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ON THE MONADIC NATURE OF CATEGORIES OF ORDERED SETS

by Gavin J. SEAL

Résumé. Si $\mathbb{S}$ est une monade sur Set avec une factorisation au travers de la catégorie des ensembles ordonnés et des fonctions adjointes à gauche, alors un morphisme de monades $\tau : \mathbb{S} \to \mathbb{T}$ induit une factorisation similaire sur $\mathbb{T}$. La catégorie de Eilenberg-Moore de $\mathbb{T}$ est alors monadique sur la catégorie des monoïdes dans la catégorie de Kleisli de $\mathbb{S}$.

Abstract. If $\mathbb{S}$ is an order-adjoint monad, that is, a monad on Set that factors through the category of ordered sets with left adjoint maps, then any monad morphism $\tau : \mathbb{S} \to \mathbb{T}$ makes $\mathbb{T}$ order-adjoint. The Eilenberg-Moore category of $\mathbb{T}$ is then monadic over the category of monoids in the Kleisli category of $\mathbb{S}$.

Keywords: order-adjoint monad, Eilenberg-Moore category, Kleisli monoid, monadic functor

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1 Introduction

A monadic functor from A to X determines a unique (up to isomorphism) monad $\mathbb{T}$ on X, and such a monad yields a category of Eilenberg-Moore algebras $\mathsf{X}^{\mathbb{T}}$ that is equivalent to A; however, A can be monadic over a range of different categories. Illustrations of this fact that stem from distributive laws [1] spring to mind, but other examples originate from different contexts.

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For instance, the category Cnt of continuous lattices that is strictly monadic over Set, as well as over the category Top of topological spaces, the category Sup of complete sup-lattices, and the category CHaus of compact Hausdorff spaces ([4], [17]). The last two examples can be obtained as consequences of the following result (see for example Corollary 4.5.10 in [3]):

> *A morphism $\tau : \mathbb{S} \to \mathbb{T}$ between monads on* Set *induces a strictly monadic functor* $\mathsf{Set}^\tau : \mathsf{Set}^\mathbb{T} \to \mathsf{Set}^\mathbb{S}$.

Indeed, taking $\mathbb{T}$ to be the filter monad $\mathbb{F}$, and $\mathbb{S}$ the powerset monad $\mathbb{P}$ or the ultrafilter monad $\mathbb{B}$, the principal filter monad morphism $\tau : \mathbb{P} \to \mathbb{F}$ yields the monadicity of $\mathsf{Cnt} \cong \mathsf{Set}^\mathbb{F}$ over $\mathsf{Sup} \cong \mathsf{Set}^\mathbb{P}$, and the embedding morphism $\mathbb{B} \hookrightarrow \mathbb{F}$ leads to the monadicity of Cnt over $\mathsf{CHaus} \cong \mathsf{Set}^\mathbb{B}$ (see [8]). However, the presence of Top in this context remains somewhat idiosyncratic.

The aim of the present work is to describe a setting that leads to a systematic display of monadic functors induced by monad morphisms over categories such as Top or Ord, rather than Eilenberg-Moore categories (and turns out, contrarily to the cited result, not to require recourse to the Axiom of Choice). The main ingredient is an order-adjoint monad, that is, a monad on Set whose extension operation factorizes through the category of ordered sets and left adjoint maps (see 2.3). Another ingredient is inspired by the description of an object in Top as a monoid in the ordered hom-set $\mathsf{Set}(X, FX)$ (where FX is the set of all filters on X, ordered by reverse inclusion): a topological space can be defined as a set X with a neighborhood map $\alpha : X \to FX$ such that

$$\eta_X \leq \alpha \qquad \text{and} \qquad \mu_X \cdot F\alpha \cdot \alpha = \alpha \ ,$$

where η and μ are respectively the unit and multiplication of $\mathbb{F}$ (see [5]). The second identity—idempotency of α in the hom-set of the Kleisli category of $\mathbb{F}$—is central to our construction of a monad $\mathbb{T}'$ from $\mathbb{T}$. In fact, the category $\mathsf{Set}(\mathbb{S})$ of *Kleisli monoids* associated to an order-adjoint monad $\mathbb{S}$ plays the same role as $\mathsf{Set}^\mathbb{S}$ previously and leads to an isomorphism

$$\mathsf{Set}^\mathbb{T} \cong \mathsf{Set}(\mathbb{S})^{\mathbb{T}'}$$

(Theorem 4.8). After illustrating this result with a number of scattered—and previously unrelated—results occurring in the literature, we show how the intrinsic order-adjoint nature of algebra structures contributes to the study of relevant Eilenberg-Moore categories.

2 Order-adjoint monads

In this section, we recall basic facts and terminology pertaining to order-adjoint monads, and settle a number of notations. Further details can be found in [15].

2.1 Monads. A monad $\mathbb{T}$ on a category X is a triple (T, η, μ) formed by a functor $T : \mathsf{X} \to \mathsf{X}$, and two natural transformations: the *unit* $\eta : \mathrm{Id} \to T$ and *multiplication* $\mu : TT \to T$ of the monad that must satisfy

$$\mu \cdot T\eta = 1 = \mu \cdot \eta T \qquad \text{and} \qquad \mu \cdot T\mu = \mu \cdot \mu T \ .$$

We say that a pair $(R, \sigma) : \mathbb{S} \to \mathbb{T}$ is a *monad morphism* from a monad $\mathbb{S} = (S, \delta, \nu)$ on A to a monad $\mathbb{T} = (T, \eta, \mu)$ on X, if $R : \mathsf{X} \to \mathsf{A}$ is a functor and $\sigma : SR \to RT$ a natural transformation such that

$$R\eta = \sigma \cdot \delta R \qquad \text{and} \qquad R\mu \cdot \sigma T \cdot S\sigma = \sigma \cdot \nu R \ .$$

In the case where $\mathsf{A} = \mathsf{X}$ and R is the identity, we write $\sigma : \mathbb{S} \to \mathbb{T}$ instead of $(1_{\mathsf{X}}, \sigma) : \mathbb{S} \to \mathbb{T}$.

A monad can also be described by way of a *Kleisli triple* $(T, \eta, (-)^{\mathbb{T}})$ on X (Exercise 1.3.12 in [9]), that is,

(i) a function $T : \mathrm{ob}\,\mathsf{X} \to \mathrm{ob}\,\mathsf{X}$,

(ii) for every X-object X, an X-morphism $\eta_X : X \to TX$,

(iii) an *extension operation* $(-)^{\mathbb{T}}$ that sends an X-morphism $f : X \to TY$ to an X-morphism $f^{\mathbb{T}} : TX \to TY$,

subject to the conditions

$$\eta_X^{\mathbb{T}} = 1_{TX} \quad , \qquad f^{\mathbb{T}} \cdot \eta_X = f \qquad \text{and} \qquad g^{\mathbb{T}} \cdot f^{\mathbb{T}} = (g^{\mathbb{T}} \cdot f)^{\mathbb{T}} \ . \qquad (*)$$

Every Kleisli triple $(T, \eta, (-)^{\mathbb{T}})$ yields a monad $\mathbb{T} = (T, \eta, \mu)$ via

$$Tf := (\eta_Y \cdot f)^{\mathbb{T}} \qquad \text{and} \qquad \mu_X := (1_{TX})^{\mathbb{T}} \ ,$$

and every monad $\mathbb{T} = (T, \eta, \mu)$ defines a Kleisli triple thanks to

$$f^{\mathbb{T}} := \mu_Y \cdot Tf \ .$$

These processes are inverse of one another, and we freely switch between the two descriptions: not only is the extension operation $(-)^{\mathbb{T}}$ ubiquitous in our context, but the three Kleisli triple conditions are very economical to verify.

In the case where two Kleisli triples $(S, \delta, (-)^{\mathbb{S}})$ and $(T, \eta, (-)^{\mathbb{T}})$ are defined on the same category X, a family $(\sigma_X : SX \to TX)_{X \in \mathrm{ob}\,\mathsf{X}}$ defines a monad morphism $\sigma : \mathbb{S} \to \mathbb{T}$ if and only if the equalities

$$\eta_X = \sigma_X \cdot \delta_X \qquad \text{and} \qquad (\sigma_Y \cdot f)^{\mathbb{T}} \cdot \sigma_X = \sigma_Y \cdot f^{\mathbb{S}}$$

hold for all X-objects X and X-morphisms $f : X \to SY$.

2.2 Eilenberg-Moore and Kleisli categories. Given a monad $\mathbb{T} = (T, \eta, \mu)$ on a category X, an *Eilenberg-Moore algebra* (or a $\mathbb{T}$-*algebra*) is a pair (X, a), with X an object of X, and $a : TX \to X$ a *structure morphism* that satisfies

$$1_X = a \cdot \eta_X \qquad \text{and} \qquad a \cdot Ta = a \cdot \mu_X \ .$$

In particular, the pair (TX, μ_X) forms an Eilenberg-Moore algebra, the *free* $\mathbb{T}$-*algebra* on X. A *morphism* of Eilenberg-Moore algebras $f : (X, a) \to (Y, b)$ is an X-morphism $f : X \to Y$ such that

$$f \cdot a = b \cdot Tf \ .$$

Thus, a $\mathbb{T}$-algebra structure a is itself such a morphism $a : (TX, \mu_X) \to (X, a)$. The category of Eilenberg-Moore algebras and their morphisms is denoted by $\mathsf{X}^{\mathbb{T}}$ and is also called the *Eilenberg-Moore category* of $\mathbb{T}$. If $\mathbb{T}$ is given by a Kleisli triple, the conditions for an X-morphism $a : TX \to X$ to form an Eilenberg-Moore structure can be expressed as

$$1_X = a \cdot \eta_X \qquad \text{and} \qquad \forall f, g \in \mathsf{X}(Y, TX)\, (a \cdot f = a \cdot g \implies a \cdot f^{\mathbb{T}} = a \cdot g^{\mathbb{T}}) \, .$$

If $\mathbb{S} = (S, \delta, \nu)$ is a monad on A and $\mathbb{T}$ a monad on X, then a functor $\overline{R} : \mathsf{X}^{\mathbb{T}} \to \mathsf{A}^{\mathbb{S}}$ is said to be *algebraic* over a functor $R : \mathsf{X} \to \mathsf{A}$ if it makes the diagram

$$\begin{array}{ccc} \mathsf{X}^{\mathbb{T}} & \xrightarrow{\overline{R}} & \mathsf{A}^{\mathbb{S}} \\ \downarrow & & \downarrow \\ \mathsf{X} & \xrightarrow{R} & \mathsf{A} \end{array}$$

commute (the vertical arrows represent the respective forgetful functors). Any monad morphism $(R, \sigma) : \mathbb{S} \to \mathbb{T}$ from a monad $\mathbb{S}$ on A to a monad $\mathbb{T}$ on X induces such an algebraic functor; this is defined on objects by

$$\overline{R}(X, a) = (RX, Ra \cdot \sigma_X) \ ,$$

and necessarily sends an X-morphism f to Rf. Conversely, every functor $\overline{R} : \mathsf{X}^{\mathbb{T}} \to \mathsf{A}^{\mathbb{S}}$ that is algebraic over $R : \mathsf{X} \to \mathsf{A}$ is induced by a monad morphism (R, σ): if $\overline{\mu}_X : SRTX \to RTX$ denotes the A-morphism given by $\overline{R}(TX, \mu_X) = (RTX, \overline{\mu}_X)$, then one can define the components of $\sigma : SR \to RT$ by

$$\sigma_X := \overline{\mu}_X \cdot SR\eta_X \ .$$

The objects of the *Kleisli category* $\mathsf{X}_{\mathbb{T}}$ associated to the monad $\mathbb{T}$ are the objects of X, and morphisms $f : X \rightharpoonup Y$ in $\mathsf{X}_{\mathbb{T}}$ are those X-morphisms $f : X \to TY$. The *Kleisli composition* of $f : X \rightharpoonup Y$ and $g : Y \rightharpoonup Z$ in $\mathsf{X}_{\mathbb{T}}$ is defined via the composition in X as

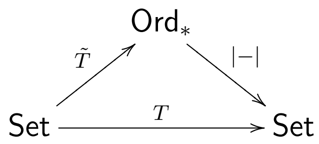
$$g \circ f := \mu_Z \cdot Tg \cdot f = g^{\mathbb{T}} \cdot f \ .$$

The identity $1_X : X \rightharpoonup X$ in this category is just the component $\eta_X : X \to TX$ of the unit.

2.3 Order-adjoint monads. Let Ord denote the category of ordered sets (that is, sets equipped with a reflexive, transitive and antisymmetric relation) with monotone maps, and Ord_* the subcategory of Ord with same objects but whose maps are left adjoint. Explicitly a map $f : X \to Y$ is a morphism of Ord_* if it is monotone and there exists a monotone map, denoted by $f^* : Y \to X$, satisfying

$$1_X \leq f^* \cdot f \qquad \text{and} \qquad f \cdot f^* \leq 1_Y \ .$$

A functor T : $\mathsf{Set} \to \mathsf{Set}$ *factors through* Ord_* if there is a functor $\tilde{T} : \mathsf{Set} \to \mathsf{Ord}_*$ that makes the diagram



commute (where $|-|$ denotes the forgetful functor). For convenience, such a functor T is understood to be *given with* a fixed $\tilde{T}$ that is moreover identified with T; for example, we talk about "the right adjoint $(Tf)^*$ of $Tf : TX \to TY$" to mean "the underlying function of the right adjoint $(\tilde{T}f)^*$ of $\tilde{T}f : \tilde{T}X \to \tilde{T}Y$". The hom-sets $\mathsf{Set}(X, TY)$ are then equipped with the pointwise order, so that for $f, f' \in \mathsf{Set}(X, TY)$, one has

$$f \leq f' \iff \forall x \in X \ (f(x) \leq f'(x)) .$$

A monad $\mathbb{T} = (T, \eta, \mu)$ on Set is *order-adjoint* if the components of its extension operation $(-)^{\mathbb{T}}$ take values in Ord_*, that is, if and only if T factors through Ord_* and every component μ_X of the monad multiplication is a morphism in Ord_*.

Without any additional assumption on a monad $\mathbb{T}$ on Set whose extension operation takes values in Ord_*, composition in the Kleisli category $\mathsf{Set}_{\mathbb{T}}$ is only monotone in the second variable:

$$f \leq f' \implies h \circ f \leq h \circ f'$$

for all $f, f' \in \mathsf{Set}_{\mathbb{T}}(X, Y)$, $h \in \mathsf{Set}_{\mathbb{T}}(Y, Z)$. We say that an order-adjoint monad $\mathbb{T}$ is *enhanced* if moreover $(-)^{\mathbb{T}}$ preserves the order on the hom-sets $\mathsf{Set}_{\mathbb{T}}(X, Y)$:

$$f \leq f' \implies f^{\mathbb{T}} \leq (f')^{\mathbb{T}}$$

for all $f, f' \in \mathsf{Set}_{\mathbb{T}}(X, Y)$. This condition is equivalent to requiring that composition in $\mathsf{Set}_{\mathbb{T}}$ is monotone in the first variable, so an enhanced order-adjoint monad makes the Kleisli category $\mathsf{Set}_{\mathbb{T}}$ into an ordered category. Note that even if an order-adjoint monad is enhanced, its functor needs not preserve adjoint situations, that is, $T(Tf)^* = (TTf)^*$ does not hold in general.

2.4 Lemma. *An* Ord_**-morphism* $f : X \to Y$ *is split epic in* Set *if and only if* $f \cdot f^* = 1_Y$.

Proof. If $f \cdot f^* = 1_Y$, then f is split epic by definition. If there is a map $g : Y \to X$ with $f \cdot g = 1_Y$, then $g \le f^*$. Therefore, $1_Y \le f \cdot f^* \le 1_Y$ and equality holds. □

2.5 Proposition. *A monad* $\mathbb{T}$ *on* Set *is order-adjoint if and only if the forgetful functor from* $\mathsf{Set}^{\mathbb{T}}$ *to* Set *factors through* Ord_*.

Hence, if $\mathbb{T}$ *is order-adjoint, then all* $\mathbb{T}$*-algebras* (X, a) *are ordered sets, and all* $\mathbb{T}$*-algebra morphisms* $f : (X, a) \to (Y, b)$*, in particular* $a : (TX, \mu_X) \to (X, a)$*, are left adjoint maps.*

Proof. We give a brief outline of the proof, while details can be found in [15]. If $\mathbb{T}$ is order-adjoint, the order on the underlying set of a $\mathbb{T}$-algebra (X, a) is inherited from TX via the map $a^\circ := \mu_X \cdot (Ta)^* \cdot \eta_X$:

$$x \le y \iff a^\circ(x) \le a^\circ(y)$$

for all $x, y \in X$; this order makes a° into the right adjoint a^* of a (and returns the original order on TX via μ_X°). For a $\mathbb{T}$-algebra morphism $f : (X, a) \to (Y, b)$, one defines the map $f^\circ := a \cdot (Tf)^* \cdot b^*$, which turns out to be the right adjoint f^* of f (relatively to the orders on X and Y induced by a and b respectively, as described above). Conversely, if $\mathsf{Set}^{\mathbb{T}} \to \mathsf{Set}$ factors through Ord_*, one observes that $\mathbb{T}$ is order-adjoint by exploiting that all $Tf : (TX, \mu_X) \to (TY, \mu_Y)$ and $\mu_X : (TTX, \mu_{TX}) \to (TX, \mu_X)$ are $\mathsf{Set}^{\mathbb{T}}$-morphisms. □

2.6 Proposition. *Let* $\tau : \mathbb{S} \to \mathbb{T}$ *be a monad morphism from an order-adjoint monad* $\mathbb{S} = (S, \delta, \nu)$ *to a monad* $\mathbb{T} = (T, \eta, \mu)$ *on* Set*. Then* $\mathbb{T}$ *is order-adjoint, and the components* $\tau_X : SX \to TX$ *are left adjoints (with respect to the induced order on* TX*).*

Proof. As in the proof of Proposition 2.5, the $\mathbb{S}$-algebra structure $\mu_X \cdot \tau_{TX} : STX \to TX$ defines an order on TX via

$$\chi \le y \iff (\mu_X \cdot \tau_{TX})^\circ(\chi) \le (\mu_X \cdot \tau_{TX})^\circ(y) \,,$$

where $(\mu_X \cdot \tau_{TX})^\circ := \nu_X \cdot (S(\mu_X \cdot \tau_{TX}))^* \cdot \delta_{TX}$, and one then has $(\mu_X \cdot \tau_{TX})^\circ = (\mu_X \cdot \tau_{TX})^*$. For monotonicity of Tf, we use Lemma 2.4 in

$$Tf = Tf \cdot (\mu_X \cdot \tau_{TX}) \cdot (\mu_X \cdot \tau_{TX})^* = (\mu_Y \cdot \tau_{TY}) \cdot STf \cdot (\mu_X \cdot \tau_{TX})^*$$

to observe that Tf is a composite of monotone maps. Monotonicity of μ_X and τ_X are proved similarly. The respective right adjoints are easily seen to be given by

$$\begin{aligned} (Tf)^* &:= \mu_X \cdot \tau_{TX} \cdot (STf)^* \cdot (\mu_Y \cdot \tau_{TY})^* , \\ \mu_X^* &:= \mu_{TX} \cdot \tau_{TTX} \cdot (S\mu_X)^* \cdot (\mu_X \cdot \tau_{TX})^* , \\ \tau_X^* &:= \nu_X \cdot (S\tau_X)^* \cdot (\mu_X \cdot \tau_{TX})^* . \end{aligned}$$

□

For the explicit description of the monads mentioned in the following examples, we refer to [15] or [14]. Further references are given in 4.10.

2.7 Examples. The Eilenberg-Moore algebras of the powerset monad $\mathbb{P}$ on Set are complete lattices, with their morphisms sup-maps:

$$\mathsf{Set}^{\mathbb{P}} \cong \mathsf{Sup} .$$

Since $\mathbb{P}$ is order-adjoint, any monad morphism $\tau : \mathbb{P} \to \mathbb{T}$ makes $\mathbb{T}$ order-adjoint (Proposition 2.6). Thus, the filter monad $\mathbb{F}$ on Set becomes order-adjoint via the principal-filter monad morphism $\tau : \mathbb{P} \to \mathbb{F}$; the same statement holds for up-set monad $\mathbb{U}$, and the double-dualization monad $\mathbb{D}$, since there is a chain of monad morphisms

$$\mathbb{P} \to \mathbb{F} \to \mathbb{U} \to \mathbb{D}$$

(the last two simply given by the inclusions $FX \hookrightarrow UX \hookrightarrow DX$ for all sets X). There is also a monad morphism from $\mathbb{P}$ into the monad $\mathbb{U}_{\text{fin}}$ of finitely generated up-sets (obtained by sending an element $A \in PX$ to $\{B \in PX \mid B \cap A \neq \emptyset\}$) that leads to another chain of monad morphisms

$$\mathbb{P} \to \mathbb{U}_{\text{fin}} \to \mathbb{U} \to \mathbb{D} .$$

In this case, the order induced by $\mathbb{P}$ on $U_{\mathrm{fin}}X$, UX, or DX, is given by set-inclusion—rather than its opposite as in the previous filter case. There is also a monad morphism

$$\mathbb{P} \to \mathbb{P}_{\mathbf{P}_+}$$

of the powerset monad, into the $\mathbf{P}_+$-based powerset monad, where $\mathbf{P}_+$ denotes the extended real half-line $[0, \infty]$ equipped with its quantale structure, in which the tensor is given by extended addition, and the order is opposite to the natural order (the components of the morphism are given by the maps $\chi_{(-)} : PX \to P_{\mathbf{P}_+}X$ that send $A \subseteq X$ to its characteristic function $\chi_A : X \to \mathbf{P}_+$ given by $\chi_A(x) = 0$ if $x \in A$ and $\chi_A(x) = \infty$ otherwise). With the sets $P_{\mathbf{P}_+}X$ ordered pointwise, $\mathbb{P}_{\mathbf{P}_+}$ becomes order-adjoint. For future reference, note that the monad morphism from $\mathbb{P}$ to $\mathbb{P}_{\mathbf{P}_+}$ has a left inverse

$$\mathbb{P}_{\mathbf{P}_+} \to \mathbb{P}$$

(whose component at X is left adjoint to $\chi_{(-)} : PX \to P_{\mathbf{P}_+}X$, and sends a map $\phi : X \to \mathbf{P}_+$ to the set $\{x \in X \mid \phi(x) < \infty\}$).

