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DOUBLE ADJUNCTIONS AND FREE MONADS

by Thomas M. FIORE, Nicola GAMBINO and Joachim KOCK

Résumé. Nous caractérisons les adjonctions doubles en termes de préfaisceaux et carrés universels, puis appliquons ces caractérisations aux monades libres et aux objets d'Eilenberg–Moore dans les catégories doubles. Nous améliorons notre resultat paru dans [13] comme suit : si une catégorie double munie d'un co-pliage admet la construction des monades libres dans sa 2-catégorie horizontale, alors elle admet aussi la construction des monades libres en tant que catégorie double. Nous y démontrons aussi qu'une catégorie double admet les objets d'Eilenberg–Moore si et seulement si un certain préfaisceau paramétrisé est représentable. Pour ce faire, nous développons une notion de préfaisceaux paramétrisés sur les catégories doubles et démontrons un lemme de Yoneda pour icelles.

Abstract. We characterize double adjunctions in terms of presheaves and universal squares, and then apply these characterizations to free monads and Eilenberg–Moore objects in double categories. We improve upon our earlier result in [13] to conclude: if a double category with cofolding admits the construction of free monads in its horizontal 2-category, then it also admits the construction of free monads as a double category. We also prove that a double category admits Eilenberg–Moore objects if and only if a certain parameterized presheaf is representable. Along the way, we develop parameterized presheaves on double categories and prove a double-categorical Yoneda Lemma.

Keywords. Double categories, adjunctions, monads, free monads, folding, cofolding, parameterized presheaf, Yoneda, Eilenberg–Moore.

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Contents

1. Introduction

The notion of double category was introduced by Ehresmann [8] in 1963, as an instance of the concept of internal category from [9], and was developed in the context of a general theory of structure, as synthesized in his book *Catégories et structures* [11] (published in 1965), which in many regards was ahead of its time. Meanwhile, Bénabou in his thesis work (under Ehresmann's supervision) emphasized the simpler notion of 2-category, discovered that Cat itself is an example, and derived the notion from that of enrichment (Catégories relatives) [2]. 2-categories rather than double categories became the standard setting for 2-dimensional structures in category theory, not only because of a more generous supply of examples, but also because 2-categories behave and feel a lot more like 1-categories, whereas double categories present certain strange phenomena. For example not every compatible arrangement of squares in a double category is composable, see Dawson–Paré [7]. The past decade, however, with the proliferation of higher-categorical viewpoints and methods, has seen a certain renaissance of double categories, and double-categorical structures are being discovered and studied more and more frequently in many different areas, while also traditional 2-categorical situations are being revisited in the new light of double categories.

We became interested in double categories through work in conformal field theory, topological quantum field theory, operad theory, and categorical logic. In all these cases, the double-categorical structures come about in situations where there are two natural kinds of morphisms, typically some complicated morphisms (like spans of sets or bimodules) and some more elementary ones (like functions between sets or ring homomorphisms), and the double-categorical aspects concern the interplay between such different kinds of morphisms. While it often provides great conceptual insight to have everything encompassed in a double category, one is often confronted with the lack of machinery for dealing with double categories, and a need is being felt for a more systematic theory of double categories.

This paper can be seen as a small step in that direction: although our work is motivated by some concrete questions about monads, we develop further the basics of adjunctions between double categories: we introduce parameterized presheaves, prove a double Yoneda Lemma, characterize adjunctions in several ways, and go on to study double categories with further structure — foldings or cofoldings — for which we study the question of existence of free monads and Eilenberg–Moore objects. This was our original motivation, and in that sense the present paper is a sequel to our previous paper [13] about *monads in double categories*, although logically it is rather a precursor: with the theory we develop here, some of the results from [13] can be strengthened and simplified at the same time.

The notion of adjunction we consider is that of internal adjunction in $\mathbf{Cat}$. There are two such notions: horizontal and vertical, depending on the interpretation of double categories as internal categories. A more general notion of vertical double adjunction was studied by Grandis and Paré [19]; we comment on the relationship in Section 5. Although horizontal and vertical adjunctions are abstractly equivalent notions, under transposition of double categories, often the double categories have extra structure which breaks the symmetry and makes the two notions different. In this paper we need both notions.

In some regards, double adjunctions express universality in the ways one expects based on experience with 1-categories, as we prove in Theorem 5.2: a horizontal double adjunction may be given by double functors F and G with horizontal natural transformations η and ε satisfying the two triangle identities, or by double functors F and G with a universal horizontal natural transformation (η or ε), or by a single double functor F or G equipped with appropriate universal squares compatible with vertical composition, or by a bijection between sets of squares compatible with vertical composition.

This article primarily deals with *strict* double categories and strict double adjunctions, and the unmodified term "double category" always means "*strict* double category". However, we do develop a result about horizontal adjunctions between normal, vertically weak double categories in Theorem 5.4. Its transpose applies to the free–forgetful adjunction between endomorphisms and monads in the normal, horizontally weak double category $\mathbb{S}\text{pan}$ of horizontal spans, see the final paragraphs of Section 2 for more on "pseudo" versus "strict" and the example in Section 8.

Although double adjunctions express universality in some of the ways one expects, the characterizations of adjointness in 1-category theory in terms of representability do not carry over to double category theory in a straightforward way, and instead require a new notion of *presheaf on a double category*. Namely, to prove that an ordinary 1-functor $F\colon \mathbf{A} \to \mathbf{X}$ admits a right adjoint, it is sufficient to show that the presheaf $\mathbf{A}(F-, A)$ is representable for each object A separately. But to establish that a *double* functor F admits a horizontal right *double* adjoint, two new requirements arise: first, we must consider how the analogous presheaves *vertically combine*, and second, we must consider the representability of all the analogous presheaves *simultaneously rather than separately*. The first requirement forces presheaves on double categories to be *vertically lax* and to *take values in the normal, vertically weak double category* $\mathbb{S}\text{pan}^t$ *of vertical spans*, as opposed to the 1-category $\mathbf{Set}$. We prove a Yoneda Lemma for such $\mathbb{S}\text{pan}^t$-valued presheaves in Proposition 3.10. The second requirement leads us to consider *parameterized* presheaves on double categories. With these notions we establish the double-categorical analogue of the representability characterization of adjunctions in Theorem 5.5, namely a double functor admits a horizonal right adjoint if and only if a certain parameterized $\mathbb{S}\text{pan}^t$-valued presheaf is representable. Parameterized presheaves also play a role in the proof of Theorem 5.2.

Yoneda theory for double categories has been studied also in a recent paper by Paré [23]. He independently obtains our Examples 3.3 and 3.4 (his Section 2.1), Proposition 3.10 on the Double Yoneda Lemma (his Theorem 2.3), and Theorem 5.2 (vi) (his Theorem 2.8).

Many double categories of interest have additional structure that allows one to reduce certain questions about the double category to questions about the horizontal 2-category. Two such structures are *folding* and *cofolding*, recalled in Definitions 6.2 and 6.7. Double categories with both folding and cofolding are essentially the same as *framed bicategories* in the sense of Shulman [24]. In this article we work with foldings and cofoldings separately because some examples, including our motivating examples, admit one or the other but not both.

As an example of the principle of reduction to the horizontal 2-category in the presence of a folding or cofolding, Proposition 6.10 states that two double functors F and G compatible with foldings (or cofoldings) are horizontal double adjoints if and only if their underlying horizontal 2-functors are 2-adjoints.

It is a much more subtle question to deduce a *vertical* double adjunction from a 2-adjunction in the horizontal 2-category. We discuss the special cases of quintet double categories in the second half of Section 6. Surprisingly such a deduction is possible in the case of our main result, Theorem 9.6, which concerns monads in double categories and the free-monad adjunction, as we proceed to explain. In our earlier paper [13] we showed how to associate to a double category $\mathbb{D}$ a double category $\mathbb{E}\mathrm{nd}(\mathbb{D})$ of endomorphisms in $\mathbb{D}$ and a double category $\mathbb{M}\mathrm{nd}(\mathbb{D})$ of monads in $\mathbb{D}$. The double categories $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$ are extensions of Street's 2-categories of endomorphisms and monads in [26] in the sense that if $\mathbf{K}$ is a 2-category and $\mathbb{H}(\mathbf{K})$ is $\mathbf{K}$ viewed as a vertically trivial double category, then the horizontal 2-categories of $\mathbb{E}\mathrm{nd}(\mathbb{H}(\mathbf{K}))$ and $\mathbb{M}\mathrm{nd}(\mathbb{H}(\mathbf{K}))$ are Street's 2-categories $\mathrm{End}(\mathbf{K})$ and $\mathrm{Mnd}(\mathbf{K})$. In [13, Theorem 3.7] we established a fairly technical criterion which allows one to conclude the existence of free monads in a double-categorical sense from the existence of free monads in the underlying horizontal 2-category. The basic assumptions were that the double category is a framed bicategory and the appropriate substructures admit 1-categorical equalizers and coproducts. In the present paper we clarify and generalize this, using the theory of double adjunctions and cofoldings.

A double category $\mathbb{D}$ is said to *admit the construction of free monads* if the forgetful double functor $\mathbb{M}\text{nd}(\mathbb{D}) \to \mathbb{E}\text{nd}(\mathbb{D})$ admits a *vertical* left double adjoint such that the underlying vertical morphism of each unit component is the identity. This is somewhat more stringent than our earlier definition in [13], where we required only a vertical left double adjoint. Our main application, Theorem 9.6, states that a double category with cofolding admits the construction of free monads if its horizontal 2-category admits the construction of free monads. This improves [13, Theorem 3.7], since it removes most of the technical hypotheses and also strengthens the conclusion. A main step is Proposition 7.2, which states that a cofolding on a double category $\mathbb{D}$ induces cofoldings on $\mathbb{E}\text{nd}(\mathbb{D})$ and $\mathbb{M}\text{nd}(\mathbb{D})$. The corresponding statement for foldings does not seem to be true.

To illustrate the theory, we consider in detail the example of the normal, horizontally weak double category $\mathbb{S}\text{pan}$ of horizontal spans. In $\mathbb{S}\text{pan}$, the endomorphisms are directed graphs and monads are categories. The vertical, double-categorical free–forgetful adjunction between the normal, horizontally weak double categories $\mathbb{E}\text{nd}(\mathbb{S}\text{pan})$ and $\mathbb{M}\text{nd}(\mathbb{S}\text{pan})$ extends the classical construction of the free category on a graph.

Returning to general double categories without cofolding, we now describe our second main application. Theorem 10.3 states that a double category $\mathbb{D}$ admits Eilenberg–Moore objects if and only if the parameterized presheaf is representable which assigns to a monad (X, S) and an object I in $\mathbb{D}$ the set $S\text{-Alg}_I$ of S-algebra structures on I. The proof is quite short, since most of the work was done in the earlier sections.

Outline of the paper. Section 2 presents our notational conventions. In Section 3 we introduce parameterized presheaves on double categories and their representability, and prove the Double Yoneda Lemma. In Sections 4 and 5 we introduce universal squares, and prove the various characterizations of horizontal double adjunctions. Section 6 is concerned with the case of horizontal double adjunctions compatible with foldings and cofoldings. In Section 7 we prove that $\mathbb{E}\text{nd}(\mathbb{D})$ and $\mathbb{M}\text{nd}(\mathbb{D})$ admit cofoldings when $\mathbb{D}$ does. Section 8 works out the vertical double adjunction between $\mathbb{E}\text{nd}(\mathbb{S}\text{pan})$ and $\mathbb{M}\text{nd}(\mathbb{S}\text{pan})$ explicitly. Sections 9 and 10 are applications of the results on double adjunctions to the construction of free monads in double categories with cofolding and to a characterization of the existence of Eilenberg–Moore objects in a general double category.

2. Notational Conventions

We begin by fixing some notation concerning double categories.

A *double category* is a categorical structure consisting of objects, horizontal morphisms, vertical morphisms, squares, the relevant domain and codomain functions, compositions, and units, subject to a few axioms [8]. Succinctly, a double category is an internal category in $\mathbf{Cat}$ [9], and in particular involves a diagram of categories and functors

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{m} \mathbb{D}_1 \xleftarrow{u} \mathbb{D}_0.$$

Here $\mathbb{D}_0$ is the category of objects and vertical arrows of $\mathbb{D}$, and $\mathbb{D}_1$ is the category of horizontal arrows and squares, and m and u express horizontal composition and identity cells.

The notion was introduced by C. Ehresmann in the mid sixties and investigated by A. Ehresmann and C. Ehresmann in the 60's and 70's; among those pioneering works on the subject, the most relevant for the present paper are [8, 9, 10, 11]. We refer to Bastiani–Ehresmann [1], Brown–Mosa [4], Fiore–Paoli–Pronk [16], and Grandis–Paré [18] for more modern treatments, each starting with a short introduction to double categories. The homotopy theory of double categories has been investigated by Fiore–Paoli [15] and Fiore–Paoli–Pronk [16].

We indicate double categories with blackboard letters, such as $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{E}$, and denote horizontal respectively vertical composition of squares by

$$[\alpha \;\; \beta] \quad \text{and} \quad \begin{bmatrix} \alpha \\ \gamma \end{bmatrix}, \tag{1}$$

when they are defined. The double category axiom called *interchange law* then states the equality

$$\begin{bmatrix} [\alpha & \beta] \\ [\gamma & \delta] \end{bmatrix} = \left[\begin{bmatrix} \alpha \\ \gamma \end{bmatrix} \;\; \begin{bmatrix} \beta \\ \delta \end{bmatrix} \right]. \tag{2}$$

We simply denote this composite by

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}. \tag{3}$$

The notation in (1) similarly applies to horizontal and vertical morphisms, for instance, $[f \quad g]$ and $\left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right]$ denote the composites $g \circ f$ and $k \circ j$ in the usual orthography. The horizontal and vertical identity morphisms on an object C in $\mathbb{C}$ are denoted 1^h_C and 1^v_C respectively. The horizontal identity square for a vertical morphism j is denoted by i^h_j, while the vertical identity square for a horizontal morphism f is indicated with i^h_f.

If $\mathbb{D}$ is a double category, then $\operatorname{Hor}\mathbb{D}, \operatorname{Ver}\mathbb{D}$, and $\operatorname{Sq}\mathbb{D}$, signify the collections of horizontal morphisms, vertical morphisms, and squares in $\mathbb{D}$. To specify the set of horizontal respectively vertical morphisms from an object D_1 to an object D_2, we write $\operatorname{Hor}\mathbb{D}(D_1, D_2)$ and $\operatorname{Ver}\mathbb{D}(D_1, D_2)$. Similarly, the notation $\operatorname{Hor}\mathbb{D}(f, g)\colon \operatorname{Hor}\mathbb{D}(D_1, D_2) \to \operatorname{Hor}\mathbb{D}(D_1', D_2')$ indicates the function obtained by pre- and postcomposition with the horizontal morphisms f and g. The function $\operatorname{Ver}\mathbb{D}(j, k)$ is defined analogously. To indicate the collection of squares with fixed left vertical boundary j and fixed right vertical boundary k, we write

$$\mathbb{D}(j,k) = \left\{ \alpha \in \operatorname{Sq}\mathbb{D} \;\middle|\; \alpha \text{ has the form } \begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ \cdot & \longrightarrow & \cdot \end{array} \right\}. \tag{4}$$

For example, for the vertical identities $1^v_{D_1}$ and $1^v_{D_2}$, the set $\mathbb{D}(1^v_{D_1}, 1^v_{D_2})$ consists of the 2-cells between morphisms $D_1 \to D_2$ in the horizontal 2-category of $\mathbb{D}$. In general, the squares in $\mathbb{D}(j, k)$ may not compose vertically. Also in analogy to the hom-notation, the notation $\mathbb{D}(\alpha, \beta)$ means horizontal pre- and postcomposition by squares α and β.

For any double category $\mathbb{D}$, the *horizontal opposite* $\mathbb{D}^{\text{horop}}$ is formed by switching horizontal domain and codomain for both horizontal morphisms and squares in $\mathbb{D}$. More precisely, the horizontal 1-category of $\mathbb{D}^{\text{horop}}$ is equal to the opposite of the horizontal 1-category of $\mathbb{D}$, the vertical 1-category of $\mathbb{D}^{\text{horop}}$ is the same as that of $\mathbb{D}$, and the category $(\operatorname{Ver}\mathbb{D}^{\text{horop}}, \operatorname{Sq}\mathbb{D}^{\text{horop}})$ is equal to the opposite category of $(\operatorname{Ver}\mathbb{D}, \operatorname{Sq}\mathbb{D})$.

The *transpose* of a double category is obtained by switching the vertical and horizontal directions. The symmetric nature of the notion of double category means that each double category has two different interpretations as an internal category; these two interpretations are interchanged by transposition. We shall always stick with the "horizontal" interpretation outlined initially.

Double functors are just internal functors, and the same notion results from the two possible interpretations of double categories as internal categories. We shall also need *vertically lax double functors*: these strictly preserve horizontal composition, but provide non-invertible comparison 2-cells for composition of vertical arrows. We refer to Grandis–Paré [19] for the details. A *horizontal natural transformation* is an internal natural transformation in $\mathbf{Cat}$ (for our preferred internal interpretation). In particular, a horizontal natural transformation $\theta\colon F \Rightarrow G$ for $F, G\colon \mathbb{D} \to \mathbb{E}$ assigns to each object A of $\mathbb{D}$ a horizontal morphism $\theta A\colon FA \to GA$, and assigns to each vertical morphism j in $\mathbb{D}$ a square θj bounded on the left and right by Fj and Gj respectively, such that

$$\theta 1^v_A = i^h_{1^v_A} \qquad \theta \begin{bmatrix} j_1 \\ j_2 \end{bmatrix} = \begin{bmatrix} \theta j_1 \\ \theta j_2 \end{bmatrix} \qquad [F\alpha \ \ \theta k] = [\theta j \ \ G\alpha] \tag{5}$$

for all objects A of $\mathbb{D}$, composable vertical morphisms j_1 and j_2 of $\mathbb{D}$, and squares α in $\mathbb{D}(j, k)$. A *vertical natural transformation* can be defined as an internal natural transformation for the transposed internal interpretation, which is the same as the transpose of the horizontal notion above, but can also be described succinctly as follows: a vertical natural transformation θ between two double functors $F, G\colon \mathbb{A} \to \mathbb{X}$ consists of two natural transformations $\theta_0\colon F_0 \Rightarrow G_0$ and $\theta_1\colon F_1 \Rightarrow G_1$ compatible with horizontal composition and identity cells.

Double categories, double functors and horizontal natural transformations form a 2-category $\mathbf{DblCat_h}$, and there is a canonical 2-functor

$$\mathbf{H}\colon \mathbf{DblCat_h} \longrightarrow \mathbf{2Cat}$$

which to a double category associates its horizontal 2-category, i.e. which consists of objects, horizontal arrows and squares whose vertical sides are identities. Similarly there is a 2-category $\mathbf{DblCat_v}$ of double categories, double functors, and vertical natural transformations, and a canonical 2-functor $\mathbf{V}\colon \mathbf{DblCat_v} \to \mathbf{2Cat}$ defined similarly as $\mathbf{H}$.

The double category $\mathbb{V}_1\mathbb{D}$ has vertical 1-category the vertical 1-category of $\mathbb{D}$ and everything else trivial, that is, there are no non-trivial squares and no non-trivial horizontal morphisms in $\mathbb{V}_1\mathbb{D}$. The subscript 1 in $\mathbb{V}_1\mathbb{D}$ reminds us that we retain only the vertical 1-category part of $\mathbb{D}$, and also distinguishes $\mathbb{V}_1\mathbb{D}$ for a double category $\mathbb{D}$ from $\mathbb{V}\mathbf{K}$ for a 2-category $\mathbf{K}$, which we define momentarily.

A 2-category $\mathbf{K}$ gives rise to various double categories. The double category $\mathbb{H}\mathbf{K}$ has $\mathbf{K}$ as its horizontal 2-category and only trivial vertical morphisms. Similarly, the double category $\mathbb{V}\mathbf{K}$ has $\mathbf{K}$ as its vertical 2-category and only trivial horizontal morphisms. Double categories of quintets of a 2-category will be introduced in Examples 6.1 and 6.6.

In this paper, the term "double category" always means "*strict* double category." We predominantly work with strict double categories, except for a few specified passages: in Section 3 the normal, vertically weak double category $\mathbb{S}\text{pan}^t$ is the codomain of presheaves on double categories, Theorem 5.4 concerns double adjunctions of strict double functors between horizontally weak double categories, and Section 8 treats the main example of the free–forgetful double adjunction between the normal, vertically weak double categories $\mathbb{E}\text{nd}(\mathbb{S}\text{pan})$ and $\mathbb{M}\text{nd}(\mathbb{S}\text{pan})$.

To explain the meaning of this terminology, recall that a *pseudo double category* is like a double category, except one of the two morphism compositions (vertical or horizontal) is associative and unital up to coherent invertible squares, rather than strictly, cf. Grandis–Paré [18], see also Chamaillard [6], Fiore [12], Martins-Ferreira [22]. In this article we specify the weak direction in a given pseudo category by our usage of the terms *horizontally weak double category* and *vertically weak double category*. In either case, the interchange law in (2) holds strictly.

All of the pseudo double categories we work with will also be *normal*, that is, the coherent unit squares are actually identity squares, so that the identity morphisms in the weak direction are strict identities. As mentioned in [18, page 172], this is easily arranged for pseudo double categories in which the weakly associative composition is given by some kind of choice (e.g. choice of pullbacks in the case of $\mathbb{S}\text{pan}$ in Example 2.1).

