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OBJECTIVE CATEGORIES AND SCHEMES

by Wolfgang RUMP

Dedicated to B. V. M.

RESUME. Dans ce travail nous considérons les faisceaux quasi-cohérents sur un schéma comme des modules sur une catégorie "objective". On montre que la catégorie **Obj** des catégories objectives est duale de la catégorie des schémas. Nous exhibéons **Obj** comme une sous-catégorie pleine reflexive de la catégorie **POb** (catégories préobjectives) dont les objets sont des foncteurs contravariants d'un ensemble ordonné dans la catégorie des anneaux commutatifs tandis que les morphismes de **POb** sont relatifs à la structure responsable de la génération des schémas. De cette façon, la définition des morhismes des schémas prend une forme assez simple comme foncteurs entre des catégories objectives qui préservent la structure pertinente. Le résultat principal est une reconstruction des schémas plus explicite que celle due à Rosenberg (Noncommutative schemes, Compos. Math. 112 (1998), 93-125).

Abstract. Quasi-coherent sheaves over a scheme are regarded as modules over an *objective* category. The category **Obj** of objective categories is shown to be dual to the category of schemes. We exhibit **Obj** as a reflective full subcategory of a category **POb** (pre-objective categories) whose objects are contravariant functors from a poset to the category of commutative rings while the morphisms of **POb** take care of the structure responsible for the generation of schemes. In this context, morhisms of schemes just turn into functors between objective categories preserving the relevant structure. Our main result gives a more explicit version of Rosenberg's reconstruction of schemes (Noncommutative schemes, Compositio Math. 112 (1998), 93-125).

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Introduction

The most natural approach toward non-commutative algebraic geometry is based on suitable categories generalizing the abelian category $\mathbf{Qcoh}(X)$ of quasi-coherent sheaves over a scheme X. After Gabriel's reconstruction [4] of noetherian schemes X in terms of $\mathbf{Qcoh}(X)$, this approach was fully justified by Rosenberg [11] who extends Gabriel's result to arbitrary schemes.

For an affine scheme X with structure sheaf $\mathscr{O}_X$, the category of quasi-coherent sheaves coincides with the module category $\mathbf{Mod}(R)$ over the ring $R = \mathscr{O}_X(X)$ of global sections. If R is non-commutative, R can be recovered from $\mathbf{Mod}(R)$ up to Morita equivalence, i. e. instead of R itself, the category $\mathbf{proj}(R)$ of finitely generated projective R-modules can be recovered from $\mathbf{Mod}(R)$. Moreover, the objects of $\mathbf{Mod}(R)$ are additive functors $\mathscr{C}^{\mathrm{op}} \to \mathbf{Ab}$, where $\mathscr{C}$ can be chosen to be either $\mathbf{proj}(R)$ or the one-object full subcategory $\{{}_RR\}$ of $\mathbf{proj}(R)$ which can be identified with the ring R.

If the scheme X is non-affine, a reconstruction via projectives fails dramatically, even in the most simple case of a projective line X, where non-zero projectives in $\mathbf{Qcoh}(X)$ no longer exist. Nevertheless, the affine case suggests that it should be possible to associate a category $\mathscr{O}$ to any scheme X such that quasi-coherent sheaves over X become certain modules over $\mathscr{O}$. In the present article, we define a category $\mathbf{Obj}$ of such categories $\mathscr{O}$ and prove that $\mathbf{Obj}$ is dual to the category of schemes. Since objects of $\mathscr{O}$ play a particular rôle, we call the categories $\mathscr{O} \in \mathbf{Obj}$ *objective*.

More generally, we introduce *pre-objective* categories as a class of small skeletal preadditive categories $\mathscr{O}$. We stick to the classical case and assume that the endomorphism rings $\mathscr{O}(U)$ in $\mathscr{O}$ are commutative. The two axioms (O1) and (O2) for a pre-objective category $\mathscr{O}$ are based on the concept of *short monomorphism* $i\colon U \to V$, which means that every morphism $f\colon U \to V$ is of the form $f = ig$ for some $g \in \mathscr{O}(U)$. Then (O1) states that there is a short monomorphism $U \to V$ for any pair of objects U, V with $\mathrm{Hom}_{\mathscr{O}}(U,V) \neq 0$, and (O2) asserts that short monomorphisms are closed under composition. In other words, (O1) and (O2) state that the short monomorphisms form a subcategory which is fibered over a partially ordered set on $\mathrm{Ob}\,\mathscr{O}$. Using the relationship between fibered categories and pseudo-functors [6], a pre-objective category can be conceived as a functor $\rho\colon \Omega^{\mathrm{op}} \to \mathbf{CRi}$ from a partially ordered set Ω into the category $\mathbf{CRi}$ of commutative rings, together with a cohomology class $\gamma \in H^2(\Omega, \rho^\times)$.

This cohomology class γ vanishes for pre-objective categories $\mathscr{O}$ with a greatest object (Proposition 3), which are just our concern here. To make such pre-objective categories $\mathscr{O}$ into a suitable category $\mathbf{POb}$, we introduce the concept of *short limit* $S(f)$ of an endomorphism $f \in \mathscr{O}(U)$ in $\mathscr{O}$ and call an $\mathscr{O}$-module M is *quasi-coherent* if $M\colon \mathscr{O}^{\mathrm{op}} \to \mathbf{Ab}$ respects short limits of arbitrary endomorphisms in $\mathscr{O}$. On the other hand, we say that an object $U \in \mathrm{Ob}\,\mathscr{O}$ is *affine* if the representable functor $\mathrm{Hom}_{\mathscr{O}}(-,U)$ respects short limits of endomorphisms of U and if U satisfies two other properties related to the partially ordered structure of $\mathrm{Ob}\,\mathscr{O}$. Now a *morphism* between pre-objective categories is just a functor $F\colon \mathscr{O} \to \mathscr{O}'$ which respects the relevant structure of $\mathscr{O}$, namely, short monomorphisms, short limits, finite meets, and joins of affine objects - as far as they exist.

We call a pre-objective category $\mathscr{O}$ *objective* if joins of objects exist and the full subcategory $\mathscr{O}_{\mathrm{aff}}$ of affine objects is dense [9] in $\mathscr{O}$. The latter categorical property is closely related to the recollement of schemes. With the benefit of hindsight, our categorification of schemes appears to be quite natural and almost inevitable. As already indicated above, we prove that the full subcategory $\mathbf{Obj} \subset \mathbf{POb}$ of objective categories is dual to the category of schemes (Theorem 2). In this context, the awkward definition of morphisms between schemes takes a more pleasant form. Recall that such a morphism consists of a continuous map $\varphi\colon X \to Y$ between the base spaces together with a morphism $\vartheta\colon \mathscr{O}_Y \to \varphi_*\mathscr{O}_X$ of sheaves into the opposite direction which induces a local ring homomorphism $\vartheta_x^\#\colon \mathscr{O}_{Y,\varphi(x)} \to \mathscr{O}_{X,x}$ at the stalks. By contrast, morphisms between objective categories are just functors which respect the structure that ought to be respected.

Note that as a (dual) objective category, a scheme is nothing else than a pre-objective category with joins and enough affines, while a pre-objective category is given by a fibering of unit groups of commutative rings over a poset. Conversely, we associate a scheme to any pre-objective category (Theorem 1) and show that the full subcategory **Obj** of **POb** is reflective (Theorem 3). Thus in a sense, schemes are related to pre-objective categories like sheaves are related to presheaves.

Within this framework, a quasi-coherent sheaf over a scheme with corresponding objective category $\mathscr{O}$ becomes an $\mathscr{O}_{\mathrm{aff}}$-module which respects short limits. An application to flat covers (see [3]) will be reserved to a subsequent publication. Here we just give a brief discussion of Rosenberg's result [11] and show how to reconstruct an objective category $\mathscr{O}$ from the abelian category $\mathbf{Qcoh}(\mathscr{O})$ of quasi-coherent $\mathscr{O}_{\mathrm{aff}}$-modules in a quite explicit way. More generally, we associate an objective category $\mathscr{O}_{\mathscr{A}}$ to any abelian category $\mathscr{A}$ such that $\mathscr{O}_{\mathscr{A}} \cong \mathscr{O}$ in the special case $\mathscr{A} = \mathbf{Qcoh}(\mathscr{O})$. To this end, a *point* of an abelian category $\mathscr{A}$ is defined to be a quasi-injective object P with $\mathrm{End}_{\mathscr{A}}(P)$ a field such that every non-zero subobject of P generates P. In contrast to Gabriel's reconstruction of schemes which makes use of injective objects, we confine ourselves to quasi-injectives. If R is a commutative ring, the points of $\mathbf{Mod}(R)$ correspond to the prime ideals of R. Using points of $\mathscr{A}$, we introduce objects E of *finite type*, and to any such E, we associate a subset U_E of the set $\mathrm{Spec}\,\mathscr{A}$ of points. Then the U_E define a topology on $\mathrm{Spec}\,\mathscr{A}$, and every open set U in $\mathrm{Spec}\,\mathscr{A}$ gives rise to a Serre subcategory $\mathscr{T}_U$ of $\mathscr{A}$. If $\mathscr{O}_{\mathscr{A}}(U)$ denotes the center of the abelian quotient category $\mathscr{A}/\mathscr{T}_U$, we obtain a pre-objective category $\mathscr{O}_{\mathscr{A}}$ which is objective and isomorphic to $\mathscr{O}$ whenever $\mathscr{A} = \mathbf{Qcoh}(\mathscr{O})$ for a given $\mathscr{O} \in \mathbf{Obj}$.

1 Short limits

Let $\mathscr{C}$ be a category. We call a monomorphism $i\colon X \to Y$ *short* if every morphism $f\colon X \to Y$ is of the form $f = ig$ for some endomorphism

g of X. Clearly, a short monomorphism $X \to Y$ is unique up to an automorphism of X. So we could speak of a *short subobject* X of Y.

Consider a functor $C\colon \mathscr{I} \to \mathscr{C}$ with $\mathscr{I}$ small. We define a *short cone* c over C to be a collection of short monomorphisms $c_i\colon X \to C_i$ (with $i \in \operatorname{Ob}\mathscr{I}$) such that for every $\alpha\colon i \to j$ in $\mathscr{I}$ there is a commutative square

$$\begin{array}{ccc} X & = & X \\ \downarrow{\scriptstyle c_i} & & \downarrow{\scriptstyle c_j} \\ C_i & \xrightarrow{C_\alpha} & C_j \end{array} \tag{1}$$

in $\mathscr{C}$. We call X (together with c) a *short limit* of C if every short cone $c'\colon X' \to C$ factors uniquely through c, i. e. there is a unique $f\colon X' \to X$ with $c'_i = c_i f$ for all $i \in \operatorname{Ob}\mathscr{I}$. We denote it by $\underleftarrow{\text{shlim}}\, C_i$. For $\mathscr{I} = \varnothing$, a short limit $\underleftarrow{\text{shlim}}\, \varnothing$ is just a terminal object.

Example 1. For a ring R, the short submodules of an R-module M are fully invariant, and every sum of short submodules of M is again a short submodule. So the short submodules form a complete lattice. The short limit $\underleftarrow{\text{shlim}}\, M_i$ of a non-empty family of submodules $M_i \subset M$ is given by the largest submodule L of $\bigcap M_i$ such that every inclusion $L \hookrightarrow M_i$ is a short monomorphism.

Definition 1. We call a preadditive category $\mathscr{C}$ *commutative* if the ring $\operatorname{End}_{\mathscr{C}}(X)$ is commutative for each $X \in \operatorname{Ob}\mathscr{C}$.

