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THE ZIQQURATH OF EXACT SEQUENCES OF n-GROUPOIDS

by S. KASANGIAN, G. METERE and E.M. VITALE

RÉSUMÉ. Dans ce travail nous étudions la notion de suite exacte dans la sesqui-catégorie des n-groupoïdes. En utilisant les produits fibrés homotopiques, à partir d'un n-foncteur entre n-groupoïdes pointés nous construisons une suite de six (n-1)-groupoïdes. Nous montrons que cette suite est exacte dans un sens qui généralise les notions usuelles d'exactitude pour les groupes et les gr-catégories. En réitérant le processus, nous obtenons une ziggourat[1] de suites exactes de longueur croissante et dimension décroissante. Pour $n = 1$, nous retrouvons un résultat classic du à R. Brown et, pour $n = 2$, nous retrouvons ses généralisations dues à Hardie, Kamps et Kieboom et à Duskin, Kieboom et Vitale.

RÉSUMÉ. In this work we study exactness in the sesqui-category of n-groupoids. Using homotopy pullbacks, we construct a six term sequence of (n-1)-groupoids from an n-functor between pointed n-groupoids. We show that the sequence is exact in a suitable sense, which generalizes the usual notions of exactness for groups and categorical groups. Moreover, iterating the process, we get a ziqqurath[2] of exact sequences of increasing length and decreasing dimension. For $n = 1$, we recover a classical result due to R. Brown and, for $n = 2$, its generalizations due to Hardie, Kamps and Kieboom and to Duskin, Kieboom and Vitale.

1. Introduction

This work is a contribution to the general theory of higher dimensional categorical structures, like n-categories and n-groupoids. Examples and applications of higher dimensional categorical structures abound in mathematics and mathematical physics; the reader in search of good

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[1]Les Ziggourats (ou Ziggurats) étaient des temples en forme de pyramide à gradins répandus auprès des habitants de l'ancienne Mésopotamie [14].

[2]Ziqquraths (or Ziggurats) were a type of step pyramid temples common to the inhabitants of ancient Mesopotamia [14].

motivations can consult the books [5, 10, 11]. We focalize our attention on the study of homotopy pullbacks in the sesqui-category of n-groupoids and, more precisely, on the notion of exact sequence that can be expressed using homotopy pullbacks.
The problem we take as our guide-line is to generalize to n-groupoids a result established by R. Brown in the context of groupoids. Let us recall Brown's result from [1]: consider a fibration of groupoids

$$F\colon \mathbb{A} \to \mathbb{B}$$

and, for a_0 a fixed object of $\mathbb{A}$, consider the fibre $\mathcal{F}_{a_0}$ of F over a_0. There is an exact sequence

$$\Pi_1(\mathcal{F}_{a_0}) \to \Pi_1(\mathbb{A}) \to \Pi_1(\mathbb{B}) \to \Pi_0(\mathcal{F}_{a_0}) \to \Pi_0(\mathbb{A}) \to \Pi_0(\mathbb{B})$$

where $\Pi_1(-)$ is the group of automorphisms of the base point and $\Pi_0(-)$ is the pointed set of isomorphism classes of objects.
The interest of Brown's result is that, despite its simplicity, it covers several quite different particular cases. We quote some of them:

1. A fibration $f\colon X \to Y$ of pointed topological spaces induces a fibration
$$F = \Pi_1(f)\colon \Pi_1(X) \to \Pi_1(Y)$$
on the homotopy groupoids; the sequence given by F is the first part of the homotopy sequence of f.

2. For G a fixed group, any extension $A \to B \to C$ of G-groups induces a fibration
$$F\colon \mathcal{Z}^1(G,B) \to \mathcal{Z}^1(G,C)$$
on the groupoids of derivations; the sequence given by F is the fundamental exact sequence in non-abelian cohomology of groups.

3. Let R be a commutative ring with unit, and consider

   $\mathbb{A} =$ the groupoid of Azumaya R-algebras and isomorpshisms of R-algebras.

$\mathbb{B}$ = the groupoid of Azumaya R-algebras and isomorphism classes of invertible bimodules.

As fibration F we can consider the functor $F\colon \mathbb{A} \to \mathbb{B}$ which sends an isomorphism $f\colon A \to B$ to the invertible A-B-bimodule ${}_fB$, where the action of A on ${}_fB$ is given by

$$A \otimes_R B \xrightarrow{f\otimes 1} B \otimes_R B \longrightarrow B$$

For X a fixed Azumaya R-algebra, the sequence given by F has the form

$$InnX \to AutX \to PicX \simeq PicR \to \pi_0(\mathcal{F}_X) \to \pi_0(\mathbb{A}) \to BrR$$

(Pic and Br stay for Picard and Brauer, Aut and Inn are automorphisms and inner automorphisms of R-algebras). Such a sequence is an extension of the classical Rosenberg-Zelinsky exact sequence.

These examples suggest to look for an higher dimensional version of Brown's result. Indeed:

1. A fibration of pointed topological spaces also induces a morphism between the homotopy bigroupoids; a convenient generalization of Brown's result gives then the first 9 terms of the homotopy sequence (see [7] for more details).

2. Instead of an extension of G-groups, one can consider an extension of $\mathbb{G}$-crossed modules for $\mathbb{G}$ a fixed crossed module, or an extension of $\mathbb{G}$-categorical groups for $\mathbb{G}$ a fixed categorical group, and construct a morphism between the 2-groupoids of derivations; from such a morphism one can then obtain an exact sequence in non-symmetric cohomology of crossed modules or categorical groups (see [4] for more details).

3. The functor $F\colon \mathbb{A} \to \mathbb{B}$ of Example 3 can be easily modified so to have a morphism of bigroupoids:

   $\mathbb{B}$ = the bigroupoid of Azumaya R-algebras, invertible bimodules, and isomorphisms of bimodules.

$\mathbb{A}$ = the bigroupoid of Azumaya R-algebras, isomorphisms of R-algebras, and natural isomorphisms. A natural isomorphism

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \quad \Downarrow\beta$$

is an element β of B invertible with respect to the product and such that $\beta \cdot f(a) = g(a) \cdot \beta$ for all $a \in A$.

The morphism $F\colon \mathbb{A} \to \mathbb{B}$ is defined on a natural isomorphism β by

$$F(\beta)\colon {}_fB \to {}_gB \quad F(\beta)(x) = \beta \cdot x.$$

(More in general, one can consider as F the inclusion of enriched categories, equivalences and natural isomorphisms into enriched categories, invertible distributors and natural isomorphisms.)

With these examples in mind, we have developed the theory needed to state and prove our generalization of Brown's result: consider an n-functor $F\colon \mathbb{A} \to \mathbb{B}$ between pointed n-groupoids and its homotopy kernel $K\colon \mathbb{K} \to \mathbb{A}$

i- there is an exact sequence of (n-1)-pointed groupoids

$$\Pi_1(\mathbb{K}) \to \Pi_1(\mathbb{A}) \to \Pi_1(\mathbb{B}) \to \Pi_0(\mathbb{K}) \to \Pi_0(\mathbb{A}) \to \Pi_0(\mathbb{B})$$

ii- since Π_0 and Π_1 preserve exact sequences and commute each other, we can iterate the process and we get a ziqqurath of exact sequences

three pointed n-groupoids
six pointed (n-1)-groupoids
nine pointed (n-2)-groupoids
$\vdots$
$3 \cdot n$ pointed groupoids
$3 \cdot (n+1)$ pointed sets

In particular, for $n = 1$ we obtain the two-level ziqqurath of Brown and, for $n = 2$, the three-level ziqqurath of [7] and [4].
The paper is organized as follows:

- In Section 2 we give the inductive (= enriched style) definition of n-Cat. We recall the definition of homotopy pullback and we prove that n-Cat is a sesqui-category with homotopy pullbacks.

- Section 3 is devoted to the definition of the sub-sesqui-category n-Gpd of n-groupoids, which is closed in n-Cat under homotopy pullbacks.

- In Section 4 we define exactness in the sesqui-category of pointed n-groupoids.

- The sesqui-functor $\Pi_0^{(n)}\colon n\text{-Gpd} \to (n\text{-}1)\text{-Gpd}$ is constructed in Section 5, and it is proved that it preserves exact sequences.

- Lax n-modifications are introduced in Section 6. We prove that homotopy pullbacks in n-Cat also satisfy a more sophisticated universal property expressed using lax n-modifications. This new universal property is needed in Sections 7, 8 and 9.

- In Section 7 we construct two sesqui-functors $\Pi_1^{(n)}\colon n\text{-Gpd}_* \to (n\text{-}1)\text{-Gpd}_*$ and $\Omega^{(n)}\colon n\text{-Gpd}_* \to n\text{-Gpd}_*$, and we prove that they preserve exact sequences. (Proposition 7.3 is proved using a result contained in the Appendix.)

- Finally, in Sections 8 and 9 we prove the main results: the fibration sequence and the ziqqurath of exact sequences associated with an n-functor between pointed n-groupoids.

Sections 2 and 6 are a survey of results from [13], they are inserted here for the reader's convenience. All along the paper, several proofs are omitted. Some of them are something more than a strightforward exercise. The interested reader can find all the details in [12].

2. The sesqui-category n-Cat

In this section we describe the sesqui-category n-Cat of strict n-categories, strict n-functors and lax n-transformations. We also describe homotopy pullbacks (h-pullbacks for short) in n-Cat. The definition of sesqui-category can be found in [15], for h-pullbacks see also [6].

2.1. Definition.

1. 0-Cat is the category of small sets and maps.

2. 1-Cat is the category of small categories and functors.

3. For $n > 1$, n-Cat has (strict) n-categories as objects and (strict) n-functors as morphisms.
   An n-category $\mathbb{C}$ consists of a set of objects $\mathbb{C}_0$, and for every pair $c_0, c_0' \in \mathbb{C}_0$, a ($n$-1)-category $\mathbb{C}_1(c_0, c_0')$. The structure is given by morphisms of (n-1)-categories:

$$\mathbb{I} \xrightarrow{u^0(c_0)} \mathbb{C}_1(c_0, c_0) \qquad \mathbb{C}_1(c_0, c_0') \times \mathbb{C}_1(c_0', c_0'') \xrightarrow{\circ^0_{c_0, c_0', c_0''}} \mathbb{C}_1(c_0, c_0'')$$

   called respectively 0-units and 0-compositions, with c_0, c_0', c_0'' any triple of objects of $\mathbb{C}$. Axioms are the usual ones for strict unit and strict associativity.
   An n-functor $\mathbb{F}\colon \mathbb{C} \to \mathbb{D}$ consists of a map $F_0\colon \mathbb{C}_0 \to \mathbb{D}_0$ together with morphisms of (n-1)-categories

$$F_1^{c_0, c_0'}\colon \mathbb{C}_1(c_0, c_0') \to \mathbb{D}_1(F_0 c_0, F_0 c_0')$$

   with c_0, c_0' any pair of objects of $\mathbb{C}$, such that usual strict functoriality axioms are satisfied.

2.2. Remark. The previous definition makes sense because one can prove by induction that n-Cat is a category with binary products and a terminal object $\mathbb{I}$.

2.3. NOTATION. Cell dimension will be often recalled as subscript: c_k is a k-cell in the n-category $\mathbb{C}$. Moreover, if

$$c_k : c_{k-1} \to c'_{k-1} : c_{k-2} \to c'_{k-2} : \cdots \to \cdots : c_1 \to c'_1 : c_0 \to c'_0,$$

c_k can be considered as an object of the (n-k)-category

$$\Big[\cdots \big[[\mathbb{C}_1(c_0, c'_0)]_1(c_1, c'_1) \big]_1 \cdots \Big]_1 (c_{k-1}, c'_{k-1}).$$

In order to avoid this quite uncomfortable notation, the latter will be renamed more simply $\mathbb{C}_k(c_{k-1}, c'_{k-1})$, while $\mathbb{C}_k$ denotes the set of all k-cells in $\mathbb{C}$. Finally, 0-subscript of the underlying set of an n-category and 0-superscript of unit u and composition $\circ$ will be often omitted.

2.4. DEFINITION. Let $F, G\colon \mathbb{C} \to \mathbb{D}$ be morphisms of n-categories. By a 2-morphism $\alpha : F \Rightarrow G$ is meant:

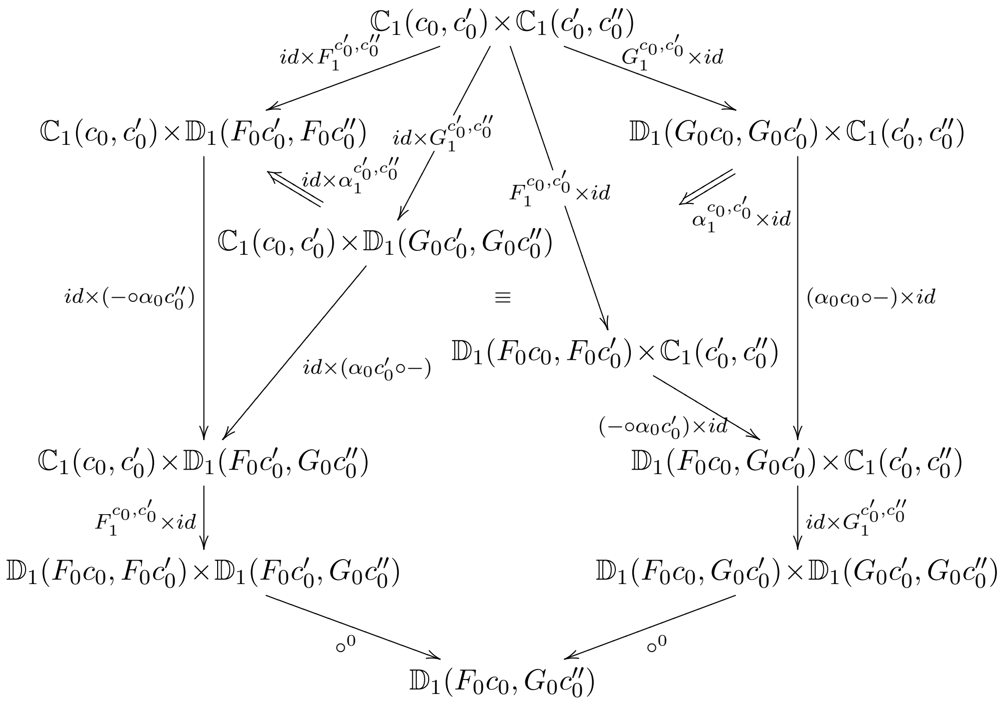
1. The equality $F = G$ if $n = 0$.
2. A natural transformation $\alpha : F \Rightarrow G$ if $n = 1$.
3. A lax n-transformation $\alpha : F \Rightarrow G$ if $n > 1$, that is, a pair (α_0, α_1) where $\alpha_0\colon \mathbb{C}_0 \to \mathbb{D}_1$ is a map such that $\alpha_0(c_0) = \alpha_{c_0} : Fc_0 \to Gc_0$, and $\alpha_1 = \{\alpha_1^{c_0,c'_0}\}_{c_0,c'_0 \in \mathbb{C}_0}$ is a collection of 2-morphisms of (n-1)-categories

$$\begin{array}{ccccc} & & \mathbb{C}_1(c_0, c'_0) & & \\ & \swarrow^{F_1^{c_0,c'_0}} & & \searrow^{G_1^{c_0,c'_0}} & \\ \mathbb{D}_1(F_0c_0, F_0c'_0) & & \overset{\alpha_1^{c_0,c'_0}}{\Longleftarrow} & & \mathbb{D}_1(G_0c_0, G_0c'_0) \\ & \searrow_{-\circ\alpha_0 c'_0} & & \swarrow_{\alpha_0 c_0 \circ -} & \\ & & \mathbb{D}_1(F_0c_0, G_0c'_0) & & \end{array} \qquad (1)$$

satisfying the following axioms:

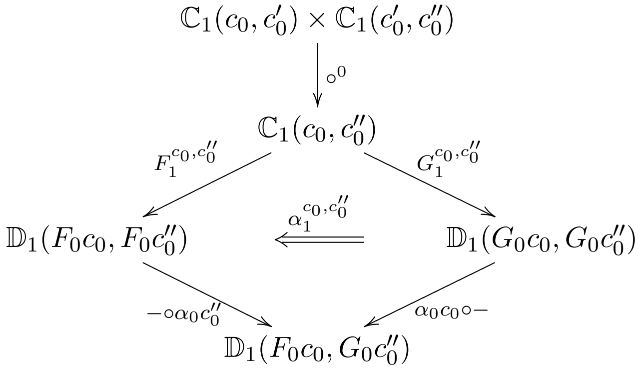
– (*functoriality w.r.t. composition*) for every triple of objects

c_0, c_0', c_0'' of $\mathbb{C}_0$,

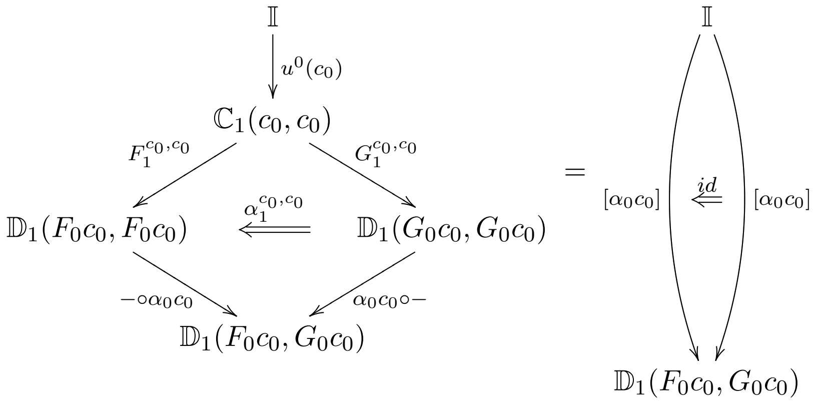


(2)

$=$



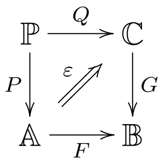
– (*functoriality w.r.t. units*) for every object c_0 of $\mathbb{C}_0$,



(3)

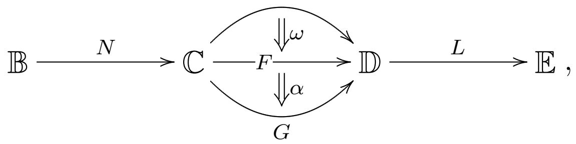
2.5. PROPOSITION. *The category* n-Cat *equipped with lax* n*-transformations is a sesqui-category with* h*-pullbacks.*

Before proving the above proposition, let us recall the universal property of the h-pullback: consider two n-functors $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{C} \to \mathbb{B}$. An h-pullback of F and G is a four-tuple $(\mathbb{P}, P, Q, \varepsilon)$



such that for any other four-tuple $(\mathbb{X}, M, N, \lambda : M \cdot F \Rightarrow N \cdot G)$ there exists a unique $L\colon \mathbb{X} \to \mathbb{P}$ satisfying $L \cdot P = M$, $L \cdot Q = N$, $L \cdot \varepsilon = \lambda$. This universal property defines h-pullbacks up to isomorphism.

Proof. We need vertical composition of lax n-transformations and reduced horizontal composition (whiskering). In fact, according to the following reference diagram



we let:

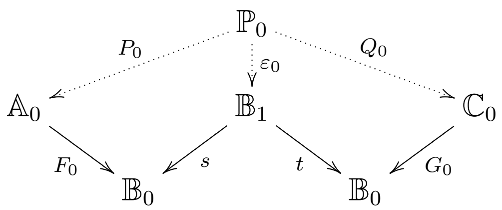
$$[\omega \cdot \alpha]_0(c_0) = \omega_0 c_0 \circ \alpha_0 c_0,$$

$$[\omega \cdot \alpha]_1^{c_0,c_0'} = \left(\alpha_1^{c_0,c_0'} \cdot (\omega_0 c_0 \circ -)\right) \cdot \left(\omega_1^{c_0,c_0'} \cdot (- \circ \alpha_0 c_0')\right);$$

$$[N \cdot \alpha]_0 = \alpha_0(N(b_0)), \qquad [N \cdot \alpha]_1^{b_0,b_0'} = N_1^{b_0,b_0'} \cdot \alpha_1^{Nb_0,Nb_0'};$$

$$[\alpha \cdot L]_0 = L(\alpha_0(c_0)), \qquad [\alpha \cdot L]_1^{c_0,c_0'} = \alpha_1^{c_0,c_0'} \cdot L_1^{Fc_0,Gc_0'}.$$

These constructions make n-Cat a sesqui-category.
An h-pullback in n-Cat can be described as follows.
For n=0, the usual pullback in Set is an instance of h-pullback, with the 2-morphism ε being an identity.
For $n > 0$, we give an inductive construction of the standard h-pullback satisfying the universal property recalled above. The set $\mathbb{P}_0$ is the following limit in Set

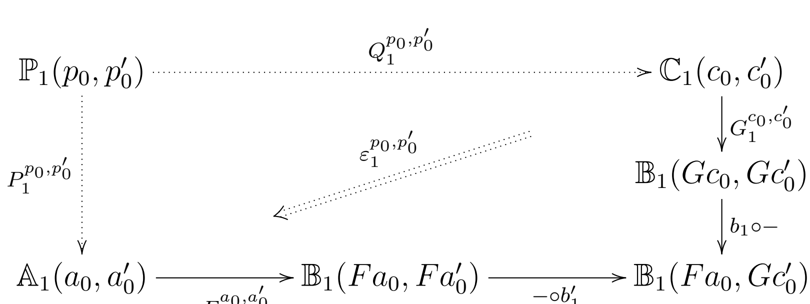


where s, t are *source* and *target* maps of 1-cells. More explicitly,

$$\mathbb{P}_0 = \{(a_0, b_1, c_0) \quad \text{s.t. } a_0 \in \mathbb{A}_0, c_0 \in \mathbb{C}_0, b_1 : Fa_0 \to Gc_0 \in \mathbb{B}_1\}$$

$$P_0((a_0, b_1, c_0)) = a_0, \quad Q_0((a_0, b_1, c_0)) = c_0, \quad \varepsilon_0((a_0, b_1, c_0)) = b_1$$

Let us fix two elements $p_0 = (a_0, b_1, c_0)$ and $p_0' = (a_0', b_1', c_0')$ of $\mathbb{P}_0$. The (n-1)-category $\mathbb{P}_1(p_0, p_0')$ is described by the following h-pullback in (n-1)-Cat:



The units and the compositions in $\mathbb{P}$ are determined by the universal property of the h-pullbacks $\mathbb{P}_1(p_0, p_0')$. ■

3. The sesqui-category n-Gpd

We first define equivalences of n-categories, and we use them to define n-groupoids.

3.1. Definition. An n-functor $F\colon \mathbb{C} \to \mathbb{D}$ is an equivalence if

(a) F is essentially surjective on objects: for each object $d_0 \in \mathbb{D}_0$, there exist $c_0 \in \mathbb{C}_0$ and $d_1\colon Fc_0 \to d_0$ such that, for each $d_0' \in \mathbb{D}_0$, the (n-1)-functors

$$d_1 \circ - : \mathbb{D}_1(d_0, d_0') \to \mathbb{D}_1(Fc_0, d_0')$$

$$- \circ d_1 : \mathbb{D}_1(d_0', Fc_0) \to \mathbb{D}_1(d_0', d_0)$$

are equivalences of (n-1)-categories, and

(b) for each $c_0, c_0' \in \mathbb{C}_0$, the ($n$-1)-functor

$$F_1^{c_0,c_0'}\colon \mathbb{C}_1(c_0, c_0') \to \mathbb{D}_1(Fc_0, Fc_0')$$

is an equivalence of (n-1)-categories.

Essentially surjective n-functors and equivalences are closed under composition and stable under h-pullback.

3.2. Definition. A 1-cell $c_1\colon c_0 \to c_0'$ of an n-category $\mathbb{C}$ is an equivalence if, for each $c_0'' \in \mathbb{C}_0$, the (n-1)-functors

$$c_1 \circ - : \mathbb{C}_1(c_0', c_0'') \to \mathbb{C}_1(c_0, c_0'') \qquad - \circ c_1 : \mathbb{C}_1(c_0'', c_0) \to \mathbb{C}_1(c_0'', c_0')$$

are equivalences of (n-1)-categories.

