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ON THE HOMOTOPY TYPE OF A (CO)FIBRED CATEGORY

by Matias L. del HOYO

Abstract: In this paper we describe two ways on which (co)fibred categories give rise to bisimplicial sets. The *fibred nerve* is a natural extension of Segal's classical nerve of a category, and it constitutes an alternative simplicial description of the homotopy type of the total category. If the fibration is splitting, then one can construct the *cleaved nerve*, a smaller variant which emerges from a closed cleavage. We interpret some classical theorems by Thomason and Quillen in terms of our constructions, and use the fibred and cleaved nerve to establish new results on homotopy and homology of small categories.

Résumé: Dans cet article on décrit deux façons par lesquelles des catégories (co)fibrées donnent lieu à des ensembles bisimpliciaux. Le nerf fibré est une extension naturelle de la notion du nerf de Segal d'une catégorie. Si la fibration est scindée, alors on peut construire le nerf clivé, une petite variante qui émerge d'un clivage fermé. On interprète quelques théorèmes classiques de Thomason et Quillen en termes de cette construction, et on utilise le nerf fibré et clivé pour établir de nouveaux résultats en théorie de l'homotopie et de l'homologie de petites catégories.

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Introduction

The classifying space functor associates to every small category C a topological space BC, namely the geometric realization of its nerve [15]. The classical homotopy theory of categories is lifted from spaces by using this functor. For instance, a *weak equivalence* between small categories is a map $f : C \to C'$ such that Bf is a homotopy equivalence.

A fundamental fact concerning this construction is that for every space X there is a small category C such that X and BC have the same weak homotopy type (cf. [11, VI,3.3.1], see also [6]). This way small categories constitute models for homotopy types, and one seeks to characterize the discrete invariants of X in terms of its underlying category C.

It is natural to expect that a small category C endowed with extra structure would give rise to a space BC equipped with some additional data. That is our motivation for introducing the *fibred nerve* and the *cleaved nerve*. These are bisimplicial sets with the homotopy type of the total category of a Grothendieck fibration, and constitute combinatorial descriptions that preserve in some sense the fibred structure.

By a Grothendieck fibration, or just a *fibration*, we mean what is usually called a *cofibred category*. We adopt this terminology for simplicity, and to emphasize the analogy with the topological case. Other notions of fibrations between small categories have been studied, for instance, in [13, 7].

Grothendieck fibrations have played an important role in homotopy theory. Among others, they were used by Thomason to describe homotopy colimits of small categories [16], and Quillen's Theorems A and B – that lead to long exact sequences of higher K-theory groups – may be stated in terms of Grothendieck fibrations [14]. We believe that the nerve constructions studied here will help in further applications, such as explicit constructions of $K(G, n)$ categories and Postnikov towers in Cat.

Organization

Section 1 deals with preliminaries. We fix some notations and recall some results about the classifying space functor and a key proposition on simplicial sets (1.2.1). The reader is referred to [14] for an introduction to homotopy of small categories, and to [8] for a comprehensive treatment of bisimplical objects.

The principal reference on Grothendieck fibrations is [9, VI]. A more recent one is [3]. In section 2 we set the definitions, recall some facts about fibrations and develop some others which will be needed later, such as the correspondence 2.2.3.

In section 3 we introduce both the fiber and the cleaved nerve, in the same fashion as the classical nerve is defined. We establish some fundamental facts (cf. 3.1.2, 3.2.2) and prove that for a splitting fibration the two constructions yield the same homotopy type (cf. 3.2.3).

We prove that the fibred nerve is homotopy equivalent to the classic nerve in section 4 (cf. 4.1.3). From these we derive the original and the relative versions of Quillen's theorem A. In addition, we show how to recover the classic nerve of a splitting fibration using the codiagonal construction over a bisimplicial set (cf. 4.3.3).

The last section summarizes applications and relations between the fibred nerve and some other constructions.

- The cleaved nerve and Bousfield-Kan construction for homotopy colimits are related in 5.1.1. We derive Thomason's theorem on homotopy colimits of small categories as a consequence.
- We develop a Leray-Serre style spectral sequence (cf. 5.2.1) relating the homology groups of the base, the fibers and the total category. We deduce as a corollary a homology version of Quillen's Theorem A (cf. 5.2.3).
- We introduce Quillen fibrations, which are families of categories with the same homotopy type, and show that Quillen's Theorem B might be interpreted as the following conceptual fact: the fibred classifying space functor maps Quillen fibrations into quasifibrations (5.3.1).
- Finally, we associate to a category endowed with a group action a splitting fibration, and prove that its cleaved nerve is a twisted cartesian product as defined in [12].

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1 Preliminaries

We denote by Cat, SSet and Top the categories of small categories, simplicial sets and topological spaces, respectively. If C is a small category, then we denote by $\mathrm{ob}(C)$ its set of objects and by $\mathrm{fl}(C)$ its set of arrows. As usual we denote the category of (non-empty) finite ordinals by Δ and by $\underline{\mathbf{n}} = \{0, \dots, n\}$, the ordinal with $n+1$ elements. We write I for the simplicial set represented by $\underline{\mathbf{1}}$. Sometimes $\underline{\mathbf{n}}$ will be regarded as a category in the usual way.

1.1 About homotopy of small categories

Given C a small category, its *nerve* NC is the simplicial set whose n-simplices are the chains

$$\underline{c} = (c_0 \to c_1 \to \dots \to c_n)$$

of n composable arrows in C, and its *classifying space* BC is the geometric realization of its nerve, namely $BC = |NC|$. It is a CW-complex with one n-cell for each chain of n composable arrows in C which does not involve an identity [15].

A functor $f : C \to C'$ in Cat is a *weak equivalence* if Bf is a homotopy equivalence in Top, and a small category C is *contractible* if BC is so. From the homeomorphism $B(C \times I) \cong BC \times BI$ it follows that a functor $C \times I \to C'$ induces a continuous map $BC \times [0,1] \to BC'$ and therefore a natural transformation $h : f \Rightarrow g : C \to C'$ yields a homotopy $Bh : Bf \Rightarrow Bg : BC \to BC'$. This leads to the following results [14].

1.1.1 Lemma. *If a functor admits an adjoint, then it is a weak equivalence.*

1.1.2 Lemma. *A category having an initial or final object is contractible.*

It is well known that given a commutative triangle in Top, if two of the three arrows involved are weak homotopy equivalences, then so does the third. It follows immediately that the same statement holds for weak equivalences in SSet and weak equivalences in Cat. We will refer to this fact as the *3-for-2-property*.

1.2 About bisimplicial sets

The nerve of a category is a simplicial set. We shall extend this concept by constructing the fibred nerve of a fibration, which is a bisimplicial set. A *bisimplicial set* is a functor $K : \Delta^\circ \times \Delta^\circ \to \mathrm{Set}$, where Δ° denotes the opposite category of Δ. A bisimplicial set K can be regarded as a family of sets $\{K_{m,n}\}_{m,n\geq 0}$ equipped with horizontal and vertical faces and degeneracies operators satisfying the simplicial identities, and such that the horizontal and vertical operators commute [8]. We denote by bSSet the category of bisimplicial sets and morphisms between them.

A bisimplicial set is the same as a simplicial object in SSet. Given K a bisimplicial set, let $\{m \mapsto K_{m,n}\}$ be the n-th *vertical* simplicial set, which is obtained from K by setting the second coordinate equal to n. The m-th *horizontal* simplicial set $\{n \mapsto K_{m,n}\}$ is defined analogously. We denote by $d(K)$ the *diagonal* of K, namely the simplicial set which is the composition of K with the diagonal functor $\Delta^\circ \to \Delta^\circ \times \Delta^\circ$.

We define the *geometric realization* of K as the space $|d(K)|$, which is naturally homeomorphic to the spaces obtained by first realizing on one direction and then on the other [14, p.10].

$$|n \mapsto |m \mapsto K_{m,n}|| \cong |d(K)| \cong |m \mapsto |n \mapsto K_{m,n}||$$

If $f : K \to L$ is a map of bisimplicial sets, then we say that it is a *weak equivalence* if its geometric realization $f_* : |d(K)| \to |d(L)|$ is a homotopy equivalence. The following is a very useful criterion to establish when a map is a weak equivalence (see e.g. [4, XII,2.3] or [8, IV,1.9]).

1.2.1 Proposition. *Let $f : X \to Y$ be a map in* bSSet *such that for all n the induced map $f_* : \{m \mapsto X_{m,n}\} \to \{m \mapsto Y_{m,n}\}$ is a weak equivalence in* SSet*. Then f is a weak equivalence.*

2 Fibrations

If $u : A \to B$ is a map between small categories, we say that $f \in \mathrm{fl}(A)$ is *over* $\phi \in \mathrm{fl}(B)$ if $u(f) = \phi$, and we say that $f \in \mathrm{fl}(A)$ is *over* $b \in \mathrm{ob}(B)$ if $u(f) = \mathrm{id}_b$. Given $b \in \mathrm{ob}(B)$, the *fiber* u_b is the subcategory of A of arrows over b, and the *homotopy fiber* u/b is the category whose objects are pairs (a, ϕ), $a \in \mathrm{ob}(A)$ and $\phi : u(a) \to b \in \mathrm{fl}(B)$, and whose arrows $f : (a, \phi) \to (a', \phi')$ are maps $f : a \to a'$ in A such that $\phi' u(f) = \phi$. By an abuse of notation we shall write A_b and A/b instead of u_b and u/b. Note that there is a canonical fully faithful inclusion $A_b \to A/b$, defined by $a \mapsto (a, id_{u(a)})$.

2.1 Basic definitions and examples

Let $p : E \to B$ a map between small categories. An arrow $f : e \to e'$ in E is said to be *cartesian* if it satisfies the following universal property: for all $g : e \to e''$ over $p(f)$ there is a unique $h : e' \to e''$ over $p(e')$ such that $hf = g$.

$$\begin{array}{ccc} e & \xrightarrow{f} & e' \\ & {\scriptstyle \forall g} \searrow & \downarrow {\scriptstyle \exists ! h} \\ & & e'' \end{array}$$

$$p(e) \xrightarrow{p(f)} p(e')$$

A map $p : E \to B$ is a *prefibration* if for any e object of E and any $\phi : p(e) \to b$ arrow of B there is a cartesian arrow $f : e \to e'$ over ϕ. It is not hard to see that $p : E \to B$ is a prefibration if and only if the inclusion $E_b \to E/b$ of the actual fiber into the homotopy fiber admits a left adjoint for all objects b in B. Therefore, if $p : E \to B$ is a prefibration then the inclusion $E_b \to E/b$ is a weak equivalence for all b (cf. 1.1.1).

A prefibration $p : E \to B$ is called a *fibration* if cartesian arrows are closed under composition. We say that B is the *base category* and that E is the *total category* of the fibration.

Examples.

- The projection $\pi : F \times B \to B$ is a fibration, since the arrows $(\mathrm{id}, \phi) \in \mathrm{fl}(F \times B)$ are cartesian.

- Given B a small category and $F : B \to \mathrm{Cat}$ a functor, the projection $F \rtimes B \to B$ is a fibration, whose fibers are the values of F. Here $F \rtimes B$ denotes the *Grothendieck construction* over F (see e.g. section 5.1).

- We denote by B^I the category of functors $I \to B$. Its objects are the arrows of B and its maps $(u, v) : f \to g$ are the commutative squares $vf = gu$ in B. The functor *cod* : $B^I \to B$ which assigns to each arrow its codomain is a fibration. The fibers of *cod* are the *slice* categories B/b.

- If $A \subset B$ is a *coideal* (cf. [10]), then the inclusion $A \to B$ is a fibration, whose fibers are either $\emptyset$ or pt, the final object of Cat.

- Given $p : E \to B$, an isomorphism $f \in \mathrm{fl}(E)$ is always cartesian. Thus, a functor between groupoids that is onto on arrows is a fibration.

The cartesian arrows in a fibration satisfy the following stronger universal property (cf. [3]).

2.1.1 Lemma. *Let $p : E \to B$ be a fibration and $f : e \to e'$ a cartesian arrow in E. Given $g : e \to e''$ such that $p(g) = \phi p(f)$ for some $\phi : p(e') \to p(e'')$, there exists a unique arrow $h : e' \to e''$ over ϕ satisfying $hf = g$.*

$$
\begin{array}{ccccc}
e & \xrightarrow{f} & e' & \overset{\exists! h}{\dashrightarrow} & e'' \\
& & \searrow_{\forall g} & &
\end{array}
$$

$$
p(e) \xrightarrow{p(f)} p(e') \xrightarrow{\phi} p(e'')
$$

Moreover, h is cartesian if and only if g is so.

Given a prefibration $p : E \to B$, a *cleavage* Σ is a choice of cartesian arrows. More precisely, a cleavage is a subset $\Sigma \subset \mathrm{fl}(E)$ whose elements are cartesian arrows and such that for all $e \in \mathrm{ob}(E)$ and $\phi : p(e) \to b \in \mathrm{fl}(B)$ there exists a unique arrow $\Sigma_{e,\phi} : e \to e'$ in Σ over ϕ.

The cleavage Σ is said to be *normal* if it contains the identities, and is said to be *closed* if it is closed under composition. Every prefibration admits a normal cleavage, but not every prefibration admits a closed one. A fibration which admits a closed cleavage is called a *splitting fibration*.

Example. Let E, B be groups, regarded as categories with a single object, and let $p : E \to B$ be a map between them. Then every map of E is cartesian as it is an isomorphism. It follows that p is a fibration if and only if p is an epimorphism of groups. A cleavage Σ for p is a set-theoretic section for p. The cleavage is normal if Σ preserves the neutral element, and the cleavage is closed if it is a morphism of groups. This example shows in particular that "only a few" fibrations are splitting.

From here on we will assume that all the cleavages are normal. The following lemma, whose proof is straight-forward, gives an alternative description of closed cleavages.

2.1.2 Lemma. *A cleavage Σ is closed if and only if $f \in \Sigma$ and $f'f \in \Sigma$ imply that $f' \in \Sigma$ for all pair f, f' of composable arrows of E.*

Next we discuss two notions of morphism between fibrations, and describe the corresponding categories.

Given $\xi = (p : E \to B)$ and $\xi' = (p' : E' \to B')$ fibrations, a *fibred map* $(f, g) : \xi \to \xi'$ is a pair $f : E \to E'$, $g : B \to B'$ of maps in Cat such that f preserves cartesian arrows and $p'f = gp$. We denote by $\mathrm{Fib}(\xi, \xi')$ the set of fibred maps $\xi \to \xi'$, and by Fib the category of fibrations and fibred maps between them.

Now suppose that cleavages Σ and Σ' of ξ and ξ' are given. A *cleaved map* $(f, g) : (\xi, \Sigma) \to (\xi', \Sigma')$ is a fibred map $(f, g) : \xi \to \xi'$ such that $f(\Sigma) \subset \Sigma'$. By $\mathrm{Cliv}((\xi, \Sigma), (\xi', \Sigma'))$ we mean the set of cleaved maps $(\xi, \Sigma) \to (\xi', \Sigma')$, and by Cliv the category of pairs (ξ, Σ) and cleaved maps.

Finally, we denote by Esc the full subcategory of Cliv whose objects are the pairs (ξ, Σ) with Σ a closed cleavage of ξ.

We have the following diagram, where the first is a full inclusion and the arrow Cliv $\to$ Fib is the forgetful functor $(\xi, \Sigma) \mapsto \xi$.

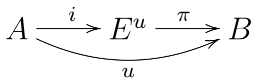
$$\mathrm{Esc} \subset \mathrm{Cliv} \to \mathrm{Fib} \subset \mathrm{Cat}^I$$

With the notations of above, we will say that $f : E \to E'$ is a *fibred map over* B if $B = B'$ and $(f, \mathrm{id}_B) : \xi \to \xi'$ is a fibred map. A *cleaved map over* B is defined similarly.

2.2 Fibration associated to a map

Given $u : A \to B$ a map between small categories, we define the *mapping category* E^u as the fiber product $A \times_B B^I$ over u and $dom : B^I \to B$ in Cat. The objects of E^u are pairs $(a, u(a) \to b)$, with a an object of A and $u(a) \to b$ an arrow of B, and the arrows are pairs (f, g) which induce a commutative square in B.

The functor u factors through E^u as πi, where i is the inclusion $a \mapsto (a, \mathrm{id}_{u(a)})$, and π is the projection $(a, u(a) \to b) \mapsto b$.



The functor i is fully faithful and admits a right adjoint, the retraction $r : E^u \to A$, which maps $(a, u(a) \to b)$ into a. This implies the following (cf. 1.1.1).

2.2.1 Lemma. *The map* $i : A \to E^u$ *is a weak equivalence.*

The functor π is a fibration. The set $\Sigma^u \subset \mathrm{fl}(E^u)$ of arrows whose first coordinate is an identity

$$\Sigma^u = \{(\mathrm{id}_a, \phi) : (a, u(a) \to b) \to (a, u(a) \to b')\},$$

is a closed cleavage for π, so it is a splitting fibration. We say that $\pi : E^u \to B$ is the *fibration associated to* u, and we endow it with the cleavage Σ^u. Note that if b is an object of B, then the fiber E^u_b of π is isomorphic to the homotopy fiber A/b of u.

Except in very special situations, the retraction $r : E^u \to A$ does not commute with the projections, namely (r, id_B) is not a map in Cat^I.

We shall describe how to replace r by others well-behaved retractions when the map u is already a fibration.

Let $p : E \to B$ be a fibration, and let $\pi : E^p \to B$ be its associated fibration. We say that a map $s : E^p \to E$ is *good* if $si = \mathrm{id}_E$, $ps = \pi$ and s preserves cartesian arrows.

$$\begin{array}{ccc} E & \underset{s}{\overset{i}{\rightleftarrows}} & E^p \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & B & \end{array}$$

If s is good, then s is a fibred map over B.

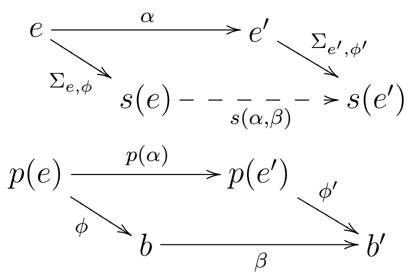
2.2.2 Lemma. *A good map s is a weak equivalence, and it induces a weak equivalence $s : E^p_b \to E_b$ for all object b in B.*

Proof. The first statement holds by 2.2.1 since s is left inverse to i. About the second, note that under the isomorphism $E^p_b \cong E/b$ the induced map $E^p_b \to E_b$ identifies with a left inverse to the inclusion $E_b \to E/b$, which is a weak equivalence indeed. □

2.2.3 Proposition. *Given $p : E \to B$ a fibration, there is a 1-1 correspondence between (normal) cleavages of E and good maps $s : E^p \to E$.*

Proof. Let $s : E^p \to E$ be a good map. For each $e \in \mathrm{ob}(E)$ and $\phi : p(e) \to b \in \mathrm{fl}(B)$ the arrow $(id_e, \phi) : (e, \mathrm{id} : p(e) \to p(e)) \to (e, \phi : p(e) \to b)$ is cartesian in E^p. Therefore, $s(id_e, \phi)$ is a cartesian arrow of E over ϕ with domain $s(i(e)) = e$. It follows that the family $\Sigma = \{s(id_e, \phi)\}_{e,\phi}$ is a cleavage of E, and it is normal because $s(\mathrm{id}_e, \mathrm{id}_{p(e)}) = s(i(\mathrm{id}_e)) = \mathrm{id}_e$.

Conversely, if Σ is a normal cleavage of E, then we shall construct a good map $s = s(\Sigma) : E^p \to E$ as follows. An object $(e, \phi : p(e) \to b)$ in E^p is mapped by s into the codomain of $\Sigma_{e,\phi} \in \mathrm{ob}(E)$. An arrow $(\alpha, \beta) : (e, \phi : p(e) \to b) \to (e', \phi' : p(e') \to b')$ of E^p is mapped by s into the unique arrow over β which makes the following diagram commutative.



The uniqueness of $s(\alpha,\beta)$ follows from 2.1.1. It also follows from 2.1.1 that s preserves cartesian arrows. As it respects identities and compositions, s is indeed a functor, and $ps = \pi$ by construction. The map s defined this way is a retraction for $i : E \to E^p$ because Σ is normal.

It is straightforward to check that these procedures are mutually inverse. □

If E is endowed with a cleavage Σ and $s : E^p \to E$ is a good map such that $s(\Sigma^u) \subset \Sigma$, then we say that s is *very good*. If s is very good, then s is a cleaved map over B.

2.2.4 Corollary. *If s and Σ are related as in 2.2.3, then Σ is closed if and only if the map s is very good.*

Proof. Let Σ be a closed cleavage and s its induced good map. If (id_e,β) is an arrow in Σ^u, then the diagram of above gives $s(\mathrm{id}_e,\beta)\Sigma_{e,\phi} = \Sigma_{e,\beta\phi}$. It follows from 2.1.2 that $s(\mathrm{id}_e,\beta) \in \Sigma$ and hence the map s is very good.

On the other hand, given Σ a cleavage which is not closed, by 2.1.2 one can find f and f' cartesian arrows of E such that $f' = gf$ with $g \notin \Sigma$. Since $g = s(id, p(g))$ it follows that s is not very good. □

3 Bisimplicial sets from fibrations

3.1 Fibred nerve

For $m, n \geqslant 0$ let $\square_{m,n}$ denotes the fibration $pr_2 : \underline{\mathbf{m}} \times \underline{\mathbf{n}} \to \underline{\mathbf{n}}$. These are the fibrations which play the role of simplices in Fib. They define a covariant functor $\square : \Delta \times \Delta \to \mathrm{Fib}$.

