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A CONSTRUCTION OF 2-COFILTERED BILIMITS OF TOPOI

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Résumé. Nous montrons l'existence des bilimites de diagrammes 2-cofiltrés de topos, généralisant la construction de bilimites cofiltrées développée dans [2]. Nous montrons qu'un tel diagramme peut être représenté par un diagramme 2-cofiltré de petits sites avec limites finies, and nous construisons un petit site pour le topos bilimite. Nous faisons ceci en considérant le 2-filtré bicolimite des catégories sous-jacentes et leurs foncteurs image inverse. Nous appliquons la construction de cette bicolimite, développée dans [4], où il est montré que si les catégories dans un diagramme ont des limites finies et les foncteurs de transition sont exacts, alors la catégorie bicolimite a aussi des limites finies et les foncteurs du pseudocone sont exacts. Une application de notre résultat est que tout topos de Galois a des points [3].

Abstract. We show the existence of bilimits of 2-cofiltered diagrams of topoi, generalizing the construction of cofiltered bilimits developed in [2]. For any given such diagram represented by any 2-cofiltered diagram of small sites with finite limits, we construct a small site for the bilimit topos (there is no loss of generality since we also prove that any such diagram can be so represented). This is done by taking the 2-filtered bicolimit of the underlying categories and inverse image functors. We use the construction of this bicolimit developed in [4], where it is proved that if the categories in the diagram have finite limits and the transition functors are exact, then the bicolimit category has finite limits and the pseudocone functors are exact. An application of our result here is the fact that every Galois topos has points [3].

Key words. 2-cofiltered, bilimits, topos.

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1 Background, terminology and notation

In this section we recall some 2-category and topos theory that we shall explicitly need, and in this way fix notation and terminology. We also include some in-edit proofs when it seems necessary. We distinguish between *small* and *large* sets. Categories are supposed to have small hom-sets. A category with large hom-sets is called *illegitimate*.

Bicolimits

By a *2-category* we mean a *Cat* enriched category, and *2-functors* are *Cat* functors, where *Cat* is the category of small categories. Given a 2-category, as usual, we denote horizontal composition by juxtaposition, and vertical composition by a "o". We consider juxtaposition more binding than "o" (thus $xy \circ z$ means $(xy) \circ z$). If \mathcal{A}, \mathcal{B} are 2-categories (\mathcal{A} small), we will denote by $[[\mathcal{A}, \mathcal{B}]]$ the 2-category which has as objects the 2-functors, as arrows the *pseudonatural transformations*, and as 2-cells the *modifications* (see [5] I,2.4.). Given $F, G, H : \mathcal{A} \rightarrow \mathcal{B}$, there is a functor:

$$(1.1) \quad [[\mathcal{A}, \mathcal{B}]](G, H) \times [[\mathcal{A}, \mathcal{B}]](F, G) \rightarrow [[\mathcal{A}, \mathcal{B}]](F, H)$$

To have a handy reference we will explicitly describe these data in the particular cases we use.

A *pseudocone* of a diagram given by a 2-functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ to an object $X \in \mathcal{B}$ is a pseudonatural transformation $F \xrightarrow{h} X$ from F to the 2-functor which is constant at X . It consists of a family of arrows $(h_A : FA \rightarrow X)_{A \in \mathcal{A}}$, and a family of invertible 2-cells $(h_u : h_A \rightarrow h_B \circ Fu)_{(A \xrightarrow{u} B) \in \mathcal{A}}$. A morphism $g \xrightarrow{\varphi} h$ of pseudocones (with same vertex) is a modification, as such, it consists of a family of 2-cells $(g_A \xrightarrow{\varphi_A} h_A)_{A \in \mathcal{A}}$. These data is subject to the following:

1.2 (Pseudocone and morphism of pseudocone equations).

$$\begin{array}{ll}
 pc0. & h_{id_A} = id_{h_A}, \quad \text{for each object } A \\
 pc1. & h_v Fu \circ h_u = h_{vu}, \quad \text{for each pair of arrows } A \xrightarrow{u} B \xrightarrow{v} C \\
 pc2. & h_B F\gamma \circ h_v = h_u, \quad \text{for each 2-cell } A \begin{array}{c} \xrightarrow{u} \\ \gamma \Downarrow \\ \xrightarrow{v} \end{array} B \\
 pcM. & h_u \circ \varphi_A = \varphi_B Fu \circ g_u, \quad \text{for each arrow } A \xrightarrow{u} B
 \end{array}$$

We state and prove now a lemma which, although expected, needs nevertheless a proof, and for which we do not have a reference in the literature. As the reader will realize, the statement concerns general pseudonatural transformations, but we treat here the particular case of pseudocones.

1.3 Lemma. *Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be a 2-functor and $F \xrightarrow{g} X$ a pseudocone. Let $FA \xrightarrow{h_A} X$ be a family of morphisms together with invertible 2-cells $g_A \xrightarrow{\varphi_A} h_A$. Then, conjugating by φ determines a pseudocone structure for h , unique such that φ becomes an isomorphism of pseudocones.*

Proof. If φ is to become a pseudocone morphism, the equation pcM. $\varphi_B Fu \circ g_u = h_u \circ \varphi_A$ must hold. Thus, $h_u = \varphi_B Fu \circ g_u \circ \varphi_A^{-1}$ determines and defines h . The pseudocone equations 1.2 for h follow from the respective equations for g :

$$\text{pc0. } h_{id_A} = \varphi_A \circ g_{id_A} \circ \varphi_A^{-1} = \varphi_A \circ id_{g_A} \circ \varphi_A^{-1} = id_{h_A}$$

$$\text{pc1. } A \xrightarrow{u} B \xrightarrow{v} C:$$

$$\begin{aligned} h_v Fu \circ h_u &= (\varphi_C Fv \circ g_v \circ \varphi_B^{-1}) Fu \circ \varphi_B Fu \circ g_u \circ \varphi_A^{-1} = \\ &\varphi_C F(vu) \circ g_v Fu \circ \varphi_B^{-1} Fu \circ \varphi_B Fu \circ g_u \circ \varphi_A^{-1} = \\ &\varphi_C F(vu) \circ g_v Fu \circ g_u \circ \varphi_A^{-1} = \\ &\varphi_C F(vu) \circ g_{vu} \circ \varphi_A^{-1} = h_{vu} \end{aligned}$$

$$\text{pc2. For } A \xrightarrow[\uparrow \gamma]{u} B \text{ we must see } h_B F\gamma \circ h_v = h_u. \text{ This is the same}$$

as $h_B F\gamma \circ \varphi_B Fv \circ g_v \circ \varphi_A^{-1} = \varphi_B Fu \circ g_u \circ \varphi_A^{-1}$. Canceling φ_A^{-1} and composing with $(\varphi_B Fu)^{-1}$ yields (1) $(\varphi_B Fu)^{-1} \circ h_B F\gamma \circ \varphi_B Fv \circ g_v = g_u$. From the compatibility between vertical and horizontal composition it follows $(\varphi_B Fu)^{-1} \circ h_B F\gamma \circ \varphi_B Fv = (\varphi_B^{-1} \circ h_B \circ \varphi_B)(Fu \circ F\gamma \circ Fv) = g_B F\gamma$. Thus, after replacing, (1) becomes $g_B F\gamma \circ g_v = g_u$. \square

Given a small 2-diagram $\mathcal{A} \xrightarrow{F} \mathcal{B}$, the category of pseudocones and its morphisms is, by definition, $pc\mathcal{B}(F, X) = [[\mathcal{A}, \mathcal{B}]](F, X)$. Given

a pseudocone $F \xrightarrow{f} Z$ and a 2-cell $Z \xrightarrow[\downarrow t]{s} X$, it is clear and

straightforward how to define a morphism of pseudocones $F \xrightarrow[\downarrow \xi f]{sf} X$

which is the composite $F \xrightarrow{f} Z \xrightarrow[\xi\psi]{s} X$. This is a particular case of 1.1, thus composing with f determines a functor (denoted ρ_f) $\mathcal{B}(Z, X) \xrightarrow{\rho_f} pc\mathcal{B}(F, X)$.

1.4 Definition. A pseudocone $F \xrightarrow{\lambda} L$ is a bicolimit of F if for every object $X \in \mathcal{B}$, the functor $\mathcal{B}(L, X) \xrightarrow{\rho_\lambda} pc\mathcal{B}(F, X)$ is an equivalence of categories. This amounts to the following:

bl) Given any pseudocone $F \xrightarrow{h} X$, there exists an arrow $L \xrightarrow{\ell} X$ and an invertible morphism of pseudocones $h \xrightarrow{\theta} \ell\lambda$. Furthermore, given any other $L \xrightarrow{t} X$ and $h \xrightarrow{\varphi} t\lambda$, there exists a unique 2-cell $\ell \xrightarrow{\xi} t$ such that $\varphi = (\xi\lambda) \circ \theta$ (if φ is invertible, then so it is ξ).

1.5 Definition. When the functor $\mathcal{B}(L, X) \xrightarrow{\rho_\lambda} pc\mathcal{B}(F, X)$ is an isomorphism of categories, the bicolimit is said to be a pseudocolimit.

It is known that the 2-category \mathcal{Cat} of small categories has all small pseudocolimits, then a “fortiori” all small bicolimits (see for example [7]). Given a 2-functor $\mathcal{A} \xrightarrow{F} \mathcal{Cat}$ we denote by $\underline{\mathcal{L}im} F$ the vertex of a bicolimit cone.

In [4] a special construction of the pseudocolimit of a 2-filtered diagram of categories (not necessarily small) is made, and using this construction it is proved a result (theorem 1.6 below) which is the key to our construction of small 2-filtered bilimits of topoi. Notice that even if the categories of the system are large, condition bl) in definition 1.4 makes sense and it defines the bicolimit of large categories.

We denote by \mathcal{CAT}_{fl} the *illegitimate* (in the sense that its hom-sets are large) 2-category of finitely complete categories and exact (that is, finite limit preserving) functors.

1.6 Theorem ([4] Theorem 2.5). $\mathcal{CAT}_{fl} \subset \mathcal{CAT}$ is closed under 2-filtered pseudocolimits. Namely, given any 2-filtered diagram $\mathcal{A} \xrightarrow{F} \mathcal{CAT}_{fl}$, the pseudocolimit pseudocone $FA \xrightarrow{\lambda_A} \underline{\mathcal{L}im} F$ taken in \mathcal{CAT} is a pseudocolimit cone in \mathcal{CAT}_{fl} . If the index 2-category \mathcal{A} as well as all the categories FA are small, then $\underline{\mathcal{L}im} F$ is a small category. \square

Topoi

By a *site* we mean a category furnished with a (Grothendieck) topology, and a small set of objects capable of covering any object (called *topological generators* in [1]). *To simplify we will consider only sites with finite limits.* A *morphism* of sites with finite limits $\mathcal{D} \xrightarrow{f} \mathcal{C}$ is a *continuous* (that is, cover preserving) and exact functor in the other direction $\mathcal{C} \xrightarrow{f^*} \mathcal{D}$. A 2-cell $\mathcal{D} \begin{array}{c} \xrightarrow{f} \\ \gamma \Downarrow \\ \xrightarrow{g} \end{array} \mathcal{C}$ is a natural transformation

$$\mathcal{C} \begin{array}{c} \xrightarrow{g^*} \\ \gamma \Downarrow \\ \xrightarrow{f^*} \end{array} \mathcal{D} \quad ^1.$$

Under the presence of topological generators it can be easily seen there is only a small set of natural transformations between any two continuous functors. We denote by $\mathcal{S}it$ the resulting 2-category of sites with finite limits. We denote by $\mathcal{S}it^*$ the 2-category whose objects are the sites, but taking as arrows and 2-cells the functors f^* and natural transformations respectively. Thus $\mathcal{S}it$ is obtained by formally inverting the arrows and the 2-cells of $\mathcal{S}it^*$. We have by definition $\mathcal{S}it(\mathcal{D}, \mathcal{C}) = \mathcal{S}it^*(\mathcal{C}, \mathcal{D})^{op}$.

A topos (also “Grothendieck topos”) is a category equivalent to the category of sheaves on a site. Topoi are considered as sites furnishing them with the canonical topology. This determines a full subcategory $\mathcal{T}op^* \subset \mathcal{S}it^*$, $\mathcal{T}op^*(\mathcal{F}, \mathcal{E}) = \mathcal{S}it^*(\mathcal{F}, \mathcal{E})$.

A morphism of topoi (also “geometric morphism”) $\mathcal{E} \xrightarrow{f} \mathcal{F}$ is a pair of adjoint functors $f^* \dashv f_*$ (called inverse and direct image respectively) $\mathcal{E} \begin{array}{c} \xrightarrow{f_*} \\ \xleftarrow{f^*} \end{array} \mathcal{F}$ together with an adjunction isomorphism

$[f^*C, D] \xrightarrow{\cong} [C, f_*D]$. Furthermore, f^* is required to preserve finite limits. Let $\mathcal{T}op$ be the 2-category of topos with geometric morphisms. 2-arrows are pairs of natural transformations $(f^* \Rightarrow g^*, g_* \Rightarrow f_*)$ compatible with the adjunction (one of the natural transformations completely determines the other). The inverse image f^* of a morphism is an arrow in $\mathcal{T}op^* \subset \mathcal{S}it^*$. This determines a forgetful 2-functor (identity on the objects) $\mathcal{T}op \longrightarrow \mathcal{S}it$ which establish

¹Notice that 2-cells are also taken in the opposite direction. This is Grothendieck original convention, later changed by some authors.

an equivalence of categories $\mathcal{T}op(\mathcal{E}, \mathcal{F}) \cong \mathcal{S}it(\mathcal{E}, \mathcal{F})$. Notice that $\mathcal{T}op(\mathcal{E}, \mathcal{F}) \cong \mathcal{T}op^*(\mathcal{F}, \mathcal{E})^{op}$, not an equality.

We recall a basic result in the theory of morphisms of Grothendieck topoi [1] expose IV, 4.9.4. (see for example [6] Chapter VII, section 7).

1.7 Lemma. *Let \mathcal{C} be a site with finite limits, and $\mathcal{C} \xrightarrow{\epsilon^*} \tilde{\mathcal{C}}$ the canonical morphism of sites to the topos of sheaves $\tilde{\mathcal{C}}$. Then for any topos \mathcal{F} , composing with ϵ^* determines a functor $\mathcal{T}op^*(\tilde{\mathcal{C}}, \mathcal{F}) \xrightarrow{\cong} \mathcal{S}it^*(\mathcal{C}, \mathcal{F})$ which is an equivalence of categories. Thus, $\mathcal{T}op(\mathcal{F}, \tilde{\mathcal{C}}) \xrightarrow{\cong} \mathcal{S}it(\mathcal{F}, \mathcal{C})$.*

By the comparison lemma [1] Ex. III 4.1 we can state it in the following form, to be used in the proof of lemma 2.3.

1.8 Lemma. *Let \mathcal{E} be any topos and \mathcal{C} any small set of generators closed under finite limits (considered as a site with the canonical topology). Then, for any topos \mathcal{F} , the inclusion $\mathcal{C} \subset \mathcal{E}$ induce a restriction functor $\mathcal{T}op^*(\mathcal{E}, \mathcal{F}) \xrightarrow{\rho} \mathcal{S}it^*(\mathcal{C}, \mathcal{F})$ which is an equivalence of categories.*

2 2-cofiltered bilimits of topoi

Our work with sites is auxiliary to prove our results for topoi, and for this all we need are sites with finite limits. The 2-category $\mathcal{S}it$ has all small 2-cofiltered pseudolimits, which are obtained by furnishing the 2-filtered pseudocolimit in $\mathcal{C}AT_{fl}$ (1.6) of the underlying categories with the coarsest topology making the cone injections site morphisms. Explicitly:

2.1 Theorem. *Let \mathcal{A} be a small 2-filtered 2-category, and $\mathcal{A}^{op} \xrightarrow{F} \mathcal{S}it$ ($\mathcal{A} \xrightarrow{F} \mathcal{S}it^*$) a 2-functor. Then, the category $\underline{\mathcal{L}im}_{\rightarrow} F$ is furnished with a topology such that the pseudocone functors $F\mathcal{A} \xrightarrow{\lambda_{\mathcal{A}}^*} \underline{\mathcal{L}im}_{\rightarrow} F$ become continuous and induce an isomorphism of categories $\mathcal{S}it^*[\underline{\mathcal{L}im}_{\rightarrow} F, \mathcal{X}] \xrightarrow{\rho_{\lambda}} \mathcal{P}CSit^*[F, \mathcal{X}]$. The corresponding site is then a pseudocolimit of F in the 2-category $\mathcal{S}it^*$. If each $F\mathcal{A}$ is a small category, then so it is $\underline{\mathcal{L}im}_{\rightarrow} F$.*

Proof. Let $FA \xrightarrow{\lambda_A} \underline{\mathcal{L}im} F$ be the colimit pseudocone in \mathcal{CAT}_{fl} . We give $\underline{\mathcal{L}im} F$ the topology generated by the families $\lambda_{Ac_\alpha} \rightarrow \lambda_{Ac}$, where $c_\alpha \rightarrow c$ is a covering in some FA , $A \in \mathcal{A}$. With this topology, the functors λ_A become continuous, thus they correspond to site morphisms. This determines the upper horizontal arrow in the following diagram (where the vertical arrows are full subcategories and the lower horizontal arrow is an isomorphism):

$$\begin{array}{ccc} \mathit{Sit}[\underline{\mathcal{L}im} F, \mathcal{X}] & \longrightarrow & pc\mathit{Sit}[F, \mathcal{X}] \\ \downarrow & & \downarrow \\ \mathit{Cat}_{fl}[\underline{\mathcal{L}im} F, \mathcal{X}] & \xrightarrow{\cong} & pc\mathit{Cat}_{fl}[F, \mathcal{X}] \end{array}$$

To show that the upper horizontal arrow is an isomorphism we have to check that given a pseudocone $h \in pc\mathit{Sit}[F, \mathcal{X}]$, the unique functor $f \in \mathit{Cat}_{fl}[\underline{\mathcal{L}im} F, \mathcal{X}]$, corresponding to h under the lower arrow, is continuous. But this is clear since from the equation $f\lambda = h$ it follows that it preserves the generating covers, and thus all covers as well. Finally, by the construction of $\underline{\mathcal{L}im} F$ in [4] we know that every object in $\underline{\mathcal{L}im} F$ is of the form λ_{Ac} for some $A \in \mathcal{A}$, $c \in FA$. It follows then that the collection of objects of the form λ_{Ac} , with c varying on the set of topological generators of each FA , is a set of topological generators for $\underline{\mathcal{L}im} F$. \square

In the next proposition we show that any 2-diagram of topoi restricts to a 2-diagram of small sites with finite limits by means of a 2-natural (thus a fortiori pseudonatural) transformation.

2.2 Proposition. *Given a 2-functor $\mathcal{A}^{op} \xrightarrow{\mathcal{E}} \mathcal{Top}$ there exists a 2-functor $\mathcal{A}^{op} \xrightarrow{\mathcal{C}} \mathit{Sit}$ such that:*

i) For any $A \in \mathcal{A}$, \mathcal{C}_A is a small full generating subcategory of \mathcal{E}_A closed under finite limits, considered as a site with the canonical topology.

ii) The arrows and the 2-cells in the \mathcal{C} diagram are the restrictions of those in the \mathcal{E} diagram: For any 2 cell $A \begin{array}{c} \xrightarrow{u} \\ \gamma \Downarrow \\ \xrightarrow{v} \end{array} B$ in \mathcal{A} , the

following diagram commutes (where we omit notation for the action of the 2 functors on arrows and 2-cells):

$$\begin{array}{ccc}
 \mathcal{E}_A & \xrightarrow{u^*} & \mathcal{E}_B \\
 \uparrow i_A & \begin{array}{c} \xrightarrow{\gamma \Downarrow} \\ \xrightarrow{v^*} \end{array} & \uparrow i_B \\
 \mathcal{C}_A & \xrightarrow{u^*} & \mathcal{C}_B \\
 & \begin{array}{c} \xrightarrow{\gamma \Downarrow} \\ \xrightarrow{v^*} \end{array} &
 \end{array}$$

Proof. It is well known that any small set \mathcal{C} of generators in a topos can be enlarged so as to determine a (non canonical) small full subcategory $\overline{\mathcal{C}} \supset \mathcal{C}$ closed under finite limits: Choose a limit cone for each finite diagram, and repeat this in a denumerable process. On the other hand, for the validity of condition ii) it is enough that for each transition functor $\mathcal{E}_A \xrightarrow{u^*} \mathcal{E}_B$ and object $c \in \mathcal{C}_A$, we have $u^*(c) \in \mathcal{C}_B$ (with this, natural transformations restrict automatically).