Normality has useful consequences. For each vertical morphism j, the square i^h_j is an identity for the horizontal composition of squares (in a general pseudo category, i^h_j is merely a distinguished square compatible with vertical composition). This small detail is needed in the proof of Theorem 5.4. Another consequence of normality is that $\mathbf{V}\mathbb{D}$ is a strict 2-category when $\mathbb{D}$ is a normal, horizontally weak double category. If $\mathbb{D}$ is horizontally weak and not normal, then $\mathbf{V}\mathbb{D}$ is neither a bicategory nor a 2-category (however the vertical composition of 2-cells in $\mathbf{V}\mathbb{D}$ can be redefined to make a 2-category). See pages 44-46 of [12], especially Remark 6.2, for a discussion of these topics.

Note also that (strict) horizontal natural transformations make sense between double functors of normal, vertically weak double categories (see the requirements in (5)).

Example 2.1. The normal, horizontally weak double category $\mathbb{S}\text{pan}$ will play a special role in this paper. Its objects are sets, its horizontal morphisms are spans of sets, its vertical morphisms are functions, and its squares are morphisms of spans. The horizontal composition of morphisms is by pullback combined with function composition: for the composite of two nontrivial horizontal morphisms, we choose the usual model for a set-theoretic pullback, which is a subset of the Cartesian product, and then compose the projections with remaining maps in the spans. However, for the composite of a horizontal morphism $B \leftarrow A \rightarrow C$ with an identity, we *choose the pullback* to be simply A. This choice of pullback makes the horizontally weak double category $\mathbb{S}\text{pan}$ *normal*, that is, the horizontal identities are actually strict horizontal identities. Consequently, for any two vertical morphisms j and k in $\mathbb{S}\text{pan}$, the horizontal identity squares i^h_j and i^h_k actually satisfy $\begin{bmatrix} i^h_j & \alpha \end{bmatrix} = \alpha = \begin{bmatrix} \alpha & i^h_k \end{bmatrix}$.

The normal, vertically weak double category $\mathbb{S}\text{pan}^t$ is the transpose of $\mathbb{S}\text{pan}$. Note that $\mathbb{S}\text{pan}$ is horizontally weak while $\mathbb{S}\text{pan}^t$ is vertically weak.

3. Parameterized Presheaves and the Double Yoneda Lemma

In this section we introduce and study parameterized presheaves, and prove a Yoneda Lemma for double categories. The Double Yoneda Lemma in Proposition 3.10 and the characterization of horizontal left double adjoints in Theorem 5.5 require parameterized $\mathbb{S}\text{pan}^t$-valued presheaves, as explained in the Introduction. The covariant Double Yoneda Lemma for presheaves on a double category $\mathbb{D}$ says that morphisms from the represented presheaf

$\mathbb{D}(R,-)$ to a presheaf K on $\mathbb{D}^{\text{horop}}$ are in bijective correspondence with the set $K(R)$.

A presheaf on a double category assigns to objects sets, to horizontal morphisms functions, to vertical morphisms spans of sets, and to squares morphisms of spans. Moreover, these image spans are equipped with a kind of composition provided by the vertical laxness of the presheaf, see for example equation (6).

Definition 3.1. Let $\mathbb{D}$ be a double category.

(i) A *presheaf* on $\mathbb{D}$ is a vertically lax double functor $\mathbb{D}^{\text{horop}} \to \mathbb{S}\text{pan}^t$.

(ii) A *morphism of presheaves* is a horizontal natural transformation of vertically lax double functors $\mathbb{D}^{\text{horop}} \to \mathbb{S}\text{pan}^t$.

Definition 3.2. Let $\mathbb{D}$ and $\mathbb{E}$ be double categories.

(i) A *presheaf on* $\mathbb{D}$ *parameterized by* $\mathbb{E}$ is a vertically lax double functor $\mathbb{D}^{\text{horop}} \times \mathbb{E} \to \mathbb{S}\text{pan}^t$. We synonymously use the term *presheaf on* $\mathbb{D}$ *indexed by* $\mathbb{E}$.

(ii) A *morphism of presheaves on* $\mathbb{D}$ *parameterized by* $\mathbb{E}$ is just a horizontal natural transformation between them.

Example 3.3. The most basic example is delivered by the hom-sets of a double category $\mathbb{D}$. Namely, a presheaf on $\mathbb{D}$ indexed by $\mathbb{D}$ is defined on objects and horizontal morphisms by

$$\begin{aligned} \mathbb{D}(-,-)\colon\ \mathbb{D}^{\text{horop}} \times \mathbb{D} &\longrightarrow \mathbb{S}\text{pan}^t \\ (D_1, D_2) &\longmapsto \operatorname{Hor}\mathbb{D}(D_1, D_2) \\ (f,g) &\longmapsto \operatorname{Hor}\mathbb{D}(f,g)\ . \end{aligned}$$

On vertical morphisms (j,k), it is the vertical span

$$\begin{array}{c} \operatorname{Hor}\mathbb{D}(s^v j, s^v k) \\ \uparrow{\scriptstyle s^v} \\ \mathbb{D}(j,k) \\ \downarrow{\scriptstyle t^v} \\ \operatorname{Hor}\mathbb{D}(t^v j, t^v k), \end{array}$$

which we often denote simply by $\mathbb{D}(j,k)$. On squares (α,β), the vertically lax double functor $\mathbb{D}(-,-)$ is the morphism of vertical spans induced by $\mathbb{D}(\alpha,\beta)(\gamma) = [\alpha\ \gamma\ \beta]$ and the functions $\operatorname{Hor}\mathbb{D}(s^v\alpha, s^v\beta)$ and $\operatorname{Hor}\mathbb{D}(t^v\alpha, t^v\beta)$.

For the vertically lax double functor $\mathbb{D}(-,-)$, the composition coherence square in $\mathbb{S}\mathrm{pan}^t$

$$\begin{bmatrix}\mathbb{D}(j,k)\\ \mathbb{D}(\ell,m)\end{bmatrix} \longrightarrow \mathbb{D}\left(\left[\begin{smallmatrix} j\\ \ell\end{smallmatrix}\right], \left[\begin{smallmatrix} k\\ m\end{smallmatrix}\right]\right)$$

is simply composition in $\mathbb{D}$. More precisely, on elements we have

$$\begin{array}{ccc} & \longrightarrow & \\ j\big\downarrow & \xi_1 & \big\downarrow k \\ & \longrightarrow & \\ \ell\big\downarrow & \xi_2 & \big\downarrow m \\ & \longrightarrow & \end{array} \quad\longmapsto\quad \begin{array}{ccc} & \longrightarrow & \\ \left[\begin{smallmatrix} j\\ \ell\end{smallmatrix}\right]\Big\downarrow & \left[\begin{smallmatrix} \xi_1\\ \xi_2\end{smallmatrix}\right] & \Big\downarrow\left[\begin{smallmatrix} k\\ m\end{smallmatrix}\right] \\ & \longrightarrow & \end{array} \quad . \tag{6}$$

The unit coherence square in $\mathbb{S}\mathrm{pan}^t$ of the vertically lax double functor $\mathbb{D}(-,-)$ is simply the vertical identity square embedding

$$1^v_{\mathbb{D}(D_1,D_2)} \xrightarrow{i^v} \mathbb{D}(1^v_{D_1}, 1^v_{D_2})$$

$$f \longmapsto \begin{array}{ccc} D_1 & \xrightarrow{f} & D_2 \\ \Big\| & i^v_f & \Big\| \\ D_1 & \xrightarrow[f]{} & D_2 \end{array} \quad .$$

The presheaf $\mathbb{D}(-,-)$ may also be considered as a presheaf on $\mathbb{D}^{\text{horop}}$ indexed by $\mathbb{D}^{\text{horop}}$. This completes the example $\mathbb{D}(-,-)$.

Example 3.4. As a special case of Example 3.3, we may fix the first variable to be an object R in $\mathbb{D}$ and we obtain a presheaf on $\mathbb{D}^{\text{horop}}$, namely

$$\mathbb{D}(R,-)\colon\ \mathbb{D} \longrightarrow \mathbb{S}\mathrm{pan}^t\ .$$

This presheaf is *represented* by the object R. We shall discuss a notion of representability for parameterized presheaves in Definition 3.8, as they will be a key ingredient in our characterizations of horizontal double adjunctions in Theorem 5.2 (vi) and Theorem 5.5.

We write out the features of Example 3.3 for this special case, since we will need these represented presheaves in the Double Yoneda Lemma. Like any double functor, this presheaf consists of an object functor and a morphism functor

$$\mathbb{D}(R,-)^{\mathsf{Obj}}\colon\ (\mathrm{Obj}\,\mathbb{D}_0, \mathrm{Obj}\,\mathbb{D}_1) \longrightarrow (\text{Sets}, \text{Functions})$$

$$\mathbb{D}(R,-)^{\mathsf{Mor}}\colon\ (\mathrm{Mor}\,\mathbb{D}_0, \mathrm{Mor}\,\mathbb{D}_1) \longrightarrow (\text{Spans}, \text{Morphisms of Spans})\ .$$

The object functor is the usual represented presheaf on the horizontal 1-category, namely

$$\begin{aligned} \mathbb{D}(R,D)^{\mathsf{Obj}} &:= \{f\colon\ R\to D \mid f \text{ horizontal morphism in } \mathbb{D}\} \\ &= \mathrm{Hor}\,\mathbb{D}(R,D) \\ \mathbb{D}(R,g)^{\mathsf{Obj}}(f) &:= [f\ \ g]\,. \end{aligned}$$

The morphism functor, on the other hand, takes a vertical morphism $j\colon\ D\to D'$ in $\mathbb{D}$ to the (vertical) span $\mathbb{D}(R,j)^{\mathsf{Mor}}$ defined as

$$\begin{array}{c} \mathbb{D}(R,D)^{\mathsf{Obj}} \\ \Big\uparrow{\scriptstyle s^v} \\ \mathbb{D}(1^v_R, j) \\ \Big\downarrow{\scriptstyle t^v} \\ \mathbb{D}(R,D')^{\mathsf{Obj}}, \end{array}$$

and on a square β we have the morphism of spans $\mathbb{D}(R,\beta)^{\mathsf{Mor}}$ induced by $\mathbb{D}(R,\beta)^{\mathsf{Mor}}(\alpha) = [\alpha\ \ \beta]\,.$

The composition coherence square in $\mathbb{S}\mathrm{pan}^t$

$$\begin{bmatrix} \mathbb{D}(R,j)^{\mathsf{Mor}} \\ \mathbb{D}(R,k)^{\mathsf{Mor}} \end{bmatrix} \longrightarrow \mathbb{D}(R, \left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right])$$

of the vertically lax double functor $\mathbb{D}(R,-)$ is simply composition in $\mathbb{D}$.

More precisely, on elements we have

$$\begin{array}{ccc} R & \longrightarrow & \\ \| & \xi_1 & \downarrow j \\ R & \longrightarrow & \\ \| & \xi_2 & \downarrow k \\ R & \longrightarrow & \end{array} \quad \longmapsto \quad \begin{array}{ccc} R & \longrightarrow & \\ \| & \left[\begin{smallmatrix}\xi_1 \\ \xi_2\end{smallmatrix}\right] & \downarrow \left[\begin{smallmatrix} j \\ k\end{smallmatrix}\right] \\ R & \longrightarrow & \end{array} \ .$$

The unit coherence square in $\mathbb{S}\mathrm{pan}^t$ of the vertically lax double functor $\mathbb{D}(R,-)$ is simply the identity embedding

$$1^v_{\mathbb{D}(R,D)^{\mathrm{Obj}}} \xrightarrow{i^v} \mathbb{D}(R, 1^v_D)^{\mathrm{Mor}}$$

$$f \longmapsto \begin{array}{ccc} R & \xrightarrow{f} & D \\ \| & i^v_f & \| \\ R & \xrightarrow[f]{} & D \end{array} \ .$$

Example 3.5. If $\mathbf{C}$ is a 1-category, then a classical presheaf on $\mathbf{C}$ may be considered a presheaf on $\mathbb{H}\mathbf{C}$ in the following way. A classical presheaf on $\mathbf{C}$ is the same thing as a strictly unital double functor $F\colon (\mathbb{H}\mathbf{C})^{\mathrm{horop}} \to \mathbb{S}\mathrm{pan}^t$ which has composition coherence morphism for $F(1^v_C) \circ F(1^v_C) \to F(1^v_C)$ given by the projection of the diagonal of $FC \times FC$ to FC. Any presheaf on $\mathbb{H}\mathbf{C}$ restricts to a classical presheaf on $\mathbf{C}$ by forgetting $F(1^v_C)$ for each C and the composition and identity coherences.

Example 3.6. A presheaf on the (opposite of the) terminal double category 1 is the same as a category, since a vertically lax double functor from 1 into $\mathbb{S}\mathrm{pan}^t$ is the same as a (horizontal) monad in $\mathbb{S}\mathrm{pan}$, which is the same as a category. Note also that morphisms of such presheaves are horizontal natural transformations of vertically lax double functors, hence are the same as functors (see [13]).

Example 3.7. Let $\mathbf{C}$ be a 1-category. Then $\mathbf{C}(-,-)$ is a presheaf on $\mathbf{C}$ indexed by $\mathrm{Obj}\,\mathbf{C}$. This is a way to consider all the presheaves $\mathbf{C}(-,C)$ simultaneously. Similarly, by parameterizing via the vertical 1-category of $\mathbb{D}$, the indexed presheaf $\mathbb{D}(-,-)\colon\ \mathbb{D}^{\text{horop}} \times \mathbb{V}_1\mathbb{D} \to \mathbb{S}\text{pan}^t$ is a way of considering all presheaves $\mathbb{D}(-,R)$ *simultaneously and how they combine vertically* (recall the notation $\mathbb{V}_1\mathbb{D}$ from Section 2). This point of view will become important for our characterization of horizontal double adjunctions in Theorems 5.2 and 5.5.

Definition 3.8. A parameterized presheaf $F\colon\ \mathbb{D}^{\text{horop}} \times \mathbb{E} \to \mathbb{S}\text{pan}^t$ in the sense of Definition 3.2 is *representable* if there exists a double functor $G\colon\ \mathbb{E} \to \mathbb{D}$ such that F is isomorphic to $\mathbb{D}(-,G-)\colon\ \mathbb{D}^{\text{horop}} \times \mathbb{E} \to \mathbb{S}\text{pan}^t$ as parameterized presheaves.

Example 3.9. The presheaf $\mathbb{D}(-,R)\colon\ \mathbb{D}^{\text{horop}} \to \mathbb{S}\text{pan}^t$ is represented by the double functor $* \to \mathbb{D}$ that is constant R. The indexed presheaf

$$\mathbb{D}(-,-)\colon\ \mathbb{D}^{\text{horop}} \times \mathbb{V}_1\mathbb{D} \longrightarrow \mathbb{S}\text{pan}^t$$

is represented by the inclusion of the vertical 1-category of $\mathbb{D}$ into $\mathbb{D}$.

We next prove the Double Yoneda Lemma. For simplicity, we do the covariant version rather than the contravariant version.

Proposition 3.10 (Double Yoneda Lemma)**.** *Let $\mathbb{D}$ be a small double category, R an object of $\mathbb{D}$, $K\colon\ \mathbb{D} \to \mathbb{S}\text{pan}^t$ a vertically lax double functor, and* $\text{HorNat}(\mathbb{D}(R,-),K)$ *the set of horizontal natural transformations from $\mathbb{D}(R,-)$ to K. Then the map*

$$\theta_{R,K}\colon\ \text{HorNat}(\mathbb{D}(R,-),K) \longrightarrow KR$$

$$\alpha \longmapsto \alpha_R(1^h_R)$$

is a bijection. Further, this bijection is a horizontal natural isomorphism of double functors N and E

$$N,E\colon\ \mathbb{D} \times \mathbb{D}\text{blCat}_{\text{vert.lax}}(\mathbb{D},\mathbb{S}\text{pan}^t) \longrightarrow \mathbb{S}\text{pan}^t$$

$$N(R,K) := \text{HorNat}(\mathbb{D}(R,-),K)$$

$$E(R,K) := K(R).$$

Proof. This is an extension of the proof of Borceux [3, Theorem 1.3.3]. We define $\theta_{R,K}(\alpha) = \alpha(1^h_R) \in K(R)$ and for $a \in K(R)$ we define a horizontal natural transformation $\tau(a)\colon \mathbb{D}(R,-) \Rightarrow K$. To each object $D \in \mathbb{D}$ we have the horizontal morphism in $\mathbb{S}\mathrm{pan}^t$

$$\begin{array}{rcl} \tau(a)_D\colon \mathbb{D}(R,D) & \longrightarrow & KD \\ f & \longmapsto & K(f)(a). \end{array}$$

and to each vertical morphism j in $\mathbb{D}$ we have the square $\tau(a)_j$ in $\mathbb{S}\mathrm{pan}^t$

$$\begin{array}{ccc} \mathbb{D}(R,D)^{\mathsf{Obj}} & \xrightarrow{\tau(a)_D} & K(D) \\ \uparrow & & \uparrow \\ \mathbb{D}(1^v_R, j) & \xrightarrow{\tau(a)_j} & K(j) \\ \downarrow & & \downarrow \\ \mathbb{D}(R,D')^{\mathsf{Obj}} & \xrightarrow[\tau(a)_{D'}]{} & K(D') \end{array} \quad = \quad \begin{array}{ccc} \mathbb{D}(R,D)^{\mathsf{Obj}} & \xrightarrow{K(-)(a)} & K(D) \\ \uparrow & & \uparrow \\ \mathbb{D}(1^v_R, j) & \xrightarrow{K(-)(\delta^K_R(a))} & K(j) \\ \downarrow & & \downarrow \\ \mathbb{D}(R,D')^{\mathsf{Obj}} & \xrightarrow[K(-)(a)]{} & K(D'). \end{array} \tag{7}$$

These squares commute, because for $\begin{array}{ccc} R & \longrightarrow & D \\ \| & \xi & \downarrow j \\ R & \longrightarrow & D' \end{array} \in \mathbb{D}(1^v_R, j)$ the squares

$$\begin{array}{ccccc} K(R) & = & K(R) & \xrightarrow{K(f)} & K(D) \\ \| & & \uparrow & & \uparrow \\ K(R) & \xrightarrow{\delta^k_R} & K(1^v_R) & \xrightarrow{K(\xi)} & K(j) \\ \| & & \downarrow & & \downarrow \\ K(R) & = & K(R) & \xrightarrow[K(f')]{} & K(D') \end{array} \tag{8}$$

commute. For example, the top square in (7) evaluated on ξ is the same as the top half of (8) evaluated on a.

The naturality of $\tau(a)$, τ, and θ is proved as in Borceux [3, Theorem 1.3.3]. □

Corollary 3.11. *For objects $R, S \in \mathbb{D}$, each horizontal natural transformation $\mathbb{D}(R, -) \Rightarrow \mathbb{D}(S, -)$ has the form $\mathbb{D}(h, -)$ for a unique horizontal arrow $h\colon S \to R$.*

Remark 3.12. If k is a vertical morphism in $\mathbb{D}$, then

$$\mathbb{D}(k, -)\colon (\mathrm{Ver}\,\mathbb{D}, \mathrm{Sq}\,\mathbb{D}) \longrightarrow (\mathrm{Sets}, \mathrm{functions})$$

$$\ell \longmapsto \mathbb{D}(k, \ell)$$

is an ordinary presheaf on $(\mathrm{Ver}\,\mathbb{D}, \mathrm{Sq}\,\mathbb{D})^{\mathrm{op}}$.

4. Universal Squares in a Double Category

The components of the unit or counit of any 1-adjunction are universal arrows. Conversely, a 1-adjunction can be described in terms of such universal arrows. In this section we introduce universal squares in a double category, with a view towards the analogous characterizations of horizontal double adjunctions in Theorem 5.2.

Definition 4.1. If $S\colon \mathbb{D} \to \mathbb{C}$ is a double functor, then a *(horizontally) universal square from the vertical morphism j to S* is a square μ in $\mathbb{C}$ of the form

$$\begin{array}{ccc} C_1 & \xrightarrow{u_1} & SR_1 \\ {\scriptstyle j}\downarrow & \mu & \downarrow{\scriptstyle Sk} \\ C_2 & \xrightarrow[u_2]{} & SR_2 \end{array}$$

such that the map

$$\begin{array}{ccc} \mathbb{D}(k, \ell) & \longrightarrow & \mathbb{C}(j, S\ell) \\ \beta' & \longmapsto & [\mu \ \ S\beta'] \end{array} \tag{9}$$

is a bijection for all vertical morphisms ℓ. There is of course a dual notion of *(horizontally) universal square from a double functor S to a vertical morphism j.*

Proposition 4.2. *Suppose* $S\colon \mathbf{K}' \to \mathbf{K}$ *is a 2-functor and* $u\colon C \to SR$ *is a morphism in* $\mathbf{K}$. *Then* $\mu := i^v_u$ *is universal from* 1^v_C *to* $\mathbb{H}S$ *if and only if the functor*

$$\begin{array}{ccc} \mathbf{K}'(R,D) & \xrightarrow{S(-)\circ u} & \mathbf{K}(C,SD) \\ f' & \longmapsto & [u\ \ Sf'] \end{array}$$

is an isomorphism of categories. In other words, the square i^v_u *in* $\mathbb{H}\mathbf{K}$ *is universal if and only if the morphism* u *of* $\mathbf{K}$ *is 2-universal.*

Proof. In this situation the assignment $\beta' \mapsto [\mu\ \ \mathbb{H}S\beta']$ is a functor, namely whiskering with u. Then the claim follows from the observation that the morphism part of a functor is bijective if and only if the functor is an isomorphism of categories. □

Proposition 4.3. *The bijection in* (9) *is a natural transformation of functors*

$$\mathbb{D}(k,-) \Longrightarrow \mathbb{C}(j,S-)\ . \tag{10}$$

Conversely, given k *and* j, *any natural bijection of functors as in* (10) *arises in this way from a unique square* $\mu \in \mathbb{C}(j,Sk)$ *which is universal from* j *to* S.

