cahiers de topologie et géométrie différentielle catégoriques

créés par CHARLES EHRESMANN en 1958
dirigés par Andrée CHARLES EHRESMANN
VOLUME LIV-3, 3e trimestre 2013

SOMMAIRE

2nd Volume dédié à René GUITART

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES Vol. LIV-3 (2013)

A CHARACTERISATION OF THE "SMITH IS HUQ" CONDITION IN THE POINTED MAL'TSEV SETTING

by Dominique BOURN, Nelson MARTINS-FERREIRA and Tim VAN DER LINDEN

dedicated to René Guitart on the occasion of his sixty-fifth birthday

Résumé. Nous donnons une caractérisation de la condition « Smith is Huq » pour une catégorie de Mal'tsev pointée $\mathbb{C}$ au moyen d'une propriété de la fibration des points $\P_{\mathbb{C}}\colon \mathsf{Pt}(\mathbb{C}) \to \mathbb{C}$, à savoir : tout foncteur changement de base $h^*\colon \mathsf{Pt}_Y(\mathbb{C}) \to \mathsf{Pt}_X(\mathbb{C})$ reflète la commutation des sous-objets normaux.

Abstract. We give a characterisation of the "Smith is Huq" condition for a pointed Mal'tsev category $\mathbb{C}$ by means of a property of the fibration of points $\P_{\mathbb{C}}\colon \mathsf{Pt}(\mathbb{C}) \to \mathbb{C}$, namely: any change of base functor $h^*\colon \mathsf{Pt}_Y(\mathbb{C}) \to \mathsf{Pt}_X(\mathbb{C})$ reflects commuting of normal subobjects.

Keywords. Fibration of points, Mal'tsev and protomodular category, commutation of subobjects, centralisation of equivalence relations, commutator theory, topological Mal'tsev model

Mathematics Subject Classification (2010). 08B05, 18C99, 18D30, 18D35, 20F12, 54H11

Introduction

It is well known that, given a group G and two subgroups H and K, they commute inside G (i.e., we have $h \cdot k = k \cdot h$, $\forall (h,k) \in H \times K$) if and only if the function $H \times K \to G\colon (h,k) \mapsto h \cdot k$ is a group homomorphism. When H and K are normal subgroups of G, and if R_H and R_K denote their associated

The second author was supported by IPLeiria/ESTG-CDRSP and Fundação para a Ciência e a Tecnologia (under grant number PTDC/MAT/120222/2010). The third author works as *chargé de recherches* for Fonds de la Recherche Scientifique–FNRS.

equivalence relations on G, this is the case if and only if the equivalence relations R_H and R_K centralise each other (see [22, 21, 11]), namely if and only if the function $R_H \times_G R_K \to G\colon (xR_HyR_Kz) \mapsto x \cdot y^{-1} \cdot z$ is a group homomorphism, where $R_H \times_G R_K$ is defined by the following pullback:

$$\begin{array}{ccc} R_H \times_G R_K & \underset{\delta_1^K}{\overset{\longleftarrow}{\longrightarrow}} & R_K \\ {\scriptstyle \delta_0^H}\downarrow\uparrow & & {\scriptstyle d_0^K}\downarrow\uparrow \\ R_H & \underset{d_1^H}{\overset{\longleftarrow}{\longrightarrow}} & G \end{array}$$

The commutation condition on subobjects is said to be *à la Huq* from [17], while the commutation condition on equivalence relations is said to be *à la Smith* from [22]. In the category Gp of groups, we just recalled that, in the case of normal subobjects, the two types of commutation are equivalent. This is the meaning of the "Smith is Huq" condition, which is far from being true in general.

It turns out that the right environment for the conceptual notion of centralisation of equivalence relations is the context of Mal'tsev categories [13, 14]. It was first shown in [11, Proposition 3.2] that, in a pointed Mal'tsev category, "Smith implies Huq", namely that if two equivalence relations R and S centralise each other (which we denote by $[R,S] = 0$), then necessarily their associated normal subobjects commute. But the converse is not true, as shown in [6, Proposition 6.1], from an example introduced by G. Janelidze in the pointed Mal'tsev category of *digroups*, namely sets endowed with two group structures only coinciding on the unit element.

The first conceptual setting where the "Smith is Huq" condition (SH) holds was pointed out in [11]: it is the context of of pointed *strongly protomodular* categories, of which the category Gp is an example. These are pointed categories $\mathbb{C}$ such that any change of base functor with respect to the fibration of points $\P_{\mathbb{C}} : \mathsf{Pt}(\mathbb{C}) \to \mathbb{C}$ is normal, i.e., conservative and reflecting normal subobjects. Further observations on the condition (SH) have been given in [19, 15, 16, 20].

So it is quite natural to ask for a characterisation of the (SH) condition, and more precisely to ask it in terms of a property of the change of base functors of the fibration $\P_{\mathbb{C}}$. Here we give an answer in the pointed Mal'tsev context: the property of *reflection of commutation of normal subobjects*. We show moreover that when a variety $\mathsf{Set}^{\mathbb{T}}$ of algebras over a Mal'tsev theory $\mathbb{T}$

satisfies this last condition, so does the category $\mathsf{Top}^{\mathbb{T}}$ of topological models of this theory, which implies that the category $\mathsf{Gp}(\mathsf{Top})$ of topological groups satisfies (SH).

We then extend some results already known for strongly protomodular categories [4] to the (SH) context. In particular, we show that, when they are defined, the Huq commutator and the Smith commutator coincide.

1 Unital categories and Mal'tsev categories

1.1 Unital categories and Huq commutation

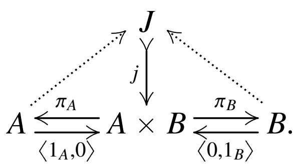
In this section, $\mathbb{C}$ will be a pointed category, i.e., a category with a zero object 0. Let us recall from [3]:

Definition 1.1. Let $\mathbb{C}$ be a pointed category with finite products. Given two objects A and B in $\mathbb{C}$, consider the diagram

$$A \underset{\langle 1_A,0\rangle}{\overset{\pi_A}{\leftrightarrows}} A \times B \underset{\langle 0,1_B\rangle}{\overset{\pi_B}{\rightleftarrows}} B.$$

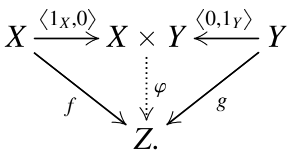
The category $\mathbb{C}$ is said to be *unital* if, for every pair of objects $A, B \in \mathbb{C}$, the morphisms $\langle 1_A, 0\rangle$ and $\langle 0, 1_B\rangle$ are jointly strongly epimorphic.

In any finitely complete category this is equivalent to saying that the object $A \times B$ is the supremum of the two subobjects $\langle 1_A, 0\rangle$ and $\langle 0, 1_B\rangle$; namely, any monomorphism $j\colon J \rightarrowtail A \times B$ containing the two previous ones:



is an isomorphism. From this last remark, it is clear that the category Mon of monoids is unital. Unital categories give a setting where it is possible to express a categorical notion of commutation *à la Huq* [5]:

Definition 1.2 (Commutation à la Huq)**.** Let $\mathbb{C}$ be a unital category. Two morphisms with the same codomain, $f\colon X \to Z$ and $g\colon Y \to Z$, are said to cooperate (or to commute) if there exists a morphism $\varphi\colon X \times Y \to Z$ such that both triangles in the following diagram commute:



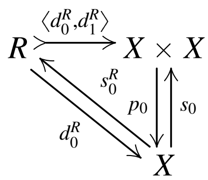
The morphism φ is necessarily unique, because $\langle 1_X, 0\rangle$ and $\langle 0, 1_Y\rangle$ are jointly epimorphic, and it is called the *cooperator* of f and g.

The uniqueness of the cooperator makes commutation a property, rather than an additional structure on the category $\mathbb{C}$.

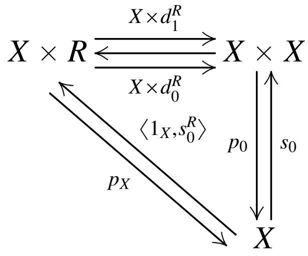
1.2 Mal'tsev categories and Smith commutation

A Mal'tsev category is a category in which every reflexive relation is an equivalence relation [13, 14]. The category Gp of groups is Mal'tsev. It is shown in [3] that a finitely complete category $\mathbb{C}$ is Mal'tsev if and only if any (necessarily pointed) fibre $\mathsf{Pt}_Y(\mathbb{C})$ of the fibration of points $\P_{\mathbb{C}}\colon \mathsf{Pt}(\mathbb{C}) \to \mathbb{C}$ is unital. Here $\mathsf{Pt}(\mathbb{C})$ is the category whose objects are the split epimorphisms in $\mathbb{C}$ and whose arrows are the commuting squares between such split epimorphisms, and $\P_{\mathbb{C}}\colon \mathsf{Pt}(\mathbb{C}) \to \mathbb{C}$ is the functor associating its codomain with any split epimorphism.

In this context, an equivalence relation R on an object X, coinciding with a reflexive relation on X, is just a subobject of the object $(p_0, s_0)\colon X \times X \rightleftarrows X$ in the fibre $\mathsf{Pt}_X(\mathbb{C})$:



Actually it is a normal subobject in this fibre since it is the *normalisation* (i.e., the class of the initial object in the pointed fibre $\mathsf{Pt}_X(\mathbb{C})$) of the following equivalence relation:



We call this normal subobject the *local representation* of the equivalence relation R. Let us recall Proposition 3.4 of [6]:

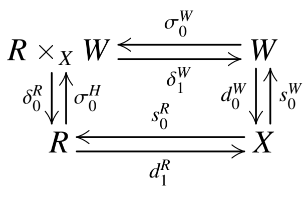
Proposition 1.1 (Commutation à la Smith)**.** *Let $\mathbb{C}$ be a finitely complete Mal'tsev category, and (R, W) a pair of equivalence relations on an object X. The equivalence relations R and W centralise each other in $\mathbb{C}$ if and only if their (normal) local representations commute in the unital fibre $\mathsf{Pt}_X(\mathbb{C})$.*

Proof. In the unital fibre $\mathsf{Pt}_X(\mathbb{C})$, the subobjects

$$\langle d_1^R, d_0^R \rangle \colon R \rightarrowtail X \times X \quad \text{and} \quad \langle d_0^W, d_1^W \rangle \colon W \rightarrowtail X \times X$$

commute if there is a cooperator $R \times_X W \to X \times X$ in the fibre; it is necessarily of the form $\phi(xRyWz) = (x, p(xRyWz))$, satisfying the two equations $p(xRxWy) = y$ and $p(xRyWy) = x$. The morphism $p \colon R \times_X W \to X$ which, satisfying these equations, characterises the property that the equivalence relations R and W centralise each other in $\mathbb{C}$, is nothing but what is called the *connector* between R and W. (See [11] and also [21, 13, 14].) ◻

As usual, we denote this situation by $[R, W] = 0$. It is worth noticing that, by construction of the pullback $R \times_X W$:



(*)

the existence of the connector p does not depend on the possibly fibered context, namely on the fact that R and W are possibly in a fibre $\mathsf{Pt}_Y(\mathbb{C})$.

2 A characterisation of the “Smith is Huq” condition (SH)

2.1 Reflections of commutation

Let us introduce the following conditions:

(C) any change of base functor with respect to the fibration of points reflects the commutation of normal subobjects;

($\bar{\text{C}}$) any change of base functor with respect to the fibration of points reflects the centralisation of equivalence relations.

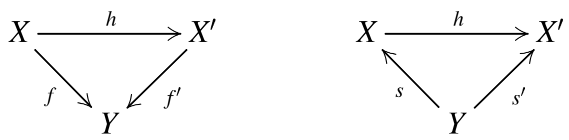
Recall that a protomodular category is such that any change of base functor with respect to the fibration of points reflects isomorphisms, and that any protomodular category is Mal’tsev. So, any protomodular category is such that any change of base functor with respect to the fibration of points reflects the inclusion of subobjects and, accordingly, the inclusion of equivalence relations.

Example 2.1. 1) According to Proposition 4.1 in [12], any *locally algebraically cartesian closed* (lacc: i.e., such that any change of base functor with respect to the fibration of points admits a right adjoint) protomodular category is such that any change of base functor with respect to the fibration of points *reflects the commutation of subobjects*, hence satisfies condition (C). The categories Gp of groups, ${}_R\mathsf{Lie}$ of Lie R-algebras, and $\mathsf{Gp}(\mathbb{E})$ of internal groups in a cartesian closed category $\mathbb{E}$ are examples of lacc protomodular categories.

2) According to Proposition 5.10 in [8], any *functorially action distinctive* protomodular category in the sense of [8] (again defined by a property of the change of base functors with respect to the fibration of points which we shall not detail here) is such that any change of base functor with respect to the fibration of points preserves the centralisers of equivalence relations, and, accordingly, satisfies condition ($\bar{\text{C}}$).

Proposition 2.1. *Let $\mathbb{C}$ be a finitely complete Mal’tsev category. Conditions* (C) *and* ($\bar{\text{C}}$) *are stable by slicing and coslicing, and consequently are still valid in any fibre* $\mathsf{Pt}_Y(\mathbb{C})$.

Proof. It is clear that given any morphism h in $\mathbb{C}/Y$ or in $Y/\mathbb{C}$ as below:



we have:

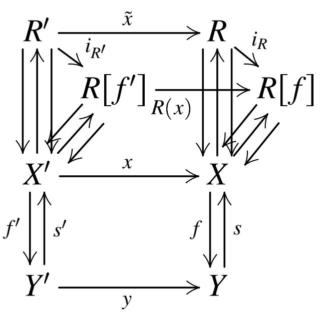
$$\begin{array}{ccc} \mathsf{Pt}_{f'}(\mathbb{C}/Y) & \xrightarrow{h^*} & \mathsf{Pt}_{f}(\mathbb{C}/Y) \\ \| & & \| \\ \mathsf{Pt}_{X'}(\mathbb{C}) & \xrightarrow[h^*]{} & \mathsf{Pt}_{X}(\mathbb{C}) \end{array} \qquad \begin{array}{ccc} \mathsf{Pt}_{s'}(Y/\mathbb{C}) & \xrightarrow{h^*} & \mathsf{Pt}_{s}(Y/\mathbb{C}) \\ \| & & \| \\ \mathsf{Pt}_{X'}(\mathbb{C}) & \xrightarrow[h^*]{} & \mathsf{Pt}_{X}(\mathbb{C}) \end{array}$$

So the result is a consequence, on the one hand, of the fact that, as we recalled above, the condition $[R,W]= 0$ does not depend on the fibered context and, on the other hand, of the fact that the normality of a subobject in $\mathsf{Pt}_f(\mathbb{C}/Y)$ or $\mathsf{Pt}_s(Y/\mathbb{C})$ is given by a pullback condition in $\mathbb{C}$ which, accordingly, is still valid in $\mathsf{Pt}_X(\mathbb{C})$. The same observation holds for the commutation condition. □

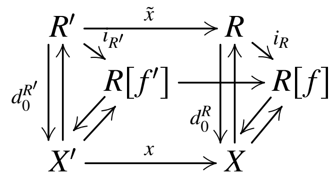
Unlike in the stricter context of protomodular categories, a normal subobject in a pointed Mal'tsev category could be the normalisation of several equivalence relations; so the following, though it is not surprising, does deserve a proof:

Proposition 2.2. *Let* $\mathbb{C}$ *be a finitely complete Mal'tsev category. Condition* (C) *implies condition* ($\bar{\text{C}}$).

Proof. Consider the following diagram in which R is an equivalence relation on the object (f,s) in $\mathsf{Pt}_Y(\mathbb{C})$, the kernel pair of f is denoted by $R[f]$ and any commutative square is a pullback:



By Proposition 1.1, the inclusions $i_R\colon R \rightarrowtail R[f]$ and $i_{R'}\colon R' \rightarrowtail R[f']$ are normal subobjects in the fibres $\mathsf{Pt}_X(\mathbb{C})$ and $\mathsf{Pt}_{X'}(\mathbb{C})$. In addition, since any commutative square is a pullback, we have $R' = y^*(R)$, and also $i_{R'} = x^*(i_R)$ in the following diagram:



Now suppose we have another equivalence relation W on (f,s) in $\mathsf{Pt}_Y(\mathbb{C})$ with $W' = y^*(W)$ such that $[W',R'] = 0$ in $\mathsf{Pt}_{Y'}(\mathbb{C})$. This last property is equivalent to the commutation of the normal monomorphisms $i_{W'} = x^*(i_W)$ and $i_{R'} = x^*(i_R)$ in the fibre $\mathsf{Pt}_{X'}(\mathbb{C})$. Since the category $\mathbb{C}$ satisfies condition (C), the normal monomorphisms i_W and i_R commute in the fibre $\mathsf{Pt}_X(\mathbb{C})$ which means that we have $[W,R] = 0$ in $\mathsf{Pt}_Y(\mathbb{C})$. □

Even though the condition ($\bar{\mathrm{C}}$) may be weaker than (C), it is certainly not automatically satisfied, as shows the following result.

Proposition 2.3. *Let $\mathbb{C}$ be a finitely complete pointed regular Mal'tsev category. Condition ($\bar{\mathrm{C}}$) implies that in $\mathbb{C}$, all extensions with abelian kernel are abelian extensions.*

Proof. We first consider the case of split epimorphisms. Let $(f,s)\colon X \rightleftarrows Y$ be an object in $\mathsf{Pt}_Y(\mathbb{C})$ such that the kernel K of f is abelian, meaning that the discrete equivalence relation Δ_K on K centralises itself. Then by ($\bar{\mathrm{C}}$) the kernel relation $R[f]$ of f—the relation associated to the kernel pair—centralises itself, which means that the extension f is abelian.

If now $g\colon Y \to Z$ is an extension with abelian kernel, i.e., a regular epimorphism in $\mathbb{C}$ of which the kernel K is abelian, then the kernel pair projection $g_0\colon R[g] \to Y$ is an abelian extension by the above. Hence also g is abelian by [10, Proposition 4.1]. □

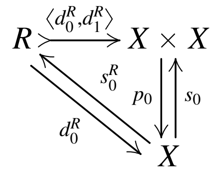
As a consequence, the counterexample from [6] in the category of digroups shows that a category may be semi-abelian without satisfying ($\bar{\mathrm{C}}$).

2.2 The characterisation

We are now ready for the characterisation:

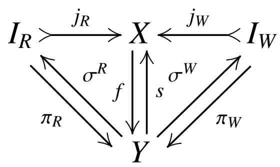
Theorem 2.1. *Let* $\mathbb{C}$ *be a finitely complete pointed Mal'tsev category. The condition* (C) *is equivalent to the "Smith is Huq" condition* (SH).

Proof. The normalisation in $\mathbb{C}$ of an equivalence relation R on X is the image by the change of base along the initial morphism $\alpha_X \colon 1 \rightarrowtail X$ of the normal local representation in $\mathsf{Pt}_X(\mathbb{C})$:

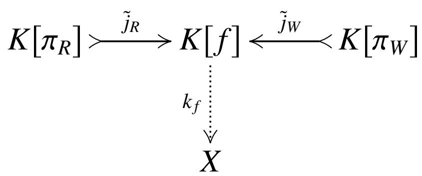


So when $\mathbb{C}$ satisfies condition (C), we have $[R, W] = 0$, i.e., the local representations of R and W commute in $\mathsf{Pt}_X(\mathbb{C})$ as soon as their normalisations commute in $\mathbb{C}$.

Conversely, suppose that the condition (SH) holds. Let $(f, s) \colon X \rightleftarrows Y$ be an object in $\mathsf{Pt}_Y(\mathbb{C})$ and (R, W) a pair of equivalence relations on it. Denote by j_R and j_W their normalisations in $\mathsf{Pt}_Y(\mathbb{C})$:



Supposing that their images by some change of base functor y^* commute implies that their images $\tilde{j}_R$ and $\tilde{j}_W$ by α_Y^*—that is to say, the respective kernels in the diagram below—commute in $\mathbb{C}$:



Accordingly the two monomorphisms $k_f \circ \tilde{j}_R$ and $k_f \circ \tilde{j}_W$ do commute in $\mathbb{C}$. But $k_f \circ \tilde{j}_R$ and $k_f \circ \tilde{j}_W$ are the normalisations of R and W in $\mathbb{C}$. Now, since $\mathbb{C}$

satisfies (SH), then we get $[R, W] = 0$ in $\mathbb{C}$ and thus in $\mathsf{Pt}_Y(\mathbb{C})$, which implies that their normalisations j_R and j_W in $\mathsf{Pt}_Y(\mathbb{C})$ commute. □

Corollary 2.1. *If $\mathbb{C}$ is a finitely complete Mal'tsev category which satisfies* (C)*, then any fibre $\mathsf{Pt}_Y(\mathbb{C})$ satisfies* (SH)*. When, in addition, $\mathbb{C}$ is pointed, if it satisfies* (SH)*, then so does any fibre $\mathsf{Pt}_Y(\mathbb{C})$.*

Proof. This is a straightforward consequence of the previous theorem and of Proposition 2.1. □

2.3 Topological Mal'tsev models

Let $\mathbb{T}$ be a (finitary) Mal'tsev theory, $\mathsf{Set}^{\mathbb{T}}$ the corresponding variety of $\mathbb{T}$-algebras and $\mathsf{Top}^{\mathbb{T}}$ the category of topological $\mathbb{T}$-algebras. Recall that $\mathsf{Top}^{\mathbb{T}}$ is then a regular Mal'tsev category, see [18], whose regular epimorphisms are the open surjective morphisms. It is clearly finitely complete and cocomplete. In this section we shall show that when the variety $\mathsf{Set}^{\mathbb{T}}$ satisfies condition (C), so does $\mathsf{Top}^{\mathbb{T}}$. In particular, this will imply the well-known fact that the category $\mathsf{Gp}(\mathsf{Top})$ of topological groups (= $\mathsf{Top}^{\mathbb{T}}$ for $\mathbb{T}$ the theory of groups) satisfies (SH).