One readily checks that all of these order-adjoint monads—with the notable exception of the double-dualization monad $\mathbb{D}$—are enhanced.

3 Kleisli monoids

3.1 The category of Kleisli monoids. The definition of a Kleisli monoid is given by way of a category X that is not quite an ordered category, as our main goal is to study the case of order-adjoint monads $\mathbb{T}$ on $\mathsf{X} = \mathsf{Set}$ (see also Proposition 5.3).

Let $\mathbb{T} = (T, e, m)$ be a monad on a category X whose hom-sets $\mathsf{X}(X, TY)$ are equipped with an order that is preserved by composition on the right:

$$f \le f' \implies f \cdot g \le f' \cdot g$$

for all $f, f' : X \to TY$, $g : Z \to X$. A *Kleisli monoid* (or $\mathbb{T}$-*monoid*) in X is a pair (X, α) made up of an X-object X and a structure X-morphism $\alpha : X \to TX$ that is *extensive* and *idempotent*:

$$e_X \le \alpha\,, \quad \alpha \circ \alpha \le \alpha$$

(composition is taken in the Kleisli category $\mathsf{X}_{\mathbb{T}}$). In the presence of extensivity, idempotency may be legitimately expressed as an equality $\alpha \circ \alpha = \alpha$; furthermore, $\alpha^{\mathbb{T}} : TX \to TX$ is also idempotent:

$$\alpha^{\mathbb{T}} \cdot \alpha^{\mathbb{T}} = (\alpha^{\mathbb{T}} \cdot \alpha)^{\mathbb{T}} = (\alpha \circ \alpha)^{\mathbb{T}} = \alpha^{\mathbb{T}} .$$

A *Kleisli morphism* (that is, a *morphism of* $\mathbb{T}$*-monoids*) $f : (X, \alpha) \to (Y, \beta)$ is an X-morphism $f : X \to Y$ such that

$$Tf \cdot \alpha \leq \beta \cdot f$$

and that composes with a Kleisli morphism $g : (Y, \beta) \to (Z, \gamma)$ as in X. The category of Kleisli monoids in X with their morphisms is denoted by $\mathsf{X}(\mathbb{T})$.

In the case where $\mathsf{X} = \mathsf{Set}$, the underlying set of a Kleisli monoid (X, α) can be equipped with the *initial preorder* induced by $\alpha : X \to TX$: for $x, y \in X$

$$x \leq y \iff \alpha(x) \leq \alpha(y) .$$

This preorder becomes an order exactly when $\alpha : X \to TX$ is a monomorphism; in this case, the Kleisli monoid (X, α) is said to be *separated*. The full subcategory of $\mathsf{Set}(\mathbb{T})$ whose objects are separated Kleisli monoids is denoted by $\mathsf{Set}(\mathbb{T})_0$.

3.2 Examples. The categories of Kleisli monoids of the monads given in 2.7 are the following (see [15] or [14]).

$\mathsf{Set}(\mathbb{P}) \cong \mathsf{PrOrd}$**:** category of preordered sets with monotone maps.

$\mathsf{Set}(\mathbb{P})_0 \cong \mathsf{Ord}$**:** category of ordered sets with monotone maps.

$\mathsf{Set}(\mathbb{F}) \cong \mathsf{Top}$**:** category of topological spaces with continuous maps [5].

$\mathsf{Set}(\mathbb{F})_0 \cong \mathsf{Top}_0$**:** category of T_0 topological spaces with continuous maps.

$\mathsf{Set}(\mathbb{U}) \cong \mathsf{Cls}$**:** category of closure spaces with continuous maps.

$\mathsf{Set}(\mathbb{D}) \cong \mathsf{Cls}_{\dagger}$**:** category "non-monotone" closure spaces with continuous maps.

$\mathsf{Set}(\mathbb{U}_{\mathrm{fin}}) \cong \mathsf{Cls}_{\mathrm{fin}}$**:** category of finitary closure spaces with continuous maps.

$\mathsf{Set}(\mathbb{P}_{\mathbf{P}_+}) \cong \mathsf{Met}$**:** category of generalized metric spaces with contractions.

3.3 $\mathbb{T}$-algebras and $\mathbb{T}$-monoids. If $\mathbb{T}$ is an order-adjoint monad, a $\mathbb{T}$-monoid structure α on X is in particular a Kleisli morphism $\alpha : (X, \alpha) \to (TX, \mu_X^*)$. Moreover, for a $\mathbb{T}$-algebra (X, a), the pair (X, a^*) defines a Kleisli monoid. Indeed, the structure a has a right adjoint a^* by Proposition 2.5, so $a \cdot \eta_X \leq 1_X$ and $a \cdot \mu_X \leq a \cdot Ta$ imply

$$\eta_X \leq a^* \qquad \text{and} \qquad \mu_X \cdot T(a^*) \cdot a^* \leq (a^* \cdot a \cdot Ta) \cdot T(a^*) \cdot a^* = a^* .$$

Similarly, a morphism $f : (X, a) \to (Y, b)$ of $\mathbb{T}$-algebras yields a morphism $f : (X, a^*) \to (Y, b^*)$. The right adjoint operation on structures therefore defines a functor $L : \mathsf{Set}^{\mathbb{T}} \to \mathsf{Set}(\mathbb{T})$. In fact, since $a \cdot a^* = 1_X$ (Lemma 2.4), the structure a^* is a monomorphism, so L factors through the category of separated Kleisli monoids, and can be seen as having $\mathsf{Set}(\mathbb{T})_0$ as codomain. This functor is both faithful and injective on objects (by unicity of the right adjoint of a structure a), so that $\mathsf{Set}^{\mathbb{T}}$ can be considered as a subcategory of $\mathsf{Set}(\mathbb{T})_0$ or of $\mathsf{Set}(\mathbb{T})$.

3.4 Proposition. *Let $\mathbb{S}$ be an order-adjoint monad. A monad morphism $\tau : \mathbb{S} \to \mathbb{T}$ yields a faithful functor $\mathsf{Set}(\tau) : \mathsf{Set}(\mathbb{S}) \to \mathsf{Set}(\mathbb{T})$ obtained by sending an $\mathbb{S}$-monoid (X, α) to $(X, \tau_X \cdot \alpha)$ and leaving maps untouched.*

Proof. Proposition 2.6 shows that in the given situation, $\mathbb{T}$ factors through Ord, so that $\mathsf{Set}(\mathbb{T})$ can be defined. The claim then follows by straightforward verifications using that τ_X is monotone. □

3.5 Proposition. *Let $\mathbb{S}$ be an order-adjoint monad, and $\tau : \mathbb{S} \to \mathbb{T}$ a monad morphism. There is a functor $Q : \mathsf{Set}^{\mathbb{T}} \to \mathsf{Set}(\mathbb{S})$ that sends a $\mathbb{T}$-algebra (X, a) to $(X, (a \cdot \tau_X)^*)$ and commutes with the underlying-set functors.*

Proof. A $\mathbb{T}$-algebra (X, a) yields an $\mathbb{S}$-algebra $(X, a \cdot \tau_X)$ via the functor $\mathsf{Set}^{\tau} : \mathsf{Set}^{\mathbb{T}} \to \mathsf{Set}^{\mathbb{S}}$, and therefore an $\mathbb{S}$-monoid $(X, (a \cdot \tau_X)^*)$ thanks to the functor $\mathsf{Set}^{\mathbb{S}} \to \mathsf{Set}(\mathbb{S})$ of 3.3. These operation therefore describe a functor $Q : \mathsf{Set}^{\mathbb{T}} \to \mathsf{Set}(\mathbb{S})$ that commutes with the underlying-set functors. □

4 Monads on $\mathsf{Set}(\mathbb{S})$

4.1 The equalizer construction. Consider a morphism $\tau : \mathbb{S} \to \mathbb{T}$ between monads on Set with $\mathbb{S}$ order-adjoint. We proceed to describe the components T', η', and $(-)^{\mathbb{T}'}$ of a Kleisli triple on $\mathsf{Set}(\mathbb{S})$ (Proposition 4.4).

(i) For an object (X, α) of $\mathsf{Set}(\mathbb{S})$, one sets $\beta := \tau_X \cdot \alpha$ and defines $T'X$, the set of $\beta^{\mathbb{T}}$-*invariants*, as the equalizer in Set of the pair $(\beta^{\mathbb{T}}, 1_{TX})$:

$$T'X \xrightarrow{s_X} TX \underset{1_{TX}}{\overset{\beta^{\mathbb{T}}}{\rightrightarrows}} TX\ .$$

The universal property of s_X yields the existence of a map $r_X : TX \to T'X$ such that

$$s_X \cdot r_X = \beta^{\mathbb{T}} \qquad \text{and} \qquad r_X \cdot s_X = 1_{T'X}\ .$$

The set $T'X$ can be equipped with the $\mathbb{S}$-monoid structure $\omega_X : T'X \to ST'X$ given by

$$\omega_X := Sr_X \cdot (\mu_X \cdot \tau_{TX})^* \cdot s_X\ .$$

Lemma 4.3 below shows that $s_X : (T'X, \omega_X) \to (TX, (\mu_X \cdot \tau_{TX})^*)$ is also an equalizer in $\mathsf{Set}(\mathbb{S})$. This implies that the maps η'_X and $f^{\mathbb{T}'}$ defined in the following points are morphisms of $\mathbb{S}$-monoids.

(ii) Since $\beta^{\mathbb{T}} \cdot \beta = \beta$ (Proposition 3.4), there exists a map $\eta'_X : X \to T'X$ with $s_X \cdot \eta'_X = \beta$:

$$\begin{array}{ccccc} X & & & & \\ {\scriptstyle \eta'_X}\downarrow & \searrow^{\beta} & & & \\ T'X & \xrightarrow[s_X]{} & TX & \underset{1_{TX}}{\overset{\beta^{\mathbb{T}}}{\rightrightarrows}} & TX\ . \end{array}$$

This yields a morphism of $\mathbb{S}$-monoids $\eta'_X : (X, \alpha) \to (T'X, \omega_X)$. Let us point out that since $s_X \cdot \eta'_X = \beta$ and $r_X \cdot s_X = 1_{T'X}$, one can equivalently obtain η'_X as either

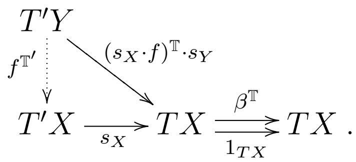
$$\eta'_X = r_X \cdot \beta \qquad \text{or} \qquad \eta'_X = r_X \cdot \eta_X$$

because $r_X \cdot \beta = r_X \cdot \beta^{\mathbb{T}} \cdot \eta_X = r_X \cdot \eta_X$.

(iii) If (Y, α_Y) is another $\mathbb{S}$-monoid, and $f : (Y, \alpha_Y) \to (T'X, \omega_X)$ is a $\mathsf{Set}(\mathbb{S})$-morphism, then one observes

$$\beta^{\mathbb{T}} \cdot (s_X \cdot f)^{\mathbb{T}} = (\beta^{\mathbb{T}} \cdot s_X \cdot f)^{\mathbb{T}} = (s_X \cdot f)^{\mathbb{T}} .$$

Thus, there exists a unique map $f^{\mathbb{T}'} : T'Y \to T'X$ making the following diagram commute:



This yields a morphism of $\mathbb{S}$-monoids $f^{\mathbb{T}'} : (T'Y, \omega_Y) \to (T'X, \omega_X)$ that can also be described directly as

$$f^{\mathbb{T}'} = r_X \cdot (s_X \cdot f)^{\mathbb{T}} \cdot s_Y .$$

4.2 Remark. The monad morphism $\tau : \mathbb{S} \to \mathbb{T}$ does not need to make $\mathbb{T}$ enhanced (Proposition 2.6), so it is not clear in general whether $r_X : (TX, (\mu_X \cdot \tau_{TX})^*) \to (T'X, \omega_X)$ is a Kleisli morphism or not.

4.3 Lemma. *For a monad morphism $\tau : \mathbb{S} \to \mathbb{T}$ with $\mathbb{S}$ an order-adjoint monad, the map*

$$s_X : (T'X, \omega_X) \to (TX, (\mu_X \cdot \tau_{TX})^*)$$

(defined in the previous construction) is an equalizer in $\mathsf{Set}(\mathbb{S})$*. As a consequence,*

$$\eta'_X : (X, \alpha) \to (T'X, \omega_X) \qquad \textit{and} \qquad f^{\mathbb{T}'} : (T'Y, \omega_Y) \to (T'X, \omega_X)$$

are $\mathsf{Set}(\mathbb{S})$*-morphisms.*

Proof. To verify that $\omega_X : T'X \to ST'X$ is an $\mathbb{S}$-monoid structure, observe that

$$\delta_{T'X} = \delta_{T'X} \cdot r_X \cdot s_X = Sr_X \cdot \delta_{TX} \cdot s_X \le Sr_X \cdot (\mu_X \cdot \tau_{TX})^* \cdot s_X$$

by Lemma 2.4, adjunction, and the fact that τ is a monad morphism,

$$\begin{aligned}
\omega^{\mathbb{S}}_X \cdot \omega_X &= Sr_X \cdot \nu_{TX} \cdot S(\mu_X \cdot \tau_{TX})^* \cdot S\beta^{\mathbb{T}} \cdot (\mu_X \cdot \tau_{TX})^* \cdot s_X \\
&\leq Sr_X \cdot \nu_{TX} \cdot (S(\mu_X \cdot \tau_{TX}))^* \cdot S\beta^{\mathbb{T}} \cdot (\mu_X \cdot \tau_{TX})^* \cdot s_X \\
&\leq Sr_X \cdot \nu_{TX} \cdot (S\mu_X \cdot S\tau_{TX})^* \cdot (\mu_X \cdot \tau_{TX})^* \cdot \beta^{\mathbb{T}} \cdot s_X \\
&= Sr_X \cdot \nu_{TX} \cdot (\mu_X \cdot \tau_{TX} \cdot \nu_{TX})^* \cdot \beta^{\mathbb{T}} \cdot s_X \\
&= Sr_X \cdot (\mu_X \cdot \tau_{TX})^* \cdot s_X \\
&= \omega_X \ .
\end{aligned}$$

One can reason similarly to obtain

$$Ss_X \cdot \omega_X = S\beta^{\mathbb{T}} \cdot (\mu_X \cdot \tau_{TX})^* \cdot s_X \leq (\mu_X \cdot \tau_{TX})^* \cdot \beta^{\mathbb{T}} \cdot s_X = (\mu_X \cdot \tau_{TX})^* \cdot s_X \ ,$$

so that $s_X : (T'X, \omega_X) \to (TX, (\mu_X \cdot \tau_{TX})^*)$ is a $\mathsf{Set}(\mathbb{S})$-morphism. Suppose now that $g : (Y, \alpha_Y) \to (TX, (\mu_X \cdot \tau_{TX})^*)$ is a $\mathsf{Set}(\mathbb{S})$-morphism satisfying $\beta^{\mathbb{T}} \cdot g = g$. Since $s_X : T'X \to TX$ is an equalizer of $(\beta^{\mathbb{T}}, 1_{TX})$ in Set, there exists a unique map $h : Y \to T'X$ with $g = s_X \cdot h$; moreover,

$$Sh \cdot \alpha_Y = Sr_X \cdot Sg \cdot \alpha_Y \leq Sr_X \cdot (\mu_X \cdot \tau_{TX})^* \cdot g = \omega_X \cdot h \ ,$$

which shows that $h : (Y, \alpha_Y) \to (T'X, \omega_X)$ is a $\mathsf{Set}(\mathbb{S})$-morphism. As a consequence, s_X is an equalizer in $\mathsf{Set}(\mathbb{S})$, and η'_X, $f^{\mathbb{T}'}$ are the underlying maps of the corresponding unique $\mathsf{Set}(\mathbb{S})$-morphisms into $(T'X, \omega_X)$. □

4.4 Proposition. *If $\tau_X : \mathbb{S} \to \mathbb{T}$ is a monad morphism and $\mathbb{S}$ an order-adjoint monad, then the construction detailed in* (i)–(iii) *above defines a Kleisli triple $(T', \eta', (-)^{\mathbb{T}'})$ on* $\mathsf{Set}(\mathbb{S})$.

Proof. Lemma 4.3 insures that the construction yields components T', η', and $(-)^{\mathbb{T}'}$ of a Kleisli triple on $\mathsf{Set}(\mathbb{S})$. The conditions $(*)$ of 2.1 follow from a straightforward verification, as in [15]. □

4.5 Proposition. *Let (X, α) be an $\mathbb{S}$-monoid. The initial preorder induced by $\omega_X : T'X \to ST'X$ on $T'X$ is an order that moreover makes $s_X : T'X \to TX$ into an order-embedding, and $r_X : TX \to T'X$ into a monotone map. If $\mathbb{T}$ is enhanced, then $r_X : (TX, (\mu_X \cdot \tau_{TX})^*) \to (T'X, \omega_X)$ is a Kleisli morphism and the pair (r_X, s_X) forms an adjoint situation $r_X \dashv s_X$.*

Proof. The cited results follow from straightforward verifications using the fact that

$$\mu_X \cdot \tau_{TX} \cdot Ss_X \cdot \omega_X = s_X \ .$$

See [15] for further details. □

4.6 Examples. If $\mathbb{S} = \mathbb{T} = \mathbb{P}$, then a $\mathbb{P}$-monoid is a pair (X, α), where $\alpha = \downarrow_X : X \to PX$ is the down-set map of X; that is, X is a preordered set (as mentioned in 3.2). An element $A \in PX$ is an $\alpha^{\mathbb{P}}$-invariant precisely when $\alpha^{\mathbb{P}}(A) = A$, that is, when A is down-closed:

$$\bigcup_{x \in A} \downarrow_X x = A \ .$$

Hence, the monad $\mathbb{P}'$ yields the *down-set monad* $\mathbb{P}_\downarrow = (P_\downarrow, \downarrow, \bigcup)$ on PrOrd.

If $\mathbb{S} = \mathbb{T} = \mathbb{F}$ is the filter monad, then an $\mathbb{F}$-monoid is a topological space (X, ν), where $\nu : X \to FX$ is the neighborhood filter map. A filter $f \in FX$ is $\nu^{\mathbb{F}}$-invariant if and only if f is spanned by open sets of X:

$$A \in \nu^{\mathbb{F}}(f) \iff \nu^{-1}(A^{\mathbb{F}}) \in f \iff \{x \in X \mid A \in \nu(x)\} \in f$$

for all $A \in PX$ (and where $A^{\mathbb{F}} = \{\chi \in FX \mid A \in \chi\}$), so $\nu^{\mathbb{F}}(f) = f$ means that if $A \in f$ then its interior must also be in f. The monad $\mathbb{F}'$ is the *open-filter monad* on Top, obtained by considering the neighborhood maps $\nu = e'_X : X \to F'X$ to form its unit, and the restriction of the filtered sum of $\mathbb{F}$ for its multiplication.

Consider now the principal filter natural transformation $\tau : \mathbb{P} \to \mathbb{F}$. The previous examples show that the construction of $\mathbb{F}'$ associates to a preordered set $(X, \downarrow_X)$ the topological space (X, ν) whose neighborhood map is given at each $x \in X$ by the principal filter of $\downarrow_X x \in PX$:

$$\nu(x) = \uparrow_{PX}(\downarrow_X x) \ ,$$

that is, (X, ν) is the Alexandroff space associated to a preordered set X, and open sets are down-closed sets. The set of $\nu^{\mathbb{F}}$-invariant filters can be identified with the set of filters on $P_\downarrow X$, and one obtains the *down-set-filter monad* $\mathbb{F}_\downarrow$ on PrOrd.

