cahiers de topologie et géométrie différentielle catégoriques

créés par CHARLES EHRESMANN en 1958
dirigés par Andrée CHARLES EHRESMANN
VOLUME LIV-1, 1er trimestre 2013

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CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Vol. LIV-1 (2013)

A DIRECT PROOF THAT THE CATEGORY OF 3-COMPUTADS IS NOT CARTESIAN CLOSED

by Eugenia CHENG

Abstract

Nous démontrons par contre-exemple que la catégorie des 3-computads (ou "3-polygraphes") n'est pas cartésienne fermée; ce résultat était démontré premièrement par Makkai et Zawadowski. Nous donnons un 3-computad B et nous démontrons que le foncteur $\_ \times B$ n'a pas d'adjoint à droite de la façon suivante: nous donnons un conoyau qui n'est pas respecté par ce foncteur.

We prove by counterexample that the category of 3-computads is not cartesian closed, a result originally proved by Makkai and Zawadowski. We give a 3-computad B and show that the functor $\_ \times B$ does not have a right adjoint, by giving a coequaliser that is not preserved by it.

Keywords: computad, cartesian closed, presheaf.

MSC2000: 18D05

Introduction

Makkai and Zawadowski proved in [7] that the category of (strict) 3-computads is not cartesian closed and hence is not a presheaf category. The result can be considered surprising—for example, the opposite was erroneously claimed in [3] (and corrected after Makkai and Zawadowski, in [4]).

The reason is related to the Eckmann-Hilton argument, but the proof given in [7], while having this reason at its heart, uses some sophisticated technology to bring this "reason" to fruition—some technical results of [3] for Artin glueing, which in turn rely on some technical results of Day [6].

In this paper we give a direct counterexample, that is, we give a 3-computad B and a coequaliser

$$E \rightrightarrows A \longrightarrow C$$

that is not preserved by the functor $\_ \times B$, hence $\_ \times B$ does not have a right adjoint.

The idea behind this counterexample is the same as the idea behind the proof in [7], and the result is, evidently, not new. However, we believe it is of value to provide this direct argument.

The root of the problem is that 2-cells having 1-cell identities as source and target do not behave "geometrically"—by an Eckmann-Hilton argument, horizontal and vertical composition for such cells must be the same and commutative. Intuitively, this means that cells do not have well-defined "shape"; a little more precisely, this means for example that if we have 2-cells a and b with identity source and target, then a 3-cell with source ab $(= ba)$ cannot have well-defined faces, as we cannot put the putative faces a and b in any order.

This argument obviously does not constitute a proof, but it is the idea at the root of the argument in [7] and at the root of the argument we give here. We begin in Section 1 by recalling the basic definitions; in Section 2 we give the counterexample, and in Section 3 we give the justification. Experts will only need to read Section 2.

Note that unless otherwise stated, all n-categories are strict.

Acknowledgements

I would like to thank François Métayer for asking me for this counterexample. I would also like to thank Albert Burroni and Yves Guiraud for lively discussions around the subject.

1 Basic definitions

We begin by recalling the definition of the category of 3-computads. However, we will only need a small fragment of it for our counterexample, so we will focus on that part. 2-computads are defined by Street in

[8]; the higher-dimensional generalisation is given by Burroni under the name "polygraphs" in [2] (see also [1]).

The idea is that a 3-computad is a 3-category that is "level-wise free". From another point of view it is the underlying data for a 3-category in which k-cells are allowed to have source and target that are pasting diagrams of $(k-1)$-cells, rather than the single $(k-1)$-cells that are the only allowed source and target for globular sets. Crucially for us, this means in particular that the source and target can be degenerate, that is, identities.

The definition proceeds inductively. At each dimension we must specify the k-cells and then generate pasting diagrams freely in order to specify the boundaries of cells at the next dimension. This is done using a free 3-category functor and is the technically tricky part of the definition. However, we will not actually need the full construction of this functor.

Definition 1.1. A **3-computad** A is given by, for each $0 \le k \le 3$

- a set A_k of k-cells, and
- a boundary map $A_k \longrightarrow PA_{k-1}$.

Here PA_{k-1} denotes the set of parallel pairs of formal composites of $(k-1)$-cells of A. A **morphism of 3-computads** $A \longrightarrow B$ is given by, for each $0 \le k \le 3$ a morphism

$$f_k : A_k \longrightarrow B_k$$

making the obvious squares commute. We write **3Comp** for the category of 3-computads and their morphisms.

In general it is quite complicated to make P precise, but each of the computads involved in our counterexample will have only one 0-cell and no 1-cells. In this case, the free 2-category on the 2-dimensional data is simply the free commutative monoid on A_2 (regarded as a doubly degenerate 2-category). We use the following terminology.

Definition 1.2. A 3-computad A is called **2-degenerate** if A_0 is terminal and A_1 is empty. Thus by the Eckmann-Hilton argument it consists of

- sets A_2 and A_3, equipped with
- source and target maps
$$A_3 \underset{t}{\overset{s}{\rightrightarrows}} A_2^*$$
where A_2^* denotes the free commutative monoid on A_2.

A morphism $A \longrightarrow B$ of such 3-computads is given by morphisms

$$A_2 \xrightarrow{f_2} B_2$$
$$A_3 \xrightarrow{f_3} B_3$$

such that the following diagram commutes serially.

$$\begin{array}{ccc} A_3 & \underset{t}{\overset{s}{\rightrightarrows}} & A_2^* \\ {\scriptstyle f_3}\downarrow & & \downarrow{\scriptstyle f_2^*} \\ B_3 & \underset{t}{\overset{s}{\rightrightarrows}} & B_2^* \end{array}$$

2 The counterexample

All the 3-computads involved here will be 2-degenerate. When we check universal properties we will of course need to check them against all computads *a priori*, but we quickly see that the diagrams will ensure 2-degeneracy of any 3-computads involved.

We will write 2-cells as $a, b, \ldots$ and the commutative composition as

$$a.b = b.a.$$

In all that follows, every 3-cell will have a single 2-cell as target, but this is largely to ease the notation; a "smaller" counterexample would be possible with empty targets, eliminating the need for the 2-cells a_3 and y.

To show that **3Comp** is not cartesian closed we need to show that there exists $B \in$ **3Comp** such that $\_ \times B$ does not have a right adjoint, so it suffices for $\_ \times B$ not to preserve all colimits. So we exhibit a coequaliser

$$E \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} A \xrightarrow{\beta} C$$

and a computad B such that the functor $\_ \times B$ does not preserve it.

Step 1: the coequaliser

1. Let A be the 2-degenerate 3-computad with 2-cells a_1, a_2, a_3 and a single 3-cell
$$a_1.a_2 \xrightarrow{f} a_3.$$

2. Let E be the 2-degenerate 3-computad with 2-cells x, y and no 3-cells.

3. Define the morphism α_1 by
$$\begin{array}{ccc} x & \longmapsto & a_1 \\ y & \longmapsto & a_3 \end{array}$$
and define α_2 by
$$\begin{array}{ccc} x & \longmapsto & a_2 \\ y & \longmapsto & a_3 \end{array}$$

4. Thus the coequaliser C simply identifies a_1 and a_2; it has 2-cells $\bar{a}, a_3$ and a single 3-cell
$$\bar{a}.\bar{a} \xrightarrow{\bar{f}} a_3.$$

Step 2: the functor _ $\times B$

5. Let B be the 2-degenerate 3-computad (isomorphic to A) with 2-cells b_1, b_2, b_3 and a single 3-cell
$$b_1.b_2 \xrightarrow{g} b_3.$$

6. $E \times B$ has 2-cells (x, b_j) and (y, b_j) for $j = 1, 2, 3$. It has no 3-cells.

7. $A \times B$ is the key structure. It has 2-cells (a_i, b_j) for $i, j = 1, 2, 3$ and *two* 3-cells
$$\begin{array}{ccc} (a_1, b_1).(a_2, b_2) & \xrightarrow{(f,g)_1} & (a_3, b_3) \\ (a_2, b_1).(a_1, b_2) & \xrightarrow{(f,g)_2} & (a_3, b_3) \end{array}$$
This is probably the most interesting part of the argument; we give the full proof later.

8. $C \times B$ has 2-cells $(\bar{a}, b_j)$ and (a_3, b_j) for $j = 1, 2, 3$ and a single 3-cell

$$(\bar{a}, b_1).(\bar{a}, b_2) \xrightarrow{(\bar{f},g)} (a_3, b_3).$$

Step 3: non-preservation

9. We now examine the coequaliser

$$E \times B \underset{\alpha_2 \times 1}{\overset{\alpha_1 \times 1}{\rightrightarrows}} A \times B \longrightarrow P$$

and show that it is not isomorphic to $C \times B$.

Now the morphism $\alpha_1 \times 1$ is given by

$$\begin{array}{ccc} (x, b_j) & \longmapsto & (a_1, b_j) \\ (y, b_j) & \longmapsto & (a_3, b_j) \end{array}$$

and $\alpha_2 \times 1$ by

$$\begin{array}{ccc} (x, b_j) & \longmapsto & (a_2, b_j) \\ (y, b_j) & \longmapsto & (a_3, b_j) \end{array}$$

Thus the coequaliser P simply identifies (a_1, b_j) with (a_2, b_j) for each j. So it has 2-cells which we may call $(\bar{a}, b_j)$ and (a_3, b_j) (which is to be expected as the coequaliser is preserved up to 2 dimensions).

P has two distinct 3-cells

$$\begin{array}{ccc} (\bar{a}, b_1).(\bar{a}, b_2) & \xrightarrow{(f,g)_1} & (a_3, b_3) \\ (\bar{a}, b_1).(\bar{a}, b_2) & \xrightarrow{(f,g)_2} & (a_3, b_3). \end{array}$$

Since $C \times B$ has only one 3-cell it is clear that $C \times B$ is not isomorphic to this coequaliser P, that is, $\_ \times B$ does not preserve the original coequaliser.

Note that the canonical factorisation

$$P \longrightarrow C \times B$$

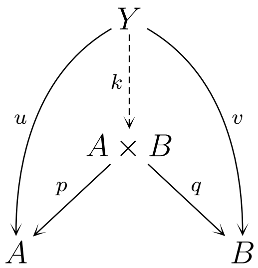
identifies the 3-cells $(f, g)_1$ and $(f, g)_2$.

3 Universal properties

In this section we check all the universal properties required for the counterexample. In principle we only need to check the 3-cells, as 2-computads form a presheaf category so we know that the lower dimensions behave pointwise. However we include the full argument for completeness, and because it is straightforward.

Lemma 3.1. *The product $A \times B$ is as given in the previous section, with the obvious projections.*

Proof. We exhibit its universal property. Consider a 3-computad Y and morphisms



We seek to exhibit a unique factorisation k as shown. On 0-, 1- and 2-cells, $A \times B$ is just a product, so we define the factorisation at these dimensions as for products ie

$$k(t) = (u(t), v(t)).$$

Note in particular that A and B have no 1-cells, so for the morphisms u and/or v to exist, Y cannot have any 1-cells either. So this map respects boundaries trivially.

We now discuss the factorisation on 3-cells. Let e be a 3-cell in Y. Now A and B have only one 3-cell each, f and g respectively. So we must have

$$\begin{aligned} u(e) &= f \\ v(e) &= g \end{aligned}$$

thus e must have boundary as follows

$$y_1 . y_2 \xrightarrow{e} y_3$$

for some 2-cells $y_1, y_2, y_3 \in Y$. Then since the action of u and v respect the boundary of e we know y_3 must be sent to a_3 and b_3 respectively. However considering the source there is some ambiguity as the product is commutative, so for each of u and v there are two possibilities—either the subscripts are left the same, or they are switched. That is, on ordered pairs the action of u is

$$\begin{array}{rccc} \text{either} & (y_1, y_2) & \longmapsto & (a_1, a_2) \\ \text{or} & (y_1, y_2) & \longmapsto & (a_2, a_1) \end{array}$$

and similarly the action of v is

$$\begin{array}{rccc} \text{either} & (y_1, y_2) & \longmapsto & (b_1, b_2) \\ \text{or} & (y_1, y_2) & \longmapsto & (b_2, b_1). \end{array}$$

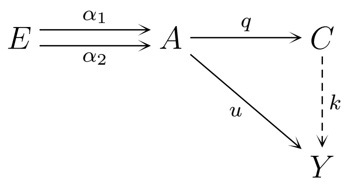
There are thus 4 cases, but in each case $k(e)$ is uniquely determined to be either $(f, g)_1$ or $(f, g)_2$ by the condition that k preserves boundary. Explicitly, $k(e)$ is specified by examining the action of u and v as shown by the following table.

		v: $(y_1, y_2) \longmapsto (b_1, b_2)$	v: $(y_1, y_2) \longmapsto (b_2, b_1)$
u	$(y_1, y_2) \longmapsto (a_1, a_2)$	$(f, g)_1$	$(f, g)_2$
u	$(y_1, y_2) \longmapsto (a_2, a_1)$	$(f, g)_2$	$(f, g)_1$

□

The other products follow similarly, but more easily. It remains to check the universal properties of the two coequalisers in question, which is much more straightforward.

Consider a diagram



with $u\alpha_1 = u\alpha_2$. We seek a unique factorisation k as shown.

- On 0-cells: A and C only have one 0-cell each; writing each as $*$ we must have $k(*) = u(*) \in Y$.

- On 1-cells: A and C have no 1-cells, so as before Y cannot have any either.

- On 2-cells: To make the triangle commute we must put

$$\begin{aligned} k(\bar{a}) &= u(a_1)\ [= u(a_2)] \\ k(a_3) &= u(a_3). \end{aligned}$$

  This respects boundaries as all 2-cells involved are degenerate.

- On 3-cells: To make the triangle commute, we must have $k(\bar{f}) = u(f)$. This respects boundaries, by our definition of k on 2-cells.

The other coequaliser proceeds in the same way, but with two 3-cells.

Remark 3.2. Note that this sort of counterexample cannot arise for 2-computads, as 2 is the lowest dimension of cell for which the Eckmann-Hilton argument can be used. Note also that this problem does not arise for weak 3-computads as weak identity 1-cells impede the Eckmann-Hilton argument on degenerate 2-cells. This difference between the commutativity of degenerate 3-cells in weak and strict structures also arises in [5].

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Vol. LIV-1 (2013)

TRIJUNCTIONS AND TRIADIC GALOIS CONNECTIONS

by René GUITART

Résumé. Dans cet article sont introduites les *trijonctions*, qui sont aux connexions galoisiennes triadiques ce que les adjonctions sont aux connexions galoisiennes. Nous décrivons le tripode trifibré associé à une trijonction, la trijonction entre topos de préfaisceaux associée a une trifibration discrète, et l'engendrement de toute trijonction par un bi-adjoint. À côté des exemples associés aux connexions galoisiennes triadiques, aux relations ternaires, d'autres le sont à des tenseurs symétriques, aux topos et univers algébriques.

Abstract. In this paper we introduce the notion of a *trijunction*, which is related to a triadic Galois connection just as an adjunction is to a Galois connection. We construct the trifibered tripod associated to a trijunction, the trijunction between toposes of presheaves associated to a discrete trifibration, and the generation of any trijunction by a bi-adjoint functor. While some examples are related to triadic Galois connections, to ternary relations, others are associated to some symmetric tensors, to toposes and algebraic universes.
Keywords. Galois connection, adjunction, bi-adjunction, trijunction, trifibration, topos, algebraic universes.
Mathematics Subject Classification (2010). 06A15, 18A40, 18B10, 18D30.

1. Introduction

A *trijunction* (definition 2.1) was introduced in [7] as a categorification of a triadic Galois connection [1], just as an adjunction [9] could be understood as a categorification of a Galois connection [13]: triadic Galois connections and Galois connections are trijunctions and adjunctions reduced to the case of posets (section 3). Any trijunction is generated by a bi-adjoint and determines a trifibration (section 2.1), and conversely a discrete trifibration determines a trijunction between toposes of presheaves. We give examples of trijunctions associated to adjunctions with parameters related to a symmetric tensor, and the constitutive auto-trijunctions of toposes or algebraic universes (section 4), which allow to reproduce internally triadic Galois connections.

2. Trijunctions, bi-adjunctions, discrete trifibrations

2.1 Trijunctions

Definition 2.1. *A* trijunction *between 3 categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, is the datum (γ, β, α) of 3 contravariant functors between any product of two of these categories and the third, i.e. 3 covariant functors as:*

$$\gamma : \mathcal{A} \times \mathcal{B} \to \mathcal{C}^{\mathrm{op}},\ \beta : \mathcal{C} \times \mathcal{A} \to \mathcal{B}^{\mathrm{op}},\ \alpha : \mathcal{B} \times \mathcal{C} \to \mathcal{A}^{\mathrm{op}}$$

and 3 natural equivalences with a circular condition

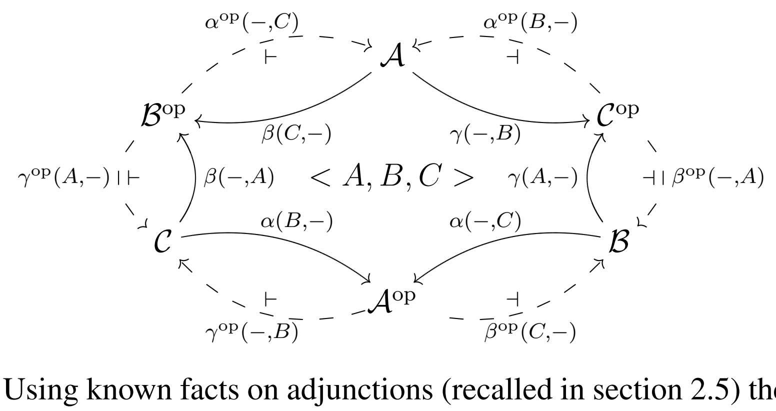
$$(-)^{\alpha,\gamma}(-)^{\gamma,\beta} = (-)^{\alpha,\beta} :$$

$$(-)^{\alpha,\gamma} : \mathrm{Hom}_{\mathcal{C}}(C, \gamma(A,B)) \simeq \mathrm{Hom}_{\mathcal{A}}(A, \alpha(B,C)) : (-)^{\gamma,\alpha} = ((-)^{\alpha,\gamma})^{-1},$$

$$(-)^{\gamma,\beta} : \mathrm{Hom}_{\mathcal{B}}(B, \beta(C,A)) \simeq \mathrm{Hom}_{\mathcal{C}}(C, \gamma(A,B)) : (-)^{\beta,\gamma} = ((-)^{\gamma,\alpha})^{-1},$$

$$(-)^{\beta,\alpha} : \mathrm{Hom}_{\mathcal{A}}(A, \alpha(B,C)) \simeq \mathrm{Hom}_{\mathcal{B}}(B, \beta(C,A)) : (-)^{\alpha,\beta} = ((-)^{\beta,\alpha})^{-1}.$$

Proposition 2.2. *Given a trijunction (γ, β, α) as in definition 2.1 and an object (A, B, C) of $\mathcal{A} \times \mathcal{B} \times \mathcal{C}$ we get 12 functors of one variable, in the bi-hexagon $< A, B, C >$, in which an exterior dotted line indicates a right adjoint to the corresponding internal unbroken line:*



Proof. Using known facts on adjunctions (recalled in section 2.5) the equivalences in definition 2.1 provide equivalences of adjunction when one argument is fixed, hence the adjunctions in the hexagon. □

Proposition 2.3. *Associated to adjunctions in the hexagon of proposition 2.2, there are 6 unit transformations which are* natural *in lower arguments and* dinatural *in upper arguments:*