As usual, we regard a partially ordered set Ω as a small skeletal category with at most one morphism $a \to b$ between any pair of objects. If such a morphism exists, we write $a \leqslant b$. An ordinal λ will be regarded as a well-ordered set

$$\lambda = \{\alpha \in \mathbf{Ord} \mid \alpha < \lambda\},$$

i. e. a full subcategory of the category **Ord** of all ordinals.

For a preadditive category $\mathscr{C}$, every morphism $f\colon X_0 \to X_1$ in $\mathscr{C}$ gives rise to a functor $2 \to \mathscr{C}$, also denoted by f. The short limit $S(f)$

of an endomorphism $f\colon X \to X$ (viewed as a functor $2 \to \mathscr{C}$) is given by a commutative diagram

$$\begin{array}{ccc} S(f) & & \\ \big\downarrow i & \searrow j & \\ X & \xrightarrow{f} & X \end{array} \tag{2}$$

with short monomorphisms i, j. Thus $j = if^{\times}$ with an automorphism $f^{\times}$ of $S(f)$. This gives a commutative square

$$\begin{array}{ccc} S(f) & \xrightarrow{f^{\times}} & S(f) \\ \big\downarrow i & & \big\downarrow i \\ X & \xrightarrow{f} & X, \end{array} \tag{3}$$

and $S(f) = \underleftarrow{\text{shlim}}\, f$ means that every short monomorphism $i'\colon Y \to X$ with $fi' = i'e$ for some automorphism e of Y factors through i. We call $S(f)$ the *support* of f. If $\mathscr{C}$ is commutative and skeletal, the automorphism $f^{\times}$ in (3) is unique. In fact, if we replace i by ie with an isomorphism $e\colon Y \xrightarrow{\sim} S(f)$, then $Y = S(f)$, and thus $f \cdot ie = if^{\times}e = ie \cdot f^{\times}$.

Regarding $\mathbb{Z}$ as a partially ordered set, let $f^{\mathbb{Z}}\colon \mathbb{Z} \to \mathscr{C}$ denote the functor with $f^{\mathbb{Z}}(n) := X$ and $f^{\mathbb{Z}}(n \to n+1) := f$, i. e. the diagram

$$\cdots \to X \xrightarrow{f} X \xrightarrow{f} X \to \cdots \tag{4}$$

in $\mathscr{C}$. Then

$$S(f) = \underleftarrow{\text{shlim}}\, f^{\mathbb{Z}}. \tag{5}$$

We denote the natural morphism $S(f) \to f^{\mathbb{Z}}(0)$ by i_f. So we have a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & S(f) & \xrightarrow{f^{\times}} & S(f) & \xrightarrow{f^{\times}} & S(f) & \longrightarrow & \cdots \\ & & \big\downarrow & & \big\downarrow i_f & & \big\downarrow & & \\ \cdots & \longrightarrow & X & \xrightarrow{f} & X & \xrightarrow{f} & X & \longrightarrow & \cdots \end{array} \tag{6}$$

Definition 2. We define a commutative preadditive category $\mathscr{O}$ to be *pre-objective* if it is small and skeletal such that the following hold.

(O1) For every pair of objects U, V with $\mathrm{Hom}_{\mathscr{O}}(U, V) \neq 0$, there exists a short monomorphism $U \to V$.

(O2) The set of short monomorphisms is closed under composition.

Note that the factorization in (O1) is unique up to isomorphism, i. e. if $U \to U \xrightarrow{j} V$ is a second factorization of f, we have a commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & U & \xrightarrow{i} & V \\ \| & & \wr\downarrow e & & \| \\ U & \longrightarrow & U & \xrightarrow{j} & V \end{array}$$

with an automorphism e. In what follows, we write $\mathscr{O}(U)$ instead of $\mathrm{End}_{\mathscr{O}}(U)$ for an object U of $\mathscr{O}$.

Proposition 1. *Every short monomorphism $i\colon U \to V$ in a pre-objective category $\mathscr{O}$ defines a ring homomorphism $\rho^V_U\colon \mathscr{O}(V) \to \mathscr{O}(U)$ given by a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{f|_U} & U \\ \downarrow i & & \downarrow i \\ V & \xrightarrow{f} & V \end{array} \tag{7}$$

where $f|_U := \rho^V_U(f)$ merely depends on $f \in \mathscr{O}(V)$ and $U \in \mathrm{Ob}\,\mathscr{O}$.

Proof. Let $f \in \mathscr{O}(V)$ be given. By (O1), the morphism fi has a factorization $fi\colon U \to U \xrightarrow{j} V$ with a short monomorphism j. Furthermore, $j = ie$ for some automorphism e of U. Hence fi factors through i, and so we get a commutative diagram (7). Since i is monic, $f|_U$ is unique. If i is replaced by $j = ie$, we have $fj = fie = i \cdot f|_U \cdot e = ie \cdot f|_U = j \cdot f|_U$. Thus $f|_U$ is determined by f and U. $\square$

For short monomorphisms $U \to V \to W$ in $\mathscr{O}$, the ring homomorphism of Proposition 1 satisfies

$$\rho^V_U \rho^W_V = \rho^W_U \ ; \quad \rho^U_U = 1_{\mathscr{O}(U)}. \tag{8}$$

Proposition 2. *Let $i\colon U \to V$ and $j\colon V \to W$ be morphisms in a pre-objective category $\mathscr{O}$ such that ji is a short monomorphism. Then i is a short monomorphism. Every short monomorphism $U \to U$ is invertible.*

Proof. If $ji = 0$, then $U = 0$, and thus i is a short monomorphism. Otherwise, $i \neq 0$, and so there is a short monomorphism $i'\colon U \to V$ such that $i = i'e$ with an endomorphism e of U. Hence $ji' = ji \cdot f$ for some $f\colon U \to U$, and thus $ji \cdot fe = ji$, which gives $fe = 1$. Since $\mathscr{O}$ is commutative, e is invertible, whence i is a short monomorphism. The second assertion is trivial. □

Let $\mathscr{O}$ be a pre-objective category. For $U, V \in \operatorname{Ob}\mathscr{O}$, we write $U \leqslant V$ if there exists a monomorphism $U \to V$. By (O1), (O2), and Proposition 2, this makes $\operatorname{Ob}\mathscr{O}$ into a partially ordered set. In fact, if $U \leqslant V \leqslant U$, there are short monomorphisms $U \to V \to U$ by (O1). So $V \to U$ is a split epimorphism by Proposition 2, hence invertible.

If $U \in \operatorname{Ob}\mathscr{O}$ and $f \in \mathscr{O}(U)$, then a short limit $S(f)$ is equivalent to a greatest $V \leqslant U$ in $\operatorname{Ob}\mathscr{O}$ such that $f|_V$ is invertible. In other words, $S(f)$ exists if and only if the join $U_f := \bigvee\{V \leqslant U \mid \rho^U_V(f) \in \mathscr{O}(V)^\times\}$ exists and $f|_{U_f}$ is invertible.

For a pre-objective category $\mathscr{O}$, the subset $\operatorname{sh}(\mathscr{O})$ of short monomorphisms can be regarded as a fibered category [6] over the partially ordered set $\operatorname{Ob}\mathscr{O}$ such that the fiber over each $U \in \operatorname{Ob}\mathscr{O}$ consists of a single object, and every morphism in $\operatorname{sh}(\mathscr{O})$ is cartesian. In other words, $\operatorname{sh}(\mathscr{O}) \to \operatorname{Ob}\mathscr{O}$ is a linear extension in the sense of Baues and Wirsching [1].

By Eqs. (8), a pre-objective category $\mathscr{O}$ gives rise to a functor

$$\rho\colon \Omega^{\mathrm{op}} \to \mathbf{CRi} \tag{9}$$

from the dual of the partially ordered set $\Omega := \operatorname{Ob}\mathscr{O}$ to the category **CRi** of commutative rings. So we get a functor $\rho^\times\colon \Omega^{\mathrm{op}} \to \mathbf{Ab}$ into the category **Ab** of abelian groups which maps $U \in \Omega$ to the unit group $\rho(U)^\times$. Furthermore, we obtain a 2-cocycle. Namely, if we assign a short monomorphism $i_U^V\colon U \to V$ to each morphism $U \to V$ in Ω, any relation $U \leqslant V \leqslant W$ in Ω leads to an equation

$$i_V^W i_U^V = i_U^W \cdot c_{UVW} \tag{10}$$

with $c_{UVW} \in \mathscr{O}(U)^\times$. The associativity of composition yields

$$\rho_U^V(c_{VWY}) \cdot c_{UWY}^{-1} \cdot c_{UVY} \cdot c_{UVW}^{-1} = 1 \tag{11}$$

for $U \leqslant V \leqslant W \leqslant Y$, which means that c is a 2-cocycle with respect to the functor ρ. (In the terminology of [1], we have to regard $\rho^\times$ as a natural system which assigns $\rho(U)^\times$ to $U \leqslant V$.) If we set $i_U^U := 1_{\mathscr{O}(U)}$ for all $U \in \operatorname{Ob} \mathscr{O}$, then c will be *normalized*, i. e. it satisfies

$$c_{UUV} = c_{UVV} = 1. \tag{12}$$

If the i_U^V are replaced by $i_U^V \cdot d_{UV}$ with $d_{UV} \in \mathscr{O}(U)^\times$, then c changes by a 2-boundary. This leads to the following explicit description of pre-objective categories.

Proposition 3. *Up to isomorphism, there is a one-to-one correspondence between pre-objective categories and pairs (ρ, γ), where ρ is a functor* (9) *with a partially ordered set Ω such that $\rho(U) = 0$ for at most one U, and $\gamma \in H^2(\Omega, \rho^\times)$.*

Proof. Let (ρ, γ) be given. We define a pre-objective category $\mathscr{O}$ with $\operatorname{Ob} \mathscr{O} := \Omega$ as follows. For $U, V \in \Omega$, we set $\operatorname{Hom}_{\mathscr{O}}(U, V) = 0$ in case that $U \nleqslant V$. If $U \leqslant V$, we define $\operatorname{Hom}_{\mathscr{O}}(U, V) := \rho(U)$. Let $\gamma \in H^2(\Omega, \rho^\times)$ be represented by a 2-cocycle c. If we set $U = V = W$ or $V = W = Y$ in Eq. (11), we get $c_{UUU} = c_{UUV}$ and $\rho_U^V(c_{VVV}) = c_{UVV}$. Setting $d_{UV} := c_{UUU}$, we have a 2-boundary $(\delta d)_{UVW} = \rho_U^V(d_{VW}) \cdot d_{UW}^{-1} \cdot d_{UV} = \rho_U^V(c_{VVV})$. Therefore, we can normalize c by multiplying with $(\delta d)^{-1}$.

For $U \leqslant V \leqslant W$ in Ω and morphisms $f \in \operatorname{Hom}_{\mathscr{O}}(U, V)$ and $g \in \operatorname{Hom}_{\mathscr{O}}(V, W)$ in $\mathscr{O}$, we define

$$gf := \rho_U^V(g) \cdot f \cdot c_{UVW}. \tag{13}$$

With this composition, $\mathscr{O}$ becomes a small commutative preadditive category. Since $\rho(U) = 0$ occurs at most once, the category $\mathscr{O}$ is skeletal.