3.3. Definition. An n-category $\mathbb{C}$ is an n-groupoid if

(a) every 1-cell of $\mathbb{C}$ is an equivalence, and

(b) for each $c_0, c_0' \in \mathbb{C}_0$, the ($n$-1)-category $\mathbb{C}_1(c_0, c_0')$ is an (n-1)-groupoid.

3.4. Remark. In the context of strict n-categories, the previous definition of n-groupoid is equivalent to those given by Street [15] and Kapranov and Voevodsky [9]. This fact is not easy to prove and a detailed proof can be found in [8]. In the same paper we also show that Definition 3.1 and Definition 3.2 are indeed redundant. In fact

- in Definition 3.1, $d_1 \circ -$ is an equivalence if, and only if, $- \circ d_1$ is;
- in Definition 3.2, $c_1 \circ -$ is an equivalence if, and only if, $- \circ c_1$ is.

We denote by n-Gpd the full sub-sesqui-category of n-Cat having as objects n-groupoids. The following result is straightforward.

3.5. Proposition. *The sesqui-category* n-Gpd *is closed in* n-Cat *under h-pullbacks.*

We denote by n-Gpd$_\star$ the sequi-category of pointed n-groupoids: a pointed n-groupoid is an n-groupoid $\mathbb{C}$ together with a fixed object $\star \in \mathbb{C}_0$, an n-functor $F\colon \mathbb{C} \to \mathbb{D}$ is pointed if $F(\star_\mathbb{C}) = \star_\mathbb{D}$, a lax n-transformation $\alpha\colon F \Rightarrow G$ is pointed if $\alpha(\star_\mathbb{C}) = u^0_{\star_\mathbb{D}}$. Once again, h-pullbacks in n-Gpd$_\star$ are constructed as in n-Cat.

4. Exact sequences

To define exactness, we need a notion of surjectivity suitable for n-categories.

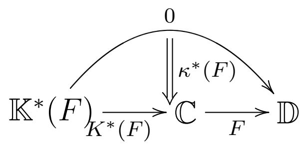
4.1. Definition. An n-functor $F\colon \mathbb{C} \to \mathbb{D}$ is h-surjective if

(a) it is essentially surjective on objects (see Definition 3.1), and

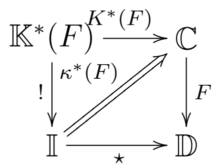
(b) for each $c_0, c_0' \in \mathbb{C}_0$, the ($n$-1)-functor $F_1^{c_0,c_0'}$ is h-surjective.

Once again, h-surjective functors are closed under composition and stable under h-pullbacks. Moreover, an n-functor is an equivalence iff it is h-surjective and faithful (i.e., injective on n-cells).

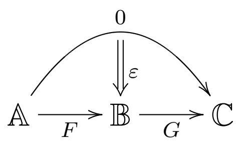
If $F\colon \mathbb{C} \to \mathbb{D}$ is an n-functor in n-Gpd$_\star$, we denote by



its h-kernel, that is the h-pullback



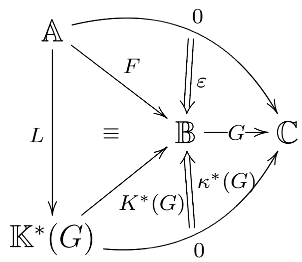
4.2. Definition. Let the following diagram in n-Gpd$_\star$ be given:



We call the triple (F, ε, G) *exact in* $\mathbb{B}$ if the comparison n-functor

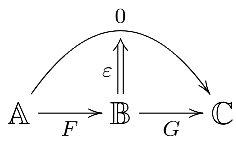
$$L\colon \mathbb{A} \to \mathbb{K}^*(G)$$

given by the universal property of the h-kernel $(\mathbb{K}^*(G), K^*(G), \kappa^*(G))$, is h-surjective.

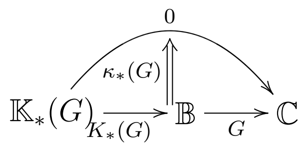


Observe that for $n = 0$ this is the usual definition of exact sequence of pointed sets, and for $n = 1$ this is the notion of 2-exactness introduced in [17] for categorical groups. In fact, for $n = 1$ h-surjective precisely means full and essentially surjective.

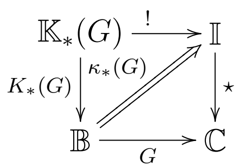
4.3. Remark. Analogously, we say that the triple (F, ε, G)



is exact in $\mathbb{B}$ if the comparison n-functor $L\colon \mathbb{A} \to \mathbb{K}_*(G)$ is h-surjective, where



is the h-pullback



5. The sesqui-functors $\Pi_0^{(n)}$ and $D^{(n)}$

In this section we define two sesqui-functors

$$n\text{-Gpd} \underset{D^{(n)}}{\overset{\Pi_0^{(n)}}{\rightleftarrows}} (n\text{-}1)\text{-Gpd}$$

(see [16] for the definition of sesqui-functor).

5.1. Definition. (The functor $\Pi_0^{(n)}$)

1. $\Pi_0^{(1)}$ is the functor Gpd $\to$ Set assigning to a groupoid $\mathbb{C}$ the set $|\mathbb{C}|$ of isomorphism classes of objects of $\mathbb{C}$.

2. For $n > 1$, let an n-groupoid $\mathbb{C}$ be given. Then
$$\Pi_0^{(n)}\mathbb{C} = ([\Pi_0^{(n)}\mathbb{C}]_0, [\Pi_0^{(n)}\mathbb{C}]_1(-,-))$$
where $[\Pi_0^{(n)}\mathbb{C}]_0 = \mathbb{C}_0$ and $[\Pi_0^{(n)}\mathbb{C}]_1(c_0, c_0') = \Pi_0^{(n-1)}(\mathbb{C}_1(c_0, c_0'))$. Compositions and units are obtained inductively: ${}^{\Pi_0^{(n)}\mathbb{C}}\circ = \Pi^{(n-1)}({}^{\mathbb{C}}\circ)$,

$u(c_0) = \Pi_0^{(n-1)}(u(c_0))$.
Now let an n-functor $F : \mathbb{C} \to \mathbb{D}$ be given. Then $[\Pi_0^{(n)} F]_0 = F_0$ and $[\Pi_0^{(n)} F]_1^{c_0,c'_0} = \Pi_0^{(n-1)}(F_1^{c_0,c'_0})$ define $\Pi_0^{(n)}$ on morphisms.

Note that the previous definition makes sense because one can prove inductively that $\Pi_0^{(n)}$ preserves binary products and the terminal object $\mathbb{I}$.

5.2. Definition. (The sesqui-functor $\Pi_0^{(n)}$)

1. Since $[\Pi_0^{(2)}\mathbb{D}]_1 = \mathbb{D}_1/\sim$, the quotient of $\mathbb{D}_1$ under the equivalence relation $\sim$ identifying 1-cells $d_1, d'_1 \colon d_0 \to d'_0$ if there exists a 2-cell $d_2 \colon d_1 \to d'_1$, we define $[\Pi''_0\alpha]_0 = \alpha_0 \cdot p$, where p is the canonic projection on the quotient.

2. For $n > 2$, let $\alpha : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$ be a 2-morphism. We define $\Pi_0^{(n)}\alpha$ by $[\Pi_0^{(n)}\alpha]_0 = \alpha_0$ and $[\Pi_0^{(n)}\alpha]_1^{c_0,c'_0} = \Pi_0^{(n-1)}(\alpha_1^{c_0,c'_0})$.

A careful use of induction shows that $\Pi_0^{(n)}$ is well-defined and is indeed a sesqui-functor.

The definition of the sesqui-functor $D^{(n)}$ is easier. We make it explicit only on objects.

5.3. Definition. (The sesqui-functor $D^{(n)}$)

1. $D^{(1)}$ is the functor (= trivial sesqui-functor) $D^{(1)} : \mathrm{Set} \to \mathrm{Gpd}$ assigning to a set C the discrete groupoid $D(C)$ with objects the elements of C and only identity arrows.

2. For $n > 1$, let an (n-1)-groupoid $\mathbb{C}$ be given. Then $D^{(n)}$ is given by $[D^{(n)}\mathbb{C}]_0 = \mathbb{C}_0$ and $[D^{(n)}\mathbb{C}]_1(c_0, c'_0) = D^{(n-1)}(\mathbb{C}_1(c_0, c'_0))$.

It is an exercise for the reader to prove the following fact which, in particular, implies that $D^{(n)}$ preserves h-pullbacks.

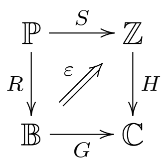
5.4. Proposition. *For all $n > 1$, there is an adjunction of sesqui-functors*

$$\Pi_0^{(n)} \dashv D^{(n)} ,$$

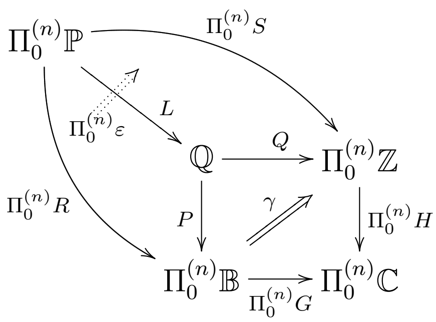
and therefore an adjunction of the underlying functors.

We are going to prove the main result of this section: the sesqui-functor $\Pi_0^{(n)}$ preserves exact sequences. Two preliminary lemmas clarify the relations between preservation of exactness and its main ingredients: h-surjectivity and the notion of h-pullback.

5.5. Lemma. *Let us consider the following h-pullback diagram:*



The comparison $L : \Pi_0^{(n)}\mathbb{P} \to \mathbb{Q}$ *with the h-pullback of* $\Pi_0^{(n)}(G)$ *and* $\Pi_0^{(n)}(H)$ *is h-surjective.*



Proof. By induction on n.
1) For $n = 1$, the h-pullback $\mathbb{P}$ has objects and arrows

$$(b_0,\ Gb_0 \xrightarrow{c_1} Hz_0\ , z_0), \quad (b_1, =, z_1) : (b_0, c_1, z_0) \to (b'_0, c'_1, z'_0)$$

where the "=" stays for the commutative square $c_1 \circ Hz_1 = Gb_1 \circ c'_1$. Hence the set $\Pi_0^{(1)}(\mathbb{P})$ has elements the classes $[b_0, c_1, z_0]_\sim$. On the other side, the set $\mathbb{Q}$ is a usual pullback in Set. It has elements the pairs $([b_0]_\sim, [z_0]_\sim)$ such that $\Pi_0^{(1)}G([b_0]_\sim) = \Pi_0^{(1)}H([z_0]_\sim)$, *i.e.* $[Gb_0]_\sim = [Hz_0]_\sim$, *i.e.* such that there exists $c_1 : Gb_0 \to Hz_0$. Then the comparison $L = L_0 : [b_0, c_1, z_0]_\sim \mapsto ([b_0]_\sim, [z_0]_\sim)$ is clearly surjective.
2) For $n = 2$, the h-pullback $\mathbb{P}$ is a 2-groupoid with objects

$$(b_0,\ Gb_0 \xrightarrow{c_1} Hz_0\ , z_0).$$

Arrows and 2-cells are of the form

$$(b_1, c_2, z_1) : (b_0, c_1, z_0) \to (b'_0, c'_1, z'_0), \quad (b_2, =, z_2) : (b_1, c_2, z_1) \to (b'_1, c'_2, z'_1)$$

Therefore the groupoid $\Pi_0^{(2)}\mathbb{P}$ has objects (b_0, c_1, z_0) and arrows $[b_1, c_2, z_1]_\sim$. On the other side, the groupoid $\mathbb{Q}$ has objects and arrows

$$(b_0,\ Gb_0 \xrightarrow{[c_1]_\sim} Hz_0\ , z_0), \quad ([b_1]_\sim, =, [z_1]_\sim)$$

with $[b_1]_\sim : b_0 \to b'_0$ in $\Pi_0^{(2)}\mathbb{B}$ and $[z_1]_\sim : z_0 \to z'_0$ in $\Pi_0^{(2)}\mathbb{Z}$ such that the diagram

$$\begin{array}{ccc} Gb_0 & \xrightarrow{[c_1]_\sim} & Hz_0 \\ {\scriptstyle [Gb_1]_\sim}\downarrow & & \downarrow{\scriptstyle [Hz_1]_\sim} \\ Gb'_0 & \xrightarrow[{[c'_1]_\sim}]{} & Hz'_0 \end{array} \qquad (4)$$

commutes. Hence the comparison

$$\begin{aligned} L \ : \ & (b_0, c_1, z_0) \mapsto (b_0, c_1, z_0) \\ & [b_1, c_2, z_1]_\sim \mapsto ([b_1]_\sim, [z_1]_\sim) \end{aligned}$$

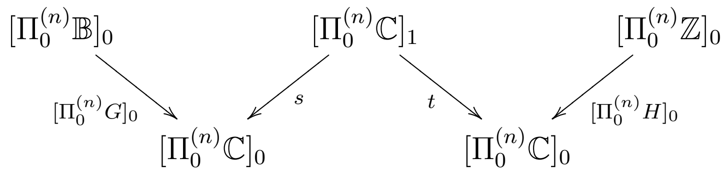
is h-surjective. In fact it is an identity on objects, and full on homs. Let us fix a pair of objects (b_0, c_1, z_0) and (b'_0, c'_1, z'_0) in the domain, and an arrow $([b_1]_\sim, =, [z_1]_\sim)$ in $\mathbb{Q}$, where the "$=$" express the commutativity of the diagram (4) above. Then $[c_1 \circ Hz_1]_\sim = [Gb_1 \circ c'_1]_\sim$ if, and only if, there exists

$$c_2 : c_1 \circ Hz_1 \to Gb_1 \circ c'_1.$$

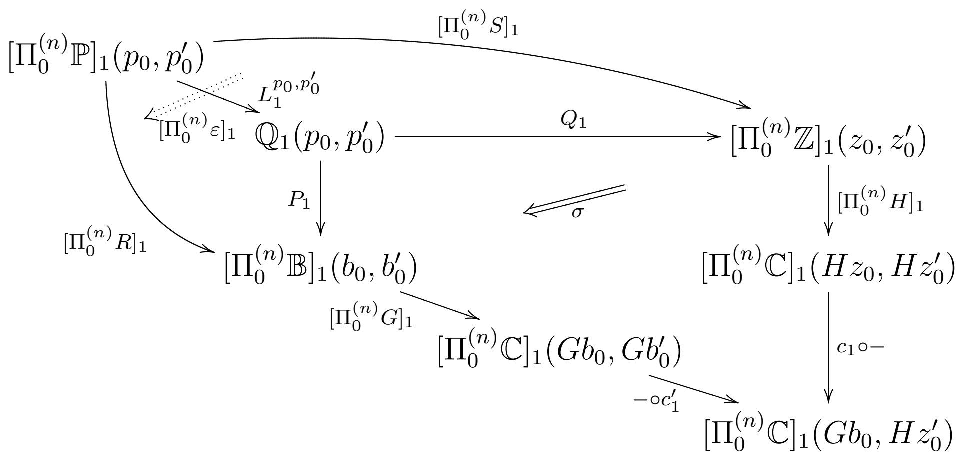
In other words we get an arrow $[b_1, c_2, z_1]_\sim$ of $\Pi_0^{(2)}\mathbb{P}$ sent by L to $([b_1]_\sim, =, [z_1]_\sim)$, *i.e.* L is full.
3) Finally, let $n > 2$. On objects, $L_0 : [\Pi_0^{(n)}\mathbb{P}]_0 \to \mathbb{Q}_0$ is the identity. In fact, $[\Pi_0^{(n)}\mathbb{P}]_0 = \mathbb{P}_0$, the set-theoretical limit over the diagram

$$\begin{array}{ccccccc} \mathbb{B}_0 & & & \mathbb{C}_1 & & & \mathbb{Z}_0 \\ & {\scriptstyle G_0}\searrow & \swarrow{\scriptstyle s} & & {\scriptstyle t}\searrow & \swarrow{\scriptstyle H_0} & \\ & & \mathbb{C}_0 & & \mathbb{C}_0 & & \end{array}$$

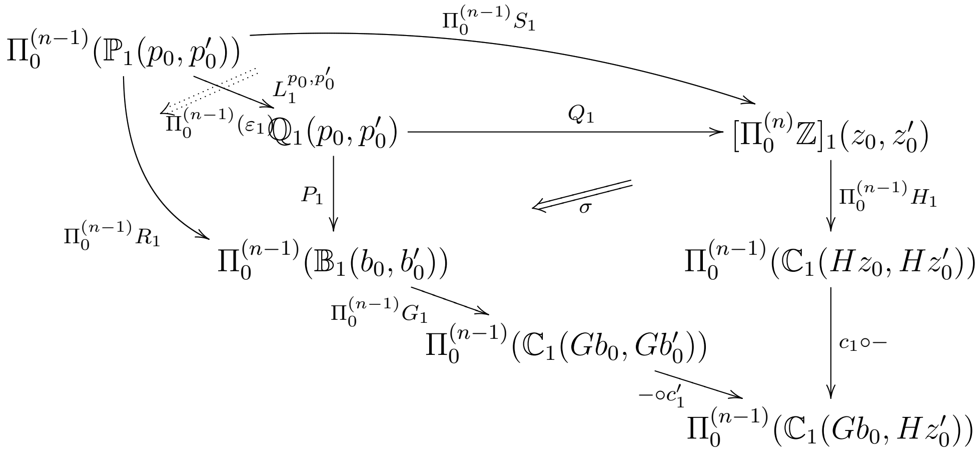
and, for $n > 2$, this diagram coincides with the one defining $\mathbb{Q}_0$:



On homs, let us fix two objects $p_1 = (b_0, c_1, z_0)$ and $p'_1 = (b'_0, c'_1, z'_0)$ of $[\Pi_0^{(n)}\mathbb{P}]_0 = \mathbb{P}_0$ and compute $L_1^{p_1,p'_1}$ by means of universal property of h-pullbacks. The diagram



is the same as (and determined by)



This shows that $L_1^{p_0,p_0'}$ is itself a comparison between Π_0 of an h-pullback and an h-pullback of a Π_0 of a diagram (of (n-1)-groupoids), hence it is h-surjective by induction hypothesis. ∎

5.6. Lemma. *If an n-functor $L : \mathbb{A} \to \mathbb{K}$ is h-surjective, then also $\Pi_0^{(n)}(L)$ is h-surjective.*

Proof. By induction on n.
1) For $n = 1$, let L be an h-surjective functor between groupoids, *i.e.* L is full and essentially surjective on objects. Therefore, for an element $[k_0]_\sim \in \Pi_0^{(1)}\mathbb{K}$ there exists a pair $(a_0, k_1 : La_0 \to k_0)$. Hence $(\Pi_0^{(1)}L)([a_0]_\sim) = [La_0]_\sim = [k_0]_\sim$.
2) For $n = 2$, let L be an h-surjective morphism between 2-groupoids. Explicitely, this means that

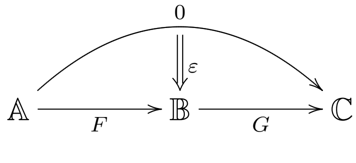
(i) for any k_0 there exist $(a_0, k_1 : La_0 \to k_0)$, and

(ii) for any pair a_0, a_0', $L_1^{a_0,a_0'}$ is h-surjective.

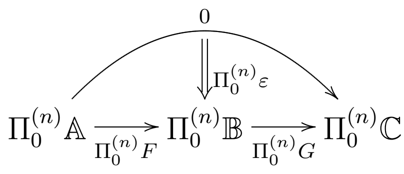
Since $[\Pi_0^{(2)}L]_0 = L_0$, for any k_0 one has $[k_1]_\sim : La_0 \to k_0$, and this proves the first condition on $\Pi_0^{(2)}L$. Moreover, once we fix a pair a_0, a_0' of objects, by definition one has $[\Pi_0^{(2)}L]_1^{a_0,a_0'} = \Pi_0^{(1)}(L_1^{a_0,a_0'})$. Hence it is h-surjective by the previous case.
3) Finally, let $n > 2$. A morphism L of n-groupoids is h-surjective when conditions (i) and (ii) above hold. Since $[\Pi_0^{(n)}L]_0 = L_0$ and $[\Pi_0^{(n)}(\mathbb{K})]_1 = \Pi_0^{(n-1)}(\mathbb{K}_1)$, condition (i) for $\Pi_0^{(n)}L$ precisely is condition (i) for L, hence it holds. Moreover, whence we fix a pair a_0, a_0' of objects, by definition one has $[\Pi_0^{(n)}L]_1^{a_0,a_0'} = \Pi_0^{(n-1)}(L_1^{a_0,a_0'})$. Hence it is h-surjective by induction hypothesis. ∎

We are ready to state and prove the main result of this section.

5.7. Proposition. *Given an exact sequence in n-*Gpd$_*$

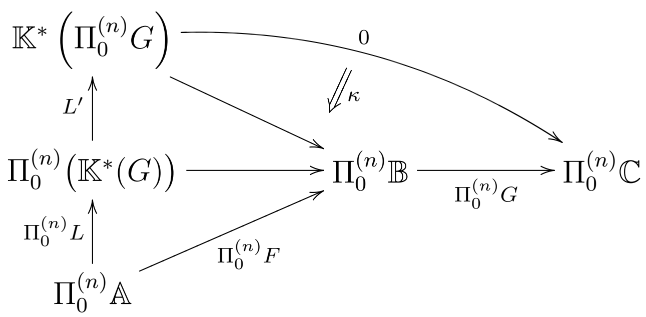


the sequence



is exact in (*n-1*)-Gpd$_*$.

Proof. Let us consider the diagram

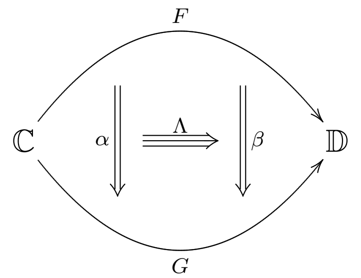


where L is the comparison in n-Gpd (notations as in Definition 4.2). L is h-surjective by hypothesis. Therefore $\Pi_0^{(n)}L$ is h-surjective by Lemma 5.6. Moreover, L' is the comparison in (n-1)-Gpd, so that it is h-surjective by Lemma 5.5. Finally, their composition is again h-surjective, and it is the comparison between $\Pi_0^{(n)}\mathbb{A}$ and the kernel of $\Pi_0^{(n)}G$ by uniqueness in the universal property of h-kernels. ∎

6. Lax n-modifications

We already have two sesqui-functors $\Pi_0^{(n)}$ and $D^{(n)}$. In Section 7 we will construct two other sesqui-functors $\Pi_1^{(n)}$ and $\Omega^{(n)}$. In order to define $\Omega^{(n)}$ on lax n-transformations, we use the fact that h-pullbacks in n-Gpd$_\star$ satisfay another universal property. To express this new universal property we introduce here lax n-modifications between lax n-transformations.

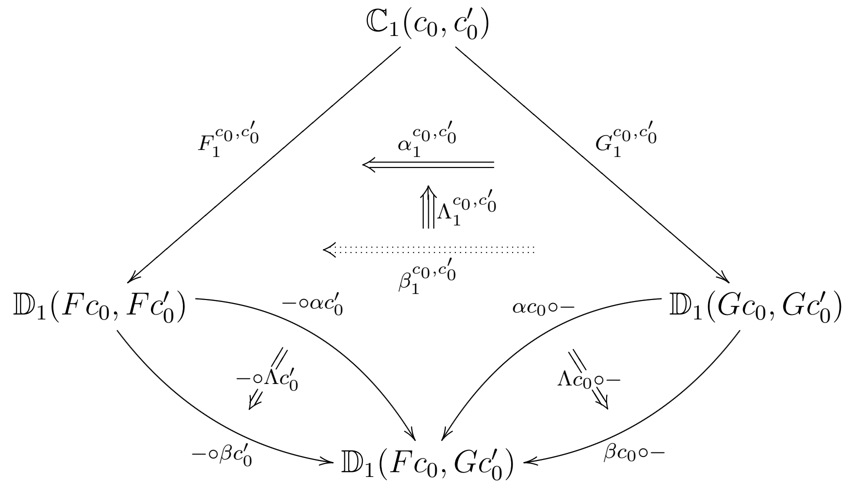
6.1. DEFINITION. Let $\alpha, \beta \colon F \Rightarrow G \colon \mathbb{C} \to \mathbb{D}$ be 2-morphisms of n-categories. By a 3-morphism $\Lambda : \alpha \Rrightarrow \beta$



is meant:

1. The equality $\alpha = \beta$ if $n = 1$.

2. A lax n-modification $\Lambda : \alpha \Rrightarrow \beta$ if $n > 1$, that is, a pair $\langle \Lambda_0, \Lambda_1 \rangle$, where $\Lambda_0 \colon \mathbb{C}_0 \to \mathbb{D}_2$ is a map such that, for every c_0 in $\mathbb{C}_0$, $\Lambda_0(c_0) \colon \alpha_0(c_0) \to \beta_0(c_0)$, and $\Lambda_1 = \{\Lambda_1^{c_0,c_0'}\}_{c_0,c_0' \in \mathbb{C}_0}$ is a collection of 3-morphisms of (n-1)-categories that fill the following diagrams:



i.e.