Proof. The proof is very similar to that of Mac Lane [21, Proposition 1, page 59]. The bijection is natural because

$$[\mu\ \ S\,[\beta'\ \ \gamma']] = [\mu\ \ [S\beta'\ \ S\gamma']]\,.$$

For the converse, let $\phi\colon \mathbb{D}(k,-) \Rightarrow \mathbb{C}(j,S-)$ be a natural bijection, and define $\mu := \phi_k(i^h_k)$. The naturality diagram for ϕ and β' yields $[\mu\ \ S\beta'] = \phi_\ell(\beta')$, which in turn implies that (9) is a bijection, since ϕ_ℓ is a bijection. □

For later use, we record the dual to Proposition 4.3 using the inverse bijection.

Proposition 4.4. *Universal squares in* $\mathbb{C}(Sk,j)$ *from* $S\colon \mathbb{D} \to \mathbb{C}$ *to* j *are in bijective correspondence with natural bijections*

$$\mathbb{C}(S-,j) \Longrightarrow \mathbb{D}(-,k)\ .$$

5. Double Adjunctions

For any 2-category $\mathbf{K}$, there is a notion of adjunction in $\mathbf{K}$ [20]. Namely, two 1-morphisms $f\colon A \to B$ and $g\colon B \to A$ in $\mathbf{K}$ are adjoint if there exist 2-cells $\eta\colon 1_A \Rightarrow gf$ and $\varepsilon\colon fg \Rightarrow 1_B$ satisfying the triangle identities. From the 2-categories $\mathbf{DblCat_h}$ and $\mathbf{DblCat_v}$ we thus get two notions of adjunction between double categories.

Definition 5.1. A *horizontal double adjunction* is an adjunction in the 2-category $\mathbf{DblCat_h}$. A *vertical double adjunction* is an adjunction in the 2-category $\mathbf{DblCat_v}$.

The notions of horizontal and vertical adjunctions are of course transpose to each other, so the result we list in this section for horizontal adjunctions are also valid for vertical adjunctions. However, as soon as the involved double categories have further structure, like the foldings and cofoldings we consider from Section 6 and onwards, the two notions behave differently. In this paper we need both notions.

A more general notion of vertical adjunction was introduced and studied by Grandis and Paré [19] (cf. further comments below). Vertical adjunctions were also studied by Garner [17, Appendix A] and Shulman [24, Section 8].

For the basic theory, which we treat in this section, we work only with horizontal adjunctions. The 2-category $\mathbf{DblCat_h}$ is the same as the 2-category $\mathsf{Cat}(\mathbf{Cat})$ of internal categories in $\mathbf{Cat}$, internal functors, and internal natural transformations, which leads to various characterizations of horizontal double adjunctions in terms of universal arrows and bijections of hom-sets, along the lines of Mac Lane [21, Theorem 2, p.83]. Our results in this vein in Theorem 5.2 can be deduced from more general results of Grandis–Paré [19], but we have included the proofs since they are quite natural from the internal viewpoint (which is not mentioned in [19]). The first novelty comes when trying to characterize adjunctions in terms of presheaves: here it turns out we need parameterized presheaves, which is the content of Theorem 5.5.

In Section 8 we present a completely worked example of a *vertical* double adjunction: the free and forgetful double functors between endomorphisms and monads in $\mathbb{S}$pan. This is an extension of the classical adjunction between small directed graphs and small categories.

Let $\mathbb{A}$ and $\mathbb{X}$ be double categories. Since a horizontal double adjunction is precisely an internal adjunction, an explicit description is this: a horizontal double adjunction from $\mathbb{X}$ to $\mathbb{A}$ consists of double functors

$$\mathbb{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{A} \tag{11}$$

and horizontal natural transformations

$$\eta\colon\ 1_{\mathbb{X}} \Longrightarrow GF$$

$$\varepsilon\colon\ FG \Longrightarrow 1_{\mathbb{A}}$$

such that the composites

$$G \xRightarrow{\eta * i_G} GFG \xRightarrow{i_G * \varepsilon} G$$

$$F \xRightarrow{i_F * \eta} FGF \xRightarrow{\varepsilon * i_F} F$$

are the respective identity horizontal natural transformations. Here F is the *horizontal left adjoint*, G is the *horizontal right adjoint*, and we use the notation $F \dashv G$ to indicate this horizontal adjunction. In this section we consider only horizontal adjunctions, and suppress the adjective "horizontal" for brevity.

Theorem 5.2 (Characterizations of horizontal double adjunctions)**.** *A horizontal double adjunction $F \dashv G$ is completely determined by the items in any one of the following lists.*

(i) Double functors F, G as in (11) *and a horizontal natural transformation $\eta\colon 1_{\mathbb{X}} \Rightarrow GF$ such that for each vertical morphism j in $\mathbb{X}$, the square η_j is universal from j to G.*

(ii) A double functor G as in (11) *and functors*

$$F_0\colon\ (\operatorname{Obj}\mathbb{X}, \operatorname{Ver}\mathbb{X}) \longrightarrow (\operatorname{Obj}\mathbb{A}, \operatorname{Ver}\mathbb{A})$$

$$\eta\colon\ (\operatorname{Obj}\mathbb{X}, \operatorname{Ver}\mathbb{X}) \longrightarrow (\operatorname{Hor}\mathbb{X}, \operatorname{Sq}\mathbb{X})$$

such that for each vertical morphism j in $\mathbb{X}$ the square η_j is of the form

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & GF_0X \\ {\scriptstyle j}\downarrow & \eta_j & \downarrow{\scriptstyle GF_0 j} \\ Y & \xrightarrow[\eta_Y]{} & GF_0Y \end{array}$$

and is universal from j to G. Then the double functor F is defined on vertical arrows by F_0 and on squares χ by universality via the equation $[\eta_{s\chi}\ \ GF\chi] = [\chi\ \ \eta_{t\chi}]$.

(iii) Double functors F, G as in (11) *and a horizontal natural transformation $\varepsilon\colon\ FG \Rightarrow 1_{\mathbb{A}}$ such that for each vertical morphism k in $\mathbb{A}$, the square ε_k is universal from F to k.*

(iv) A double functor F as in (11) *and functors*

$$G_0\colon\ (\mathrm{Obj}\,\mathbb{A}, \mathrm{Ver}\,\mathbb{A}) \longrightarrow (\mathrm{Obj}\,\mathbb{X}, \mathrm{Ver}\,\mathbb{X})$$

$$\varepsilon\colon\ (\mathrm{Obj}\,\mathbb{A}, \mathrm{Ver}\,\mathbb{A}) \longrightarrow (\mathrm{Hor}\,\mathbb{A}, \mathrm{Sq}\,\mathbb{A})$$

such that for each vertical morphism k in $\mathbb{A}$ the square ε_k is of the form

$$\begin{array}{ccc} FG_0A & \xrightarrow{\varepsilon_A} & A \\ {\scriptstyle FG_0 k}\downarrow & \varepsilon_k & \downarrow{\scriptstyle k} \\ FG_0B & \xrightarrow[\varepsilon_B]{} & B \end{array}$$

and is universal from F to k. Then the double functor G is defined on vertical morphisms by G_0 and on squares α by universality via the equation $[FG\alpha\ \ \varepsilon_{t\alpha}] = [\varepsilon_{s\alpha}\ \ \alpha]$.

(v) Double functors F, G as in (11) *and a bijection*

$$\varphi_{j,k}\colon\ \mathbb{A}(Fj, k) \longrightarrow \mathbb{X}(j, Gk)$$

natural in the vertical morphisms j and k and compatible with vertical composition.

Naturality here means natural as a functor

$$(\mathrm{Ver}\,\mathbb{X}, \mathrm{Sq}\,\mathbb{X})^{\mathrm{op}} \times (\mathrm{Ver}\,\mathbb{A}, \mathrm{Sq}\,\mathbb{A}) \longrightarrow \mathbf{Set}\,.$$

That is, for squares $\sigma \in \mathbb{X}(j', j)$, $\alpha \in \mathbb{A}(Fj, k)$, $\tau \in \mathbb{A}(k, k')$ and squares σ, we have

$$\varphi\left([F\sigma \;\; \alpha]\right) = [\sigma \;\; \varphi\left(\alpha\right)]$$

$$\varphi\left([\alpha \;\; \tau]\right) = [\varphi\left(\alpha\right) \;\; G\tau]\,.$$

Compatibility with vertical composition means

$$\varphi\left(\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right) = \begin{bmatrix}\varphi(\alpha)\\ \varphi(\beta)\end{bmatrix}.$$

(vi) Double functors F, G as in (11) *and a horizontal natural isomorphism between the vertically lax double functors (parameterized presheaves)*

$$\mathbb{A}(F-,-)\colon\ \mathbb{X}^{\mathrm{horop}} \times \mathbb{A} \longrightarrow \mathbb{S}\mathrm{pan}^t$$

$$\mathbb{X}(-,G-)\colon\ \mathbb{X}^{\mathrm{horop}} \times \mathbb{A} \longrightarrow \mathbb{S}\mathrm{pan}^t\,.$$

Remark 5.3. As mentioned, Grandis and Paré [19] have introduced a more general notion of double adjunction, which mixes colax and lax double functors, and due to this mixture, this notion is *not* an instance of an adjunction in a bicategory. However, they observe that if at least one of the functors is pseudo (so that both functors can be considered colax or both lax), then the notion is the 2-categorical notion from the 2-category of double categories, either colax or lax double functors, and *vertical* natural transformations. We just add to their observations that in the strict case we can transpose, and find that the strict version of their notion specializes to Definition 5.1 above. Under these relationships, Theorem 5.2 becomes essentially a special case of results of Grandis–Paré: characterization (v) is the transpose of the strict version of [19, Theorem 3.4], and characterization (iv) is the transpose of the strict version of [19, Theorem 3.6]. The other characterizations in Theorem 5.2 are variations, but (vi) appears to be new.

Proof. We first prove Definition 5.1 is equivalent to (v), then we use this equivalence to prove the other equivalences (we provide much detail in the equivalence Definition 5.1 ⇔ (v) because we will need these details for a pseudo version in Theorem 5.4). In each equivalence, we omit the proof that the two procedures are inverse to one another.

Definition 5.1 ⇒ (v). Suppose $\langle F, G, \eta, \varepsilon\rangle$ is a double adjunction. Then for any square γ of the form

$$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ {\scriptstyle j}\big\downarrow & \gamma & \big\downarrow{\scriptstyle \ell} \\ \cdot & \longrightarrow & \cdot \end{array}$$

we have $[\eta_j \;\; GF\gamma] = [\gamma \;\; \eta_\ell]$ by the horizontal naturality of η. We define $\varphi_{j,k}$ and $\varphi^{-1}_{j,k}$ by

$$\begin{aligned} \varphi_{j,k}(\alpha) &:= [\eta_j \;\; G\alpha] \\ \varphi^{-1}_{j,k}(\beta) &:= [F\beta \;\; \varepsilon_k]\,. \end{aligned}$$

Then we have

$$\begin{aligned} \varphi\varphi^{-1}\beta &= \varphi\,[F\beta \;\; \varepsilon_k] \\ &= [\eta_j \;\; GF\beta \;\; G\varepsilon_k] \\ &= [\beta \;\; \eta_{Gk} \;\; G\varepsilon_k] \text{ (by horizontal naturality)} \\ &= \beta \text{ (by triangle identity)} \end{aligned}$$

and similarly $\varphi^{-1}\varphi(\alpha) = \alpha$.

For the naturality of $\varphi_{j,k}$ in k, we have

$$\varphi\,([\alpha \;\; \tau]) \overset{\text{def}}{=} [\eta_j \;\; G[\alpha \;\; \tau]] = [\eta_j \;\; G\alpha \;\; G\tau] \overset{\text{def}}{=} [\varphi\,(\alpha) \;\; G\tau]\,.$$

Naturality of $\varphi_{j,k}$ in j is similar, but additionally uses the naturality of η.

For the compatibility of $\varphi_{j,k}$ with vertical composition, we must use the interchange law from (2) and the resulting convention (3), as well as the compatibility of the horizontal natural transformation η with vertical composition.

$$\begin{bmatrix} \varphi(\alpha) \\ \varphi(\beta) \end{bmatrix} = \begin{bmatrix} \eta_j & G\alpha \\ \eta_m & G\beta \end{bmatrix} = \begin{bmatrix} \eta_{\left[\begin{smallmatrix} j \\ m \end{smallmatrix}\right]} & G\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] \end{bmatrix}$$

We now have $\langle F, G, \varphi\rangle$ as in (v).

(v) $\Rightarrow$ Definition 5.1. From $\langle F, G, \varphi\rangle$ as in (v), we define horizontal natural transformations by

$$\eta_j := \varphi(i^h_{Fj})$$
$$\varepsilon_k := \varphi^{-1}(i^h_{Gk}).$$

The assignment η is natural because i^h_- is a horizontal identity square

$$[\eta_j \;\; GF\gamma] \overset{\text{def}}{=} \left[\varphi(i^h_{Fj}) \;\; GF\gamma\right] = \varphi\left[i^h_{Fj} \;\; F\gamma\right] = \varphi(\gamma)$$

$$[\gamma \;\; \eta_\ell] \overset{\text{def}}{=} \left[\gamma \;\; \varphi(i^h_{F\ell})\right] = \varphi\left[F\gamma \;\; i^h_{F\ell}\right] = \varphi(\gamma).$$

For the compatibility of η with vertical composition, we use the fact that i^h_- is compatible with vertical composition

$$\eta_{\left[\substack{j\\m}\right]} \overset{\text{def}}{=} \varphi(i^h_{F\left[\substack{j\\m}\right]}) = \varphi\begin{bmatrix} i^h_{Fj} \\ i^h_{Fm}\end{bmatrix} = \begin{bmatrix}\varphi(i^h_{Fj}) \\ \varphi(i^h_{Fm})\end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix}\eta_j \\ \eta_m\end{bmatrix}.$$

The assignment ε is similarly a horizontal natural transformation.

To verify that $G \xRightarrow{\eta * i_G} GFG \xRightarrow{i_G * \varepsilon} G$ is the identity horizontal natural transformation on G we have

$$[\eta_{Gk} \;\; G(\varepsilon_k)] \overset{\text{def}}{=} \left[\varphi(i^h_{FGk}) \;\; G\varphi^{-1}(i^h_{Gk})\right] = \varphi\left[i^h_{FGk} \;\; \varphi^{-1}(i^h_{Gk})\right] = i^h_{Gk}.$$

The proof of the other triangle identity is similar.

Finally, we now have $\langle F, G, \eta, \varepsilon\rangle$ as in Definition 5.1. We acknowledge the exposition of Mac Lane [21, pages 81–82] for this proof.

(i) $\Rightarrow$ (v). Suppose we have $\langle F, G, \eta\rangle$ as in (i). The universality of η_j says that

$$\begin{aligned} \mathbb{A}(Fj, k) &\longrightarrow \mathbb{X}(j, Gk) \\ \alpha &\longmapsto [\eta_j \;\; G\alpha] \end{aligned} \tag{12}$$

is a bijection. Clearly this bijection is natural in j and k, and compatible with vertical composition, so we obtain $\langle F, G, \varphi\rangle$ as in description (v).

(v) $\Rightarrow$ (i). From the first part, we know that Definition 5.1 is equivalent to (v) and that $\varphi_{j,k}(\alpha) = [\eta_j \;\; G\alpha]$. This gives us F, G, and η. The universality of η_j then follows, because the map in (12) is equal to $\varphi_{j,k}$ and is therefore bijective.

(i) $\Rightarrow$ (ii). The data in (ii) are just a restriction of the data in (i).

(ii) $\Rightarrow$ (i). The universality of η_j guarantees that for each square χ in $\mathbb{X}$ there is a unique square $F\chi$ such that $[\eta_{s\chi} \;\; GF\chi] = [\chi \;\; \eta_{t\chi}]$. This defines F on squares χ in $\mathbb{X}$, and we take F to be F_0 on the vertical morphisms of $\mathbb{X}$. Then F is a double functor by the universality and the hypothesis that F_0 and η are functors. Finally, η is natural because of the defining equation $[\eta_{s\chi} \;\; GF\chi] = [\chi \;\; \eta_{t\chi}]$.

5.1 $\Leftrightarrow$ (iii). The proof of the equivalence Definition 5.1 $\Leftrightarrow$ (iii) is dual to the proof the equivalence Definition 5.1 $\Leftrightarrow$ (i).

(iii) $\Leftrightarrow$ (iv). The proof of the equivalence (iii) $\Leftrightarrow$ (iv) is dual to the proof of the equivalence (i) $\Leftrightarrow$ (ii).

(v) $\Leftrightarrow$ (vi). We first point out that the data of (v) and (vi) are the same: to obtain the outer maps of the span 2-cells for the horizontal natural isomorphism in (vi), we take j and k to be 1_X and 1_A and obtain bijections $\mathbb{A}(FX, A) \cong \mathbb{X}(X, GA)$. To obtain the middle maps of the span 2-cells for (vi), we directly take the $\varphi_{j,k}$'s. Conversely, to obtain the bijections $\varphi_{j,k}$ in (v) from the horizontal natural isomorphism in (vi), we simply take the middle maps of the span 2-cells. So the data of (v) and (vi) are the same. As to the conditions: for the data to form the horizontal natural transformation of (vi), two compatibilities are required: one horizontal compatibility equation for each square, which amounts precisely to naturality of $\varphi_{j,k}$ in (v), and one compatibility condition with respect to the coherence squares of the vertically lax double functors. Since these coherence squares are given by vertical composition (cf. Example 3.3), this condition amounts precisely to φ being compatible with vertical composition.

This completes the proof of the equivalence of Definition 5.1 with each of (i), (ii), (iii), (iv), (v), and (vi). □

We next prove a slightly weakened version of the equivalence Definition 5.1$\Leftrightarrow$(v). The transpose of this slightly weakened version will be used in the proof of the *vertical* double adjunction between $\mathbb{E}\text{nd}(\mathbb{S}\text{pan})$ and $\mathbb{M}\text{nd}(\mathbb{S}\text{pan})$ in Proposition 8.1.

Theorem 5.4 (Pseudo version of Theorem 5.2 (v))**.** *Let $\mathbb{A}$ and $\mathbb{X}$ be normal, vertically weak double categories. Let $F\colon \mathbb{X} \to \mathbb{A}$ and $G\colon \mathbb{A} \to \mathbb{X}$ be* *strict double functors, that is, F and G strictly preserve all compositions and identities of $\mathbb{X}$ respectively $\mathbb{A}$. Then there exist strict horizontal natural transformations $\eta\colon 1_{\mathbb{X}} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow 1_{\mathbb{A}}$ satisfying the two triangle identities if and only if statement (v) of Theorem 5.2 holds.*

Proof. The proof is the same as the proof of Definition 5.1 $\Leftrightarrow$ (v) in Theorem 5.2, only we must verify that the arguments there still make sense for the present hypotheses.

For the direction Definition 5.1 $\Rightarrow$ (v), we note i) the horizontal composition of squares is strictly associative (since the pseudo double categories are weak only vertically), ii) G strictly preserves horizontal compositions, and iii) the interchange law holds in $\mathbb{A}$ and $\mathbb{X}$ as in any pseudo double category [18, page 210].

For the direction (v) $\Rightarrow$ Definition 5.1, we note that i^h_- is a horizontal identity square because $\mathbb{A}$ and $\mathbb{X}$ are normal (recall the discussion before Example 2.1). □

In ordinary 1-category theory, a functor $F\colon \mathbf{A} \to \mathbf{X}$ admits a right adjoint if and only if the presheaf $\mathbf{A}(F-, A)$ is representable for each A. But for double categories and double functors $F\colon \mathbb{A} \to \mathbb{X}$, we must consider the representability of the parameterized $\mathbb{S}\mathrm{pan}^t$-valued presheaf $\mathbb{A}(F-,-)$. We arrive at the following characterization of horizontal left double adjoints in terms of parameterized representability.

Theorem 5.5. *A double functor $F\colon \mathbb{X} \to \mathbb{A}$ admits a horizontal right double adjoint if and only if the parameterized presheaf on $\mathbb{X}$*

$$\mathbb{A}(F-,-)\colon \mathbb{X}^{\mathrm{horop}} \times \mathbb{V}_1\mathbb{A} \longrightarrow \mathbb{S}\mathrm{pan}^t$$

is represented by a double functor $G_0\colon \mathbb{V}_1\mathbb{A} \to \mathbb{X}$.

Remark 5.6. Recalling the definition of $\mathbb{V}_1$ from Section 2, and the parameterized presheaves from Definitions 3.2 and 3.8, we see that Theorem 5.5 essentially says that a double functor F admits a horizontal right double adjoint if and only if for every vertical morphism k in $\mathbb{A}$, the classical presheaf

$$\mathbb{A}(F-,k)\colon (\mathrm{Ver}\,\mathbb{X}, \mathrm{Sq}\,\mathbb{X})^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

is representable in a way compatible with vertical composition.

Proof. Suppose that a horizontal right double adjoint G exists. Then by Theorem 5.2 (vi) the parameterized presheaves $\mathbb{A}(F-,-)$ and $\mathbb{X}(-,G-)$ are horizontally naturally isomorphic as vertically lax double functors on $\mathbb{X}^{\text{horop}} \times \mathbb{A}$, so their restrictions to $\mathbb{X}^{\text{horop}} \times \mathbb{V}_1\mathbb{A}$ are also horizontally naturally isomorphic. The double functor G_0 is simply the restriction of G. We have represented $\mathbb{A}(F-,-)$ by G_0.