Let's start with any set of generators $\mathcal{R}_A \subset \mathcal{E}_A$ for all $A \in \mathcal{A}$. We will naively add objects to these sets to remedy the failure of each condition alternatively. In this way we achieve simultaneously the two conditions:

Define $\mathcal{C}_A^0 = \overline{\mathcal{R}_A} \supset \mathcal{R}_A$. Define $\mathcal{R}_A^{n+1} = \bigcup_{X \xrightarrow{u} A} u^*(\mathcal{C}_X^n)$. \mathcal{R}_A^{n+1} is small because \mathcal{A} is small. $\mathcal{C}_X^n \subset \mathcal{R}_A^{n+1}$ due to id_A . Suppose now $c \in \mathcal{R}_A^{n+1}$, $c = u^*(d)$ with $d \in \mathcal{C}_X^n$, and let $A \xrightarrow{v} B$ in \mathcal{A} . We have $v^*(c) = v^*u^*(d) = (vu)^*(d)$, thus $v^*(c) \in \mathcal{R}_B^{n+1}$. Define $\mathcal{C}_A^{n+1} = \overline{\mathcal{R}_A^{n+1}} \supset \mathcal{R}_A^{n+1}$. Then, it is straightforward to check that $\mathcal{C}_A = \bigcup_{n \in \mathbb{N}} \mathcal{C}_A^n$ satisfy the two conditions. \square

A generalization of lemma 1.8 to pseudocones holds.

2.3 Lemma. *Given any 2-diagram of topoi $\mathcal{A}^{op} \xrightarrow{\mathcal{E}} \mathcal{T}op$, a restriction $\mathcal{A}^{op} \xrightarrow{\mathcal{C}} \mathcal{S}it$ as before, and any topos \mathcal{F} , the inclusions $\mathcal{C}_A \subset \mathcal{E}_A$ induce a restriction functor $pc\mathcal{T}op^*(\mathcal{E}, \mathcal{F}) \xrightarrow{\rho} pc\mathcal{S}it^*(\mathcal{C}, \mathcal{F})$ which is an equivalence of categories.*

Proof. The restriction functor ρ is just a particular case of 1.1, so it is well defined. We will check that it is essentially surjective and fully-

faithful. The following diagram illustrates the situation:

$$\begin{array}{ccccc}
 & & g_A^* & & \\
 & \curvearrowright & & \curvearrowleft & \\
 \mathcal{C}_A & \xrightarrow{i_A} & \mathcal{E}_A & \xrightarrow{h_A^*} & \mathcal{F} \\
 \downarrow u^* \equiv & & \downarrow u^* & \Downarrow h_u & \\
 \mathcal{C}_B & \xrightarrow{i_B} & \mathcal{E}_B & \xrightarrow{h_B^*} & \mathcal{F} \\
 & \curvearrowleft & & \curvearrowright & \\
 & & g_B^* & &
 \end{array}$$

$\cong \varphi_A$ above $\mathcal{E}_A \rightarrow \mathcal{F}$, $\cong \varphi_B$ below $\mathcal{E}_B \rightarrow \mathcal{F}$.

essentially surjective: Let $g \in pcSit^*(\mathcal{C}, \mathcal{F})$. For each $A \in \mathcal{A}$, take by lemma 1.8 $\mathcal{E}_A \xrightarrow{h_A^*} \mathcal{F}$, φ_A , $h_A^* i_A \xrightarrow{\varphi_A} g_A^*$. By lemma 1.3, $h^* i$ inherits a pseudocone structure such that φ becomes a pseudocone isomorphism. For each arrow $A \xrightarrow{u} B$ we have $(h^* i)_A \xrightarrow{(h^* i)_u} (h^* i)_B u^*$. Since ρ_A is fully-faithful, there exists a unique $h_A^* \xrightarrow{h_u} h_B^* u^*$ extending $(h^* i)_u$. In this way we obtain data $h^* = (h_A^*, h_u)$ that restricts to a pseudocone. Again from the fully-faithfulness of each ρ_A it is straightforward to check that it satisfies the pseudocone equations 1.2.

fully-faithful: Let $h^*, l^* \in pcTop^*(\mathcal{E}, \mathcal{F})$ be two pseudocones, and let $\tilde{\eta}$ be a morphism between the pseudocones $h^* i$ and $l^* i$. We have natural transformations $h_A^* i_A \xrightarrow{\tilde{\eta}_A} l_A^* i_A$. Since the inclusions i_A are dense, we can extend $\tilde{\eta}_A$ uniquely to $h_A^* \xrightarrow{\eta_A} l_A^*$ such that $\tilde{\eta} = \eta i$. As before, from the fully-faithfulness of each ρ_A it is straightforward to check that $\eta = (\eta_A)$ satisfies the morphism of pseudocone equation 1.2. \square

2.4 Theorem. *Let \mathcal{A}^{op} be a small 2-filtered 2-category, and $\mathcal{A}^{op} \xrightarrow{\varepsilon} Top$ be a 2-functor. Let $\mathcal{A}^{op} \xrightarrow{\mathcal{C}} Sit$ be a restriction to small sites as in 2.2. Then, the topos of sheaves $\widetilde{\mathcal{L}im_{\mathcal{C}} \mathcal{C}}$ on the site $\mathcal{L}im_{\mathcal{C}} \mathcal{C}$ of 2.1 is a bilimit of \mathcal{E} in Top , or, equivalently, a bicolimit in Top^* .*

Proof. Let λ^* be the pseudocolimit pseudocone $\mathcal{C}_A \xrightarrow{\lambda_A^*} \mathcal{L}im_{\mathcal{C}} \mathcal{C}$ in the 2-category Sit^* (2.1). Consider the composite pseudocone $\mathcal{C}_A \xrightarrow{\lambda_A^*} \mathcal{L}im_{\mathcal{C}} \mathcal{C} \xrightarrow{\varepsilon} \widetilde{\mathcal{L}im_{\mathcal{C}} \mathcal{C}}$ and let l^* be a pseudocone from \mathcal{E} to $\widetilde{\mathcal{L}im_{\mathcal{C}} \mathcal{C}}$

such that $l^*i \simeq \epsilon^*\lambda^*$ given by lemma 2.3. We have the following diagrams commuting up to an isomorphism:

$$\begin{array}{ccc}
 \mathcal{F} \longleftarrow \widetilde{\mathcal{L}im} \mathcal{C} \xleftarrow{\epsilon^*} \mathcal{L}im \mathcal{C} & \mathcal{T}op^*(\widetilde{\mathcal{L}im} \mathcal{C}, \mathcal{F}) \xrightarrow{\rho_\epsilon} \mathcal{S}it^*(\mathcal{L}im \mathcal{C}, \mathcal{F}) \\
 \uparrow l^* \quad \cong \quad \uparrow \lambda^* & \downarrow \rho_l \quad \cong \quad \downarrow \rho_\lambda \\
 \mathcal{E} \longleftarrow i \mathcal{C} & pc\mathcal{T}op^*(\mathcal{E}, \mathcal{F}) \xrightarrow{\rho} pc\mathcal{S}it^*(\mathcal{C}, \mathcal{F})
 \end{array}$$

In the diagram on the right the arrows ρ_ϵ , ρ_λ and ρ are equivalences of categories (1.7, 2.1 and 2.3 respectively), so it follows that ρ_l is an equivalence. This finishes the proof. \square

This theorem shows the existence of small 2-cofiltered bilimits in the 2-category of topoi and geometric morphisms. But, it shows more, namely, that given any small 2-filtered diagram of topoi represented by a 2-cofiltered diagram of small sites with finite limits, a small site with finite limits for the bilimit topoi can be constructed by taking the 2-cofiltered bicolimit of the underlying categories of the small sites. If the 2-filtered diagram of topoi does not arise represented in this way, the existence of the bilimit seems to depend on the *axiom of choice* (needed for Proposition 2.2). We notice for the interested reader that if we allow large sites (as in Theorem 2.1), we can take the topoi themselves as sites, and the proof of theorem 2.4 with $\mathcal{C} = \mathcal{E}$ is independent of Proposition 2.2. Thus, without the use of choice we have:

2.5 Theorem. *Let \mathcal{A}^{op} be a small 2-filtered 2-category, and $\mathcal{A}^{op} \xrightarrow{\mathcal{E}} \mathcal{T}op$ be a 2-functor. Then, the topos of sheaves $\widetilde{\mathcal{L}im} \mathcal{E}$ on the site $\mathcal{L}im \mathcal{E}$ of 2.1 is a bilimit of \mathcal{E} in $\mathcal{T}op$, or, equivalently, a bicolimit in $\mathcal{T}op^*$.*

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KÄHLER CATEGORIES

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Résumé. Dans cet article, on établit une relation entre la notion de *catégorie codifférentielle* et la théorie, plus classique, des différentielles de Kähler, qui appartient à l'algèbre commutative. Une catégorie codifférentielle est une catégorie monoïdale additive, ayant une monade T qui est en outre une modalité d'algèbre, c.à.d. une attribution naturelle d'une structure d'algèbre associative à chaque objet de la forme $T(C)$. Enfin, une catégorie codifférentielle est équipée d'une transformation dérivante, qui satisfait quelques axiomes typiques de différentiation, exprimés algébriquement.

La notion classique de différentielle de Kähler définit celle d'un module des formes A -différentielles par rapport à A , où A est une k -algèbre commutative. Ce module est équipé d'une A -dérivation universelle. Une *catégorie Kähler* est une catégorie monoïdale additive, ayant une modalité d'algèbre et un *objet des formes différentielles* associé à chaque objet. Suivant l'hypothèse que la monade algèbre libre existe et que l'application canonique vers T est épimorphique, les catégories codifférentielles sont Kähler.

Abstract. This paper establishes a relation between the notion of a *codifferential category* and the more classic theory of Kähler differentials in commutative algebra. A codifferential category is an additive symmetric monoidal category with a monad T , which is furthermore an algebra modality, *i.e.* a natural assignment of an associative algebra structure to each object of the form $T(C)$. Finally, a codifferential category comes equipped with a deriving transformation satisfying typical differentiation axioms, expressed algebraically.

The traditional notion of Kähler differentials defines the notion of a module of A -differential forms with respect to A , where A is a commutative k -algebra. This module is equipped with a universal A -derivation. A *Kähler category* is an additive monoidal category with an algebra modality and an *object of differential forms* associated to every object. Under the assumption that the free algebra monad exists and that the canonical map to T is epimorphic, codifferential categories are Kähler.

Keywords. Differential categories, Kähler differential, Kähler category

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1. Introduction

Differential categories were introduced in [3] in part to categorify work of Ehrhard and Regnier on differential linear logic and the differential λ -calculus [10, 11]. In the present paper, we shall work with the dual notion of a codifferential category. The notion was also introduced with an eye towards capturing the interaction in certain monoidal categories between an abstract differentiation operator and a (possibly monoidal) monad or comonad. We require our monads to be equipped with *algebra modalities*, *i.e.* each object naturally obtains the structure of an algebra with respect to the monoidal structure. The primary examples of differential and codifferential categories were the categories of vector spaces, relations and sup-lattices, each with some variation of the symmetric algebra monad. Differentiation is formal differentiation of polynomials. The notion of algebra modality is also fundamental in the categorical formulation of linear logic [4]. Thus both the work of Ehrhard and Regnier as well as our work can be seen as an attempt to extend linear logic to include differential structure.

The logical and semantic consequences of this sort of extension of linear logic look to be very promising, likely establishing connections to such areas as functional analysis, as in the *Köthe spaces* or *finiteness spaces* introduced by Ehrhard, [8, 9]. Recent work [5] shows that the category of *convenient vector spaces* [12] is also a differential category. This category is of great interest as it provides underlying linear structure for the category of *smooth spaces* [12], a cartesian closed category in which one can consider infinite-dimensional manifolds.

Two significant areas in which there is a well-established notion of abstract differentiation is algebraic geometry and commutative algebra, where *Kähler differentials* are of great significance. There the Kähler module of differential forms is introduced, for instance see [13, 14]. This is similar in concept to various aspects of the definition of differential category; in particular, the notion of differentiation must satisfy the usual *Leibniz rule*. But, in addition, Kähler differentials have a universal property that the notion of differential category seems to be lacking. Roughly, given a commutative algebra A , the Kähler A -module of differential forms is a module equipped with a derivation satisfying Leibniz, which is universal in the sense that to any other A -module equipped with a derivation, there is a unique A -module map commuting with this differential structure. There is no such (explicit) universal structure in the definition of differential category.

With this in mind, we introduce the new notion of a *Kähler category*. A Kähler category is an additive symmetric monoidal category equipped with a monad T and an algebra modality. We further require that each object be assigned an object of differential forms, *i.e.* an object equipped with a derivation and satisfying a universal property analogous to that arising from the Kähler theory in commutative algebra.

Our main result is that every codifferential category, satisfying a minor structural property, is Kähler. In retrospect, this perhaps should not have been surprising. In any symmetric monoidal category, one can define both the notions of associative algebra and module over an associative algebra. Furthermore if A is any associative algebra in a symmetric monoidal category and C is an arbitrary object, then one can form the free A -module generated by C , as $A \otimes C$. This satisfies the usual universal property of free A -modules. So in a codifferential category, TC is automatically an associative algebra, and thus $TC \otimes C$ is the free TC -module generated by C . This

is what we will take to be our object of differential forms.

The difficulty in the proof is in demonstrating that the map of TC -modules arising from the freeness of $TC \otimes C$ also commutes with the differential structure. This is where an additional property, which we call *Property K*, becomes necessary. We assume that our category has sufficient coproducts to construct free associative algebras. As such, there is a canonical morphism of monads between this free algebra monad and the monad giving the differential structure. Property K requires that this morphism be an epimorphism. In many codifferential categories, this is indeed the case. The proof that this condition suffices reveals additional structure in the definition of codifferential category.

A different approach to capturing the universality of Kähler differentials is contained in [7]. There the work is grounded in the notion of Lawvere algebraic theory, as opposed to linear logic in the present framework. A comparison of the two approaches would be interesting.

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2. Codifferential categories

We here review the basic definition in the paper [3]. The emphasis there was on differential categories. We here need the dual definition of codifferential category. We refer the reader to [3] for more details and motivations.

2.1 Basic definitions

Definition 2.1. 1. A symmetric monoidal category \mathcal{C} is additive if it is enriched over commutative monoids¹. Note that in an additive symmetric monoidal category, the tensor distributes over the sum.

2. An additive symmetric monoidal category has an algebra modality if it is equipped with a monad (T, μ, η) such that for every object C in

¹In particular, we only need addition on Hom-sets, rather than abelian group structure.

\mathcal{C} , the object, $T(C)$, has a commutative associative algebra structure

$$m: T(C) \otimes T(C) \rightarrow T(C), \quad e: I \rightarrow T(C)$$

and this family of associative algebra structures satisfies evident naturality conditions.

3. An additive symmetric monoidal category with an algebra modality is a codifferential category if it is also equipped with a deriving transformation², i.e. a natural transformation

$$d: T(C) \rightarrow T(C) \otimes C$$

satisfying the following four equations³:

- (d1) $e; d = 0$ (Derivative of a constant is 0.)
- (d2) $m; d = (id \otimes d); (m \otimes id) + (d \otimes id); c; (m \otimes id)$ (where c is the appropriate symmetry) (Leibniz Rule)
- (d3) $\eta; d = e \otimes id$ (Derivative of a linear function is constant.)
- (d4) $\mu; d = d; \mu \otimes d; m \otimes id$ (Chain Rule)

For a diagrammatic presentation of (the duals of) these equations, see [3].

We will need an iterated version of the Leibniz rule, which we state now. (The proof is straightforward.)

Lemma 2.2. *In any codifferential category, the composite:*

$$TC^{\otimes n} \xrightarrow{m} TC \xrightarrow{d} TC \otimes C$$

is equal to the sum over i of the composites:

$$\begin{array}{ccc} TC^{\otimes n} & \xrightarrow{id \otimes id \cdots d \cdots \otimes id} & TC \otimes \cdots TC \otimes C \otimes \cdots TC \\ & \xrightarrow{c} & TC \otimes \cdots TC \otimes \cdots TC \otimes C \\ & \xrightarrow{m \otimes id} & TC \otimes C \end{array}$$

In this composite the d occurs in the i -th position. The c is the appropriate symmetry to move the C to the final position without changing the order of the TC terms.

²We use the terminology of a *deriving transformation* in both differential and codifferential categories.

³For simplicity, we assume the monoidal structure is strict

2.2 The polynomial example

We review the canonical example of a codifferential category, as this construction will be generalized in a number of different ways. Let k be a field, and Vec the category of k -vector spaces. It is well-established that Vec is an additive, symmetric monoidal category, and further that the free symmetric algebra construction determines an algebra modality. Specifically, if V is a vector space, set

$$T(V) = k + V + (V \otimes_s V) + (V \otimes_s V \otimes_s V) \dots,$$

where \otimes_s denotes the usual symmetrized tensor product.

An equivalent, basis-dependent description is obtained as follows. Let J be a basis for V , then

$$T(V) \cong k[x_j \mid j \in J],$$

in other words, $T(V)$ is the polynomial ring generated by the basis J . We have that $T(V)$ provides the free commutative k -algebra generated by the vector space V , and as such provides an adjoint to the forgetful functor from the category of commutative k -algebras to Vec . The adjunction determines a monad on Vec , and the usual polynomial multiplication makes $T(V)$ an associative commutative algebra, and endows T with an algebra modality.

Furthermore Vec is a codifferential category [3]. It is probably easiest to see using the basis-dependent definition. Noting that, even if V is infinite-dimensional, any polynomial only has finitely many variables appearing, the coderiving transformation is defined by

$$f(x_{j_1}, x_{j_2}, \dots, x_{j_n}) \mapsto \sum_{i=1}^n \frac{\partial f}{\partial x_{j_i}}(x_{j_1}, x_{j_2}, \dots, x_{j_n}) \otimes j_i$$

where $\frac{\partial f}{\partial x_{j_i}}$ is defined in the usual way for polynomial functions.

Theorem 2.3. (See [3]) *The above construction makes Vec a codifferential category.*

By similar arguments, we can state:

Theorem 2.4.

1. *The category \mathbf{Rel} of sets and relations is a differential and codifferential category⁴.*
2. *The category \mathbf{Sup} of sup-semi lattices and homomorphisms is a codifferential category.*

Further details can be found in [3].

3. Review of Kähler differentials

To see the origins of our theory of Kähler categories and introduce our main example, we now consider the classical case of Kähler differentials; see [13, 14] and many other sources, for details.

Let k be a field, A a commutative k -algebra, and M an A -module⁵.

Definition 3.1. *An A -derivation from A to M is a k -linear map $\partial: A \rightarrow M$ such that $\partial(aa') = a\partial(a') + a'\partial(a)$.*

One can readily verify under this definition that $\partial(1) = 0$ and hence $\partial(r) = 0$ for any $r \in k$.

Definition 3.2. *Let A be a k -algebra. A module of A -differential forms is an A -module Ω_A together with an A -derivation $\partial: A \rightarrow \Omega_A$ which is universal in the following sense: for any A -module M , for any A -derivation $\partial': A \rightarrow M$, there exists a unique A -module homomorphism $f: \Omega_A \rightarrow M$ such that $\partial' = \partial f$.*

Lemma 3.3. *For any commutative k -algebra A , a module of A -differential forms exists.*

There are several well-known constructions. The most straightforward, although the resulting description is not that useful, is obtained by constructing the free A -module generated by the symbols $\{\partial a \mid a \in A\}$ divided out by the evident relations, most significantly $\partial(aa') = a\partial(a') + a'\partial(a)$. Of more value is the following description, found, for instance, as Proposition 8.2A of [13].

⁴Noting the self-duality which commutes with the monoidal structure.

⁵All modules throughout the paper will be *left* modules.

Lemma 3.4. *Let A be an k -algebra. Consider the multiplication of A :*

$$\mu: A \otimes A \rightarrow A.$$

Let I be the kernel of μ and set $\Omega_A = I/I^2$. Define a map $\partial: A \rightarrow \Omega_A$ by

$$\partial b = [1 \otimes b - b \otimes 1]$$

where we use square brackets to represent the equivalence class. The pair (Ω_A, ∂) acts as a module of differential forms. \square

Example 3.5. For the key example, let $A = k[x_1, x_2, \dots, x_n]$, then Ω_A is the free A -module generated by the symbols dx_1, dx_2, \dots, dx_n , so a typical element of Ω_A looks like

$$f_1(x_1, x_2, \dots, x_n)dx_1 + f_2(x_1, x_2, \dots, x_n)dx_2 + \dots + f_n(x_1, x_2, \dots, x_n)dx_n.$$

Note how this compares to our polynomial example of a codifferential category. If V is an n -dimensional space, then there is a canonical isomorphism:

$$\Omega_{T(V)} \cong T(V) \otimes V.$$

This provides the basis for our main theorem on Kähler categories below.

4. Kähler categories

In all of the following, the category \mathcal{C} will be symmetric, monoidal and additive. Unless otherwise stated, all algebras will be assumed to be both associative and commutative for the remainder of the paper.

Definition 4.1. *Let A be an algebra, and $M = \langle M, \cdot_M: A \otimes M \rightarrow M \rangle$ an A -module. Then an A -derivation to M is an arrow $\partial: A \rightarrow M$ such that*

$$\mu; \partial = c; id \otimes \partial; \cdot_M + id \otimes \partial; \cdot_M \quad \text{and} \quad \partial(1) = 0$$

Note that if we are enriched over abelian groups, the second condition may be dropped.