Every morphism $U \to V$ in Ω can be associated to the short morphism $U \to V$ in $\mathscr{O}$ given by $1 \in \rho(U)$. This implies (O1). Assume that $U \xrightarrow{f} U \xrightarrow{i} V$ is a non-zero short monomorphism in $\mathscr{O}$. Then $i = if \cdot g$ for some $g \colon U \to U$. Thus $fg = 1$, and $if \cdot gf = if \cdot 1$ implies that $gf = 1$. Since $\rho(i)$ maps automorphisms to automorphisms, we get (O2). Now it is straighforward to verify that the correspondence is one-to-one. □

Corollary. *Let $\mathscr{O}$ be a pre-objective category with a greatest object X. Then the corresponding $\gamma \in H^2(\mathrm{Ob}\,\mathscr{O}, \rho^{\times})$ is trivial, i. e. $\mathscr{O}$ is just given by the functor* (9).

Proof. For $U \in \mathrm{Ob}\,\mathscr{O}$, we choose a short monomorphism $i_U \colon U \to X$. Therefore, if $U \leqslant V$ in $\mathrm{Ob}\,\mathscr{O}$, there is a unique short monomorphism $i_U^V \colon U \to V$ with $i_U = i_V \cdot i_U^V$. With this normalization of short monomorphisms, we get

$$i_V^W i_U^V = i_U^W \, ; \quad i_U^U = 1_U \tag{14}$$

for $U \leqslant V \leqslant W$. Hence γ is trivial. □

Example 2. In particular, the preceding corollary shows that every (commutatively) ringed space can be regarded as a pre-objective category with trivial 2-cocycle.

2 Affine objects

In what follows, we write $\mathbf{Mod}(\mathscr{O})$ for the category of *$\mathscr{O}$-modules*, i. e. additive functors $M \colon \mathscr{O}^{\mathrm{op}} \to \mathbf{Ab}$. We identify $\mathscr{O}$ with the full subcategory of representable functors via the Yoneda embedding

$$\mathscr{O} \hookrightarrow \mathbf{Mod}(\mathscr{O}) \tag{15}$$

which maps $U \in \mathrm{Ob}\,\mathscr{O}$ to $\mathrm{Hom}_{\mathscr{O}}(-, U)$.

Definition 3. Let $f: U \to U$ be an endomorphism in a pre-objective category $\mathscr{O}$. Assume that the short limit $S(f)$ exists. We call an $\mathscr{O}$-module M *regular* with respect to f or simply *f-regular* if the natural morphism

$$\varinjlim(M \circ f^{\mathbb{Z}}) \longrightarrow M(S(f)) \tag{16}$$

is invertible. We call M is *quasi-coherent* if M is f-regular for every morphism $f: U \to U$ in $\mathscr{O}$, provided that $S(f)$ exists for all such f.

Explicitly, the condition for $M \in \mathbf{Mod}(\mathscr{O})$ to be regular with respect to $f \in \mathscr{O}(U)$ states that the following are satisfied.

(L1) If a morphism $g: U \to M$ satisfies $gi_f = 0$, then $gf^n = 0$ for some $n \in \mathbb{N}$.

(L2) For any $g: S(f) \to M$ in $\mathbf{Mod}(\mathscr{O})$, there is an $n \in \mathbb{N}$ such that $g(f^{\times})^n$ factors through i_f.

Definition 4. We say that an object U of a pre-objective category $\mathscr{O}$ is *covered* by a set $\mathscr{V} \subset \operatorname{Ob}\mathscr{O}$ if every $W \in \operatorname{Ob}\mathscr{O}$ with $V \leqslant W$ for all $V \in \mathscr{V}$ satisfies $U \leqslant W$. We call U *affine* if the short limit (5) exists for every $f \in \mathscr{O}(U)$ and the following are satisfied.

(A1) As an $\mathscr{O}$-module, U is f-regular for all $f \in \mathscr{O}(U)$.

(A2) If $V \leqslant U$, then $\bigvee\{S(f) \leqslant V \mid f \in \mathscr{O}(U)\} = V$.

(A3) If $f \in \mathscr{O}(U)$, then $S(f)$ is covered by $\mathscr{V} \subset \operatorname{Ob}\mathscr{O}$ if and only if the ideal of $\mathscr{O}(U)$ generated by the $g \in \mathscr{O}(U)$ with $S(g) \leqslant S(f)$ and $S(g) \leqslant V$ for some $V \in \mathscr{V}$ contains a power of f.

The full subcategory of affine objects in $\mathscr{O}$ will be denoted by $\mathscr{O}_{\text{aff}}$.

Proposition 4. *Let $\mathscr{O}$ be a pre-objective category, and let f, g be endomorphisms of $U \in \operatorname{Ob}\mathscr{O}$. Assume that the short limit $S(e)$ exists for all $e \in \mathscr{O}(U)$. Then*

$$S(fg) = S(f) \wedge S(g). \tag{17}$$

If U is affine, then $S(f+g)$ is covered by $\{S(f), S(g)\}$.

Proof. First, there are short monomorphisms $i_f\colon S(f) \to U$ and $i_g\colon S(g) \to U$. Since $f|_{S(fg)}$ is invertible, the short monomorphism $i_{fg}\colon S(fg) \to U$ factors through i_f and i_g. Hence $S(fg) \leqslant S(f), S(g)$ by Proposition 2. Now assume that $V \leqslant S(f), S(g)$. Then $f|_V$ and $g|_V$ are invertible. Hence $(fg)|_V$ is invertible, and thus $V \leqslant S(fg)$.

Let U be affine. Then we have $S((f+g)\cdot f) = S(f+g) \wedge S(f)$ and $S((f+g)\cdot g) = S(f+g) \wedge S(g)$. Since $(f+g)\cdot f + (f+g)\cdot g = (f+g)^2$, (A3) implies that $S(f+g)$ is covered by $\{S(f), S(g)\}$. □

In particular, Proposition 4 shows that $S(fg) = S(f)$ holds if g is invertible. The next proposition shows that for any $f \in \mathscr{O}(U)$, the ring homomorphism $\rho^U_{S(f)}$ of Proposition 1 can be regarded as a localization with respect to f.

Proposition 5. *Let $\mathscr{O}$ be a pre-objective category. For any $U \in \operatorname{Ob} \mathscr{O}_{\mathrm{aff}}$ and $f \in \mathscr{O}(U)$, there is a natural isomorphism*

$$\mathscr{O}(S(f)) \cong \mathscr{O}(U)_f. \tag{18}$$

Proof. For a given $h \in \mathscr{O}(S(f))$, there is an $n \in \mathbb{N}$ such that $h(f^\times)^n = g|_{S(f)}$ for some $g \in \mathscr{O}(U)$. We define a map $\varepsilon\colon \mathscr{O}(S(f)) \to \mathscr{O}(U)_f$ by $\varepsilon(h) := \frac{g}{f^n}$. By (L1), this map is well-defined, and it is easily checked that ε is a ring isomorphism. □

Proposition 6. *Let $\mathscr{O}$ be a pre-objective category, and let $f\colon U \to U$ be an endomorphism in $\mathscr{O}_{\mathrm{aff}}$. Then $S(f)$ is affine.*

Proof. Let $g \in \mathscr{O}(S(f))$ be given. To verify (A1) for $S(f)$, let $h\colon S(f) \to S(f)$ be an endomorphism with $hi_g = 0$. Since f satisfies (L2), there is an $n \in \mathbb{N}$ with $i_f \cdot g(f^\times)^n = u \cdot i_f$ and $i_f \cdot h(f^\times)^n = v \cdot i_f$. Hence $i_f \cdot g(f^\times)^{n+1} = f \cdot i_f \cdot g(f^\times)^n = fu \cdot i_f$. Since $S(fu) \leqslant S(f)$, this gives $S(fu) = S(g(f^\times)^{n+1}) = S(g)$. Therefore, $v \cdot i_{fu} \cong v \cdot i_f i_g = i_f \cdot h(f^\times)^n i_g = 0$ implies that $v(fu)^m = 0$ for some $m \in \mathbb{N}$. Hence $i_f h g^m (f^\times)^{mn+m+n} = v \cdot i_f g^m (f^\times)^{mn+m} = vu^m i_f (f^\times)^m = vu^m f^m i_f = 0$, and thus $hg^m = 0$. This proves (L1). Next let $h\colon S(g) \to S(f)$ be given. Then $S(fu) = S(g)$ and $fu|_{S(g)} = g^\times \cdot (f^\times)^{n+1}|_{S(g)}$ implies that $i_f h(g^\times)^k \cdot (f^\times)^{(n+1)k}|_{S(g)}$ factors through $i_f i_g$ for a suitable $k \in \mathbb{N}$. Hence $h(g^\times)^k \cdot (f^\times)^{(n+1)k}|_{S(g)}$ factors through i_g, and thus $h(g^\times)^k$ factors through i_g. This shows that $S(f)$ satisfies (A1).

If $V \leqslant S(f)$, then there is a subset $F \subset \mathscr{O}(U)$ with $\bigvee\{S(g) \mid g \in F\} = V$. Hence $\bigvee\{S(g|_{S(f)}) \mid g \in F\} = V$, which proves (A2) for $S(f)$.

Finally, let $g \in \mathscr{O}(S(f))$ and $\mathscr{V} \subset \operatorname{Ob}\mathscr{O}$ be given. By multiplying g with a suitable power of $f^\times$, we can assume that $g = g'|_{S(f)}$ for some $g' \in \mathscr{O}(U)$, and $S(g) = S(g')$. Let I be the ideal of $\mathscr{O}(S(f))$ generated by the $h \in \mathscr{O}(S(f))$ with $S(g) \geqslant S(h) \leqslant V$ for some $V \in \mathscr{V}$, and let I' be the ideal of $\mathscr{O}(U)$ generated by the $h \in \mathscr{O}(U)$ with $S(g') \geqslant S(h) \leqslant V$ for some $V \in \mathscr{V}$. Then the ideal I is generated by $\rho^U_{S(f)}(I')$. Hence I contains a power of g if and only if I' contains a power of $g'f$. This proves (A3) for $S(f)$. □

Definition 5. We call an additive functor $F\colon \mathscr{O} \to \mathscr{O}'$ between pre-objective categories *objective* if the following are satisfied.

(F1) F maps short monomorphisms to short monomorphisms.
(F2) If $U \in \operatorname{Ob}\mathscr{O}$ and $f \in \mathscr{O}(U)$ such that $S(f)$ exists, then $FS(f) = S(Ff)$.
(F3) F respects finite meets whenever they exist.
(F4) For any $V \in \operatorname{Ob}\mathscr{O}$, the FU with affine $U \leqslant V$ cover every affine $W \leqslant FV$.

Let $\mathscr{C}$ be a preadditive category. Recall that a full subcategory $\mathscr{D}$ is said to be *dense* if every object C of $\mathscr{C}$ satisfies

$$C = \operatorname{Colim}(\mathscr{D}/C \to \mathscr{C}). \tag{19}$$

Here the *slice category* $\mathscr{D}/C$ has objects $f\colon D \to C$ with $D \in \operatorname{Ob}\mathscr{D}$. If $f'\colon D' \to C$ is a second object, a morphism $f \to f'$ in $\mathscr{D}/C$ is a morphism $g\colon D \to D'$ in $\mathscr{D}$ with $f'g = f$. The natural functor $\mathscr{D}/C \to \mathscr{C}$ maps $f\colon D \to C$ to D. Note that $\mathscr{D} \subset \mathscr{C}$ is dense if and only if the functor

$$\mathscr{C} \longrightarrow \mathbf{Mod}(\mathscr{D}) \tag{20}$$

which maps C to $\operatorname{Hom}_{\mathscr{D}}(-, C)$ is fully faithful (see [9], X.6, dual of Proposition 2).

