p* → M *associée à p. D'où une comonade D* sur Cat.*

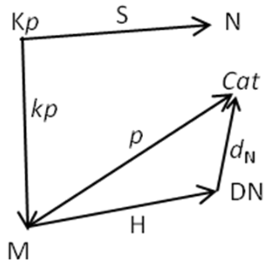
2. DC *s'étend en une 2-catégorie* **D**C *qui est la lax-cocomplétion de* C ; *et grâce à K,* **D** *donne une 2-monade (à isomorphisme près) notée* ***D*** *sur la 2-catégorie* ***Cat*** *; la 2-catégorie de Kleisli associée est notée* **Kl*D***, et la catégorie sous-jacente Kl***D***.

La catégorie de Kleisli Kl***D*** joue le rôle d'une catégorie des relations sur *Cat.*

2.3. *Interprétation en termes de machines* [1974, 1978, 1980a], *et tenseurs* [1980a]

La 2-catégorie **Kl*D*** peut s'interpréter en utilisant la notion suivante de machine :

Définition. Une *machine* d'entrée M et sortie N est définie par un span (*kp*, S), où *kp*: K*p* → M est la fibration associée à un foncteur *p*: M → *Cat* et S: K*p* → N est un foncteur.



On retrouve une machine de Mealy si M et N sont des monoïdes, *p* est à valeurs dans *Ens* et K*p* se réduit à un produit M×E. Les machines sont les 1-morphismes d'une 2-catégorie ***Mac.***

Théorème. *On a un* 2-i*somorphisme* Γ *de* ***Mac*** sur **Kl*D*** *et un plongement plein et 2-plein* Π *de* ***Mac*** *dans la 2-catégorie des 2-distributeurs.*

Γ associe à la machine (*kp*, S) le foncteur H: M → DN tel que

$$H(m) = S|p(m) \quad \text{et} \quad H(f) = (p(f), tf),$$

où *tf* est la transformation naturelle *tf(e) = S(e, f)*. Et Π associe à (*kp*,S) le distributeur composé S⊗*kp*°.

Guitart montre que l'analyse du passage des machines de Mealy déterministes aux machines non-déterministes dans les catégories monoïdales s'explique par la construction de produits tensoriels d'algèbres, en tant que classifiant de bimorphismes.

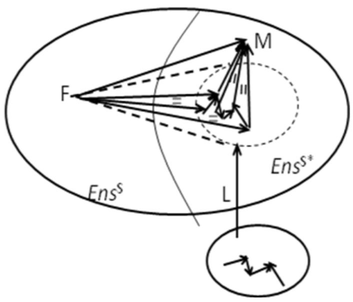
3. Esquisses. Logique

Dans une série de travaux (dont certains avec Christian Lair), René cherche à utiliser la théorie des esquisses pour prouver que logique = homologie.

3.1 *Esquisses mixtes et théorème du diagramme localement libre* [1981a, 1982a, 1982b]

Rappelons qu'une *esquisse mixte* S* = (S, P, I) est la donnée d'une catégorie S, d'un ensemble P de cônes projectifs sur S et d'un ensemble I de cônes inductifs sur S. Un modèle de S* dans une catégorie C est un foncteur de S vers C qui transforme chaque cône de P en un cône-limite projective et chaque cône de I en un cône limite inductive.

Théorème *(Guitart-Lair* 1981). *Soit* S* = (S, P, I) *une esquisse mixte. A tout foncteur* F *de* S *vers Ens est associé un petit* ***diagramme localement libre*** L *dans la catégorie* Ens^{S^*} *des modèles de* S* *vérifiant* : L *est base d'un cône projectif de sommet* F *qui est validé par tout modèle* M *de* S* *au sens que tout* F → M *se factorise dans le cône, par une génératrice unique à un zig-zag près.*



L n'est pas unique comme diagramme ; il généralise la notion de spectre d'un anneau local. Dans [1994a, 1997c], Guitart en donne une preuve plus 'effective' dans un article en collaboration avec Gerner.