1. $\beta(C,-) \dashv \alpha^{\mathrm{op}}(-C)$ *and* $\gamma(-,B) \dashv \alpha^{\mathrm{op}}(B,-)$ *give on* $\mathcal{A}$*:*
$$\alpha\beta CAC \xleftarrow{\alpha_A^C} A \xrightarrow{\alpha_A^B} \alpha B\gamma AB$$
2. $(\alpha(-,C) \dashv \beta^{\mathrm{op}}(C,-)$ *and* $\gamma(A,-) \dashv \beta^{\mathrm{op}}(-,A)$ *give on* $\mathcal{B}$*:*
$$\beta C\alpha BC \xleftarrow{\beta_B^C} B \xrightarrow{\beta_B^A} \beta\gamma ABA$$
3. $\beta(-,A) \dashv \gamma^{\mathrm{op}}(A,-)$ *and* $\alpha(B,-) \dashv \gamma^{\mathrm{op}}(-,B)$ *give on* $\mathcal{C}$*:*
$$\gamma A\beta CA \xleftarrow{\gamma_C^A} C \xrightarrow{\gamma_C^B} \gamma\alpha BCB$$

We recover the equivalences $(-)^{\alpha,\beta}$ *etc., by:*
$$a : A \to \alpha(B,C) = b^{\alpha,\beta} = \alpha(b,C)\alpha_A^C = c^{\alpha,\gamma} = \alpha(B,c)\alpha_A^B,$$
$$b : B \to \beta(C,A) = c^{\beta,\gamma} = \beta(c,A)\beta_B^A = a^{\beta,\alpha} = \beta(C,a)\beta_B^C,$$
$$c : C \to \gamma(A,B) = a^{\gamma,\alpha} = \gamma(a,B)\gamma_C^B = b^{\gamma,\beta} = \gamma(A,b)\gamma_C^A.$$

Proof. For $\beta(C,-) \dashv \alpha^{\mathrm{op}}(-C)$ the unit is
$$\alpha_A^C = (1_{\beta(C,A)})^{\alpha,\beta} : A \to \alpha\beta CAC := \alpha(\beta(C,A),C)$$
This α_A^C is *natural* in A, i.e. such that, for any $u : A \to A'$,
$$\alpha_{A'}^C u = \alpha(\beta(C,u),C)\alpha_A^C,$$
and is *dinatural* in C, i.e. such that, for any $w : C \to C'$, we have:
$$\alpha(\beta(w,A),C)\alpha_A^C = \alpha(\beta(C',A),w)\alpha_A^{C'}.$$
The situation here is an "adjunction with a parameter" (see [11, p. 100]) in $\mathcal{C}$ between α and β, and the naturality and dinaturality are proved in [11, p. 216]; in fact the converse holds: if α_A^C is natural in A and dinatural in C, then $(-)^{\beta,\alpha}$ (or its inverse $(-)^{\alpha,\beta}$) is natural in its three arguments. This is indicated in [11] (exercise 2 p. 100 and exercise 1 p. 218): the unit $\eta_A^B : A \to R(B, L(A,B))$ of an adjunction with parameter is dinatural in B, and this is equivalent to the naturality of the adjunction τ itself in B. $\square$

Proposition 2.4. *With the hypothesis and notations of propositions 2.2 and 2.3 we have* 6 equations of adjunction*:*

$$\alpha(B,\gamma_C^B)\alpha^B_{\alpha(B,C)} = 1_{\alpha(B,C)} = \alpha(\beta_B^C, C)\alpha^C_{\alpha(B,C)},$$
$$\beta(C,\alpha_A^C)\beta^C_{\beta(C,A)} = 1_{\beta(C,A)} = \beta(\gamma_C^A, A)\beta^A_{\beta(C,A)},$$
$$\gamma(A,\beta_B^A)\gamma^A_{\gamma(A,B)} = 1_{\gamma(A,B)} = \gamma(\alpha_A^B, B)\gamma^B_{\gamma(A,B)};$$

and we have the condition of circularity*, expressible in 6 equivalent ways:*

$$\alpha_A^B = \alpha(\beta_B^A, \gamma(A,B))\alpha_A^{\gamma(A,B)}, \quad \alpha_A^C = \alpha(\beta(C,A), \gamma_C^A)\alpha_A^{\beta(C,A)},$$
$$\beta_B^C = \beta(\gamma_C^B, \alpha(B,C))\beta_B^{\alpha(B,C)}, \quad \beta_B^A = \beta(\gamma(A,B), \alpha_A^B)\beta_B^{\gamma(A,B)},$$
$$\gamma_C^A = \gamma(\alpha_A^C, \beta(C,A))\gamma_C^{\beta(C,A)}, \quad \gamma_C^B = \gamma(\alpha(B,C), \beta_B^C)\gamma_C^{\alpha(B,C)}.$$

Proof. For example, between the unit α_A^C and the corresponding co-unit β_B^C we have the known equations of adjunctions recalled in proposition 2.11.
For example, as $\beta(C,A)$ is a functor in each variable, and as β_B^C is dinatural in C (proposition 2.3), the fourth circularity condition, expressing β_B^A, allows to deduce for any $c : C \to \gamma(A,B)$ that

$$\beta(c,A)\beta_B^A = \beta(C, \alpha(B,c)\alpha_A^B)\beta_B^C,$$

which (cf. proposition 2.3) is equivalent to $(-)^{\beta,\gamma} = (-)^{\beta,\alpha}(-)^{\alpha,\gamma}$. This implies conversely the fourth condition.
By the equations of adjunction, the six natural transformations $(-)^{\alpha,\beta}$ etc. are invertible (equivalence), and from the last equation we get the five analogs, and then any equation of circularity. □

2.2 Bi-adjunction

Definition 2.5. *A bi-functor $\gamma : \mathcal{A}\times\mathcal{B} \to \mathcal{C}^{\mathrm{op}}$ is a* left bi-adjunction *if for every A in $\mathcal{A}$ the functor $\gamma(A,-) : \mathcal{B} \to \mathcal{C}^{\mathrm{op}}$ is a left adjoint, and for every B in $\mathcal{B}$ the functor $\gamma(-,B) : \mathcal{A} \to \mathcal{C}^{\mathrm{op}}$ is a left adjoint.*

Proposition 2.6. *1 — A bi-functor $\gamma : \mathcal{A}\times\mathcal{B} \to \mathcal{C}^{\mathrm{op}}$ is a* left bi-adjunction *if and only if there is a trijunction (γ,β,α), in the sense of definition 2.1. In this case, β and γ are unique up to natural isomorphisms.*

2 — A trijunction is completely determined up to isomorphisms by a functor $\gamma : \mathcal{A} \times \mathcal{B} \to \mathcal{C}^{\mathrm{op}}$ with the datum for each object C of two objects αBC and βCA, with two morphisms

$$\gamma A\beta CA \xleftarrow{\gamma_C^A} C \xrightarrow{\gamma_C^B} \gamma\alpha BCB\ ,$$

such that for any $c : C \to \gamma AB$ there are two unique maps $a : A \to \alpha BC$ and $b : B \to \beta CA$ such that

$$c = \gamma(a, B)\gamma_C^B, \quad c = \gamma(A, b)\gamma_C^A.$$

Proof. 1 — The proposition is an application of known results (recalled in proposition 2.13 later). So we introduce β and α by $\gamma(A, -) \dashv \beta^{\mathrm{op}}(-, A)$ and $\gamma(-, B) \dashv \alpha^{\mathrm{op}}(B, -)$. With the formula $(\star)$ in the proof of proposition 2.13 we get bi-functors β and γ, with natural equivalences $(-)^{\gamma,\beta}$ and $(-)^{\alpha,\gamma}$, and we define $(-)^{\alpha,\beta}$ as the composition $(-)^{\alpha,\gamma}(-)^{\gamma,\beta}$.
2 — This results from the determination of adjoints by free objects.
So, all the data and equations in a trijunction (cf. propositions 2.2, 2.3 and 2.4) are consequences of these two "free object" properties. □

2.3 Discrete trifibration associated to a trijunction

A triadic Galois connection is known to be a generalization of a ternary relation (recalled in proposition 3.5 later); a similar understanding for a trijunction is in terms of trifibrations.

Definition 2.7. *Given a trijunction (γ, β, α) we construct its "trigraph", the category $\mathcal{G} = \mathcal{G}(\gamma, \beta, \alpha)$with objects $G = (a, b, c)$ as in*

$$a : A \to \alpha(B, C)$$

$$c : C \to \gamma(A, B) \qquad\qquad b : B \to \beta(C, A)$$

with

$$b = a^{\beta,\alpha},\ c = b^{\gamma,\beta},\ a = c^{\alpha,\gamma},$$

as in proposition 2.3; a morphism from (a, b, c) to (a', b', c') is a $g = (u, v, w) : (A, B, C) \to (A', B', C')$ with one of the equivalent conditions:

$$\alpha(v, w)a'u = a, \quad \beta(w, u)b'v = b, \quad \gamma(u, v)c'w = c.$$

Proposition 2.8. *We have a discrete fibration given by:*

$$\pi = \pi_{\gamma,\beta,\alpha} : \mathcal{G}(\gamma, \beta, \alpha) \to \mathcal{A} \times \mathcal{B} \times \mathcal{C} : (a, b, c) \mapsto (A, B, C)$$

$$\begin{array}{c} \mathcal{G} \\ \downarrow \pi \\ \mathcal{A} \times \mathcal{B} \times \mathcal{C} \end{array} \qquad \begin{array}{ccc} & \mathcal{A} & \\ & \uparrow \pi_{\mathcal{A}} & \\ & \mathcal{G} & \\ \swarrow \pi_{\mathcal{C}} & & \searrow \pi_{\mathcal{B}} \\ \mathcal{C} & & \mathcal{B} \end{array}$$

Proof. In fact $\mathcal{G}(\gamma, \beta, \alpha) = \mathcal{G}$ is isomorphic to the discrete fibration $\int \alpha$ associated to $\mathrm{Hom}_{\mathcal{A}}(\mathrm{Id}_{\mathcal{A}^{\mathrm{op}}} \times \alpha^{\mathrm{op}}) : (\mathcal{A} \times \mathcal{B} \times \mathcal{C})^{\mathrm{op}} \to \mathrm{Ens}$, as well as the one $\int \beta$ associated to $\mathrm{Hom}_{\mathcal{B}}(\mathrm{Id}_{\mathcal{B}^{\mathrm{op}}} \times \beta^{\mathrm{op}}) : (\mathcal{B} \times \mathcal{C} \times \mathcal{A})^{\mathrm{op}} \to \mathrm{Ens}$ or the one $\int \gamma$ associated to $\mathrm{Hom}_{\mathcal{C}}(\mathrm{Id}_{\mathcal{C}^{\mathrm{op}}} \times \gamma^{\mathrm{op}}) : (\mathcal{C} \times \mathcal{A} \times \mathcal{B})^{\mathrm{op}} \to \mathrm{Ens}$. So in the category of fibrations over $\mathcal{A} \times \mathcal{B} \times \mathcal{C}$ we have three isomorphisms

$$\begin{array}{ccc} & \int \alpha & \\ & \downarrow \simeq & \\ & \pi_{\gamma,\beta,\alpha} & \\ \overset{\simeq}{\nearrow} & & \underset{\simeq}{\nwarrow} \\ \int \gamma & & \int \beta \end{array}$$

In fact the isomorphisms between these fibrations exactly correspond to equivalences in the definition (2.1) of the trijunction. □

2.4 From discrete trifibrations to trijunctions between presheaves

Proposition 2.9. *Given a functor* $R : (\mathcal{A} \times \mathcal{B} \times \mathcal{C})^{\mathrm{op}} \to \mathrm{Ens}$ *with* $\mathcal{A}$*,* $\mathcal{B}$*,* $\mathcal{C}$ *any small categories, or the associated discrete fibration* $\pi_R : \mathcal{G} \to \mathcal{A} \times \mathcal{B} \times \mathcal{C}$ *(called a discrete trifibration), there is an associated trijunction* $(\gamma_R, \beta_R, \alpha_R)$ *between toposes of presheaves* $\hat{\mathcal{A}} := \mathrm{Ens}^{\mathcal{A}^{\mathrm{op}}}$*,* $\hat{\mathcal{B}} := \mathrm{Ens}^{\mathcal{B}^{\mathrm{op}}}$*, and* $\hat{\mathcal{C}} := \mathrm{Ens}^{\mathcal{C}^{\mathrm{op}}}$*. Especially any bi-functor* $\mathcal{A} \times \mathcal{B} \to \mathcal{C}^{\mathrm{op}}$ *determines such a trijunction.*

Proof. With $R_{\mathcal{C}}(C)(A, B) = R_{\mathcal{B}}(B)(C, A) = R_{\mathcal{A}}(A)(B, C) = R(A, B, C)$, $R_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ens}^{(\mathcal{A}\times\mathcal{B})^{\mathrm{op}}}$, $R_{\mathcal{B}} : \mathcal{B}^{\mathrm{op}} \to \mathrm{Ens}^{(\mathcal{C}\times\mathcal{A})^{\mathrm{op}}}$, $R_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \to \mathrm{Ens}^{(\mathcal{B}\times\mathcal{C})^{\mathrm{op}}}$. For F, G and H in $\hat{\mathcal{A}}$, $\hat{\mathcal{B}}$, and $\hat{\mathcal{C}}$ we define $F \boxtimes G(A, B) = F(A) \times G(B)$, $H \boxtimes F(C, A) = H(C) \times F(A)$ and $G \boxtimes H(B, C) = G(B) \times H(C)$. Then

$$\gamma_R(F, G)(C) = \mathrm{Hom}_{\mathrm{Ens}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{B}^{\mathrm{op}}}}(F \boxtimes G, R_{\mathcal{C}}(C)),$$
$$\beta_R(H, F)(B) = \mathrm{Hom}_{\mathrm{Ens}^{\mathcal{C}^{\mathrm{op}}\times\mathcal{A}^{\mathrm{op}}}}(H \boxtimes F, R_{\mathcal{B}}(B)),$$
$$\alpha_R(G, H)(A) = \mathrm{Hom}_{\mathrm{Ens}^{\mathcal{B}^{\mathrm{op}}\times\mathcal{C}^{\mathrm{op}}}}(G \boxtimes H, R_{\mathcal{A}}(A)).$$

Then for example we associate to $\theta : F \boxtimes G \boxtimes H \to R$ a $\nu : H \to \gamma(F, G)$ by $(\nu_C(z))_{(A,B)}(x, y) = \theta_{(A,B,C)}(x, y, z)$. □

2.5 Annex 1: Classical facts on adjunctions

In 1958, Daniel Kan [9] introduced the notion of *adjoint functors*; then Peter Freyd (Princeton thesis, 1960) and William Lawvere (Columbia thesis, 1963) "emphasized the dominant position of adjunctions" [11, p. 103]:

Definition 2.10. *Let $\mathcal{A}$ and $\mathcal{C}$ be categories. Then a covariant functor $L : \mathcal{A} \to \mathcal{C}$ is called* left adjoint *to a covariant functor $R : \mathcal{C} \to \mathcal{A}$ (notation $\tau : L \dashv R$) if there exists a natural equivalence*

$$\tau : \mathrm{Hom}_{\mathcal{C}}(L(A), C) \simeq \mathrm{Hom}_{\mathcal{A}}(A, R(C)).$$

Proposition 2.11. *$\tau : L \dashv R$ is equivalent to $L \dashv R(\epsilon, \eta)$, with 2 natural transformations $\epsilon := \tau^{-1}(1_R) : LR \to \mathrm{Id}_{\mathcal{C}}$ and $\eta := \tau(1_L) : \mathrm{Id}_{\mathcal{A}} \to RL$ with the equations:*
$(\epsilon L)(L\eta) = \mathrm{Id}_L, \quad (R\epsilon)(\eta R) = \mathrm{Id}_R$. *Furthermore we get τ and τ^{-1} by:*
$\tau(c : LA \to C) = R(c)\eta_A, \quad \tau^{-1}(a : A \to RC) = \epsilon_C L(a).$

Proof. This is coming from lemmas 6.2 p.306 and 6.2* p.307 in [9]. See also [11, chap. IV, p. 80-81]. □

Definition 2.12. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be categories. Then a covariant functor $L : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is called* left adjoint *— with a parameter in $\mathcal{B}$ — to a functor $R : \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{A}$ contravariant in $\mathcal{B}$ and covariant in $\mathcal{C}$ if there exists a natural equivalence*

$$\tau : \mathrm{Hom}_{\mathcal{C}}(L(A, B), C) \simeq \mathrm{Hom}_{\mathcal{A}}(A, R(B, C)).$$

Proposition 2.13. *Given $L : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ and for each object B in $\mathcal{B}$ a right adjoint R_B to $L(-, B)$, with $\tau_B : L(-, B) \dashv R_B$, then these functors determine a unique functor $R : \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{A}$ with an equivalence τ as in definition 2.12, with for every $c : C \to C'$, $R(B, c) = R_B(c)$, and with $\tau(A, B, C) = \tau_B(A, C)$.*

Proof. This is proved as theorem 4.1 p. 300 in [9]. See also [11, p. 100]. With $\epsilon_B = \tau_B^{-1}(1_{R_B})$ and $\eta_{B'} = \tau_{B'}(1_{L(-,B')})$, an explicit formula for $R(b, c)$ with $b : B' \to B$ and $c : C \to C'$ is

$$R(b, c) = R_{B'}(c) R_{B'}(\epsilon_B(C)) R_{B'}(L(R_B(C), b)) \eta_{B'}(R_B(C)) \qquad (\star)$$

□

Kan was especially motivated by the case of $\otimes$:

Proposition 2.14. *The functor* $\otimes : \mathrm{Ab} \times \mathrm{Ab} \to \mathrm{Ab}$ *is left adjoint to* $\mathrm{Hom} : \mathrm{Ab}^{\mathrm{op}} \times \mathrm{Ab} \to \mathrm{Ab}$*, in the sense of definition 2.12.*

3. Triadic Galois Connections and ternary relations

3.1 Triadic Galois connections and residuations

In relation with the calculus of ternary relations between sets and the "triadic concept analysis" as introduced in [10] and [14], and the notion of "*trilattice*", the notion of a *triadic Galois connection* has been introduced in 1997 by Klaus Biedermann [1], [2], [3]. We adapt his definition, without references to trilattices, and with a slightly different system of notations, in order to show that this notion is a particular case of a trijunction.
NB: In this section 3.1 we use and extend the classical properties of Galois connections (see 3.3) to triadic Galois connections. So we get a mini-model of the theory of trijunctions, namely its reduction to the case of posets.