Given $\xi = (p : E \to B)$ a fibration, we define the *fibred nerve of* ξ as the bisimplicial set $N_f\xi$ whose m, n-simplices are given by

$$N_f\xi_{m,n} = \mathrm{Fib}(\square_{m,n}, \xi).$$

We define the *fibred classifying space* $B_f\xi$ as the geometric realization $|d(N_f\xi)|$ of the fibred nerve. These constructions are functorial. For short, we shall write N_fE and B_fE instead of $N_f\xi$ and $B_f\xi$.

The fibred nerve extends the classical nerve in the sense that there exists a natural isomorphism

$$d(N_f(\mathrm{id}_B)) = d(N_fB) \cong NB.$$

A m, n-simplex of N_fE consists of a pair $s = (s_0, s_1)$, where $s_0 : \underline{\mathbf{m}} \times \underline{\mathbf{n}} \to E$ and $s_1 : \underline{\mathbf{n}} \to B$ are such that the induced square commutes. We say that $s_1 \in NB_n$ is the *base* of the simplex s, and that $s_0|_{pr_2^{-1}(0)} \in (NE_{b_0})_m$ is the *mast* of s. Of course, s_0 completely determines s.

We visualize s as an array of arrows of E going down and right. The horizontal arrows are cartesian and the vertical arrows are over identities.

$$\begin{array}{ccccccc}
e_{0,0} & \longrightarrow & e_{0,1} & \longrightarrow & \cdots & \longrightarrow & e_{0,n} \\
\downarrow & & \downarrow & & & & \downarrow \\
e_{1,0} & \longrightarrow & e_{1,1} & \longrightarrow & \cdots & \longrightarrow & e_{1,n} \\
\downarrow & & \downarrow & & & & \downarrow \\
\cdots & & \cdots & & \cdots & & \cdots \\
\downarrow & & \downarrow & & & & \downarrow \\
e_{m,0} & \longrightarrow & e_{m,1} & \longrightarrow & \cdots & \longrightarrow & e_{m,n} \\
 & & & & & & \\
b_0 & \longrightarrow & b_1 & \longrightarrow & \cdots & \longrightarrow & b_n
\end{array}$$

Sometimes we will write $e^s_{i,j}$ to denote $s_0((i,j))$, and $e^s_{i,j} \to e^s_{i',j'}$ to denote $s_0((i,j) \to (i',j'))$.

The next technical result plays a key role hereafter. Fix $\underline{b} \in NB_n$, and let $N_fE_{\underline{b}}$ be the simplicial set whose simplices are those of N_fE with base $\underline{b}$, with faces and degeneracies in the vertical direction.

3.1.1 Lemma. *The map $\mu : N_fE_{\underline{b}} \to NE_{b_0}$ which assigns to each simplex s its mast is a weak equivalence of simplicial sets.*

Proof. We choose a cleavage Σ and construct a homotopy inverse $\nu : NE_{b_0} \to N_f E_{\underline{b}}$ for μ as follows. The map ν associates to a simplex $\underline{a}$ the unique simplex $s = \nu(\underline{a})$ with mast $\underline{a}$ and base $\underline{b}$ and such that $e^s_{i,j} \to e^s_{i,j+1} \in \Sigma$ for all i, j. It is clear that $\mu\nu = \text{id}$. We shall describe a simplicial homotopy $h : N_f E_{\underline{b}} \times I \to N_f E_{\underline{b}}$ between $\nu\mu$ and id, which induces a continuous homotopy $|h|$ and completes the proof.

We have that $(N_f E_{\underline{b}} \times I)_m = (N_f E_{\underline{b}})_m \times I_m$, and that $I_m = \{t : \underline{\mathbf{m}} \to \underline{\mathbf{1}}\}$. Given $(s, t) \in (N_f E_{\underline{b}} \times I)_m$ we define $h(s, t)$ as the unique m-simplex of $N_f E_{\underline{b}}$ with the same mast as s and such that

$$e^{h(s,t)}_{i,j} \to e^{h(s,t)}_{i,j+1} = \begin{cases} e^s_{i,j} \to e^s_{i,j+1} & \text{if } t(i) = 0 \\ e^{\nu\mu(s)}_{i,j} \to e^{\nu\mu(s)}_{i,j+1} & \text{if } t(i) = 1 \end{cases}$$

It is easy to see that h defined as above is a simplicial map, that $h(s, 0) = s$ and that $h(s, 1) = \nu\mu(s)$. □

The main feature of the fibred nerve is that it satisfies the following homotopy preserving property.

3.1.2 Proposition. *Let $\xi = (p : E \to B)$ and $\xi' = (p' : E' \to B)$ be fibrations, and let $f : E \to E'$ be a fibred map over B. If $f : E_b \to E'_b$ is a weak equivalence for all objects b of B, then $f_* : N_f E \to N_f E'$ is a weak equivalence.*

Proof. By proposition 1.2.1 it suffices to prove that the map $f_* : \{m \mapsto N_f E_{m,n}\} \to \{m \mapsto N_f E'_{m,n}\}$ is a weak equivalence for each n. Faces and degeneracies in direction m preserve the base of a simplex, thus we have decompositions

$$\{m \mapsto N_f E_{m,n}\} = \coprod_{\underline{b}=(b_0 \to \cdots \to b_n)} N_f E_{\underline{b}}$$

and

$$\{m \mapsto N_f E'_{m,n}\} = \coprod_{\underline{b}=(b_0 \to \cdots \to b_n)} N_f E'_{\underline{b}}.$$

Moreover, f_* also preserves the base of a simplex, and therefore it can be written as the coproduct of the maps $f_* : N_f E_{\underline{b}} \to N_f E'_{\underline{b}}$. Now consider the following commutative square.

$$\begin{array}{ccc} N_f E_{\underline{b}} & \xrightarrow{f_*} & N_f E'_{\underline{b}} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ N E_{b_0} & \xrightarrow{f_*} & N E'_{b_0} \end{array}$$

The vertical maps are weak equivalences by 3.1.1, and the bottom one is so by hypothesis. It follows from the 3-for-2 property that the upper one is also a weak equivalence and thus the proposition follows. $\square$

3.2 Cleaved nerve

The fibration $\square_{m,n}$ is splitting, since its unique cleavage $\Sigma = \{(id, \alpha)\}$ is closed. We consider $\square_{m,n}$ as equipped with this cleavage, and we obtain a covariant functor $\square : \Delta \times \Delta \to \mathrm{Esc} \subset \mathrm{Cliv}$.

Given $\xi = (p : E \to B)$ a fibration endowed with a cleavage Σ, we define the *cleaved nerve of* (ξ, Σ) as the bisimplicial set $N_c(\xi, \Sigma)$ whose m, n-simplices are given by

$$N_c(\xi, \Sigma)_{m,n} = \mathrm{Cliv}(\square_{m,n}, (\xi, \Sigma))$$

We define the *cleaved classifying space* of $B_c(\xi, \Sigma)$ as the geometric realization $|d(N_c(\xi, \Sigma))|$ of the cleaved nerve. These constructions are functorial. As before, we shall write $N_c E$ and $B_c E$ instead of $N_c(\xi, \Sigma)$ and $B_c(\xi, \Sigma)$ when there is no place to confusion.

The cleaved nerve extends the classical nerve in the sense that there is a natural isomorphism

$$d(N_c(\mathrm{id}_B)) = d(N_c B) \cong NB,$$

where $\mathrm{id} : B \to B$ is equipped with the cleavage $\Sigma = \mathrm{fl}(B)$.

Note that, if we forget the cleavage Σ, then we can form the fibred nerve $N_f E$ and there is a natural inclusion in bSSet

$$i : N_c E \to N_f E.$$

3.2.1 Lemma. *Let $\xi = (p : E \to B)$ be a fibration with cleavage Σ. If s and s' are simplices in N_cE with the same base and the same mast, then $s = s'$. If Σ is closed, then for all $\underline{b} \in NB_n$ and $\underline{a} \in (NE_{b_0})_m$ there exists a unique m,n-simplex $s \in N_cE$ with base $\underline{b}$ and mast $\underline{a}$.*

Proof. Note that $(e^s_{i,0} \to e^s_{i,1}) = (e^{s'}_{i,0} \to e^{s'}_{i,1})$ since they are arrows in Σ over $b_0 \to b_1$ with the same domain. We see that $(e^s_{i,j} \to e^s_{i,j+1}) = (e^{s'}_{i,j} \to e^{s'}_{i,j+1})$ by iterating this argument. Finally, $(e^s_{i,j} \to e^s_{i+1,j}) = (e^{s'}_{i,j} \to e^{s'}_{i+1,j})$ by the universal property of cartesian arrows. This proves the first assertion.

It is not hard to see that there exists a unique simplex $s \in N_fE$ with base $\underline{b}$, mast $\underline{a}$, and such that $e^s_{i,j} \to e^s_{i,j+1} \in \Sigma$ for all i, j. If Σ is closed then $e^s_{i,j} \to e^s_{i,k} \in \Sigma$ for all i, j, k and thus s is in N_cE and the second statement holds. □

If the fibration is splitting, then N_cE satisfies a homotopy preserving property analogous to 3.1.2.

3.2.2 Proposition. *Let $\xi = (p : E \to B)$ and $\xi' = (p' : E' \to B)$ be splitting fibrations with closed cleavages Σ and Σ', and let $f : E \to E'$ be a cleaved map over B. If $f : E_b \to E'_b$ is a weak equivalence for all object b of B then $f_* : N_cE \to N_cE'$ is a weak equivalence.*

Proof. This is analogous to that of 3.1.2, using the restriction $\mu : N_cE_{\underline{b}} \to NE_{b_0}$, which is also a weak equivalence by 3.2.1 – actually, it is an isomorphism. □

The following result asserts that the cleaved nerve suffices to describe the homotopy type of the fibred nerve when the cleavage is closed.

3.2.3 Theorem. *If $\xi = (p : E \to B)$ is a splitting fibration with closed cleavage Σ, then the inclusion $i : N_cE \to N_fE$ is a weak equivalence.*

Proof. Again by proposition 1.2.1, we only must show that for each n the inclusion induces a weak equivalence $i_* : \{m \mapsto N_cE_{m,n}\} \to \{m \mapsto N_fE_{m,n}\}$. For fixed n, the map i_* can be written as the coproduct of

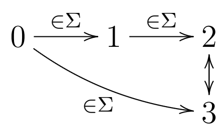
$$i_* : N_cE_{\underline{b}} \to N_fE_{\underline{b}}$$

where $\underline{b}$ runs over all n-simplices of NB. The composition $\mu i_* : N_cE_{\underline{b}} \to NE_{b_0}$ is an isomorphism by 3.2.1. It follows by 3.1.1 and the 3-for-2-property that i_* is a weak equivalence and thus the proposition. □

If the cleavage Σ is not closed, then N_cE and N_fE do not necessarily have the same homotopy type. Let us illustrate this with an example.

Example. Let E be the category obtained from the ordinal $\underline{\mathbf{3}}$ by formally inverting the arrow $2 \to 3$. Note that E has an initial element and hence BE is contractible (cf. 1.1.2). We shall see E as the total category of a fibration endowed with a cleavage Σ in such a way that N_cE is not contractible. Since $d(N_fE)$ and NE have the same homotopy type (see 4.1.3), we conclude that in this example the inclusion $i : N_cE \to N_fE$ is not a weak equivalence.

Let $B = \underline{\mathbf{2}}$ and let $p : E \to B$ be the surjection which twice takes the value 2. Clearly it is a fibration. Let Σ be the normal cleavage which contains the arrow $0 \to 3$.



If a simplex $s \in N_cE$ is not contained in the fiber E_2, then its mast must be trivial. Since a simplex in N_cE is determined by its mast and its base (cf. 3.2.1), it follows that the non-degenerate simplices of N_cE are $0 \to 1, 0 \to 3, 1 \to 2 \in N_cE_{0,1}$ and some others included in the fiber E_2. Thus, the loop $0 \to 1 \to 2 \to 3 \leftarrow 0$ gives a non-trivial element of $\pi_1(B_cE, 0)$ and therefore B_cE is not contractible.

4 Relation with the classic nerve

4.1 The main result

Let $\xi = (p : E \to B)$ be a fibration, and let $s = (s_0, s_1)$ be an element of $N_fE_{n,n}$. The composition $s_0 \circ diag : \underline{\mathbf{n}} \to E$ gives a n-simplex of NE, which we denote by

$$k(s) = (e^s_{0,0} \to e^s_{1,1} \to \cdots \to e^s_{n,n}).$$

This way we get a natural map of simplicial sets $k : d(N_f E) \to NE$ and its geometric realization $k_* : B_f E \to BE$. We shall see that it is a weak equivalence, so the fibred nerve becomes an alternative model for the homotopy type of E.

We prove that k is a weak equivalence first for splitting fibrations and then for any fibration.

4.1.1 Proposition. *Let $\xi = (p : E \to B)$ be a splitting fibration, with closed cleavage Σ. Then the map $k|_{d(N_c E)} = ki : d(N_c E) \to NE$ is a weak equivalence.*

Proof. (Compare with [16, 1.2]) From 2.2.4 we know that the cleavage Σ induces a very good map $s : E^p \to E$ and hence a commutative square

$$\begin{array}{ccc} d(N_c(E^p)) & \xrightarrow{ki} & N(E^p) \\ \downarrow{s_*} & & \downarrow{s_*} \\ d(N_c E) & \xrightarrow{ki} & NE \end{array}$$

by the naturality of k. In this square the vertical arrows are weak equivalences (cf. 2.2.2, 3.2.2), so in order to prove that the bottom arrow is a weak equivalence, by the 3-for-2-property it only remains to show that the upper arrow is one as well. To do that, we define a map $l : d(N_c E^p) \to NE^p$, prove that there is a simplicial homotopy $ki \cong l$, and prove that l is a weak equivalence.

A simplex $s = (s_0, s_1)$ of $N_c E^p_{m,n}$ is uniquely determined by its mast and its base (cf. 3.2.1), so it essentially consists of the following data

$$s = (e_0 \to e_1 \to \cdots \to e_m, p(e_m) \to b_0 \to b_1 \to \cdots \to b_n).$$

For $i = 0, \dots, m$, $j = 0, \dots, n$, we have $e^s_{i,j} = (e_i, p(e_i) \to b_j)$, with all the arrows induced by the sequence of above. Given $i = 0, \dots, m$ we define $e^s_{i,-1}$ as the object $(e_i, p(e_i) \to p(e_m))$ of E^p induced by s. These new objects lay at the mast of the following simplex of $N_c E^p_{m,n+1}$ induced by s.

$$\tilde{s} = (e_0 \to e_1 \to \cdots \to e_m, p(e_m) \xrightarrow{\text{id}} p(e_m) \to b_0 \to b_1 \to \cdots \to b_n)$$

Using $\tilde{s}$ we define $l : d(N_cE^p) \to NE^p$ by

$$l(s) = (e^s_{0,-1} \to e^s_{1,-1} \to \cdots \to e^s_{n,-1}).$$

In the same fashion, the homotopy $h : d(N_cE^p) \times I \to NE^p$ is given by

$$h(s,t) = (e^s_{0,-1} \to \cdots \to e^s_{i-1,-1} \to e^s_{i,i} \to \cdots \to e^s_{n,n})$$

where $s \in N_cE^p_{n,n}$, $t \in I_n$, $h(s,t)_j = e^s_{j,-1}$ if $t(j) = 0$ and $h(s,t)_j = e^s_{j,j}$ if $t(j) = 1$. One verifies that h is a map, that $h(s,0) = l(s)$ and that $h(s,1) = ki(s)$.

Finally, let us prove that l is a weak equivalence. We regard NE^p as a bisimplicial set constant in direction n, so $NE^p_{m,n} = NE^p_m$. The map l is the diagonalization of a bisimplicial map $L : N_cE^p \to NE^p$, defined with the same formula than l. The m-th component $L_{m,-}$ of L can be identified with the coproduct

$$\coprod_{e_0 \to \cdots \to e_m} N(p(e_m)/B) \to \coprod_{e_0 \to \cdots \to e_m} \text{pt}$$

which is a weak equivalence because $p(e_m)/B$ has an initial element and therefore is contractible (1.1.2). The map L is a weak equivalence by 1.2.1 and thus the result. □

4.1.2 Corollary. *If $\xi = (p : E \to B)$ is a splitting fibration, then $k : d(N_fE) \to NE$ is a weak equivalence.*

Proof. Fix a closed cleavage Σ and then use 3.2.3 and 4.1.1. □

Now we extend 4.1.2 to a non-necessarily splitting fibration.

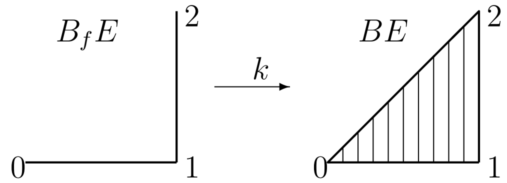
4.1.3 Theorem. *If $\xi = (p : E \to B)$ is a fibration, then the map $k : d(N_fE) \to NE$ is a weak equivalence.*

Proof. Let Σ be a cleavage of ξ. The good map $s : E^p \to E$ induced by Σ (cf. 2.2.3) gives a commutative square

$$\begin{array}{ccc} d(N_fE^p) & \xrightarrow{k} & NE^p \\ \downarrow{s} & & \downarrow{s} \\ d(N_fE) & \xrightarrow{k} & NE \end{array}$$

Since the fibration $E^p \to B$ is always splitting, it follows from 4.1.2 that the upper arrow is a weak equivalence. The vertical arrows are also weak equivalences (cf. 2.2.2, 3.1.2) and then the result follows from the 3-for-2-property. □

Example. The surjection $s : \underline{\mathbf{2}} \to \underline{\mathbf{1}}$ which takes the value 1 twice is a fibration. Down below we show the spaces $B_f E$ and BE. The map k is in this case the obvious inclusion.



Even when this example is quite simple, it is useful to understand some of the differences between the two constructions. Many of the diagonal arrows in the total category do not provide relevant homotopy information, and the fibred nerve omits them.

The cleaved nerve is smaller than the fibred nerve, and therefore a more effective codification of the homotopy type of the total category. On the other hand, it only works when the fibration is splitting, while the fibred nerve is useful for any fibration.

4.2 Quillen's Theorem A and its relative version

Quillen's Theorem A states sufficient conditions for a functor to be a weak equivalence. It was proved to be very useful not only in the work of Quillen but also in many other situations. We derive it here from our framework.

The good behaviour of fibred nerve with respect to homotopy (cf. 3.1.2) together with theorem 4.1.3 gives the following result.

4.2.1 Proposition. *If $f : E \to E'$ is a fibred map over B such that $f : E_b \to E'_b$ is a weak equivalence for all object b of B, then f is a weak equivalence.*

We deduce both Theorem A and its relative version from this proposition.

4.2.2 Corollary (Relative Quillen's Theorem A). *Let $u : A \to B$ and $u' : A' \to B$ be small categories over B. If $f : A \to A'$ is a map over B such that the induced map $A/b \to A'/b$ is a weak equivalence for all $b \in \mathrm{ob}(B)$, then f is a weak equivalence.*

Proof. Consider the following commutative square of categories over B, where E^u and $E^{u'}$ are the associated fibrations for u and u', and the bottom arrow is induced by f in a natural way.

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ E^u & \xrightarrow{f_*} & E^{u'} \end{array}$$

Since the actual fiber E^u_b identifies with the homotopy fiber A/b, the last proposition asserts that the bottom arrow is a weak equivalence, and the vertical ones are also weak equivalences by 2.2.1. The result follows from this and the 3-for-2-property. □

4.2.3 Corollary (Quillen's Theorem A). *A map $u : A \to B$ between small categories whose homotopy fibers A/b are contractible is a weak equivalence.*

Proof. Take $u' = id_B$ in the relative version. □

4.3 Fibred nerve, cleaved nerve and the codiagonal construction

In [2] the following construction is introduced. Given K a bisimplicial set, its *codiagonal* (or *bar construction*) is the simplicial set $\nabla(K)$ whose n-simplices are

$$\nabla(K)_n = \big\{(x_0, x_1, \ldots, x_n) : x_i \in K_{i,n-i},\ d^h_0 x_i = d^v_{i+1} x_{i+1} \text{ for } 0 \leqslant i < n\big\}$$

and whose faces and degeneracies are

$$d_i(x_0, \ldots, x_n) = (d^h_i x_0, d^h_{i-1} x_1, \ldots, d^h_1 x_{i-1}, d^v_i x_{i+1}, \ldots, d^v_i x_n)$$

and

$$s_j(x_0,\dots,x_n) = (s_j^h x_0, s_{j-1}^h x_1, \dots, s_0^h x_j, s_j^v x_j, \dots, s_j^v x_n).$$

There is a natural weak equivalence $\theta : d(K) \to \nabla(K)$ defined as follows,

$$\theta(x) = ((d_1^v)^n x, (d_2^v)^{n-1} d_0^h x, \dots, (d_{i+1}^v)^{n-i} (d_0^h)^i x, \dots, (d_0^h)^n x)$$

where x is a n-simplex of $d(K)$ (cf. [5]).

In the case of the fibred nerve, both the codiagonal $\nabla N_f E$ and the map θ can be described in terms of *singular functors* of fibrations. We shall give these description, and prove that for a splitting fibration there is an isomorphism between the codiagonal of the cleaved nerve and the classic nerve of the total category.