3.2. *Présentation logique et algorithmique des esquisses* [1980c, 1986a, 1986b, 1987, 1988, 1990, 2008a]

Une *esquisse concrète* **S** = (S, P, Q) est la donnée d'une esquisse projective (S, P) et d'une famille Q = (Q_i) de cônes projectifs dans Ens^S, appelés **S**-*formules.* Un *modèle* de **S** est un modèle R de l'esquisse projective (S, P) qui valide tous les Q_i.

A toute esquisse S* = (S, P, I) on peut associer une esquisse concrète **S** = (S, P, Y(I)) ayant pour **S**-formules les images des cônes dans I par le foncteur Y: $S^{op} \to Ens^S$.

Théorème. S* *et* **S** *ont les même modèles. Esquisses concrètes et esquisses déterminent les mêmes catégories 'esquissables'.*

Les formules du 1^{er} ordre peuvent s'exprimer sous forme de **S**-formules. On en déduit qu'une théorie logique du 1^{er} ordre est esquissable, ce qui permet de développer un calcul logique sur les formules internes basé sur les seules notions de (co)limites et carrés exacts. D'où :

Théorème. *Toute théorie logique du 1^{er} ordre est esquissable.*

Des applications sont données à l'étude des programmes et algorithmes en termes d'esquisses. Ensuite une analyse de la satisfaction des formules du premier ordre en terme d'homologie est proposée.

Dans un article récent [2008a] Guitart expose comment, du point de vue des esquisses, on peut comprendre toute théorie comme algébrique.

4. Carrés exacts. Cohomologie

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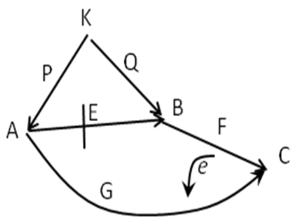
La notion de carré exact est introduite dans sa thèse, où Guitart définit des relations à partir de carrés exacts. Cette notion interviendra de manière essentielle dans plusieurs de ses travaux postérieurs. Récemment un complément notable est apporté par la notion de carré exact contractible.

4.1. *Carrés exacts* [1979a, 1980b, 1981b et c, 2013b]

Guitart introduit les *carrés exacts* de foncteurs et transformations naturelles, d'abord comme généralisations des extensions de Kan absolues, puis il montre que du côté abélien cela redonne les carrés exacts de Hilton. En principe la composition des relations, la déduction par 'diagram-chasing', et l'homologie sont définissables à partir des carrés exacts. De plus le calcul des satellites par résolutions se comprend comme la manipulation de carrés exacts en tant que présentations de distributeurs ou bimodules.

4.2. *Satellites. Cohomologie non abélienne* [1983a, 1983b, 1989]

Pour construire un cadre général pour une (co)homologie non abélienne, Guitart introduit d'une notion de *satellite d'un foncteur relativement à un distributeur ou à un carré exact.*



Soit E un distributeur de A vers B. Etant donné un foncteur F: B → C, Guitart définit le *satellite* de F pour E comme étant un couple (G, *e*) universel tel que

$$G: A \to C \text{ et } e: F \otimes E \Rightarrow G.$$

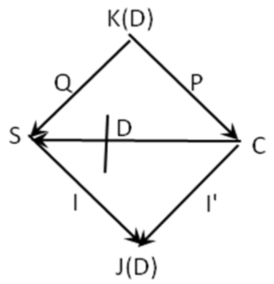
Pour rendre effectif le calcul des satellites, il utilise des *présentations* de E. Ainsi lorsque E = Q⊗P° où P: K → B et Q: K → A, alors G est le satellite de F si, et seulement si, il existe une transformation F.Q => G.P universelle, présentant G comme extension de Kan de FQ le long de P.

Exemple : Si A = B est abélienne et si K est formé de ses suites exactes courtes, alors E = EXT et le satellite de F: A → C devient le satellite gauche usuel.

Pour obtenir une notion de *cohomologie dans Cat*, Guitart construit un distributeur BIM = J⊗K° de *Cat* dans *Cat* comme suit :

J est un foncteur de la catégorie des distributeurs dans *Cat* qui associe au distributeur D de C vers S la *catégorie joint* J(D) : elle a C et S pour sous-catégories et J(D)(*c*, *s*) = D(*c*, *s*).