Definition 4.2. A Kähler category is an additive symmetric monoidal category with

- a monad T ,
- a (commutative) algebra modality for T ,
- for all objects C , a module of $T(C)$ -differential forms $\partial_C: T(C) \rightarrow \Omega_C$, viz a $T(C)$ -module Ω_C , and a $T(C)$ -derivation, $\partial_C: T(C) \rightarrow \Omega_C$, which is universal in the following sense: for every $T(C)$ -module M , and for every $T(C)$ -derivation $\partial': T(C) \rightarrow M$, there exists a unique $T(C)$ -module map $h: \Omega_C \rightarrow M$ such that $\partial; h = \partial'$.

$$\begin{array}{ccc}
 T(C) & \xrightarrow{\partial} & \Omega_C \\
 & \searrow \partial' & \downarrow h \\
 & & M
 \end{array}$$

Remark 4.3. We remark that Ω is functorial, indeed, is left adjoint to a forgetful functor, in the following sense. Consider the category $\mathcal{D}er(T)$ of “ T -derivations”: its objects are tuples (C, M, ∂) , for C an object of \mathcal{C} , M a $T(C)$ -module, and $\partial: T(C) \rightarrow M$ a derivation. A morphism $(C, M, \partial) \rightarrow (C', M', \partial')$ is a pair $(f: C \rightarrow C', g: M \rightarrow M')$, where f is a morphism in \mathcal{C} and g is a $T(C)$ -module morphism, satisfying $\partial; g = T(f); \partial': T(C) \rightarrow M'$. The universal property of Ω allows us to regard it as a functor $\mathcal{C} \rightarrow \mathcal{D}er(T)$, since given $f: C \rightarrow C', T(f); \partial': T(C) \rightarrow \Omega_{C'}$ is a derivation if ∂' is, and hence f induces $\Omega_f: \Omega_C \rightarrow \Omega_{C'}$. Moreover Ω is easily seen to be left adjoint to the forgetful functor $\mathcal{D}er(T) \rightarrow \mathcal{C}$ given by the first projection.

Theorem 4.4. The category of vector spaces over an arbitrary field is a Kähler category, with structure as described in the previous section.

We would like to show that codifferential categories are Kähler, but are not in a position to do so at the moment, although we do not have a counterexample. The difficulty in getting a general result lies in the fact that in the definition of differential or codifferential category, there is no *a priori* universal property; evidently universality is fundamental in Kähler theory.

However there is a universal property at our disposal: since our monad is equipped with an algebra modality, we can use the fact that $T(C) \otimes C$ is the free $T(C)$ module generated by C .

Now suppose that \mathcal{C} is a Kähler category. For each object C , we wish to construct an object Ω_C , with a universal derivation. As already suggested, we will define $\Omega_C = T(C) \otimes C$.

So suppose we have a $T(C)$ -derivation $\partial: T(C) \rightarrow M$. We must construct the unique $T(C)$ -module map $h: T(C) \otimes C \rightarrow M$ with the required property. But because of the universal property of the free left $T(C)$ -module generated by C , we already know there is a unique $T(C)$ -module map $h: T(C) \otimes C \rightarrow M$.

It remains to verify that $d; h = \partial$, which is the focus of the remainder of the paper. The key to our approach is that there must be an interaction between the T -algebra structure and the associative algebra structure.

4.1 Free associative algebras vs. algebra modalities

We assume we have a symmetric monoidal additive category with an algebra modality and with finite biproducts and countable coproducts. We will also need to consider the tensor algebra, *i.e.*

$$F(C) = I + C + C \otimes C + C \otimes C \otimes C \dots$$

As always, this is the free (not-necessarily-commutative) associative algebra generated by C . As such, the functor induces a monad $(F, \bar{\mu}, \bar{\eta})$ on our category, and that monad has its own (noncommutative) algebra modality.

Because of the existence of biproducts, we are able to establish close connections between the tensor algebra monad and the associative algebras arising from our algebra modality. These are expressed as a collection of natural transformations.

By the universality of F , we have the following natural transformations: $\alpha: FT \rightarrow T$ (given by the lifting of the identity $T \rightarrow T$), and $\varphi: F \rightarrow T$ (given by the lifting of the unit $\eta: I \rightarrow T$). More explicitly, these are given by the following constructions.

For any object C , $\alpha_C: FT(C) \rightarrow T(C)$ can be built out of each component (since its domain is a coproduct). So we want a map $\alpha_n: T(C)^{\otimes n} \rightarrow T(C)$, but this is just the n -fold multiplication on $T(C)$. In the case

where $n = 0$, there is the canonical map $\eta: I \rightarrow T(C)$. The map α_C is the usual quotient of the free associative algebra generated by (the underlying object of) $T(C)$ onto $T(C)$.

Also we observe that $\varphi_C: FC \rightarrow TC$ is simply $F\eta_C$; $\alpha_C: FC \rightarrow FTC \rightarrow TC$.

Lemma 4.5. *φ is a morphism of monads*

Proof. This follows immediately from Proposition 6.1, Chapter 3 of [1] (where the reader can also find the definition of a morphism of monads). That proposition states that φ will be a morphism of monads if the following diagrams commute:

$$\begin{array}{ccc}
 T(C) & \xrightarrow{\eta} & FT(C) \\
 & \searrow 1 & \downarrow \alpha \\
 & & T(C)
 \end{array}
 \qquad
 \begin{array}{ccccc}
 FFT(C) & \xrightarrow{\mu} & FT(C) & \xleftarrow{F\mu} & FTT(C) \\
 F\alpha \downarrow & & \alpha \downarrow & & \downarrow \alpha \\
 FT(C) & \xrightarrow{\alpha} & T(C) & \xleftarrow{\mu} & TT(C)
 \end{array}$$

These are straightforward, and in fact are an immediate consequence of the universal property of F , since the individual morphisms in these diagrams are all associative algebra maps (and so each composite is the unique lifting of the obvious map). More concretely, since objects of the form $F(C)$ are all coproducts, it suffices to check the equations componentwise, which is a simple exercise. \square

Definition 4.6. *The monad T satisfies Property K if the natural transformation $\varphi: F \rightarrow T$ is a componentwise epimorphism.*

If we are working in a category in which there is an evident monad, we will say that the category satisfies Property K, rather than the monad.

Proposition 4.7. *The categories of vector spaces, relations and sup-lattices, as described in Theorems 2.3, 2.4, satisfy Property K.*

Proof. (Sketch) For vector spaces, for example, this is the usual quotient by symmetrizing. The other two examples are similar. \square

4.2 Codifferential categories satisfying \mathbf{K} are Kähler

We now present the main result of the paper. In fact, we offer two proofs to illustrate different aspects of the notions involved.

Theorem 4.8. *If \mathcal{C} is a codifferential category, whose monad satisfies Property \mathbf{K} , then \mathcal{C} is a Kähler category, with $\Omega_{\mathcal{C}} = T(\mathcal{C}) \otimes \mathcal{C}$.*

Proof. We consider the “inclusion” map $\eta; d: \mathcal{C} \rightarrow T(\mathcal{C}) \otimes \mathcal{C}$. By equation 1 in the definition of codifferential category, we have $\eta; d = u; e \otimes id_{\mathcal{C}}$.

Hence by the freeness of $T(\mathcal{C}) \otimes \mathcal{C}$, for any $T(\mathcal{C})$ -module M and for any morphism $h: \mathcal{C} \rightarrow M$, there exists a unique map of $T(\mathcal{C})$ -modules, $\hat{h}: T(\mathcal{C}) \otimes \mathcal{C} \rightarrow M$ such that $\eta; d; \hat{h} = u; e \otimes id_{\mathcal{C}}; \hat{h} = h$. Suppose as in the definition of Kähler category that we have a $T(\mathcal{C})$ -module M and a $T(\mathcal{C})$ -derivation $\partial: T(\mathcal{C}) \rightarrow M$. Taking $h = \eta; \partial$, we thus have a unique $T(\mathcal{C})$ -module map $\hat{h}: T(\mathcal{C}) \otimes \mathcal{C} \rightarrow M$ such that $\eta; d; \hat{h} = h = \eta; \partial$

So our goal is to show that we can cancel the η 's in the previous equation.

Proof #1 The first proof is a straight calculation. We consider the morphisms:

$$\Phi = F\eta; \alpha; d; \hat{h} \quad \text{and} \quad \Psi = F\eta; \alpha; \partial$$

If we can show these two maps are equal, we are done given that Property \mathbf{K} gives that $F\eta; \alpha$ is surjective and thus $d; \hat{h} = \partial$.

Since the domain of Φ and Ψ is a coproduct, it suffices to show that the maps are equal on each component.

For the I component, both composites are 0, by definition.

For the \mathcal{C} component, we have $\eta; d; \hat{h} = \eta; \partial$, which has already been shown.

We next argue the binary $\mathcal{C} \otimes \mathcal{C}$ component, to demonstrate the techniques for the n -ary case. We wish to show that the composite

$$\Phi_2 = \mathcal{C} \otimes \mathcal{C} \xrightarrow{\eta \otimes \eta} TC \otimes TC \xrightarrow{m} TC \xrightarrow{d} TC \otimes \mathcal{C} \xrightarrow{\hat{h}} M$$

is equal to:

$$\Psi_2 = \mathcal{C} \otimes \mathcal{C} \xrightarrow{\eta \otimes \eta} TC \otimes TC \xrightarrow{m} TC \xrightarrow{d'} M$$

Proceed as follows. Throughout the proof, we assume strict associativity. Any unit isomorphism is denoted by u and c always denotes a symmetry. It will always be clear from the context what the relevant symmetry is.

$$\begin{aligned}\Phi_2 &= \eta \otimes \eta; id \otimes d; m \otimes id; \hat{h} + \eta \otimes \eta; d \otimes id; c; m \otimes id; \hat{h} \\ &= \eta \otimes u; id \otimes e \otimes id; m \otimes id; \hat{h} + u \otimes \eta; id \otimes e \otimes id; c; m \otimes id; \hat{h} \\ &= \eta \otimes id; \hat{h} + id \otimes \eta; c; \hat{h}\end{aligned}$$

Now note that

$$\begin{aligned}\Psi_2 &= \eta \otimes \eta; id \otimes \partial; \cdot_M + \eta \otimes \eta; \partial \otimes id; \cdot_M \\ &= \eta \otimes h; \cdot_M + h \otimes \eta; c; \cdot_M\end{aligned}$$

The result then follows from the universal property of $(\hat{-})$. In particular, $id_{TC} \otimes h; \cdot_M = \hat{h}$.

This calculation shows the structure for the general n -ary case, which requires the n -ary Leibniz rule of Section 2. The n -ary versions of Φ and Ψ are

$$\Phi_n = \eta^{\otimes n}; m^{\otimes n-1}; d; \hat{h} \quad \Psi_n = \eta^{\otimes n}; m^{\otimes n-1}; \partial$$

Expanding, we obtain

$$\Phi_n = \sum_{i=1}^n \eta^{\otimes i-1} \otimes id \otimes \eta^{\otimes n-i}; c; m^{\otimes n-2}; \hat{h}$$

and

$$\Psi_n = \sum_{i=1}^n \eta^{\otimes i-1} \otimes h \otimes \eta^{\otimes n-i}; c; m^{\otimes n-2}; \cdot_M$$

The result again follows from the definition of \hat{h} . □

We now give a more conceptual proof, using the universality of F (as the free associative algebra functor), rather than its explicit construction.

Suppose that A is a (commutative) algebra, and M an A -module. Then in fact $A + M$ has the structure of an algebra, in the following way. The unit is $I \xrightarrow{\langle e, 0 \rangle} A + M$, and the multiplication $(A + M) \otimes (A + M) \rightarrow A + M$

is induced by the following three maps:

$$\begin{array}{ccccc} A \otimes A & \xrightarrow{m} & A & \rightarrow & A + M \\ A \otimes M & \xrightarrow{\cdot} & M & \rightarrow & A + M \\ M \otimes M & \xrightarrow{0} & M & \rightarrow & A + M \end{array}$$

Moreover, this construction is functorial in M , so given a module morphism $M \rightarrow N$, the map $A + M \rightarrow A + N$ is an algebra morphism.

The following well-known observation [6] was used in the early work of Beck [2].

Lemma 4.9. *If A is a (commutative) algebra, M an A -module, then $A \xrightarrow{\partial} M$ is a derivation iff $A \xrightarrow{\langle 1, \partial \rangle} A + M$ is an algebra morphism.*

Proof #2 We note that $d; \hat{h} = \partial$ if and only if

$$\begin{array}{ccc} T(C) & \xrightarrow{\langle 1, d \rangle} & T(C) + T(C) \otimes C \\ & \searrow \langle 1, \partial \rangle & \downarrow 1 + \hat{h} \\ & & T(C) + M \end{array} \quad (*)$$

Now, given property K, this previous diagram commutes if and only if

$$\begin{array}{ccc} & & F(C) \\ & \swarrow & \swarrow \\ T(C) & \xrightarrow{\langle 1, d \rangle} & T(C) + T(C) \otimes C \\ & \searrow \langle 1, \partial \rangle & \downarrow 1 + \hat{h} \\ & & T(C) + M \end{array}$$

Note that a $T(C)$ -derivation followed by a $T(C)$ -module map is a derivation. So in the diagram above, every morphism is a morphism of algebras. Since $F(C)$ is the free algebra generated by C , this diagram commutes if

and only if it commutes on the image of C .

$$\begin{array}{c}
 & & & F(C) \xleftarrow{\eta} C \\
 & & \swarrow & \nearrow \\
 T(C) & \xleftrightarrow{\langle 1, d \rangle} & T(C) + T(C) \otimes C & \\
 & \searrow \langle 1, \partial \rangle & \downarrow 1 + \hat{h} & \\
 & & T(C) + M &
 \end{array}$$

But this amounts to the equation $\eta; d; \hat{h} = \eta; \partial$, which is already established. \square

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OPERADIC DEFINITION OF NON-STRICT CELLS

by *Camell KACHOUR*

Résumé

Dans [16] nous avons étendu le travail de Jacques Penon sur les ω -catégories non-strictes en définissant leurs ω -foncteurs non-stricts, leurs ω -transformations naturelles non-strictes, etc. tout ceci en utilisant des extensions de ces "étirements catégoriques" que l'on a appelés " n -étirements catégoriques" ($n \in \mathbb{N}^*$). Dans cet article nous poursuivons le travail de Michael Batanin sur les ω -catégories non-strictes [2] en définissant leurs ω -foncteurs non-stricts, leurs ω -transformations naturelles non-strictes, etc. en utilisant des extensions de son ω -opérade contractile universelle K , i.e en construisant des ω -opérades colorées contractiles universelles B^n ($n \in \mathbb{N}^*$) adaptés.

Abstract

In [16] we pursue Penon's work in higher dimensional categories by defining weak ω -functors, weak natural ω -transformations, and so on, all that with Penon's frameworks i.e with the "étirements catégoriques", where we have used an extension of this object, namely the " n -étirements catégoriques" ($n \in \mathbb{N}^*$). In this article we are pursuing Batanin's work in higher dimensional categories [2] by defining weak ω -functors, weak natural ω -transformations, and so on, using Batanin's frameworks i.e by extending his universal contractible ω -operad K , by building the adapted globular colored contractible ω -operads B^n ($n \in \mathbb{N}^*$).

Keywords. Higher Weak Omega Transformations, Grothendieck Weak Omega Groupoids, Colored Operads, Abstract Homotopy Theory.

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One of the fundamental but still conjectural properties of any theory of higher categories has to be the statement that n -categories as a totality have a structure of an $(n + 1)$ -category. Or taking the limit : there must exist an ∞ -category of ∞ -categories. This means that we should be able to define functors between ∞ -categories, transformations between such functors, transformations between transformations etc..

A difficulty here is that these functors and transformations must be as weak as possible, meaning that they are functors, transformations etc. only up to all higher cells. There are approaches to this problem which attempt to avoid the direct construction of higher transformations using methods of homotopy theory ([8, 12, 19, 22, 23]).

Even though there are some serious advantages to such approaches I believe it is of fundamental importance to have a precise notion of n -transformation, especially for the so called algebraic model of higher category theory (see [2, 20, 21]) where an ∞ -category is defined as an algebra of a special monad or algebraic theory. The very spirit of these approaches, which I believe, coincides with Grothendieck's original vision of higher category theory, requires a similar definition of higher transformations.

The first step in this direction was undertaken in [16], where I have introduced the globular complex of higher transformations for Penon ∞ -categories. In this paper I construct such a complex for Batanin ∞ -categories. As it was shown by Batanin [3], Penon's ∞ -categories are a special case of Batanin, so this work can be considered as a generalization of my previous work. The methods of this work, apply also to Leinster's ∞ -categories which is a slight variation of Batanin's original definition. I leave as an exercise for a reader interested in Leinster's n -transformations to make the necessary changes in definitions.

In my paper I use the language of the theory of T -categories invented by A.Burroni [7] and rediscovered later by Leinster and Hermida [10, 18]. I refer the reader to the book of Leinster for the main definitions. I also use the following terminology: weak ∞ -Functors are called 1-Transformations, weak ∞ -natural transformations are called 2-Transformations, weak ∞ -modifications are called 3-Transformations, etc.

A new technique is the use of 2-colored operads. This is reminiscent to the use of 2-colored operads in the classical operad theory to define coherent maps between operadic algebras. For this purpose I develop a necessary generalisation of Batanin's techniques [2] to handle colored operads.

Batanin built his weak ∞ -categories with a contractible operad equipped with a composition system. I adopt the same point of view and construct a sequence of contractible globular operads with "bicolored composition systems" (called operation systems). Like in [2], these operads are initial in

an appropriate sense. This property happens to be crucial for constructing the sources and targets of the underlying graphs of the probable Weak Omega Category of Weak Omega Categories.

In more detail the construction proceeds in 4 stages: one first constructs a co- ∞ -graph of operation systems, followed by a co- ∞ -graph of globular colored operads, which will successively lead to an ∞ -graph in the category of categories equipped with a monad, and finally to the ∞ -graph of their algebras. These algebras will contain all Batanin's n -Transformations ($n \in \mathbb{N}^*$). This work was exposed in Calais in June 2008 in the International Category Theory Conference [15].

In "pursuing stacks" [9] Alexander Grothendieck gave his own definition of weak omega groupoids in which he saw them as models of some specific theories called "cohérateurs", and a slight modification of this definition led to a notion of weak omega category [20]. Thus in the spirit of Grothendieck, weak and higher structures should be seen as models of certain kinds of theories. Section 7 is devoted to showing, thanks to the Abstract Nerve Theorem of Mark Weber ([25]), that our approach of weak omega transformations can be seen also from the point of view of theories and their models. According to [1], our approach and that of Grothendieck seem to be very similar.

In a forthcoming paper I will show that this globular complex of higher transformations has a natural action of a globular operad. The contractibility of this operad will be studied in the third paper of this series. This will complete the proof of the hypothesis of the existence of an algebraic model of the Weak ∞ -Category of the Weak ∞ -Categories.

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I dedicate this work to my uncle Mohamed-Kommandar Mezouar who taught us important life lessons even while struggling with a difficult illness.

1 Pointed and Contractibles T -Graphs

From here $\mathbb{T} = (T, \mu, \eta)$ refers to the cartesian monad of strict ∞ -categories. Its cartesian feature permits us to build the bigategory $\text{Span}(T)$ of spans. The various concepts in this article are defined in this bicategory, which is described in Leinster [18, 4.2.1 page 138]. In all this paper if \mathbb{C} is a category then $\mathbb{C}(0)$ is the class of its objects (but we often omit "0" when there is no confusion) and $\mathbb{C}(1)$ is the class of its morphisms. The symbol $:=$ means "by definition is".

1.1 T -Graphs

A T -graph (C, d, c) is a datum of a diagram of $\infty\text{-Gr}$ such as

$$T(G) \xleftarrow{d} C \xrightarrow{c} G$$

T -graphs are endomorphisms of $\text{Span}(T)$ and they form a category $T\text{-Gr}$ (described in Leinster [18, definition 4.2.4 page 140]). If we choose $G \in \infty\text{-Gr}(0)$, the endomorphisms on G (in $\text{Span}(T)$) forms a subcategory of $T\text{-Gr}$ which will be noted $T\text{-Gr}_G$, and it is well-known that $T\text{-Gr}_G$ is a monoidal category such as the definition of its tensor:

$$(C, d, c) \otimes (C', d', c') := (T(C) \times_{T(G)} C', \mu(G)T(d)\pi_0, c\pi_1),$$

and its unity object $I(G) = (G, \eta(G), 1_G)$. We can remember that $I(G)$ is also an identity morphism of $\text{Span}(T)$. The ∞ -graph G is called the graph of globular arities.

1.2 Pointed T -Graphs

A T -graph (C, d, c) equipped with a morphism $I(G) \xrightarrow{p} (C, d, c)$ is called a pointed T -graph. Also we note $(C, d, c; p)$ for a pointed T -graph. That also means that one has a 2-cell $I(G) \xrightarrow{p} (C, d, c)$ of $\text{Span}(T)$ such as $dp = \eta(G)$ and $cp = 1_G$. We define in a natural way the category $T\text{-Gr}_p$ of pointed T -graphs and the category $T\text{-Gr}_{p,G}$ of G -pointed T -graphs: Their morphisms keep pointing in an obvious direction.