In the other direction, suppose that the parameterized presheaf on $\mathbb{X}$

$$\mathbb{A}(F-,-)\colon\ \mathbb{X}^{\text{horop}} \times \mathbb{V}_1\mathbb{A} \longrightarrow \mathbb{S}\text{pan}^t$$

is representable by a double functor $G_0\colon\ \mathbb{V}_1\mathbb{A} \to \mathbb{X}$, and let

$$\varphi\colon\ \mathbb{A}(F-,-) \Longrightarrow \mathbb{X}(-,G_0-)$$

be a horizontally natural isomorphism between vertically lax functors. For vertical morphisms (j,k), we then have an isomorphism of spans in $\mathbf{Set}$.

$$\begin{array}{ccc}
\mathbb{A}(Fs^vj, s^vj) & \xrightarrow{\varphi(s^vj,s^vj)} & \mathbb{X}(s^vj, G_0s^vj) \\
{\scriptstyle s^v}\uparrow & & \uparrow{\scriptstyle s^v} \\
\mathbb{A}(Fj, j) & \xrightarrow{\varphi(j,k)} & \mathbb{X}(j, G_0j) \\
{\scriptstyle t^v}\downarrow & & \downarrow{\scriptstyle t^v} \\
\mathbb{A}(Ft^vj, t^vj) & \xrightarrow[\varphi(t^vj,t^vj)]{} & \mathbb{X}(t^vj, G_0t^vj)
\end{array}$$

Since $\mathbb{V}_1\mathbb{A}$ has no nontrivial horizontal morphisms or squares, the condition of horizontal naturality in k is satisfied vacuously. So, essentially we have horizontally natural bijections $\varphi(-,k)\colon\ \mathbb{A}(F-,k) \Rightarrow \mathbb{X}(-,G_0k)$, and these correspond to universal squares from F to k of the form

$$\begin{array}{ccc}
FG_0A & \xrightarrow{\varepsilon(A)} & A \\
{\scriptstyle FG_0k}\downarrow & \varepsilon(k) & \downarrow{\scriptstyle k} \\
FG_0B & \xrightarrow[\varepsilon(B)]{} & B
\end{array}$$

by Proposition 4.4. The assignments of $\varepsilon(A)$ and $\varepsilon(k)$ to A and k form a functor

$$\varepsilon\colon\ (\text{Obj}\,\mathbb{A}, \text{Ver}\,\mathbb{A}) \longrightarrow (\text{Hor}\,\mathbb{X}, \text{Sq}\,\mathbb{X})$$

because of the compatibility of φ with the vertical laxness of the parameterized presheaves. Finally, the characterization in Theorem 5.2 (iv) tells us that G_0 extends to a horizontal right adjoint G, defined on squares α using universality and the equation $\big[FG\alpha \;\; \varepsilon(t^h\alpha)\big] = \big[\varepsilon(s^h\alpha) \;\; \alpha\big]$. $\square$

Remark 5.7. In this section we have treated horizontal double adjunctions. By transposition, all the results are equally valid for vertical double adjunctions. In practice, however, the two notions are very different, as further properties or structure of the double categories in question may break the symmetry. An instructive example is given by one-object/one-vertical-arrow double categories: these are monoids internal to $\mathbf{Cat}$, i.e. monoidal categories (with strictness according to the strictness of the double categories). Double functors between such are precisely monoidal functors (again with according strictness). Vertical natural transformations are precisely monoidal natural transformations. Horizontal natural transformations are something quite different, some sort of intertwiners: for two double functors $F, G\colon \mathbb{D} \to \mathbb{C}$ between one-object/one-vertical-arrow double categories, a horizontal natural transformation gives to a horizontal arrow S of $\mathbb{C}$ (i.e. an object of the corresponding monoidal category **C**) and an equation (or 2-cell) $S \otimes F = G \otimes S$ (where $\otimes$ denotes horizontal composition, i.e. the tensor product in **C**).

6. Compatibility with Foldings or Cofoldings

Many double categories of interest have additional structure that allows one to reduce certain questions about the double category to questions about the horizontal 2-category. There are several different, but closely related, formalisms for this sort of situation, cf. Brown–Mosa [4], Brown–Spencer [5], Fiore [12], Grandis–Paré [18], Shulman [24]; comparisons between the different formalisms can be found in [12] and [24]. In this section we investigate how the additional structure of *folding* or *cofolding* on double categories allows us to reduce questions concerning adjunctions to their horizontal 2-categories.

The notion of folding was introduced in [12], extending notions from [4]. A *folding* associates to every vertical morphism a horizontal morphism in a way that gives a bijection between certain squares in the double category and certain 2-cells in the horizontal 2-category. The precise definition is given below. In Example 6.3), we illustrate the folding for the double category of spans, which to a set map (vertical morphism) $j\colon\ A \to C$ associates the span (horizontal morphism) $A \stackrel{1^h_A}{\leftarrow} A \stackrel{j}{\to} C$. The double category of spans was discussed in Example 2.1.

A folding can be seen as a kind of covariant action of the vertical 1-category on the horizontal 2-category, a sort of pushforward operation; see [12, Section 4]. A *cofolding* is similar to a folding but constitutes instead a contravariant kind of action of the vertical 1-category on the horizontal 2-category, a sort of pullback operation. In Example 6.8, we illustrate the cofolding for the double category of spans, which to a vertical map $j\colon\ A \to C$ associates the horizontal morphism $C \stackrel{j}{\leftarrow} A \stackrel{1^h_A}{\to} A$.

Folding together with cofolding is equivalent to having a framing in the sense of Shulman [24], the category of spans being an archetypical example. However, some important double categories admit either a folding or a cofolding but not both, and it is necessary to study the two notions separately. This is the case for the double categories of endomorphisms and monads, $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$, in Section 7: if $\mathbb{D}$ admits a cofolding, then so do $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$ (cf. Proposition 7.2), but the analogous statement for foldings does not seem to be true.

The main result in this section, Proposition 6.10, states that if F and G are double functors between double categories with foldings, and F and G preserve the foldings, then F and G are horizontally double adjoint if and only if the horizontal 2-functors $\mathbf{H}F$ and $\mathbf{H}G$ are 2-adjoint. For the special case of quintet double categories, which we characterize in terms of folding with fully faithful holonomy in Lemma 6.13 and Proposition 6.15, we establish stronger characterizations of double adjunctions: briefly, all notions of adjunction agree in this case, see Corollary 6.16.

We begin the detailed discussion of foldings and cofolding with the notion of quintets.

Example 6.1 (Direct quintets)**.** With a 2-category $\mathbf{K}$ is associated a double category $\mathbb{Q}\mathbf{K}$, called the double category of *direct quintets*: its objects are the objects of $\mathbf{K}$, horizontal and vertical morphisms are the morphisms of

$\mathbf{K}$, and the squares

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \tag{13}$$

are the 2-cells $\alpha\colon\ k \circ f \Rightarrow g \circ j$ in $\mathbf{K}$. The horizontal 2-category of $\mathbb{Q}\mathbf{K}$ is $\mathbf{K}$. The vertical 2-category of $\mathbb{Q}\mathbf{K}$ is $\mathbf{K}$ with the 2-cells reversed. The terminology "quintet" is due to Ehresmann [10] for the case $\mathbf{K} = \mathbf{Cat}$. We add the word "direct" to distinguish from the "inverse quintets" introduced in Example 6.6, as we shall need both variants.

The double category $\mathbb{Q}\mathbf{K}$ is entirely determined by its horizontal 2-category, in fact, a quintet square α is by definition a 2-cell in $\mathbf{K}$ between appropriate composites of boundary components of α. Similarly, any double category with folding, as in the following definition, is determined by its vertical 1-category and horizontal 2-category in the sense that squares with a given boundary are in bijective correspondence with 2-cells in the horizontal 2-category between appropriate "boundary composites".

Definition 6.2. (Cf. Brown–Mosa [4] for the edge-symmmetric case and Fiore [12] for the general case.) A *folding* on a double category $\mathbb{D}$ is a double functor $\Lambda\colon\ \mathbb{D} \to \mathbb{Q}\mathbf{H}\mathbb{D}$ which is the identity on the horizontal 2-category $\mathbf{H}\mathbb{D}$ of $\mathbb{D}$ and is fully faithful on squares. We proceed to spell out the details.

A *folding on a double category* $\mathbb{D}$ consists of the following.

(i) A 2-functor $\overline{(-)}\colon\ (\mathbf{V}\mathbb{D})_0 \to \mathbf{H}\mathbb{D}$ which is the identity on objects. Here, the notation $(\mathbf{V}\mathbb{D})_0$ denotes the vertical 1-category of $\mathbb{D}$. In other words, to each vertical morphism $j\colon\ A \to C$, there is associated a horizontal morphism $\overline{j}\colon\ A \to C$ with the same domain and codomain in a functorial way. We call this 2-functor $j \mapsto \overline{j}$ the *holonomy*, following the terminology of Brown-Spencer in [5], who first distinguished the notion.

(ii) Bijections $\Lambda^{f,k}_{j,g}$ from squares in $\mathbb{D}$ with boundary

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle k} \\ C & \xrightarrow[\;g\;]{} & D \end{array} \tag{14}$$

to squares in $\mathbb{D}$ with boundary

$$\begin{array}{ccc} A & \xrightarrow{[f\ \bar{k}]} & D \\ \Big\| & & \Big\| \\ A & \xrightarrow[{[\bar{j}\ g]}]{} & D. \end{array} \tag{15}$$

These bijections are required to satisfy the following axioms.

(i) Λ is the identity if j and k are vertical identity morphisms.

(ii) Λ preserves horizontal composition of squares, that is,

$$\Lambda\left(\begin{array}{ccccc} A & \xrightarrow{\;f_1\;} & B & \xrightarrow{\;f_2\;} & C \\ {\scriptstyle j}\big\downarrow & \alpha & \big\downarrow{\scriptstyle k} & \beta & \big\downarrow{\scriptstyle \ell} \\ D & \xrightarrow[\;g_1\;]{} & E & \xrightarrow[\;g_2\;]{} & F \end{array}\right) \quad = \quad \begin{array}{ccc} A & \xrightarrow{[f_1\ f_2\ \bar{\ell}]} & F \\ \Big\| & [i^v_{f_1}\ \Lambda(\beta)] & \Big\| \\ A & \xrightarrow{[f_1\ \bar{k}\ g_2]} & F \\ \Big\| & [\Lambda(\alpha)\ i^v_{g_2}] & \Big\| \\ A & \xrightarrow[{[\bar{j}\ g_1\ g_2]}]{} & F. \end{array}$$

(iii) Λ preserves vertical composition of squares, that is,

$$\Lambda\left(\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j_1}\downarrow & \alpha & \downarrow{\scriptstyle k_1} \\ C & \xrightarrow{g} & D \\ {\scriptstyle j_2}\downarrow & \beta & \downarrow{\scriptstyle k_2} \\ E & \xrightarrow[h]{} & F, \end{array}\right) = \begin{array}{ccc} A & \xrightarrow{[f\ \overline{k}_1\ \overline{k}_2]} & F \\ \| & [\Lambda(\alpha)\ i^v_{\overline{k}_2}] & \| \\ A & \xrightarrow{[\overline{j}_1\ g\ \overline{k}_2]} & F \\ \| & [i^v_{\overline{j}_1}\ \Lambda(\beta)] & \| \\ A & \xrightarrow[{[\overline{j}_1\ \overline{j}_2\ h]}]{} & F. \end{array}$$

(iv) Λ preserves identity squares, that is,

$$\Lambda\left(\begin{array}{ccc} A & = & A \\ {\scriptstyle j}\downarrow & i^h_j & \downarrow{\scriptstyle j} \\ B & = & B \end{array}\right) = \begin{array}{ccc} A & \xrightarrow{\overline{j}} & B \\ \| & i^v_{\overline{j}} & \| \\ A & \xrightarrow[\overline{j}]{} & B. \end{array}$$

Example 6.3. The double category $\mathbb{S}\mathrm{pan}$ admits a folding. The holonomy is

$$\left(\begin{array}{c} A \\ {\scriptstyle j}\downarrow \\ C \end{array}\right) \longmapsto \left(A \xleftarrow{1^h_A} A \xrightarrow{j} C\right)$$

and the folding is

$$\left(\begin{array}{ccccc} A & \xleftarrow{f_0} & Y & \xrightarrow{f_1} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle k} \\ C & \xleftarrow[g_0]{} & Z & \xrightarrow[g_1]{} & D \end{array}\right) \longmapsto \left(\begin{array}{ccccc} A & \xleftarrow{f_0} & Y & \xrightarrow{k\circ f_1} & D \\ \| & & \downarrow{\scriptstyle (f_0,\alpha)} & & \| \\ A & \xleftarrow[\mathrm{pr}_1]{} & A\times_C Z & \xrightarrow[g_1\circ \mathrm{pr}_2]{} & D \end{array}\right).$$

Remark 6.4. If a double category $\mathbb{D}$ is equipped with a folding, then 2-cell composition in the vertical 2-category $\mathbf{V}\mathbb{D}$ corresponds to 2-cell composition in the horizontal 2-category $\mathbf{H}\mathbb{D}$. More precisely, if f_1, f_2, g_1, g_2 are identities in Definition 6.2 (ii), then $[\ \alpha\ \ \beta\ ]$ is the vertical composition $\beta \odot \alpha$ in the 2-category $\mathbf{V}\mathbb{D}$, and compatibility with horizontal composition says $\Lambda(\beta \odot \alpha) = \Lambda(\alpha) \odot \Lambda(\beta)$. Concerning vertical composition in the 2-category $\mathbf{V}\mathbb{D}$, if f, g, h in Definition 6.2 (iii), then $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ is the horizontal composition $\beta * \alpha$ in the 2-category $\mathbf{V}\mathbb{D}$, and $\Lambda(\beta * \alpha) = \Lambda(\beta) * \Lambda(\alpha)$.

Definition 6.5 (Compatibility with folding)**.** Let $\mathbb{C}$ and $\mathbb{D}$ be double categories with folding.

(i) A double functor $F\colon\ \mathbb{C} \to \mathbb{D}$ is *compatible with the foldings* if

$$F(\overline{j}) = \overline{F(j)} \qquad \text{and} \qquad F(\Lambda^{\mathbb{C}}(\alpha)) = \Lambda^{\mathbb{D}}(F(\alpha))$$

for all vertical morphisms j and squares α in $\mathbb{C}$.

(ii) Let $F, G\colon\ \mathbb{C} \to \mathbb{D}$ be double functors compatible with the foldings. A horizontal natural transformation $\theta\colon\ F \Rightarrow G$ is *compatible with the foldings* if for all vertical morphisms j in $\mathbb{C}$ the following equation holds.

$$\Lambda\left(\begin{array}{ccc} FA & \xrightarrow{\theta A} & GA \\ {\scriptstyle Fj}\downarrow & \theta j & \downarrow{\scriptstyle Gj} \\ FC & \xrightarrow[\theta C]{} & GC \end{array}\right) = \begin{array}{ccc} FA & \xrightarrow{[\theta A\ G\overline{j}]} & GC \\ \Vert & i^v_{[\theta A\ G\overline{j}]} & \Vert \\ FA & \xrightarrow[{[F\overline{j}\ \theta C]}]{} & GC \end{array} \qquad (16)$$

(iii) Let $F, G\colon\ \mathbb{C} \to \mathbb{D}$ be double functors compatible with the foldings. A vertical natural transformation $\sigma\colon\ F \Rightarrow G$ is *compatible with the foldings* if for all vertical morphisms j the following equation holds.

$$\Lambda\left(\begin{array}{ccc} FA & \xrightarrow{F\overline{j}} & FC \\ {\scriptstyle \sigma A}\downarrow & \sigma\overline{j} & \downarrow{\scriptstyle \sigma C} \\ GA & \xrightarrow[G\overline{j}]{} & GC \end{array}\right) = \begin{array}{ccc} FA & \xrightarrow{[F\overline{j}\ \overline{\sigma C}]} & GC \\ \Vert & i^v_{[F\overline{j}\ \overline{\sigma C}]} & \Vert \\ FA & \xrightarrow[{[\overline{\sigma A}\ G\overline{j}]}]{} & GC \end{array} \qquad (17)$$

Some double categories admit a cofolding rather than a folding, as the following variant of the quintets of Example 6.1 illustrates. For double categories of monads and endomorphisms (in the sense of [13] and Section 7 below), cofoldings are more relevant than foldings, since cofoldings are inherited from the underlying double category (cf. Proposition 7.2) whereas foldings are not.

Example 6.6 (Inverse quintets)**.** For $\mathbf{K}$ a 2-category, the double category of *inverse quintets* $\overline{\mathbb{Q}}\mathbf{K}$ is the double category in which the objects are the objects of $\mathbf{K}$, the horizontal 1-category is the underlying 1-category of $\mathbf{K}$, the vertical 1-category is the *opposite* of the underlying 1-category of $\mathbf{K}$, and the squares

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j^{\mathrm{op}}}\big\downarrow & \alpha & \big\downarrow{\scriptstyle k^{\mathrm{op}}} \\ C & \xrightarrow[g]{} & D \end{array}$$

are 2-cells of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\big\uparrow & \overset{\alpha}{\Searrow} & \big\uparrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

in $\mathbf{K}$. The double category $\overline{\mathbb{Q}}\mathbf{K}$ admits a *cofolding* in the following sense.

Definition 6.7. A *cofolding* is a double functor $\Lambda\colon \mathbb{D} \to \overline{\mathbb{Q}}\mathbf{H}\mathbb{D}$ which is the identity on the horizontal 2-category $\mathbf{H}\mathbb{D}$ of $\mathbb{D}$ and is fully faithful on squares. We proceed to spell out the details.

A *cofolding on a double category* $\mathbb{D}$ consists of the following.

(i) A 2-functor $(-)^*\colon (\mathbf{V}\mathbb{D})_0^{\mathrm{op}} \to \mathbf{H}\mathbb{D}$ which is the identity on objects. Here, the notation $(\mathbf{V}\mathbb{D})_0^{\mathrm{op}}$ denotes the opposite of the vertical 1-category of $\mathbb{D}$. In other words, to each vertical morphism $j\colon A \to C$, there is associated a horizontal morphism $j^*\colon C \to A$ in a functorial way. We call the 2-functor $j \mapsto j^*$ the *coholonomy*.

(ii) Bijections $\Lambda^{f,k}_{j,g}$ from squares in $\mathbb{D}$ with boundary

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \tag{18}$$

to squares in $\mathbb{D}$ with boundary

$$\begin{array}{ccc} C & \xrightarrow{[j^* \ f]} & B \\ \Big\| & & \Big\| \\ C & \xrightarrow[{[g \ k^*]}]{} & B. \end{array} \tag{19}$$

These bijections are required to satisfy the following axioms.

(i) Λ is the identity if j and k are vertical identity morphisms.

(ii) Λ preserves horizontal composition of squares, that is,

$$\Lambda\left(\begin{array}{ccccc} A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C \\ {\scriptstyle j}\big\downarrow & \alpha & \big\downarrow{\scriptstyle k} & \beta & \big\downarrow{\scriptstyle \ell} \\ D & \xrightarrow[g_1]{} & E & \xrightarrow[g_2]{} & F \end{array}\right) \quad = \quad \begin{array}{ccc} D & \xrightarrow{[j^* \ f_1 \ f_2]} & C \\ \Big\| & [\Lambda(\alpha) \ i^v_{f_2}] & \Big\| \\ D & \xrightarrow{[g_1 \ k^* \ f_2]} & C \\ \Big\| & [i^v_{g_1} \ \Lambda(\beta)] & \Big\| \\ D & \xrightarrow[{[g_1 \ g_2 \ \ell^*]}]{} & C. \end{array}$$

(iii) Λ preserves vertical composition of squares, that is,

$$\Lambda\left(\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j_1}\big\downarrow & \alpha & \big\downarrow{\scriptstyle k_1} \\ C & \xrightarrow{g} & D \\ {\scriptstyle j_2}\big\downarrow & \beta & \big\downarrow{\scriptstyle k_2} \\ E & \xrightarrow[h]{} & F, \end{array}\right) \quad = \quad \begin{array}{ccc} E & \xrightarrow{[j_2^*\ j_1^* f]} & B \\ \| & [i^v_{j_2^*}\ \Lambda(\alpha)] & \| \\ E & \xrightarrow{[j_2^*\ g\ k_1^*]} & B \\ \| & [\Lambda(\beta)\ i^v_{k_1^*}] & \| \\ E & \xrightarrow[{[h\ k_2^*\ k_1^*]}]{} & B. \end{array}$$

(iv) Λ preserves identity squares, that is,

$$\Lambda\left(\begin{array}{ccc} A & = & A \\ {\scriptstyle j}\big\downarrow & i^h_j & \big\downarrow{\scriptstyle j} \\ B & = & B \end{array}\right) \quad = \quad \begin{array}{ccc} B & \xrightarrow{j^*} & A \\ \| & i^v_{j^*} & \| \\ B & \xrightarrow[j^*]{} & A. \end{array}$$

Example 6.8. The double category $\mathbb{S}\mathrm{pan}$ admits a cofolding. The coholonomy is

$$\left(\begin{array}{c} A \\ {\scriptstyle j}\big\downarrow \\ C \end{array}\right) \longmapsto \left(C \stackrel{j}{\leftarrow} A \stackrel{1^h_A}{\rightarrow} A\right)$$

and the cofolding is

$$\left(\begin{array}{ccccc} A & \xleftarrow{f_0} & Y & \xrightarrow{f_1} & B \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle k} \\ C & \xleftarrow[g_0]{} & Z & \xrightarrow[g_1]{} & D \end{array}\right) \longmapsto \left(\begin{array}{ccccc} C & \xleftarrow{j\circ f_0} & Y & \xrightarrow{f_1} & B \\ \| & & \big\downarrow{\scriptstyle (\alpha, f_1)} & & \| \\ C & \xleftarrow[g_0\circ \mathrm{pr}_1]{} & Z\times_D B & \xrightarrow[\mathrm{pr}_2]{} & B \end{array}\right).$$

Definition 6.9 (Compatibility with cofolding)**.** Let $\mathbb{C}$ and $\mathbb{D}$ be double categories with cofolding.