Definition 6. We define an *objective* category to be a pre-objective category $\mathscr{O}$ which satisfies

(O3) Arbitrary joins exist in $\mathrm{Ob}\,\mathscr{O}$.

(O4) The full subcategory $\mathscr{O}_{\mathrm{aff}}$ is dense in $\mathscr{O}$.

(O3) implies that $\mathrm{Ob}\,\mathscr{O}$ is a complete lattice. In particular, there is a greatest object X. So the corollary of Proposition 3 implies that the morphisms $U \to V$ in the partially ordered set $\Omega := \mathrm{Ob}\,\mathscr{O}$ can be regarded as short monomorphisms $i_U^V \in \mathscr{O}$ via (14), i. e. Ω becomes a full subcategory of $\mathscr{O}$. In the sequel, we choose a fixed embedding $i\colon \Omega \hookrightarrow \mathscr{O}$ with (14) for any objective category $\mathscr{O}$. In particular, if $U \in \mathrm{Ob}\,\mathscr{O}$ and $f \in \mathscr{O}(U)$, we set $i_f := i_{S(f)}^U$.

Axiom (O4) is related to a recollement of sheaves.

Proposition 7. *Let $\mathscr{O}$ be an objective category. For every $Y \in \mathrm{Ob}\,\mathscr{O}$,*

$$Y = \bigvee \{U \in \mathrm{Ob}\,\mathscr{O}_{\mathrm{aff}} \mid U \leqslant Y\} \tag{21}$$

$$\mathscr{O}(Y) = \varprojlim \{\mathscr{O}(U) \mid Y \geqslant U \in \mathrm{Ob}\,\mathscr{O}_{\mathrm{aff}}\}. \tag{22}$$

Proof. By definition, $Y = \mathrm{Colim}(\mathscr{O}_{\mathrm{aff}}/Y \to \mathscr{O})$. Let $Z \in \mathrm{Ob}\,\mathscr{O}$ be an object with $U \leqslant Z$ for all affine $U \leqslant Y$. Then the map $i_U^Y f \mapsto i_U^Z f$ defines a cocone over $\mathscr{O}_{\mathrm{aff}}/Y \to \mathscr{O}$. Hence there is a unique morphism $h\colon Y \to Z$ with $h \cdot i_U^Y = i_U^Z$ for all affine $U \leqslant Y$. If $Y = 0$, then $Y \leqslant Z$. Otherwise, there exists a non-zero affine $U \leqslant Y$. Therefore, $i_U^Z \neq 0$, which gives $h \neq 0$. Thus $Y \leqslant Z$. This proves (21).

To verify (22), we have to show that the restrictions $\rho_U^Y\colon \mathscr{O}(Y) \to \mathscr{O}(U)$ with $Y \geqslant U \in \mathrm{Ob}\,\mathscr{O}_{\mathrm{aff}}$ form a limit cone. Thus let $f_U \in \mathscr{O}(U)$ be given for each affine $U \leqslant Y$ such that $f_U|_V = f_V$ for affine $V \leqslant U$. Then $i_U^Y f \mapsto i_U^Y f \cdot f_U$ defines a cocone $(\mathscr{O}_{\mathrm{aff}}/Y \to \mathscr{O}) \to Y$. So there is a unique $g\colon Y \to Y$ with $g i_U^Y = i_U^Y f_U$ for all affine $U \leqslant Y$. Whence $g|_U = f_U$ for all U. □

Note that Definition 5(F4) can be regarded as a relative version of (O4).

3 The associated scheme

In this section, we associate a scheme $\operatorname{Spec}\mathscr{O}$ to any pre-objective category $\mathscr{O}$. In Section 4, we will prove that $\operatorname{Spec}\mathscr{O}$ determines $\mathscr{O}$ if $\mathscr{O}$ is objective.

Definition 7. We define a *point* of a pre-objective category $\mathscr{O}$ to be a non-empty subset x of $\operatorname{Ob}\mathscr{O}_{\text{aff}}$ such that the following are satisfied.

(P0) $0 \notin x$.
(P1) $U \geqslant V \in x \;\Rightarrow\; U \in x$.
(P2) $U, V \in x \;\Rightarrow\; \exists W \in x\colon W \leqslant U, V$.
(P3) $U \in x,\ f \in \mathscr{O}(U) \;\Rightarrow\; \big(S(f) \in x \;\text{ or }\; S(1-f) \in x\big)$.

The set of points of $\mathscr{O}$ will be denoted by $\operatorname{Spec}\mathscr{O}$.

Let $\mathscr{O}$ be a pre-objective category. We introduce a topology on $\operatorname{Spec}\mathscr{O}$ with basic open sets

$$\widetilde{U} := \{x \in \operatorname{Spec}\mathscr{O} \mid U \in x\} \tag{23}$$

for each $U \in \operatorname{Ob}\mathscr{O}_{\text{aff}}$. If a point $x \in \operatorname{Spec}\mathscr{O}$ satisfies $x \in \widetilde{U} \cap \widetilde{V}$ for two objects U, V of $\mathscr{O}_{\text{aff}}$, then $U, V \in x$, and so there exists some $W \in x$ with $W \leqslant U, V$. Thus $x \in \widetilde{W} \subset \widetilde{U} \cap \widetilde{V}$. So the $\widetilde{U}$ form a basis of open sets. For arbitrary $V \in \operatorname{Ob}\mathscr{O}$, we define

$$\widetilde{V} := \{x \in \operatorname{Spec}\mathscr{O} \mid \exists\, U \in x\colon U \leqslant V\}. \tag{24}$$

If V is affine, this definition coincides with (23).

Proposition 8. *Let $\mathscr{O}$ be a pre-objective category, and let $U \in \operatorname{Ob}\mathscr{O}_{\text{aff}}$ be covered by $\mathscr{V} \subset \operatorname{Ob}\mathscr{O}$. For every $x \in \widetilde{U}$, there exists some $V \in \mathscr{V}$ with $x \in \widetilde{V}$.*

Proof. Since $U = S(1_U)$ is affine, (A3) implies that the $g \in \mathscr{O}(U)$ with $S(g) \leqslant V$ for some $V \in \mathscr{V}$ generate $\mathscr{O}(U)$. Hence $1_U = \sum_{i=1}^{n} a_i g_i$ with $a_i, g_i \in \mathscr{O}(U)$ and $S(g_i) \leqslant V_i \in \mathscr{V}$. Suppose that $S(a_i g_i) \notin x$ for all $i \in \{1, \dots, n\}$. We set $f_k := \sum_{i=1}^{k} a_i g_i$. Since $S(1_U) \in x$, there is a minimal $m > 1$ with $S(f_m) \in x$. With $a := f_{m-1}|_{S(f_m)} \cdot (f_m^{\times})^{-1}$ and $b := a_m g_m|_{S(f_m)} \cdot (f_m^{\times})^{-1}$, we have $a + b = 1$. Therefore, (P3) implies that $S(a) \in x$ or $S(b) \in x$. From $S(a) = S(f_{m-1}|_{S(f_m)}) \leqslant S(f_{m-1})$ and $S(b) \leqslant S(a_m g_m)$, we get $S(f_{m-1}) \in x$ or $S(a_m g_m) \in x$, a contradiction. Hence $S(a_i g_i) \in x$ for some $i \in \{1, \dots, n\}$. Since $S(a_i g_i) \leqslant S(g_i) \leqslant V_i$, it follows that $x \in \widetilde{V}_i$. □

For any $x \in \widetilde{U}$ with $U \in \operatorname{Ob} \mathscr{O}_{\text{aff}}$, we define

$$\mathfrak{p}_x := \{f \in \mathscr{O}(U) \mid S(f) \notin x\}. \tag{25}$$

Proposition 9. *Let $\mathscr{O}$ be a pre-objective category. Then $x \mapsto \mathfrak{p}_x$ gives a homeomorphism*

$$\mathfrak{p} \colon \widetilde{U} \longrightarrow \operatorname{Spec} \mathscr{O}(U) \tag{26}$$

for every affine object U of $\mathscr{O}$.

Proof. We show first that $\mathfrak{p}_x$ is a prime ideal of $\mathscr{O}(U)$ for any $x \in \widetilde{U}$. Thus let $f, g \in \mathscr{O}(U)$ be given. Assume that $f \in \mathfrak{p}_x$. Then $S(f) \notin x$, and $S(fg) \leqslant S(f)$ by Proposition 4. Hence (P1) of Definition 7 gives $S(fg) \notin x$ and thus $fg \in \mathfrak{p}_x$. In particular, $-f \in \mathfrak{p}_x$. Furthermore, (P0) implies that $0 \in \mathfrak{p}_x$. To show that $\mathfrak{p}_x$ is an ideal of $\mathscr{O}(U)$, suppose that $f, g \in \mathfrak{p}_x$ and $f + g \notin \mathfrak{p}_x$. Then $S(f), S(g) \notin x$ but $S(f + g) \in x$. Proposition 4 implies that $S(f + g)$ is covered by $\{S(f), S(g)\}$. So we get a contradiction to Proposition 8.

To show that $\mathfrak{p}_x$ is a prime ideal, assume that $f, g \in \mathscr{O}(U)$ satisfy $fg \in \mathfrak{p}_x$. Then $S(fg) \notin x$. Suppose that $f, g \notin \mathfrak{p}_x$, i. e. $S(f), S(g) \in x$. By (P2), we find an object $W \in x$ with $W \leqslant S(f), S(g) \leqslant U$. Proposition 4 implies that $W \leqslant S(f) \wedge S(g) = S(fg)$. Thus $S(fg) \in x$, a contradiction. Finally, $x \in \widetilde{U}$ implies that $S(1_U) = U \in x$, whence $1 \notin \mathfrak{p}_x$.

Conversely, let $\mathfrak{p}$ be a prime ideal of $\mathscr{O}(U)$. We define

$$x := \{V \in \mathrm{Ob}\,\mathscr{O}_{\mathrm{aff}} \mid \exists f \in \mathscr{O}(U) \smallsetminus \mathfrak{p} \colon S(f) \leqslant V\} \tag{27}$$

and show that x is a point of $\mathscr{O}$. If $f \in \mathscr{O}(U) \smallsetminus \mathfrak{p}$, then f is not nilpotent. Hence, the (L1) part of (A1) implies that $f \cdot i_f \neq 0$, which yields $S(f) \neq 0$. This proves (P0). As (P1) is trivial, let us prove (P2). If $U, V \in x$, there are $f, g \in \mathscr{O}(U) \smallsetminus \mathfrak{p}$ with $S(f) \leqslant U$ and $S(g) \leqslant V$. Hence $fg \notin \mathfrak{p}$ and $U, V \geqslant S(f) \wedge S(g) = S(fg) \in x$.

To verify (P3), let $V \in x$ and $f \in \mathscr{O}(V)$ be given. So there exists some $g \in \mathscr{O}(U) \smallsetminus \mathfrak{p}$ with $S(g) \leqslant V$. By the (L2) part of (A1), there is some $n \in \mathbb{N}$ with $f|_{S(g)} \cdot (g^{\times})^n = h|_{S(g)}$ for some $h \in \mathscr{O}(U)$. Since $g^{n+1} \notin \mathfrak{p}$, it follows that $hg \notin \mathfrak{p}$ or $g^{n+1} - hg \notin \mathfrak{p}$. Hence $S(hg) \in x$ or $S(g^{n+1} - hg) \in x$. Furthermore, $S(hg) \leqslant S(g)$ and $S(g^{n+1} - hg) \leqslant S(g)$ implies that $S(hg) = S(f|_{S(g)} \cdot (g^{\times})^{n+1}) \leqslant S(f|_{S(g)}) \leqslant S(f)$ and $S(g^{n+1} - hg) = S((g^{\times})^{n+1} - f|_{S(g)} \cdot (g^{\times})^{n+1}) \leqslant S(1 - f|_{S(g)}) \leqslant S(1 - f)$. Therefore, we get $S(f) \in x$ or $S(1 - f) \in x$, which completes the proof of (P3). Thus x is a point of $\mathscr{O}$. Since $1_U \in \mathscr{O}(U) \smallsetminus \mathfrak{p}$ and $S(1_U) = U$, we have $U \in x$, i. e. $x \in \widetilde{U}$. Next we show that $\mathfrak{p}_x = \mathfrak{p}$.