Let T_n be the full subcategory of $\underline{\mathbf{n}} \times \underline{\mathbf{n}}$ whose objects are the pairs (i,j) satisfying $i \leqslant j$. The restriction $pr_2|_{T_n} : T_n \to \underline{\mathbf{n}}$ is a fibration as one can easily check. This way we get a covariant functor $T : \Delta \to \mathrm{Fib}$, $\underline{\mathbf{n}} \mapsto T_n$, as the restriction of $\square \circ diag$.

4.3.1 Proposition. *Let $\xi = (p : E \to B)$ be a fibration. Then there is a canonical isomorphism of simplicial sets*

$$(\nabla N_f E)_n \cong \mathrm{Fib}(T_n, \xi)$$

where the right hand side is the singular functor *induced by $T : \Delta \to \mathrm{Fib}$. Under this isomorphisms, the map θ is identified with the restriction $s \mapsto s|_{T_n}$.*

Proof. Let S be the simplicial set $n \mapsto \mathrm{Fib}(T_n, \xi)$. For $k = 0, \dots, n$ let $\alpha^k = (\alpha_0^k, \alpha_1^k) : \square_{k,n-k} \to T_n$ be the fibred map satisfying $\alpha_0^k(i,j) = (i, j+k)$ for all $(i,j) \in \mathrm{ob}(\square_{k,n-k})$. We define $\lambda : S \to \nabla N_f E$ by mapping an n-simplex $x : T_n \to \xi$ to $\lambda(x) = (x\alpha^0, x\alpha^1, \dots, x\alpha^n)$. It is straightforward to check that λ is well defined, i.e. the coordinates of $\lambda(x)$ satisfy the compatibility conditions of the codiagonal, and that λ respects the faces and degeneracies.

To see that λ is actually an isomorphism, we remark that a simplex $s \in N_f E_{m,n}$ can be presented as an array of $m \times n$ commutative little squares of E

$$\begin{array}{ccc} e^s_{i,j} & \longrightarrow & e^s_{i,j+1} \\ \downarrow & & \downarrow \\ e^s_{i+1,j} & \rightarrow & e^s_{i+1,j+1} \end{array}$$

on which the vertical arrows are over identities and the horizontal ones are cartesian arrows. If $y_k \in N_f E_{k,n-k}$, $k = 0, \ldots, n$, the equation $d^h_0 y_k = d^v_{i+1} y_{k+1}$ says that the array of $k \times (n-k-1)$ little squares obtained from y_k by deleting the first column equals the array obtained from y_{k+1} by deleting the last row.

It is clear from these descriptions that a simplex $x \in S_n$ identifies with a sequence $(y_0, \ldots, y_n)$, $y_k \in N_f E_{k,n-k}$, under the compatibility conditions that impose the codiagonal. □

A similar statement holds for the cleaved nerve. Its proof is essentially that of 4.3.1. Note that T_n inherits the closed cleavage from $\Box_{n,n}$ and hence T can be considered as a functor $\Delta \to \mathrm{Esc} \subset \mathrm{Cliv}$.

4.3.2 Proposition. *Let $\xi = (p : E \to B)$ be a fibration with cleavage Σ. Then there is a canonical isomorphism of simplicial sets*

$$(\nabla N_c E)_n \cong \mathrm{Cliv}(T_n, (\xi, \Sigma))$$

where the right hand side is the singular functor *induced by $T : \Delta \to$* Esc $\subset$ Cliv. *Under this isomorphisms, the map θ identifies with the restriction $s \mapsto s|_{T_n}$.*

Given $\xi = (p : E \to B)$ a fibration endowed with a cleavage Σ, we have the following diagram of simplicial sets, where $\bar{k}$ is defined below.

$$\begin{array}{ccccc} dN_cE & \xrightarrow{i} & dN_fE & \overset{k}{\searrow} & \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} & & NE \\ \nabla N_cE & \xrightarrow{i} & \nabla N_fE & \underset{\bar{k}}{\nearrow} & \end{array}$$

If $x = (x_0, x_1) \in \mathrm{Fib}(T_n, E)$, then the composition $x_0 \circ diag : \underline{\mathbf{n}} \to E$ defines a simplex in NE_n. Under the identification of 4.3.1 this gives the map $\bar{k} : \nabla N_f E \to NE$, $\bar{k}(x) = x_0 \circ diag$.

The following shows how to recover the classic nerve of the total category of a splitting fibration from the cleaved nerve.

4.3.3 Theorem. *If the cleavage Σ is closed, then the map $\bar{k}i : \nabla N_c E \to NE$ is an isomorphism.*

Proof. The proof is similar to that of lemma 3.2.1. To see that $\bar{k}i$ is injective, consider a simplex $x \in NE_n$, view $x : \underline{\mathbf{n}} \to E$ as defined over the diagonal of $\Box_{n,n}$ and note that an extension $s : T_n \to E$ of x is necessarily unique: The horizontal arrows must belong to the cleavage, and the vertical ones are uniquely determined by the universal property of cartesian arrows.

If Σ is closed, then the unique functor $s : T_n \to E$ such that $s \circ diag = x$ and $e^s_{i,j} \to e^s_{i,j+1} \in \Sigma$ determines a cleaved map $T_n \to E$ and hence a simplex $s \in (\nabla N_c E)_n$ satisfying $\bar{k}i(s) = x$. Thus the surjectivity. $\Box$

5 Other examples and applications

5.1 Homotopy colimits

Bousfield and Kan [4] give a construction of a representing object for the homotopy colimit of a diagram of simplicial sets. Given $Z : I \to \mathrm{SSet}$, let $hc(Z)$ be the bisimplicial set whose m, n-simplices are

$$hc(Z)_{m,n} = \coprod_{i_0 \to \cdots \to i_n} Z(i_0)_m$$

where the coproduct runs over all simplices of dimension n of NI, and faces and degeneracies are defined in the obvious way. Then $hc(Z)$ satisfies the homotopy universal property of homotopy colimits (cf. [4]).

In [16] Thomason uses the Bousfield-Kan construction to describe homotopy colimits in Cat in terms of the Grothendieck construction for a functor. We recall Grothendieck construction over a functor $F : B \to$ Cat, compare the Bousfield-Kan construction with the cleaved nerve and derive Thomason's theorem from this.

Given $F : B \to \mathrm{Cat}$ a diagram of small categories, its *Grothendieck construction* is a splitting fibration $F \rtimes B \to B$ whose fibers are the values of F. The objects of the total category $F \rtimes B$ are pairs (x, b) with b an object of B and x and object of $F(b)$. An arrow $(f, \phi) : (x, b) \to$

(x', b') in $F \rtimes B$ is a pair $\phi : b \to b'$, $f : F(\phi)(e) \to e'$. Composition is given by $(\psi, g) \circ (\phi, f) = (\psi\phi, gF(\psi)(f))$. The map $F \rtimes B \to B$ is the projection, and the arrows (id, ϕ) form a distinguished closed cleavage.

5.1.1 Theorem. *Given $F : B \to \mathrm{Cat}$, there is an isomorphism $N_c(F \rtimes B) \xrightarrow{\sim} hc(NF)$ between the cleaved nerve of the Grothendieck construction of F and the Bousfield-Kan construction for homotopy colimits.*

Proof. The isomorphism $N_c(F \rtimes B) \to hc(NF)$ maps a m, n-simplex s of $N_c(F \rtimes B)$ to the element $\underline{a}$ in the summand indexed by $\underline{b}$, where $\underline{a}$ is the mast of s and $\underline{b}$ is the base of s. This is indeed a morphism of bisimplicial sets, and it is invertible because of 3.2.1. □

If E is any fibration, then one can define a function $N_f E \to hc(NF)$ in a similar fashion. However, this function is not a morphism in general, as it does not respect the 0-th face operator.

5.1.2 Corollary (Thomason's theorem)**.** *The Grothendieck construction $F \rtimes B$ over a functor $F : B \to \mathrm{Cat}$ is a representing object for the homotopy colimit of F.*

Proof. This is a consequence of 4.1.1 and 5.1.1. □

5.2 Spectral sequence of a fibration

A bisimplicial set gives rise to a bisimplicial abelian group and hence to a bicomplex. In this section we study the spectral sequence associated to the bicomplex coming from the fibred nerve. We recall some definitions on homology of categories from [14]. Then we describe how a fibration gives rise to a pseudofunctor, and define the modules $H_m(F)$. Finally we state and prove theorem 5.2.1 and derive a homology version of Quillen's Theorem A as a corollary.

Given a small category C, a *module over C* is a functor $A : C \to \mathrm{Ab}$, where Ab denotes the category of abelian groups. The m-th *homology group of C with coefficients in a module A* is defined as the m-th left derived functor of *colim* : $\mathrm{Ab}^C \to \mathrm{Ab}$.

$$H_m(C, A) = colim^m_C A$$

The groups $H_m(C, A)$ can be computed as the homology of the following simplicial abelian group

$$C_m(C, A) = \bigoplus_{c_0 \to \cdots \to c_m} A(c_0)$$

and, in the case that A is morphism inverting, they agree with the homology of the classifying space BC with local coefficients induced by A. We write $H_m(C)$ instead of $H_m(C, A)$ when A is the constant functor $\mathbb{Z}$. It follows that

$$H_m(C) = H_m(BC)$$

where the right side denotes the singular homology of the space BC.

Let $\xi = (p : E \to B)$ be a fibration, and let Σ be a cleavage of ξ. For each arrow $\phi : b \to b'$ in B a *base-change functor* $\phi_* : E_b \to E_{b'}$ is defined as follows: If e is an object of E_b, then $\phi_*(e)$ is the codomain of $\Sigma_{e,\phi}$, and if $f : e \to e'$ is an arrow of E_b, then $\phi_*(f)$ is the unique arrow in $E_{b'}$ such that $\phi_*(f) \circ \Sigma_{e,\phi} = \Sigma_{e',\phi} \circ f$.

Of course, ϕ_* depends on the cleavage, but different cleavages give rise to naturally isomorphic base-change functors, as follows from the universal property of cartesian arrows. In the same fashion, given ϕ, ψ composable arrows of B, there is a natural isomorphism $\psi_*\phi_* \Rightarrow (\psi\phi)_*$. The set of data

$$b \mapsto E_b \qquad \phi \mapsto \phi_* \qquad \psi_*\phi_* \Rightarrow (\psi\phi)_*$$

defines a *pseudofunctor* $B \dashrightarrow \text{Cat}$ (cf. [9]). Note that if Σ is closed then the isomorphisms $\psi_*\phi_* \Rightarrow (\psi\phi)_*$ are identities and one has a true functor $B \to \text{Cat}$.

Given $m \geq 0$, let $H_m(F) : B \to \text{Ab}$ be the functor which assigns to each $b \in B$ the group $H_m(E_b)$, and to each arrow $\phi : b \to b'$ the map induced by ϕ_*. Since isomorphic functors yields homotopic maps between the classifying spaces, the module $H_m(F)$ is well defined (i.e. is a functor) and does not depend on the cleavage Σ.

5.2.1 Theorem. *There is a spectral sequence $\{X^r_{m,n}\}$ which converges to the homology of the total category E and whose second sheet consists of the homology of the base with coefficients in the homology of the fibers.*

$$X^2_{m,n} = H_n(B, H_m(F)) \Rightarrow H_{m+n}(E)$$

Proof. From the bisimplicial set N_fE we construct the free bisimplicial abelian group $\mathbb{Z}N_fE$, and the bicomplex C_fE, whose m,n-th group equals that of $\mathbb{Z}N_fE$ and whose horizontal and vertical differential maps are the alternate sum of the horizontal and vertical faces, respectively. We have that

$$C_fE_{m,n} = \bigoplus_{\underline{b}=(b_0\to\cdots\to b_n)} \mathbb{Z}[(N_fE_{\underline{b}})_m]$$

where $(N_fE_{\underline{b}})_m$ is the set of m,n-simplices of N_fE with base $\underline{b}$. Filtering the bicomplex C_fE in the horizontal direction gives a spectral sequence

$$H_n(H_m(C_fE)) \Rightarrow H_{m+n}(Tot(C_fE)).$$

The first sheet of this spectral sequence is obtained by computing the vertical homology (m-direction) of C_fE. In degree m,n this is equal to

$$H_m(C_fE)_{m,n} = \bigoplus_{\underline{b}} H_m(N_fE_{\underline{b}}) \cong \bigoplus_{\underline{b}} H_m(E_{b_0}),$$

where the isomorphism $\cong$ is that induced by μ (cf. 3.1.1). The second sheet of this spectral sequence is obtained by computing the horizontal homology (n-direction). In degree m,n this is equal to $H_nH_m(C_fE) = H_n(B, H_m(F))$. Finally, by the generalized Eilenberg-Zilber theorem (cf. [8, IV,2.5]) the homology of the total complex $H_{m+n}(Tot(C_fE))$ is isomorphic to the homology of the diagonal $H_{m+n}(d(\mathbb{Z}N_fE))$, which equals $H_{m+n}(E)$ since $d(N_fE)$ and NE are homotopic (cf. 4.1.3). This completes the proof. □

Suppose now that $\xi = (p : E \to B)$ is a fibration whose fibers are homologically trivial, namely $H_m(E_b) = 0$ if $m > 0$ and $H_0(E_b) = \mathbb{Z}$ for all objects b of B. If $m > 0$ then the functors $H_m(F)$ are constant and equal to 0, so the second sheet of the spectral sequence X of 5.2.1 is

$$X^2_{m,n} = \begin{cases} 0 & \text{if } m > 0 \\ H_n(B) & \text{if } m = 0 \end{cases}$$

It follows that $X^\infty = X^2$ and thus the homology of E is that of B. It is not hard to see that $p_* : H_n(E) \to H_n(B)$ is actually the isomorphism.

5.2.2 Corollary. *If a fibration* $\xi = (p : E \to B)$ *is such that the fibers* E_b *are homologically trivial, then* $p_* : H_n(E) \xrightarrow{\sim} H_n(B)$ *is an isomorphism for all* $n \geq 0$.

5.2.3 Corollary (Homology version of Quillen's Theorem A)**.** *Let* $u : A \to B$ *be a map between small categories whose homotopy fibers* A/b *are homologically trivial. Then* $u_* : H_n(A) \xrightarrow{\sim} H_n(B)$ *is an isomorphism for all* $n \geq 0$.

Proof. If $\pi : E^u \to B$ is the fibration associated to u, then its fibers are isomorphic to the homotopy fibers of u and thus π induces isomorphisms in homology by 5.2.2. Since $u = \pi \circ i$ and $i : A \to E^u$ is a weak equivalence (cf. 2.2.1) the result follows. □

5.3 Quillen fibrations and Theorem B

We have seen in many examples that different fibers of a Grothendieck fibration need not have the same homotopy type. This remark shows that in general the map $B_f E \to B_f B$ is not a fibration, nor a quasi-fibration. It is remarkable that this is the only obstruction. In this section we define Quillen fibrations, discuss the monodromy action and reformulate Quillen's Theorem B in terms of the fibred nerve.

We say that a fibration $\xi = (p : E \to B)$ is a *Quillen fibration* if for each arrow $\phi : b \to b'$ in B the base-change functor $\phi_* : E_b \to E_{b'}$ is a weak equivalence. Note that this definition does not depend on the cleavage, for two base-change functors over ϕ must be homotopic.

In a Quillen fibration the induced functor $B \to [\mathrm{Top}]$, $b \mapsto BE_b$ is morphism inverting, therefore it induces a map

$$\pi_1(B) \to [\mathrm{Top}].$$

Here $\pi_1(B)$ denotes the fundamental groupoid of B, namely the groupoid obtained by formally inverting all the arrows of B, and [Top] denotes the category of topological spaces and homotopy classes of continuous maps. We call $\pi_1(B) \to [\mathrm{Top}]$ the *monodromy action* of the fibration. The monodromy action is a first tool to classify Quillen fibrations. In very special situations, it suffices to recover the whole fibration, as we can see in the following example.

Example. (cf. [14]) If $p : E \to B$ is a Quillen fibration with discrete fibers (the only arrows in the fibers are identities), then the base-change functors $E_b \to E_{b'}$ must be bijections. It follows that a Quillen fibration with discrete fibers is essentially the same as a functor $B \to \text{Set} \subset \text{Cat}$ which is morphism inverting, or what is the same, a functor $\pi_1(B) \to$ Set.

A Quillen fibration $p : E \to B$ with discrete fibers should be thought of as a covering of categories. Indeed, they yield coverings after applying the classifying space functor.

One is interest in understand how fibrations behave with respect to the classifying space functor. The next example shows that $B_f E \to B_f B$ need not be a fibration, so we shall look for a notion weaker than that.

Example. Let E be the full subcategory of $I \times I$ with objects $(1, 0)$, $(0, 1)$ and $(1, 1)$. Then the second projection $E \to I$ is a fibration. Since the fibers are contractible it is, in fact, a Quillen fibration. Despite this, the induced map of topological spaces is not a fibration, as one can easily check.

Recall that a *quasifibration* of topological spaces $f : X \to Y$ is a map such that the inclusions of the actual fibers into the homotopy fibers are weak homotopy equivalences. They extend the notion of fibration, and their most important feature is that they yield long exact sequences relating the homotopy groups of the fibre, the total space and the base space.

5.3.1 Theorem. *If $p : E \to B$ is a Quillen fibration, then the induced map $p_* : B_f E \to B_f B$ is a quasifibration of topological spaces.*

Proof. It is essentially that of [14, lemma p.14]. We endow $B_f B = BB$ with the canonical cellular structure. We prove that the restriction of p_* to the n-th skeleton $sk_n(BB)$ is a quasifibration by induction on n, from which the result follows. To prove the inductive step we write $sk_n(BB)$ as the union $U \cup V$, where U is obtained by removing the barycenters of the n-cells and V is the union of the interiors of the n-cells, and prove that p_* is a quasifibration when restricted to U, V and $U \cap V$.

We denote $|N_f E_{\underline{b}}|$ by $B_f E_{\underline{b}}$ (see 3.1). Realizing first in the m-direction, the restriction of p_* to the interior of the n-cell indexed by $\underline{b} \in NB_n$ can be identified with the restriction of the projection $B_f E_{\underline{b}} \times \Delta^n \to \Delta^n$ to the interior of the topological n-simplex Δ^n. It follows from this that $p_*|_V$ and $p_*|_{U\cap V}$ are quasifibrations.

We deform $p_*^{-1}(U)$ into $p_*^{-1}(sk_{n-1}(B_f B))$ by using the radial deformation of Δ^n minus its barycenter into $\partial\Delta^n$, and use the inductive assumption to conclude that $p_*|_U$ is a quasifibration. We must verify that if this deformation carries x into x', then the map $g : p_*^{-1}(x) \to p_*^{-1}(x')$ induced by the deformation is a weak homotopy equivalence. Let x be a point in the interior of the n-cell indexed by $\underline{b} \in NB_n$. If the radial deformation push x into the open cell indexed by the face $\underline{b}'$ of $\underline{b}$, then $p_*^{-1}(x) = B_f E_{\underline{b}}$ and $p_*^{-1}(x') = B_f E_{\underline{b}'}$. Fixed a cleavage Σ, the composition (cf. 3.1.1)

$$BE_{b_0} \xrightarrow{\nu} B_f E_{\underline{b}} \xrightarrow{g} B_f E_{\underline{b}'} \xrightarrow{\mu} BE_{b_0'},$$

(with $\nu = \nu(\underline{b})$ and $\mu = \mu(\underline{b}')$) equals a base-change functor over the arrow $b_0 \to b_0'$ of $\underline{b}$ (more precisely, it is the composition of the base-changes given by Σ over the arrows $b_i \to b_{i+1}$). Since p is a Quillen fibration, and ν and μ are weak equivalences, it follows from the 3-for-2-property that g is a homotopy equivalence and thus the result. □

The last theorem shows an interesting feature of the fibred nerve: it carries Quillen fibrations into quasifibrations. The question of whether or not $BE \to BB$ is a quasifibration is rather unclear, and this can be understood as a disadvantage of the classic nerve when dealing with fibrations.

5.3.2 Corollary (Quillen's Theorem B). *If $u : A \to B$ is a map between small categories such that $A/b \to A/b'$ is a weak equivalence for all $b \to b' \in \mathrm{fl}(B)$, then there is a long exact sequence*

$$\dots \xrightarrow{\partial} \pi_k(A/b, \overline{a}) \to \pi_k(A, a) \xrightarrow{u_*} \pi_k(B, b) \xrightarrow{\partial} \pi_{k-1}(A/b, \overline{a}) \to \dots$$

where $a \in \mathrm{ob}(A)$, $b = u(a)$ and $\overline{a} = (a, \mathrm{id}_b)$.

Proof. Let $i : A \to E^u$ be the canonical map into the associated fibration, r its right adjoint and $w : A/b \to A$ be the map $(a, u(a) \to b) \mapsto a$.

In the diagram

$$\begin{array}{ccccc} A/b & \xrightarrow{w} & A & \xrightarrow{u} & B \\ \| & & \uparrow r & & \| \\ E^u_b & \hookrightarrow & E^u & \xrightarrow{\pi} & B \end{array}$$

the left square commutes, and since $\pi i = u$ and r is a homotopy inverse to i the right square commutes up to homotopy. We conclude that the homotopy groups of A/b, A and B can be identified naturally with that of the base, the fiber and the total category of the associated fibration $E^u \to B$. It is a Quillen fibration, and thus the result follows from 5.3.1. □

5.4 Group actions and TCP

Regarding small categories as combinatorial models for homotopy types, it is natural to investigate how they behave under the action of a group. In this section we derive a splitting fibration from a small category endowed with a group action, and relate its cleaved nerve with a twisted cartesian product in the sense of [12]. We also study the spectral sequence 5.2.1 in this particular case.