Definition 3.1. *A* triadic Galois connection *between 3 posets* $\mathcal{A} = (A, \leq)$*,* $\mathcal{B} = (B, \leq)$ *and* $\mathcal{C} = (C, \leq)$ *is the datum* (γ, β, α) *of 3* decreasing *functions* $\gamma : A \times B \to C,\ \beta : C \times A \to B,\ \ \alpha : B \times C \to A$*, such that for all* $a \in A$*,* $b \in B$*,* $c \in C$*:*

$$c \leq \gamma(\alpha(b, c), b), \quad c \leq \gamma(a, \beta(c, a)),$$

$$b \leq \beta(\gamma(a, b), a), \quad b \leq \beta(c, \alpha(b, c)),$$

$$a \leq \alpha(\beta(c, a), c), \quad a \leq \alpha(b, \gamma(a, b)).$$

Proposition 3.2. *A triadic Galois connection is equivalent to the datum* (γ, β, α) *of 3* decreasing *functions* $\gamma : A \times B \to C,\ \beta : C \times A \to B,$ $\alpha : B \times C \to A$*, such that*

$$\forall a \in A\, \forall b \in B\, \forall c \in C \left[c \leq \gamma(a, b) \Leftrightarrow b \leq \beta(c, a) \Leftrightarrow a \leq \alpha(b, c) \right].$$

Proposition 3.3. *A* triadic Galois connection *is exactly the special case of a* trijunction *according to definition 2.1 in which* $\mathcal{A}$*,* $\mathcal{B}$ *and* $\mathcal{C}$ *are posets.*

Proposition 3.4. *1 — Let $(M, \leq)$ be a sup-lattice and let $\otimes : M \times M \to M$ a binary law compatible with* sup*. Then with $\mathcal{A} = (M, \leq)$, $\mathcal{B} = (M, \leq)$, $\mathcal{C} = (M, \geq)$, and with $\gamma(a, b) = a \otimes b$, we get a triadic Galois connection (γ, β, α) in the sense of (def. 3.1).*
2 — Let $(M, \leq)$ a sup-lattice and a triadic Galois connection (γ, β, α) between $\mathcal{A} = (M, \leq)$, $\mathcal{B} = (M, \leq)$, $\mathcal{C} = (M, \geq)$. Then γ is a binary law compatible with sup*.*

Proof. 1 — We take $\beta(c, a) = {}^a c := \sup_{a\otimes b\leq c} b$, $\alpha(b, c) = c^b := \sup_{a\otimes b\leq c} a$, i.e. (see [4, p. 325]) the right and left residuals c/a of c by a, $c\backslash b$ of c by b.
2 — $\gamma(a, -)$ is a left adjoint, and $\gamma(-, b)$ is a left adjoint too. □

3.2 Functional counterpart of a ternary relation

Proposition 3.5. *A triadic Galois connection (γ, β, α) between the posets $(\mathcal{P}(A), \subseteq)$, $(\mathcal{P}(B), \subseteq)$ and $(\mathcal{P}(C), \subseteq)$ is equivalent to the datum of a ternary relation $R \subset A \times B \times C$, according to the association:*

$$R = R_\gamma := \{(a, b, c); c \in \gamma(\{a\}, \{b\})\},$$

$$R = R_\beta := \{(a, b, c); b \in \beta(\{c\}, \{a\})\},$$

$$R = R_\alpha := \{(a, b, c); a \in \alpha(\{b\}, \{c\})\},$$

$$\gamma(A', B') = \gamma_R(A', B') := \{c; \forall a' \in A' \,\forall b' \in B' \;(a', b', c) \in R\},$$

$$\beta(C', A') = \beta_R(C', A') := \{b; \forall c' \in C' \,\forall a' \in A' \;(a', b, c') \in R\},$$

$$\alpha(B', C') = \alpha_R(B', C') := \{a; \forall b' \in B' \,\forall c' \in C' \;(a, b', c') \in R\}.$$

Furthermore

$$C' \leq \gamma(A', B') \Leftrightarrow B' \leq \beta(C', A') \Leftrightarrow A' \leq \alpha(B', C') \Leftrightarrow A' \times B' \times C' \subseteq R.$$

Proof. It is an immediate reformulation of Biedermann [1], [2], [3]. □

Proposition 3.6. *Given a ternary relation $R \subseteq A \times B \times C$, and subsets $A' \subseteq A$, $B' \subseteq B$, $C' \subseteq C$, we get, with the notations of 3.1 and with*

$$R^*_C(C') = \{(a, b); \forall c' \in C' \;(a, b, c') \in R\},$$

$$R^*_B(B') = \{(c, a); \forall b' \in B' \;(a, b', c) \in R\},$$

$$R_A^*(A') = \{(b,c); \forall a' \in A' \ (a',b,c) \in R\},$$

an hexagonal picture of seven equivalent conditions:

$$\begin{array}{ccccc}
 & & A' \subseteq \alpha_R(B',C') & & \\
 & \nearrow & & \nwarrow & \\
C' \times A' \subseteq R_B^*(B') & & & & A' \times B' \subseteq R_C^*(C') \\
\downarrow & & A' \times B' \times C' \subseteq R & & \downarrow \\
C' \subseteq \gamma_R(A',B') & & & & B' \subseteq \beta_R(C',A') \\
 & \nwarrow & & \nearrow & \\
 & & B' \times C' \subseteq R_A^*(A') & &
\end{array}$$

Furthermore each of the six operators α_R, R_A^*, β_R, R_B^*, γ_R, R_C^*, determines the five others, and the relation R itself.

Proof. It is a direct complement to proposition 3.5, in the style of [8]. For the last point starting for example from the datum of α_R, we get R_A^* by $R_A^*(A') = \cup_{A' \subseteq \alpha_R(B',C')} B' \times C'$, etc. □

Proposition 3.7. *A triadic Galois connection between $\mathcal{A} = (\mathcal{P}(E), \subseteq)$, $\mathcal{B} = (\mathcal{P}(E), \subseteq)$, $\mathcal{C} = (\mathcal{P}(E), \supseteq)$ is equivalent to the datum of a ternary relation $R \subset E^3$.*

Proof. A sup-compatible binary law $\gamma : \mathcal{P}(E)^2 \to \mathcal{P}(E)$ is equivalent to a map $r : E^2 \to \mathcal{P}(E)$, i.e. a ternary relation $R \subset E^3$. □

3.3 Annex 2: Classical facts on Galois connections

Clearly a posteriori an adjunction could be understood as a categorification of a Galois connection in the following sense of definition 3.8.

In his talk at the Summer Meeting of AMS at Chicago in 1941, Oystein Ore introduced — as a tool for the calculus of binary relations — the notion of a *Galois connexion* [13] (see also Garrett Birkhoff [4, p.124]) — or equivalently *Galois connection* (also named *Galois correspondence*) —, as follows.

Definition 3.8. *A [dyadic]* Galois connection *between 2 posets $\mathcal{A} = (A, \leq)$ and $\mathcal{B} = (B, \leq)$ is the datum (β, α) of two* decreasing *functions $\beta : A \to B$ and $\alpha : B \to A$ such that*

$$\forall a \in A \left[a \leq \alpha(\beta(a)) \right], \quad \forall b \in B \left[b \leq \beta(\alpha(b)) \right].$$

Proposition 3.9. *It is equivalent for a Galois connection to assume that α and β are* ordinary *functions such that*

$$\forall a \in A \forall b \in B \left[b \leq \beta(a) \Leftrightarrow a \leq \alpha(b) \right].$$

Proposition 3.10. *A decreasing function $\beta : A \to B$ between two posets $\mathcal{A}$ and $\mathcal{B}$ determines two increasing functions $\beta^l : \mathcal{A} \to \mathcal{B}^{\text{op}}$ and $\beta^r : \mathcal{A}^{\text{op}} \to \mathcal{B}$, with $\beta^l = \beta^{r\,\text{op}}$ and $\beta^r = \beta^{l\,\text{op}}$; a* Galois connection *as (β, α) in 3.8 is exactly an* adjunction *in the sense of 2.10, namely $\alpha^l \dashv \beta^r$, or, equivalently, $\beta^l \dashv \alpha^r$.*

Proposition 3.11. *A Galois connection (β, α) between the posets $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(B), \subseteq)$ is equivalent to the datum of a binary relation $R \subset A \times B$, according to the association:*

$$R = \{(a, b); b \in \beta(\{a\})\} = \{(a, b); a \in \alpha(\{b\}),$$

$$\beta(A') = \{b; \forall a' \in A' \ (a', b) \in R\}, \quad \alpha(B') = \{a; \forall b' \in B' \ (a, b') \in R\}.$$

Furthermore

$$A' \subseteq \alpha(B') \Leftrightarrow B' \subseteq \beta(A') \Leftrightarrow A' \times B' \subseteq R.$$

Proof. See Ore [13, thm.10, p.503]. □

4. The auto-trijunction on a topos or an algebraic universe

4.1 Algebraic universe

We recall the definition of an *algebraic universe*, a notion we have developed in the 70's (see [5], [6]).

An *algebraic universe* is a category $\mathcal{X}$ with finite limits and colimits equipped with a contravariant functor $P : \mathcal{X} \to \mathcal{X}^{\text{op}}$ such that $P \dashv P^{\text{op}}$, this adjunction being monadic (analogous to *Stone duality*); we assume also that for any X in $\mathcal{X}$, the map $\eta_X : X \to PPX$ is factorized as $\psi_X a_X$ with $\psi_X : PX \to PPX$ (meeting map) and $a_X : X \to PX$ (atom map), and there are also $\pi_X : PX \to PPX$ (inclusion map), $\nu_X : PX \to PX$ (negation map) and $c_X : X^2 \to PX$ (pairing map); among these data a precise system of equations is assumed.

In any algebraic universe the construction P on objects is extended in two ways in a covariant functor: for $f : X \to Y$ we take:

$$\exists f = P(Pf.a_Y)\psi_X, \quad \forall f = P(Pf.a_Y)\pi_X,$$

$$\bigcup_X = P\eta_X\psi_{PX}, \quad \bigcap_X = P\eta_X P\pi_{PX}\psi_{PX}.$$

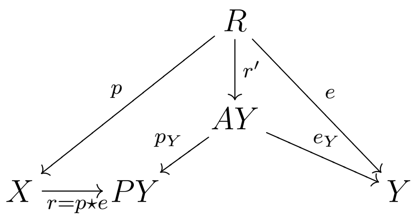
Given a relation $\rho = (p, e) : R \to A \times B$ we introduce its "characteristic map":

$$r = p \star e = (\exists e)(Pp)a_A : A \to PB.$$

Proposition 4.1. *(See [5]) Given a complete lattice equipped with a* sup-*compatible abelian monoid law* $\mathbb{L} = (L \leq, \otimes)$ *there is a structure of algebraic universe on* Ens *in which* $PX = L^X$*, and this generates the calculus of* $\mathbb{L}$*-fuzzy relations.*

4.2 Topos as an algebraic universe

An elementary *topos* (in the sense of Lawvere-Tierney, see [12]) is a category $\mathcal{E}$ with finite limits and colimits, with exponentials and subobject classifier. This is reducible to the conditions that $\mathcal{E}$ is with finite limits and colimits, and is such that for all object Y in $\mathcal{E}$ there is $(PY \overset{p_Y}{\longleftarrow} AY \overset{e_Y}{\longrightarrow} Y)$ such that for every $(X \overset{p}{\longleftarrow} R \overset{e}{\longrightarrow} Y)$ there is a unique $r = p \star e : X \longrightarrow PY$ and a unique $r' : R \longrightarrow AY$ with a pullback $(p, r'; r, p_Y)$ with $e = e_Y.r'$:



Proposition 4.2. *In a topos* $\mathcal{E}$ *the construction* P *is a contravariant functor which is its own adjoint:*

$$(P : \mathcal{E} \to \mathcal{E}^{\mathrm{op}}) \dashv (P^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}),$$

and in fact with this P *we get a structure of an algebraic universe.*

Proof. It is well known. Given a morphism $f : Y \to X$, we get $P(f) : PX \to PY$ by $aX := 1_X \star 1_X$ and $Pf = ((a_X f) \star 1_Y$. Starting with $r : X \to PY$, $r = p \star e$, we get its "converse" $s : Y \to PX$ with $p = r^*(p_Y)$, $r' = p^*(r)$, $e = e_Y r'$, and $s = e \star p$. Then a structure of an algebraic universe is given by this P, with ψ and π in the internal language:

$$\psi_X(A) = \{B; \exists x(x \in A \,\&\, x \in B)\},\ \pi_X(A) = \{B; \forall x(x \in B \Rightarrow x \in A)\}.$$

□

4.3 Symmetric tensors with right adjoints

Proposition 4.3. *With $\mathcal{A} = \mathrm{Ab}$, $\mathcal{B} = \mathrm{Ab}$, $\mathcal{C} = \mathrm{Ab}^{\mathrm{op}}$, we get a trijunction (def. 2.1) with $\gamma(A, B) = A \otimes B$, $\beta(C, A) = \mathrm{Hom}(A, C)$, and with $\alpha(B, C) = \mathrm{Hom}(B, C)$.*

Proof. This proposition results of proposition 2.14 , by imitation of proposition 3.4. Details of the proof arise also from proposition 2.6. □

Proposition 4.4. *In a symmetric monoidal closed category $\mathcal{E}$, there is a trijunction between $\mathcal{E}$, $\mathcal{E}$ and $\mathcal{E}^{\mathrm{op}}$ with*

$$\gamma(A, B) = A \otimes B, \quad \beta(C, A) = C^A, \quad \alpha(B, C) = C^B.$$

Proof. Analogous to the case in proposition 4.3. In a monoidal closed category, for any object B the functor $(-) \otimes B$ has a right adjoint $(-)^B$, and for any A the functor $A \otimes (-)$ has a right adjoint $(-)^A$. We conclude by proposition 2.6. □

Proposition 4.5. *In a symmetric monoidal closed category $\mathcal{E}$, with any object L, there is an associated (auto-)trijunction between $\mathcal{E}$, $\mathcal{E}$ and $\mathcal{E}$ with*

$$\gamma(A, B) = L^{A \otimes B}, \quad \beta(C, A) = L^{C \otimes A}, \quad \alpha(B, C) = L^{B \otimes C}.$$

Proof. $\mathrm{Hom}_{\mathcal{E}}(X, L^Y) \simeq \mathrm{hom}_{\mathcal{E}}(X \otimes Y, L) \simeq \mathrm{Hom}_{\mathcal{E}}(Y, L^X)$, so the functor $L^{(-)} : \mathcal{E} \to \mathcal{E}^{\mathrm{op}}$ is left adjoint to L^{op}. One of the equivalences in a trijunction (definition 2.1) is given by: $\mathrm{Hom}_{\mathcal{E}}(A, L^{B \otimes C}) \simeq \mathrm{Hom}_{\mathcal{E}}(A \otimes (B \otimes C), L) \simeq \mathrm{Hom}_{\mathcal{E}}(B \otimes (A \otimes C), L) \simeq \mathrm{Hom}_{\mathcal{E}}(B, L^{A \otimes C})$. □

4.4 Canonical auto-trijunction on an algebraic universe

Proposition 4.6. *Given an algebraic universe $\mathcal{X}$ — for example a topos or a category of fuzzy sets (cf. propositions 4.1 and 4.2) — we get an auto-trijunction (γ, β, α) between $\mathcal{A} = \mathcal{X}$, $\mathcal{B} = \mathcal{X}$,$\mathcal{C} = \mathcal{X}$, with*

$$\gamma(A, B) = P(A \times B), \quad \beta(C, A) = P(C \times A), \quad \alpha(B, C) = P(B \times C).$$

Proof. An algebraic universe is a cartesian closed category, and we have $PX = P(1)^X$. So we have just to apply proposition 4.5. □

4.5 Toward a calculus of triadic Galois connections in a topos

In fact the auto-trijunction in proposition 4.6 does not depend on ψ, π, etc., but only on the composition $\psi.a = \eta$, the cartesian closed structure on the topos or the algebraic universe, and the object $P(1)$. Nevertheless:

Proposition 4.7. *In a topos $\mathcal{E}$, using the canonical auto-trijunction (proposition 4.6) and the data ψ, π, etc., we can internally recover a theory of Galois connections and triadic Galois connections.*

Proof. We indicate only the starting point. From a ternary relation $(p, q, r) : R \to A \times B \times C$, we can construct the different terms in the hexagon pictured in proposition 3.6 in the case of the category Ens.
We consider $c = r \star (p, q) : C \to P(A \times B)$, we know how to construct $\exists c : PC \to PP(A \times B)$, $\bigcap_{A \times B} : PP(A \times B) \to P(A \times B)$, and the composition $R^*_C = \bigcap_{A \times B} \exists c : PC \to P(A \times B)$.
We consider also $a' = (r, q) \star p : C \times B \to PA$, $\alpha_R = \bigcap_A \exists(a')$.
A calculus of ternary relations in terms of *internal triadic Galois connections* is available in any topos; this works also in any category of fuzzy sets.

□

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES
Vol. LIV-1 (2013)

DERIVED HALL ALGEBRAS FOR STABLE HOMOTOPY THEORIES

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Résumé. Dans cet article, nous étendons la construction de l'algèbre de Hall dérivée de Toen, dans laquelle il obtient des algèbres associatives avec unité à partir de certaines categories de modèles stables, au cas où ces algèbres sont obtenues à partir de théories homotopiquesstables plus génèrales, en particulier espaces de Segal complets stables satisfaisant des hypothèses de finitude appropriées.

Abstract. In this paper we extend Toën's derived Hall algebra construction, in which he obtains unital associative algebras from certain stable model categories, to one in which such algebras are obtained from more general stable homotopy theories, in particular stable complete Segal spaces satisfying appropriate finiteness assumptions.

Keywords. derived Hall algebras, homotopy theories, complete Segal spaces, $(\infty, 1)$-categories
Mathematics Subject Classification (2010). 55U35, 55U40, 18G55, 18E30, 16S99

1 Introduction

Hall algebras associated to abelian categories play an important role in representation theory. In particular, when the abelian category in question is the category of $\mathbb{F}_q$-representations of a quiver associated to a simply-laced Dynkin diagram, there is a close relationship between the Hall algebra and the quantum enveloping algebra of the Lie algebra associated to the same Dynkin diagram. Recent attempts to strengthen this relationship have led to the problem of associating some kind of Hall algebra to categories which are triangulated rather than abelian. In particular, it is conjectured that one could recover the quantum enveloping algebra from an appropriate Hall-type algebra associated to Peng and Xiao's root category, which is, roughly speaking, the derived category of the category of this abelian category of representations, modulo a double shift relation [15].

In [22], Toën constructs "derived Hall algebras" associated to triangulated categories arising as homotopy categories of model categories whose objects are modules over a sufficiently finitary differential graded category over $\mathbb{F}_q$. In doing so, he develops a formula for the multiplication in this algebra in such a way that it can be regarded as a generalization of the formula for the multiplication in an ordinary Hall algebra. This formula was verified for more general triangulated categories, still satisfying certain finiteness conditions, by Xiao and Xu [24]. However, none of these methods can yet be applied to the root category, as it does not satisfy these finiteness assumptions.

In this paper, we seek to generalize Toën's development of derived Hall algebras. Specifically, we modify his proof to establish derived Hall algebras corresponding to triangulated categories arising as homotopy categories for more general stable homotopy theories. Most triangulated categories can be realized as homotopy categories of such stable homotopy theories. Although such triangulated categories are covered by Xiao and Xu's work, our objective is rather to broaden the context in which we can make use of homotopy-theoretic methods. We expect that these ideas will shed light on the question of how to find a similar algebra arising from a triangulated category which is not finitary. Also, it seems that this more flexible setting should be more amenable than the model category world for finding a coalgebra or even a Hopf algebra structure on derived Hall algebras, extending these structures which are significant in the study of ordinary Hall algebras. This idea will be the subject of future work in collaboration with Robertson.

We expect that the methods of this paper will be applicable to other settings, enabling one to use more general stable homotopy theories in settings in which the additional structure of stable model categories is too restrictive. For example, not all derived categories arise from actual model categories, but they do always come from a stable homotopy theory. It is expected that the ability to work with such homotopy theories, which contain more information than their associated derived categories, will facilitate progress in the many areas in which derived categories appear.