If $f \in \mathfrak{p}_x$, then $S(f) \notin x$, which gives $f \in \mathfrak{p}$. Conversely, assume that $f \notin \mathfrak{p}_x$. Then $S(f) \in x$. So there exists some $g \in \mathscr{O}(U) \smallsetminus \mathfrak{p}$ with $S(g) \leqslant S(f)$. Thus $f|_{S(g)}$ is invertible. By (A1), there exists some $n \in \mathbb{N}$ such that $i_g \cdot f|_{S(g)}^{-1} \cdot (g^{\times})^n = hi_g$ for some $h \in \mathscr{O}(U)$. Hence $fhi_g = i_g \cdot (g^{\times})^n = g^n i_g$. By (A1), this gives $fhg^m = g^{m+n}$ for a suitable $m \in \mathbb{N}$. Therefore, we get $f \notin \mathfrak{p}$, which proves that $\mathfrak{p}_x = \mathfrak{p}$.

For the bijectivity of (26), it remains to be shown that

$$x = \{V \in \mathrm{Ob}\,\mathscr{O}_{\mathrm{aff}} \mid \exists f \in \mathscr{O}(U) \smallsetminus \mathfrak{p}_x \colon S(f) \leqslant V\} \tag{28}$$

holds for any point $x \in \widetilde{U}$. The inclusion "$\supset$" follows by (25). Conversely, assume that $V \in x$. Then (P2) yields an object $W \in x$ such that $W \leqslant U, V$. By (A2), it follows that W is covered by the $S(f)$ with $f \in \mathscr{O}(U)$ and $S(f) \leqslant W$. Furthermore, Proposition 6 implies that these $S(f)$ are affine. Hence Proposition 8 yields some $f \in \mathscr{O}(U)$ with $S(f) \leqslant W$ and $S(f) \in x$. Thus (25) gives $f \in \mathscr{O}(U) \smallsetminus \mathfrak{p}_x$ and $S(f) \leqslant V$. This proves (28).

Finally, (A2) of Definition 4 implies that the $\widetilde{S(f)}$ with $f \in \mathscr{O}(U)$ form a basis of $\widetilde{U}$. Therefore, Proposition 5 shows that the map (26) is a homeomorphism. □

By Proposition 9, every affine object U of a pre-objective category $\mathscr{O}$ gives rise to an embedding

$$\operatorname{Spec} \mathscr{O}(U) \hookrightarrow \operatorname{Spec} \mathscr{O} \tag{29}$$

such that $\operatorname{Spec} \mathscr{O}(U)$ can be identified with $\widetilde{U} \subset \operatorname{Spec} \mathscr{O}$. To any basic open set $\widetilde{U}$ of $\operatorname{Spec} \mathscr{O}$, we associate the commutative ring

$$\mathscr{O}(\widetilde{U}) := \mathscr{O}(U). \tag{30}$$

By Eqs. (8), this makes $\mathscr{O}$ into a presheaf on $\operatorname{Spec} \mathscr{O}$. So we obtain

Theorem 1. *For a pre-objective category $\mathscr{O}$, the associated presheaf makes* $\operatorname{Spec} \mathscr{O}$ *into a scheme.*

Proof. For an open set $V \subset \operatorname{Spec} \mathscr{O}$, we define

$$\mathscr{O}(V) := \varprojlim_{\widetilde{U} \subset V} \mathscr{O}(\widetilde{U}), \tag{31}$$

where U runs through $\operatorname{Ob} \mathscr{O}_{\text{aff}}$. By Proposition 5, the $\widetilde{U}$ are affine schemes. Hence $\operatorname{Spec} \mathscr{O}$ is a scheme with structure sheaf (30) by [5], 0.3.2.2. □

4 Objective categories

In the sequel, we write **POb** for the category of pre-objective categories with a greatest object and with objective functors as morphisms. By the corollary of Proposition 3, the objects of **POb** can be regarded as functors $\rho \colon \Omega^{\text{op}} \to \mathbf{CRi}$ such that the partially ordered set Ω has a greatest element, and $\rho(a) = 0$ for at most one element $a \in \Omega$. Note that by (F3) of Definition 5, a morphism $\mathscr{O} \to \mathscr{O}'$ in **POb** respects the greatest object $\bigwedge \varnothing$ of $\mathscr{O}$. By **Obj** we denote the full subcategory of **POb** consisting of the objective categories. The category of schemes as locally ringed spaces (see [7], II.2) will be denoted by **Sch**.

Proposition 10. *Let $\mathscr{O}$ be a pre-objective category. For $U, V \in \operatorname{Ob}\mathscr{O}$ with U affine,*

$$U \leqslant V \iff \widetilde{U} \subset \widetilde{V}. \tag{32}$$

If $\mathscr{O}$ is objective, the equivalence holds for all $U, V \in \operatorname{Ob}\mathscr{O}$, and every open set of $\operatorname{Spec}\mathscr{O}$ is of the form $\widetilde{V}$ for some $V \in \operatorname{Ob}\mathscr{O}$.

Proof. The implication "⇒" follows by (24). Assume that $\widetilde{U} \subset \widetilde{V}$, and let $\mathfrak{p}$ be any prime ideal of $\mathscr{O}(U)$. By Proposition 9, this implies that $\mathfrak{p} = \mathfrak{p}_x$ for some $x \in \widetilde{U} \subset \widetilde{V}$. So there is an affine $W \in x$ with $W \leqslant V$. By Eq. (27), we find some $f \in \mathscr{O}(U) \smallsetminus \mathfrak{p}$ with $S(f) \leqslant W \leqslant V$. Therefore, the $f \in \mathscr{O}(U)$ with $S(f) \leqslant V$ generate $\mathscr{O}(U)$. Thus (A3) implies that $U = S(1_U) \leqslant V$. If $\mathscr{O}$ is objective, the restriction on U can be dropped by virtue of Proposition 7.

Now let $\mathscr{O}$ be objective, and let V' be an open set of $\operatorname{Spec}\mathscr{O}$. Then $V' = \bigcup\{\widetilde{U} \mid U \in \mathscr{V}\}$ for some $\mathscr{V} \subset \operatorname{Ob}\mathscr{O}_{\text{aff}}$. We show that $V := \bigvee \mathscr{V}$ satisfies $\widetilde{V} = V'$. For all $U \in \mathscr{V}$, we have $U \leqslant V$, hence $\widetilde{U} \subset \widetilde{V}$, and thus $V' \subset \widetilde{V}$. Conversely, assume that $x \in \widetilde{V}$. By (24), there is some $U \in x$ with $U \leqslant V$. Hence $U = S(1_U)$ is covered by $\mathscr{V}$. So Proposition 8 implies that $x \in \widetilde{W}$ for some $W \in \mathscr{V}$. Hence $x \in \widetilde{W} \subset V'$. □

As an immediate consequence, Proposition 10 yields

Corollary. *For an objective category $\mathscr{O}$, the map $V \mapsto \widetilde{V}$ is a lattice isomorphism between the complete lattice $\operatorname{Ob}\mathscr{O}$ and the set of open sets of $\operatorname{Spec}\mathscr{O}$.*

Now we are ready to prove

Theorem 2. *The category* **Obj** *of objective categories is dual to the category* **Sch** *of schemes.*

Proof. We show first that the map which associates a scheme to an objective category (Theorem 1) extends to a functor

$$\mathrm{Spec}\colon \mathbf{Obj} \longrightarrow \mathbf{Sch}^{\mathrm{op}}. \tag{33}$$

Thus let $F\colon \mathscr{O} \to \mathscr{O}'$ be a morphism in **Obj**. We define

$$\mathrm{Spec}\, F\colon \mathrm{Spec}\, \mathscr{O}' \to \mathrm{Spec}\, \mathscr{O} \tag{34}$$

as follows. For $x' \in \mathrm{Spec}\, \mathscr{O}'$, we set

$$(\mathrm{Spec}\, F)(x') := \{U \in \mathscr{O}_{\mathrm{aff}} \mid x' \in \widetilde{FU}\}. \tag{35}$$

We show that $x := (\mathrm{Spec}\, F)(x') \in \mathrm{Spec}\, \mathscr{O}$. Since F is additive, (P0) holds for x. As F is monotonous by (F3), we get (P1). If $U, V \in x$, then $x' \in \widetilde{FU} \cap \widetilde{FV}$. So there exists some $W' \in x'$ with $W' \leqslant FU \wedge FV = F(U \wedge V)$. By (F4), the FW with affine $W \leqslant U \wedge V$ cover W'. Therefore, Proposition 8 implies that $x' \in \widetilde{FW}$ for some affine $W \leqslant U \wedge V$. Hence $W \in x$, which proves (P2) for x. To verify (P3), let $U \in x$ and $f \in \mathscr{O}(U)$ be given. Then $FU \geqslant U'$ for some $U' \in x'$. Hence $g := (Ff)|_{U'}$ satisfies $S(g) \in x'$ or $S(1-g) \in x'$. Now $S(g) \leqslant S(Ff) = FS(f)$, and similarly, $S(1-g) \leqslant FS(1-f)$. Hence $S(f) \in x$ or $S(1-f) \in x$. So the map (34) is well-defined.

To show that $\mathrm{Spec}\, F$ is continuous, let $V \in \mathrm{Ob}\, \mathscr{O}$ be given. Then (24) yields

$$\begin{aligned} x' \in (\mathrm{Spec}\, F)^{-1}(\widetilde{V}) &\Leftrightarrow (\mathrm{Spec}\, F)(x') \in \widetilde{V} \\ &\Leftrightarrow \exists\, U \in \mathrm{Ob}\, \mathscr{O}_{\mathrm{aff}}\colon V \geqslant U \in (\mathrm{Spec}\, F)(x') \\ &\Leftrightarrow \exists\, U \in \mathrm{Ob}\, \mathscr{O}_{\mathrm{aff}}\colon U \leqslant V, x' \in \widetilde{FU}. \end{aligned}$$

By Eq. (21), we have $V = \bigvee\{U \in \mathrm{Ob}\, \mathscr{O}_{\mathrm{aff}} \mid U \leqslant V\}$. Therefore, (F4) gives $FV = \bigvee\{FU \mid V \geqslant U \in \mathrm{Ob}\, \mathscr{O}_{\mathrm{aff}}\}$, and the corollary of Proposition 10 yields $\widetilde{FV} = \bigcup\{\widetilde{FU} \mid V \geqslant U \in \mathrm{Ob}\, \mathscr{O}_{\mathrm{aff}}\}$. So we get

$$(\mathrm{Spec}\, F)^{-1}(\widetilde{V}) = \widetilde{FV} \tag{36}$$

which shows that the map (34) is continuous.