4.7 Lemma. *If $R : \mathsf{Set}(\mathbb{S}) \to \mathsf{Set}$ denotes the functor that forgets the structure of objects, then the maps r_X form the components of a natural transformation $r : TR \to RT'$, and the pair $(R, r) : \mathbb{T} \to \mathbb{T}'$ defines a monad morphism.*

Proof. An $\mathbb{S}$-monoid morphism $f:(Y,\alpha_Y)\to(X,\alpha)$ yields a $\mathbb{T}$-monoid morphism $f:(Y,\beta_Y)\to(TX,\beta)$ (where $\beta_Y=\tau_Y\cdot\alpha_Y$ and $\beta=\tau_X\cdot\alpha$), so that

$$\beta\cdot f=(\beta\cdot f)^{\mathbb{T}}\cdot\eta_Y\leq(\beta\cdot f)^{\mathbb{T}}\cdot\beta_Y=\beta^{\mathbb{T}}\cdot Tf\cdot\beta_Y\leq\beta^{\mathbb{T}}\cdot\beta\cdot f=\beta\cdot f\ .$$

Therefore, one has $(\beta\cdot f)^{\mathbb{T}}\cdot\beta_Y=\beta\cdot f$, and using that $T'f=(\eta'_X\cdot f)^{\mathbb{T}'}$ one obtains

$$\begin{aligned}T'f\cdot r_Y=r_X\cdot(\beta\cdot f)^{\mathbb{T}}\cdot\beta_Y^{\mathbb{T}}&=r_X\cdot((\beta\cdot f)^{\mathbb{T}}\cdot\beta_Y)^{\mathbb{T}}\\&=r_X\cdot(\beta\cdot f)^{\mathbb{T}}=r_X\cdot\beta^{\mathbb{T}}\cdot Tf=r_X\cdot Tf\ ,\end{aligned}$$

which proves that $r:TR\to RT'$ is a natural transformation. Moreover, for $\mu'_X=(1_{T'X})^{\mathbb{T}'}=r_X\cdot(s_X)^{\mathbb{T}}\cdot s_{T'X}$, one has

$$\begin{aligned}\mu'_X\cdot r_{T'X}\cdot Tr_X&=r_X\cdot(s_X)^{\mathbb{T}}\cdot(\tau_{T'X}\cdot\omega_X)^{\mathbb{T}}\cdot Tr_X\\&=r_X\cdot(\mu_{TX}\cdot T\beta^{\mathbb{T}}\cdot\tau_{TX}\cdot(\mu_X\cdot\tau_{TX})^*\cdot s_X)^{\mathbb{T}}\cdot Tr_X\\&=r_X\cdot(\beta^{\mathbb{T}}\cdot\mu_X\cdot\tau_{TX}\cdot(\mu_X\cdot\tau_{TX})^*\cdot s_X)^{\mathbb{T}}\cdot Tr_X\\&=r_X\cdot(\beta^{\mathbb{T}}\cdot s_X)^{\mathbb{T}}\cdot Tr_X\\&=r_X\cdot(s_X)^{\mathbb{T}}\cdot Tr_X\\&=r_X\cdot\beta^{\mathbb{T}}\cdot\mu_X\\&=r_X\cdot\mu_X\ .\end{aligned}$$

Since $\eta'_X=r_X\cdot\eta_X$, the pair $(R,r):\mathbb{T}\to\mathbb{T}'$ forms a monad morphism. □

4.8 Theorem. *If $\tau_X:\mathbb{S}\to\mathbb{T}$ is a monad morphism from an order-adjoint monad $\mathbb{S}$, then there is an isomorphism of Eilenberg-Moore categories that is identical on morphisms:*

$$\mathsf{Set}^{\mathbb{T}}\cong\mathsf{Set}(\mathbb{S})^{\mathbb{T}'}\ .$$

Proof. Suppose first that (X,a) is a $\mathbb{T}$-algebra. One obtains an $\mathbb{S}$-monoid (X,α), with $\alpha=(a\cdot\tau_X)^*$, that can be equipped with the structure $a':(T'X,\omega_X)\to(X,\alpha)$ defined by

$$a':=a\cdot s_X\ .$$

By Proposition 3.5, a' is indeed a morphism of $\mathbb{S}$-monoids. To see that a' satisfies the algebra conditions for the monad $\mathbb{T}'$, we first use the definition of η'_X and Lemma 2.4 to obtain

$$a' \cdot \eta'_X = a \cdot s_X \cdot \eta'_X = a \cdot \beta = a \cdot \tau_X \cdot (a \cdot \tau_X)^* = 1_X \ .$$

Suppose now that $f, g : (Y, \beta) \to (T'X, \omega_X)$ are $\mathsf{Set}(\mathbb{S})$-morphisms satisfying $a' \cdot f = a' \cdot g$, or equivalently, $a \cdot s_X \cdot f = a \cdot s_X \cdot g$; since a is a $\mathbb{T}$-algebra structure, one has $a \cdot (s_X \cdot f)^{\mathbb{T}} = a \cdot (s_X \cdot g)^{\mathbb{T}}$ (see 2.2), so that

$$a' \cdot f^{\mathbb{T}'} = a \cdot s_X \cdot f^{\mathbb{T}'} = a \cdot (s_X \cdot f)^{\mathbb{T}} \cdot s_Y = a \cdot (s_X \cdot g)^{\mathbb{T}} \cdot s_Y = a \cdot s_X \cdot g^{\mathbb{T}'} = a' \cdot g^{\mathbb{T}'} \ .$$

Therefore, $((X, \alpha), a')$ is a $\mathbb{T}'$-algebra. A morphism $f : (X, a_X) \to (Y, a_Y)$ of $\mathbb{T}$-algebras yields a $\mathsf{Set}(\mathbb{S})$-morphism $f : (X, (a_X \cdot \tau_X)^*) \to (Y, (a_Y \cdot \tau_Y)^*)$. Since a_Y is a $\mathbb{T}$-algebra structure, one has

$$\begin{aligned} a_Y \cdot (\tau_Y \cdot (a_Y \cdot \tau_Y)^* \cdot f)^{\mathbb{T}} &= a_Y \cdot \mu_X \cdot T(\tau_Y \cdot (a_Y \cdot \tau_Y)^*) \cdot Tf \\ &= a_Y \cdot Ta_Y \cdot T(\tau_Y \cdot (a_Y \cdot \tau_Y)^*) \cdot Tf \\ &= a_Y \cdot Tf \end{aligned}$$

by Lemma 2.4. To verify that $a'_Y \cdot (\eta'_Y \cdot f)^{\mathbb{T}'} = f \cdot a'_X$, we use the previous observation in

$$\begin{aligned} a'_Y \cdot (\eta'_Y \cdot f)^{\mathbb{T}'} &= a_Y \cdot (s_Y \cdot \eta'_Y \cdot f)^{\mathbb{T}} \cdot s_X \\ &= a_Y \cdot (\tau_Y \cdot (a_Y \cdot \tau_Y)^* \cdot f)^{\mathbb{T}} \cdot s_X \\ &= f \cdot a_X \cdot s_X \\ &= f \cdot a'_X \ , \end{aligned}$$

which proves that $f : ((X, \alpha_X), a'_X) \to ((Y, \alpha_Y), a'_Y)$ is a morphism of $\mathbb{T}'$-algebras. Thus, the assignment of $((X, (a \cdot \tau_X)^*), a \cdot s_X)$ to a $\mathbb{T}$-algebra (X, a) yields a functor $\overline{Q} : \mathsf{Set}^{\mathbb{T}} \to \mathsf{Set}(\mathbb{S})^{\mathbb{T}'}$ that leaves maps untouched.

The monad morphism $(R, r) : \mathbb{T} \to \mathbb{T}'$ (Lemma 4.7) yields a functor $\overline{R} : \mathsf{Set}(\mathbb{S})^{\mathbb{T}'} \to \mathsf{Set}^{\mathbb{T}}$ by the discussion in 2.2. This functor sends a $\mathbb{T}'$-algebra $((X, \alpha), a')$ to (X, a), where $a : TX \to X$ is defined by

$$a := a' \cdot r_X \ ,$$

and is invariant on maps.

Given a $\mathbb{T}$-algebra (X, a), the structure of $\overline{R}\,\overline{Q}(X, a)$ is described by

$$a \cdot s_X \cdot r_X = a \cdot \mu_X \cdot T(\tau_X \cdot (a \cdot \tau_X)^*) = a \cdot T(a \cdot \tau_X) \cdot T(a \cdot \tau_X)^* = a \ .$$

To study the image of a $\mathbb{T}'$-algebra $((X, \alpha), a')$ via $\overline{Q}\,\overline{R}$, note first that $a' : (T'X, \omega_X) \to (X, \alpha)$ is a $\mathsf{Set}(\mathbb{S})$-morphism. Thus, after setting $\beta = \tau_X \cdot \alpha$ and observing that $(\mu_X \cdot \tau_{TX}) \cdot S(\tau_X \cdot \alpha) = \beta^{\mathbb{T}} \cdot \tau_X$, one obtains

$$\begin{aligned} 1_{SX} &= S(a' \cdot r_X) \cdot S(\tau_X \cdot \alpha) \\ &\le S(a' \cdot r_X) \cdot (\mu_X \cdot \tau_{TX})^* \cdot \beta^{\mathbb{T}} \cdot \tau_X \\ &= Sa' \cdot \omega_X \cdot r_X \cdot \tau_X \\ &\le \alpha \cdot (a' \cdot r_X \cdot \tau_X) \ . \end{aligned}$$

This inequality, combined with $(a' \cdot r_X \cdot \tau_X) \cdot \alpha = 1_X$ and the fact that both α and $(a' \cdot r_X \cdot \tau_X)$ are monotone, yields

$$\alpha = (a' \cdot r_X \cdot \tau_X)^* \ .$$

Hence, the image via $\overline{Q}\ \overline{R}$ of the $\mathbb{T}'$-algebra $((X, \alpha), a')$ returns the $\mathbb{T}'$-algebra whose underlying $\mathbb{S}$-monoid is $(X, (a' \cdot r_X \cdot \tau_X)^*) = (X, \alpha)$; its structure is therefore given by

$$a' \cdot r_X \cdot s_X = a' \ ,$$

so that $\overline{Q}$ and $\overline{R}$ are inverse of one another, and $\mathsf{Set}^{\mathbb{T}} \cong \mathsf{Set}(\mathbb{S})^{\mathbb{T}'}$. $\square$

4.9 Corollary. *Given a morphism $\tau : \mathbb{S} \to \mathbb{T}$ with $\mathbb{S}$ an order-adjoint monad, the monad $\mathbb{T}'$ restricts to $\mathsf{Set}(\mathbb{S})_0$, and the isomorphism of Theorem 4.8 becomes*

$$\mathsf{Set}^{\mathbb{T}} \cong \mathsf{Set}(\mathbb{S})_0^{\mathbb{T}'} \ .$$

Proof. The functor $\overline{R}$ restricts to $\mathsf{Set}(\mathbb{S})_0^{\mathbb{T}}$, and $\overline{Q}$ factors through $\mathsf{Set}(\mathbb{S})_0^{\mathbb{T}'}$ since the functor $Q : \mathsf{Set}^{\mathbb{S}} \to \mathsf{Set}(\mathbb{S})$ of 3.1 factors through $\mathsf{Set}(\mathbb{S})_0$. $\square$

4.10 Examples. The Eilenberg-Moore algebras of the monads mentioned in 2.7 have been described as follows (although most results are classical, we try to give the original printed source in each case and refer to [9] and [6] for further details).

$\mathsf{Set}^{\mathbb{F}} \cong \mathsf{Cnt}$**:** category of continuous lattices with continuous sup-maps [4]; see also 5.4 below.

$\mathsf{Set}^{\mathbb{U}} \cong \mathsf{Ccd}$**:** category of constructive completely distributive lattices with maps that preserve all suprema and infima [12].

$\mathsf{Set}^{\mathbb{D}} \cong \mathsf{CaBool}$**:** category of complete atomistic Boolean algebras with ring homomorphisms that preserve all suprema and infima [9].

$\mathsf{Set}^{\mathbb{U}_{\mathrm{fin}}} \cong \mathsf{Frm}$**:** category of frames with sup-maps that preserve finite infima, see [2] (in fact, Bénabou describes free frames over meet-semilattices; in conjunction with the free meet-semilattice construction over sets, one obtains monadicity over Set as in [6]).

$\mathsf{Set}^{\mathbb{P}_{\mathbf{P}_+}} \cong \mathbf{P}_+\text{-}\mathsf{Mod}$**:** category of left $\mathbf{P}_+$-modules with sup-maps that commute with the action of $\mathbf{P}_+$ on Sup, see [11].

Hence, one obtains the following table of strict monadicities (of the categories in the entry line over categories displayed in the entry column) using Theorems 4.8 and Corollary 4.9. Previous explicit references are mentioned to the best of our knowledge, though we make absolutely *no* originality claim in their absence. For example, monadicity of Cnt over Ord can hardly be considered novel, but we were not able to find this particular instance in the literature; similarly, the column for CaBool is not surprising in view of [16], even though the results presented therein refer to not-necessarily-strict monadicity.

	Sup	Cnt	Frm	Ccd	CaBool	$\mathbf{P}_+$-Mod
Set	[7]	[4]	[6]	[12]	[9]	[11]
PrOrd	4.8	[13]	4.8	[13]	4.8	4.8
Ord	4.9	4.9	Linton ([10])	[10]	4.9	4.9
Top		[17]		4.8	4.8	
Top_0		[4]		4.9	4.9	
$\mathsf{Cls}_{\mathrm{fin}}$			4.8	4.8	4.8	
Cls				4.8	4.8	
$\mathsf{Cls}_{\dagger}$					4.8	
Met	4.8	4.8	4.8	4.8	4.8	[11]

5 Algebras of enhanced order-adjoint monads

A morphism $\tau_X : \mathbb{S} \to \mathbb{T}$ between monads on Set induces a functor $\mathsf{Set}^\tau : \mathsf{Set}^\mathbb{T} \to \mathsf{Set}^\mathbb{S}$, so that a $\mathbb{T}$-algebra (X, a) is an $\mathbb{S}$-algebra $(X, a \cdot \tau_X)$. Theorem 4.8 can be used in the same way to identify categories of $\mathbb{T}$-algebras. We illustrate this further on by giving an original proof of the isomorphism $\mathsf{Set}^\mathbb{F} \cong \mathsf{Cnt}$.

5.1 Lemma. *Let $\mathbb{S} = (S, \delta, \nu)$ be an order-adjoint monad and $\tau : \mathbb{S} \to \mathbb{T}$ a monad morphism that makes $\mathbb{T} = (T, \eta, \mu)$ enhanced (via the order described in Proposition 2.6). Given an $\mathbb{S}$-monoid (X, α) and a map $\lambda : X \to T'X$, one has that $((X, \alpha), \lambda)$ is a $\mathbb{T}'$-monoid if and only if $(X, s_X \cdot \lambda)$ is a $\mathbb{T}$-monoid (using the notations of Section 4).*

Proof. Let us first verify that a $\mathbb{T}'$-monoid structure λ on (X, α) yields a $\mathbb{T}$-monoid structure $s_X \cdot \lambda$. From $r_X \cdot \eta_X = \eta'_X \le \lambda$, one obtains extensivity of $s_X \cdot \lambda$:

$$\eta_X = \tau_X \cdot \delta_X \le \tau_X \cdot \alpha = (\tau_X \cdot \alpha)^\mathbb{T} \cdot \eta_X = s_X \cdot r_X \cdot \eta_X \le s_X \cdot \lambda \, .$$

Idempotency is then a consequence of

$$(s_X \cdot \lambda)^\mathbb{T} \cdot s_X \cdot \lambda = s_X \cdot \lambda^{\mathbb{T}'} \cdot \lambda \le s_X \cdot \lambda \, .$$

Suppose now that $s_X \cdot \lambda$ is a $\mathbb{T}$-monoid structure with $\tau_X \cdot \alpha \le s_X \cdot \lambda$. As $\mathbb{T}$ is enhanced, one immediately obtains $\eta'_X \le \lambda$ and $\lambda^{\mathbb{T}'} \cdot \lambda \le \lambda$ from extensivity and idempotency of $s_X \cdot \lambda$. We are therefore left to verify that $\lambda : (X, \alpha) \to (T'X, \omega)$ is an $\mathbb{S}$-monoid morphism; by composing each side of the extensivity condition with $(\tau_X \cdot \alpha)^\mathbb{T} = s_X \cdot r_X$ on the left, we obtain $\tau_X \cdot \alpha = (\tau_X \cdot \alpha)^\mathbb{T} \cdot \eta_X \le s_X \cdot \lambda$, so that

$$\mu_X \cdot \tau_{TX} \cdot S(s_X \cdot \lambda) \cdot \alpha = (s_X \cdot \lambda)^\mathbb{T} \cdot \tau_X \cdot \alpha \le (s_X \cdot \lambda)^\mathbb{T} \cdot s_X \cdot \lambda \le s_X \cdot \lambda \, .$$

After composing these expressions with $Sr_X \cdot (\mu_X \cdot \tau_{TX})^*$ on the left, we obtain the desired inequality. $\square$

5.2 Lemma. *Given an enhanced order-adjoint monad $\mathbb{T}$, a $\mathbb{T}$-monoid (X, α) is of the form (X, a^*) for a $\mathbb{T}$-algebra structure $a : TX \to X$ if and only if α has a left adjoint α_* with $\alpha_* \cdot \alpha = 1_X$.*

Proof. See Corollary 4.11 in [15]. □

5.3 Proposition. *Given an order-adjoint monad* $\mathbb{S}$ *and a monad morphism* $\tau : \mathbb{S} \to \mathbb{T}$ *that makes* $\mathbb{T}$ *enhanced, a morphism* $a' : (T'X, \omega_X) \to (X, \alpha)$ *of* $\mathbb{S}$*-monoids is a* $\mathbb{T}'$*-algebra structure if and only if* $a' : T'X \to X$ *has a right adjoint* $(a')^*$ *with* $a' \cdot (a')^* = 1_X$ *that moreover makes* $(X, (a')^*)$ *into a* $\mathbb{T}'$*-monoid.*

Proof. Let (X, α) be an $\mathbb{S}$-monoid. If $a' : (T'X, \omega) \to (X, \alpha)$ is a $\mathbb{T}'$-algebra structure, then $a = a' \cdot r_X : TX \to X$ is a $\mathbb{T}$-algebra structure by the correspondence between algebras described in the proof of Theorem 4.8. Proposition 2.5 states that a has a right adjoint $a^* : X \to TX$, and one observes that

$$1_{T'X} \leq r_X \cdot a^* \cdot a \cdot s_X = (r_X \cdot a^*) \cdot a' \quad , \qquad a' \cdot (r_X \cdot a^*) = a \cdot a^* = 1_X \ ,$$

so $(a')^* = r_X \cdot (a' \cdot r_X)^*$ is the required right adjoint of a'. The proof of Theorem 4.8 shows that $(a \cdot \tau_X)^*$ is the $\mathbb{S}$-monoid structure α; since $\mathbb{T}$ is enhanced, one has $a^* \leq (\tau_X \cdot \alpha)^{\mathbb{T}} \cdot a^* \leq a^*$, so $s_X \cdot (a')^*$ is also a $\mathbb{T}$-monoid structure:

$$s_X \cdot (a')^* = s_X \cdot r_X \cdot a^* = (\tau_X \cdot \alpha)^{\mathbb{T}} \cdot a^* = a^* \ .$$

It then follows from Lemma 5.1 that $(a')^*$ is a $\mathbb{T}'$-monoid structure on (X, α).

Suppose now that $a' : T'X \to X$ has a right adjoint $\mathbb{T}'$-monoid structure $(a')^* : X \to T'X$ with $a' \cdot (a')^* = 1_X$. Lemma 5.1 yields that $(X, s_X \cdot (a')^*)$ is a $\mathbb{T}$-monoid, and setting $a := a' \cdot r_X$, one has

$$1_{TX} \leq s_X \cdot (a')^* \cdot a' \cdot r_X = (s_X \cdot (a')^*) \cdot a \ , \quad a \cdot (s_X \cdot (a')^*) = a' \cdot (a')^* = 1_X \ .$$

Hence, we can apply Lemma 5.2 to the right adjoint $\mathbb{T}$-monoid structure $s_X \cdot (a')^*$ to conclude. □

5.4 Continuous lattices. The monad $\mathbb{F}_{\downarrow}$ on PrOrd can equivalently described using both the down-set monad $\mathbb{P}_{\downarrow} = (P_{\downarrow}, \downarrow, \bigcup)$ (see Examples 4.6), and the ordered-filter monad $\mathbb{P}_{\Uparrow} = (P_{\Uparrow}, \uparrow, \bigcup)$ on PrOrd, whose functor $P_{\Uparrow}$ is the restriction of $P_{\uparrow}$ to filters in X (that is, to up-closed down-directed sets in X). For the up-set map $\uparrow_X : X \to P_{\Uparrow}X$ to be monotone, the set $P_{\Uparrow}X$

is ordered by reverse inclusion. It will be convenient to use the following notations for the units and multiplications of the respective monads:

$$d_X(x) = {\downarrow_X} x\,, \qquad \sup_{P_\downarrow X}(\mathcal{A}) = \bigcup \mathcal{A}\,,$$
$$u_X(x) = {\uparrow_X} x\,, \qquad \inf_{P_\Uparrow X}(\mathcal{B}) = \bigcup \mathcal{B}\,,$$

for all $x \in X$, $\mathcal{A} \in P_\downarrow P_\downarrow X$, and $\mathcal{B} \in P_\Uparrow P_\Uparrow X$. One observes that the down-set-filter monad can be written as

$$\mathbb{F}_\downarrow = (F_\downarrow, \eta', \mu') = (P_\Uparrow P_\downarrow, uP_\downarrow \cdot d, \inf_{P_\Uparrow P_\downarrow} \cdot P_\Uparrow \sup_{P_\Uparrow P_\downarrow})\,.$$

We say that a complete lattice X is *continuous* if the infimum map $\inf_X : P_\Uparrow X \to X$ has a right adjoint $\Uparrow_X : X \to P_\Uparrow X$ sending x to a filter $\Uparrow_X x = \Uparrow x$; since $P_\Uparrow$ is monotone, we have

$$X \underset{\Uparrow_X}{\overset{\inf_X}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} P_\Uparrow X \underset{P_\Uparrow d_X}{\overset{P_\Uparrow \sup_X}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} P_\Uparrow P_\downarrow X = F_\downarrow X\,. \qquad (*)$$

A sup-map $f : X \to Y$ is *continuous* if it preserves infima of down-directed sets. Recall from 4.10 that the category of continuous lattices and continuous sup-maps is denoted by Cnt.