In this paper, we use the complete Segal space model for homotopy theories. If we regard a homotopy theory as a category with weak equivalences, then there are several equivalent models for homotopy theories as mathematical objects, in particular objects of model categories with appropriate weak equivalences. Complete Segal spaces were developed by Rezk [17]; they are simplicial spaces satisfying conditions enabling one to regard them as something like a simplicial category up to homotopy. Their associated model category is in fact equivalent to the model structure on the category of simplicial categories [3], as well as to the model structures for Segal categories [3] and quasi-categories [10]. While any one of these models could be used, we prefer the complete Segal space model here because it is particularly well-suited for understanding fiber products of model categories [2], one of the key tools used by Toën in his proof of the associativity of derived Hall algebras. Specifically, we are able to use homotopy pullbacks of complete Segal spaces where he used the homotopy fiber product of model categories.

There is, in fact, another perspective on complete Segal spaces (and equivalent objects); they are also models for $(\infty, 1)$-categories, or ∞-categories with n-morphisms invertible for $n > 1$. While the motivation for using complete Segal spaces in this paper arises from the viewpoint that they are generalizations of model categories, it is also useful, in particular when we need to define categorical notions such as colimits within them, to remember that they can be thought of as generalizations of ordinary categories in this way.

In Section 2, we give a review of stable model categories. These ideas are generalized in Section 3, where we explain how Lurie's methods for stable quasi-categories can be translated to stable complete Segal spaces. We review our main tool of interest, homotopy fiber products of model categories and homotopy pullbacks of complete Segal spaces, in Section 4, then introduce Toën's derived Hall algebras in Section 5. The main results of the paper can be found in Section 6, where we establish derived Hall algebras for stable complete Segal spaces.

Acknowledgments. The ideas in this paper have benefitted enormously from conversations with many people over the last few years. These people include Zongzhu Lin, Bertrand Toën, Clark Barwick, David Gepner, John Baez, and Christopher Walker. Thanks also to the referee for a close reading of the paper. The author was partially supported by NSF grant DMS-0805951. Support from the Fields Institute in Spring 2007 and the CRM

Barcelona in Spring 2008, where some of this work was completed, is also gratefully acknowledged.

2 Stable model categories

Recall that a *model category* $\mathcal{M}$ is a category with three distinguished classes of morphisms: weak equivalences, fibrations, and cofibrations, satisfying five axioms [5, 3.3]. Given a model category structure, one can pass to the *homotopy category* $\mathrm{Ho}(\mathcal{M})$, which is a localization of $\mathcal{M}$ with respect to the class of weak equivalences [8, 1.2.1]. In particular, the weak equivalences, as the morphisms that we wish to invert, make up the most important part of a model category. An object x in a model category $\mathcal{M}$ is *fibrant* if the unique map $x \to *$ to the terminal object is a fibration. Dually, an object x in $\mathcal{M}$ is *cofibrant* if the unique map $\phi \to x$ from the initial object is a cofibration.

The standard notion of equivalence of model categories is given by the following definitions. First, recall that an *adjoint pair* of functors $F\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon G$ satisfies the property that, for any objects X of $\mathcal{C}$ and Y of $\mathcal{D}$, there is a natural isomorphism

$$\varphi : \mathrm{Hom}_{\mathcal{D}}(FX, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, GY).$$

The functor F is called the *left adjoint* and G the *right adjoint* [14, IV.1].

Definition 2.1. [8, 1.3.1] An adjoint pair of functors $F\colon \mathcal{M} \leftrightarrows \mathcal{N}\colon G$ between model categories is a *Quillen pair* if F preserves cofibrations and G preserves fibrations. The left adjoint F is called a *left Quillen functor*, and the right adjoint G is called the *right Quillen functor*.

Definition 2.2. [8, 1.3.12] A Quillen pair of model categories is a *Quillen equivalence* if for all cofibrant X in $\mathcal{M}$ and fibrant Y in $\mathcal{N}$, a map $f\colon FX \to Y$ is a weak equivalence in $\mathcal{D}$ if and only if the map $\varphi f\colon X \to GY$ is a weak equivalence in $\mathcal{M}$.

We also consider model categories with the additional data that their homotopy categories are triangulated. Recall that a *triangulated category* T is an additive category, together with an equivalence $\Sigma\colon T \to T$ called a *shift functor*, and a collection of *distinguished triangles*

$$x \xrightarrow{\alpha} y \xrightarrow{\beta} z \xrightarrow{\gamma} \Sigma x$$

satisfying four axioms [11, §2.1].

For a model category to have a triangulated homotopy category, it must first be *pointed*, in that its initial and terminal objects coincide. Such an object is called a *zero object*.

Definition 2.3. [8, 7.1.1] A pointed model category $\mathcal{M}$ is *stable* if its homotopy category Ho$(\mathcal{M})$ is triangulated.

Example 2.4. Let $\mathcal{R}$ be a ring and $Ch(R)$ the category of chain complexes of R-modules. Then the model category structure on $Ch(R)$ is triangulated. In fact, its homotopy category is equivalent to the *derived category* $\mathcal{D}(R)$, formed by taking $Ch(R)$ modulo the equivalence relation given by chain homotopies of maps, and formally inverting the quasi-isomorphisms [11, §1.2].

3 Stable complete Segal spaces

3.1 Simplicial spaces and complete Segal spaces

Recall that the simplicial indexing category $\mathbf{\Delta}^{op}$ is defined to be the category with objects finite ordered sets $[n] = \{0 \to 1 \to \cdots \to n\}$ and morphisms the opposites of the order-preserving maps between them. A *simplicial set* is then a functor

$$K \colon \mathbf{\Delta}^{op} \to \mathcal{S}ets.$$

We denote by $\mathcal{SS}ets$ the category of simplicial sets, and this category has a natural model category structure equivalent to the standard model structure on topological spaces [6, I.10].

One can consider more general simplicial objects; in this paper we work with *simplicial spaces* (also called bisimplicial sets), or functors

$$X \colon \mathbf{\Delta}^{op} \to \mathcal{SS}ets.$$

Given a simplicial set K, we also denote by K the simplicial space which has the simplicial set K at every level. We denote by K^t, or "K-transposed", the constant simplicial space in the other direction, where $(K^t)_n = K_n$, where on the right-hand side K_n is regarded as a discrete simplicial set. The category of simplicial spaces has a model category structure called the *Reedy* *structure* in which weak equivalences are given levelwise and all objects are cofibrant [16].

Specifically, we consider simplicial spaces satisfying additional conditions, namely, those inducing a notion of composition up to homotopy. These Segal spaces and complete Segal spaces were first introduced by Rezk [17], and the name is meant to be suggestive of similar ideas first presented by Segal [21].

Definition 3.1. [17, 4.1] A *Segal space* is a Reedy fibrant simplicial space W such that the Segal maps

$$\varphi_n \colon W_n \to \underbrace{W_1 \times_{W_0} \cdots \times_{W_0} W_1}_{n}$$

are weak equivalences of simplicial sets for all $n \geq 2$.

Given a Segal space W, we can consider its *objects* $\text{ob}(W) = W_{0,0}$, and, between any two objects x and y, the *mapping space* $\text{map}_W(x, y)$, given by the homotopy fiber of the map $W_1 \to W_0 \times W_0$ given by the two face maps $W_1 \to W_0$. The Segal condition stated above guarantees that a Segal space has a notion of n-fold composition of mapping spaces, up to homotopy.

The *homotopy category* of W, denoted $\text{Ho}(W)$, has as objects the elements of the set $W_{0,0}$, and

$$\text{Hom}_{\text{Ho}(W)}(x, y) = \pi_0 \text{map}_W(x, y).$$

A *homotopy equivalence* in W is a 0-simplex of W_1 whose image in $\text{Ho}(W)$ is an isomorphism. We consider the subspace of W_1 whose components contain homotopy equivalences, denoted W_{hoequiv}. Notice that the degeneracy map $s_0 \colon W_0 \to W_1$ factors through W_{hoequiv}; hence we may make the following definition.

Definition 3.2. [17, §6] A *complete Segal space* is a Segal space W such that the map $W_0 \to W_{\text{hoequiv}}$ is a weak equivalence of simplicial sets.

Given this definition, we can describe the complete Segal space model structure on the category of simplicial spaces.

Theorem 3.3. *[17, 7.2] There is a model structure $\mathcal{CSS}$ on the category of simplicial spaces such that the fibrant and cofibrant objects are precisely the complete Segal spaces. Furthermore, $\mathcal{CSS}$ has the additional structure of a cartesian closed model category.*

The fact that $\mathcal{CSS}$ is cartesian closed allows us to consider, for any complete Segal space W and simplicial space X, the complete Segal space W^X. In particular, using the simplicial structure, the simplicial set at level n is given by

$$(W^X)_n = \text{Map}(X \times \Delta[n]^t, W).$$

3.2 Stable quasi-categories and stable complete Segal spaces

As with model categories, we need to consider complete Segal spaces which are *stable*, in the sense that their homotopy categories are triangulated. It should be noted that, although we have given this simple definition of a stable complete Segal space, one could define it in a more technical way which permits a better understanding of the structure of a stable complete Segal space; Lurie has explained these ideas extensively for stable quasi-categories in [12], and they can fairly easily be translated into the equivalent setting of complete Segal spaces.

Although we do not go into this level of detail on this point in this paper, there are other notions that have been developed for quasi-categories which are useful here for complete Segal spaces. Thus, we give a very brief summary of quasi-categories and their relationship with complete Segal spaces.

Recall that a *quasi-category* X is a simplicial set satisfying the inner Kan condition, so that for any $n \geq 1$ and $0 < k < n$, a dotted arrow lift exists in any diagram of the form

$$\begin{array}{ccc} V[n,k] & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta[n]. & & \end{array}$$

The notion of quasi-category goes back to Boardman and Vogt [4], but is has received extensive attention more recently, especially by Joyal [9] and Lurie [13]. In particular, Joyal proves that there is a model structure on the category of simplicial sets such that the fibrant and cofibrant objects are precisely the quasi-categories. We denote this model category $\mathcal{QC}at$. Furthermore, Joyal and Tierney have proved that the model category $\mathcal{QC}at$ is Quillen equivalent to Rezk's model category $\mathcal{CSS}$ [10]. Remarkably, they prove that there are actually two different Quillen equivalences between these two model categories. Here, we make use of the one that is particularly easy to describe, the right adjoint $\mathcal{CSS} \rightarrow \mathcal{QC}at$ given by $W \mapsto W_{*,0}$.

Using this relationship we return to the matter of explaining some necessary structures on complete Segal spaces. For a complete Segal space to be stable, we need it to be pointed, or to have a zero object, denoted 0. As we have seen, in an ordinary category, a zero object is one which is both initial and terminal, so for any object x, there are unique morphisms $x \rightarrow 0$ and $0 \rightarrow x$. As a complete Segal space is a homotopical generalization of a category, we require a homotopical notion of initial and terminal objects. The following definitions, given by Joyal [9] and Lurie [13, 1.2.12.1, 1.2.12.6] for quasi-categories, are easy to reformulate for complete Segal spaces.

Definition 3.4. An object $x \in W_{0,0}$ of a complete Segal space is *initial* if it is initial as an object of $\text{Ho}(W)$, i.e., if $\text{map}_W(x, y)$ is weakly contractible for any $y \in W_{0,0}$. Dually, x is *terminal* if it is terminal as an object of $\text{Ho}(W)$, i.e., if $\text{map}_W(y, x)$ is weakly contractible for any y. An object is a *zero object* of W if it is both initial and terminal.

In addition to having a zero object, we need to have a notion of "pushout" within a complete Segal space, another analogue of a standard categorical idea within this generalized setting. Fortunately, formal definitions of limits and colimits within quasi-categories have been established by Lurie [13, 1.2.13.4]. We give a brief exposition here, enough to translate his definition into the world of complete Segal spaces; see [13, 1.2.8, 1.2.13] for a detailed treatment.

Let X and Y be simplicial sets. We can define their *join* $X \star Y$ by

$$(X \star Y)_n = X_n \amalg Y_n \amalg \coprod_{i+j=n-1} X_i \times Y_j.$$

Note that the operation defines a monoidal product on $\mathcal{SS}ets$ with unit the empty simplicial set ϕ. Then, for a fixed simplicial set X, we can define a functor

$$X \star (-) \colon \mathcal{SS}ets \rightarrow \mathcal{SS}ets$$

by

$$Y \mapsto X \star Y$$

and notice that the map $\phi \to Y$ is sent to the map $X \star \phi = X \to X \star Y$. Thus, the simplicial set $X \star Y$ comes equipped with a canonical map $X \to X \star Y$, and so we can regard $X \star Y$ as an object of the *undercategory* or *category of simplicial sets under* X [14, II.6], denoted $X \downarrow \mathcal{SS}ets$. In doing so, we can think of our functor as

$$X \star (-) \colon \mathcal{SS}ets \to X \downarrow \mathcal{SS}ets.$$

This functor has a right adjoint given by

$$(p \colon X \to Y) \mapsto Y.$$

To remember that Y has come from some map $p \colon X \to Y$, Lurie denotes the image of this functor $Y_{p/}$. We can think of $Y_{p/}$ as the simplicial set Y with a specified X-shaped diagram inside it.

Such an object can be used to define colimits in a quasi-category. If Y is a quasi-category and $p \colon X \to Y$ is a map of simplicial sets, then a *colimit* for p is an initial object of $Y_{p/}$. Dually, one could use the functor $(-) \star X$, its right adjoint, and the resulting definition of $Y_{/p}$ to define a limit in a quasi-category Y.

Now, we translate this definition into $\mathcal{CSS}$.

Definition 3.5. Let W be a complete Segal space and X a simplicial set, together with a map $p \colon X^t \to W$. A *colimit* for p in W is an initial object of $(W_{*,0})_{p/}$, regarded as an object of W.

In this paper, we consider the case where the simplicial set X is $\Delta[1] \amalg_{\Delta[0]} \Delta[1]$, forming the diagram $\cdot \leftarrow \cdot \rightarrow \cdot$, so that the colimit is a "pushout" in the complete Segal space W. One can show that if W is stable, the fact that Ho(W) is triangulated guarantees that colimits must always exist in W. Again, we refer the reader to Lurie's manuscript on stable quasi-categories [12] for greater depth on this point.

3.3 Model categories and complete Segal spaces

We conclude this section with a brief exposition on the relationship between model categories and complete Segal spaces. Since we are translating a construction on model categories to one on complete Segal spaces, we need to understand how to regard a model category as a specific kind of complete Segal space.

As described by Rezk [17], any category with weak equivalences gives rise to a complete Segal space via the functor we denote L_C; given such a category $\mathcal{C}$, $L_C\mathcal{C}$ is given by

$$(L_C\mathcal{C})_n = \text{nerve}(\text{we}(\mathcal{C}^{[n]}))$$

where $\text{we}(\mathcal{C}^{[n]})$ denotes the category of weak equivalences of chains of n composable morphisms in $\mathcal{C}$.

If $\mathcal{M}$ is a model category, then we can apply this construction, but, as explained in [2], it is only a functor when the morphisms between model categories preserve weak equivalences. Since we want a construction which is functorial on the category of model categories with left Quillen functors between them, we can modify the construction by restricting to the full subcategory of $\mathcal{M}$ whose objects are cofibrant.

The main result of [1] is that this construction is well-behaved with respect to other natural ways of getting a complete Segal space from a model category; in particular, the resulting complete Segal space is weakly equivalent to the one obtained from taking the simplicial localization and then applying any one of several functors from simplicial categories to complete Segal spaces. There is an up-to-homotopy characterization of the resulting complete Segal space as well. While we do not make use of this description explicitly in this paper, it is key to the proof of Theorem 4.1 below.

4 Fiber products of model categories and homotopy pullbacks of complete Segal spaces

A key tool in Toën's proof that his derived Hall algebras are associative is the fiber product of model categories. We begin with his definition as given in [22]. First, suppose that

$$\mathcal{M}_1 \xrightarrow{F_1} \mathcal{M}_3 \xleftarrow{F_2} \mathcal{M}_2$$

is a diagram of left Quillen functors of model categories. Define their *fiber product* to be the model category $\mathcal{M} = \mathcal{M}_1 \times^h_{\mathcal{M}_3} \mathcal{M}_2$ whose objects are given by 5-tuples $(x_1, x_2, x_3; u, v)$ such that each x_i is an object of $\mathcal{M}_i$ fitting into a diagram

$$F_1(x_1) \xrightarrow{u} x_3 \xleftarrow{v} F_2(x_2).$$

A morphism of $\mathcal{M}$, say $f\colon (x_1, x_2, x_3; u, v) \to (y_1, y_2, y_3; z, w)$, is given by maps $f_i : x_i \to y_i$ such that the following diagram commutes:

$$\begin{array}{ccccc} F_1(x_1) & \xrightarrow{u} & x_3 & \xleftarrow{v} & F_2(x_2) \\ \downarrow{\scriptstyle F_1(f_1)} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle F_2(f_2)} \\ F_1(y_1) & \xrightarrow{z} & y_3 & \xleftarrow{w} & F_2(y_2). \end{array}$$

This category $\mathcal{M}$ can be given the structure of a model category, where the weak equivalences and cofibrations are given levelwise. In other words, f is a weak equivalence (or cofibration) if each map f_i is a weak equivalence (or cofibration) in $\mathcal{M}_i$.

A more restricted definition of this construction requires that the maps u and v be weak equivalences in $\mathcal{M}_3$. Unfortunately, if we impose this additional condition, the resulting category cannot be given the structure of a model category because it does not have sufficient limits and colimits. However, it is still a perfectly good category with weak equivalences, and in some cases we can localize $\mathcal{M}$ so that the fibrant-cofibrant objects of the localized model category have u and v weak equivalences [2]. Although Toën uses the model structure given above, at the point where he really makes use of the fiber product he restricts to the case where the maps u and v are weak equivalences. Thus, we assume here this extra structure.

Consider the functor L_C, described in the previous section, which takes a model category (or category with weak equivalences) to a complete Segal space. Given a fiber square of model categories where we require the maps u and v to be weak equivalences, we can apply this functor to obtain a commutative square

$$\begin{array}{ccc} L_C\mathcal{M} & \longrightarrow & L_C\mathcal{M}_2 \\ \downarrow & & \downarrow \\ L_C\mathcal{M}_1 & \longrightarrow & L_C\mathcal{M}_3. \end{array}$$

Alternatively, we could apply the functor L_C only to the original diagram and take the homotopy pullback, which we denote P, and obtain the following diagram:

$$\begin{array}{ccc} P & \longrightarrow & L_C\mathcal{M}_2 \\ \downarrow & & \downarrow \\ L_C\mathcal{M}_1 & \longrightarrow & L_C\mathcal{M}_3. \end{array}$$

Theorem 4.1. *[2] The complete Segal spaces $L_C\mathcal{M}$ and $P = L_C\mathcal{M}_1 \times^h_{L_C\mathcal{M}_3} \L_C\mathcal{M}_2$ are weakly equivalent.*

This theorem allows us to use the homotopy pullback of complete Segal spaces to generalize the situations in which Toën uses the fiber product of model categories. In particular, we generalize a scenario given by Toën [22, 4.2] as follows.

Let

$$\begin{array}{ccc} W & \xrightarrow{H_1} & X \\ {\scriptstyle H_2}\downarrow & & \downarrow{\scriptstyle F_1} \\ Y & \xrightarrow{F_2} & Z \end{array}$$

be diagram of complete Segal spaces equipped with an isomorphism $\alpha\colon F_1 \circ H_1 \Rightarrow F_2 \circ H_2$, and define a map

$$F\colon W \to V = X \times^h_Z Y$$

by

$$w \mapsto (H_1(w), H_2(w); \alpha_w).$$

Lemma 4.2. *If Ho$(W) \to$ Ho(V) is an equivalence of categories, then the diagram*

$$\begin{array}{ccc} \textit{nerve}(\textit{Ho}(wW)) & \longrightarrow & \textit{nerve}(\textit{Ho}(wX)) \\ \downarrow & & \downarrow \\ \textit{nerve}(\textit{Ho}(wY)) & \longrightarrow & \textit{nerve}(\textit{Ho}(wY)) \end{array}$$

is homotopy cartesian.