(i) A double functor $F\colon \mathbb{C} \to \mathbb{D}$ is *compatible with the cofoldings* if

$$F(j^*) = F(j)^* \qquad \text{and} \qquad F(\Lambda^{\mathbb{C}}(\alpha)) = \Lambda^{\mathbb{D}}(F(\alpha))$$

for all vertical morphisms j and squares α in $\mathbb{C}$.

(ii) Let $F, G\colon \mathbb{C} \to \mathbb{D}$ be double functors compatible with the cofoldings. A horizontal natural transformation $\theta\colon F \Rightarrow G$ is *compatible with the cofoldings* if for all vertical morphisms j in $\mathbb{C}$ the following equation holds.

$$\Lambda\left(\begin{array}{ccc} FA & \xrightarrow{\theta A} & GA \\ {\scriptstyle Fj}\downarrow & \theta j & \downarrow{\scriptstyle Gj} \\ FC & \xrightarrow[\theta C]{} & GC \end{array}\right) = \begin{array}{ccc} FA & \xrightarrow{[Fj^*\ \theta A]} & GC \\ \| & i^v_{[Fj^*\ \theta A]} & \| \\ FA & \xrightarrow[{[\theta C\ Gj^*]}]{} & GC \end{array} \tag{20}$$

(iii) Let $F, G\colon \mathbb{C} \to \mathbb{D}$ be double functors compatible with the cofoldings. A vertical natural transformation $\sigma\colon F \Rightarrow G$ is *compatible with the cofoldings* if for all vertical morphisms $j\colon A \to C$ the following equation holds.

$$\Lambda\left(\begin{array}{ccc} FC & \xrightarrow{Fj^*} & FA \\ {\scriptstyle \sigma C}\downarrow & \sigma j^* & \downarrow{\scriptstyle \sigma A} \\ GC & \xrightarrow[Gj^*]{} & GA \end{array}\right) = \begin{array}{ccc} FC & \xrightarrow{[(\sigma C)^*\ Fj^*]} & GA \\ \| & i^v_{[(\sigma C)^*\ Fj^*]} & \| \\ FC & \xrightarrow[{[Gj^*\ (\sigma A)^*]}]{} & GA \end{array} \tag{21}$$

We now come to the main result of this section.

Proposition 6.10. *Let $\mathbb{A}$ and $\mathbb{X}$ be double categories with folding (respectively cofolding) and consider double functors F and G compatible with the foldings (respectively cofoldings).*

$$\mathbb{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{A} \tag{22}$$

Then F and G are horizontal double adjoints if and only if their horizontal 2-functors $\mathbf{H}F$ and $\mathbf{H}G$ are 2-adjoints.

Proof. If F and G are horizontal double adjoints, then $\mathbf{H}F$ and $\mathbf{H}G$ are 2-adjoints, since the 2-functor $\mathbf{H}\colon \mathbf{DblCat_h} \to$ **2-Cat** preserves adjoints, as does any 2-functor.

For the converse, suppose that F and G are compatible with the foldings and $\varphi_{X,A}\colon \mathbf{H}\mathbb{A}(FX,A) \to \mathbf{H}\mathbb{X}(X,GA)$ is a natural isomorphism of categories. We use the double adjunction characterization in Theorem 5.2 (v). For vertical morphisms j and k in $\mathbb{X}$ and $\mathbb{A}$ respectively, we define a bijection

$$\varphi_{j,k}\colon\ \mathbb{A}(Fj,k) \longrightarrow \mathbb{X}(j,Gk)$$

$$\varphi_{j,k}(\alpha) := \left(\Lambda^{f^\dagger,Gk}_{j,g^\dagger}\right)^{-1} \varphi_{sj,tk}\left(\Lambda^{f,k}_{Fj,g}(\alpha)\right).$$

Here $f^\dagger$ and $g^\dagger$ are the transposes of the horizontal morphisms f and g with respect to the underlying 1-adjunction. The naturality of $\varphi_{X,A}$ guarantees that the boundaries are correct.

The bijection $\varphi_{j,k}$ is compatible with vertical composition for the following reasons:

(i) $\varphi_{X,A}$ is compatible with the vertical composition of 2-cells in $\mathbf{H}\mathbb{X}$ and $\mathbf{H}\mathbb{A}$

(ii) the isomorphism $\varphi_{X,A}$ is natural in X and A, and

(iii) the foldings are compatible with vertical composition as in Definition 6.2 (iii).

The naturality of $\varphi_{j,k}$ in j and k similarly follows from (i) and (ii) above, and the compatibility of the foldings with horizontal composition in Definition 6.2 (ii).

These natural bijections $\varphi_{j,k}$ compatible with vertical composition are equivalent to a unit η and counit ε in a horizontal double adjunction by Theorem 5.2 (v), so we are finished.

The analogous proof works for the cofolding claim. □

Remark 6.11. In Proposition 6.10, note that the horizontal natural transformations η and ε which make F and G into horizontal double adjoints are not required to be compatible with the foldings, though if η and ε exist, they can be replaced by horizontal natural transformations compatible with the foldings. Note also that the holonomy (respectively coholonomy) is not required to be fully faithful.

Proposition 6.10 allows us to draw conclusions about horizontal double adjointness when both double functors F and G are already given, and are compatible with the foldings. It would be useful to have criteria for concluding the existence of a horizontal right double adjoint for a given double functor F (compatible with foldings) given the existence of a right 2-adjoint for $\mathbf{H}F$, without referencing G at the outset. One criterion that comes to mind is to require the holonomy to be fully faithful, but this happens only for double categories of direct quintets, as we now proceed to explain. A subtler criterion for a special case of interest will be derived in Proposition 7.3.

Example 6.12. If $\mathbf{K}$ is a 2-category, the canonical folding of the double category of direct quintets $\mathbb{Q}\mathbf{K}$ of Example 6.1 has fully faithful holonomy. Similarly, the canonical cofolding on the double category of inverse quintets $\overline{\mathbb{Q}}\mathbf{K}$ of Example 6.6 has fully faithful coholonomy.

Lemma 6.13. *If $\mathbb{D}$ is a double category with folding and fully faithful holonomy, then the folding $\Lambda\colon\ \mathbb{D} \to \mathbb{Q}\mathbf{H}\mathbb{D}$ is an isomorphism of double categories.*

Proof. Indeed, Λ is the identity on the horizontal 2-category, fully faithful on the vertical 1-category, and fully faithful on squares. □

Lemma 6.14. *If $\mathbb{D}$ and $\mathbb{C}$ are double categories with fully faithful holonomy, and F and G are double functors $\mathbb{D} \to \mathbb{C}$ compatible with the holonomies, then the holonomy and folding provide a 1-1 correspondence between 2-natural transformations $\mathbf{V}F \Rightarrow \mathbf{V}G$ and 2-natural transformations $\mathbf{H}F \Rightarrow \mathbf{H}G$.*

Proof. This is a consequence of the compatibility with horizontal composition of 2-cells in the vertical 2-category, cf. Remark 6.4. □

In fact, we can refine Lemma 6.13 to an equivalence of 2-categories. Let $\mathbf{DblCatFoldHol_h}$ denote the 2-category of small double categories with folding and fully faithful holonomy, double functors compatible with foldings, and horizontal natural transformations compatible with folding (see Definitions 6.2 and 6.5). Let $\mathbf{DblCatFoldHol_v}$ denote the 2-category of small double categories with folding and fully faithful holonomy, double functors compatible with foldings, and vertical natural transformations compatible with folding.

Proposition 6.15. *The forgetful 2-functors*

$$\mathbf{H}\colon \mathbf{DblCatFoldHol_h} \longrightarrow \mathbf{2Cat}$$
$$\mathbf{V}\colon \mathbf{DblCatFoldHol_v} \longrightarrow \mathbf{2Cat}$$

are equivalences of 2*-categories.*

Proof. Note first that $\mathbf{H}$ and $\mathbf{V}$ are essentially surjective by Examples 6.1 and 6.12. Suppose $F, G\colon \mathbb{C} \to \mathbb{D}$ are double functors compatible with foldings, and in particular compatible with the fully faithful holonomy, and suppose $\mathbf{H}F = \mathbf{H}G$. Then the double functors F and G agree on the horizontal 2-categories. If j is a vertical morphism in $\mathbb{C}$, then $\overline{F(j)} = F(\bar{j}) = G(\bar{j}) = \overline{G(j)}$, and $F(j) = G(j)$ by the faithfulness of the holonomy. The double functors F and G similarly agree on squares because of the folding bijections. Conversely, if a 2-functor is defined on horizontal 2-categories, then it can be extended to the double categories using the bijective holonomy and then the foldings. Thus $\mathbf{H}\colon \mathbf{DblCatFoldHol_h} \to \mathbf{2Cat}$ is bijective on the objects of hom-categories. Similarly, $\mathbf{V}$ is bijective on the objects of hom-categories (here the fullness of the holonomy plays a role).

Similar arguments hold for injectivity on horizontal respectively vertical natural transformations.

For fullness of $\mathbf{H}$ for 2-natural transformations, suppose $\theta\colon \mathbf{H}F \Rightarrow \mathbf{H}G$ is a 2-natural transformation. We extend θ to a horizontal natural transformation: for a vertical morphism j in $\mathbb{C}$, define θj by equation (16). We verify double naturality for θ, namely the equation $[\; F\alpha \;\; \theta k \;] = [\; \theta j \;\; G\alpha \;]$ for any square α in $\mathbb{C}$ with boundary as in equation (13). By the definition of θj and θk via equation (16), we have $\Lambda(\theta j) = i^v_{[\;\theta A \;\; G\bar{j}\;]}$ and $\Lambda(\theta k) = i^v_{[\;\theta B \;\; G\bar{k}\;]}$, so that the equation

$$\begin{bmatrix} i^v_{Ff} & \Lambda(\theta k) \\ \Lambda(F\alpha) & i^v_{\theta D} \end{bmatrix} = \begin{bmatrix} i^v_{\theta A} & \Lambda(G\alpha) \\ \Lambda(\theta j) & i^v_{Gg} \end{bmatrix} \tag{23}$$

holds by 2-naturality of θ. The double naturality then follows from an application of Λ^{-1} to (23) using axiom (ii) of Definition 6.2.

For fullness of $\mathbf{V}$ on 2-natural transformations, suppose $\sigma\colon \mathbf{V}F \Rightarrow \mathbf{V}G$ is a 2-natural transformation. We extend σ to a vertical natural transformation: for any horizontal morphism $\overline{j}$ in $\mathbb{C}$, define $\sigma\overline{j}$ by equation (17). Recall that the holonomy is fully faithful, so any horizontal morphism is of the form $\overline{j}$ for a unique vertical morphism j. The proof for surjectivity of $\mathbf{V}$ on 2-natural transformations proceeds like that of $\mathbf{H}$, using Lemma 6.14. □

Corollary 6.16. *Let $\mathbb{A}$ and $\mathbb{X}$ be double categories with folding and fully faithful holonomies. Let $F\colon \mathbb{X} \to \mathbb{A}$ be a double functor compatible with the foldings. Then the following are equivalent.*

(i) The double functor F admits a horizontal right double adjoint (not necessarily compatible with the foldings).

(ii) The 2-functor $\mathbf{H}F\colon \mathbf{H}\mathbb{X} \to \mathbf{H}\mathbb{A}$ admits a right 2-adjoint.

(iii) The double functor F admits a vertical right double adjoint (not necessarily compatible with the foldings).

(iv) The 2-functor $\mathbf{V}F\colon \mathbf{V}\mathbb{X} \to \mathbf{V}\mathbb{A}$ admits a right 2-adjoint.

Proof. By Proposition 6.15, the 2-functor $\mathbf{H}\colon \mathbf{DblCatFoldHol_h} \to \mathbf{2Cat}$ is 2-fully faithful, so F admits a horizontal right double adjoint compatible with the foldings if and only if $\mathbf{H}F$ admits a right 2-adjoint. But if F admits a horizontal right double adjoint G not necessarily compatible with the foldings, then $\mathbf{H}G$ is still a right 2-adjoint to $\mathbf{H}F$, and Proposition 6.15 applies to extend the 2-adjunction $\mathbf{H}F \dashv \mathbf{H}G$ to a horizontal double adjunction with horizontal left double adjoint F. Thus (i)⇔(ii) and similarly (iii)⇔(iv).

To complete the proof, we observe (ii)⇔(iv), because the fully faithful holonomy and folding provide a 1-1 correspondence between 2-natural transformations $\mathbf{V}F_1 \Rightarrow \mathbf{V}F_2$ and 2-natural transformations $\mathbf{H}F_1 \Rightarrow \mathbf{H}F_2$, by Lemma 6.14. □

For completeness, we also state the analogues of Lemma 6.13, Proposition 6.15 and Corollary 6.16 for double categories with cofoldings and fully faithful coholonomies.

Lemma 6.17. *If $\mathbb{D}$ is a double category with cofolding and fully faithful coholonomy, then the cofolding $\Lambda\colon\ \mathbb{D} \to \overline{\mathbb{Q}}\mathbf{H}\mathbb{D}$ is an isomorphism of double categories.*

With self-explanatory notation as in Proposition 6.15, we have:

Proposition 6.18. *The forgetful 2-functors*

$$\mathbf{H}\colon\ \mathbf{DblCatCofoldCohol_h} \longrightarrow \mathbf{2Cat}$$
$$\mathbf{V}\colon\ \mathbf{DblCatCofoldCohol_v}^{\mathrm{co}} \longrightarrow \mathbf{2Cat}$$

are equivalences of 2*-categories.*

The reversal of 2-cells by $\mathbf{V}$ (indicated with the superscript $^{\mathrm{co}}$) stems from the contravariant nature of the cofolding.

Proof. The entire proof is very similar to that of Proposition 6.15. The only small difference is in the fullness of $\mathbf{H}$ and $\mathbf{V}$ for 2-natural transformations. Suppose $\theta\colon\ \mathbf{H}F \Rightarrow \mathbf{H}G$ is a 2-natural transformation. We extend θ to a horizontal natural transformation: for a vertical morphism j in $\mathbb{C}$, define θj by equation (20). By the definition of θj and θk via equation (20), we have $\Lambda(\theta j) = i^v_{[\,Fj^*\ \ \theta A\,]}$ and $\Lambda(\theta k) = i^v_{[\,Fk^*\ \ \theta B\,]}$, so that the equation

$$\begin{bmatrix} \Lambda(F\alpha) & i^v_{\theta B} \\ i^v_{Fg} & \Lambda(\theta k) \end{bmatrix} = \begin{bmatrix} \Lambda(\theta j) & i^v_{Gf} \\ i^v_{\theta C} & \Lambda(G\alpha) \end{bmatrix} \tag{24}$$

holds by 2-naturality of θ. The double naturality equation $[\,F\alpha\ \ \theta k\,] = [\,\theta j\ \ G\alpha\,]$ for θ then follows from an application of Λ^{-1} to (24) using axiom (ii) of Definition 6.7. $\square$

The contravariant nature of the cofolding also affects the direction of the vertical adjunction in the following cofolding analog of Corollary 6.16:

Corollary 6.19. *Let $\mathbb{A}$ and $\mathbb{X}$ be double categories with cofolding and fully faithful coholonomies. Let $F\colon\ \mathbb{X} \to \mathbb{A}$ be a double functor compatible with the cofoldings. Then the following are equivalent.*

(i) The double functor F admits a horizontal right double adjoint (not necessarily compatible with the cofoldings).

(ii) The 2-functor $\mathbf{H}F\colon\ \mathbf{H}\mathbb{X} \to \mathbf{H}\mathbb{A}$ admits a right 2-adjoint.

(iii) The double functor F admits a vertical left double adjoint (not necessarily compatible with the cofoldings).

(iv) The 2-functor $\mathbf{V}F\colon\ \mathbf{V}\mathbb{X} \to \mathbf{V}\mathbb{A}$ admits a left 2-adjoint.

7. Endomorphisms and Monads in a Double Category

The notions of endomorphism and monad in a double category were introduced in [13], the main theorem of which gave sufficient conditions for the existence of free monads in a double category. One of the goals of this paper is to simultaneously remove several hypotheses from our main theorem in [13] and strengthen its conclusion to obtain Theorem 9.6 of this paper, which says that if a double category $\mathbb{D}$ with cofolding admits the construction of free monads in its horizontal 2-category, then $\mathbb{D}$ admits the construction of free monads as a double category. Towards that goal, we prove in this section that a cofolding on $\mathbb{D}$ induces a cofolding on the double categories $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$ of endomorphisms and monads in $\mathbb{D}$, see [13, Definitions 2.3 and 2.4]. Another goal of this paper is Theorem 10.3, the characterization of the existence of Eilenberg–Moore objects in a double category in terms of representability of certain parameterized presheaves. For that we also need an understanding of the double category $\mathbb{M}\mathrm{nd}(\mathbb{D})$.

Following [13], by endomorphism and monad in a double category we mean horizontal endomorphism and horizontal monad. Hence an *endomorphism* in a double category is a pair (X, P) where X is an object and $P : X \to X$ is a horizontal morphism. A *monad structure* on (X, P) consists of squares

$$\begin{array}{ccccc} X & \xrightarrow{P} & X & \xrightarrow{P} & X \\ \Vert & & \mu_P & & \Vert \\ X & & \xrightarrow[P]{} & & X \end{array} \qquad\qquad \begin{array}{ccc} X & = & X \\ \Vert & \eta_P & \Vert \\ X & \xrightarrow[P]{} & X \end{array}$$

satisfying obvious laws of associativity and unitality. In other words, endomorphisms and monads are the same as endomorphisms and monads in the horizontal 2-category.

A *horizontal map* between endomorphisms (X, P) and (Y, Q) is a horizontal morphism $F\colon\ X \to Y$ together with a square

$$\begin{array}{ccccc} X & \xrightarrow{F} & Y & \xrightarrow{Q} & Y \\ \| & & \phi & & \| \\ X & \xrightarrow[P]{} & X & \xrightarrow[F]{} & Y. \end{array} \tag{25}$$

A *vertical map* $(u, \bar{u})\colon\ (X, P) \to (X', P')$ consists of a vertical morphism $u\colon\ X \to X'$ and a square

$$\begin{array}{ccc} X & \xrightarrow{P} & X \\ {\scriptstyle u}\downarrow & \bar{u} & \downarrow{\scriptstyle u} \\ X' & \xrightarrow[P']{} & X'. \end{array}$$

The definitions of horizontal and vertical maps between monads are similar, but the squares ϕ and $\bar{u}$ are then subject to some evident compatibility conditions with respect to the monad structures. There are also notions of endomorphism square and monad square (which we shall not recall here) making $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$ into double categories, cf. [13]. See Examples 8.2 and 8.3.

The direction of the square ϕ in the definition of horizontal endomorphism map and horizontal monad map is chosen so as to agree with the convention of Street [26] for endomorphism maps and monad maps in the horizontal 2-category, which in turn is motivated among other things by the desire to pullback algebras for monads. This choice has some consequences for some other choices in this paper, and we pause to explain this. For brevity we talk only about monads, the case of endomorphisms being analogous.

The other natural choice for horizontal monad maps $(X, P) \to (Y, Q)$ is with squares of the form

$$\begin{array}{ccccc} X & \xrightarrow{P} & X & \xrightarrow{F} & Y \\ \| & & \phi & & \| \\ X & \xrightarrow[F]{} & Y & \xrightarrow[Q]{} & Y, \end{array}$$

which for fun we call Avenue monad maps in the following discussion. We temporarily denote by $\mathbb{M}\mathrm{nd}^{\mathrm{st}}(\mathbb{D}) = \mathbb{M}\mathrm{nd}(\mathbb{D})$ the double category whose horizontal morphisms are Street monad maps (the convention used elsewhere in this paper), and by $\mathbb{M}\mathrm{nd}^{\mathrm{av}}(\mathbb{D})$ the double category with Avenue monad maps. The two double categories have the same vertical morphisms.

Both notions of monad map refer only to the horizontal 2-category and make sense already for 2-categories, so for a 2-category $\mathbf{K}$ we have two different 2-categories of monads, $\mathrm{Mnd}^{\mathrm{st}}(\mathbf{K})$ and $\mathrm{Mnd}^{\mathrm{av}}(\mathbf{K})$. The two notions of monad maps for 2-categories can be combined into a single double category that has Street monad maps as horizontal morphisms and Avenue monad maps as vertical morphisms; there is a unique natural choice of what square should be taken to be to make this into a double category. This double category is naturally isomorphic to $\mathbb{M}\mathrm{nd}^{\mathrm{st}}(\mathbb{Q}\mathbf{K})$, which is different from $\mathbb{Q}(\mathrm{Mnd}^{\mathrm{st}}(\mathbf{K}))$: both double categories have $\mathrm{Mnd}^{\mathrm{st}}(\mathbf{K})$ as horizontal 2-category, but while the vertical 2-category of $\mathbb{M}\mathrm{nd}^{\mathrm{st}}(\mathbb{Q}\mathbf{K})$ is $\mathrm{Mnd}^{\mathrm{av}}(\mathbf{K})$ with 2-cells reversed, the vertical 2-category of $\mathbb{Q}(\mathrm{Mnd}^{\mathrm{st}}(\mathbf{K}))$ is $\mathrm{Mnd}^{\mathrm{st}}(\mathbf{K})$ with the 2-cells reversed. In contrast we have the following result, whose proof is a straightforward but tedious verification.