A simplicial group G *operates on a simplicial set* K *(from the left)* if there is a simplicial map $G \times K \to K$, $(g, k) \mapsto g \cdot k$ satisfying $1_n \cdot k = k$ for all $k \in K_n$ and $g_1 \cdot (g_2 \cdot k) = (g_1 g_2) \cdot k$ for all $k \in F_n$ and $g_1, g_2 \in G_n$. Here 1_n denotes the unit of G_n. Given A, B simplicial sets and G a simplicial group which operates on A, a *twisted cartesian product (TCP)* with fibre A, base B and group G is a simplicial set $A \times_\tau B$ with simplices $(A \times_\tau B)_n = A_n \times B_n$ and faces and degeneracies given by

$$d_i(a,b) = \begin{cases} (d_i a, d_i b) & i > 0 \\ (\tau(b) \cdot d_0 a, d_0 b) & i = 0 \end{cases} \qquad s_i(a,b) = (s_i a, s_i b), \quad i \geq 0.$$

Here $\tau : B_n \to G_{n-1}$ is a function which must satisfy some standard identities in order to make $A \times_\tau B$ a simplicial set. This τ is called the *twisting function*.

Let G be a group, and let A be a small category on which G acts. This action can be seen as a group morphism $G \to Aut(A)$, $g \mapsto u_g$, or equivalently, as a functor $G \to \text{Cat}$ that maps the unique object of G to $A \in \text{ob(Cat)}$. The Grothendieck construction over this functor is a splitting fibration $p : G \rtimes A \to G$ over G.

The constant simplicial group G (in which every face and degeneracy operator is the identity) operates on NA from the left via the formula

$$g \cdot (a_0 \to \cdots \to a_n) = (u_g(a_0) \to \cdots \to u_g(a_n)).$$

5.4.1 Proposition. *The diagonal of the cleaved nerve $dN_c(G \rtimes A)$ can be regarded as a TCP between the nerves of A and G, namely $NA \times_\tau NG$.*

Proof. Let $\tau : NG_n \to G_{n-1} = G$ be the projection $\tau(* \xrightarrow{g_1} * \xrightarrow{g_2} \dots \xrightarrow{g_n} *) = g_1$, and let $NA \times_\tau NG$ be the TCP with twisting function τ. We define a simplicial map $\varphi : dN_c(G \rtimes A) \to NA \times_\tau NG$ by giving to each simplex $s \in N_c(G \rtimes A)_{n,n}$ the pair (a, b), where a is the mast of s and b is its base. One checks easily that φ is actually a simplicial map, and it is an isomorphism because $G \rtimes A \to G$ is splitting, together with 3.2.1. □

Given G acting on A, the fibration $G \rtimes A \to G$ has a unique fiber, which is isomorphic to A. Thus the modules $H_m(F)$ of the spectral sequence 5.2.1 are just the homology groups of A endowed with the action of G. Writing $A//G = G \rtimes A$ for the homotopy theoretic quotient (cf. 5.1.2) we obtain the following version of the Eilenberg-Moore spectral sequence (cf. [1, p.775]) as an application of 5.2.1.

5.4.2 Proposition. *There is a spectral sequence $\{X^r_{m,n}\}$ which converges to the homology of the homotopy theoretic quotient $A//G$ and whose second sheet consists of the group homology of G with coeficients in the homology of A.*

$$X^2_{m,n} = H_n(G, H_m(A)) \Rightarrow H_{m+n}(A//G)$$

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A LAX SYMMETRIC CUBICAL CATEGORY ASSOCIATED TO A DIRECTED SPACE*

by Marco GRANDIS

Résumé. Le domaine récent de la topologie algébrique dirigée étudie les "espaces dirigés", où chemins et homotopies peuvent être non réversibles. Les applications principales concernent la programmation parallèle.

On introduit ici, pour un espace dirigé, une catégorie fondamentale de *dimension infinie*, de type cubique *lax*: les cubes singuliers de l'espace ont une structure cubique, où les concatenations sont associatives à une re-paramétrisation invertible près, mais les dégénérescences sont seulement lax-unitaires. En outre cette structure est *symétrique*, par permutation des variables des cubes singuliers, ce qui simplifie les propriétés de cohérence.

Les "cubes de Moore" de l'espace donnent une catégorie cubique *stricte*, moyennant une construction similaire.

Abstract. The recent domain of directed algebraic topology studies 'directed spaces', where paths and homotopies need not be reversible. The main applications are concerned with concurrency.

We introduce here, for a directed space, an *infinite dimensional* fundamental category, of a *lax* cubical type: the singular cubes of the space have a cubical structure, where concatenations are associative up to invertible reparametrisation while degeneracies are only lax-unital. Moreover, this structure is *symmetric*, by permuting the variables of singular cubes; this simplifies the coherence properties.

By a similar construction, the 'Moore cubes' of the space give a *strict* symmetric cubical category.

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Introduction

Directed algebraic topology studies structures with *privileged directions*, like 'directed spaces' in some sense: for instance, ordered or locally ordered topological spaces, 'spaces with distinguished paths' (examined below), simplicial and cubical sets, etc. Such objects have *directed* paths and homotopies, which need not be reversible. The present applications deal mostly with the analysis of concurrent processes, see [FGR1, FGR2, FRGH, Ga1, Ga2, GG, GH, Go, R1, R2], but the theory aims to model non-reversible phenomena in any domain. Directed algebraic topology is the subject of a recent issue of the journal 'Applied Categorical Structures', guest-edited by the present author (vol. 15, no. 4, 2007), and of a recent book [G10]. The ideas at the basis of the present paper have been exposed at the conference 'Applied Topological Methods in Computer Sciences III', Paris 2008.

Directed spaces can be studied with homology and homotopy theories, modified to keep an account of privileged directions: namely, *preordered* homology groups [G3] and fundamental higher *categories* (in some sense) instead of the classical homology groups and fundamental higher *groupoids* of algebraic topology. Thus, directed algebraic topology is more clearly linked with higher dimensional category theory, and can also yield some geometric intuition to the latter.

Here, we want to study an infinite dimensional version of the fundamental category of a *d-space*, or *space with distinguished paths* (1.1), our main notion of directed space, which was introduced in [G2] and also studied in various works by various authors [G4-G6, G10, FhR, FjR, R2, Bu, Ga3]. While there is no problem in defining the *fundamental category* $\uparrow\Pi_1(X)$ of a d-space [G2], the construction of higher versions is complicated, even in dimension 2: see [G4] for a strict 2-categorical version and [G5, G6] for lax ones.

The present approach is cubical, rather than globular, and follows a study of weak cubical categories begun by the author in [G7-G9, G11]. We start from the standard n-dimensional cube $\mathbf{I}^n = [0, 1]^n$ and its *directed* version $\uparrow\mathbf{I}^n$ (1.2). The singular cubes of a d-space X, i.e. the maps $\uparrow\mathbf{I}^n \rightarrow X$ of d-spaces, form a 'basic symmetric pre-cubical category' $\uparrow\square X$ (Section 1), i.e. a symmetric cubical set equipped with concatenation laws in all directions, satisfying various geometrical properties and linked by transposition symmetries; the term 'basic' means that these operations are not (yet) required to satisfy associativity, interchange and unitarity, in any sense - even weak or lax.

To take these aspects into account, we formalise in Section 2 the notion of a *u-lax symmetric cubical category*. The previous framework (a basic symmetric pre-cubical category) is enriched with *transversal maps* between n-dimensional objects; these maps include *comparisons* for associativity, interchange and unitarity, which are only assumed to be invertible in the first two cases. The new structure is thus a generalisation of a *weak symmetric cubical category* introduced in [G7, G9]; it is very similar to the 'quasi cubical' case considered in [G8] for higher cospans composed with homotopy pushouts, in relation with higher dimensional cobordism.

Then, in Sections 3 and 4, we make the previous structure $\uparrow\Box X$ into the singular u-lax symmetric cubical category $\uparrow\mathbb{S}ng(X)$ of the d-space X, by adding transversal maps and comparisons. Here, a transversal map $f: x \to y$ between two singular cubes $x, y: \uparrow\mathbf{I}^n \to X$ is given by a *reparametrisation mapping* $f: \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n$ such that $x = yf$; the obvious *transversal composition* of such maps is strictly categorical. The operations of concatenation of the singular cubes become thus *weakly associative* (up to invertible reparametrisations) and *lax unital* (up to non-invertible reparametrisations), while interchange - here - works strictly. The non-directed structure $\mathbb{S}ng(X)$ associated to a topological space X is briefly described in 4.6.

In Section 5 we outline a *strict* version of the previous framework. It is based on the *Moore* directed cubes of a d-space, defined on products of directed intervals of variable length $a_h \geq 0$

(1) $I(a_1,..., a_n) = \prod_{h=1,...,n} \uparrow[0, a_h].$

These have operations of concatenation that are *strictly associative and unital*, also because we allow these intervals to be degenerate. Transversal maps are given by 'Moore reparametrisations', but their role is less evident here, since no comparisons are needed: we get a strict symmetric cubical category $\uparrow\mathbb{M}\mathrm{Sng}(X)$.

We end, in Section 6, with a few hints to a family $\mathbb{T}(A)$ of u-lax symmetric cubical categories, depending on a topological space A, and related to higher categories of tangles, as considered in [BL, Ch]. This family is constructed starting from $\mathbb{T} = \mathbb{S}ng(\mathbf{S}^0)$, the u-lax symmetric cubical category associated to the discrete space on two points $\mathbf{S}^0$, where a singular n-cube $x: \mathbf{I}^n \to \mathbf{S}^0$ can be identified to a subset of $\mathbf{I}^n$.

As a general principle of higher category theory, weak structures seem to be more important than the strict ones; this is why the strict structure of Moore cubes has here a marginal position. Let us also recall that interest is arising in category theory and algebraic topology for categorical structures (possibly higher dimensional) with lax units or 'no units' (see [MBB, Ko1, Ko2, JK, G8, G12]).

References to the rich literature on higher categories can be found in two recent books, by T. Leinster [Le] and E. Cheng, A. Lauda [CL]; but these works are mostly developed in the globular approach, rather than the cubical one. Strict cubical categories with connections (and no transversal maps) are studied in [ABS], and proved to be equivalent to the ordinary (globular) ω-categories. Weak symmetric cubical categories have been studied by the present author [G7-G9, G11]; pseudo double categories are a truncated version of the latter, studied in [GP] and three subsequent papers by the same authors.

Cubical sets have been extensively studied by R. Brown and P.J. Higgins, which introduced their connections in [BH1, BH2]. The present author began a systematic use of their symmetries in [G1]. There is a recent preprint on symmetric cubical sets, by S.B. Isaacson [Is], which investigates their non-directed homotopy theory.

For a (non-directed) topological space X, a recent preprint by R. Brown [Br] deals with a cubical structure $M_*(X)$ based on 'Moore hyperrectangles', equivalent to our Moore cubes (see a note at the end of Section 5.3). With respect to the present structure $\uparrow\mathbb{M}\mathrm{Sng}(X)$, the 'strict cubical category' $M_*(X)$ has *connections* and no *transpositions* nor *transversal maps*; it might be called a 'basic cubical category with connections', in the present terminology - where a 'cubical category' is always assumed to have transversal maps.

One dimensional reparametrisation mappings $f: \uparrow\mathbf{I} \to \uparrow\mathbf{I}$, in the same sense as here, have been studied in [G5, G6, FhR, R3].

As a matter of notation, the indices α, β take the values 0, 1, that are more often written as $-, +$. $\mathbf{I}$ denotes the standard interval $[0, 1]$ with euclidean topology.

1. The singular cubes of a d-space and their concatenations

We briefly recall the notion of d-space, introduced in [G2]. Then we show that the singular (directed) cubes of a d-space X, with obvious concatenations in all directions, form a 'basic symmetric pre-cubical category' $\uparrow\Box X$. These operations satisfy a strict middle-four interchange; their weak associativity and lax unitarity properties will be studied in Section 4, after developing adequate structures.

1.1. Spaces with distinguished paths. A *d-space* X, or *space with distinguished paths*, is a topological space equipped with a set dX of (continuous) maps $a: \mathbf{I} \to X$, called *distinguished paths* or *directed paths* or *d-paths*, satisfying three axioms:

(i) (*constant paths*) every constant map $\mathbf{I} \to X$ is distinguished,

(ii) (*partial reparametrisation*) dX is closed under composition with every (weakly) increasing map $\mathbf{I} \to \mathbf{I}$,

(iii) (*concatenation*) dX is closed under path-concatenation: if the d-paths a, b are consecutive in X (i.e. $a(1) = b(0)$), then their ordinary concatenation $a + b$ is also a d-path.

A *directed map* $f: X \to Y$ (or *d-map*, or *map* of d-spaces) is a continuous mapping between d-spaces which preserves the directed paths: if $a \in dX$, then $fa \in dY$.

The category of d-spaces is written as d**Top**. It has all limits and colimits, constructed as in **Top** and equipped with the initial or final d-structure for the structural maps; for instance a path $\mathbf{I} \to \Pi X_j$ with values in a product is directed if and only if all its components $\mathbf{I} \to X_j$ are so. The forgetful functor U: d**Top** $\to$ **Top** preserves thus all limits and colimits; a topological space is generally viewed as a d-space by its *natural* structure, where all paths are directed (via the right adjoint to U).

Reversing d-paths, by the involution $r(t) = 1 - t$, yields the *opposite* d-space $RX = X^{op}$, where $a \in d(X^{op})$ if and only if ar is in dX. This defines the *reversor* endofunctor

(1) R: d**Top** $\to$ d**Top**, $\quad\quad RX = X^{op}$.

A d-space X is said to be *reversible* if it coincides with X^{op}, and *reflexive* if it is isomorphic to the latter.

1.2. Standard objects. The *directed real line*, or *d-line* $\uparrow\mathbf{R}$, is the euclidean line with directed paths given by the (weakly) increasing maps $\mathbf{I} \to \mathbf{R}$. Its cartesian power in d**Top**, the n-*dimensional real d-space* $\uparrow\mathbf{R}^n$ is similarly described (with respect to the product order of $\mathbf{R}^n$, $x \leq y$ if $x_i \leq y_i$ for all i). The *standard d-interval* $\uparrow\mathbf{I} = \uparrow[0, 1]$ has the subspace structure of the d-line; the *standard d-cube* $\uparrow\mathbf{I}^n$ is its n-th power, and a subspace of $\uparrow\mathbf{R}^n$ (with the induced structure). These d-spaces are not reversible (for $n > 0$), but they are all reflexive.

The *standard directed circle* $\uparrow\mathbf{S}^1$ will be the standard circle with the *anticlockwise structure*, where the directed paths $a: \mathbf{I} \to \mathbf{S}^1$ move this way, in the oriented plane $\mathbf{R}^2$: $a(t) = (\cos\vartheta(t), \sin\vartheta(t))$, with an increasing (continuous) argument $\vartheta: \mathbf{I} \to \mathbf{R}$.

$\uparrow\mathbf{S}^1$ can be obtained as the coequaliser in d**Top** of the following pair of maps

(1) $\partial^-, \partial^+: \{*\} \rightrightarrows \uparrow\mathbf{I}, \qquad \partial^-(*) = 0, \quad \partial^+(*) = 1.$

Indeed, the ordinary construction of this coequaliser is the quotient $\uparrow\mathbf{I}/\partial\mathbf{I}$, which identifies the endpoints; the d-structure of the quotient (*generated* by the projected paths) is the desired one precisely because of the axioms on concatenation and reparametrisation of d-paths.

The directed circle can also be described as an orbit d-space

(2) $\uparrow\mathbf{S}^1 = \uparrow\mathbf{R}/\mathbf{Z},$

with respect to the action of the group of integers on the directed line $\uparrow\mathbf{R}$, by translations; in this quotient, the distinguished paths of $\uparrow\mathbf{S}^1$ are simply the projections of the increasing paths in the line.

The *directed n-dimensional sphere* is defined, for $n > 0$, as the quotient of the directed cube $\uparrow\mathbf{I}^n$ modulo the equivalence relation which collapses its (ordinary) boundary $\partial\mathbf{I}^n$ to a single point, while $\uparrow\mathbf{S}^0$ has the discrete topology and the natural d-structure (obviously discrete)

(3) $\uparrow\mathbf{S}^n = (\uparrow\mathbf{I}^n)/(\partial\mathbf{I}^n) \quad (n > 0), \qquad \uparrow\mathbf{S}^0 = \mathbf{S}^0 = \{-1, 1\}.$

All directed spheres are reflexive.

1.3. Directed interval and paths. A (standard) *path* in a d-space X is a d-map a: $\uparrow\mathbf{I} \to X$ defined on the standard d-interval. Plainly, this is the same as a structural map $a \in dX$, and will *also* be called a *directed* path when we want to stress the difference from ordinary paths in the underlying space UX.

The basic, 'first order' structure of $\uparrow\mathbf{I}$ consists of four maps, linking its 0-th cartesian power, the singleton $\uparrow\mathbf{I}^0 = \{*\}$, to $\uparrow\mathbf{I}$ or to the opposite d-space $\uparrow\mathbf{I}^{op}$

(1) $\partial^\alpha : \{*\} \rightrightarrows \uparrow\mathbf{I}$, $\quad \partial^-(*) = 0, \quad \partial^+(*) = 1$ (*faces*),

$e: \uparrow\mathbf{I} \to \{*\}$, $\quad e(t) = *$ (*degeneracy*),

$r: \uparrow\mathbf{I} \to \uparrow\mathbf{I}^{op}$, $\quad r(t) = 1 - t$ (*reflection*).

Identifying a point x of the space X with the corresponding map $x: \{*\} \to X$, this basic structure determines:

(a) the endpoints of a path $a: \uparrow\mathbf{I} \to X$, i.e. $\partial^-(a) = a\partial^- = a(0)$, $\partial^+(a) = a\partial^+ = a(1)$,

(b) the trivial path at the point x, i.e. $0_x = e(x) = xe$,

(c) the *reflected path of* a *in* X^{op}, i.e. $r(a) = (Ra).r: \uparrow\mathbf{I} \to \uparrow\mathbf{I}^{op} \to X^{op}$.

Two consecutive paths $a, b: \uparrow\mathbf{I} \to X$ $(\partial^+(a) = \partial^-(b)$, i.e. $a(1) = b(0))$ have a concatenated path $a + b$, which is distinguished, by definition of d-structure. This amounts to saying that, in d**Top**, the *standard concatenation pushout* – pasting two copies of the d-interval, one after the other – can be realised as $\uparrow\mathbf{I}$ itself (as for spaces: pasting two copies of **I** gives **I**)

(2)
$$\begin{array}{ccc} \{*\} & \xrightarrow{\partial^+} & \uparrow\mathbf{I} \\ {\scriptstyle\partial^-}\downarrow & & \downarrow{\scriptstyle c^-} \\ \uparrow\mathbf{I} & \xrightarrow[c^+]{} & \uparrow\mathbf{I} \end{array} \qquad c^-(t) = t/2, \quad c^+(t) = (t+1)/2.$$

This pushout is preserved by cartesian product with any fixed d-space ([G2], Lemma 1.8).

Finally, there is a 'second order' structure which involves the standard directed square $\uparrow\mathbf{I}^2 = [0,1] \times [0,1]$ and is used to construct (directed) homotopies of (directed) paths

(3) $g^-: \uparrow\mathbf{I}^2 \to \uparrow\mathbf{I}$, $\quad g^-(t, t') = \max(t, t')$ (*lower connection*),

$g^+: \uparrow\mathbf{I}^2 \to \uparrow\mathbf{I}$, $\quad g^+(t, t') = \min(t, t')$ (*upper connection*),

$s: \uparrow\mathbf{I}^2 \to \uparrow\mathbf{I}^2$, $\quad s(t, t') = (t', t)$ (*transposition*).

Together with (1), these maps complete the structure of $\uparrow\mathbf{I}$ as a *lattice* in **Top** (isomorphic to the opposite lattice, via r). The choice of the superscripts of g^-, g^+ comes from the fact that the unit of g^α is $\partial^\alpha(*)$. Within homotopy theory, the importance of these binary operations has been highlighted by R. Brown and P.J. Higgins [BH1, BH2], which introduced the term of *connection*, or *higher degeneracy* (with a notation similar to the previous one for faces, degeneracy and connections: ∂^α, ε, Γ^α; notice that for simplicial sets the letter s generally denotes degeneracies).

Here, we will use the transposition symmetry s, but not the connections; the article [Br] shows as the latter can be used in the context of Moore cubes (or standard cubes, of course).

1.4. The singular symmetric cubical set of a d-space. Every d-space X has an associated *symmetric cubical set* - a notion whose general definition will be recalled below (see 1.5)

(1) $\uparrow\square X = ((\uparrow\square_n X), (\partial_i^\alpha), (e_i), (s_i)).$

Firstly, the component of $\uparrow\square X$ in degree $n \geq 0$ is the set of *singular (directed) n-cubes* of X, which will also be called *n-cubes* of X

(2) $\uparrow\square_n X = d\mathbf{Top}(\uparrow\mathbf{I}^n, X).$

In particular, a 0-cube $x: \uparrow\mathbf{I}^0 \to X$ is identified with a point of X, and a 1-cube $x: \uparrow\mathbf{I} \to X$ is a (directed) path.