Proof. We want to show that the map

$$\text{nerve}(\text{Ho}(wW)) \to \text{nerve}(\text{Ho}(wX)) \times^h_{\text{nerve}(\text{Ho}(wZ))} \text{nerve}(\text{Ho}(wY))$$

is a weak equivalence of simplicial sets. By our assumption, we know that the map
$$\mathrm{Ho}(W) \to \mathrm{Ho}(X \times^h_Z Y)$$
is an equivalence of categories. Notice that the homotopy category $\mathrm{Ho}(wW)$ is the maximal subgroupoid of $\mathrm{Ho}(W)$ and analogously for the other complete Segal spaces in the diagram. Hence, we have an equivalence of categories

$$\begin{aligned}\mathrm{Ho}(wW) \to \mathrm{Ho}(w(X \times^h_Z Y)) &\simeq \mathrm{Ho}(wX \times^h_{wZ} wY) \\ &\simeq \mathrm{Ho}(wX) \times^h_{\mathrm{Ho}(wZ)} \mathrm{Ho}(wY).\end{aligned}$$

Since nerves of equivalent categories are weakly equivalent simplicial sets, the lemma follows. □

5 Hall algebras and derived Hall algebras

5.1 Classical Hall algebras

Let $\mathcal{A}$ be an abelian category. Throughout this section, we assume that $\mathcal{A}$ is *finitary*, in that, for any objects x and y of $\mathcal{A}$, the groups $\mathrm{Hom}(x, y)$ and $\mathrm{Ext}^1(x, y)$ are finite.

Definition 5.1. [20] Given an abelian category $\mathcal{A}$, its *Hall algebra* $\mathcal{H}(\mathcal{A})$ is defined as

1. the vector space with basis isomorphism classes of objects in $\mathcal{A}$, with
2. multiplication given by
$$[x]\cdot [y] = \sum_{[z]} g^z_{x,y}[z]$$
where the *Hall numbers* $g^z_{x,y}$ are given by
$$g^z_{x,y} = \frac{|\{0 \to x \to z \to y \to 0 \text{ exact}\}|}{|\mathrm{Aut}(x)|\cdot |\mathrm{Aut}(y)|}.$$

Notice that our assumptions on $\mathcal{A}$ guarantee that each Hall number really is a finite number. It can be shown that this definition gives $\mathcal{H}(\mathcal{A})$ the structure of a unital associative algebra [18].

Although Hall algebras have been investigated for a number of purposes, recent interest in them has arisen from the close relationship between Hall algebras and quantum groups in the following situation. Suppose that $\mathfrak{g}$ is a Lie algebra of type A, D, or E. Then $\mathfrak{g}$ has an associated simply-laced Dynkin diagram, which is just an unoriented graph with no cycles. Assigning an orientation to each of the edges in this graph gives a quiver, or oriented graph, which we denote Q. Given a finite field $\mathbb{F}_q$, let $\mathcal{A}$ be the category of $\mathbb{F}_q$-representations of this quiver Q. It can be shown that $\mathcal{A}$ is in fact an abelian category satisfying our finiteness assumptions, and hence we have an associated Hall algebra $\mathcal{H}(\mathcal{A})$ [18]. The Hall algebra as we have defined it is not independent of the chosen orientation on the quiver, but a slight modification by Ringel makes it so; this algebra is often called the *Ringel-Hall algebra* [19].

However, another algebra can be obtained from $\mathfrak{g}$, namely the quantum enveloping algebra $U_q(\mathfrak{g})$. This algebra can be given its triangular decomposition

$$U_q(\mathfrak{g}) = U_q(\mathfrak{n}^+) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^-).$$

Work of Ringel, further developed by Green, has shown that there is a close relationship between the Hall algebra $\mathcal{H}(\mathcal{A})$ and the positive part of the quantum enveloping algebra,

$$U_q(\mathfrak{b}^+) = U_q(\mathfrak{n}^+) \otimes U_q(\mathfrak{h})$$

[7], [18].

A natural question to ask is whether there is some kind of enlarged version of the Hall algebra from which one could recover not just $U_q(\mathfrak{b}^+)$, but all of $U_q(\mathfrak{g})$. Work of Peng and Xiao [15] has led to the conjecture that such an algebra should be obtained from the following category. Using the abelian category $\mathcal{A}$ of quiver representations as above, consider its bounded derived category $\mathcal{D}^b(\mathcal{A})$, which is no longer abelian, but is instead a triangulated category. As such, it has a shift functor $\Sigma\colon \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{A})$. We then define the *root category* of $\mathcal{A}$ to be $\mathcal{D}^b(\mathcal{A})/\Sigma^2$, the triangulated category obtained from $\mathcal{D}^b(\mathcal{A})$ by identifying an object with its double shift.

It is still an open question how to find a "Hall algebra" associated to this root category. To begin with, the usual definition does not apply because the root category is not abelian. It is, however, triangulated, and recent efforts in this area have focused on finding Hall algebras for triangulated categories. In the rest of this section, we describe derived Hall algebras, defined by Toën, which can be obtained from certain triangulated categories. Thus far the necessary restrictions on these triangulated categories prohibit us from being able to define a derived Hall algebra for the root category.

5.2 Derived Hall algebras

Recall that a *differential graded category* or *dg category*, is a category enriched over $Ch(R)$, the category of cochain complexes of modules over a ring R. Thus, given any objects x and y in a dg category $\mathcal{T}$, we have a cochain complex $\mathcal{T}(x, y)$. Here, we assume that $R = \mathbb{F}_q$, the finite field with q elements. Toën defines a dg category $\mathcal{T}$ to be *locally finite* if for any objects x and y in $\mathcal{T}$, the cochain complex $\mathcal{T}(x, y)$ is cohomologically bounded and has all cohomology groups finite dimensional [22, 3.1].

Given a locally finite dg category $\mathcal{T}$, we consider $\mathcal{M}(\mathcal{T})$, the category of dg $\mathcal{T}^{op}$-modules, or functors $\mathcal{T} \to Ch(\mathbb{F}_q)$. This category has the structure of a stable model category, with levelwise weak equivalences and fibrations [23, §3]. We have made finiteness assumptions about the dg category $\mathcal{T}$, but in taking the module category, we may have cochain complexes in the image which do not satisfy these kinds of conditions. If we restrict to functors which are appropriately finitary, we no longer have a model structure, since this subcategory does not possess enough limits and colimits. So, we work with the model category $\mathcal{M}(\mathcal{T})$ of all modules but consider also the full subcategory $\mathcal{P}(\mathcal{T})$ of perfect objects. A module in $\mathcal{M}(\mathcal{T})$ is *perfect* if it belongs to the smallest subcategory of $\mathrm{Ho}(\mathcal{M}(\mathcal{T}))$ containing the quasi-representable modules (see [23, 3.6] for a definition) and which is stable by retracts, homotopy pushouts, and homotopy pullbacks [22]. Perfect objects coincide with the compact objects in the triangulated category $\mathrm{Ho}(\mathcal{M}(\mathcal{T}))$. (Recall that if T is a triangulated category with arbitrary coproducts, then an object x of T is *compact* if any map $x \to \amalg_i y_i$ factors through a finite coproduct [11, 6.5].)

Since $\mathrm{Ho}\mathcal{M}(\mathcal{T})$ is a triangulated category, it has a shift functor; we denote maps from x to the ith shift of y in this category by $[x, y[i]]$ or by $\mathrm{Ext}^i(x, y)$. Notice that for perfect modules, these Ext groups are all finite.

Theorem 5.2. *[22, 1.1, 5.1] Let $\mathcal{T}$ be a locally finite dg category over a finite field $\mathbb{F}_q$. Define $\mathcal{DH}(\mathcal{T})$ to be the $\mathbb{Q}$-vector space with basis the characteristic functions χ_x, where x runs through the set of weak equivalence classes of perfect objects in $\mathcal{M}(\mathcal{T})$. Then there exists an associative and unital product*

$$\mu\colon \mathcal{DH}(\mathcal{T}) \otimes \mathcal{DH}(\mathcal{T}) \to \mathcal{DH}(\mathcal{T})$$

such that

$$\mu(\chi_x, \chi_y) = \sum_z g^z_{x,y} \chi_z$$

and these derived Hall numbers $g^z_{x,y}$ are given by the formula

$$g^z_{x,y} = \frac{|[x,z]_y| \cdot \prod_{i>0} |\mathit{Ext}^{-i}(x,z)|^{(-1)^i}}{|\mathit{Aut}(x)| \cdot \prod_{i>0} |\mathit{Ext}^{-i}(x,x)|^{(-1)^i}},$$

where $[x, z]_y$ denotes the subset of $[x, z]$ of morphisms $f\colon x \to z$ whose cone is isomorphic to y in $Ho(\mathcal{M}(\mathcal{T}))$.

6 More general derived Hall algebras

In this section, we establish the existence of derived Hall algebras for sufficiently finitary stable complete Segal spaces. Our strategy follows that of Toën, and some proofs of his continue to hold without change. However, without the restrictions of a model structure, some of the proofs are greatly simplified.

Throughout this section, suppose that W is a pointed stable complete Segal space, so that $\mathrm{Ho}(W)$ is a triangulated category with a zero object. As in the previous section, we define for any objects x, y in W

$$\mathrm{Ext}^i(x, y) = [x, y[i]]$$

where the outside brackets denote maps in $\mathrm{Ho}(W)$ and the inside brackets denote the shift functor giving the triangulated structure of $\mathrm{Ho}(W)$.

Definition 6.1. A stable complete Segal space W is *finitary* if in $\mathrm{Ho}(W)$ we have that $\mathrm{Ext}^i(x, y)$ is finite for all pairs of objects (x, y) and all values of i, and zero for sufficiently large values of i.

We assume for the rest of the paper that all our stable complete Segal spaces are finitary.

Since the model category $\mathcal{CSS}$ is cartesian closed, the simplicial space $W^{\Delta[1]}$ is also a complete Segal space. Notice that W itself is isomorphic to the mapping object $W^{\Delta[0]}$, and so we can use the two maps $\Delta[0] \to \Delta[1]$ to define "source" and "target" maps $s, t \colon W^{\Delta[1]} \to W$. Since an object of $W^{\Delta[1]}$ is a 0-simplex $u \in \mathrm{map}_W(x, y)$ for some x and y objects of W, these two maps can be defined by $s(u) = x$ and $t(u) = y$. We also have a "cone" map $c \colon W^{\Delta[1]} \to W$ given by $c(u) = y \amalg_x 0$, where such a cone object exists because we have required that W be stable; in the homotopy category, it is just the completion of $u \colon x \to y$ to a distingushed triangle.

Using these maps, we can put together the diagram

$$\begin{array}{ccc} W^{\Delta[1]} & \xrightarrow{t} & W \\ {\scriptstyle s\times c}\downarrow & & \\ W \times W & & \end{array}$$

analogous to Toën's diagram of model categories [22, §4].

Because we are no longer working with model categories, a number of aspects of this diagram have been simplified, compared to the analogous one in Toën's paper. Because the objects are complete Segal spaces, rather than model categories, we no longer have to be concerned with whether these maps are left Quillen functors. Furthermore, we are able to impose conditions on W from the beginning so that its objects are already "perfect" in that all the necessary finiteness conditions are already satisfied.

A word on this point would perhaps be helpful here. It is likely that a stable complete Segal space that would arise in nature would not have all pairs of objects x and y satisfying the necessary finiteness conditions on $\mathrm{Ext}^i(x, y)$. However, we can show that restricting to the sub-complete Segal space with objects satisfying such conditions is still a complete Segal space. Explicitly, given a complete Segal space W, consider the doubly constant simplicial space $W_{0,0}$, and the sub-simplicial space $Z_{0,0}$ given by the perfect objects of W. Then define Z to be the simplicial spaces given by the pullback

$$\begin{array}{ccc} Z & \longrightarrow & W \\ \downarrow & & \downarrow \\ Z_{0,0} & \longrightarrow & W_{0,0}. \end{array}$$

Since $W_{0,0}$ is discrete, the map $W \to W_{0,0}$ is a fibration in $\mathcal{CSS}$, from which it follows that the map $Z \to Z_{0,0}$ is a fibration also. Thus, Z is a fibrant simplicial space in $\mathcal{CSS}$, or a complete Segal space. Furthermore, since compact objects of a triangulated category form a triangulated subcategory [11, 6.5], $\mathrm{Ho}(Z)$ is triangulated and Z is stable. Thus, we can restrict to the appropriate setting without losing the structure that we need, and so we always assume that, given an arbitrary stable complete Segal space W, we have implicitly restricted to Z.

Now, as Toën does, we restrict to the sub-complete Segal spaces of W and $W^{\Delta[1]}$, whose mapping spaces are sent to isomorphisms in the homotopy category; we call these spaces wW and $wW^{\Delta[1]}$, respectively. Taking the nerve of the homotopy categories, we obtain a diagram

$$\begin{array}{ccc} \mathrm{nerve}(\mathrm{Ho}(wW^{\Delta[1]})) & \xrightarrow{t} & \mathrm{nerve}(\mathrm{Ho}(wW)) \\ \downarrow{\scriptstyle s\times c} & & \\ \mathrm{nerve}(\mathrm{Ho}(wW)) \times \mathrm{nerve}(\mathrm{Ho}(wW)). & & \end{array}$$

For simplicity of notation, we write this diagram

$$\begin{array}{ccc} X^{(1)} & \xrightarrow{t} & X^{(0)} \\ \downarrow{\scriptstyle s\times c} & & \\ X^{(0)} \times X^{(0)}. & & \end{array}$$

To get an algebra with a well-defined multiplication, we need to show that this diagram of spaces satisfies some properties.

Definition 6.2. [22, 2.1] An object X in the homotopy category of spaces is *locally finite* if it satisfies the conditions

1. for any base point $x \in X$ and $i > 0$, the group $\pi_i(X, x)$ is finite, and

2. for any base point $x \in X$, there is some n, depending on x, such that $\pi_i(X, x) = 0$ for all $i > n$.

Lemma 6.3. *The spaces $X^{(0)}$ and $X^{(1)}$ are locally finite.*

Proof. For any $x \in \pi_0(X^{(0)})$, we use the facts that

$$\pi_1(X^{(0)}) \subseteq \text{Ext}^0(x, x) = [x, x]$$

and

$$\pi_i(X^{(0)}) = \text{Ext}^{1-i}(x, x)$$

for $i > 1$ [22, 3.2]. Our assumption on W guarantees that these groups are all finite, and that they are zero for sufficiently large i. Thus, $X^{(0)}$ is locally finite.

To show that $X^{(1)} = \text{nerve}(\text{Ho}(wW^{\Delta[1]}))$ is locally finite, notice that this space is weakly equivalent to

$$\text{nerve}(\text{Ho}(wW)) \times \Delta[1] = X^{(0)} \times \Delta[1]$$

which is also locally finite. □

Definition 6.4. [22, 2.5] A morphism $f : X \to Y$ of locally finite homotopy types is *proper* if, for any $y \in \pi_0(Y)$, there are only finitely many $x \in \pi_0(X)$ with $f(x) = y$.

Notice that f is proper if and only if, for any $y \in \pi_0(Y)$, the set $\pi_0(F_y)$ is finite. The proof of the following lemma follows just as it does in Toën's paper [22, 3.2].

Lemma 6.5. *The map $s \times c$ is proper.*

With these properties established for our diagram, we can use it to define an algebra analogous to that of Toën [22, §4].

Definition 6.6. [22, 2.2] Let X be a space. The $\mathbb{Q}$-vector space of *rational functions with finite support* on X is the $\mathbb{Q}$-vector space of functions on the set $\pi_0(X)$ with values in $\mathbb{Q}$ and finite support, and is denoted by $\mathbb{Q}_c(X)$.

Definition 6.7. As a vector space, the *derived Hall algebra $\mathcal{DH}(W)$* of W is given by $\mathbb{Q}_c(X^{(0)})$.

Given a morphism $f\colon X \to Y$ of locally finite spaces, we define a push-forward morphism $f_!\colon \mathbb{Q}_c(X) \to \mathbb{Q}_c(Y)$ as follows. Given $y \in \pi_0(Y)$, let F_y denote the homotopy fiber of f over y, and let $i\colon F_y \to X$ be the natural map. Using the long exact sequences of homotopy groups, one can see that for any $z \in \pi_0(F_y)$, the group $\pi_i(F_y, z)$ is finite for all $i > 0$ and zero for sufficiently large i. Furthermore, the fibers of the map $\pi_0(F_y) \to \pi_0(X)$ are all finite. Then, for any $\alpha \in \mathbb{Q}_c(X)$ and $y \in \pi_0(Y)$, define the function $f_!$ by

$$f_!(\alpha)(y) = \sum_{z \in \pi_0(F_y)} \alpha(i(z)) \cdot \prod_{i>0} |\pi_i(F_y, z)|^{(-1)^i}.$$

The assumption that α have finite support guarantees that $f_!$ is well-defined.

If $f\colon X \to Y$ is a proper map of locally finite spaces, then we have a well-defined pullback $f^*\colon \mathbb{Q}_c(Y) \to \mathbb{Q}_c(X)$ defined in the usual way as $f^*(\alpha)(x) = \alpha(f(x))$ for any $\alpha \in \mathbb{Q}_c(Y)$ and $x \in \pi_0(X)$. The requirement that f be proper guarantees that $f^*(\alpha)$ has finite support, so that f^* is in fact well-defined.

The following lemma is key for establishing associativity.

Lemma 6.8. *[22, 2.6] Consider a homotopy pullback diagram of locally finite spaces*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{u} & Y \end{array}$$

with u proper. Then the map v is also proper, and

$$u^* \circ f_! = g_! \circ v^*\colon \mathbb{Q}_c(X) \to \mathbb{Q}_c(Y').$$

To define the multiplication on $\mathcal{DH}(W)$, first notice that we have an isomorphism

$$\mathcal{DH}(W) \otimes \mathcal{DH}(W) \to \mathbb{Q}_c(X^{(0)} \times X^{(0)})$$

given by

$$(f, g) \mapsto ((x, y) \mapsto f(x) \cdot g(x)).$$

Then we can consider the map

$$\mu = t_! \circ (s \times c)^*\colon \mathcal{DH}(W) \otimes \mathcal{DH}(W) \to \mathcal{DH}(W).$$

The algebra structure on $\mathcal{DH}(W)$ is then given by

$$x \cdot y = \sum_z g^z_{x,y} z$$

where

$$g^z_{x,y} = \mu(\chi_x, \chi_y)(z)$$

where χ_x denotes the characteristic function of x.

Proposition 6.9. *With this multiplication, $\mathcal{DH}(W)$ is a unital algebra.*

Our proof essentially follows the one given by Toën [22, 4.1], with the necessary changes being made as we translate to the complete Segal space setting.

Proof. Given any object x in W, let χ_x denote its characteristic function; in particular, consider χ_0, the characteristic function of the zero object of W.

Notice that the set $\pi_0(X^{(1)})$ is isomorphic to the set of isomorphism classes of objects in $\operatorname{Ho}(wW^{\Delta[1]})$. Thus, fix some 0-simplex $u\colon x \to y$ of $\operatorname{map}_W(x,y)$, regarded as an object of $\operatorname{Ho}(wW^{\Delta[1]})$. Then

$$(s \times c)^*(u) = \begin{cases} 1 & \text{if } y \cong 0 \text{ and } x \cong z \text{ in } \operatorname{Ho}(wW) \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $(s \times c)^*(\chi_0, \chi_x)$ is the characteristic function of the subset of $\pi_0(X^{(1)})$ consisting of maps $0 \to z$ with $z \cong x$ in $\operatorname{Ho}(wW)$.

