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THE PERIODIC TABLE OF n-CATEGORIES II: DEGENERATE TRICATEGORIES

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Abstract

Nous continuons le travail commencé en [5] en étudiant les tricatégories dégénérees et en les comparant avec les structures prédites par le tableau périodique des n-catégories. Pour les tricatégories trois fois dégénérées nous démontrons une triéquivalence avec la tricatégorie partiellement discrète des monoïdes commutatifs. Pour les tricatégories deux fois dégénérées nous expliquons comment on peut construire une catégorie monoïdale tressée d'une tricatégorie deux fois dégénérée donnée, mais nous démontrons que cette construction n'induit pas une comparaison simple entre **BrMonCat** et **Tricat**. Nous discutons comment on peut itérer la construction des "icônes" pour produire un équivalence, mais nous espérons à la suite pour donner les détails. Finalement nous étudions les tricatégories dégénérées pour donner la première définition de bicatégorie monoïdale complètement algébrique et la structure entière de tricatégorie de **MonBicat**.

We continue the project begun in [5] by examining degenerate tricategories and comparing them with the structures predicted by the Periodic table. For triply degenerate tricategories we exhibit a triequivalence with the partially discrete tricategory of commutative monoids. For the doubly degenerate case we explain how to construct a braided monoidal category from a given doubly degenerate category, but show that this does not induce a straightforward comparison between **BrMonCat** and **Tricat**. We indicate how to iterate the icon construction to produce an equivalence, but leave the details to a sequel. Finally we study degenerate tricategories in order to give the first fully algebraic definition of monoidal bicategories and the full tricategory structure **MonBicat**.

Keywords: tricategory, degenerate tricategory, braided monoidal category, monoidal bicategory, icon.

MSC2000: 18A05, 18D05, 18D10

Introduction

This work is a continuation of the work begun in [5], studying the "Periodic Table" of n-categories proposed by Baez and Dolan [1]. The idea of the Periodic Table is to study "degenerate" n-categories, that is, n-categories in which the lowest dimensions are trivial. For small n this is supposed to yield well-known algebraic structures such as commutative monoids or braided monoidal categories; this helps us understand some specific part of the whole n-category via better-known algebraic structures, and also helps us to try to predict what n-categories should look like for higher n.

More precisely, the idea of degeneracy is as follows. Consider an n-category in which the lowest non-trivial dimension is the kth dimension, that is, there is only one cell of each dimension lower than k. We call this a "k-degenerate n-category". We can then perform a "dimension shift" and consider the k-cells of the old n-category to be 0-cells of a new $(n-k)$-category, as shown in the schematic diagram in Figure 1.

This yields a "new" $(n-k)$-category, but it will always have some

Figure 1: Dimension-shift for k-fold degenerate n-categories

"old" n-category	$\longrightarrow$	**"new" $(n-k)$-category**
0-cells 1-cells $\vdots$ $(k-1)$-cells } trivial		
k-cells	$\longrightarrow$	0-cells
$(k+1)$-cells	$\longrightarrow$	1-cells
$\vdots$	$\vdots$	$\vdots$
n-cells	$\longrightarrow$	$(n-k)$-cells

special extra structure: the k-cells of the old n-category have k different compositions defined on them (along bounding cells of each lower dimension), so the 0-cells of the "new" $(n-k)$-category must have k multiplications defined on them, interacting via the interchange laws from the old n-category. Likewise every cell of higher dimension will have k "extra" multiplications defined on them as well as composition along bounding cells.

In [1], Baez and Dolan *define* a "k-tuply monoidal $(n-k)$-category" to be a k-degenerate n-category, but *a priori* it should be an $(n-k)$-category with k monoidal structures on it, interacting via coherent pseudo-invertible cells. A direct definition has not yet been made for general n and k. Balteanu et al [3] study a lax version of this, where the monoidal structures interact via non-invertible cells; this gives different structures, which we will discuss later.

The Periodic Table seeks to answer the question: exactly what sort of $(n-k)$-category structure does the degeneracy process produce? Figure 2 shows the first few columns of the hypothesised Periodic Table: the (n,k)th entry predicts what a k-degenerate n-category "is". (In this table we follow Baez and Dolan and omit the word "weak" understanding that all the n-categories in consideration are weak.)

One consequence of the present work is that although k-tuply monoidal $(n-k)$-categories and k-degenerate n-categories are related, we see that the relationship is not straightforward. So in fact we need to consider three possible structures for each n and k:

- k-degenerate n-categories
- k-tuply monoidal $(n-k)$-categories
- the (n,k)th entry of the Periodic Table.

In [5] we examined the top left hand corner of the table, that is, degenerate categories and degenerate bicategories. We found that we had to be careful about the exact meaning of "is". The main problem is the presence of some unwanted extra structure in the "new" $(n-k)$-categories in the form of distinguished elements, arising from the structure constraints in the original n-categories — a specified k-cell structure

Figure 2: The hypothesised Periodic Table of n-categories

set	category	2-category	3-category	...
monoid ≡ category with only one object	monoidal category ≡ 2-category with only one object	monoidal 2-category ≡ 3-category with only one object	monoidal 3-category ≡ 4-category with only one object	...
commutative monoid ≡ 2-category with only one object only one 1-cell	braided monoidal category ≡ 3-category with only one object only one 1-cell	braided monoidal 2-category ≡ 4-category with only one object only one 1-cell	braided monoidal 3-category ≡ 5-category with only one object only one 1-cell	...
" ≡ 3-category with only one object only one 1-cell only one 2-cell	**symmetric monoidal category** ≡ 4-category with only one object only one 1-cell only one 2-cell	sylleptic monoidal 2-category ≡ 5-category with only one object only one 1-cell only one 2-cell	sylleptic monoidal 3-category ≡ 6-category with only one object only one 1-cell only one 2-cell	...
" ≡ 4-category with only one object only one 1-cell only one 2-cell only one 3-cell	" ≡ 5-category with only one object only one 1-cell only one 2-cell only one 3-cell	**symmetric monoidal 2-category** ≡ 6-category with only one object only one 1-cell only one 2-cell only one 3-cell	? ≡ 7-category with only one object only one 1-cell only one 2-cell only one 3-cell	...
"	"	"	**symmetric monoidal 3-category**	...
⋮	⋮	⋮	⋮	

constraint in the "old" n-category will appear as a distinguished 0-cell in the "new" $(n-k)$-category under the dimension-shift depicted in Figure 1. (For $n = 2$ this phenomenon is mentioned by Leinster in [17] and was further described in a talk [18].)

This problem becomes worse when considering functors, transformations, modifications, and so on, as we will discuss in the next section.

0.1 Totalities of structures

Broadly speaking we have two aims:

1. Object level: to find the structures predicted by the Periodic Table arising from degenerate tricategories.

2. Structure level: to make precise statements about the claims of the Periodic Table by examining the *totalities* of the structures involved, that is, not just the degenerate n-categories but also all the higher morphisms between them.

The point of (1) is that in practice we may simply want to know that a given doubly degenerate tricategory is a braided monoidal category, or that a given functor is a braided monoidal functor, for example, without needing to know if the *theory* of doubly degenerate tricategories corresponds to the theory of braided monoidal categories. The motivating example discussed in [1] is the degenerate n-category of "manifolds with corners embedded in n-cubes"; work towards constructing such a structure appears in [2] and [6].

In this work we see that although the tricategories and functors behave more-or-less as expected, the higher morphisms are much more general than the ones we want. Moreover, for (2) we see that the overall dimensions of the totalities do not match up. On the one hand we have k-degenerate n-categories, which naturally organise themselves into an $(n+1)$-category—the full sub-$(n+1)$-category of **nCat**; by contrast, the structure predicted by the Periodic Table is an $(n-k)$-category with extra structure, and these organise themselves into an $(n-k+1)$-category—the full sub-$(n-k+1)$-category of **(n-k)Cat**. In order to compare an $(n+1)$-category with an $(n-k+1)$-category we either need to remove some dimensions from the former or add some to the latter.

The most obvious thing to do is add dimensions to the latter in the form of higher identity cells. However, we quickly see that this does not yield an equivalence of $(n+1)$-categories because the $(n+1)$-cells of **nCat** are far from trivial. Instead we try to reduce the dimensions of **nCat**. We cannot in general apply a simple truncation to j-dimensions as this will not result in a j-category. Besides, we would also like to restrict the remaining morphisms in order to achieve a better comparison with the structures given in the Periodic Table—*a priori* our morphisms are too general.

The most efficacious way to deal with this is to perform a construction analogous to the construction of "icons" [16]. The idea of icons is to organise bicategories into a *bicategory* rather than a tricategory, by discarding the modifications, selecting only those transformations that have all their components the identity, and altering their composition to ensure closure. This gives us a bicategory **Icon**; the full sub-bicategory whose 0-cells are the degenerate bicategories is then biequivalent to the 2-category of monoidal categories, monoidal functors and monoidal transformations. Note that this is not a sub-tricategory of **Bicat** (but is implicitly a quotient of one). In [5] a somewhat ad hoc approach was taken to yield this structure; icons were introduced in [16] shortly afterwards, and give the right framework for this analysis, as shown by the following results.

For degenerate tricategories, a straightforward generalisation produces the tricategory **MonBicat** of monoidal bicategories, and higher monoidal cells. The idea is that we can organise tricategories into a *tricategory* rather than a tetracategory, by discarding the perturbations, and selecting only those transformations and modifications whose components on objects are the identity; as for icons, we must then alter the composition to ensure closure. The full sub-tricategory whose 0-cells are the degenerate tricategories can then be taken as a definition of the tricategory **MonBicat**. We explicitly construct this tricategory in some detail in Section 3. As in the case of icons, this tricategory does not arise as a full sub-tetracategory of **Tricat**, but is a quotient of one.

For doubly degenerate tricategories, we must *iterate* the icon construction in order to give the correct 2-category **BrMonCat** of braided monoidal categories and braided monoidal higher cells. The idea is that given a monoidal bicategory K we can consider categories weakly enriched in K. These might be expected to organise themselves into a tricategory; however the "icon construction" produces a *bicategory* of these, by restricting the transformations to those with identity components. Starting with $K = \mathbf{Cat}$ and applying this construction once gives the original bicategory **Icon** as described above; applying this construction again (that is, with $K = \mathbf{Icon}$) gives a bicategory whose objects are special kinds of tricategories. The full sub-bicategory whose 0-cells are the doubly degenerate (special kinds of) tricategories is then biequivalent to **BrMonCat**.

An added advantage of the icon construction is that it becomes possible to consider lax maps. This is not possible in general as whiskering fails to be coherent, but modifying the composition as for icons solves this problem. This opens up the possibility of studying lax k-tuply monoidal structures such as the n-fold monoidal categories of [3]; we will discuss this in the sequel to this work.

Note that the structure produced by iterating the icon construction is not the same as that given in [8]. In that work, tricategories are organised into a bicategory by a modified icon construction that restricts the transformations further, whereas iterating the standard icon construction also restricts the tricategories and functors.

To keep this paper to a reasonable length, we will defer the details of this construction to a sequel; furthermore, this generalisation of icons is of independent interest. In the present work we will just give a brief explanation of why a more naive approach fails.

0.2 Results

The main results of [5] can be summed up as follows. (Here we write "degenerate" for "1-degenerate", and "doubly degenerate" for "2-degenerate", although in general we also use "degenerate" for any level of degeneracy.)

- Comparing each degenerate category with the monoid formed by its 1-cells, we exhibit an equivalence of categories of these structures, but not a biequivalence of bicategories.

- Comparing each doubly degenerate bicategory with the commutative monoid formed by its 2-cells, we exhibit a biequivalence of bicategories of these structures, but not an equivalence of categories or a triequivalence of tricategories.

- Comparing each degenerate bicategory with the monoidal category formed by its 1-, 2-, and 3-cells, we exhibit an equivalence of categories of these structures, but not a biequivalence of bicategories or a triequivalence of tricategories.

In the present work we proceed to the next dimension and study degenerate tricategories. We use the fully algebraic definition of tricategory given in [12]; this is based on the definition given in [9] which is not fully algebraic. The results can be summed up as follows, but cannot be stated quite so succinctly.

- Comparing each triply degenerate tricategory with the commutative monoid formed by its 3-cells, we exhibit a triequivalence of tricategories of these structures, but not an equivalence of categories, a biequivalence of bicategories, or a tetra-equivalence of tetra-categories.

- We show how doubly degenerate tricategories give rise to braided monoidal categories. The process of producing the braiding is complicated, and there is a great deal of "extra structure" on the resulting braided monoidal category. The disparity is even greater for functors, transformations and modifications.

- A degenerate tricategory gives, by definition, a monoidal bicategory formed by its 1-cells, 2-cells and 3-cells. The totality of monoidal bicategories has not previously been understood; here we consider the tricategory of tricategories described above, and use this to define a tricategory **MonBicat** of monoidal bicategories, in which the higher-dimensional structure is not directly inherited from **Tricat**.

The organisation of the paper is as follows; it is worth noting that each section is significant for different reasons, as we will point out. In Section 1 we examine triply degenerate tricategories; the significance of this section is that this is a "stable" case, and the results therefore have implications for the Stabilisation Hypothesis. In Section 2 we examine doubly degenerate tricategories. We show that these give braided monoidal categories with extra structure, and briefly discuss how a naive approach fails to handle this structure correctly.

In Section 3 we examine degenerate tricategories (i.e. 1-degenerate tricategories). The main purpose of this section is to give the first full definition of algebraic monoidal bicategories, together with their functors, transformations and modifications, and to organise them into a tricategory **MonBicat**.

The case of doubly degenerate tricategories shows us that a k-degenerate n-category does not give rise to a k-tuply monoidal structure on the associated $(n-k)$-category in a straightforward way. In the sequel to this paper we will see that iterating the icon construction produces special kinds of n-categories whose k-degenerate versions more naturally give rise to k-tuply monoidal structures as required. The problem of turning a k-tuply monoidal structure into the desired entry in the Periodic Table is then a separate issue.

1 Triply degenerate tricategories

In this section, we will study triply degenerate tricategories and the higher morphisms between them—functors, transformations, modifications and perturbations. By the Periodic Table, triply degenerate tricategories are expected to be commutative monoids; by results of [5] we now expect them to be commutative monoids *equipped with some distinguished invertible elements* arising from the structure constraints in the tricategory. The process of finding how many such elements there are is highly technical and not particularly enlightening; we simply examine the data and axioms for a tricategory and calculate which constraints determine the others in the triply degenerate case. The importance of these results is not in the exact number of distinguished invertible elements, but rather in the fact that there are any at all, and more than in the bicategory case. We expect n-degenerate n-categories to have increasing numbers of distinguished invertible elements as n increases, and thus for the precise algebraic situation to become more and more intractible in a somewhat uninteresting way.

The other important part of this result examines whether the higher morphisms between triply degenerate tricategories rectify the situation—if any higher morphisms essentially ignore the distinguished invertible elements already specified, then we can still have a structure equivalent to commutative monoids. For doubly degenerate bicategories, this happened at the transformation level; for triply degenerate tricategories, this happens at the modification level. As expected from results of [5], the top level morphisms, that is the perturbations, destroy the possibility of an equivalence on the level of tetracategories.

Throughout this section we use results of [5] to characterise the (single) doubly degenerate hom-bicategory of a triply degenerate tricategory.

1.1 Basic results

The overall results for triply degenerate tricategories are as follows; we will discuss the calculations that lead to these results in the following sections. We should also point out that the results in this section show that the higher-dimensional hypotheses we made in [5] are incorrect.

Theorem 1.1.

1. *A triply degenerate tricategory T is precisely a commutative monoid X_T together with eight distinguished invertible elements $d, m, a, l, r, u, \pi, \mu$.*

2. *Extending the above correspondence, a weak functor $S \to T$ is precisely a monoid homomorphism $F : S \to T$ together with four distinguished invertible elements m_F, χ, ι, γ.*

3. *Extending the above correspondence, a tritransformation $\alpha : F \to G$ is precisely the assertion that $(F, m_F) = (G, m_G)$ together with distinguished invertible elements Π and α_T.*

4. *Extending the above correspondence, a trimodification $m : \alpha \Rightarrow \beta$ is precisely the assertion that α and β are parallel.*

5. *Extending the above correspondence, a perturbation $\sigma : m \Rrightarrow n$ is precisely an element σ in T.*

1.2 Tricategories

In this section we perform the calculations for the triply degenerate tricategories themselves. First we prove a useful lemma concerning adjoint equivalences. The data for a tricategory involves the specification of various adjoint equivalences whose components are themselves adjoint equivalences in the doubly-degenerate hom-bicategories. We are thus interested in adjoint equivalences in doubly degenerate bicategories.

Lemma 1.2. *Let B be a doubly degenerate bicategory. Then an adjoint equivalence $(f, g, \eta, \varepsilon)$ in B consists of an invertible element $\eta \in X_B$ with $\varepsilon = \eta^{-1}$.*

Proof. The triangle identities yield the following equation in any bicategory.

$$\eta * 1_g = a^{-1} \circ (1_g * \varepsilon^{-1}) \circ r_g^{-1} \circ l_g$$

Using the fact that B is doubly degenerate, we see that in the commutative monoid X_B (with unit written as 1) $a = 1, 1_g = 1$, and $r = l$. We also note that $* = \circ$, so the above equation reduces to the fact that η and ε are inverse to each other. □

A priori, a triply degenerate tricategory T consists of the following data, which we will need to try to "reduce":

- a single object $\star$;
- a doubly degenerate bicategory $T(\star, \star)$, which will be considered as a commutative monoid with distinguished invertible element, (T, d_T);
- a weak functor $T(\star, \star) \times T(\star, \star) \to T(\star, \star)$, which will be considered as a monoid homomorphism together with a distinguished invertible element, $(\otimes, m_T)$;

- a weak functor $I : 1 \longrightarrow T(\star, \star)$, which will be considered as the unique monoid homomorphism $1 \longrightarrow T$ together with a distinguished invertible element u_T;
- an adjoint equivalence $\mathbf{a} : \otimes \circ \otimes \times 1 \Rightarrow \otimes \circ 1 \times \otimes$, which is the assertion that $\otimes$ is strictly associative as a binary operation on T together with a distinguished invertible element a_T;
- adjoint equivalences $\mathbf{l} : \otimes \circ I \times 1 \Rightarrow 1, \mathbf{r} : \otimes \circ 1 \times I \Rightarrow 1$, which is the assertion that 1 is a unit for $\otimes$ as a binary operation on T, together with distinguished invertible elements l_T, r_T;
- and four distinguished invertible elements $\pi_T, \mu_T, \lambda_T, \rho_T$.