To see this, let us first recall that the functor $U\colon \mathsf{Top}^{\mathbb{T}} \to \mathsf{Set}^{\mathbb{T}}$ forgetting the topological data is topological [23] and, consequently, left exact. Hence it is cotopological [2, Proposition 7.3.6] and, consequently, right exact. This implies that the functor U is faithful.

Lemma 2.1. *Let $\mathbb{T}$ be a Mal'tsev theory and the following diagram a pullback of split epimorphisms in $\mathsf{Top}^{\mathbb{T}}$:*

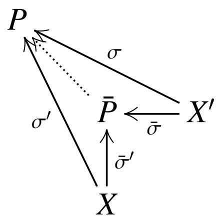
$$\begin{array}{ccc}
P & \overset{\sigma}{\underset{\phi}{\leftrightarrows}} & X' \\
{\scriptstyle \phi'}\downarrow\uparrow{\scriptstyle \sigma'} & & {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} \\
X & \overset{s}{\underset{f}{\leftrightarrows}} & Y
\end{array}$$

then P is endowed with the final topology with respect to the pair

$$\begin{array}{ccc}
U(P) & \overset{U(\sigma)}{\longleftarrow} & U(X'), \\
{\scriptstyle U(\sigma')}\uparrow & & \\
U(X) & &
\end{array}$$

namely, P is the $\mathbb{T}$-algebra $U(P)$ endowed with the finest topology making $U(P)$ a topological algebra and the pair $(U(\sigma), U(\sigma'))$ a pair of continuous homomorphisms.

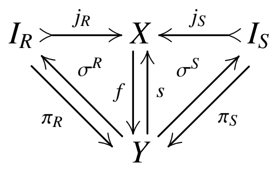
Proof. Since the functor U is cotopological, we can endow the $\mathbb{T}$-algebra $U(P)$ with the final topology with respect to the pair in question. This defines the object $\bar{P}$ and the following lower diagram in $\mathsf{Top}^{\mathbb{T}}$ above the given pair:



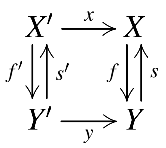
By the universal property of the final topology, there exists a factorisation $\iota\colon \bar{P} \to P$ making the diagram above commute. In other words, the topology $\bar{P}$ on $U(P)$ is finer than the topology P and $\iota = 1_{U(P)}\colon \bar{P} \to P$ is continuous. This morphism is clearly a monomorphism in $\mathsf{Top}^{\mathbb{T}}$. Now $\mathsf{Top}^{\mathbb{T}}$ is a Mal'tsev category, the fibre $\mathsf{Pt}_Y(\mathsf{Top}^{\mathbb{T}})$ is unital, and the pair (σ, σ') is jointly strongly epic, which implies that the monomorphism ι is an isomorphism, and means that the topologies P and $\bar{P}$ on $U(P)$ coincide. □

Proposition 2.4. *Let $\mathbb{T}$ be a Mal'tsev theory such that the variety $\mathsf{Set}^{\mathbb{T}}$ satisfies condition* (C). *Then so does the category $\mathsf{Top}^{\mathbb{T}}$.*

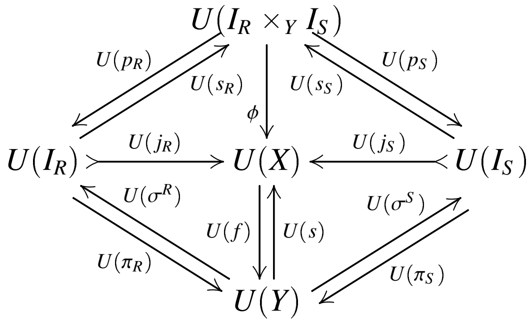
Proof. Let us consider the following pair of normal monomorphisms in the fibre $\mathsf{Pt}_Y(\mathsf{Top}^{\mathbb{T}})$:



and the following pullback in $\mathsf{Top}^{\mathbb{T}}$:



Suppose that $y^*(j_R)$ and $y^*(j_S)$ commute in the fibre $\mathsf{Pt}_{Y'}(\mathsf{Top}^{\mathbb{T}})$; then this is the case for their images by the functor U. Since $\mathsf{Set}^{\mathbb{T}}$ satisfies condition (C), the images $U(j_R)$ and $U(j_S)$ commute in $\mathsf{Set}^{\mathbb{T}}$. This means that there is a $\mathbb{T}$-homomorphism ϕ such that $\phi \circ U(s_R) = U(j_R)$ and $\phi \circ U(s_S) = U(j_S)$ in the following diagram, where the whole quadrangle is the image by U of a pullback of split epimorphisms in $\mathsf{Top}^{\mathbb{T}}$:



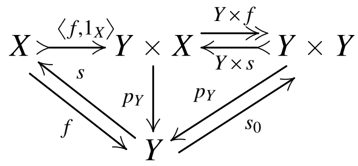
This means that the "restrictions" $\phi \circ U(s_R)$ and $\phi \circ U(s_S)$ of the $\mathbb{T}$-homomorphism ϕ to the "subobjects" $U(I_R)$ and $U(I_S)$ are the continuous $\mathbb{T}$-homomorphisms j_R and j_S. By the previous lemma, $I_R \times_Y I_S$ is endowed with the final topology with respect to the pair $U(s_R)$ and $U(s_S)$, which implies that the $\mathbb{T}$-homomorphism ϕ is itself continuous: $I_R \times_Y I_S \to X$ and actually lies in $\mathsf{Top}^{\mathbb{T}}$. This means precisely that j_R and j_S commute in the fibre $\mathsf{Pt}_Y(\mathsf{Top}^{\mathbb{T}})$. $\square$

3 Applications of the condition (SH)

In this section we shall extend some results known for strongly protomodular categories to a context merely satisfying the condition (SH).

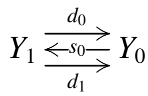
3.1 Discrete fibrations of reflexive graphs

First observe that in a finitely complete category $\mathbb{E}$, any split epimorphism $(f, s)\colon X \rightleftarrows Y$ is actually the domain of the kernel of a split epimorphism in the pointed fibre $\mathsf{Pt}_Y(\mathbb{E})$:

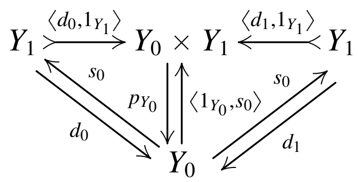


and thus it produces the normal monomorphism $\langle f, 1_X\rangle$ in $\mathsf{Pt}_Y(\mathbb{E})$.

On the other hand, it is known from [14] that, in a Mal'tsev category $\mathbb{C}$, a reflexive graph is endowed with at most one structure of internal category, and that any internal category is a groupoid. Let us recall (from [9]) another proof of this result which sheds a new light on the nature of the uniqueness of the groupoid structure. From any reflexive graph



in $\mathbb{C}$, we get two normal subobjects in $\mathsf{Pt}_{Y_0}(\mathbb{C})$:

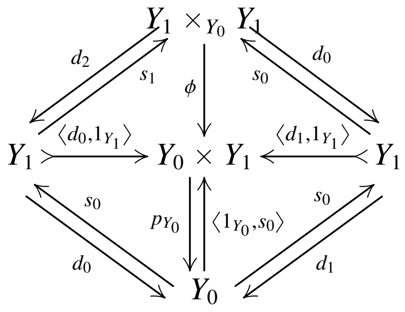


We can now assert the following:

Proposition 3.1. *Let $\mathbb{C}$ be a finitely complete Mal'tsev category. The reflexive graph in question is a groupoid if and only if these two normal subobjects commute in $\mathsf{Pt}_{Y_0}(\mathbb{C})$.*

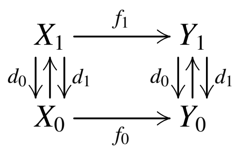
Proof. The two subobjects commute in $\mathsf{Pt}_{Y_0}(\mathbb{C})$ if and only if they have a cooperator $\phi\colon Y_1 \times_{Y_0} Y_1 \to Y_0 \times Y_1$, i.e. a morphism satisfying $\phi\circ s_0 \not\preccurlyeq\ d_1, 1_{Y_1}\rangle$

and $\phi \circ s_1 = \langle d_0, 1_{Y_1} \rangle$:



where the whole quadrangle is a pullback in $\mathbb{C}$. Hence the morphism ϕ is a pair $\langle d_0 \circ d_2, d_1 \rangle$, where $d_1 \colon Y_1 \times_{Y_0} Y_1 \to Y_1$ is such that $d_1 \circ s_0 = 1_{Y_1}$ and $d_1 \circ s_1 = 1_{Y_1}$. Since the morphism d_1 satisfies these two identities, it makes the reflexive graph in question *multiplicative* in the sense of [14]. And, according to Theorem 2.2 in [14], in a Mal'tsev category, any multiplicative reflexive graph is a groupoid. Conversely, the composition morphism $d_1 \colon Y_1 \times_{Y_0} Y_1 \to Y_1$ of an internal category satisfies the previous two identities and produces the cooperator $\phi = \langle d_0 \circ d_2, d_1 \rangle$. □

Now let us consider a morphism of reflexive graphs



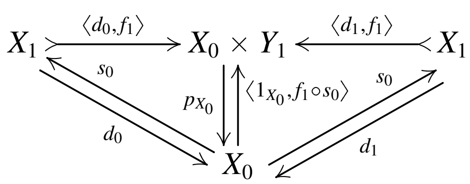
and recall the following result from [3, Proposition 14]:

Proposition 3.2. *When $\mathbb{C}$ is a finitely complete Mal'tsev category, then, given any morphism of reflexive graphs as above, the square indexed by* 0 *is a pullback if and only if the square indexed by* 1 *is a pullback. In such a situation this morphism is said to be a* discrete fibration *between reflexive graphs.*

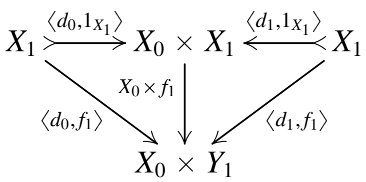
We can now extend a result already known in strongly protomodular categories, see [4, Consequence B, p. 216]:

Proposition 3.3. *Let $\mathbb{C}$ be a finitely complete Mal'tsev category satisfying condition* (C)*. Given any discrete fibration of reflexive graphs, the codomain reflexive graph Y_1 is a groupoid as soon as so is the domain reflexive graph X_1.*

Proof. Since $\mathbb{C}$ satisfies condition (C), it is enough to show that the images under the change of base functor along f_0 of the two normal monomorphisms associated with the codomain reflexive graph Y_1 do commute in the fibre $\mathsf{Pt}_{X_0}(\mathbb{C})$. The two images in question are the following ones:



since the morphism of reflexive graphs is a discrete fibration. They do commute in $\mathsf{Pt}_{X_0}(\mathbb{C})$, being given by the following composition in this fibre, where the horizontal part commutes since the reflexive graph X_1



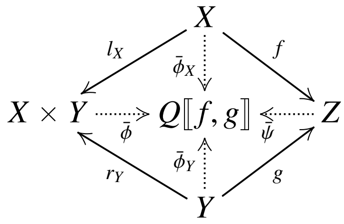
is a groupoid. □

3.2 The condition (SH) and commutators

In this section we shall prove that, as expected, in the Mal'tsev context, under (SH) the Smith and the Huq commutators in the sense of [6] do coincide.

3.2.1 The Huq commutator in a unital category

We shall suppose here that $\mathbb{C}$ is a unital category which is moreover finitely cocomplete. In this context, in [6] there was given a construction, for any pair $f\colon X \to Z$, $g\colon Y \to Z$ of morphisms with the same codomain, of a morphism which universally makes them cooperate. Indeed consider the following diagram, where $Q[\![f,g]\!]$ is the colimit of the diagram made of the plain arrows:



Clearly the morphisms $\bar{\phi}_X$ and $\bar{\phi}_Y$ are completely determined by the pair $(\bar{\phi}, \bar{\psi})$, and clearly the morphism $\bar{\phi}$ is the cooperator of the pair $(\bar{\psi}\circ f, \bar{\psi}\circ g)$. On the other hand, the strong epimorphism $\bar{\psi}$ measures how far the pair (f,g) is from cooperating, and we have [6]:

Proposition 3.4. *Suppose $\mathbb{C}$ finitely cocomplete and unital. Then $\bar{\psi}$ is the universal morphism which, by composition, makes the pair (f,g) cooperate. The morphism $\bar{\psi}$ is an isomorphism if and only if the pair (f,g) cooperates.*

Since the morphism $\bar{\psi}$ is a strong epimorphism, its distance from being an isomorphism is its distance from being a monomorphism, which is exactly measured by its kernel relation $R[\bar{\psi}]$, whence the following definition:

Definition 3.1 (Huq commutator)**.** Given any pair (f,g) of morphisms with the same codomain in a finitely cocomplete unital category $\mathbb{C}$, their *Huq commutator* $[\![f,g]\!]$ is the kernel relation $R[\bar{\psi}]$.

When the category $\mathbb{C}$ is moreover regular [1], i.e., such that the strong epimorphisms are stable by pullback and any effective equivalence relation (= kernel pair) admits a quotient, we can add some piece of information. First, any morphism $f\colon X \to Z$ has a canonical regular epi/mono factorisation $X \twoheadrightarrow f(X) \rightarrowtail Z$, and the morphism $f(X) \rightarrowtail Z$ is then called the *image* of the morphism f. Secondly, two morphisms f and g cooperate if and only if their images $f(X) \rightarrowtail Z$ and $g(Y) \rightarrowtail Z$ do.

3.2.2 The Smith commutator in a Mal'tsev category

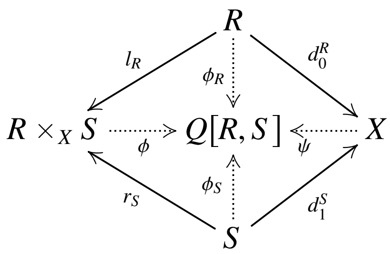
We shall suppose here that $\mathbb{C}$ is finitely complete and cocomplete, regular Mal'tsev category. In a regular Mal'tsev category, given a regular epimorphism $f\colon X \twoheadrightarrow Y$, any equivalence relation R on X has a *direct image* $f(R)$

along f on Y. It is given by the regular epi/mono factorisation of the morphism

$$\langle f\circ d_0, f\circ d_1\rangle\colon R \twoheadrightarrow f(R) \rightarrowtail Y \times Y$$

Clearly in any regular category $\mathbb{C}$, the relation $f(R)$ is reflexive and symmetric; when moreover $\mathbb{C}$ is Mal'tsev, $f(R)$ is an equivalence relation.

Now let us recall the following results and definition from [6]: first consider the following diagram, in which $Q[R,S]$ is the colimit of the plain arrows:



Notice that, here, in consideration of the pullback defining $R \times_X S$ (diagram $(*)$), the roles of the projections d_0 and d_1 have been interchanged. As in the section above, the morphisms ϕ_R and ϕ_S are completely determined by the pair (ϕ, ψ) and the morphism ψ is a strong epimorphism (and thus a regular epimorphism in our regular context). This morphism ψ measures how far the equivalence relations R and S are from centralising each other:

Proposition 3.5. *Let $\mathbb{C}$ be a finitely complete and cocomplete, regular Mal'tsev category. The morphism ψ is the universal regular epimorphism which makes the direct images $\psi(R)$ and $\psi(S)$ centralise each other (i.e. $[R,S]=0$). The equivalence relations R and S centralise each other if and only if ψ is an isomorphism.*

Since the morphism ψ is a regular epi, its distance from being an isomorphism is its distance from being a monomorphism, which is exactly measured by its kernel relation $R[\psi]$. Accordingly, it is meaningful to introduce the following definition:

Definition 3.2 (Smith commutator)**.** Let $\mathbb{C}$ be a finitely complete and cocomplete, regular Mal'tsev category. Consider in $\mathbb{C}$ two equivalence relations $(d_0^R, d_1^R)\colon R \rightrightarrows X$ and $(d_0^S, d_1^S)\colon S \rightrightarrows X$ on the same object X. The kernel relation $R[\psi]$ of the morphism ψ is called the *Smith commutator* of R and S. We shall use the classical notation $[R,S]$ for this commutator.

Example 3.1. If we suppose moreover that the category $\mathbb{C}$ is Barr exact [1]—namely such that any equivalence relation is effective, i.e., the kernel relation of some morphism—then, thanks to Theorem 3.9 in [21], the previous definition is equivalent to the definition of [21], and accordingly to the definition of Smith [22] in the Mal'tsev-varietal context. On the other hand, one of the advantages of this definition is that it extends the meaning and existence of commutator from the exact Mal'tsev context to the regular Mal'tsev context, enlarging the range of examples to the Mal'tsev quasi-varieties and to the topological Mal'tsev models, as the category $\mathsf{Gp}(\mathsf{Top})$ of topological groups for instance.

The example $\mathsf{Gp}(\mathsf{Top})$ is also interesting for the following reason. In a Mal'tsev category $\mathbb{C}$, given any pair (R, S) of equivalence relations on an object X, we obtain $[R, S] = 0$ as soon as the intersection $R \cap S$ is the discrete equivalence relation Δ_X. When the Mal'tsev $\mathbb{C}$ is not only regular, but also exact (which means that any equivalence relation is *effective*, i.e., the kernel relation of some morphism), this implies that we necessarily have $[R, S] \leqslant R \cap S$. Indeed, since $\mathbb{C}$ is exact, we may take the quotient $q\colon X \twoheadrightarrow Q$ of the equivalence relation $R \cap S$; then we see that $q(R) \cap q(S) = \Delta_Q$ as in any regular category. Accordingly, $[q(R), q(S)] = 0$. When, in addition, $\mathbb{C}$ is finitely cocomplete Mal'tsev, we have a factorisation $\xi\colon Q[R, S] \twoheadrightarrow Q$ and the inclusion $[R, S] \leqslant R \cap S$. The regular (but not exact) Mal'tsev category $\mathsf{Gp}(\mathsf{Top})$ provides a setting in which this inclusion does not hold: see [7, Proposition 5.3].

3.3 Commutators in the pointed Mal'tsev setting

From now on $\mathbb{C}$ will be a regular pointed Mal'tsev category. Recall from [6] that, on the one hand, if $f\colon X \twoheadrightarrow Y$ is a regular epimorphism and R an equivalence relation on X, then the normal subobject $j(f(R))$ associated with $f(R)$ is the direct image $f(j(R))$ along f of the normal subobject $j(R)$ associated with R. On the other hand, we get:

Proposition 3.6. *Let $\mathbb{C}$ be a finitely complete and cocomplete, regular, pointed Mal'tsev category. Then, given any pair (R, S) of equivalence relations on an object X, there is a natural comparison $\zeta\colon Q[\![j(R), j(S)]\!] \twoheadrightarrow Q[R, S]$,* *and consequently we have* $[\![j(R), j(S)]\!] \leq [R, S]$*, namely an inclusion of the Huq commutator into the Smith commutator.*

Proof. Consider the morphism $\psi\colon X \twoheadrightarrow Q[R,S]$. We have $[\psi(R), \psi(S)] = 0$, so that

$$[\![\psi(j(R)), \psi(j(S))]\!] = [\![j(\psi(R)), j(\psi(S))]\!] = 0.$$

Hence the two morphisms $\psi \circ j(R)$ and $\psi \circ j(S)$ commute. Now thanks to the universal property of the morphism $\bar{\psi}\colon X \twoheadrightarrow Q[\![j(R), j(S)]\!]$, there is a unique factorisation $\zeta\colon Q[\![j(R), j(S)]\!] \to Q[R,S]$ such that $\zeta \circ \bar{\psi} = \psi$, and thus an inclusion $[\![j(R), j(S)]\!] \leqslant [R,S]$ of the Huq commutator into the Smith commutator. □

Exactly in the same way as for strongly protomodular categories [6], we can now assert:

Theorem 3.1. *Let* $\mathbb{C}$ *be a finitely complete and cocomplete, pointed and regular Mal'tsev category satisfying* (SH)*. Then, given any pair* (R,S) *of equivalence relations on an object X, the natural comparison* $\zeta\colon Q[\![j(R), j(S)]\!] \to Q[R,S]$ *is an isomorphism, and consequently we have* $[\![j(R), j(S)]\!] = [R,S]$*, namely the Smith and the Huq commutators coincide.*

Proof. Consider the morphism $\bar{\psi}\colon X \twoheadrightarrow Q[\![j(R), j(S)]\!]$. Then we get:

$$[\![j(\bar{\psi}(R)), j(\bar{\psi}(S))]\!] = [\![\bar{\psi}(j(R)), \bar{\psi}(j(S))]\!] = 0$$

Now thanks to condition (SH), we have that $[\bar{\psi}(R), \bar{\psi}(S)] = 0$. Then the universal property of the morphism $\psi\colon X \twoheadrightarrow Q[R,S]$ produces a unique factorisation $\theta\colon Q[R,S] \to Q[\![j(R), j(S)]\!]$ which is necessarily an inverse of ζ (see Proposition 3.6), and thus an isomorphism $[R,S] \cong [\![j(R), j(S)]\!]$. Hence the two notions of commutator coincide. □

References

[1] M. Barr, *Exact categories*, Exact categories and categories of sheaves, Lecture Notes in Math., vol. 236, Springer, 1971, pp. 1–120.