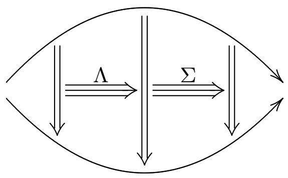
$$\begin{array}{ccc} G_1^{c_0,c_0'}\cdot(-\circ\alpha c_0') & \overset{\alpha_1^{c_0,c_0'}}{\Longrightarrow} & F_1^{c_0,c_0'}\cdot(-\circ\alpha c_0') \\ {\scriptstyle G_1^{c_0,c_0'}\cdot(\Lambda c_0\circ -)}\Downarrow & \Lambda_1^{c_0,c_0'} \Rrightarrow & \Downarrow{\scriptstyle F_1^{c_0,c_0'}\cdot(-\circ\Lambda c_0)} \\ G_1^{c_0,c_0'}\cdot(\beta c_0\circ -) & \underset{\beta_1^{c_0,c_0'}}{\Longrightarrow} & F_1^{c_0,c_0'}\cdot(-\circ\beta c_0') \end{array}$$

These data must obey to *functoriality* axioms described by equations in (n-1)-Cat, see [13].

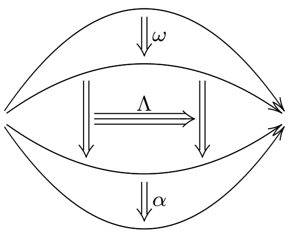
In the pointed case we ask moreover that $\Lambda_0(\star)$ is the identity 2-cell.

Once equipped with lax n-modifications, the sequi-categories n-Cat, n-Gpd and n-Gpd$_\star$ are sesqui^2-categories, see [13]. This essentially means that:

- there are 2-compositions of lax n-modifications as $\Lambda\cdot\Sigma$:

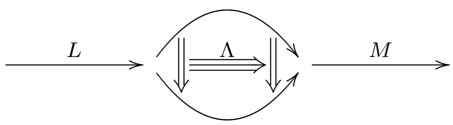


  and reduced 1-compositions of lax n-modifications as $\omega\cdot\Lambda$ and $\Lambda\cdot\alpha$:



  so that homs are sequi-categories;

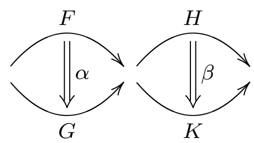
- there is reduced horizontal 0-composition of lax n-modifications as $L\cdot\Lambda$ and $\Lambda\cdot M$:



so that composing with an n-functor gives a sesqui-functor between homs;

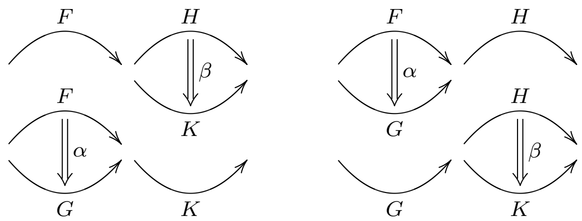
- there is horizontal 0-composition of lax n-transformations

$$\alpha * \beta \colon \alpha \setminus \beta \Longrightarrow \alpha/\beta \colon F \cdot H \Longrightarrow G \cdot K$$



where domain and codomain 2-morphisms are respectively

$$\alpha \setminus \beta := (F \cdot \beta) \cdot (\alpha \cdot K) \qquad (\alpha \cdot H) \cdot (G \cdot \beta) =: \alpha/\beta$$



(5)

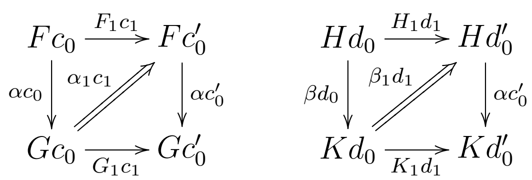
so that 0-composing with a 2-morphism gives a lax natural transformation of sesquifunctors.

The full definition of sesqui^2-category, with the remaining compatibility axioms, and the fact that n-Cat is a sesqui^2-category can be found in [12, 13], see also [2, 3] for the horizontal 0-composition of lax n-transformations. Here we just recall the inductive definition of $\alpha * \beta$:

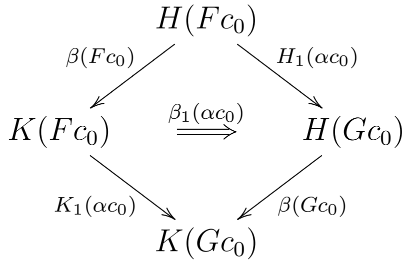
(a) for any $c_0 \in \mathbb{C}_0$, $[\alpha * \beta]_0(c_0) = \beta_1(\alpha_0(c_0))$

(b) for any $c_0, c_0' \in \mathbb{C}_0$, $[\alpha * \beta]_1^{c_0,c_0'} = \alpha_1^{c_0,c_0'} * \beta_1^{Fc_0,Gc_0'}$

6.2. Remark. For sake of clarity, let us write explicitly $\alpha * \beta$ for $n = 2$. Given $c_1 \colon c_0 \to c_0'$ in $\mathbb{C}$ and $d_1 \colon d_0 \to d_0'$ in $\mathbb{D}$, the lax 2-transformations α and β are specified by

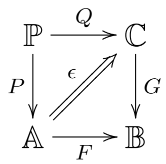


and the lax 2-modification $\alpha * \beta$ is given by

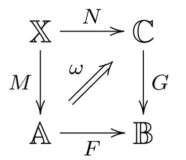


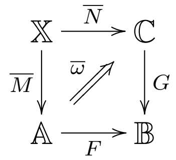
The following proposition holds in n-Cat as well as in n-Gpd and in n-Gpd$_\star$ (for h^2-pullbacks see also [6]).

6.3. PROPOSITION. *The h-pullback described in Section 2*

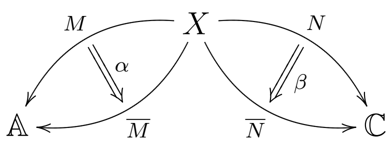


satisfies the following universal property: for every pair of four-tuples $(\mathbb{X}, M, N, \omega)$ *and* $(\mathbb{X}, \overline{M}, \overline{N}, \overline{\omega})$

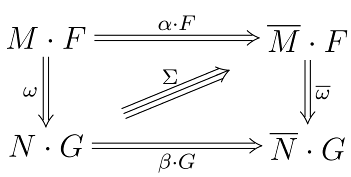




for every pair of lax n-transformations α *and* β



and for every lax n-modification Σ



there exists a unique lax n-transformation $\lambda\colon\ L \Rightarrow \overline{L}\colon\ \mathbb{X} \to \mathbb{P}$ such that

$$\lambda \cdot P = \alpha, \qquad \lambda \cdot Q = \beta, \qquad \lambda * \varepsilon = \Sigma.$$

We will recall this property as the universal property of the h^2-pullback.

Proof. The n-functors L and $\overline{L}$ are given by the universal property of the h-pullback applied respectively to $(\mathbb{X}, M, N, \omega)$ and $(\overline{\mathbb{X}}, \overline{M}, \overline{N}, \overline{\omega})$. As far as λ is concerned, one has:

- for any $x_0 \in \mathbb{X}_0$, $\lambda x_0 = (\alpha x_0, \Sigma x_0, \beta x_0)\colon Lx_0 \to \overline{L}x_0$
- for any $x_0, x_0' \in \mathbb{X}_0$, $\lambda_1^{x_0,x_0'}$ is given by the universal property of the h^2-pullback $\mathbb{P}_1(Lx_0, \overline{L}x_0')$ in $(n$-1)-Cat:

  $$\lambda_1^{x_0,x_0'}\colon \overline{L}_1^{x_0,x_0'} \cdot (\lambda_0 x_0 \circ -) \Rightarrow L_1^{x_0,x_0'} \cdot (- \circ \lambda_0 x_0')$$

  is the unique lax $(n$-1)-transformation such that

$$\lambda_1^{x_0,x_0'} \cdot Q_1^{Lx_0,\overline{L}x_0'} = \beta_1^{x_0,x_0'},\ \lambda_1^{x_0,x_0'} \cdot P_1^{Lx_0,\overline{L}x_0'} = \alpha_1^{x_0,x_0'},\ \lambda_1^{x_0,x_0'} * \epsilon_1^{Lx_0,\overline{L}x_0'} = \Sigma_1^{x_0,x_0'}$$

■

7. The sesqui-functors $\Pi_1^{(n)}$ and $\Omega^{(n)}$

We start with a first, easy description of the sesqui-functor

$$\Pi_1^{(n)}\colon n\text{-Gpd}_\star \to (n\text{-}1)\text{-Gpd}_\star$$

7.1. Definition. (The sesqui-functor $\Pi_1^{(n)}$)

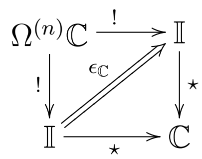
- Let $\mathbb{C}$ be a pointed n-groupoid. We define $\Pi_1^{(n)}\mathbb{C} = \mathbb{C}_1(\star, \star)$. It is a pointed $(n$-1)-groupoid with the identity 1-cell as base point.
- Let $F\colon \mathbb{C} \to \mathbb{D}$ be an n-functor in n-Gpd$_\star$. Since $F(\star) = \star$, we get an $(n$-1)-functor $\Pi_1^{(n)}F = F_1^{\star,\star}\colon \mathbb{C}_1(\star,\star) \to \mathbb{D}_1(\star,\star)$
- Let $\alpha\colon F \Rightarrow G\colon \mathbb{C} \to \mathbb{D}$ be a lax n-transformation in n-Gpd$_\star$. Since $\alpha_0(\star) = u(\star)$, we get a lax $(n$-1)-transformation $\Pi_1^{(n)}\alpha = \alpha_1^{\star,\star}\colon G_1^{\star,\star} \to F_1^{\star,\star}$.

It is easy to check that $\Pi_1^{(n)} \colon n\text{-Gpd}_\star \to (n\text{-}1)\text{-Gpd}_\star$ is a sesqui-functor contravariant on lax n-transformations.

Despite its simplicity, the previous definition is quite difficult to use in our setting, because it is not an inductive definition. For this reason, we look for a different decription of $\Pi_1^{(n)}$.

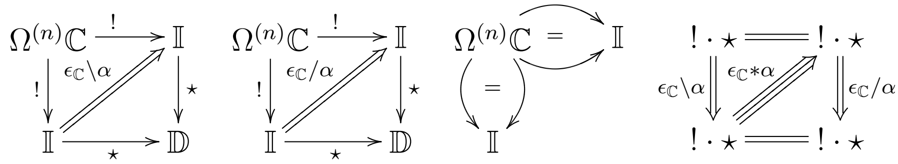
7.2. Definition. (The sesqui-functor $\Omega^{(n)}$)

- Let $\mathbb{C}$ be a pointed n-groupoid. We define $\Omega^{(n)}\mathbb{C}$ by the following h-pullback



- Let $F \colon \mathbb{C} \to \mathbb{D}$ be an n-functor in n-Gpd$_\star$. The universal property of the h-pullback $\Omega^{(n)}\mathbb{D}$ gives a unique n-functor $\Omega^{(n)}F \colon \Omega^{(n)}\mathbb{C} \to \Omega^{(n)}\mathbb{D}$ such that $\Omega^{(n)}F \cdot \epsilon_{\mathbb{D}} = \epsilon_{\mathbb{C}} \cdot F$.

- Let $\alpha \colon F \Rightarrow G \colon \mathbb{C} \to \mathbb{D}$ be a lax n-transformation in n-Gpd$_\star$. Consider the following situation



The universal property of the h^2-pullback $\Omega^{(n)}\mathbb{D}$ gives a unique lax n-transformation

$$\Omega^{(n)}\alpha \colon \Omega^{(n)}G \Rightarrow \Omega^{(n)}F \colon \Omega^{(n)}\mathbb{C} \to \Omega^{(n)}\mathbb{D}.$$

such that $\Omega^{(n)}\alpha * \epsilon_{\mathbb{D}} = \epsilon_{\mathbb{C}} * \alpha$.

It is easy to check that the previous data give a sesqui-functor

$$\Omega^{(n)} \colon n\text{-Gpd}_\star \to n\text{-Gpd}_\star$$

contravariant on lax n-transformations.

7.3. Proposition. *There are two strict natural isomorphisms of sesqui-functors*

$$\begin{array}{ccccc} & & (n\text{-}1)\text{-Gpd}_\star & & \\ & {\scriptstyle \Pi_1^{(n)}}\nearrow & \simeq & \searrow{\scriptstyle D^{(n)}} & \\ n\text{-Gpd}_\star & & \xrightarrow[\Omega^{(n)}]{} & & n\text{-Gpd}_\star \end{array}$$

$$\begin{array}{ccccc} n\text{-Gpd}_\star & & \xrightarrow{\Pi_1^{(n)}} & & (n\text{-}1)\text{-Gpd}_\star \\ & {\scriptstyle \Omega^{(n)}}\searrow & \simeq & \nearrow{\scriptstyle \Pi_0^{(n)}} & \\ & & n\text{-Gpd}_\star & & \end{array}$$

Proof. The second isomorphism follows from the first one composing with $\Pi_0^{(n)}$. As far as the first one is concerned, we recover a natural isomorphism of pointed n-groupoids $\theta_{\mathbb{C}}\colon D^{(n)}(\Pi_1^{(n)}\mathbb{C}) \to \Omega^{(n)}\mathbb{C}$ as a special case of that given in Proposition A.1. It suffices to let $\theta_{\mathbb{C}} = \mathfrak{S}_{\mathbb{C}}^{\star;\star}$. ■

We are going to prove the main result of this section: the sesqui-functors $\Omega^{(n)}$ and $\Pi_1^{(n)}$ preserve exact sequences. We need two preliminary lemmas.

7.4. Lemma. *The sesqui-functor $\Omega^{(n)}\colon n\text{-}Gpd_\star \to n\text{-}Gpd_\star$ preserves h-pullbacks.*

Proof. (Throughout the proof we will omit the superscripts (n).) Let us consider an h-pullback

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{Q} & \mathbb{C} \\ {\scriptstyle P}\downarrow & \overset{\phi}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle G} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

in n-Gpd$_\star$. By the universal property of h^2-pullbacks we get $\Omega(\phi)$ as in the diagram

$$\begin{array}{ccc} \Omega(\mathbb{Q}) & \xrightarrow{\Omega(P)} & \Omega(\mathbb{A}) \\ {\scriptstyle \Omega(Q)}\downarrow & \underset{\Omega(\phi)}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle \Omega(F)} \\ \Omega(\mathbb{C}) & \xrightarrow[\Omega(G)]{} & \Omega(\mathbb{B}) \end{array}$$

Further let us consider the diagram

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\;M\;} & \Omega(\mathbb{A}) \\ {\scriptstyle N}\downarrow & \overset{\omega}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle \Omega(F)} \\ \Omega(\mathbb{C}) & \xrightarrow[\Omega(G)]{} & \Omega(\mathbb{B}) \end{array}$$

that by Proposition 7.3 can be redrawn

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\;M\;} & D(\mathbb{A}_1(\alpha,\alpha)) \\ {\scriptstyle N}\downarrow & \overset{\omega}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle D(F_1^{\alpha,\alpha})} \\ D(\mathbb{C}_1(\star,\star)) & \xrightarrow[D(G_1^{\star,\star})]{} & D(\mathbb{B}_1(\star,\star)) \end{array}$$

Applying Π_0 one gets

$$\begin{array}{ccc} \Pi_0(\mathbb{X}) & \xrightarrow{\;\Pi_0(M)\;} & \mathbb{A}_1(\star,\star) \\ {\scriptstyle \Pi_0(N)}\downarrow & \overset{\Pi_0(\omega)}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle F_1^{\star,\star}} \\ \mathbb{C}_1(\star,\star) & \xrightarrow[G_1^{\star,\star}]{} & \mathbb{B}_1(\star,\star) \end{array}$$

Since $\mathbb{Q}_1(\star,\star)$ is defined as an h-pullback, its universal property yields a unique morphism $L\colon \Pi_0(\mathbb{X}) \to \mathbb{Q}_1(\star,\star)$ such that

$$L \cdot P_1^{\star,\star} = \Pi_0(M), \quad L \cdot Q_1^{\star,\star} = \Pi_0(N), \quad L \cdot \phi_1^{\star,\star} = \Pi_0(\omega)$$

Using Proposition 5.4 and Proposition 7.3, we obtain from L the required factorization $\mathbb{X} \to D(\mathbb{Q}_1(\star,\star)) \simeq \Omega(\mathbb{Q})$. ■

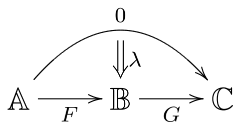
7.5. LEMMA. *The sesqui-functor $\Omega^{(n)}\colon n\text{-}Gpd_\star \to n\text{-}Gpd_\star$ preserves h-surjective morphisms.*

Proof. This is straightforward. Let $L\colon \mathbb{K} \to \mathbb{A}$ be an h-surjective morphism. Then

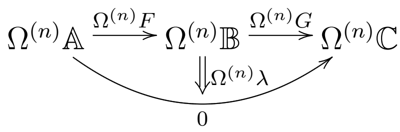
$$\Omega^{(n)}(L) = D^{(n)}(L_1^{\star,\star})$$

Now $L_1^{\star,\star}$ is h-surjective by definition since L is, and clearly $D^{(n)}$ preserves h-surjective morphisms. ■

7.6. Proposition. *Given an exact sequence in n-$Gpd_\star$*



the sequence



is exact in $(n$-$1)$-$Gpd_\star$.

Proof. Let $L\colon \mathbb{A} \to \mathbb{K}^*(G)$ be the h-surjective comparison with the h-kernel of G. By Lemma 7.4 above, $\Omega^{(n)}L$ is the comparison with the h-kernel of $\Omega^{(n)}G$, and it is h-surjective by Lemma 7.5. ∎

7.7. Corollary. *The sesqui-functor $\Pi_1^{(n)}\colon n$-$Gpd_\star \to (n$-$1)$-$Gpd_\star$ preserves exact sequences, reversing the direction of the 2-morphism.*

8. The fibration sequence of an n-functor

In this section we construct an exact sequence of the form

$$\Omega^{(n)}\mathbb{B} \xrightarrow{\Omega^{(n)}F} \Omega^{(n)}\mathbb{C} \xrightarrow{\nabla} \mathbb{K}^*(F) \xrightarrow{K^*(F)} \mathbb{B} \xrightarrow{F} \mathbb{C}$$

starting from a pointed n-functor F. We need some lemmas.

8.1. Lemma.

1. *Let $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{B} \to \mathbb{C}$ be morphisms of n-groupoids. If G is an equivalence and $F \cdot G$ is h-surjective, then F is h-surjective.*

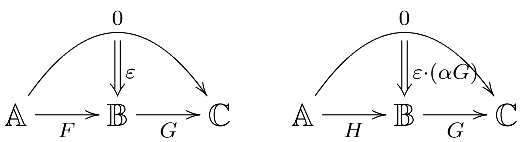
2. *Let $\alpha\colon F \Rightarrow G\colon \mathbb{B} \to \mathbb{C}$ be a 2-morphism of n-groupoids. F is h-surjective if, and only if, G is h-surjective.*

Proof. The proof of the first part, by induction, is strightforward. As far as the second part is concerned, observe that applying the inductive step we have that

$$\mathbb{C}_1(c_0, c_0') \xrightarrow{G_1^{c_0,c_0'}} \mathbb{D}_1(Gc_0, Gc_0') \xrightarrow{\alpha_0 c_0 \circ -} \mathbb{D}_1(Fc_0, Gc_0')$$

is h-surjective. Since $\alpha_0 c_0 \circ -$ is an equivalence, we conclude using the first part of the lemma. ∎

8.2. Lemma. *Let $\alpha\colon F \Rightarrow H\colon \mathbb{A} \to \mathbb{B}$ be a 2-morphism of pointed n-groupoids. Consider also the following diagrams*



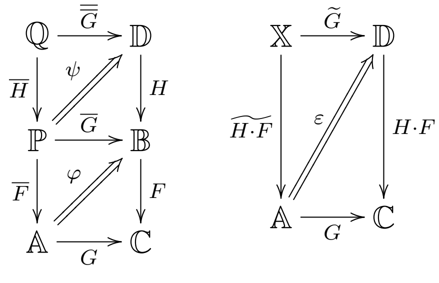
If (F, ε, G) is exact, then $(H, \varepsilon \cdot (\alpha G), G)$ is exact.

Proof. Let $F', H'\colon \mathbb{A} \to \mathbb{K}^*(G)$ be the canonical factorizations of F and H through the h-kernel. Using the universal property of the h^2-kernel $\mathbb{K}^*(G)$ we get a 2-morphism $\alpha'\colon F' \Rightarrow H'$ and we conclude using the second part of Lemma 8.1. ∎

8.3. Lemma. *Consider*

$$\mathbb{A} \xrightarrow{G} \mathbb{C} \xleftarrow{F} \mathbb{B} \xleftarrow{H} \mathbb{D}$$

*and the following h-pullbacks in n-*Gpd*:*



Consider also the canonical morphisms

$$\nabla\colon \mathbb{X} \to \mathbb{P} \;\; \textit{such that} \;\; \nabla \cdot \overline{F} = \widetilde{H \cdot F},\ \nabla \cdot \overline{G} = \widetilde{G} \cdot H,\ \nabla \cdot \varphi = \varepsilon$$

$$L\colon \mathbb{X} \to \mathbb{Q} \;\; \textit{such that} \;\; L \cdot \overline{H} = \nabla,\ L \cdot \overline{\overline{G}} = \widetilde{G},\ L \cdot \psi = id$$

Then $L\colon \mathbb{X} \to \mathbb{Q}$ is an equivalence.

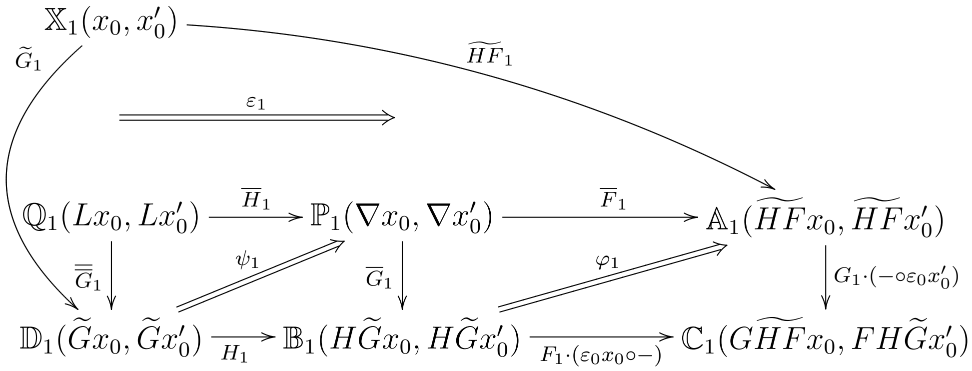
Proof. By induction on n.
1) For $n = 1$, the result immediately follows from the fact that in this case h-pullbacks are bilimits, so that they are defined up to equivalence.
2) For $n > 1$, let us check that L is essentially surjective. An object in $\mathbb{Q}$ has the form

$$q_0 = ((a_0, c_1 \colon Ga_0 \to Fb_0, b_0), b_1 \colon b_0 \to Hd_0, d_0)$$

whereas an object in $\mathbb{X}$ has the form

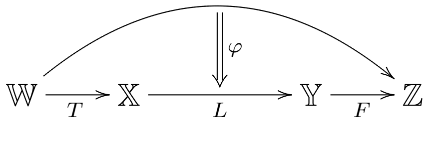
$$x_0 = (a_0, \gamma_1 \colon Ga_0 \to F(Hd_0), d_0)$$

with $L(x_0) = ((a_0, \gamma_1, Hd_0), =, d_0)$. Therefore, for a given object $q_0 \in \mathbb{Q}$, we put $x_0 = (a_0, c_1 \cdot F(b_1), d_0)$ and we have a 1-cell $q_0 \to L(x_0)$ with the identity as components in $\mathbb{A}$ and $\mathbb{D}$, and b_1 as component in $\mathbb{B}$.
Finally, to prove that $L_1^{x_0,x_0'} \colon \mathbb{X}_1(x_0, x_0') \to \mathbb{Q}_1(Lx_0, Lx_0')$ is an equivalence of (n-1)-groupoids we can apply the inductive hypothesis. Indeed, since $L \cdot \psi = id$ and $\nabla \cdot \varphi = \varepsilon$, $L_1^{x_0,x_0'}$ is constructed using h-pullbacks precisely as L starting from



∎

8.4. Lemma. *Consider the following diagram in n-Gpd$_\star$:*



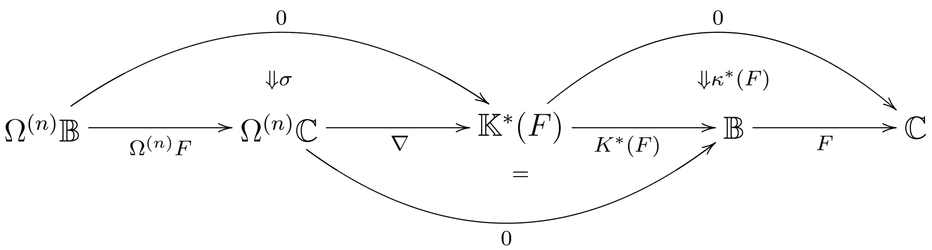
If $(T \cdot L, \varphi, F)$ is exact and L is an equivalence, then $(T, \varphi, L \cdot F)$ is exact.