The diagram $(*)$ suggests that $\inf_X \cdot P_\Uparrow \sup_X$ is the structure of a $\mathbb{F}_\downarrow$-algebra on X, and this is confirmed in the following result. Proposition 5.3 and Lemma 5.1 therefore state that the left adjoint $P_\Uparrow d_X \cdot \Uparrow_X : X \to F_\downarrow X$ is the neighborhood map $P_\Uparrow d_X \cdot \Uparrow_X : X \to FX$ of a topology on the set X: the Scott topology on a continuous lattice.

5.5 Proposition. *There is an isomorphism*

$$\mathsf{Cnt} \cong \mathsf{PrOrd}^{\mathbb{F}_\downarrow}$$

that commutes with the underlying-functors to PrOrd.

Proof. Let us first check that for a continuous lattice X, the map $a' := \inf_X \cdot P_\Uparrow \sup_X$ defines the structure morphism of a $\mathbb{P}_\Uparrow\mathbb{P}_\downarrow$-algebra. We already have $a' \cdot \eta'_X = a' \cdot P_\Uparrow d_X \cdot u_X = 1_X$, so we only need to verify that

$a' \cdot P_{\Uparrow} P_{\downarrow} a' = a' \cdot \mu'_X$. This follows from

$$\begin{aligned} a' \cdot P_{\Uparrow} P_{\downarrow} a' &= \inf_X \cdot P_{\Uparrow}(\sup_X \cdot P_{\downarrow} a') \\ &= \inf_X \cdot P_{\Uparrow}(a' \cdot \sup_{P_{\Uparrow} P_{\downarrow} X}) \\ &= \inf_X \cdot \inf_{P_{\Uparrow} X} \cdot P_{\Uparrow} P_{\Uparrow} \sup_X \cdot P_{\Uparrow} \sup_{P_{\Uparrow} P_{\downarrow} X} \\ &= \inf_X \cdot P_{\Uparrow} \sup_X \cdot \inf_{P_{\Uparrow} P_{\downarrow} X} \cdot P_{\Uparrow} \sup_{P_{\Uparrow} P_{\downarrow} X} \\ &= a' \cdot \mu'_X \end{aligned}$$

because a' is left adjoint (see $(*)$) and therefore preserves suprema, $\inf_X$ preserves infima, and $\inf_{P_{\Uparrow}} : P_{\Uparrow} P_{\Uparrow} \to P_{\Uparrow}$ is a natural transformation.

Consider now an $\mathbb{F}_{\downarrow}$-algebra $(X, a' : P_{\Uparrow} P_{\downarrow} X \to X)$. There is a monad morphism $uP_{\downarrow} : \mathbb{P}_{\downarrow} \to \mathbb{F}_{\downarrow}$, so the preordered set X is a $\mathbb{P}_{\downarrow}$-algebra, that is, a complete lattice with supremum given by $\sup_X = a' \cdot u_{P_{\downarrow} X}$. Proposition 5.3 yields that a' has a right adjoint $(a')^* : X \to P_{\Uparrow} P_{\downarrow} X$, so we are in the presence of the following adjunctions:

$$X \underset{(a')^*}{\overset{a'}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} P_{\Uparrow} P_{\downarrow} X \underset{P_{\Uparrow}(a' \cdot u_{P_{\downarrow} X})}{\overset{P_{\Uparrow} d_X}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} P_{\Uparrow} X \ .$$

Since the components of the up-set monad's multiplication are $\inf_{P_{\Uparrow} X}$, we have in particular $\inf_{P_{\Uparrow} P_{\downarrow} X} \cdot P_{\Uparrow} u_{P_{\downarrow} X} = 1_{P_{\Uparrow} P_{\downarrow} X}$. Consequently, by using that $a' \cdot P_{\Uparrow} P_{\downarrow} a' = a' \cdot \mu'_X$ we may write

$$\begin{aligned} a' \cdot P_{\Uparrow} d_X \cdot P_{\Uparrow}(a' \cdot u_{P_{\downarrow} X}) &= a' \cdot \mu'_X \cdot P_{\Uparrow} d_{P_{\Uparrow} P_{\downarrow} X} \cdot P_{\Uparrow} u_{P_{\downarrow} X} \\ &= a' \cdot \inf_{P_{\Uparrow} P_{\downarrow} X} \cdot P_{\Uparrow} \sup_{P_{\Uparrow} P_{\downarrow} X} \cdot P_{\Uparrow} d_{P_{\Uparrow} P_{\downarrow} X} \cdot P_{\Uparrow} u_{P_{\downarrow} X} = a' \ . \end{aligned}$$

This shows that $a' \cdot P_{\Uparrow} d_X$ admits $P_{\Uparrow}(a' \cdot u_{P_{\downarrow} X}) \cdot (a')^*$ as a right adjoint (and also proves that $a' = \inf_X \cdot P_{\Uparrow} \sup_X$). But the infimum operation (obtained via the monad morphism $P_{\Uparrow} d : \mathbb{P}_{\Uparrow} \to \mathbb{F}_{\downarrow}$) is precisely $\inf_X = a' \cdot P_{\Uparrow} d_X$, so X is a continuous lattice.

Finally, a continuous lattice morphism $f : X \to Y$ is also an $\mathbb{F}_{\downarrow}$-algebra morphism, and a morphism $f : (X, a') \to (Y, b')$ of $\mathbb{F}_{\downarrow}$-algebras naturally preserves both suprema and down-directed infima because it is both a $\mathbb{P}_{\downarrow}$-algebra and a $\mathbb{P}_{\Uparrow}$-algebra morphism. □

5.6 Corollary. *There is an isomorphism*

$$\mathsf{Cnt} \cong \mathsf{Set}^{\mathbb{F}}$$

that commutes with the underlying-functors to Set.

Proof. Since the monad $\mathbb{F}'$ obtained from $\mathbb{F}$ is the down-set-filter monad $\mathbb{F}_{\downarrow}$ (Examples 4.6), the results follows from Proposition 5.5 combined with Theorem 4.8. □

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COVERING MORPHISMS OF CROSSED COMPLEXES AND OF CUBICAL OMEGA-GROUPOIDS ARE CLOSED UNDER TENSOR PRODUCT

by Ronald BROWN and Ross STREET

Résumé

Le but de cet article est de démontrer les théorèmes mentionnés dans le titre, ainsi que le corollaire disant que le produit tensoriel de deux résolutions croisées libres, en groupes ou en groupoïdes, est aussi une résolution croisée libre, en groupes ou en groupoïdes. Ce corollaire est obtenu en utilisant l'équivalence entre la catégorie des complexes croisés et celle des omega-groupoïdes cubiques, avec connexion, dans laquelle on donne la définition initiale du produit tensoriel. D'autre part, c'est dans cette deuxième catégorie qu'on peut appliquer les techniques de sous-catégories denses pour reconnaitre qu'un produit tensoriel de revêtements est un revêtement.

Abstract

The aim is the proof of the theorems of the title and the corollary that the tensor product of two free crossed resolutions of groups or groupoids is also a free crossed resolution of the product group or groupoid. The route to this corollary is through the equivalence of the category of crossed complexes with that of cubical ω-groupoids with connections where the initial definition of the tensor product lies. It is also in the latter category that we are able to apply techniques of dense subcategories to identify the tensor product of covering morphisms as a covering morphism.

Mots-clés / Keywords: crossed complexes, cubical omega-groupoids, monoidal closed, density, covering morphisms.

Classification MSC 2010: 18B40, 18D05, 18D10, 18G40, 55U40

Introduction

A series of papers by R. Brown and P.J. Higgins, surveyed in [Bro99, Bro09], has shown how the category Crs of crossed complexes is a useful tool for certain nonabelian higher dimensional local-to-global problems in algebraic topology, for example the calculation of homotopy 2-types of unions of spaces; and also that crossed complexes are suitable coefficients for nonabelian cohomology, generalising an earlier use of crossed modules as coefficients. While crossed complexes have a long history in algebraic topology, particularly in the reduced case, i.e. when C_0 is a singleton, the extended use in these papers made them a tool whose properties could be developed independently of classical tools in algebraic topology such as simplicial approximation. A key new tool for this approach was cubical, using the notion of cubical ω–groupoids with connections. A book is in press on these topics, [BHS11].

One aspect of this work is that it leads to specific calculations of homotopical and group theoretical invariants; as an example, the notion of identities among relations for a presentation of groups combines both of these fields, since it also concerns the second homotopy group $\pi_2(K(\mathcal{P}))$ of the 2-complex determined by a presentation $\mathcal{P}$ of a group. Calculations of this module were obtained in [BRS99] not through 'killing homotopy groups', or its homological equivalent, finding generators of a kernel, but through the notion of 'constructing a home for a contracting homotopy'. To this end we had to work by constructing a free crossed resolution $\widetilde{F}$ of the universal covering crossed complex of a group or groupoid. Any construction of a contracting homotopy of $\widetilde{F}$ breaks the symmetry of the situation, as is necessary, and also may rely on rewriting methods, such as determining a maximal tree in the Cayley graph. Thus we see covering crossed complexes as a basic tool in the application of crossed complex methods, in analogy to the application of covering spaces in algebraic topology.

A major tool for dealing with homotopies is the construction of a monoidal closed structure on the category Crs of crossed complexes giving an exponential law of the form

$$\mathsf{Crs}(A \otimes B, C) \cong \mathsf{Crs}(A, \mathsf{CRS}(B, C))$$

for all crossed complexes A, B, C, [BH87].

This monoidal closed structure and the notion of classifying space BC of a crossed complex C is applied in [BH91] to give the homotopy classification result

$$[X, BC] \cong [\Pi X_*, C]$$

where on the left hand side with X a CW-complex, we have topology, and on the right hand side, with ΠX_* the fundamental crossed complex of the skeletal filtration of X, we have the algebra of crossed complexes.

Tonks proved in [Ton94, Theorem 3.1.5] that the tensor product of free crossed resolutions of a group is a free crossed resolution: his proof used the crossed complex Eilenberg-Zilber Theorem, [Ton94, Theorem 2.3.1], which was published in [Ton03]. The result on resolutions is applied in for example [BP96] to construct some small free crossed resolutions of a product of groups. We give here an alternative approach to this result.

The PhD thesis [Day70] of Brian Day addressed the problem of extending a promonoidal structure on a category $\mathcal{A}$ along a dense functor $J : \mathcal{A} \to \mathcal{X}$ into a suitably complete category $\mathcal{X}$ to obtain a closed monoidal structure on $\mathcal{X}$. The two published papers [Day70a, Day72] are only part of the thesis and represent components towards the density result. The formulas in, and the spirit of, Day's work suggested our approach to the present paper. However, here the category $\mathcal{A}$ is actually small (consisting of cubes) and monoidal, and so is an easy case of Day's general setting. The same simplification occurs in the approach to the Gray tensor product of 2-categories in [Str88], and of globular ∞-categories in [Cra99, Proposition 4.1].

One advantage of cubical methods is the standard formula

$$I_*^m \otimes I_*^n \cong I_*^{m+n} \tag{1}$$

where I_*^m is the standard topological m-cube with its standard skeletal filtration. This equation is modelled in the category ω-Gpd by the formula

$$\mathbb{I}^m \otimes \mathbb{I}^n \cong \mathbb{I}^{m+n} \tag{2}$$

where for $m \geqslant 0$ $\mathbb{I}^m$ is the free ω-Gpd on one generator c_m of dimension m. We apply (2) by proving in Theorem 5.1 that the full subcategory of ω-Gpd on these objects $\mathbb{I}^m, m \geqslant 0$, is dense in ω-Gpd. The proof requires a further property of ω-groupoids, that they are T-complexes [BH81, BH81c]. We then use the methods of Brian Day [Day72] to characterise the tensor product on ω-Gpd as determined by the formula (2).

We use freely the notions and properties of ends and coends, for which see [ML71].

The final ingredient we need is the fact that if $p\colon \widetilde{C} \to C$ is a covering morphism of crossed complexes then $p^*\colon \mathsf{Crs}/C \to \mathsf{Crs}/\widetilde{C}$ preserves colimits, since it has a right adjoint. This result is due to Howie [How79], in fact for the case of a fibration rather than just a covering morphism. Because of the equivalence of categories, this applies also to the case of the category ω-Gpd. However we need to characterise fibrations and coverings in the category ω-Gpd. This is done in Section 4. It is possible that the covering morphisms are part of a factorization system as are the discrete fibrations in the contexts of [Bou87] and [SV10].

The use of crossed complexes continues work of J.H.C. Whitehead, [Whi49, Whi50], and of J. Huebschmann, [Hue80], all for the single vertex case.

1 Crossed complexes

For the purposes of algebraic topology the most important feature of the category Crs of crossed complexes is the *fundamental crossed complex functor*, [BH81a],

$$\Pi\colon \mathsf{FTop} \to \mathsf{Crs}$$

from the category of filtered spaces

$$X_*\colon X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty.$$

An extra assumption is commonly made that X_∞ is the union of all the X_n, but we do not use that condition. For such a filtered space X_*, various relative homotopy groups

$$(\Pi X_*)_n(x) = \pi_n(X_n, X_{n-1}, x)$$

for $x \in X_0$ and $n \geqslant 2$, may be combined with the fundamental groupoid $(\Pi X_*)_1 = \pi_1(X_1, X_0)$ on the set X_0 to give a crossed complex ΠX_*. There are boundary operations $\delta_n \colon (\Pi X_*)_n \to (\Pi X_*)_{n-1}$ and operations of $(\Pi X_*)_1$ on $(\Pi X_*)_n, n \geqslant 2$, satisfying axioms which are characteristic for crossed complexes. This last fact follows because for every crossed complex C there is a filtered space X_* such that $C \cong \Pi X_*$ [BH81a, Corollary 9.3].

2 Fibrations and covering morphisms of crossed complexes

The definition of fibration of crossed complexes we are using is due to Howie in [How79]; it requires the definition of fibration of groupoids given in [Bro70, Bro06], generalising the definition of covering morphism of groupoids given in [Hig71]. The notion of fibration of crossed complexes given in this Section leads to a Quillen model structure on the category Crs, as shown by Brown and Golasiński in [BG89], and compared with model structures on related categories in [ArMe10].

First recall that for a groupoid G and object x of G we write $\mathrm{Cost}_G\, x$ for the union of the $G(u, x)$ for all objects u of G. A morphism of groupoids $p \colon H \to G$ is called a *fibration (covering morphism)*, [Bro70], if the induced map $\mathrm{Cost}_H\, y \to \mathrm{Cost}_G\, py$ is a surjection (bijection) for all objects y of H. (Here we use the conventions of [BHS11] rather than of [Bro06].)

Definition 2.1 A morphism $p \colon D \to C$ of crossed complexes is a *fibration (covering morphism)* if

(i) the morphism $p_1 \colon D_1 \to C_1$ is a *fibration (covering morphism)* of groupoids;

(ii) for each $n \geqslant 2$ and $y \in D_0$, the morphism of groups $p_n \colon D_n(y) \to C_n(py)$ is surjective (bijective).

The morphism p is a *trivial fibration* if it is a fibration, and also a weak equivalence, by which is meant that p induces a bijection on π_0 and isomorphisms

$\pi_1(D, y) \to \pi_1(C, py)$, $H_n(D, y) \to H_n(C, py)$ for all $y \in D_0$ and $n \geqslant 2$.
□

Remark 2.2 It is worth remarking that the notion of covering morphism of groupoids appears in the paper [Smi51, (7.1)] under the name 'regular morphism'. Strong applications of covering morphisms to combinatorial group theory are given in [Hig71], and a full exposition is also given in [Bro06, Chapter 10].

A fibration of groupoids gives rise to a family of exact sequences, [Bro70, Bro06], which are extended in [How79] to a family of exact sequences arising from a fibration of crossed complexes. These latter exact sequences have been applied to the classification of nonabelian extensions of groups in [BM94], and to the homotopy classification of maps of spaces in [Bro08a]. □

In Section 4 we will need the following result, which is an analogue for crossed complexes of known results for groupoids [Bro06, 10.3.3] and for spaces.

Proposition 2.3 *Let $p \colon \widetilde{C} \to C$ be a covering morphism of crossed complexes, and let $y \in \widetilde{C}_0$. Let F be a connected crossed complex, let $x \in F_0$, and let $f \colon F \to C$ be a morphism of crossed complexes such that $f(x) = p(y)$. Then the following are equivalent:*

(i) *f lifts to a morphism $\widetilde{f} \colon F \to \widetilde{C}$ such that $\widetilde{f}(x) = y$ and $p\widetilde{f} = f$;*

(ii) *$f(F_1(x)) \subseteq p(\widetilde{C}_1(y))$;*

(iii) *$f_*(\pi_1(F, x)) \subseteq p_*(\pi_1(\widetilde{C}, y))$.*

Further, if the lifted morphism as above exists, then it is unique.

Proof That (i)$\Rightarrow$ (ii) $\Rightarrow$ (iii) is clear.

So we assume (iii) and prove (i).

We first assume F_0 consists only of x. Then the value of $\widetilde{f}$ on x is by assumption defined to be y.

Next let $a \in F_1(x)$. By the assumption (iii) there is $c \in C_2(py)$ and $b \in \widetilde{C}(y)$ such that $f(a) = p(b) + \delta_2(c)$. Since p is a covering morphism there is a unique $d \in \widetilde{C}_2(y)$ such that $p(d) = c$. Thus $f(a) = p(b + \delta_2(d))$. So we define $\widetilde{f}(a) = b + \delta_2(d) \in \widetilde{C}_2(y)$. It is easy to prove from the definition of covering morphism of groupoids that this makes $\widetilde{f}$ a morphism $F_1(x) \to \widetilde{C}_1(y)$ such that $p\widetilde{f} = f$.

For $n \geqslant 2$ we define $\widetilde{f}\colon F_n(x) \to \widetilde{C}_n(y)$ to be the composition of f in dimension n and the inverse of the bijection $p\colon \widetilde{C}_n(y) \to C_n(py)$.

It is now straightforward to check that this defines a morphism $\widetilde{f}\colon F, x \to \widetilde{C}, y$ of crossed complexes as required.

If F_0 has more than one point, then we choose for each u in F_0 an element $\tau_u \in F_1(u, x)$ with $\tau_x = 1_x$. Then $f(\tau_u)$ lifts uniquely to $\bar{\tau}_u \in \mathrm{Cost}_{\widetilde{C}}\, y$: any lift $\widetilde{f}\colon F, x \to \widetilde{C}, y$ of f must satisfy $\widetilde{f}(\tau_u) = \bar{\tau}_u$ so we take this as a definition of $\widetilde{f}$ on these elements.

If $a \in F_1(u, v)$ then $a = \tau_u + a' - \tau_v$ where $a' \in F_1(x)$ and so we define $\widetilde{f}(a) = \bar{\tau}_u + \widetilde{f}(a') - \bar{\tau}_v$. If $n \geqslant 2$ and $\alpha \in F_n(u)$ then $\alpha^{\tau_u} \in F_n(x)$ and we define $\widetilde{f}(\alpha) = \widetilde{f}(\alpha^{\tau_u})^{-\bar{\tau}_u}$.

It is straightforward to check that these definitions give a morphism $\widetilde{f}$: $F, x \to \widetilde{C}, y$ of crossed complexes lifting f, and the uniqueness of such a lift is also easy to prove. □

We will use the above result in the following form.

Corollary 2.4 *Let $p\colon \widetilde{C} \to C$ be a covering morphism of crossed complexes, and let F be a connected and simply connected crossed complex. Then the following diagram, in which each ε is an evaluation morphism, is a pullback in the category of crossed complexes:*

$$\begin{array}{ccc} \mathsf{Crs}(F, \widetilde{C}) \times F & \xrightarrow{\varepsilon} & \widetilde{C} \\ {\scriptstyle p_* \times 1}\downarrow & & \downarrow{\scriptstyle p} \\ \mathsf{Crs}(F, C) \times F & \xrightarrow[\varepsilon]{} & C, \end{array}$$

where the sets of morphisms of crossed complexes have the discrete crossed complex structure.

Proof This is simply a restatement of a special case of the existence and uniqueness of liftings of morphisms established in the Proposition. □

Remark 2.5 Because the category Crs is equivalent to that of strict globular ω-groupoids, as shown in [BH81b], the methods of this paper are also relevant to that category; see also [Bro08b]. However we are not able to make use of the globular case, nor even the 2-groupoid case. □

Let C be a crossed complex. We write CrsCov/C for the full subcategory of the slice category Crs/C whose objects are the covering morphisms of C. The following Theorem, which is proved in [BRS99], shows that the classification of covering morphisms of crossed complexes, reduces to that of covering morphisms of groupoids.