[2] F. Borceux, *Handbook of categorical algebra 2: Categories and structures*, Encyclopedia Math. Appl., vol. 51, Cambridge Univ. Press, 1994.

[3] D. Bourn, *Mal'cev categories and fibration of pointed objects*, Appl. Categ. Structures **4** (1996), 307–327.

[4] D. Bourn, *Normal functors and strong protomodularity*, Theory Appl. Categ. **7** (2000), no. 9, 206–218.

[5] D. Bourn, *Intrinsic centrality and associated classifying properties*, Journal of Algebra, 256 , 2002, 126-145.

[6] D. Bourn, *Commutator theory in strongly protomodular categories*, Theory Appl. Categ. **13** (2004), no. 2, 27–40.

[7] D. Bourn, *Normal subobjects of topological groups and topological semi-abelian algebras*, Topology Appl. **153** (2006), 1341–1364.

[8] D. Bourn, *Two ways to centralizers of equivalence relations*, Appl. Categ. Structures, accepted for publication, 2012.

[9] D. Bourn, *On the monad of internal groupoids*, Theory Appl. Categ. **28** (2013), no. 5, 150–165.

[10] D. Bourn and M. Gran, *Centrality and connectors in Maltsev categories*, Algebra Universalis **48** (2002), 309–331.

[11] D. Bourn and M. Gran, *Centrality and normality in protomodular categories*, Theory Appl. Categ. **9** (2002), no. 8, 151–165.

[12] D. Bourn and J. R. A. Gray, *Aspects of algebraic exponentiation*, Bull. Belg. Math. Soc. Simon Stevin **19** (2012), 823–846.

[13] A. Carboni, J. Lambek, and M. C. Pedicchio, *Diagram chasing in Mal'cev categories*, J. Pure Appl. Algebra **69** (1991), 271–284.

[14] A. Carboni, M. C. Pedicchio, and N. Pirovano, *Internal graphs and internal groupoids in Mal'cev categories*, Proceedings of Conf. Category Theory 1991, Montreal, Am. Math. Soc. for the Canad. Math. Soc., Providence, 1992, pp. 97–109.

[15] M. Gran, G. Janelidze, and A. Ursini, *Weighted commutators in semi-abelian categories*, J. Algebra, accepted for publication, 2013.

[16] M. Hartl and T. Van der Linden, *The ternary commutator obstruction for internal crossed modules*, Adv. Math. **232** (2013), no. 1, 571–607.

[17] S. A. Huq, *Commutator, nilpotency and solvability in categories*, Q. J. Math. **19** (1968), no. 2, 363–389.

[18] P. T. Johnstone and M. C. Pedicchio, *Remarks on continuous Mal'cev algebras*, Rend. Istit. Mat. Univ. Trieste **25** (1995), 277–297.

[19] N. Martins-Ferreira and T. Van der Linden, *A note on the "Smith is Huq" condition*, Appl. Categ. Structures **20** (2012), no. 2, 175–187.

[20] N. Martins-Ferreira and T. Van der Linden, *A decomposition formula for the weighted commutator*, Appl. Categ. Structures, accepted for publication, 2013.

[21] M. C. Pedicchio, *A categorical approach to commutator theory*, J. Algebra **177** (1995), 647–657.

[22] J. D. H. Smith, *Mal'cev varieties*, Lecture Notes in Math., vol. 554, Springer, 1976.

[23] O. Wyler, *On the categories of general topology and topological algebra*, Arch. Math. (Basel) **22** (1971), no. 1, 7–17.

Dominique Bourn
Université du Littoral, Laboratoire de Mathématiques Pures et Appliquées
B.P. 699, 62228 Calais Cedex, France.
Email: *bourn@lmpa.univ-littoral.fr*

Nelson Martins-Ferreira
Escola Superior de Tecnologia e Gestão
Centro para o Desenvolvimento Rápido e Sustentado do Produto
Instituto Politécnico de Leiria
Leira, Portugal.
Email: *martins.ferreira@ipleira.pt*

Tim Van der Linden
Université catholique de Louvain
Institut de Recherche en Mathématique et Physique
Chemin du Cyclotron 2 bte L7.01.02, 1348 Louvain-la-Neuve, Belgium.
Email: *tim.vanderlinden@uclouvain.be*

THE FUNDAMENTAL GROUP FUNCTOR AS A KAN EXTENSION

by Tomas EVERAERT, Julia GOEDECKE and Tim VAN DER LINDEN

dedicated to René Guitart on the occasion of his sixty-fifth birthday

Résumé. On montre que le foncteur *groupe fondamental* considéré en théorie de Galois catégorique peut être calculé comme une extension de Kan.

Abstract. We prove that the fundamental group functor from categorical Galois theory may be computed as a Kan extension.

Keywords. Homology, categorical Galois theory, semi-abelian category, higher central extension, satellite, Kan extension, fundamental group

Mathematics Subject Classification (2010). 18G50, 18G60, 18G15, 20J, 55N

1 Introduction

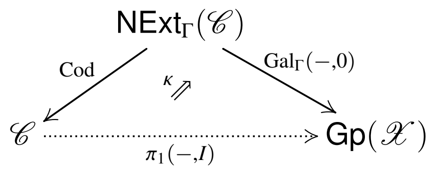
The main aim of this paper is to prove that the fundamental group from categorical Galois theory [20] may be computed as a Kan extension:

$$\begin{array}{ccc} & \mathsf{NExt}_{\Gamma}(\mathscr{C}) & \\ {\scriptstyle \mathrm{Cod}} \swarrow & \overset{\delta}{\Uparrow} & \searrow {\scriptstyle \mathrm{Ker}} \\ \mathscr{C} & \xrightarrow[\pi_1(-,I)]{\cdots\cdots} & \mathscr{X} \end{array} \qquad \textbf{(A)}$$

The first author's research was supported by Fonds voor Wetenschappelijk Onderzoek (FWO-Vlaanderen). The third author works as *chargé de recherches* for Fonds de la Recherche Scientifique–FNRS. Both would like to thank the DPMMS for its kind hospitality during their stays in Cambridge.

This makes it a *satellite* in the sense of Janelidze [17], Guitart–Van den Bril [13, 12] and two authors of the present paper [10]. Here Γ is a Galois structure, consisting of an adjunction $I \dashv H \colon \mathscr{C} \to \mathscr{X}$ and certain classes of morphisms, $\mathsf{NExt}_\Gamma(\mathscr{C})$ is the category of *normal extensions*, which are defined via the Galois structure Γ, Ker is the kernel functor and Cod is the codomain functor.

In fact, we will see this in two steps. First we show that the following is a Kan extension:



(B)

Here $\mathrm{Gal}_\Gamma(-,0)$ gives the Galois group of a normal extension, as defined in the context of categorical Galois theory by Janelidze [20]. This step uses that the Galois group functor is a *Baer invariant* with respect to the codomain functor, in the following sense: any two morphisms between objects in $\mathsf{NExt}_\Gamma(\mathscr{C})$ which agree on the codomain of the objects are sent to the *same* morphism between the Galois groups. This makes it possible to define $\pi_1(B,I)$ by taking a *weakly universal* normal extension $u \colon U \to B$ of B, and then applying the Galois group functor to it. The above property ensures that this assignment is well defined, i.e. independent of the choice of u, and functorial in B.

To attain the first-mentioned Kan extension from this one, we use the fact that the underlying object of the Galois group of a normal extension $p \colon E \to B$ can be computed as the intersection of the kernel of p with the kernel of the unit $\eta_E \colon E \to HI(E)$. This makes it a subobject of $\mathrm{Ker}(p)$, and so gives a component-wise monic natural transformation $\iota \colon \mathrm{Gal}_\Gamma(-,0) \Rightarrow \mathrm{Ker}$. We then show that, for any given functor $F \colon \mathscr{C} \to \mathscr{X}$, any natural transformation $F \circ \mathrm{Cod} \Rightarrow \mathrm{Ker}$ lifts over this ι. This implies that the universal property of the Kan extension (**B**) carries over to (**A**).

Our arguments go through under fairly weak assumptions on the Galois structure Γ, and can moreover be adapted to situations where the fundamental group functor is not everywhere defined. In the latter case, we obtain a Kan extension similar to (**A**) and (**B**), by replacing $\mathscr{C}$ with its full subcategory of objects B for which $\pi_1(B,I)$ *is* defined, and restrict $\mathsf{NExt}_\Gamma(\mathscr{C})$

accordingly.

When $\mathscr{C}$ is pointed, exact and Mal'tsev, and $\mathscr{X}$ is a Birkhoff subcategory of $\mathscr{C}$, we show that (**A**) induces a Kan extension

$$\begin{array}{ccc} & \mathsf{Ext}_\Gamma(\mathscr{C}) & \\ {}^{\mathrm{Cod}}\swarrow & \Uparrow & \searrow^{\mathrm{Ker}\circ I_1} \\ \mathscr{C} & \xrightarrow[\pi_1(-,I)]{\cdots\cdots\cdots} & \mathscr{X} \end{array}$$

where $\mathsf{Ext}_\Gamma(\mathscr{C})$ is the category of regular epimorphisms (=extensions) in $\mathscr{C}$, and I_1 is left adjoint to the inclusion functor $\mathsf{NExt}_\Gamma(\mathscr{C}) \to \mathsf{Ext}_\Gamma(\mathscr{C})$. In the case of a semi-abelian $\mathscr{C}$, this Kan extension was first obtained in [10], where it was also shown that, for a given extension p, the p-component of the universal natural transformation defining it is a connecting homomorphism in the long exact homology sequence induced by p.

The latter result, we will see, has a topological counterpart: for a certain Galois structure, the components of the universal natural transformation δ defining the Kan extension (**A**) (or, actually, the "restricted" version, since here the fundamental group functor is not everywhere defined) are connecting maps in an exact *homotopy* sequence.

Note that we have used the same notation $\pi_1(-,I)$ for functors $\mathscr{C} \to \mathsf{Gp}(\mathscr{X})$ and $\mathscr{C} \to \mathscr{X}$ and have called both "fundamental group functor", while the image of an object $B \in |\mathscr{C}|$ under the latter is actually the *underlying object* of the fundamental group $\pi_1(B,I)$. A similar remark can be made regarding the Galois group functor $\mathrm{Gal}_\Gamma(-,0)$. This does not pose any problems when $\mathscr{X}$ is Mal'tsev, since then any internal group is determined, up to isomorphism, by its underlying object. However, the latter is of course not true in general, and it is in particular false for the topological example just referred to.

2 Galois structures

To define the ingredients of the Kan extensions considered in this paper, we need a *Galois structure* and the concept of *normal extension* arising from it, as introduced by Janelidze [18, 19].

Definition 2.1. A **Galois structure** $\Gamma = (\mathscr{C}, \mathscr{X}, H, I, \eta, \epsilon, \mathscr{E}, \mathscr{F})$ on a category $\mathscr{C}$ consists of an adjunction

$$\mathscr{C} \underset{H}{\overset{I}{\rightleftarrows}} \mathscr{X} \quad (I \dashv H)$$

with unit $\eta\colon 1_{\mathscr{C}} \Rightarrow HI$ and counit $\epsilon\colon IH \Rightarrow 1_{\mathscr{X}}$, as well as classes of morphisms $\mathscr{E}$ in $\mathscr{C}$ and $\mathscr{F}$ in $\mathscr{X}$ such that

(i) $\mathscr{E}$ and $\mathscr{F}$ contain all isomorphisms;

(ii) $\mathscr{E}$ and $\mathscr{F}$ are pullback-stable, meaning here that the pullback of a morphism in $\mathscr{E}$ (resp. $\mathscr{F}$) along any morphism *exists* and is in $\mathscr{E}$ (resp. $\mathscr{F}$);

(iii) $\mathscr{E}$ and $\mathscr{F}$ are closed under composition;

(iv) $H(\mathscr{F}) \subseteq \mathscr{E}$;

(v) $I(\mathscr{E}) \subseteq \mathscr{F}$.

We will use the terminology of [19] and call the morphisms in $\mathscr{E}$ **fibrations**.

Given such a Galois structure, some fibrations have some additional useful and interesting properties. We write $(\mathscr{E} \downarrow B)$ for the full subcategory of the slice category $(\mathscr{C} \downarrow B)$ determined by morphisms in $\mathscr{E}$.

Definition 2.2. A **trivial covering** is a morphism $f\colon A \to B$ in $\mathscr{E}$ such that

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HI(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle HI(f)} \\ B & \xrightarrow[\eta_B]{} & HI(B) \end{array}$$

is a pullback. A **monadic extension** is a fibration $p\colon E \to B$ such that the pullback functor $p^*\colon (\mathscr{E} \downarrow B) \to (\mathscr{E} \downarrow E)$ is monadic. A **covering** (sometimes called **central extension**) is a fibration $f\colon A \to B$ whose pullback $p^*(f)$ along *some* monadic extension p is trivial. A **normal extension** is a monadic extension p such that $p^*(p)$ is a trivial covering, i.e. a monadic extension with trivial kernel pair projections.

The trivial coverings are exactly those fibrations which are cartesian with respect to the functor $I\colon \mathscr{C} \to \mathscr{X}$.

For many uses of such Galois structures, we need Γ to satisfy an extra property called *admissibility*. For this we consider the induced adjunction

$$(\mathscr{E} \downarrow B) \underset{H^B}{\overset{I^B}{\rightleftarrows}} (\mathscr{F} \downarrow I(B)) \quad (\bot)$$

for any object $B \in \mathscr{C}$; here $I^B\colon (\mathscr{E} \downarrow B) \to (\mathscr{F} \downarrow I(B))$ is the restriction of I, and H^B sends a fibration $g\colon X \to I(B)$ to the pullback of $H(g)$ along η_B:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & H(X) \\ {\scriptstyle H^B(g)}\downarrow & \lrcorner & \downarrow{\scriptstyle H(g)} \\ B & \xrightarrow[\eta_B]{} & HI(B) \end{array}$$

Definition 2.3. A Galois structure $\Gamma = (\mathscr{C}, \mathscr{X}, H, I, \eta, \epsilon, \mathscr{E}, \mathscr{F})$ is **admissible** when all functors H^B are full and faithful.

An important consequence of admissibility is

Lemma 2.4. *[22, Proposition 2.4] If Γ is admissible, then $I\colon \mathscr{C} \to \mathscr{X}$ preserves pullbacks along trivial coverings. In particular, the trivial coverings are pullback-stable.* □

So if the Galois structure is admissible, we can view the class of all trivial coverings as the pullback-closure of $H(\mathscr{F})$, while the coverings are *locally trivial*. In certain situations the coverings are also pullback-stable:

Lemma 2.5. *If Γ is admissible and monadic extensions are pullback-stable, then normal extensions and coverings are pullback-stable.*

Proof. The proof of [21, Proposition 4.3] remains valid under our assumptions. □

Examples 2.6. There are many different kinds of categorical Galois structures; we list a few which are relevant for the present article.

(i) Take $\mathscr{C} = \mathsf{Gp}$ and $\mathscr{X} = \mathsf{Ab}$, the subcategory of abelian groups in the category of groups, and let I be the abelianisation functor sending a group G to the quotient $G/[G, G]$, which is left adjoint to the inclusion H. Then choosing $\mathscr{E}$ and $\mathscr{F}$ to be the classes of surjective group homomorphisms defines an admissible Galois structure Γ as above. Here every map in $\mathscr{E}$ is a monadic extension, the trivial coverings are those surjective homomorphisms $A \to B$ whose restriction to the commutator subgroups $[A, A] \to [B, B]$ is an isomorphism, and the coverings are the central extensions in the usual sense: surjective homomorphisms whose kernel lies in the centre of the domain. Normal extensions and coverings coincide. (See [18].)

(ii) More generally, taking for $\mathscr{C}$ an exact Mal'tsev (or Goursat) category and for $\mathscr{X}$ a Birkhoff subcategory (= a full reflective subcategory closed under subobjects and regular quotients), and all regular epimorphisms for $\mathscr{E}$ and $\mathscr{F}$, defines an admissible Galois structure Γ, whose coverings are studied in [21]. Normal extensions and coverings still coincide, and every regular epimorphism is a monadic extension. In particular, $\mathscr{C}$ could be a Mal'tsev *variety* and $\mathscr{X}$ its subvariety of abelian algebras, in which case the coverings are the central extensions arising from commutator theory in universal algebra: those surjective homomorphisms $f\colon A \to B$ for which the commutator $[\mathrm{Eq}(f), A \times A]$ of the kernel congruence $\mathrm{Eq}(f)$ of f with the largest congruence $A \times A$ on A is trivial (see [23, 11]). Or, $\mathscr{C}$ could be a variety of Ω-groups [15] and $\mathscr{X}$ an arbitrary subvariety of $\mathscr{C}$. Now the coverings are the (relative) central extensions studied by Fröhlich and others (see [21]).

(iii) Consider $\mathscr{C} = \mathsf{LoCo}$ to be the category of locally connected topological spaces and $\mathscr{X} = \mathsf{Set}$ the category of sets. Take $I = \pi_0$, the connected components functor, $H = \mathrm{Dis}$ the discrete topology functor, $\mathscr{E}$ the class of étale maps (= local homeomorphisms), and $\mathscr{F}$ the class of all maps in Set. This gives another admissible Galois structure. Here the monadic extensions are exactly the *surjective* local homeomorphisms, the trivial coverings and the coverings are, respectively, the disjoint unions of trivial covering maps, and the covering maps, in the usual topological sense. For connected A and B, a normal extension $f\colon A \to B$ is the same as a regular covering map: a covering map $f: A \to B$ such that for every pair of elements $x, y \in A$ which are in the same fibre of f there is a unique continuous map $a: A \to A$ (actually, a covering) such that $f = f \circ a$ and $a(x) = y$. See [1, Chapter 6] for more details.

(iv) Similarly, take $\mathscr{C}$ to be the category of simplicial sets and $\mathscr{X} = \mathsf{Set}$ with the adjunction consisting of $I = \pi_0$ and H giving the discrete simplicial set on a given set. Then taking $\mathscr{E}$ and $\mathscr{F}$ to be the classes of all morphisms gives an admissible Galois structure. For this example, monadic extensions are degree-wise surjective functions. The coverings are precisely the coverings in the sense of Gabriel–Zisman [9]: Kan fibrations whose "Kan liftings" are uniquely determined. See [1, A.3.9] for more details.

(v) For a different Galois structure Γ on the category $\mathscr{C}$ of simplicial sets, let $\mathscr{X}$ be the category of groupoids, and I and H be the fundamental groupoid and nerve functors, and take for $\mathscr{E}$ and $\mathscr{F}$ the classes of Kan fibrations, and of fibrations in the sense of Brown [2], respectively. This particular Γ is studied in [3] where its covering morphisms are called *second order covering maps*. It is *not* admissible.

(vi) Example (iii) has an obvious "pointed" version, obtained by replacing LoCo and Set by the categories LoCo_* and Set_* of pointed locally connected spaces and of pointed sets, respectively. $\mathscr{E}$ and $\mathscr{F}$ now consist of those étale maps and maps that preserve the basepoint. Clearly, this is still an (admissible) Galois structure; the monadic extensions, trivial coverings, coverings and normal extensions are "the same" as in the non-pointed case, only now they are required to be basepoint-preserving.