Secondly, after the basic structure recalled above, the *higher faces*, *degeneracies* and *transpositions* of the standard cubes are defined as follows

(3) $\partial_i^\alpha \ = \ \uparrow\mathbf{I}^{i-1} \times \partial^\alpha \times \uparrow\mathbf{I}^{n-i}\colon \ \uparrow\mathbf{I}^{n-1} \to \uparrow\mathbf{I}^n, \qquad \partial_i^\alpha(t_1,..., t_{n-1}) \ = \ (t_1,..., \alpha,..., t_{n-i}),$

$e_i \ = \ \uparrow\mathbf{I}^{i-1} \times e \times \uparrow\mathbf{I}^{n-i}\colon \ \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^{n-1}, \qquad e_i(t_1,..., t_n) \ = \ (t_1,..., \hat{t}_i,..., t_n),$

$s_i \ = \ \uparrow\mathbf{I}^{i-1} \times s \times \uparrow\mathbf{I}^{n-i}\colon \ \uparrow\mathbf{I}^{n+1} \to \uparrow\mathbf{I}^{n+1}, \qquad s_i(t_1,..., t_{n+1}) \ = \ (t_1,..., t_{i+1}, t_i,..., t_{n+1}),$

where $\alpha = 0, 1$, $i = 1,..., n$ (and, as usual, $\hat{t}_i$ means to omit the coordinate t_i).

These maps produce (contravariantly, by pre-composition) the *faces*, *degeneracies* and *transpositions* of our symmetric cubical set $\uparrow\square X$, which will be denoted by the *same* symbols

(4) $\partial_i^\alpha\colon \uparrow\square_n X \to \uparrow\square_{n-1}X, \qquad \partial_i^\alpha(x) \ = \ x.\partial_i^\alpha,$

$e_i\colon \uparrow\square_{n-1}X \to \uparrow\square_n X, \qquad e_i(x) \ = \ x.e_i,$

$s_i\colon \uparrow\square_{n+1}X \to \uparrow\square_{n+1}X \qquad s_i(x) \ = \ x.s_i \qquad (\alpha = 0, 1, \ i = 1,..., n).$

Every n-cube $x\colon \uparrow\mathbf{I}^n \to X$ has 2^n vertices: $\partial_1^\alpha\partial_2^\beta\partial_3^\gamma(x) = \partial_1^\gamma\partial_1^\beta\partial_1^\alpha(x)$, for $n = 3$.

The contravariant action of the transpositions $s_1,..., s_{n-1}$ on $\uparrow\square_n X$ can obviously be extended to a (*right*) action of the group of permutations of the coordinates of $\mathbf{I}^n$. This amounts to saying that the transpositions s_i satisfy the *Moore relations*, under which they generate the symmetric group S_n

(5) $s_i.s_i \ = \ 1, \qquad s_i.s_j.s_i \ = \ s_j.s_i.s_j \quad (i = j-1), \qquad s_i.s_j \ = \ s_j.s_i \quad (i < j-1),$

(see Coxeter-Moser [CM], 6.2; or Johnson [Jo], Section 5, Thm. 3).

Notice also that we have applied the functors

(6) $(-)_i^n \ = \ \uparrow\mathbf{I}^{i-1} \times - \times \uparrow\mathbf{I}^{n-i}\colon \ \mathrm{d}\mathbf{Top} \to \mathrm{d}\mathbf{Top} \qquad (i = 1,..., n),$

to deduce the higher structural maps (3) from the basic ones, ∂^α, e, s, introduced in 1.3. This procedure is usual in homotopy theory based on a standard interval, and will be repeatedly used below.

1.5. Symmetric cubical sets. Let us recall some points on the classical notion of cubical set (see [K1, K2, BH1, BH2]) and the less known notion of *symmetric* cubical set.

A cubical set $X = ((X_n), (\partial_i^\alpha), (e_i))$ is a sequence of sets $(X_n)_{n\geq 0}$ equipped with *faces* (∂_i^α) and *degeneracies* (e_i)

(1) $\partial^\alpha_i: X_n \rightleftarrows X_{n-1} :e_i$ (i = 1,..., n; $\alpha = \pm$),

satisfying the cubical relations :

(2) $\partial^\alpha_i.\partial^\beta_j = \partial^\beta_j.\partial^\alpha_{i+1}$ $(j \leq i)$, $e_j.e_i = e_{i+1}.e_j$ $(j \leq i)$,

$\partial^\alpha_i.e_j = e_j.\partial^\alpha_{i-1}$ $(j < i)$, *or* id $(j = i)$, *or* $e_{j-1}.\partial^\alpha_i$ $(j > i)$.

A *morphism* $f = (f_n): X \to Y$ is a sequence of mappings $f_n: X_n \to Y_n$ commuting with faces and degeneracies. All this forms a category **Cub**, which is a category of presheaves: a cubical set can be viewed as a functor $X: \mathbb{I}^{op} \to$ **Set**, where $\mathbb{I}$ is the subcategory of **Set** consisting of the *elementary cubes* $2^n = \{0, 1\}^n$, together with the maps $\{0, 1\}^m \to \{0, 1\}^n$ which delete some coordinates and insert some 0's and 1's, without modifying the order of the remaining coordinates [GM]. Therefore, **Cub** has all limits and colimits and is cartesian closed. However, the important monoidal structure is the Kan tensor product, which is non-symmetric and biclosed [BH2] (but this is not used here).

A *symmetric cubical set* [GM, G7] is a cubical set which is further equipped with *transpositions*

(3) $s_i: X_n \to X_n$ (i = 1,..., n–1; $n \geq 2$).

which satisfy the Moore relations (1.4.5) and the following coherence conditions:

		$j < i$	$j = i$	$j = i+1$	$j > i+1$
(4)	$\partial^\alpha_j.s_i =$	$s_{i-1}.\partial^\alpha_j$	∂^α_{i+1}	∂^α_i	$s_i.\partial^\alpha_j$,
	$s_i.e_j =$	$e_j.s_{i-1}$	e_{i+1}	e_i	$e_j.s_i$.

Because of the Moore relations, the symmetric group S_n operates on X_n.

A *morphism of symmetric cubical sets* $f = (f_n): X \to Y$ is a sequence of mappings $f_n: X_n \to Y_n$ commuting with faces, degeneracies and transpositions. The resulting category s**Cub** is again a category of presheaves $X: \mathbb{I}_s{}^{op} \to$ **Set**, for the *symmetric cubical site* $\mathbb{I}_s$. The latter can be defined as the subcategory of **Set** consisting of the elementary cubes $2^n = \{0, 1\}^n$ together with the maps $2^m \to 2^n$ which delete some coordinates, permute the remaining ones and insert some 0's and 1's. It is a subcategory of the *extended cubical site* $\mathbb{K}$ of [GM], which also contains the connections.

Again, s**Cub** has all limits and colimits and is cartesian closed; moreover, it inherits from **Cub** a *symmetric* monoidal closed structure [G9] and *one* internal hom (that is not used here).

1.6. An equivalent presentation. The presence of transpositions makes all faces and degeneracies determined by those belonging to a fixed direction, e.g. the 1-indexed ones, ∂_1^α and e_1. In fact, from $\partial_{i+1}^\alpha = \partial_i^\alpha.s_i$ and $e_{i+1} = s_i.e_i$, we deduce that:

(1) $\partial_{i+1}^\alpha \ = \ \partial_1^\alpha.s_1. \ldots .s_i, \qquad\qquad e_{i+1} \ = \ s_i. \ldots .s_1.e_1.$

Thus, as proved in [G8],1.2, a symmetric cubical set can be equivalently defined as a system

(2) $X \ = \ ((X_n), (\partial_1^\alpha), (e_1), (s_i)),$

$\partial_1^\alpha \colon X_n \rightleftarrows X_{n-1} \colon e_1, \qquad\qquad s_i \colon X_{n+1} \to X_{n+1} \qquad\qquad (n \geq 1),$

under the Moore relations for transpositions (1.4.5) and the axioms:

(3) $\partial_1^\alpha.\partial_1^\beta \ = \ \partial_1^\beta.\partial_1^\alpha.s_1, \qquad s_i.\partial_1^\alpha \ = \ \partial_1^\alpha.s_{i+1}, \qquad \partial_1^\alpha.e_1 \ = \ \mathrm{id},$

$e_1e_1 \ = \ s_1.e_1e_1, \qquad e_1.s_i \ = \ s_{i+1}.e_1.$

1.7. A basic symmetric pre-cubical category. The symmetric cubical set $\uparrow\square X$ can be further equipped with partial operations of *concatenation in direction i*, or *i-concatenation*, or *i-composition* (with $i = 1,\ldots, n$ for n-dimensional cubes); globally, we will speak of *cubical compositions* (as opposed to the transversal composition that will be introduced later).

Indeed, acting on the concatenation pushout (1.3.2), the functors $(-)_i^n$ (1.4.6) produce the *n-dimensional i-concatenation pushout*, with embeddings $c_i^\alpha \colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n$

(1)
$$\begin{array}{ccc} \uparrow\mathbf{I}^{n-1} & \xrightarrow{\ \partial_i^+\ } & \uparrow\mathbf{I}^n \\ {\scriptstyle\partial_i^-}\downarrow & & \downarrow{\scriptstyle c_i^-} \\ \uparrow\mathbf{I}^n & \xrightarrow[\ c_i^+\ ]{} & \uparrow\mathbf{I}^n \end{array} \qquad \begin{array}{l} c_i^\alpha \ = \ \uparrow\mathbf{I}^{i-1} \times c^\alpha \times \uparrow\mathbf{I}^{n-i} \colon \ \uparrow\mathbf{I}^n \to \ \uparrow\mathbf{I}^n, \\ c^-(\ldots, t_i,\ldots) \ = \ (\ldots, t_i/2,\ldots), \\ c^+(\ldots, t_i,\ldots) \ = \ (\ldots, (t_i + 1)/2,\ldots). \end{array}$$

(We have already recalled that the basic concatenation pushout is preserved by products with fixed d-spaces.) Now, given two *i-consecutive* n-cubes $x, y\colon \uparrow\mathbf{I}^n \to X$ (with $\partial_i^+ x = \partial_i^- y$), their *i-concatenation* $z = x +_i y$ is computed on the previous pushout

(2) $z\colon \uparrow\mathbf{I}^n \to \mathbf{X}, \qquad\qquad z.c_i^- = x, \quad z.c_i^+ = y.$

$\uparrow\Box X$ becomes thus a *basic symmetric pre-cubical category*, i.e. a symmetric cubical set with 'geometrically consistent' cubical compositions. (This structure was called a 'reduced symmetric pre-cubical category' in [G9], Section 3.5.) More precisely, this means that $\uparrow\Box X$ is a symmetric cubical set with the following additional structure.

For $1 \leq i \leq n$, the i-*concatenation* $x +_i y$ (or i-composition) of two n-cubes x, y is defined when x, y are *i-consecutive*, i.e. $\partial_i^+(x) = \partial_i^-(y)$, and satisfies the following 'geometric' relations with faces, degeneracies and transpositions:

(3) $\partial_i^-(x +_i y) = \partial_i^-(x), \qquad \partial_i^+(x +_i y) = \partial_i^+(y),$

$\partial_j^\alpha(x +_i y) = \partial_j^\alpha(x) +_{i-1} \partial_j^\alpha(y) \qquad (j < i),$

$= \partial_j^\alpha(x) +_i \partial_j^\alpha(y) \qquad (j > i),$

(4) $e_j(x +_i y) = e_j(x) +_{i+1} e_j(y) \qquad (j \leq i \leq n),$

$= e_j(x) +_i e_j(y) \qquad (i < j \leq n+1)$ (*nullary interchange*).

(5) $s_{i-1}(x +_i y) = s_{i-1}(x) +_{i-1} s_{i-1}(y), \qquad s_i(x +_i y) = s_i(x) +_{i+1} s_i(y),$

$s_j(x +_i y) = s_j(x) +_i s_j(y) \qquad (j \neq i-1, i).$

There are no other conditions: in the definition of a basic symmetric *pre*-cubical category we are not assuming that the i-compositions behave in a categorical way or satisfy the binary interchange law, in any sense - strict or weak or lax.

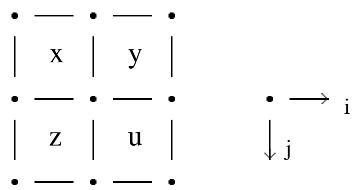
However, for the singular structure $\uparrow\Box X$ which we are studying, the *binary interchange law* holds strictly. Indeed, for $1 \leq i < j \leq n$, and n-cubes x, y, z, u, we obviously have

(6) $(x +_i y) +_j (z +_i u) = (x +_j z) +_i (y +_j u)$ (*middle-four interchange*),

whenever these compositions make sense:

(7) $\partial_i^+(x) = \partial_i^-(y), \quad \partial_i^+(z) = \partial_i^-(u),$

$\partial_j^+(x) = \partial_j^-(z), \quad \partial_j^+(y) = \partial_j^-(u),$



Comparisons for associativity and unitarity of singular cubes will be introduced in Section 4.

2. Weak symmetric cubical categories with lax units

We now define a notion of cubical structure adapted to the present situation, and called a *u-lax symmetric cubical category*. It is a generalisation of the weak case introduced in [G7, G9] and is similar to the 'quasi cubical' case considered in [G8] for higher cospans composed with homotopy pushouts (the latter is even more relaxed, with weaker cubical relations for degeneracies).

Here we allow the comparisons for left and right unitarity to be *non-invertible* and *directed towards simpler expressions*, while we require that the comparisons for associativity and interchange be invertible; indeed, this is the situation that we find in our leading examples (like singular cubes, here, or cubical cospans in [G8]). One should also notice that - for associativity and interchange - there seems to be no formal reason that might distinguish a particular direction, while - for unitarity - a rewriting rule would normally point towards simplification.

2.1. Symmetric pre-cubical categories. As a first step, let us recall that a *symmetric pre-cubical category* is a *category object* $\mathbb{A}$ within the category of *basic* symmetric pre-cubical categories and their morphisms (1.7)

$$(1) \qquad \mathbb{A}^{(0)} \underset{e_0}{\overset{\partial_0^\alpha}{\leftleftarrows\!\!\rightarrow}} \mathbb{A}^{(1)} \overset{c_0}{\longleftarrow} \mathbb{A}^{(2)} \qquad\qquad (\alpha = \pm).$$

Explicitly, this means the following data and axioms.

(wcub.1) A basic symmetric pre-cubical category $\mathbb{A}^{(0)} = ((A_n), (\partial_i^\alpha), (e_i), (s_i), (+_i))$, whose entries are called *n-cubes*, or *n-dimensional objects* of $\mathbb{A}$.

(wcub.2) A basic symmetric pre-cubical category $\mathbb{A}^{(1)} = ((M_n), (\partial_i^\alpha), (e_i), (s_i), (+_i))$, whose entries are called *n-maps* of $\mathbb{A}$, or also *(n+1)-cells*.

(wcub.3) Symmetric cubical functors ∂_0^α and e_0, called 0-*faces* and 0-*degeneracy*, with $\partial_0^\alpha.e_0 = \mathrm{id}$.

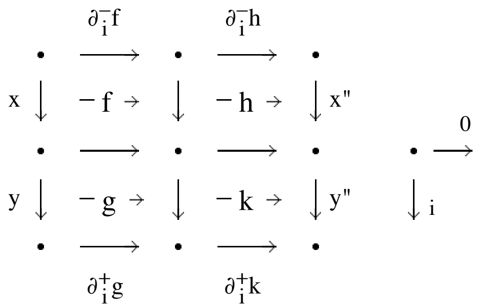
Typically, an n-map will be written as $f: x \to x'$, where $\partial_0^- f = x$, $\partial_0^+ f = x'$ are n-cubes. Every n-dimensional object x has an *identity* $e_0(x): x \to x$. Note that ∂_0^α and e_0 preserve cubical faces (∂_i^α, with $i > 0$), cubical degeneracies (e_i), transpositions (s_i) and cubical concatenations ($+_i$). In particular, given two i-consecutive n-maps f, g, their 0-faces are also i-consecutive and we have:

(2) $f +_i g: \; x +_i y \to x' +_i y'$ (for $f: x \to x'$, $g: y \to y'$; $\partial_i^+ f = \partial_i^- g$).

(wcub.4) A composition law c_0 which assigns to 0-consecutive n-maps $f: x \to x'$, $h: x' \to x''$ (of the same dimension), an n-map $hf: x \to x''$ (also written $h.f$). This composition law is (strictly) categorical, and forms a category $\mathbb{A}_n = (A_n, M_n, \partial_0^\alpha, e_0, c_0)$. It is also consistent with the basic symmetric pre-cubical structure, in the following sense

(3) $\partial_i^\alpha(hf) = (\partial_i^\alpha h).(\partial_i^\alpha f), \quad e_i(hf) = (e_i h)(e_i f), \quad s_i(hf) = (s_i h)(s_i f),$

$(h +_i k).(f +_i g) = hf +_i kg$



The last condition is the (strict) middle-four interchange *between the strict composition* c_0 *and any weak one*. An n-map $f: x \to x'$ is said to be *special* if its 2^n vertices are identities

(4) $\partial^{\boldsymbol{\alpha}} f: \; \partial^{\boldsymbol{\alpha}} x \to \partial^{\boldsymbol{\alpha}} x'$ $\qquad \partial^{\boldsymbol{\alpha}} = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n}$ $\qquad (\alpha_i = \pm).$

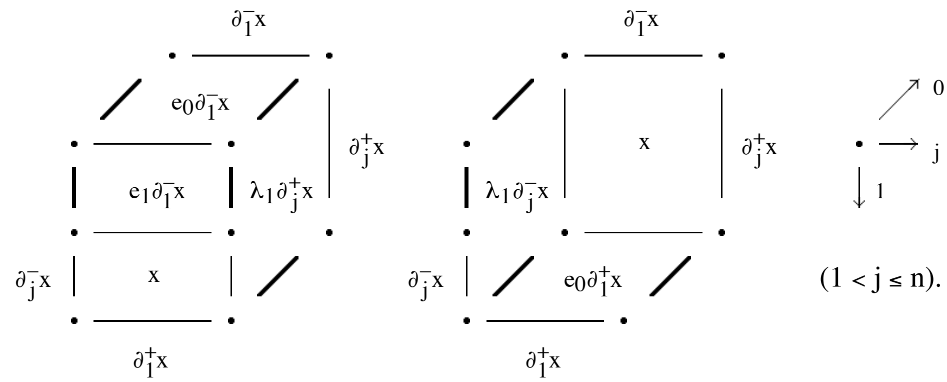
In degree 0, this just means an identity.

2.2. Comparisons. We now define a *u-lax symmetric cubical category* $\mathbb{A}$ as a symmetric pre-cubical category (2.1), which is further equipped with some *special* transversal maps, playing the role of comparisons for units, associativity and cubical interchange, as follows. (We only assign the comparisons in direction 1; all the others can be obtained with transpositions.)

(ucub.5) For every n-cube x, we have a special n-map $\lambda_1 x$, which is natural on n-maps and has the following faces (for $n > 0$)

(1) $\lambda_1 x\colon (e_1\partial_1^- x) +_1 x \;\to\; x$ (*left-unit 1-comparison*),

$\partial_1^\alpha \lambda_1 x \;=\; e_0 \partial_1^\alpha x, \qquad \partial_j^\alpha \lambda_1 x \;=\; \lambda_1 \partial_j^\alpha x \qquad (1 < j \leq n),$



$(1 < j \leq n).$

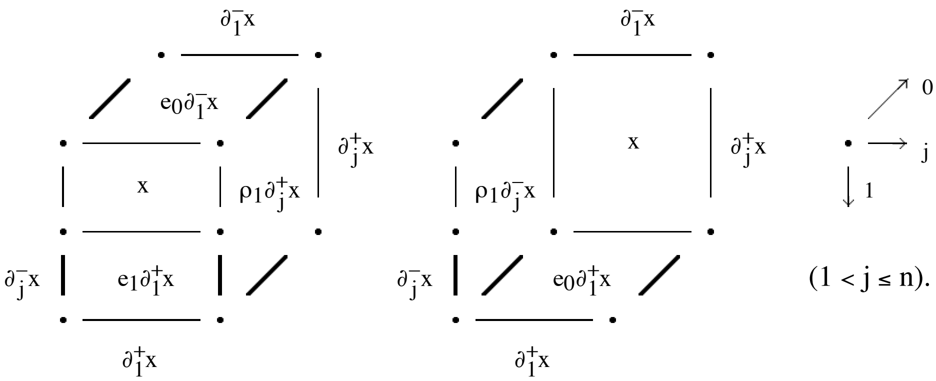
The naturality condition means that, for every n-map $f\colon x \to x'$, the following square of n-maps commutes

$$(2) \qquad \begin{array}{ccc} (e_1\partial_1^- x) +_1 x & \xrightarrow{\lambda_1 x} & x \\ {\scriptstyle (e_1\partial_1^- f) +_1 f}\downarrow & & \downarrow {\scriptstyle f} \\ (e_1\partial_1^- x') +_1 x' & \xrightarrow[\lambda_1 x']{} & x' \end{array}$$

(ucub.6) For every n-cube x, we have a special n-map $\rho_1 x$, which is natural on n-maps and has the following faces (the naturality diagram, similar to diagram (2), is not written down)

(3) $\rho_1 x \colon x +_1 (e_1 \partial_1^+ x) \ \to\ x,$ (*right-unit 1-comparison*),

$\partial_1^\alpha \rho_1 x \ = \ e_0 \partial_1^\alpha x, \qquad \partial_j^\alpha \rho_1 x \ = \ \rho_1 \partial_j^\alpha x \qquad (1 < j \leq n),$



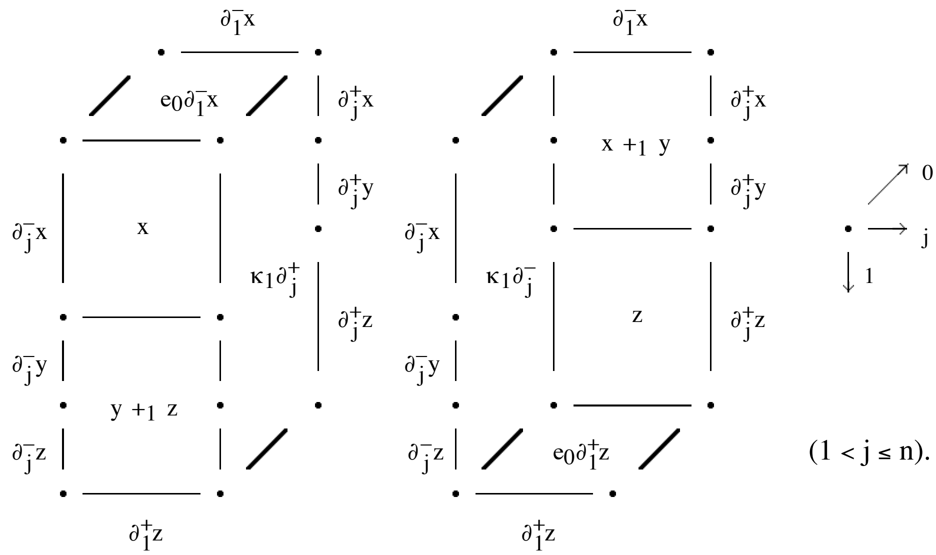
(1 < j ≤ n).