Lemma 7.1. *For any* 2*-category* $\mathbf{K}$*, we have natural identifications*

$$\mathbb{E}\mathrm{nd}^{\mathrm{st}}(\overline{\mathbb{Q}}(\mathbf{K})) = \overline{\mathbb{Q}}(\mathrm{End}^{\mathrm{st}}(\mathbf{K})) \qquad \mathbb{M}\mathrm{nd}^{\mathrm{st}}(\overline{\mathbb{Q}}(\mathbf{K})) = \overline{\mathbb{Q}}(\mathrm{Mnd}^{\mathrm{st}}(\mathbf{K}))$$
$$\mathbb{E}\mathrm{nd}^{\mathrm{av}}(\mathbb{Q}(\mathbf{K})) = \mathbb{Q}(\mathrm{End}^{\mathrm{av}}(\mathbf{K})) \qquad \mathbb{M}\mathrm{nd}^{\mathrm{av}}(\mathbb{Q}(\mathbf{K})) = \mathbb{Q}(\mathrm{Mnd}^{\mathrm{av}}(\mathbf{K})).$$

The fact that Street monad maps are more compatible with the inverse quintet construction $\overline{\mathbb{Q}}$ of Example 6.6 than with the direct quintet construction $\mathbb{Q}$ (Example 6.1) explains to some extent why in the following it is cofolding rather than folding that goes well with monads. With the Avenue convention on monad maps, the following results would have concerned folding instead of cofolding.

The following is the main point of this section: a cofolding on a double category $\mathbb{D}$ induces a cofolding on $\mathbb{M}\mathrm{nd}(\mathbb{D})$ and $\mathbb{E}\mathrm{nd}(\mathbb{D})$.

Proposition 7.2. *If* $(\mathbb{D}, \Lambda^{\mathbb{D}})$ *is a double category with cofolding, then the double categories* $\mathbb{M}\mathrm{nd}(\mathbb{D})$ *and* $\mathbb{E}\mathrm{nd}(\mathbb{D})$ *inherit cofoldings from* $\mathbb{D}$*, and the forgetful double functor* $U\colon \mathbb{M}\mathrm{nd}(\mathbb{D}) \to \mathbb{E}\mathrm{nd}(\mathbb{D})$ *preserves them.*

Proof. We first construct the cofolding on $\mathbb{E}\mathrm{nd}(\mathbb{M})$: if $(u, \bar{u})\colon\ (X, P) \to (X', P')$ is a vertical endomorphism map, then the corresponding horizontal endomorphism map $(u, \bar{u})^*$ under the coholonomy is

$$(u^*, \Lambda^{\mathbb{D}}(\bar{u}))\colon\ (X', P') \to (X, P),$$

if α is an endomorphism square, then the corresponding endomorphism 2-cell is the $\mathbb{D}$-cofolding of α, namely $\Lambda^{\mathbb{D}}(\alpha)$. It is straightforward to check, using the functoriality of the coholonomy on $\mathbb{D}$ and the compatibility of $\Lambda^{\mathbb{D}}$ with horizontal and vertical composition of squares, that these assignments constitute a cofolding on $\mathbb{E}\mathrm{nd}(\mathbb{D})$.

Next we verify that the same construction of the cofolding works for monads: if (X, P) and (X', P') are monads, and $(u, \bar{u})$ is vertical monad map, then $(u, \bar{u})^* = (u^*, \Lambda^{\mathbb{D}}(\bar{u}))$ is a horizontal monad map, and if α is a monad square, then $\Lambda^{\mathbb{D}}(\alpha)$ is a monad 2-cell. This follows readily from the compatibility of $\Lambda^{\mathbb{D}}$ with horizontal and vertical composition of squares. Since the two cofoldings are given by the same construction, it is clear that the forgetful functor preserves them. □

In Proposition 7.2, note that if $\mathbb{D}$ has fully faithful coholonomy, then the induced coholonomies on $\mathbb{M}\mathrm{nd}(\mathbb{D})$ and $\mathbb{E}\mathrm{nd}(\mathbb{D})$ are again fully faithful. This follows from Lemma 6.17 and Lemma 7.1. We have seen in Corollary 6.19 that when the coholonomy is fully faithful, all questions about adjunction can be settled in the horizontal 2-category, but we noted also that this requirement is a very restrictive condition. The following technical result can be interpreted as saying that in the situation of the preceding proposition, although $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$ do not often have fully faithful coholonomies, they do have some fully faithfulness relative to $\mathbb{D}$: for a *fixed* vertical morphism u in $\mathbb{D}$, we do get certain bijections. This result, which generalizes [13, Lemma 3.4], will play an important role in the proofs of Proposition 9.5 and Theorem 9.6.

Proposition 7.3. *In the situation of Proposition 7.2, if* $u\colon\ X \to X'$ *is a fixed vertical morphism in* $\mathbb{D}$*, then*

$$(u, \bar{u}) \mapsto (u^*, \Lambda^{\mathbb{D}}(\bar{u}))$$

is a bijection between vertical endomorphism maps $(X, P) \to (X', P')$ *with underlying vertical morphism* u *and horizontal endomorphism maps*

$(X', P') \to (X, P)$ with underlying horizontal morphism u^*. If (X, P) and (X', P') are monads, we have a similar bijection between vertical monad maps with underlying morphism u and horizontal monad maps with underlying morphism u^*.

Proof. Vertical endomorphism maps over u from (X, P) to (X', P') are squares

$$\begin{array}{ccc} X & \xrightarrow{P} & X \\ {\scriptstyle u}\downarrow & \bar{u} & \downarrow{\scriptstyle u} \\ X' & \xrightarrow[P']{} & X', \end{array}$$

which under $\Lambda^{\mathbb{D}}$ correspond to squares

$$\begin{array}{ccccc} X' & \xrightarrow{u^*} & X & \xrightarrow{P} & X \\ \| & & \Lambda^{\mathbb{D}}(\bar{u}) & & \| \\ X' & \xrightarrow[P']{} & X' & \xrightarrow[u^*]{} & X. \end{array}$$

which are precisely the horizontal endomorphism maps over u^* from (X', P') to (X, P). The assertion about monad maps is similar. $\square$

8. Example: Endomorphisms and Monads in $\mathbb{S}$pan

We consider the normal, horizontally weak double category $\mathbb{S}$pan of spans in **Set** from Example 2.1 in order to exemplify the notions of endomorphism and monad in a double category, to illustrate the local description of double adjunctions in Theorem 5.4 (a slightly weak version of Theorem 5.2 (v)), and to motivate Theorem 9.6 below. We establish by hand the following result, which is a special case of [13, Proposition 3.8].

Proposition 8.1. *The forgetful double functor $G\colon\ \mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan}) \to \mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ has a vertical double left adjoint F,*

$$\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan}) \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan}). \quad \bot \tag{26}$$

Note that although $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ and $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ are horizontally weak double categories, the double functors F and G strictly preserve all compositions and identities. The 1-adjunction

$$\mathbf{DirGraph} \underset{Forget}{\overset{Free}{\rightleftarrows}} \mathbf{Cat} \quad (\bot)$$

is the *vertical* 1-category part of (26).

We next spell out the double categories $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ and $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$.

Example 8.2 (Endomorphisms in $\mathbb{S}$pan)**.** Objects and vertical morphisms of $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ are directed graphs $G_0 \leftarrow G_1 \rightarrow G_0$ and morphisms of directed graphs. A horizontal morphism $(U, \phi)\colon\ G_* \rightarrow G'_*$ in $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ is a span $U\colon\ G_0 \leftarrow U_1 \rightarrow G'_0$ equipped with a chosen (not necessarily vertically invertible) square in $\mathbb{S}$pan as below.[1]

$$\begin{array}{ccccccccc}
G_0 & \longleftarrow & U_1 & \longrightarrow & G'_0 & \longleftarrow & G'_1 & \longrightarrow & G'_0 \\
\| & & & & \phi & & & & \| \\
G_0 & \longleftarrow & G_1 & \longrightarrow & G_0 & \longleftarrow & U_1 & \longrightarrow & G'_0.
\end{array} \tag{27}$$

Horizontal composition of horizontal morphisms is by pullback, with the usual choice made for identities as in Example 2.1 (ϕ is then the identity on G_1). The associated ϕ-part of the composite is the vertical composite of the following squares.

$$\begin{array}{ccccccccccccc}
G_0 & \longleftarrow & U_1 & \longrightarrow & G'_0 & \longleftarrow & V_1 & \longrightarrow & G''_0 & \longleftarrow & G''_1 & \longrightarrow & G''_0 \\
\| & & 1_U & & \| & & & & \psi & & & & \| \\
G_0 & \longleftarrow & U_1 & \longrightarrow & G'_0 & \longleftarrow & G'_1 & \longrightarrow & G'_0 & \longleftarrow & V_1 & \longrightarrow & G''_0 \\
\| & & & & \phi & & & & \| & & 1_V & & \| \\
G_0 & \longleftarrow & G_1 & \longrightarrow & G_0 & \longleftarrow & U_1 & \longrightarrow & G'_0 & \longleftarrow & V_1 & \longrightarrow & G''_0.
\end{array} \tag{28}$$

[1]If $U\colon\ G_0 \leftarrow U_1 \rightarrow G'_0$ is not an identity span, then a square as in (27) is a (not necessarily bijective) function $\phi\colon\ U_1 \times_{G'_0} G'_1 \rightarrow G_1 \times_{G_0} U_1$ making the relevant squares commute. If U is an identity span, then a square as in (27) is a (not necessarily bijective) function $\phi\colon\ G'_1 \rightarrow G_1$. Recall the choice of pullback described in Example 2.1.

A square in $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$

$$\begin{array}{ccc} G_* & \xrightarrow{U} & G'_* \\ {\scriptstyle J_*}\downarrow & \alpha & \downarrow{\scriptstyle J'_*} \\ H_* & \xrightarrow[V]{} & H'_* \end{array} \tag{29}$$

is a square in $\mathbb{S}\mathrm{pan}$

$$\begin{array}{ccccc} G_0 & \longleftarrow & U_1 & \longrightarrow & G'_0 \\ \downarrow & & {\scriptstyle\alpha}\downarrow & & \downarrow \\ H_0 & \longleftarrow & V_1 & \longrightarrow & H'_0 \end{array}$$

such that the cube with ϕ on top and ϕ' on bottom commutes. Horizontal and vertical composition of squares in $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ are the horizontal and vertical compositions of the underlying squares in $\mathbb{S}\mathrm{pan}$, for example, horizontal composition is defined via pullback.

Example 8.3 (Monads in $\mathbb{S}\mathrm{pan}$)**.** Objects and vertical morphisms of $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ are categories and functors. The horizontal morphisms of $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ are the same as Street's morphisms of monads in a 2-category [26]. Namely, a horizontal monad morphism $U\colon\ C_* \to D_*$ is a span $C_0 \leftarrow U_1 \to D_0$ and a square in $\mathbb{S}\mathrm{pan}$

$$\begin{array}{ccccccccc} C_0 & \longleftarrow & U_1 & \longrightarrow & D_0 & \longleftarrow & D_1 & \longrightarrow & D_0 \\ \| & & & & \phi & & & & \| \\ C_0 & \longleftarrow & C_1 & \longrightarrow & C_0 & \longleftarrow & U_1 & \longrightarrow & D_0 \end{array}$$

such that

$$\begin{bmatrix} [\ 1^v_U \ \ \eta^D\ ] \\ \phi \end{bmatrix} = [\ \eta^C \ \ 1^v_U\ ]$$

and

$$\begin{bmatrix} & \phi & & 1^v_D \\ 1^v_C & & \phi & \\ & \mu^C & & 1^v_U \end{bmatrix} = \begin{bmatrix} [\ 1^v_U \ \ \mu^D\ ] \\ \phi \end{bmatrix}.$$

In other words, we have a function $\phi\colon\ U_1 \times_{D_0} D_1 \to C_1 \times_{C_0} U_1$ such that

$$\phi(u, 1_{tu}) = (1_{su}, u) \tag{30}$$

for all $u \in U_1$ and

$$\phi^C(\phi^U(u,d),d') \circ \phi^C(u,d) = \phi^C(u,d' \circ d) \tag{31}$$

$$\phi^U(\phi^U(u,d),d') = \phi^U(u,d' \circ d). \tag{32}$$

Note that if D and K have just one object, then equation (32) and the unit equation (30) essentially say ϕ^U defines a left monoid action of D_1 on U_1. Horizontal composition of horizontal morphisms in $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ is by pullback, and the ϕ-parts compose as in equation (28). The horizontal identities are as in span, with ϕ the identity on C_1.

Finally, a square

$$\begin{array}{ccc} A_* & \xrightarrow{(U,\phi)} & B_* \\ {\scriptstyle (J_1,J_0)}\downarrow & \alpha & \downarrow{\scriptstyle (K_1,K_0)} \\ C_* & \xrightarrow[(V,\psi)]{} & D_* \end{array} \tag{33}$$

in $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ is a square α in $\mathbb{S}\mathrm{pan}$ such that

$$\begin{bmatrix} \phi \\ [\ J_1\ \ \alpha\ ] \end{bmatrix} = \begin{bmatrix} [\ \alpha\ \ K_1\ ] \\ \psi \end{bmatrix},$$

in other words

$$(J_1(\phi^A(u,b)),\ \alpha(\phi^U(u,b))) = (\psi^C(\alpha(u),K(b)),\ \psi^V(\alpha(u),K(b)).$$

Remark 8.4. One way to think of a horizontal endomorphism map ϕ is as an assignment that converts a path

$$\xrightarrow{\in U_1}\ \xrightarrow{\in D_1}$$

to a path

$$\xrightarrow{\in C_1}\ \xrightarrow{\in U_1}$$

in a way compatible with unit and composition.

Now that we understand the double categories involved, we can give the proof of Proposition 8.1. Since the double adjunction (26) is *vertical* rather than horizontal, we use the transpose of the characterizations in Theorem 5.4. We cannot simply transpose the double categories and double functors in (26) in order to apply the non-transposed Theorem 5.4, because our notions of monads in a double category and their various morphisms prefer the horizontal direction as distinguished.

Proof. Proof of Proposition 8.1. We first describe F, then check the conditions of (transposed) Theorem 5.4. On objects and vertical morphisms (that is, on directed graphs and their morphisms), F is the free category functor. On a horizontal morphism $(U, \phi)\colon\ G_* \to G'_*$ in $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ as in (27), we have $F(U)_1 := U_1$. The function ϕ extends to $F(\phi)$ by Remark 8.4 and the fact that morphisms in the free category on a (non-reflexive) graph are paths of edges. On $F(U)_1 \times_{G'_0} G'_1$, the function $F(\phi)$ is simply ϕ. On $F(U)_1 \times_{G'_0} F(G'_*)_1$, the function $F(\phi)$ is defined by moving the element of U_1 across the path, one edge at a time using ϕ. For example,

$$\begin{array}{ccc} \xrightarrow{\;u\;} & \xrightarrow{\;g\;} & \xrightarrow{\;h\;} \\[2ex] \xrightarrow{\phi^G(u,g)} & \xrightarrow{\phi^U(u,g)} & \xrightarrow{\;h\;} \\[2ex] \xrightarrow{\phi^G(u,g)} & \xrightarrow{\phi^G(\phi^U(u,g),h)} & \xrightarrow{\phi^U(\phi^U(u,g),h)} \end{array} \tag{34}$$

which is the same as below.

$$\begin{array}{cc} \xrightarrow{\;u\;} & \xrightarrow{\;h\circ g\;} \\[2ex] \xrightarrow{\phi^G(u,h\circ g)} & \xrightarrow{\phi^U(u,h\circ g)} \end{array} \tag{35}$$

The equality of the composites in the last lines of the respective displays (34) and (35) shows that $F(\phi)$ satisfies the composition rules in (31) and (32) by definition. Similarly, (30) holds by definition and the fact that our directed graphs are non-reflexive. Concerning the definition of F on squares, the double functor F takes a square α in $\mathbb{E}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ as in (29) to the square $F\alpha$ in $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$ as in (33) which has the same middle function $U_1 \to V_1$ as α, but the left and right vertical morphisms are the unique functors on the free categories that extend the directed graph morphisms on the left and right of α. For this reason, F clearly preserves vertical composition of vertical morphisms and squares. It also preserves horizontal composition because the horizontal composition in both double categories is defined via pullback. Also the ϕ part of $F(V \circ U)$ is the appropriate composite of the ϕ-parts of U and V by an inductive verification using the "switching" point of view on ϕ as just discussed. Thus F is a *strict* double functor.

We use the transpose of the local description of double adjunctions in Theorem 5.4 to prove that $F \dashv G$ is a vertical double adjunction. To simplify our work with the transposed characterization, we introduce the notations

$$\mathbb{M}\text{nd}(\mathbb{S}\text{pan})\begin{pmatrix} FU \\ V \end{pmatrix} \quad \text{and} \quad \mathbb{E}\text{nd}(\mathbb{S}\text{pan})\begin{pmatrix} U \\ GV \end{pmatrix}$$

to mean the set of squares in $\mathbb{M}\text{nd}(\mathbb{S}\text{pan})$ with vertical domain FU and vertical codomain V, and the set of squares in $\mathbb{E}\text{nd}(\mathbb{S}\text{pan})$ with vertical domain U and vertical codomain GV. This notation is the transpose of the notation in equation (4). We define a bijection

$$\varphi^U_V\colon\ \mathbb{M}\text{nd}(\mathbb{S}\text{pan})\begin{pmatrix} FU \\ V \end{pmatrix} \longrightarrow \mathbb{E}\text{nd}(\mathbb{S}\text{pan})\begin{pmatrix} U \\ GV \end{pmatrix} \tag{36}$$

$$\begin{array}{ccc} FA_* & \xrightarrow{F(U,\phi)} & FB_* \\ {\scriptstyle J}\downarrow & \alpha & \downarrow{\scriptstyle K} \\ C_* & \xrightarrow[(V,\psi)]{} & D_* \end{array} \quad \longmapsto \quad \begin{array}{ccc} A_* & \xrightarrow{(U,\phi)} & B_* \\ {\scriptstyle J_{\text{res}}}\downarrow & \alpha_{\text{res}} & \downarrow{\scriptstyle K_{\text{res}}} \\ GC_* & \xrightarrow[G(V,\psi)]{} & GD_* \end{array}$$

that is compatible with horizontal composition. The subscript res means restriction: the maps J_{res} and K_{res} are the restrictions of the functors J and K to the directed graphs A_* and B_*, while α_{res} has the same exact middle function $U_1 \to V_1$ as α does. The square α_{res} is restricted only in the sense that its horizontal domain and codomain are restricted. Since the middle function of α is the same as that of α_{res}, the function φ^U_V is manifestly injective. If α' is a square in $\mathbb{E}\text{nd}(\mathbb{S}\text{pan})\begin{pmatrix} U \\ GV \end{pmatrix}$, then we use the bijection $J \leftrightarrow J_{\text{res}}$ to find the horizontal domain and codomain of $(\varphi^U_V)^{-1}(\alpha')$, and define the middle function of $(\varphi^U_V)^{-1}(\alpha')$ to be that of α'. This proves the surjectivity of φ^U_V.

To see that $\varphi([\ \alpha\ \ \beta\ ]) = [\ \varphi(\alpha)\ \ \varphi(\beta)\ ]$, we only need to observe that $(\alpha \times_{K_0} \beta)_{\text{res}}$ is the same as $\alpha_{\text{res}} \times_{(K_{\text{res}})_0} \beta_{\text{res}}$ because the diagrams, from which we are forming the pullbacks, are exactly the same. Namely,

$$\begin{array}{ccccccccc}
(FA_*)_0 & \longleftarrow & F(U)_1 & \longrightarrow & (FB_*)_0 & \longleftarrow & F(W)_1 & \longrightarrow & (FH_*)_1 \\
{\scriptstyle J_0}\big\downarrow & & {\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle K_0} & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle L_0} \\
C_0 & \longleftarrow & V_1 & \longrightarrow & D_0 & \longleftarrow & X_1 & \longrightarrow & D_0
\end{array}$$

is exactly the same as

$$\begin{array}{ccccccccc}
A_0 & \longleftarrow & U_1 & \longrightarrow & B_0 & \longleftarrow & W_1 & \longrightarrow & H_1 \\
{\scriptstyle (J_{\text{res}})_0}\big\downarrow & & {\scriptstyle \alpha_{\text{res}}}\big\downarrow & & \big\downarrow{\scriptstyle (K_{\text{res}})_0} & & \big\downarrow{\scriptstyle \beta_{\text{res}}} & & \big\downarrow{\scriptstyle (L_{\text{res}})_0} \\
(GC_*)_0 & \longleftarrow & (GV)_1 & \longrightarrow & (GD_*)_0 & \longleftarrow & (GX)_1 & \longrightarrow & (GD_*)_0.
\end{array}$$

It only remains to check the naturality of φ^U_V in U and V, but that is similar to the naturality of the ordinary free category functor-forgetful functor adjunction, the only difference is that here we use *vertical* pre- and post-composition of *squares*.

In summary, the bijection ϕ^U_V in (36) is compatible with horizontal composition and natural in the horizontal morphisms U and V, so F is vertical double left adjoint to G by the transpose of Theorem 5.4. $\square$

In the next section we analyze the free-monad adjunction in a more general setting. In Section 10 we study another important example of double adjunction, namely an Eilenberg–Moore type adjunction.

9. Free Monads in Double Categories with Cofolding

In this section we remove several hypotheses from our main theorem in [13] and strengthen its conclusion to obtain Theorem 9.6, which says that if a double category $\mathbb{D}$ with cofolding admits the construction of free monads in its horizontal 2-category, then $\mathbb{D}$ admits the construction of free monads as a double category. Since the free–forgetful double adjunction is a *vertical* adjunction, it is remarkable that it can be inferred from the free–forgetful adjunction in the *horizontal* 2-category. We first recall free monads on endomorphisms in a 2-category in Definition 9.1, which is due to Staton [25, Theorem 6.1.5] in the case $\mathbf{K} = \mathbf{Cat}$, and is treated in general in our previous paper [13, Theorem 1.1].

Definition 9.1. Let $\mathbf{K}$ be a 2-category. We say $\mathbf{K}$ *admits the construction of free monads* if either of the two following equivalent conditions hold.