Define X to be the simplicial set contained in $X^{(1)}$ consisting of all the support of $(s \times c)^*(\chi_0, \chi_x)$, and notice that X is a connected simplicial set. Then using the definition of the product map μ, we get

$$\mu(\chi_0, \chi_x)(x) = \prod_{i>0} \left(|\pi_i(X)|^{(-1)^i} \cdot |\pi_i(X^{(0)}, x)|^{(-1)^{i+1}} \right).$$

Notice in particular that whenever $y \neq x$,

$$\mu(\chi_0, \chi_x)(y) = 0.$$

Restricting the target map $t\colon W^{\Delta[1]} \to W$ to the maps $y \to z$ such that $y \cong 0$ in $\operatorname{Ho}(wW)$, we see that on such objects t is fully faithful, up to homotopy. Thus, the induced map $t\colon X \to X^{(0)}$ induces isomorphisms $t_*\colon \pi_i(X) \to \pi_i(X^{(0)})$ for all $i > 0$, and the simplicial set X can be identified with a connected component of $X^{(0)}$. Hence, $\mu(\chi_0, \chi_x)(x) = 1$, so that $\mu(\chi_0, \chi_x) = \chi_x$.

Changing the order and following the same argument, one can see that we also have $\mu(\chi_x, \chi_0) = \chi_x$, thus proving that χ_0 is a unit element for $\mathcal{DH}(W)$. □

Theorem 6.10. *With this multiplication, $\mathcal{DH}(W)$ is an associative algebra.*

Proof. Consider the complete Segal space $W^{\Delta[2]}$, and, as with $W^{\Delta[1]}$ and W, denote by $X^{(2)}$ the simplicial set nerve(Ho$(wW^{\Delta[2]})$). Notice that there are three natural maps

$$f, g, h\colon W^{\Delta[2]} \to W^{\Delta[1]}$$

induced by the three inclusion maps $\Delta[1] \to \Delta[2]$, where f sends $x \to y \to z$ to $x \to y$, g sends it to $y \to z$, and h sends it to $x \to z$. There is also a cone map

$$k\colon W^{\Delta[2]} \to W^{\Delta[1]}$$

given by

$$(x \to y \to z) \mapsto (y \amalg_x 0 \to z \amalg_x 0),$$

with the pushouts defined as before in a stable complete Segal space, and a map between the two given by the universal property. This map may not be unique, but all such maps form a weakly contractible space.

Using these maps, we get two diagrams:

$$\begin{array}{ccccc}
X^{(2)} & \xrightarrow{\quad g \quad} & X^{(1)} & \xrightarrow{\ t\ } & X^{(0)} \\
\big\downarrow{\scriptstyle f\times(c\circ k)} & & \big\downarrow{\scriptstyle s\times c} & & \\
X^{(1)} \times X^{(0)} & \xrightarrow{\ t\times \mathrm{id}\ } & X^{(0)} \times X^{(0)} & & \\
\big\downarrow{\scriptstyle (s\times c)\times \mathrm{id}} & & & & \\
(X^{(0)} \times X^{(0)}) \times X^{(0)} & & & &
\end{array}$$

and

$$\begin{array}{ccccc}
X^{(2)} & \xrightarrow{h} & X^{(1)} & \xrightarrow{t} & X^{(0)} \\
\Big\downarrow{\scriptstyle (s\circ f)\times k} & & \Big\downarrow{\scriptstyle s\times c} & & \\
X^{(0)}\times X^{(1)} & \xrightarrow{\mathrm{id}\times t} & X^{(0)}\times X^{(0)} & & \\
\Big\downarrow{\scriptstyle \mathrm{id}\times(s\times c)} & & & & \\
X^{(0)}\times(X^{(0)}\times X^{(0)}) & & & &
\end{array}$$

which both give the same result taking composites across the top and down the left side:

$$\begin{array}{ccc}
X^{(2)} & \longrightarrow & X^{(0)} \\
\Big\downarrow & & \\
X^{(0)}\times X^{(0)}\times X^{(0)}. & &
\end{array}$$

Thus, to prove associativity of $\mathcal{DH}(W)$, it suffices by Lemma 6.8 to prove that the square in each of these diagrams is homotopy cartesian. In fact, it suffices to show that the diagrams

$$\begin{array}{ccc}
X^{(2)} & \xrightarrow{g} & X^{(1)} \\
{\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle s} \\
X^{(1)} & \xrightarrow{t} & X^{(0)}
\end{array}
\qquad\qquad
\begin{array}{ccc}
X^{(2)} & \xrightarrow{h} & X^{(1)} \\
\Big\downarrow{\scriptstyle k} & & \Big\downarrow{\scriptstyle c} \\
X^{(1)} & \xrightarrow{t} & X^{(0)}
\end{array}$$

are homotopy cartesian. For the first diagram, this fact follows immediately from the fact that the original diagram

$$\begin{array}{ccc}
W^{\Delta[2]} & \xrightarrow{g} & W^{\Delta[1]} \\
{\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle s} \\
W^{\Delta[1]} & \xrightarrow{t} & W
\end{array}$$

is a homotopy pullback diagram of complete Segal spaces. To show that the second diagram is homotopy cartesian requires more effort.

In this second diagram, let Z denote the homotopy pullback $W^{\Delta[1]} \times^h_W W^{\Delta[1]}$. Using Lemma 4.2, it suffices to prove that $\mathrm{Ho}(W^{\Delta[2]}) \to \mathrm{Ho}(Z)$ is fully faithful and essentially surjective. We begin with the argument for the latter. Suppose we have an object $(x \to z, w \to z \amalg_x 0)$ in $\mathrm{Ho}(Z)$; we want to find an object y of W such that $x \to y \to z$ is an object of $\mathrm{Ho}(W^{\Delta[2]})$ with $y \amalg_x 0 = w$. Such a y can be found by applying the axioms for a triangulated category to the diagram

$$\begin{array}{ccccccc} x & \dashrightarrow & y & \dashrightarrow & w & \longrightarrow & x[1] \\ {\scriptstyle =}\downarrow & & \vdots & & \downarrow & & \downarrow{\scriptstyle =} \\ x & \longrightarrow & z & \longrightarrow & z \amalg_x 0 & \longrightarrow & x[1]. \end{array}$$

To prove that the functor is fully faithful, we need to prove that, for any objects $x \to y \to z$ and $x' \to y' \to z'$ in $\mathrm{Ho}(W^{\Delta[2]})$, the map

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Ho}(W^{\Delta[2]})}(x \to y \to z, x' \to y' \to z') \to \\ &\quad \mathrm{Hom}_{\mathrm{Ho}(Z)}((x \to z, y \amalg_x 0 \to z \amalg_x 0), (x' \to z', y' \amalg_{x'} 0 \to z' \amalg_{x'} 0)) \end{aligned}$$

is an isomorphism. Elements of the set on the left-hand side are triples of maps making the diagram

$$\begin{array}{ccccc} x & \longrightarrow & y & \longrightarrow & z \\ \downarrow & & \downarrow & & \downarrow \\ x' & \longrightarrow & y' & \longrightarrow & z' \end{array}$$

commute, where elements of the set on the right-hand side are 4-tuples of maps making the pair of diagrams

$$\begin{array}{ccc} x & \longrightarrow & z \\ \downarrow & & \downarrow \\ x' & \longrightarrow & z', \end{array} \qquad \begin{array}{ccc} y \amalg_x 0 & \longrightarrow & z \amalg_x 0 \\ \downarrow & & \downarrow \\ y' \amalg_{x'} 0 & \longrightarrow & z' \amalg_{x'} 0 \end{array}$$

commute. Given an element of the right-hand set, we can use the axioms for a triangulated category to find a map $y \to y'$ compatible with the maps $x \to x'$ and $z \to z'$ to obtain an element of the left-hand set. Thus, the map is surjective. A similar argument can be used to prove that it is injective. $\square$

The proof of the following formula is essentially the same as the one given by Toën [22, 5.1]; we give it here with the necessary changes to our situation.

Proposition 6.11. *The derived Hall numbers are given by*

$$g^z_{x,y} = \frac{|[x,z]_y| \cdot \prod_{i>0} |Ext^{-i}(x,z)|^{(-1)^i}}{|Aut(x)| \cdot \prod_{i>0} |Ext^{-i}(x,x)|^{(-1)^i}},$$

where $[x,z]_y$ denotes the subset of $[x,z]$ of morphisms $f\colon x \to z$ whose cone is isomorphic to y in Ho(W).

Proof. Given the target map $t\colon X^{(1)} \to X^{(0)}$ and an object z of $Ho(W)$, let F^z denote the homotopy fiber of t over z. Using the definitions of $X^{(1)}$ and $X^{(0)}$, notice that F^z is weakly equivalent to the nerve of the category equiv$(W \downarrow z)$ whose objects are maps from arbitrary objects of W to z, and whose morphisms are the homotopy equivalences of W, making the resulting triangular diagram commute.

Given two other objects x and y of W, let $F^z_{x,y}$ denote the nerve of the full subcategory of equiv$(W \downarrow z)$ whose objects are the maps $u\colon x' \to z$, where $x' \simeq x$, and whose cofiber is equivalent to y. Notice that $F^z_{x,y}$ is locally finite, since both $X^{(1)}$ and $X^{(0)}$ are; moreover, $\pi_0(F^z_{x,y})$ is finite, and it is isomorphic to $[x,z]_y/\text{Aut}(x)$.

Using $F^z_{x,y}$, we can reformulate our definition of the derived Hall number $g^z_{x,y}$ as

$$g^z_{x,y} = \sum_{(u\colon x' \to y) \in \pi_0(F^z_{x,y})} \prod_{i>0} |\pi_i(F^z_{x,y}, u)|^{(-1)^i}.$$

We first prove that

$$\prod_{i>0} |\pi_i(F^z_{x,y}, u)|^{(-1)^i} = |\text{Aut}(f/z)|^{-1} \cdot \prod_{i>0} |\text{Ext}^{-i}(x,z)|^{(-1)^i} \cdot |\text{Ext}^{-i}(x,x)|^{(-1)^{i+1}},$$

where Aut(f/z) denotes the stabilizer of a map $f \in [x,z]_y$ under the action of Aut(x).

Notice that we get a homotopy cartesian square of mapping spaces

$$\begin{array}{ccc} \text{map}_{W\downarrow z}(x,x) & \longrightarrow & \text{map}_W(x,x) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \text{map}_W(x,z) \end{array}$$

where the bottom horizontal map specifies the map $u\colon x \to z$. Thus, we have a fibration of simplicial sets, and hence a long exact sequence of homotopy groups

$$\begin{aligned}\cdots \to \pi_2(\text{map}(x,z)) \to \pi_1(\text{map}_{W\downarrow z}(x,x)) \to \pi_1(\text{map}_W(x,x)) \to \pi_1(\text{map}_W(x,z))\\ \to \pi_0(\text{map}_{W\downarrow z}(x,x)) \to \pi_0(\text{map}_W(x,x)) \to \pi_0(\text{map}_W(x,z)) \to 0.\end{aligned}$$

Composing the last two maps between nontrivial sets, we get a surjection

$$\pi_0(\text{map}_{W\downarrow z}(x,x)) \to \text{Aut}(f/z).$$

Furthermore, notice that $\pi_i(\text{map}_W(x,z)) = [x, z[-i]] = \text{Ext}^{-i}(x,z)$ and, similarly, that $\pi_i(\text{map}_W(x,x)) = \text{Ext}^{-i}(x,x)$. Finally, observe that $\pi_i(\text{map}_{W\downarrow z}(x,x))$ is weakly equivalent to $\pi_{i+1}(\text{nerve}(\text{equiv}(W \downarrow z)), u)$, which, as we have noted previously, is equivalent to $\pi_{i+1}(F^z_{x,y}, u)$. Thus, we have a long exact sequence

$$\begin{aligned}\cdots \to \text{Ext}^{-2}(x,z) \to \pi_2(F^z_{x,y}, u) \to \text{Ext}^{-1}(x,x) \to \text{Ext}^{-1}(x,z)\\ \to \pi_1(F^z_{z,y}, u) \to \text{Aut}(f/z) \to 0.\end{aligned}$$

Using properties of long exact sequences, we obtain the equation given above.

To prove the statement of the proposition, we use the fact that, since $\text{Aut}(x)$ is a finite group and $[x,z]_y$ is a finite set, we get that

$$\frac{|[x,z]_y|}{|\text{Aut}(x)|} = \sum_{f \in ([x,z]_y/\text{Aut}(x))} |\text{Aut}(f/x)|.$$

The formula follows. □

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CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Vol. LIV-1 (2013)

LINEAR GROUPS AND PRIMITIVE POLYNOMIALS OVER F_p

by Jean-Yves DEGOS

Résumé. En nous inspirant du groupe de Klein $\mathrm{GL}_3(\mathbb{F}_2)$ (voir l'introduction), nous introduisons les nouvelles notions de groupes n-cyclables et de groupes n-brunniens de type I et II (voir section 1). Nous montrons ensuite que les groupes $\mathrm{SL}_n(\mathbb{F}_p)$ et $\mathrm{GL}_n(\mathbb{F}_p)$ jouissent d'une structure de groupes n-brunniens de type I pour p premier et $n \geq 3$ (voir section 2). Dans la section 3, nous énonçons deux conjectures, à savoir les conjectures $A(n,p,P)$ et $B(n,p,P)$ concernant les polynômes primitifs sur $\mathbb{F}_p$, et nous donnons des résultats partiels dans la section 4.

Abstract. Motivated by the case of Klein's group $\mathrm{GL}_3(\mathbb{F}_2)$ (see the introduction), we introduce the new notions of n-cyclable groups and n-brunnian groups of type I and II (see section 1). We then prove that the groups $\mathrm{SL}_n(\mathbb{F}_p)$ and $\mathrm{GL}_n(\mathbb{F}_p)$ enjoy a structure of n-brunnian groups of type I for p prime and $n \geq 3$ (see section 2). In section 3, we state two conjectures, namely the conjectures $A(n,p,P)$ and $B(n,p,P)$ about primitive polynomials over $\mathbb{F}_p$, and we give some evidence in section 4.

Keywords. borromean groups, brunnian groups, primitive polynomials, linear groups, finite fields.

Mathematics Subject Classification (2010). 12Y05, 20H30.

Introduction

The group $\mathrm{GL}_3(\mathbb{F}_2) \simeq \mathrm{PGL}_3(\mathbb{F}_2)$ is known to be the automorphism group of Klein's quartic ([6]):

$$X(7) = \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) \,,\, x^3y + y^3z + z^3x = 0\} \,.$$

According to the literature, this group is generated by a generator of order 2, a generator of order 3, and a generator of order 7 (see [1]). But, in 2005, Guitart showed that it could be generated by the following three matrices

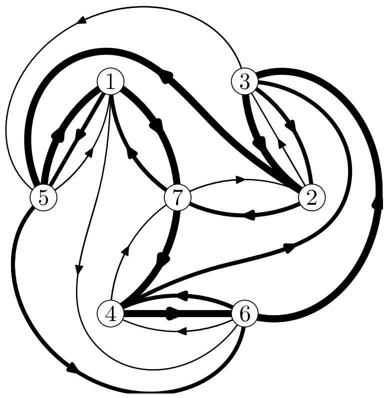

Figure 1: Action of $\mathrm{GL}_3(\mathbb{F}_2)$ on $\{1, 2, 3, 4, 5, 6, 7\}$

([4] and [5], **5**):

$$r = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, s = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \text{ and } i = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix};$$

and that it could be viewed as a subgroup of the symmetric group $\mathfrak{S}_7$, which acts on $\{1, 2, 3, 4, 5, 6, 7\}$ as the permutations $r = (1746325)$, $s = (5164723)$, and $i = (1564327)$ do ([5], Proposition 10), like in Figure 1.

The group $\mathrm{GL}_3(\mathbb{F}_2)$ is thus called a borromean group.

In front of this situation, we can ask the following questions:

(i) How could we make this threefold geometrical symmetry visible in the algebraic description of $\mathrm{GL}_3(\mathbb{F}_2)$ as a matrix group?

(ii) Could we generalize the notion of a borromean group to dimension n?

In the following, we are going to give partial answers to these questions.

1. A few definitions and generalizations

In knot theory, the borromean rings consist of three topological circles which are linked and form a brunnian link, i.e., removing any ring results in two unlinked rings. A brunnian link is a nontrivial link that becomes trivial if any component is removed. In other words, cutting any loop frees all the other loops (so that no two loops can be directly linked).

Imitating these notions, we can define the notions of brunnian groups in two ways.

1.1 The notion of an n-cyclable group

Definition 1.1. *Let $n \geq 1$ be an integer. A group G is n-cyclable if it can be generated by n elements $g_1, g_2, \ldots, g_n$ satisfying the following axiom: if $M(g_1, g_2, \ldots, g_n) = 1$ (with M a word in $g_1, g_2, \ldots, g_n$), then*

$$M(g_{\gamma^k(1)}, g_{\gamma^k(2)}, \ldots, g_{\gamma^k(n)}) = 1\,,$$

where γ is the n-cycle $(1, 2, \ldots, n)$, and $1 \leq k \leq n-1$.

1.2 The notion of an n-brunnian group of type I

Definition 1.2. *A group G is n-brunnian of type I if:*

(i) it is n-cyclable;

(ii) for all $1 \leq i \leq n$, if we set $g_i = 1$, the group generated by $g_1, g_2, \ldots, g_n$ is trivial.

1.3 The notion of an n-brunnian group of type II

Definition 1.3. *A group G is n-brunnian of type II if:*

(i) it is n-cyclable;

(ii) for all $1 \leq i \leq n$, the group generated by $g_1, g_2, \ldots, g_n$ except g_i does not generate G.

2. The groups $\mathrm{SL}_n(\mathbb{F}_p)$ and $\mathrm{GL}_n(\mathbb{F}_p)$ as brunnian groups

To state the theorems, we need two definitions.

Definition 2.1. *Let $n \geq 2$ an integer. For $1 \leq i, j \leq n$ and $i \neq j$, we denote by $T_{i,j}$ the transvection matrix $(t_{k,l})$ with $t_{k,k} = 1$ for $1 \leq k \leq n$, $t_{i,j} = 1$*

and $t_{k,l} = 0$ *if* $k \neq l$ *and* $(k,l) \neq (i,j)$*, namely:*

$$T_{i,j} = \begin{bmatrix} 1 & 0 & \dots & & & 0 \\ 0 & 1 & \dots & & & \vdots \\ \vdots & \vdots & \ddots & 1_{i,j} & & \\ & & & \ddots & & \vdots \\ & & & & \ddots & 0 \\ 0 & \dots & & \dots & 0 & 1 \end{bmatrix}.$$

Definition 2.2. *If* $n \geq 2$ *is an integer,* p *is a prime, and*

$$f(X) = X^n + a_{n-1}X^{n-1} + \cdots a_1 X + a_0 \in \mathbb{F}_p[X]$$

then we denote by $\mathrm{Comp}\,(f(X))$ *the matrix:*

$$\mathrm{Comp}\,(f(X)) = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}.$$

Theorem 2.3. *Let* $n \geq 3$ *be an integer and* p *be a prime number.*
We set $G_1 = T_{1,2}$*,* $G = \mathrm{Comp}\,(X^n - 1)$*, and*
$G_{i+1} = GG_iG^{-1}$ *for* $1 \leq i \leq n-1$*. Then*

$$SL_n(\mathbb{F}_p) = \langle G_1, G_2, \dots, G_n \rangle .$$

The group $SL_n(\mathbb{F}_p)$ *is therefore* n*-cyclable, and* n*-brunnian of type I with respect to these generators. It is also* n*-brunnian of type II.*

Proof. The fact that $\mathrm{SL}_n(\mathbb{F}_p)$ is n-cyclable (and n-brunnian of type I) is an easy consequence of the main lemma, which is proved in section 4.