(ucub.7) For three 1-consecutive n-cubes x, y, z, we have an invertible special n-map $\kappa_1(x, y, z)$, which is natural on n-maps and has the following faces

(4) $\kappa_1(x, y, z) \colon \ x +_1 (y +_1 z) \ \to\ (x +_1 y) +_1 z$ (*associativity 1-comparison*),

$\partial_1^- \kappa_1(x, y, z) \ = \ e_0 \partial_1^- x, \qquad \partial_1^+ \kappa_1(x, y, z) \ = \ e_0 \partial_1^+ z,$

$\partial_j^\alpha \kappa_1(x, y, z) \ = \ \kappa_1(\partial_j^\alpha x, \partial_j^\alpha y, \partial_j^\alpha z) \qquad (1 < j \leq n),$



(1 < j ≤ n).

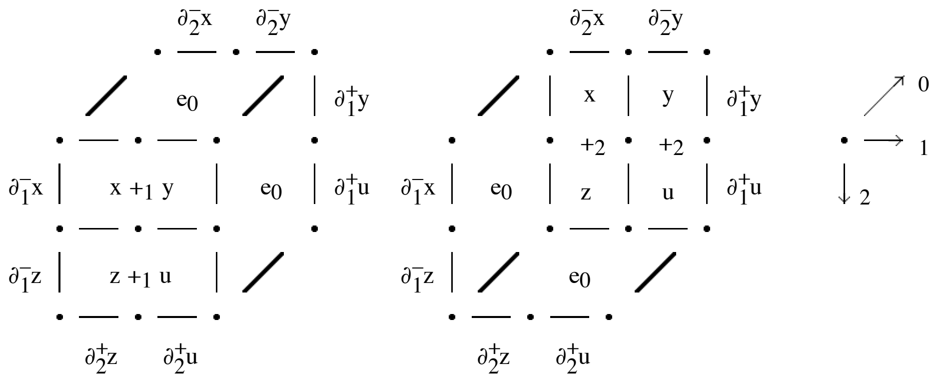
(ucub.8) Given four n-cubes x, y, z, u which satisfy the boundary conditions making the following concatenations possible, we have an invertible n-map χ_1 (*interchange 1-comparison*) which is natural on n-maps and has the following faces (partially displayed below)

(5) $\chi_1(x, y, z, u)$: $(x +_1 y) +_2 (z +_1 u) \;\to\; (x +_2 z) +_1 (y +_2 u)$,

$$\partial_1^-\chi_1(x, y, z, u) \;=\; e_0(\partial_1^- x +_2 \partial_1^- z), \qquad \partial_1^+\chi_1(x, y, z, u) \;=\; e_0(\partial_1^+ y +_2 \partial_1^+ u),$$

$$\partial_2^-\chi_1(x, y, z, u) \;=\; e_0(\partial_2^- x +_1 \partial_2^- y), \qquad \partial_2^+\chi_2(x, y, z, u) \;=\; e_0(\partial_2^+ z +_1 \partial_2^+ u),$$

$$\partial_j^\alpha\chi_1(x, y, z, u) \;=\; \chi_1(\partial_j^\alpha x, \partial_j^\alpha y, \partial_j^\alpha z, \partial_j^\alpha u) \qquad (2 < j \leq n),$$



(ucub.9) Finally, these comparisons must satisfy some conditions of coherence, listed below (2.3).

$\mathbb{A}$ is a *weak symmetric cubical category*, as defined in [G7-G9], if the unit comparisons λ, ρ are also invertible. (In this version, the axioms above are denoted as (wcub.5-9).) Among the examples studied in such papers are: the weak symmetric cubical category $\omega\mathbb{S}\mathrm{p}(\mathbf{X})$ (resp. $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$) of cubical spans (resp. cospans) on a category $\mathbf{X}$ with pullbacks (resp. pushouts); the strict symmetric cubical category $\omega\mathbb{R}\mathrm{el}$ of cubical relations; structures of 'collared cospans' related to higher cobordism.

2.3. Coherence. The coherence axiom (ucub.9) means that the following diagrams of transversal maps commute (assuming that all the cubical compositions make sense):

(i) *coherence pentagon for* $\kappa = \kappa_1$:

$$
(1) \qquad
\begin{array}{ccccc}
 & & (x +_1 y) +_1 (z +_1 u) & & \\
 & \overset{\kappa}{\nearrow} & & \overset{\kappa}{\searrow} & \\
x +_1 (y +_1 (z +_1 u)) & & & & ((x +_1 y) +_1 z) +_1 u \\
 & \underset{1+\kappa}{\searrow} & & \underset{\kappa+1}{\nearrow} & \\
 & x +_1 ((y +_1 z) +_1 u) & \xrightarrow{\ \kappa\ } & (x +_1 (y +_1 z)) +_1 u &
\end{array}
$$

(ii) *coherence conditions for* $\kappa = \kappa_1$, $\lambda = \lambda_1$ *and* $\rho = \rho_1$

$$
(2) \qquad
\begin{array}{ccc}
e_1\partial_1^- x +_1 (x +_1 y) & \xrightarrow{\ \kappa\ } & (e_1\partial_1^- x +_1 x) +_1 y \\
{}_{\lambda}\searrow & & \swarrow_{\lambda+1} \\
 & x +_1 y &
\end{array}
$$

$$
(3) \qquad
\begin{array}{ccc}
x +_1 (e_1\partial_1^- y +_1 y) & \xrightarrow{\ \kappa\ } & (x +_1 e_1\partial_1^+ x) +_1 y \\
{}_{1+\lambda}\searrow & & \swarrow_{\rho+1} \\
 & x +_1 y &
\end{array}
$$

$$
(4) \qquad
\begin{array}{ccc}
x +_1 (y +_1 e_1\partial_1^+ y) & \xrightarrow{\ \kappa\ } & (x +_1 y) +_1 e_1\partial_1^+ y \\
{}_{1+\rho}\searrow & & \swarrow_{\rho} \\
 & x +_1 y &
\end{array}
$$

(iii) *coherence hexagon for* $\kappa = \kappa_1$ *and* $\chi = \chi_1$ (writing $+$ for $+_1$)

$$
(5) \qquad
\begin{array}{ccc}
(x + (y + z)) +_2 (x' + (y' + z')) & \xrightarrow{\ \kappa+\kappa\ } & ((x + y) + z) +_2 ((x' + y') + z') \\
\chi \downarrow & & \downarrow \chi \\
(x +_2 x') + ((y + z) +_2 (y' + z')) & & ((x + y) +_2 (x' + y')) + (z +_2 z') \\
1+\chi \downarrow & & \downarrow \chi+ \\
(x +_2 x') + ((y +_2 y') + (z +_2 z')) & \xrightarrow[\ \kappa\ ]{} & ((x +_2 x') + (y +_2 y')) + (z +_2 z')
\end{array}
$$

(iv) *coherence conditions for* $\chi = \chi_1$, $\lambda = \lambda_1$ *and* $\rho = \rho_1$ (writing $+$ for $+_1$)

$$
(6) \qquad \begin{array}{ccccc}
(e_1\partial_1^- x + x) +_2 (e_1\partial_1^- y + y) & \xrightarrow{\lambda+\lambda} & x +_2 y & \xleftarrow{\rho+\rho} & (x + e_1\partial_1^+ x) +_2 (y + e_1\partial_1^+ y) \\
\chi \downarrow & & \| & & \downarrow \chi \\
(e_1\partial_1^- x +_2 e_1\partial_1^- y) + (x +_2 y) & & \| & & (x +_2 y) + (e_1\partial_1^+ x +_2 e_1\partial_1^+ y) \\
\| & & \| & & \| \\
e_1\partial_1^-(x +_2 y) + (x +_2 y) & \xrightarrow[\lambda]{} & x +_2 y & \xleftarrow[\rho]{} & (x +_2 y) + e_1\partial_1^+(x +_2 y)
\end{array}
$$

The equality in the left (or right) column of this diagram follows from the 'geometric relations' 1.7.4 (nullary interchange) and 1.7.3. Notice also that we do not require the condition $\lambda_1 e_1 x = \rho_1 e_1 x \colon e_1 x +_1 e_1 x \to e_1 x$, which is not satisfied in our case (cf. 4.3), even though $e_1 x +_1 e_1 x$ *does coincide* with $e_1 x$.

3. Reparametrisation mappings

We study now the reparametrisation mappings $\uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n$ and their interaction with the singular cubes of a d-space, as a first step in the construction of the cubical structure $\uparrow\mathbb{S}\mathrm{ng}(X)$.

3.1. Directed reparametrisation mappings. An *n-dimensional (directed) reparametrisation mapping* $f \colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n$ will be a d-map (i.e. an order-preserving continuous mapping) which sends each face of the domain to the corresponding face of the codomain, i.e. satisfies the following equivalent conditions (for $i = 1,\dots, n$ and $\alpha = 0,1$)

(a) $f(\partial_i^\alpha(\uparrow\mathbf{I}^n))) \subset \partial_i^\alpha(\uparrow\mathbf{I}^n)$,

(b) $f.\partial_i^\alpha = \partial_i^\alpha.e_i.f.\partial_i^\alpha$.

As a consequence, f sends each vertex of its domain to the corresponding vertex of the codomain; more generally, the 'lower' faces of any dimension are transformed into the corresponding ones. But actually, *onto them*, *and* f *itself is surjective*, as we prove below, by an Intermediate Value Theorem on the Cube (see 3.5).

Reparametrisation mappings of dimension n form a monoid $\mathcal{S}_n$, under the usual composition.

Moreover, there are faces, transpositions and degeneracies (which will be proved to form a symmetric cubical object $\mathcal{S}$ within monoids, in 3.2)

(1) $\underline{\partial}_i^\alpha\colon \mathcal{S}_n \to \mathcal{S}_{n-1}, \qquad \underline{s}_i\colon \mathcal{S}_n \to \mathcal{S}_n, \qquad \underline{e}_i\colon \mathcal{S}_n \to \mathcal{S}_{n+1},$

$$\underline{\partial}_i^\alpha(f) = e_i.f.\partial_i^\alpha\colon \uparrow\mathbf{I}^{n-1} \to \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^{n-1} \qquad (f \in \mathcal{S}_n),$$

$$\underline{s}_i(f) = s_i.f.s_i\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n \qquad (f \in \mathcal{S}_n),$$

$$\underline{e}_i(f)\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n \qquad (f \in \mathcal{S}_{n-1}),$$

$$\underline{e}_1(f)(t_1,..., t_n) = (t_1, f(t_2,..., t_n)), \qquad \underline{e}_{i+1}(f) = \underline{s}_i(....\underline{s}_2(\underline{s}_1(\underline{e}_1(f)))...).$$

We use *underlined symbols* to avoid confusing the face $\underline{\partial}_i^\alpha(f) = e_i.f.\partial_i^\alpha$ of f *as a reparametrisation mapping* with its face $\partial_i^\alpha(f) = f.\partial_i^\alpha\colon \uparrow\mathbf{I}^{n-1} \to \uparrow\mathbf{I}^n$ *as an n-cube* of its codomain; likewise for degeneracies and transpositions.

Notice also that we have defined all degeneracies $\underline{e}_i$ using $\underline{e}_1$ and the transpositions (according to the formula 1.6.1). Explicitly, if $f \in \mathcal{S}_{n-1}$, the reparametrisation $\underline{e}_i(f)$ operates by setting apart the i-th coordinate t_i, then applying $f \in \mathcal{S}_{n-1}$ to the remaining n–1 coordinates and finally reinserting t_i at the original i-th place:

$$\underline{e}_i(f)(t_1,..., t_n) = (f_1(t_1,..., \hat{t}_i,..., t_n),..., t_i,..., f_{n-1}(t_1,..., \hat{t}_i,..., t_n)).$$

In other words, $\underline{e}_i(f)$ is determined by the following two conditions

(2) $e_i.\underline{e}_i(f) = f.e_i, \qquad p_i.\underline{e}_i(f) = p_i \qquad (f \in \mathcal{S}_{n-1}),$

where $p_i\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}$ denotes the i-th projection (the one omitted by $e_i\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^{n-1}$). For instance, if $f\colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}$ is in $\mathcal{S}_1$, its two degeneracies in $\mathcal{S}_2$ are computed by the following formulas:

(3) $\underline{e}_1(f)(t_1, t_2) = (t_1, f(t_2)), \qquad \underline{e}_2(f)(t_1, t_2) = (f(t_1), t_2) \qquad (t_1, t_2 \in \mathbf{I}).$

Reparametrisation mappings will be used to reparametrise the singular cubes $x\colon {\uparrow}\mathbf{I}^n \to X$ of a d-space. The interactions of the two 'algebras' will be developed in 3.4.

Notice that the following squares commute (also because of (2))

$$
(4)\qquad
\begin{array}{ccc}
{\uparrow}\mathbf{I}^n & \xrightarrow{\ f\ } & {\uparrow}\mathbf{I}^n \\
{\scriptstyle \partial^\alpha_i}\uparrow & & \uparrow{\scriptstyle \partial^\alpha_i} \\
{\uparrow}\mathbf{I}^{n-1} & \xrightarrow[\underline{\partial}^\alpha_i f]{} & {\uparrow}\mathbf{I}^{n-1}
\end{array}
\qquad
\begin{array}{ccc}
{\uparrow}\mathbf{I}^{n-1} & \xrightarrow{\ f\ } & {\uparrow}\mathbf{I}^{n-1} \\
{\scriptstyle e_i}\uparrow & & \uparrow{\scriptstyle e_i} \\
{\uparrow}\mathbf{I}^{n} & \xrightarrow[\underline{e}_i f]{} & {\uparrow}\mathbf{I}^{n}
\end{array}
\qquad
\begin{array}{ccc}
{\uparrow}\mathbf{I}^n & \xrightarrow{\ f\ } & {\uparrow}\mathbf{I}^n \\
{\scriptstyle s_i}\uparrow & & \uparrow{\scriptstyle s_i} \\
{\uparrow}\mathbf{I}^{n} & \xrightarrow[\underline{s}_i f]{} & {\uparrow}\mathbf{I}^{n}
\end{array}
$$

3.2. Theorem (The structure of reparametrisation mappings). *Reparametrisation mappings, with the faces, degeneracies and transpositions defined above, form a symmetric cubical object in the category of monoids.*

Proof. Faces and degeneracies preserve the composition of reparametrisation mappings (and - plainly - the identity). Indeed, applying 3.1(b) and the characterisation 3.1.2 for $\underline{e}_i$, we have

$$\underline{\partial}^\alpha_i(gf) \;=\; e_i.gf.\partial^\alpha_i \;=\; e_i g.\partial^\alpha_i e_i.f\partial^\alpha_i \;=\; \underline{\partial}^\alpha_i(g).\underline{\partial}^\alpha_i(f),$$

$$e_i.\underline{e}_i(gf) \;=\; (gf)e_i, \;=\; g(e_ie_i(f)) \;=\; e_i.\underline{e}_i(g).\underline{e}_i(f),$$

$$p_i.\underline{e}_i(gf) \;=\; p_i \;=\; p_i.\underline{e}_i(g).\underline{e}_i(f),$$

$$\underline{s}_i(gf) \;=\; s_i.gf.s_i \;=\; s_igs_i.s_ifs_i \;=\; \underline{s}_i(g).\underline{s}_i(f).$$

Finally, we verify the symmetric cubical identities, working with the simpler presentation of 1.6.3 to reduce computations, and taking into account the fact that the structural maps of cubes satisfy the follwing *dual* conditions

(1) $\partial^\beta_1\partial^\alpha_1 \;=\; s_1.\partial^\alpha_1\partial^\beta_1,\qquad \partial^\alpha_1.s_i \;=\; s_{i+1}.\partial^\alpha_1,\qquad e_1.\partial^\alpha_1 \;=\; id,$

$e_1e_1 \;=\; e_1e_1.s_1,\qquad s_i.e_1 \;=\; e_1.s_{i+1}.$

Now, we have:

- $\underline{\partial}^\alpha_1\underline{\partial}^\beta_1(f) \;=\; (e_1e_1).f.(\partial^\beta_1\partial^\alpha_1) \;=\; (e_1e_1s_1).f.(s_1.\partial^\alpha_1.\partial^\beta_1) \;=\; \underline{\partial}^\beta_1\underline{\partial}^\alpha_1\underline{s}_1(f),$
- $\underline{s}_i.\underline{\partial}^\alpha_1(f) \;=\; s_ie_1.f.\partial^\alpha_1 s_i \;=\; e_1s_{i+1}.f.s_{i+1}.\partial^\alpha_1 \;=\; \underline{\partial}^\alpha_1.\underline{s}_{i+1}(f),$

- $\underline{\partial}_1^\alpha \underline{e}_1(f) \;=\; e_1.\underline{e}_1(f).\partial_1^\alpha \;=\; f.e_1\partial_1^\alpha \;=\; f,$
- $e_1(\underline{e}_1\underline{s}_i(f)) \;=\; (s_i.f.s_i).e_1 \;=\; s_i.f.e_1.s_{i+1} \;=\; s_i.e_1.\underline{e}_1(f).s_{i+1} \;=\; e_1.s_{i+1}.\underline{e}_1(f).s_{i+1} \;=\; e_1(\underline{s}_{i+1}\underline{e}_1(f)),$
- $p_1(\underline{e}_1\underline{s}_i(f)) \;=\; p_1 \;=\; s_{i+1}.p_1.s_{i+1} \;=\; s_{i+1}.p_1.\underline{e}_1(f).s_{i+1} \;=\; p_1.s_{i+1}.\underline{e}_1(f).s_{i+1} \;=\; p_1(\underline{s}_{i+1}\underline{e}_1(f)).$

The n-dimensional reparametrisation mapping

$$(\underline{e}_1\underline{e}_1(f))(t_1,..., t_n) = (t_1, t_2, f(t_3,..., t_n)),$$

is plainly invariant under $\underline{s}_1 = s_1.(-).s_1$. Finally, the Moore relations for the transpositions $\underline{s}_i$ follow trivially from those of the original s_i. For instance, for $i = j-1$:

- $\underline{s}_i.\underline{s}_j.\underline{s}_i(f) \;=\; (s_i.(s_j.(s_i.f.s_i).s_j).s_i) \;=\; \underline{s}_j.\underline{s}_i.\underline{s}_j(f).$ □

3.3. Concatenating reparametrisation mappings. The cubical set $\mathcal{S}$ has the following i-concatenation, or i-composition.

If $f, g \in \mathcal{S}_n$ are i-consecutive $(\underline{\partial}_i^+ f = \underline{\partial}_i^- g)$, we define:

$$(1) \quad (f +_i g)(..., t_i,...) \;=\; \begin{cases} u_i(f(t_1,..., 2t_i,..., t_n)), & \text{if } \; 0 \le t_i \le 1/2, \\ v_i(g(t_1,..., 2t_i - 1,..., t_n)), & \text{if } \; 1/2 \le t_i \le 1, \end{cases}$$

where the map $u_i\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n$ halves the i-th coordinate, while $v_i\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n$ operates on this coordinate as $t \mapsto (t + 1)/2$ (all the other coordinates staying unchanged).

Finally, it is obvious that $f +_i g$ is again a reparametrisation mapping.

3.4. Proposition (The interaction of cubes and reparametrisation mappings). *For a d-space* X, *the reparametrisations of its singular cubes agree with faces, degeneracies, transpositions and concatenations, in the following sense*

(1) $\partial_i^\alpha(xf) \;=\; \partial_i^\alpha(x).\underline{\partial}_i^\alpha(f), \qquad e_i(xf) \;=\; e_i(x).\underline{e}_i(f), \qquad s_i(xf) \;=\; s_i(x).\underline{s}_i(f),$

(2) $(x +_i y).(f +_i g) \;=\; xf +_i yg,$

where x, y: $\uparrow\mathbf{I}^n \to X$ *are* i-*consecutive singular* n-*cubes and* f, g *are* i-*consecutive mappings in* $\mathcal{S}_n$.