1.3 Contractible T -Graphs

Let (C, d, c) be a T -graph. For any $k \in \mathbb{N}$ we consider

$$D_k = \{(\alpha, \beta) \in C(k) \times C(k) / s(\alpha) = s(\beta), t(\alpha) = t(\beta) \text{ and } d(\alpha) = d(\beta)\}$$

A contraction on that T -graph, is the datum, for all $k \in \mathbb{N}$, of a map

$$D_k \xrightarrow{[\cdot, \cdot]_k} C(k+1)$$

such that

- $s([\alpha, \beta]_k) = \alpha, t([\alpha, \beta]_k) = \beta,$
- $d([\alpha, \beta]_k) = 1_{d(\alpha)=d(\beta)}.$

This maps $[\cdot, \cdot]_k$ form the bracket law (as the terminology in [16]). A T -graph which is equipped with a contraction will be called contractible and we note $(C, d, c; ([\cdot, \cdot]_k)_{k \in \mathbb{N}})$ for a contractible T -graph. Nothing prevents a contractible T -graph from being equipped with several contractions. So here $CT\text{-Gr}$ is a category of contractible T -graphs equipped with a specific contraction. The morphisms of this category preserves the contractions and one can also refer to the category $CT\text{-Gr}_G$ where contractible T -graphs are only taken on a specific ∞ -graph of globular arities G .

Remark 1 If $(\alpha, \beta) \in D_k$ then this does not lead to $c(\alpha) = c(\beta)$, but this equality will be verified for constant ∞ -graphs (see below) and in particular for collections with two colours (These are the most important T -graphs in this article). We should also bear in mind $CT\text{-Gr}_p$, the category of pointed and contractible T -graphs resulting from the previous definitions. A pointed and contractible T -graph will be noted $(C, d, c; ([,]_k)_{k \in \mathbb{N}}, p)$. \square

1.4 Constant ∞ -Graphs

A constant ∞ -graph is a ∞ -graph G such as $\forall n, m \in \mathbb{N}$ we have $G(n) = G(m)$ and such as source and target maps are identity. We note $\infty\text{-Gr}_c$ the corresponding category of constant ∞ -graphs. Constant ∞ -graph are important because it is in this context that we have an adjunction result (theorem 1) that we used to produce free colored contractibles operads of n -Transformations ($n \in \mathbb{N}^*$). We write $T\text{-Gr}_c$ for the subcategory of $T\text{-Gr}$ consisting of T -graphs with underlying ∞ -graphs of globular arity which are constant ∞ -graphs, $T\text{-Gr}_{c,p}$ for the subcategory of $T\text{-Gr}_p$ consisting of pointed T -graphs with underlying ∞ -graphs of globular arity which are constant ∞ -graphs, and we write $T\text{-Gr}_{c,p,G}$ for the fiber subcategory in $T\text{-Gr}_{c,p}$ (for a given G in $\infty\text{-Gr}_c$).

2 Contractible T -Categories

2.1 T -Categories

A T -category is a monad of the bigategory $\text{Span}(T)$ or in a equivalent way a monoid of the monoidal category $T\text{-Gr}_G$ (for a specific G). The definition of T -categories are in Leinster [18, definition 4.2.2 page 140], and their category will be noted $T\text{-Cat}$ and that of T -categories of the same ∞ -graph of globular arities G will be noted $T\text{-Cat}_G$. A T -category $(B, d, c; \gamma, u) \in T\text{-Cat}$ is specifically given by the morphism of (operadic) composition $(B, d, c) \otimes (B, d, c) \xrightarrow{\gamma}$

(B, d, c) and the (operadic) unit $I(G) \xrightarrow{u} (B, d, c)$ fitting axioms of associativity and unity [see 18]. Note that $(B, d, c; \gamma, u)$ has $(B, d, c; u)$ as natural underlying pointed T -graph.

2.2 Contractibles T -Categories and the Theorem of Initial Objects

A T -category $(B, d, c; \gamma, u)$ will be said to be contractible if its underlying T -graph is contractible. To specify the underlying contraction of contractible T -categories we eventually noted it $(B, d, c; \gamma, u, ([,]_k)_{k \in \mathbb{N}})$. The category of contractible T -categories will be noted $CT\text{-Cat}$, that of contractible T -categories of the same ∞ -graph of globular arities G will be noted $CT\text{-Cat}_G$. We also write $CT\text{-Cat}_c$ for the subcategory of $CT\text{-Cat}$ whose objects are contractible T -categories whose underlying ∞ -graph of globular arities is a constant ∞ -graph. Besides there is an obvious forgetful functor

$$CT\text{-Cat}_{c,G} \xrightarrow{O} T\text{-Gr}_{c,p,G}$$

and there is the

Theorem 1 (Theorem of Initial Objects) *O has a left adjoint $F: F \dashv O$. \square*

PROOF The first monad (L, m, l) , resulting from the adjunction

$$T\text{-Cat}_{c,G} \begin{array}{c} \xrightarrow{U} \\ \dashv \\ \xleftarrow{M} \end{array} T\text{-Gr}_{c,p,G}$$

and the second monad (C, m, c) , resulting from the adjunction

$$CT\text{-Gr}_{c,p,G} \begin{array}{c} \xrightarrow{V} \\ \dashv \\ \xleftarrow{H} \end{array} T\text{-Gr}_{c,p,G}$$

are built as in [2];

The hypotheses of the section 6 are satisfied because the forgetful functors U and V are monadic, $T\text{-Cat}_{c,G}$ and $CT\text{-Gr}_{c,p,G}$ have coequalizers and $\vec{\mathbb{N}}$ -colimits and it is easy to notice that the forgetful functors U and V are faithful

and preserve $\vec{\mathbb{N}}$ -colimits as well. Thus this two adjunctions are fusionable which permits, through theorem 2, to make the fusion

$$\begin{array}{ccccc}
 T\text{-Cat}_{c,G} & \xrightleftharpoons[U]{M} & T\text{-Gr}_{c,p,G} & \xrightleftharpoons[V]{H} & CT\text{-Gr}_{c,p,G} \\
 & \searrow^{p_1} & \uparrow O \quad \downarrow F & \swarrow_{p_2} & \\
 & & CT\text{-Cat}_{c,G} & &
 \end{array}$$

where trivially

$$CT\text{-Cat}_{c,G} \simeq T\text{-Cat}_{c,G} \times_{T\text{-Gr}_{c,p,G}} CT\text{-Gr}_{c,p,G} \quad \blacksquare$$

The monad of this adjunction $F \dashv O$ is noted $\mathbb{B} = (B, \rho, b)$.

Remark 2 We can also prove that the forgetful functor

$$CT\text{-Cat}_c \xrightarrow{O} T\text{-Gr}_{c,p}$$

has a left adjoint. A way to prove it is to extend the work of [6] on "Surcatégories", and it is done in [13]. But it seems that this result is too much strong for this article where we use no more than 2 colours. However we will use this adjunction for a future paper, after the talk [17] where we need to use more than two colors. □

2.3 T -Categories equipped with a System of Operations

Consider $(B, \bar{d}, \bar{c}; \gamma, u) \in T\text{-Cat}_G$ and $(C, d, c) \in T\text{-Gr}_G$. If there exists a diagram of $T\text{-Gr}_G$

$$(I(G), \eta_G, id) \xrightarrow{p} (C, d, c) \xrightarrow{k} (B, \bar{d}, \bar{c})$$

such as $k \circ p = u$, then (C, d, c) is qualified system of operations, and one can say that $(B, \bar{d}, \bar{c}; \gamma, u)$ is equipped with the system of operations (C, d, c) . With this definition and the previous theorem it is clear that all pointed T -graphs $(C, d, c; p)$ induces a free contractible T -category $F(C)$, which has (C, d, c) as a system of operations. See also section 3.

3 Systems of Operations of the n -Transformations ($n \in \mathbb{N}^*$)

3.1 Preliminaries

The 2-coloured collection of the n -Transformations ($n \in \mathbb{N}^*$) are just noted C^n without specified its underlying structure, and we do the same simplification for its free contractible 2-coloured operads B^n .

From here on only the contractible 2-coloured operads of n -Transformations will be studied. All these operads are obtained applying the free functor of the theorem 1 to specific 2-coloured collections. These 2-coloured collections will be those of the n -Transformations and they count an infinite countable number of elements. Thus for each $n \in \mathbb{N}$ there is the 2-coloured collection of n -Transformations, C^n , which freely produces the free contractible 2-coloured operad B^n of n -Transformations. The pointed collection C^0 is the system of composition of Batanin's operad of weak ∞ -categories, i.e. the collection gathering all the symbols of atomic operations necessary for the weak ∞ -categories, plus the symbols of operadic units (the latter are given by pointing). The pointed 2-coloured collection C^1 is adapted to weak ∞ -functors, i.e. it gathers all the symbols of operations of the source and target weak ∞ -categories (which will be composed of different colours whether they concern the source or the target). It also brings together the unary symbols of functors as well as the symbols of operadic units. Thus as we will see, the unary symbols of functors have a domain with the same colour as the domains and codomains of the symbols of operations of source weak ∞ -categories and they have a codomain with the same colour as the domains and codomains of the symbols of operations of target weak ∞ -categories. However these symbols of functors have domains and codomains with different colours. The pointed 2-coloured collection C^2 is adapted to weak natural ∞ -transformations, etc.

3.2 Pointed 2-Coloured Collections $C^n (n \in \mathbb{N})$

In order to clearly see the bicolour feature of these symbols of operations, we write $(1+1)(n) := \{1(n), 2(n)\}$, which enables to identify $T(1) \sqcup T(1)$ with $T(1) \cup T(2)$ and $1 \sqcup 1$ with $1 \cup 2$. So the colour 1 and the colour 2 will be referred to. Let us move to the definition of $C^n (n \in \mathbb{N})$. In the diagram

$$T(1) \cup T(2) \xleftarrow{d} C^n \xrightarrow{c} 1 \cup 2$$

C^n is a ∞ -graph so that it contains both source and target maps which will be noted $C^n(m+1) \xrightleftharpoons[t_m^{m+1}]{s_m^{m+1}} C^n(m)$, $(m \in \mathbb{N})$.

3.2.1 Definition of C^0

C^0 is Batanin's system of composition, i.e. there is the following collection $T(1) \xleftarrow{d^0} C^0 \xrightarrow{c^0} 1$ such as C^0 precisely contains the symbols of the compositions of weak ∞ -categories $\mu_p^m \in C^0(m) (0 \leq p < m)$, plus the operadic unary symbols $u_m \in C^0(m)$. More specifically:

$$\forall m \in \mathbb{N}, C^0 \text{ contains the } m\text{-cell } u_m \text{ such as: } s_{m-1}^m(u_m) = t_{m-1}^m(u_m) = u_{m-1} \\ (\text{if } m \geq 1); d^0(u_m) = 1(m) (= \eta(1 \cup 2)(1(m))), c^0(u_m) = 1(m).$$

$$\forall m \in \mathbb{N} - \{0, 1\}, \forall p \in \mathbb{N}, \text{ such that } m > p, C^0 \text{ contains the } m\text{-cell } \mu_p^m \text{ such} \\ \text{as: If } p = m - 1, s_{m-1}^m(\mu_{m-1}^m) = t_{m-1}^m(\mu_{m-1}^m) = u_{m-1}. \text{ If } 0 \leq p < m - 1, \\ s_{m-1}^m(\mu_p^m) = t_{m-1}^m(\mu_p^m) = \mu_p^{m-1}. \text{ Also } d^0(\mu_p^m) = 1(m) \star_p^m 1(m), \text{ and} \\ \text{inevitably } c^0(\mu_p^m) = 1(m).$$

$$\text{Furthermore } C^0 \text{ contains the } 1\text{-cell } \mu_0^1 \text{ such as } s_0^1(\mu_0^1) = t_0^1(\mu_0^1) = u_0, d^0(\mu_0^1) = \\ 1(1) \star_0^1 1(1), \text{ also inevitably } c^0(\mu_0^1) = 1(1).$$

The system of composition C^0 has got a well-known pointing λ^0 which is defined as $\forall m \in \mathbb{N}, \lambda^0(1(m)) = u_m$.

3.2.2 Definition of C

Firstly we will define a collection (C, d, c) which will be useful to build the collections of n -Transformations ($n \in \mathbb{N}^*$). C contains two copies of the symbols of C^0 , each having a distinct colour: The symbols formed with the letters μ and u are those of the colour 1, and those formed with the letters ν and v are those of the colour 2. Let us be more precise:

$\forall m \in \mathbb{N}$, C contains the m -cell u_m such as: $s_{m-1}^m(u_m) = t_{m-1}^m(u_m) = u_{m-1}$ (if $m \geq 1$) and $d(u_m) = 1(m)$, $c(u_m) = 1(m)$.

$\forall m \in \mathbb{N} - \{0, 1\}$, $\forall p \in \mathbb{N}$, such as $m > p$, C contains the m -cell μ_p^m such as: If $p = m - 1$, $s_{m-1}^m(\mu_{m-1}^m) = t_{m-1}^m(\mu_{m-1}^m) = u_{m-1}$. If $0 \leq p < m - 1$, $s_{m-1}^m(\mu_p^m) = t_{m-1}^m(\mu_p^m) = \mu_p^{m-1}$. Also $d(\mu_p^m) = 1(m) \star_p^m 1(m)$, $c(\mu_p^m) = 1(m)$.

Furthermore C contains the 1-cell μ_0^1 such as $s_0^1(\mu_0^1) = t_0^1(\mu_0^1) = u_0$ and $d(\mu_0^1) = 1(1) \star_0^1 1(1)$, $c(\mu_0^1) = 1(1)$.

Besides, $\forall m \in \mathbb{N}$, C contains the m -cellule v_m such that: $s_{m-1}^m(v_m) = t_{m-1}^m(v_m) = v_{m-1}$ (if $m \geq 1$) and $d(v_m) = 2(m)$, $c(v_m) = 2(m)$.

$\forall m \in \mathbb{N} - \{0, 1\}$, $\forall p \in \mathbb{N}$, such that $m > p$, C contains the m -cell ν_p^m such as: If $p = m - 1$, $s_{m-1}^m(\nu_{m-1}^m) = t_{m-1}^m(\nu_{m-1}^m) = v_{m-1}$. If $0 \leq p < m - 1$, $s_{m-1}^m(\nu_p^m) = t_{m-1}^m(\nu_p^m) = \nu_p^{m-1}$. Also $d(\nu_p^m) = 2(m) \star_p^m 2(m)$, $c(\nu_p^m) = 2(m)$.

Furthermore C contains the 1-cell ν_0^1 such as $s_0^1(\nu_0^1) = t_0^1(\nu_0^1) = v_0$ and $d(\nu_0^1) = 2(1) \star_0^1 2(1)$, $c(\nu_0^1) = 2(1)$.

3.2.3 Definition of $C^i (i = 1, 2)$

C^1 is the system of operations of weak ∞ -functors. It is built on the basis of C adding to it a single symbol of functor (for each cell level): $\forall m \in \mathbb{N}$ the F^m

m -cell is added, which is such as: If $m \geq 1$, $s_{m-1}^m(F^m) = t_{m-1}^m(F^m) = F^{m-1}$. Also $d^1(F^m) = 1(m)$ and $c^1(F^m) = 2(m)$.

C^2 is the system of operations of weak natural ∞ -transformations. C^2 is built on C , adding to it two symbols of functor (for each cell level) and a symbol of natural transformation. More precisely

$\forall m \in \mathbb{N}$ we add the m -cell F^m such as: If $m \geq 1$, $s_{m-1}^m(F^m) = t_{m-1}^m(F^m) = F^{m-1}$. Also $d^2(F^m) = 1(m)$ and $c^2(F^m) = 2(m)$.

Then $\forall m \in \mathbb{N}$ we add the m -cell H^m such as: If $m \geq 1$, $s_{m-1}^m(H^m) = t_{m-1}^m(H^m) = H^{m-1}$. Also $d^2(H^m) = 1(m)$ and $c^2(H^m) = 2(m)$.

And finally we add 1-cell τ such as: $s_0^1(\tau) = F^0$ and $t_0^1(\tau) = H^0$. Also $d^2(\tau) = 1_{1(0)}$ and $c^2(\tau) = 2(1)$.

We can point out that the 2-coloured collections C^i ($i = 1, 2$) are naturally equipped with a pointing λ^i defined by $\lambda^i(1(m)) = u_m$ and $\lambda^i(2(m)) = v_m$.

3.2.4 Definition of C^n for $n \geq 3$

In order to define the general theory of n -Transformations ($n \in \mathbb{N}^*$), it is necessary to define the systems of operations C^n for the superior n -Transformations ($n \geq 3$). This paragraph can be left out in the first reading. Each collection C^n is built on C , adding to it the required cells. They contain four large groups of cells: The symbols of source and target weak ∞ -categories, the symbols of operadic units (obtained on the basis of C), the symbols of functors (sources and targets), and the symbols of n -Transformations (natural transformations, modification, etc). More precisely, on the basis of C :

Symbols of Functors $\forall m \in \mathbb{N}$, C^n contains the m -cells α_0^m and β_0^m such as:

If $m \geq 1$, $s_{m-1}^m(\alpha_0^m) = t_{m-1}^m(\alpha_0^m) = \alpha_0^{m-1}$ and $s_{m-1}^m(\beta_0^m) = t_{m-1}^m(\beta_0^m) = \beta_0^{m-1}$. Furthermore $d^n(\alpha_0^m) = d^n(\beta_0^m) = 1(m)$ and $c^n(\alpha_0^m) = c^n(\beta_0^m) = 2(m)$.

Symbols of the Higher n -Transformations $\forall p$, with $1 \leq p \leq n-1$, C^n contains the p -cells α_p and β_p which are such as: $\forall p$, with $2 \leq p \leq n-1$, $s_{p-1}^p(\alpha_p) = s_{p-1}^p(\beta_p) = \alpha_{p-1}$ and $t_{p-1}^p(\alpha_p) = t_{p-1}^p(\beta_p) = \beta_{p-1}$. If $p = 1$, $s_0^1(\alpha_1) = s_0^1(\beta_1) = \alpha_0^0$ and $t_0^1(\alpha_1) = t_0^1(\beta_1) = \beta_0^0$. What's more, $\forall p$, with $1 \leq p \leq n-1$, $d^n(\alpha_p) = d^n(\beta_p) = 1_p^0(1(0))$ and $c^n(\alpha_p) = c^n(\beta_p) = 2(p)$. Finally C^n contains the n -cell ξ_n such as $s_{n-1}^n(\xi_n) = \alpha_{n-1}$, $b_{n-1}^n(\xi_n) = \beta_{n-1}$ and $d^n(\xi_n) = 1_n^0(1(0))$ and $c^n(\xi_n) = 2(n)$ (Here 1_n^0 is the map resulting from the reflexive structure of $T(1 \cup 2)$). See [16]).

We can see that $\forall n \in \mathbb{N}^*$, the 2-colored collection C^n is naturally equipped with the pointing $1 \cup 2 \xrightarrow{\lambda^n} (C^n, d, c)$ defined as $\forall m \in \mathbb{N}$, $\lambda^n(1(m)) = u_m$ and $\lambda^n(2(m)) = v_m$.

3.3 The Co- ∞ -Graph of Coloured Operads of the n -Transformations ($n \in \mathbb{N}^*$)

In order not to make heavy notations we can write with the same notation δ_{n+1}^n and κ_{n+1}^n , sources and targets of the co- ∞ -graph of coloured collections, the co- ∞ -graph of coloured operads, and the ∞ -graph in $\mathbb{M}nd$ below. There is no risk of confusion. The set $\{C^n/n \in \mathbb{N}\}$ has got a natural structure of co- ∞ -graph. This co- ∞ -graph is generated by diagrams

$$C^n \begin{array}{c} \xrightarrow{\delta_{n+1}^n} \\ \xrightarrow{\kappa_{n+1}^n} \end{array} C^{n+1}$$

of pointed 2-coloured collections. For $n \geq 2$, these diagrams are defined as follows: First the $(n+1)$ -coloured collection contains the same symbols of operations as C^n for the j -cells, $0 \leq j \leq n-1$ or $n+2 \leq j < \infty$. For the n -cells and the $(n+1)$ -cells the symbols of operations will change: C^n contains the n -cell ξ_n whereas C^{n+1} contains the n -cells α_n and β_n , in addition contains the $(n+1)$ -cell ξ_{n+1} . If one notes $C^n - \xi_n$ the n -coloured collection

obtained on the basis of C^n by taking from it the n -cell ξ_n , then δ_{n+1}^n is defined as follows: $\delta_{n+1}^n|_{C^n - \xi_n}$ (i.e the restriction of δ_{n+1}^n to $C^n - \xi_n$) is the canonical injection $C^n - \xi_n \hookrightarrow C^{n+1}$ and $\delta_{n+1}^n(\xi_n) = \alpha_n$. In a similar way κ_{n+1}^n is defined as follows: $\kappa_{n+1}^n|_{C^n - \xi_n} = \delta_{n+1}^n|_{C^n - \xi_n}$ and $\kappa_{n+1}^n(\xi_n) = \beta_n$. We can notice that δ_{n+1}^n and κ_{n+1}^n keeps pointing, i.e we have for all $n \geq 1$ the equalities $\delta_{n+1}^n \lambda^n = \lambda^{n+1}$ and $\kappa_{n+1}^n \lambda^n = \lambda^{n+1}$.