(vii) *Categorical Galois theory* does indeed capture classical Galois theory, as the name suggests. For this, let k be some fixed field and take $\mathscr{C}$ to be the dual of the category of finite-dimensional commutative k-algebras with $\mathscr{E}^{\mathrm{op}}$ all algebra morphisms, $\mathscr{X}$ the category of finite sets with $\mathscr{F}$ the class of all functions, and $I: \mathscr{C} \to \mathscr{X}$ defined through idempotent decomposition. See [1, A.2] or [18] for further details.

For the rest of this paper, we will assume that our Galois structures are admissible and that H is in fact an inclusion of a full reflective subcategory $\mathscr{X}$ into $\mathscr{C}$. We will also assume that monadic extensions are pullback-stable. Note that this is the case for each of the examples above, with the exception of (v).

One of the important concepts in categorical Galois theory is the *Galois groupoid*:

Definition 2.7. [18, 20] Let $p\colon E \to B$ be a normal extension of B. Then the **Galois groupoid** $\mathrm{Gal}_\Gamma(p)$ of p is the image under I of the kernel pair $\mathrm{Eq}(p)$ of p.

$$\begin{array}{ccccc} \mathrm{Eq}(p) & \underset{c}{\overset{d}{\rightrightarrows}} & E & \xrightarrow{p} & B \\ {\scriptstyle \eta_{\mathrm{Eq}(p)}}\downarrow & & \downarrow{\scriptstyle \eta_E} & & \\ I(\mathrm{Eq}(p)) & \underset{I(c)}{\overset{I(d)}{\rightrightarrows}} & I(E) & & \end{array}$$

Note that this image of the kernel pair is indeed a groupoid: since the functor I preserves pullbacks along trivial coverings (by Lemma 2.4), the image of any groupoid with trivial domain and codomain morphisms is again a groupoid (see the definition of groupoids 3.1). And since p is normal, its kernel pair projections are indeed trivial coverings.

3 Internal groupoids

We have already seen groupoids enter the picture above, so we recall the definition.

Definition 3.1. An **internal category** in a category $\mathscr{C}$ is a diagram

$$R_1 \;\underset{c}{\overset{d}{\rightrightarrows}}\!\!\!\overset{e}{\longleftarrow}\; R_0$$

such that $de = 1_{R_0} = ce$, together with a **multiplication** (or composition)

$$m\colon R_1 \times_{R_0} R_1 \to R_1$$

making the following diagrams commute, where the pullback **(1)** defines the object $R_1 \times_{R_0} R_1$ of "composable arrows":

$$\begin{array}{ccc} R_1 \times_{R_0} R_1 & \xrightarrow{p_1} & R_1 \\ {\scriptstyle p_2}\downarrow & \textbf{(1)} & \downarrow{\scriptstyle c} \\ R_1 & \xrightarrow[d]{} & R_0 \end{array} \qquad \begin{array}{ccc} R_1 \times_{R_0} R_1 & \xrightarrow{m} & R_1 \\ {\scriptstyle p_1}\downarrow & \textbf{(2)} & \downarrow{\scriptstyle d} \\ R_1 & \xrightarrow[d]{} & R_0 \end{array} \qquad \begin{array}{ccc} R_1 \times_{R_0} R_1 & \xrightarrow{m} & R_1 \\ {\scriptstyle p_2}\downarrow & \textbf{(3)} & \downarrow{\scriptstyle c} \\ R_1 & \xrightarrow[c]{} & R_0; \end{array}$$

furthermore, the composition m makes the diagrams

$$\begin{array}{ccccc} R_1 & \xrightarrow{\langle 1_{R_1}, sc\rangle} & R_1 \times_{R_0} R_1 & \xleftarrow{\langle sd, 1_{R_1}\rangle} & R_1 \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle m} & \diagup\!\!\diagup & \\ & & R_1 & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} R_1 \times_{R_0} R_1 \times_{R_0} R_1 & \xrightarrow{1\times m} & R_1 \times_{R_0} R_1 \\ {\scriptstyle m\times 1}\downarrow & & \downarrow{\scriptstyle m} \\ R_1 \times_{R_0} R_1 & \xrightarrow[m]{} & R_1 \end{array}$$

commute. An internal category R is an **internal groupoid** when there exists a morphism $s\colon R_1 \to R_1$ such that $ds = c$ and $cs = d$ and both squares

$$\begin{array}{ccc} R_1 & \xrightarrow{\langle 1_{R_1}, s\rangle} & R_1 \times_{R_0} R_1 \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle m} \\ R_0 & \xrightarrow[e]{} & R_1 \end{array} \qquad \begin{array}{ccc} R_1 & \xrightarrow{\langle s, 1_{R_1}\rangle} & R_1 \times_{R_0} R_1 \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle m} \\ R_0 & \xrightarrow[e]{} & R_1 \end{array}$$

commute. Such an s is necessarily unique. In fact, it is well known that an internal category R is an internal groupoid if and only if **(2)** and **(3)** are also pullbacks.

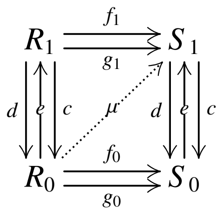
An **internal functor** between two internal categories R and S is a pair of morphisms (f_0, f_1) making the three squares with d, c and e as on the left

$$\begin{array}{ccc} R_1 & \xrightarrow{f_1} & S_1 \\ {\scriptstyle d}\downarrow\uparrow{\scriptstyle e}\downarrow{\scriptstyle c} & & {\scriptstyle d}\downarrow\uparrow{\scriptstyle e}\downarrow{\scriptstyle c} \\ R_0 & \xrightarrow{f_0} & S_0 \end{array} \qquad \begin{array}{ccc} R_1 \times_{R_0} R_1 & \xrightarrow{f_1\times f_1} & S_1 \times_{R_0} S_1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ R_1 & \xrightarrow[f_1]{} & S_1 \end{array}$$

as well as the right hand square commute.

An internal groupoid R with $R_0 = 1$, the terminal object, is called an **internal group**. We shall write $\mathsf{Gp}(\mathscr{C})$ for the category of internal groups and internal functors.

Definition 3.2 (Internal natural transformations and isomorphisms)**.** Given two internal functors $f, g\colon R \to S$ between internal categories R and S, an **internal natural transformation** from f to g is a morphism $\mu\colon R_0 \to S_1$ as in

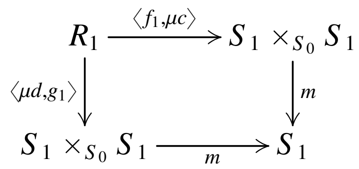


satisfying

(i) $d\mu = f_0$,

(ii) $c\mu = g_0$,

(iii) $m\langle f_1, \mu c\rangle = m\langle \mu d, g_1\rangle$.



For fixed internal categories R and S, the internal functors $R \to S$ and the internal natural transformations between them form a category: the composition of two natural transformations $\mu\colon f \to g$ and $\nu\colon g \to h$ is given by the morphism $m\langle \nu, \mu\rangle$; the identity on f is given by the morphism ef_0. In particular, an internal natural transformation μ is an **internal natural isomorphism** when there is a (unique) internal natural transformation ν from g to f such that $m\langle \mu, \nu\rangle = ef_0$ and $m\langle \nu, \mu\rangle = eg_0$.

Remark 3.3. When R and S are internal groupoids, an internal natural transformation is automatically a natural isomorphism between f and g.

Remark 3.4. If S is a relation, then d and c are jointly monic, so (iii) is automatically satisfied.

In particular, for effective equivalence relations we have

Lemma 3.5. *Given two morphisms $f = (f_1, f_0)$ and $g = (g_1, g_0)$ from $b\colon B_1 \to B_0$ to $c\colon C_1 \to C_0$ satisfying $f_0 = g_0$, there is an internal natural isomorphism between the induced internal functors from* $\mathrm{Eq}(b)$ *to* $\mathrm{Eq}(c)$.

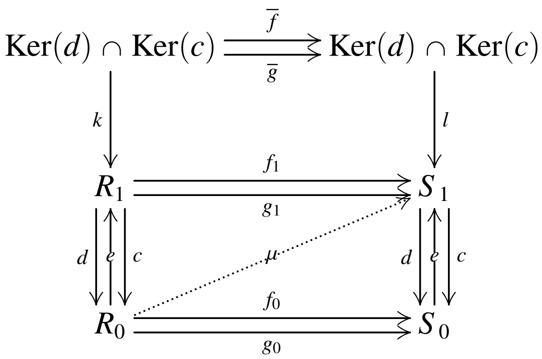
Proof. The condition $f_0 = g_0$ implies that $cf_1 = f_0 b = g_0 b = cg_1$. So let $\mu = \langle f_1, g_1 \rangle$. Then (i) and (ii) from Definition 3.2 are satisfied by definition, and (iii) is satisfied automatically, as $\mathrm{Eq}(c)$ is a kernel pair and so a relation. □

In the special case that $B_0 = C_0 = A$ and $f_0 = 1_A$, we say that f is a **morphism over** A.

From now on, let $\mathscr{C}$ be a finitely complete pointed category. Then for any groupoid R in $\mathscr{C}$, we may restrict R_0 to the zero object 0, and R_1 to $\mathrm{Ker}(d) \cap \mathrm{Ker}(c)$, which gives us the internal group of "loops at 0" or "internal automorphisms at 0", which we denote by $\mathrm{Aut}_R(0)$. When we restrict to this group of internal automorphisms, natural isomorphisms as above collapse the two functors onto each other:

Lemma 3.6. *Any two naturally isomorphic functors $f, g\colon R \to S$ between internal categories induce the same morphism* $\mathrm{Aut}_R(0) \to \mathrm{Aut}_S(0)$.

Proof. Consider the diagram



in which k and l are the inclusions of $\mathrm{Ker}(d) \cap \mathrm{Ker}(c)$ into R_1 and S_1, respectively. We wish to show that $\overline{f} = \overline{g}$, or equivalently, that $l\overline{f} = l\overline{g}$, as l is a monomorphism. From Condition (iii) we know that $m\langle f_1 k, \mu ck \rangle = m\langle \mu dk, g_1 k \rangle$. But since $dk = 0 = ck$ and $dl = 0 = cl$, we can reformulate this as

$$m\langle f_1 k, \mu ck \rangle = m\langle l\overline{f}, ecl\overline{f} \rangle = m\langle 1_{S_1}, ec \rangle l\overline{f} = l\overline{f},$$
$$m\langle \mu dk, g_1 k \rangle = m\langle edl\overline{g}, l\overline{g} \rangle = m\langle ed, 1_{S_1} \rangle l\overline{g} = l\overline{g}$$

giving $l\overline{f} = l\overline{g}$ as required. □

4 The Galois group and the fundamental group

Let $\Gamma = (\mathscr{C}, \mathscr{X}, H, I, \eta, \epsilon, \mathscr{E}, \mathscr{F})$ be an admissible Galois structure on a finitely complete pointed category $\mathscr{C}$ with H a full inclusion, and assume that monadic extensions are pullback stable. Note that this excludes the classical Galois theory Example 2.6 (vii), but it includes Examples 2.6 (i) and (vi), as well as all the Galois structures of Example 2.6 (ii) for which $\mathscr{C}$ is pointed.

Definition 4.1. [20] For a normal extension $p\colon E \to B$, its **Galois group**

$$\mathrm{Gal}_\Gamma(p, 0) = \mathrm{Aut}_{\mathrm{Gal}_\Gamma(p)}(0)$$

is the group of automorphisms at 0 of the Galois groupoid:

$$\begin{array}{ccccc} & & \mathrm{Eq}(p) & \overset{d}{\underset{c}{\rightrightarrows}} & E \xrightarrow{\;p\;} B \\ & & \downarrow{\scriptstyle \eta_{\mathrm{Eq}(p)}} & \lrcorner & \downarrow{\scriptstyle \eta_E} \\ \mathrm{Gal}_\Gamma(p, 0) = \mathrm{Ker}(Id) \cap \mathrm{Ker}(Ic) & \longrightarrow & I(\mathrm{Eq}(p)) & \overset{I(d)}{\underset{I(c)}{\rightrightarrows}} & I(E) \end{array}$$

The resulting functor

$$\mathrm{Gal}_\Gamma(-, 0)\colon \mathsf{NExt}_\Gamma(\mathscr{C}) \to \mathsf{Gp}(\mathscr{X})$$

has some very useful properties: it is a *Baer invariant* [7, 8] with respect to the codomain functor $\mathrm{Cod}\colon \mathsf{NExt}_\Gamma(\mathscr{C}) \to \mathscr{C}$, in the sense that any two maps between normal extensions which agree on the codomains also induce the same map between the Galois groups. To show this, we will use some properties of Section 3.

Lemma 4.2. *If two internal functors $f, g\colon R \to S$ between internal categories with source and target morphisms d, c being trivial coverings are naturally isomorphic, then the functors $I(f), I(g)\colon I(R) \to I(S)$ are still naturally isomorphic.*

Proof. Recall that I preserves pullbacks along trivial coverings, so $I(R)$ and $I(S)$ are still internal categories. In particular,

$$I(S_1 \times_{S_0} S_1) = I(S_1) \times_{I(S_0)} I(S_1)$$

and $I(m)$ is the multiplication of $I(S)$.

Let $\mu\colon R_0 \to S_1$ be an internal natural isomorphism between f and g. Then functoriality of I and the preservation of the multiplication ensures that $I(\mu)$ is still an internal natural transformation. □

Proposition 4.3. *Let $p\colon E \to B$ and $p'\colon E' \to B'$ be normal extensions. Any two morphisms $(f,b)\colon p \to p'$ and $(g,b)\colon p \to p'$ in $\mathsf{NExt}_\Gamma(\mathscr{C})$ with the same codomain component induce the same morphism*

$$\mathrm{Gal}_\Gamma(p,0) \to \mathrm{Gal}_\Gamma(p',0)$$

on the Galois groups.

Proof. This follows from Lemmas 3.5, 4.2 and 3.6. □

In particular, this means that any endomorphism $(f, 1_B)\colon p \to p$ induces the *identity* on the Galois group $\mathrm{Gal}_\Gamma(p,0)$. This means that we can now sensibly introduce the following definition. Recall that a normal extension $u\colon U \to B$ is called **weakly universal** if it is a weak initial object in the full subcategory $\mathsf{NExt}_\Gamma(B)$ of $(\mathscr{C} \downarrow B)$ given by all normal extensions of B, i.e. for every normal extension $p\colon E \to B$ there exists a morphism $e\colon U \to E$ such that $p\circ e = u$.

Definition 4.4. [20] Given an object B of $\mathscr{C}$, its **fundamental group (with coefficient functor** I**)** is the Galois group

$$\pi_1(B,I) = \mathrm{Gal}_\Gamma(u,0)$$

of some weakly universal normal extension $u\colon U \to B$, assuming such exists.

Note that $\pi_1(B,I)$ is independent of the choice of weakly universal normal extension $u\colon U \to B$, by Proposition 4.3 and weak universality of u. Assuming a weakly universal normal extension $u\colon U \to B$ exists for *every* B, we moreover have:

Proposition 4.5. *The above definition of fundamental group gives a functor*

$$\pi_1(-,I)\colon \mathscr{C} \to \mathsf{Gp}(\mathscr{X}).$$

Proof. Consider $f\colon A \to B$ in $\mathscr{C}$, and let $u\colon U \to B$ and $v\colon V \to A$ be weakly universal normal extensions of B and A, respectively. Pulling back u along f gives another normal extension of A by Lemma 2.5, so v factors over it, giving a morphism $v \to u$ which need not be unique. However, Proposition 4.3 ensures that any two such morphisms induce the same morphism on π_1. It is then clear that $\pi_1(-, I)$ preserves identities and composition. □

Remark 4.6. Not every Galois structure has the property that every object admits a weakly universal normal extension into it. Note, however, that even when this is not the case, the fundamental group still defines a functor, but its domain is restricted to the full subcategory of $\mathscr{C}$ of those B for which $\pi_1(B, I)$ is defined.

Examples 4.7. (i) For the Galois structure Γ of Example 2.6 (i), there is a weakly universal normal extension for every group B: if $p\colon P \to B$ is a surjective group homomorphism with a free domain P, then the induced central extension $P/[\mathrm{Ker}(p), P] \to B$ is easily seen to be weakly universal. The fundamental group $\pi_1(B, I) = \mathrm{H}_2(B)$ in this case is the second (integral) homology group of B. (See [20].)

(ii) More generally, for Galois structures Γ of the type considered in Example 2.6 (ii), $\mathsf{NExt}_\Gamma(B)$ is a reflective subcategory of $(\mathscr{E} \downarrow B)$ for every B (see [5, 23]), and the reflection into $\mathsf{NExt}_\Gamma(B)$ of any regular epimorphism $P \to B$ with a projective domain P is weakly universal. Hence, if $\mathscr{C}$ is pointed with enough projectives, $\pi_1(B, I)$ is well defined for every B.

When $\mathscr{C}$ is a semi-abelian category with a monadic forgetful functor to Set, then $\pi_1(B, I) = \mathrm{H}_2(B, I)$ is the second Barr-Beck homology group of B with coefficient functor I (see [6]).

(iii) For Example 2.6 (iii), not every locally connected topological space B admits a weakly universal normal extension $u\colon U \to B$. However, it is well known that there exists a (surjective) covering map $u\colon U \to B$ with a simply connected domain U for every connected, locally path-connected and semi-locally simply connected space B (see, for instance, [14, 26]). Such a u has the following property: for every covering map $f\colon A \to B$ and every pair of elements $x \in U$ and $y \in A$

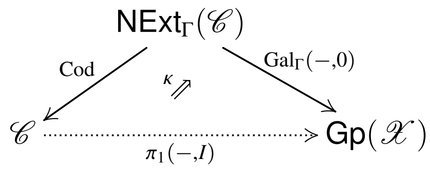
in corresponding fibres there is a unique continuous map $a\colon U \to A$ (actually, a covering map) such that $u = fa$ and $a(x) = y$. Hence such a u is in particular a regular covering map which is clearly a weakly universal normal extension.

Choosing base points $x \in U$ and $y \in B$ such that $u(x) = y$, the map $u\colon (U,x) \to (B,y)$ becomes a weakly universal normal extension with respect to the Galois structure Γ of Example 2.6 (vi). In fact, in this case it is even an *initial* object of $\mathsf{NExt}_\Gamma(B)$ (rather than merely a weakly initial one), which agrees with the usual terminology of calling such a u a *universal* covering map. Now $\pi_1((B,y),I)$ is the classical Poincaré fundamental group of (B,x) (see [1, Chapter 6]).

5 The fundamental group functor as a Kan extension of the Galois group functor

Throughout this section and the next, $\Gamma = (\mathscr{C}, \mathscr{X}, H, I, \eta, \epsilon, \mathscr{E}, \mathscr{F})$ will, as before, be an admissible Galois structure on a finitely complete pointed category $\mathscr{C}$ with H a full inclusion, and such that monadic extensions are pullback stable. For simplicity we shall moreover assume that every object of $\mathscr{C}$ admits a weakly universal normal extension into it. However, our results can easily be adapted to situations where this is not the case (see Section 8).

In the diagram



we now know all ingredients except the natural transformation

$$\kappa\colon \pi_1(-,I)\circ \mathrm{Cod} \Rightarrow \mathrm{Gal}_\Gamma(-,0).$$

For a normal extension $p\colon E \to B$, we define the component

$$\kappa_p\colon \pi_1(B,I) \to \mathrm{Gal}_\Gamma(p,0)$$

as $\mathrm{Gal}_\Gamma((h,1_B),0)\colon \mathrm{Gal}_\Gamma(u,0) \to \mathrm{Gal}_\Gamma(p,0)$ for a weakly universal normal extension $u\colon U \to B$ and any induced

$$\begin{array}{ccc} U & \overset{h}{\dashrightarrow} & E \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle p} \\ B & = & B \end{array}$$

in $\mathsf{NExt}_\Gamma(\mathscr{C})$. Again by Proposition 4.3, any such $(h,1_B)$ will induce the *same* morphism $\mathrm{Gal}_\Gamma((h,1_B),0) = \kappa_p$. It is easy to check that κ is natural.

To prove that the above diagram really is a Kan extension, we just have to show that this natural transformation κ is universal.