Proof. Let $S\colon \mathbb{K}^*(L\cdot F)\to\mathbb{K}^*(F)$ be the factorization of $(K^*(L\cdot F)\cdot L,\kappa^*(L\cdot F))$ through the h-kernel of F. Clearly S is the composite of two equivalences (the first one coming from Lemma 8.3 applied to

$$\mathbb{I}\xrightarrow{\ \star\ }\mathbb{Z}\xleftarrow{\ F\ }\mathbb{Y}\xleftarrow{\ L\ }\mathbb{X}$$

and the second one obtained by pulling back L along $K^*(F)$), thus it is itself an equivalence. Consider now the factorization $T'\colon \mathbb{W}\to\mathbb{K}^*(L\cdot F)$ of (T,φ) through the h-kernel of $L\cdot F$, and the factorization $G\colon \mathbb{W}\to\mathbb{K}^*(F)$ of $(T\cdot L,\varphi)$ thorugh the h-kernel of F. By uniqueness in the universal property of the h-kernel, $T'\cdot S=G$. Since S is an equivalence and, by assumption, G is h-surjective, we conclude by Lemma 8.1 that T' is h-surjective. ∎

8.5. Proposition. *Let $F\colon\mathbb{B}\to\mathbb{C}$ be an n-functor in n-*Gpd$_\star$. *There is a sequence*



which is exact in $\Omega^{(n)}\mathbb{C}$, $\mathbb{K}^*(F)$, and $\mathbb{B}$.

Proof. 1) Exactness in $\mathbb{B}$: obvious.
2) Exactness in $\mathbb{K}^*(F)$: applying Lemma 8.3 to $\mathbb{I}\xrightarrow{\ \star\ }\mathbb{C}\xleftarrow{\ F\ }\mathbb{B}\xleftarrow{\ \star\ }\mathbb{I}$ we get an equivalence

$$L\colon \Omega^{(n)}\mathbb{C}\to\mathbb{K}_*(K^*F)\text{ such that } L\cdot K_*(K^*F)=\nabla,\ L\cdot\kappa_*(K^*F)=id,$$

and the exactness of $(\nabla,id,K^*(F))$.
3) Exactness in $\Omega^{(n)}\mathbb{C}$: applying Lemma 8.3 to

$$\mathbb{I}\xrightarrow{\ \star\ }\mathbb{K}^*(F)\xrightarrow{K^*(F)}\mathbb{B}\xleftarrow{\ \star\ }\mathbb{I}$$

we get an exact sequence

$$\xymatrix{\Omega^{(n)}\mathbb{B} \ar[r]_{\Theta} \ar@/^2pc/[rr]^{0}_{\Downarrow id} & \mathbb{K}_*(K^*F) \ar[r]_{K_*(K^*F)} & \mathbb{K}^*(F)}$$

where Θ is the unique morphism such that $\Theta \cdot K_*(K^*F) = 0$ and $\Theta \cdot \kappa_*(K^*F) = \epsilon_{\mathbb{B}}$. By the universal property of the h^2-kernel $\mathbb{K}^*(F)$ we get a 2-morphism $\sigma \colon 0 \Rightarrow \Omega^{(n)}F \cdot \nabla$ such that $\sigma \cdot K^*(F) = \epsilon_{\mathbb{B}}$ and $\sigma * \kappa^*(F) = id$. By the universal property of the h^2-kernel $\mathbb{K}_*(K^*F)$ we get a 2-morphism $\lambda \colon \Theta \Rightarrow \Omega^{(n)}F \cdot L$ such that $\lambda \cdot K_*(K^*F) = \sigma$ and $\lambda * \kappa_*(K^*F) = id$. Therefore, following Lemma 8.2, we get the sequence

$$\xymatrix{\Omega^{(n)}\mathbb{B} \ar[r]_{\Omega^{(n)}F} \ar@/^2pc/[rrr]^{0}_{\Downarrow \lambda\cdot K_*(K^*F)} & \Omega^{(n)}\mathbb{C} \ar[r]_{L} & \mathbb{K}_*(K^*F) \ar[r]_{K_*(K^*F)} & \mathbb{K}^*(F)}$$

exact in $\mathbb{K}_*(K^*F)$. Finally, since $L \cdot K_*(K^*F) = \nabla$ and $\lambda \cdot K_*(K^*F) = \sigma$, following Lemma 8.4 we get the exact sequence

$$\xymatrix{\Omega^{(n)}\mathbb{B} \ar[r]_{\Omega^{(n)}F} \ar@/^2pc/[rr]^{0}_{\Downarrow \sigma} & \Omega^{(n)}\mathbb{C} \ar[r]_{\nabla} & \mathbb{K}^*(F)}$$

■

Since the sesqui-functor $\Omega^{(n)}$ preserves exact sequences, if we apply it to the sequence of Proposition 8.5 we get another exact sequence

$$\Omega^{(n)}\Omega^{(n)}\mathbb{B} \longrightarrow \Omega^{(n)}\Omega^{(n)}\mathbb{C} \longrightarrow \Omega^{(n)}\mathbb{K}^*(F) \longrightarrow \Omega^{(n)}\mathbb{B} \longrightarrow \Omega^{(n)}\mathbb{C}$$

This sequence and the sequence of Proposition 8.5 can be pasted together. Therefore, iterating the process, we obtain a long exact sequence (which trivializes after n applications).

9. The Ziqqurath of a pointed n-functor

A different perspective is gained by considering the sesqui-functor Π_1 in place of Ω. In fact in the longer exact sequences obtained at the end of the previous section, repeated applications of Ω give structures which are discrete in higher dimensional cells. Their exactness can be investigated in lower dimensional settings, *i.e.* after repeated applications of Π_0. This is a consequence of the following easy to prove

9.1. LEMMA. *The sesqui-functor* Π_0 *commutes with the sesqui-functor* Π_1, i.e. *for every integer* $n > 1$ *the following diagram is commutative*

$$\begin{array}{ccc} n\text{-Gpd}_\star & \xrightarrow{\Pi_0^{(n)}} & (n\text{-}1)\text{-Gpd}_\star \\ {\scriptstyle \Pi_1^{(n)}}\downarrow & & \downarrow{\scriptstyle \Pi_1^{(n-1)}} \\ (n\text{-}1)\text{-Gpd}_\star & \xrightarrow[\Pi_0^{(n-1)}]{} & (n\text{-}2)\text{-Gpd}_\star \end{array}$$

9.2. REMARK. In the language of loops, we can restate the above lemma in other terms:

$$\Pi_0(\Pi_0(\Omega(-))) = \Pi_0(\Omega(\Pi_0(-)))$$

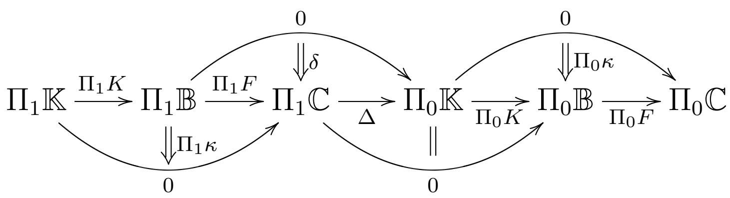
Let now a morphism $F : \mathbb{C} \to \mathbb{D}$ of pointed n-groupoids be given. Then the h-kernel exact sequence

$$\mathbb{K} \xrightarrow[K]{} \mathbb{B} \xrightarrow[F]{} \mathbb{C} \qquad \left(0 : \mathbb{K} \to \mathbb{C}, \quad \kappa : 0 \Rightarrow F \cdot K\right)$$

gives two exact sequences of pointed $(n$-$1)$-groupoids:

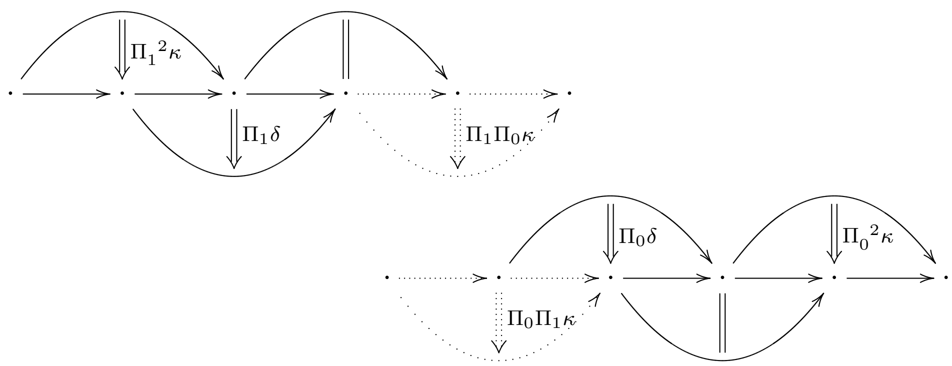
$$\Pi_1\mathbb{K} \xrightarrow{\Pi_1 K} \Pi_1\mathbb{B} \xrightarrow{\Pi_1 F} \Pi_1\mathbb{C} \quad \left(\Pi_1\kappa : 0 \Rightarrow \Pi_1 F \cdot \Pi_1 K\right) \qquad \Pi_0\mathbb{K} \xrightarrow[\Pi_0 K]{} \Pi_0\mathbb{B} \xrightarrow[\Pi_0 F]{} \Pi_0\mathbb{C} \quad \left(\Pi_0\kappa : 0 \Rightarrow \Pi_0 F \cdot \Pi_0 K\right)$$

These can be connected together in order to give a six term exact sequence of pointed (n-1)-groupoids

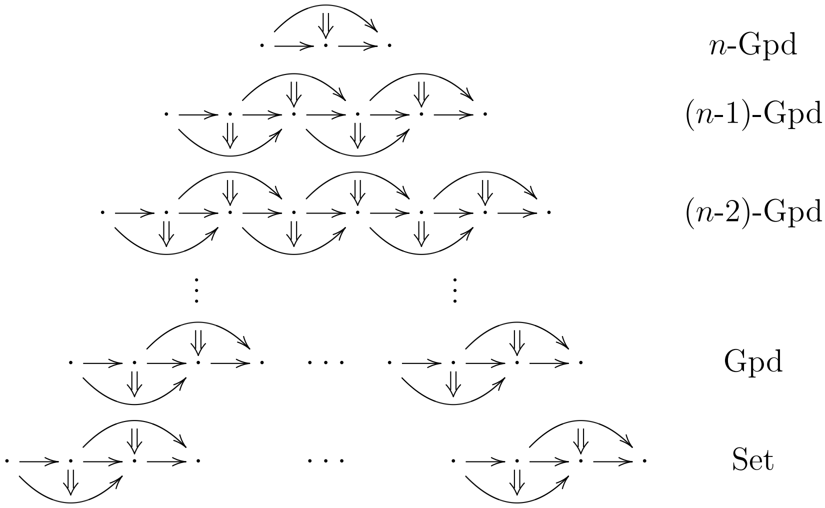


where $\Delta = \Pi_0(\nabla)$ and $\delta = \Pi_0(\sigma)$ (see 8.5 for ∇ and σ).

Applying Π_0 and Π_1, we get two six-term exact sequences. Using the previous lemma, these can be pasted in a nine-term exact sequence of (n-2)-groupoids (cells to be pasted are dotted in the diagram):



Iterating the process we obtain a sort of tower, a Ziqqurath, in which the lower is the level, the lower is the dimension and the longer is the length of the sequence.



In particular, the last row counts $3(n{+}1)$ terms. From left to right, there are $3(n\text{-}1)$ abelian groups, 3 groups and 3 pointed sets.

A. Paths sesqui-functor in n-Cat

This section is quite technical. Its aim is to give some explicit constructions that specialize in order to have a good description of the sesqui-functor $\Pi_1^{(n)}$. An observation which can help throughout this section is the analogy between the hom-(n-1)-groupoid of an n-groupoid $\mathbb{C}$ and the paths of a topological space. Given an n-groupoid (n-category) $\mathbb{C}$ and two objects c_0, c_0', we define $\mathbb{P}_{c_0,c_0'}(\mathbb{C})$ by means of the following h-pullback:

$$\begin{array}{ccc} \mathbb{P}_{c_0,c_0'}(\mathbb{C}) & \xrightarrow{\;!\;} & \underset{(n)}{\mathbb{I}} \\ {\scriptstyle !}\downarrow & \nearrow\!\!\!\!\nearrow \; \varepsilon_{\mathbb{C}}^{c_0,c_0'} & \downarrow{\scriptstyle [c_0']} \\ \underset{(n)}{\mathbb{I}} & \xrightarrow[{[c_0]}]{} & \mathbb{C} \end{array} \tag{6}$$

$\mathbb{P}_{c_0,c_0'}$ easily extends to morphisms. In fact for $F : \mathbb{C} \to \mathbb{D}$ one defines

$$\mathbb{P}_{c_0,c_0'}(F) : \; \mathbb{P}_{c_0,c_0'}(\mathbb{C}) \to \mathbb{P}_{c_0,c_0'}(\mathbb{D})$$

by means of the universal property of h-pullbacks yielding $\mathbb{P}_{c_0,c_0'}(\mathbb{D})$, for the four-tuple $\langle \mathbb{P}_{c_0,c_0'}(\mathbb{C}), !, !, \varepsilon_{\mathbb{C}}^{c_0,c_0'} \cdot F \rangle$. It is easy to see that this makes

$\mathbb{P}_{c_0,c_0'}(-)$ functorial. Unfortunately this does not extend straightforward to 2-morphisms. In fact for a pair of parallel morphisms $F, G : \mathbb{C} \to \mathbb{D}$, $\mathbb{P}_{c_0,c_0'}(F)$ and $\mathbb{P}_{c_0,c_0'}(G)$ are no longer parallel.
Indeed in applying the same argument as for defining $\mathbb{P}_{c_0,c_0'}(-)$ on morphisms, the corresponding diagram suggests to consider the 0-composition of 2-morphisms

$$\varepsilon_{\mathbb{C}}^{c_0,c_0'} * \alpha\colon\ \varepsilon_{\mathbb{C}}^{c_0,c_0'} \setminus \alpha \Longrightarrow \varepsilon_{\mathbb{C}}^{c_0,c_0'}/\alpha\,.$$

Hence we can consider the four-tuples

$$\langle \mathbb{P}_{c_0,c_0'}(\mathbb{C}), !, !, \varepsilon_{\mathbb{C}}^{c_0,c_0'} \setminus \alpha \rangle \quad \text{and} \quad \langle \mathbb{P}_{c_0,c_0'}(\mathbb{C}), !, !, \varepsilon_{\mathbb{C}}^{c_0,c_0'}/\alpha \rangle$$

together with $id_! : ! \Rightarrow !$ (taken two times) and the 3-morphism $\varepsilon_{\mathbb{C}}^{c_0,c_0'} * \alpha$. Applying the universal property of h^2-pullbacks we get a 2-morphism

$$\mathbb{P}_{c_0,c_0'}(\alpha) :\ \mathbb{P}_{[\alpha_{c_0}]} \circ \mathbb{P}_{c_0,c_0'}(G)) \Rightarrow \mathbb{P}_{c_0,c_0'}(F) \circ \mathbb{P}_{[\alpha_{c_0'}]} :\ \mathbb{P}_{c_0,c_0'}(C) \to \mathbb{P}_{Fc_0,Gc_0'}(\mathbb{D})$$

such that $\mathbb{P}_{c_0,c_0'}(\alpha) * \varepsilon_{\mathbb{D}}^{Fc_0,Gc_0'} = \varepsilon_{\mathbb{C}}^{c_0,c_0'} * \alpha$.
We have denoted by $\mathbb{P}_{[\alpha_{c_0}]} \circ \mathbb{P}_{c_0,c_0'}(G)$ and $\mathbb{P}_{c_0,c_0'}(F) \circ \mathbb{P}_{[\alpha_{c_0'}]}$ the morphisms obtained by applying the 1-dimensional universal property to $\varepsilon_{\mathbb{C}}^{c_0,c_0'} \setminus \alpha$ and $\varepsilon_{\mathbb{C}}^{c_0,c_0'}/\alpha$ respectively. Therefore the symbol $\circ$ involved should be considered just as a typographical suggestion. (Indeed it can be shown that it is a 0-composition of morphisms, but this would lead us far from the point.)

A.1. Proposition. *For every n-category $\mathbb{C}$, and every two objects c_0, c_0' in $\mathbb{C}$, there exists a canonical isomorphism*

$$\mathfrak{S}_{\mathbb{C}}^{c_0,c_0'} : D(\mathbb{C}_1(c_0, c_0')) \to \mathbb{P}_{c_0,c_0'}(\mathbb{C})$$

In the case of pointed n-groupoids, this gives a natural isomorphism with components

$$\mathfrak{S}_{\mathbb{C}}^{*,*} : D(\Pi_1(\mathbb{C})) \to \Omega(\mathbb{C})$$

where $\Omega(\mathbb{C}) = \mathbb{P}_{*,*}(\mathbb{C})$

We start by making explicit the h-pullback, but first we need to be more precise on units.

A.2. Remark. Let $\mathbb{C}$ be an n-category. For a fixed object c_0 of $\mathbb{C}$, let us consider the unit $(n\text{-}1)$-functor ${}^{\mathbb{C}}u^0(c_0) : \underset{(n-1)}{\mathbb{I}} \to \mathbb{C}_1(c_0, c_0)$, that is a pair

$$\begin{array}{lcl} [{}^{\mathbb{C}}u^0(c_0)]_0 & : & * \mapsto id(c_0) \quad \in [\mathbb{C}_1(c_0, c_0')]_0 \\ [{}^{\mathbb{C}}u^0(c_0)]_1 & : & \underset{(n-2)}{\mathbb{I}} \mapsto [\mathbb{C}_1(c_0, c_0')]_1(id(c_0), id(c_0)) \end{array}$$

Now, by functoriality we get the interchange

$$[{}^{\mathbb{C}}u^0(c_0)]_1 = {}^{\mathbb{C}}u^1(id(c_0)) = {}^{\mathbb{C}_1(c_0,c_0)}u^0(id(c_0))$$

and this allows the following explicit definition:

$${}^{\mathbb{C}}u^0(c_0) = \langle u^{(1)}(c_0), u^{(2)}(c_0), \cdots, u^{(n)}(c_0)\rangle$$

where $u^{(k)}(c_0)$ is the identity k-cell over c_0.

In the rest of this section the n-category $\mathbb{P}_{c_0,c_0'}(\mathbb{C})$ will be denoted by $\mathbb{Q}$.

A.3. Proposition. *Given the h-pullback of n-categories*

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{\ !\ } & \underset{(n)}{\mathbb{I}} \\ {\scriptstyle !}\downarrow & \nearrow\!\!\!\nearrow \varepsilon & \downarrow{\scriptstyle [c_0']} \\ \underset{(n)}{\mathbb{I}} & \xrightarrow[{[c_0]}]{} & \mathbb{C} \end{array} \qquad (7)$$

and $c_k, c_k' \colon c_{k-1} \to c_{k-1}'$, the hom-(n-k)-category

$$\mathbb{Q}_k\left(\left(u^{(k-1)}(*), c_k, u^{(k-1)}(*)\right), \left(u^{(k-1)}(*), c_k', u^{(k-1)}(*)\right)\right)$$

is well-defined and it is given by h-pullback over the pair $\langle [c_k], [c_k']\rangle$.

The proof by induction can be found in [13], and it yields immediately:

A.4. Corollary. *The 2-morphism ε is given explicitly by*

$$\varepsilon = \langle \varepsilon_0, [\varepsilon_1^{-,-}]_0, \ldots, [\varepsilon_{n-1}^{-,-}]_0, =\rangle$$

where

$$\begin{array}{rcl} [\varepsilon_k^{(*,c_{k-1},*),(*,c_{k-1}',*)}]_0 : \mathbb{Q}_k((*, c_{k-1}, *), (*, c_{k-1}', *)) & \to & \mathbb{C}_k(c_{k-1}, c_{k-1}') \\ (*,\ c_{k-1} \xrightarrow{c_k} c_{k-1}',\ *) & \mapsto & c_k. \end{array}$$

A.5. Corollary. *With notation as above,*

$$D(\Pi_0(\mathbb{Q})) = \mathbb{Q}.$$

In order to describe all compositions of the n-category $\mathbb{Q}$, it suffices to study the 0-composition in $\mathbb{Q} = \mathbb{P}_{c_0,c'_0}(\mathbb{C})$, because k-compositions (for $k > 0$) are implicit in the inductive definition. *Ditto* for units. This is reported in the lemma below, which together with the following one establishes a link between the globular and the inductive point of view. The interested reader can find the proofs in [13].

A.6. Lemma. *Let $c_1, c'_1, c''_1 : c_0 \to c'_0$ be fixed in $\mathbb{C}$. Given*

$$c_k : c_1 \Longrightarrow c'_1\,, \qquad c'_k : c_1 \Longrightarrow c'_1$$

with $1 < k \leq n$, the following equations hold:

$$(*, c_k, *) \,{}^{\mathbb{Q}}\!\circ^0\, (*, c_k, *) = (*, c_k \,{}^{\mathbb{C}}\!\circ^1\, c'_k, *);$$

$$\left[{}^{\mathbb{Q}}u^0((*, c_1, *))\right]_k = \left(*, \left[{}^{\mathbb{C}}u^1(c_1)\right], *\right).$$

A.7. Lemma. *Let c_0, c'_0 be objects of an n-category $\mathbb{C}$. The assignment*

$$\mathfrak{S}^{c_0,c'_0}_{\mathbb{C}} = \mathfrak{S} : D(\mathbb{C}_1(c_0, c'_0)) \to \mathbb{P}_{c_0,c'_0}(\mathbb{C}) = \mathbb{Q}$$

given explicitly by $\mathfrak{S} = \langle \mathfrak{S}_0, \mathfrak{S}_1, \ldots, \mathfrak{S}_n \rangle$ with $\mathfrak{S}_{i-1}(c_i) = (*, c_i, *)$ for $i = 1, 2, \ldots, n$, and $\mathfrak{S}_n = \mathfrak{S}_{n-1}$, is an isomorphism of discrete n-categories.

Now that we have developed the machinery, we are able to prove the main result of the section.