The morphisms of 2-colored pointing collections of the diagram

$$C^0 \begin{array}{c} \xrightarrow{\delta_1^0} \\ \xrightarrow{\kappa_1^0} \end{array} C^1 \begin{array}{c} \xrightarrow{\delta_2^1} \\ \xrightarrow{\kappa_2^1} \end{array} C^2 \begin{array}{c} \xrightarrow{\delta_3^2} \\ \xrightarrow{\kappa_3^2} \end{array} C^3$$

have a similar definition:

By considering notation of section 3.2, we have for all integer $0 \leq p < n$ and for all $\forall m \in \mathbb{N}$:

$$\delta_1^0(\mu_p^n) = \mu_p^n; \delta_1^0(u_m) = u_m; \kappa_1^0(\mu_p^n) = v_p^n; \kappa_1^0(u_m) = v_m.$$

$$\text{Also: } \delta_2^1(\mu_p^n) = \mu_p^n; \delta_2^1(u_m) = u_m; \delta_2^1(v_p^n) = v_p^n; \delta_2^1(v_m) = v_m; \delta_2^1(F^m) = F^m. \text{ And } \kappa_2^1(\mu_p^n) = \mu_p^n; \kappa_2^1(u_m) = u_m; \kappa_2^1(v_p^n) = v_p^n; \kappa_2^1(v_m) = v_m; \kappa_2^1(F^m) = H^m.$$

$$\text{Finally: } \delta_3^2(\mu_p^n) = \mu_p^n; \delta_3^2(u_m) = u_m; \delta_3^2(v_p^n) = v_p^n; \delta_3^2(v_m) = v_m; \delta_3^2(F^m) = \alpha_0^m; \delta_3^2(H^m) = \beta_0^m; \delta_3^2(\tau) = \alpha_1. \text{ And } \kappa_3^2(\mu_p^n) = \mu_p^n; \kappa_3^2(u_m) = u_m; \kappa_3^2(v_p^n) = v_p^n; \kappa_3^2(v_m) = v_m; \kappa_3^2(F^m) = \alpha_0^m; \kappa_3^2(H^m) = \beta_0^m; \kappa_3^2(\tau) = \beta_1.$$

The pointed 2-coloured collections C^n ($n \in \mathbb{N}^*$) are the sytems of operations of the n -Transformations. Each of them freely produces the contractible 2-colored operads B^n ($n \in \mathbb{N}^*$). Each of these contractible operads is equipped with a system of operations given by the pointed 2-coloured collection C^n . These operads B^n are the operads of the n -Transformations ($n \in \mathbb{N}^*$) and are the most important objects in this article. They produce the monads \underline{B}^n whose algebras are the sought-after n -Transformations (see section 4 below). Due to

the universal property of the unit b of the monad \mathbb{B} , $C^n \xrightarrow{b(C^n)} B^n = B(C^n)$, one obtains the co- ∞ -graph B^\bullet of the coloured operads of the n -Transformations.

$$\begin{array}{ccccccc}
 B^0 & \xrightleftharpoons[\kappa_1^0]{\delta_1^0} & B^1 & \xrightleftharpoons[\kappa_2^1]{\delta_2^1} & B^2 & \cdots & B^{n-1} & \xrightleftharpoons[\kappa_n^{n-1}]{\delta_n^{n-1}} & B^n & \cdots \\
 \uparrow b(C^0) & & \uparrow b(C^1) & & \uparrow b(C^2) & & \uparrow b(C^{n-1}) & & \uparrow b(C^n) & \\
 C^0 & \xrightleftharpoons[\kappa_1^0]{\delta_1^0} & C^1 & \xrightleftharpoons[\kappa_2^1]{\delta_2^1} & C^2 & \cdots & C^{n-1} & \xrightleftharpoons[\kappa_n^{n-1}]{\delta_n^{n-1}} & C^n & \cdots
 \end{array}$$

The commutativity property of these diagrams is important for the consistency of algebras (see section 4.5). In particular morphisms

$$B^0 \xrightleftharpoons[\kappa_1^0]{\delta_1^0} B^1$$

are obtain with the following way:

First we consider "morphisms of colors" (in the category of ω -graphs)

$$1 \xrightleftharpoons[i_2]{i_1} 1 \cup 2$$

such as $\forall n \in \mathbb{N}, i_1(1(n)) = 1(n)$ and $i_2(1(n)) = 2(n)$

Then we build for each $j \in \{1, 2\}$ the following diagram

$$\begin{array}{ccccc}
 B^0 & \xrightarrow{u_j} & (T(i_j) \times i_j)^*(B^1) & \xrightarrow{v_j} & B^1 \\
 & \searrow (d^0, c^0) & \downarrow & & \downarrow (d^1, c^1) \\
 & & T(1) \times 1 & \xrightarrow{T(i_j) \times i_j} & T(1 \cup 2) \times (1 \cup 2)
 \end{array}$$

where the right square is cartesian (we change the color of the operad B^1 by pullback) and where the new operads $(T(i_j) \times i_j)^*(B^1)$ has a composition system and is contractible as well. So by universality, for each $j \in \{1, 2\}$, we get the unique morphism u_j and we write $v_1 \circ u_1 = \delta_1^0$ and $v_2 \circ u_2 = \kappa_1^0$. Also it is not difficult to see the co-globularity property of the diagram

$$B^0 \xrightleftharpoons[\kappa_1^0]{\delta_1^0} B^1 \xrightleftharpoons[\kappa_2^1]{\delta_2^1} B^2$$

4 Monads and Algebras of the n -Transformations ($n \in \mathbb{N}^*$)

$\mathbb{M}nd$ is the category of the categories equipped with a monad, and $\mathbb{A}dj$ is the category of the adjunction pairs. These categories are defined in [16].

4.1 Monads \underline{B}^n of the n -Transformations ($n \in \mathbb{N}^*$).

If \mathcal{C} is a topos we shall note $\mathcal{C}/B \xrightarrow{f^*} \mathcal{C}/A$ the pullback functor associated with an arrow $A \xrightarrow{f} B$ of \mathcal{C} , and $\mathcal{C}/A \xrightarrow{\Sigma_f} \mathcal{C}/B$ the composition functor. We have the usual adjunctions: $\Sigma_f \dashv f^* \dashv \pi_f$, where π_f is the internal product functor.

Each T -category produces a monad which is described in [18, 4.3 page 150]. Hence $\forall n \in \mathbb{N}^*$, the operad B^n of the n -Transformations produce a monad \underline{B}^n on $\infty\text{-Gr}/1 \cup 2$. More precisely, if we note (B^n, d^n, c^n) its underlying T -graph we have: $\underline{B}^n := \Sigma_{c^n}(d^n)^* \widehat{T}$ (where we put $\widehat{T}(C, d, c) := (T(C), T(d), T(c))$). A bicolour ∞ -graph $G \xrightarrow{g} 1 \cup 2$ is often noted G because there is no risk of confounding. We can therefore write $\underline{B}^n(G)$ instead of $\underline{B}^n(g)$, and it will be the same for the natural transformations δ_n^{n-1} and κ_n^{n-1} (see below) and we write $\underline{B}^n(G) := T(G) \times_{T(1 \cup 2)} B^n$ (implied $\underline{B}^n(g) = c^n \circ \pi_1$) and the definition of \underline{B}^n on morphisms is as easy. Projection on $T(G) \times_{T(1 \cup 2)} B^n$ are noted π_0 and π_1 . The definition of \underline{B}^0 is similar.

4.2 The ∞ -graph of $\mathbb{M}nd$ of Monads of n -Transformations ($n \in \mathbb{N}^*$)

Considering $G \xrightarrow{g} 1 \cup 2$, a bicolour ∞ -graph. If we apply to it the monads \underline{B}^n and \underline{B}^{n-1} we obtain the equalities $d^n \pi_1 = T(g) \pi_0$, $d^{n-1} \pi_1 = T(g) \pi_0$. We also have $d^{n-1} = d^n \delta_n^{n-1}$ (To remove any confusion on our abuses of notations, the reader is encouraged to draw corresponding diagram). Thus we have

$d^n \circ \delta_n^{n-1} \circ \pi_1 = 1_{T(1 \cup 2)} \circ d^{n-1} \circ \pi_1 = 1_{T(1 \cup 2)} \circ T(g) \circ \pi_0 = T(g) \circ 1_{T(G)} \circ \pi_0$.
 Hence the existence of a single morphism of ∞ -graph

$$T(G) \times_{T(1 \cup 2)} B^{n-1} \xrightarrow{\delta_n^{n-1}(G)} T(G) \times_{T(1 \cup 2)} B^n$$

such as $\delta_n^{n-1} \pi_1 = \pi_1 \delta_n^{n-1}(G)$ and $\pi_0 = \pi_0 \delta_n^{n-1}(G)$. In particular we obtain the equality $c^n \pi_1 \delta_n^{n-1}(G) = c^{n-1} \pi_1$. It is then easy to see that to each bicolour ∞ -graph is associated the morphism (of $\infty\text{-Gr}/1 \cup 2$): $\underline{B}^{n-1}(G) \xrightarrow{\delta_n^{n-1}(G)} \underline{B}^n(G)$ (these morphisms are still simply called $\delta_n^{n-1}(G)$). It is very easy to see that the set of these morphisms produce a natural transformation $\underline{B}^{n-1} \xrightarrow{\delta_n^{n-1}} \underline{B}^n$. It is shown that δ_n^{n-1} fits the axioms $\mathbb{M}nd1$ and $\mathbb{M}nd2$ of the morphisms of monads (these axioms are in [16]; particularly because $B^{n-1} \xrightarrow{\delta_n^{n-1}} B^n$ is a morphism of operads). Hence we get the morphism of $\mathbb{M}nd$

$$(\infty\text{-Gr}/1 \cup 2, \underline{B}^n) \xrightarrow{\delta_n^{n-1}} (\infty\text{-Gr}/1 \cup 2, \underline{B}^{n-1})$$

Thus the morphisms of coloured operads $B^{n-1} \xrightarrow[\kappa_n^{n-1}]{\delta_n^{n-1}} B^n$ ($n \geq 2$), create natural transformations $\underline{B}^{n-1} \xrightarrow[\kappa_n^{n-1}]{\delta_n^{n-1}} \underline{B}^n$ which fits into the axioms $\mathbb{M}nd1$ and $\mathbb{M}nd2$ of morphisms of monads. So we get the diagrams of $\mathbb{M}nd(n \geq 2)$

$$(\infty\text{-Gr}/1 \cup 2, \underline{B}^n) \xrightarrow[\kappa_n^{n-1}]{\delta_n^{n-1}} (\infty\text{-Gr}/1 \cup 2, \underline{B}^{n-1})$$

Similarly the morphisms $B^0 \xrightarrow[\kappa_1^0]{\delta_1^0} B^1$ produce two natural transformations $\underline{B}^0 \circ i_1^* \xrightarrow{\delta_1^0} i_1^* \circ \underline{B}^1$, $\underline{B}^0 \circ i_2^* \xrightarrow{\kappa_1^0} i_2^* \circ \underline{B}^1$ (i_1^* and i_2^* are the colour functors) which also fits $\mathbb{M}nd1$ and $\mathbb{M}nd2$, which leads to the diagram of $\mathbb{M}nd$

$$(\infty\text{-Gr}/1 \cup 2, \underline{B}^1) \xrightarrow[\kappa_1^0]{\delta_1^0} (\infty\text{-Gr}/1 \cup 2, \underline{B}^0)$$

It is generally appeared that the building of the monad associated to a T -category is functorial, so the diagram of $\mathbb{M}nd$

$$\cdots \rightrightarrows (\infty\text{-Gr}/1 \cup 2, \underline{B}^n) \cdots \rightrightarrows (\infty\text{-Gr}/1 \cup 2, \underline{B}^1) \rightrightarrows (\infty\text{-Gr}/1 \cup 2, \underline{B}^0)$$

is a ∞ -graph: The ∞ -graph \underline{B}^\bullet in $\mathbb{M}nd$ of the monads of the n -Transformations ($n \in \mathbb{N}^*$).

4.3 The ∞ -Graph of $\mathbb{C}AT$ of Batanin's Algebras of n -Transformations ($n \in \mathbb{N}^*$)

As in [16, § 4.3] we know that we have the functors

$$\mathbb{M}nd \xrightarrow{A} \mathbb{A}dj \xrightarrow{D} \mathbb{C}AT$$

where A is the functor, which is linked with any monad, its pair of adjunction functors and where D is the projection functor which associates X with any adjunction $X \begin{matrix} \xrightarrow{G} \\ \leftarrow F \end{matrix} Y$. So it is easy to see that $D \circ A$ associates its category of Eilenberg-Moore algebras to any monads. Particularly the functor $\mathbb{M}nd \xrightarrow{D \circ A} \mathbb{C}AT$ produces the following ∞ -graph of $\mathbb{C}AT$

$$\cdots \rightrightarrows \mathbb{A}lg(\underline{B}^n) \begin{matrix} \xrightarrow{\sigma_{n-1}^n} \\ \leftarrow \beta_{n-1}^n \end{matrix} \mathbb{A}lg(\underline{B}^{n-1}) \cdots \rightrightarrows \mathbb{A}lg(\underline{B}^1) \begin{matrix} \xrightarrow{\sigma_0^1} \\ \leftarrow \beta_0^1 \end{matrix} \mathbb{A}lg(\underline{B}^0)$$

which is the ∞ -graph $\mathbb{A}lg(\underline{B}^\bullet)$ of algebras of n -Transformations ($n \in \mathbb{N}$). It is the most important ∞ -graph of this article since it contains all Batanin's n -Transformations ($n \in \mathbb{N}$).

4.4 Domains and Codomains of Algebras

Let us remember the morphisms of $\mathbb{M}nd$: $(C, T) \xrightarrow{(Q, t)} (C', T')$ are given by functors $C \xrightarrow{Q} C'$ and natural transformations $T' \circ Q \xrightarrow{t} Q \circ T$ whose fits $\mathbb{M}nd1$ and $\mathbb{M}nd2$. If we apply the functor $\mathbb{M}nd \xrightarrow{D \circ A} \mathbb{C}AT$ to these morphisms, one

can get the functor, $\mathbb{A}lg(T) \rightarrow \mathbb{A}lg(T')$, defined on the objects as $(G, \nu) \mapsto (Q(G), Q(\nu) \circ t(G))$. We can now describe the functors σ_{n-1}^n and β_{n-1}^n ($n \geq 1$):

- If $n \geq 2$ then $\mathbb{A}lg(\underline{B}^n) \xrightarrow{\sigma_{n-1}^n} \mathbb{A}lg(\underline{B}^{n-1})$, $(G, \nu) \mapsto (G, \nu \circ \delta_n^{n-1}(G))$
and $\mathbb{A}lg(\underline{B}^n) \xrightarrow{\beta_{n-1}^n} \mathbb{A}lg(\underline{B}^{n-1})$, $(G, \nu) \mapsto (G, \nu \circ \kappa_n^{n-1}(G))$.
- If $n = 1$ then $\mathbb{A}lg(\underline{B}^1) \xrightarrow{\sigma_0^1} \mathbb{A}lg(\underline{B}^0)$, $(G, \nu) \mapsto (i_1^*(G), i_1^*(\nu) \circ \delta_1^0(G))$
and $\mathbb{A}lg(\underline{B}^1) \xrightarrow{\beta_0^1} \mathbb{A}lg(\underline{B}^0)$, $(G, \nu) \mapsto (i_2^*(G), i_2^*(\nu) \circ \kappa_1^0(G))$.

4.5 Consistence of Algebras

As Penon's [16], Batanin's n -Transformations ($n \in \mathbb{N}^*$) are particular in that they describe the hole semantics of their domain and codomain algebras as follows: If we have an algebra (G, ν) of n -Transformations, then a symbol of operation of the operad B^n which has its counterpart in the operad B^p ($0 \leq p < n$) will be semantically interpreted similarly via this algebra (G, ν) or via the algebra $\sigma_p^n(G, \nu)$ or the algebra $\beta_p^n(G, \nu)$.

Remark 3 This terminology is taken from measure theory where different coverings of a measurable subset are measured with the same value by a determined measure, which makes sense to that measure. \square

This is the simple consequence of the commutative property of diagrams in section 3.3 applied to a bicolour ∞ -graph.

So as to illustrate this property of consistence, let us take for example the symbol of operation H^m of the operad B^2 (identified with $b(C^2)(H^m)$). It will be semantically interpreted by an algebra $(G, \nu) \in \mathbb{A}lg(\underline{B}^2)$ on a m -cell $a \in G(m)$ (of colour 1), similarly to how the F^m symbol of the B^1 operad is interpreted by the target algebra $\beta_1^2(G, \nu) \in \mathbb{A}lg(\underline{B}^1)$. Indeed the equalities $\kappa_2^1 \pi_1 = \pi_1 \kappa_2^1(G)$ and $\kappa_2^1 b(C^1) = b(C^2) \kappa_2^1$ immediately suggests that: $(a, F^m) \xrightarrow{\kappa_2^1(G)} (a, H^m)$, then $\nu(a, H^m) = (\nu \circ \kappa_2^1(G))(a, F^m) =$

$\beta_1^2(G, \nu)(a, F^m)$, which expresses consistence. In short, we will say that Batanin's algebras (as Penon's algebras) are consistent.

5 Dimension 2

5.1 Dimension of Algebras

The dimension of Penon's algebras is defined in [21] and in [16]. The dimension of Batanin's algebras is totally similar, but we must precisely define the structures of the underlying ∞ -magmas of these algebras so as to have a reflexive structure. So we can note $B^n \times_{T(1 \cup 2)} T(G) \xrightarrow{\nu} G$ a \underline{B}^n -algebra i.e a weak n -transformation ($n \geq 1$). The two ∞ -magmas ([16]) of this algebra are defined as follows: $\alpha \circ_p^n \beta := \nu(\mu_p^n; \eta(\alpha) \star_p^n \eta(\beta))$ and $1_\alpha := \nu([u_n, u_n]; 1_{\eta(\alpha)})$, if $\alpha, \beta \in G(n)$ and are with colour 1. Furthermore $\alpha \circ_p^n \beta := \nu(\nu_p^n; \eta(\alpha) \star_p^n \eta(\beta))$ and $1_\alpha := \nu([v_n, v_n]; 1_{\eta(\alpha)})$, if $\alpha, \beta \in G(n)$ and are with colour 2. Then (G, ν) has dimension 2 if its two underlying ∞ -magmas has dimension 2. We have the same definition for \underline{B}^0 -algebras (i.e weak ∞ -categories).