(i) For every endomorphism (Y, Q) there exists a monad (Y, Q^{free}) and a 2-cell $\iota\colon\ Q \Rightarrow Q^{\text{free}}$ in $\mathbf{K}$ such that the endomorphism map

$$(1_Y, \iota_Q)\colon\ (Y, Q^{\text{free}}) \longrightarrow (Y, Q)$$

is universal in the sense that for every monad (X, P), post-composition with $(1_Y, \iota_Q)$ induces an isomorphism of categories

$$\mathrm{Mnd}_{\mathbf{K}}((X, P), (Y, Q^{\text{free}})) \xrightarrow{(1_Y, \iota_Q)\circ U(-)} \mathrm{End}_{\mathbf{K}}(U(X, P), (Y, Q)),$$

where $U\colon\ \mathrm{Mnd}(\mathbf{K}) \to \mathrm{End}(\mathbf{K})$ is the forgetful 2-functor.

(ii) The forgetful functor $U\colon\ \mathrm{Mnd}(\mathbf{K}) \to \mathrm{End}(\mathbf{K})$ admits a *right* 2-adjoint $R\colon\ \mathrm{End}(\mathbf{K}) \to \mathrm{Mnd}(\mathbf{K})$ with a counit ε such that for each endomorphism (Y, Q), the underlying morphism in $\mathbf{K}$ of the counit component $\varepsilon_{(Y,Q)}\colon\ UR(Y, Q) \to (Y, Q)$ is 1_Y.

Remark 9.2. The reason Definition 9.1 requires a *right* adjoint to the forgetful functor (as opposed to an expected *left* adjoint) is the choice of the direction of 2-cell in the definition of endomorphism map and monad map, as we now explain. Briefly, this right adjoint restricts to a left adjoint when we consider monads and endomorphisms on a fixed object Y. In detail, consider a fixed object Y of the 2-category $\mathbf{K}$. The *category of endomorphisms on* Y, denoted $\mathrm{End}(Y)$, has objects endomorphisms on Y. The morphisms in $\mathrm{End}(Y)$ are endomorphism maps with underlying morphism the identity on Y, that is, endomorphism maps of the form $(1_Y, \phi)\colon\ (Y, Q_1) \to (Y, Q_2)$. We follow the convention of Street [26] for the 2-cell ϕ, namely $\phi\colon\ Q_2 1_Y \Rightarrow 1_Y Q_1$. There are no compatibility requirements on ϕ. The *category of monads on* Y, denoted $\mathrm{Mnd}(Y)$, has objects monads on Y. The morphisms in $\mathrm{Mnd}(Y)$ are monad maps with underlying morphism the identity on Y, that is, morphisms are monad maps of the form $(1_Y, \psi)\colon\ (Y, M_1) \to (Y, M_2)$. Again, we follow Street's convention in [26] for the 2-cell ψ, namely ψ :

$M_2 1_Y \Rightarrow 1_Y M_1$. The 2-cell ψ is required to be compatible with the unit and multiplication of the monads M_1 and M_2.

The variance in Definition 9.1 restricts to the expected one for monads on the fixed object Y, that is, the 2-category $\mathbf{K}$ *is said to admit the construction of free monads on the object* Y if the forgetful functor $U_Y \colon \mathrm{Mnd}(Y) \to \mathrm{End}(Y)$ admits a *left* adjoint. If $\mathbf{K}$ admits the construction of free monads in the sense of Definition 9.1, then $\mathbf{K}$ admits the construction of free monads on each object Y.

Remark 9.3. In Definition 9.1 (i), the isomorphism of categories commutes with the evident forgetful functors

$$\begin{array}{ccc} \mathrm{Mnd}(\mathbf{K})((X,P),(Y,Q^{\text{free}})) & \xrightarrow{\cong} & \mathrm{End}(\mathbf{K})(U(X,P),(Y,Q)) \\ & \searrow \quad \swarrow & \\ & \mathbf{K}(X,Y), & \end{array}$$

since the underlying morphisms and 2-cells in $\mathbf{K}$ are composed with (whiskered with) 1_Y.

The following definition is slightly different from [13, Definition 2.8] in that it insists on the vertical triviality of the unit.

Definition 9.4. A double category $\mathbb{D}$ is said to *admit the construction of free monads* if the forgetful double functor $U \colon \mathbb{M}\mathrm{nd}(\mathbb{D}) \to \mathbb{E}\mathrm{nd}(\mathbb{D})$ admits a vertical *left* double adjoint R with a unit η such that the underlying vertical morphism in $\mathbb{D}$ of each unit component $\eta_{(Y,Q)} \colon (Y,Q) \to UR(Y,Q)$ is 1^v_Y.

We shall shortly prove that if $\mathbb{D}$ has a cofolding, then the existence of free monads in $\mathbf{H}\mathbb{D}$ implies the existence of free monads in $\mathbb{D}$. This amounts to extending an adjunction from the horizontal 2-categories to a *vertical* double adjunction. We first extend the 2-adjunction of horizontal 2-categories to a horizontal double adjunction. For both results, observe that the double-categorical notions of endomorphism, monad, and the forgetful double functor $U \colon \mathbb{M}\mathrm{nd}(\mathbb{D}) \to \mathbb{E}\mathrm{nd}(\mathbb{D})$ are essentially notions of the horizontal 2-category. More precisely we can identify $\mathbf{H}U \colon \mathbf{H}\mathbb{M}\mathrm{nd}(\mathbb{D}) \to \mathbf{H}\mathbb{E}\mathrm{nd}(\mathbb{D})$ with the forgetful 2-functor $\mathrm{Mnd}(\mathbf{H}\mathbb{D}) \to \mathrm{End}(\mathbf{H}\mathbb{D})$.

Proposition 9.5. *Let $\mathbb{D}$ be a double category with cofolding Λ. Suppose that the horizontal 2-category $\mathbf{H}\mathbb{D}$ admits the construction of free monads in the sense of Definition 9.1. Then the 2-adjunction*

$$\mathrm{Mnd}(\mathbf{H}\mathbb{D}) \underset{R}{\overset{U}{\rightleftarrows}} \mathrm{End}(\mathbf{H}\mathbb{D}) \quad (\perp)$$

extends to a horizontal double adjunction

$$\mathbb{M}\mathrm{nd}(\mathbb{D}) \underset{R}{\overset{U}{\rightleftarrows}} \mathbb{E}\mathrm{nd}(\mathbb{D}). \quad (\perp)$$

Proof. By the above remark, U automatically extends to a double functor. The main point is to extend R, which relies on the cofoldings on $\mathbb{E}\mathrm{nd}(\mathbb{D})$ and $\mathbb{M}\mathrm{nd}(\mathbb{D})$ guaranteed by Proposition 7.2, and the crucial fact that the counit of the 2-adjunction $U \dashv R$ has components of the form $\varepsilon_{(Y,Q)} = (1^h_Y, \iota_Q)$. The 2-functor R is defined on (horizontal) endomorphism maps $(F, \phi)\colon (X, P) \to (Y, Q)$ and endomorphism 2-cells $\alpha\colon (F_1, \phi_1) \Rightarrow (F_2, \phi_2)$ by the equations

$$\big[\, UR(F, \phi) \;\; (1^h_Y, \iota_Q)\big] = \big[\; (1^h_X, \iota_P) \;\; (F, \phi) \;\big] \tag{37}$$

$$\Big[\, UR\alpha \;\; i^v_{(1^h_Y,\iota_Q)}\Big] = \Big[\; i^v_{(1^h_X,\iota_P)} \;\; \alpha \;\Big]. \tag{38}$$

If $(u, \overline{u})$ is a vertical endomorphism map, then the horizontal monad map $R(u^*, \Lambda(\overline{u})) =: (Ru^*, R\Lambda(\overline{u}))$ is defined by (37). We see from (37) that the underlying horizontal morphism of Ru^* is u^*, so by Proposition 7.3 we may apply Λ^{-1} to $R\Lambda(\overline{u})$ to obtain $R(u, \overline{u}) := (u, \Lambda^{-1}R\Lambda(\overline{u}))$ with underlying vertical morphism u. A similar argument using equation (38) defines R on squares of $\mathbb{E}\mathrm{nd}(\mathbb{D})$. By construction, the double functors R and U are compatible with the cofoldings, so the 2-adjunction $\mathbf{H}U \dashv \mathbf{H}R$ extends to a horizontal double adjunction by Proposition 6.10. □

Theorem 9.6 (Reduction of construction of free monads to horizontal 2-category)**.** *Let $\mathbb{D}$ be a double category with cofolding. If the horizontal 2-category $\mathbf{H}\mathbb{D}$ admits the construction of free monads in the sense of Definition 9.1, then the double category $\mathbb{D}$ admits the construction of free monads in the sense of Definition 9.4.*

Proof. By Proposition 9.5 the 2-functor R of Definition 9.1 extends to a double functor $R\colon \mathbb{E}\mathrm{nd}(\mathbb{D}) \to \mathbb{M}\mathrm{nd}(\mathbb{D})$. We shall check that R is vertical left double adjoint to $U\colon \mathbb{M}\mathrm{nd}(\mathbb{D}) \to \mathbb{E}\mathrm{nd}(\mathbb{D})$ using the transpose of Theorem 5.2 (ii), which requires functors

$$R_0\colon (\mathrm{Obj}\,\mathbb{E}\mathrm{nd}(\mathbb{D}), \mathrm{Hor}\,\mathbb{E}\mathrm{nd}(\mathbb{D})) \longrightarrow (\mathrm{Obj}\,\mathbb{M}\mathrm{nd}(\mathbb{D}), \mathrm{Hor}\,\mathbb{M}\mathrm{nd}(\mathbb{D}))$$

$$\eta\colon (\mathrm{Obj}\,\mathbb{E}\mathrm{nd}(\mathbb{D}), \mathrm{Hor}\,\mathbb{E}\mathrm{nd}(\mathbb{D})) \longrightarrow (\mathrm{Ver}\,\mathbb{E}\mathrm{nd}(\mathbb{D}), \mathrm{Sq}\,\mathbb{E}\mathrm{nd}(\mathbb{D}))$$

such that for each horizontal morphism (F,ϕ) in $\mathbb{E}\mathrm{nd}(\mathbb{D})$ the square $\eta_{(F,\phi)}$ is of the form

$$\begin{array}{ccc} (X,P) & \xrightarrow{(F,\phi)} & (Y,Q) \\ {\scriptstyle \eta_{(X,P)}}\downarrow & \eta_{(F,\phi)} & \downarrow{\scriptstyle \eta_{(Y,Q)}} \\ UR_0(X,P) & \xrightarrow[UR_0(F,\phi)]{} & UR_0(Y,Q) \end{array}$$

and is universal from (F,ϕ) to U.

We define R_0 as the horizontal 1-adjoint already present, namely $R_0(X,P) := (X, P^{\mathrm{free}})$ and $R_0(F,\phi)\colon (X,P^{\mathrm{free}}) \to (Y,Q^{\mathrm{free}})$ is the unique (horizontal) monad morphism such that $(1^h_Y, \iota_Q) \circ UR_0(F,\phi) = (F,\phi) \circ (1^h_X \iota_P)$.

The functor η on objects is $\eta_{(X,P)} := (1^v_X, \left(\Lambda^{\mathbb{D}}\right)^{-1}(\iota_P)) = (1^v_X, \iota_P)$. Here $\Lambda^{\mathbb{D}}$ is the cofolding on $\mathbb{D}$, and we are using Proposition 7.2 for the cofolding on $\mathbb{E}\mathrm{nd}(\mathbb{D})$, the bijection in Proposition 7.3 for the fixed vertical morphism (1^v_X), and the fact that $(1^v_X)^* = 1^h_X$. For a horizontal endomorphism map (F,ϕ), we define $\eta_{(F,\phi)}$ to be $\left(\Lambda^{\mathbb{E}\mathrm{nd}(\mathbb{D})}\right)^{-1}$ of the vertical identity square

$$\begin{array}{ccccc} UL_0(X,P) & \xrightarrow{(1^h_X,\iota_P)} & (X,P) & \xrightarrow{(F,\phi)} & (Y,Q) \\ \| & & i^v & & \| \\ UL_0(X,P) & \xrightarrow[UL_0(F,\phi)]{} & UL_0(Y,Q) & \xrightarrow[(1^h_Y,\iota_Q)]{} & (Y,Q) \end{array}$$

in $\mathbb{E}\mathrm{nd}(\mathbb{D})$.

For the universality of $\eta_{(Y,Q)}$ concerning vertical morphisms, we must prove for each endomorphism (Y,Q) and each monad (X,P) that

$$\mathrm{Ver}_{\mathbb{M}\mathrm{nd}(\mathbb{D})}(Y,Q^{\mathrm{free}}),(X,P)) \xrightarrow{U(-)\circ(1^v_Y,\iota_Q)} \mathrm{Ver}_{\mathbb{E}\mathrm{nd}(\mathbb{D})}((Y,Q),U(X,P))$$

is a bijection. For injectivity, if $U(u,\overline{u})\circ(1^v_Y,\iota_Q)=U(v,\overline{v})\circ(1^v_Y,\iota_Q)$, then $u=v$, and the coholonomy on $\mathbb{E}\mathrm{nd}(\mathbb{D})$ gives us

$$(1^h_Y,\iota_Q)\circ U(u^*,\Lambda(\overline{u}))=(1^h_Y,\iota_Q)\circ U(v^*,\Lambda(\overline{v})),$$

so $\Lambda(\overline{u})=\Lambda(\overline{v})$ by horizontal universality of $(1^h_Y,\iota_Q)$. Finally, $\overline{u}=\overline{v}$ by Proposition 7.3. For surjectivity, if $(w,\overline{w})\colon (Y,Q)\to U(X,P)$ is a vertical endomorphism map, the horizontal universality of $(1^h_Y,\iota_Q)$ guarantees a horizontal monad map $(F,\phi)\colon (X,P)\to(Y,Q^{\text{free}})$ such that $(1^h_Y,\iota_Q)\circ U(F,\phi)=(w^*,\Lambda(\overline{w}))$. Then $F=w^*$, and we may take $(u,\overline{u})=(w,\Lambda^{-1}([\ \phi\ \ \iota_Q\ ])$ so that $U(u,\overline{u})\circ(1^v_Y,\iota_Q)=(w,\overline{w})$, again by Proposition 7.3.

We next prove that the square $\eta_{(F,\phi)}$ is vertically universal, that is, the map

$$\mathbb{M}\mathrm{nd}(\mathbb{D})\begin{pmatrix}R_0(F,\phi)\\(F',\phi')\end{pmatrix}\longrightarrow\mathbb{E}\mathrm{nd}(\mathbb{D})\begin{pmatrix}(F,\phi)\\U(F',\phi')\end{pmatrix}\tag{39}$$

$$\beta\longmapsto\begin{bmatrix}\eta_{(F,\phi)}\\U\beta\end{bmatrix}.$$

is a bijection (recall Definition 4.1.). The notation $\mathbb{M}\mathrm{nd}(\mathbb{D})\begin{pmatrix}R_0(F,\phi)\\(F',\phi')\end{pmatrix}$ indicates the set of monad squares with top horizontal arrow $R_0(F,\phi)$ and bottom horizontal arrow (F',ϕ'). The notation $\mathbb{E}\mathrm{nd}(\mathbb{D})\begin{pmatrix}(F,\phi)\\U(F',\phi')\end{pmatrix}$ indicates the set of endomorphism squares with top horizontal arrow (F,ϕ) and bottom horizontal arrow $U(F',\phi')$.

Since we have already checked the universality of $\eta_{(Y,Q)}$ with respect to vertical morphisms, and since squares with distinct vertical arrows are distinct, it suffices to prove a bijection for monad squares which additionally have the left and right vertical arrows fixed, so we consider monad squares of the form

$$\begin{array}{ccc}(X,P)&\xrightarrow{R_0(F,\phi)}&(Y,Q^{\text{free}})\\ {\scriptstyle(u,\bar{u})}\downarrow&\beta&\downarrow{\scriptstyle(v,\bar{v})}\\ (X',P')&\xrightarrow[(F',\phi')]{}&(Y',Q').\end{array}$$

We factor the map in (39) (for fixed $(u, \bar{u})$ and $(v, \bar{v})$), into a sequence of bijections.

$$\begin{aligned} \beta &\leftrightarrow \Lambda^{\mathbb{M}\mathrm{nd}(\mathbb{D})}(\beta) \\ &\leftrightarrow \begin{bmatrix} U\Lambda^{\mathbb{M}\mathrm{nd}(\mathbb{D})}(\beta) & i^v_{(1_Y,\iota_Q)} \end{bmatrix} \\ &\leftrightarrow \begin{bmatrix} i^v_{(u,\bar{u})^*} & i^v \\ U\Lambda^{\mathbb{M}\mathrm{nd}(\mathbb{D})}(\beta) & i^v_{(1_Y,\iota_Q)} \end{bmatrix} \\ &\leftrightarrow \begin{bmatrix} \eta_{(F,\phi)} \\ U\beta \end{bmatrix} \end{aligned}$$

The last bijection is $(\Lambda^{\mathbb{E}\mathrm{nd}(\mathbb{D})})^{-1}$ and relies on the fact that U is compatible with the cofoldings $\Lambda^{\mathbb{M}\mathrm{nd}(\mathbb{D})}$ and $\Lambda^{\mathbb{E}\mathrm{nd}(\mathbb{D})}$. □

Remark 9.7. Note that the conclusion of Theorem 9.6, that $\mathbb{D}$ admits the construction of free monads, amounts to a vertical double adjunction, the free-monad double functor R being the *left* double adjoint. Since $\mathbf{V}$: $\mathbf{DblCat_v} \to \mathbf{Cat}$ is a 2-functor, we obtain (in the situation of the Theorem) also a 2-adjunction

$$\mathbf{V}\mathbb{E}\mathrm{nd}(\mathbb{D}) \underset{\mathbf{V}U}{\overset{\mathbf{V}R}{\rightleftarrows}} \mathbf{V}\mathbb{M}\mathrm{nd}(\mathbb{D}). \quad (\bot)$$

10. Existence of Eilenberg–Moore Objects

The double functor $\mathbb{M}\mathrm{nd}(\mathbb{D}) \to \mathbb{D}$ which to a monad associates its underlying object, has a horizontal double right adjoint $\mathrm{Inc}_{\mathbb{D}}$ which to an object in $\mathbb{D}$ associates the trivial monad on it:

$$\mathbb{M}\mathrm{nd}(\mathbb{D}) \underset{\mathrm{Inc}_{\mathbb{D}}}{\overset{\mathrm{Und}}{\rightleftarrows}} \mathbb{D}. \quad (\bot)$$

In this final section we analyze when $\mathrm{Inc}_{\mathbb{D}}$ has a further right double adjoint.

In Street's article [26], a 2-category $\mathbf{K}$ is said to *admit the construction of algebras* if the inclusion 2-functor $\mathrm{Inc}_{\mathbf{K}} \colon \mathbf{K} \to \mathrm{Mnd}(\mathbf{K})$ admits a right 2-adjoint $\mathrm{Alg}_{\mathbf{K}}\colon\ \mathrm{Mnd}(\mathbf{K}) \to \mathbf{K}$. Synonymously, we say $\mathbf{K}$ *admits Eilenberg–Moore objects*. For a monad (X, S) in $\mathbf{K}$, the object $\mathrm{Alg}_{\mathbf{K}}(X, S)$ is denoted X^S. A right 2-adjoint $\mathrm{Alg}_{\mathbf{K}}$ exists if and only if for each monad (X, S), the presheaf $\mathrm{Mnd}_{\mathbf{K}}\left(\mathrm{Inc}_{\mathbf{K}} -, (X, S)\right)$ is representable. The representing object is then X^S.

The situation for monads in a double category $\mathbb{D}$ is more subtle, as representability of the individual presheaves $\mathrm{Mnd}_{\mathbb{D}}\left(\mathrm{Inc}_{\mathbb{D}}(-), (X, S)\right)$ does not suffice, and we must consider parameterized presheaves.

Definition 10.1. Let $\mathbb{D}$ be a double category and let $\mathrm{Inc}_{\mathbb{D}}\colon\ \mathbb{D} \to \mathbb{M}\mathrm{nd}(\mathbb{D})$, $I \mapsto (I, \mathrm{id}_I)$ be the inclusion double functor. We say that the double category $\mathbb{D}$ *admits Eilenberg–Moore objects* if $\mathrm{Inc}_{\mathbb{D}}$ admits a horizontal right double adjoint.

Remark 10.2. To an object I and a monad (X, S) in $\mathbb{D}$, we may associate the set $S\text{-}\mathrm{Alg}_I$ of S-algebra structures on I, which is the set of horizontal monad morphisms from (I, id_I) to (X, S). This assignment extends to a parameterized presheaf on $\mathbb{D}$ in the sense of Definition 3.2, namely

$$\mathbb{M}\mathrm{nd}(\mathbb{D})(\mathrm{Inc}_{\mathbb{D}} -, -)\colon\ \mathbb{D}^{\mathrm{horop}} \times \mathbb{V}_1\mathbb{M}\mathrm{nd}(\mathbb{D}) \longrightarrow \mathbb{S}\mathrm{pan}^t\,. \qquad (40)$$

Recall that $\mathbb{V}_1\mathbb{M}\mathrm{nd}(\mathbb{D})$ is the double category which has the same vertical 1-category as $\mathbb{M}\mathrm{nd}(\mathbb{D})$, but everything else is trivial, as in Section 2.

Theorem 10.3 (Characterization of existence of Eilenberg–Moore objects)**.** *The inclusion double functor*

$$\mathrm{Inc}_{\mathbb{D}}\colon\ \mathbb{D} \longrightarrow \mathbb{M}\mathrm{nd}(\mathbb{D})$$

$$I \longmapsto (I, \mathrm{id})$$

admits a horizontal right double adjoint if and only if the parameterized presheaf

$$\text{-}\mathrm{Alg}\_\colon\ \mathbb{D}^{\mathrm{horop}} \times \mathbb{V}_1\mathbb{M}\mathrm{nd}(\mathbb{D}) \longrightarrow \mathbb{S}\mathrm{pan}^t$$

is (horizontally) representable in the sense of Definition 3.8.