Theorem 5.1. *The following is a Kan extension:*

$$\begin{array}{ccc} & \mathsf{NExt}_\Gamma(\mathscr{C}) & \\ {\scriptstyle \mathrm{Cod}}\swarrow & {\scriptstyle \kappa}\Nearrow & \searrow{\scriptstyle \mathrm{Gal}_\Gamma(-,0)} \\ \mathscr{C} & \underset{\pi_1(-,I)}{\dashrightarrow} & \mathsf{Gp}(\mathscr{X}) \end{array}$$

Proof. Given another functor $F\colon \mathscr{C} \to \mathsf{Gp}(\mathscr{X})$ with a natural transformation

$$\begin{array}{ccc} & \mathsf{NExt}_\Gamma(\mathscr{C}) & \\ {\scriptstyle \mathrm{Cod}}\swarrow & {\scriptstyle \gamma}\Nearrow & \searrow{\scriptstyle \mathrm{Gal}_\Gamma(-,0)} \\ \mathscr{C} & \underset{F}{\dashrightarrow} & \mathsf{Gp}(\mathscr{X}), \end{array}$$

define $\alpha\colon F \Rightarrow \pi_1(-,I)$ by $\alpha_B = \gamma_u$ for some weakly universal normal extension u of B. This α is really natural: given $f\colon A \to B$ in $\mathscr{C}$, the morphism

$$\pi_1(f,I)\colon \pi_1(A,I) \to \pi_1(B,I)$$

is defined as in Proposition 4.5 using a morphism

$$\begin{array}{ccc} V & \overset{g}{\longrightarrow} & U \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ A & \underset{f}{\longrightarrow} & B \end{array}$$

between weakly universal normal extensions of A and B. Using naturality of γ on this morphism in $\mathsf{NExt}_\Gamma(\mathscr{C})$ gives naturality of α, because

$$\pi_1(f,I) = \mathrm{Gal}_\Gamma((g,f),0)\colon \mathrm{Gal}_\Gamma(v,0) = \pi_1(A,I) \to \mathrm{Gal}_\Gamma(u,0) = \pi_1(B,I).$$

Naturality of γ also implies that $\kappa\circ\alpha_{\mathrm{Cod}} = \gamma$: For each normal extension $p\colon E \to B$, any morphism

$$\begin{array}{ccc} U & \overset{h}{\dashrightarrow} & E \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle p} \\ B & = & B \end{array} \qquad \text{gives} \qquad \begin{array}{ccc} FB & = & FB \\ {\scriptstyle \gamma_u = \alpha_B}\downarrow & & \downarrow{\scriptstyle \gamma_p} \\ \pi_1(B,I) & \xrightarrow[\kappa_p = Gal((h,1_B),0)]{} & \mathrm{Gal}_\Gamma(p,0) \end{array}$$

and so $\kappa_p\circ\alpha_B = \gamma_p$.

To see that α is unique, notice that, for a weakly universal normal extension u, the component κ_u is an isomorphism. So if $\beta\colon F \Rightarrow \pi_1(-,I)$ also satisfies $\kappa\circ\beta_{\mathrm{Cod}} = \gamma$, taking a weakly universal normal extension of B immediately implies $\alpha_B = \beta_B$, for all B. □

Remark 5.2. In fact, in the definition of $\pi_1(-,I)$ and the above proof of the universality of κ, we have only used the following properties of $\mathrm{Gal}_\Gamma(-,0)$ and Cod:

Given two functors

$$\mathscr{C} \xleftarrow{F} \mathscr{N} \xrightarrow{G} \mathscr{D}$$

such that

(i) for all $f, g \in \mathscr{N}$, $F(f) = F(g)$ implies $G(f) = G(g)$;

(ii) for all $C \in \mathscr{C}$, there exists $U \in \mathscr{N}$ such that $F(U) = C$ and, for all $N \in \mathscr{N}$, the function

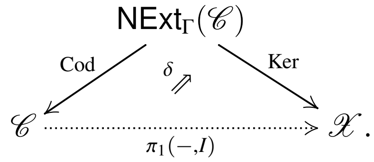
$$\mathrm{Hom}_{\mathscr{N}}(U,N) \to \mathrm{Hom}_{\mathscr{C}}(C,FN)$$

giving the action of F is surjective.

Then it is possible to define a functor $H\colon \mathscr{C} \to \mathscr{D}$ via $H(C) = G(U)$ and a natural transformation $\kappa\colon HF \Rightarrow G$ giving a Kan extension as we have done in our specific case above.

6 The fundamental group functor as a Kan extension of the kernel functor

To compare this construction of the fundamental group given in the context of categorical Galois theory with other viewpoints on semi-abelian homology or with universality properties of connecting homomorphisms in long exact sequences, we actually need a slightly different Kan extension, namely



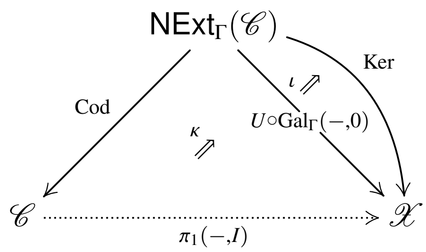
In this section we construct this Kan extension from the one we have already obtained. We first recall that the underlying object of a Galois group can also be calculated in another way:

Lemma 6.1. *[20, Theorem 2.1] Given a normal extension $p\colon E \to B$, the underlying object of its Galois group can be computed as the intersection* $\mathrm{Gal}_\Gamma(p,0) = \mathrm{Ker}(p) \cap \mathrm{Ker}(\eta_E)$. □

This lemma implies that there is a component-wise monic natural transformation

$$\iota\colon U \circ \mathrm{Gal}_\Gamma(-,0) \Rightarrow \mathrm{Ker}$$

from the functor giving the underlying object of the Galois group to the kernel functor.



It is clear that the big triangle in this diagram is still a Kan extension, forgetting only the internal group structure in the Kan extension of Section 5, since this internal group structure is not used anywhere in the proof. We now show that, for any functor $F\colon \mathscr{C} \to \mathscr{X}$, any natural transformation

$\gamma\colon F\circ\mathrm{Cod} \Rightarrow \mathrm{Ker}$ factors over ι. Then universality of κ implies that $\delta = \iota\circ\kappa$ also defines a Kan extension. However, we need a small extra condition to make this work: we now assume that

all morphisms of the kind $IE \to 0$ are in the class $\mathscr{F}$.

Being split epimorphisms, this implies that they are monadic extensions (see [24]), hence normal extensions, since the kernel pair projections are clearly trivial coverings, as they are in $\mathscr{X}$. Notice that this is indeed the case for all of our examples.

Lemma 6.2. *Let $F\colon \mathscr{C} \to \mathscr{X}$ be a functor and $\gamma\colon F\circ\mathrm{Cod} \Rightarrow \mathrm{Ker}$ a natural transformation. For any normal extension $p\colon E \to B$, the component γ_p factors over the inclusion* $\mathrm{Ker}(p) \cap \mathrm{Ker}(\eta_E) \to \mathrm{Ker}(p)$.

Proof. Since the above inclusion is the kernel of $\mathrm{Ker}(p) \xrightarrow{\ker p} E \xrightarrow{\eta_E} IE$, it is sufficient to show that the composite

$$FB \xrightarrow{\gamma_p} \mathrm{Ker}(p) \xrightarrow{\ker p} E \xrightarrow{\eta_E} IE$$

is zero. To do this, consider the three normal extensions

$$\begin{array}{ccccc} E & \xrightarrow{\eta_E} & IE & \longleftarrow & 0 \\ {\scriptstyle p}\downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & 0 & = & 0 \end{array}$$

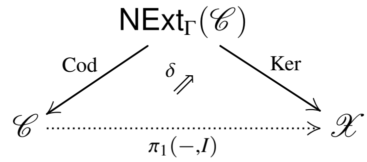
with the given morphisms between them. Naturality of γ gives

$$\begin{array}{ccccc} FB & \longrightarrow & F0 & = & F0 \\ {\scriptstyle \gamma_p}\downarrow & & \downarrow{\scriptstyle 0} & & \downarrow \\ \mathrm{Ker}(p) & \xrightarrow[\eta_E\circ\ker p]{} & IE & \longleftarrow & 0 \end{array}$$

which shows that γ_p does indeed factor over $\mathrm{Ker}(p) \cap \mathrm{Ker}(\eta_E) \to \mathrm{Ker}(p)$. □

So, using universality of κ and this lemma, we obtain

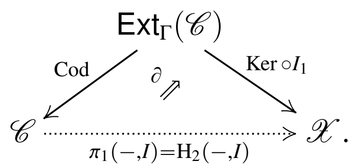
Theorem 6.3. *The diagram*



is a Kan extension. □

7 When normal extensions are reflective

Assume that $\mathscr{C}$ is a semi-abelian category with enough regular projectives, that $\mathscr{X}$ is a Birkhoff subcategory of $\mathscr{C}$, and that $\mathscr{E}$ and $\mathscr{F}$ consist of all regular epimorphisms (so we are in the situation of Example 2.6 (ii)). It was shown in [10] that there is a Kan extension



Here $\mathsf{Ext}_\Gamma(\mathscr{C})$ is the full subcategory of $\mathsf{Arr}(\mathscr{C})$ given by all monadic extensions,

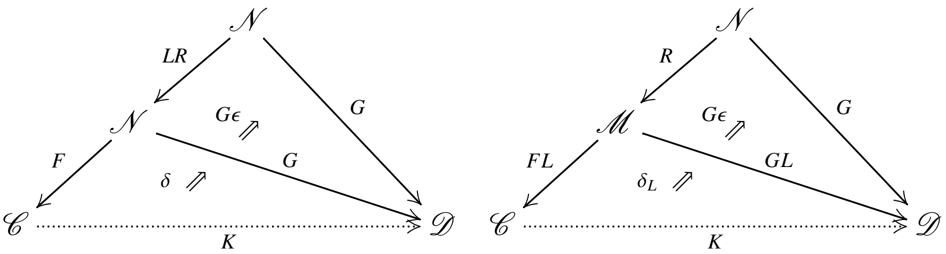
$$I_1 \colon \mathsf{Ext}_\Gamma(\mathscr{C}) \to \mathsf{NExt}_\Gamma(\mathscr{C})$$

is left adjoint to the inclusion functor $\mathsf{NExt}_\Gamma(\mathscr{C}) \to \mathsf{Ext}_\Gamma(\mathscr{C})$ and, for every monadic extension $p \colon E \to B$, the morphism $\partial_p \colon \mathrm{H}_2(B,I) \to \mathrm{Ker}(I_1(f))$ is a connecting morphism in the long exact homology sequence associated with f and I. In order to deduce this result from ours, we need a lemma.

Lemma 7.1. *If the left hand triangle*

is a Kan extension and the functor $L \colon \mathscr{M} \to \mathscr{N}$ admits a fully faithful right adjoint, then the right hand triangle is a Kan extension as well.

Proof. Write R for the fully faithful right adjoint of L, and $\epsilon\colon LR \Rightarrow 1_{\mathscr{M}}$ for the counit. By [25, Proposition 3 in X.7], the natural transformation $G\epsilon\colon GLR \Rightarrow G$ defines a Kan extension, as pictured in the top triangle of the right hand diagram:



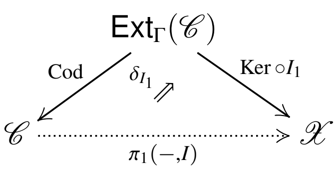
We want the bottom triangle in the right hand diagram to be a Kan extension as well. Since ϵ is a natural isomorphism, this will be the case if the outer triangle and the natural transformation $G\epsilon\circ\delta_{LR}\colon KFLR \Rightarrow G$ form a Kan extension. And indeed this is true, since the two outer triangles coincide, and in the left hand diagram both triangles are Kan extensions: the bottom one by assumption and the top one again by [25, Proposition 3 in X.7], because $LR\colon \mathscr{N} \to \mathscr{N}$ is right adjoint to the identity functor, since R is fully faithful. □

Theorem 6.3 and Lemma 7.1 imply in particular:

Corollary 7.2. *Under the assumptions of Section 6, and when, moreover, the inclusion functor* $\mathsf{NExt}_\Gamma(\mathscr{C}) \to \mathsf{Ext}_\Gamma(\mathscr{C})$ *admits a left adjoint*

$$I_1\colon \mathsf{Ext}_\Gamma(\mathscr{C}) \to \mathsf{NExt}_\Gamma(\mathscr{C}),$$

the diagram



is a Kan extension.

Proof. It suffices to observe that I_1 leaves the codomains intact since every identity morphism is a normal extension and $\mathsf{NExt}_\Gamma(\mathscr{C})$ is a replete subcategory of $\mathsf{Ext}_\Gamma(\mathscr{C})$ (see Corollary 5.2 in [16]). □

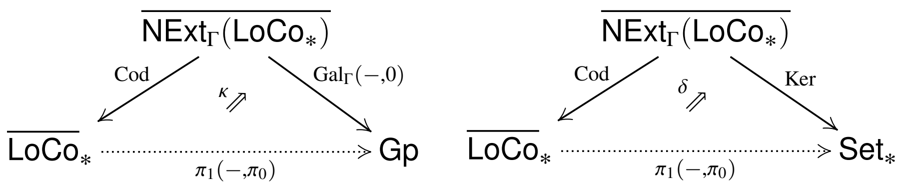
The inclusion functor $\mathsf{NExt}_\Gamma(\mathscr{C}) \to \mathsf{Ext}_\Gamma(\mathscr{C})$ admits a left adjoint not only in the semi-abelian case mentioned above, but more generally, whenever $\mathscr{C}$ is an exact Mal'tsev category, $\mathscr{X}$ is a Birkhoff subcategory and $\mathscr{E}$ and $\mathscr{F}$ consist of all regular epimorphisms (see [5]). Another class of examples is given in [4].

8 Exact homotopy sequence

As remarked above, the Galois structure Γ of Example 2.6 (vi) satisfies all conditions assumed in Sections 5 and 6, except one: it is admissible, the category LoCo_* is finitely complete and pointed, the discrete topology functor

$$\mathrm{Dis}\colon \mathsf{Set}_* \to \mathsf{LoCo}_*$$

is fully faithful, monadic extensions are pullback stable and, for every pointed set (X, x), the map $(X, x) \to 0$ is in $\mathscr{F}$ (since here $\mathscr{F}$ consists of *all* base-point preserving maps); yet not every pointed topological space admits a weakly universal normal extension into it. We do know, however, that a *universal* normal extension exists for every connected, locally path connected, semi-locally simply connected space B with base-point $y \in B$, namely its universal covering map in the usual topological sense: a covering map $u\colon (U, w) \to (B, y)$ with U connected and simply connected. Theorems 5.1 and 6.3 and their proofs can easily be adapted to this situation. Thus we obtain Kan extensions



where $\overline{\mathsf{LoCo}_*}$ is the full subcategory of LoCo_* consisting of all connected, locally path connected, semi-locally simply connected pointed spaces, and the full subcategory $\overline{\mathsf{NExt}_\Gamma(\mathsf{LoCo}_*)}$ of $\mathsf{NExt}_\Gamma(\mathsf{LoCo}_*)$ is determined by those normal extensions whose codomain is in $\overline{\mathsf{LoCo}_*}$. Notice that $\mathsf{Gp}(\mathsf{Set}_*) \simeq \mathsf{Gp}(\mathsf{Set}) \simeq \mathsf{Gp}$.

Now let $p\colon (E,x) \to (B,y)$ be a Γ-normal extension of a connected, locally path-connected, semi-locally simply connected pointed space (B,y) with kernel (F,x) (meaning in this context of course the fibre over y). Let u be the universal covering map $(U,w) \to (B,y)$, write e for the unique continuous base-point preserving map $(U,w) \to (E,p)$ such that $pe = u$ and recall that it is a covering map. Since U is connected, the image of e is contained in the connected component E_x of x and the left hand triangle restricts to the commutative right hand triangle

$$\begin{array}{ccc} (U,w) & & \\ {\scriptstyle e}\downarrow & \searrow^{u} & \\ (E,x) & \xrightarrow[p]{} & (B,y) \end{array} \qquad\qquad \begin{array}{ccc} (U,w) & & \\ {\scriptstyle e'}\downarrow & \searrow^{u} & \\ (E_x,x) & \xrightarrow[p']{} & (B,y) \end{array}$$

Now e' is still a covering map, and it is surjective since its codomain is connected—the image of a covering map is always both open and closed. Moreover, since U is connected and simply connected, e' is the universal covering map of (E_x,x). Taking kernels yields an exact sequence of pointed sets

$$0 \to \mathrm{Ker}(e') \to \mathrm{Ker}(u) \to \mathrm{Ker}(p') \to 0$$

hence an exact sequence of groups

$$0 \to \pi_1(E,x) \to \pi_1(B,y) \to (F \cap E_x, x) \to 0$$

where $(F \cap E_x, x)$ is the Galois group of the normal extension p'. As we clearly have an exact sequence of pointed sets

$$0 \to (F \cap E_x, x) \to (F,x) \to \pi_0(E,x) \to 0$$

and because $(F,x) = \pi_0(F,x)$ since F is a discrete space, we can paste the two sequences together to obtain an exact sequence

$$0 \to \pi_1(E,x) \to \pi_1(B,y) \to \pi_0(F,x) \to \pi_0(E,x) \to 0$$

and this is the low-dimensional part of the usual exact homotopy sequence induced by the fibration

$$(F,x) \to (E,x) \to (B,y).$$

Notice that $\pi_0(B,y) = 0$ as B is connected. What we would like to point out here is that the morphism $\pi_1(B,y) \to \pi_0(F,x) = (F,x)$ is the p-component δ_p of the natural transformation defining the right hand Kan extension pictured above. Hence, we are in a similar situation as with the algebraic case studied in the previous section, where the Kan extension of Corollary 7.2 expresses a universal property of the connecting morphisms in an exact homology sequence.

References

[1] F. Borceux and G. Janelidze, *Galois theories*, Cambridge Stud. Adv. Math., vol. 72, Cambridge Univ. Press, 2001.

[2] R. Brown, *Fibrations of groupoids*, J. Algebra **15** (1970), 103–132.

[3] R. Brown and G. Janelidze, *Galois theory of second order covering maps of simplicial sets*, J. Pure Appl. Algebra **135** (1999), 23–31.

[4] M. Duckerts-Antoine, *Fundamental groups in $\mathbb{E}$-semi-abelian categories*, Ph.D. thesis, Université catholique de Louvain, 2013.

[5] T. Everaert, *Higher central extensions in Mal'tsev categories*, preprint `arXiv:1209.4398`, 2012.

[6] T. Everaert, M. Gran, and T. Van der Linden, *Higher Hopf formulae for homology via Galois Theory*, Adv. Math. **217** (2008), no. 5, 2231–2267.

[7] T. Everaert and T. Van der Linden, *Baer invariants in semi-abelian categories I: General theory*, Theory Appl. Categ. **12** (2004), no. 1, 1–33.

[8] A. Fröhlich, *Baer-invariants of algebras*, Trans. Amer. Math. Soc. **109** (1963), 221–244.

[9] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Springer, 1967.

[10] J. Goedecke and T. Van der Linden, *On satellites in semi-abelian categories: Homology without projectives*, Math. Proc. Cambridge Philos. Soc. **147** (2009), no. 3, 629–657.

[11] M. Gran, *Applications of categorical Galois theory in universal algebra*, Galois Theory, Hopf Algebras, and Semiabelian Categories (G. Janelidze, B. Pareigis, and W. Tholen, eds.), Fields Inst. Commun., vol. 43, Amer. Math. Soc., 2004, pp. 243–280.

[12] R. Guitart, *An anabelian definition of abelian homology*, Cah. Topol. Géom. Différ. Catég. **XLVIII** (2007), no. 4, 261–269.

[13] R. Guitart and L. Van den Bril, *Calcul des satellites et présentations des bimodules à l'aide des carrés exacts, I and II*, Cah. Topol. Géom. Différ. Catég. **XXIV** (1983), no. 3 and 4, 299–330 and 333–369.

[14] A. Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[15] P. J. Higgins, *Groups with multiple operators*, Proc. Lond. Math. Soc. (3) **6** (1956), no. 3, 366–416.

[16] G. B. Im and G. M. Kelly, *On classes of morphisms closed under limits*, J. Korean Math. Soc. **23** (1986), 19–33.

[17] G. Janelidze, *On satellites in arbitrary categories*, Bull. Acad. Sci. Georgian SSR **82** (1976), no. 3, 529–532, in Russian, English translation `arXiv:0809.1504v1`.

[18] G. Janelidze, *Pure Galois theory in categories*, J. Algebra **132** (1990), no. 2, 270–286.

[19] G. Janelidze, *Categorical Galois theory: revision and some recent developments*, Galois connections and applications, Math. Appl., vol. 565, Kluwer Acad. Publ., 2004, pp. 139–171.

[20] G. Janelidze, *Galois groups, abstract commutators and Hopf formula*, Appl. Categ. Structures **16** (2008), 653–668.

[21] G. Janelidze and G. M. Kelly, *Galois theory and a general notion of central extension*, J. Pure Appl. Algebra **97** (1994), no. 2, 135–161.

[22] G. Janelidze and G. M. Kelly, *The reflectiveness of covering morphisms in algebra and geometry*, Theory Appl. Categ. **3** (1997), no. 6, 132–159.

[23] G. Janelidze and G. M. Kelly, *Central extensions in Mal'tsev varieties*, Theory Appl. Categ. **7** (2000), no. 10, 219–226.

[24] G. Janelidze and W. Tholen, *Facets of descent II*, Appl. Categ. Structures **5** (1997), 229–248.

[25] S. Mac Lane, *Categories for the working mathematician*, second ed., Grad. Texts in Math., vol. 5, Springer, 1998.