Proof of Proposition A.1. From the previous lemmas we have the existence of the canonical isomorphism of n-categories $\mathfrak{S}^{c_0,c'_0}_{\mathbb{C}}$ for any pair of objects c_0, c'_0. Further, for an n-functor $F\colon \mathbb{C} \to \mathbb{D}$ we get a (c_0, c'_0)-indexed family of commutative squares:

$$\begin{array}{ccc}
D(\mathbb{C}_1(c_0, c'_0)) & \xrightarrow{D(F_1^{c_0,c'_0})} & D(\mathbb{D}_1(Fc_0, Fc'_0)) \\
{\scriptstyle \mathfrak{S}^{c_0,c'_0}_{\mathbb{C}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{S}^{Fc_0,Fc'_0}_{\mathbb{D}}} \\
\mathbb{P}_{c_0,c'_0}(\mathbb{C}) & \xrightarrow[\mathbb{P}_{c_0,c'_0}(F)]{} & \mathbb{P}_{Fc_0,Fc'_0}(\mathbb{D})
\end{array} \tag{8}$$

We prove this by induction.
For $n = 1$ it is just a diagram of discrete categories. It suffices to verify commutativity on objects. To this end, let us choose a $c_1 : c_0 \to c_0'$. Then $\mathfrak{S}_{\mathbb{D}}(DF(c_1)) = \mathfrak{S}_{\mathbb{D}}(Fc_1) = (*, Fc_1, *)$ and $\mathbb{P}F(\mathfrak{S}_{\mathbb{C}}(c_1)) = \mathbb{P}F(*, c_1, *) = (*, Fc_1, *)$.
For $n > 1$, first we have to show that diagram (8) commutes on objects, but this amounts exactly to what we have just shown for $n = 1$. Thus for $c_1, c_1' : c_0 \to c_0'$ we consider homs:

$$\begin{array}{ccc}
[D(\mathbb{C}_1(c_0, c_0'))]_1(c_1, c_1') & \xrightarrow{[D(F_1^{c_0,c_0'})]_1^{c_1,c_1'}} & [D(\mathbb{D}_1(Fc_0, Fc_0'))]_1(Fc_1, Fc_1') \\
\Big\downarrow {\scriptstyle [\mathfrak{S}_{\mathbb{C}}^{c_0,c_0'}]_1^{c_1,c_1'}} & & \Big\downarrow {\scriptstyle [\mathfrak{S}_{\mathbb{D}}^{Fc_0,Fc_0'}]_1^{Fc_1,Fc_1'}} \\
[\mathbb{P}_{c_0,c_0'}(\mathbb{C})]_1((*, c_1, *), (*, c_1', *)) & \xrightarrow{[\mathbb{P}_{c_0,c_0'}(F)]_1^{\diamond,\diamond}} & [\mathbb{P}_{Fc_0,Fc_0'}(\mathbb{D})]_1((*, Fc_1, *), (*, Fc_1', *))
\end{array}$$

The definition of D and the previous discussion give

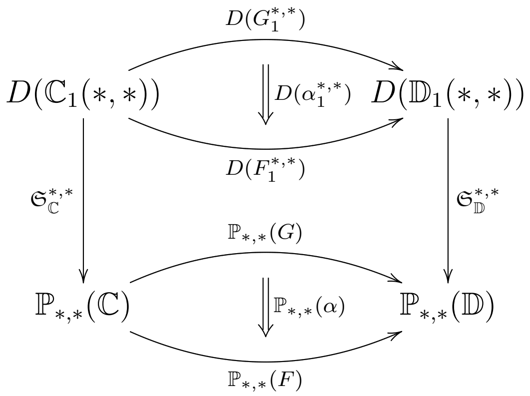
$$\begin{array}{ccc}
D\big([\mathbb{C}_1(c_0, c_0')]_1(c_1, c_1')\big) & \xrightarrow{D\left([F_1^{c_0,c_0'}]_1^{c_1,c_1'}\right)} & D\big([\mathbb{D}_1(c_0, c_0')]_1(Fc_1, Fc_1')\big) \\
{\scriptstyle T_{\mathbb{C}}^{c_1,c_1'}} \Big\downarrow & & \Big\downarrow {\scriptstyle T_{\mathbb{D}}^{Fc_1,Fc_1'}} \\
\mathbb{P}_{c_1,c_1'}\big(\mathbb{C}_1(c_0, c_0')\big) & \xrightarrow[\mathbb{P}_{c_1,c_1'}(F_1^{c_0,c_0'})]{} & \mathbb{P}_{Fc_1,Fc_1'}\big(\mathbb{D}_1(Fc_0, Fc_0')\big)
\end{array}$$

Now, as the T's are just the $\mathfrak{S}$'s given for n-1, *i.e.*

$$T_{\mathbb{C}}^{c_1,c_1'} = \mathfrak{S}_{\mathbb{C}_1(c_0,c_0')}^{c_1,c_1'}, \quad T_{\mathbb{D}}^{Fc_1,Fc_1'} = \mathfrak{S}_{\mathbb{D}_1(Fc_0,Fc_0')}^{Fc_1,Fc_1'},$$

the last diagram commutes by induction hypothesis.
All this obviously restricts to n-groupoids. Moreover, in the pointed case we obtain a 2-contravariant natural isomorphism of sesqui-functors $\mathfrak{S}\colon \Pi_1 D \Rightarrow \Omega : n\text{-Gpd}_* \to n\text{-Gpd}_*$, *i.e.* a strict natural transformation of sesqui-functors that reverses the direction of 2-morphisms and in

which the assignments on objects are isomorphisms. In fact in n-Gpd$_*$ for a 2-morphism $\alpha : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$, we can express the (strict) naturality condition



$$D(\alpha_1^{*,*}) \cdot \mathfrak{S}_{\mathbb{D}}^{*,*} = \mathfrak{S}_{\mathbb{C}}^{*,*} \cdot \mathbb{P}_{*,*}(\alpha)$$

The proof that this last condition indeed holds is a consequence of a more general (non-pointed) lemma which can be found in [13]. ■

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Dipartimento di Matematica
Università di Milano
Via Saldini 50
I 20133 Milano, Italia

Institut de Recherche en Mathématique et Physique
Université catholique de Louvain
Chemin du Cyclotron 2
B 1348 Louvain-la-Neuve, Belgique
Email: Stefano.Kasangian@unimi.it, giuseppe.metere@unimi.it,
enrico.vitale@uclouvain.be

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES *Vol. LII-1 (2011)*

SINGULARITIES AND REGULAR PATHS (AN ELEMENTARY INTRODUCTION TO SMOOTH HOMOTOPY)

by Marco GRANDIS,

Résumé. Cet article est une introduction élémentaire à la topologie algébrique lisse, suivant une approche particulière: notre but est d'étudier des "espaces lisses avec singularités', par des méthodes d'homotopie adaptées à cette tâche. On explore ici des régions euclidiennes, moyennant des chemins de classe C^k, *en tenant compte du nombre de leurs arrêts* en fonction de k. Le groupoïde fondamental de l'espace acquiert ainsi une séquence de poids qui dépend d'un index de classe C^k et qui peut distinguer l'ordre des singularités "linéaires". On peut envisager d'appliquer ces méthodes à la théorie des réseaux.

Abstract. This article is a basic introduction to a particular approach within smooth algebraic topology: our aim is to study 'smooth spaces with singularities', by methods of homotopy theory adapted to this task. Here we explore euclidean regions by paths of (variable) class C^k, *counting their stops*. The fundamental groupoid of the space acquires thus a sequence of integral C^k-weights that depend on a smoothness index; a sequence that can distinguish 'linear' singularities and their order. These methods can be applied to the theory of networks.

Mathematics Subject Classifications (2000): 58A40, 58KXX, 58A20, 55Q05.

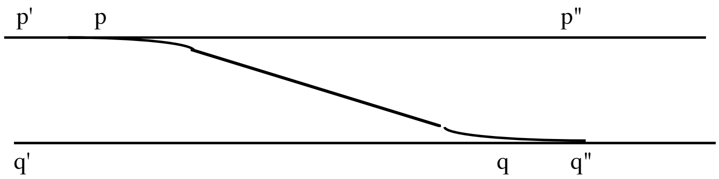
Key words: differential space, jet, singularity, homotopy groups.

Introduction

We want to explore a 'smooth space with singularities', in order to distinguish its singularities and their order. Now, it is obvious and well-known that smooth paths can go through singularities, by braking at the crossing. Therefore, it will be important to count these stops, or - alternatively - to consider smooth paths that never stop.

As a concrete situation which can be analysed in this way, consider a piece of railway network as represented below, with two main tracks linked by switches at p and q

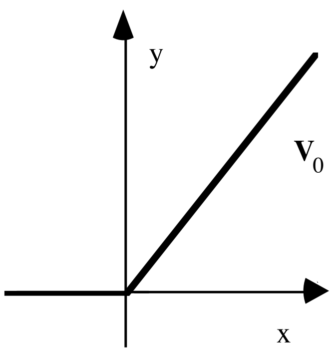
(1)

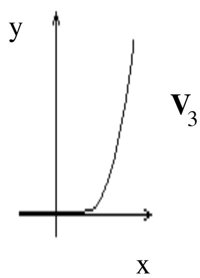


Plainly, a train can move from p' to q" (or vice versa) without stopping, but cannot do the same from q' to p" (or vice versa). Moreover, a route through p' and q" should require a slowing down (with respect to a straight route), which can be expressed by letting the diverging junctions be (only) of class C^1. This example will be further analysed in 2.3(c).

Let us consider here a more basic space of this kind, the *standard deviation* $\mathbf{V}_k$, *of class* C^k $(0 \leq k < \infty)$, in the euclidean plane

(2) $\mathbf{V}_k = \{(x, y) \in \mathbf{R}^2 \mid (x \leq 0,\ y = 0) \ \text{ or } \ (x \geq 0,\ y = x^{k+1})\},$





Take a curve $c: \mathbf{R} \to \mathbf{V}_k$ that crosses the singularity (0, 0)

(3) $c(t) = (x(t), 0) \ \ \textit{or} \ \ (x(t), x^{k+1}(t)), \qquad \textit{for} \ \ t \leq 0 \ \ \textit{or} \ \ t \geq 0,$

where $x: \mathbf{R} \to \mathbf{R}$ is a strictly increasing C^∞-function that annihilates at 0. It is easy to see that $c: \mathbf{R} \to \mathbf{R}^2$ is always of class C^k; moreover the right (k+1)-derivative of a at 0 is the vector

$$(D^{k+1}_{+}c)(0) = (x^{(k+1)}(0), (k+1)!\,(x'(0))^{k+1}) \in \mathbf{R}^2,$$

so that c is of class C^{k+1} if and only if $x'(0) = 0$, and then $c'(0) = 0$. In other words, the class C^k of the singularity of $\mathbf{V}_k$ can be determined as the highest class of paths that go through the singularity without stopping.

In Section 1 we begin by considering a *euclidean space* X as a topological subspace of some standard euclidean space $\mathbf{R}^m$; we define a *C^k-path* $a: \mathbf{I} \to X$, in the usual way (1.2). For $k > 0$, a is said to be *C^k-regular* if it is constant or $a'(t) \in \mathbf{R}^m$ never vanishes on $\mathbf{I}$. More generally, for a continuous path a, we introduce a 'penalty' for each stop or breaking of C^k-smoothness, counted by an (extended) integral *C^k-weight* $w^k(a) \in \mathbf{N} \cup \{\infty\}$, so that a path a is C^k-regular if and only if $w^k(a) \leq 1$; the precise definition of the weight can be found in 1.3. This also defines a C^k-weight $w^k: \Pi_1(X) \to \mathbf{N} \cup \{\infty\}$ on the fundamental groupoid of the euclidean space X (1.6).

In the next section we analyse X at the basic level of the *existence* of paths, by a sequence of *tolerance relations* (reflexive and symmetric) on the set X itself, indexed by an *extended* natural number $k = 0, 1, \ldots, \infty$

(4) $x \,!_k\, y \quad \Leftrightarrow \quad$ (x and y are connected by a C^k-regular path in X).

For $k = 0$, this is the equivalence relation of path-connectedness and gives the classical quotient set $\Pi_0(X) = |X|/!_0$. More generally, we have a structure $R^k\Pi_0(X) = (X, !_k)$, consisting of a set equipped with a tolerance relation, which is better analysed in a *reduced form* $\mathrm{red}(R^k\Pi_0(X))$ (2.2).

For instance, in $\mathbf{V}_k$, the origin (0, 0) is in relation $!_\infty$ with any other point, but the points (– 1, 0) and (1, 1) are only in relation $!_h$ for $h \leq k$. For every $h > k$, the set $\mathrm{red}(R^h\Pi_0(X))$ consists of three equivalence classes: $\{(0, 0)\}$, the left open arm [(1, 0)] and the right open arm [(1, 0)]; the first is in relation with the other two, that are unrelated.

In Section 3 we consider the initial and terminal *k-jets* of a C^k-path in a euclidean space X, and define the *effective* k-jets at a point. Then, in Section 4, we extend the fundamental groupoid $\Pi_1(X)$ of the space (and its fundamental groups), introducing the *fundamental C^k-regular semicategory* $R^k\Pi_1(X)$ of X (see 4.4): its vertices are the 'regular' k-jets of X, its arrows are classes $[a]: j \to j'$ of C^k-regular paths; the homotopy relation used to define an arrow works at fixed initial and terminal k-jets. (A *semicategory* is the obvious generalisation of a category, without assumption of identities.)

In Section 5, we study the fundamental C^1-regular semicategory $R^1\Pi_1(X)$, and compare it with the fundamental groupoid $\Pi_1(T^*X)$ of the space of *non-zero* *tangent vectors*, isomorphic to the fundamental groupoid $\Pi_1(UTX)$ of the space of *unit tangent vectors* of X. We prove that this comparison is an isomorphism when X is a C^1-embedded manifold of dimension ≥ 2 (Theorem 5.4). Thus, the fundamental monoid $R^1\pi_1(\mathbf{R}^2, j)$ is isomorphic to the group of integers, by the winding number of a regular path, and expresses the possible 'shapes' of a planar loop realised with a smooth elastic wire. We also give examples where the comparison is not full (cf. 5.5).

We end with a more detailed study of tolerance sets, in Section 6.

This subject can be developed, working with 'convenient smooth structures', like C^∞-rings, Frölicher spaces, Chen spaces, diffeological spaces or other objects of synthetic differential geometry (cf. [MR, Fr, Ch, So, BH, Ko]). Smooth and directed algebraic topology [G3] can also be combined, to study 'directed smooth spaces'.

With respect to the existing literature about 'smooth paths' and 'smooth homotopy groups', our goals and results are completely different from those of Cherenack [Ck], or Caetano and Picken [CP], or Schreiber and Waldorf [SW1, SW2], where paths are allowed to stop or even required to be locally constant at the end-points. Such approaches have a smooth concatenation based on points (instead of jets), but do not distinguish what we want to explore. A recent paper by Sati, Schreiber and Stasheff [SSS] has applications of smooth cohomology to theoretical physics.

Jets of differentiable functions were introduced by C. Ehresmann, as equivalence classes of functions [E1, E2]. The notion of a tolerance set was introduced by E.C. Zeeman [Ze], in connection with mathematical models of the brain; the original name is 'tolerance space'. The interest of semicategories in category theory is recent: see [MBB].

1. Euclidean spaces and regular maps

We consider 'euclidean spaces' and maps of class C^k between them, where $k \leq \infty$ is an extended natural number, i.e. $k \in \bar{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$. The standard interval is $\mathbf{I} = [0, 1]$. The usual concatenation of consecutive paths is written as $a * b$.

1.1. Euclidean spaces. A *euclidean space* will be just a topological subspace X of some standard euclidean space $\mathbf{R}^m$. (The standard euclidean spaces are viewed as naturally embedded, identifying $\mathbf{R}^m$ with $\mathbf{R}^m \times \{0\} \subset \mathbf{R}^{m+1}$; their union is a vector space of countable dimension, that can be equipped with the finest topology making the inclusions of all $\mathbf{R}^m$ continuous.)

Let us fix some examples, for future use.

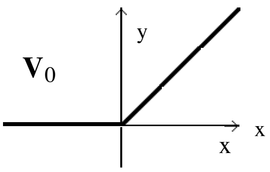
(a) Any real C^k-manifold M can be C^k-embedded in a suitable $\mathbf{R}^m$, and any subspace of M can be viewed as a subspace of $\mathbf{R}^m$.

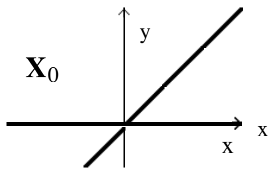
(b) The *standard C^k-deviation* $\mathbf{V}_k$, the *standard C^k-crossing* $\mathbf{X}_k$ and the *standard C^k-switch* $\mathbf{Y}_k$ will be the following subspaces of the real plane (for $k \in \mathbf{N}$):

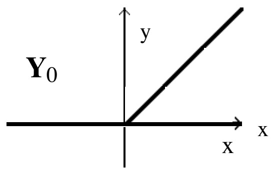
(1) $\mathbf{V}_k = \{(x, y) \in \mathbf{R}^2 \mid (x \leq 0,\ y = 0) \text{ or } (x \geq 0,\ y = x^{k+1})\}$,

(2) $\mathbf{X}_k = \{(x, y) \in \mathbf{R}^2 \mid y = 0 \text{ or } y = x^{k+1}\}$,

(3) $\mathbf{Y}_k = \{(x, y) \in \mathbf{R}^2 \mid y = 0 \text{ or } (x \geq 0,\ y = x^{k+1})\}$,



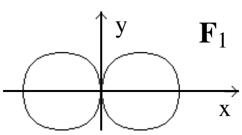




(c) We choose a lemniscate $\mathbf{E}_0$ as the *standard figure-eight curve of class C^0*. More generally, we denote by $\mathbf{E}_k$ a *standard figure-eight curve of class C^k*; it can be constructed in $\mathbf{R}^2$ from the *bounded C^k-crossing* $\mathbf{X}_k \cap \mathbf{B}^2$, linking smoothly its left arms together and its right ones as well. ($\mathbf{B}^2$ denotes the standard compact disc of $\mathbf{R}^2$.)

(d) We write $\mathbf{F}_k$ the *standard spectacles of class C^k*, that can be described as two smooth simple loops meeting at a point, with a contact of order k (precisely, i.e. not higher). Actually, the name of 'spectacles' is only appropriate for k *odd*, when there is a simple model in the plane, given by the union of two algebraic closed curves

(4) $\mathbf{F}_k = \{(x, y) \in \mathbf{R}^2 \mid (x \pm 1)^{k+1} + y^{k+1} = 1\} \subset \mathbf{R}^2$,



For k *even*, $\mathbf{F}_k$ can be constructed in $\mathbf{R}^3$, starting from the bounded C^k-crossing $\mathbf{X}_k \cap \mathbf{B}^2$ and smoothly linking together its arms on $y = 0$, and the other arms (on $y = x^{k+1}$) as well. $\mathbf{F}_0$ can also be realised as two circles in $\mathbf{R}^3$, which meet in (only) one point, with different tangent lines.

It is easy to see that the even case has no model in the plane. Indeed, two smooth simple loops c, c' in the plane which meet at a point p, with a contact of even order k, must 'cross each other' at the meeting point (see below). Now, the complement of c in the plane has two connected components; if c, c' have no other meeting point, the complement of p in c' should stay in both components, which is absurd.

(If the loops c, c' have different tangent lines at p - only possible when $k = 0$ - they necessarily cross each other. Otherwise, in suitable cartesian coordinates, these curves can be locally represented around p as the graphs of two smooth functions $f, g: \mathbf{R} \to \mathbf{R}$ with $f(0) = g(0) = p$. Then the Taylor polynomial of degree $k+1$ of $h = f - g$, at 0, is a monomial of odd degree; therefore $h(x)$ changes of sign around the origin, and again our curves must cross each other.)

1.2. Smooth cubes. Smooth maps between euclidean spaces will be tested over smooth cubes. An n-*cube* of $X \subset \mathbf{R}^m$ is a continuous mapping

(1) $a: \mathbf{I}^n \to X$,

that will be viewed as a mapping $\mathbf{I}^n \to \mathbf{R}^m$ (with image in X) whenever useful.

This cube is said to be *of class* C^k, or a C^k*-cube*, if it has a C^k-extension $U \to \mathbf{R}^m$ over some open neighbourhood of $\mathbf{I}^n$ in $\mathbf{R}^n$ (or, equivalently, over $\mathbf{R}^n$); notice that this extension is *not* required to stay in X. For $t \in \mathbf{I}^n$, and a multi-index $i = (i_1,..., i_m) \in \mathbf{N}^m$ of height $|i| = i_1 +... + i_m$, the partial derivative of the component $a_j: \mathbf{I}^n \to \mathbf{R}$ is well defined (using any C^k-extension)

(2) $(\partial^{|i|}a_j / \partial t^i)(t) = (\partial^{|i|}a_j / \partial t_1^{i_1}... \partial t_n^{i_n})(t)$.

Equivalently, a C^k-cube can be defined as a mapping $a: \mathbf{I}^n \to X \subset \mathbf{R}^m$ that has continuous partial derivatives up to order k in the interior of $\mathbf{I}^n$, so that all such real functions have a continuous extension to $\mathbf{I}^n$. The equivalence can be proved using adequate extension theorems; for instance, Whitney's theorems as stated in Malgrange [Ma], Chapter 1.

1.3. Smooth weights and regular paths. Let $a: \mathbf{I} \to X$ be a continuous mapping with values in a euclidean space. It will also be called a C^0*-path*, or a C^0*-regular path*.

Let now $k \in \bar{\mathbf{N}}$ be positive. We let $St^k(a) \subset \mathbf{I}$ be the (possibly infinite) *set of* C^k*-stops* of the path a (including every breaking of C^k-smoothness):

(1) $\{0, 1\} \cup \{t \in]0, 1[\mid$ either a is not C^k near t, or it is and $a'(t) = 0\}$,

where near t means on a convenient neighbourhood of t.

Then, we introduce a C^k*-weight* $w^k(a) \in \bar{\mathbf{N}}$, which - loosely speaking - counts each breaking of C^k-smoothness and each stop. Namely:

- $w^k(a) = \infty$ if a is not piecewise C^k;
- otherwise, $w^k(a)$ = (# of connected components of $St^k(a)$) – 1.

We say that the path a is *C^k-regular* if $w^k(a) \leq 1$. This condition includes two cases:

- $w^k(a) = 0$ means that a is constant,
- $w^k(a) = 1$ means that a is of class C^k on **I** and $St^k(a)$ has precisely two connected components, those of 0 and 1; in other words the C^k-stops of a reduce to two disjoint closed intervals, possibly degenerate: $St^k(a) = [0, t_0] \cup [t_1, 1]$, where a is constant.

Any further unit in $w^k(a)$ means an internal C^k-stop point or an additional non-degenerate stop-interval. In brief, *a constant path costs nothing; otherwise, there is a fixed cost of 1/2 at departure and arrival, and a fixed cost of 1 at each C^k-stop* (independently of duration).

Notice that (always for $k > 0$) a path of class C^k is C^k-regular if and only if it is C^1-regular (i.e. it does not stop). Notice also that

(2) $w^k(a*b) \leq w^k(a) + w^k(b)$,

since there are two cases:

- $w^k(a) + w^k(b) = w^k(a*b) + 1$ if $c = a*b$ is C^k at a neighbourhood of $t = 1/2$ and $c'(1/2) \neq 0$,
- $w^k(a) + w^k(b) = w^k(a*b)$, otherwise.

1.4. Smooth maps. As an elementary way of introducing 'smooth spaces with singularities', let us introduce the category C^k**Euc** of subspaces of all the standard euclidean spaces $\mathbf{R}^m$, with *C^k-maps* $f: X \to Y$ between them; by this we mean a continuous mapping f that takes, by composition, the C^h-cubes of X into C^h-cubes of Y, for all $h \leq k$. Thus

(1) C^∞**Euc** $\subset$ C^k**Euc** $\subset$ $C^{k'}$**Euc** $\subset$ C^0**Euc** $\qquad (0 \leq k' \leq k \leq \infty)$.

This definition agrees with the usual one when X and Y are C^k-manifolds. (Let us also recall that, by Boman's theorem [Bo], a map $f: X \to Y$ between C^∞-manifolds is C^∞ if and only if it preserves C^∞-paths.)

A C^k-map $\mathbf{I}^n \to Y$ is the same as a C^k-cube (according to the previous definition, 1.2). For general euclidean spaces ($X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$) we get a broader and (perhaps) better definition than by asking that f can be extended to a C^k-map

$U \to \mathbf{R}^m$ over some open neighbourhood of X in $\mathbf{R}^n$. For instance, if $X \subset \mathbf{R}^2$ is the union of three distinct (oriented) lines r_i through the origin, a C^1-map $f: X \to \mathbf{R}$ (as defined above) needs only to be separately C^1 on each line, whereas the other condition would also impose a relation on the directional derivatives $\partial f/\partial r_i$ at the origin.