Proof. By Theorem 5.5, the double functor $\mathrm{Inc}_{\mathbb{D}}$ admits a horizontal right double adjoint if and only if the parameterized presheaf (40) is representable, but -Alg_ is (40) by definition. □

Example 10.4. Suppose $\mathbf{K}$ is a 2-category which admits Eilenberg–Moore objects in the sense of 2-category theory, that is, the 2-functor $\mathrm{Inc}_{\mathbf{K}}\colon \mathbf{K} \to \mathrm{Mnd}(\mathbf{K})$ admits a right 2-adjoint. Then the double category $\overline{\mathbb{Q}}\mathbf{K}$ admits Eilenberg–Moore objects since $\overline{\mathbb{Q}}\mathbf{K}$ and $\mathbb{M}\mathrm{nd}(\overline{\mathbb{Q}}\mathbf{K}) = \overline{\mathbb{Q}}\,\mathrm{Mnd}(\mathbf{K})$ both have cofoldings with fully faithful coholonomies, $\mathrm{Inc}_{\overline{\mathbb{Q}}\mathbf{K}}$ preserves them, and $\mathbf{H}\,\mathrm{Inc}_{\overline{\mathbb{Q}}\mathbf{K}} = \mathrm{Inc}_{\mathbf{K}}$ admits a right 2-adjoint. See Example 6.6, Proposition 7.2, and Corollary 6.19. The representing functor $G\colon\ \mathbb{V}_1\mathbb{M}\mathrm{nd}(\overline{\mathbb{Q}}\mathbf{K}) \to \mathbb{S}\mathrm{pan}^t$ for $\text{-}\mathrm{Alg}\_$ is the transposed opposite of the right adjoint to $\mathrm{Inc}_{\mathbf{K}}$.

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SATURATION FOR CLASSES OF MORPHISMS

by E. BRODOLONI and L. STRAMACCIA

Résumé. Une classe Σ extérieurement saturée de morphismes d'une catégorie $\mathcal{C}$ est une classe de morphismes qui sont inversés par un foncteur $F : \mathcal{C} \to \mathcal{D}$. Par ailleurs, Σ est intérieurement saturée si elle coincide avec son double orthogonal au sens de Freyd-Kelly. Dans cette courte note nous prouvons qu'une classe $\Sigma \subset Mor\,\mathcal{C}$ est intérieurement saturée si et seulement si elle est extérieurement saturée et admet un calcul de fractions á gauche.

Abstract. An externally saturated class Σ of morphisms in a category $\mathcal{C}$ is the class of morphisms that are inverted by some functor $F : \mathcal{C} \to \mathcal{D}$. On the other hand, Σ is internally saturated if it coincides with its double orthogonal in the sense of Freyd-Kelly. In this short note we prove that $\Sigma \subset Mor\,\mathcal{C}$ is an internally saturated class if and only if it is externally saturated and admits a calculus of left fractions.

Keywords. orthogonality, saturation, calculus of fractions, shape, shape equivalences..
Mathematics Subject Classification (2010). 18A40, 18A25, 54B30, 55P55.

1. Introduction

Given a class Σ of morphisms in a category $\mathcal{C}$, there are currently at hand wo different notions of saturation for it. Σ is called externally saturated if it s saturated in the sense of category of fractions, e.g. as defined in the book by Gabriel and Zisman [6]. On the other hand Σ is called internally saturated f it coincides with its double orthogonal in the sense of Freyd and Kelly [5]. The internal-external terminology is due, as far as we know, to Casacuberta and Frei [1]. In that paper it was shown that every internal saturated class is also externally saturated, using a suitable shape functor. There the main result asserts that if Σ is the class of morphisms inverted by some functor $F : \mathcal{C} \to \mathcal{D}$ having a right adjoint, then Σ is is both internally and externally saturated. This is the case when F is part of a monad. In a previous paper the second author pointed out that every internally saturated class has also a calculus of left fractions, here we prove that the converse holds true provided that the category $\mathcal{C}$ has finite colimits and a terminal object, that is, internally saturated classes are the same as externally saturated classes admitting a calculus of left fractions. It turns out that internally saturated classes of morphisms are exactly what was needed in [4] in the context of the Adams completion with respect to a homology theory.

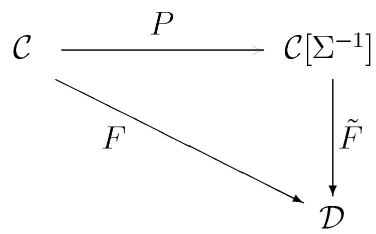
2. Preliminaries

Let $\mathcal{C}$ be be a fixed category, that we assume to be finitely cocomplete and endowed with a terminal object $\mathtt{T}$, throughout the paper. Moreover, given a class Σ morphisms of $\mathcal{C}$, let us denote by $\mathcal{C}[\Sigma^{-1}]$ the category of fractions of $\mathcal{C}$ with respect to Σ and let $P : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ be the canonical functor [6]. Let us recall that the pair $(\mathcal{C}[\Sigma^{-1}], P)$ is uniquely determined by the following properties:

(a) $P(s)$ is an isomorphism, for all $s \in \Sigma$,

(b) if $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $F(s)$ is an isomorphism, for all $s \in \Sigma$, then there is a unique functor $\tilde{F} : \mathcal{C}[\Sigma^{-1}] \to \mathcal{D}$ such that $\tilde{F} \circ P = F$.

Diagrammatically



The *external saturation* of Σ is the class $\overline{\Sigma}$ of all morphisms in $\mathcal{C}$ that are taken to isomorphisms by P. Σ is *externally saturated* when $\Sigma = \overline{\Sigma}$.

It is easy to realize that Σ is externally saturated iff there is some functor $F : \mathcal{C} \to \mathcal{D}$ such that

$$\Sigma = \mathcal{S}(F) = \{s \in Mor\,\mathcal{C} \mid F(s) \text{ is an isomorphism}\},$$

see, for instance [4], Prop.1.1.

Proposition 2.1. *An externally saturated class Σ has the following properties :*

- *(a) it contains all isomorphisms of $\mathcal{C}$,*
- *(b) it has the "two out of three property", that is, given morphisms $u : A \to B$ and $v : B \to C$, if two of $u, v, v \circ u$ are in Σ, then the third is also in Σ,*
- *(c) given morphisms $u : A \to B, v : B \to C$ and $t : C \to D$, if $v \circ u, t \circ v \in \Sigma$, then $v \in \Sigma$.*

Proof. (a) and (b) are obvious. For (c) see [9], Proposition 19.3.3 (a). □

The *orthogonal* of a class $\Sigma \subset Mor\,\mathcal{C}$ is the class of objects in $\mathcal{C}$ defined as follows

$$\Sigma^{\perp} = \{P \in Ob\,\mathcal{C} \mid \mathcal{C}(s, P) \text{ is bijective for all } s \in \Sigma\}$$

where $s : X \to Y$ and $\mathcal{C}(s, P) : \mathcal{C}(Y, P) \to \mathcal{C}(X, P)$ is defined by composition with s.
Symmetrically, the orthogonal of a class $\mathcal{K} \subset Ob\,\mathcal{C}$ is given by

$$\mathcal{K}^{\top} = \{s \in Mor\,\mathcal{C} \mid \mathcal{C}(s, P) \text{ is bijective for all } P \in \mathcal{K}\}.$$

Notice that, in general, the following relations hold:

1. $\Sigma \subseteq \Sigma^{\perp\top}$,
2. $\mathcal{K} \subseteq \mathcal{K}^{\top\perp}$.

Σ (resp. $\mathcal{K}$) will be called *internally saturated* whenever $\Sigma = \Sigma^{\perp\top}$ (resp. $\mathcal{K} = \mathcal{K}^{\top\perp}$) [1, 2].
By a standard abuse of notation we often denote by $\mathcal{K}$ the class of objects and the full subcategory of $\mathcal{C}$ with these objects.
If $\mathcal{K}$ is a subcategory of $\mathcal{C}$, the shape category [7] of the pair $(\mathcal{C}, \mathcal{K})$ is the category $SH(\mathcal{C}, \mathcal{K})$ having the same objects as $\mathcal{C}$ and morphisms given by

$$SH(\mathcal{C}, \mathcal{K})(X, Y) = Nat(\mathcal{C}(Y, E(-)), \mathcal{C}(X, E(-))),$$

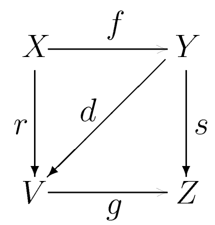
where $E : \mathcal{K} \to \mathcal{C}$ is the inclusion functor. There is a *shape functor*

$$Sh : \mathcal{C} \to SH(\mathcal{C}, \mathcal{K})$$

which takes objects fixed and sends a morphism $f : X \to Y$ to the natural transformation $\mathcal{C}(f, E(-)) : \mathcal{C}(Y, E(-)) \to \mathcal{C}(X, E(-))$ defined by composition with f. f is called a *shape equivalence* for the pair $(\mathcal{C}, \mathcal{K})$ whenever $Sh(f) = \mathcal{C}(f, E(-))$ is a natural isomorphism. It follows at once that the class of shape equivalences for $(\mathcal{C}, \mathcal{K})$ coincides with the orthogonal class $\mathcal{K}^{\top}$. On the other hand, it is clear that, given an internally saturated class Σ of morphisms of $\mathcal{C}$, then $\Sigma = \mathcal{S}(Sh)$ is the class of shape equivalences for the pair $(\mathcal{C}, \Sigma^{\perp})$ [1, 10]. Such arguments show that

Proposition 2.2. *[1] Every internally saturated class $\Sigma \subset Mor\,\mathcal{C}$ is also externally saturated.*

Given morphisms $f : X \to Y$ and $g : V \to Z$ in $\mathcal{C}$, we write $f \uparrow g$ to mean that every commutative diagram



has a unique diagonal $d : Y \to V$ such that $g \circ d = s$ and $d \circ f = r$.
For $\Sigma \subset Mor\,\mathcal{C}$ one defines

$$\Sigma^{\uparrow} = \{f \in Mor\,\mathcal{C} \mid f \uparrow g,\ \text{for all } g \in \Sigma\}$$

Let $\mathcal{K} \subset \mathcal{C}$ and denote by $\mathcal{K}_{\mathtt{T}}$ the class of all morphisms $K \longrightarrow \mathtt{T}$, for $K \in \mathcal{K}$. Then, it is clear that

Proposition 2.3. $\mathcal{K}^{\perp} = \mathcal{K}_{\mathtt{T}}^{\uparrow}$.

It follows that, for any subcategory $\mathcal{K} \subset \mathcal{C}$, the class $\mathcal{K}^{\perp}$ is the left part of a *prefactorization system* on $\mathcal{C}$ and consequently it has a number of properties, among which we record the fact that $\mathcal{K}^{\perp}$ is closed under pushouts [5], Propositions 2.1.1 and 2.1.3.

3. Calculus of Fractions

Let us recall [9, 6] that a class of morphisms $\Sigma \subset Mor\,\mathcal{C}$ is said to admit *a calculus of left fractions* when it has the following properties

(a) contains all identities,

(b) is closed under composition,

(c) every span

$$X \xleftarrow{\;s\;} X' \xrightarrow{\;f\;} Y$$

with $s \in \Sigma$, can be inserted in a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\;f\;} & Y \\ {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle s'} \\ X & \xrightarrow[\;f'\;]{} & Y' \end{array}$$

where $s' \in \Sigma$,

(d) if $f \circ s = g \circ s,\ s \in \Sigma$, then there exists a $t \in \Sigma$ such that $t \circ f = t \circ g$.

Proposition 3.1. *[10] Every internally saturated class $\Sigma \subset Mor\,\mathcal{C}$ admits a calculus of left fractions.*

Proof. In view of Proposition 1.1, properties (a) and (b) for a calculus of left fractions are satisfied. Property (c) holds because of Proposition 1.3, since $\mathcal{C}$ has finite colimits and a terminal object. As for (d), consider a pair $f, g : Y \to Z$ and an $s : X \to Y, s \in \Sigma$ with $f \circ s = g \circ s$. One can form the pushout

$$\begin{array}{ccc} X & \xrightarrow{\;s\;} & Y \\ {\scriptstyle f \circ s = g \circ s}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ Z & \xrightarrow[\;t\;]{} & W \end{array}$$

where $t \in \Sigma$, by Proposition 1.3. By the universal property of the pushout there are uniquely determined morphisms $u, v : W \to Z$ such that $u \circ t = 1_Z,\ u \circ h = g$ and $v \circ t = 1_Z,\ v \circ h = f$. Notice that, by Proposition 1.1 (b) it follows that $u, v \in \Sigma$, then in the pushout

$$\begin{array}{ccc} W & \xrightarrow{\;u\;} & Z \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Z & \xrightarrow[\;q\;]{} & Q \end{array}$$

also $p, q \in \Sigma$ and, moreover $p = q$, since $p = p \circ u \circ t = q \circ v \circ t = q$. Finally, $p \circ f = p \circ v \circ h = q \circ u \circ h = q \circ g$, which concludes the proof. We have taken the last argument from the proof of Theorem 1.3 of [4] for the sake of completeness. □

Corollary 3.2. *Every internally saturated class $\Sigma \subset Mor\,\mathcal{C}$ is externally saturated and admits a calculus of left fractions.*

Recall now the following result from [9], Theorem 19.3.1 (a)

Theorem 3.3. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $\Sigma = \mathcal{S}(F)$. Let $\tilde{F} : \mathcal{C}[\Sigma^{-1}] \to \mathcal{D}$ be the unique functor such that $\tilde{F} \circ P = F$. If Σ admits a calculus of left fractions, then $\tilde{F}$ reflects isomorphisms.*

It allows us to prove our main result

Theorem 3.4. *Let $\Sigma \subset Mor\,\mathcal{C}$ be an externally saturated class admitting a calculus of left fractions. Then Σ is internally saturated.*

Proof. $\Sigma^{\perp\top}$ is the class of shape equivalences for the pair $(\mathcal{C},\mathcal{K})$, where $\mathcal{K} = \Sigma^{\perp}$. Let $Sh : \mathcal{C} \to SH(\mathcal{C},\mathcal{K})$ be the corresponding shape functor. Then there is a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{P} & \mathcal{C}[\Sigma^{-1}] \\ & \searrow^{Sh} & \downarrow \tilde{Sh} \\ & & SH(\mathcal{C},\mathcal{K}) \end{array}$$

where, by Theorem 2.3, the functor Sh reflects isomorphisms. Let s be a shape equivalence, then $Sh(s) = \tilde{Sh} \circ P(s)$ is iso in $SH(\mathcal{C},\mathcal{K})$. It follows that $P(s)$ has to be an isomorphism in $\mathcal{C}[\Sigma^{-1}]$, hence $s \in \Sigma$, because Σ is externally saturated. This shows that $\Sigma^{\perp\top} \subseteq \Sigma$, hence the theorem. $\square$

From [4], Theorem 1.3 and Theorem 1.5 we obtain

Corollary 3.5. *An externally saturated class $\Sigma \subset Mor\,\mathcal{C}$ is internally saturated if and only if it satisfies the following "weak pushout property":*

every span

$$X \xleftarrow{\;s\;} X' \xrightarrow{\;f\;} Y$$

with $s \in \Sigma$, can be inserted in a weak pushout diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ Y' & \xrightarrow[f']{} & W \end{array}$$

where $s' \in \Sigma$.

Corollary 3.6. *Let $\Sigma \subset Mor\, \mathcal{C}$ be an internally saturated class of morphisms. The following are equivalent for an object $X \in \mathcal{C}$:*

(a) *the functor $\mathcal{C}[\Sigma^{-1}](-, X) : \mathcal{C} \to Sets$ is represented by $Z \in \mathcal{C}$,*

(b) *$Z \in \Sigma^{\perp}$ and there is an $s : X \to Z,\ s \in \Sigma$,*

(c) *there is an $s : X \to Z,\ s \in \Sigma$, which is terminal in the comma category X/Σ.*

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RESUMES DES ARTICLES PUBLIES dans le Volume LIII (2012)

G. MALTSINIOTIS, Carrés exacts homotopiques et dérivateurs, 3-63.
Le but de ce texte est d'introduire une variante homotopique de la notion de carré exact, étudiée par René Guitart, et d'expliquer le rapport de cette généralisation avec la théorie des dérivateurs.

E. MANES, Varieties generated by compact metric spaces, 64-79.
Un ultrafiltre nonprincipal *r* sur ω choisit un point de convergence pour chaque séquence dans un espace métrique compact. La classe d'algèbres produites par cette opération est une sous-catégorie pleine d'espaces topologiques dénombrablement tendus et d'applications continues qui contient tous les espaces métriques compacts. Les équations qui déterminent cette classe sont précisément celles satisfaites par la fonction caractéristique $2^{\omega} \to 2$ de *r*.

M.L. DEL HOYO, On the homotopy type of a (co)fibred category, 82-114.
Dans cet article on décrit deux façons par lesquelles des catégories (co)fibrées donnent lieu à des ensembles bisimpliciaux. Le nerf fibré est une extension naturelle de la notion du nerf de Segal d'une catégorie. Si la fibration est scindée, alors on peut construire le nerf clivé, une petite variante qui émerge d'un clivage fermé. On interprète quelques théorèmes classiques de Thomason et Quillen à l'aide de cette construction, et on utilise le nerf fibré et clivé pour établir de nouveaux résultats en théorie de l'homotopie et de l'homologie de petites catégories.

M. GRANDIS, A lax symmetric cubical category associated to a directed space, 115-156**.**
Le domaine récent de la topologie algébrique dirigée étudie les "espaces dirigés", où chemins et homotopies peuvent être non réversibles. Les applications principales concernent la programmation parallèle. On introduit ici, pour un espace dirigé, une catégorie fondamentale de *dimension infinie*, de type cubique *lax* : les cubes singuliers de l'espace ont une structure cubique, où les concatenations sont associatives à une re-paramétrisation inversible près, mais les dégénérescences sont seulement lax-unitaires. En outre cette structure est *symétrique* par permutation des variables des cubes singuliers, ce qui simplifie les propriétés de cohérence. Les "cubes de Moore" de l'espace donnent une catégorie cubique *stricte* moyennant une construction similaire.

K. WALDORF, Transgression to loop spaces and its inverse, I: Diffeological bundles and fusion maps, 162-210.
On montre que les classes d'isomorphisme de fibrés principaux sur un espace difféologique sont en bijection avec certaines applications sur son espace des lacets, aussi bien dans le cas d'une configuration avec connexions que dans celui d'une configuration sans connexions. Les applications sur l'espace des lacets sont lisses et satisfont une propriété "fusion" à l'égard de triplets de chemins. Les bijections sont établies par des isomorphismes explicites, qui seront appelés 'transgression' et 'régression'. Réduits au cas d'une variété différentielle, nos résultats étendent des résultats précédents de J.W. Barrett.

BLUTE, EHRHARD & TASSON, A convenient differential category, 211-232.
Les auteurs montrent que les espaces vectoriels convenables au sens de Frölicher et Kriegl forment une catégorie différentielle. Ces catégories ont été introduites par Blute, Cockett et Seely en tant que modèles de la logique linéaire différentielle de Ehrhard et Regnier. Ils montrent que la catégorie en question rend parfaitement compte des intuitions de cette logique. Il était déjà clair dans l'ouvrage de Frölicher et Kriegl que la catégorie des espaces vectoriels convenables a une structure remarquable. Ici on donne une interprétation catégorique d'une partie importante de cette structure. Ainsi, on montre que cette catégorie possède une comonade dont la catégorie de co-Kleisli coïncide avec la catégorie des fonctions infiniment différentiables et que cette comonade modélise la modalité exponentielle de la logique linéaire. Le système logique suggère de nouvelles structures. Notamment on met en évidence l'existence d'un morphisme "codereliction" qui permet d'obtenir la dérivée de n'importe quel morphisme par simple précomposition.

R. GUITART, Pierre Damphousse, mathématicien (1947-2012), 233-240.
Pierre Damphousse fut un membre actif de la communauté catégoricienne. Dans ce court article, René Guitart (avec qui il a travaillé) parcourt son itinéraire mathématique, où trois sujets dominent : la pureté dans les modules, les cartes cellulaires, les "fixob" et le foncteur parties sur Ens.

FIORE, GAMBINO & J. KOCK, Double adjunctions and free monads, 242-306**.**
Les auteurs caractérisent les adjonctions doubles en termes de préfaisceaux et carrés universels, puis appliquent ces caractérisations aux monades libres et aux objets d'Eilenberg–Moore dans les catégories doubles. Ils améliorent un de leurs résultats antérieurs comme suit: si une catégorie double munie d'un co-pliage admet la construction des monades libres dans sa 2-catégorie horizontale, alors elle admet aussi la construction des monades libres en tant que catégorie double.

Ils démontrent aussi qu'une catégorie double admet les objets d'Eilenberg–Moore si et seulement si un certain préfaisceau paramétrisé est représentable. Pour ce faire, ils développent une notion de préfaisceau paramétrisé sur les catégories doubles et démontrent un lemme de Yoneda pour les catégories doubles.

BRODOLONI & STRAMACCIA, Saturation for classes of morphisms, 307-315.
Une classe Σ de morphismes extérieurement saturée d'une catégorie C est une classe de morphismes qui sont inversés par un foncteur de C dans D. Par ailleurs elle est intérieurement saturée si elle coïncide avec son double orthogonal au sens de Freyd-Kelly. Dans cette courte note, on prouve qu'une classe Σ est intérieurement saturée si et seulement si elle est extérieurement saturée et admet un calcul de fractions à gauche.

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