Thus we have a commutative monoid T, a monoid homomorphism

$$\otimes : T \times T \to T,$$

and distinguished invertible elements $d_T, m_T, u_T, a_T, l_T, r_T, \pi_T, \mu_T, \lambda_T, \rho_T$. The fact that $\otimes$ is a monoid homomorphism is expressed in the following equation, where we have written the monoid structure on T as concatenation.

$$(ab) \otimes (cd) = (a \otimes c)(b \otimes d)$$

The adjoint equivalences $\mathbf{l}, \mathbf{r}$ each imply that 1 is a unit for $\otimes$. Using this and the equation above, the Eckmann-Hilton argument immediately implies that $a \otimes b = ab$.

We will later need to use the naturality isomorphisms; it is simple to compute that that the naturality isomorphism for the transformation a is 1, and the naturality isomorphisms for l and r are both m_T.

There are three tricategory axioms that we must now check to find the dependence between distinguished invertible elements. Using the above, it is straightforward to check that the first tricategory axiom is vacuous, the second gives the equation

$$\lambda\pi = d^2 m_T^4,$$

and the third gives the equation

$$\rho\pi = d^2 m_T^4.$$

Since λ, ρ, π, and d are invertible,

$$\lambda = \rho = \pi^{-1} d^2 m_T^4.$$

Thus λ and ρ are determined by the remaining data, hence we have the result as summarised above.

1.3 Weak functors

In this section we characterise weak functors between triply degenerate tricategories. *A priori* a weak functor $F : S \to T$ between triply degenerate tricategories consists of the following data, which we will try to simplify:

- a weak functor $F_{\star,\star} : S(\star,\star) \to T(\star,\star)$, which by the results of [5] is a monoid homomorphism $F : S \to T$ together with a distinguished invertible element $m_F \in T$;
- an adjoint equivalence $\chi : \otimes' \circ (F \times F) \Rightarrow F \circ \otimes$, which is the trivial assertion that $F(a \otimes b) = Fa \otimes' Fb$ together with a distinguished invertible element $\chi \in T$;
- an adjoint equivalence $\iota : I'_\star \Rightarrow F \circ I_\star$, which is the trivial assertion that $F1 = 1$ together with a distinguished invertible element $\iota \in T$;
- and invertible modifications ω, γ, and δ.

Thus we have a monoid homomorphism F and six distinguished invertible elements $m_F, \chi, \iota, \omega, \gamma$, and δ. It is straightforward to compute that the naturality isomorphism for χ is given by the invertible element $Fm_S \cdot (m_T m_F)^{-1}$ and the naturality isomorphism for ι is given by m_F.

There are two axioms for weak functors for tricategories. In the case of triply degenerate tricategories, the first axiom reduces to the equation

$$\omega \cdot \pi_T \cdot Fm_S^2 \cdot m_T^{-2} \cdot Fd_S^2 \cdot d_T^{-2} = F\pi_S$$

thus by invertibility ω is determined by the rest of the data. The second axiom reduces to the equation

$$\omega \cdot \delta \cdot \gamma \cdot \mu_T \cdot Fm_S^2 \cdot m_T^{-2} \cdot Fd_S^2 \cdot d_T^{-2} = F\mu_S.$$

By the previous equation and the invertibility of all terms involved, δ and γ determine each other once the rest of the data is fixed, hence we have the result as summarised above.

1.4 Tritransformations

In this section we characterise tritransformations for triply degenerate tricategories. First we need the following lemma, which is a simple calculation.

Lemma 1.3. *Let T be a triply degenerate tricategory. Then the functor*

$$T(1, I_\star) = I_\star \circ - : T(\star, \star) \to T(\star, \star)$$

is given by the identity homomorphism together with the distinguished invertible element $d^{-1}m$. Additionally, $T(1, I_\star) = T(I_\star, 1)$.

A priori, the data for a tritransformation $\alpha : F \to G$ of triply degenerate tricategories consists of:

- an adjoint equivalence $\boldsymbol{\alpha} : T(1, I_\star) \circ F \Rightarrow T(I_\star, 1) \circ G$, which consists of the assertion that $F = G$ as monoid homomorphisms together with a distinguished invertible element α_T; and
- distinguished invertible elements Π and M.

It is easy to compute that the naturality isomorphism for the transformation α is $m_F^{-1} m_G$. The first transformation axiom reduces to the equation

$$m_G = m_F,$$

the second axiom reduces to the equation

$$\Pi \mu_T l_T \gamma_F = M m_T^4 d_T^2 a_T^{-1} \gamma_G,$$

and the third to the equation

$$\Pi \delta_F = a_T^{-1} l_T^{-1} d_T^2 m_T^4 \mu^{-1} M \delta_G.$$

Thus we see that Π determines M, and that the second and third axioms combine to yield no new information. So we have remaining distinguished invertible elements Π and α_T, giving the results as summarised above.

1.5 Trimodifications and perturbations

The data for a trimodification $m : \alpha \Rightarrow \beta$ consists of a single invertible element m in T, and there are two axioms. The first is the equation

$$m^2 \cdot \Pi \cdot Gd_S = \Pi \cdot Fd_S \cdot m$$

which reduces to $m = 1$ since $F = G$ as monoid homomorphisms. The second axiom also reduces to $m = 1$, thus there is a unique trimodification between any two parallel transformations. Note that this means that *any* diagram of trimodifications in this setting commutes, a fact that will be useful later.

The data for a perturbation $\sigma : m \Rrightarrow n$ consists of an element σ in T. The single axiom is vacuous so a perturbation is precisely an element $\sigma \in T$.

1.6 Overall structure

We now compare the totalities of, on the one hand triply degenerate tricategories, and on the other hand commutative monoids. Recall that for the case of doubly degenerate bicategories we were able to attempt comparisons at the level of categories, bicategories and tricategories of such, simply by truncating the full sub-tricategory of **Bicat** to the required dimension. However, for triply degenerate tricategories we show that truncating the full sub-tetracategory of **Tricat** does not yield a category or a bicategory; truncation does yield a tricategory, and this is the only level that yields an equivalence with commutative monoids. As in [5] we compare with the discrete j-categories of commutative monoids obtained by adding higher identity cells to **CMon**.

Note that we do not actually prove that we have a tetracategory of triply degenerate tricategories; for the comparison, we simply prove that the obvious putative functor is not full and faithful and therefore cannot be an equivalence.

We have a 4-dimensional structure with

0-cells:	triply degenerate tricategories
1-cells:	weak functors between them
2-cells:	tritransformations between those
3-cells:	trimodifications between those
4-cells:	perturbations between those.

We write $\mathbf{Tricat}(3)_j$ for the truncation of this structure to a j-dimensional structure, and $\mathbf{CMon}_j$ for the j-category of commutative monoids and their morphisms (and higher identities where necessary).

There are obvious assignments

triply degenerate tricategory	$\mapsto$	underlying commutative monoid
weak functor	$\mapsto$	underlying homomorphism of monoids

which, together with the unique maps on higher cells, form the underlying morphism on j-globular sets for putative functors

$$\xi_j : \mathbf{Tricat}(3)_j \to \mathbf{CMon}_j.$$

Theorem 1.4.

1. $\mathbf{Tricat}(3)_1$ *is not a category.*
2. $\mathbf{Tricat}(3)_2$ *is not a bicategory.*
3. $\mathbf{Tricat}(3)_3$ *is a tricategory, and* ξ_3 *defines a functor which is a triequivalence.*
4. ξ_4 *does not give a tetra-equivalence of tetra-categories.*

The rest of this section will constitute a gradual proof of the various parts of this theorem. We begin by constructing the hom-bicategories for a tricategory structure on $\mathbf{Tricat}(3)_3$.

Proposition 1.5. *Let* X, Y *be triply degenerate tricategories. Then there is a bicategory* $\mathbf{Tricat}(3)_3(X,Y)$ *with 0-cells weak functors* $F : X \to Y$*, 1-cells tritransformations* $\alpha : F \Rightarrow G$*, and 2-cells trimodifications* $m : \alpha \Rrightarrow \beta$*.*

Proof. To give the bicategory structure, we need only provide unit 1-cells and 1-cell composition since there is a unique trimodification between every pair of parallel tritransformations. It is simple to read off the required distinguished invertible elements from the corresponding formulae for composites of tritransformations and from the data for the unit tritransformation. □

Remark 1.6. Note that composition of 1-cells in $\mathbf{Tricat}(3)_3(X, Y)$ is strictly associative, but is not strictly unital. In particular, this shows that $\mathbf{Tricat}(3)_2$ is not a bicategory, proving Theorem 1.4, part 2.

We now construct the composition functor

$$\otimes : \mathbf{Tricat}(3)_3(Y, Z) \times \mathbf{Tricat}(3)_3(X, Y) \to \mathbf{Tricat}(3)_3(X, Z).$$

for any triply degenerate tricategories X, Y, Z. We define the composite GF of functors $F : X \to Y$, $G : Y \to Z$ by the following formulae which can be read off directly from the formulae giving the composite of functors between tricategories.

$$\begin{array}{rcl} m_{GF} & = & m_G G m_F \\ \chi_{GF} & = & \chi_G G(\chi_F d_Y) d_Z^{-2} \\ \iota_{GF} & = & \iota_G G(\iota_F d_Y) d_Z^{-2} \\ \gamma_G & = & d_Z^{-2} m_Z^2 m_G^2 \gamma_G G(\gamma_F d_Y m_Y) \end{array}$$

The formulae for the composite $\beta \otimes \alpha$ of two transformations are derived similarly, and thus we have a weak functor $\otimes$ for composition as required.

Similarly, there is a unit functor

$$I_X : 1 \to \mathbf{Tricat}(3)_3(X, X)$$

whose value on the unique 0-cell is the identity functor on X.

Remark 1.7. The formulae above make it obvious that $\otimes$ is not strictly associative on 0-cells, and that the identity functor is not a strict unit for $\otimes$. This shows that $\mathbf{Tricat}(3)_1$ is not a category, proving Theorem 1.4, part 1.

Next we need to specify the required constraint adjoint equivalences. It is straightforward to find adjoint equivalences

$$\begin{aligned} \mathcal{A} &: \otimes \circ \otimes \times 1 \Rightarrow \otimes \circ 1 \times \otimes \\ \mathcal{L} &: \otimes \circ I \times 1 \Rightarrow 1 \\ \mathcal{R} &: \otimes \circ 1 \times I \Rightarrow 1 \end{aligned}$$

in the appropriate functor bicategories; the actual choice of adjoint equivalence is irrelevant, since there is a unique modification between any pair of parallel transformations.

Finally, to finish constructing the tricategory $\mathbf{Tricat}(3)_3$ we must define invertible modifications π, μ, λ, ρ and check three axioms. However since there are unique trimodifications between parallel tritransformations, these modifications are uniquely determined and the axioms automatically hold.

We now examine the morphism ξ_3 of 3-globular sets and show that it defines a functor

$$\mathbf{Tricat}(3)_3 \longrightarrow \mathbf{CMon}_3;$$

in fact functoriality is trivial as $\mathbf{CMon}_3$ has discrete hom-2-categories. Furthermore we show it is an equivalence as follows. The functor is clearly surjective on objects, and the functor on hom-bicategories

$$\mathbf{Tricat}(3)_3(X,Y) \to \mathbf{CMon}_3(\xi_3 X, \xi_3 Y)$$

is easily seen to be surjective on objects as well. This functor on hom-bicategories is also a local equivalence since $\mathbf{CMon}_3$ is discrete at dimensions two and three and $\mathbf{Tricat}(3)_3$ has unique 3-cells between parallel 2-cells. This finishes the proof of Theorem 1.4, part 3.

For part 4, we observe that the morphism ξ_4 of 4-globular sets is clearly not locally faithful on 4-cells. This finishes the proof of Theorem 1.4.

2 Doubly degenerate tricategories

We now compare doubly degenerate tricategories with braided monoidal categories. As described informally in the Introduction the comparison is not straightforward. Therefore we begin by directly listing the structure that we get on the monoidal category given by the (unique) degenerate hom-bicategory; this is simply a matter of writing out the definitions as nothing simplifies in this case. Afterwards, we show how to extract a braided monoidal category from this structure. Essentially, all of the data listed in Section 2.2 can be thought of as "extra structure" that arises on the braided monoidal category we will construct.

We will begin with an informal overview of this whole section as we feel that for many readers the ideas will be at least as important as the technical details.

2.1 Overview

It is widely accepted that a doubly degenerate bicategory "is" a commutative monoid, and that a doubly degenerate tricategory "is" a braided monoidal category. Moreover, it is widely accepted that the proof of the bicategory case is "simply" a question of applying the Eckmann-Hilton argument to the multiplications given by horizontal and vertical composition, and that the tricategory result is proved by doing this process up to isomorphism. In this section we give an informal overview of the extent to which this is and is not the case. We believe that this is important because the disparity will increase as dimensions increase, and because this issue seems to lie at the heart of various critical phenomena in higher-dimensional category theory, such as:

1. why we do not expect every weak n-category to be equivalent to a strict one
2. why weak n-categories are expected to model homotopy n-types while strict ones are known not to do so [10, 1, 22]
3. why some diagrams of constraints in a tricategory do not in general commute, and why these do not arise in free tricategories [12]
4. why strict computads do not form a presheaf category [19]
5. why the existing definitions of n-categories based on reflexive globular sets fail to be fully weak [7]

6. why a notion of semistrict n-category with weak units but strict interchange may be weak enough to model homotopy n-types and give coherence results [21, 15, 13].

A doubly degenerate bicategory B has only one 0-cell $\star$ and only one 1-cell $I_\star$. To show that the 2-cells form a commutative monoid we first use the fact that they are the morphisms of the single hom-category $B(\star, \star)$; since this hom-category has only one object $I_\star$ we know it is a monoid, with multiplication given by vertical composition of 2-cells. To show that it is a *commutative* monoid, we apply the Eckmann-Hilton argument to the two multiplications defined on the set of 2-cells: vertical composition and horizontal composition.

Recall that the Eckmann-Hilton argument says: Let A be a set with two binary operations $*$ and $\circ$ such that

1. $*$ and $\circ$ are unital with the same unit
2. $*$ and $\circ$ distribute over each other i.e. $\forall a, b, c, d \in A$

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d).$$

Then $*$ and $\circ$ are in fact equal and this operation is commutative.

However, in our case a difficulty arises because horizontal composition in a bicategory is not strictly unital. The situation is rescued by the fact that $l_I = r_I$ in any bicategory. This, together with the naturality of l and r, enables us to prove, albeit laboriously, that horizontal composition is strictly unital for 2-cells in a *doubly degenerate* bicategory, and moreover that the vertical 2-cell identity also acts as a horizontal identity. Thus we can in fact apply the Eckmann-Hilton argument.

Generalising this argument to doubly degenerate tricategories directly is tricky. There are various candidates for a "categorified Eckmann-Hilton argument" provided by Joyal and Street [14, 4]. The idea is to replace all the equalities in the argument by isomorphisms, but as usual we need to take some care over *specifying* these isomorphisms rather than merely asserting their existence; see Definition 2.8.

However, when we try and apply this result to a doubly degenerate tricategory we have some further difficulties: composition along bounding 0-cells is difficult to manipulate as a multiplication, because we cannot use coherence results for tricategories. Coherence for tricategories

[11] tells us that "every diagram of constraints in a free tricategory (on a category-enriched 2-graph) commutes". In particular this means that if we need to use cells that do not arise in a free tricategory, then we cannot use coherence results to check axioms. This is the case if we attempt to build a multiplication out of composition along 0-cells; we have to use the fact that we only have one 1-cell in our tricategory, and therefore that various composites of 1-cells are all "accidentally" the same. This comes down to the fact that the free tricategory on a doubly degenerate tricategory is not itself doubly degenerate; it is not clear how to construct a "free doubly degenerate tricategory".

However, to rectify this situation we can look at an alternative way of proving the result for degenerate bicategories that does not make such identifications. We still use the Eckmann-Hilton argument but instead of attempting to apply it using horizontal composition of 2-cells, we define a new binary operation on 2-cells that is derived from horizontal composition as follows:

$$\beta \odot \alpha = r \circ (\beta * \alpha) \circ l^{-1}$$

(Essentially this is what we used to prove that horizontal composition is strictly unital in the previous argument.) Unlike horizontal composition, this operation does "categorify correctly", that is, given a doubly degenerate tricategory we can define a multiplication on its associated monoidal category by using the above formula (this is the content of Theorem 2.10), and we can manipulate it using coherence for tricategories.

To extract a braiding from this we then have to follow the steps of the Eckmann-Hilton argument and keep track of all the isomorphisms used; this is Proposition 2.9.

We see that we use instances of the following cells, in a lengthy composite:

- naturality constraints for l_I and r_I
- constraints for weak interchange of 2-cells
- isomorphisms showing that $l_I \cong r_I$

This indicates why a theory with weak units but strict interchange can still produce braidings that are not necessarily symmetries—the braiding is built from all of the above structure contraints, so if any one of them is weak then braidings can still arise. As mentioned above we do, however, get a certain amount of extra structure on the braided monoidal category that arises; an iterated icon construction enables us to rectify this situation completely, but we defer the details of this to the sequel.