Theorem 2.6 *If C is a crossed complex, then the functor* $\pi_1\colon \mathsf{Crs} \to \mathsf{Gpd}$ *induces an equivalence of categories*

$$\pi_1'\colon \mathsf{CrsCov}/C \to \mathsf{GpdCov}/(\pi_1 C).$$

An alternative descriptions of the category GpdCov/G for a groupoid G in terms of actions of G on sets is well known and of course gives the classical theory of covering maps of spaces, see [Bro06, Chapter 10]. Consequently, if the crossed complex C is connected, and $x \in C_0$, then connected covering morphisms of C are determined up to isomorphism by conjugacy classes of subgroups of $\pi_1(C, x)$. In particular, a universal cover $\widetilde{C} \to C$ of a connected crossed complex is constructed up to isomorphism from a base point $x \in C_0$ and the trivial subgroup of $\pi_1(C, x)$.

The monoidal closed structure and many other major properties of crossed complexes are obtained by working through another algebraic category, that of *cubical ω-groupoids* with connections which we abbreviate here to *ω-groupoids*. The category of these, which we write ω-Gpd, is a natural home for these deeper properties. The equivalence with crossed complexes proved in [BH81] is a foundation for this whole project. Indeed the definition of tensor product for ω-groupoids is much easier to deal with than that for crossed complexes, and we find it easier to give a dense subcategory for ω-groupoids than for crossed complexes.

3 Cubical omega-groupoids with connection

We recall from [BH81] that a *cubical ω-groupoid with connection* is in the first instance a cubical set $\{K_n \mid n \geqslant 0\}$, so that it has face maps $\{\partial_i^{\pm}\colon K_n \to K_{n-1} \mid i = 1, \dots, n; n \geqslant 1\}$ and degeneracy maps $\{\varepsilon_i\colon K_n \to K_{n+1} \mid i = 1, \dots, n; n \geqslant 0\}$ satisfying the usual rules. Further there are *connections* $\{\Gamma_i^{\pm}\colon K_n \to K_{n+1} \mid i = 1, \dots, n; n \geqslant 1\}$ which amount to an additional family of 'degeneracies' and which in the case of the singular cubical complex of a space derive from the monoid structures $\max, \min$ on the unit interval $[0, 1]$. Finally there are n groupoid structures $\{\circ_i \mid i = 1, \dots, n\}$, defined on K_n with initial, final and identity maps $\partial_i^-, \partial_i^+, \varepsilon_i$ maps respectively.

The laws satisfied by all these structures are given in several places, such as [AABS02, GM03], and we do not repeat them here. Note that because we are dealing with groupoid operations $\circ_i$ we can set $\Gamma_i = \Gamma_i^-$ so that $\Gamma_i^+ = -_i -_{i+1} \Gamma_i$. In this case the laws were first given in [BH81].

A major example of this structure is constructed from a filtered space X_* as follows. One first forms the cubical set with connections RX_* which in dimension n is the set of filtered maps $I_*^n \to X_*$ where I_*^n is the standard n-cube with its skeletal filtration. Then ρX_* is the quotient of RX_* by the relation of homotopy through filtered maps and relative to the vertices of I^n. It is easy to see that ρX_* inherits the structure of cubical set with connection, and it is proved in [BH81a, Theorem A] that the obvious compositions on RX_* are also inherited by ρX_* to make it what is called the fundamental ω-groupoid ρX_* of the filtered space X_*.

The main result of [BH81] is that the category ω-Gpd is equivalent to the category Crs of crossed complexes, and in [BH81a, Theorem 5.1] it is proved that this equivalence takes ρX_* to ΠX_*.

As said in the Introduction, the free ω-groupoid on a generator c_n of dimension n is written $\mathbb{I}^n$. More generally, the free ω-groupoid on a cubical set K is written $\rho' K$: this is a purely algebraic definition. A major result is that $\rho' K$ is equivalent to $\rho|K|_*$ where $|K|_*$ is the skeletal filtration of the geometric realisation of K and ρ is defined above; so we write both as ρK. This equivalence is proved in [BH81a, Proposition 9.5] for the case $K = \mathsf{I}^n$, and the general case follows by similar methods.

We shall also need the properties of thin elements in an ω–groupoid G. An element t of G_n is called *thin* if it has a decomposition as a multiple composition of elements $\varepsilon_i x, \Gamma_j y$, or their repeated negatives in various directions. Clearly a morphism of ω–groupoids preserves thin elements.

A family B of elements of $\mathbb{I}^n$ is called an $(n-1)$*-box in* $\mathbb{I}^n$ if they form all faces $\partial_i^\pm c_n$ but one of c_n. An element x is called a *filler* of the box if these all-but-one faces $\partial_i^\pm x$ are exactly the elements of B.

Then B generates a sub-ω-groupoid $\bar{B}$ of $\mathbb{I}^n$. The image family $\widehat{b}(B)$ of this by a morphism of ω–groupoids $\widehat{b}\colon \bar{B} \to G$ is called an $(n-1)$-box in G. Again we have the notion of a filler of a box in G. A basic result on ω–groupoids [BH81, Proposition 7.2] is:

Proposition 3.1 (Uniqueness of thin fillers) *A box in an ω–groupoid has a unique thin filler.*

The thin elements in an ω-groupoid satisfy Keith Dakin's axioms, [Dak77]:

D1) a degenerate element is thin;

D2) every box has a unique thin filler;

D3) if all faces but one of a thin element are thin, then so is the remaining face.

These axioms for a thin structure in fact give a structure equivalent to that of an ω–groupoid, as shown in [BH81c]. That is, the connections *and the compositions* are determined by the thin structure: we will use this fact in the proof of Theorem 5.1. The following Lemma is also used there.

Lemma 3.2 *If $t \in G_n$ is a thin element of an ω–groupoid G , then there is a thin element $b_t \in \mathbb{I}^n$ such that $\widehat{t}(b_t) = \widehat{t}(c_n)$.*

Proof Let $\widehat{t} : \mathbb{I}^n \to G$ be the morphism such that $\widehat{t}(c_n) = t$. We can find a box B in $\mathbb{I}^n$ and such that t is a filler of $\widehat{t}|\colon \bar{B} \to G$. This box B in $\mathbb{I}^n$ also has a unique thin filler b_t in $\mathbb{I}^n$. Since $\widehat{t}$ is a morphism of ω-groupoids, it preserves thin elements and so $\widehat{t}(b_t)$ is thin and also a filler of the box B in G. By uniqueness of thin fillers $\widehat{t}(b_t) = t = \widehat{t}(c_n)$. □

Remark 3.3 Thin elements in higher categorical rather than groupoid situations are also used in [Str87, Hig05, Ste06, Ver08]. □

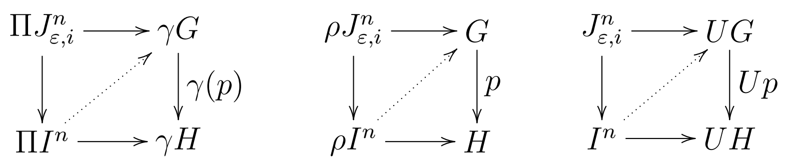
4 Fibrations and coverings of omega-groupoids

We now transfer to cubical ω–groupoids the definition in Section 2 of fibration and covering morphism of crossed complex.

Theorem 4.1 *Let $p\colon G \to H$ be a morphism of ω-Gpds. Then the corresponding morphism of crossed complexes $\gamma(p)\colon \gamma(G) \to \gamma(H)$ is a fibration (covering morphism) if and only if $p\colon G \to H$ is a Kan fibration (covering map) of cubical sets.*

Proof Let $J^n_{\varepsilon,i}$ for $\varepsilon = \pm, i = 1, \ldots, n$, be the subcubical set of the cubical set I^n generated by all faces of I^n except ∂^ε_i.

We consider the following diagrams:



(i) (ii) (iii)

By a simple modification of the simplicial argument in [BH91], we find that the condition that diagrams of the first type have the completion shown by the dotted arrow is necessary and sufficient for γp to be a fibration of crossed complexes (with uniqueness for a covering morphism). In the second diagram, $\rho(K)$ is the free cubical ω-groupoid on the cubical set K, and the equivalence of the first and the second diagram is one of the results of

[BH81a, Section 9]. Finally, the equivalence with the third diagram, in which U gives the underlying cubical set, follows from freeness of ρ. □

Corollary 4.2 *Let $p\colon K \to L$ be a morphism of ω-Gpds such that the underlying map of cubical sets is a Kan fibration. Then the pullback functor*

$$f^*\colon \omega\text{-}\mathsf{Gpd}/L \to \omega\text{-}\mathsf{Gpd}/K$$

has a right adjoint and so preserves colimits.

Proof This is immediate from Theorem 4.1 and the main result of Howie [How79]. □

Corollary 4.3 *A covering crossed complex of a free crossed complex is also free.*

Proof A free crossed complex is given by a sequence of pushouts, analogously to the definition of CW-complexes, see [BH91, BHS11]. □

5 Dense subcategories

Our aim in this section is to explain and prove the theorem:

Theorem 5.1 *The full subcategory $\mathcal{I}$ of ω-Gpd on the objects $\mathbb{I}^n$ is dense in ω-Gpd.*

We recall from [ML71] the definition of a dense subcategory. First, in any category C, a morphism $f\colon C \to D$ induces a natural transformation $f_*\colon \mathsf{C}(-,C) \Rightarrow \mathsf{C}(-,D)$ of functors $\mathsf{C}^{op} \to \mathsf{Set}$. Conversely, any such natural transformation is induced by a (unique) morphism $C \to D$.

If $\mathcal{I}$ is a subcategory of C, then each object C of C gives a functor

$$\mathsf{C}^{|\mathcal{I}}(-,C)\colon \mathcal{I}^{op} \to \mathsf{Set}$$

and a morphism $f\colon C \to D$ of C induces a natural transformation of functors $f_*\colon \mathsf{C}^{|\mathcal{I}}(-,C) \Rightarrow \mathsf{C}^{|\mathcal{I}}(-,D)$. The subcategory $\mathcal{I}$ is *dense* in C if every such natural transformation arises from a morphism. More precisely, there is a functor $\eta\colon \mathsf{C} \to \mathsf{CAT}(\mathcal{I}^{op}, \mathsf{Set})$ defined in the above way, and $\mathcal{I}$ is *dense* in C if η is full and faithful.

Example 5.2 Consider the Yoneda embedding

$$\Upsilon\colon \mathsf{C} \to \mathsf{C}^{op}\text{-}\mathsf{Set} = \mathsf{CAT}(\mathsf{C}^{op}, \mathsf{Set})$$

where C is a small category. Then each object $K \in \mathsf{C}^{op}$-Set is a colimit of objects in the image of Υ and this is conveniently expressed in terms of coends as that the natural morphism

$$\int^{c} (\mathsf{C}^{op}\text{-}\mathsf{Set}(\Upsilon c, K) \times \Upsilon c) \quad \to \quad K$$

is an isomorphism. Thus the Yoneda image of C is dense in C^{op}-Set. For more on the relation between density and the Yoneda Lemma, see [Pra09]. □

Example 5.3 Let $\mathbb{Z}$ be the cyclic group of integers. Then $\{\mathbb{Z}\}$ is a generating set for the category Ab of abelian groups, but the full subcategory of Ab on this set is not dense in Ab. In order for a natural transformation to specify not just a function $f\colon A \to B$ but a morphism in Ab, we have to enlarge this to a full subcategory including $\mathbb{Z} \oplus \mathbb{Z}$. □

Proof of Theorem 5.1 We will use the main result of [BH81c], that the compositions in a cubical ω-groupoid are determined by its thin elements.

Let G, H be ω–groupoids and let $\mathsf{f}\colon \omega\text{-}\mathsf{Gpd}_{\mathcal{I}}(-,G) \to \omega\text{-}\mathsf{Gpd}_{\mathcal{I}}(-,H)$ be a natural transformation. We define $f\colon G \to H$ as follows.

Let $x \in G_n$. Then x defines $\widehat{x}\colon \mathbb{I}^n \to G$. We set $f(x) = \mathsf{f}(\widehat{x})(c_n) \in H_n$. We have to prove f preserves all the structure.

For example, we prove that $f(\partial_i^{\pm} x) = \partial_i^{\pm} f(x)$. Let $\bar{\partial}_i^{\pm}\colon \mathbb{I}^{n-1} \to \mathbb{I}^n$ be given by having value $\partial_i^{\pm} c_n$ on c_{n-1}. The natural transformation condition implies that $\mathsf{f}(\bar{\partial}_i^{\pm})^* = (\bar{\partial}_i^{\pm})^*\mathsf{f}$. On evaluating this on $\widehat{x}$ we obtain

$f(\partial_i^{\pm} x) = \partial_i^{\pm} f(x)$ as required. In a similar way, we prove that f preserves the operations ε_i, Γ_i.

Now suppose that $t \in G_n$ is thin in G. We prove that $f(t)$ is thin in H. By Lemma 3.2, there is a thin element $b_t \in \mathbb{I}^n$ such that $\widehat{t}(b_t) = t$. Let $\bar{b} \colon \mathbb{I}^n \to \mathbb{I}^n$ be the unique morphism such that $\bar{b}(c_n) = b_t$. Then the natural transformation condition implies $f(t) = \mathsf{f}(\widehat{t})(c_n) = \mathsf{f}(\widehat{t})(b_t)$. Since b_t is thin, it follows that $f(t)$ is thin. Thus f preserves the thin structure.

The main result of [BH81c] now implies that the operations $\circ_i$ are preserved by f. □

We can also conveniently represent each ω–groupoid as a coend.

Corollary 5.4 *The subcategory $\mathcal{I}$ of ω-*Gpd *is dense and for each object G of ω-*Gpd *the natural morphism*

$$\int^n \omega\text{-}\mathsf{Gpd}(\mathbb{I}^n, G) \times \mathbb{I}^n \to G$$

is an isomorphism.

Proof This is a standard consequence of the property of $\mathcal{I}$ being dense. □

Corollary 5.5 *The full subcategory of* Crs *generated by the objects ΠI_*^n is dense in* Crs.

Proof This follows from the fact that the equivalence $\gamma \colon \omega\text{-}\mathsf{Gpd} \to \mathsf{Crs}$ takes $\mathbb{I}^n$ to ΠI_*^n, [BH81a, Theorem 5.1]. □

Remark 5.6 The paper [BH81b] gives an equivalence between the category Crs of crossed complexes and the category there called ∞-groupoids and now commonly called globular ω-groupoids. Thus the above Corollary yields also a dense subcategory, based on models of cubes, in the latter category. □

Remark 5.7 It is easy to find a generating set of objects for the category Crs, namely the free crossed complexes on single elements, given in fact by

ΠE^n_*, where E^n_* is the usual cell decomposition of the unit ball, with one cell for $n = 0$ and otherwise three cells. It is not so obvious how to construct directly from this generating set a dense subcategory closed under tensor products. $\Box$

6 The tensor product of covering morphisms

Our aim is to prove the following:

Theorem 6.1 *The tensor product of two covering morphisms of crossed complexes is a covering morphism.*

Remark 6.2 The reason why we have to give an indirect proof of this result is that the definition of covering morphism involves *elements* of crossed complexes; but it is difficult to specify exactly the elements of a tensor product whose definition is perforce by generators and relations. $\Box$

It is sufficient to assume that all the crossed complexes involved are connected. We will also work in the category of ω–groupoids, and prove the following:

Theorem 6.3 *Let G, H be connected ω–groupoids with base points x, y respectively, and let $p : \widetilde{G} \to G$ be the covering morphism determined by the subgroup M of $\pi_1(G, x)$. Let $\phi : C \to G \otimes H$ be the covering morphism determined by the subgroup $M \times \pi_1(H, y)$ of*

$$\pi_1(G \otimes H, (x, y)) \cong \pi_1(G, x) \times \pi_1(H, y).$$

Then there is an isomorphism $\psi : C \to \widetilde{G} \otimes H$ such that $(p \otimes 1_H)\psi = \phi$, and, consequently,

$$p \otimes 1_H : \widetilde{G} \otimes H \to G \otimes H$$

is a covering morphism.

Proof Here we were inspired by the formulae of Brian Day [Day70].

First we know from [BH87] that the tensor product of ω-Gpds satisfies $\mathbb{I}^m \otimes \mathbb{I}^n \cong \mathbb{I}^{m+n}$, showing that $\mathcal{I}$ is a full monoidal subcategory of ω-Gpds.

Since also from [BH87] the tensor preserves colimits in each variable, it follows from Corollary 5.4 that the tensor product $G \otimes H$ of ω-groupoids G and H satisfies

$$G \otimes H \cong \int^{m,n} \omega\text{-}\mathsf{Gpd}(\mathbb{I}^m, G) \times \omega\text{-}\mathsf{Gpd}(\mathbb{I}^n, H) \times (\mathbb{I}^m \otimes \mathbb{I}^n). \tag{3}$$

Let $p\colon \widetilde{G} \to G$ be the covering morphism determined by the subgroup M and let $\phi\colon C \to G \otimes H$ be the covering morphism determined by the subgroup $M \times \pi_1(H, y)$ of

$$\pi_1(G, x) \times \pi_1(H, y) \cong \pi_1(G \otimes H, (x, y)).$$

By Corollary 4.2, pullback ϕ^* by ϕ preserves colimits. Hence

$$\begin{aligned} C &\cong \phi^* \left(\int^{m,n} \omega\text{-}\mathsf{Gpd}(\mathbb{I}^m, G) \times \omega\text{-}\mathsf{Gpd}(\mathbb{I}^n, H) \times (\mathbb{I}^m \otimes \mathbb{I}^n) \right) \\ &\cong \int^{m,n} \phi^*(\omega\text{-}\mathsf{Gpd}(\mathbb{I}^m, G) \times \omega\text{-}\mathsf{Gpd}(\mathbb{I}^n, H)) \times (\mathbb{I}^m \otimes \mathbb{I}^n) \end{aligned}$$

and so because of the construction of C by the specified subgroup:

$$\begin{aligned} &\cong \int^{m,n} \omega\text{-}\mathsf{Gpd}(\mathbb{I}^m, \widetilde{G}) \times \omega\text{-}\mathsf{Gpd}(\mathbb{I}^m, H) \times (\mathbb{I}^m \otimes \mathbb{I}^n) \\ &\cong \widetilde{G} \otimes H. \end{aligned}$$ □

Corollary 6.4 *The tensor product of covering morphisms of ω-groupoids is again a covering morphism.*

Proof Because tensor product commutes with disjoint union, it is sufficient to restrict to the connected case. Since the composition of covering morphisms is again a covering morphism, it is sufficient to restrict to the case of $p \otimes 1_H$, and that is proved in Theorem 6.3. □

The proof of Theorem 6.1 follows immediately.

Corollary 6.5 *If F, F' are free and aspherical crossed complexes, then so also is $F \otimes F'$.*

Proof It is sufficient to assume F, F' are connected. Since F, F' are aspherical, their universal covers $\widetilde{F}, \widetilde{F'}$ are acyclic. Since they are also free, they are contractible, by a Whitehead type theorem, [BG89, Theorem 3.2]. But the tensor product of free crossed complexes is free, by [BH91, Cor. 5.2]. Therefore $\widetilde{F} \otimes \widetilde{F'}$ is contractible, and hence acyclic. Therefore $F \otimes F'$ is aspherical. $\square$

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CATEGORIES AS MONOIDS IN *Span, Rel* AND *Sup*

by Toby KENNEY and Robert PARE

Résumé. Nous étudions les représentations de petites catégories comme les monoïdes dans trois bicatégoies monoïdales, étroitement liées. Les catégories peuvent être exprimées comme certains types de monoïdes dans la catégorie $\mathcal{S}pan$. En fait, ces monoïdes sont aussi dans $\mathcal{R}el$. Il y a une équivalence bien connue, entre $\mathcal{R}el$ et une sous-catégorie pleine de la catégorie des treillis complets et des morphismes qui préservent les sups. Cela nous permet de représenter une catégorie comme un monoïde dans $\mathcal{S}up$. Les monoïdes dans $\mathcal{S}up$ s'appellent des quantales, et sont intéressants dans plusieurs domaines. Nous étudions aussi dans ce contexte la représentation d'autres structures catégoriques, par exemple, les foncteurs, les transformations naturelles, et les profoncteurs.

Abstract. We study the representation of small categories as monoids in three closely related monoidal bicategories. Categories can be expressed as special types of monoids in the category $\mathcal{S}pan$. In fact, these monoids also live in $\mathcal{R}el$. There is a well-known equivalence between $\mathcal{R}el$, and a full subcategory of the category $\mathcal{S}up$, of complete lattices and sup-preserving morphisms. This allows us to represent categories as a special kind of monoid in $\mathcal{S}up$. Monoids in $\mathcal{S}up$ are called quantales, and are of interest in a number of different areas. We will also study the appropriate ways to express other categorical structures such as functors, natural transformations and profunctors in these categories.

Keywords. Category, Monoid, Span, Quantale

Mathematics Subject Classification (2010). 18B10, 18D35, 06F07

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1. Introduction

This research was originally conceived as an attempt to understand the following natural construction:

From a category C, we can form a quantale QC as follows:

– Elements of QC are sets of morphisms in C.
– The product of elements A and B in QC is the set $\{ab|a \in A, b \in B\}$ of composites.
– Join is union.