K est un foncteur de la catégorie des distributeurs dans *Cat* qui associe à D la catégorie K(D) ayant pour objets les flèches de C vers S dans J(D), et pour morphismes des carrés commutatifs de J(D).



Le distributeur BIM = J⊗K°: *Cat* → *Cat* redonne EXT dans le cas abélien. Le calcul des satellites relatifs à BIM est à la base d'une cohomologie non abélienne sur *Cat*, calculable à l'aide de carrés exacts.

5. Travaux mathématico-philosophiques

Dans la dernière décade, René a cherché à "Comprendre le sens" par différentes approches.

5.1. *Compléments sur le foncteur "Parties"* [1992, 1979b, 1994b, 1999, 2001b, 1999]

(i) *L'enveloppe karoubienne* d'une catégorie C est la catégorie ayant pour objets les (*c*, *e*) où *e: c* → *e* est un idempotent.

***Théorème** (Guitart-Riguet, 1992). L'enveloppe karoubienne de la catégorie des relations KlP est isomorphe à la catégorie des treillis complets complètement distributifs avec applications sup-compatibles. KlP s'identi-*

fie à la sous-catégorie pleine dont les objets sont booléens.

(ii) Guitart caractérise des sous-foncteurs B de *P* tels que B(A) contienne le vide pour tout A (appelés *bornes*). En vue d'une théorie de la traduction, avec Damphousse ils caractérisent aussi des transformations naturelles de *P'* vers *P'P"*, où *P'* et *P"* sont soit *P*, soit l'un des 2 autres foncteurs (dont l'un contravariant) associant *P*(E) à E.

(iii) Ensuite ils montrent que la monade des parties admet des relèvements sur les catégories d'espaces qualifiés, et que cela produit une dualité entre le calcul des applications continues et celui des applications ouvertes.

(iv) Enfin ils classifient les *fixob*, ou foncteurs F: S $\rightarrow$ S fixant les objets, où S est une sous-catégorie pleine de *Ens*.

Théorème *(Guitart-Damphousse, 2001).* Si *les objets de* S *sont finis, un* fixob *est isomorphe à l'identité ssi* S *a au moins trois objets non-vides non-isomorphes*

5.2. *Analyse des discours* [1991, 1997a, 1999b, 2000, 2001c, 2003, 2009a]

Depuis 1992, une partie importante du travail de Guitart est consacrée à des travaux de nature philosophique, épistémologique ou psychanalytique sur l'analyse des discours, considérés comme 'êtres 'vivants'.

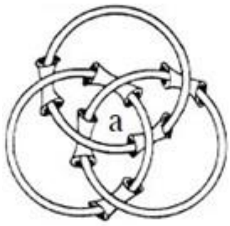
Pour ceci, René propose un modèle mathématique (basé sur des foncteurs adjoints) de la *logique spéculaire* qui intervient dans les discours, prenant en compte l'ambiguité et le jeu local/global. En particulier il souligne la *pulsation mathématique* à la base de toute activité mathématique.

Le sens d'un discours, conçu au travers de l'auto-mouvement entre les différentes logiques de ses présupposés, est représenté par une classe de cohomologie d'un espace déterminé par le discours.

L'étude de la notion d'assimilation, vue comme "nouage des verbes remplacer, substituer, incorporer" en un nœud borroméen, est à la base de plusieurs travaux récents sur les objets borroméens (e.g. groupe de Klein) et la logique borroméenne.

5.3 *Objets borroméens* [2005, 2008b, 2009 c, 2012]

L'idée d'objet borroméen est une catégorification précise de ce que l'on voit dans le cas géométrique de l'entrelac borroméen : trois objets isomorphes qui se tiennent ensemble à trois mais pas deux par deux.



Notamment il y a des groupes borroméens, comme le groupe de Klein $GL_3(F_2)$, il y a aussi des logiques borroméennes. Guitart montre qu'une logique borroméenne réside sur le corps de Galois à 4 éléments.