5.2 The \underline{B}^1 -Algebras of dimension 2 are Pseudo-2-Functors

Let (G, ν) be a \underline{B}^1 -algebra of dimension 2. The \underline{B}^0 -algebra's source of (G, ν) : $\sigma_0^1(G, \nu) = (i_1^*(G), i_1^*(\nu) \circ \delta_1^0(G))$ put on $i_1^*(G)$ a bicategory structure which coincides with the one produced by (G, ν) on $i_1^*(G)$. In the same way, the \underline{B}^0 -algebra target of (G, ν) : $\beta_0^1(G, \nu) = (i_2^*(G), i_2^*(\nu) \circ \kappa_1^0(G))$ put on $i_2^*(G)$ a bicategory structure which coincides with that one produced by (G, ν) on $i_2^*(G)$. All these coincidences come from the consistence of algebras, and so we can therefore make all our calculations merely with the \underline{B}^1 -algebra (G, ν) to show the given below axiom of associativity-distributivity (that we call *AD*-axiom) of pseudo-2-functors. For other axioms of the pseudo-2-

functors, which are easier, we proceed in the same way. Let $F^m (m \in \mathbb{N})$ be the unary operations symbols of functors of the operad B^1 . The B^1 -algebra of dimension 2 interprets these symbols into pseudo-2-functors. Indeed if $B^1 \times_{T(1 \cup 2)} T(G) \xrightarrow{v} G$ is a B^1 -algebra of dimension 2 then we get: $\forall m \in \mathbb{N}$, $F(a) := v(F^m; \eta(a))$ if $a \in G(m)$ (a has the colour 1), which defines a morphism of ∞ -graphs $i_1^*(G) \xrightarrow{F} i_2^*(G)$ where $i_1^*(G)$ and $i_2^*(G)$ are bicategories. So we will show that this morphism F fits the AD -axiom of pseudo-2-functors. Let $x \xrightarrow{a} y \xrightarrow{b} z \xrightarrow{c} t$ be a 1-cellules diagram of $i_1^*(G)$, we are going to check that we get the following commutativity

$$\begin{array}{ccc}
 F(a) \circ_0^1 (F(b) \circ_0^1 F(c)) & \xrightarrow{1_{F(a)} \circ_0^2 d(b,c)} & F(a) \circ_0^1 F(b \circ_0^1 c) \\
 \uparrow a(F(a), F(b), F(c)) \parallel & & \downarrow d(a, b \circ_0^1 c) \\
 (F(a) \circ_0^1 F(b)) \circ_0^1 F(c) & \square & F(a \circ_0^1 (b \circ_0^1 c)) \\
 \downarrow d(a,b) \circ_0^2 1_{F(c)} \parallel & & \uparrow F(a(a,b,c)) \\
 F(a \circ_0^1 b) \circ_0^1 F(c) & \xrightarrow{d(a \circ_0^1 b, c)} & F((a \circ_0^1 b) \circ_0^1 c)
 \end{array}$$

where $a \circ_0^1 (b \circ_0^1 c) \xrightarrow{a(a,b,c)} (a \circ_0^1 b) \circ_0^1 c$ is an associativity coherence cell and $F(a) \circ_0^1 F(b) \xrightarrow{d(a,b)} F(a \circ_0^1 b)$ is a distributivity coherence cell (particular to pseudo-2-functors). The strategy to demonstrate the AD -axiom is simple: We build a diagram of 3-cells of B^1 which will be semantically interpreted by the B^1 -algebras of dimension 2 as the AD -axiom. To be clearer, the operadic multiplication of the coloured operad B^1

$$B^1 \times_{T(1 \cup 2)} T(B^1) \xrightarrow{\gamma} B^1$$

will be noted γ_i for each i -cellular level. Let the following 2-cells in B^1 :

$$d := [\gamma_1(v_0^1; \eta(F^1) \star_0^1 \eta(F^1)); \gamma_1(F^1; \eta(\mu_0^1))];$$

$$a_1 := [\gamma_1(\mu_0^1; \eta(\mu_0^1) \star_0^1 \eta(u_1)); \gamma_1(\mu_0^1; \eta(u_1) \star_0^1 \eta(\mu_0^1))];$$

$$a_2 := [\gamma_1(v_0^1; \eta(v_0^1) \star_0^1 \eta(v_1)); \gamma_1(v_0^1; \eta(v_1) \star_0^1 \eta(v_0^1))].$$

Remark 4 The operation symbol d is interpreted by the algebra as the distributivity coherence cells of the pseudo-2-functors. The symbols a_1 and a_2 are interpreted as the associativity coherence cells, the first one for the weak ∞ -category source the second one for the weak ∞ -category target. \square

Then we can consider the following 2-cells of B^1 :

$$\rho_1 = \gamma_2(v_0^2; \eta([F^1; F^1]) \star_0^2 \eta(d));$$

$$\rho_2 = \gamma_2(d; 1_{\eta(u_1)} \star_0^2 1_{\eta(\mu_0^1)});$$

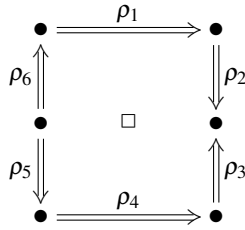
$$\rho_3 = \gamma_2(F^2; \eta(a_1));$$

$$\rho_4 = \gamma_2(d; 1_{\eta(\mu_0^1)} \star_0^2 1_{\eta(u_1)});$$

$$\rho_5 = \gamma_2(v_0^2; \eta(d) \star_0^2 \eta([F^1; F^1]));$$

$$\rho_6 = \gamma_2(a_2; 1_{\eta(F^1)} \star_0^2 1_{\eta(F^1)} \star_0^2 1_{\eta(F^1)}).$$

This 2-cells are the conglomerations of operation symbols that are interpreted by algebras as the coherence 2-cells of the diagram of the AD -axiom of pseudo-2-functors



Then we consider the following 2-cells of B^1

$$\Lambda_1 = \gamma_2(v_1^2; \eta(\gamma_2(v_1^2; \eta(\rho_2) \star_1^2 \eta(\rho_1))) \star_1^2 \eta(\rho_6));$$

$$\Lambda'_1 = \gamma_2(v_1^2; \eta(\rho_2) \star_1^2 \eta(\gamma_2(v_1^2; \eta(\rho_1) \star_1^2 \eta(\rho_6))));$$

$$\Lambda_2 = \gamma_2(v_1^2; \eta(\gamma_2(v_1^2; \eta(\rho_3) \star_1^2 \eta(\rho_4)))) \star_1^2 \eta(\rho_5);$$

$$\Lambda'_2 = \gamma_2(v_1^2; \eta(\rho_3) \star_1^2 \eta(\gamma_2(v_1^2; \eta(\rho_4) \star_1^2 \eta(\rho_5)))).$$

We can show that these 2-cells are parallels and with the same domain, so they are connected with coherences 3-cells

$$\Theta_1 = [\Lambda_1, \Lambda'_1], \Theta_2 = [\Lambda'_1, \Lambda_2], \Theta_3 = [\Lambda_2, \Lambda'_2],$$

and the interpretation by \underline{B}^1 -algebras of dimension 2 of this 3-cells gives the *AD*-axiom of pseudo-2-functors.

5.3 The \underline{B}^2 -Algebras of dimensions 2 are Natural Pseudo-2-Transformations

Let (G, v) be a \underline{B}^2 -algebra of dimension 2. The \underline{B}^0 -algebra source of (G, v) : $\sigma_1^2(\sigma_0^1(G, v)) = (i_1^*(G), i_1^*(v \circ \delta_2^1(G)) \circ \delta_1^0(G))$ put in $i_1^*(G)$ a bicategory structure which coincides with the one produced by (G, v) on $i_1^*(G)$. In the same way, the \underline{B}^0 -algebra target of (G, v) : $\beta_1^2(\beta_0^1(G, v)) = (i_2^*(G), i_2^*(v \circ \kappa_2^1(G)) \circ \kappa_1^0(G))$ put in $i_2^*(G)$ a bicategory structure which coincides with the one produced by (G, v) on $i_2^*(G)$. And the \underline{B}^1 -algebra source of (G, v) : $\sigma_1^2(G, v) = (G, v \circ \delta_2^1(G))$ produces a pseudo-2-functor F_1 (see above) which coincides with the one produced by (G, v) i.e the one built with the ∞ -graph morphism $i_1^*(G) \xrightarrow{F_1} i_2^*(G)$ defined as: $F_1(a) := v(F^m; \eta(a))$ if $a \in i_1^*(G)(m)$. Besides the \underline{B}^1 -algebra target of (G, v) : $\beta_1^2(G, v) = (G, v \circ \kappa_2^1(G))$ produces a pseudo-2-functor H_1 which coincides with the one produced by (G, v) i.e the one built with the ∞ -graph morphism $i_1^*(G) \xrightarrow{H_1} i_2^*(G)$ defined as: $H_1(a) := v(H^m; \eta(a))$ if $a \in i_1^*(G)(m)$. All these coincidences come from the consistence of algebras, and we can therefore make all our calculations merely with the \underline{B}^2 -algebra (G, v) (without using its source algebra or its target algebra) to show the axiom below of compatibility with associativity-distributivity of natural pseudo-2-transformations (that we call *CAD*-axiom). Then let τ be the unary operation symbol of natural transformation of the

operad B^2 . This symbol is interpreted by the \underline{B}^2 -algebras of dimension 2 as natural pseudo-2-transformations. Indeed if $B^2 \times_{T(1 \cup 2)} T(G) \xrightarrow{v} G$ is an \underline{B}^2 -algebra of dimension 2 then we write

$$\tau_1(a) := v(\tau; 1_{\eta(a)}), \text{ if } a \in G(0) (a \text{ has colour } 1),$$

We can see that it defines a 1-cells family τ_1 in $i_2^*(G)$ indexed by $i_1^*(G)(0)$

$$i_1^*(G) \begin{array}{c} \xrightarrow{F_1} \\ \Downarrow \tau_1 \\ \xrightarrow{H_1} \end{array} i_2^*(G)$$

We are going to show that the previous family τ_1 fits the *CAD*-axiom of natural pseudo-2-transformations. For other axioms of natural pseudo-2-transformations, which are easier, we proceed in the same way. Let $x \xrightarrow{a} y \xrightarrow{b} z$ be an 1-cells diagram of $i_1^*(G)$, we are going to prove that we have the following commutativity

$$\begin{array}{ccc} H_1(b) \circ_0^1 (H_1(a) \circ_0^1 \tau_1(x)) & \xrightarrow{1_{H_1(b)} \circ_0^2 \omega(a)} & H_1(b) \circ_0^1 (\tau_1(y) \circ_0^1 F_1(a)) \\ \Downarrow a(H_1(b), H_1(a), \tau_1(x)) & & \Downarrow a(H_1(b), \tau_1(y), F_1(a)) \\ (H_1(b) \circ_0^1 H_1(a)) \circ_0^1 \tau_1(x) & & (H_1(b) \circ_0^1 \tau_1(y)) \circ_0^1 F_1(a) \\ \Downarrow d_1(a,b) \circ_0^2 1_{\tau_1(x)} & \square & \Downarrow \omega(b) \circ_0^2 1_{F_1(a)} \\ H_1(b \circ_0^1 a) \circ_0^1 \tau_1(x) & & (\tau_1(z) \circ_0^1 F_1(b)) \circ_0^1 F_1(a) \\ \Downarrow \omega(b \circ_0^1 a) & & \Downarrow a(\tau_1(z), F_1(b), F_1(a)) \\ \tau_1(z) \circ_0^1 F_1(b \circ_0^1 a) & \xleftarrow{1_{\tau_1(z)} \circ_0^2 d_0(b,a)} & \tau_1(z) \circ_0^1 (F_1(b) \circ_0^1 F_1(a)). \end{array}$$

where in particular $H_1(a) \circ_0^1 \tau_1(x) \xrightarrow{\omega(a)} \tau_1(y) \circ_0^1 F_1(a)$ is a coherence cell specific to natural pseudo-2-transformations. The strategy to demonstrate the *CAD*-axiom is similar to the previous demonstration (for the *AD*-axiom of pseudo-2-functors): We build a diagram of 3-cells of B^2 that will be

semantically interpreted by the \underline{B}^2 -algebras of dimension 2 as the *CAD*-axiom. Like before operadic composition is

$$B^2 \times_{T(1 \cup 2)} T(B^2) \xrightarrow{\gamma} B^2$$

will be noted γ_i for each i -cellular level. So we can consider the following 2-cells of B^2

$$\omega := [\gamma_1(v_0^1; \eta(H^1) \star_0^1 \eta(\tau)); \gamma_1(v_0^1; \eta(\tau) \star_0^1 \eta(F^1))];$$

$$d^F := [\gamma_1(v_0^1; \eta(F^1) \star_0^1 \eta(F^1)); \gamma_1(F^1; \eta(\mu_0^1))];$$

$$d^H := [\gamma_1(v_0^1; \eta(H^1) \star_0^1 \eta(H^1)); \gamma_1(H^1; \eta(\mu_0^1))];$$

$$a := [\gamma_1(v_0^1; \eta(v_1) \star_0^1 \eta(v_0^1)); \gamma_1(v_0^1; \eta(v_0^1) \star_0^1 \eta(v_1))];$$

$$b := [\gamma_1(v_0^1; \eta(v_0^1) \star_0^1 \eta(v_1)); \gamma_1(v_0^1; \eta(v_1) \star_0^1 \eta(v_0^1))].$$

Then we consider the following 2-cells

$$\rho_1 = \gamma_2(v_0^2; \eta([H^1; H^1]) \star_0^2 \eta(\omega));$$

$$\rho_2 = \gamma_2(a; 1_{\eta(H^1)} \star_0^2 1_{\eta(\tau)} \star_0^2 1_{\eta(F^1)});$$

$$\rho_3 = \gamma_2(v_0^2; \eta(\omega) \star_0^2 \eta([F^1; F^1]));$$

$$\rho_4 = \gamma_2(b; 1_{\eta(\tau)} \star_0^2 1_{\eta(F^1)} \star_0^2 1_{\eta(F^1)});$$

$$\rho_5 = \gamma_2(v_0^2; \eta([\tau; \tau]) \star_0^2 \eta(d^F));$$

$$\rho_6 = \gamma_2(\omega; 1_{\eta(\mu_0^1)});$$

$$\rho_7 = \gamma_2(v_0^2; \eta(d) \star_0^2 \eta([\tau; \tau]));$$

$$\rho_8 = \gamma_2(a; 1_{\eta(H^1)} \star_0^2 1_{\eta(H^1)} \star_0^2 1_{\eta(\tau)}).$$

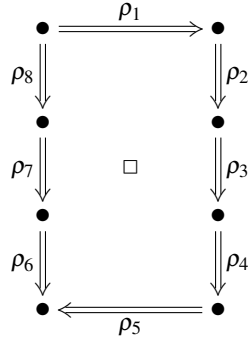
We also consider one 2-cell ρ'_5 built as follows:

$$\delta^F := [\gamma_1(F^1; \eta(\mu_0^1)); \gamma_1(v_0^1; \eta(F^1) \star_0^1 \eta(F^1))].$$

In that case we define

$$\rho'_5 = \gamma_2(v_0^2; \eta([\tau; \tau]) \star_0^2 \eta(\delta^F)).$$

These 2-cells are the conglomeration of operation symbols that are interpreted by algebras as the coherence 2-cells of the diagram of the *CAD*-axiom of natural pseudo-2-transformations



To build the ten coherence 2-cells Λ_i ($1 \leq i \leq 10$) below, which enables to conclude, we need the following additional 2-cells

$$\Theta_1 = \gamma_2(v_1^2; \eta(\gamma_2(v_1^2; \eta(v_1^2) \star_1^2 \eta(v_2)))) \star_1^2 \eta(v_2));$$

$$\Theta_2 = \gamma_2(v_1^2; \eta(\gamma_2(\mu_1^2; \eta(v_2) \star_1^2 \eta(v_1^2)))) \star_1^2 \eta(v_2));$$

$$\Theta_3 = \gamma_2(v_1^2; \eta(v_2) \star_1^2 \eta(\gamma_2(v_1^2; \eta(v_1^2) \star_1^2 \eta(v_2))));$$

$$\Theta_4 = \gamma_2(v_1^2; \eta(v_2) \star_1^2 \eta(\gamma_2(v_1^2; \eta(v_2) \star_1^2 \eta(v_1^2))));$$

$$\Theta_5 = \gamma_2(v_1^2; \eta(v_1^2) \star_1^2 \eta(v_1^2)).$$

The 2-cells Λ_i ($1 \leq i \leq 10$) are then defined in the following way

$$\Lambda_1 = \gamma_2(\Theta_1; \eta(\rho_4) \star_1^2 \eta(\rho_3) \star_1^2 \eta(\rho_2) \star_1^2 \eta(\rho_1));$$

$$\Lambda_2 = \gamma_2(\Theta_2; \eta(\rho_4) \star_1^2 \eta(\rho_3) \star_1^2 \eta(\rho_2) \star_1^2 \eta(\rho_1));$$

$$\Lambda_3 = \gamma_2(\Theta_3; \eta(\rho_4) \star_1^2 \eta(\rho_3) \star_1^2 \eta(\rho_2) \star_1^2 \eta(\rho_1));$$

$$\Lambda_4 = \gamma_2(\Theta_4; \eta(\rho_4) \star_1^2 \eta(\rho_3) \star_1^2 \eta(\rho_2) \star_1^2 \eta(\rho_1));$$

$$\Lambda_5 = \gamma_2(\Theta_5; \eta(\rho_4) \star_1^2 \eta(\rho_3) \star_1^2 \eta(\rho_2) \star_1^2 \eta(\rho_1)).$$

We can note as well $\lambda = \eta(\rho'_5) \star_1^2 \eta(\rho_6) \star_1^2 \eta(\rho_7) \star_1^2 \eta(\rho_8)$. And consider $\Lambda_6 = \gamma_2(\Theta_1; \lambda)$, $\Lambda_7 = \gamma_2(\Theta_2; \lambda)$; $\Lambda_8 = \gamma_2(\Theta_3; \lambda)$; $\Lambda_9 = \gamma_2(\Theta_4; \lambda)$; $\Lambda_{10} = \gamma_2(\Theta_5; \lambda)$.

We can prove that these 2-cells are parallels and with the same domain, so they are connected with coherences 3-cells: $\zeta_i := [\Lambda_i; \Lambda_{i+1}]$ ($1 \leq i \leq 9$). And the interpretation by \underline{B}^2 -algebras of dimension 2 of these 3-cells gives the *CAD*-axiom of natural pseudo-2-transformations.

6 Fusion of Adjunctions

As we saw in theorem 1 we need to do the "fusion" of two monads to obtain a new monad, which inherits at the same time properties of these two monads. This monad is the contractible monoids monad $\mathbb{B} = (B, \rho, b)$ of the theorem 1 which permits us to build the operads of n -Transformations ($n \in \mathbb{N}$). The fusion between adjunctions require some hypotheses (see below) and naturally we shall see that our two adjunctions fill these hypotheses.

The following "fusion theorem" is a generalization of techniques used by Batanin in [2]. This theorem is going to be shown especially powerful because the required hypotheses are so simple. As a result the fusion product of two adjunctions is possible under conditions that we can often run into.

Lemma 1 *Let us consider the adjunction $\mathcal{C} \begin{array}{c} \xrightarrow{U} \\ \dashv \\ \xleftarrow{F} \end{array} \mathcal{B}$ such as \mathcal{C} has a co-equalizer and U is faithful. Let the diagram $B \begin{array}{c} \xrightarrow{d_0} \\ \rightrightarrows \\ \xrightarrow{d_1} \end{array} U(C)$ in \mathcal{B} , then there*

is a unique morphism $C \xrightarrow{q} Q$ of \mathcal{C} verifying $U(q)d_0 = U(q)d_1$ and which is universal for this property, i.e if we give ourselves another morphism $C \xrightarrow{q'} Q'$ of \mathcal{C} such as $U(q')d_0 = U(q')d_1$, then there is a unique morphism $Q \xrightarrow{h} Q'$ of \mathcal{C} such as $U(h)U(q) = U(q')$. \square

PROOF Given \bar{d}_0, \bar{d}_1 the morphisms of \mathcal{C} which are the extensions of d_0 and d_1 , and let us put $\hat{d}_0 = U(\bar{d}_0)$ and $\hat{d}_1 = U(\bar{d}_1)$. Let us note $C \xrightarrow{q} Q$ the co-equalizer of \bar{d}_0 and \bar{d}_1 . We get $U(q)d_0 = U(q)U(\bar{d}_0)\eta_X = U(q)U(\bar{d}_1)\eta_X = U(q)d_1$. We can show that q is universal for this property. Let $C \xrightarrow{q'} Q'$ another morphism of \mathcal{C} verifying $U(q')d_0 = U(q')d_1$. So $U(q')U(\bar{d}_0)\eta_X = U(q')U(\bar{d}_1)\eta_X$, i.e $U(q'\bar{d}_0)\eta_X = U(q'\bar{d}_1)\eta_X$, Therefore we have $q'\bar{d}_0 = q'\bar{d}_1$ with $q = \text{coker}(\bar{d}_0, \bar{d}_1)$, which shows that there is a unique morphism $Q \xrightarrow{h} Q'$ of \mathcal{C} such as $hq = q'$ and also this morphism is unique such as $U(h)U(q) = U(q')$, because U is faithful. \blacksquare

Let the following adjunction be: $(\mathcal{C}, A) \xrightleftharpoons[U]{U} (\mathcal{B}, A)$. It is fusionnable if the following properties are verified:

- \mathcal{C} has coequalizers and $\overrightarrow{\mathbb{N}}$ -colimits.
- \mathcal{B} have $\overrightarrow{\mathbb{N}}$ -colimits.
- U is faithful and preserves $\overrightarrow{\mathbb{N}}$ -colimits.

Remark 5 Here $\overrightarrow{\mathbb{N}}$ -colimits is the notation used in [6] for directed colimits. \square

Let us go to the fusion theorem.

Theorem 2 Let us consider the adjunction $\mathcal{C} \xrightleftharpoons[M]{U} \mathcal{B}$ with monad (L, m, l) , and the adjunction $\mathcal{D} \xrightleftharpoons[H]{V} \mathcal{B}$ with monad (C, m, c) . We suppose that these

adjunctions are fusionnable. In this case, if we consider the cartesian square of categories

$$\begin{array}{ccc} \mathcal{C} \times_{\mathcal{B}} \mathcal{D} & \xrightarrow{p_2} & \mathcal{D} \\ p_1 \downarrow & & \downarrow v \\ \mathcal{C} & \xrightarrow{U} & \mathcal{B} \end{array}$$

then the forgetful functor $\mathcal{C} \times_{\mathcal{B}} \mathcal{D} \xrightarrow{O} \mathcal{B}$ has a left adjoint: $F \dashv O$. \square

PROOF • Let $X \in \mathcal{B}(0)$. At first, we are going to build by induction an object $B(X)$ of \mathcal{B} and secondly we shall reveal that $B(X)$ has got the expected universal property.