Therefore, we just have to show that $\mathrm{SL}_n(\mathbb{F}_p)$ is n-brunnian of type II. However, it can be shown that the group generated by $G_1, G_2, \dots, G_{n-1}$ is the group of all matrices of the following form:

$$\begin{bmatrix} 1 & \times & \cdots & \times \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \times \\ 0 & \cdots & 0 & 1 \end{bmatrix},$$

where the $\times$ symbols stand for any element of $\mathbb{F}_p$. This group is a Sylow subgroup of $\mathrm{SL}_n(\mathbb{F}_p)$ and has order

$$p^{\frac{n(n-1)}{2}},$$

and is therefore not equal to $\mathrm{SL}_n(\mathbb{F}_p)$. □

Corollary 2.4. *With the notations of Theorem* 2.3*, let* $s : \mathrm{SL}_n(\mathbb{F}_p) \to \mathrm{PSL}_n(\mathbb{F}_p)$ *be the canomial map,* $H_i = s(G_i)$ *for* $1 \leq i \leq n$*, and*

$$\begin{array}{rcl} \theta : \mathrm{PSL}_n(\mathbb{F}_p) & \to & \mathrm{PSL}_n(\mathbb{F}_p) \\ s(M) & \mapsto & s(G)s(M)s(G)^{-1} \end{array}.$$

Then:

(i) $H_{i+1} = \theta(H_i)$ *for* $1 \leq i \leq n-1$*;*

(ii) $\mathrm{PSL}_n(\mathbb{F}_p) = \langle H_1, H_2, \ldots, H_n \rangle$.

The group $\mathrm{PSL}_n(\mathbb{F}_p)$ *is therefore an* n*-cyclable, and* n*-brunnian of type I, with respect to these generators.*

Proof. The proof is the same as that of Corollary 2.6 below. □

Theorem 2.5. *Let* $n \geq 3$ *be an integer,* p *be a prime number and* d *be a generator of* $\mathbb{F}_p^\times$.

We denote by $G_1 = (g_{i,j})$ *the matrix defined by* $g_{i,i} = 1$ *for* $1 \leq i \leq n$ *and* $i \neq 3$*,* $g_{3,3} = d$*,* $g_{1,2} = 1$ *and* $g_{i,j} = 0$ *if* $i \neq j$ *and* $(i,j) \neq (1,2)$*. We set* $G = \mathrm{Comp}\,(X^n - 1)$.

We set $G_{i+1} = GG_iG^{-1}$ *for* $1 \leq i \leq n-1$*. Then*

$$\mathrm{GL}_n(\mathbb{F}_p) = \langle G_1, G_2, \ldots, G_n \rangle .$$

The group $\mathrm{GL}_n(\mathbb{F}_p)$ *is therefore an* n*-cyclable and* n*-brunnian group of type I with respect to these generators. It is also and* n*-brunnian group of type II.*

Proof. The fact that $\mathrm{GL}_n(\mathbb{F}_p)$ is n-cyclable (and n-brunnian of type I) is an easy consequence of the main lemma, which is proved in section 4.

Therefore, we just have to show that $\mathrm{GL}_n(\mathbb{F}_p)$ is n-brunnian of type II. However, it can be shown that the group generated by $G_1, G_2, \ldots, G_{n-1}$ is the group of all matrices which are upper triangular, with a 1 in position $(n-1, n-1)$. This group has order

$$p^{\frac{n(n-1)}{2}}(p-1)^{n-1},$$

and is therefore not equal to $\mathrm{GL}_n(\mathbb{F}_p)$. □

Corollary 2.6. *With the notations of Theorem* 2.5*, let* $s : \mathrm{GL}_n(\mathbb{F}_p) \to \mathrm{PGL}_n(\mathbb{F}_p)$ *be the canomial map,* $H_i = s(G_i)$ *for* $1 \leq i \leq n$*, and*

$$\begin{array}{rcl}\theta : \mathrm{PGL}_n(\mathbb{F}_p) & \to & \mathrm{PGL}_n(\mathbb{F}_p) \\ s(M) & \mapsto & s(G)s(M)s(G)^{-1}\end{array} .$$

Then:

(i) $H_{i+1} = \theta(H_i)$ *for* $1 \leq i \leq n-1$*;*

(ii) $\mathrm{PGL}_n(\mathbb{F}_p) = \langle H_1, H_2, \ldots, H_n \rangle$*.*

The group $\mathrm{PGL}_n(\mathbb{F}_p)$ *is therefore an* n*-cyclable, and* n*-brunnian of type I, with respect to these generators.*

Proof. First, the points (i) and (ii) are obvious. We only have to check that the automorphism θ has order n. Let k be the order of θ in $\mathrm{PGL}_n(\mathbb{F}_p)$. Then k divides n, because G has order n in $\mathrm{GL}_n(\mathbb{F}_p)$. Then, we have:

$$\begin{array}{rcl}\theta^k = 1_{\mathrm{PGL}_n(\mathbb{F}_p)} & \Rightarrow & \forall M \in \mathrm{GL}_n(\mathbb{F}_p), \theta^k(s(M)) = s(M) \\ & \Rightarrow & \forall M \in \mathrm{GL}_n(\mathbb{F}_p), s(G)^k s(M) s(G)^{-k} = s(M) \\ & \Rightarrow & \forall M \in \mathrm{GL}_n(\mathbb{F}_p), \exists \lambda \in \mathbb{F}_p^\times, G^k M G^{-k} = \lambda M.\end{array}$$

But if $k \neq n$, this last property is false for $M = \mathrm{Comp}(Q(X))$ for any irreducible polynomial $Q(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$. We now are going to prove that.

Indeed, the eigenvalues of $G^k M G^{-k}$ are the eigenvalues of M, namely the elements of the set:

$$\Lambda := \left\{\alpha^{p^i} \text{ for } 0 \leq i \leq n-1\right\},$$

α being a root of $Q(X)$.

The equality $G^k M G^{-k} = \lambda M$ implies that $x \mapsto \lambda x$ is a bijection of Λ. If it is not the identity map, there are two integers i and j with $i \neq j$ and $\lambda \alpha^{p^i} = \alpha^{p^j}$, and we deduce from this fact that $\lambda \notin \mathbb{F}_p^\times$. Consequently, this bijection is the identity map, and $\lambda = 1$. Thus, $G^k M G^{-k} = M$. However, this is impossible, as we prove it below. Indeed, we have:

$$G^k M G^{-k} = (m_{\gamma^{-k}(i), \gamma^{-k}(j)})_{i,j} \text{ where } M = (m_{i,j})_{i,j} .$$

Hence, for all $1 \leq i, j \leq n$, $m_{\gamma^{-k}(i), \gamma^{-k}(j)} = m_{i,j}$. Using this with $i = 1$ and $j = n$, we obtain:

$$-a_0 = 1 \text{ and } -a_{i-1} = 0 \text{ for } 2 \leq i \leq n .$$

Therefore $Q(X) = X^n - 1$, which is a contradiction, because $Q(X)$ is irreducible.

We conclude that $k = n$. □

3. Two conjectures on primitive polynomials

Definition 3.1. *Let $n \geq 1$ an integer, $p \geq 2$ a prime number, and $P(X) \in \mathbb{F}_p[X]$ with $\deg P = n$. The polynomial $P(X)$ is said to be primitive if it is the minimal polynomial of a primitive element of $\mathbb{F}_{p^n}$.*

Example 3.2. If $n = 2$ and $p = 2$, $P(X) = X^2 + X + 1$ is the only primitive polynomial of degree n over $\mathbb{F}_p[X]$.

Example 3.3. If $n = 8$ and $p = 2$, $P(X) = X^8 + X^4 + X^3 + X + 1$ is an irreducible polynomial of degree n over $\mathbb{F}_p[X]$, but it is not a primitive one.

Conjecture 3.4 (A(n,p,P))**.** *Let p be a prime number, $n \geq 2$ be an integer, and $P(X) \in \mathbb{F}_p[X]$ be a primitive polynomial of degree n. Let $G = \mathrm{Comp}(X^n - 1)$ and $C = \mathrm{Comp}(P(X))$. Then*

$$GL_n(\mathbb{F}_p) = \langle G, C \rangle \, .$$

Remark 3.5. Conjecture $A(n, p, P)$ results from Conjecture $B(n, p, P)$ that follows.

Conjecture 3.6 (B(n,p,P))**.** *Let p be a prime number and $n \geq 2$ be an integer. Let $P(X) \in \mathbb{F}_p[X]$ be a primitive polynomial of degree n.*
Let $G = \mathrm{Comp}(X^n - 1)$, $C = \mathrm{Comp}(P(X))$ and let us define $(G_i)_{1 \leq i \leq n}$ by

$$\begin{cases} G_1 & = & C \, , \\ G_{i+1} & = & G G_i G^{-1} \text{ for } 1 \leq i \leq n-1 \, . \end{cases}$$

Then $GL_n(\mathbb{F}_p) = \langle G_1, G_2, \ldots, G_n \rangle$. So the group $GL_n(\mathbb{F}_p)$ is n-cyclable and n-brunnian of type I with respect to these generators.

4. Some evidence

4.1 Three theorems

Theorem 4.1. *Given $P(X) = X^2 + a_1X + a_0$ a primitive polynomial of degree 2 over $\mathbb{F}_p$ with $p \in \{2, 3\}$, we have the following results:*
(i) $B(2, 3, P)$ is true; therefore $A(2, 3, P)$ is true;
(ii) $A(2, 2, P)$ is true, but $B(2, 2, P)$ is false.

The proof of Theorem 4.1 uses elementary operations.

Proof. (i) $p = 3$.
As a_0 generates $\mathbb{F}_p^\times$, we only have to show that the following matrices:

$$T := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, T' := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } M := \begin{bmatrix} 1 & 0 \\ 0 & a_0 \end{bmatrix}$$

are in $\langle G_1, G_2 \rangle$.
We start from

$$H := G_2 {a_0}^{p-2} G_1 = \begin{bmatrix} \frac{1}{a_0} & a_1 - \frac{a_1}{a_0} \\ 0 & a_0 \end{bmatrix}.$$

As $p = 3$, we have ${a_0}^2 = 1$, so $a_0 = -1$ and

$$H^2 = \begin{bmatrix} 1 & -a_1 \\ 0 & 1 \end{bmatrix}.$$

As $-a_1 \neq 0$, there is an integer k such that $H^{2k} = T$. Then $T \in \langle G_1, G_2 \rangle$.
Starting from ${a_0}^{p-1} G_1 G_2$, we could show that $T' \in \langle G_1, G_2 \rangle$.
Then $\langle G_1, G_2 \rangle$ contains all the tranvections, and so contains $\mathrm{SL}_2(\mathbb{F}_p)$.
The matrices M and G_1 have the same determinant: a_0. Then, they are equivalent modulo $\mathrm{SL}_2(\mathbb{F}_p)$.
We can conclude that $M \in \langle G_1, G_2 \rangle$ and $\langle G_1, G_2 \rangle = \mathrm{GL}_2(\mathbb{F}_p)$.
(ii) $p = 2$. Then we have:

$$GC = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } CG = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

so $\langle G, C\rangle = \mathrm{SL}_2(\mathbb{F}_2) = \mathrm{GL}_2(\mathbb{F}_2)$.

But:

$$G_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } G_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = {G_1}^2 ,$$

then $\langle G_1, G_2\rangle = \langle G_1\rangle \neq \mathrm{GL}_2(\mathbb{F}_2)$. □

The following lemma is the heart of this paper, and will be useful to prove Theorem 4.3 and Theorem 4.4.

Lemma 4.2 (Main lemma)**.** *Let $n \geq 3$ be an integer, p be a prime number, and $H \subset GL_n(\mathbb{F}_p)$ a subgroup satisfying the following properties:*

(i) for every $h \in H$, then $GhG^{-1} \in H$, where $G = \mathrm{Comp}\,(X^n - 1)$;

(ii) the group H contains a matrix D the determinant of which is d and generates $\mathbb{F}_p^\times$;

(iii) the group H contains a tranvection matrix $T_{i,j}$ with $j = \gamma(i)$ or $i = \gamma(j)$.

Then $H = GL_n(\mathbb{F}_p)$.

Proof. Let g be the isomorphism of $GL_n(\mathbb{F}_p)$ defined by $g(M) = GMG^{-1}$. Set $T_1 := T_{i,j}$ and $T_k := g^{-1}(T_{k-1})$ for $2 \leq k \leq n$. Then T_k is a transvection matrix, and $T_k \in H$, because it is a conjugate of T_1 by $G^{-(k-1)}$. More precisely, for $1 \leq k \leq n$, we have:

$$T_k = T_{\gamma^{-(k-1)}(i),\gamma^{-(k-1)}(j)} \, .$$

Then, as $j = \gamma(i)$ or $i = \gamma(j)$, there is an n-cycle $(j_1, j_2, \ldots j_n)$ such that the set

$$\{T_k \,|\, 1 \leq k \leq n\}$$

can be rewritten as the set:

$$T_{j_1,j_2},\, T_{j_2,j_3},\, \ldots, T_{j_n,j_1} \, .$$

As $n \geq 3$, we can use the well-known formula ([7], proof of Theorem 9.2, XIII, 9, page 541), and deduce that:

$$T_{k_1,k_2} T_{k_2,k_3} {T_{k_1,k_2}}^{p-1} {T_{k_2,k_3}}^{p-1} = T_{k_1,k_3} \text{ for } k_2 \notin \{k_1, k_3\}$$

to show that H contains all the matrices $T_{k,l}$ with $1 \leq k, l \leq n$ and $k \neq l$. We conclude that $SL_n(\mathbb{F}_p) \subset H$.

Now, as H contains a matrix of determinant d, which is equivalent modulo $SL_n(\mathbb{F}_p)$ (therefore modulo H) to a dilatation matrix of determinant d, and as H is stable by conjugation by G and d generates $\mathbb{F}_p^\times$, then H contains all the dilatations matrices.

Therefore $H = GL_n(\mathbb{F}_p)$. □

Theorem 4.3. *Let us suppose that* $p = 2$*,* n *is* odd*, and there is an* i *in* $\{1, n-1\}$ *such that* $P(X) = X^n + X^i + 1$ *is primitive.*
Then $B(n, p, P)$ *is true, therefore* $A(n, p, P)$ *is true.*

Proof. To prove Theorem 4.3, we just have to check that the subgroup

$$H = \langle G_1, \dots, G_n \rangle$$

satisfies the three points (i), (ii), (iii) of the main lemma.

(i) : the group H is stable by conjugation by G;

(ii) : the group H contains $G_1 = \mathrm{Comp}(P(X))$, the determinant of which is 1, and generates $\mathbb{F}_2^\times$;

(iii) : we have $G^{-1}C = G^{-1}G_1 = T_{i,n}$, which is a transvection matrix of order 2, with $\gamma(i) = n$ or $\gamma(n) = i$. Moreover, we have $G^2 = G_2G_1 \in H$. As n and 2 are coprime, there are integers u and v such that $2u + nv = 1$. Therefore, $G = (G^2)^u \in H$, and $T_{i,n} \in H$. □

Theorem 4.4. *Let us suppose that* $p = 2$*,* n *is* even*, and there is an* i *in* $\{1, n-1\}$ *such that* $P(X) = X^n + X^i + 1$ *is primitive.*
Then $A(n, p, P)$ *is true.*

Proof. To prove Theorem 4.4, we just have to check that the subgroup

$$H = \langle G, C \rangle$$

satisfies the three points (i), (ii), (iii) of the main lemma.

(i) : the group H is stable by conjugation by G;

(ii) : the group H contains $G_1 = \mathrm{Comp}(P(X))$, the determinant of which is 1, and generates $\mathbb{F}_2^\times$;

(iii) : we have $G^{-1}C = G^{-1}G_1 = T_{i,n}$, which is a transvection matrix of order 2, with $\gamma(i) = n$ or $\gamma(n) = i$. Moreover, $T_{i,n} \in H$. □

We can find in [2] the irreducible polynomials of the form $x^n + x + 1$ over $\mathbb{F}_2$, up to $n = 30000$. There are only 33.

4.2 The centers of $\langle G, C\rangle$ and $\langle G_1, G_2, \ldots, G_n\rangle$

According to Conjecture $A(n,p,P)$, we should have $\mathrm{GL}_n(\mathbb{F}_p) = \langle G, C\rangle$, with the notations of Conjecture 3.4. Thus, we should have also the equality between the centers of these groups. This is the case.

According to Conjecture $B(n,p,P)$, the group $\langle G_1, G_2, \ldots, G_n\rangle$ equals the group $\mathrm{GL}_n(\mathbb{F}_p)$, with the notations of Conjecture 3.6. Thus, we should have also the equality between the centers of these groups. This is the case, provided $G \in \langle G_1, G_2, \ldots, G_n\rangle$.

To prove these results, we need a lemma.

Lemma 4.5. *We have:*

$$G_1{}^{1+p+\cdots+p^{n-1}} = \begin{bmatrix} (-1)^n a_0 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (-1)^n a_0 \end{bmatrix},$$

therefore $\langle G, C\rangle$ and $\langle G_1, G_2, \ldots, G_n\rangle$ both contain all the homotheties.

Proof. If $\sigma : x \mapsto x^p$ is the Frobenius automorphism, there is a matrix Q with coefficients in $\mathbb{F}_p(\alpha)$ (where α is a root of $P(X)$) such that $Q^{-1}CQ = D$, with

$$D = \begin{bmatrix} \sigma^0(\alpha) & 0 & \ldots & 0 \\ 0 & \sigma^1(\alpha) & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & 0 & \sigma^{n-1} \end{bmatrix},$$

therefore: $G_1{}^{1+p+\cdots+p^{n-1}} = QD^{1+p+\cdots+p^{n-1}}Q^{-1} = (-1)^n a_0 Id$. But as $P(X)$ is a primitive polynomial, $(-1)^n a_0$ generates $\mathbb{F}_p^\times$, QED. □

Theorem 4.6. *We have the following results (the notation $Z(\Gamma)$ stands for the center of the group Γ):*

(i) $Z(\langle G, C\rangle) = \{xId, x \in \mathbb{F}_p^\times\}$;

(ii) if $G \in \langle G_1, G_2, \ldots, G_n\rangle$, $Z(\langle G_1, G_2, \ldots, G_n\rangle) = \{xId, x \in \mathbb{F}_p^\times\}$;

Proof. We know from the previous lemma that all the homotheties are contained in $\langle G, C\rangle$ and $\langle G_1, G_2, \ldots, G_n\rangle$. Now, if a matrix M is in the center of $\langle G_1, G_2, \ldots, G_n \rangle$, it commutes with G^{-1} and G_1. As it commutes with G^{-1}, it has the following form:

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \alpha_n & \alpha_1 & \ldots & \alpha_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_2 & \alpha_3 & \ldots & \alpha_1 \end{bmatrix},$$

and as it commutes with G_1, the following equations hold:

$$-a_0\alpha_i = \alpha_i \text{ for } 2 \leq i \leq n$$

and

$$-a_i\alpha_j = 0 \text{ for } 1 \leq i \leq n-2 \text{ and } 2 \leq j \leq n.$$

Consequently, for a given $2 \leq j \leq n$, if $\alpha_j \neq 0$, we have:

$$a_1 = a_2 = \cdots = a_{n-1} = 0 \text{ and } a_0 = -1$$

hence $P(X) = X^n - 1$. This is a contradiction, because $P(X)$ is supposed to be primitive, hence irreducible. Thus, we have $\alpha_j = 0$ for $2 \leq j \leq n$. Therefore, the matrix M is that of an homothety. □

4.3 Experimental checkings

We used a Sage worksheet to do computations to check the conjectures on Langevin's table of primitive polynomials (see [8]). In the next subsubsections, we give the functions of our worksheet, and we give the results we obtained.