5.3. *Travaux historiques* [1979a, 1998, 2000 2001a, 2009b, 2013a]

René a écrit divers articles de nature mathématico-historique.

En 1979, il avait fait un travail de synthèse sur la théorie du potentiel au travers des travaux d'axiomatisation à la Perron-Wiener-Brelot (2nde thèse donnée par Gustave Choquet). Plus tard, il a développé un travail historique sur la théorie du potentiel en relation avec les fonctions elliptiques.

Avec Evelyne Barbin (son épouse), il a écrit 2 articles historiques sur les ovales cartésiennes, donnant diverses descriptions et constructions des ovales et étudiant les relations entre ovales et fonctions elliptiques.

Il a aussi fait un travail historique sur Lamé, montrant comment il est conduit par des problèmes de physique à l'introduction des coordonnées curvilignes et à leurs relations avec les fonctions elliptiques.

Avec Evelyne Barbin ce travail a été prolongé par une étude sur l'émergence de la physique mathématique au XIXème siècle, telle que signifiée dans l'œuvre de Mathieu.

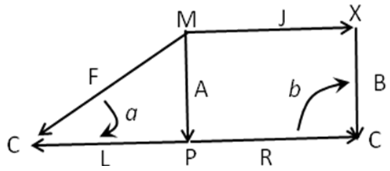
6. Cohomologie anabélienne [2007]

Pour terminer, je voudrais attirer l'attention sur un petit article assez récent (publié dans ces "Cahiers" XXXXVIII-4, p. 261-269) où il propose une approche particulièrement simple et efficace pour l'étude de la cohomologie non nécessairement abélienne.

Le but est de définir l'homologie et la cohomologie des objets d'une catégorie X à valeurs dans une catégorie C relativement à une donnée Φ = [J; L,R] de 3 foncteurs : J : M → X décrivant les 'modèles' dans X, L et R de P vers C donnant des présentations des objets de C.

En particulier si C = *Ab*, P = EXA^n(Ab), L (resp. R) associe à une suite son 1er (resp. dernier) élément.

Pour F: M → C la *catégorie d'homologie* H_{*F} pour F a pour objets les quadruples μ = (A, B, *a*, b), où :



On a un foncteur U_F: $H_{*F} \to C^X$ associant B à μ.

Théorème. *Si les limites existent, l'homologie pour* Φ = [J; L,R] est donnée par

$$H_*(\Phi): X \times C^M \to C \quad \text{où} \quad H_*(\Phi)(-, F) = \lim U_F$$