Now F induces a ring homomorphism $\mathscr{O}(U) \to \mathscr{O}'(FU)$ for any $U \in \operatorname{Ob}\mathscr{O}$. For a short monomorphism $i\colon V \to U$, we have a commutative diagram

$$\begin{array}{ccc} \mathscr{O}(U) & \longrightarrow & \mathscr{O}'(FU) \\ \Big\downarrow \rho_V^U & & \Big\downarrow \rho_{FV}^{FU} \\ \mathscr{O}(V) & \longrightarrow & \mathscr{O}'(FV) \end{array}$$

by (F1). Therefore, Eq. (36) implies that F induces a morphism $\mathscr{O} \to (\operatorname{Spec} F)_* \mathscr{O}'$ of sheaves, which yields a morphism of schemes. This establishes the functor (33).

Conversely, we construct a functor

$$\mathbf{Sch}^{\mathrm{op}} \longrightarrow \mathbf{Obj}. \tag{37}$$

Let X be a scheme with structure sheaf $\mathscr{O}_X$. According to Proposition 3 and its corollary, the presheaf X defines a pre-objective category $\mathscr{O}$ with trivial 2-cocycle. Therefore, we can regard every inclusion $U \subset V$ with $U, V \in \operatorname{Ob}\mathscr{O}$ as a short morphism $i_U^V\colon U \to V$ in $\mathscr{O}$. Furthermore, $\mathscr{O}$ has a unique zero object $\varnothing$.

Let $f\colon U \to U$ be an endomorphism in $\mathscr{O}$, i. e. $f \in \mathscr{O}_X(U)$. We define $S(f)$ to be the set of points $x \in U$ such that the germ f_x of f at x is invertible in $\mathscr{O}_{X,x}$. Thus $S(f)$ is the maximal open subset V of U such that $f|_V$ is invertible. Hence $S(f) = \underleftarrow{\mathrm{shlim}}\, f^{\mathbb{Z}}$ (see [7], chap. II, Exercise 2.16). Furthermore, $\mathscr{O}$ satisfies (O3).

Now let $U \subset X$ be an affine open set. For any $f \in \mathscr{O}(U)$, the set $S(f) \subset U$ consists of the prime ideals $\mathfrak{p}$ of $\mathscr{O}(U)$ with $f \notin \mathfrak{p}$. Hence $\mathscr{O}(S(f)) = \mathscr{O}(U)_f$, and thus U satisfies (A1) of Definition 4. Moreover, (A2) and (A3) are easily verified. Hence $U \in \operatorname{Ob}\mathscr{O}_{\mathrm{aff}}$. Conversely, the open sets in $\operatorname{Ob}\mathscr{O}_{\mathrm{aff}}$ are affine by Proposition 5. To verify (O4), let $V, W \subset X$ be open such that for any affine open $U \subset V$ and a morphism $i_U^V \cdot f\colon U \to V$, there is a corresponding morphism $i_U^W \cdot f'\colon U \to W$. Assume that these maps form a cocone over $\mathscr{O}_{\mathrm{aff}}/V \to \mathscr{O}$. This means that for each affine open $U \subset V$, there is a section $f_U := (1_U)' \in \mathscr{O}_X(U)$ such that for every affine open $U' \subset U$, $f_{U'} = f_U|_{U'}$. Since $\mathscr{O}_X$ is a sheaf, there exists a unique section $f \in \mathscr{O}_X(V)$ with $f|U = f_U$ for all affine open $U \subset V$. So we get a morphism $i_V^W f\colon V \to W$ which completes the proof of (O4).

Next let $(\varphi, \vartheta)\colon X' \to X$ be a morphism of schemes, i. e. $\varphi\colon X' \to X$ is continuous, and $\vartheta\colon \mathscr{O}_X \to \varphi_* \mathscr{O}_{X'}$ is a morphism of sheaves. Let $\mathscr{O}', \mathscr{O}$ be the corresponding objective categories. We define a functor $F\colon \mathscr{O} \to \mathscr{O}'$ as follows. For $U \in \operatorname{Ob} \mathscr{O}$, we set $FU := \varphi^{-1}(U)$, and for a morphism $i_U^V \cdot f\colon U \to V$ in $\mathscr{O}$, we define $F(i_U^V \cdot f) := i_{FU}^{FV} \cdot \vartheta_U(f)$. Thus F respects addition of morphisms, and for a morphism $i_V^W \cdot g\colon V \to W$, we have $F(i_V^W g \cdot i_U^V f) = F(i_U^W \cdot g|_U \cdot f) = i_{FU}^{FW} \vartheta_U(g|_U \cdot f) = i_{FU}^{FW} \vartheta_U(g|_U) \vartheta_U(f) = i_{FU}^{FW} \vartheta_V(g)|_{FU} \cdot \vartheta_U(f) = i_{FV}^{FW} \vartheta_V(g) \cdot i_{FU}^{FV} \vartheta_U(f) = F(i_V^W \cdot g) F(i_U^V \cdot f)$. Since $F(i_U^V) = i_{FU}^{FV}$, it follows that F is an additive functor which satisfies (F1) of Definition 5. For $f \in \mathscr{O}_X(U)$, we have $FS(f) = \varphi^{-1}(S(f)) = S(\vartheta_U(f)) = S(Ff)$ since (φ, ϑ) is a morphism of locally ringed spaces. This proves (F2). As (F3) and (F4) are trivial, the functor F is objective. It is straightforward to check that the functor (37) is inverse to (33). $\square$

Remark. The preceding proof shows that for a scheme X, an open subset U of X is affine if and only if the object U of the corresponding objective category is affine in the sense of Definition 4.

By virtue of Theorem 2, the following result locates the category of schemes within the category of pre-objective categories.

Theorem 3. *The category* **Obj** *of objective categories is a reflective full subcategory of the category* **POb** *of preobjective categories with a greatest object.*

Proof. For a pre-objective category $\mathscr{O}$ with a greatest object X, Theorem 2 implies that the associated scheme $\operatorname{Spec} \mathscr{O}$ corresponds to an objective category $\widetilde{\mathscr{O}}$. The objects of $\widetilde{\mathscr{O}}$ are the open sets of $\operatorname{Spec} \mathscr{O}$. By the corollary of Proposition 3, $\mathscr{O}$ admits a subcategory of short monomorphisms $i_U^V\colon U \to V$ for $U \leqslant V$ such that the relations (14) are satisfied. For any $V \in \operatorname{Ob} \mathscr{O}$, Eq. (24) can be rewritten as

$$\widetilde{V} = \bigcup \{\widetilde{U} \mid V \geqslant U \in \mathscr{O}_{\text{aff}}\}. \tag{38}$$

Every $f \in \mathscr{O}(V)$ gives rise to a system of $f|_U \in \mathscr{O}(U) = \mathscr{O}(\widetilde{U})$ for all affine $U \leqslant V$. By recollement, this yields an endomorphism $\widetilde{f} \in \widetilde{V}$ with $\widetilde{f}|_{\widetilde{U}} = f|_U$ for all U. So we get an additive functor $H\colon \mathscr{O} \to \widetilde{\mathscr{O}}$ which maps $i_V^W \cdot f\colon V \to W$ with $f \in \mathscr{O}(V)$ to $\widetilde{f} \in \mathscr{O}(\widetilde{V}) = \mathrm{Hom}_{\widetilde{\mathscr{O}}}(\widetilde{V}, \widetilde{W})$ according to Proposition 3. By construction, H satisfies (F1) of Definition 5. Assume that $S(f)$ exists for some $f \in \mathscr{O}(V)$ with $V \in \mathrm{Ob}\,\mathscr{O}$. To verify (F2), we have to show that $S(Hf) \subset HS(f)$. This means that $\widetilde{U} \subset HS(f)$ for every $U \in \mathscr{O}_{\mathrm{aff}}$ with $\widetilde{U} \subset S(Hf) = S(\widetilde{f})$. For such U, the restriction $\widetilde{f}|_{\widetilde{U}} = f|_U$ is invertible. Hence $U \subset S(f)$, and thus $\widetilde{U} \subset HS(f)$. Properties (F3) and (F4) are immediate consequences of (38). Thus $H\colon \mathscr{O} \to \widetilde{\mathscr{O}}$ is a morphism in **POb**.

Now let $F\colon \mathscr{O} \to \mathscr{O}'$ be a morphism in **POb** with $\mathscr{O}'$ objective. For $Y \in \mathrm{Ob}\,\widetilde{\mathscr{O}}$, we define

$$F'Y := \bigvee \{FU \mid U \in \mathscr{O}_{\mathrm{aff}},\ \widetilde{U} \subset Y\}. \tag{39}$$

For any $V \in \mathrm{Ob}\,\mathscr{O}$, Proposition 10 gives $F'\widetilde{V} = \bigvee\{FU \mid U \in \mathscr{O}_{\mathrm{aff}},\ \widetilde{U} \subset \widetilde{V}\} = \bigvee\{FU \mid V \geqslant U \in \mathscr{O}_{\mathrm{aff}}\} = FV$. Choose a system of short monomorphisms $i_V\colon V \to FX$ for any $V \in \mathrm{Ob}\,\mathscr{O}'$ such that $i_{FU} = Fi_U^X$ holds for $U \in \mathrm{Ob}\,\mathscr{O}$. So there is a unique set of short monomorphisms $i_V^W\colon V \to W$ in $\mathscr{O}'$ which satisfy (14) and $i_{FU}^{FV} = F(i_U^V)$ for $U \leqslant V$ in $\mathscr{O}$. Every morphism $f \in \widetilde{\mathscr{O}}(Y)$ restricts to a system of endomorphisms $f|_{\widetilde{U}} \in \mathscr{O}(\widetilde{U}) = \mathscr{O}(U)$ with $U \in \mathrm{Ob}\,\mathscr{O}_{\mathrm{aff}}$ and $\widetilde{U} \subset Y$. By Theorem 2, $\mathscr{O}'$ can be regarded as a scheme such that the short monomorphisms $i_U^V \in \mathscr{O}'$ are to be viewed as inclusions $U \hookrightarrow V$. Therefore, the $F(f|_{\widetilde{U}}) \in \mathscr{O}(FU)$ admit a recollement $f' \in \mathscr{O}'(F'Y)$. For any inclusion $i\colon Y \subset Z$ in $\mathrm{Ob}\,\widetilde{\mathscr{O}}$, we set $F'(if) := i_{F'Y}^{F'Z} \cdot f'$. This gives an additive functor $F'\colon \widetilde{\mathscr{O}} \to \mathscr{O}'$ with $F'H = F$ which satisfies (F1).

Assume that $f \in \widetilde{\mathscr{O}}(Y)$. With $\mathscr{V} := \{U \in \mathrm{Ob}\,\mathscr{O}_{\mathrm{aff}} \mid \widetilde{U} \subset Y\}$, we have $F'S(f) = \bigvee\{FU \mid U \in \mathscr{V},\ f|_{\widetilde{U}} \in \mathscr{O}(U)^\times\} = \bigvee\{FS(f|_{\widetilde{U}}) \mid U \in \mathscr{V}\}$. Since $FS(f|_{\widetilde{U}}) = S(F(f|_{\widetilde{U}})) = S(F'f|_{FU})$, we get

$$F'S(f) = \bigvee\{S(F'f|_{FU}) \mid U \in \mathscr{V}\} = \bigvee_{U \in \mathscr{V}} \big(S(F'f) \wedge FU\big) = S(F'f).$$

Thus F' satisfies (F2). Furthermore, Eq. (39) implies that (F3) and (F4) for F carry over to F'. Hence F' is objective and unique. □

5 Quasi-coherent sheaves

Let X be a scheme with structure sheaf $\mathscr{O}_X$, and let $\mathscr{O}$ be the corresponding objective category. By the remark of section 4, the scheme X is affine if and only if the largest object X of $\mathscr{O}$ is affine. For an object Y of $\mathscr{O}$, we denote the full subcategory of objects $Z \leqslant Y$ by $\mathscr{O}|_Y$. Thus $\mathscr{O}|_Y$ is affine if and only if $Y \in \operatorname{Ob}\mathscr{O}_{\text{aff}}$.