4.3.1 The Sage functions

We used the following Sage functions.

```
def Comp(n,p,f):
    A=GL(n,p)
    Fp=GF(p)
```

```
    FpX.<x>=PolynomialRing(Fp,'x')
    M=Matrix(n,n,range(n*n))
    for i in range(1,n+1):
        for j in range (1,n+1):
            M[i-1,j-1]=0
        M[i-1,n-1]=-FpX(f)[i-1]
    for i in range (1,n):
        M[i,i-1]=1
    return M

def G(n,p):
    return Comp(n,p,x^n-1)

def C(n,p,P):
    return Comp(n,p,P)

def Gi(k,n,p,P):
    if k==1:
       return C(n,p,P)
    else:
       return G(n,p)*Gi(k-1,n,p,P)*G(n,p)^(-1)

def ConjA(n,p,P):
    print n,p,P
    gens_A=[GL(n,p)(C(n,p,P)),GL(n,p)(G(n,p))]
    H_A=MatrixGroup(gens_A)
    return GL(n,p).order()==H_A.order()

def ConjB(n,p,P):
    print n,p,P
    gens_B=[GL(n,p)(Gi(k,n,p,P)) for k in range(1,n+1)]
    print n,p,P
    H_B=MatrixGroup(gens_B)
    return GL(n,p).order()==H_B.order()
```

4.3.2 The results

The results we obtained are given below. We have tested each primitive polynomial of Langevin's table (see [8]) of degree n over $\mathbb{F}_p$ for which $p^n \leq 50000$ with our personnal MacBook.

Primitive polynomials over $\mathbb{F}_2$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	2	x^2+x+1	True	False
3	2	x^3+x+1	True	True
4	2	x^4+x+1	True	True
5	2	x^5+x^2+1	True	True
6	2	x^6+x+1	True	True
7	2	x^7+x+1	True	True
8	2	$x^8+x^7+x^2+x+1$	True	True
9	2	x^9+x^4+1	True	True
10	2	$x^{10}+x^3+1$	True	True
11	2	$x^{11}+x^2+1$	True	True
12	2	$x^{12}+x^8+x^2+x+1$	True	True
13	2	$x^{13}+x^5+x^2+x+1$	True	True
14	2	$x^{14}+x^{12}+x^2+x+1$	True	True
15	2	$x^{15}+x+1$	True	True
16	2	$x^{16}+x^5+x^3+x^2+1$	?	?

Primitive polynomials over $\mathbb{F}_3$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	3	x^2+x+2	True	True
3	3	x^3+2x^2+1	True	True
4	3	x^4+x^3+2	True	True
5	3	$x^5+x^4+x^2+1$	True	True
6	3	x^6+x^5+2	True	True
7	3	$x^7+x^6+x^4+1$	True	True
8	3	x^8+x^5+2	True	True
9	3	$x^9+x^7+x^5+1$	True	True
10	3	$x^{10}+x^9+x^7+2$	?	?

Primitive polynomials over $\mathbb{F}_5$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	5	x^2+x+2	True	True
3	5	x^3+x^2+2	True	True
4	5	x^4+x^3+x+3	True	True
5	5	x^5+x^2+2	True	True
6	5	x^6+x^5+2	True	True
7	5	x^7+x^6+2	?	?

Primitive polynomials over $\mathbb{F}_7$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	7	x^2+x+3	True	True
3	7	x^3+x^2+x+2	True	True
4	7	$x^4+x^3+x^2+3$	True	True
5	7	x^5+x^4+4	True	True
6	7	$x^6+x^5+x^4+3$	?	?

Primitive polynomials over $\mathbb{F}_{11}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	11	x^2+x+7	True	True
3	11	x^3+x^2+3	True	True
4	11	x^4+x^3+8	True	True
5	11	$x^5+x^4+x^3+3$	?	?

Primitive polynomials over $\mathbb{F}_{13}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	13	x^2+x+2	True	True
3	13	x^3+x^2+2	True	True
4	13	$x^4+x^3+x^2+6$	True	True
5	13	$x^5+x^4+x^3+6$	?	?

Primitive polynomials over $\mathbb{F}_{17}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	17	x^2+x+3	True	True
3	17	x^3+x^2+7	True	True
4	17	x^4+x^3+5	?	?

Primitive polynomials over $\mathbb{F}_{19}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	19	x^2+x+2	True	True
3	19	x^3+x^2+6	True	True
4	19	x^4+x^3+2	?	?

Primitive polynomials over $\mathbb{F}_{23}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	23	x^2+x+7	True	True
3	23	x^3+x^2+6	True	True
4	23	x^4+x^3+20	?	?

Primitive polynomials over $\mathbb{F}_{29}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	29	x^2+x+3	True	True
3	29	x^3+x^2+3	True	True
4	29	x^4+x^3+2	?	?

Primitive polynomials over $\mathbb{F}_{31}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	31	x^2+x+12	True	True
3	31	x^3+x^2+9	True	True
4	31	x^4+x^3+13	?	?

Primitive polynomials over $\mathbb{F}_{37}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	37	x^2+x+5	True	True
3	37	x^3+x^2+17	?	?

Primitive polynomials over $\mathbb{F}_{41}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	41	x^2+x+12	True	True
3	41	x^3+x^2+11	?	?

Primitive polynomials over $\mathbb{F}_{43}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	43	x^2+x+3	True	True
3	43	x^3+x^2+9	?	?

Primitive polynomials over $\mathbb{F}_{47}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	47	x^2+x+13	True	True
3	47	x^3+x^2+2	?	?

Primitive polynomials over $\mathbb{F}_{53}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	53	x^2+x+5	True	True
3	53	x^3+x^2+2	?	?

Primitive polynomials over $\mathbb{F}_{59}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	59	x^2+x+2	True	True
3	59	x^3+x^2+9	?	?

Primitive polynomials over $\mathbb{F}_{61}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	61	x^2+x+2	True	True
3	61	x^3+x^2+6	?	?

Primitive polynomials over $\mathbb{F}_{67}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	67	x^2+x+12	True	True
3	67	x^3+x^2+6	?	?

Primitive polynomials over $\mathbb{F}_{71}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	71	x^2+x+11	True	True
3	71	x^3+x^2+8	?	?

Primitive polynomials over $\mathbb{F}_{73}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	73	x^2+x+11	True	True
3	73	x^3+x^2+5	?	?

Primitive polynomials over $\mathbb{F}_{79}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	79	x^2+x+3	True	True
3	79	x^3+x^2+2	?	?

Primitive polynomials over $\mathbb{F}_{83}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	83	x^2+x+2	True	True
3	83	x^3+x^2+11	?	?

Primitive polynomials over $\mathbb{F}_{89}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	89	x^2+x+6	True	True
3	89	x^3+x^2+6	?	?

Primitive polynomials over $\mathbb{F}_{97}$

n	p	$P(x)$	$A(n,p,P)$	$B(n,p,P)$
2	97	x^2+x+5	True	True
3	97	x^3+x^2+5	?	?

Conclusion

In this paper, we introduced the new notions of n-cyclable groups and n-brunnian groups of type I and II (see section 1). We then proved that the groups $\mathrm{SL}_n(\mathbb{F}_p)$, $\mathrm{PSL}_n(\mathbb{F}_p)$, $\mathrm{GL}_n(\mathbb{F}_p)$, and $\mathrm{PGL}_n(\mathbb{F}_p)$ enjoy a structure of n-brunnian groups of type I for p prime and $n \geq 3$ (see section 2). In section 3, we state two conjectures, namely the conjectures $A(n,p,P)$ and $B(n,p,P)$ about primitive polynomials over $\mathbb{F}_p$, and we give some evidence in section 4.

Unfortunately, the conjectures $A(n,p,P)$ and $B(n,p,P)$ do not characterize primitive polynomials, because, they are both true for the polynomial of Example 3.3.

It is altogether interesting to find some significant counterexamples, or to find a conceptual proof of them.

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CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES
Vol. LIV-1 (2013)

CORRECTIONS TO THE ARTICLE OPERADIC DEFINITION OF THE NON-STRICT CELLS published in Volume LII-4 (2011), pp. 269-316

by Camell KACHOUR

Résumé

Dans ces notes nous proposons une nouvelle approche de la contractibilité pour les ω-opérades colorées telle que définie dans l'article publié dans les Cahiers de Topologie et de Géométrie Différentielle Catégorique (2011), volume 4. Nous proposons aussi une autre façon de construire la monade des ω-opérades contractiles colorées libres.

Abstract

In this short notes we propose a new notion of contractibility for coloured ω-operad defined in the paper published in Cahiers de Topologie et de Géométrie Différentielle Catégorique (2011), volume 4. Also we propose an alternative direction to build the monad for free contractible coloured ω-operads,

Keywords. ω-operads, weak higher transformations .

Mathematics Subject Classification (2010). 18B40,18C15, 18C20, 18G55, 20L99, 55U35, 55P15.

Introduction

Steve Lack has suggested to me to use the more common name *weak higher transformations* instead of *Non-strict cells* which were defined in [2]. More precisely, in this article we defined a coglobular complex of ω-operads

$$B^0 \underset{\kappa_0^1}{\overset{\delta_0^1}{\rightrightarrows}} B^1 \underset{\kappa_1^2}{\overset{\delta_1^2}{\rightrightarrows}} B^2 \cdots\cdots \rightrightarrows B^{n-1} \underset{\kappa_{n-1}^n}{\overset{\delta_{n-1}^n}{\rightrightarrows}} B^n \cdots\cdots$$

such that algebras for B^0 are the weak ω-categories, algebras for B^1 are the weak ω-functors, algebras for B^2 are the weak ω-natural transformations, etc. However André Joyal has pointed out to us that there are too many coherence cells for each B^n when $n \geqslant 2$, and gave us a simple example of a natural transformation which cannot be an algebra for the 2-coloured ω-operad B^2. In this section we propose a notion of contractibilty, slightly different from those used in [1, 2]. This new approach excludes the counterexample of André Joyal.

Furthermore the main theorem of the section 6 in [2] is false. I am indebted to Michael Batanin and to Mark Weber, to have shown us a counterexample which invalid this result. However this false theorem has no impact to main ideas of the article [2]. I am indebted to Michael Batanin who told us that the technics of the coproduct of monads was adapted to substitute technically the role of this false theorem, and to Steve Lack who gave us the precise result and references that we needed for this correction.

Acknowledgement. I am grateful to André Joyal, Michael Batanin, and to Mark Weber to have pointed out to me these imperfections.

Corrections

Here $\mathbb{T}$ design the monad of strict ω-categories on ω-graphs. Notions of $\mathbb{T}$-graphs, $\mathbb{T}$-categories, constant ω-graphs, can be found in [2, 5]. The category $T\text{-}\mathbb{G}\mathrm{r}_{p,c}$ of pointed T-graphs over constant ω-graphs, and the category $T\text{-}\mathbb{C}\mathrm{at}_c$ of $\mathbb{T}$-categories over constant ω-graphs are both defined in [2].

Definition 1 For any $\mathbb{T}$-graph (C,d,c) over a constant ω-graph G, a pair of cells (x,y) of $C(n)$ has the *the loop property* if: $s^n_0(x) = s^n_0(y) = t^n_0(x) = t^n_0(y)$ □

Remark 1 If G is a constant ω-graph (see section 1.4 of the article [2]) A p-cell of G is denoted by $g(p)$ and this notation has the following meaning: The symbol g indicates the "colour", and the symbol p point out that we must see $g(p)$ as a p-cell of G, because G has to be seen as an ω-graph even though it is just a set. $\square$

Definition 2 For any $\mathbb{T}$-graph (C,d,c) over a constant ω-graph G, we call *the root cells* of (C,d,c), those cells whose arities are the reflexivity of a 0-cell $g(0)$ of G, where here "g" indicates the colour (see section 1), or in other words, those cells $x \in C(n)$ $(n \geqslant 1)$ such that $d(x) = 1^0_n(g(0))$. $\square$

Here 1^0_n design the reflexivity operators of free strict ω-category $\mathbb{T}(G)$ (see also [2]). These notions of *root cells* and *loop condition* are the keys for our new approach to contractibility. These observations motivate us to put the following definition of what should be a contractible $\mathbb{T}$-graphs (C,d,c). For each integers $k \geqslant 1$ let us note $\tilde{C}(k) = \{(x,y) \in C(k) \times C(k) : x \| y$ and $d(x) = d(y)$, and if also (x,y) is a pair of root cells then they also need to verify the *loop property*: $s^k_0(x) = t^k_0(y)\}$. Also we denote $\tilde{C}(0) = \{(x,x) \in C(0) \times C(0)\}$.

Definition 3 A contraction on the $\mathbb{T}$-graph (C,d,c), is the datum, for all $k \in \mathbb{N}$, of a map $\tilde{C}(k) \xrightarrow{[,]_k} C(k+1)$ such that

- $s([\alpha,\beta]_k) = \alpha, t([\alpha,\beta]_k) = \beta$,
- $d([\alpha,\beta]_k) = 1_{d(\alpha)=d(\beta)}$. $\square$

A $\mathbb{T}$-graph which is equipped with a contraction will be called contractible and we use the notation $(C,d,c;([,]_k)_{k\in\mathbb{N}})$ for a contractible $\mathbb{T}$-graph. Nothing prevents a contractible $\mathbb{T}$-graph from being equipped with several contractions. So here $CT\text{-}\mathbb{G}\mathrm{r}_c$ is the category of the contractible $\mathbb{T}$-graphs equipped with a specific contraction, and morphisms of this category preserves the contractions. One can also refer to the category $CT\text{-}\mathbb{G}\mathrm{r}_{c,G}$, where here contractible $\mathbb{T}$-graphs are only taken over a specific constant ∞-graph G. A pointed contractible $\mathbb{T}$-graphs (see section 1.2 of the article [2]) is denoted

$(C,d,c;p,([,]_k)_{k\in\mathbb{N}})$, and morphisms between two pointed contractible $\mathbb{T}$-graphs preserve contractibilities and pointings. The category of pointed contractible $\mathbb{T}$-graphs is denoted $CT\text{-}\mathbb{G}\mathrm{r}_{p,c}$. The categories $T\text{-}\mathbb{G}\mathrm{r}_{p,c}$ and $CT\text{-}\mathbb{G}\mathrm{r}_{p,c}$ are both locally finitely presentable and the forgetful functor V

$$H \dashv V : CT\text{-}\mathbb{G}\mathrm{r}_{p,c} \longrightarrow T\text{-}\mathbb{G}\mathrm{r}_{p,c}$$

is monadic, with induced monad $\mathbb{T}_C$ is finitary.

Also the category $T\text{-}\mathbb{C}\mathrm{at}_c$ is locally finitely presentable and the forgetful functor U

$$M \dashv U : T\text{-}\mathbb{C}\mathrm{at}_c \longrightarrow T\text{-}\mathbb{G}\mathrm{r}_{p,c}$$

is monadic, with induced monad $\mathbb{T}_M$ is finitary.

A $\mathbb{T}$-category is contractible if its underlying pointed $\mathbb{T}$-graph lies in $CT\text{-}\mathbb{G}\mathrm{r}_{p,c}$. Morphisms between two contractible $\mathbb{T}$-categories are morphisms of $\mathbb{T}$-categories which preserve contractibilities. Let us write $CT\text{-}\mathbb{C}\mathrm{at}_c$ for the category of contractible $\mathbb{T}$-categories. Also consider the pullback in $\mathbb{C}AT$

$$\begin{array}{ccc} CT\text{-}\mathbb{G}\mathrm{r}_{p,c} \underset{T\text{-}\mathbb{G}\mathrm{r}_{p,c}}{\times} T\text{-}\mathbb{C}\mathrm{at}_c & \xrightarrow{p_1} & T\text{-}\mathbb{C}\mathrm{at}_c \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle U} \\ CT\text{-}\mathbb{G}\mathrm{r}_{p,c} & \xrightarrow[V]{} & T\text{-}\mathbb{G}\mathrm{r}_{p,c} \end{array}$$

We have an equivalence of categories

$$CT\text{-}\mathbb{G}\mathrm{r}_{p,c} \underset{T\text{-}\mathbb{G}\mathrm{r}_{p,c}}{\times} T\text{-}\mathbb{C}\mathrm{at}_c \simeq CT\text{-}\mathbb{C}\mathrm{at}_c$$

Furthermore we have the general fact (which can be found in the articles [3, 4])

Proposition 1 (Max Kelly) *Let K be a locally finitely presentable category, and $Mnd_f(K)$ the category of finitary monads on K and strict morphisms of* *monads. Then $Mnd_f(K)$ is itself locally finitely presentable. If T and S are object of $Mnd_f(K)$, then the coproduct $T \amalg S$ is algebraic, which means that $K^T \underset{K}{\times} K^S$ is equal to $K^{T \amalg S}$ and the diagonal of the pullback square*

$$\begin{array}{ccc} K^T \underset{K}{\times} K^S & \xrightarrow{p_1} & K^S \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle U} \\ K^T & \xrightarrow[V]{} & K \end{array}$$

is the forgetful functor $K^{T \amalg S} \longrightarrow K$. Furthermore the projections $K^T \underset{K}{\times} K^S \longrightarrow K^T$ and $K^T \underset{K}{\times} K^S \longrightarrow K^S$ are monadic. □

Remark 2 According to Steve Lack this result can be easily generalise for monads having ranks in the context of locally presentable category. □

We apply this proposition to the diagram above which shows that $CT\text{-}\mathbb{C}\mathrm{at}_c$ is a locally presentable category, and also that the forgetful functor

$$CT\text{-}\mathbb{C}\mathrm{at}_c \xrightarrow{O} T\text{-}\mathbb{G}\mathrm{r}_{p,c}$$

is monadic. Denote by F the left adjoint of O. If we apply the functor F to the coglobular complex of $T\text{-}\mathbb{G}\mathrm{r}_{p,c}$ build in the article [2]

$$C^0 \underset{\kappa_0^1}{\overset{\delta_0^1}{\rightrightarrows}} C^1 \underset{\kappa_1^2}{\overset{\delta_1^2}{\rightrightarrows}} C^2 \cdots\cdots \rightrightarrows C^{n-1} \underset{\kappa_{n-1}^n}{\overset{\delta_{n-1}^n}{\rightrightarrows}} C^n \cdots\cdots$$

we obtain the coglobular complex of the coloured ω-operads of the weak higher transformations with our corrected notion of contractibility

$$B_C^0 \underset{\kappa_0^1}{\overset{\delta_0^1}{\rightrightarrows}} B_C^1 \underset{\kappa_1^2}{\overset{\delta_1^2}{\rightrightarrows}} B_C^2 \cdots\cdots \rightrightarrows B_C^{n-1} \underset{\kappa_{n-1}^n}{\overset{\delta_{n-1}^n}{\rightrightarrows}} B_C^n \cdots\cdots$$

Remark 3 It is evident that the ω-operad B_C^0 of Michael Batanin is still initial in the category of contractible ω-operads equipped with a composition system, where our new approach of contractibility is considered. □

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