We also notice that cubes can test smoothness in a 'finer' way than maps defined on euclidean *open* sets. For instance, if $X = \mathbf{V}_0$, a C^1-map $c: \mathbf{R} \to X$ with $c(0) = (0, 0)$ must have $c'(0) = 0$ and would test C^1-smoothness of functions defined over X in a less effective way than paths $a: \mathbf{I} \to X$ with initial or terminal point at the origin.

1.5. Pathwise regular maps. Let $f: X \to Y$ be a mapping between euclidean spaces. To say that it is *pathwise C^0-regular* will just mean that it is continuous.

For $k > 0$, we say that f is *pathwise C^k-regular* if it is a C^k-map and preserves, by composition, the *C^1-regular paths*. Then, it also preserves by composition the *C^h-regular paths*, for all $h \leq k$. If X and Y are C^k-embedded C^k-manifolds, a C^k-map is pathwise regular if and only if it is an immersion, i.e. the linear mapping $T_x f: T_x X \to T_{fx} Y$ is injective, for every $x \in X$.

These maps define the subcategory $C^k\mathbf{Reg} \subset C^k\mathbf{Euc}$ of *euclidean spaces and pathwise C^k-regular maps*

(1) $C^0\mathbf{Reg} = C^0\mathbf{Euc}, \qquad C^k\mathbf{Reg} = C^1\mathbf{Reg} \cap C^k\mathbf{Euc} \qquad (k > 0),$

$C^\infty\mathbf{Reg} \subset C^k\mathbf{Reg} \subset C^{k'}\mathbf{Reg} \subset C^0\mathbf{Reg} = C^0\mathbf{Euc} \qquad (0 \leq k' \leq k \leq \infty).$

Notice that $C^k\mathbf{Reg}$ lacks products, for $k > 0$; indeed, a cartesian projection $p_i: \mathbf{R}^2 \to \mathbf{R}$ is not immersion, and takes a regular, circular path to a path with stops.

1.6. The weighted fundamental groupoid. For a euclidean space X and an extended integer $k > 0$, the fundamental groupoid $\Pi_1(X)$ has a *C^k-weight* inherited from that of paths

(1) $w^k: \Pi_1(X) \to \bar{\mathbf{N}}, \qquad w^k[a] = \min \{w^k(b) \mid b \in [a]\}.$

Plainly, the identity at a point has C^k-weight 0; moreover, by1.3.2, a composed arrow $[a] + [b] = [a*b]$ gives:

(2) $w^k([a] + [b]) \leq w^k[a] + w^k[b].$

If $f: X \to Y$ is a pathwise C^k-regular map (1.5) between euclidean spaces

(3) $w^k(fa) = w^k(a), \qquad w^k[fa] \leq w^k[a].$

2. Regular 0-homotopy

A euclidean space X is now equipped with an extended sequence of *k-regular 0-homotopy* objects $R^k\Pi_0(X)$. These are sets equipped with a tolerance relation, called *C^k-regular connectedness*.

2.1. Tolerance sets. A *tolerance set* X will be a set equipped with a *tolerance relation* x!y, reflexive and symmetric. A *tolerance morphism* f: X → Y is a mapping between such sets which preserves the tolerance relation.

The category **Tol** of tolerance sets and morphisms is complete and cocomplete, with limits and colimits created by the forgetful functor U: **Tol** → **Set**. It will be analysed more deeply in the last section.

A tolerance set X has an *associated equivalence relation*

(1) $x \sim y \iff (\forall z \in X,\ z \,!\, x \iff z \,!\, y)$,

and we say that X (or its tolerance relation) is *reduced* if (1) is the identity relation.

The quotient $X/\sim$, equipped with the induced relation $\xi \,!\, \eta$ (denoted by the same symbol)

(2) $\mathrm{red}(X) = X/\sim$,

$$\xi \,!\, \eta \iff (\exists x \in \xi,\ \exists y \in \eta,\ x \,!\, y) \iff (\forall x \in \xi,\ \forall y \in \eta,\ x \,!\, y),$$

will be called the *reduced* tolerance set associated to X.

Indeed, it is easy to see that the induced relation $\xi \,!\, \eta$ on red(X) is necessarily reduced: $\xi \sim \eta$ implies $\xi = \eta$. (Let $[x] \sim [y]$; from $z \,!\, x$ it follows that $[z] \,!\, [x]$; then $[z] \,!\, [y]$ and $z \,!\, y$; the symmetric argument gives $x \sim y$.)

If X is *transitive*, i.e. its tolerance relation is an equivalence, then the associated equivalence relation coincides with !, and the quotient red(X) = X/! 'is a mere set' (in the sense that its induced tolerance relation is the identity).

The tolerance set red(X) is an effective description of X, which reduces its redundancy. However, *this reduction is not functorial*, and should be used with care: indeed, a tolerance morphism f: X → Y need not preserve the associated equivalence relation. (This is trivially true when X is transitive.)

2.2. Tolerance relations of regular connectedness. In the euclidean space X, every extended natural number $k \leq \infty$ defines a tolerance relation, called *C^k-regular connectedness* in X

(1) $x \,!_k\, y \iff$ (x and y are connected by a C^k-regular path in X).

This relation is preserved by pathwise C^k-regular maps. Generally, it is not transitive (for $k > 0$); but it is for a C^k-embedded C^k-manifold. Obviously, $x\ !_k\ y$ implies $x\ !_h\ y$, for $h \leq k$ in $\bar{\mathbf{N}}$.

By definition, the *k-regular* 0-*homotopy* object of X will be the tolerance set:

(2) $R^k\Pi_0(X) = (X, !_k)$ $(k \in \bar{\mathbf{N}})$.

We have thus an (extended) sequence of functors $R^k\Pi_0\colon C^k\mathbf{Reg} \to \mathbf{Tol}$, with values in the category of tolerance sets and tolerance maps. In particular, the tolerance set $R^0\Pi_0(X) = (X, !_0)$ is transitive and its reduction yields the usual set $\Pi_0(X)$ of path-components of X

(3) $\Pi_0(X) = \mathrm{red}(R^0\Pi_0(X))$.

We will often use the reduced tolerance set $\mathrm{red}(R^k\Pi_0(X))$ to describe $R^k\Pi_0(X)$, even though this quotient cannot be made into a functor on $C^k\mathbf{Reg}$, for $k > 0$.

2.3. Examples. (a) For the standard deviation $X = \mathbf{V}_k$ (1.1.1), $!_h$ is the chaotic relation (that links all pairs of points) when $h \leq k$. For $h > k$, we have $x\ !_h\ y$ if and only if x and y both belong to the left closed arm or the right closed arm of $\mathbf{V}_k$; these 'arms' meet at the origin, which is in relation $!_\infty$ with any other point.

Thus, for $h > k$, $\mathrm{red}(R^h\Pi_0(\mathbf{V}_k))$ has three elements, corresponding to the singularity and the two open arms ξ, η of $\mathbf{V}_k$; the singularity is $!_h$-related to the other two elements, which are not related

(1) $0 = [(0, 0)]$, $\xi = [(-1, 0)]$, $\eta = [(1, 1)]$,

(2) $0\ !_h\ \xi$, $0\ !_h\ \eta$ $(h > k)$.

The euclidean sets $\mathbf{X}_k$ and $\mathbf{Y}_k$ (1.1.2-3) yield similar results.

(b) For the C^k-figure eight $\mathbf{E}_k$ (1.1(c)), the set $\mathrm{red}(R^h\Pi_0(\mathbf{E}_k))$ has one element, for all h.

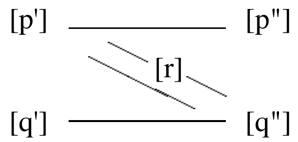
The space $\mathbf{F}_k$ (1.1(d)) gives a different result. The set $\mathrm{red}(R^h\Pi_0(\mathbf{F}_k))$ has three elements as soon as $h > k$ (and just one for $h \leq k$). These elements are the singularity at the origin and two 'punctured circles' ξ, η (i.e. circles without a point); the tolerance relation of $\mathrm{red}(R^k\Pi_0(\mathbf{F}_k))$ is described as above, in (2).

(c) Let us come back to the railway example, in figure (1) of the Introduction, where the route p', p, q, q" is assumed to be - precisely - of class C^1 at p and q. Then:

- the space is path-connected, and all pair of points are in relation $!_0$;
- p' and q', p" and q", q' and p" are not in relation $!_1$;
- no point of the upper line is in relation $!_2$ with any point of the lower line.

More precisely, the tolerance set $\mathrm{red}(R^1\Pi_0(X))$ consists of five equivalence classes, with the tolerance relation expressed by the following (non oriented) graph

(3)



For $k > 1$, $R^k\Pi_0(X)$ is transitive, and $red(R^k\Pi_0(X))$ has three *unrelated* classes: [p], [q], [r].

2.4. Remarks. A path in the euclidean space X is C^∞-regular if and only if it is C^k-regular, for all $k < \infty$. But the relation $a\ !_\infty\ b$ is strictly stronger than the conjunction of the relations $a\ !_k\ b$, for $k < \infty$; in other words, two points can be linked by suitable (different) paths of any possible C^k-class with $k < \infty$, without being linked by a regular C^∞-path.

For instance this happens in the euclidean space union of all deviations $\mathbf{V}_k$ (1.1.1)

(1) $X = \bigcup_{k\in\mathbf{N}} \mathbf{V}_k \subset \mathbf{R}^2$,

where $(-1, 0)\ !_k\ (1, 1)$ if and only if $k < \infty$.

3. Jets and paths

After a brief review of formal series, and their k-truncated versions, we consider the initial and terminal k-jets of a C^k-path in a euclidean space X, and define the *effective* k-jets at a point. Of course, jets can also be defined as equivalence classes of smooth functions, as in the original definition of C. Ehresmann [E1, E2].

3.1. Formal series and truncated polynomials. We begin by recalling the formalism of k-jets, as formal series, for $k = \infty$, or truncated series (i.e. truncated polynomials) for $k < \infty$.

Formal series $S = \Sigma\, a_i\tau^i$ in one variable τ, with coefficients in the real field, form a well-known $\mathbf{R}$-algebra $A_\infty = \mathbf{R}[[\tau]]$. They have a composition law (cf. Cartan [Ca])

(1) $S{\circ}T = \Sigma_i\, a_iT^i \qquad (S = \Sigma_i\, a_i\tau^i,\ \ T = \Sigma_j\, b_j\tau^j)$,

provided the initial term b_0 *of* T *is zero*, so that the sum $\Sigma_i\, a_iT^i$ makes sense. (Indeed, the *order* $\omega(T^i)$, defined as the degree of the *lowest* non null coefficient of T^i, is $i.\omega(T) \geq i$; consequently, each coefficient of $S{\circ}T$ is computed by a finite sum of terms $a_i b_{j_1} ... b_{j_k}$.)

Notice that $S \circ 0 = a_0$ is the initial term of S, that is also written as $S(0)$. The algebraic properties of the composition law can be seen in [Ca].

For $k < \infty$, the algebra of k-*truncated series* (or k-truncated polynomials)

(2) $A_k = \mathbf{R}[[\tau]] / (\tau^{k+1}) = \mathbf{R}[\tau] / (\tau^{k+1})$,

has an induced composition law $[S] \circ [T] = [S \circ T]$ (when $T(0) = 0$).

A class $[S] = \Sigma_{i \leq k} a_i [\tau]^i$ will also be written as $\Sigma_{i \leq k} a_i \tau^i$, by abuse of notation. Operations in A_k are thus performed as with polynomials in an algebraic element τ such that $\tau^{k+1} = 0$: one omits all the terms of degree $> k$ ('the higher-order infinitesimals'), that come out of operations like product or composition of polynomials.

There are obvious *truncation epimorphisms*

(3) $tr_{kk'}: A_k \to A_{k'}$ $\qquad (0 \leq k' \leq k \leq \infty)$,

ending with $A_0 = \mathbf{R}$. We will refer to A_k as the algebra of *k-truncated series in one variable*, also when $k = \infty$ (and truncation is trivial).

We view A_k as the fibre bundle of k-jets of the real line

(4) $T_k\mathbf{R} = A_k$, $\qquad p: T_k\mathbf{R} \to \mathbf{R}$, $p(S) = S(0)$.

Of course, it is a trivial bundle, and can be identified with the product of the line and its fibre at 0, the subalgebra $T_{k0}\mathbf{R}$ of k-truncated series with $S(0) = 0$

(5) $\mathbf{R} \times T_{k0}\mathbf{R} = T_k\mathbf{R}$, $\qquad (x, S) \mapsto x + S$.

Each fibre $\{x\} \times T_{k0}\mathbf{R}$ is a real vector space *at fixed* x, with the operations of $T_{k0}\mathbf{R}$:

(6) $(x + S) + (x + T) = x + S + T$, $\qquad \lambda.(x + S) = x + \lambda S$.

Composition is everywhere defined on the fibre $T_{k0}\mathbf{R}$. It *can* be extended, as in (1), letting:

(7) $(x + S) \circ T = x + S \circ T$.

3.2. Series in many variables. More generally, we will use formal series in the variables $\tau_1, ..., \tau_m$

(1) $S = \Sigma_i a_i \tau^i$, $\qquad i = (i_1, ..., i_m) \in \mathbf{N}^m$, $\qquad \tau^i = \tau_1^{i_1} \cdot ... \cdot \tau_m^{i_m}$,

and their $\mathbf{R}$-algebra $A_{m\infty} = \mathbf{R}[[\tau_1, ..., \tau_m]]$.

Now, there is a composition $S \circ T$, where T is a family of formal series $T_1, ...,$

T_m *with initial term zero*, on the same n variables, say $\vartheta_1,..., \vartheta_n$

(2) $S = \Sigma_i a_i\tau^i \in \mathbf{R}[[\tau_1,..., \tau_m]]$, $\quad T = (T_1,..., T_m) \in (\mathbf{R}[[\vartheta_1,..., \vartheta_n]])^m$,

$S{\circ}T = \Sigma_i a_i T^i$, $\quad T^i = T_1^{i_1} \cdot ... \cdot T_m^{i_m}$.

Again, the sum in $S{\circ}T$ makes sense because

$\omega(T^i) = i_1\omega(T_1) + ... + i_m\omega(T_1) \geq i_1 + ... + i_m = |i|$.

The k-truncated version

(3) $A_{mk} = \mathbf{R}[[\tau_1,..., \tau_m]] / I_k$ $\quad (= \mathbf{R}[\tau_1,..., \tau_m] / J_k$, for $k < \infty$),

annihilates the ideal I_k (or J_k) of all series (or polynomials) whose order is $> k$, that is generated by the monomials $\tau^i = \tau_1^{i_1} \cdot ... \cdot \tau_m^{i_m}$ with $|i| = k+1$. Of course, $I_\infty = (0)$.

Again, we will refer to A_{mk} as the algebra of *k-truncated series* in m variables, even for $k = \infty$.

3.3. Jets of functions. Let A_{nk} be the algebra of k-truncated series in n variables $\vartheta_1,..., \vartheta_n$ (3.2), with $k \leq \infty$. For a C^k-mapping $f: U \to V$ between open euclidean spaces of dimensions n, m, we define the *k-jet* of f at a point $x \in U$

(1) $(j^kf)(x) \in (A_{nk})^m$, $\quad ((j^kf)(x))_i \in A_{nk}$ $\quad (i = 1, ... m)$,

$((j^kf)(x))_i = \Sigma_{s\leq k} (s!)^{-1} D_sf_i(x){\cdot}\vartheta^s$,

$D_sf_i(x) = (\partial^sf_i/\partial x_1^{h_1}...\partial x_n^{h_n})_{|h|=s}$, $\quad \vartheta^s = (\vartheta_1^{h_1},..., \vartheta_n^{h_n})_{|h|=s}$,

where $\cdot$ is the scalar product of vectors indexed on all the n-tuples $h = (h_1,..., h_n)$ with a fixed sum $s = |h| = h_1 + ... + h_n$.

For instance, for $f: \mathbf{R}^2 \to \mathbf{R}$, and leaving the variable $x \in \mathbf{R}^2$ understood, j^2f yields the Taylor formula of f, of order 2, in the form:

$$j^2f = f + \vartheta_1.\partial f/\partial x_1 + \vartheta_2.\partial f/\partial x_2 + 1/2(\vartheta_1^2.\partial^2f/\partial x_1^2 + 2\vartheta_1\vartheta_2.\partial^2f/\partial x_1\partial x_2 + \vartheta_2^2.\partial^2f/\partial x_2^2).$$

The jet of a composition can be expressed in a useful compact form, using the composition of formal series. Namely, if $f: U \to V$ and $g: V \to W$ are C^k-mappings between open euclidean spaces, so is gf and:

(2) $j^k(gf)(x) = ((j^kg)(f(x)))\circ((j^kf)(x) - f(x))$.

Notice that $(j^kf)(x) - f(x)$ is a (possibly truncated) formal series with initial term 0, so that the composition makes sense. Concretely, we must replace the variable $\tau = (\tau_1,..., \tau_m)$ of the jet $(j^kg)(f(x))$ with the increment of j^kf.

Thus, for $k = 2$ and in one variable, the second term of (2) is a 2-truncated polynomial whose coefficients are (indeed) the derivatives at x of the composed function gf, up to the second:

$$(gf(x) + g'f(x).\tau + g''f(x).\tau^2)\circ(f'(x).\vartheta + f''(x).\vartheta^2)$$
$$= gf(x) + g'f(x).f'(x).\vartheta + (g'f(x).f''(x) + g''f(x).f'^2(x)).\vartheta^2.$$

We shall apply this formula also when f or g are C^k-cubes, taking into account the fact that they have C^k-extensions to open subsets containing their domain.

3.4. Initial and terminal jets of a path. Let us fix a euclidean space $X \subset \mathbf{R}^m$ and an extended natural number $k \leq \infty$.

Take a C^k-path $a: \mathbf{I} \to X \subset \mathbf{R}^m$ with $a(0) = x$ and $a(1) = x'$. Its *initial k-jet* (at $t = 0$) has components in the trivial fibre bundle $T_k\mathbf{R} = \mathbf{R} \times T_{k0}\mathbf{R}$ (3.1)

(1) $((j^ka)(0))_i = \Sigma_{h\leq k} (h!)^{-1} (a^{(h)}(0))_i . \tau^h \in T_k\mathbf{R} = \mathbf{R} \times T_{k0}\mathbf{R}$ $(i = 1, ... m)$.

It gives an element of the trivial fibre bundle $T_k\mathbf{R}^m = \mathbf{R}^m \times T_{k0}\mathbf{R}^m$

(2) $\partial_k^- a = (j^ka)(0) = x + \Sigma_{0<h\leq k} (h!)^{-1} a^{(h)}(0).\tau^h \in \{x\} \times T_{k0}\mathbf{R}^m$,

that will also be written in the form $x + (v^ka)(0)$.

Similarly, we have a *terminal k-jet* (at $t = 1$)

(3) $\partial_k^+ a = (j^ka)(1) = x' + (v^ka)(1)$
$= x' + \Sigma_{0<h\leq k} (h!)^{-1} a^{(h)}(1) . \tau^h \in \{x'\} \times T_{k0}\mathbf{R}^m$.

Initial and terminal jets determine each others. Indeed, let $b: \mathbf{I} \to X$ denote the reversed path of a, namely $b(t) = a(1 - t)$; then the initial k-jet of a and the terminal k-jet of b

(4) $\partial_k^- a = (j^ka)(0)$, $\qquad \partial_k^+ b = (j^kb)(1) = \rho_k((j^ka)(0))$,

are linked by the involution ρ_k which changes sign to the derivatives of odd degree

(5) $\rho_k: T_k\mathbf{R}^m \to T_k\mathbf{R}^m$, $\qquad \rho_k(\Sigma_{h\leq k} a_h\tau^h) = \Sigma_{h\leq k} (-1)^h a_h\tau^h$.

3.5. Effective and virtual jets. We want now to define the set E_xT_kX of effective k-jets of X at x, as a subset of the vector space $\{x\} \times T_{k0}\mathbf{R}^m$.

In the same hypotheses as above (3.4), any initial k-jet of a C^k-path $a: \mathbf{I} \to X \subset \mathbf{R}^m$ with $a(0) = x$ will be called a *lower-effective k-jet of* X *at* x. Their set will be written as

(1) $E_x^-T_kX \subset \{x\} \times T_{k0}\mathbf{R}^m$.

The set of terminal k-jets of all paths ending at x will be written as

(2) $E^+_x T_k X = \rho_k(E^-_x T_k X)$,

(cf. 3.4.4) and called the set of *upper-effective k-jets of* X *at* x. Finally, we write

(3) $B_x T_k X = E^-_x T_k X \cap E^+_x T_k X$, $\quad E_x T_k X = E^-_x T_k X \cup E^+_x T_k X$,

the sets of *bilateral* and *effective k-jets of* X *at* x. (The term 'effective' can be left understood.)

We will see (in 3.7) that the bilateral jets are precisely those that can be obtained as jets $(j^k a)(t)$ (of C^k-paths of X), at an *internal* point $t \in]0, 1[$. As a consequence, the effective jets are those that can be obtained at some $t \in \mathbf{I}$.

There are inclusions

(4) $B_x T_k X \subset E^\alpha_x T_k X \subset E_x T_k X \subset \{x\} \times T_{k0}\mathbf{R}^m \quad (\alpha = \pm)$.

Letting x vary, we get five 'fibred sets' on X

(5) $BT_k X \subset E^\alpha T_k X \subset ET_k X \subset T_k\mathbf{R}^m \quad (\alpha = \pm)$.

For $k = 0$, we just have: $BT_0X = ET_0X = X \subset \mathbf{R}^m$.

For $k = 1$, the vector subspace spanned by $E^-_x T_1 X$ (or $E^+_x T_1 X$) in the real vector space of tangent vectors $\{x\} \times \mathbf{R}^m$ will be written as $W_x T_1 X$ and called the vector space of *virtual tangent vectors of* X *at* x. Notice that their collection WT_1X is not a fibre bundle, generally: the vector spaces $W_x T_1 X$ can have variable dimension, as is easy to see in the examples of 1.1.

The subset $E^\alpha_x T_1 X$ inherits a *multiplication by real scalars* $\lambda \geq 0$, and will be viewed as a union of *semilinear subspaces* (semimodules on the semiring of weakly positive real numbers). Indeed, if $u: \mathbf{I} \to \mathbf{I}$ is any C^∞-function whose ∞-jet at 0 is $\lambda\tau$, the initial 1-jet of the reparametrised path au is

(6) $j^1(au)(0) = ((j^1a)(0)\circ((j^1u)(0)) = a(0) + \lambda.(v^1a)(0)$.

The sets $B_x T_1 X$ and $E_x T_1 X$ (of bilateral and effective tangent vectors) inherit thus, from the vector space $W_x T_1 X$, a *multiplication by real scalars*, and will be viewed as unions of linear subspaces of $W_x T_1 X$.

Because of these multiplications, the following topological spaces

(7) $BT_1X \subset E^\alpha T_1X \subset ET_1X \subset WT_1X \subset X \times \mathbf{R}^m \quad (\alpha = \pm)$,

admit $X \times \{0\}$ as a deformation retract, and are homotopically equivalent to X. *We also write* TX *for* ET_1X, *the fibred set of (effective) tangent vectors.*

Finally, we write E^*T_kX the set of the *regular* k-jets of X

(8) $E^*T_kX \subset ET_kX \subset T_k\mathbf{R}^m$,

i.e. the effective k-jets of X *with a non-zero term of degree 1*. These are also characterised below, in 3.7.