A presheaf of $\mathscr{O}_X$-modules is just an object of $\mathbf{Mod}(\mathscr{O})$, i. e. an additive functor $M : \mathscr{O}^{\text{op}} \to \mathbf{Ab}$. In fact, if $U \in \operatorname{Ob}\mathscr{O}$, then the ring homomorphism $\mathscr{O}(U) \to \operatorname{End}(M(U))$ makes $M(U)$ into an $\mathscr{O}(U)$-module, and for $V \leqslant U$ in $\operatorname{Ob}\mathscr{O}$, the restriction $M(i_V^U) : M(U) \to M(V)$ is $\mathscr{O}(U)$-linear if $M(V)$ is regarded as an $\mathscr{O}(U)$-module via $\rho_V^U : \mathscr{O}(U) \to \mathscr{O}(V)$. Furthermore, any pair of $\mathscr{O}$-modules M, N has a tensor product $M \otimes_{\mathscr{O}} N$ given by $(M \otimes_{\mathscr{O}} N)(U) := M(U) \otimes_{\mathscr{O}(U)} N(U)$ for all $U \in \operatorname{Ob}\mathscr{O}$ and the obvious restrictions. In the sequel, we write $\operatorname{Hom}_{\mathscr{O}}(M, N)$ instead of $\operatorname{Hom}_{\mathbf{Mod}(\mathscr{O})}(M, N)$.

Proposition 11. *Let X be a scheme with structure sheaf $\mathscr{O}_X$, and let $\mathscr{O}$ be the corresponding objective category. Up to isomorphism, there is a natural bijection between quasi-coherent sheaves on X and quasi-coherent $\mathscr{O}_{\text{aff}}$-modules (see Definition 3).*

Proof. Every quasi-coherent sheaf on X restricts to an $\mathscr{O}_{\text{aff}}$-module M. For an endomorphism $f : U \to U$ in $\mathscr{O}_{\text{aff}}$, Proposition 5 implies that $\mathscr{O}(S(f)) \cong \mathscr{O}(U)_f$. As an $\mathscr{O}(U)$-module, $\mathscr{O}(U)_f \cong \mathscr{O}(U)[t]/(1 - ft)$ is the direct limit of the diagram

$$\cdots \to \mathscr{O}(U) \xrightarrow{f} \mathscr{O}(U) \xrightarrow{f} \mathscr{O}(U) \to \cdots$$

Hence $\varinjlim(M \circ f^{\mathbb{Z}}) \cong \mathscr{O}(U)_f \otimes_{\mathscr{O}(U)} M(U) \cong M(S(f))$.

Conversely, let $M \in \mathbf{Mod}(\mathscr{O}_{\mathrm{aff}})$ be quasi-coherent. To show that M defines a sheaf of $\mathscr{O}_X$-modules via (31), we use [5], 0.3.2.2. By [5], I, Theorem 1.4.1, the conditions (L1) and (L2) after Definition 3 imply that for any $U \in \mathrm{Ob}\, \mathscr{O}_{\mathrm{aff}}$, the restriction of M to $\mathbf{Mod}(\mathscr{O}_{\mathrm{aff}}|_U)$ coincides with the associated sheaf of an $\mathscr{O}(U)$-module. Hence M defines a sheaf of $\mathscr{O}_X$-modules which is quasi-coherent by [5], I, Proposition 2.2.1. □

Proposition 11 shows that the category of quasi-coherent sheaves on X can be identified with the full subcategory $\mathbf{Qcoh}(\mathscr{O}) \subset \mathbf{Mod}(\mathscr{O}_{\mathrm{aff}})$ of quasi-coherent $\mathscr{O}_{\mathrm{aff}}$-modules. Since direct limits in $\mathbf{Mod}(\mathscr{O}_{\mathrm{aff}})$ are exact, the full subcategory $\mathbf{Qcoh}(\mathscr{O})$ is closed under kernels and colimits (cf. [5], I, Corollary 2.2.2). Furthermore, $\mathbf{Qcoh}(\mathscr{O})$ is closed with respect to the tensor product, and the greatest object X of $\mathscr{O}$ (which corresponds to the structure sheaf $\mathscr{O}_X$) belongs to $\mathbf{Qcoh}(\mathscr{O})$. Hence $\mathbf{Qcoh}(\mathscr{O})$ is a cocomplete abelian tensor category.

For $M \in \mathbf{Qcoh}(\mathscr{O})$ and $x \in \mathrm{Spec}\, \mathscr{O}$, the localization

$$M_x := \varinjlim_{U \in x} M(U) \tag{40}$$

can be regarded as an object of $\mathbf{Qcoh}(\mathscr{O})$, given by the skyscraper sheaf

$$M_x(U) := \begin{cases} M_x & \text{for } x \in \widetilde{U} \\ 0 & \text{for } x \notin \widetilde{U}. \end{cases} \tag{41}$$

Moreover, there is a natural morphism $M \to M_x$ in $\mathbf{Qcoh}(\mathscr{O})$.

Now we briefly discuss how to recover an objective category $\mathscr{O}$ from the abelian category $\mathbf{Qcoh}(\mathscr{O})$. Our method is more explicit than the reconstruction of Rosenberg [11] who considered various non-commutative generalizations [11, 12]. Recall that an object Q of an abelian category is said to be *quasi-injective* [10] if for morphisms $f, i\colon A \to Q$ with i monic there is an endomorphism e of Q with $f = ei$.

Definition 8. Let $\mathscr{A}$ be an abelian category. We call an object P of $\mathscr{A}$ a *point* of $\mathscr{A}$ if P is quasi-injective, $\mathrm{End}_{\mathscr{A}}(P)$ is a field, and every subobject $A \neq 0$ of P generates P. By $\mathrm{Spec}\, \mathscr{A}$ we denote a skeleton of the full subcategory of points.

For a non-commutative generalization in the affine case, see [2].

Proposition 12. *Let $\mathscr{O}$ be an objective category. There is a natural bijection $\kappa\colon \operatorname{Spec}\mathscr{O} \xrightarrow{\sim} \operatorname{Spec}\mathbf{Qcoh}(\mathscr{O})$.*

Proof. Let X denote the corresponding scheme with structure sheaf $\mathscr{O}_X$, and let P be a point of $\mathbf{Qcoh}(\mathscr{O})$. Choose an affine $U \in \operatorname{Ob}\mathscr{O}$ with $P(U) \neq 0$. By Proposition 9, there exists a point $x \in \widetilde{U}$ with $P_x \neq 0$. So we have an exact sequence

$$0 \to P' \to P \to P_x$$

in $\mathbf{Qcoh}(\mathscr{O})$. Since P' is invariant under $\operatorname{End}_{\mathscr{O}}(P)$ and $P' \neq P$, we have $P' = 0$. Hence $P \cong P_x$. Thus P can be regarded as a module over the local ring $\mathscr{O}_{X,x} = \varinjlim_{x\in\widetilde{U}} \mathscr{O}(U)$. For any $f \in \mathscr{O}_{X,x}$, the submodule $fP \subset P$ is fully invariant. Hence $fP = P$ or $fP = 0$. Therefore, the annihilator $\mathfrak{p} := \operatorname{Ann}(P) \subset \mathscr{O}_{X,x}$ is prime. If $fP = P$, then f is invertible on P since every non-zero submodule of P generates P. So we can assume that $\mathfrak{p} = \operatorname{Rad}\mathscr{O}_{X,x}$, and P is a vector space over the residue field $\kappa(x)$ of $\mathscr{O}_{X,x}$. Since $\operatorname{End}_{\mathscr{O}}(P)$ is a field, P must be one-dimensional over $\kappa(x)$. Conversely, the skyscraper sheaf with stalk $\kappa(x)$ at x is a point in $\mathbf{Qcoh}(\mathscr{O})$. So we get a bijection $\operatorname{Spec}\mathscr{O} \xrightarrow{\sim} \operatorname{Spec}\mathbf{Qcoh}(\mathscr{O})$. □

Let $\mathscr{A}$ be an abelian category. For a subset $U \subset \operatorname{Spec}\mathscr{A}$, let $\mathscr{T}_U$ denote the full subcategory of $\mathscr{A}$ consisting of the objects X such that $\operatorname{Hom}_{\mathscr{A}}(Y, P) = 0$ for all subobjects Y of X and $P \in U$. Thus $\mathscr{T}_U$ is a Serre subcategory of $\mathscr{A}$. In particular, we write $\mathscr{T}_P := \mathscr{T}_{\{P\}}$ for any $P \in \operatorname{Spec}\mathscr{A}$.

Definition 9. Let $\mathscr{A}$ be an abelian category. For an object X of $\mathscr{A}$ and $P \in \operatorname{Spec}\mathscr{A}$, we call a set F of morphisms $f\colon Y \to X$ *P-epic* if every epimorphism $g\colon X \to Z$ with $gf = 0$ for all $f \in F$ satisfies $Z \in \mathscr{T}_P$. We say that $X \in \operatorname{Ob}\mathscr{A}$ is of *finite type* if for every $P \in \operatorname{Spec}\mathscr{A}$, any P-epic set F of morphisms $Y \to X$ has a finite P-epic subset.

For $X \in \operatorname{Ob}\mathscr{A}$ of finite type, we define a subset $U_X \subset \operatorname{Spec}\mathscr{A}$ by

$$U_X := \{P \in \operatorname{Spec}\mathscr{A} \mid X \in \mathscr{T}_P\}. \tag{42}$$

If $X, Y \in \operatorname{Ob}\mathscr{A}$ are of finite type, then $X \oplus Y$ is of finite type, and

$$U_X \cap U_Y = U_{X\oplus Y}. \tag{43}$$

Therefore, with respect to inclusion, the U_X form a partially ordered set

$$\Omega_{\mathscr{A}} := \{U_X \mid X \in \operatorname{Ob}\mathscr{A} \text{ of finite type}\}. \tag{44}$$

which is a basis of open sets for a topology on $\operatorname{Spec}\mathscr{A}$. We endow $\operatorname{Spec}\mathscr{A}$ with this topology. In particular, $\Omega_{\mathscr{A}}$ has a greatest element $U_0 = \operatorname{Spec}\mathscr{A}$.

Recall that the *center* $Z(\mathscr{C})$ of a preadditive category $\mathscr{C}$ is the ring of natural endomorphisms of the identity functor $1\colon \mathscr{C} \to \mathscr{C}$. If $\mathscr{C}$ is small, then $Z(\mathscr{C}) \in \mathbf{CRi}$. We define

$$\mathscr{O}_{\mathscr{A}}(U) := Z(\mathscr{A}/\mathscr{T}_U) \tag{45}$$

for any $U \in \Omega_{\mathscr{A}}$. If $U \subset V$ holds in $\Omega_{\mathscr{A}}$, then $\mathscr{T}_V \subset \mathscr{T}_U$, which induces an additive functor $\mathscr{A}/\mathscr{T}_V \to \mathscr{A}/\mathscr{T}_U$, and thus a ring homomorphism $\rho^V_U\colon \mathscr{O}_{\mathscr{A}}(V) \to \mathscr{O}_{\mathscr{A}}(U)$. So we get a functor

$$\rho\colon \Omega^{\mathrm{op}}_{\mathscr{A}} \to \mathbf{CRi}. \tag{46}$$

If $\rho(U) = 0$, then $\mathscr{A}/\mathscr{T}_U = 0$, which implies that $U = \varnothing$. Hence by the corollary of Proposition 3, the functor (46) defines a pre-objective category $\mathscr{O}_{\mathscr{A}} \in \mathbf{POb}$.