Proof. The formulas (1) are an easy consequence of the definitions (in 3.1)

$$\text{(3)} \quad \partial_i^\alpha(xf) = (xf)\partial_i^\alpha = x.\partial_i^\alpha e_i.f\partial_i^\alpha = \partial_i^\alpha(x).\underline{\partial}_i^\alpha(f),$$

$$e_i(xf) = (xf)e_i, = xe_i.\underline{e}_i(f) = e_i(x).\underline{e}_i(f),$$

$$s_i(xf) = (xf)s_i = x.s_is_i.fs_i = s_i(x).\underline{s}_i(f).$$

The first point also proves that $xf +_i yg$ makes sense, in (2). Then, this formula is easily verified, with the definitions of concatenations of cubes and reparametrisations (in 1.7.2 and 3.3.1). □

3.5. Theorem (Intermediate Value Theorem on the Cube). *Let* $f: \mathbf{I}^n \to \mathbf{I}^n$ *be a continuous mapping which sends each* (n–1)*-dimensional face* to *itself. Then* f *is surjective and sends each 'lower' face (of any dimension)* onto *itself.*

Proof. Let us begin by considering the affine homotopy $h: f \simeq \mathrm{id}: \mathbf{I}^n \to \mathbf{I}^n$

$$\text{(1)} \quad h(t_1,..., t_n, t) = (1 - t).f(t_1,..., t_n) + t.(t_1,..., t_n),$$

and note that it sends each face of $\mathbf{I}^n$ into itself, because f and id both do, and each face is convex. Now, let us prove that f is surjective, by induction on n. Our thesis being trivial for $n = 0$, let us assume it holds for $n-1$ and prove it for $n > 0$.

Every restriction of f to an (n–1)-dimensional face of the cube gives a map $\mathbf{I}^{n-1} \to \mathbf{I}^{n-1}$ which satisfies the hypothesis, and is surjective; whence, the restriction $f': \partial\mathbf{I}^n \to \partial\mathbf{I}^n$ to the boundary of the cube is surjective.

Collapsing the boundary $\partial\mathbf{I}^n$ to a point, we get an induced endomap of the sphere, $f'': \mathbf{S}^n \to \mathbf{S}^n$, which is still homotopic to the identity, by a homotopy induced by h; therefore f" is also surjective, or its image would be contained in a contractible space and f" would be homotopic to a constant map. Therefore, the image of f also contains the interior points of $\mathbf{I}^n$ and f is surjective. □

3.6. Remarks. The previous statement is trivial for $n = 0$, and amounts to the classical Intermediate Value Theorem for $n = 1$. For $n = 2$, one might describe the statement as follows: in order to cover a picture with a rectangular piece of cloth, it is sufficient to ensure that each edge of the cloth is placed on the 'corresponding' edge of the picture (so that vertices are necessarily placed at vertices and each edge covers an edge).

Notice also that, for $n \geq 2$, it is not sufficient to assume that f covers the boundary of the cube, as simple examples can show. The crucial assumption is that the restriction of f to the boundary is not homotopically trivial. This can be formulated as follows.

Intermediate Value Theorem on the Ball. Let $f: \mathbf{B}^n \to \mathbf{B}^n$ be a map which sends the boundary $\mathbf{S}^{n-1}$ into itself. If the restriction $f': \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ is not homotopic to a constant map (or, equivalently, if its homological degree is not null), then f is surjective.

An equivalent formulation can be found in Agoston's text [Ag], Section 7.4. (We thank Sibe Mardešić for this reference.)

4. Transversal maps and comparisons

Reparametrisation mappings are now used to define the transversal maps of the singular cubes of a d-space X. These include comparisons for the associativity and unitarity of the operations of concatenation, yielding the u-lax symmetric cubical category $\uparrow\mathbb{S}\mathrm{ng}(X)$.

4.1. Transversal maps. For a d-space X, a *transversal map* $f: x \to y$ between two singular n-cubes x, y of X will be a reparametrisation mapping $f: \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n$ such that $x = yf$.

More precisely, a transversal map should be defined as a triple $\hat{f} = (f, x, y)$, and we will use this notation when useful. Notice that y always determines x, while x determines y *if* f is bijective.

The choice of the direction of f, *from x to y,* is formal but has the advantage of agreeing with the composition of reparametrisations. In fact, the n-cubes of X and their transversal maps form a category $\uparrow Sng_n(X)$, with obvious faces, identities and composition

(1) $\partial_0^-(f, x, y) = x, \quad \partial_0^+(f, x, y) = y, \qquad e_0(x) = (id, x, x),$

$c_0(f, g) = gf: x \to z \qquad$ (for $g: y \to z$, so that $x = yf = zgf$).

$\uparrow Sng_0(X)$ is a discrete category: the only transversal maps between 0-cubes are the identities.

Transversal maps also form a symmetric cubical set, using the cubical structure of reparametrisation maps (defined in 3.1) and that of singular cubes

(2) $\partial_i^\alpha(f, x, y) = (\underline{\partial}_i^\alpha f, \ \partial_i^\alpha x, \ \partial_i^\alpha y), \qquad e_i(f, x, y) = (\underline{e}_i f, \ e_i x, \ e_i y),$

$s_i(f, x, y) = (\underline{s}_i f, \ s_i x, \ s_i y).$

This is legitimate, since the relation $x = yf$ implies

(3) $\partial_i^\alpha(x) = yf.\partial_i^\alpha = y.\partial_i^\alpha.e_i.f.\partial_i^\alpha = \partial_i^\alpha(y).\underline{\partial}_i^\alpha(f),$

$e_i(x) = yf.e_i = y.e_i.\underline{e}_i(f) = e_i(y).\underline{e}_i(f),$

$s_i(x) = yf.s_i = y.s_i.s_i.f.s_i = s_i(y).\underline{s}_i(f).$

Finally, we define the i-concatenation of i-consecutive transversal maps as

(4) $(f, x, y) +_i (g, z, u) = (f +_i g, \ x +_i z, \ y +_i u) \qquad (\partial_i^+(f, x, y) = \partial_i^-(g, z, u)),$

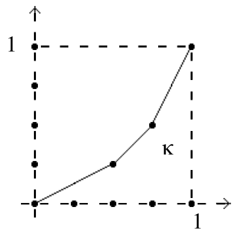
where $f +_i g: \mathbf{I}^n \to \mathbf{I}^n$ is the i-concatenation of reparametrisation maps (3.3.1), and the relations $x = yf$ and $z = ug$ imply that

(5) $(y +_i u)(f +_i g) = yf +_i ug = x +_i z.$

4.2. Associativity comparison. Given three consecutive paths (1-cubes) $x, y, z: \uparrow\mathbf{I} \to X$, the two ternary concatenations $w' = x +_1 (y +_1 z)$ and $w'' = (x +_1 y) +_1 z$

(1) $w'(t) = \begin{cases} x(2t) & (0 \leq t \leq 1/2), \\ y(4t-2) & (1/2 \leq t \leq 3/4), \\ z(4t-3) & (3/4 \leq t \leq 1), \end{cases} \qquad w''(t) = \begin{cases} x(4t) & (0 \leq t \leq 1/4), \\ y(4t-1) & (1/4 \leq t \leq 1/2), \\ z(2t-1) & (1/2 \leq t \leq 1), \end{cases}$

can be turned one into the other by a suitable *invertible* reparametrisation of the interval. Namely, we have an invertible transversal map $\kappa: w' \to w''$ $(w' = w''\kappa)$, where $\kappa: \uparrow\mathbf{I} \to \uparrow\mathbf{I}$ is the following reparametrisation function

(2) 

$$\kappa(t) \;=\; \begin{cases} t/2 & (0 \leq t \leq 1/2), \\ t - 1/4 & (1/2 \leq t \leq 3/4), \\ 2t - 1 & (3/4 \leq t \leq 1). \end{cases}$$

In degree n, we shall use the reparametrisation maps obtained from κ in the usual way

(3) $\kappa_i \;=\; \uparrow\mathbf{I}^{i-1} \times \kappa \times \uparrow\mathbf{I}^{n-i}\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n.$

It follows that the i-concatenation of singular n-cubes is associative up to the following family of *invertible* transversal maps

(4) $\kappa_i(x, y, z)\colon\; x +_i (y +_i z) \to (x +_i y) +_i z.$

This family is natural, with respect to transversal maps: given three n-maps

$f\colon x' \to x, \qquad g\colon y' \to y, \qquad h\colon z' \to z,$

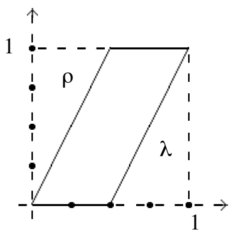
that are consecutive in direction i, we must verify that the following square commutes

(5)
$$\begin{array}{ccc} x' +_i (y' +_i z') & \xrightarrow{\;\kappa_i\;} & (x' +_i y') +_i z' \\ {\scriptstyle f +_i (g +_i h)} \downarrow & & \downarrow {\scriptstyle (f +_i g) +_i h} \\ x +_i (y +_i z) & \xrightarrow[\;\kappa_i'\;]{} & (x +_i y) +_i z \end{array}$$

Plainly, it is sufficient to check this for $n = 1$ (and $i = 1$). Then, both composites transform the partition $(0, 1/2, 3/4, 1)$ of $\mathbf{I}$ into the partition $(0, 1/4, 1/2, 1)$, by a pasting of 'affine modifications' of f, g, h on domain and codomain. The *common result* of both compositions is thus the transversal map defined by the following reparametrisation function

$$k(t) \;=\; \begin{cases} f(2t)/4 & (0 \leq t \leq 1/2), \\ 1/4 + g(4t - 2)/4 & (1/2 \leq t \leq 3/4), \\ 1/2 + h(4t - 3)/2 & (3/4 \leq t \leq 1). \end{cases}$$

4.3. Identity comparisons. Given a path $x: \mathbf{I} \to X$ with endpoints $x_0 = x(0)$, $x_1 = x(1)$, the two concatenations x', x'' of x with trivial paths can be obtained from the original path x by non-invertible reparametrisations of the interval, with two piecewise affine functions λ, ρ:

(1) 

$$x' = e_1(x_0) +_1 x = x\lambda, \quad \lambda(t) = \max(0, 2t - 1),$$

$$x'' = x +_1 e_1(x_1) = x\rho, \quad \rho(t) = \min(2t, 1).$$

In degree n, we shall use the reparametrisation maps

(2) $\lambda_i = \uparrow\mathbf{I}^{i-1} \times \lambda \times \uparrow\mathbf{I}^{n-i}: \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n, \qquad \rho_i = \uparrow\mathbf{I}^{i-1} \times \rho \times \uparrow\mathbf{I}^{n-i}: \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n.$

We have thus two natural transversal maps

(3) $\lambda_i(x)\colon x\lambda_i \to x, \qquad \rho_i(x)\colon x\rho_i \to x,$

$x\lambda_i = e_i(\partial_i^- x) +_i x, \qquad x\rho_i = x +_i e_i(\partial_i^+ x).$

We do not need other comparisons: we have already remarked, at the end of 1.7, that $\uparrow\square X$ has a *strict* interchange of concatenations (binary and nullary).

4.4. The u-lax cubical category of a directed space. For a d-space X, we have thus defined the u-lax symmetric cubical category $\uparrow\mathbb{S}\mathrm{ng}(X)$: it consists of the basic symmetric pre-cubical category $\uparrow\square X$ (1.7), with the addition of:

- transversal maps given by reparametrisations (4.1),
- invertible comparisons for pseudo associativity (4.2),
- comparisons for lax unitarity (4.3),
- identity comparisons for strict interchange.

The coherence conditions of 2.3 are satisfied. To verify this point for the pentagon, it is sufficient to note that the five associativity comparisons of diagram 2.3.1 are produced by the mapping

(1) $\kappa_1 = \kappa \times \uparrow\mathbf{I}^{n-1}: \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n.$

Their action on the first coordinate is piecewise affine, and determined by the following partitions of the interval $[0, 1]$

$$\begin{array}{ccccc} & & (0, 1/4, 1/2, 3/4, 1) & & \\ & {}^{\kappa}\nearrow & & \searrow^{\kappa} & \\ (0, 1/2, 3/4, 7/8, 1) & & & & (0, 1/8/, 1/4, 1/2, 1) \\ & {}^{1+\kappa}\searrow & & \nearrow^{\kappa+1} & \\ & (0, 1/2, 5/8, 3/4, 1) & \xrightarrow{\kappa} & (0, 1/4, 3/8, 1/2, 1) & \end{array} \tag{2}$$

Now, both composites coincide with the piecewise affine mapping which transforms the left-hand partition into that at the right. The other axioms of coherence are verified in the same way.

Notice also that, when χ is the identity, the coherence hexagon 2.3.5 reduces to a condition of consistency of the associativity comparison $\kappa = \kappa_1$ with 2-concatenation:

(3) $\kappa(x, y, z) +_2 \kappa(x', y', z') = \kappa(x +_2 x', y +_2 y', z +_2 z')$.

4.5. Remarks. It would be interesting to quotient singular cubes up to invertible reparametrisations, but this is not easily done because - of course - we want to have induced concatenations. Now, if $x +_i y$ and $x' +_i y'$ are defined in $\uparrow Sng_n(X)$, and there exist two invertible reparametrisations $f: x' \to x$, $g: y' \to y$, these need not be i-consecutive, and there are cases where there is *no* reparametrisation at all from $x' +_i y'$ to $x +_i y$.

We give such an example in dimension 2. Let us start from two 1-consecutive 2-cubes x, y of the ordinary plane $\mathbf{R}^2$, that have constant faces ∂_1^α and are injective outside of such faces, like

$$x, y\colon \uparrow\mathbf{I}^2 \to \mathbf{R}^2, \qquad x(t, t') = (t, t'.t(1-t)), \qquad y(t, t') = (t+1, t'.t(1-t)).$$

There is precisely one transversal endomap $f: x \to x$, namely the identity, because its reparametrisation $f: \uparrow\mathbf{I}^2 \to \uparrow\mathbf{I}^2$ must be the identity on a dense subset of the standard square.

Let now $g = id \times \varphi: \uparrow\mathbf{I}^2 \to \uparrow\mathbf{I}^2$ be an invertible reparametrisation map given by a directed homeomorphism $\varphi: \uparrow\mathbf{I} \to \uparrow\mathbf{I}$ other than the identity (for instance, $\varphi(t) = t^2$). There is only one transversal endomap $yg \to y$, and is given by $g: \uparrow\mathbf{I}^2 \to \uparrow\mathbf{I}^2$.

The 2-cubes x, y and $y' = yg$ have the same (constant) 1-indexed faces, whence there are concatenated cubes $x +_1 y$ and $x +_1 y'$, with values in $\mathbf{R}^2$.

However, there is no transversal map $x +_1 y' \to x +_1 y$: indeed, its reparametrisation mapping $h: \uparrow\mathbf{I}^2 \to \uparrow\mathbf{I}^2$ should restrict to the identity on $[0, 1/2] \times \mathbf{I}$ and to $id \times \varphi$ on $[1/2, 1] \times \mathbf{I}$, which gives a contradiction on the intersection $\{1/2\} \times \mathbf{I}$ of these rectangles.

One might think of solving this problem by considering 'piecewise reparametrisations' on 'multi-partitions' of the singular cubes, but this leads to two problems.

(a) If $f: \partial_i^+(x) \to \partial_i^-(y)$ is a (global) invertible transversal map, it is easy to show that there exists a cube y' and an invertible transversal map $f': y' \to y$ such that $x +_i y'$ is defined; but $y' = yf'$ is constructed by extending the reparametrisation mapping f; a 'piecewise reparametrisation' of our faces would not allow us to construct a cube y'.

(b) The relation between cubes obtained this way is not transitive.

Extending our relation by transitivity would solve the second point but still stumble on the first. The same happens with a different approach, by *partial reparametrisation mappings* defined on convenient dense open subsets.

4.6. The non-directed case. Let now X be a (non-directed) topological space. As already said, we view it as a d-space by its natural (reversible) structure, where all paths are directed.

Now, the singular symmetric cubical set $\uparrow\square X$ (1.4) will be written as $\square X$ and equipped with *reversions* produced, contravariantly, by the reversions of the standard cube

(1) $r_i = \mathbf{I}^{i-1} \times r \times \mathbf{I}^{n-i}: \mathbf{I}^n \to \mathbf{I}^n, \qquad r_i(t_1,..., t_{n+1}) = (t_1,..., 1 - t_i,..., t_n).$

$r_i: \square_n X \to \square_n X \qquad r_i(x) = x.r_i \qquad (i = 1,..., n),$

We now replace the weak symmetric cubical category $\uparrow\mathbb{S}$ng(X) with a *larger* structure $\mathbb{S}$ng(X), where the transversal maps (f, x, y): x → y are given by reparametrisation mappings f: $\mathbf{I}^n \to \mathbf{I}^n$ that need not preserve the natural ordering (but, of course, still have to send each face onto itself).

The weak symmetric cubical category $\mathbb{S}$ng(X) is also equipped with *reversions*, extending those of its cubes: if (f, x, y): x → y is a transversal map between singular n-cubes of X, we let

(2) $\underline{r}_i(f) = r_i.f.r_i\colon \mathbf{I}^n \to \mathbf{I}^n$, $\quad\quad r_i(f, x, y) = (\underline{r}_i f,\ r_i x,\ r_i y)$.

The theory of *reversible symmetric cubical sets* is sketched in [GM], Section 9. (The site $!\mathbb{K}$ considered there also contains the connections, that can be discarded.)

The definition of a *weak reversible symmetric cubical category* is not difficult to set up, blending the theory mentioned above with the non-reversible notion studied above. Of course, new consistency and coherence conditions must be added, like:

(3) $r_1(x +_1 y) = r_1(y) +_1 r_1(x)$,

$r_1(\lambda_1 x) = \rho_1(r_1(x))$, $\quad\quad r_1(\kappa_1(x, y, z)) = \kappa_1(r_1(z), r_1(y), r_1(x))$.

5. The Moore symmetric cubical category of a d-space

In this section we briefly consider a strict version of the previous construction. It is based on the *Moore* directed cubes of a d-space, defined on 'multi-intervals'. Their cubical compositions are *strictly* associative and unital. Reparametrisation mappings of multi-intervals provide transversal maps and the (extended) Moore symmetric cubical category $\uparrow\mathbb{M}$Sng(X) of a d-space.

5.1. Multi-intervals. A (*directed*) *multi-interval* will be a product of directed intervals, possibly degenerate, of variable length $a \geq 0$ (or $a_h \geq 0$)

(1) $I(a) = \uparrow[0, a]$, $\quad\quad I(a_1,..., a_n) = \prod_{h=1,...,n} \uparrow[0, a_h]$.

The topological dimension of this parallelepiped can be any integer between 0 and n, but we say that it has *formal dimension* n and *span* $(a_1,..., a_n) \in [0, \infty[^n$.

There is precisely one directed multi-interval of formal dimension 0, i.e. the empty product $\{*\}$; its span is the empty family.

The *faces*, *degeneracies* and *transpositions* of multi-intervals are defined as follows

(2) $\partial_i^\alpha\colon \Pi_{h\neq i}\, I(a_h) \to \Pi_h\, I(a_h)$, $\quad\partial_i^\alpha(t_1,..., t_{n-1}) = (t_1,..., \alpha a_i,..., t_{n-1})$,

$e_i\colon \Pi_h\, I(a_h) \to \Pi_{h\neq i}\, I(a_h)$, $\quad e_i(t_1,..., t_n) = (t_1,..., \hat{t}_i,..., t_n)$,

$s_i\colon \Pi_h\, I(a_{\sigma_i(h)}) \to \Pi_h\, I(a_h)$, $\quad s_i(t_1,..., t_n) = (t_1,..., t_{i+1}, t_i,..., .., t_n)$ $\quad (i < n)$.

Here, $\sigma_i\colon \{1,..., n\} \to \{1,..., n\}$ denotes the involution that interchanges i and i+1. Notice also that - as a consequence of using multi-intervals instead of standard cubes:

- the upper face ∂_i^+ is determined by its codomain, or equivalently by its domain *and the number* a_i,

- the degeneracy e_i is determined by its domain, or equivalently by its codomain *and* a_i,

- ∂_i^- and s_i are determined by their domain, or equivalently by their codomain.

In order to reparametrise cubes, we will need degeneracies whose *codomain* is known; in such a case, the degeneracy e_i in (2) can be written as e_i^a, to mean that it is determined by its codomain $\Pi_{h\neq i}\, I(a_h)$ *together with* $a_i = a \geq 0$.

Working with the singular cubes, in the previous sections, we have used the *standard degeneracy* e_i^1, which preserves them. Below, working with Moore cubes, we will use the *strict degeneracy* e_i^0, that has the advantage of giving strict identities for i-concatenation

(3) $e_i^0\colon \Pi_h\, I(a_h) \to \Pi_{h\neq i}\, I(a_h)$ $\quad (a_i = 0)$.