[26] E. H. Spanier, *Algebraic topology*, McGraw-Hill series in higher mathematics, McGraw-Hill, New York, 1966.

Tomas Everaert
Vakgroep Wiskunde, Vrije Universiteit Brussel
Pleinlaan 2, 1050 Brussel, Belgium
teveraer@vub.ac.be

Julia Goedecke
Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
Cambridge CB3 0WB, United Kingdom
julia.goedecke@cantab.net

Tim Van der Linden
Institut de Recherche en Mathématique et Physique
Université catholique de Louvain
chemin du cyclotron 2 bte L7.01.02, 1348 Louvain-la-Neuve, Belgium
tim.vanderlinden@uclouvain.be

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES *Vol. LIV-3 (2013)*

BORROMEANITE DU GROUPE PULSATIF

par Jean-Yves DEGOS

En hommage amical à René Guitart,
à l'occasion de ses 65 ans.

Résumé. Dans la section 1, nous définissons le groupe pulsatif, noté Pul, comme le groupe de symétries de la sphère entière de rayon $\sqrt{5}$, et donnons une description de ce groupe. Dans la section 2, nous en donnons une présentation borroméenne, et nous faisons le lien avec le groupe spécial orthogonal. Dans la section 3, nous nous intéressons aux possibilités de représenter graphiquement ce groupe, en utilisant la méthode de Newton. Enfin, dans la section 4, nous en proposons une interprétation et une généralisation.

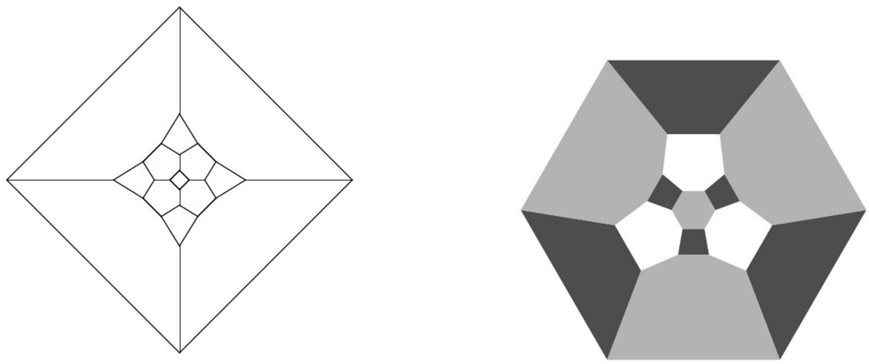
Abstract. In the section 1, we define the pulsative group, denoted by Pul, as the group of symetries of the integral sphere of radius $\sqrt{5}$, and we give a description of this group. In the section 2, we give a borromean presentation of it, and we give a link with the special orthogonal group. In the section 3, we focus on different ways of representing this group graphically, using Newton's method. At last, in the section 4, we give an interpretation and a generalization of it.

Keywords. borromean groups, brunnian groups, mathematical pulsation, pulsative group.

Mathematics Subject Classification (2010). 12Y05, 20H30.

1. Introduction : le groupe pulsatif

Dans [6], §13, René Guitart introduit la notion de *site pulsatif*. Pour l'auteur, “un acte mathématique réel est une façon de faire tenir ensemble (...) 24 postures”, qu'il place sur un *hexagramme pulsatif*, qui, selon l'auteur “est image dans le plan de l'octaèdre régulier adouci, qui est l'un des 13 solides archimédiens, encore appelé polyèdre de Kelvin”.



Les deux figures ci-dessus représentent deux projections du polyèdre de Kelvin dans le plan d'Argand-Cauchy :

– celle de gauche est la projection stéréographique sur le plan d'équation $z = 0$, le pôle nord étant le point $N(0,0,1)$, et le pôle sud le point $S(0,0,-1)$;
– celle de droite est la projection stéréographique sur le plan d'équation $x + y + z + \sqrt{15} = 0$, le pôle nord étant le point $N\left(\frac{\sqrt{5}}{\sqrt{3}}, \frac{\sqrt{5}}{\sqrt{3}}, \frac{\sqrt{5}}{\sqrt{3}}\right)$, et le pôle sud le point $S\left(-\frac{\sqrt{5}}{\sqrt{3}}, -\frac{\sqrt{5}}{\sqrt{3}}, -\frac{\sqrt{5}}{\sqrt{3}}\right)$.

Proposition 1.1. *Le polyèdre de Kelvin est inscriptible dans une sphère de rayon* 1*. Ses* 24 *sommets ont pour coordonnées les éléments de l'ensemble :*

$$\mathcal{S} = \left\{ \left(\frac{x}{\sqrt{5}}, \frac{y}{\sqrt{5}}, \frac{z}{\sqrt{5}} \right) \text{ avec } (x,y,z) \in \mathbb{Z}^3 \text{ et } x^2 + y^2 + z^2 = 5 \right\}. \quad (1)$$

Définition 1.2. *On appelle* groupe pulsatif *le groupe de symétries de la sphère entière de rayon $\sqrt{5}$, c'est-à-dire le groupe des isométries qui laissent globalement invariante cette sphère. On note ce groupe* Pul.

Nous allons donner une description matricielle du groupe Pul.

Définition 1.3. *(i) On considère les matrices suivantes :*

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ et } G = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, V = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$T_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, T_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, T_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(ii) On appelle K le groupe (isomorphe au groupe de Klein) :

$$K = \{I, U, V, W\}.$$

(iii) On appelle S le groupe (isomorphe au groupe symétrique $\mathfrak{S}_3$) :

$$S = \{I, T_{12}, T_{23}, T_{31}, G, G^{-1}\}.$$

Ceci nous permet d'énoncer le théorème qui suit.

Théorème 1.4. *Avec les notations de la Définition 1.3, on a les résultats suivants :*
(i) Le produit $KS = \{MN, M \in K, N \in S\}$ est un groupe d'ordre 24.
(ii) Le groupe Pul *est égal à KS.*

2. Borroméanité du groupe pulsatif et lien avec le groupe spécial orthogonal

Nous allons établir que le groupe Pul a une structure de groupe borroméen ([7]), ou plus exactement de groupe 3-brunnien de type I, pour reprendre la terminologie de [3]. Nous commençons par une définition ; ensuite nous montrons que Pul et $SO_3(\mathbb{F}_3)$ sont les deux seuls sous-groupes du groupe orthogonal sur $\mathbb{F}_3$ à satisfaire une certaine propriété (Théorème 2.3).

Définition 2.1. *On considère les trois matrices suivantes :*

$$R_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, R_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \textit{ et } R_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Nous donnons le théorème annoncé.

Théorème 2.2. *On a les résultats suivants :*
(i) $R_2 = GR_1G^{-1}, R_3 = GR_2G^{-1}, R_1 = GR_3G^{-1}$.
(ii) $\mathrm{Pul} = \langle R_1, R_2, R_3 \rangle$.
(iii) Pul *est* 3*-brunnien de type I.*

Démonstration. Le point (i) est évident. Pour le point (ii), on dresse la table de multiplication de K par S, et on essaie d'écrire chaque produit, donc chaque élément de Pul, comme un mot sur R_1, R_2, R_3. La table qu'on obtient, grâce à l'aide un système de calcul formel comme Maple ou Sage, est la suivante :

$\times$	I	T_{12}	T_{23}	T_{31}	G	G^{-1}
I	I	$R_3{R_2}^2$	$R_1{R_3}^2$	$R_2{R_1}^2$	R_1R_2	${R_2}^{-1}{R_1}^{-1}$
U	${R_1}^2$	R_3	$R_1{R_2}^2$	${R_2}^{-1}$	${R_1}^{-1}R_2$	$R_1{R_3}^{-1}$
V	${R_2}^2$	${R_3}^{-1}$	R_1	$R_2{R_3}^2$	${R_2}^{-1}R_3$	$R_2{R_1}^{-1}$
W	${R_3}^2$	$R_3{R_1}^2$	${R_1}^{-1}$	R_2	${R_3}^{-1}R_2$	$R_3{R_2}^{-1}$

,

ce qu'il fallait démontrer. Pour montrer le point (iii), on remarque que (i) et (ii) entraînent que Pul est 3-cyclable (voir [3]). Il faut donc montrer que :

1. Si $R_1 = I$ (resp. $R_2 = I$, $R_3 = I$), le groupe Pul devient trivial.
2. Si on supprime R_1 (resp. R_2, R_3) le groupe $\langle R_2, R_3 \rangle$ (resp. $\langle R_3, R_1 \rangle$, $\langle R_1, R_2 \rangle$) est quand même Pul.

Cela ne pose aucune difficulté. □

Théorème 2.3. *Soit* H *un sous-groupe de* $\mathrm{O}_3(\mathbb{F}_3)$ *(le groupe des isométries d'un cube régulier, c'est-à-dire formé des* 48 *matrices de permutations signées), vérifiant la propriété que :*

$$\forall N \in \mathrm{O}_3(\mathbb{F}_3), N \in H \text{ ou } -N \in H\,.$$

Alors $H = \mathrm{Pul}$ *ou* $H = \mathrm{SO}_3(\mathbb{F}_3)$.

Démonstration. Le groupe H est d'indice 2, donc d'ordre 24.

(i) Si $R_1 \in H$ et $G \in H$, alors :

$$R_2 = GR_1G^{-1} \in H, \text{ et } R_3 = GR_2G^{-1} \in H$$

donc $\mathrm{Pul} = \langle R_1, R_2, R_3 \rangle \subset H$, et comme les deux groupes ont même ordre, ils sont égaux.

(ii) Si $R_1 \in H$ et $-G \in H$, alors :

$$R_2 = (-G)R_1(-G)^{-1} \in H, \text{ et } R_3 = (-G)R_2(-G)^{-1} \in H$$

donc $\mathrm{Pul} = \langle R_1, R_2, R_3 \rangle \subset H$, et comme les deux groupes ont même ordre, ils sont égaux.

(iii) Si $R_1 \notin H$ alors $-R_1 \in H$; si $G \in H$, alors :

$$-R_2 = G(-R_1)G^{-1} \in H, \text{ et } -R_3 = G(-R_2)G^{-1} \in H$$

donc $\mathrm{SO}_3(\mathbb{F}_3) = \langle -R_1, -R_2, -R_3 \rangle \subset H$, et comme les deux groupes ont même ordre, ils sont égaux.

(iv) Si $R_1 \notin H$ alors $-R_1 \in H$; si $-G \in H$, alors :

$$-R_2 = (-G)(-R_1)(-G)^{-1} \in H, \text{ et } -R_3 = (-G)(-R_2)(-G)^{-1} \in H$$

donc $\mathrm{SO}_3(\mathbb{F}_3) = \langle -R_1, -R_2, -R_3 \rangle \subset H$, et comme les deux groupes ont même ordre, ils sont égaux. □

3. Représentations graphiques du groupe pulsatif

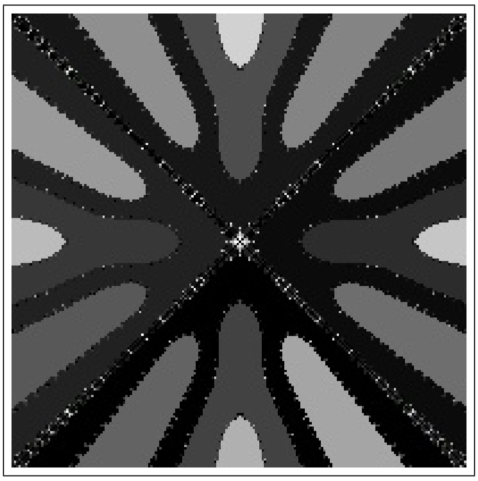
Nous nous intéressons dans cette section à la possibilité de représenter graphiquement le groupe pulsatif, dans le plan (sous-section 3.1) ou dans l'espace (sous-section 3.2). Enfin la sous-section 3.3 comporte un tableau des couleurs utilisées pour produire la figure de la sous-section 3.1.

3.1 Méthode de Newton en dimension 2

Une première solution consiste à effectuer une projection stéréographique (comme indiqué dans [5], pages 318-319) des éléments de $\mathcal{S}$ (voir l'égalité (1) de la Proposition 1.1). Nous obtenons alors des points du plan, que nous identifions à leurs affixes complexes. Nous formons enfin le polynôme de degré 24 qui admet ces affixes pour racines. Nous obtenons :

$$P(X) = X^{24} - \tfrac{8211}{25}X^{20} + \tfrac{51819}{25}X^{16} + \tfrac{15134}{25}X^{12} + \tfrac{51819}{25}X^{8} - \tfrac{8211}{25}X^{4} + 1\,.$$

Grâce au tableaux de couleurs de la sous-section 3.3, nous obtenons, avec Fractal Domains 2.0.11 (voir [4]), la figure suivante [1].



Ce n'est donc pas très satisfaisant : la représentation à plat rend mal compte des symétries que l'on devrait intuitivement visualiser. Il faut donc tâcher de représenter graphiquement le groupe pulsatif en dimension 3. Dans la section suivante, nous allons étudier cette possibilité.

3.2 Méthode de Newton en dimension 3

Proposition 3.1. *Les éléments de $\sqrt{5}\mathcal{S}$, la sphère entière de rayon $\sqrt{5}$, sont exactement les solutions réelles du système :*

$$\begin{cases} x^2+y^2+z^2-5=0 \\ ((x+y+z)^2-1)((x+y+z)^2-9)=0 \\ xyz=0 \end{cases} .$$

Preuve succincte. Le système d'équation est invariant par permutation circulation sur x, y, z. On peut donc, en utilisant la troisième équation, supposer que $z = 0$. La deuxième équation permet alors d'affirmer que $x + y$ vaut $-1, 1, -3$ ou 3. D'autre part, en calculant $xy = \frac{1}{2}\big((x+y)^2 - (x^2+y^2)\big)$,

1. Les couleurs à l'écran ont toutefois été remplaçées par des niveaux de gris pour une meilleure qualité d'impression. Le lecteur peut obtenir la version colorisée en écrivant à l'auteur.

et en utilisant la première équation, on obtient : $xy \in \{-2, 2\}$. Cela laisse pour (x, y) les seules valeurs possibles : $(-2, 1)$; $(1, -2)$; $(-1, 2)$; $(2, -1)$; $(-2, -1)$; $(-1; -2)$; $(1, 2)$; $(2, 1)$. $\square$

Théorème 3.2. *Considérons l'application*

$$\begin{array}{rccc} F: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & (x,y,z) & \longmapsto & (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)) \end{array}$$

avec

$$\begin{array}{rcl} f_1(x,y,z) & = & S_1(x,y,z)^2 - 2S_2(x,y,z) - 5\,, \\ f_2(x,y,z) & = & S_1(x,y,z)^4 - 10S_1(x,y,z)^2 + 9\,, \\ f_3(x,y,z) & = & S_3(x,y,z)\,, \end{array}$$

où S_1, S_2, S_3 désignent les polynômes symétriques élémentaires en les variables X, Y, Z. Alors :
(i) L'application F est différentiable en tout point de $\mathbb{R}^3$.
(ii) Au voisinage des zéros de F, la différentielle F' est inversible.

Démonstration. Le point (i) est évident, puisque f_1, f_2, f_3 sont des applications polynomiales. Pour vérifier le point (ii), on détermine $J(F)(x,y,z)$, la matrice jacobienne de F au point (x,y,z), et on regarde où s'annule son jacobien $j(F)(x,y,z)$. Tous calculs faits on trouve :

$$J(F)(x,y,z) = \begin{bmatrix} 2x & 2y & 2z \\ 4{S_1}^3 - 20S_1 & 4{S_1}^3 - 20S_1 & 4{S_1}^3 - 20S_1 \\ yz & zx & xy \end{bmatrix}$$

puis

$$\begin{array}{rcl} j(F)(x,y,z) & = & -2\big(4S_1(x,y,z)^3 - 20S_1(x,y,z)\big)(y-z)(z-x)(z-y) \\ & = & -8S_1(x,y,z)\big(S_1(x,y,z)^2 - 5\big)(y-z)(z-x)(z-y)\,. \end{array}$$

Il est alors clair que si $F(x,y,z) = 0$, alors $j(F)(x,y,z) \neq 0$, donc F' est inversible au voisinage des zéros de F. $\square$

Il est donc possible de représenter graphiquement le groupe pulsatif en utilisant la méthode de Newton en dimension 3 appliquée à la fonction F sur la sphère. Ce sera l'objet d'un travail ultérieur.

3.3 Annexe : tableaux associant les codes couleurs aux racines

#	Racine	Codage
23	(0,1,2)	(0,1,2)
21	(0,-1,2)	(0,5,2)
3	(0,1,-2)	(0,1,4)
1	(0,-1,-2)	(0,5,4)

#	Racine	Codage
20	(0,2,1)	(0,2,1)
17	(0,-2,1)	(0,4,1)
8	(0,2,-1)	(0,2,5)
7	(0,-2,-1)	(0,4,5)

#	Racine	Codage
22	(1,0,2)	(1,0,2)
24	(-1,0,2)	(5,0,2)
2	(1,0,-2)	(1,0,4)
4	(-1,0,-2)	(5,0,4)

#	Racine	Codage
13	(1,2,0)	(1,2,0)
14	(-1,2,0)	(5,2,0)
16	(1,-2,0)	(1,4,0)
10	(-1,-2,0)	(5,4,0)

#	Racine	Codage
19	(2,0,1)	(2,0,1)
18	(-2,0,1)	(4,0,1)
5	(2,0,-1)	(2,0,5)
6	(-2,0,-1)	(4,0,5)

#	Racine	Codage
12	(2,1,0)	(2,1,0)
15	(-2,1,0)	(4,1,0)
11	(2,-1,0)	(2,5,0)
9	(-2,-1,0)	(4,5,0)

Les tableaux ci-dessus doivent se décoder de la manière suivante : pour obtenir le “codage”, on prend les composantes de la “racine” et on les réduit modulo 6. Ce codage modulo 6 est celui de la couleur associée à la racine (de façon canonique) : les composantes de ce codage, multipliées chacune par 51, donnent la composition de la couleur en Rouge, Vert, Bleu. Le “#” est le numéro de la racine pour Fractal Domains 2.0.11 ([4]).

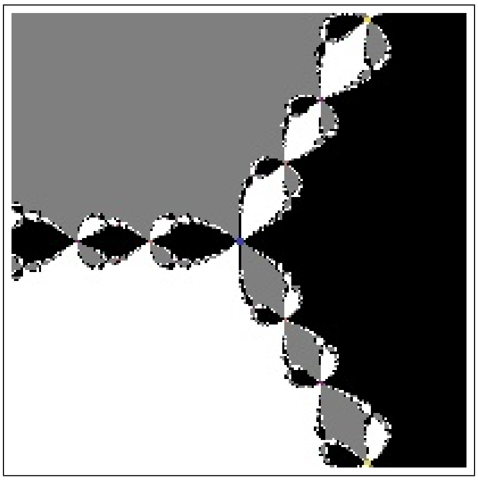
4. Conclusion : quelques remarques et prolongements

Remarque 4.1 (Lien Pulsation-Borroméanité)**.** L’intérêt du groupe pulsatif Pul est qu’il établit un lien formel entre deux soucis de René Guitart :

1. la notion de pulsation mathématique (voir [5]) ;
2. la notion de groupe borroméen (voir [7]).

Remarque 4.2 (Motivation de la section 3)**.** Dans [2], Jean-Guy Degos se sert de la figure ci-dessous pour illustrer la complexité des frontières entre les trois notions de solvabilité, flexibilité et rentabilité d’une entreprise. Il écrit en particulier ([2], page 39) :

Si on approfondit l'analyse des relations "flexibilité et long terme", "solvabilité et moyen terme" et "rentabilité et court terme", elles ne sont pas tout à fait aussi simples que nous les avons exposées. Comme dans beaucoup de domaines du monde réel et comme l'a montré M. F. Barnsley sur le plan théorique en illustrant ses propos par la résolution d'équations à racines complexes par la méthode de Newton [2] il n'y a jamais de frontières nettes entre les causes et les résultats de plusieurs éléments arbitrairement, ou pédagogiquement dissociés.



De la même façon, nous aimerions bien donner à voir la complexité du va-et-vient entre les 24 postures du site pulsatif.

Remarque 4.3 (Généralisations du groupe pulsatif)**.** On pourrait généraliser le groupe pulsatif dans deux directions.

– La première direction consiste à observer que la sphère entière de rayon $\sqrt{5}$ se généralise en la notion de *permutoèdre* : le permutoèdre d'ordre n est un polytope de dimension $n - 1$ plongé dans un espace de dimension n, dont les sommets sont obtenus en permutant les coordonnées du vecteur $(1, 2, \ldots, n)$. On peut montrer que le polyèdre de Kelvin correspond au permutoèdre d'ordre 4. Il serait donc légitime de définir le groupe pulsatif d'ordre n comme le groupe de symétries du permutoèdre d'ordre n.

2. Voir [1], pp. 280-283.

– La deuxième direction consisterait à définir le groupe pulsatif d'ordre n comme le groupe de symétries de la sphère entière de rayon $\sqrt{n+1}$.