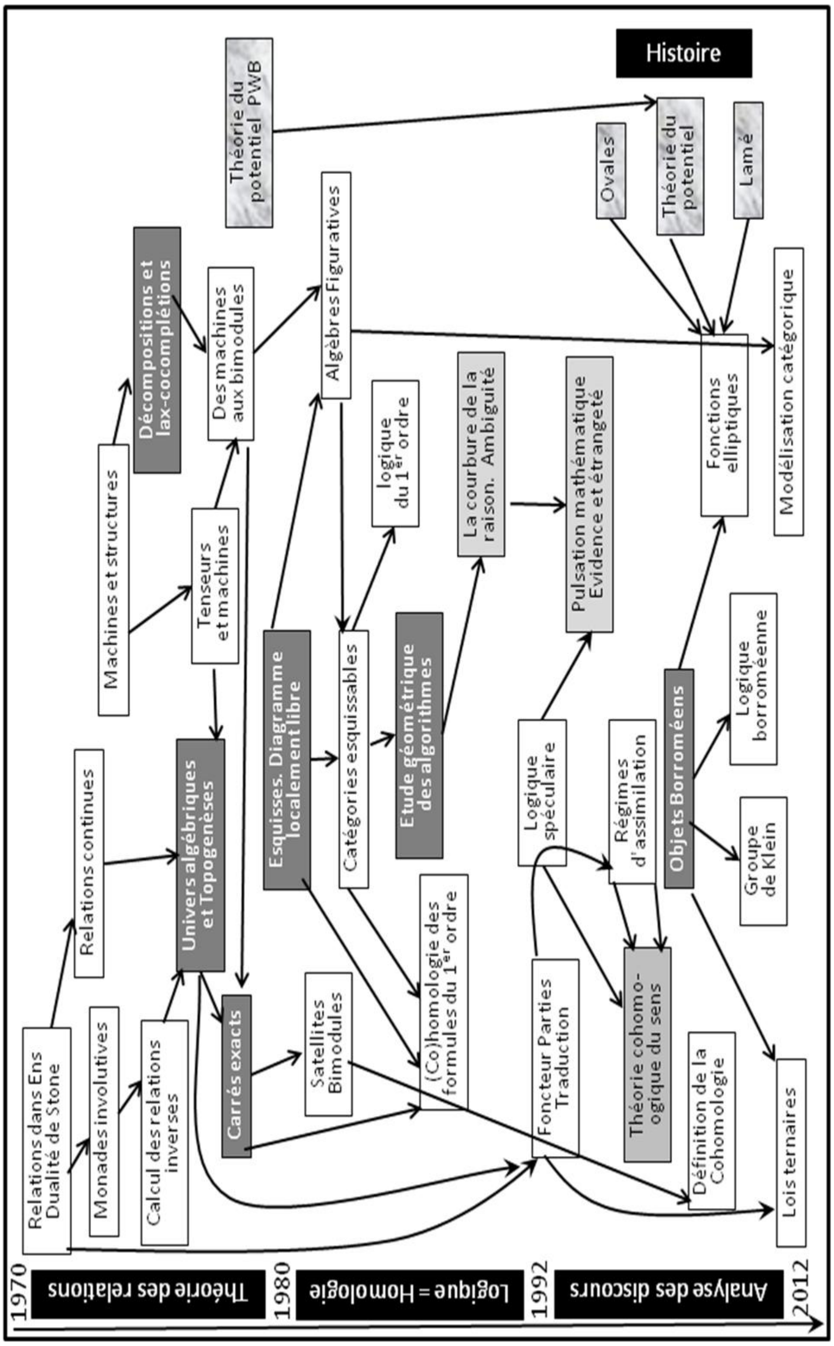
La cohomologie pour Φ *est définie dualement (lim remplacé par colim).*

On retrouve la (co)homologie usuelle dans le cas abélien.

Conclusion

René Guitart est un mathématicien fécond, qui a obtenu d'importants résultats dans différents domaines purement mathématiques concernant la théorie des catégories, la logique, la cohomologie, mais aussi dans des domaines à cheval sur l'épistémologie et la sémiotique, avec l'analyse des discours vus comme "systèmes vivants", où il développe des modèles mathématiques. Et divers travaux d'histoire des mathématiques.

Le tableau ci-après donne le "Système Evolutif" de ses travaux, c'est-à-dire la liste plus ou moins chronologique des domaines qu'il a abordés de 1970 2012, ainsi que les liens entre ces domaines. Ces liens qui sont très nombreux et s'entremêlent montrent que, malgré ses multiples facettes, René a toujours poursuivi une même ligne de pensée qui oriente tous ses travaux.


Histoire
Théorie du potentiel PWB
Ovales
Théorie du potentiel
Lamé
Décompositions et lax-cocomplétions
Des machines aux bimodules
Algèbres Figuratives
Modélisation catégorique
Fonctions elliptiques
Machines et structures
Tenseurs et machines
logique du 1er ordre
La courbure de la raison. Ambiguité
Pulsation mathématique Evidence et étrangeté
Relations continues
Univers algébriques et Topogenèses
Esquisses. Diagramme localement libre
Catégories esquissables
Etude géométrique des algorithmes
Logique spéculaire
Régimes d'assimilation
Objets Borroméens
Logique borroméenne
Groupe de Klein
Relations dans Ens Dualité de Stone
Monades involutives
Calcul des relations inverses
Carrés exacts
Satellites Bimodules
(Co)homologie des formules du 1er ordre
Foncteur Parties Traduction
Théorie cohomologique du sens
Définition de la Cohomologie
Lois ternaires
1970
Théorie des relations
1980
Logique = Homologie
1992
Analyse des discours
2012

Références aux travaux de R. Guitart

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TAC: THEORY AND APPLICATIONS OF CATEGORIES

Hereafter we give some information about the electronic Journal**:**
Theory and Applications of Categories **(T_AC),** ISSN 1201-561X

Contents of VOLUME 27 - CT2011

A special volume of articles from the *CT2011 Conference*. The Editors of T_AC wish to to thank Marino Gran, George Janelidze, Stephen Lack, John MacDonald and Walter Tholen who acted as guest editors for this volume.