- If $n = 0$ we give ourselves the following diagram of \mathcal{B} :

$$C_0 = X \xrightarrow{l_0=l(C_0)} L(C_0) \xrightarrow{\phi_0=1} L_0 \xrightarrow{c_0=c(L_0)} C(L_0) \xrightarrow{\psi_0=1} C_1 \xrightarrow{l_1=l(C_1)} L(C_1)$$

Thanks to the lemma, we obtain the morphism ϕ_1 with the diagram

$$L(C_0) \begin{array}{c} \xrightarrow{d_0=l_1 \psi_0 c_0 \phi_0} \\ \xrightarrow{d_1=L(j_0)=L(\psi_0 c_0 \phi_0 l_0)} \end{array} L(C_1) \xrightarrow{\phi_1} L_1$$

What allows to extend the previous diagram

$$C_1 \xrightarrow{l_1} L(C_1) \xrightarrow{\phi_1} L_1 \xrightarrow{c_1=c(L_1)} C(L_1)$$

And it allows again to obtain the morphism ψ_1

$$C(L_0) \begin{array}{c} \xrightarrow{\delta_0=c_1 \phi_1 l_1 \psi_0} \\ \xrightarrow{\delta_1=C(k_0)=C(\phi_1 l_1 \psi_0 c_0)} \end{array} C(L_1) \xrightarrow{\psi_1} C_2$$

and thus to prolong once more the previous diagram

$$C_1 \xrightarrow{l_1} L(C_1) \xrightarrow{\phi_1} L_1 \xrightarrow{c_1} C(L_1) \xrightarrow{\psi_1} C_2 \xrightarrow{l_2} L(C_2)$$

We do an induction. We can suppose that up to the rank n we can build these diagrams. In particular we give ourselves the following diagram

$$C_n \xrightarrow{l_n} L(C_n) \xrightarrow{\phi_n} L_n \xrightarrow{c_n} C(L_n) \xrightarrow{\psi_n} C_{n+1} \xrightarrow{l_{n+1}} L(C_{n+1})$$

where we especially note $j_n = \psi_n c_n \phi_n l_n$. We are going to show that we can prolong this type of diagram in the rank $n + 1$. Thanks to the Lemma, we consider the morphism ϕ_{n+1}

$$L(C_n) \xrightarrow[\delta_1=L(j_n)=L(\psi_n c_n \phi_n l_n)]{d_0=l_{n+1} \psi_n c_n \phi_n} \cong L(C_{n+1}) \xrightarrow{\phi_{n+1}} L_{n+1}$$

what allows to prolong the previous diagram

$$C_{n+1} \xrightarrow{l_{n+1}} L(C_{n+1}) \xrightarrow{\phi_{n+1}} L_{n+1} \xrightarrow{c_{n+1}=c(L_{n+1})} C(L_{n+1})$$

where we can particularly note $k_n = \phi_{n+1} l_{n+1} \psi_n c_n$. Then we consider, due to to the lemma, the morphism ψ_{n+1}

$$C(L_n) \xrightarrow[\delta_1=C(k_n)=C(\phi_{n+1} l_{n+1} \psi_n c_n)]{\delta_0=c_{n+1} \phi_{n+1} l_{n+1} \psi_n} \cong C(L_{n+1}) \xrightarrow{\psi_{n+1}} C_{n+2}$$

and thus to prolong still the previous diagram

$$C_{n+1} \xrightarrow{l_{n+1}} L(C_{n+1}) \xrightarrow{\phi_{n+1}} L_{n+1} \xrightarrow{c_{n+1}} C(L_{n+1}) \xrightarrow{\psi_{n+1}} C_{n+2} \xrightarrow{l_{n+2}} L(C_{n+2})$$

Thus for all $n \in \mathbb{N}$ we have this construction, what brings to light the filtered diagram built with these diagrams. This filtered diagram is noted B_* . In particular the diagrams

$$\begin{array}{ccccc} L(C_{n-1}) & \xrightarrow[\delta_1=L(\psi_{n-1} c_{n-1} \phi_{n-1} l_{n-1})]{d_0=l_n \psi_{n-1} c_{n-1} \phi_{n-1}} \cong & L(C_n) & \xrightarrow[\delta_1=L(\psi_n c_n \phi_n l_n)]{d_0=l_{n+1} \psi_n c_n \phi_n} \cong & L(C_{n+1}) \\ & & \downarrow \phi_n & & \downarrow \phi_{n+1} \\ & & L_n & \xrightarrow{\lambda_n} & L_{n+1} \end{array}$$

show that

$$\phi_{n+1}l_{n+1}\psi_n c_n \phi_n l_n \psi_{n-1} c_{n-1} \phi_{n-1} = \phi_{n+1}l_{n+1}\psi_n c_n \phi_n L(\psi_{n-1}c_{n-1}\phi_{n-1}l_{n-1}).$$

Thus according to the lemma, there is a unique morphism $L_n \xrightarrow{\lambda_n} L_{n+1}$, which is the forgetting of a morphism of \mathcal{C} , returning commutative these diagrams. Thus we obtain the filtered diagram L_* of \mathcal{B} which is the forgetting of a diagram filtered of \mathcal{C}

$$L_0 \xrightarrow{\lambda_0} L_1 \xrightarrow{\lambda_1} \dots \xrightarrow{\lambda_n} L_{n+1} \xrightarrow{\lambda_{n+1}} \dots$$

where B_* is an expanded diagram of L_* i.e we have

$$\overbrace{C_0 \xrightarrow{l_0} L(C_0) \xrightarrow{\phi_0} L_*}^{B_*}$$

We also have the diagram

$$\begin{array}{ccccc} C(L_{n-2}) & \xrightarrow[\delta_1=C(\phi_{n-1}l_{n-1}\psi_{n-2}c_{n-2})]{\delta_0=c_{n-1}\phi_{n-1}l_{n-1}\psi_{n-2}} & \rightleftarrows & C(L_{n-1}) & \xrightarrow[\delta_1=C(\phi_n l_n \psi_{n-1} c_{n-1})]{\delta_0=c_n \phi_n l_n \psi_{n-1}} & \rightleftarrows & C(L_n) \\ & & & \psi_{n-1} \downarrow & & & \downarrow \psi_n \\ & & & C_n & \xrightarrow{\kappa_n} & & C_{n+1} \end{array}$$

which shows that

$$\psi_n c_n \phi_n l_n \psi_{n-1} c_{n-1} \phi_{n-1} l_{n-1} \psi_{n-2} = \psi_n c_n \phi_n l_n \psi_{n-1} C(\phi_{n-1} l_{n-1} \psi_{n-2} c_{n-2}).$$

Thus according to the lemma, there is a unique morphism $C_n \xrightarrow{\kappa_n} C_{n+1}$ which is the forgetting of a morphism of \mathcal{D} returning commutative these diagrams. Therefore we obtain the filtered diagram C_* of \mathcal{B} which is the forgetting of a diagram filtered of \mathcal{D}

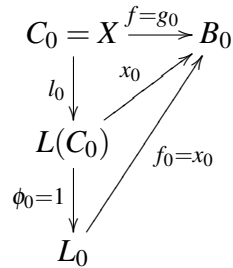
$$C_1 \xrightarrow{\kappa_1} C_2 \xrightarrow{\kappa_2} \dots \xrightarrow{\kappa_n} C_{n+1} \xrightarrow{\kappa_{n+1}} \dots$$

where B_* is an expanded diagram of C_* , i.e we have

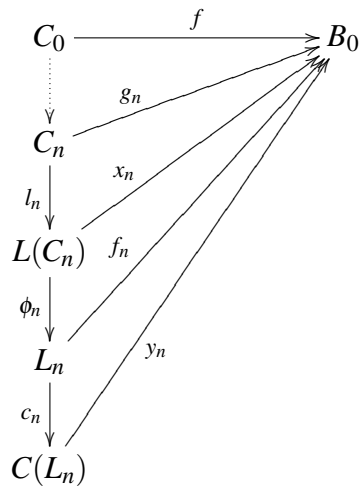
$$\overbrace{C_0 \xrightarrow{c_0 \phi_0 l_0} C(L_0) \xrightarrow{\psi_0} C_*}^{B_*}$$

Thus these diagrams B_* , L_* and C_* have the same colimit $B(X)$ in \mathcal{B} . We put $L_* = U(M_*)$ and $M_* \rightarrow \Delta M_X$ its colimit (in \mathcal{C}), $C_* = V(H_*)$ and $H_* \rightarrow \Delta H_X$ its colimit (in \mathcal{D}). The functors U and V preserving $\vec{\mathbb{N}}$ -colimits, therefore $B(X)$ is the forgetting of the pair (M_X, H_X) which is an object of $\mathcal{C} \times_{\mathcal{B}} \mathcal{D}$: $B(X) = O((M_X, H_X)) = U(M_X) = V(H_X)$. We put $F(X) = (M_X, H_X)$ which gives, as we are going to see, the desired left adjoint of the forgetful functor O , and where (B, ρ, b) is the associated monad. $B(X)$ inherits at the same time the structure of the object M_X (which lives in \mathcal{C}) and the structure of the object H_X (which lives in \mathcal{D}). It is the reason why the monad (B, ρ, b) can be called "fusion" of monads (L, m, l) and (C, m, c) . We note b_X the produced arrow $X \xrightarrow{b_X} B(X)$ The continuation consists in showing the universal character of b_X . We are going to show that if we give ourselves a morphism $X \xrightarrow{f} B_0$ of \mathcal{B} such as B_0 is the forgetting of an object (M_0, H_0) of $\mathcal{C} \times_{\mathcal{B}} \mathcal{D}$, then there is a unique morphism $(M_X, H_X) \xrightarrow{(h,k)} (M_0, H_0)$ of $\mathcal{C} \times_{\mathcal{B}} \mathcal{D}$ such as $O(h, k)b_X = f$. For that, we are going to use the filtered diagram B_* with which we are going to build by induction a cocone $B_* \rightarrow \Delta B_0$, and it will display the existence of the pair (h, k) .

- Let $g_0 = f$ and f_0 which is the extension of f from $L_0 = L(X)$:



- We can suppose that this construction is up to the rank n . Thus in particular we have the following diagram



Also the natural transformation $1_{\mathcal{B}} \xrightarrow{c} C$ applied to

$$C(L_{n-1}) \xrightarrow{\phi_n l_n \psi_{n-1}} L_n$$

gives the equality

$$C(\phi_n l_n \psi_{n-1})c(C(L_{n-1})) = c_n \phi_n l_n \psi_{n-1} = \delta_0$$

thus $y_n \delta_0 = y_n C(\phi_n l_n \psi_{n-1})c(C(L_{n-1}))$. On the other hand

$$y_n \delta_0 = y_n \delta_0 m(L_{n-1})c(C(L_{n-1}))$$

(unity axiom of monads), which leads to the equality

$$y_n C(\phi_n l_n \psi_{n-1}) = y_n \delta_0 m(L_{n-1})$$

(do not forget that $y_n \delta_0$ is the forgetting of a morphism of \mathcal{D} because $y_n \delta_0 = y_{n-1}$). What allows to write

$$\begin{aligned} y_n \delta_1 &= y_n C(k_{n-1}) = y_n C(\phi_n l_n \psi_{n-1} c_{n-1}) \\ &= y_n C(\phi_n l_n \psi_{n-1}) C(c_{n-1}) = y_n \delta_0 m(L_{n-1}) C(c_{n-1}) \\ &= y_n \delta_0 \text{ (unity axiom of monads)} \end{aligned}$$

So the universality of ψ_n implies the existence of a unique morphism of \mathcal{D} that the forgetting g_{n+1} is such as $g_{n+1} \psi_n = y_n$. We also have the extension x_{n+1} of g_{n+1} from $L(C_{n+1})$. Then the natural transformation $1_{\mathcal{B}} \xrightarrow{l} L$ applied to $L(C_n) \xrightarrow{\psi_n c_n \phi_n} C_{n+1}$ gives the equality

$$L(\psi_n c_n \phi_n) l(L(C_n)) = l_{n+1} \psi_n c_n \phi_n = d_0$$

thus $x_{n+1} d_0 = x_{n+1} L(\psi_n c_n \phi_n) l(L(C_n))$, and

$$x_{n+1} d_0 = x_{n+1} d_0 m(C_n) l(L(C_n)) \text{ (unity axiom of monads)}$$

which leads to the equality

$$x_{n+1} L(\psi_n c_n \phi_n) = x_{n+1} d_0 m(C_n)$$

(do not forget that $x_{n+1} d_0$ is the forgetting of a morphism of \mathcal{C} because $x_{n+1} d_0 = x_n$). What allows to write

$$\begin{aligned} x_{n+1} d_1 &= x_{n+1} L(j_n) = x_{n+1} L(\psi_n c_n \phi_n l_n) \\ &= x_{n+1} L(\psi_n c_n \phi_n) L(l_n) = x_{n+1} d_0 m(C_n) L(l_n) \\ &= x_{n+1} d_0 \text{ (unity axiom of monads)} \end{aligned}$$

Then the universality of ϕ_{n+1} implies the existence of a unique morphism of \mathcal{C} which the forgetting f_{n+1} is such as $f_{n+1} \phi_{n+1} = x_{n+1}$. We also have the extension y_{n+1} of f_{n+1} from $C(L_{n+1})$.

- Thus we obtain a cone $B_* \rightarrow \Delta B_0$, with $B_0 = O(M_0, H_0) = U(M_0) = V(H_0)$. We have the two cocones as well $L_* \rightarrow \Delta U(M_0)$ and $C_* \rightarrow \Delta V(H_0)$. The functor U preserving the $\overrightarrow{\mathbb{N}}$ -colimits, the diagram of \mathcal{B}

$$\begin{array}{ccc} L_* & \longrightarrow & \Delta U(M_0) \\ & \searrow & \\ & & \Delta U(M_X) \end{array}$$

results of the diagram of \mathcal{C}

$$\begin{array}{ccc} M_* & \longrightarrow & \Delta M_0 \\ & \searrow & \\ & & \Delta M_X \end{array}$$

such as $M_* \rightarrow \Delta M_X$ is a colimit. There is consequently a unique morphism h of \mathcal{C} such as the triangle commutes

$$\begin{array}{ccc} M_* & \longrightarrow & \Delta M_0 \\ & \searrow & \uparrow \Delta h \\ & & \Delta M_X \end{array}$$

In the same way the functor V preserves $\overrightarrow{\mathbb{N}}$ -colimits, so the diagram of \mathcal{B}

$$\begin{array}{ccc} C_* & \longrightarrow & \Delta V(H_0) \\ & \searrow & \\ & & \Delta V(H_X) \end{array}$$

results of the diagram of \mathcal{D}

$$\begin{array}{ccc} H_* & \longrightarrow & \Delta H_0 \\ & \searrow & \\ & & \Delta H_X \end{array}$$

such as $H_* \rightarrow \Delta H_X$ is a colimit. Therefore there is a unique morphism k of \mathcal{D} such as the following triangle commutes

$$\begin{array}{ccc} H_* & \longrightarrow & \Delta H_0 \\ & \searrow & \uparrow \Delta k \\ & & \Delta H_X \end{array} .$$

It shows the existence of the unique morphism (h, k) of $\mathcal{C} \times_{\mathcal{B}} \mathcal{D}$ such as

$$\begin{array}{ccc} B_* & \longrightarrow & \Delta B_0 \\ & \searrow & \uparrow O(h, k) \\ & & \Delta B(X) \end{array} .$$

In consequence we obtain the morphism (h, k) of $\mathcal{C} \times_{\mathcal{B}} \mathcal{D}$ such as $O(h, k)b_X = f$. Let (h', k') another morphism of $\mathcal{C} \times_{\mathcal{B}} \mathcal{D}$ making the following triangle commute

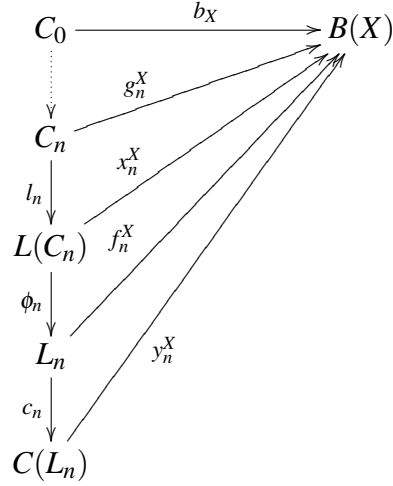
$$\begin{array}{ccc} X & \xrightarrow{f} & B_0 = O(M_0, H_0) \\ b_X \downarrow & \nearrow O(h', k') & \\ B(X) = O(M_X, H_X) & & \end{array}$$

We are going to prove by induction that it makes commutative the following triangle of natural transformations

$$\begin{array}{ccc} B_* & \longrightarrow & \Delta B_0 \\ & \searrow & \uparrow O(h', k') \\ & & \Delta B(X) \end{array} .$$

then it will immediatly prove the unicity of (h, k) .

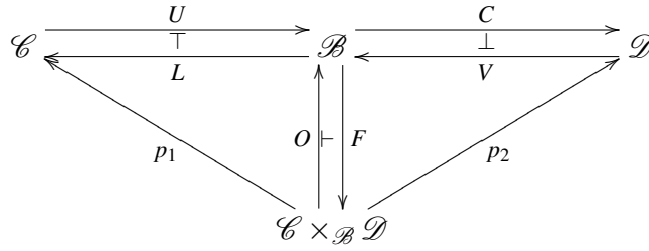
The cocone $B_* \rightarrow \Delta B(X)$ is explicitly given by the following diagram



We need to prove that $\forall n \in \mathbb{N}$ we have the equalities: $O(h', k')g_n^X = g_n$, $O(h', k')f_n^X = f_n$, $O(h', k')x_n^X = x_n$, $O(h', k')y_n^X = y_n$.

- If $n = 0$ we have $V(k')x_0^X l_0 = V(k')b_X = f = x_0 l_0$ (don't forget that $V(k') = U(h') = O(h', k')$) thus $V(k')x_0^X = x_0$. We trivially have $V(k')f_0^X = f_0$ because $f_0 = x_0$ and $f_0^X = x_0^X$. Also, $V(k')y_0^X c_0 = V(k')f_0^X = f_0 = y_0 c_0$, so $V(k')y_0^X = y_0$. And g_1 is unique such as $g_1 \psi_0 = y_0$. However $V(k')g_1^X \psi_0 = V(k')y_0^X = y_0$, thus $V(k')g_1^X = g_1$.
- We can suppose that until $n \geq 1$, we have these equalities; g_{n+1} is unique such as $g_{n+1} \psi_n = y_n$. But $V(k')g_{n+1}^X \psi_n = V(k')y_n^X = y_n$, thus $V(k')g_{n+1}^X = g_{n+1}$. Also $V(k')x_{n+1}^X l_{n+1} = V(k')g_{n+1}^X = g_{n+1} = x_{n+1} l_{n+1}$. Thus $V(k')x_{n+1}^X = x_{n+1}$. And f_{n+1} is unique such as $f_{n+1} \phi_{n+1} = x_{n+1}$. Nevertheless $V(k')f_{n+1}^X \phi_{n+1} = V(k')x_{n+1} = x_{n+1}$, thus $V(k')f_{n+1}^X = f_{n+1}$. So we have $V(k')y_{n+1}^X c_{n+1} = V(k')f_{n+1}^X = f_{n+1} = y_{n+1} c_{n+1}$, which proves that $V(k')y_{n+1}^X = y_{n+1}$.

Finally we obtain the following fusion diagram



■

7 Theories of the n -Transformations ($n \in \mathbb{N}^*$) and their Models.

The goal of this section is to build, thanks to the Nerve Theorem ([25]), the equivalence in $\mathbb{G}lob(\mathbb{C}AT)$ of 7.3 which shows that n -Transformations can be seen as models for some very elegant theories which are colored in a precise sense (see 7.2). We refer to the papers [14], [5] for materials that we are going to use here. Here $\mathbb{A}r$ is the category of categories with arities, $\mathbb{A}r\mathbb{M}nd$ is the category of categories with arities equipped with monads, and $\mathbb{M}nd\mathbb{A}r$ is the category of monads with arities. More specifically objects of $\mathbb{A}r$ are noted $(\Theta_0, i_0, \mathcal{A})$ where $\Theta_0 \xrightarrow{i_0} \mathcal{A}$ is a fully faithful functor, and objects of $\mathbb{A}r\mathbb{M}nd$ and of $\mathbb{M}nd\mathbb{A}r$ are noted $((\Theta_0, i_0, \mathcal{A}), (T, \eta, \mu))$ or $(\Theta_0, i_0, \mathcal{A})$ when there is no confusion about monads T which act on \mathcal{A} . Strongly cartesian monads [5] are the most important example of monads with arities for our purpose, because all monads arising from operads of the n -transformations are strongly cartesian (see proposition 2). But before this easy but important proposition 2, we are going to show some interesting objects of $co\mathbb{G}lob(\mathbb{C}AT)$ (in 7.1 and 7.2), the category of coglobular objects in $\mathbb{C}AT$.