We will also show that every braided monoidal category gives rise to a doubly degenerate tricategory in a canonical way, and moreover, that every doubly degenerate tricategory is triequivalent to one arising in this way.

2.2 Basic results

Many of the diagrams needed in the theorems below are excessively large, and since they are all obtained by simply rewriting the appropriate definitions from [11] using the results of [5], we have omitted them.

Just as we began the previous section by characterising adjoints in doubly degenerate bicategories, we begin this section by recalling the definition of "dual pair" of objects in a monoidal category, since this characterises adjoints for 1-cells in degenerate bicategories; eventually we will of course be interested in adjoint equivalences, not just adjoints.

Definition 2.1. *Let M be a monoidal category. Then a* dual pair *in M consists of a pair of objects $X, X^{\cdot}$ together with morphisms $\varepsilon : X \otimes X^{\cdot} \to I, \eta : I \to X^{\cdot} \otimes X$ satisfying the two equations below, where all unmarked isomorphisms are given by coherence isomorphisms.*

$$\begin{array}{ccccccccc} X & \xrightarrow{\cong} & XI & \xrightarrow{1\eta} & X(X^{\cdot}X) & \xrightarrow{\cong} & (XX^{\cdot})X & \xrightarrow{\varepsilon 1} & IX \\ & & & & & & & & \downarrow{\scriptstyle \cong} \\ & & & & \xrightarrow{\quad 1 \quad} & & & & X \end{array}$$

$$\begin{array}{ccc} X^{\cdot} \xrightarrow{\cong} IX^{\cdot} \xrightarrow{\eta 1} (X^{\cdot}X)X^{\cdot} \xrightarrow{\cong} X^{\cdot}(XX^{\cdot}) \xrightarrow{1\varepsilon} & X^{\cdot}I \\ {\scriptstyle 1} \searrow & \downarrow {\scriptstyle \cong} \\ & X^{\cdot} \end{array}$$

Theorem 2.2. *A doubly degenerate tricategory B is precisely*

- *a monoidal category $(B, \otimes, U, a, l, r)$ given by the single degenerate hom-bicategory;*
- *a monoidal functor $\boxtimes : B \times B \to B$ from composition;*
- *a monoid I in B and an isomorphism $I \cong U$ as monoids in B; this comes from the functor for units $I \longrightarrow B(\star, \star)$*
- *a dual pair $(A, A^{\cdot}, \varepsilon_A, \eta_A)$ with ε_A, η_A both invertible, and natural isomorphisms*

$$A \otimes \Big((X \boxtimes Y) \boxtimes Z) \Big) \cong \Big(X \boxtimes (Y \boxtimes Z) \Big) \otimes A$$
$$A^{\cdot} \otimes \Big(X \boxtimes (Y \boxtimes Z) \Big) \cong \Big((X \boxtimes Y) \boxtimes Z) \Big) \otimes A^{\cdot};$$

  subject to diagrams omitted as discussed above.

- *a dual pair $(L, L^{\cdot}, \varepsilon_L, \eta_L)$ with with ε_L, η_L both invertible, and natural isomorphisms*

$$L \otimes (I \boxtimes X) \cong X \otimes L$$
$$L^{\cdot} \otimes X \cong (I \boxtimes X) \otimes L^{\cdot}$$

  subject to diagrams omitted as discussed above,

- *a dual pair $(R, R^{\cdot}, \varepsilon_R, \eta_R)$ with with ε_L, η_L both invertible, and natural isomorphisms*

$$R \otimes (X \boxtimes I) \cong X \otimes R$$
$$R^{\cdot} \otimes X \cong (X \boxtimes I) \otimes R^{\cdot};$$

  subject to diagrams omitted as discussed above,

- *and isomorphisms*

$$\Big((U \boxtimes A) \otimes \big(A \otimes (A \boxtimes U)\big)\Big) \overset{\pi}{\cong} A \otimes A$$
$$\Big((U \boxtimes L) \otimes \big(A \otimes (R^{\cdot} \boxtimes U)\big)\Big) \overset{\mu}{\cong} U$$
$$L \boxtimes U \overset{\lambda}{\cong} L \otimes A$$
$$U \boxtimes R^{\cdot} \overset{\rho}{\cong} A \otimes R^{\cdot};$$

all subject to three axioms omitted as discussed above.

Remark 2.3. It is important to note that $\boxtimes$ does not *a priori* give a monoidal structure on the category B; the obstruction is that lax transformations between weak functors of degenerate tricategories are more general than monoidal transformations between the associated monoidal functors (see [5]). As noted in Section 2.1 it may be possible to prove that $\boxtimes$ is a valid monoidal structure, but since we cannot use coherence for tricategories to help us, the proof is not very evident. Thus to extract a braiding from all this structure, we will not simply apply an Eckmann-Hilton-style argument to $\otimes$ and $\boxtimes$ (see Section 2.3).

We now describe functors, transformations, modifications and perturbations in a similar spirit.

Theorem 2.4. *A weak functor $F : B \to B'$ between doubly degenerate tricategories is precisely*

- *a monoidal functor $F : B \to B'$;*
- *a dual pair $(\chi, \chi^{\cdot}, \varepsilon_\chi, \eta_\chi)$ in B' with $\varepsilon_\chi, \eta_\chi$ both invertible, and natural isomorphisms*

$$\chi \otimes' (FX \boxtimes' FY) \cong F(X \boxtimes Y) \otimes' \chi$$
$$\chi^{\cdot} \otimes' F(X \boxtimes Y) \cong (FX \boxtimes' FY) \otimes' \chi^{\cdot}$$

subject to diagrams omitted as discussed above,

- *a dual pair* $(\iota, \iota^{\cdot}, \varepsilon_\iota, \eta_\iota)$ *with* $\varepsilon_\iota, \eta_\iota$ *both invertible, and natural isomorphisms*
$$\iota \otimes' I' \cong FI \otimes' \iota$$
$$\iota^{\cdot} \otimes' FI \cong I' \otimes' \iota^{\cdot}$$
subject to diagrams omitted as discussed above,

- *and isomorphisms*

$$FA \otimes' \Big(\chi \otimes' (\chi \boxtimes' U')\Big) \overset{\omega}{\cong} \chi \otimes' \Big((U' \boxtimes' \chi) \otimes' A'\Big)$$
$$FL \otimes' \Big(\chi \otimes' (\iota \boxtimes' U')\Big) \overset{\gamma}{\cong} L'$$
$$FR^{\cdot} \overset{\delta}{\cong} \chi \otimes' \Big((U' \boxtimes' \iota) \otimes' (R')^{\cdot}\Big);$$

all subject to axioms omitted as discussed above.

Theorem 2.5. *A weak transformation* $\alpha : F \to G$ *in the above setting is precisely*

- *a dual pair* $(\alpha, \alpha^{\cdot}, \varepsilon_\alpha, \eta_\alpha)$ *with* $\varepsilon_\alpha, \eta_\alpha$ *both invertible, and natural isomorphisms*

$$\alpha \otimes' (U' \boxtimes' FX) \cong (GX \boxtimes' U') \otimes' \alpha$$
$$\alpha^{\cdot} \otimes' (GX \boxtimes' U') \cong (U' \boxtimes' FX) \otimes' \alpha^{\cdot}$$

subject to diagrams omitted as discussed above,

- *and isomorphisms*

$$(\chi_G \otimes' U') \otimes' \left((A')^{\cdot} \otimes' \Big((U' \boxtimes' \alpha) \otimes' \big(A' \otimes' (\alpha \boxtimes' U')\big)\Big)\right)$$
$$\overset{\Pi}{\cong} \alpha \otimes' \Big((U' \boxtimes' \chi_F) \otimes' A'\Big)$$

$$\alpha \otimes' \Big((U' \boxtimes' \iota_F) \otimes' (R')^{\cdot}\Big) \overset{M}{\cong} (\iota_G \boxtimes' U') \otimes' (L')^{\cdot};$$

all subject to three axioms omitted as discussed above.

The analogous result for lax transformations should be obvious, with dual pair replaced by distinguished object since in the lax case we have a noninvertible morphism instead of an adjoint equivalence.

Theorem 2.6. *A modification $m : \alpha \Rightarrow \beta$ is precisely*

- *an object $m \in B'$ and*
- *an isomorphism*

$$(U' \boxtimes m) \otimes' \alpha \cong \beta \otimes' (m \boxtimes' U')$$

subject to two axioms omitted as discussed above.

Theorem 2.7. *A perturbation $\sigma : m \Rrightarrow n$ is precisely a morphism $\sigma : m \to n$ in B' satisfying the single axiom omitted as discussed above.*

2.3 Braidings

In this section we show that the underlying monoidal category of a doubly degenerate tricategory does have a braiding on it. To show this, we use the fact that to give a braiding for a monoidal structure, it suffices to give the structure of a multiplication on the monoidal category in question. We give the relevant definitions below; for additional details, see [14].

Definition 2.8. *Let M be a monoidal category, and equip $M \times M$ with the componentwise monoidal structure. Then a multiplication φ on M consists of a monoidal functor $\varphi : M \times M \to M$ and invertible monoidal transformations $\rho : \varphi \circ (id \times I) \Rightarrow id, \lambda : \varphi \circ (I \times id) \Rightarrow id$ where $I : 1 \to M$ is the canonical monoidal functor whose value on the single object is the unit of M and whose structure constraints are given by unique coherence isomorphisms.*

The following result, due to Joyal and Street [14], says that a multiplication naturally gives rise to a braiding.

Proposition 2.9. *Let M be a monoidal category with multiplication φ. Then M is braided with braiding given by the composite below.*

$$ab \xrightarrow{\lambda^{-1}\rho^{-1}} \phi(I,a)\phi(b,I) \xrightarrow{\cong} \phi(Ib,aI) \xrightarrow{\phi(l,r)} \phi(b,a)$$
$$\xrightarrow{\phi(r^{-1},l^{-1})} \phi(bI,Ia) \xrightarrow{\cong} \phi(b,I)\phi(I,a) \xrightarrow{\rho\lambda} ba$$

We will use this construction to provide a braiding for the monoidal category associated to a doubly degenerate tricategory. As can be seen from the above formula, this braiding is "natural" but not exactly "simple".

Theorem 2.10. *Let B be a doubly degenerate tricategory, and also denote by B the monoidal category associated to the single (degenerate) hom-bicategory. Then there is a multiplication φ on B with*

$$\varphi(X,Y) = R \otimes \big((X \boxtimes Y) \otimes L^{\cdot}\big).$$

This result is a lengthy but routine 2-dimensional diagram chase that requires repeated use of the coherence theorm for tricategories as well as coherence for bicategories and functors. We thus omit it, and only record the following crucial corollary.

Corollary 2.11. *Let B be a doubly degenerate tricategory, and also denote by B the monoidal category associated to the single (degenerate) hom-bicategory. Then B is a braided monoidal category.*

The situation for functors is similar, with braided monoidal functors arising from "multiplicative" functors as follows.

Definition 2.12. *Let (M,φ) and (N,ψ) be monoidal categories equipped with multiplications. A multiplicative functor $F : (M,\varphi) \to (N,\psi)$ consists of a monoidal functor $F : M \to N$ and an invertible monoidal transformation $\chi : \psi \circ (F \times F) \Rightarrow F \circ \phi$, satisfying unit axioms.*

Proposition 2.13. *Let (M,φ) and (N,ψ) be monoidal categories equipped with multiplications, and let $F : (M,\varphi) \to (N,\psi)$ be a multiplicative functor between them. Then the underlying monoidal functor F is braided when M and N are equipped with the braidings induced by their respective multiplications.*

The following theorem says that functors between doubly degenerate tricategories do give rise to multiplicative functors, and as a corollary, braided monoidal functors. The proof of the theorem is another long but routine calculation involving coherence.

Theorem 2.14. *Let B and B' be doubly degenerate tricategories, and let $F : B \to B'$ be a functor between them. Then the monoidal functor F between the monoidal categories B and B' can be given the structure of a multiplicative functor when we equip B and B' with the multiplications of Theorem 2.10.*

Corollary 2.15. *Let B and B' be doubly degenerate tricategories, and let $F : B \to B'$ be a functor between them. Then the monoidal functor F is braided with respect to the braided monoidal categories B and B' as in Corollary 2.11.*

The situation for transformations does not lend itself to the same sort of analysis: a transformation of doubly degenerate tricategories is rather different from a monoidal transformation. This also occurs in the study of degenerate bicategories, where transformations of degenerate bicategories are rather different from monoidal transformations. Thus, as discussed in the introduction, the best approach is to iterate the icon construction. We defer the details of this to the sequel; here we will just include a brief discussion to show how problematic a more naive approach would be.

An ad hoc or "naive" approach would be to strictify the doubly degenerate tricategories a little in order to make the "extra structure" on the associated braided monoidal category trivial. This may seem like a straightforward case of insisting that some coherence constraints are identities, but in order to organise the resulting tricategories into a bicategory we quickly see that we must make at least the following restrictions.

1. Restrict to those transformations whose component is I,

2. To ensure closure under composition, restrict to those tricategories in which $I \circ I = I$ with $l_I = r_I = 1$, and those functors F satisfying $FI = I$ and coherence constraint $\phi_I^F = 1$.

We must then check that the resulting structure is a bicategory.

There are various ways in which this approach is unsatisfactory; its ad hoc nature means that it does not generalise easily to higher dimensions, nor does it provide any insight into the relationships between degenerate structures and predictions of the periodic table. However, the most compelling way in which it is unsatisfactory is that a much more elegant approach exists, that is, the iterated icon approach.

In the iterated icon approach the correct totality of degenerate structures arises naturally, with no contrived restrictions necessary. Furthermore, it is clear how to generalise this to higher dimensions. Finally, we observe a further benefit in that the icon approach enables us to deal with fully lax situations, which we cannot otherwise do.

2.4 Strictification

While it is beyond the scope of this work to treat the totalities of structures in full, it is useful to consider the following "local" results.

Theorem 2.16.

1. *Given a braided monoidal category B, we can construct a doubly degenerate tricategory $\Sigma^2 B$ such that applying the construction in Corollary 2.11 returns the braided monoidal category B.*

2. *Every doubly degenerate tricategory T is triequivalent to one obtained from a braided monoidal category in the above way.*

Proof.

1. First choose $\boxtimes$ to be the tensor product of B; this is a monoidal functor since B is braided. Now choose all the dual pairs to be given by the unit, and all isomorphisms to be coherence isomorphisms. The axioms all follow from coherence for braided monoidal categories.

2. Let T be a doubly degenerate tricategory. Recall from [11] that there is a Gray-category $\mathbf{Gr}T$ and a functor $e : \mathbf{Gr}T \longrightarrow T$ with the following properties.

- The 0-cells of $\mathbf{Gr}T$ are just the 0-cells of T.
- The 1-cells of $\mathbf{Gr}T$ are formal strings of 1-cells in T.
- e is a triequivalence of tricategories.

Consider the full sub-Gray-category $T' \hookrightarrow \mathbf{Gr}T$ with a single 0-cell and single 1-cell given by the identity in $\mathbf{Gr}T$. We show that T' comes from a braided monoidal category as in (1) and that the inclusion is a triequivalence.

First observe that strict braided monoidal categories give rise to doubly degenerate Gray categories by the construction in (1), and that every doubly degenerate Gray category arises in this way. Thus since T' is constructed as a doubly degenerate Gray-category, we know it must come from a braided monoidal category.

Now consider the inclusion $T' \hookrightarrow \mathbf{Gr}T$. It is trivially surjective on 0-cells so we only need to show that it is locally a biequivalence of bicategories.

To show that the map on hom-bicategories is locally an equivalence of categories we note that it is actually the identity by construction, since T' is a full sub-Gray-category of T.

To show that the map on hom-bicategories is bi-essentially surjective, we must show that every 1-cell in $\mathbf{Gr}T$ is equivalent to the identity $\mathbf{Gr}T$. Since T only has a single 1-cell, namely the identity I, every 1-cell in $\mathbf{Gr}T$ is a formal string of I's; the string of length 0 is the identity in $\mathbf{Gr}T$. Any string of I's in $\mathbf{Gr}T$ is sent by e to an actual composite of I's in T, and these are all equivalent in T via left or right unit constraints; in particular, the string of length 0 is sent to I. Now e is a triequivalence, so 1-cells in $\mathbf{Gr}T$ are equivalent if and only if they are equivalent in T after applying e, hence all 1-cells in $\mathbf{Gr}T$ are equivalent. This shows that the map on hom-bicategories is bi-essentially surjective.

This completes the proof that the inclusion $T' \hookrightarrow \mathbf{Gr}T$ is a triequivalence; finally we conclude that the composite map

$$T' \hookrightarrow \mathbf{Gr}T \xrightarrow{e} T$$

exhibits the triequivalence required.

3 Degenerate tricategories

We now study degenerate tricategories, and use them to make a definition of monoidal bicategory. The difference between these structures becomes more significant at the level of transformation, where we take an "iconic" approach in order to obtain monoidal transformations between monoidal bicategories. Since we will *define* monoidal bicategories to be degenerate tricategories, a process of "comparison" would be rather circular. We just observe that our definition of transformation is significantly different from that inherited from **Tricat**, just as in the case of transformations between degenerate bicategories [5].

First we characterise degenerate tricategories and functors between them; this is straightforward, as we can simply rewrite the appropriate definitions using the results of [5]. Our definitions differ from existing definitions [9, 20] only in that they are fully algebraic. As with degenerate bicategories, we only need to modify the structures at the level of transformations and above.

Theorem 3.1. *A degenerate tricategory B is precisely*

- *a single hom-bicategory which we will also call B;*
- *a functor $\otimes : B \times B \to B$;*
- *a functor $I : 1 \to B$;*
- *adjoint equivalence $\boldsymbol{a}, \boldsymbol{l}$, and $\boldsymbol{r}$ as in the definition of a tricategory; and*
- *invertible modifications π, μ, λ, and ρ as in the definition of a tricategory*

all subject to the tricategory axioms.