3.6. Theorem and Definition (Smooth concatenation of paths). *Let* $a, b\colon \mathbf{I} \to X$ *be two* C^k*-consecutive* C^k*-paths, which means that*

(1) $\partial_k^+ a = \partial_k^- b,$

i.e. $a^{(h)}(1) = b^{(h)}(0)$, *for all* $h \leq k$. *Then there is a* smoothly concatenated C^k*-path*

(2) $a + b\colon \mathbf{I} \to X,$

$$(j^k(a + b))(0) = (j^k a)(0), \qquad (j^k(a + b))(1) = (j^k b)(1),$$

$$(a + b)(t) = a(\kappa(t)) \text{ or } b(\kappa(t - {}^1/_2)), \qquad \text{for } 0 \leq t \leq {}^1/_2 \text{ or } {}^1/_2 \leq t \leq 1.$$

We are using a concatenating C^∞-function $\kappa \in C^\infty(\mathbf{R}, \mathbf{R})$, *chosen once for all and satisfying*

(3) $(j^\infty\kappa)(0) = \tau, \qquad (j^\infty\kappa)({}^1/_4) = {}^1/_2 + \tau,$

$$(j^\infty\kappa)({}^1/_2) = 1 + \tau, \qquad \kappa'(t) > 0.$$

Moreover, if a *and* b *are* C^k*-regular, so is the concatenated path.*

Note. The function κ restricts to a strictly increasing diffeomorphism $[0, {}^1/_2] \to \mathbf{I}$, and replaces here the function 2t used for the usual concatenation $a*b$ (that is homotopic to the former, with fixed endpoints). *As an advantag*e, it has jet $2t + \tau$ (instead of $2t + 2\tau$) at the endpoints $(t = 0, {}^1/_2)$, *and leaves unchanged the initial jet of* a *and the terminal jet of* b. The similar condition at $t = {}^1/_4$ will be useful for associativity (in the proof of Theorem 4.3.)

Proof. The existence of a smooth function κ as above is obvious (or see Lemma 4.5, at the end of the next section).

Now, a + b is of class C^k, because so is the 'pasting' of a, b at ${}^1/_2$. It suffices to apply the formula 3.3.2 for the jet of a composite (at the left and at the right of ${}^1/_2$)

$$j^k_-(a + b)({}^1/_2) = j^k(a\kappa)({}^1/_2) = (j^k a)(\kappa({}^1/_2))\circ((j^k\kappa)({}^1/_2)) - \kappa({}^1/_2))$$
$$= (j^k a)(1)\circ(\tau) = (j^k a)(1),$$

$$j^k_+(a + b)({}^1/_2) = j^k(b\kappa(\tau - {}^1/_2))({}^1/_2) = ((j^k b)(\kappa(0))\circ((j^k\kappa)(0)))\circ(\tau - {}^1/_2)$$
$$= (j^k b)(0)\circ(\tau) = (j^k b)(0).$$

Similarly, a + b satisfies the initial and terminal conditions stated in (2).

Suppose now that a and b are C^k-regular, with $k > 0$. Then a and b never stop, and so does the concatenated path; indeed, $(a + b)'(\tau)$ is computed by one of the following formulas, and does not vanish at any $\tau \in \mathbf{I}$

$$a'(\kappa(\tau)).\kappa'(\tau), \qquad\qquad b'(\kappa(\tau - {}^1/_2)).\kappa'(\tau - {}^1/_2). \qquad \square$$

3.7. Corollary. *For a euclidean space X and $k \geq 0$:*

(a) the bilateral *k-jets of X are precisely those that can be obtained as jets $(j^k a)(t_0)$ at an internal point $t_0 \in\ ]0, 1[$, for some C^k-path a of X;*

(b) the effective *k-jets of X are precisely those that can be obtained as jets $(j^k a)(t_0)$ at some point $t_0 \in [0, 1]$, for some C^k-path a of X;*

(c) the regular *k-jets of X are precisely those that can be obtained as jets $(j^k a)(t_0)$ at some point $t_0 \in [0, 1]$, for some* C^k-regular *path a of X.*

Proof. Point (a) is obvious, using concatenation and smooth reparametrisation; then, (b) follows immediately. For (c), let a be a C^k-path of X , and suppose that $j = (j^k a)(t_0)$ has a non-zero term of degree 1. Then a satisfies the same property on a suitable neighbourhood of t_0 in $\mathbf{I}$; and we can restrict a to a suitable subinterval, and reparametrise it, so to obtain a C^k-regular path b which has the same k-jet at some point. $\square$

4. Fundamental smooth semicategories

After defining $R^k\Pi_0(X)$ in Section 2, we now want to analyse the fundamental groupoid $\Pi_1(X)$ of a euclidean space X. For $k > 0$, we use C^k-regular paths to get a *fundamental C^k-regular semicategory* $R^k\Pi_1(X)$: its vertices are the regular k-jets of X (3.5), and the homotopy relation used to define an arrow $[a]: j \to j'$ works at fixed initial and terminal k-jets.

4.1. Graphs of smooth paths. X is always a euclidean space. We now define its *graph of C^k-paths* C^kPX, and the *subgraph of C^k-regular paths* $R^kPX \subset C^kPX$.

For $k = 0$, $R^0PX = C^0PX = PX$ is just the graph of paths of X, with vertices the points of X and arrows $a: x \to x'$ the paths of X, from x to x'. It is a reflexive graph with composition; the latter is associative up to homotopy with fixed endpoints, and the quotient modulo this equivalence relation is the fundamental groupoid $\Pi_1(X)$ of the topological space X.

For $k > 0$, the vertices of C^kPX are the elements of ET_kX, i.e. the effective k-jets of X. An arrow $a: j \to j'$ is a C^k-path of X with the given initial and terminal jets

(1) $\partial_k^- a \ = \ (j^k a)(0) \ = \ j, \qquad\qquad \partial_k^+ a \ = \ (j^k a)(1) \ = \ j'.$

But we are more interested in the subgraph *of C^k-regular paths* $R^kPX \subset C^kPX$. Its vertices are the elements of E^*T_kX, i.e. the *regular* k-jets of X (with a non-zero term of degree 1, cf. 3.5 and 3.7). An arrow $a: j \to j'$, between two such jets $j, j' \in E^*T_kX$, is a *regular* C^k-path of X between the given end-jets.

The graph R^kPX has the composition described above (Theorem 3.6), that we prove now to be associative up to the appropriate notion of homotopy.

4.2. Regular homotopy. Let two C^k-regular paths $a, b: \mathbf{I} \to X \subset \mathbf{R}^m$ be given. A *C^k-regular homotopy with fixed end jets*, from a to b, will be a C^k-cube $\varphi: \mathbf{I}^2 \to X$ (1.2) such that the C^k-paths $\varphi_t = \varphi(-, t): \mathbf{I} \to X$ *are regular* and satisfy the following conditions

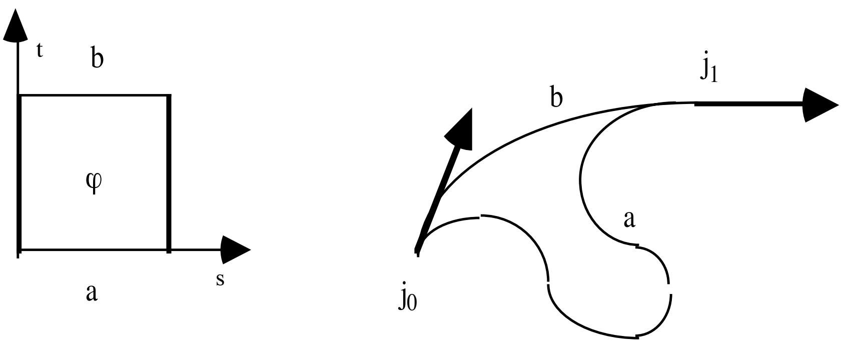
(i) $\varphi_0 \ = \ a, \quad \varphi_1 \ = \ b,$

(ii) $(j^k\varphi_t)(0)$ and $(j^k\varphi_t)(1)$ are independent of $t \in \mathbf{I}$ (*fixed end jets*).

In the presence of (i), condition (ii) can be equivalently expressed as

(ii') $(j^k\varphi_t)(s) \ = \ (j^k a)(s) \ = \ (j^k b)(s), \qquad$ for $t \in \mathbf{I}$ and $s = 0, 1$.

The picture below represents the case $k = 1$, where the initial and terminal jets are (bound) vectors j_0, j_1; the homotopy φ is constant on the vertical edges of $\mathbf{I}^2$

(1) 

If such a 'distinguished' homotopy exists, we write $a \simeq_k b$. It is an equivalence relation for C^k-regular paths of X, because - plainly - these homotopies can be vertically reversed and pasted, and include the degenerate homotopy of a C^k-regular path: $0_a(s, t) = a(s)$.

Of course, $a \simeq_0 b$ is the ordinary relation of homotopy with fixed end-points. Moreover

(2) $a \simeq_k b \;\Rightarrow\; a \simeq_{k'} b \qquad (k' \leq k).$

4.3. Theorem. In the graph of C^k-regular paths R^kPX, the equivalence relation $\simeq_k$ agrees with concatenation and induces an associative operation on the quotient. This equivalence relation is preserved by pathwise C^k-regular maps.

Proof. (a) First we prove that, given four C^k-regular paths a, b, c, d, if

$$a,\ b\colon j' \to j, \qquad c,\ d\colon j \to j'', \qquad a \simeq_k b, \quad c \simeq_k d$$

then $a + b \simeq_k c + d$.

Let φ and ψ be our distinguished homotopies, that we want to paste horizontally. Since $\varphi(1, t) = a(1) = b(1)$ is constant, each partial derivative of a component φ_i annihilates at $\mathbf{I}\times\{1\}$, unless it only concerns derivation with respect to the first variable; the same holds for ψ at $\mathbf{I}\times\{0\}$; the relevant derivatives form the jet

$$(j^k\varphi_t)(1) \;=\; j \;=\; (j^k\psi_t)(0).$$

Therefore we can concatenate φ and ψ (horizontally), and obtain an (obviously) 'distinguished' homotopy; therefore $a + b \simeq_k c + d$.

(b) We prove now that, for three consecutive C^k-regular paths a, b, c

$$(a + b) + c \;\simeq_k\; a + (b + c).$$

Recall that the diffeomorphism $\kappa\colon [0, 1/2] \to [0, 1]$ that defines concatenation (3.6) was chosen to satisfy

(1) $\kappa[0, 1/4] \;=\; [0, 1/2], \qquad \kappa[1/4, 1/2] \;=\; [1/2, 1],$

so that the two ternary composites are computed as

$$(2) \quad (a + b) + c \;=\; \begin{cases} a\kappa\kappa(t) & \textit{for}\ \ t \in [0, 1/4], \\ b\kappa(\kappa(t) - 1/2)) & \textit{for}\ \ t \in [1/4, 1/2], \\ c\kappa(t - 1/2) & \textit{for}\ \ t \in [1/2, 1], \end{cases}$$

$$(3) \quad a + (b + c) \;=\; \begin{cases} a\kappa(t) & \textit{for}\ \ t \in [0, 1/2], \\ b\kappa\kappa(t - 1/2) & \textit{for}\ \ t \in [1/2, 3/4], \\ c\kappa(\kappa(t - 1/2) - 1/2) & \textit{for}\ \ t \in [3/4, 1]. \end{cases}$$

This can be re-written using the Moore concatenation $<abc>$ over $[0, 3]$ and two C^∞-functions $\lambda, \mu\colon \mathbf{I} \to [0, 3]$ (reparametrisations)

$$(4) \quad <abc>(t) \;=\; \begin{cases} a(t) & \textit{for}\ \ t \in [0, 1], \\ b(t - 1) & \textit{for}\ \ t \in [1, 2], \\ c(t - 2) & \textit{for}\ \ t \in [2, 3], \end{cases}$$

(5) $(a + b) + c = \langle abc \rangle \circ \lambda$, $\qquad a + (b + c) = \langle abc \rangle \circ \mu$,

$$\lambda(t) = \begin{cases} \kappa\kappa(t) & \textit{for} \;\; t \in [0, 1/4], \\ 1 + \kappa(\kappa(t) - 1/2)) & \textit{for} \;\; t \in [1/4, 1/2], \\ 2 + \kappa(t - 1/2) & \textit{for} \;\; t \in [1/2, 1], \end{cases}$$

$$\mu(t) = \begin{cases} \kappa(t) & \textit{for} \;\; t \in [0, 1/2], \\ 1 + \kappa\kappa(t - 1/2) & \textit{for} \;\; t \in [1/2, 3/4], \\ 2 + \kappa(\kappa(t - 1/2) - 1/2) & \textit{for} \;\; t \in [3/4, 1]. \end{cases}$$

Now, the function λ is C^∞, because at each pasting point $t = 1/4$ or $1/2$ we get $(j^\infty\lambda)(t) = t + \tau$ (using the composition of jets and the hypotheses 3.6.3 on the jets of κ at these points). Similarly, μ is C^∞, and so is the affine homotopy ξ

(6) $\xi: \mathbf{I}^2 \to \mathbf{R}$, $\qquad \xi(s, t) = (1 - t).\lambda(s) + t.\mu(s)$,

with $\xi_0 = \xi(-, 0) = \lambda$ and $\xi_1 = \xi(-, 1) = \mu$. Its end-jets are fixed (independent of t)

$$(j^k\xi_t)(0) = (1 - t).j^k\lambda(0) + t.j^k\mu(0) = (1 - t).\tau + t.\tau = \tau,$$
$$(j^k\xi_t)(1) = (1 - t).j^k\lambda(1) + t.j^k\mu(1) = (1 - t).(3 + \tau) + t.(3 + \tau) = 3 + \tau.$$

At fixed t, $\xi_t = \xi(-, t)$ is an affine combination of λ, μ; since these are strictly increasing, so is ξ_t. Therefore $\xi: \mathbf{I}^2 \to \mathbf{R}$ is a C^k-regular homotopy from λ to μ, and $\langle abc \rangle \circ \xi$ is a regular homotopy from $(a + b) + c$ to $a + (b + c)$.

(c) Let $f: X \to Y$ be a pathwise C^k-regular map: by definition, it preserves C^k-regular paths and cubes. Let now $\varphi: \mathbf{I}^2 \to X$ be a C^k-regular homotopy satisfying the conditions (i), (ii) of 4.2. Then $f\varphi: \mathbf{I}^2 \to Y$ is a C^k-regular cube, with $(f\varphi)_0 = f\varphi_0 = fa$ and $(f\varphi)_1 = fb$. It has fixed end-jets, by applying the composition formula 3.3.2, for $s = 0, 1$

$$j^k(f\varphi_t)(s) = ((j^kf)(\varphi_t(s))\circ((j^k\varphi_t)(s) - \varphi(s)),$$

where $\varphi_t(s) = a(s)$ and $(j^k\varphi_t)(s) = (j^ka)(s)$ are both independent of t. □

4.4. The fundamental C^k-regular semicategory. For a euclidean space X, we will write

(1) $R^k\Pi_1(X) = R^kPX / \simeq_k$,

the quotient of the graph R^kPX (4.1) modulo the equivalence relation $\simeq_k$ (4.2), with the induced, associative concatenation.

$R^k\Pi_1(X)$ will be called the *fundamental C^k-regular semicategory of* X; where a *semicategory* is the obvious generalisation of a category, without assuming the existence of identities (cf. [MBB]).

We have thus defined a functor

(2) $R^k\Pi_1$: $C^k\mathbf{Reg} \to s\mathbf{Cat}$,

with values in the category of small semicategories and *semifunctors* between them (preserving composition).

For any regular k-jet $j \in E^*T_kX$, there is a semigroup

(3) $R^k\pi_1(X, j) = R^k\Pi_1(X)(j, j)$.

Some computations of such semicategories and semigroups will be given in the next section.

If $k' \leq k$, let U: $C^k\mathbf{Reg} \subset C^{k'}\mathbf{Reg}$ be the inclusion (1.5). There are natural transformations

(4) $tr_{kk'}$: $R^k\Pi_1 \to (R^{k'}\Pi_1)\circ U$: $C^k\mathbf{Reg} \to s\mathbf{Cat}$ $(k \geq k')$,

whose component on the euclidean space X is the obvious functor

(5) $tr_{kk'}(X)$: $R^k\Pi_1(X) \to R^{k'}\Pi_1(X)$, $j \to tr_{kk'}(j)$,

that operates on k-jets by truncation (3.1.3), and on equivalence classes of path by 'inclusion' (taking 4.2.1 into account).

4.5. Lemma. *Let (a_n), (b_n) be two sequences of real numbers $(n \geq 0)$. Then there is a C^∞-function f: $\mathbf{R} \to \mathbf{R}$ whose sequences of derivatives at 0 and 1 are the given ones. Moreover*:

(a) if $a_0, b_0 > 0$, one can choose f so that $f(t) > 0$ over $\mathbf{R}$;

(b) if $a_0 < b_0$ and $a_1, b_1 > 0$, one can choose f so that $f'(t) > 0$ over $\mathbf{R}$.

Proof. By a well-known Borel's Lemma, there exist two C^∞-functions g, h: $\mathbf{R} \to \mathbf{R}$ whose sequences of derivatives at 0 and 1 are, respectively, the given sequences. Take a smooth 'bell' function β: $\mathbf{R} \to \mathbf{R}$ vanishing outside $]-\varepsilon, \varepsilon[$ $(\varepsilon < {}^1/_3)$ and satisfying

$$\beta(0) = 1, \quad \beta^{(n)}(0) = 0, \qquad \beta(t) \geq 0 \qquad (n > 0, \ t \in \mathbf{R}),$$

then $f(t) = \beta(t).g(t) + \beta(t-1).h(t)$ satisfies our conditions.

In case (a), take $f(t) = c + \beta(t).(g(t) - c) + \beta(t-1).(h(t) - c)$, after choosing a positive $c < a_0, b_0$ and ε sufficiently small so that $g(t)$, $h(t+1) \geq c$ in $]-\varepsilon, \varepsilon[$.

In case (b), first consider the shifted sequences (a_{n+1}), (b_{n+1}) starting at $a_1, b_1 > 0$, and let u be a positive solution for them, as in the previous case. Now, the function

$$f(t) = a_0 + U(t), \qquad U(t) = \int_0^t u,$$

is a solution, provided that u is constructed so that the positive number $U(1)$ coincides with $b_0 - a_0 > 0$. This can always be done, either modifying β (to make $U(1)$ smaller) or adding to u a third bell-function with support contained in $[\varepsilon, 1 - \varepsilon]$ (to make $U(1)$ bigger). □

5. Comparison with the fundamental groupoid of tangent versors

We study in more detail the semicategory $R^1\Pi_1(X)$ of a euclidean space X, and compare it with the fundamental groupoid $\Pi_1(T^*X)$ of the space of *non-zero (effective) tangent vectors*, isomorphic to the fundamental groupoid $\Pi_1(UTX)$ of the space of *unit tangent vectors* of X. Under convenient hypotheses, this comparison is an isomorphism (Theorem 5.4).

5.1. The comparison. Let $X \subset \mathbf{R}^m$ be a euclidean space. Consider the obvious embedding

(1) $R^1PX \to C^0P(T^*X), \qquad a \mapsto \hat{a} = (a, a'),$

of the graph of regular C^1-paths into the graph of paths of the subspace of *non-zero* (effective) tangent vectors:

(2) $T^*X \subset TX = ET_1X, \qquad T^*X = TX \cap (X \times (\mathbf{R}^m \setminus \{0\})).$

Plainly, this embedding is the identity on vertices and consistent with concatenation. It is also consistent with the appropriate notions of homotopy: if $a \simeq_1 b$ in R^1PX (4.2), the C^1-regular homotopy $\varphi: \mathbf{I}^2 \to X$ consists of a family of regular C^1-paths $\varphi_t = \varphi(-, t): \mathbf{I} \to X$ $(t \in \mathbf{I})$, and can be lifted to a homotopy of paths in T^*X, with fixed endpoints

(3) $\hat{\varphi}: \mathbf{I}^2 \to T^*X, \qquad \hat{\varphi}(s, t) = \hat{\varphi}_t(s) = (\varphi(t, s), \partial\varphi/\partial s(t, s)),$
$\hat{\varphi}(0, -) = \hat{a}, \qquad \hat{\varphi}(1, -) = \hat{b},$

$\hat{\varphi}(s, t) = \hat{\varphi}_t(s) = (j^1\varphi_t)(s) \qquad$ (independent of $t \in \mathbf{I}$, for $s = 0, 1$).

Therefore, there is a canonical comparison semifunctor, that is the identity on the objects, the (bound) vectors of T^*X

(4) $\mathbf{u}: R^1\Pi_1(X) \to \Pi_1(T^*X), \qquad [a] \mapsto [\hat{a}] = [(a, a')].$

(Let us recall, from 3.5.7, that $TX \simeq X$ cannot give here a 'good' comparison.)

Now, the subspace $UTX \subset T^*X$ of *unit tangent vectors* is a strong deformation retract of T^*X. We identify their fundamental groupoids, by the canonical isomorphism

(5) $\Pi_1(UTX) \cong \Pi_1(T^*X)$,

induced by the embedding $UTX \subset T^*X$ and its retraction $p: T^*X \to UTX$ (the normalisation of non-zero vectors).

The comparison **u** need not be full (cf. 5.5). But we prove that it is an isomorphism when X is a C^1-embedded manifold of dimension ≥ 2 (Theorem 5.4, after the following two lemmas).

5.2. Lemma. *Let $f: S \to G$ be a semifunctor from a semicategory to a groupoid. Then f is full and faithful if and only if the following conditions hold:*

(a) for every object x of S, f restricts to an isomorphism of semigroups $S(x, x) \to G(f(x), f(x))$ (which is thus an isomorphism of groups);

(b) for every pair of objects x, y in S such that f(x), f(y) are connected in G, there is some arrow $a: x \to y$ in S.

Moreover, if all this holds true, S is a groupoid and f is actually a functor.

Proof. The necessity of these conditions is obvious, as well as the last remark. Conversely, let us suppose they hold and fix a pair of objects x, y of objects of S. Composition in S and G is written in additive notation.

If f(x), f(y) are connected in G, there is some arrow $a: x \to y$ in S. Since G is a groupoid, all the arrows of G(f(x), f(y)) can be expressed as $g = h + f(a)$, for some endomap $h \in G(f(x), f(x))$. Applying (a), we have that $g = f(b)$ for some $b: x \to y$ in S. Therefore, f is full.

Suppose now that $a, b: x \to y$ in S are identified by f, and choose some $a': y \to x$ such that $f(a') = -f(a)$. Then $f(b + a') = f(a + a') = 0_{fx}$ and $b + a' = a + a' = 0_x$, by (a); similarly, $a' + b = a' + a = 0_y$, and finally $a = b$. □

5.3. Lemma. *For $X = \mathbf{R}^m$ and $m \geq 2$, the canonical comparison semifunctor*

(1) $\mathbf{u}: R^1\Pi_1(X) \to \Pi_1(T^*X), \qquad [a] \mapsto [\hat{a}] = [(a, a')],$

(cf. 5.1.4) is an isomorphism of groupoids. These are codiscrete for $m > 2$.

For $m = 2$, we have a connected groupoid whose groups of endoarrows are infinite cyclic

(2) $R^1\Pi_1(X)(j, j) \cong \Pi_1(T^*X)(j, j) \cong \Pi_1(\mathbf{S}^1)(j_0, j_0) \cong \mathbf{Z}.$

Proof. For $m > 2$,

$\Pi_1(UTX) = \Pi_1(\mathbf{R}^m \times \mathbf{S}^{m-1}) = \Pi_1(\mathbf{S}^{m-1})$,

is a codiscrete groupoid, i.e. between any two vertices there is precisely one arrow. The same is obviously true of $R^1\Pi_1(X)$, since any two C^1-regular paths $j \to j'$ can be deformed one into the other ($\mathbf{R}^3$ has 'sufficient room' to do that).

We now take $X = \mathbf{R}^2$ and apply the previous lemma. Its condition (b) is obviously satisfied: for any two vectors $j, j' \in T^*\mathbf{R}^2$ there exists a C^1-regular path a that gives an arrow $[a]: j \to j'$ in $R^1\Pi_1(X)$. We are left with considering the endoarrows of the semicategories in (1).

The space $X = \mathbf{R}^2$ will be given the usual orientation, by its embedding in $\mathbf{R}^3$ (with normal versor $(0, 0, 1)$).