7.1 Coglobular Complex of Kleisli of the n -Transformations ($n \in \mathbb{N}^*$).

Here categories \mathbb{Mnd} and \mathbb{Adj} are slightly different from those which were defined in 4 (see [14, 24] for their definitions) and are adapted for building the coglobular complex of Kleisli of the n -Transformations ($n \in \mathbb{N}^*$). Consider the functor $\mathbb{Mnd} \xrightarrow{\mathbb{K}} \mathbb{Adj}$ which send the monad $(\mathcal{G}, (T, \eta, \mu))$ to the adjunction $(Kl(T), \mathcal{G}, L_T, U_T, \eta_T, \varepsilon_T)$ coming from the Kleisli construction. Objects of $Kl(T)$ are objects of \mathcal{G} and morphisms $G \xrightarrow{f} G'$ of $Kl(T)$ are given by morphisms $G \xrightarrow{f} T(G')$ of \mathcal{G} . Also if $G \xrightarrow{g} G'$ lives in \mathcal{G} then $L_T(g) = \eta(G') \circ g$ and if $G \xrightarrow{f} G'$ lives in $Kl(T)$ then $U_T(f) = \mu(G') \circ T(f)$. Finally \mathbb{K} send the morphism (Q, q) of \mathbb{Mnd} to the morphism (P, Q) of \mathbb{Adj} such that if $G \xrightarrow{f} G'$ is a morphism of $Kl(T)$ then $P(f) = q(G')Q(f)$. Then consider the coglobular complex of $CT\text{-Cat}_c$ of the globular contractible colored operads of the n -Transformations 3.3

$$B^0 \begin{array}{c} \xrightarrow{\delta_1^0} \\ \xrightarrow{\kappa_1^0} \end{array} B^1 \begin{array}{c} \xrightarrow{\delta_2^1} \\ \xrightarrow{\kappa_2^1} \end{array} B^2 \cdots \xrightarrow{\delta_{n-1}^{n-1}} B^{n-1} \begin{array}{c} \xrightarrow{\delta_n^{n-1}} \\ \xrightarrow{\kappa_n^{n-1}} \end{array} B^n \cdots$$

For each $j \in \mathbb{N}$ we note $(\underline{B}^j, \mu^j, \eta^j)$ the corresponding monads (see 4). Given the following functors "choice of a color" $\omega - \mathbb{Gr} \xrightarrow{i_{j*}} \omega - \mathbb{Gr}/1 \cup 2$ for each $j \in \{1, 2\}$ which send the ω -graph G to the bicolored ω -graph $i_{j*} \circ !_G$ and which send a morphism f to f . It result from the morphisms of color $1 \xrightarrow{i_j} 1 \cup 2$ (see 3.3). By definition of the monads \underline{B}^0 and \underline{B}^1 we have the following natural transformations $i_{1*} \underline{B}^0 \xrightarrow{\delta_1^0} \underline{B}^1 i_{1*}$ and $i_{2*} \underline{B}^0 \xrightarrow{\kappa_1^0} \underline{B}^1 i_{2*}$ and furthermore we have for each $j \geq 1$ the following natural transformations $\underline{B}^j \xrightarrow{\delta_{j+1}^j} \underline{B}^{j+1}$ and $\underline{B}^j \xrightarrow{\kappa_{j+1}^j} \underline{B}^{j+1}$ and it is easy to see that these natural transformations fit well the axioms of morphisms of \mathbb{Mnd} (and it is similar to the construction in [16]). The functoriality of the building a monad from a

\mathbb{T} -Category implied that we can build the corresponding coglobular complex of $\mathbb{M}\text{nd}$ (similar to 4.2)

$$\underline{B}^0 \underset{\kappa_1^0}{\overset{\delta_1^0}{\rightrightarrows}} \underline{B}^1 \underset{\kappa_2^1}{\overset{\delta_2^1}{\rightrightarrows}} \underline{B}^2 \cdots \underset{\kappa_{n+1}^n}{\overset{\delta_{n+1}^n}{\rightrightarrows}} \underline{B}^n \cdots$$

If $\text{Adj} \xrightarrow{\mathbb{P}} \text{CAT}$ is the projection functor, then the functor

$$\mathbb{M}\text{nd} \xrightarrow{\mathbb{K}} \text{Adj} \xrightarrow{\mathbb{P}} \text{CAT}$$

brings to light the following coglobular complex of Kleisli of the n -Transformations ($n \in \mathbb{N}^*$)

$$\text{Kl}(\underline{B}^0) \underset{\kappa_1^0}{\overset{\delta_1^0}{\rightrightarrows}} \text{Kl}(\underline{B}^1) \underset{\kappa_2^1}{\overset{\delta_2^1}{\rightrightarrows}} \text{Kl}(\underline{B}^2) \cdots \underset{\kappa_n^{n-1}}{\overset{\delta_n^{n-1}}{\rightrightarrows}} \text{Kl}(\underline{B}^{n-1}) \cdots$$

7.2 Coglobular Complex of the Theories of the n -Transformations ($n \in \mathbb{N}^*$).

We are going to exhibit the categories of arities for the n -Transformations where we can immediately see their colored nature. Then we construct the theories of the n -Transformations where in particular we can see again their bicolored features and then we describe these colored theories as full subcategories of their Kleisli categories. Finally we exhibit the coglobular complex of the theories of the n -Transformations.

Given Θ_0 the category of graphic trees (see [2], [11], [4]). Theories build with sums $\Theta_0 \sqcup \dots \sqcup \Theta_0$ are called n -colored if the sum use Θ_0 n times.

We have the following easy proposition

Proposition 1 *For all $n \in \mathbb{N}^*$ the following canonical inclusion functors*

$$\Theta_0 \sqcup \dots \sqcup \Theta_0 \xrightarrow{i_0} \omega - \text{Gr}/1 \cup 2 \cup \dots \cup n$$

produce categories with arities. □

For the n -Transformations the following morphisms of $\mathbb{A}r$ are important

$$\begin{array}{ccc}
 \Theta_0 & \begin{array}{c} \xrightarrow{i_{1*}} \\ \xrightarrow{i_{2*}} \end{array} & \Theta_0 \sqcup \Theta_0 \\
 \downarrow i_0 & & \downarrow i_0 \\
 \omega - \mathbb{G}r & \begin{array}{c} \xrightarrow{i_{1*}} \\ \xrightarrow{i_{2*}} \end{array} & \omega - \mathbb{G}r/1 \cup 2
 \end{array}$$

where i_{1*} and i_{2*} are the functors "choice of a color" (see section 7.1).

Let us consider the case of the categories with arities equipped with monads of the n -Transformations $((\Theta_0, i_0, \omega - \mathbb{G}r), (\underline{B}^0, \eta^0, \mu^0))$ and $((\Theta_0 \sqcup \Theta_0, i_0, \omega - \mathbb{G}r/1 \cup 2), (\underline{B}^i, \eta^i, \mu^i))$ if $i \geq 1$

We have the following factorisation

$$\begin{array}{ccc}
 \Theta_0 & \begin{array}{c} \xrightarrow{i_0} \\ \searrow j \end{array} & \omega - \mathbb{G}r \xrightarrow{L^0} \underline{B}^0 - Alg \\
 & & \nearrow i \\
 & & \Theta_{\underline{B}^0}
 \end{array}$$

and for each $i \geq 1$ we have the following factorisations

$$\begin{array}{ccc}
 \Theta_0 \sqcup \Theta_0 & \begin{array}{c} \xrightarrow{i_0} \\ \searrow j \end{array} & \omega - \mathbb{G}r/1 \cup 2 \xrightarrow{L^i} \underline{B}^i - Alg \\
 & & \nearrow i \\
 & & \Theta_{\underline{B}^i}
 \end{array}$$

where the functors j are identity on the objects and the functors i are fully faithful (see [14, 25]). The categories $\Theta_{\underline{B}^0}, \Theta_{\underline{B}^1}, \dots, \Theta_{\underline{B}^i}, \dots$ etc. are the theories of the n -Transformations (by abuse we call $\Theta_{\underline{B}^0}$ the theory of the 0-Transformations, which is actually the theory built by Clemens Berger in [4]). We can also give to them the following alternative definition: Each $\Theta_{\underline{B}^i}$ can be seen as the full subcategory of the Kleisli category $Kl(\Theta_{\underline{B}^i})$ (see the paragraph section 7.1) which objects are the bicolored trees if $i \geq 1$ (i.e belong in $\Theta_0 \sqcup \Theta_0$), and which objects are the trees if $i = 0$. With this description we

obtain the coglobular complex of the theories of the n -Transformations which is seen as a subcomplex of the coglobular complex of the Kleisli categories of the n -Transformations

$$\begin{array}{ccccccc}
 \Theta_{\underline{B}^0} & \xrightarrow[\kappa_1^0]{\delta_1^0} & \Theta_{\underline{B}^1} & \xrightarrow[\kappa_2^1]{\delta_2^1} & \Theta_{\underline{B}^2} & \cdots & \Theta_{\underline{B}^{n-1}} & \xrightarrow[\kappa_n^{n-1}]{\delta_n^{n-1}} & \Theta_{\underline{B}^n} & \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 Kl(\underline{B}^0) & \xrightarrow[\kappa_1^0]{\delta_1^0} & Kl(\underline{B}^1) & \xrightarrow[\kappa_2^1]{\delta_2^1} & Kl(\underline{B}^2) & \cdots & Kl(\underline{B}^{n-1}) & \xrightarrow[\kappa_n^{n-1}]{\delta_n^{n-1}} & Kl(\underline{B}^n) & \cdots
 \end{array}$$

7.3 An application of the Nerve Theorem.

Given \mathcal{A} a category with a final object 1, and a functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$

We have the following factorisation:

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\
 \searrow F_1 & & \nearrow cod \\
 & \mathcal{B}/F(1) &
 \end{array}$$

where $F_1(a) := F(!_a)$. In that case we have the following important definition

Definition 1 (Street 2001) The last F is qualified as Parametric Right Adjoint (p.r.a for short) if F_1 has a left adjoint. □

Definition 2 A monad $(\mathcal{G}, (T, \eta, \mu))$ is a strongly cartesian monad if T is p.r.a. and if its unit and multiplication are cartesian. □

Remark 6 In 2001 Ross Street has called them p.r.a monads, Mark Weber in [25] has called them locally right adjoint monads (l.r.a monads), but we adopt here the terminology of the paper [5]. □

Monads of the n -Transformations are in fact strongly cartesian monads (see the proposition 2 below, where the proof is left to the reader) which allow us to exhibit the coglobular complex in $\mathbb{M}nd \mathbb{A}r$ of the n -Transformations and thus, thanks to the Nerve Theorem [25] we get the globular complex of nerves of the n -Transformations and finally the equivalence in $\mathbb{G}lob(\mathbb{C}AT)$, which express the definition of the n -Transformations as models for theories, that is the outcome of this section. It is well known that $(\omega - \mathbb{G}r, (\underline{B}^0, \eta^0, \mu^0))$ is a strongly cartesian monad [25]. In fact all monads of the n -Transformations ($n \in \mathbb{N}^*$) have this property

Proposition 2 *For all $i \geq 1$ the monad $(\omega - \mathbb{G}r/1 \cup 2, (\underline{B}^i, \eta^i, \mu^i))$ is strongly cartesian. Furthermore $(\Theta_0 \sqcup \Theta_0, i_0, \omega - \mathbb{G}r/1 \cup 2)$ is their canonical arities (see remark 2.10 in [5]).*

So we obtain the coglobular complex in $\mathbb{M}nd \mathbb{A}r$ of the n -Transformations

$$\begin{aligned} (\Theta_0, i_0, \omega - \mathbb{G}r) \xrightarrow[\kappa_1^0]{\delta_1^0} (\Theta_0 \sqcup \Theta_0, i_0, \omega - \mathbb{G}r/1 \cup 2) \xrightarrow[\kappa_2^1]{\delta_2^1} \dots \\ (\Theta_0 \sqcup \Theta_0, i_0, \omega - \mathbb{G}r/1 \cup 2) \xrightarrow[\kappa_{i+1}^i]{\delta_{i+1}^i} \dots \end{aligned}$$

which brings to light the globular complex of nerves of the n -Transformations

$$\begin{array}{ccccccc} \cdots \rightrightarrows \underline{B}^n - \mathbb{A}lg & \xrightarrow[\beta_{n-1}^n]{\sigma_{n-1}^n} & \underline{B}^{n-1} - \mathbb{A}lg & \cdots \rightrightarrows & \underline{B}^1 - \mathbb{A}lg & \xrightarrow[\beta_0^1]{\sigma_0^1} & \underline{B}^0 - \mathbb{A}lg \\ N_{\underline{B}^n} \downarrow & & \downarrow N_{\underline{B}^{n-1}} & & \downarrow N_{\underline{B}^1} & & \downarrow N_{\underline{B}^0} \\ \cdots \rightrightarrows \widehat{\Theta}_{\underline{B}^n} & \xrightarrow[\beta_{n-1}^n]{\sigma_{n-1}^n} & \widehat{\Theta}_{\underline{B}^{n-1}} & \cdots \rightrightarrows & \widehat{\Theta}_{\underline{B}^1} & \xrightarrow[\beta_0^1]{\sigma_0^1} & \widehat{\Theta}_{\underline{B}^0} \end{array}$$

which finally achieve the goal of this section by showing the following equivalence in $\mathbb{G}lob(\mathbb{C}AT)$ given by the nerves functors, i.e each nerve

functor $N_{\underline{B}^n}$ of the commutative diagram below is an equivalence of categories

$$\begin{array}{ccccccc}
 \cdots \rightrightarrows \underline{B}^n - \mathbb{A}lg & \xrightarrow[\beta_{n-1}^n]{\sigma_{n-1}^n} & \underline{B}^{n-1} - \mathbb{A}lg & \cdots \rightrightarrows & \underline{B}^1 - \mathbb{A}lg & \xrightarrow[\beta_0^1]{\sigma_0^1} & \underline{B}^0 - \mathbb{A}lg \\
 \downarrow N_{\underline{B}^n} & & \downarrow N_{\underline{B}^{n-1}} & & \downarrow N_{\underline{B}^1} & & \downarrow N_{\underline{B}^0} \\
 \cdots \rightrightarrows \mathbb{M}od(\Theta_{\underline{B}^n}) & \xrightarrow[\beta_{n-1}^n]{\sigma_{n-1}^n} & \mathbb{M}od(\Theta_{\underline{B}^{n-1}}) & \cdots \rightrightarrows & \mathbb{M}od(\Theta_{\underline{B}^1}) & \xrightarrow[\beta_0^1]{\sigma_0^1} & \mathbb{M}od(\Theta_{\underline{B}^0})
 \end{array}$$

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RESUMES DES ARTICLES PUBLIES dans le Volume LII (2011)

KASANGIAN, METERE and VITALE, The Ziqqurath of exact sequences of n -groupoids, 2-44

Les auteurs étudient la notion de suite exacte dans la sesqui-catégorie des n -groupoïdes. En utilisant les produits fibrés homotopiques, à partir d'un n -foncteur entre n -groupoïdes pointés, ils construisent une suite de six $(n-1)$ -groupoïdes. Ils montrent que cette suite est exacte en un sens qui généralise les notions usuelles d'exactitude pour les groupes et les gr-catégories. En réitérant le processus, ils obtiennent une ziggourath de suites exactes de longueur croissante et dimension décroissante. Pour $n = 1$, ils retrouvent un résultat classique dû à R. Brown et, pour $n = 2$, ses généralisations dues à Hardie, Kamps et Kieboom et à Duskin, Kieboom et Vitale.

M. GRANDIS, Singularities and regular paths (an elementary introduction to smooth homotopy), 45-76.

Cet article est une introduction élémentaire à la topologie algébrique lisse, suivant une approche particulière : le but est d'étudier des "espaces lisses avec singularités" par des méthodes d'homotopie adaptées à cette tâche. On explore ici des régions euclidiennes, moyennant des chemins de classe C^k , en tenant compte du nombre de leurs arrêts en fonction de k . Le groupoïde fondamental de l'espace acquiert ainsi une séquence de poids qui dépend d'un index de classe C^k et qui peut distinguer l'ordre des singularités "linéaires". Ces méthodes pourraient s'appliquer à la théorie des réseaux.

HARTL & LOISEAU, A characterization of finite cocomplete homological and of semi-abelian categories, 77- 80.

Les catégories semi-abéliennes et homologiques finiment cocomplètes sont définies via quatre, respectivement trois, axiomes simples exprimés en termes de notions catégoriques de base.

CHENG & GURSKI, The periodic table of n -categories II: Degenerate tricategories, 82-125.

Suivant un travail précédent, les auteurs étudient les tricatégories dégénérées en les comparant aux structures prédites par le tableau périodique des n -catégories. Pour les tricatégories trois fois dégénérées, ils démontrent une tri-équivalence avec la tricatégorie partiellement discrète des monoïdes commutatifs. Pour les tricatégories deux fois dégénérées ils expliquent comment construire une catégo-

rie monoïdale tressée à partir d'une tricatégorie deux fois dégénérée, mais cette construction n'induit pas une comparaison simple entre BrMonCat et Tricat. Ils discutent comment on peut itérer la construction des "icones" pour obtenir une équivalence, renvoyant à plus tard pour les détails. Finalement ils étudient les tricatégories dégénérées, donnant la première définition des bicatégories monoïdales complètement algébriques et toute la structure de tricatégorie de MonBicat.

BARILE, BARONE & TULCZYJEW, The total exterior differential, 126-160.

La définition des jets mixtes inclut les suites finies de vecteurs verticaux tangents à des fibrés de jets. Ceci permet de définir des opérateurs différentiels sur des formes verticales à un fibré de jets, en utilisant les prolongements de jets mixtes. La différentielle extérieure totale en est un cas particulier.

G. SEAL, On the monadic nature of categories of ordered sets, 163-187.

Si S est une monade sur Set avec une factorisation au travers de la catégorie des ensembles ordonnés et des fonctions adjointes à gauche, alors un morphisme de monades entre S et T induit une factorisation similaire sur T . La catégorie de Eilenberg-Moore de T est alors monadique sur la catégorie des monoïdes dans la catégorie de Kleisli de S .

BROWN & STREET, Covering morphisms of crossed complexes and of cubical omega-groupoids are closed under tensor product, 188-208.

Le but de cet article est de démontrer les théorèmes mentionnés dans le titre, ainsi que le corollaire disant que le produit tensoriel de deux résolutions croisées libres, en groupes ou en groupoïdes, est aussi une résolution croisée libre, en groupes ou en groupoïdes. Ce corollaire est obtenu en utilisant l'équivalence entre la catégorie des complexes croisés et celle des omega-groupoïdes des cubiques avec connexion, dans laquelle on donne la définition initiale du produit tensoriel. D'autre part, c'est dans cette deuxième catégorie qu'on peut appliquer les techniques de sous-catégories denses pour reconnaître qu'un produit tensoriel de revêtements est un revêtement.

KENNEY & PARE, Categories as monoids in Span, Rel and Sup, 209-240.

Les auteurs étudient les représentations de petites catégories comme les monoïdes dans trois bicatégories monoïdales étroitement liées. Les catégories peuvent être exprimées comme certains types de monoïdes dans la catégorie Span. En fait, ces monoïdes sont aussi dans Rel. Il y a une équivalence bien connue entre Rel et une sous-catégorie pleine de la catégorie des treillis complets et des morphismes qui préservent les sups. Cela permet de représenter une catégorie comme un monoïde dans Sup. Les monoïdes dans Sup s'appellent des quantales, et sont intéressants dans plusieurs domaines. Les auteurs étudient aussi dans ce contexte la représen-

tation d'autres structures catégoriques, par exemple les foncteurs, les transformations naturelles et les profoncteurs.

DUBUC & YUHJTMAN, A construction of 2-cofiltered bilimits of topoi, 242-252.

Les auteurs montrent l'existence des bilimites de diagrammes 2-cofiltrés de topos, généralisant la construction de bilimites cofiltrées développée précédemment. Ils montrent qu'un tel diagramme peut être représenté par un diagramme 2-cofiltré de petits sites avec limites finies, and ils construisent un petit site pour le topos bilimite. Pour ceci ils considèrent le 2-filtre bicolimite des catégories sous-jacentes et leurs foncteurs image inverse. Appliquant la construction de cette bicolimite développée dans un article antérieur, ils montrent que si les catégories du diagramme ont des limites finies et si les foncteurs de transition sont exacts, alors la catégorie bicolimite a aussi des limites finies et les foncteurs du pseudo-cone sont exacts. Comme application ils retrouvent que tout topos de Galois a des points.

BLUTE, COCKETT, PORTER & SEELY, Kähler categories, 253-268.

Dans cet article, on établit une relation entre la notion de catégorie codifférentielle et la théorie, plus classique, des différentielles de Kähler en algèbre commutative. Une catégorie codifférentielle est une catégorie monoïdale additive, ayant une monade T avec une 'modalité d'algèbre', i.e. avec donnée d'une structure d'algèbre associative pour chaque objet de la forme $T(C)$; elle est aussi équipée d'une transformation dérivée, avec axiomes de différentiation sous forme algébrique. La notion classique de différentielle de Kähler définit celle d'un module des formes A -différentielles par rapport à une k -algèbre commutative A , qui est équipé d'une A -dérivation universelle. Une catégorie de Kähler est une catégorie monoïdale additive, ayant une modalité d'algèbre et un objet des formes différentielles associé à chaque objet. Si la monade algèbre libre existe et si l'application canonique vers T est épimorphique, les catégories codifférentielles sont Kähler.

C. KACHOUR, Operadic definition of non-strict cells, 269-231.

Dans un article précédent, l'auteur avait étendu le travail de J. Penon sur les ω -catégories non-strictes en définissant leurs ω -foncteurs non-stricts, leurs ω -transformations naturelles non-strictes, etc., en utilisant des extensions de ses "étirements catégoriques" appelés " n -étirements catégoriques". Ici il poursuit le travail de M. Batanin sur les ω -catégories non-strictes en définissant leurs ω foncteurs non-stricts, leurs ω -transformations naturelles non-strictes, etc., en utilisant des extensions de son ω -opérade contractile universelle K , i.e en construisant des ω -opérades colorées contractiles universelles B^n adaptées.

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