Ces deux définitions possibles n'ont pas de raison de coïncider si $n \neq 4$. Peut-on décrire ces groupes ? Sont-ils brunniens ? Peut-on déterminer une "fonction F" (voir section 3) pour les représenter graphiquement ?

Certains résultats de cet article ont été utilisés dans [8].

Références

[1] [Barnsley M. F., 1988] Fractals Everywhere, *Academic Press*, 1988.

[2] [Degos J.-G., 1991] Contribution à l'étude du diagnostic financier des petites et moyennes entreprises, *Thèse d'État en Sciences de gestion*, Université Bordeaux I, 1991.

[3] [Degos J.-Y., 2013] Linear groups and primitive polynomials over $\mathbb{F}_p$, *Cah. Top. Géo. Diff. Cat.*, LIV-1 (2013) 56–74.

[4] [De Mars D. C., 1994] Fractal Domains 2.0.11, `http://www.fractaldomains.com`, Copyright 1994–2011.

[5] [Guitart R., 1999] La pulsation mathématique. Rigueur et ambiguïté, la nature de l'activité mathématique. Ce dont il s'agit d'instruire, *L'Harmattan*, 1999.

[6] [Guitart R., 2002] Sur les places du sujet et de l'objet dans la pulsation mathématique, *Questions éducatives*, revue du Centre de Recherche en éducation de l'Université Jean Monnet de Saint-Étienne, numéro 22, 2002.

[7] [Guitart R., 2009] Klein's group as a borromean object, *Cah. Top. Géo. Diff. Cat.*, **L-2**, (2009) 144–155.

[8] [Guitart R., 2012] Borromean and Circular Aspects of Rotations and Elliptic Functions, *preprint*, 20 février 2012, 23 pages.

Jean-Yves Degos
Rés. Les Lotus, appt 19
22, avenue de Chiquet
33600 Pessac (France)
jydegos@gmail.com

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Vol. LIV-3 (2013)

FAITHFUL CALCULUS OF FRACTIONS

by O. ABBAD and E.M. VITALE

Dedicated to René Guitart on the occasion of his 65th birthday

Résumé. Dans cet article, nous développons un argument simple sur les bicatégories de fractions qui montre que, si Σ est la classe des équivalences faibles entre groupoides internes à une catégorie régulière $\mathcal{A}$ qui admet suffisamment d'objets projectifs réguliers, alors la description de $\mathrm{Grpd}(\mathcal{A})[\Sigma^{-1}]$ peut être considérablement simplifiée.

Résumé. The aim of this note is to develop a simple argument on bicategories of fractions showing that, if Σ is the class of weak equivalences between groupoids internal to a regular category $\mathcal{A}$ with enough regular projective objects, then the description of $\mathrm{Grpd}(\mathcal{A})[\Sigma^{-1}]$ can be considerably simplified.

1. Introduction

Bicategories of fractions, the 2-dimensional analogue of Gabriel and Zisman's categories of fractions [9], have been introduced by D. Pronk [14] and used mainly to study fractions of 2-categories of internal functors between various kinds of internal structures (internal categories, internal groupoids, internal crossed modules, etc.), see for example [16] for recent applications. Recently, general results on bicategories of fractions of internal functors with respect to internal weak equivalences have been obtained in [1, 10, 15]. In particular, in [1] the bicategory of fractions of crossed modules internal to a semi-abelian category $\mathcal{A}$ has been described in terms of "butterflies". This description generalizes the case where the base category $\mathcal{A}$ is the category of groups, which

2000 Mathematics Subject Classification: 18D05,18A22, 18D35, 18D40, 18G05.
Key words and phrases: bicategory of fractions, weak equivalence, projective objects, internal groupoid.

is the case studied by B. Noohi in [11, 12] (see also [2]). It is interesting to notice that bicategories of fractions do not appear explicitly in [11, 12], where the main result is stated in terms of an equivalence between hom-categories

$$\mathcal{B}(A, B) \simeq \mathcal{C}(X, B)$$

where $\mathcal{C}$ is the 2-category of crossed modules of groups, $\mathcal{B}$ is the bicategory of butterflies in groups, and X is a cofibrant replacement of A. In [1, Proposition 8.1], we explain that this equivalence of hom-categories easily follows from the fact that $\mathcal{B}$ is indeed the bicategory of fractions of $\mathcal{C}$ and the fact that the category of groups has enough regular projective objects. Moreover, a general argument on bicategories of fractions, subsuming the previous equivalence, is announced [1, Remark 8.2].

The aim of this note is to fully develop such an argument: we will show that, if the class Σ of arrows to be inverted has a "faithful calculus of fractions", a condition stronger than Pronk's right calculus of fractions, and if $\mathcal{C}$ has enough Σ-projective objects, then the description of the bicategory of fractions $\mathcal{C}[\Sigma^{-1}]$ can be drastically simplified and the equivalence

$$\mathcal{C}[\Sigma^{-1}](A, B) \simeq \mathcal{C}(X, B)$$

becomes almost tautological. The surprise is that, despite the fact that the condition of having a faithful calculus of fractions is a very strong condition (so strong that its 1-dimensional version for categories of fractions is probably totally uninteresting), it is satisfied by the prominent example where $\mathcal{C}$ is the 2-category of groupoids and functors internal to a regular category, and Σ is the class of weak equivalences. Moreover, the fact that $\mathcal{C}$ has enough Σ-projective objects holds if the base category has enough regular projective objects. This covers the case of groups and of Lie algebras studied in [11, 12, 2, 17].

Notation: the composite of $f\colon A \to B$ and $g\colon B \to C$ is written $f \cdot g$.

2. Calculus of fractions

The reader can consult [4] or [6, Chapter 7] for an introduction to Bénabou's notion of bicategory. In this paper, bicategory means bicategory with invertible 2-cells. Moreover, for the sake of readability, we write diagrams and equations as in a 2-category. Let us start with a point of standard terminology

Definition 2.1 Let $f\colon X \to Y$ be a 1-cell in a bicategory $\mathcal{C}$. We say that f is

1. full (faithful) if, for every object $C \in \mathcal{C}$, the functor

$$\mathcal{C}(C,f)\colon \mathcal{C}(C,X) \to \mathcal{C}(C,Y)$$

   is full (faithful); in other words, for every 2-cell $\beta\colon h \cdot f \Rightarrow k \cdot f$, there exists at least (at most) a 2-cell $\alpha\colon h \Rightarrow k$ such that $\alpha \cdot f = \beta$;

2. an equivalence if, for every object $C \in \mathcal{C}$, the functor

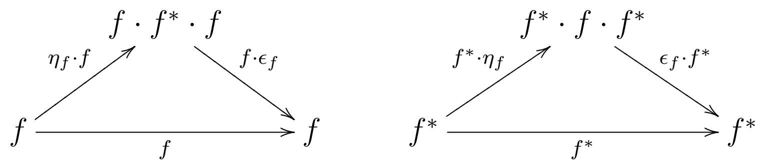
$$\mathcal{C}(C,f)\colon \mathcal{C}(C,X) \to \mathcal{C}(C,Y)$$

   is an equivalence of categories; in other words, there exist a 1-cell $f^*\colon Y \to X$ and two 2-cells $\epsilon_f\colon f^* \cdot f \Rightarrow 1_Y$ and $\eta_f\colon 1_X \Rightarrow f \cdot f^*$.

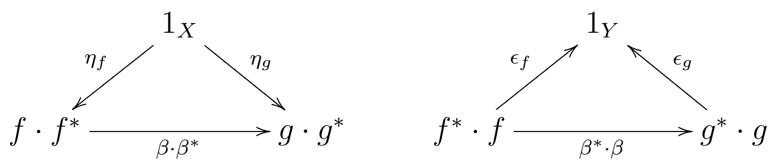
Remark 2.2

1. If f is full and faithful and there exists $\epsilon_f\colon f^* \cdot f \Rightarrow 1_Y$, then f is an equivalence.

2. If f is an equivalence, it is always possible to choose η_f and ϵ_f so that the usual triangular identities are satisfied:



3. If $f, g\colon X \to Y$ are equivalences, $\beta\colon f \Rightarrow g$ is a 2-cell, and $(f^*, \eta_f, \epsilon_f)$ and $(g^*, \eta_g, \epsilon_g)$ satisfy the triangular identities, then there exists a unique $\beta^*\colon f^* \Rightarrow g^*$ making commutative the following diagrams:



4. If f is an equivalence, for every object C the functor

$$\mathcal{C}(f,C)\colon \mathcal{C}(Y,C) \to \mathcal{C}(X,C)$$

is an equivalence of categories (use the triangular identities to check that it is full).

5. If $f\colon X \to Y$ and $g\colon Y \to Z$ are full (faithful) (equivalences), then so is the composite $f \cdot g\colon X \to Z$.

Now we recall from [14] the general definition of bicategory of fractions and we introduce the notion of faithful calculus of fractions.

Definition 2.3 (Pronk) Let Σ be a class of 1-cells in a bicategory $\mathcal{C}$. The bicategory of fractions of $\mathcal{C}$ with respect to Σ is a homomorphism of bicategories

$$P_\Sigma\colon \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$$

universal among all homomorphisms $\mathcal{F}\colon \mathcal{C} \to \mathcal{A}$ such that $\mathcal{F}(s)$ is an equivalence for all $s \in \Sigma$. In other words, for every bicategory $\mathcal{A}$,

$$P_\Sigma \cdot - \; : \; Hom(\mathcal{C}[\Sigma^{-1}], \mathcal{A}) \to Hom_\Sigma(\mathcal{C}, \mathcal{A})$$

is a biequivalence of bicategories, where $Hom_\Sigma(\mathcal{C}, \mathcal{A})$ is the bicategory of those homomorphisms $\mathcal{F}$ such that $\mathcal{F}(s)$ is an equivalence for all $s \in \Sigma$.

Definition 2.4 Let Σ be a class of 1-cells in a bicategory $\mathcal{C}$. The class Σ has a faithful calculus of fractions if the following conditions hold:

FF1. Σ contains all equivalences;

FF2. Given 1-cells $f\colon X \to Y$ and $g\colon Y \to Z$ with $g \in \Sigma$, then $f \cdot g \in \Sigma$ iff $f \in \Sigma$;

FF3. For every $\begin{array}{ccc} & & C \\ & & \downarrow{\scriptstyle g\in\Sigma} \\ A & \xrightarrow[f]{} & B \end{array}$ there exists $\begin{array}{ccc} P & \xrightarrow{f'} & C \\ {\scriptstyle g'\in\Sigma}\downarrow & \Rightarrow & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array}$

FF4. If there exists a 2-cell $f \Rightarrow g$, then $f \in \Sigma$ iff $g \in \Sigma$;

FF5. Σ is contained in the class of full and faithful 1-cells.

Remark 2.5 In (FF3), if $f \in \Sigma$, then $f' \in \Sigma$. Indeed, $g', f \in \Sigma$, so that, by (FF2), $g' \cdot f \in \Sigma$ and then, by (FF4), $f' \cdot g \in \Sigma$. Since $g \in \Sigma$, (FF2) implies now $f' \in \Sigma$.

It is easy to compare the conditions defining a faithful calculus of fractions with those defining a right calculus of fractions in the sense of [14].

Proposition 2.6 *Let Σ be a class of 1-cells in a bicategory $\mathcal{C}$. If Σ has a faithful calculus of fractions, then it has a right calculus of fractions.*

Proof. We have to check the following condition:

RF. For every $\alpha\colon f \cdot w \Rightarrow g \cdot w$ with $w \in \Sigma$, there exist $v \in \Sigma$ and $\beta\colon v \cdot f \Rightarrow v \cdot g$ such that $v \cdot \alpha = \beta \cdot w$, and for any other $v' \in \Sigma$ and $\beta'\colon v' \cdot f \Rightarrow v' \cdot g$ such that $v' \cdot \alpha = \beta' \cdot w$, there exist u, u' and $\varepsilon\colon u \cdot v \Rightarrow u' \cdot v'$ such that $u \cdot v \in \Sigma$ and

$$\begin{array}{ccc} u \cdot v \cdot f & \xrightarrow{u \cdot \beta} & u \cdot v \cdot g \\ {\scriptstyle \varepsilon \cdot f}\downarrow & & \downarrow{\scriptstyle \varepsilon \cdot g} \\ u' \cdot v' \cdot f & \xrightarrow[u' \cdot \beta']{} & u' \cdot v' \cdot g \end{array}$$

commutes.

As far as the existence of (v, β) is concerned, we can take $v = 1_X \in \Sigma$ and, since w is full, there exists $\beta\colon f \Rightarrow g$ such that $\beta \cdot w = \alpha$.
Let now (v, β) and (v', β') be as in condition (RF); by (FF3), there exists $\epsilon\colon u \cdot v \Rightarrow u' \cdot v'$ with $u \in \Sigma$ and then $u \cdot v \in \Sigma$. It remains to show that the diagram in condition (RF) commutes. Since w is faithful, it is enough to check the commutativity of the diagram obtained by composing with w

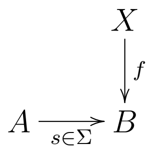
$$\begin{array}{ccc} u \cdot v \cdot f \cdot w & \xrightarrow{u \cdot \beta \cdot w} & u \cdot v \cdot g \cdot w \\ {\scriptstyle \varepsilon \cdot f \cdot w}\downarrow & & \downarrow{\scriptstyle \varepsilon \cdot g \cdot w} \\ u' \cdot v' \cdot f \cdot w & \xrightarrow[u' \cdot \beta' \cdot w]{} & u' \cdot v' \cdot g \cdot w \end{array}$$

and this is obvious because we can replace $u \cdot \beta \cdot w$ by $u \cdot v \cdot \alpha$ and $u' \cdot \beta' \cdot w$ by $u' \cdot v' \cdot \alpha$. ∎

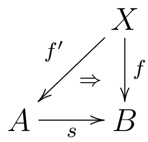
3. Σ-projective objects and Σ-covers

Definition 3.1 Let Σ be a class of 1-cells in a bicategory $\mathcal{C}$.

1. An object X is Σ-projective if, for every 1-cell $s\colon A \to B$ in Σ, the functor
$$\mathcal{C}(X,s)\colon \mathcal{C}(X,A) \to \mathcal{C}(X,B)$$
is essentially surjective; in other words, for every



there exists



2. A Σ-cover of an object A is a 1-cell $a\colon X \to A$ in Σ with X a Σ-projective object.

3. We say that $\mathcal{C}$ has enough Σ-projective objects if each object has a Σ-cover.

Remark 3.2 Assume that Σ is contained in the class of full and faithful 1-cells.

1. If $s\colon A \to X$ is in Σ and X is a Σ-projective object, then s is an equivalence. Indeed, use condition 3.1.1 with $f = 1_X$ to get s^* and ϵ_s, and conclude by Remark 2.2.1.

2. If a Σ-cover of an object exists, then it is unique up to an essentially unique equivalence.

In Example 3.5, we will state that the class of weak equivalences between groupoids internal to a regular category has a faithful calculus of fractions. The reader can consult [7, Chapter 2] for an introduction to regular categories (in the sense of M. Barr [3]), and [6, Chapter 8] for basic facts about internal category theory. If $\mathcal{A}$ is a category with finite limits, we denote by Grpd($\mathcal{A}$) the 2-category of groupoids, functors and natural transformations internal to $\mathcal{A}$. The notions of essentially surjective and of weak equivalence for internal functors come from [8].

Definition 3.3 (Bunge-Paré) Let $\mathcal{A}$ be a regular category and let

$$\begin{array}{ccc} A_1 & \xrightarrow{F_1} & B_1 \\ {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ A_0 & \xrightarrow[F_0]{} & B_0 \end{array}$$

be a functor between groupoids in $\mathcal{A}$. The functor (F_1, F_0) is:

1. essentially surjective (on objects) if

$$A_0 \times_{F_0,d} B_1 \xrightarrow{t_2} B_1 \xrightarrow{c} B_0$$

   is a regular epimorphism, where t_2 is defined by the following pullback

$$\begin{array}{ccc} A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle d} \\ A_0 & \xrightarrow[F_0]{} & B_0 \end{array}$$

2. a weak equivalence if it is full and faithful and essentially surjective.

Remark 3.4 With the notation of Definition 3.3. A functor (F_1, F_0) is:

1. full and faithful iff the following diagram is a limit diagram

$$\begin{array}{ccccccc} & & & A_1 & & & \\ & & {\scriptstyle d}\swarrow & \downarrow{\scriptstyle F_1} & \searrow{\scriptstyle c} & & \\ & A_0 & & B_1 & & A_0 & \\ & {\scriptstyle F_0}\searrow & \swarrow{\scriptstyle d} & & {\scriptstyle c}\searrow & \swarrow{\scriptstyle F_0} & \\ & & B_0 & & B_0 & & \end{array}$$

2. an equivalence iff it is full and faithful and

$$A_0 \times_{F_0,d} B_1 \xrightarrow{t_2} B_1 \xrightarrow{c} B_0$$

is a split epimorphism.

Example 3.5 Let $\mathcal{A}$ be a regular category and Σ the class of weak equivalences in the 2-category Grpd($\mathcal{A}$).

1. Σ has a faithful calculus of fractions.
The proof can be reconstructed by examining the proofs of Proposition 4.5 and Proposition 5.5 in [17]. For the reader's convenience we reproduce here some points; we refer to [17] for more details.
- Condition (FF1) immediately follows from Remark 3.4.2.
- Condition (FF2): consider two internal functors $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{B} \to \mathbb{C}$

$$\begin{array}{ccccc} A_1 & \xrightarrow{F_1} & B_1 & \xrightarrow{G_1} & C_1 \\ {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ A_0 & \xrightarrow[F_0]{} & B_0 & \xrightarrow[G_0]{} & C_0 \end{array}$$

• If F and G are essentially surjective, so is the composite $F \cdot G$: consider the following pullbacks

$$\begin{array}{ccc} A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle d} \\ A_0 & \xrightarrow[F_0]{} & B_0 \end{array} \qquad \begin{array}{ccc} B_0 \times_{G_0,d} C_1 & \xrightarrow{t_2} & C_1 \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle d} \\ B_0 & \xrightarrow[G_0]{} & C_0 \end{array}$$

$$\begin{array}{ccc} A_0 \times_{F_0 \cdot G_0,d} C_1 & \xrightarrow{\tau_2} & C_1 \\ {\scriptstyle \tau_1}\downarrow & & \downarrow{\scriptstyle d} \\ A_0 & \xrightarrow[F_0 \cdot G_0]{} & C_0 \end{array}$$

and the commutative diagram (where m is the internal composition in $\mathbb{C}$)

$$\begin{array}{ccccc}
A_0 \times_{F_0,d} B_1 \times_{G_1\cdot c,d} C_1 & \xrightarrow{t_2\times 1} & B_1 \times_{G_1\cdot c,d} C_1 & \xrightarrow{c\times 1} & B_0 \times_{G_0,d} C_1 \\
{\scriptstyle 1\times G_1\times 1}\downarrow & & & & \downarrow{\scriptstyle t_2} \\
A_0 \times_{F_0\cdot G_0,d} C_1 \times_{c,d} C_1 & & & & C_1 \\
{\scriptstyle 1\times m}\downarrow & & & & \downarrow{\scriptstyle c} \\
A_0 \times_{F_0\cdot G_0,d} C_1 & \xrightarrow[\tau_2]{} & C_1 & \xrightarrow[c]{} & C_0
\end{array}$$

In a regular category, regular epimorphisms are closed under composition and finite products; moreover, if a composite is a regular epimorphism then the last component is a regular epimorphism. Therefore, from the previous diagram we deduce that $\tau_2 \cdot c$ is a regular epimorphism, as needed.

• If $F \cdot G$ is essentially surjective and G is full and faithful, then F is essentially surjective: consider one more pullback

$$\begin{array}{ccc}
Q & \xrightarrow{\lambda_2} & B_0 \\
{\scriptstyle \lambda_1}\downarrow & & \downarrow{\scriptstyle G_0} \\
A_0 \times_{F_0\cdot G_0,d} C_1 \xrightarrow[\tau_2]{} C_1 & \xrightarrow[c]{} & C_0
\end{array}$$

We have that λ_2 is a regular epimorphism because, by assumption, $\tau_2 \cdot c$ is a regular epimorphism and regular epimorphisms are pullback stable in any regular category. Since G is full and faithful, by Remark 3.4.1 we get $\lambda\colon Q \to B_1$ such that $\lambda \cdot d = \lambda_1 \cdot \tau_1 \cdot F_0$, $\lambda \cdot G_1 = \lambda_1 \cdot \tau_2$ and $\lambda \cdot c = \lambda_2$. From the first equation on λ, we get $\mu\colon Q \to A_0 \times_{F_0,d} B_1$ such that $\mu \cdot t_1 = \lambda_1 \cdot \tau_1$ and $\mu \cdot t_2 = \lambda$. Finally, $\mu \cdot t_2 \cdot c = \lambda \cdot c = \lambda_2$, so that $t_2 \cdot c$ is a regular epimorphism, as needed.