Theorem 3.2. *A weak functor $F : B \to B'$ between degenerate tricategories is precisely*

- *a weak functor $F : B \to B'$;*
- *adjoint equivalences $\boldsymbol{\chi}$ and $\boldsymbol{\iota}$ as in the definition of weak functor between tricategories; and*
- *invertible modifications ω, δ, and γ as in the definition of weak functor, as shown below*

all subject to axioms which are identical to the functor axioms aside from source and target considerations.

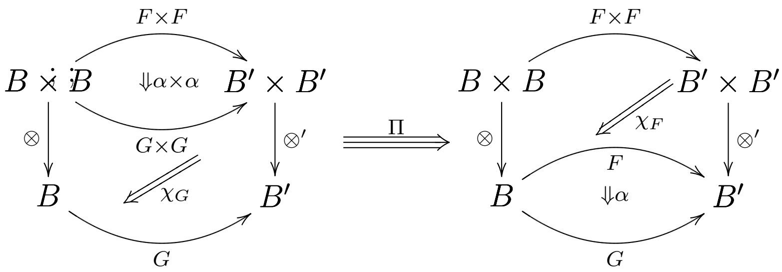
We use the above as *definitions* of monoidal bicategory and monoidal functor, and we now show how to organise the totality of these into a tricategory. As in the case of degenerate bicategories, we cannot simply take the full sub-tetracategory of **Tricat**; instead, we must perform an icon-like construction to ensure that we get the correct notions of monoidal transformation and modification. This is an immediate generalisation of the 2-dimensional version in which the bicategory of monoidal categories, monoidal functors and monoidal transformations can be found as a full sub-bicategory of the bicategory of icons. For details of the icon construction see [16]. In this case the idea is to construct a tricategory of tricategories with restricted versions of transformations and modifications as the 2-cells and 3-cells. In the present work we only give the degenerate case i.e. monoidal bicategories.

Thus we define monoidal transformations as a special case of lax transformations where the single object component is the identity, the lax transformation α is actually weak, and the two modifications Π and M are invertible. The data and axioms presented here use collapsed versions of the transformation diagrams, making use of the left and right unit adjoint equivalences to simplify the diagrams involved.

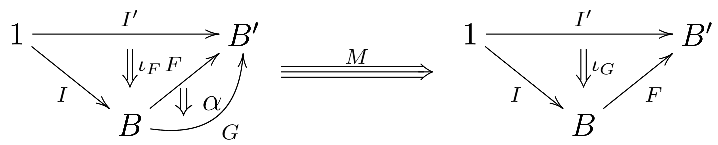
Definition 3.3. Let B, B' be monoidal bicategories and $F, G : B \to B'$ be monoidal functors between them. A *monoidal transformation* $\alpha : F \Rightarrow G$ consists of

- a weak transformation $\alpha : F \Rightarrow G$ between the underlying weak functors,

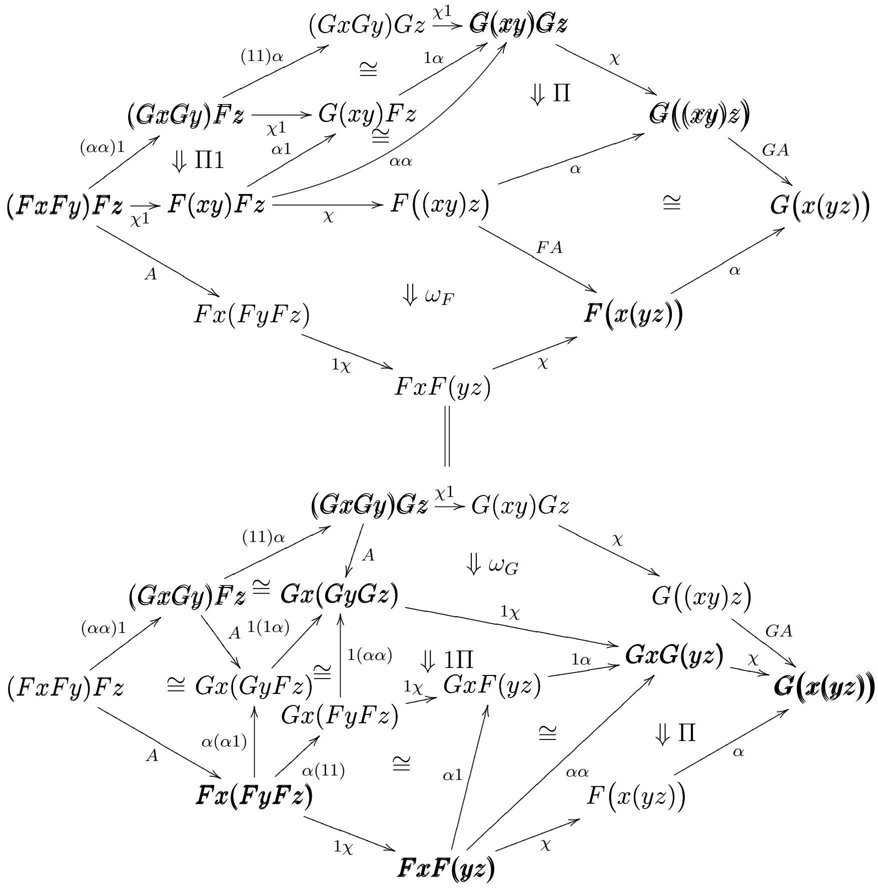
- an invertible modification as displayed below,

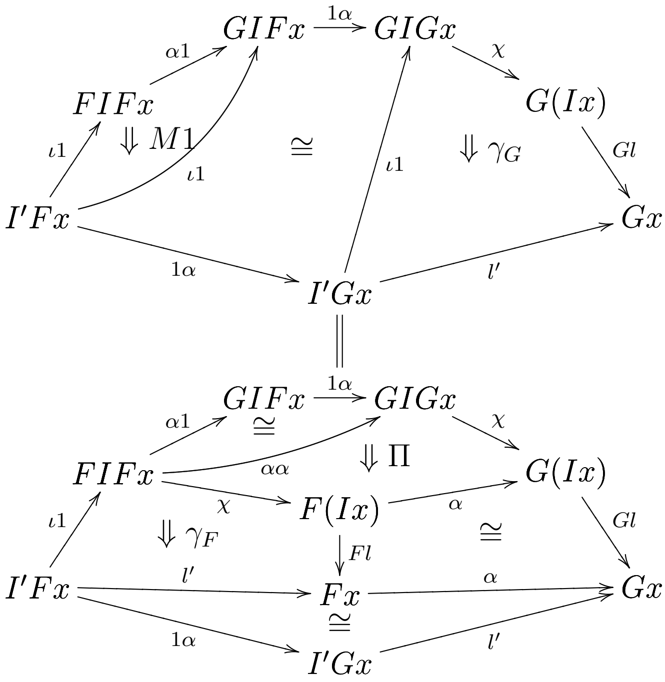


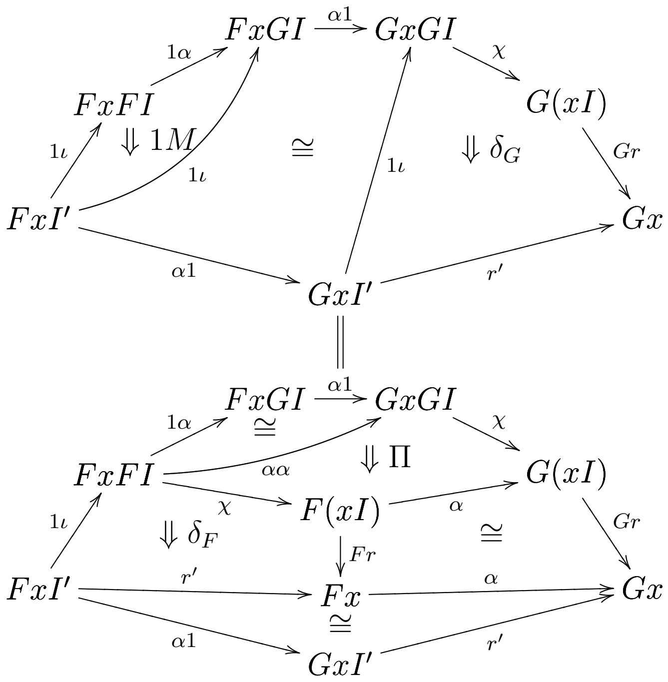
- and an invertible modification as displayed below,



all subject to the following three axioms.



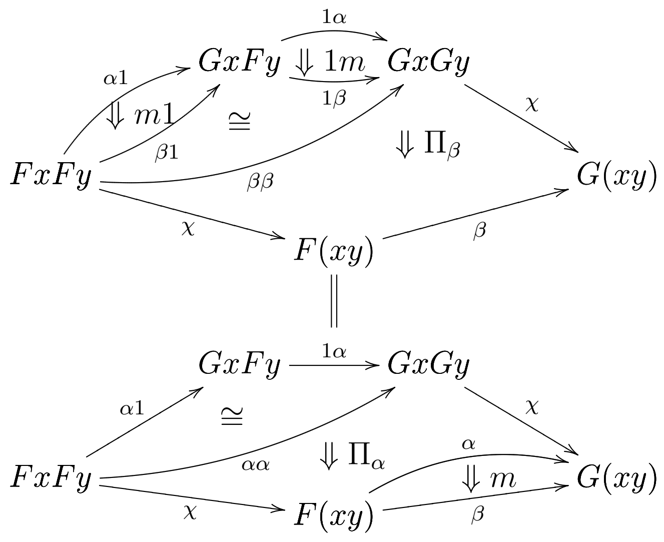


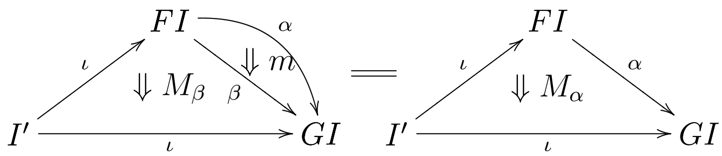


Note that in the previous diagram we have written δ_F and δ_G when in fact their mates are used.

We now define monoidal modifications between monoidal bicategories in a similar fashion, as a special case of lax modifications with the component at the single object being given by an identity. Using the left and right unit adjoint equivalences, we are then able to simplify the diagrams to those given below.

Definition 3.4. Let $\alpha, \beta : F \Rightarrow G$ be monoidal transformations between monoidal functors. A *monoidal modification* $m : \alpha \Rrightarrow \beta$ consists of a modification $m : \alpha \Rrightarrow \beta$ between the underlying transformations such that the following two axioms hold.



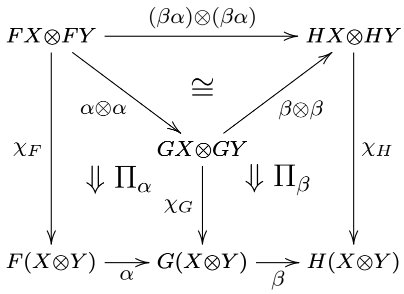


The rest of this section will be devoted to defining the structure of the tricategory **MonBicat** whose 0-cells are monoidal bicategories, 1-cells are monoidal functors, 2-cells are monoidal transformations, and 3-cells are monoidal modifications. We begin by defining the hom-bicategories for this tricategory; note that composition is not inherited directly from **Tricat** but can be thought of as a "hybrid" of the respective structures of **Tricat** and **Bicat**.

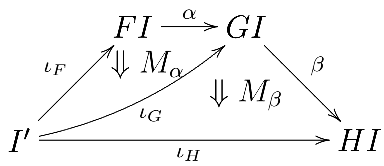
For 1-cell composition, consider monoidal transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$. We define a monoidal transformation $\beta\alpha$ as follows:

- its underlying transformation is the composite $\beta\alpha$,
- the invertible modification $\Pi_{\beta\alpha}$ has component at (X, Y) given by

the diagram below,



- and the invertible modification $M_{\beta\alpha}$ is given by the diagram below.

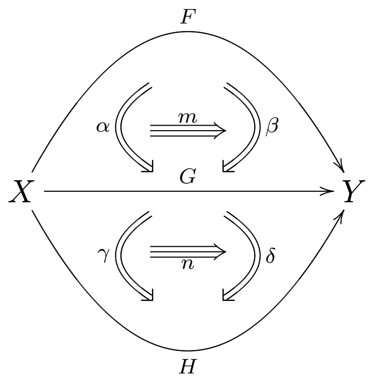


The three axioms are easily checked by a simple diagram chase.

For identity 1-cells, consider a monoidal functor F. Then the identity transformation $u : F \Rightarrow F$ can be equipped with the structure of a monoidal transformation with both Π_u and M_u being given by unique coherence isomorphisms. The axioms follow immediately from the coherence theorem for tricategories.

For vertical 2-cell composition, consider monoidal modifications $m : \alpha \Rrightarrow \beta$ and $n : \beta \Rrightarrow \gamma$. Then we can check that the composite $nm : \alpha \Rrightarrow \gamma$ in **Bicat** is in fact monoidal, and likewise the identity.

For horizontal 2-cell composition, consider monoidal modifications as displayed below.



Then we can check that the composite $n * m : \gamma\alpha \Rrightarrow \delta\beta$ in **Bicat** is in fact monoidal, and that this composition is functorial.

For coherence isomorphisms in the hom-bicategories, consider monoidal transformations $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$, and $\gamma : H \Rightarrow J$.

- Let $r : \alpha u_F \Rrightarrow \alpha$ be the modification with component at X the right unit isomorphism r_{α_X}. It follows from coherence for tricategories that r and r^{-1} are monoidal.
- Let $l : u_G\alpha \Rrightarrow \alpha$ be the modification with component at X the left unit isomorphism l_{α_X}. Observe as above that this modification and its inverse l^{-1} are monoidal.
- Let $a : (\gamma\beta)\alpha \Rrightarrow \gamma(\beta\alpha)$ be the modification with component at X the associativity isomorphism $a_{\gamma_X\beta_X\alpha_X}$ is monoidal. Observe as above that this modification and its inverse a^{-1} are monoidal.

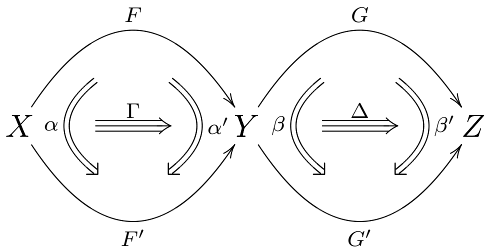
Theorem 3.5. *The above structure defines a bicategory*

$$\mathbf{MonBicat}(X, Y).$$

Proof. The axioms follow from the bicategory axioms in Y. □

We next define composition along bounding 0-cells for the tricategory **MonBicat**, which we will denote ⊠; we simply extend the definition of composition in the tricategory **Bicat** which we now recall.

Consider functors, transformations, and modifications as below.



Then we have the following formulae in **Bicat**, where $\otimes$ is horizontal composition.

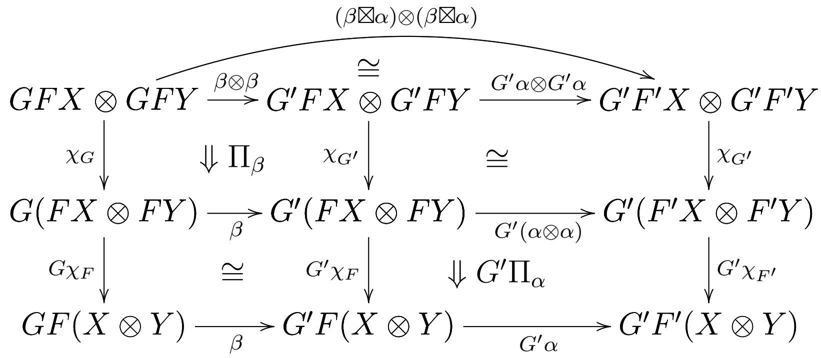
$$G \otimes F := GF$$
$$\beta \otimes \alpha := (G' * \alpha) \circ (\beta * F)$$
$$(\Delta \otimes \Gamma)_x := G'\Gamma_x * \Delta_{Fx}$$

Now suppose all of the above data are monoidal.

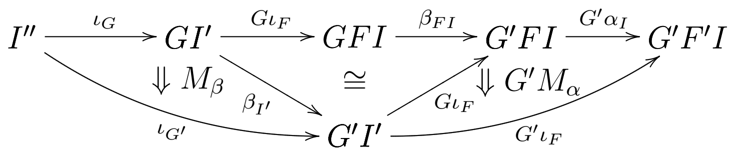
1. The composite $G \boxtimes F$ is the composite of the functors of the underlying degenerate tricategories.

2. The composite $\beta \boxtimes \alpha$ has underlying transformation $\beta \otimes \alpha$ as above together with

   - invertible modification Π given by the diagram below, and



   - invertible modification M given by the diagram below.



3. The modification $\Delta \otimes \Gamma$ is a monoidal modification, so we can put $\Delta \boxtimes \Gamma = \Delta \otimes \Gamma$.

Theorem 3.6. *The assignments above extend to a functor*

$$\boxtimes : \mathbf{MonBicat}(Y, Z) \times \mathbf{MonBicat}(X, Y) \to \mathbf{MonBicat}(X, Z).$$

Proof. The constraint modifications are the same as those given in [11]; we need only check that they are monoidal modifications, which is accomplished by a lengthy, but routine, diagram chase. The functor axioms follow from coherence and the transformation axioms. □

We now define units for the composition $\boxtimes$.