Examples 1.1.

1. When C is the indiscrete category on a set X, this is the quantale of all relations on X.
2. When C is a group, G, viewed as a 1-element category, this quantale is the quantale of all subsets of G.

These two examples are of interest because they give a deeper understanding of the well-known connection between equivalence relations on a set, and subgroups of a group. In both cases, these can be viewed as symmetric idempotent elements above 1 in their respective quantales.

This construction has also been studied in more detail in the case of étale groupoids by Resende [6]. In this case, instead of all sets of morphisms, he takes only open sets. Because he is considering only groupoids, the quantale is in addition involutive.

The question arises: which quantales occur in this way? We will answer this question indirectly by firstly producing a correspondance between categories and certain monoids in $\mathcal{R}el$. Using this, we will be able to describe which quantales correspond to categories, using a well-known equivalence:

Proposition 1.2. *The category of sets and relations is equivalent to the category of complete atomic Boolean algebras and sup-morphisms.*

Proof. On objects, there is a well-known correspondance between power sets and complete atomic Boolean algebras. We need to show that the direct image of a relation is a sup-morphism, and that every sup-morphism is the direct image of a relation. This is straightforward to check □

The correspondance with certain monoids in $\mathcal{R}el$, or in $\mathcal{S}pan$, has the additional advantage of holding for internal categories in other categories. However, there is not such a correspondence between internal relations in a category and quantales in that category, so for example, describing topological categories as quantales would need a different approach.

2. Monoids in $\mathcal{S}pan$ and $\mathcal{R}el$

We begin by listing some basic properties of general monoids in $\mathcal{S}pan$ and in $\mathcal{R}el$, and the relation between the two. These properties will be of interest later when we are studying the particular monoids in $\mathcal{S}pan$ and $\mathcal{R}el$ that correspond to categories.

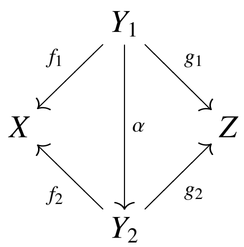
2.1 Preliminaries

To start with, we will clarify exactly what we mean by monoids in $\mathcal{S}pan$, since this could be interpreted in several different ways. Firstly, we will view $\mathcal{S}pan$ as a bicategory in the following way:

Objects Sets X

Morphisms Spans $X \xleftarrow{f} Y \xrightarrow{g} Z$ in **Set**.

2-cells Commutative diagrams:



in **Set**.

This is furthermore, a monoidal bicategory, with tensor product $\otimes$ given by cartesian product of sets, and the obvious tensor products of morphisms.

Of course, we can extend all of this to spans in an arbitrary category C, with pullbacks and products, and all the results we present are equally valid for this context. However, the equivalence between $\mathcal{R}el$ and $\mathbf{CABA}_{sup}$ is specific to **Set**, so the results about quantales cannot be applied to internal categories in a category C.

Now, by a monoid in $\mathcal{S}pan$, we really mean a pseudomonoid with respect to the tensor product (rather than the categorical product, which is disjoint union in $\mathcal{S}pan$) – i.e. a diagram $C \otimes C \xrightarrow{M} C \xleftarrow{I} 1$ of morphisms in $\mathcal{S}pan$, with the associativity and unit laws for a monoid commuting only up to isomorphism, with these isomorphisms satisfying the usual coherence axioms. The first reference for a monoid with respect to the tensor product of a tensor category appears to be [1].

Besides being the natural choice for the definition of monoid in $\mathcal{S}pan$, this definition also makes sense when we pass to the category of relations, because when we view relations as jointly monic spans, if two relations are isomorphic as spans, then the isomorphism is unique – indeed if two spans are isomorphic, and one is a relation, then the other is also a relation, and the isomorphism is unique. It will therefore be clear for the monoids which correspond to categories, that the isomorphisms present in the monoid axioms are unique, and therefore satisfy coherence conditions.

To save rewriting the same thing many times, we will begin by fixing our usual notation for monoids, in $\mathcal{S}pan$, $\mathcal{R}el$, or $\mathbf{CABA}_{sup}$. We will then use this notation without restating it each time.

We will denote monoids in these categories by $C \otimes C \xrightarrow{M} C \xleftarrow{I} 1$ and $D \otimes D \xrightarrow{N} D \xleftarrow{J} 1$. In the case of $\mathcal{S}pan$, we will furthermore use the name of a span to denote the set that is the domain of both morphisms of the span. For instance the span $C \otimes C \xrightarrow{M} C$ will denote the span $C \times C \xleftarrow{m} M \xrightarrow{m'} C$.

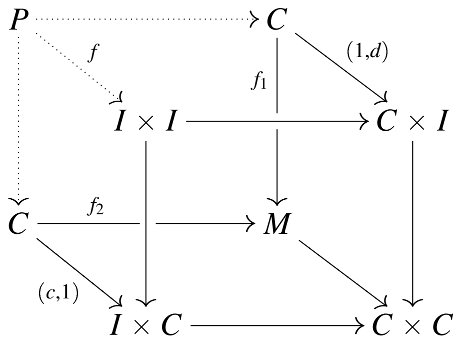
2.2 Monoids in $\mathcal{S}pan$

Proposition 2.1. *In any monoid in $\mathcal{S}pan$, the opposite of the unit is a partial function.*

Proof. By the unit laws, we get that there are pullbacks:

$$\begin{array}{ccc} C & \xrightarrow{f_1} & M \\ {\scriptstyle (1,d)}\downarrow & & \downarrow \\ C \times I & \longrightarrow & C \times C \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C & \xrightarrow{f_2} & M \\ {\scriptstyle (c,1)}\downarrow & & \downarrow \\ I \times C & \longrightarrow & C \times C \end{array}$$

for some choices of functions, $C \xrightarrow{d} I$, $C \xrightarrow{f_1} M$, $C \xrightarrow{c} I$ and $C \xrightarrow{f_2} M$ satisfying $C \xrightarrow{f_i} M \longrightarrow C$ is the identity for $i = 1, 2$. From this, in the following diagram, where the back square is a pullback, and the morphism f is the unique factorisation through the front pullback:



The front is clearly a pullback and the right and bottom squares are pullbacks by the unit laws. Therefore, by a standard argument, the top and left-hand squares are also pullbacks. We start by showing that P is isomorphic to I. In the following diagram:

$$\begin{array}{ccccc} P & \xrightarrow{f} & I \times I & \xrightarrow{\pi_1} & I \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & C \times I & \xrightarrow[\pi_1]{} & C \end{array}$$

where the left-hand square is a pullback, we know that the right-hand square is a pullback, and the bottom composite is the identity, so the whole rectangle is a pullback, and the top composite is an isomorphism, and so $P \cong I$. Thus, for the morphism f in the above cube, $\pi_1 f$ must be an isomorphism.

This means that the induced morphism $C \xrightarrow{d} I$ is a splitting (up to isomorphism) of the morphism $I \longrightarrow C$, which is therefore monic. $\square$

If (C, M, I) is a monoid in $\mathcal{S}pan$, then we have functions $M \xrightarrow{m} C \times C$ and $M \xrightarrow{m'} C$. Using these three functions from M to C, we partition M as a disjoint union of sets M_c^{ab} for $a, b, c \in C$, where

$$M_c^{ab} = \{x \in M | m(x) = (a, b), m'(x) = c\}$$

We can think of the elements of M_c^{ab} as witnesses to the fact that $ab = c$. We can then extend this to witnesses of more complicated expressions, so we can allow $M_d^{(ab)c}$ to be the set of witnesses to the fact that $(ab)c = d$. By composition in $\mathcal{S}pan$, we know that $M_d^{(ab)c} = \sum_{v \in C} M_v^{ab} \times M_d^{vc}$. We can let M' be the union of all these $M_d^{(ab)c}$. Now M' is given by the pullback

$$\begin{array}{ccc} M' & \xrightarrow{l} & M \\ {\scriptstyle k}\downarrow & & \downarrow \\ M \times C & \longrightarrow & C \times C \end{array}$$

in **Set**. By associativity, M' is also given by the following pullback in **Set**:

$$\begin{array}{ccc} M' & \xrightarrow{r} & M \\ {\scriptstyle s}\downarrow & & \downarrow \\ C \times M & \longrightarrow & C \times C \end{array}$$

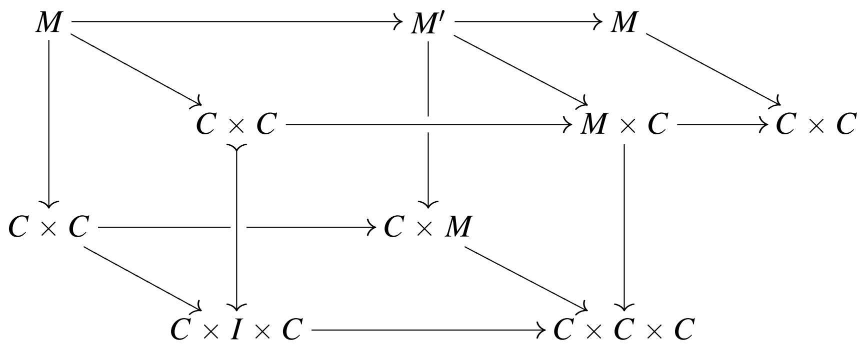
When we are trying to consider categories as monoids in $\mathcal{S}pan$, we will want the condition that $(ab)c$ (or equivalently $a(bc)$) should exist if and only if both the composites ab and bc exist, or equivalently, the following diagram in **Set**:

$$\begin{array}{ccc} M' & \xrightarrow{k} & M \times C \\ {\scriptstyle s}\downarrow & & \downarrow \\ C \times M & \longrightarrow & C \times C \times C \end{array}$$

is also a pullback. We will call a monoid in $\mathcal{S}pan$ that satisfies this condition *categorical*.

Proposition 2.2. *The multiplication of any categorical monoid in $\mathcal{S}pan$ is a partial morphism.*

Proof. Consider the pullback squares:



The front and bottom squares are pullbacks by the unit laws. The right square in the cube is the pullback in the definition of a categorical monoid. The top right square is the pullback in the definition of M'. Also by the unit law, the top front composite $C \times C \longrightarrow M \times C \longrightarrow C \times C$ is the identity, so we know that the top left arrow is isomorphic to the morphism $M \longrightarrow C \times C$. However, the front left morphism is monic, since it is split by the projection. Therefore $M \longrightarrow C \times C$ is also monic. □

Proposition 2.3. *If M is a partial function, then the following are equivalent:*

1. *The monoid is categorical*
2. *There is a (necessarily unique) 2-cell:*

$$\begin{array}{ccc} C \otimes C & \xrightarrow{\ M\ } & C \\ {\scriptstyle (M\otimes C)^{op}}\downarrow & \nearrow & \downarrow{\scriptstyle M^{op}} \\ C \otimes C \otimes C & \xrightarrow[C\otimes M]{} & C \times C \end{array}$$

in Span.

Proof. Firstly suppose the monoid is categorical. Now in the diagram in **Set**:

$$\begin{array}{ccccc} C\times C\times C & \longleftarrow & C\times M & \longrightarrow & C\times C \\ \uparrow & & \uparrow & & \uparrow \\ M\times C & \longleftarrow & M' & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ C\times C & \longleftarrow & M & \longrightarrow & C \end{array}$$

all except the lower right square are pullbacks. Because the lower right square commutes, it must factor through the pullback:

$$\begin{array}{ccc} N & \longrightarrow & M \\ \downarrow & & \downarrow \\ M & \longrightarrow & C \end{array}$$

This factorisation is exactly the 2-cell we require.

Conversely, suppose that the 2-cell exists. Now in the diagram:

$$\begin{array}{ccccc} C\times C\times C & \longleftarrow & C\times M & \longrightarrow & C\times C \\ \uparrow & & \uparrow{\scriptstyle a} & & \uparrow \\ M\times C & \underset{d}{\longleftarrow} & M' & \overset{b}{\longrightarrow} & M \\ \downarrow & & \downarrow{\scriptstyle c} & & \downarrow \\ C\times C & \longleftarrow & M & \longrightarrow & C \end{array}$$

The top right and bottom left squares are pullbacks by associativity, and the top left square must factor through the pullback:

$$\begin{array}{ccc} N & \overset{w}{\longrightarrow} & C\times M \\ {\scriptstyle z}\downarrow & & \downarrow \\ M\times C & \longrightarrow & C\times C\times C \end{array}$$

We will denote this factorisation $M' \xrightarrow{f} N$.

On the other hand, because of the 2-cell, we get a commutative diagram:

$$\begin{array}{ccccc}
C \times C \times C & \longleftarrow & C \times M & \longrightarrow & C \times C \\
\uparrow & & \uparrow{\scriptstyle w} & & \uparrow \\
M \times C & \xleftarrow[z]{} & N & \xrightarrow{x} & M \\
\downarrow & & \downarrow{\scriptstyle y} & & \downarrow \\
C \times C & \longleftarrow & M & \longrightarrow & C
\end{array}$$

The top right and bottom left squares of this diagram have M' as the pullback. We will denote the factorisation of the bottom left square through the pullback by $N \xrightarrow{g} M'$. Now we have that $dgf = zf = d$. Since $M \longrightarrow C \times C$ is monic, the pullback d is also monic, so we have that $gf = 1_{M'}$. On the other hand, we also have that $zfg = dg = z$, and z is a pullback of $C \times M \longrightarrow C \times C \times C$, which is monic. Therefore, z is also monic, showing that f and g are inverses, yielding an isomorphism between N and M'. It is straightforward to check that this extends to an isomorphism between the labelled morphisms in the diagrams. □

2.3 Premonoidal Structures

The study of monoids in *Span* is of some interest as they correspond to Day's premonoidal structures on discrete categories [2]. As such, they correspond to monoidal closed structures on products of the category of sets

$$\prod_C \mathbf{Set} = \mathbf{Set}^{\mathbf{C}} \simeq \mathbf{Set}/\mathbf{C}$$

If C is a set, then a premonoidal structure on C, considered as a discrete category with values in **Set**, consists of:

(1) a triplely indexed family of sets $\langle M_c^{ab} \rangle_{a,b,c \in C}$;

(2) a singlely indexed family of sets $\langle I_a \rangle_{a \in C}$;

(3) isomorphisms $\alpha_d^{abc} : \sum_{x \in C} M_x^{ab} \times M_d^{xc} \to \sum_{x \in C} M_x^{bc} \times M_d^{ax}$;

(4) isomorphisms $\lambda_b^a : \sum_{x \in C} I_x \times M_b^{xa} \cong \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$

(5) isomorphisms $\rho_b^a : \sum_{x \in C} I_x \times M_b^{ax} \cong \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$

satisfying the well-known coherence conditions.

If we combine all the M_c^{ab} into a single set $M = \sum_{a,b,c} M_c^{ab}$ together with indexing functions $C \times C \leftarrow M \to C$, we have a morphism $M : C \times C \to C$ in $\mathcal{S}pan$. Similarly $I = \sum_a I_a$ with $I \to C$ gives a morphism $I : 1 \to C$ in $\mathcal{S}pan$. The isomorphisms α_d^{abc}, λ_b^a, ρ_b^a fit together into isomorphisms α, λ, ρ expressing the fact that (C, M, I) is a monoid in $\mathcal{S}pan$. Thus we have an equivalence between monoids in $\mathcal{S}pan$ and discrete premonoidal structures.

We can recast the proof of Proposition 2.1 in this setting. The conclusion says that for every $a \in C$, $I_a = 0$ or 1. Suppose then that some I_c has more than one element. The isomorphisms λ_b^a imply that $M_b^{ca} = 0$ for all a and b. Then $\rho_c^c : \sum I_x \times M_c^{cx} \to 1$ is not an isomorphism as the domain is 0.

On the other hand, Day's theory of convolution products tells us that premonoidal structures on C correspond to monoidal closed structures on $\mathbf{Set}^{\mathbf{C}}$. Given two families $\langle X_a \rangle$, $\langle Y_a \rangle$ in $\mathbf{Set}^{\mathbf{C}}$,

$$\langle X_a \rangle_a \otimes \langle Y_a \rangle_a = \langle \sum_{x,y \in C} X_x \times Y_y \times M_a^{xy} \rangle_a$$

is a monoidal closed structure if and only if M_c^{ab} is part of a premonoidal structure on C. Furthermore, all monoidal closed structures on $\mathbf{Set}^{\mathbf{C}}$ arise this way.

2.4 Example – Monoid Structures on 0, 1 and 2 in *Span*

Using the results from the previous sections, we list all the monoid structures on sets with at most 2 elements, in $\mathcal{S}pan$. When looking at examples, the notation from the previous section will be useful i.e., we express M as the disjoint union of M_c^{ab}, for elements a, b, c in C, and I as a disjoint union of I_a for elements a in C. By Proposition 2.1, we know that each I_a has at most one element.

Lemma 2.4. *If all $I_a = 1$ (i.e. if $I = C$), then*

$$M_c^{ab} = \begin{cases} 1 & \text{if } a = b = c \\ 0 & \text{otherwise} \end{cases}$$

The corresponding monoidal structure in $\mathbf{Set}^{\mathbf{C}}$ is the cartesian one.

Proof. If $b \neq c$, $\lambda_c^b : \sum_x M_c^{xb} \cong 0$, so $M_c^{ab} = 0$, for all a. If $a \neq c$, then $\rho_c^a : \sum M_c^{ax} \cong 0$ so $M_c^{ab} = 0$ for all b. Thus the only non zero Ms are the M_a^{aa} and so λ_a^a gives that $M_a^{aa} = 1$.

The claim that the monoidal structure on $\mathbf{Set}^{\mathbf{C}}$ is cartesian is clear from the definition of the corresponding monoidal structure. □

We will call the monoid in the above lemma, the *discrete* monoid on C. It is the monoid which comes from a discrete category, when applying the construction of Section 3. On the other hand:

Lemma 2.5. *If $I = 0$, then also $C = 0$.*

Proof. If I were 0, then the composite $M(I \otimes C)$ would also be 0. However, by the unit law, it is isomorphic to the identity span on C. □

Example 2.6. The empty set admits a unique monoid structure, given by $I = M = 0$, since the empty set is strict initial, it is clear that this is the only possible monoid structure. It is easy to check that it is indeed a monoid structure. This monoid structure corresponds to the unique monoidal closed structure on the category $\mathbf{Set}^{\mathbf{0}} \simeq \mathbf{1}$.

Example 2.7. The set $1 = \{0\}$ admits a unique monoid structure. Indeed the unique I_0 can't be 0 so $I_0 = 1$, so Lemma 2.4 applies, and the monoid structure has M as the diagonal subset of 1×1.

Example 2.8. For the two-element set $\{0, 1\}$, there are multiple possible monoid structures in $\mathcal{S}pan$. By Proposition 2.1, we know that I must be a subset of $\{0, 1\}$. It is therefore, up to isomorphism, either a singleton (w.l.o.g. $\{0\}$) or the whole of $\{0, 1\}$ (it can't be empty, by Lemma 2.5). In the latter case, Lemma 2.4 tells us exactly what the monoid has to be.

Now suppose that the unit is the singleton $\{0\}$. By the unit laws, we know that:

$$\begin{array}{lll} M_0^{00} = 1 & M_0^{01} = 0 & M_0^{10} = 0 \\ M_1^{00} = 0 & M_1^{01} = 1 & M_1^{10} = 1 \end{array}$$

So the only choices we have are M_0^{11} and M_1^{11}. We will show that

Proposition 2.9. *For any two sets A and B, setting* $M_0^{11} = A$ *and* $M_1^{11} = B$ *gives a monoid.*

Proof. The unit laws are easily checked – they clearly don't involve M_0^{11} and M_1^{11}, so the fact that $A = 1, B = 0$ gives a monoid (in **Set**, and therefore also in $\mathcal{S}pan$) means that they hold.

For associativity, the composite $M(M \otimes 1)$ is given by the pullback:

$$\begin{array}{ccc} M' & \longrightarrow & M \times C \\ \downarrow & & \downarrow{\scriptstyle c\times 1} \\ M & \longrightarrow & C \times C \end{array}$$

We can partition M' into 16 sets $M_l^{(ij)k}$ for $i, j, k, l \in \{0, 1\}$. We can explicitly calculate the $M_l^{(ij)k}$ as follows:

(i,j) / (k,l)	00	01	10	11
00	1	0	0	A
01	0	1	1	B
10	0	A	A	AB
11	1	B	B	$A + B^2$

It is easy to check that this is also isomorphic to the composite $M(1\otimes M)$, so that the monoid is indeed associative for any A and B. Furthermore, the isomorphism is canonical – we can just take the identity function on each (i, j, k, l). It is clear that this satisfies the required coherence conditions. □

The induced monoidal structure on **Set** × **Set** is the following:

$$(X_0, X_1) \otimes (Y_0, Y_1) = (X_0Y_0 + AX_1Y_1, X_0Y_1 + X_1Y_0 + BX_1Y_1)$$

(where we have used juxtaposition to denote product). The internal hom for the tensor product on **Set** × **Set** is given by

$$(Y_0, Y_1)^{(X_0,X_1)} = ({Y_0}^{X_0}{Y_1}^{X_1}, {Y_0}^{AX_1}{Y_1}^{X_0}{Y_1}^{BX_1})$$

Remark 2.10. Note that we have determined all the monoid structures on 2 in *Span*, as far as the identity and multiplication. However, we have not shown that the isomorphisms are unique. If there are any non obvious ways of defining the α, they will give a non-symmetric tensor product on **Set** $\times$ **Set**.