5.2. The cocubical relations. The faces and degeneracies of multi-intervals satisfy cocubical relations analogous to those of a cocubical set. We display them on diagrams because the symbols ∂_i^α, e_i, s_i are far from containing the whole information

(1) $\partial^\beta_j.\partial^\alpha_i \;=\; \partial^\alpha_{i+1}\partial^\beta_j \quad (j \leq i),$

$$\begin{array}{ccc} \prod_{h\neq i+1,j} I(a_h) & \xrightarrow{\partial^\alpha_i} & \prod_{h\neq j} I(a_h) \\ {\scriptstyle \partial^\beta_j}\downarrow & & \downarrow{\scriptstyle \partial^\beta_j} \\ \prod_{h\neq i+1} I(a_h) & \xrightarrow[\partial^\alpha_{i+1}]{} & \prod_{h} I(a_h) \end{array}$$

(2) $e_i.e_j \;=\; e_j.e_{i+1} \quad (j \leq i),$

$$\begin{array}{ccc} \prod_{h} I(a_h) & \xrightarrow{e_i} & \prod_{h\neq j} I(a_h) \\ {\scriptstyle e_{i+1}}\downarrow & & \downarrow{\scriptstyle e_i} \\ \prod_{h\neq i+1} I(a_h) & \xrightarrow[e_j]{} & \prod_{h\neq i+1,j} I(a_h) \end{array}$$

(3) $e_i.\partial^\alpha_i \;=\; \mathrm{id}$

$$\prod_{h\neq i} I(a_h) \;\longrightarrow\; \prod_{h} I(a_h) \;\longrightarrow\; \prod_{h\neq i} I(a_h)$$

(4) $e_j.\partial^\alpha_i \;=\; \partial^\alpha_{i-1}.e_j \quad (j < i),$

$$\begin{array}{ccc} \prod_{h\neq i} I(a_h) & \xrightarrow{\partial^\alpha_i} & \prod_{h} I(a_h) \\ {\scriptstyle e_j}\downarrow & & \downarrow{\scriptstyle e_j} \\ \prod_{h\neq j,i-1} I(a_h) & \xrightarrow[\partial^\alpha_{i-1}]{} & \prod_{h\neq j} I(a_h) \end{array}$$

(5) $e_j.\partial^\alpha_i \;=\; \partial^\alpha_i.\, e_{j-1} \quad (j > i),$

$$\begin{array}{ccc} \prod_{h\neq i} I(a_h) & \xrightarrow{\partial^\alpha_i} & \prod_{h} I(a_h) \\ {\scriptstyle e_{j-1}}\downarrow & & \downarrow{\scriptstyle e_j} \\ \prod_{h\neq i,j-1} I(a_h) & \xrightarrow[\partial^\alpha_i]{} & \prod_{h\neq j} I(a_h) \end{array}$$

This structure is thus a sort of 'cocubical aggregate' of directed spaces, more general than a cocubical directed space: in dimension n there are various objects (all the multi-intervals of formal dimension n), instead of a single one. The cocubical set of the standard cubes $\uparrow\mathbf{I}^n$ is a 'substructure' of the present structure.

Furthermore, the transpositions $s_i\colon \prod_h I(a_{\sigma(h)}) \to \prod_h I(a_h)$ satisfy the Moore relations (1.4.5) and the coherence conditions with faces and degeneracies already stated above (in the contravariant form of *cubical* relations, see 1.5.4):

$$\begin{array}{lllllll} & & & j < i & j = i & j = i{+}1 & j > i{+}1 \\ (6) & s_i.\partial_j^\alpha & = & \partial_j^\alpha.s_{i-1} & \partial_{i+1}^\alpha & \partial_i^\alpha & \partial_j^\alpha.s_i, \\ & e_j.s_i & = & s_{i-1}.e_j & e_{i+1} & e_i & s_i.e_j. \end{array}$$

For instance, the first two rewriting rules on $s_i.\partial_j^\alpha$ (for $j < i$ and $j = i$, respectively) amount to the commutativity of the following diagrams:

$$(7)\qquad \begin{array}{ccc} \Pi_{h \neq j}\, I(a_{\sigma_i(h)}) & \xrightarrow{\partial_j^\alpha} & \Pi_h\, I(a_{\sigma_i(h)}) \\ {\scriptstyle s_{i-1}} \downarrow & & \downarrow {\scriptstyle s_i} \\ \Pi_{h \neq j}\, I(a_h) & \xrightarrow[\partial_j^\alpha]{} & \Pi_h\, I(a_h) \end{array} \qquad \begin{array}{ccc} \Pi_{h \neq i+1}\, I(a_{\sigma_i(h)}) & \xrightarrow{\partial_i^\alpha} & \Pi_h\, I(a_{\sigma_i(h)}) \\ & {\scriptstyle \partial_{i+1}^\alpha} \searrow & \downarrow {\scriptstyle s_i} \\ & & \Pi_h\, I(a_h) \end{array}$$

5.3. Moore cubes. We now introduce the *Moore symmetric cubical set* of a d-space X

(1) $\mathbb{M}(X) = ((\mathbb{M}_n X), (\partial_i^\alpha), (e_i), (s_i)).$

A *Moore (directed) n-cube* of X will be a map with values in X and defined on a multi-interval of formal dimension n

(2) $x: \Pi_{h=1,\dots,n}\, I(a_h) \to X, \qquad sp(x) = (a_1,\dots, a_n).$

Faces, *degeneracies* and *transpositions* of $\mathbb{M}(X)$ are obtained by pre-composing with those of multi-intervals, and will be denoted by the same symbols

(3) $\partial_i^\alpha: \mathbb{M}_n X \to \mathbb{M}_{n-1} X, \qquad \partial_i^\alpha(x) = x.\partial_i^\alpha: \Pi_{h \neq i}\, I(a_h) \to X,$

$e_i: \mathbb{M}_{n-1} X \to \mathbb{M}_n X, \qquad e_i(x) = x.e_i^0: \Pi_{h<i}\, I(a_h) \times \{0\} \times \Pi_{h \geq i}\, I(a_h) \to X,$

$s_i: \mathbb{M}_n X \to \mathbb{M}_n X \qquad s_i(x) = x.s_i: \Pi_h\, I(a_{\sigma_i(h)}) \to X \qquad (i \leq n-1).$

Notice, again, that the *(strict)* degeneracy $e_i(x) = x.e_i^0$ of a singular cube $x: \uparrow\mathbf{I}^{n-1} \to X$ is *not* a singular cube of X: the *standard* degeneracy used in the previous section is $x.e_i^1: \uparrow\mathbf{I}^n \to X$. We use the same notation $e_i(x)$, since the context is generally sufficient to specify whether we are working within Moore or singular cubes of X.

These maps satisfy cubical relations, dual to those considered above, so that $\mathbb{M}(X)$ is indeed a symmetric cubical set, as defined in 1.5. Again, the contravariant action of the transpositions $s_1,..., s_{n-1}$ on $\mathbb{M}_nX$ can be extended to a (*right*) action of the symmetric group S_n.

As we have already seen in 1.6, the presence of transpositions makes all faces and degeneracies determined by (say) the 1-indexed ones, ∂_1^α and e_1:

(4) $\partial_{i+1}^\alpha = \partial_1^\alpha.s_1. ... .s_i, \qquad e_{i+1} = s_i. ... .s_1.e_1.$

For a (non-directed) topological space X, R. Brown [Br] has recently given a cubical construction $M_*(X)$ similar to the present $\mathbb{M}(X)$. His n-cubes, called *Moore hyperrectangles*, are pairs (x, a), where $x: [0, \infty[^n \to X$ is a map, $a = (a_1,..., a_n) \in [0, \infty[^n$, and $x(t_1,..., t_n)$ is independent of the coordinate t_i for $t_i \geq a_i$. This is obviously equivalent to the present definition of n-cubes (letting all paths of X be distinguished), but the cubical structure considered in [Br] is different from the present one, as already mentioned in the Introduction.

5.4. A basic symmetric cubical category. The symmetric cubical set $\mathbb{M}(X)$ can be further equipped with partial operations, called *cubical compositions*, the *concatenation in direction* i, or i*-concatenation*, or i*-composition*.

In dimension n, and for $i = 1,..., n$, it is based on the following *i-concatenation pushout*, with embeddings c_i^α

(1)
$$\begin{array}{ccc} \prod_{h\neq i} I(a_h) & \xrightarrow{\partial_i^+} & \prod_h I(a_h) \\ \partial_i^- \downarrow & & \downarrow c_i^- \\ \prod_h I(b_h) & \xrightarrow[c_i^+]{} & \prod_h I(d_h) \end{array} \qquad \begin{array}{l} a_h = b_h = d_h \quad \text{for } h \neq i, \\ d_i = a_i + b_i, \end{array}$$

$$c_i^-(t_1,..., t_n) = (t_1,..., t_n), \qquad c_i^+(t_1,..., t_n) = (t_1,..., a_i + t_i,..., t_n).$$

Now, given two *i-consecutive* Moore n-cubes $x: \prod_h I(a_h) \to X$ and $y: \prod_h I(b_h) \to X$ (with $\partial_i^+x = \partial_i^-y$), their i-concatenation $z = x +_i y$ is computed on the previous pushout

(2) $z: \prod_h I(d_h) \to \mathbf{X}, \qquad z.c_i^- = x, \quad z.c_i^+ = y,$

$sp(z) = (a_1,..., a_i + b_i,..., a_n) = (b_1,..., a_i + b_i,..., b_n),$

$z(t_1,..., t_n) = x(t_1,..., t_n)$ for $t_i \leq a_i$,

$z(t_1,..., t_n) = y(t_1,..., t_i - a_i,..., t_n)$ for $t_i \geq a_i$.

$\mathbb{M}(X)$ becomes thus a *basic symmetric cubical category*, i.e. a symmetric cubical set with 'geometrically consistent' cubical compositions, that satisfy a strict interchange law, *are strictly associative and have strict identities given by degeneracies*. (Again, the term 'basic' refers to the fact that transversal maps have not yet been added.)

5.5. Moore reparametrisation mappings. An *n-dimensional (directed) reparametrisation mapping* with *domain-span* $(a_1,..., a_n)$ and *codomain-span* $(b_1,..., b_n)$.

(1) $f: I(a_1,..., a_n) \to I(b_1,..., b_n)$,

will be a d-map (i.e. an order-preserving continuous mapping) which sends each face of the domain multi-interval to the corresponding face of the codomain, i.e. satisfies the following equivalent conditions (for $i = 1,..., n$ and $\alpha = 0,1$)

(a) $f(\partial_i^\alpha(\Pi_h I(a_h))) \subset \partial_i^\alpha(\Pi_h I(b_h))$,

(b) $f.\partial_i^\alpha = \partial_i^\alpha.e_i.f.\partial_i^\alpha$,

(in the second formula, notice that e_i is determined by its domain, which is the codomain of f).

Again, the faces of any dimension are transformed *onto* the corresponding ones and f *itself is surjective* (by the obvious extension of Theorem 3.5). In particular, $a_h = 0$ implies $b_h = 0$. The topological dimension of the domain is thus greater than or equal to that of the codomain, while the formal dimensions are the same.

Multi-intervals of formal dimension n and their reparametrisation mappings form a category $\mathcal{R}_n$, under the usual composition. There are faces, transpositions and degeneracies:

(2) $\underline{\partial}_i^\alpha: \mathcal{R}_n \rightleftarrows \mathcal{R}_{n-1}$, $\quad \underline{s}_i: \mathcal{R}_n \to \mathcal{R}_n$, $\quad \underline{e}_i: \mathcal{R}_n \to \mathcal{R}_{n+1}$,

$\underline{\partial}_i^\alpha(f) = e_i.f.\partial_i^\alpha: \Pi_{h\neq i} I(a_h) \to \Pi_h I(a_h) \to \Pi_h I(b_h) \to \Pi_{h\neq i} I(b_h)$,

$\underline{s}_i(f) \; = \; s_i.f.s_i\colon \Pi_h \, I(a_{\sigma_i(h)}) \to \Pi_h \, I(a_h) \to \Pi_h \, I(b_h) \to \Pi_h \, I(b_{\sigma_i(h)}),$

$\underline{e}_i(f)\colon \Pi_{h<i} \, I(a_h) \times \{0\} \times \Pi_{h>i} \, I(a_h) \; \to \; \Pi_{h<i} \, I(b_h) \times \{0\} \times \Pi_{h>i} \, I(b_h),$

$\underline{e}_1(f)(0, t_2,..., t_n) \; = \; (0, f(t_2,..., t_n)), \qquad \underline{e}_{i+1}(f) \; = \; \underline{s}_i(....\underline{s}_2(\underline{s}_1(\underline{e}_1(f)))...).$

Moore reparametrisation mappings, with the faces, degeneracies and transpositions defined above, form *a symmetric cubical object in the category of small categories*. (The proof is similar to that of Theorem 3.2, for standard reparametrisations.)

Notice that these faces and transpositions extend those of standard reparametrisations, while *degeneracies do not*. Here $\underline{e}_i(f)$ is determined by the following condition

(3) $e^0_i.\underline{e}_i(f) \; = \; f.e^0_i.$

5.6. Concatenating reparametrisation mappings. The cubical object $\mathcal{R}$ has the following i-directed concatenation. Take two reparametrisation mappings f, g that are i-consecutive, i.e. $\underline{\partial}^+_i f = \underline{\partial}^-_i g$

(1) $f\colon \Pi_h \, I(a_h) \to \Pi_h \, I(b_h), \qquad g\colon \Pi_h \, I(c_h) \to \Pi_h \, I(d_h),$

$(a_h = c_h, \; b_h = d_h, \quad \text{for } h \neq i).$

Their i-concatenation has spans $(a_1,..., a_i + c_i,..., a_n)$ and $(b_1,..., b_i + d_i,..., b_n)$

(2) $$(f +_i g)(t_1,.., t_n) \; = \begin{cases} f(t_1,..., t_i,..., t_n), & \text{for } 0 \leq t_i \leq a_i, \\ (0,..., b_i,..., 0) + g(t_1,..., t_i - a_i,..., t_n), & \text{for } a_i \leq t_i \leq a_i + b_i. \end{cases}$$

For a d-space X, reparametrisation of its Moore cubes agrees with faces, degeneracies, transpositions and i-indexed compositions, in the following sense

(3) $\partial^\alpha_i(xf) \; = \; \partial^\alpha_i(x).\underline{\partial}^\alpha_i(f), \qquad e_i(xf) \; = \; e_i(x).\underline{e}_i(f), \qquad s_i(xf) \; = \; s_i(x).\underline{s}_i(f),$

(4) $(x +_i y).(f +_i g) \; = \; xf +_i yg,$

where f, g are i-consecutive mappings in $\mathcal{R}_n$, as in 5.6.1, and x, y are i-consecutive Moore n-cubes (such that xf and yg are defined). The proof is similar to that of Proposition 3.4, for standard reparametrisations.

5.7. Moore transversal maps. For a d-space X, a *transversal map* $f: x \to y$ between two Moore n-cubes x, y of X will be a reparametrisation mapping such that $x = yf$

(1) $f: \Pi_h I(a_h) \to \Pi_h I(b_h)$, $y: \Pi_h I(b_h) \to X$, $x = yf: \Pi_h I(a_h) \to X$.

Also here, a transversal map should be defined as a triple $\hat{f} = (f, x, y)$, and we will use this notation when useful. Again, y determines x (and conversely if f is bijective).

The Moore n-cubes of X and their transversal maps form a category $\uparrow\mathbb{M}\mathrm{Sng}_n(X)$, with obvious faces, identities and composition

(2) $\partial_0^-(f, x, y) = x$, $\partial_0^+(f, x, y) = y$, $e_0(x) = (\mathrm{id}, x, x)$,
$c_0(f, g) = gf: x \to z$ (for $g: y \to z$, so that $x = yf = zgf$).

This category is discrete in degree 0: the only transversal maps between 0-cubes are the identities.

Transversal maps also form a symmetric cubical set, using the cubical structure of Moore cubes (5.3) and of their reparametrisation maps (5.5)

(3) $\partial_i^\alpha(f, x, y) = (\underline{\partial}_i^\alpha f, \partial_i^\alpha x, \partial_i^\alpha y)$, $e_i(f, x, y) = (\underline{e}_i f, e_i x, e_i y)$,
$s_i(f, x, y) = (\underline{s}_i f, s_i x, s_i y)$.

Finally, one defines the i-concatenation of i-consecutive transversal maps as

(4) $(f, x, y) +_i (g, z, u) = (f +_i g, x +_i z, y +_i u)$ $(\partial_i^+(f, x, y) = \partial_i^-(g, z, u))$,

where $f +_i g$ is the i-concatenation of reparametrisation maps (5.6).

This completes the definition of the *Moore symmetric cubical category* $\uparrow\mathbb{M}\mathrm{Sng}(X)$ of the d-space X.

6. Some hints to lax cubical structures of tangles

We end with a few hints to a family $\mathbb{T}(A)$ of u-lax symmetric cubical categories, depending on a topological space A, and related to higher categories of tangles, as considered in [BL, Ch].

In this section the faces, degeneracies and transpositions of the standard cubes $\mathbf{I}^n$ are written as $\delta_i^\alpha: \mathbf{I}^{n-1} \to \mathbf{I}^n$, $\varepsilon_i: \mathbf{I}^n \to \mathbf{I}^{n-1}$ and $\sigma_i: \mathbf{I}^n \to \mathbf{I}^n$.

6.1. A preparatory structure. Let us come back to the u-lax symmetric cubical category $\mathbb{S}\mathrm{ng}(S)$ associated to a topological space S (4.6).

It is interesting to note that, if we start from the discrete space on two points $\mathbf{S}^0 = \{-1, 1\}$, a singular n-cube $x: \mathbf{I}^n \to \mathbf{S}^0$ can be identified with a subset $X \subset \mathbf{I}^n$, namely the counterimage $x^{-1}\{1\}$. We obtain thus a particular u-lax symmetric cubical category $\mathbb{T} = \mathbb{S}\mathrm{ng}(\mathbf{S}^0)$, which can be viewed as a starting point to define a u-lax symmetric cubical category of tangles.

Concretely, $\mathbb{T}$ can be described in the following terms.

(a) An n-cube is a subset $X \subset \mathbf{I}^n$.

(b) Faces, degeneracies and transpositions of n-cubes are obtained as counterimages of the corresponding maps δ_i^α, ε_i and σ_i between standard cubes

(1) $\partial_i^\alpha(X) = (\delta_i^\alpha)^{-1}(X), \qquad e_i(X) = (\varepsilon_i)^{-1}(X), \qquad s_i(X) = (\sigma_i)^{-1}(X) = \sigma_i(X).$

(c) The i-concatenation $X +_i Y$ of i-consecutive n-cubes is defined by the union of their images in two i-consecutive halves of $\mathbf{I}^n$:

(2) $X +_i Y = \varphi_i(X) \cup \psi_i(Y),$

$\varphi_i(t_1,..., t_n) = (t_1,..., t_i/2,..., t_n), \qquad \psi_i(t_1,..., t_n) = (t_1,..., (t_i+1)/2,..., t_n).$

(d) A transversal map $(f, X, Y): X \to Y$ is given by a reparametrisation mapping $f: \mathbf{I}^n \to \mathbf{I}^n$ (see 4.6) such that $X = f^{-1}(Y)$ (which *implies* $f(X) = Y$, because f is surjective).

(e) Their faces are (again) defined by the faces of reparametrisation mappings *and* of n-cubes; similarly for degeneracies, transpositions and concatenations

(3) $\partial_i^\alpha(f, x, y) = (\underline{\partial}_i^\alpha f,\ \partial_i^\alpha x,\ \partial_i^\alpha y), \qquad e_i(f, x, y) = (\underline{e}_i f,\ e_i x,\ e_i y),$

$s_i(f, x, y) = (\underline{s}_i f,\ s_i x,\ s_i y),$

$(f, x, y) +_i (g, z, u) = (f +_i g,\ x +_i z,\ y +_i u) \qquad (\partial_i^+(f, x, y) = \partial_i^-(g, z, u)).$

(f) Interchange is strict. Invertible comparisons for associativity and non-invertible ones for unitarity are given by the reparametrisation mappings $\kappa_i, \lambda_i, \rho_i$: $\mathbf{I}^n \to \mathbf{I}^n$ defined in 4.2. 4.3.

Replacing the discrete topology on $\{-1, 1\}$ with the *Sierpinski topology*, where the point 1 is open (resp. closed), we obtain the substructure $\mathbb{T}'$ (resp. $\mathbb{T}''$) whose n-cubes are the open (resp. closed) subsets of $\mathbf{I}^n$. Replacing $\mathbf{S}^0$ with the discrete space $S_p = \{0, 1,..., p\}$ on $p+1$ points, we obtain a u-lax symmetric cubical category $\mathbb{T}_p = \mathbb{S}\mathrm{ng}(S_p)$ where an n-cube amounts to an (ordered!) family $(X_1,..., X_p)$ of p disjoint subsets of $\mathbf{I}^n$.

6.2. Tangles. Finally, to approach the theory of tangles, we modify the previous construction obtaining a u-lax symmetric cubical category $\mathbb{T}(A)$, depending on a fixed topological space A, and giving back $\mathbb{T}$ when A is the singleton. (A standard case would be to choose the k-dimensional cube $\mathbf{I}^k$.)

$\mathbb{T}(A)$ is defined as follows.

(a) An n-cube is a subset $X \subset A \times \mathbf{I}^n$.

(b) Faces, degeneracies and transpositions of n-cubes are obtained as counterimages

(1) $\quad \partial_i^\alpha(X) = (A \times \delta_i^\alpha)^{-1}(X), \qquad e_i(X) = (A \times \varepsilon_i)^{-1}(X),$

$\quad s_i(X) = (A \times \sigma_i)^{-1}(X) = (A \times \sigma_i)(X).$

(c) The i-concatenation $X +_i Y$ of (i-consecutive) n-cubes is defined by the union of their images in two i-consecutive halves of $A \times \mathbf{I}^n$:

(2) $\quad X +_i Y = (A \times \varphi_i)(X) \cup (A \times \psi_i)(Y).$

(d) A transversal map (f, X, Y): $X \to Y$ is given by a reparametrisation mapping f: $\mathbf{I}^n \to \mathbf{I}^n$ (see 4.6) such that $X = (A \times f)^{-1}(Y)$ (which implies $(A \times f)(X) = Y$).

(e) Their faces, degeneracies, transpositions and concatenations are defined as in 6.1.3. Comparisons as in 6.1(f).

More generally, the u-lax symmetric cubical category $\mathbb{T}_p = \mathbb{S}\mathrm{ng}(S_p)$ considered above yields a structure $\mathbb{T}_p(A)$, of interest for *p-colored tangles*.

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