Proposition 3.7. *Let X be a monoidal bicategory. There is a functor $I_X : 1 \to \mathbf{MonBicat}(X, X)$ whose value on the single object is the identity monoidal functor and whose value on the single 1-cell is the identity monoidal transformation.*

Proof. Functoriality determines that the value on the single 2-cell is the identity. The unit constraint is the identity, and the composition constraint is given by the left (or right) unit isomorphism in X, which we have already determined is a monoidal modification. The axioms then follow from coherence. □

We now define the adjoint equivalences

$$\mathbf{a} : \boxtimes \circ (\boxtimes \times 1) \Rightarrow \boxtimes \circ (1 \times \boxtimes)$$
$$\mathbf{l} : \boxtimes \circ (I_X \times 1) \Rightarrow 1$$
$$\mathbf{r} : \boxtimes \circ (1 \times I_X) \Rightarrow 1.$$

The underlying adjoint equivalences of transformations are all the same as the relevant adjoint equivalences in **Bicat**. It remains to provide the component modifications, check that these choices give monoidal transformations, check that the unit and counit modifications are monoidal, and check the triangle identities. All the cells involved are coherence cells, and we can use coherence for tricategories to check that all necessary diagrams commute.

Theorem 3.8. *There is a tricategory* **MonBicat** *with*

- *0-cells monoidal bicategories;*
- *hom-bicategories given by the bicategories* $\mathbf{MonBicat}(X,Y)$ *defined above;*
- *composition functor given by* $\boxtimes$*;*
- *unit given by the functor* $I_X : 1 \to \mathbf{MonBicat}(X,X)$*;*
- *adjoint equivalences* $\mathbf{a}, \mathbf{l}, \mathbf{r}$ *as above; and*
- *invertible modifications* π, λ, ρ, μ *with each modification having components given by unique coherence cells in the target bicategory.*

Furthermore, the obvious forgetful functor $\mathbf{MonBicat} \to \mathbf{Bicat}$ *is a strict functor between tricategories.*

Proof. The tricategory axioms follow from coherence for bicategories. The fact that the modifications above are monoidal follows from coherence for tricategories. □

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THE TOTAL EXTERIOR DIFFERENTIAL

by M. BARILE, F. BARONE and W.M. TULCZYJEW

Abstract. The definition of mixed jets includes the finite sequences of vertical vectors tangent to jet bundles. This allows us to define differential operators on vertical forms on jet bundles by using mixed jets prolongations. The total exterior differential is a special case.

Résumé. La définition de jets mixtes inclut les suites finies de vecteurs verticaux tangents à des fibrés de jets. Cela nous permet de définir des opérateurs différentiels sur des formes verticales à un fibré de jets, en utilisant les prolongements de jets mixtes. Le différentiel exterieur total en est un cas particulier.

KEYWORDS: *Jet bundles, differential operators.*

A.M.S. CLASSIFICATION: 58A20

1. Introduction.
The total exterior differential presented in this paper is an operator that generalizes the total derivative [3][4] and its construction is strongly based on the notion of mixed jets introduced in Section 3. Our line of thought can be clarified by means of the following simple example. Let $\mathsf{T}M$ be a tangent bundle and

$$f : \mathsf{T}M \to \mathbb{R} \ : \ \mathsf{t}^1\gamma(0) \mapsto f\big(\mathsf{t}^1\gamma(0)\big) \tag{1}$$

be a differentiable function defined on $\mathsf{T}M$. The total derivative of f is the following differentiable function defined on the second tangent bundle T^2M:

$$\mathrm{d}_T f : \mathsf{T}^2M \to \mathbb{R} \ : \ \mathsf{t}^2\gamma(0) \mapsto \mathrm{D}(f \circ \mathsf{t}^1\gamma)(0), \tag{2}$$

where $\mathsf{t}^1\gamma : \mathbb{R} \to \mathsf{T}M$ is the prolongation of the curve γ to $\mathsf{T}M$, i.e. its tangent lift, and D is the usual derivative of real functions. In the definition (2) we explicitly use two basic facts: any second tangent vector is an equivalence class of curves, and any curve on the manifold M can be prolonged to a curve on the tangent bundle $\mathsf{T}M$.
If we now regard f as a 0-form on $\mathsf{T}M$, we may look for an extension of the total derivative to q-forms on $\mathsf{T}M$, i.e., to multilinear totally antisymmetric mappings

$$\Omega : \times^q_{\mathsf{T}M}\mathsf{T}(\mathsf{T}M) \to \mathbb{R}. \tag{3}$$

What we expect, as a result, is a mapping

$$\mathrm{d}_T\Omega : \times^q_{\mathsf{T}^2M}\mathsf{T}(\mathsf{T}^2M) \to \mathbb{R} \tag{4}$$

still multilinear and totally antisymmetric.
In order to follow the same pattern as above, we need to consider the elements of the fiber product $\times^q_{\mathsf{T}^2M}\mathsf{T}(\mathsf{T}^2M)$ as equivalence classes of (families of) curves on M, and we need the definition of their prolongations to $\times^q_{\mathsf{T}M}\mathsf{T}(\mathsf{T}M)$. This is made possible by the notion of mixed tangent vector.
Now notice that in (2) the derivative D acting on real functions could as well be interpreted as the exterior differential d. We will then obtain

a further extension of the total derivative, if the role played by D is taken over by the exterior differential d. In this passage, the role of the iterated tangent bundles will be played by the vertical bundles tangent to k-jets and the mixed tangent vectors will be replaced by mixed jets. The result will be the total exterior differential.
The paper is subdivided into two main parts and an appendix. The first part is devoted to the definition of mixed jets, restricted to the case of those mixed jets that can be identified with q-uples of vertical vectors tangent to k-jets. The second part deals with the total exterior differential, starting from the special case of the total derivative. In the appendix we will give the coordinate-based approach to the main constructions presented in the paper.
Remark. The total derivative appears in the Euler-Lagrange operator acting on Lagrangian forms defined on iterated tangent bundles. The total exterior differential will take over its role in the case of the Euler-Lagrange operator acting on Lagrangian forms defined on jet bundles. This is the topic of a forthcoming paper.

2. Preliminaries.

In this paper we will adopt the algebraic interpretation of jets [2], [6]. Unless otherwise specified, all mappings considered in the paper will be local and differentiable. Let M and N be differential manifolds. A mapping φ from M to N will be also denoted by $\varphi : M \to N$ without specifying its domain. The set of all mappings from M to N which are defined at $x \in M$ will be denoted by $\mathcal{D}(N|M, x)$.
Consider the following equivalence relation in $\mathcal{D}(N|M, x)$: φ and φ' are equivalent if they coincide on some open neighbourhood of x. The equivalence class of φ, denoted by $\mathsf{j}^{\mathfrak{c}}\varphi(x)$, is called the germ of φ at x. The set of all germs at x is denoted by $\mathsf{J}^{\mathfrak{c}}(N|M, x)$ and we set

$$\mathsf{J}^{\mathfrak{c}}(N|M) = \bigcup_{x \in M} \mathsf{J}^{\mathfrak{c}}(N|M, x). \tag{5}$$

Consider the special case $N = \mathbb{R}$. In this case the set of germs at x of real functions, denoted by $\mathsf{A}^{\mathfrak{c}}(M, x)$, is a commutative associative algebra with a unit element, and has a unique maximal ideal, namely

$$\mathsf{I^c}_0(M,x) = \{\mathsf{j^c} f(x) \in \mathsf{A^c}(M,x);\ f(x) = 0\}\,. \tag{6}$$

In the algebra $\mathsf{A^c}(M,x)$ we have the sequence of ideals

$$\mathsf{I^c}_0(M,x), \mathsf{I^c}_1(M,x), \dots, \mathsf{I^c}_k(M,x), \mathsf{I^c}_{k+1}(M,x), \dots \tag{7}$$

where, for any $k \in \mathbb{N}$,

$$\mathsf{I^c}_k(M,x) = (\mathsf{I^c}_0(M,x))^{k+1}. \tag{8}$$

Inclusion relations

$$\mathsf{I^c}_k(M,x) \subset \mathsf{I^c}_{k'}(M,x) \tag{9}$$

hold for all k' and k in $\mathbb{N}$ such that $k' \leqslant k$.
In the set $\mathcal{D}(N|M,x)$ we have, for each $k \in \mathbb{N}$, another equivalence relation: φ' and φ are equivalent if

$$\mathsf{j^c}(f \circ \varphi')(x) - \mathsf{j^c}(f \circ \varphi)(x) \in \mathsf{I^c}_k(M,x) \tag{10}$$

for any function f on N for which the compositions $(f \circ \varphi')$ and $(f \circ \varphi)$ make sense. The equivalence class of φ, denoted by $\mathsf{j}^k\varphi(x)$, is called the *k-jet* of φ at x. The set of all k-jet at x is denoted by $\mathsf{J}^k(N|M,x)$ and we set

$$\mathsf{J}^k(N|M) = \bigcup_{x \in M} \mathsf{J}^k(N|M,x). \tag{11}$$

The set $\mathsf{J}^k(N|M)$ can be endowed with a structure of a differential manifold (Cf.(A18)) such that the k-jet-source projection

$$\sigma_{k(N|M)} : \mathsf{J}^k(N|M) \to M : \mathsf{j}^k\varphi(x) \mapsto x \tag{12}$$

and the k-jet-target projection

$$\tau_{k(N|M)} : \mathsf{J}^k(N|M) \to N : \mathsf{j}^k\varphi(x) \mapsto \varphi(x) \tag{13}$$

are differentiable fibrations.

The k-jet prolongation of a mapping $\varphi : M \to N$ is the mapping

$$\mathsf{j}^k\varphi : M \to \mathsf{J}^k(N|M) : x \mapsto \mathsf{j}^k\varphi(x). \tag{14}$$

The case $M = \mathbb{R}$ is of special interest: the k-tangent fibration

$$\begin{array}{c} \mathsf{T}^k N \\ {\scriptstyle \tau_{k\ N}}\big\downarrow \\ N \end{array} \tag{15}$$

of a manifold N can be regarded as the restriction of the projection $\tau_{k\ (N|\mathbb{R})} : \mathsf{J}^k(N|\mathbb{R}) \to N$ to the fiber $\mathsf{J}^k(N|\mathbb{R}, 0)$. In view of this identification, we will always write $\mathsf{t}^k\gamma(0)$ instead of $\mathsf{j}^k\gamma(0)$ for every curve γ in N.

From the tangent fibration

$$\begin{array}{c} \mathsf{T}\mathsf{J}^k(N|M) \\ {\scriptstyle \tau_{\mathsf{J}^k(N|M)}}\big\downarrow \\ \mathsf{J}^k(N|M) \end{array} \tag{16}$$

we select the subfibration

$$\begin{array}{c} \mathsf{V}\mathsf{J}^k(N|M) \\ {\scriptstyle \upsilon_{\mathsf{J}^k(N|M)}}\big\downarrow \\ \mathsf{J}^k(N|M) \end{array} \tag{17}$$

vertical with respect to the fibration (12), i.e, for each $z \in \mathsf{J}^k(N|M)$,

$$\mathsf{V}_z\mathsf{J}^k(N|M) = (\upsilon_{\mathsf{J}^k(N|M)})^{-1}(z) = \ker T_z\sigma_{k(N|M)}, \tag{18}$$

where $T_z\sigma_{k(N|M)}$ is the tangent mapping of $\sigma_{k(N|M)}$ at z.
We finally recall that a q-form on a manifold M is a mapping

$$\Omega \colon \times^q_M \mathsf{T}M \longrightarrow \mathbb{R}, \tag{19}$$

multilinear and totally antisymmetric. It can be identified with a section of the fiber bundle

$$\begin{array}{c} \wedge^q \mathsf{T}^* M \\ \pi^q_M \Big\downarrow \\ M \end{array} \tag{20}$$

The space of all q-forms on the manifold M will be denoted by $\Lambda^q(M)$ and then $\Lambda(M)$ will be the exterior algebra on M.

3. Mixed Jets.

In this section we focus on mappings defined on a cross-product of two manifolds. The first step will be the construction of a class of ideals which describe the behaviour of the mappings on the two manifolds separately. Then we will use these ideals to define the mixed jets. For our purposes it will be sufficient to choose $\mathbb{R}^q$ as one of the two manifolds involved.
Let M be an m-dimensional differential manifold, consider the cross-product $\mathbb{R}^q \times M$ and denote by pr_1 and pr_2 the natural projections onto $\mathbb{R}^q$ and M, respectively.
Now let $(\mathbf{0}, x) \in \mathbb{R}^q \times M$ and consider the mapping

$$\mathsf{A}^c(\mathbb{R}^q, \mathbf{0}) \longrightarrow \mathsf{A}^c(\mathbb{R}^q \times M, (\mathbf{0}, x)) : \mathsf{j}^c f(\mathbf{0}) \mapsto \mathsf{j}^c(f \circ pr_1)(\mathbf{0}, x). \tag{21}$$

We denote the image of the ideal $\mathsf{I}^{\mathfrak{c}}_1(\mathbb{R}^q, \mathbf{0})$ by $\mathsf{I}^{\mathfrak{c}}_1(\mathbb{R}^q, \mathbf{0}) \circ \mathsf{j}^{\mathfrak{c}} pr_1(\mathbf{0}, x)$. Then we consider the ideal of $\mathsf{A}^{\mathfrak{c}}(\mathbb{R}^q \times M, (\mathbf{0}, x))$ generated by this image

$$\mathsf{I}^{\mathfrak{c}}_{\mathbf{1}}\big(\mathbb{R}^q; (M, x)\big)_{\mathbf{0}} := \Big(\mathsf{I}^{\mathfrak{c}}_1(\mathbb{R}^q, \mathbf{0}) \circ \mathsf{j}^{\mathfrak{c}} pr_1(\mathbf{0}, x)\Big). \tag{22}$$

Any element in the above ideal (22) is then the germ at $(\mathbf{0}, x)$ of a function on $\mathbb{R}^q \times M$ that is the sum of products

$$(f \circ pr_1)g, \tag{23}$$

where g is any function on $\mathbb{R}^q \times M$, and, owing to Proposition A1, in the notation introduced in the Appendix, $f : \mathbb{R}^q \to \mathbb{R}$ satisfies,

$$\partial_{\boldsymbol{\rho}} f(x) = 0 \tag{24}$$

for any q-multi-index $\boldsymbol{\rho}$ such that $|\boldsymbol{\rho}| \leqslant 1$.
We repeat the construction starting, this time, with the mapping

$$\mathsf{A}^{\mathfrak{c}}(M, x) \longrightarrow \mathsf{A}^{\mathfrak{c}}(\mathbb{R}^q \times M, (\mathbf{0}, x)) : \mathsf{j}^{\mathfrak{c}}\varphi(x) \mapsto \mathsf{j}^{\mathfrak{c}}(\varphi \circ pr_2)(\mathbf{0}, x). \tag{25}$$

We first consider the image of the ideal $\mathsf{I}^{\mathfrak{c}}_k(M, x)$, which will be denoted by $\mathsf{I}^{\mathfrak{c}}_k(M, x) \circ \mathsf{j}^{\mathfrak{c}} pr_2(\mathbf{0}, x)$, and then the ideal of $\mathsf{A}^{\mathfrak{c}}(\mathbb{R}^q \times M, (\mathbf{0}, x))$ generated by this image

$$\mathsf{I}^{\mathfrak{c}}_k\big((\mathbb{R}^q, \mathbf{0}); M\big)_x = \Big(\mathsf{I}^{\mathfrak{c}}_k(M, x) \circ \mathsf{j}^{\mathfrak{c}} pr_2(\mathbf{0}, x)\Big). \tag{26}$$

Any element in the above ideal (26) is still the germ at $(\mathbf{0}, x)$ of a function on $\mathbb{R}^q \times M$ that is the sum of products

$$(f \circ pr_2)g, \tag{27}$$

but, this time, $f : M \to \mathbb{R}$ satisfies

$$\partial_{\boldsymbol{\mu}} f(x) = 0 \tag{28}$$

for any m-multi-index $\boldsymbol{\mu}$ such that $|\boldsymbol{\mu}|\leqslant k$.
Finally consider the ideal sum

$$\mathsf{I}^{\mathsf{c}}_{(\mathbf{1},k)}\big(\mathbb{R}^q \times M, (\mathbf{0}, x)\big) = \mathsf{I}^{\mathsf{c}}_{\mathbf{1}}\big(\mathbb{R}^q; (M, x)\big)_{\mathbf{0}} + \mathsf{I}^{\mathsf{c}}_{k}\big((\mathbb{R}^q, \mathbf{0}); M\big)_x. \tag{29}$$

Any element in the mixed ideal (29) is then the germ at $(\mathbf{0}, x)$ of a function on $\mathbb{R}^q \times M$ that is the sum of terms

$$(f_1 \circ pr_1)g_1 + (f_2 \circ pr_2)g_2 \tag{30}$$

with

$$\begin{aligned} \partial_{\boldsymbol{\rho}} f_1(x) &= 0 \\ \partial_{\boldsymbol{\mu}} f_2(x) &= 0 \end{aligned} \tag{31}$$

for any q-multi-index $\boldsymbol{\rho}$ and m-multi-index $\boldsymbol{\mu}$ such that $|\boldsymbol{\rho}|\leqslant 1$ and $|\boldsymbol{\mu}|\leqslant k$.
We have the following inclusions

$$\mathsf{I}^{\mathsf{c}}_{\mathbf{1}}\big(\mathbb{R}^q; (M, x)\big)_{\mathbf{0}} \subset \mathsf{I}^{\mathsf{c}}_{1}\big(\mathbb{R}^q \times M, (\mathbf{0}, x)\big) \tag{32}$$

$$\mathsf{I}^{\mathsf{c}}_{k}\big((\mathbb{R}^q, \mathbf{0}); M\big)_x \subset \mathsf{I}^{\mathsf{c}}_{k}\big(\mathbb{R}^q \times M, (\mathbf{0}, x)\big), \tag{33}$$

moreover relations

$$\mathsf{I}^{\mathsf{c}}_{(\mathbf{1},k)}\big(\mathbb{R}^q \times M, (\mathbf{0}, x)\big) \subset \mathsf{I}^{\mathsf{c}}_{(\mathbf{1},k')}\big(\mathbb{R}^q \times M, (\mathbf{0}, x)\big) \tag{34}$$

hold for all k' and k in $\mathbb{N}$ such that $k'\leqslant k$.
The mixed ideals (29) will now lead to the definition of mixed jets. Let N be an n-dimensional differential manifold. In the set $\mathcal{D}\big(N|\mathbb{R}^q \times M, (\mathbf{0}, x)\big)$ we introduce, for each $k \in \mathbb{N}$, the following equivalence relation: χ' and χ are equivalent if

$$\mathsf{j}^{\mathsf{c}}(f \circ \chi')(\mathbf{0}, x) - \mathsf{j}^{\mathsf{c}}(f \circ \chi)(\mathbf{0}, x) \in \mathsf{I}^{\mathsf{c}}_{(\mathbf{1},k)}\big(\mathbb{R}^q \times M, (\mathbf{0}, x)\big) \tag{35}$$

for any function f on N for which the compositions make sense.