If j_0 is the versor of the vector $j \in T^*X$, the canonical isomorphism

(3) $w\colon \Pi_1(T^*X)(j, j) \to \Pi_1(UTX)(j_0, j_0) = \Pi_1(\mathbf{R}^2 \times \mathbf{S}^1)(j_0, j_0) \to \mathbf{Z}$,

is computed as a winding number

(4) $w[a] = w(a_2)$,

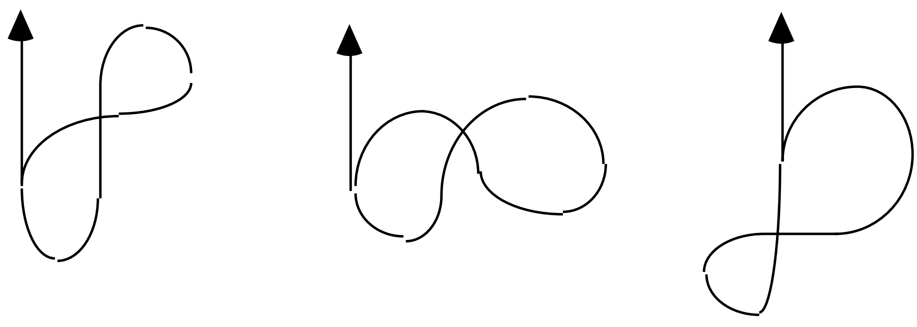
where $a = (a_1, a_2)$ and $a_2\colon \mathbf{I} \to \mathbf{R}^2 \setminus \{0\}$. It gives a winding-number homomorphism (of semigroups)

(5) $w\colon R^1\Pi_1(X)(j, j) \to \mathbf{Z}$, $\qquad [a] \mapsto w(a')$,

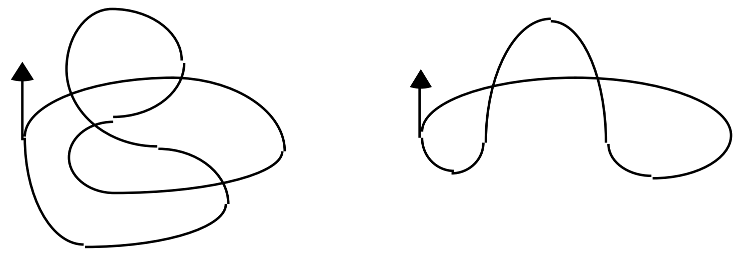
and it suffices to prove that this homomorphism is an isomorphism.

In fact, the semigroup $R^1\Pi_1(X)(j, j)$ is generated by two classes $[a], [b]$ with winding number 1 and -1, respectively. This proves that w is surjective.

But these two classes commute: $[a] + [b] = [b] + [a]$ (with winding number 0), as is shown by the following sequence of pictures



Moreover, $[b] + [a] + [b] = [a]$ (and, symmetrically, $[a] + [b] + [a] = [b]$), as proved by:



It follows that, for $n > 0$:

$$[b] + n[a] + [b] \;=\; (n-1)[a] + [b] + [a] + [b] \;=\; n[a],$$

and every class is equivalent to $n[a]$, or $n[b]$, or $0_j = [a] + [b] = [b] + [a]$ $(n > 0)$. Therefore, the homomorphism w is bijective (and an isomorphism of groups). □

5.4. Theorem (Versors in a manifold). *Let $X \subset \mathbf{R}^m$ be a C^1-embedded manifold of dimension ≥ 2. Then $R^1\Pi_1(X)$ is a groupoid, and the canonical comparison semifunctor (5.1.4)*

(1) $\mathbf{u}: R^1\Pi_1(X) \to \Pi_1(T^*X), \qquad [a] \mapsto [\hat{a}] = [(a, a')],$

is an isomorphism of groupoids.

Proof. Again, since f is the identity on the objects, it is sufficient to prove that the canonical *semi*functor (1) is full and faithful. (But Lemma 5.2 would be of no real help here.)

(a) To prove that **u** is full, let us fix two vectors $j, j' \in T^*X$ and a path $b: j \to j'$ in the graph $P(T^*X)$, with projection $a: x \to x'$ in the graph PX; notice that a is just a continuous map $\mathbf{I} \to X$.

For $s \in]0, 1]$, we write $b_s: j \to b(s)$ the restriction of b to the interval $[0, s]$, reparametrised on $\mathbf{I}$, namely $b_s(t) = b(st)$ for $0 \leq t \leq 1$. Then we let

(2) $A = \{s \in]0, 1] \mid [b_s] \in \mathbf{u}(R^1\Pi_1(X)(j, b(s))\},$

and we have to prove that $1 \in A$.

First, the set A is not empty, because there exists a neighbourhood U of x in X that is C^1-diffeomorphic to a space $\mathbf{R}^n$ of dimension ≥ 2; if s is sufficiently small, b_s is a path in T^*U, and $\mathbf{u}: R^1\Pi_1(U) \to \Pi_1(T^*U)$ is full (and faithful), by Lemma 5.3.

Let $s_0 = \sup A \leq 1$, and let us prove that $s_0 \in A$. Choose a neighbourhood U of $x_0 = a(s_0)$ with the same property as above, and some $s_1 \in A$ such that $a(s_1) \in U$. Then b_{s_1} ends at a vector $j_1 = b_{s_1}(1) = b(s_1) \in T^*U$ and $[b_{s_1}] \in \mathbf{u}(R^1\Pi_1(X)(j, j_1)$. But b stays in T^*U on some interval $[s_1, s_2]$ with $s_2 \geq s_0$, and this restriction can be replaced with a C^1-regular path in U, which can then be pasted to the one we already had on $[0, s_1]$, showing that $s_2 \in A$, and a fortiori $s_0 \in A$.

Moreover, $s_0 = 1$, otherwise in the previous argument we could take $s_2 > s_0$, and conclude $s_2 \in A$, a contradiction.

(b) Finally, to prove that f is faithful, let us take two paths $a, b\colon j \to j'$ in the graph R^1PX, such that $\mathbf{u}[a] = \mathbf{u}[b]$ in $\Pi_1(T^*X)$. This means that there exists a homotopy $\varphi\colon \mathbf{I}^2 \to T^*X$ such that

(i) $\varphi(-, 0) = (a, a')$, $\varphi(-, 1) = (b, b')$,

(ii) $(j^1\varphi(-, t))(0) = (a(0), a'(0)) = (b(0), b'(0))$, for all $t \in \mathbf{I}$,

(iii) $(j^1\varphi(-, t))(1) = (a(1), a'(1)) = (b(1), b'(1))$, for all $t \in \mathbf{I}$.

Notice that the intermediate paths $\varphi_\tau = \varphi(t, -)\colon \mathbf{I} \to T^*X$, between $\varphi_0 = (a, a')$ and $\varphi_1 = (b, b')$, have a projection on X which need not even be of class C^1. We let A be the set of $s \in]0, 1]$ such that

- there exists a homotopy $\varphi\colon \mathbf{I}^2 \to T^*X$ with fixed end jets, whose projection $p\varphi\colon \mathbf{I}^2 \to X$ restricts to a C^1-regular homotopy $a_s \to b_s$,

where, without reparametrisation, a_s and b_s are the restrictions of a, b to $[0, s]$.

Again, it is sufficient to prove that $1 \in A$. The proof is similar to the previous one, and we only write down its beginning. The set A is not empty, because there exists a neighbourhood U of $x = a(0) = b(0)$ in X that is C^1-diffeomorphic to a space $\mathbf{R}^n$ of dimension ≥ 2; if s is sufficiently small, the paths a_s and b_s are in T^*U, and one can modify φ so that the restriction $(p\varphi)_s\colon [0, s] \times \mathbf{I} \to X$ is a C^1-regular homotopy. □

5.5. The circle and other curves. In dimension 1, this comparison need not be full, even for a manifold, namely the circle $\mathbf{S}^1 \subset \mathbf{R}^2$.

Let us fix the versor $j = (1, 0) + (0, 1).\tau$ of $UT\mathbf{S}^1 \subset T\mathbf{R}^2$. Then

(1) $R^1\pi_1(\mathbf{S}^1, j) = \mathbf{N}^*$,

is the additive semigroup of positive integers, properly contained in

(2) $\pi_1(UT\mathbf{S}^1, j) = \pi_1(\mathbf{S}^1 \times \mathbf{S}^0, j) = \mathbf{Z}$.

One can start from the standard path that turns around the circle n times (with λ

$= 2n\pi$)

(3) $a_n(t) = (\cos(\lambda t), \sin(\lambda t)), \qquad t \in \mathbf{I}$.

This has end-jets $(j_1a)(0) = (j_1a)(1) = (1, 0) + (0, \lambda).\tau$. Therefore, it is sufficient to reparametrise it as $a_n\varphi$, by a diffeomorphism $\varphi: \mathbf{R} \to \mathbf{R}$ with

$$\varphi(0) = 0, \qquad \varphi(1) = 1, \qquad \varphi'(t) > 0 \ (t \in \mathbf{R}), \qquad \varphi'(0) = \varphi'(1) = 1/\lambda.$$

It is not difficult to prove that $R^k\pi_1(\mathbf{S}^1, j) = \mathbf{N}^*$ holds for all $k \geq 1$ and $j \in T_k\mathbf{S}^1$, provided - obviously - that the coefficient of j of degree 1 is not null.

The spaces $\mathbf{E}_k, \mathbf{F}_k$ (1.1) also give free semigroups $R^k\pi_1(-, j)$, which is easy to compute.

5.6. The sphere. By theorem 5.4

(1) $R^1\pi_1(\mathbf{S}^2, j) = \pi_1(V\mathbf{S}^2, j)$.

This fundamental group can be easily computed with the van Kampen Theorem: $\pi_1(UT\mathbf{S}^2, j) = \mathbf{Z}_2$. But it is also easy to see directly that $R^1\pi_1(\mathbf{S}^2, j) = \mathbf{Z}_2$, since the stereographic embedding $f: \mathbf{R}^2 \to \mathbf{S}^2$ induces a surjective homomorphism of semigroups (hence of groups)

(2) $f_*: R^1\pi_1(\mathbf{R}^2, j) \to R^1\pi_1(\mathbf{S}^2, j)$,

which identifies the generator [a] with its opposite [b] (in the notation of the proof of Lemma 5.3).

6. Tolerance relations

We end with a more complete study of sets equipped with a tolerance relation, and their category.

6.1. Limits and colimits. Recall that a tolerance set X is a set equipped with a tolerance relation $x!y$, reflexive and symmetric. A tolerance morphism $f: X \to Y$ is a mapping between such sets which preserves the tolerance relation.

The category **Tol** of tolerance sets and morphisms is complete and cocomplete, with limits and colimits created by the forgetful functor U: **Tol** → **Set**. In particular, we have the following basic cases:

(a) the *product* ΠX_i is the product of the underlying sets, with $(x_i)!(y_i)$ if and only if, for all indices i, $x_i \,!\, y_i$ in X_i;

(b) the *equaliser* of $f, g: X \to Y$ is the equaliser $E = \{x \in X \mid fx = gx\}$ in **Set**,

with the restricted tolerance relation;
(c) the *sum* ΣX_i is the sum of the underlying sets, with $x!y$ if and only if this holds in one subset X_i;
(d) the *coequaliser* of $f, g: X \to Y$ is the coequaliser $E' = Y/R$ in **Set** (R is the equivalence relation of Y generated by $fx \sim gx$, for $x \in X$), equipped with the finest tolerance relation making the projection $Y \to Y/R$ a tolerance map; in other words, $[x]![y]$ if and only if $x'!y'$ for some $x' \in [x]$ and $y' \in [y]$.

The following example will be referred to as the *test-case*: X is the union of the three coordinate planes of $\mathbf{R}^3$, and $x!y$ means that x and y are equal or lie in *one* such plane.

6.2. Tensor product and Hom. Our category **Tol** has a monoidal closed structure, with *tensor product* $X \otimes Y$ given by the cartesian product of the underlying sets, equipped with a tolerance which is *finer* than the cartesian one:

(1) $(x, y) ! (x', y')$ if $(x!x'$ and $y = y')$ or $(x = x'$ and $y!y')$.

The identity is the terminal object $\top = \{*\}$, which acts under the tensor product as under product.

The internal hom-functor

(2) $\mathrm{Hom}: \mathbf{Tol}^{\mathrm{op}} \times \mathbf{Tol} \to \mathbf{Tol}$,

is obtained by equipping the set $\mathbf{Tol}(X, Y)$ with the *pointwise* tolerance relation:

(3) $f!g \quad \Leftrightarrow \quad$ (for all $x \in X$, $fx ! gx$ in Y).

Now, it is trivial to verify that the exponential law in **Set** restricts to an isomorphism:

(4) $\mathrm{Hom}(X \otimes Y, Z) \to \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)), \quad f \mapsto (x \mapsto f(x, -))$.

A *tolerance category* **A** will be a category enriched over the monoidal closed category **Tol**. This simply means that **A** is equipped with a binary relation ! between parallel maps, which is reflexive, symmetric and consistent with composition in the following weak sense

(5) if $g!g'$ then $hgf ! hg'f$ (whenever the composition makes sense).

A will be said to be a *cartesian tolerance* category if the following stronger condition holds

(6) if $f!f'$ and $g!g'$ then $gf ! g'f'$,

corresponding to enrichment with respect to the *cartesian* structure of **Tol**.

For instance, **Tol** itself is a tolerance category, but not a cartesian one. On the other hand, any cohesive category [G1, G2] satisfies the cartesian condition: for instance, the category of sets and partial mappings, where f!f' means that the partial mappings $f, f': X \to Y$ coincide on the elements of X on which they are both defined.

6.3. Club-structures. An equivalence relation over a set X can be equivalently assigned by means of a partition. Extending this well-known fact, a tolerance relation ! over X can be equivalently assigned by means of a *club-structure*, i.e. a set $\mathcal{A} \subset \mathcal{P}X$, whose elements will be called *clubs* of X (in the *test-case* 6.1, the clubs are the three coordinate planes).

Clubs must satisfy the following axioms:

(a) $\mathcal{A}$ is a covering of X (every point lies in a club);

(b) if $A \subset X$, and every pair $a, a' \in A$ lies in a common club, then A is contained in a club;

(c) if $A \subset B$ are clubs, then $A = B$.

No club can be empty (unless $X = \emptyset$). Moreover, if $A \in \mathcal{A}$, $x \in X$ and every $a \in A$ lies in a club containing x, then $x \in A$ (because $A \cup \{x\}$ must be contained in some club, which has to coincide with A). More generally:

(d) if $A \subset B \subset X$, A is a club and all pairs of points of B lie in a common club, then $A = B$.

The bijective correspondence between our two notions is given by:

(1) $(X, !) \mapsto (X, \mathcal{A})$, the clubs being the maximal subsets of X which are pairwise !-linked,

(2) $(X, \mathcal{A}) \mapsto (X, !)$, where x!y if and only if x, y belong to a common club.

First, note that (1) is well defined by Zorn's lemma: every pairwise !-linked subset of X is contained in a maximal one. Now, it is obvious to verify that $(X, !) \mapsto (X, \mathcal{A}) \mapsto (X, !')$ produces a tolerance relation !' that coincides with !. On the other hand, consider the procedure $(X, \mathcal{A}) \mapsto (X, !) \mapsto (X, \mathcal{A}')$; if $A \in \mathcal{A}$, A is pairwise!-linked and therefore is contained in some maximal !-linked subset $A' \in \mathcal{A}'$, which has to coincide with A by (d). Conversely, if $A' \in \mathcal{A}'$, then A' is (maximal) !-linked and contained in some club $A \in \mathcal{A}$ (because of (b)); this also is !-linked, by definition of !, whence it coincides with A'.

In this correspondence, a *map* $f: X \to Y$ of club-sets is obviously a mapping of sets taking each club of X into some club of Y.

6.4. The associated equivalence relation. A tolerance set X usually contains a great redundancy, which can be cut out (as we have already seen in 2.1), much in the same way as in the procedure turning a preordered set into the associated ordered set.

For every point $x \in X$, the *star* of x will be

(1) $st(x) = \{z \in X \mid x \,!\, z\}$ = union of the clubs containing x.

The equivalence relation *associated* to the link ! is produced by the mapping $st: X \to \mathcal{P}X$

(2) $x \sim y \;\Leftrightarrow\; st(x) = st(y),$
$\Leftrightarrow$ for every $z \in X$, $z!x \Leftrightarrow z!y$,
$\Leftrightarrow$ the clubs containing x coincide with the ones containing y.

The quotient set $red(X) = X/\sim$ corresponds thus, bijectively, to the set of stars of X, but should not be confused with the latter; the stars of X form a partition if and only if the link of X is an equivalence relation, in which case clubs and stars coincide. The set red(X) has an induced tolerance relation

(3) $[x] \,!\, [y] \;\Leftrightarrow\; x!y$ (independently of the choice of representatives),

that determines the one of X and is *reduced*, in the sense that its associated equivalence relation is the identity (cf. 2.1). Let us recall that the procedure of reduction is not functorial: a tolerance map $f: X \to Y$ need not preserve the equivalence relation associated to the tolerance relation.

In the test-case, the star of the origin is X itself; the star of each other point of an axis is the union of its two coordinate planes; the star of each other point is its coordinate plane. There are 7 equivalence classes: the origin [0], the three axes without the origin $[e_i]$, the three coordinate planes without their axes $[e_i + e_j]$ $(i \neq j)$.

6.5. The associated preorder. A tolerance set X has also an *associated preorder*

(1) $x \prec y \;\Leftrightarrow\; st(x) \supset st(y),$
$\Leftrightarrow$ for every $z \in X$, $z!y \Rightarrow z!x$,
$\Leftrightarrow$ x belongs to each club containing y.

It determines the associated equivalence relation $x \sim y$ (as $x \prec y$ and $y \prec x$). Thus, the quotient $red(X) = X/\sim$ is an ordered set (anti-isomorphic to the ordered set of stars)

(2) $[x] \leq [y]$ if $st(x) \supset st(y)$.

We say that x is a *maximal* element of X if st(x) is a club, if and only if x belongs to a unique club, if and only if (y!x!z ⇒ y!z), if and only if x is maximal in the associated preorder. In our test-case, $[0] < [e_i] < [e_i+e_j]$; the maximal elements of X are all the points which do not lie on some axis.

On the other hand, each preordered set $(X, \prec)$ has an associated tolerance

(3) x ! y if x, y have a common upper bound z $(x \prec z,\ y \prec z)$.

We say that a tolerance set (X, !) is *of preorder type* if these two procedures yield back the original tolerance relation; or equivalently if the link ! satisfies

(4) if x!y then there exists some z whose star is contained in $st(x) \cap st(y)$.

In fact, the converse implication always hold (if $st(z) \subset st(x) \cap st(y)$, then z!x, whence $x \in st(z) \subset st(y)$).

Our test case is of preorder type, whereas the tolerance $|x - y| < 1$ in **R** is not; its clubs are the open intervals of length 1, while $st(x) =]x - 1, x + 1[$.

6.6. Pointed tolerance relations. A *pointed tolerance set* is a pointed set $X = (X, 0_X)$ equipped with a tolerance relation such that $x ! 0_X$, for all $x \in X$.

Equivalently, all the clubs of X contain the base point. A morphism has to respect both structures. This defines the category **Tol.** of *pointed tolerance sets* (or pointed club-sets). Again, it is complete and cocomplete and has a canonical monoidal closed structure.

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Dipartimento di Matematica
Università di Genova
via Dodecaneso 35
16146 Genova, Italy
grandis@dima.unige.it

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A CHARACTERIZATION OF FINITE COCOMPLETE HOMOLOGICAL AND OF SEMI-ABELIAN CATEGORIES

by Manfred HARTL and Bruno LOISEAU

Abstract

Semi-abelian and finitely cocomplete homological categories are characterized in terms of four resp. three simple axioms, in terms of the basic categorical notions introduced in the first few chapters of MacLane's classical book.

Résumé

Les catégories semi-abéliennes et homologiques finiment cocomplètes sont définies en quatre (respectivement trois) axiomes simples exprimés en termes de notions catégoriques de base introduites dans les premiers chapitres du livre classique de MacLane.

Key words: Homological categories, exact categories, semi-abelian categories
MSC: 18E10,18A30

After a long development, the notion of semi-abelian category as introduced in [5] turns out to be kind of a best-possible generalization of the classical notion of abelian category, in that it both includes a maximum of interesting examples and provides a convenient general framework for many concepts, in particular such arising from group theory, homological and homotopical algebra, see for instance [1], [3], [4], [8]. But also the weaker notion of homological category already allows for many interesting properties [2], in particular when combined with finite cocompleteness where it supports an alternative approach to actions, commutators, modules and crossed modules as is shown in [6] and in forthcoming joint work with Van der Linden.

So the aim of this note is to provide a short and very elementary definition of the notions of semi-abelian and finitely cocomplete homological category, only in terms of the most basic concepts found in

MacLane's classical book [7]. We emphasize that this does not mean an attempt of relapsing into "old-style" category theory but originates from purely pedagogical motivations: namely, to advertize semi-abelian category theory to non-category theorists, in particular to group theorists and algebraic topologists (such as the first author) who may definitely find strong interest in these and related further developments once the presentation of the basic notions appears sufficiently familiar to be easily understood.

Consider the following axioms about a category $\mathbb{C}$, where we denote by kernel or cokernel the corresponding injection or projection arrow, resp.

A1. $\mathbb{C}$ is pointed, finitely complete and cocomplete;

A2. For any split epimorphism $p : X \to Y$ with section $s : Y \to X$ and with kernel $k : K \hookrightarrow X$, the arrow $\langle k, s\rangle : K + Y \to X$ is a cokernel;

A3. The pullback of a cokernel is a cokernel;

A4. The image of a kernel by a cokernel is a kernel.

Note that when $\mathbb{C}$ is the category of groups, axiom A2 says that a semi-direct product group $G = N \rtimes T$ is generated by N and T. Note also that the mere formulation of Axiom A4 supposes that the category is regular, but this follows from Axioms A1, A2 and A3, as will be shown.

Theorem. Let $\mathbb{C}$ be a category. Then:

(1) $\mathbb{C}$ is finitely cocomplete homological if and only if it satisfies the three axioms A1-A3, or equivalently A'1-A2-A3, where A'1 is the axiom
A'1. *$\mathbb{C}$ is pointed, finitely complete, has binary sums and has coequalizers of equivalence relations;*

(2) $\mathbb{C}$ is semi-abelian iff it satisfies all the four axioms A1-A4.

Proof. (1): First note that condition A2 is stronger than condition PM_0^+ of [5] (which asserts that the factorization is an extremal epimorphism). Moreover, for a finitely complete pointed category with finite sums, 3.1 and 3.2 of [5] show that the latter condition is equivalent to protomodularity. Since moreover in any finitely complete pointed protomodular category any regular epimorphism is the cokernel of its kernel (see e.g.

[2, Proposition 3.1.23]), the axiom A3 is equivalent to stability by pullbacks of regular epimorphisms, and A1 of course implies existence of coequalizers of kernel pairs. So any finitely cocomplete homological category satisfies A1-A3, and any category which satisfies those axioms is regular, pointed and homological, and it just remains to show that it is finitely cocomplete, i.e. that any parallel pair of morphisms has a coequalizer since by A1 $\mathbb{C}$ has finite coproducts. But the analysis of the proof given in [2, Proposition 5.1.3] of the fact that this holds in any semi-abelian category, shows that the effectiveness hypothesis for equivalence relations is only used to show that some equivalence relation has a coequalizer, hence it can be replaced by the hypothesis that any equivalence relation has one.
(2): In [5, 3.7], one finds a proof that semi-abelianness is equivalent to the following axiom system:

- (SA'1=SA1) $\mathbb{C}$ has binary products and sums and a zero object;
- (SA'2=SA2) $\mathbb{C}$ has pullbacks of (split) monomorphisms;
- (SA'3) $\mathbb{C}$ has cokernels of kernels, and every morphism with zero kernel is a monomorphism;
- (SA'4=SA4) The (split) Short Five Lemma holds true in $\mathbb{C}$;
- (SA'5) = our A3;
- (SA'6)= our A4.

By (1) above, Axioms A1-A3 hold in any semi-abelian category, and so does A4 since it is SA'6. Conversely, Axioms A1-A4 obviously imply Axioms SA'1 and SA'2, and it follows again from [5, Paragraphs 3.1 and 3.2] that they imply SA4 (at least in its weaker form), i.e. imply the protomodularity of the category. A1 obviously implies the first part of SA'3. Finally, it remains to show that A1-A4 imply its second part, i.e. that every morphism with zero kernel is a monomorphism. But this is true in any pointed protomodular category with finite limits [2, Proposition 3.1.21].

□

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Address of the authors:
Hartl M[1235], Loiseau B[1234]
[1] Univ Lille Nord de France, F-59000 Lille, France
[2] UVHC, LAMAV, ISTV2, Mont Houy, F-59313 Valenciennes Cedex 9, France
[3] CNRS, FR 2956, F-59655 Villeneuve d'Ascq, France
[4] UArtois, IUFM, F-59313 Valenciennes, France
[5] Corresponding author - `manfred.hartl@univ-valenciennes.fr`