For our treatment of categories as monoids in *Span*, the choice of isomorphism for the pseudomonoids will be unique, because the multiplication and unit of monoids corresponding to categories are relations, so there will be a unique isomorphism, and so it will obviously satisfy the coherence conditions. Therefore, we will not have to worry about coherence conditions in that section.

Remark 2.11. In the introduction, we said that we are most interested in the monoids in *Span* that come from categories, using the construction we will give in Section 3. There are three categories with exactly two morphisms – the discrete two-object category, and two monoid structures. We already said that the discrete category corresponds to the discrete monoid structure on 2. The two monoids correspond to the cases $A = 0, B = 1$, and $A = 1, B = 0$, above. This is not surprising, because in these cases, we see that the multiplication for the monoid in *Span* actually becomes a function, and the unit is already a function, because I is a one-element set, so these monoids in *Span* actually live in the subcategory **Set**.

2.5 Monoids in *Rel*

There is a morphism of monoidal bicategories from *Span* to *Rel*, sending sets to themselves, and sending a span $A \xleftarrow{l} S \xrightarrow{r} B$ to it's underlying relation – i.e. the relation that relates an element a of A to an element b of B if and only if there is at least one element $s_{a,b}$ of S satisfying $l(s_{a,b}) = a$ and $r(s_{a,b}) = b$. This functor preserves monoids, so from a monoid in *Span*, we get a monoid in *Rel*.

In the other direction, we can view a relation as a jointly monic span. However, this is merely an oplax morphism, because the composite of two jointly monic spans need not necessarily be jointly monic. Therefore, not all monoids in *Rel* are monoids in *Span*. Being a monoid in *Span* imposes additional equations on a monoid in *Rel*. We will call a monoid in *Rel* which can be viewed as a monoid in *Span*, by sending the multiplication and unit to the corresponding jointly monic spans, *spanish*.

Example 2.12. There is a monoid in $\mathcal{R}el$, on the 4-element set $\{e, x, y, z\}$, where e is the unique identity, and multiplication is given by the following table. (The sets in the table are the collection of all elements related to the pair given by the row and the column.)

	e	x	y	z
e	$\{e\}$	$\{x\}$	$\{y\}$	$\{z\}$
x	$\{x\}$	$\{y, z\}$	$\{x, z\}$	$\{x, y\}$
y	$\{y\}$	$\{x, z\}$	$\{x, y\}$	$\{y, z\}$
z	$\{z\}$	$\{x, y\}$	$\{y, z\}$	$\{x, y, z\}$

It is straightforward to check that this is indeed associative in $\mathcal{R}el$, and so a monoid. However, it is not a monoid in $\mathcal{S}pan$, since for example,

$$M_y^{(xy)z} = \sum_{w \in \{e,x,y,z\}} M_w^{xy} \times M_y^{wz} = 2$$

by taking the values $w = x$ and $w = z$. However, on the other hand,

$$M_y^{x(yz)} = \sum_{v \in \{e,x,y,z\}} M_y^{xv} \times M_v^{yz} = 1$$

with the only non-zero value when $v = z$.

In this section, we will show that monoids in $\mathcal{R}el$ that satisfy that the multiplication is a partial morphism are spanish.

Lemma 2.13. *If* $1 \xrightarrow{I} C \xleftarrow{M} C \otimes C$ *is a monoid in* $\mathcal{R}el$*, then*

$$\begin{array}{ccc} I \otimes I & \longrightarrow & C \otimes C \\ {\scriptstyle \Delta^{\mathrm{op}}} \downarrow & & \downarrow {\scriptstyle M} \\ I & \longrightarrow & C \end{array}$$

commutes in $\mathcal{R}el$*, where* I *represents the subset of all elements in* C *that are related to the unique element of 1, and* $I \xrightarrow{\Delta} I \times I$ *is the diagonal function, viewed as a relation.*

Proof. From the unit law:

$$\begin{array}{ccccc} C & \longleftarrow & C\otimes I & \longrightarrow & C\otimes C \\ & \searrow^{1_C} & & & \downarrow{\scriptstyle M} \\ & & & & C \end{array}$$

we see that the relational composite $C \otimes I \longrightarrow C \otimes C \xrightarrow{M} C$ is less than or equal to the first projection $C \otimes I \xrightarrow{\pi_1} C$. Similarly, from the other unit law, we see that the composite $I \otimes C \longrightarrow C \otimes C \xrightarrow{M} C$ is less than or equal to the second projection $I \otimes C \xrightarrow{\pi_2} C$. Restricting to the subset $I \otimes I$, we get that the composite $I \otimes I \longrightarrow C \otimes C \xrightarrow{M} C$ is less than or equal to both projections $I \otimes I \xrightarrow{\pi_1} I \longrightarrow C$ and $I \otimes I \xrightarrow{\pi_2} I \longrightarrow C$. The intersection of these projections is $I \otimes I \xrightarrow{\Delta^{\mathrm{op}}} I \longrightarrow C$, so one inclusion in the square is proved.

Since $I \longrightarrow C$ is a function, the inequality we have proved means that we have a commutative square

$$\begin{array}{ccc} I\otimes I & \longrightarrow & C\otimes C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle M} \\ I & \longrightarrow & C \end{array}$$

for some relation $f \leqslant \Delta^{\mathrm{op}}$. We want to show that $f = \Delta^{\mathrm{op}}$. By the unit law, the composite $I \xrightarrow{\pi_1{}^{\mathrm{op}}} I \otimes I \longrightarrow C \otimes C \xrightarrow{M} C$ is the inclusion $I \longrightarrow C$. Therefore, we know that

$$I \xrightarrow{\pi_1{}^{\mathrm{op}}} I \otimes I \xrightarrow{f} I \longrightarrow C = I \xrightarrow{\pi_1{}^{\mathrm{op}}} I \otimes I \xrightarrow{\Delta^{\mathrm{op}}} I \longrightarrow C$$

Since $I \longrightarrow C$ is monic, this gives that $f\pi_1{}^{\mathrm{op}} = 1_I$, and $f \leqslant \Delta^{\mathrm{op}}$. It is easy to see that the only solution to this is $f = \Delta^{\mathrm{op}}$, giving the required commutativity. □

In the case where the multiplication M is a partial function, the commutative diagram in the above proposition lives entirely within the bicategory

$\mathcal{P}art$, of sets, partial functions, and inclusions of graphs of partial functions. However, $\mathcal{P}art$ is a subcategory of both $\mathcal{R}el$ and $\mathcal{S}pan$, so we see that the lifting of the above diagram to $\mathcal{S}pan$ also commutes.

When the multiplication is a partial function, then the associativity square lives in the subcategory $\mathcal{P}art$, so it lifts to a commutative square in $\mathcal{S}pan$. Therefore, to show that a monoid in $\mathcal{R}el$, with a partial multiplication is spanish, it is sufficient to show that the unit laws lift to commutative diagrams in $\mathcal{S}pan$, or equivalently that there are morphisms $C \xrightarrow{d} I$ and $C \xrightarrow{c} I$ such that

$$\begin{array}{ccc} C & \xrightarrow{(1,d)} & C \times I \\ \downarrow & & \downarrow \\ M & \rightarrowtail & C \times C \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C & \xrightarrow{(c,1)} & I \times C \\ \downarrow & & \downarrow \\ M & \rightarrowtail & C \times C \end{array}$$

are pullbacks in **Set**.

Proposition 2.14. *If a monoid in $\mathcal{R}el$ has a partial multiplication, then it is spanish.*

Proof. Let

$$\begin{array}{ccc} P & \overset{(i,f)}{\rightarrowtail} & I \times C \\ {\scriptstyle g}\downarrow & & \downarrow \\ M & \rightarrowtail & C \times C \end{array}$$

be a pullback in **Set**. By the unit law, we know that $P \xrightarrow{g} M \longrightarrow C$ is equal to f, and is an extremal epimorphism. Now let

$$\begin{array}{ccc} P_3 & \xrightarrow{a} & P \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f} \\ P & \xrightarrow[f]{} & C \end{array}$$

be a pullback in **Set**. We will show that $a = b$, so that f is mono. Hence we will have that f is an isomorphism, and so the unit law lifts to $\mathcal{S}pan$. A similar argument will show the same for the other unit law, giving that the monoid is spanish. We consider the diagram of pullbacks:

$$\begin{array}{ccccc}
P_3 & \rightarrowtail & P' & \longrightarrow & P \\
\downarrow & \searrow^{(j,k,f)} & \downarrow & & \downarrow{\scriptstyle (i,f)} \\
P'' & \rightarrowtail & I \times I \times C & \longrightarrow & I \times C \\
\downarrow & & \downarrow & & \downarrow \\
P & \xrightarrow[(i,f)]{} & I \times C & \longrightarrow & C
\end{array}$$

We see that we can describe P_3 entirely by the morphisms j, k, and f. We will show that $j = k$. However, $j = ia$ and $k = ib$. Furthermore, we already know that $fa = fb$, and that (i, f) is a monomorphism. Thus, we can deduce that $a = b$, and so f is a monomorphism.

To show that $j = k$, we consider the commutative diagram in $\mathcal{R}el$:

$$\begin{array}{ccccc}
I \times I \times C & \rightarrowtail & C \times C \times C & \xrightarrow{1 \times M} & C \times C \\
{\scriptstyle \Delta^{\mathrm{op}} \times 1_C}\downarrow & & {\scriptstyle M \times 1}\downarrow & & \downarrow{\scriptstyle M} \\
I \times C & \rightarrowtail & C \times C & \xrightarrow[M]{} & C
\end{array}$$

The right-hand square is associativity, while the left hand square is from Lemma 2.13. Since all morphisms are partial functions, the diagram lifts to $\mathcal{S}pan$. In $\mathcal{S}pan$, the top-right composite is

$$I \times I \times C \xleftarrow{(j,k,f)} P_3 \longrightarrow P \longrightarrow M \longrightarrow C$$

Since the diagram commutes in $\mathcal{S}pan$, the left-hand leg of this span must factor through the diagonal $I \xrightarrow{\Delta} I \times I$, and so $j = k$ as required.

Using the other unit law in a similar way, we can deduce that it also lifts to a commutative diagram in $\mathcal{S}pan$, so that the monoid is spanish. □

3. Categories, Functors, Profunctors and Natural Transformations in $\mathcal{S}pan$

3.1 Categories

In this section, we will establish a bijective correspondence between categories and certain types of monoid in $\mathcal{S}pan$, or equivalently in $\mathcal{R}el$. We will fix some notation. For a category C, the corresponding monoid will be $C \otimes C \xrightarrow{M} C \xleftarrow{I} 1$. For a category $\mathcal{D}$, the corresponding monoid will be $D \otimes D \xrightarrow{N} D \xleftarrow{J} 1$.

Given a small category C, we can form a monoid in $\mathcal{S}pan$ as follows: The underlying set is the set C of morphisms of C. Composition gives a partial function from $C \times C$ to C, defined on composable pairs, i.e. pairs (f, g) such that $\operatorname{dom} f = \operatorname{cod} g$. The identity is the opposite to the partial function from C to 1 that is defined only on identity morphisms. It is easy to check that this is indeed a monoid in $\mathcal{S}pan$. This also works for internal categories in any category with all finite limits, and the following theorems also all apply in this case, with the exception of Proposition 3.2.

Theorem 3.1. *A monoid in $\mathcal{S}pan$ can be expressed as the result of the above construction for a category if and only if the multiplication is a partial morphism and there is a (necessarily unique) 2-cell*

$$\begin{array}{ccc} C \otimes C & \xrightarrow{M} & C \\ {\scriptstyle (M\otimes C)^{\mathrm{op}}}\downarrow & \nearrow & \downarrow{\scriptstyle M^{\mathrm{op}}} \\ C \otimes C \otimes C & \xrightarrow[C\otimes M]{} & C \otimes C \end{array}$$

in $\mathcal{S}pan$.

Proof. It is easy to see that the monoid we obtain from a category using the above construction, has a partial function for its multiplication, and also has the unique 2-cell in the above theorem.

Conversely, given a monoid $C \otimes C \xrightarrow{M} C \xleftarrow{I} 1$ in $\mathcal{S}pan$, where M is a partial function and I^{op} is also a partial function, C will be the object of morphisms. The domain of I^{op} will be the object I of objects. The left identity law for the monoid:

$$C \xrightarrow{I \otimes 1_C} C \otimes C \xrightarrow{M} C = 1_C$$

says that for every element f of C, there is exactly one element j of I such that the composite $jf = f$, and no other composites are possible. We will call this unique j the *codomain* of f. Similarly, there is exactly one element j of I such that $fj = f$. We will call this the *domain* of f. These give the functions $C \xrightarrow{\text{dom}} I$ and $C \xrightarrow{\text{cod}} I$, needed for a category. Also, M must be the object of composable pairs in the category. We need to show that it is the object of pairs (f, g) such that $\text{dom}(f) = \text{cod}(g)$, as is required for a category.

The 2-cell shows that if the composites fg and gh both exist, then the composite $(fg)h$ also exists. Associativity then gives us that $f(gh)$ also exists, and is equal to $(fg)h$. Also, associativity gives us that if either $f(gh)$ or $(fg)h$ exists, then the other also exists, and they are equal. This means that in this case, both fg and gh must exist.

Finally, from the case where g is an identity, we know that the composite fh exists if and only if the domain of f is equal to the codomain of h. This is exactly what we need for a category. □

We note that since the multiplication is a partial morphism, and the unit is the opposite of a partial morphism, they are both relations. From Lemma 2.14, we see that a monoid in $\mathcal{R}el$ comes from a category if and only if the multiplication is a partial morphism, and the same 2-cell exists.

In the particular case of **Set**, it is possible to write the third condition in a different way. This will be useful when we discuss categories as quantales.

Proposition 3.2. *A monoid in $\mathcal{R}el$ can be expressed as the result of the above construction for a category if and only if the multiplication is a partial morphism and whenever the products xy and yz are both defined, then so is xyz.*

Proof. We need to show that for a monoid in $\mathcal{R}el$, whose multiplication is a partial morphism, the condition about products being defined is equivalent to the condition in Theorem 3.1.

We know that if the products xy and yz both exist, then the composite $(C \times M)(M \times C)^{\text{op}}$ relates (xy, z) to (x, yz). Therefore, by the condition in

Theorem 3.1, the composite $M^{\mathrm{op}}M$ must also relate them. For this to happen, $M(xy,z) = (xy)z$ must be defined.

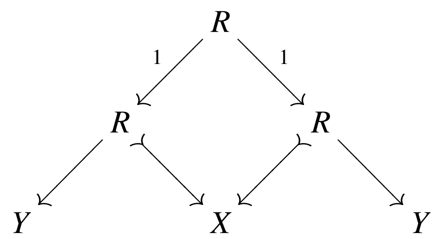
Conversely, suppose we have that whenever xy and yz are both defined, so is xyz. Now suppose that $(C \times M)(M \times C)^{\mathrm{op}}$ relates (a,b) to (c,d). This means that there is a triple (c,x,b) which is related to (a,b) by $M \times C$, and to (c,d) by $C \times M$. Thus, $cx = a$ and $xb = d$ are both defined, so the products $(cx)b$ and $c(xb)$ are both defined (and equal by associativity). Now M relates both (a,b) and (c,d) to cxb, so the composite $M^{\mathrm{op}}M$ relates (a,b) to (c,d). Since (a,b) and (c,d) were an arbitrary pair related by $(C \times M)(M \times C)^{\mathrm{op}}$, this means that $(C \times M)(M \times C)^{\mathrm{op}} \leqslant M^{\mathrm{op}}M$ in $\mathcal{R}el$. □

To make the description from Theorem 3.1 internal in $\mathcal{S}pan$, or $\mathcal{R}el$, we need to give a way of identifying which spans are partial functions.

Proposition 3.3. *A span is a relation if and only if it is a subterminal object in its hom-category in the bicategory $\mathcal{S}pan$.*

Proposition 3.4. *A relation $X \xrightarrow{R} Y$ is a partial function if and only if there is a 2-cell $RR^{\mathrm{op}} \Longrightarrow 1_Y$ in $\mathcal{R}el$, or equivalently in $\mathcal{S}pan$.*

Proof. If R is a partial function, then in $\mathcal{S}pan$, the composite RR^{op} is given by the pullback:



The function from R to Y then gives the required 2-cell in $\mathcal{S}pan$, and in $\mathcal{R}el$.

Conversely, suppose R has the required 2-cell, then the composite RR^{op} is a span in which both functions are the same. Because the composite is the pullback square, and the morphisms from R to X and to Y are jointly monic, this means that the two functions f and g of the pullback square:

$$\begin{array}{ccc} P & \xrightarrow{f} & R \\ {\scriptstyle g}\downarrow & & \downarrow \\ R & \longrightarrow & X \end{array}$$

are equal. This can only happen if $R \longrightarrow X$ is a monomorphism, so R is a partial function. ◻

3.2 Functors

Proposition 3.5. *If C and $\mathcal{D}$ are categories, corresponding to the monoids (C, M, I) and (D, N, J) in $\mathcal{S}pan$, then functors $C \xrightarrow{F} \mathcal{D}$ correspond bijectively to lax monoid homomorphisms from C to D in $\mathcal{S}pan$, which are also functions.*

Proof. Given a lax monoid homomorphism $C \xrightarrow{f} D$ in $\mathcal{S}pan$, where f is a function, one lax monoid homomorphism condition says that fm' admits a 2-cell to $n'(f \times f)$. The composite $n'(f \times f)$ is the pullback:

$$\begin{array}{ccccc} P & \longrightarrow & N & \xrightarrow{n'} & D \\ \downarrow & & \downarrow & & \\ C \times C & \xrightarrow[f \times f]{} & D \times D & & \end{array}$$

The 2-cell therefore says that there is a morphism $M \rightarrowtail P$. This means that any morphisms that compose in C are sent to morphisms that compose in $\mathcal{D}$. We get a commutative square in **Set**:

$$\begin{array}{ccc} M & \xrightarrow{f \times f} & N \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle n'} \\ C & \xrightarrow[f]{} & D \end{array}$$

This is exactly the functoriality condition involving composition. Similarly, the other lax monoid homomorphism condition for f gives the square

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ {\scriptstyle I}\downarrow & \Nearrow & \downarrow{\scriptstyle J} \\ C & \xrightarrow[f]{} & D \end{array}$$

in $\mathcal{S}pan$, with a 2-cell from fI to J. This gives a morphism from I to J, sending an identity 1_X to $F(1_X)$, which this 2-cell shows is an identity. Since f preserves composition, this must be 1_{FX}.

Conversely, suppose $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a functor. Then its action on morphisms is a function $C \xrightarrow{f} D$ such that the following diagram commutes:

$$\begin{array}{ccccc} M & \xrightarrow{m'} & C & \xleftarrow{i} & I \\ {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ N & \xrightarrow[n']{} & D & \xleftarrow[j]{} & J \end{array}$$

This induces a morphism from M to the pullback of $N \rightarrowtail D \times D$ along $f \times f$, and a morphism from I to the pullback of J along f. These give the 2-cells required to make f into a lax monoid homomorphism in $\mathcal{S}pan$. □

Composition of functors is the obvious composition of functions. We can identify morphisms as the spans with right adjoints. The situation is identical in $\mathcal{R}el$ – lax monoid homomorphisms in $\mathcal{R}el$ remain lax homomorphisms in $\mathcal{S}pan$, and functions are relations with a right adjoint.

Remark 3.6. The reader may find it strange that categories correspond to pseudomonoids, and yet functors only correspond to lax monoid homomorphisms. This leads us to consider what lax monoids correspond to. There is a correspondance between certain kinds of protocategories [3], and certain lax monoids (unbiassed in Leinster's [4] terminology). Given a protocategory $\mathcal{C}$, with at most one composite for each pair of protomorphisms, we form a lax monoid in $\mathcal{S}pan$ as follows: C is the set of protomorphisms; M is the set of composable pairs of protomorphisms; I is the set of objects.

It turns out that this is a lax monoid. However, we can view a category as a protocategory in which every morphism has exactly one source and target. From this point of view, any function between categories that preserves the protocategory structure (i.e. preserves identities and composition) is a functor. These are lax monoid homomorphisms.

Strict monoid homomorphisms are functors that are injective on objects, since for the 2-cell to be an isomorphism would require that any pair of morphisms whose images are composable in $\mathcal{D}$ must also be composable in $\mathcal{C}$.