The equivalence class of χ is denoted by $\mathsf{j}^{(\mathbf{1},k)}\chi(x)$ and is called the $(\mathbf{1},k)$-*jet* of χ at x. The set of all $(\mathbf{1},k)$-jets at x will be denoted by $\mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M, x)$ and we set

$$\mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M) = \bigcup_{x\in M} \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M, x). \tag{36}$$

The set $\mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M)$ can be endowed with a differential structure (Cf.(A.31)) such that the $(\mathbf{1},k)$-jet-source projection

$$\sigma_{(\mathbf{1},k)(N|\mathbb{R}^q\times M)} : \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M) \to M : \mathsf{j}^{(\mathbf{1},k)}\chi(x) \mapsto x \tag{37}$$

and the $(\mathbf{1},k)$-jet-target projection

$$\tau_{(\mathbf{1},k)(N|\mathbb{R}^q\times M)} : \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M) \to N : \mathsf{j}^{(\mathbf{1},k)}\chi(x) \mapsto \chi(\mathbf{0},x) \tag{38}$$

are differentiable fibrations.
The $(\mathbf{1},k)$-jet prolongation of a mapping $\chi : \mathbb{R}^q \times M \to N$ is the mapping

$$\mathsf{j}^{(\mathbf{1},k)}\chi : M \to \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M) : x \mapsto \mathsf{j}^{(\mathbf{1},k)}\chi(x). \tag{39}$$

We conclude this section by showing how q-uples of vertical vectors tangent to jet spaces can be related to mixed jets. Let us consider the fiber product of q copies of the vertical bundle (17),

$$\begin{array}{c} \times^{q}_{\mathsf{J}^k(N|M)}\mathsf{VJ}^k(N|M) \\ \upsilon^{q}_{\mathsf{J}^k(N|M)} \Big\downarrow \\ \mathsf{J}^k(N|M) \end{array} \tag{40}$$

Proposition 1. *The elements of* $\mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M)$ *are in a one-to-one correspondence with the elements of* $\times^{q}_{\mathsf{J}^k(N|M)}\mathsf{VJ}^k(N|M)$.

PROOF: Let $\mathsf{j}^{(\mathbf{1},k)}\chi(x) \in \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M)$ and $j \in \{1, \ldots, q\}$. For any real number s^j in a suitable neighbourhood of $0 \in \mathbb{R}$, we can consider the mapping

$$\chi^j(s^j, \cdot\,) = \chi(0, \ldots, 0, s^j, 0, \ldots, 0, \cdot\,) \colon M \to N \tag{41}$$

and its k-jet, $\mathsf{j}^k\chi^i(s^j, \cdot\,)(x)$, at x. In this way we define q curves

$$\gamma_x^j \colon \mathbb{R} \to \mathsf{J}^k(N|M, x) \colon s^j \mapsto \mathsf{j}^k\chi^j(s^j, \cdot\,)(x) \tag{42}$$

in the fiber $\mathsf{J}^k(N|M, x)$, whose tangent vectors $\mathsf{t}\gamma_x^j(0)$ are, therefore, vertical. We set

$$\mathsf{tj}^k\chi^j(0, x) := \mathsf{t}\gamma_x^j(0). \tag{43}$$

Note that, since

$$\chi^1(0, \cdot\,) = \ldots = \chi^q(0, \cdot\,) = \chi(\mathbf{0}, \cdot\,), \tag{44}$$

we have that

$$\mathsf{j}^k\chi^1(0, \cdot\,)(x) = \ldots = \mathsf{j}^k\chi^q(0, \cdot\,)(x) = \mathsf{j}^k\chi(\mathbf{0}, \cdot\,)(x) \tag{45}$$

and, as a consequence,

$$\big(\mathsf{tj}^k\chi^1(0, x), \ldots, \mathsf{tj}^k\chi^q(0, x)\big) \in \times^q_{\mathsf{J}^k(N|M)} \mathsf{VJ}^k(N|M). \tag{46}$$

We have constructed the mapping

$$\begin{aligned}\varphi^{k,q}_{(N|M)} \colon \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M) &\longrightarrow \times^q_{\mathsf{J}^k(N|M)} \mathsf{VJ}^k(N|M) \\ \colon \mathsf{j}^{(\mathbf{1},k)}\chi(x) &\mapsto \big(\mathsf{tj}^k\chi^1(0, x), \ldots, \mathsf{tj}^k\chi^q(0, x)\big).\end{aligned} \tag{47}$$

We will prove that it is a bijection. Let ξ and η be charts on M and N, respectively. On the one hand $\mathsf{j}^{(\mathbf{1},k)}\chi(x)$ admits the coordinates

$$\big(\xi(x), \partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\mu}}\chi\big(\mathbf{0}, x\big)\big)_{|\boldsymbol{\rho}| \leqslant 1\ ,\ |\boldsymbol{\mu}| \leqslant k}\ , \tag{48}$$

(Cf. Appendix, (A.31)), on the other hand, the coordinate expression of the j-th curve γ_x^j in $\mathsf{VJ}^k(N|M)$ is

$$\big(\xi(x), \partial_{\boldsymbol{\mu}}\chi^j(s^j,\cdot)(x)\big)_{|\boldsymbol{\mu}|\leqslant k}\ , \tag{49}$$

(Cf. Appendix, (A.18)). From (49) it follows that the tangent vector $\mathsf{t}\gamma_x^j(0)$ has coordinates

$$\Big(\xi(x), \partial_{\boldsymbol{\mu}}\chi^j(0,\cdot)(x); \mathrm{D}\big(\partial_{\boldsymbol{\mu}}\chi^j(s^j,\cdot)(x)\big)(0)\Big)_{|\boldsymbol{\mu}|\leqslant k}, \tag{50}$$

or, owing to (44),

$$\Big(\xi(x), \partial_{\boldsymbol{\mu}}\chi(\mathbf{0},x); \mathrm{D}\big(\partial_{\boldsymbol{\mu}}\chi^j(s^j,\cdot)(x)\big)(0)\Big)_{|\boldsymbol{\mu}|\leqslant k}. \tag{51}$$

The q-uple $\big(\mathsf{t}\gamma_x^1(0),\ldots,\mathsf{t}\gamma_x^q(0)\big) \in \times^q_{\mathsf{J}^k(N|M)}\mathsf{VJ}^k(N|M)$ can then be given the coordinates

$$\begin{aligned}
&\Big(\xi(x), \partial_{\boldsymbol{\mu}}\chi(\mathbf{0},x); \mathrm{D}\big(\partial_{\boldsymbol{\mu}}\chi^1(s^1,\cdot)(x)\big)(0),..,\mathrm{D}\big(\partial_{\boldsymbol{\mu}}\chi^q(s^q,\cdot)(x)\big)(0)\Big)_{|\boldsymbol{\mu}|\leqslant k}\\
&= \Big(\big(\xi(x), \partial_{\boldsymbol{\mu}}\chi(\mathbf{0},x)\big); \partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\mu}}\chi(\mathbf{0},x)\Big)_{|\boldsymbol{\rho}|=1\,,\,|\boldsymbol{\mu}|\leqslant k} \qquad (52)\\
&= \big(\xi(x), \partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\mu}}\chi(\mathbf{0},x)\big)_{|\boldsymbol{\rho}|\leqslant 1\,,\,|\boldsymbol{\mu}|\leqslant k}
\end{aligned}$$

This together with (48) shows that the mapping (47) is injective. We prove that it is also surjective.

Let $(w^1, \ldots, w^q) \in \times^q_{\mathsf{J}^k(N|M)} \mathsf{VJ}^k(N|M)$ and let

$$\left(\bar{x}^1, \ldots, \bar{x}^m, \bar{y}^1_{\boldsymbol{\mu}}, \ldots, \bar{y}^n_{\boldsymbol{\mu}}, \bar{y}^1_{\boldsymbol{\rho\mu}}, \ldots, \bar{y}^n_{\boldsymbol{\rho\mu}}\right)_{|\boldsymbol{\rho}| \leqslant 1\,,\, |\boldsymbol{\mu}| \leqslant k} \tag{53}$$

be its coordinates. For each $A = 1, \ldots, n$, consider the following polynomial function on $\mathbb{R}^q \times \mathbb{R}^m$:

$$P^A(s^1, \ldots, s^q; x^1, \ldots, x^m) =$$

$$\sum_{|\boldsymbol{\mu}| \leq k} \frac{1}{\mu_1! \ldots \mu_m!} \, \bar{y}^A_{\boldsymbol{\mu}} \, (x^1 - \bar{x}^1)^{\mu_1} \cdots (x^m - \bar{x}^m)^{\mu_m} + \tag{54}$$

$$\sum_{\substack{|\boldsymbol{\mu}| \leq k \\ |\boldsymbol{\rho}| = 1}} \frac{1}{\rho_1!..\rho_q!\mu_1!..\mu_m!} \, \bar{y}^A_{\boldsymbol{\rho\mu}} \, (s^1)^{\rho_1} \cdot\cdot (s^q)^{\rho_q} (x^1 - \bar{x}^1)^{\mu_1} \cdot\cdot (x^m - \bar{x}^m)^{\mu_m}$$

The mapping

$$(P^1, \ldots, P^n) : \mathbb{R}^q \times \mathbb{R}^m \longrightarrow \mathbb{R}^n \tag{55}$$

is the coordinate expression of the mapping

$$\chi = \eta^{-1} \circ (P^1, \ldots, P^n) \circ \tilde{\xi} : \mathbb{R}^q \times M \longrightarrow N\,, \tag{56}$$

where $\tilde{\xi} = \xi \circ \mathrm{id}_{\mathbb{R}^q}$. It is easy to check that the image of the $(\mathbf{1}, k)$-jet $\mathsf{j}^{(\mathbf{1},k)}\chi(\mathbf{0}, x)$ in the mapping (47) is the assigned q-uple $(w^1, \ldots, w^q)$. ■

The bijection $\varphi^{k,q}_{(N|M)}$ defined in the proof of the above proposition gives rise to the following commutative diagram

$$
\begin{array}{ccccccc}
\mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M) & & \xrightarrow{\varphi^{k,q}_{(N|M)}} & & \times^q_{\mathsf{J}^k(N|M)} \mathsf{VJ}^k(N|M) & & \\
\swarrow_{\tau_{(\mathbf{1},k)(N|\mathbb{R}^q\times M)}} & \searrow^{\sigma_{(\mathbf{1},k)(N|\mathbb{R}^q\times M)}} & & \swarrow^{\tau} & & \searrow^{\sigma} & \\
N & = & = & N & & & \\
& & M & = & = & = & M
\end{array}
\tag{57}
$$

with

$$\tau = \tau_{k(N|M)} \circ \upsilon^q_{\mathsf{J}^k(N|M)} \tag{58}$$

and

$$\sigma = \sigma_{k(N|M)} \circ \upsilon^q_{\mathsf{J}^k(N|M)}. \tag{59}$$

A *super-representative* of a sequence $(w^1, .., w^q) \in \times^q_{\mathsf{J}^k(N|M)} \mathsf{VJ}^k(N|M)$, is any representative of the corresponding jet $\left(\varphi^{k,q}_{(N|M)}\right)^{-1}(w^1, .., w^q) \in \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M)$.

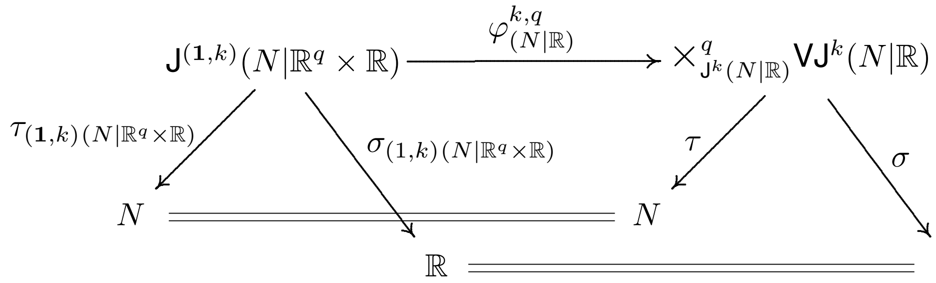
From Proposition 1 it follows that the $(\mathbf{1}, k)$-jet prolongation (39) of a mapping $\chi : \mathbb{R}^q \times M \to N$ can be identified with the mapping

$$\mathsf{j}^{(\mathbf{1},k)}\chi : M \to \times^q_{\mathsf{J}^k(N|M)} \mathsf{VJ}^k(N|M) : x \mapsto \left(\mathsf{tj}^k\chi^1(0,x), \ldots, \mathsf{tj}^k\chi^q(0,x)\right) \tag{60}$$

4. The total exterior differential.

We start the construction of the total exterior differential from a special case, i.e., the total derivative. It is a differential operator known in

the calculus of variations. An intrinsic construction of this operator, obtained as the result of a generalization of the Frölicher and Nijenhuis theory of derivations [3], was presented in [4]. We are providing here an alternative construction based on mixed jets of mappings on $\mathbb{R}^q \times \mathbb{R}$.
We will first regard q-uples in $\times^q_{\mathsf{T}^k N} \mathsf{T}\mathsf{T}^k N$ as mixed jets. Let us consider the diagram (57) in our special case $M = \mathbb{R}$,



(61)

On the one hand the bijection $\varphi^{k,q}_{(N|\mathbb{R})}$ induces a bijection between the fibers at $0 \in \mathbb{R}$,

$$\begin{aligned}\varphi^{k,q}_{(N|\mathbb{R},0)} &: \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times \mathbb{R}, 0) \longrightarrow \times^q_{\mathsf{J}^k(N|\mathbb{R},0)} \mathsf{V}\mathsf{J}^k(N|\mathbb{R}, 0) \\ &: \mathsf{j}^{(\mathbf{1},k)}\chi(0) \mapsto (\mathsf{t}\mathsf{j}^k\chi^1(0,0), \ldots, \mathsf{t}\mathsf{j}^k\chi^q(0,0)),\end{aligned} \tag{62}$$

on the other hand, as we remarked in the preliminaries, we can make the identification

$$\mathsf{J}^k(N|\mathbb{R}, 0) = \mathsf{T}^k N \tag{63}$$

and then

$$\mathsf{V}\mathsf{J}^k(N|\mathbb{R}, 0) = \mathsf{V}\mathsf{T}^k N = \mathsf{T}\mathsf{T}^k N. \tag{64}$$

If, moreover, we set

$$\begin{aligned} &\mathsf{T}^{(\mathbf{1},k)}N := \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times \mathbb{R}, 0) \\ &\mathsf{t}^{(\mathbf{1},k)}\chi(0) := \mathsf{j}^{(\mathbf{1},k)}\chi(0), \end{aligned} \tag{65}$$

the mapping (62) becomes

$$\begin{aligned} \varphi_N^{k,q} &: \mathsf{T}^{(\mathbf{1},k)}N \to \times^q_{\mathsf{T}^k N}\mathsf{T}\mathsf{T}^k N \\ &: \mathsf{t}^{(\mathbf{1},k)}\chi(0) \mapsto (\mathsf{t}\mathsf{t}^k\chi^1(0,0), \dots, \mathsf{t}\mathsf{t}^k\chi^q(0,0)). \end{aligned} \tag{66}$$

It follows that the diagram (61), when restricted to the fibers at $0 \in \mathbb{R}$, reduces to the following fiber isomorphism

$$\begin{array}{ccc} \mathsf{T}^{(\mathbf{1},k)}N & \xrightarrow{\ \varphi_N^{k,q}\ } & \times^q_{\mathsf{T}^k M}\mathsf{T}\,\mathsf{T}^k N \\ {\scriptstyle \tau_{(\mathbf{1},k)\ N}}\big\downarrow & & \big\downarrow{\scriptstyle \tau_{k\ N} \circ \tau^q_{\mathsf{T}^k N}} \\ N & = & N \end{array} \tag{67}$$

where $\tau_{(\mathbf{1},k)\ N}$ and $\tau^q_{\mathsf{T}^k N}$ are the restrictions to the fibers considered of the projections $\tau_{(\mathbf{1},k)(N|\mathbb{R}^q\times\mathbb{R})}$ and $\upsilon^q_{\mathsf{J}^k(N|\mathbb{R})}$, respectively.