Contents of VOLUME 26, 2012

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THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.
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Editorial board

RESUMES DES ARTICLES PUBLIES DANS LE VOLUME LIV

E. CHENG, A direct proof that the category of 3-computads is not cartesian closed, 3-12.
L'auteur démontre, par contre-exemple, que la catégorie des "3-computads" (ou 3-polygraphes) n'est pas cartésienne fermée ; ce résultat avait déjà été donné par Makkai et Zawadowski. Pour cela, elle construit un 3-polygraphe B tel que le foncteur $-\times B$ n'a pas d'adjoint à droite, ce qu'elle prouve en mettant en évidence un conoyau qui n'est pas préservé par ce foncteur.

R. GUITART, Trijunctions and triadic Galois connections, 13-27.
Dans cet article sont introduites les *trijonctions*, qui sont aux connexions galoisiennes triadiques ce que les adjonctions sont aux connexions galoisiennes. L'auteur décrit : le tripode trifibré associé a une trijonction, la trijonction entre topos de préfaisceaux associée a une trifibration discrète, et l'engendrement de toute trijonction par un bi-adjoint. A côté des exemples associés aux connexions galoisiennes triadiques et aux relations ternaires, d'autres le sont à des tenseurs symétriques, aux topos et univers algébriques.

J. E. BERGNER, Derived Hall algebras for stable homotopy theories, 28-55.
Cet article étend la construction de l'algèbre de Hall dérivée de Toen, dans laquelle il obtient des algèbres associatives avec unité à partir de certaines catégories de modèles stables, au cas où ces algèbres sont obtenues à partir de théories homotopiques stables plus générales, en particulier espaces de Segal complets stables satisfaisant des hypothèses de finitude appropriées.

J.-Y. DEGOS, Linear groups and primitive polynomials over $\mathbf{F}_p$, 56-74.
En s'inspirant du groupe de Klein $GL_3(F_2)$, l'auteur introduit les nouvelles notions de groupes *n*-cyclables et de groupes *n*-brunniens de type I et II. Il montre ensuite que les groupes $SL_n(F_p)$ et $GL_n(F_p)$ jouissent d'une structure de groupes *n*-brunniens de type I pour *p* premier et $n \geq 3$. Il énonce deux conjectures, à savoir les conjectures $A(n; p; P)$ et $B(n; p; P)$ concernant les polynômes primitifs sur $\mathbf{F}_p$, et donne des résultats partiels.

C. KACHOUR, Correction to the paper "Operadic definition of the non-strict cells" (2011), 75-80.
L'auteur propose une nouvelle approche de la contractibilité pour les ω-opérades colorées définies dans l'article publié dans ces "Cahiers" (Volume LII-4, 269-316). Il propose aussi une autre façon de construire la monade des ω-opérades contractiles colorées.

FOREWORD, 82-84.
Foreword to the special volume of the "Cahiers" dedicated to Professor René Guitart on the occasion of his 65th birthday**.**

PUBLICATIONS DE RENE GUITART, 85-90.

M. GRANDIS, Adjoints for symmetric cubical categories (on weak cubical categories, III), 91-136.
Etendant un article précédent (avec R. Paré) sur les adjonctions pour les catégories doubles, l'auteur traite maintenant les catégories cubiques symétriques (de dimension infinie). Ici aussi, une "adjonction cubique" générale est formée d'un foncteur cubique colax qui est adjoint à gauche d'un foncteur cubique lax. Cela ne peut pas être envisagé comme une adjonction interne à une bicatégorie, car en composant des morphismes lax et colax on détruit leur structure. Toutefois, comme dans le cas des adjonctions doubles, les adjonctions cubiques vivent dans une catégorie double intéressante, celle formée des catégories cubiques symétriques, avec les foncteurs cubiques lax et colax en tant que flèches horizontales et verticales, liées par des cellules doubles convenables.