3.3 Natural Transformations

Proposition 3.7. *Given functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\mathcal{C} \xrightarrow{G} \mathcal{D}$, corresponding to lax monoid morphisms $C \xrightarrow{f} D$ and $C \xrightarrow{g} D$ in $\mathcal{S}pan$, respectively, natural transformations correspond to functions $C \xrightarrow{a} D$ such that we have (necessarily unique) 2-cells:*

$$\begin{array}{ccc} C \otimes C & \xrightarrow{(a\otimes g)} & D \otimes D \\ {\scriptstyle M}\downarrow & \nearrow & \downarrow{\scriptstyle N} \\ C & \xrightarrow[a]{} & D \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} C \otimes C & \xrightarrow{(f\otimes a)} & D \otimes D \\ {\scriptstyle M}\downarrow & \nearrow & \downarrow{\scriptstyle N} \\ C & \xrightarrow[a]{} & D \end{array}$$

Proof. Given a natural transformation α, the function $C \xrightarrow{a} D$ sends the morphism $X \xrightarrow{h} Y$ in $\mathcal{C}$, to the morphism $FX \xrightarrow{\alpha_X} GX \xrightarrow{Gh} GY$, or equivalently $FX \xrightarrow{Fh} FY \xrightarrow{\alpha_Y} GY$. It is straightforward to check that the 2-cells above do indeed exist.

Conversely, given a morphism a such that the above 2-cells exist, in $\mathcal{S}pan$, we can form a natural transformation α by $\alpha_X = a(1_X)$. If we apply the left-hand 2-cell to $(\mathrm{cod}(h), h)$, the lower-left way around sends it to $a(h)$, while the upper-right way sends it to $a(\mathrm{cod}(h))F(h)$. We deduce that these are equal. On the other hand, if we apply the right-hand 2-cell to $(h, \mathrm{dom}(h))$, we get that $a(h) = G(h)a(\mathrm{dom}(h))$. The equality of these two is exactly the commutativity of the naturality square. □

Again, the existence of these 2-cells does not depend whether we are in $\mathcal{S}pan$ or $\mathcal{R}el$.

For composition of natural transformations, there are two types to consider. The easier type is horizontal composition. It is easy to see that this is just composition of the morphisms corresponding to the natural transformation. For vertical composition, suppose we have categories C and $\mathcal{D}$; functors F, G and H, all from C to $\mathcal{D}$; and natural transformations $F \xRightarrow{\alpha} G$ and $G \xRightarrow{\beta} H$. Let C and D be the monoids in $\mathcal{S}pan$ obtained from C and $\mathcal{D}$ respectively; let f, g, and h be the lax monoid homomorphisms corresponding to the functors F, G and H respectively; and let a and b be the functions from C to D corresponding to α and β respectively. Let $C \xrightarrow{c} D$ be the function corresponding to the composite $\beta\alpha$. For a composable pair $(x, y) \in M$, We know that $c(xy) = b(x)a(y)$. The naturality gives that the value of this composite does not change if we choose a different factorisation of the composite xy. Since every morphism has a factorisation – for example, the factorisation through an identity – we can construct c as the following composite in $\mathcal{S}pan$ or $\mathcal{R}el$:

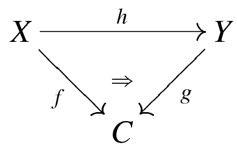
$$C \xrightarrow{M^{\mathrm{op}}} C \otimes C \xrightarrow{a \otimes b} D \otimes D \xrightarrow{N} D$$

From the above characterisation of natural transformations, it looks like they should be thought of as a kind of bimodule. To find the appropriate context in which they are bimodules, we will need the following well-known fact.

Proposition 3.8. *If $(\mathbf{C}, \otimes, I)$ is a monoidal category, and (C, M, I') is a monoid in $\mathbf{C}$, then $\mathbf{C}/C$ is a monoidal category.*

Proof. Given $X \xrightarrow{f} C$ and $Y \xrightarrow{g} C$, their tensor product is the composite $X \otimes Y \xrightarrow{f \otimes g} C \otimes C \xrightarrow{M} C$. The unit is $I \xrightarrow{I'} C$. It is straightforward to check that this satisfies all the required axioms. □

In fact, we will need to modify this for a monoidal bicategory: we define a bicategory $\mathbf{C} /\!\!/ C$ to have the same objects as $\mathbf{C}/C$, but the morphisms are now lax triangles:



and 2-cells are just 2-cells between the top morphisms in these triangles, subject to the obvious compatibility conditions with the 2-cells in the triangles. It is straightforward to see that the same argument as above makes this into a monoidal bicategory. In this monoidal bicategory, for another monoid D in C, a lax monoid homomorphism $D \xrightarrow{f} C$ becomes a monoid in this slice category $C /\!\!/ C$.

We see that when we view a category $\mathcal{C}$ as a monoid, C, in $\mathcal{S}pan$, we can view a functor with codomain $\mathcal{C}$ as a certain monoid in the slice $\mathcal{S}pan /\!\!/ C$. Now for two functors from $\mathcal{D}$ to $\mathcal{C}$, we have the corresponding monoids in the slice category $\mathcal{S}pan /\!\!/ C$. Now a natural transformation is a kind of bimodule between these monoids, in this slice category.

3.4 Profunctors

In this context, the best way to view a profunctor $P : \mathcal{C}^{\text{op}} \times \mathcal{D} \longrightarrow \mathbf{Set}$, is through the collection of elements, i.e. $\sum_{A \in \text{ob}(\mathcal{C}), B \in \text{ob}(\mathcal{D})} P(A, B)$. This collection admits a sort of left action by $\mathcal{C}$, and a right action by $\mathcal{D}$. When we look at the corresponding monoids C and D in $\mathcal{S}pan$, these actions are partial functions. We therefore see that a profunctor is a special kind of bimodule.

Proposition 3.9. *Given categories $\mathcal{C}$ and $\mathcal{D}$, and corresponding monoids C and D in $\mathcal{S}pan$, a bimodule E (a left C, right D module with the obvious coherence conditions between the actions) comes from a profunctor from $\mathcal{C}$ to $\mathcal{D}$ if and only if it satisfies the following conditions:*

1. *The actions $C \otimes E \xrightarrow{a} E$ and $E \otimes D \xrightarrow{b} E$ are partial functions.*
2. *There are (necessarily unique) 2-cells:*

$$
\begin{array}{ccc}
C \otimes E & \xrightarrow{C \otimes a^{\text{op}}} & C \otimes C \otimes E \\
{\scriptstyle a}\Big\downarrow & \Swarrow & \Big\downarrow{\scriptstyle M \otimes E} \\
E & \xrightarrow[a^{\text{op}}]{} & C \otimes E
\end{array}
\qquad \textit{and} \qquad
\begin{array}{ccc}
E \otimes D & \xrightarrow{b^{\text{op}} \otimes D} & E \otimes D \otimes D \\
{\scriptstyle b}\Big\downarrow & \Swarrow & \Big\downarrow{\scriptstyle E \otimes N} \\
E & \xrightarrow[b^{\text{op}}]{} & E \otimes D
\end{array}
$$

Proof. We will denote the actions $a(f, e)$ by $f.e$ and $b(e, g)$ by $e * g$. It is obvious that for a profunctor, the actions a and b are partial functions. Now we consider the first 2-cell in condition 2: The top-right composite is a span that above any pairs of elements (h, e) and (f, e') of $C \otimes E$, has the set of triples $(f, g, e) \in C \otimes C \otimes E$, such that $fg = h$ and $g.e = e'$. Therefore, the 2-cell in question sends all triples (f, g, e) such that fg and $g.e$ both exist, to an element e'' of E, satisfying both $f.(g.e) = e''$ and $(fg).e = e''$. Thus, the existence of this 2-cell simply indicates that if (fg) and $g.e$ both exist, then $(fg).e$ also exists (the fact that it is equal to $f.(g.e)$ is automatic by the associativity conditions required for a bimodule). Also, by the associativity conditions, we know that if $(fg).e$ exists, then both (fg) and $g.e$ must also exist.

Furthermore, by the unit laws for a bimodule, for any $e \in E$, there is a unique identity $i \in C$ such that $i.e$ exists, and for this i, we have that $i.e = e$. We will call this i, the codomain of e. Now if we substitute this codomain of e for g in the above observation, we see that $f.e$ exists if and only if $f\,\mathrm{cod}(e)$ exists, or equivalently, if and only if $\mathrm{cod}(e) = \mathrm{dom}(f)$.

By a similar argument, we see that $e * d$ exists if and only if $\mathrm{dom}(e) = \mathrm{cod}(d)$. From these conditions, it is clear that E comes from a profunctor. □

For composition of profunctors, let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories, and let $\mathcal{C} \xrightarrow{P} \mathcal{D}$ and $\mathcal{D} \xrightarrow{Q} \mathcal{E}$ be profunctors. Let the corresponding sets in $\mathcal{S}pan$ be C, D, E, P and Q respectively. We know that the composite profunctor has as elements, equivalence classes of "composable" pairs $(p \in P, q \in Q)$ under the equivalence relation that relates two pairs (p, q) and (p', q') if there is $f \in D$ such that $p = fp'$ and $q' = qf$. This is the product of P and Q over D, as bimodules in $\mathcal{S}pan$, or $\mathcal{R}el$.

4. Quantales

Just as monoids in $\mathcal{S}pan$ corresponded to premonoidal structures on discrete categories and consequently monoidal closed structures on powers of **Set**, monoids in $\mathcal{R}el$ correspond to **2**-enriched premonoidal structures on discrete sets and thus monoidal closed structures on powers of **2**. These are quantale structures on power sets, ordered by inclusion. In this way a small category gives a quantale. In this section, we study the interplay between categorical constructions and quantale ones. Niefield considers the closely related questions of the quantale of subsets of a monoid and quantales of subobjects of the unit object in certain closed categories (see [5] and the references cited there).

4.1 The Quantale of a Category

To determine which quantales occur as the quantale from a category, we just need to translate our characterisation of categories as monoids in $\mathcal{R}el$ through the equivalence between categories $\mathcal{R}el$ and $\mathbf{CABA}_{sup}$. Rosenthal [7] calls a quantale whose underlying lattice is a power set, a *power quantale*.

The most direct translation is just in terms of atoms (or equivalently join-irreducible elements). A morphism of powersets corresponding to a relation, corresponds to a partial morphism if and only if it sends atoms to either atoms or the empty set. The final condition for a monoid in $\mathcal{R}el$ to be categorical says that if the composites fg and gh both exist, then so does the triple composite fgh. (By associativity, it doesn't matter which way we express the triple composite.) We can express this for a quantale by using the contrapositive – if the triple composite fgh of three atoms is 0 (i.e. undefined) then either $fg = 0$ or $gh = 0$.

However, conditions involving atoms are not natural conditions on quantales, except in the case where the lattices are CABAs. We therefore seek to rephrase these conditions in a way that looks more natural for all quantales. We hope that these conditions might give a better guide for how we might be able to generalise our results to internal categories, for example in $\mathcal{O}rd$ or $\mathcal{L}oc$. However, such a generalisation would still require significant further work, and we would expect further conditions to be necessary. In such cases, we may find that we no longer get a strict quantale, but a lax quantale.

Lemma 4.1. *A sup-morphism between CABAs corresponds to a partial function if and only if its right adjoint preserves all non-empty sups.*

Proof. Let f be a relation. The right adjoint to its direct image is just its inverse image – i.e. it sends a subset A to the set $\{x|f(x)\subseteq A\}$, where $f(x)$ represents the set of things to which x is related. To say that this preserves non-empty sups says that if the image of a point x is contained in $\bigcup A_i$, then it is contained in one of the A_i. Since we can express any set as the union of its points, this means that the image of x is either empty or a singleton, i.e. f is a partial function. □

Remark 4.2. This lemma also applies in a constructive context if we replace CABAs by powersets, and non-empty by inhabited. However, some of the results later in this section will require complements, so the classification of which internal quantales come from internal categories in an arbitrary topos will require further work.

Remark 4.3. It is sufficient for the right adjoint to preserve binary sups, because the key point of the proof is the fact that the image of any x is join irreducible. However, in a powerset lattice, the elements that are irreducible as binary joins are exactly the singletons and the empty set. These are exactly the elements that are irreducible as non-empty joins.

An equivalent way to express this is to say that the sup-morphism sends $\bigvee$-irreducible elements to either $\bigvee$-irreducible elements or the bottom element. This will be a more natural condition for our purposes, since in the powerset, the $\bigvee$-irreducible elements are singleton sets. When we are looking at the quantale C from a category, singleton sets of C will be morphisms of the category, while singleton sets of $C \otimes C$ will be pairs of morphisms. Saying that the multiplication sends irreducible elements to irreducible elements or the bottom element therefore means that a pair of morphisms has at most one composite.

To express the final condition, we need to find an expression for the sup-morphism corresponding to the opposite of a relation. In fact, it is sufficient for our purposes to only find an expression for the opposite of a partial function.

Lemma 4.4. *If $PA \xrightarrow{f} PB$ is the sup-morphism corresponding to a partial function, then the sup-morphism corresponding to its opposite is given by $f^{\mathrm{op}}(x) = f^*(x) \backslash f^*(\bot)$, where f^* is the right adjoint to f, and $\backslash$ is the relative complementation that exists in the power set.*

Proof. The opposite of the sup-morphism $PA \xrightarrow{R} PB$ corresponding to a relation is given by $R^{\mathrm{op}}(B') = \{a \in A | (\exists b \in B')(b \in R(\{a\}))\}$, or equivalently, $R^{\mathrm{op}}(B') = \{a \in A | R(\{a\}) \cap B' \neq \emptyset\}$. In the case where the relation is a partial morphism, we can express the condition $f(\{a\}) \cap B' \neq \emptyset$ simply as $(f(\{a\}) \subseteq B') \wedge (f(\{a\}) \neq \emptyset)$. Furthermore, this can be expressed as $(\{a\} \subseteq f^*(B')) \wedge (\{a\} \nsubseteq f^*(\emptyset))$. This can be further simplified to $\{a\} \subseteq f^*(B') \backslash f^*(\emptyset)$. From this, we see that $f^{\mathrm{op}}(B') = f^*(B') \backslash f^*(\emptyset)$. □

Lemma 4.5. *The multiplication of a quantale whose underlying lattice is a CABA is the direct image of a partial function if and only if:*

- *For any* $xy \leqslant \bigvee_{i\in I} a_i$*, where* I *is non-empty, we can find a collection of pairs of elements* $(x_j, y_j)_{j\in J}$ *such that* $x \otimes y \leqslant \bigvee_{j\in J} x_j \otimes y_j$*, and for each* $j \in J$*, there is some* $i \in I$ *such that* $x_j y_j \leqslant a_i$*.*

Proof. We know that the condition that multiplication is a partial function in the monoid in $\mathcal{R}el$ corresponds to saying that the right adjoint to the quantale multiplication $Q \otimes Q \xrightarrow{m} Q$ preserves non-empty joins. The right adjoint is the function $Q \xrightarrow{m^*} Q \otimes Q$ given by $m^*(a) = \bigvee\{x \otimes y | xy \leqslant a\}$. To say that this preserves non-empty joins says that if $x \otimes y \leqslant m^*(\bigvee a_i)$, then $x \otimes y \leqslant \bigvee m^*(a_i)$, but in a tensor product, this means that $x \otimes y \leqslant \bigvee_{j\in J} x_j \otimes y_j$, where for every j there is an $i \in I$ such that $x_j \otimes y_j \leqslant m^*(a_i)$, or equivalently $m(x_j \otimes y_j) \leqslant a_i$. □

Definition 4.6. *For a quantale,* Q*, the* binary kernel, Z_2*, is the largest element of* $Q \otimes Q$ *whose product is 0. The* ternary kernel, Z_3 *is the largest element of* $Q \otimes Q \otimes Q$ *whose product is 0.*

Proposition 4.7. *A quantale* Q *is the quantale from a category if and only if it has the following properties:*

1. *The underlying lattice is a CABA.*
2. *If* $xy \leqslant \bigvee_{i\in I} a_i$*, for non-empty* I*, then we can find a collection of pairs of elements* $(x_j, y_j)_{j\in J}$ *such that* $x \otimes y \leqslant \bigvee_{j\in J} x_j \otimes y_j$*, and for each* $j \in J$*, there is some* $i \in I$ *such that* $x_j y_j \leqslant a_i$*.*
3. $Z_3 = (Z_2 \otimes \top) \vee (\top \otimes Z_2)$*.*

Proof. We just need to show that Conditions (2) and (3) are equivalent to the conditions on a monoid in $\mathcal{R}el$ from Proposition 3.2. We showed in Lemma 4.5 that the multiplication in $\mathcal{R}el$ is a partial map if and only if the powerset construction sends it to a quantale with the second property.

We want to show that Condition (3) is equivalent to the condition that for atoms x, y, and z, if the product $xyz = 0$, then either $xy = 0$ or $yz = 0$.

However, the condition $xyz = 0$ is equivalent to $x \otimes y \otimes z \leqslant Z_3$, so the condition for atoms says that if $x \otimes y \otimes z \leqslant Z_3$, then either $x \otimes y \leqslant Z_2$ or $y \otimes z \leqslant Z_2$. Since Z_3 is the join of all products of atoms $\{x \otimes y \otimes z | xyz = 0\}$, this gives that $Z_3 \leqslant (Z_2 \otimes \top) \vee (\top \otimes Z_2)$. The opposite inequality is trivial.

For the converse, we know that for atoms x, y and z, if $xyz = 0$, then $x \otimes y \otimes z \leqslant Z_3$, so by Condition (3), $x \otimes y \otimes z \leqslant (Z_2 \otimes \top) \vee (\top \otimes Z_2)$. Since $x \otimes y \otimes z$ is an atom, this means that either $x \otimes y \otimes z \leqslant Z_2 \otimes \top$ or $x \otimes y \otimes z \leqslant \top \otimes Z_2$, i.e. either $xy = 0$ or $yz = 0$. □

The condition on the binary and ternary kernel can be thought of as a variation of the condition for a ring to be an integral domain, namely, if $ab = 0$, then either $a = 0$ or $b = 0$. In a quantale, we can express the condition $a = 0$ or $b = 0$ as $a \otimes b \leqslant (0 \otimes \top) \vee (\top \otimes 0)$. (This can also be simplified to $a \otimes b = 0$, but the form we use makes the connection with our condition clearer.) Now we can take joins, to get the condition $Z_2 = (0 \otimes \top) \vee (\top \otimes 0)$. The condition from the theorem is now clearly a generalisation of this. It corresponds to the condition "if $abc = 0$ then either $ab = 0$ or $bc = 0$" in a ring. For unital rings, by substituting $b = 1$, we see that this is equivalent to the integral domain condition, but for quantales we can express b as a join of elements $b = \bigvee_{i \in i} b_i$, and we can have some of these elements satisfying $ab_i = 0$, and others satisfying $b_i c = 0$, so the condition is a weaker version of the integral domain condition. It may be of interest to study what the condition "if $abc = 0$, then either $ab = 0$ or $bc = 0$" means in the context of a non-unital ring.

In a similar manner to Theorem 4.7, we can express functors as certain kinds of morphisms between quantales.

Proposition 4.8. *For categories $\mathcal{C}$ and $\mathcal{D}$, and corresponding quantales $Q\mathcal{C}$ and $Q\mathcal{D}$, functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ correspond to lax quantale homomorphisms $Q\mathcal{C} \xrightarrow{f} Q\mathcal{D}$, where the right adjoint to f has a right adjoint of its own.*

Proof. This is automatic from the equivalence between $\mathcal{R}el$ and $\mathbf{CABA}_{sup}$. □

Proposition 4.9. *If $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\mathcal{C} \xrightarrow{G} \mathcal{D}$ are functors, with corresponding lax quantale homomorphisms f and g, then a natural transformation between F and G corresponds to a sup-morphism $C \xrightarrow{a} D$, whose right adjoint preserves sups, such that there are 2-cells*

$$\begin{array}{ccc} C \otimes C & \xrightarrow{(a \otimes g)} & D \otimes D \\ {\scriptstyle m}\downarrow & \leqslant & \downarrow{\scriptstyle N} \\ C & \xrightarrow[a]{} & D \end{array} \quad \text{and} \quad \begin{array}{ccc} C \otimes C & \xrightarrow{(f \otimes a)} & D \otimes D \\ {\scriptstyle m}\downarrow & \leqslant & \downarrow{\scriptstyle N} \\ C & \xrightarrow[a]{} & D \end{array}$$

Proof. This is automatic from the equivalence between $\mathcal{R}el$ and $\mathbf{CABA}_{sup}$. □

4.2 Retrieving the Category

We have shown that categories correspond to certain types of quantale. We now consider the inverse part of this bijection – given one of these quantales, how can we recover the category we started with?

This can actually be done fairly easily – we know that a category can be described by the collection of functors from the cocategory object

$$1 \rightleftarrows 2 \rightrightarrows 3$$

in *Cat*. More explicitly, objects of the category correspond bijectively with functors from 1 to the category. Morphisms correspond to functors from 2 to the category, and the composite of a composeable pair is calculated by looking at the functor from 3 to the category. We can describe these functors explicitly for quantales. The quantale corresponding to the one morphism category is the two-element quantale. A functor from this to another quantale must send 0 to 0, so the only question is where it must send 1. It must send 1 to a join-irreducible element below 1, which is idempotent. The objects of the category therefore correspond to join-irreducible idempotent elements below 1 in the quantale. Similarly, morphisms correspond to triples (x, y, f), where x and y are objects, and f is an irreducible element such that $yfx = f$. Finally, the composite of two morphisms (x, y, f) and (y, z, g) is (x, z, gf).

If the quantale corresponds to a category, then this will give us the corresponding category.

However, it is worth observing that while the above is a cocategory object in *Cat*, it is not a cocategory object when we extend to the category of all quantales and lax quantale homomorphisms whose right adjoint is also a sup-morphism. This means that for a general quantale, we do not get a category using this approach.

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