Finally, as a special case of (39), the $(\mathbf{1}, k)$-tangent prolongation of a local mapping $\chi : \mathbb{R}^q \times \mathbb{R} \to N$ is the mapping

$$\begin{aligned} \mathsf{t}^{(\mathbf{1},k)}\chi &: \mathbb{R} \to \times^q_{\mathsf{T}^k N}\mathsf{T}\,\mathsf{T}^k N \\ &: t \mapsto \big(\mathsf{t}\mathsf{t}^k\chi^1(0, t + \cdot)(0), \dots, \mathsf{t}\mathsf{t}^k\chi^q(0, t + \cdot)(0)\big). \end{aligned} \tag{68}$$

The representation of the elements of $\times^q_{\mathsf{T}^k N}\mathsf{T}\,\mathsf{T}^k N$ as mixed jets makes operations on forms more efficient. We introduce an operator

$$\mathrm{d}_{T(k)} \colon \Lambda(\mathsf{T}^k N) \to \Lambda(\mathsf{T}^{k+1} N) \tag{69}$$

as follows.
Let

$$f \colon \mathsf{T}^k N \to \mathbb{R} \tag{70}$$

be a 0-form on $\mathsf{T}^k N$, then $\mathrm{d}_{T(k)} f$ is the 0-form on $\mathsf{T}^{k+1} N$ given by

$$\mathrm{d}_{T(k)}\Omega \colon \mathsf{T}^{k+1} N \to \mathbb{R} \colon \mathsf{t}^{k+1}\gamma(0) \mapsto \mathrm{D}(\Omega \circ \mathsf{t}^k\gamma)(0). \tag{71}$$

If $q > 0$ and

$$\Omega \colon \times^q_{\mathsf{T}^k N}\mathsf{T}\,\mathsf{T}^k N \to \mathbb{R} \tag{72}$$

is a q-form on $\mathsf{T}^k N$, then $\mathrm{d}_{T(k)}\Omega$ is the q-form on $\mathsf{T}^{k+1} N$ given by

$$\begin{aligned} \mathrm{d}_{T(k)}\Omega \colon &\times^q_{\mathsf{T}^{k+1} N}\mathsf{T}\,\mathsf{T}^{k+1} N \to \mathbb{R} \\ &\colon (w^1, \ldots, w^q) \mapsto \mathrm{D}(\Omega \circ \varphi^{k,q}_N \circ \mathsf{t}^{(\mathbf{1},k)}\chi)(0), \end{aligned} \tag{73}$$

where χ is any super-representative of $(w^1, \ldots, w^q)$.
The operator $\mathrm{d}_{T(k)}$ is the *total derivative*. The coordinate expression of its action is presented in the Appendix.

We will now introduce a more general operator, the total exterior differential d_H, where the role played in $\mathrm{d}_{T(k)}$ by the derivation will be played by the exterior differential.
Let

$$f \colon \mathsf{J}^k(N|M) \to \mathbb{R} \tag{74}$$

be a 0-form on $\mathsf{J}^k(N|M)$, then $\mathrm{d}_H f$ is the 0-form on $\mathsf{J}^{k+1}(N|M)$ given by

$$\mathrm{d}_H f \colon \mathsf{J}^{k+1}(N|M) \to \mathsf{T}^* M \colon \mathsf{j}^{k+1}\varphi(x) \mapsto \mathrm{d}(f \circ \mathsf{j}^k\varphi)(x). \tag{75}$$

More generally, we can consider a 0-form on $\mathsf{J}^k(N|M)$ with values in $\wedge^p\mathsf{T}^*M$, i.e., a bundle morphism

$$\begin{array}{ccc} \mathsf{J}^k(N|M) & \xrightarrow{\ \Omega\ } & \wedge^p\mathsf{T}^*M \\ \Vert & & \downarrow{\scriptstyle \pi^p_M} \\ \mathsf{J}^k(N|M) & \xrightarrow{\ \sigma_{k(N|M)}\ } & M \end{array} \tag{76}$$

We obtain a 0-form on $\mathsf{J}^{k+1}(N|M)$ with values in $\wedge^{p+1}\mathsf{T}^*M$

$$\begin{array}{ccc} \mathsf{J}^{k+1}(N|M) & \xrightarrow{\ \mathrm{d}_H\Omega\ } & \wedge^{p+1}\mathsf{T}^*M \\ \Vert & & \downarrow{\scriptstyle \pi^{p+1}_M} \\ J^{k+1}(N|M) & \xrightarrow{\ \sigma_{k+1(N|M)}\ } & M \end{array} \tag{77}$$

with the mapping $\mathrm{d}_H\Omega$ defined by

$$\mathrm{d}_H\Omega\left(\mathsf{j}^{k+1}\varphi(x)\right) = \mathrm{d}(\Omega\circ\mathsf{j}^k\varphi)(x). \tag{78}$$

We finally consider a vertical q-form on $\mathsf{J}^k(N|M)$ with values in the set of p-forms on M, i.e., a bundle morphism

$$\begin{array}{ccc} \times^q_{\mathsf{J}^k(N|M)}\mathsf{VJ}^k(N|M) & \xrightarrow{\ \Omega\ } & \wedge^p\mathsf{T}^*M \\ {\scriptstyle \upsilon^q{}_{\mathsf{J}^k(N|M)}}\downarrow & & \downarrow{\scriptstyle \pi^p_M} \\ J^k(N|M) & \xrightarrow{\ \sigma_{k(N|M)}\ } & M \end{array} \tag{79}$$

Applying d_H to it we obtain

$$\begin{array}{ccc} \times^q_{\mathsf{J}^{k+1}(N|M)} \mathsf{VJ}^{k+1}(N|M) & \xrightarrow{\mathrm{d}_H\Omega} & \wedge^{p+1}\mathsf{T}^*M \\ \Big\downarrow {\scriptstyle \upsilon^q{}_{\mathsf{J}^{k+1}(N|M)}} & & \Big\downarrow {\scriptstyle \pi_M^{p+1}} \\ J^{k+1}(N|M) & \xrightarrow{\sigma_{k+1(N|M)}} & M \end{array} \tag{80}$$

where the mapping $\mathrm{d}_H\Omega$ is defined by

$$\begin{aligned} \mathrm{d}_H\Omega \colon \times^q_{\mathsf{J}^{k+1}(N|M)} \mathsf{VJ}^{k+1}(N|M) &\to \wedge^{p+1}\mathsf{T}^*M \\ \colon (w^1, \ldots, w^q) &\mapsto \mathrm{d}(\omega \circ \varphi^{k,q}_{(N|M)} \circ \mathsf{j}^{(\mathbf{1},k)}\chi)(x), \end{aligned} \tag{81}$$

and χ is a super-representative of $(w^1, \ldots, w^q)$.
The coordinate expression of the action of d_H is presented in the Appendix. We conclude this presentation by remarking that, as is evident from its construction, d_H is only one of the possible operators that can be defined by using mixed jets. Our choice is due to its future application in the definition of the Euler-Lagrange operator.

5. Appendix.

In this section we give the coordinate-based approach to the main constructions presented in the paper. First we give the differential characterizations of the ideals we have introduced, from these we derive differentiable atlases for the spaces of jets, and then we provide the coordinate expressions of the actions of the operators $\mathrm{d}_{T(k)}$ and d_H.
In the sequel all the charts on manifolds will be arbitrarily chosen within those which are compatible with the compositions involved.
We will adopt the following abridged notation.

Let $f : M \to \mathbb{R}$. We set, for any chart $\xi = (x^1, \ldots, x^m)$ on M and any $i = 1, \ldots, m$,

$$\partial_i f = \frac{\partial(f \circ \xi^{-1})}{\partial x^i} \circ \xi, \tag{A.1}$$

and for any m-multi-index $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$,

$$\partial_{\boldsymbol{\mu}} f = \frac{\partial^{|\boldsymbol{\mu}|}(f \circ \xi^{-1})}{(\partial x^1)^{\mu_1} \cdots (\partial x^m)^{\mu_m}} \circ \xi, \tag{A.2}$$

where $|\boldsymbol{\mu}| = \mu_1 + \cdots + \mu_m$.
Similarly, for any $\varphi : M \to N$, we set

$$\partial_{\boldsymbol{\mu}} \varphi = \frac{\partial^{|\boldsymbol{\mu}|}(\eta \circ \varphi \circ \xi^{-1})}{(\partial x^1)^{\mu_1} \cdots (\partial x^m)^{\mu_m}} \circ \xi, \tag{A.3}$$

where $\eta = (y^1, \ldots, y^n)$ is any chart on N.
Finally, let $\chi : \mathbb{R}^q \times M \to N$, then for any other q-multi-index $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_q)$, we set

$$\partial_{\boldsymbol{\rho}} \partial_{\boldsymbol{\mu}} \chi = \frac{\partial^{|\boldsymbol{\rho}|+|\boldsymbol{\mu}|}(\eta \circ \chi \circ \tilde{\xi}^{-1})}{(\partial s^1)^{\rho_1} \cdots (\partial s^q)^{\rho_q} (\partial x^1)^{\mu_1} \cdots (\partial x^m)^{\mu_m}} \circ \tilde{\xi} \tag{A.4}$$

where, this time $\tilde{\xi} = \mathrm{id}_{\mathbb{R}^q} \times \xi$ and $\mathrm{id}_{\mathbb{R}^q} = (s^1, \ldots, s^q)$.
In particular, for a mapping $\chi : \mathbb{R}^q \times \mathbb{R} \to N$, and any $h \in \mathbb{N}$, we set

$$\partial_{\boldsymbol{\rho}} \partial^h \chi = \frac{\partial^{|\boldsymbol{\rho}|+h}(\eta \circ \chi)}{(\partial s^1)^{\rho_1} \cdots (\partial s^q)^{\rho_q} (\partial t)^h} \tag{A.5}$$

with $(s^1, \ldots, s^q, t)$ coordinates in $\mathbb{R}^q \times \mathbb{R}$.

The symbols we adopted for partial derivatives of mappings do not contain any reference to the charts on M and N used in their definition. This is because all the claims which follow are independent of the choice of these charts, so that there is no need to mention them explicitly. The following propositions establish a link between the definition of jet based on ideals of local algebras and the standard definition of jet utilizing partial derivatives of mappings [5].

Proposition A1. *Let $f \in \mathcal{D}(\mathbb{R}|M, x)$. Then, for each $k \in \mathbb{N}$, the following conditions are equivalent.*

(i) $\mathsf{j}^{\mathsf{c}} f(x) \in \mathsf{I}_k^{\mathsf{c}}(M, x)$*;*

(ii) $\partial_{\boldsymbol{\mu}} f(x) = 0$*, for any m-multi-index $\boldsymbol{\mu}$ such that* $|\boldsymbol{\mu}| \leqslant k$.

Proof: We prove that (i) implies (ii). Each element of $\mathsf{I}_k^{\mathsf{c}}(M, x)$ is the finite sum of germs of functions of the form

$$f = g_0 g_1 \cdot \ldots \cdot g_k \tag{A.6}$$

such that

$$\mathsf{j}^{\mathsf{c}} g_h(x) \in \mathsf{I}_0^{\mathsf{c}}(M, x)\ ,\ 0 \leqslant h \leqslant k. \tag{A.7}$$

It will then suffice to prove the claim for this kind of products.
We have $\partial_i f = \sum_{h=0}^{k} g_0 g_1 \cdot \ldots \cdot g_{h-1} (\partial_i g_h) g_{h+1} \cdot \ldots \cdot g_k$, for each $i = 1, \ldots, m$, hence

$$\mathsf{j}^{\mathsf{c}} \partial_i f(x) \in \mathsf{I}_{k-1}^{\mathsf{c}}(M, x). \tag{A.8}$$

By finite iteration, we obtain

$$\mathsf{j}^{\mathsf{c}} f(x) \in \mathsf{I}_k^{\mathsf{c}}(M, x) \Rightarrow \mathsf{j}^{\mathsf{c}} \partial_{\boldsymbol{\mu}} f(x) \in \mathsf{I}_{k-|\boldsymbol{\mu}|}^{\mathsf{c}}(M, x), \tag{A.9}$$

for $|\boldsymbol{\mu}| \leqslant k$. From the inclusion relations (9) we deduce that

$$\mathsf{j}^{\mathsf{c}} \partial_{\boldsymbol{\mu}} f(x) \in \mathsf{I}_0^{\mathsf{c}}(M, x), \tag{A.10}$$

whence (ii) immediately follows.

We now prove that (ii) implies (i). Suppose that f fulfils (ii). Then consider its Taylor expansion at x, with Lagrange remainder:

$$f = f(x) + \sum_{|\boldsymbol{\mu}|=1}^{k} \frac{1}{\mu_1!..\mu_m!} \partial_{\boldsymbol{\mu}} f(x) \big(x^1 - x^1(x)\big)^{\mu_1} \cdot\ldots\cdot \big(x^m - x^m(x)\big)^{\mu_m} + R \tag{A.11}$$

Owing to our hypothesis and the properties of the Lagrange remainder,

$$\mathsf{j}^{\mathfrak{c}} f(x) = \mathsf{j}^{\mathfrak{c}} R(x) \in \mathsf{I}^{\mathfrak{c}}_k(M, x). \tag{A.12}$$

∎

Proposition A2. *Let $\varphi', \varphi \in \mathcal{D}(N|M, x)$. Then, for each $k \in \mathbb{N}$, the following conditions are equivalent.*

(i) $\mathsf{j}^k \varphi'(x) = \mathsf{j}^k \varphi(x)$;

(ii) $\partial_{\boldsymbol{\mu}} \varphi'(x) = \partial_{\boldsymbol{\mu}} \varphi(x)$, *for any m-multi-index $\boldsymbol{\mu}$ such that $|\boldsymbol{\mu}| \leqslant k$.*

PROOF: If (i) holds then, by definition, we have that, for any function f on N,

$$\mathsf{j}^{\mathfrak{c}}(f \circ \varphi' - f \circ \varphi)(x) \in \mathsf{I}^{\mathfrak{c}}_k(M, x), \tag{A.13}$$

and then Proposition A1 implies that

$$\partial_{\boldsymbol{\mu}}(f \circ \varphi')(x) = \partial_{\boldsymbol{\mu}}(f \circ \varphi)(x), \tag{A.14}$$

for $|\boldsymbol{\mu}| \leqslant k$. If we apply (A.14) to coordinate functions y^A on N, we have

$$\partial_{\boldsymbol{\mu}}(y^A \circ \varphi')(x) = \partial_{\boldsymbol{\mu}}(y^A \circ \varphi)(x), \qquad A = 1, \ldots, n, \tag{A.15}$$

whence (ii) follows.
We now prove that (ii) implies (i). Let $f : N \to \mathbb{R}$ be any function on N. By virtue of the chain rule for derivatives (cf. [1]), our hypothesis

implies that

$$\begin{aligned}\partial_{\boldsymbol{\mu}}(f\circ\varphi') &= \partial_{\boldsymbol{\mu}}(f\circ\eta^{-1}\circ\eta\circ\varphi') \\ &= \frac{\partial^{|\boldsymbol{\mu}|}\big((f\circ\eta^{-1})\circ(\eta\circ\varphi'\circ\xi^{-1})\big)}{(\partial x^1)^{\mu_1}\cdots(\partial x^m)^{\mu_m}}\circ\xi \\ &= \frac{\partial^{|\boldsymbol{\mu}|}\big((f\circ\eta^{-1})\circ(\eta\circ\varphi\circ\xi^{-1})\big)}{(\partial x^1)^{\mu_1}\cdots(\partial x^m)^{\mu_m}}\circ\xi \\ &= \partial_{\boldsymbol{\mu}}(f\circ\varphi)\end{aligned} \tag{A.16}$$

and then, from Proposition A1 we have that

$$\mathsf{j}^{\mathfrak{c}}(f\circ\varphi'-f\circ\varphi)(x)\in\mathsf{I}^{\mathfrak{c}}_k(M,x), \tag{A.17}$$

whence (i) follows from the very definition of jets. ■

Proposition A2 indicates how to construct a differential structure on $\mathsf{J}^k(N|M)$: let ξ and η be charts on M and N, respectively, then $\mathsf{j}^k\varphi(x)$ can be given the coordinates

$$\big(\xi(x),\partial_{\boldsymbol{\mu}}\varphi(x)\big)_{|\boldsymbol{\mu}|\leqslant k}\,. \tag{A.18}$$

Proposition A3. *Let $f\in\mathcal{D}(\mathbb{R}|\mathbb{R}^q\times M,(\mathbf{0},x))$. Then, for each $k\in\mathbb{N}$, the following conditions are equivalent.*

(i) $\mathsf{j}^{\mathfrak{c}}f(\mathbf{0},x)\in\mathsf{I}^{\mathfrak{c}}_{(1,k)}\big(\mathbb{R}^q\times M,(\mathbf{0},x)\big)$*;*

(ii) $\partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\mu}}f(\mathbf{0},x)=0$*, for any q-multi-index $\boldsymbol{\rho}$ and any m-multi-index $\boldsymbol{\mu}$ such that $|\boldsymbol{\rho}|\leqslant 1$ and $|\boldsymbol{\mu}|\leqslant k$.*

PROOF: We prove that (i) implies (ii). Each element of $\mathsf{I}^{\mathfrak{c}}_{(1,k)}\big(\mathbb{R}^q\times M,(\mathbf{0},x)\big)$ is the finite sum of germ at $(\mathbf{0},x)$ of functions of the form

$$(f_1\circ pr_1)g_1+(f_2\circ pr_2)g_2, \tag{A.19}$$

where $\mathsf{j}^{\mathsf{c}} f_1(\mathbf{0}) \in \mathsf{I}_1^{\mathsf{c}}(\mathbb{R}^q, \mathbf{0})$, $\mathsf{j}^{\mathsf{c}} f_2(x) \in \mathsf{I}_k^{\mathsf{c}}(M, x)$ and the projections pr_1, pr_2 refer to the cross-product $\mathbb{R}^q \times M$. It will then suffice to prove the claim for this kind of functions. We first consider the derivatives of

$$F_1 = (f_1 \circ pr_1)g_1. \qquad (A.20)$$

For all $\boldsymbol{\mu}$ and $|\boldsymbol{\rho}| \leqslant 1$, we have

$$\partial_{\boldsymbol{\rho}} \partial_{\boldsymbol{\mu}} F_1(\mathbf{0}, x) = \partial_{\boldsymbol{\rho}} \big((f_1 \circ pr_1) \partial_{\boldsymbol{\mu}} g_1 \big)(\mathbf{0}, x). \qquad (A.21)$$