S. DUGOWSON, Espaces connectifs : représentations, feuilletages, ordres et difféologies, 137-160.
Poursuivant l'étude présentée dans son article "On connectivity Spaces", l'auteur développe, après quelques rapides repères historiques, la notion d'espace de séparation et les notions adjointes de représentation et de feuilletage connectifs. Il généralise en outre la notion d'ordre connectif au cas infini, ainsi qu'aux feuilletages connectifs. Finalement, il étudie certaines relations fonctorielles entre structures connectives et structures difféologiques, caractérisant en particulier les espaces connectifs difféologisables.

BOURN, MARTINS-FERREIRA & VAN DER LINDEN, A characterisation of the "Smith is Huq" condition in the pointed Mal'tsev setting, 163-183.
Les auteurs donnent une caractérisation de la condition "Smith is Huq" pour une catégorie de Mal'tsev pointée C au moyen d'une propriété de la fibration des points $Pt(C) \to C$, à savoir : la commutation des sous-objets normaux est reflétée par tout foncteur changement de base h^*: $Pt_Y(C) \to Pt_X(C)$.

EVERAERT, GOEDECKE & VAN DER LINDEN, The fundamental group functor as a Kan extension, 184-210.
L'article montre que le foncteur groupe fondamental considéré en théorie de Galois catégorique peut être calculé comme une extension de Kan.

J.-Y. DEGOS, Borroméanité du groupe pulsatif, 211-220.
Le groupe pulsatif Pul est défini comme le groupe de symétries de la sphère entière de rayon √5, et une description de ce groupe est donnée. Ensuite l'auteur en donne une présentation borroméenne, et il le relie au groupe spécial orthogonal. Des possibilités de représenter graphiquement ce groupe sont obtenues en utilisant la méthode de Newton. L'article se termine sur une interprétation et une généralisation.

ABBAD & VITALE, Faithful calculus of fractions, 221-239.
Cet article développe un argument simple sur les bicatégories de fractions qui montre que, si Σ est la classe des équivalences faibles entre groupoïdes internes à une catégorie régulière *A* qui admet suffisamment d'objets projectifs réguliers, alors la description de Grpd(A)[Σ^{-1}] peut être considérablement simplifiée.

D. BOURN, Normality, commutation and suprema in the regular Mal'tsev and protomodular settings, 243-263.
Dans le contexte des catégories régulières de Mal'tsev et protomodulaires, l'auteur développe les conséquences d'une caractérisation, acquise dans le simple cadre des catégories unitales sans condition de colimites, du fait que le sup de deux sous-objets qui commutent est leur commun codomaine. Il retrouve ainsi, mais avec des preuves conceptuelles, quelques résultats bien connus de la catégorie Gp des groupes.

BARANOV & SOLOVIEV, Equality in Lambda Calculus. Weak universality in Category Theory and reversible computations, 264-291.
Dans cet article, les auteurs montrent d'abord comment le type d'égalité ("extensionnelle" ou "intensionnelle") du lambda calcul avec types inductifs et récursion influence les constructions universelles dans certaines catégories basées sur ce calcul (l'universalité faible est reliée à l'égalité intensionnelle). Ensuite ils établissent un lien entre universalité faible et réversibilité conditionnelle dans la théorie du calcul réversible.

J.-P. LAFFINEUR, Esquissabilité projective des espaces difféologiques, 292-297.
The author explicitly describes a projective sketch of diffeological spaces.

A.C. EHRESMANN, Parcours de René Guitart, un mathématicien aux multiples facettes, 298-316.
Brief recall of René Guitart's works, extracted from the presentation done during the "Journée en l'honneur de René Guitart" (Paris, 2012) for his 65th birthday.

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