- The stability of regular epimorphisms under pullbacks gives also that Σ is stable under bipullbacks (in the sense of bilimits introduced in [5]). This immediately implies condition (FF3).

- Condition (FF4) is a simple exercise and condition (FF5) is obvious by definition of weak equivalence.

Recall that an object X_0 of the base category $\mathcal{A}$ is regular projective if the functor $\mathcal{A}(X_0, -)\colon \mathcal{A} \to \mathrm{Set}$ preserves regular epimorphisms. The category $\mathcal{A}$ has enough regular projective objects if for every object $A_0 \in \mathcal{A}$ there exists a regular epimorphism $X_0 \to A_0$ with X_0 regular projective. Examples of regular categories with enough regular projective objects abound: monadic categories over a power of Set and their regular epireflective subcategories are of this kind. In particular, algebraic categories, varieties and quasi-varieties of universal algebras are of this kind (see for example [13]), as well as presheaf categories and categories of separated presheaves.

2. If $\mathcal{A}$ has enough regular projective objects, then $\mathrm{Grpd}(\mathcal{A})$ has enough Σ-projective objects.
   For this, start with an internal groupoid and a regular epimorphism S_0

$$\begin{array}{ccc} & & A_1 \\ & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ X_0 & \xrightarrow[S_0]{} & A_0 \end{array}$$

   with X_0 a regular projective object. Consider the limit diagram

$$\begin{array}{ccccccc} & & & X_1 & & & \\ & & {\scriptstyle d}\swarrow & \downarrow{\scriptstyle S_1} & \searrow{\scriptstyle c} & & \\ & X_0 & & A_1 & & X_0 & \\ & {\scriptstyle S_0}\searrow & \swarrow{\scriptstyle d} & & {\scriptstyle c}\searrow & \swarrow{\scriptstyle S_0} & \\ & & A_0 & & A_0 & & \end{array}$$

   The graph $d, c \colon X_1 \rightrightarrows X_0$ inherits a structure of groupoid from that of $d, c \colon A_1 \rightrightarrows A_0$, and the functor (F_1, F_0) is a weak equivalence. Indeed, it is full and faithful by construction, and it is essentially surjective because in

$$X_0 \times_{S_0,d} A_1 \xrightarrow{t_2} A_1 \xrightarrow{c} A_0$$

   t_2 is a regular epimorphism (because S_0 is a regular epimorphism) and c is a split epimorphism. Finally, since X_0 is regular projective, by Remark 3.4.2 every weak equivalence with codomain

$X_1 \rightrightarrows X_0$ is an equivalence. Since weak equivalences are stable under bipullbacks, this is enough to ensure the Σ-projectivity of $X_1 \rightrightarrows X_0$.

4. The bicategory of fractions

4.1 Let $\mathcal{C}$ be a bicategory and Σ any class of 1-cells in $\mathcal{C}$. We can construct a new bicategory

$$\mathcal{C}[\Sigma^*]$$

having Σ-covers as objects and, as hom-categories,

$$\mathcal{C}[\Sigma^*](a\colon X \to A, b\colon Y \to B) = \mathcal{C}(X, Y)$$

with identities and horizontal and vertical compositions given by those of $\mathcal{C}$.

Remark 4.2

1. If $\mathcal{C}$ is a 2-category, then $\mathcal{C}[\Sigma^*]$ is a 2-category as well.

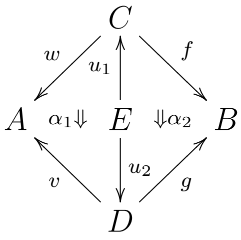
2. If $b\colon Y \to B$ is full and faithful, then the functor $\mathcal{C}(X, b)$ is full and faithful, and it is essentially surjective because X is Σ-projective, so that it induces an equivalence of categories

$$\mathcal{C}[\Sigma^*](a\colon X \to A, b\colon Y \to B) \simeq \mathcal{C}(X, B)$$

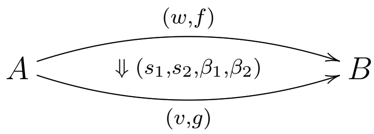
4.3 Under the assumption that the class Σ has a right calculus of fractions, the bicategory of fractions $\mathcal{C}[\Sigma^{-1}]$ has been described in [14]: objects are those of $\mathcal{C}$, 1-cells and pre-2-cells

$$A \; \underset{(v,g)}{\overset{(w,f)}{\substack{\longrightarrow \\ \Downarrow (u_1,u_2,\alpha_1,\alpha_2) \\ \longrightarrow}}} \; B$$

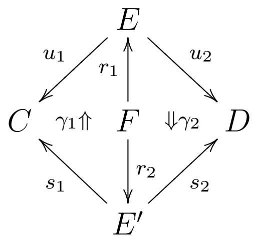
are depicted in the following diagram



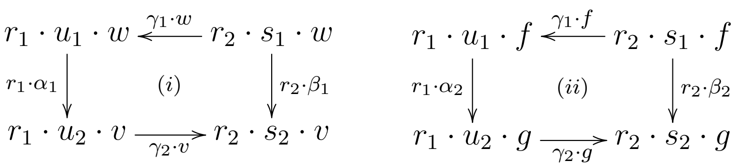
with $w, v, u_1 \cdot w \simeq u_2 \cdot v \in \Sigma$. Given another pre-2-cell



then the pre-2-cells $(u_1, u_2, \alpha_1, \alpha_2)$ and $(s_1, s_2, \beta_1, \beta_2)$ are equivalent if there exists $(r_1, r_2, \gamma_1, \gamma_2)$ as in

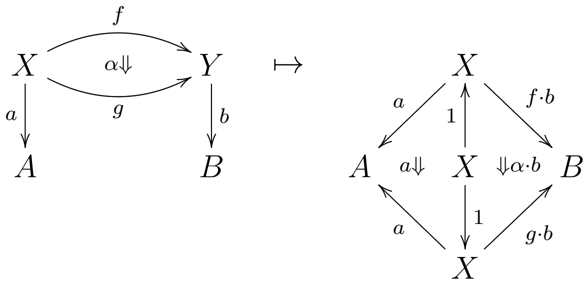


such that $r_1 \cdot u_1 \cdot w \simeq r_2 \cdot s_1 \cdot w \in \Sigma$ and such that the following diagrams commute



Clearly, there is a homomorphism of bicategories $\mathcal{E} \colon \mathcal{C}[\Sigma^*] \to \mathcal{C}[\Sigma^{-1}]$

defined by



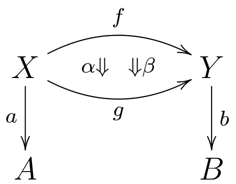
Proposition 4.4 *Let Σ be a class of 1-cells in a bicategory $\mathcal{C}$. If Σ has a faithful calculus of fractions and $\mathcal{C}$ has enough Σ-projective objects, then $\mathcal{E}\colon \mathcal{C}[\Sigma^*] \to \mathcal{C}[\Sigma^{-1}]$ is a biequivalence.*

More precisely, we are going to prove the following statements:

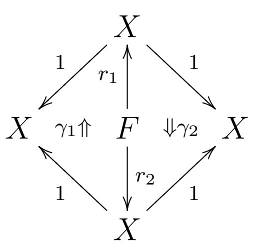
1. If Σ has a faithful calculus of fractions, then $\mathcal{E}$ is locally an equivalence.

2. If $\mathcal{C}$ has enough Σ-projective objects, then $\mathcal{E}$ is surjective on objects.

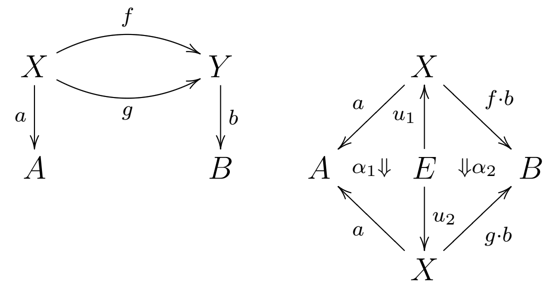
Proof. 1. $\mathcal{E}$ is locally faithful: let



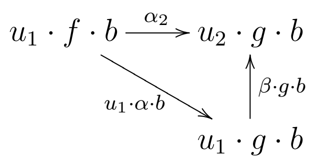
be 2-cells in $\mathcal{C}[\Sigma^*]$ and let



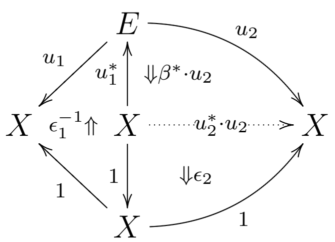
be the datum attesting that $\mathcal{E}(\alpha) = \mathcal{E}(\beta)$ in $\mathcal{C}[\Sigma^{-1}]$. Since $a, r_2 \cdot a \in \Sigma$, then by (FF2) $r_2 \in \Sigma$, and then it is an equivalence because X is Σ-projective. The first condition on $(r_1, r_2, \gamma_1, \gamma_2)$ implies that $\gamma_1 = \gamma_2^{-1}$, the second condition gives then $r_2 \cdot \alpha \cdot b = r_2 \cdot \beta \cdot b$. Since r_2 is an equivalence and b is faithful, we have $\alpha = \beta$.
$\mathcal{E}$ is locally full: consider two 1-cells f, g in $\mathcal{C}[\Sigma^*]$ and a 2-cell $\mathcal{E}(f) \Rightarrow \mathcal{E}(g)$ as follows



Since a is full and faithful, there exists a unique $\beta \colon u_1 \Rightarrow u_2$ such that $\beta \cdot a = \alpha_1$. Moreover, $a, u_1 \cdot a \in \Sigma$, so that $u_1 \in \Sigma$ by (FF2), and then u_1 is an equivalence because X is Σ-projective (the same argument holds for u_2). Since b also is full and faithful, there exists a unique $\alpha \colon f \Rightarrow g$ such that



commutes. To check that $\mathcal{E}(\alpha) = [u_1, u_2, \alpha_1, \alpha_2]$ we use the following datum, where $\beta^* \colon u_1^* \Rightarrow u_2^*$ corresponds to $\beta \colon u_1 \Rightarrow u_2$ as in Remark 2.2.3:



Condition (i) easily follows from the definition of β and Remark 2.2.3. As far as condition (ii) is concerned, since u_1 is an equivalence, by

Remark 2.2.4 it is enough to check it precomposing with u_1. Using the definition of α, condition (ii) reduces now to the commutativity of

$$\begin{array}{ccccc} u_1 \cdot u_1^* \cdot u_1 \cdot f \cdot b & & \xrightarrow{u_1 \cdot \epsilon_1 \cdot f \cdot b} & & u_1 \cdot f \cdot b \\ {\scriptstyle \beta \cdot \beta^* \cdot u_1 \cdot f \cdot b} \downarrow & & & & \downarrow {\scriptstyle \alpha_2} \\ u_2 \cdot u_2^* \cdot u_1 \cdot f \cdot b & \xrightarrow[u_2 \cdot u_2^* \cdot \alpha_2]{} & u_2 \cdot u_2^* \cdot u_2 \cdot g \cdot b & \xrightarrow[u_2 \cdot \epsilon_2 \cdot g \cdot b]{} & u_2 \cdot g \cdot b \end{array}$$

To check this last equation, past on the left side the commutative triangle

$$\begin{array}{ccc} u_1 \cdot f \cdot b & \xrightarrow{\eta_1 \cdot u_1 \cdot f \cdot b} & u_1 \cdot u_1^* \cdot u_1 \cdot f \cdot b \\ & {\scriptstyle \eta_2 \cdot u_1 \cdot f \cdot b} \searrow & \downarrow {\scriptstyle \beta \cdot \beta^* \cdot u_1 \cdot f \cdot b} \\ & & u_2 \cdot u_2^* \cdot u_1 \cdot f \cdot b \end{array}$$

and use the first triangular identity on η_1, ϵ_1 and on η_2, ϵ_2, so that both paths reduce to $\alpha_2 \colon u_1 \cdot f \cdot b \Rightarrow u_2 \cdot g \cdot g$.
$\mathcal{E}$ is locally essentially surjective: consider two objects $a \colon X \to A$ and $b \colon Y \to B$ in $\mathcal{C}[\Sigma^*]$ and a 1-cell

$$A \xleftarrow{w} C \xrightarrow{f} B$$

in $\mathcal{C}[\Sigma^{-1}]$. Using twice that X is Σ-projective, we get

$$\begin{array}{ccc} & X & \\ {\scriptstyle h} \swarrow & \varphi \Rightarrow & \searrow {\scriptstyle a} \\ C & \xrightarrow[w]{} & A \end{array} \qquad \begin{array}{ccc} & X & \\ {\scriptstyle k} \swarrow & \psi \Rightarrow & \searrow {\scriptstyle h \cdot f} \\ Y & \xrightarrow[b]{} & B \end{array}$$

This gives a 1-cell in $\mathcal{C}[\Sigma^*]$ and a 2-cell in $\mathcal{C}[\Sigma^{-1}]$

$$\begin{array}{ccc} X & \xrightarrow{k} & Y \\ {\scriptstyle a} \downarrow & & \downarrow {\scriptstyle b} \\ A & & B \end{array} \qquad \begin{array}{ccccc} & & X & & \\ & {\scriptstyle a} \swarrow & \uparrow {\scriptstyle 1} & \searrow {\scriptstyle k \cdot b} & \\ A & \varphi^{-1} \Downarrow & X & \Downarrow \psi & B \\ & {\scriptstyle w} \nwarrow & \downarrow {\scriptstyle h} & \nearrow {\scriptstyle f} & \\ & & C & & \end{array}$$

attesting that $\mathcal{E}$ is locally essentially surjective.
2. Obvious, just choose a Σ-cover $a \colon X \to A$ for every object A of $\mathcal{C}$. ■

Remark 4.5 Putting together Remark 4.2.2 and Proposition 4.4.1, we get an equivalence of hom-categories

$$\mathcal{C}[\Sigma^{-1}](A,B) \simeq \mathcal{C}[\Sigma^{*}](a\colon X \to A, b\colon Y \to B) \simeq \mathcal{C}(X,B),$$

as announced in the Introduction.

5. Extensions as fractions

In order to illustrate the difference between $\mathcal{C}[\Sigma^{-1}]$ and $\mathcal{C}[\Sigma^{*}]$, we discuss a special case of Example 3.5. We consider the bicategory $\mathrm{Grpd}(\mathcal{A})$ and we assume that $\mathcal{A}$ is semi-abelian, has split extension classifiers, and satisfies the "Huq = Smith" condition as in [1]. The typical examples of such an $\mathcal{A}$ are the category of groups (where the split extension classifier of a group H is the group of automorphisms of H) and the category of Lie algebras (where the split extension classifier of an algebra H is the Lie algebra of derivations of H).

Fix two objects G and H in $\mathcal{A}$. From [1, Section 7], we know that the groupoid of extensions $\mathrm{EXT}(G,H)$ is isomorphic to the hom-groupoid $\mathcal{B}(\mathcal{A})(D(G),[[H]])$, where $\mathcal{B}(\mathcal{A})$ is the bicategory of internal butterflies in $\mathcal{A}$ (since $\mathcal{A}$ is semi-abelian, we do not take care of the difference between internal groupoids and internal crossed modules), $D(G)$ is the discrete internal groupoid on G, and $[[H]]$ is the action groupoid, that is, the internal groupoid having the split extension classifier $[H]$ as object of objects and the holomorph $H \rtimes [H]$ as object of arrows. Since $\mathcal{B}(\mathcal{A})$ is biequivalent to the bicategory of fractions of $\mathrm{Grpd}(\mathcal{A})$ with respect to weak equivalences [1, Theorem 5.6], we have an equivalence

$$\mathrm{EXT}(G,H) \simeq \mathrm{Grpd}(\mathcal{A})[\Sigma^{-1}](D(G),[[H]])$$

and, by Remark 4.5, we also have an equivalence

$$\mathrm{EXT}(G,H) \simeq \mathrm{Grpd}(\mathcal{A})(\mathbb{X},[[H]])$$

Accordingly, we can describe an extension

$$H \xrightarrow{\iota} E \xrightarrow{\sigma} G$$

as a span of internal functors (with the left leg being a weak equivalence) or as a single internal functor. In the first case, we get the span

$$\begin{array}{ccccccc}
G & \xleftarrow{\sigma_i\cdot\sigma} & R[\sigma] & \xrightarrow{\simeq} & H \rtimes E & \xrightarrow{1\rtimes\mathcal{I}} & H \rtimes [H] \\
{\scriptstyle 1}\downarrow\downarrow{\scriptstyle 1} & & {\scriptstyle \sigma_1}\downarrow\downarrow{\scriptstyle \sigma_2} & & & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\
G & \xleftarrow[\sigma]{} & E & & \xrightarrow[\mathcal{I}]{} & & [H]
\end{array}$$

where $\sigma_1, \sigma_2 \colon R[\sigma] \rightrightarrows E$ is the kernel relation of σ, and $\mathcal{I} \colon E \to [H]$ is the action induced by the fact that $\iota \colon H \to E$ is normal. This is a "discrete fraction", in the sense that the right leg is a discrete fibration. To transform this span into a single internal functor, we fix a regular projective cover $s \colon X_0 \to G$ of G together with an extension σ_0 of s along σ as in the following commutative diagram

$$\begin{array}{ccc}
 & & X_0 \\
 & {\scriptstyle \sigma_0}\swarrow & \downarrow{\scriptstyle s} \\
E & \xrightarrow[\sigma]{} & G
\end{array}$$

Composing with the discrete fibration above, we get the internal functor

$$\begin{array}{ccccc}
R[s] & \xrightarrow{\overline{\sigma}} & R[\sigma] & \longrightarrow & H \rtimes [H] \\
{\scriptstyle s_1}\downarrow\downarrow{\scriptstyle s_2} & & {\scriptstyle \sigma_1}\downarrow\downarrow{\scriptstyle \sigma_2} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\
X_0 & \xrightarrow[\sigma_0]{} & E & \xrightarrow[\mathcal{I}]{} & [H]
\end{array}$$

where $s_1, s_2 \colon R[s] \rightrightarrows X_0$ is the kernel relation of s, and $\overline{\sigma}$ is the canonical factorization of $R[s]$ through $R[\sigma]$.

References

[1] O. Abbad, S. Mantovani, G. Metere and E.M. Vitale, Butterflies in a semi-abelian context, *Advances in Mathematics* **238** (2013) 140–183.

[2] E. Aldrovandi, B. Noohi, Butterflies I. Morphisms of 2-group stacks, Advances in Mathematics **221** (2009) 687–773.

[3] M. Barr, Exact categories, *Springer LNM* **236** (1971) 1–120.

[4] J. Bénabou, Introduction to bicategories, *Springer LNM* **40** (1967) 1–77.

[5] J. Bénabou, Some remarks on 2-categorical algebra, *Bulletin de la Société Mathématique de Belgique* **41** (1989) 127–194.

[6] F. Borceux, Handbook of Categorical Algebra 1, *Cambridge University Press* (1994).

[7] F. Borceux, Handbook of Categorical Algebra 2, *Cambridge University Press* (1994).

[8] M. Bunge and R. Paré, Stacks and equivalence of indexed categories, *Cahiers de Topologie et Géométrie Différentielle Catégorique* **20** (1979) 373–399.

[9] P. Gabriel and M. Zisman, Calculus of Fractions and Homotopy Theory, *Springer* (1967).

[10] S. Mantovani, G. Metere and E.M. Vitale, Profunctors in Mal'tsev categories and fractions of functors, *Journal of Pure and Applied Algebra* **217** (2013) 1173–1186.

[11] B. Noohi, Notes on 2-groupoids, 2-groups and crossed modules, *Homology, Homotopy and Applications* **9** (2007) 75–106.

[12] B. Noohi, On weak maps between 2-groups, arXiv:math/0506313.

[13] M.C. Pedicchio and E.M. Vitale, On the abstract characterization of quasi-varieties, *Algebra Universalis* **43** (2000) 269–278.

[14] D. Pronk, Etendues and stacks as bicategories of fractions, *Compositio Mathematica* **102** (1996) 243–303.

[15] D.M. Roberts, Internal categories, anafunctors and localisations, *Theory and Applications of Categories* **26** (2012) 788–829.

[16] M. Tommasini, A bicategory of reduced orbifolds from the point of view of differential geometry, arXiv:1304.6959.

[17] E.M. Vitale, Bipullbacks and calculus of fractions, *Cahiers de Topologie et Géométrie Différentielle Catégorique* **51** (2010) 83–113.

Institut de Recherche en Mathématique et Physique
Université catholique de Louvain
Chemin du Cyclotron 2
B 1348 Louvain-la-Neuve, Belgique
Email: oabbad@hotmail.com, enrico.vitale@uclouvain.be