Since $\mathsf{j}^{\mathsf{c}} f_1(\mathbf{0}) \in \mathsf{I}_1^{\mathsf{c}}(\mathbb{R}^q, \mathbf{0})$, it follows that $\mathsf{j}^{\mathsf{c}}(f_1 \circ pr_1)(\mathbf{0}, x) \in \mathsf{I}_1^{\mathsf{c}}(\mathbb{R}^q \times M, (\mathbf{0}, x))$ by (22) and inclusion (32). As a consequence,

$$\mathsf{j}^{\mathsf{c}}(\partial_{\boldsymbol{\rho}} \partial_{\boldsymbol{\mu}} F_1)(\mathbf{0}, x) \in \mathsf{I}_1^{\mathsf{c}}(\mathbb{R}^q \times M, (\mathbf{0}, x)), \qquad (A.22)$$

and Proposition A1 proves that

$$\partial_{\boldsymbol{\rho}} \partial_{\boldsymbol{\mu}} F_1(\mathbf{0}, x) = 0. \qquad (A.23)$$

We then consider

$$F_2 = (f_2 \circ pr_2)g_2. \qquad (A.24)$$

We have, for all $\boldsymbol{\rho}$,

$$\partial_{\boldsymbol{\rho}} F_2 = (f_2 \circ pr_2) \partial_{\boldsymbol{\rho}} g_2. \qquad (A.25)$$

Since $\mathsf{j}^{\mathsf{c}} f_2(x) \in \mathsf{I}_k^{\mathsf{c}}(M, x)$, it follows, from (26) and inclusion (33), that $\mathsf{j}^{\mathsf{c}}(f_2 \circ pr_2)(\mathbf{0}, x) \in \mathsf{I}_k^{\mathsf{c}}(\mathbb{R}^q \times M, (\mathbf{0}, x))$. Then, $\mathsf{j}^{\mathsf{c}}\big(\partial_{\boldsymbol{\rho}} F_2\big)(\mathbf{0}, x) \in \mathsf{I}_k^{\mathsf{c}}(\mathbb{R}^q \times M, (\mathbf{0}, x))$, and Proposition A1 ensures that

$$\partial_{\boldsymbol{\mu}} \partial_{\boldsymbol{\rho}} F_2\big(\mathbf{0}, x\big) = 0 \qquad (A.26)$$

for all $|\boldsymbol{\mu}| \leqslant k$. This, together with (A.23), shows that (i) implies (ii). We now show that (ii) implies (i). Let $f \in \mathcal{D}(\mathbb{R}|\mathbb{R}^q \times M, (\mathbf{0}, x))$ and consider its Taylor polynomial at $(\mathbf{0}, x)$ of order $k+1$ with Lagrange remainder. Up to a multiplicative constant, its generic term is

$$\partial_{\boldsymbol{\rho}} \partial_{\boldsymbol{\mu}} f(\mathbf{0}, x)(s^1)^{\rho_1} \cdots (s^q)^{\rho_q}(x^1 - x^1(x))^{\mu_1} \cdots (x^m - x^m(x))^{\mu_m}, \qquad (A.27)$$

where $|\boldsymbol{\rho}| + |\boldsymbol{\mu}| \leqslant k + 1$. We have

$$|\boldsymbol{\rho}| \geqslant 2 \Rightarrow \mathsf{j}^{\mathsf{c}}\big((s^1)^{\rho_1} \cdots (s^q)^{\rho_q}\big)(\mathbf{0}) \in \mathsf{I}_1^{\mathsf{c}}(\mathbb{R}^q, \mathbf{0}), \qquad (A.28)$$

$$|\boldsymbol{\mu}| \geqslant k + 1 \Rightarrow \mathsf{j}^{\mathsf{c}}\big((x^1 - x^1(x))^{\mu_1} \cdots (x^m - x^m(x))^{\mu_m}\big)(x) \in \mathsf{I}_k^{\mathsf{c}}(M, x). \qquad (A.29)$$

We distinguish three possible cases.
If $|\boldsymbol{\rho}| \geqslant 2$, by (A.28), the term (A.27) belongs to $\mathsf{I}_{\mathbf{1}}^{\mathsf{c}}\big(\mathbb{R}^q; (M, x)\big)_{\mathbf{0}}$.

If $|\boldsymbol{\mu}| \geqslant k + 1$, by (A.29), the term (A.27) belongs to $\mathsf{I}_k^{\mathsf{c}}\big((\mathbb{R}^q, \mathbf{0}); M\big)_x$.
If $|\boldsymbol{\rho}| \leqslant 1$ and $|\boldsymbol{\mu}| \leqslant k$, then, by assumption, the term (A.27) vanishes.
It follows that the above Taylor polynomial belongs to $\mathsf{I}_{(\mathbf{1},k)}^{\mathsf{c}}\big(\mathbb{R}^q \times M, (\mathbf{0}, x)\big)$, and then f fulfils (i), provided that this is the case for the Lagrange remainder. Note that this is the sum of terms of the following form:

$$C\,(s^1)^{\rho_1} \cdots (s^q)^{\rho_q}\big(x^1 - x^1(x)\big)^{\mu_1} \cdots \big(x^m - x^m(x)\big)^{\mu_m}, \qquad (A.30)$$

where $C \in \mathcal{D}(\mathbb{R}|\mathbb{R}^q \times M, (\mathbf{0}, x))$ and $|\boldsymbol{\rho}| + |\boldsymbol{\mu}| = k + 2$, which implies that $|\boldsymbol{\rho}| \geqslant 2$ or $|\boldsymbol{\mu}| \geqslant k + 1$. Hence, applying again (A.28) and (A.29) we deduce that the germ at $(\mathbf{0}, x)$ of the remainder belongs to $\mathsf{I}_{(\mathbf{1},k)}^{\mathsf{c}}\big(\mathbb{R}^q \times M, (\mathbf{0}, x)\big)$. This completes the proof. ■

In the same way as we have derived Proposition A2 from Proposition A1 one can derive the following proposition from Proposition A3.

Proposition A4. *Let* $\chi', \chi \in \mathcal{D}\big(N|\mathbb{R}^q \times M, (\mathbf{0}, x)\big)$. *Then, for each* $k \in \mathbb{N}$, *the following conditions are equivalent.*

(i) $\mathsf{j}^{(\mathbf{1},k)}\chi'(x) = \mathsf{j}^{(\mathbf{1},k)}\chi(x)$;

(ii) $\partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\mu}}\chi'\big(\mathbf{0}, x\big) = \partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\mu}}\chi\big(\mathbf{0}, x\big)$, *for any* q*-multi-index* $\boldsymbol{\rho}$ *and any* m*-multi-index* $\boldsymbol{\mu}$ *such that* $|\boldsymbol{\rho}| \leqslant 1$ *and* $|\boldsymbol{\mu}| \leqslant k$.

Proposition A4 indicates how to construct a differential structure on $\mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M)$: let ξ and η be charts on M and N, respectively, then $\mathsf{j}^{(\mathbf{1},k)}\chi(x)$ can be given the coordinates

$$\big(\xi(x), \partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\mu}}\chi(\mathbf{0}, x)\big)_{|\boldsymbol{\rho}|\leqslant 1\ ,\ |\boldsymbol{\mu}|\leqslant k}. \tag{A.31}$$

We now give the coordinate expression of the action of the total derivative $\mathrm{d}_{T(k)}$. In the sequel we will use Einstein's convention on repeated indices. We will first consider 0-forms and coordinate 1-forms and then we will derive from these the action of $\mathrm{d}_{T(k)}$ on arbitrary q-forms. For

the sake of semplicity, we will subdivide the coordinates on $\mathsf{T}^k N$ into blocks $\big(y^{(h)}\big)_{0\leqslant h\leqslant k}$ and treat each block $y^{(h)} = (y^{(h)1}, \ldots, y^{(h)n})$ as a

single coordinate.
Let $f : \mathsf{T}^k N \to \mathbb{R}$. We have, owing to (71),

$$\mathrm{d}_{T(k)} f : \mathsf{T}^{k+1} N \longrightarrow \mathbb{R}\,, \tag{A.32}$$

$$\begin{aligned} \mathrm{d}_{T(k)} f\big(\mathsf{t}^{k+1}\gamma(0)\big) &= D(f \circ \mathsf{t}^k\gamma)(0) \\ &= \sum_{h=0}^{k} \frac{\partial f}{\partial y^{(h)}}\bigg|_{\mathsf{t}^k\gamma(0)} \partial^{h+1}\gamma(0). \end{aligned} \tag{A.33}$$

Let us now consider blocks of coordinate 1-forms and treat them as single coordinate 1-forms, in particular we set $\mathrm{d}y^{(h)} : \mathsf{TT}^k N \to \mathbb{R}^n \colon u^{(k)j} \frac{\partial}{\partial y^{(k)j}} \mapsto u^{(h)}$.

We have, owing to (71),

$$\mathrm{d}_{T(k)}\mathrm{d}y^{(h)} : \mathsf{TT}^{k+1} N \longrightarrow \mathbb{R}^n\,, \tag{A.34}$$

$$\mathrm{d}_{T(k)}\mathrm{d}y^{(h)}\big(w\big) = D\big(\mathrm{d}y^{(h)} \circ \varphi_N^{k,1} \circ \mathsf{t}^{(1,k)}\chi\big)(0)\,, \tag{A.35}$$

where $\chi : \mathbb{R} \times \mathbb{R} \to N$ is a super-representative of w.
The coordinate expression of the prolongation $\mathsf{t}^{(1,k)}\chi : \mathbb{R} \to \mathsf{T}^{(1,k)}N$ is, owing to (A.31),

$$\left(\partial^h \chi(0,\cdot), \partial_1 \partial^h \chi(0,\cdot)\right)_{0 \leqslant h \leqslant k} \tag{A.36}$$

and it is also the coordinate expression of the prolongation

$$\varphi_N^{k,1} \circ \mathsf{t}^{(1,k)}\chi : \mathbb{R} \to \mathsf{TT}^k N \tag{A.37}$$

So we have

$$\begin{aligned} \mathrm{d}_{T(k)}(\mathrm{d}y^{(h)})(w) &= \mathrm{D}\left(\partial_1 \partial_h \chi(0,\cdot)\right)(0) \\ &= \partial_1 \partial_{h+1} \chi(0,0) \\ &= \mathrm{d}y^{(h+1)}(w). \end{aligned} \tag{A.38}$$

We conclude that

$$\mathrm{d}_{T(k)} \mathrm{d}y^{(h)} = \mathrm{d}y^{(h+1)}. \tag{A.39}$$

Let now

$$\Omega = \Omega_{h_1 \ldots h_q} \mathrm{d}y^{(h_1)} \wedge \ldots \wedge \mathrm{d}y^{(h_q)}, \quad 0 \leqslant h_\ell \leqslant k\,. \tag{A.40}$$

be a q-form on $\mathsf{T}^k M$. Its total derivative

$$\mathrm{d}_{T(k)}\Omega = (\mathrm{d}_{T(k)}\Omega)_{h_1 \ldots h_q} \mathrm{d}x^{(h_1)} \wedge \ldots \wedge \mathrm{d}x^{(h_q)}, \quad 0 \leqslant h_\ell \leqslant k+1, \tag{A.41}$$

is the q-form on $\mathsf{T}^{k+1}M$ whose components are

$$(\mathrm{d}_{T(k)}\Omega)_{h_1 \ldots h_q} = \mathrm{d}_{T(k)}\Omega_{h_1 \ldots h_q} + \sum_{\ell=1}^{q} \Omega_{h_1 \ldots h_{\ell-1}\, h_\ell - 1\, h_{\ell+1} \ldots h_q}, \tag{A.42}$$

where

$$\Omega_{h_1 \ldots h_q} = 0 \qquad \text{if } h_\ell = k+1 \text{ or } h_\ell < 0 \text{ for some } \ell. \tag{A.43}$$

We now give the coordinate expression of the action of the differential operator d_H. We adopt the following convention. We denote the coordinates on $\mathsf{J}^k(N|M)$ by $(x^i, y^A_{\boldsymbol{\mu}})$, where $1 \leqslant i \leqslant m$, $1 \leqslant A \leqslant n$, $\boldsymbol{\mu} \in \mathbb{N}^m$ and $|\boldsymbol{\mu}| \leqslant k$.
Let $f : \mathsf{J}^k(N|M) \to \mathbb{R}$. We have, owing to (75),

$$\mathrm{d}_H f : \mathsf{J}^{k+1}(N|M) \to \mathsf{T}^* M \tag{A.44}$$

$$\mathrm{d}_H f(\mathsf{j}^{k+1}\varphi(x)) = \mathrm{d}(f \circ \mathsf{j}^k \varphi)(x) = \left(\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial y^A_{\boldsymbol{\mu}}} \partial_{\boldsymbol{\mu}+\boldsymbol{\delta}_i} \varphi^A \right) \mathrm{d}x^i, \tag{A.45}$$

where $\boldsymbol{\delta}_i$ is the ith element of the canonical basis of $\mathbb{R}^m$, and we have used the coordinate expression of the prolongation $\mathsf{j}^k\varphi : M \to \mathsf{J}^k(N|M)$, which owing to (A.18), is $(x^i, \partial_{\boldsymbol{\mu}}\varphi^A)$, $1 \leqslant i \leqslant m$, $1 \leqslant A \leqslant n$, $\boldsymbol{\mu} \in \mathbb{N}^m$, and $|\boldsymbol{\mu}| \leqslant k$. Let us now consider the action of d_H on the exact vertical 1-form $\mathrm{d}y^A_{\boldsymbol{\mu}}$. To this end let $W \in \mathsf{VJ}^{k+1}(N|M)$ and set

$W = \mathsf{j}^{(1,k+1)}\chi(0,x)$. We have

$$\mathrm{d}_H \mathrm{d}y^A_{\boldsymbol{\mu}}(W) = \mathrm{d}\big(\mathrm{d}y^A_{\boldsymbol{\mu}} \circ \varphi^{k,1}_{(N|M)} \circ \mathsf{j}^{(1,k)}\chi\big)(x), \tag{A.46}$$

where χ is a super-representative of W. The coordinate expression of the prolongation $\mathsf{j}^{(1,k)}\chi : M \to \mathsf{J}^{(1,k)}(N|\mathbb{R} \times M)$ is, owing to (A.31),

$$(\partial_1 \partial_{\boldsymbol{\mu}} \chi^A)_{|\boldsymbol{\mu}| \leqslant k} \tag{A.47}$$

and it is also the coordinate expression of the prolongation

$$\varphi^{k,1}_{(N|M)} \circ \mathsf{j}^{(1,k)}\chi : M \to \mathsf{VJ}^k(N|M). \tag{A.48}$$

So we have

$$\begin{aligned} \mathrm{d}_H \mathrm{d}y^A_{\boldsymbol{\mu}}(W) &= \mathrm{d}\left(\partial_1 \partial_{\boldsymbol{\mu}} \chi^A\right)(x) \\ &= \left(\partial_1 \partial_{\boldsymbol{\mu}+\boldsymbol{\delta}_i} \chi^A \mathrm{d}x^i\right)(x) \\ &= W^A_{\boldsymbol{\mu}+\boldsymbol{\delta}_i} \mathrm{d}x^i(x) \\ &= \mathrm{d}y^A_{\boldsymbol{\mu}+\boldsymbol{\delta}_i}(W) \mathrm{d}x^i(x) \end{aligned} \tag{A.49}$$

This shows that

$$\mathrm{d}_H \mathrm{d}y^A_{\boldsymbol{\mu}} = \mathrm{d}y^A_{\boldsymbol{\mu}+\boldsymbol{\delta}_i} \otimes \mathrm{d}x^i. \tag{A.50}$$

Let now

$$\begin{array}{ccc} \times^q_{\mathsf{J}^k(N|M)} \mathsf{VJ}^k(N|M) & \xrightarrow{\quad \Theta \quad} & \wedge^p \mathsf{T}^* M \\ {\scriptstyle \upsilon^q{}_{\mathsf{J}^k(N|M)}} \big\downarrow & & \big\downarrow {\scriptstyle \pi^p_M} \\ J^k(N|M) & \xrightarrow{\quad \sigma_{k(N|M)} \quad} & M \end{array} \tag{A.51}$$

and set

$$\Theta = \Theta^{\boldsymbol{\mu}_1 \ldots \boldsymbol{\mu}_q}_{A_1 \ldots A_q; i_1 \ldots i_p} \, \mathrm{d}y^{A_1}_{\boldsymbol{\mu}_1} \wedge \ldots \wedge \mathrm{d}y^{A_q}_{\boldsymbol{\mu}_q} \otimes \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_p}. \tag{A.52}$$

If we apply (A.45) and (A.50) to (A.52) we obtain that

$$\mathrm{d}_H \Theta = (\mathrm{d}_H \Theta)^{\boldsymbol{\mu}_1 \ldots \boldsymbol{\mu}_q}_{A_1 \ldots A_q; i_1 \ldots i_{p+1}} \, \mathrm{d}y^{A_1}_{\boldsymbol{\mu}_1} \wedge \ldots \wedge \mathrm{d}y^{A_q}_{\boldsymbol{\mu}_q} \otimes \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_{p+1}}, \tag{A.53}$$

where

$$(\mathrm{d}_H\Theta)^{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_q}_{A_1\ldots A_q;i_1\ldots i_{p+1}} = \left(\mathrm{d}_H\Theta^{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_q}_{A_1\ldots A_q;i_1\ldots i_p}\right)_{i_{p+1}} + \sum_{\ell=1}^{q}\Theta^{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_{\ell-1}\boldsymbol{\mu}_\ell-\boldsymbol{\delta}_{i_{p+1}}\boldsymbol{\mu}_{\ell+1}\cdots\boldsymbol{\mu}_q}_{A_1\ldots A_q;i_1\ldots i_p} \qquad (A.54)$$

and

$$\Theta^{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_q}_{A_1\ldots A_q;i_1\ldots i_p} = 0 \qquad (A.55)$$

if $|\boldsymbol{\mu}_\ell| = k+1$ or $\boldsymbol{\mu}_\ell$ has a negative component for some ℓ.

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