The following theorem shows that an objective category and its corresponding scheme (cf. [11]) can be recovered from the category $\mathbf{Qcoh}(\mathscr{O})$.

Theorem 4. *Every objective category $\mathscr{O}$ is isomorphic to $\mathscr{O}_{\mathbf{Qcoh}(\mathscr{O})}$.*

Proof. We set $\mathscr{A} := \mathbf{Qcoh}(\mathscr{O})$ and $X := \operatorname{Spec}\mathscr{O}$. For a point $P = \kappa(x)$ of $\mathscr{A}$, the Serre subcategory $\mathscr{T}_P$ consists of the quasi-coherent sheaves M with $M_x = 0$. Therefore, $E \in \operatorname{Ob}\mathscr{A}$ is of finite type in the sense of Definition 9 if and only if E is of finite type as a quasi-coherent sheaf (see [5], 0.5.2). By [5], Chap. 0, Proposition 5.2.2, such an E has a closed support, i. e. $U_E \subset \operatorname{Spec} \mathscr{A}$ corresponds to an open set $\kappa^{-1}(U_E) \subset X$. This shows that the map κ of Proposition 12 is continuous.

Conversely, by Proposition 10, every open set in X is of the form $\widetilde{U}$ for some $U \in \operatorname{Ob} \mathscr{O}$. Let X_U be the corresponding quasi-coherent ideal of $\mathscr{O}_X$ which annihilates $X \smallsetminus \widetilde{U}$. Then there is a short exact sequence $0 \to X_U \to X \to E \to 0$ in $\mathbf{Qcoh}(\mathscr{O})$ with E of finite type and $U_E = \kappa(\widetilde{U})$. This shows that κ is a homeomorphism. Furthermore, $\mathscr{A}/\mathscr{T}_U \approx \mathbf{Qcoh}(\mathscr{O}|_U)$, whence $\mathscr{O}_{\mathscr{A}}(U) = \mathscr{O}(U)$. This proves that $\mathscr{O}_{\mathscr{A}} \cong \mathscr{O}$. $\square$

Note: The preceding proof shows that $\Omega_{\mathscr{A}}$ is not only a basis, but the totality of all open sets of $\operatorname{Spec} \mathscr{A}$.

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CHARACTERIZING POMONOIDS S BY COMPLETE S-POSETS

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RESUME.
Un ensemble partiellement ordonné (ou ‘poset’) muni d'une action d'un monoïde partiellement ordonné S est appelé S-poset. Pour un S-poset, il y a deux notions de complémentarité, la première en le cosidérant seulement en tant que poset, la seconde en tenant compte aussi des actions qui sont distributives sur les suprema.

Dans cet article, en cherchant à comparer ces deux notions, nous obtenons des caractérisations de certains monoïdes partiellement ordonnés.

ABSTRACT.
A poset with an action of a pomonoid S on it is called an S-poset. There are two different notions of completeness for an S-poset : one just as a poset and the other as a poset as well as the actions being distributive over the joins.

In this paper, comparing these two notions with each other, we find characterizations for some pomonoids.

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Key words: S-poset, completeness, continuous completeness.

1. **Introduction and Preliminaries**

General ordered algebraic structures play a role in a wide range of areas, including analysis, logic, theoretical computer science, and physics. One of the most important notion in any ordered algebraic structure is *completeness*. The purpose of the present article is to give some homological classification of pomonoids by *continuous completeness* of S-posets; complete posets with an action of a pomonoid S on them which is compatible with the joins (supremums).

A number of papers [2, 3, 4, 5, 8, 9, 11, 12] have studied some properties of S-posets, and the papers [8], [9], [11] also deal with completeness.

Our aim is to find some necessary or sufficient conditions on the pomonoid S under which completeness (as posets) and continuous completeness of S-posets coincide.

Introducing the notion of a *strongly indecomposable* pomonoid (having no prime ideal P satisfying $\forall p \in P, \forall s \in S\backslash P, p \nleq s$ or $\forall p \in P, \forall s \in S\backslash P, s \nleq p$) we get a necessary condition on S which plays an important role in this study.

Also, introducing the notion of a *strongly left residuated* pomonoid ($\forall s \in S, \exists t \in S, ts \leq 1 \leq st$), we prove that for such class of pomonoids, as well as for pogroups, complete and continuously complete S-posets are the same.

Finally, some classification of pomonoids by considering the additional condition "for every $s, t \in S$, $st \leq 1$ if and only if $1 \leq ts$" on S, and using completeness and continuous completeness of S-posets, are presented.

In the following we give a brief review of S-acts, posets, and S-posets needed in the sequel.

Let S be a monoid with identity 1. Recall that a (*right*) *S-act* A is a set equipped with a map $\lambda : A \times S \to A$, called its *action*, such that, denoting $\lambda(a, s)$ by as, we have $a1 = a$ and $a(st) = (as)t$, for all $a \in A$, and $s, t \in S$. The category of all S-acts, with action preserving (S-act) maps ($f : A \to B$ with $f(as) = f(a)s$, for $s \in S$, $a \in A$) between them, is denoted by **Act-S**.

An element θ of an S-act is called a *zero* or a *fixed* element if $\theta s = \theta$ for all $s \in S$. For more information about S-acts see [10] and [7].

Let **Pos** denote the category of all partially ordered sets (posets) with order preserving (monotone) maps between them. Recall that a poset is said to be *complete* if each of its subsets has an infimum and a supremum, in particular, a complete poset is bounded by the least (bottom) element $\perp$ and the greatest (top) element $\top$.

A monoid (semigroup, group) S is said to be a *pomonoid* (*posemigroup, pogroup*) if it is also a poset whose partial order $\leq$ is compatible with the binary operation($s \leq t$, $s' \leq t'$ imply $ss' \leq tt'$). In this paper S denotes a pomonoid, unless otherwise stated.

By a *monoid ideal* I of a pomonoid S, we mean a left ideal ($SI \subseteq I$) and a right ideal ($IS \subseteq I$) of S. Also, a proper nonempty monoid ideal P of S satisfying the property that $st \in P$ implies $s \in P$ or $t \in P$, for $s, t \in S$, is called a *prime ideal.*

A *right poideal* of a pomonoid S is a (possibly empty) subset I of S if it is both a monoid right ideal ($IS \subseteq I$) and a down set ($a \leq b, b \in I$ imply $a \in I$).

A (*right*) *S-poset* is a poset A which is also an S-act whose action $\lambda : A \times S \to A$ is order-preserving. Here, $A \times S$ is considered as a poset with componentwise order.

An *S-poset map* (or *morphism*) is an action preserving monotone map between S-posets. We denote the category of all right S-posets, with S-poset maps between them, by **Pos-S**.

2. Continuous completeness and Completeness

Being an ordered set as well as an algebraic structure, an S-poset may be defined to be complete in two different ways: one just as a poset and the other involving the action, too.

In this section first we define what we mean by these two notions in **Pos-S**, then we see that they are actually different and give some necessary conditions on S for them to coincide.

Definition 2.1. Let S be a pomonoid (posemigroup). An S-poset A is called

(i) *complete* if it is complete as a poset,

(ii) *continuously complete* if it is a complete poset and the actions are compatible with the joins (supremums); that is, for every $X \subseteq A$ and $s \in S$, $(\bigvee X)s = \bigvee(Xs)$.

In the following proposition and remark, we see that completeness does not necessarily implies continuous completeness.

Proposition 2.2. *If all complete S-posets are continuously complete then S must have an identity.*

Proof. On the contrary, consider a posemigroup S without 1. We construct a complete S-poset which is not continuously complete. Take the poset $A = \{\bot, a, b, c, \top\}$ with the order given by $a, b \leq c$, and a, b are incomparable. Define the action on A as $cs = \top$, for every $s \in S$, and $\bot, a, b, \top$ are fixed (zero) elements. To see that A is an S-poset, it suffices to note that $(cs)t = \top t = \top = c(st)$, for every $s, t \in S$. Now, although A is complete, it is not continuously complete. This is because $(a \vee b)s = cs = \top$ while $as \vee bs = a \vee b = c$, for every $s \in S$. □

Remark 2.3. Notice that, even if S is a pomonoid, not all complete S-posets are continuously complete. For example, consider the poset $S = (\mathbb{N}_0 = \mathbb{N} \cup \{0\}, \preceq)$ of non-negative integers ordered by division: $m \preceq n \Leftrightarrow m|n$. Then S is a lattice in which $m \vee n = \mathrm{lcm}\{m, n\}$ (the least common multiple) and $m \wedge n = \gcd\{m, n\}$ (the greatest common divisor). Also S is complete, since for $X \subseteq S$, we have $\bigvee X = \mathrm{lcm}X$ if X is finite, and $\bigvee X = 0$ if X is infinite. Note that S is a pomonoid with the multiplication $st = s \wedge t$. Therefore, S is a complete S-poset. But S is not continuously complete, because taking $X = \{3, 5, 7, ...\}$ we get $2 \wedge \bigvee X = 2 \wedge 0 = 2$, but $\bigvee\{2 \wedge x \mid x \in X\} = \bigvee\{1\} = 1$.

Since we intend to seek necessary as well as sufficient conditions on S under which all complete S-posets are continuously complete, the above proposition justifies why we have taken S to be a pomonoid.

We now give some necessary conditions on the pomonoid S under which continuous completeness coincides with completeness.

Definition 2.4. A pomonoid S is called *strongly decomposable* if $S = T \dot{\cup} I$, for some subpomonoid T and some nonempty proper monoid ideal I of S such that $\forall t \in T, \forall i \in I, t \nleq i$ or $\forall t \in T, \forall i \in I, i \nleq t$. In this case, the pair (T, I) is called a *strong decomposition* of S. Otherwise, S is said to be *strongly indecomposable*.

Remark 2.5. Notice that in the above definition, I is necessarily a prime ideal, because $T = S \backslash I$ is a monoid. Thus, the existence of a strong decomposition for S is equivalent to the existence of a prime ideal P of S with the property that $\forall p \in P, \forall s \in S \backslash P, p \nleq s$ or $\forall p \in P, \forall s \in S \backslash P, s \nleq p$.

Theorem 2.6. *If all complete S-posets are continuously complete, then the pomonoid S is strongly indecomposable.*

Proof. Let (T, I) be a strong decomposition for S. In each of the following cases, we construct a complete S-poset which is not continuously complete:

Case 1: $\forall t \in T, \forall i \in I, t \nleq i$. Consider the poset $A = \mathbf{2} \times \mathbf{2} = \{\bot, a, b, \top\}$ with the actions given by: $as = a$ if $s \in T, as = \bot$ if $s \in I$; and $\bot, b, \top$ are fixed. Then A is an S-poset. To see this, let $s, t \in S$. It suffices to show that $(as)t = a(st)$. There are 4 possible cases:

(i) Let $s, t \in T$. Then $st \in T$ and so $(as)t = at = a = a(st)$.

(ii) Let $s, t \in I$. Then $st \in I$ and so $(as)t = \bot t = \bot = a(st)$.

(iii) Let $s \in T,\ t \in I$. Then $st \in I$ and so $(as)t = at = \bot = a(st)$.

(iv) Let $s \in I,\ t \in T$. Then $st \in I$ and so $(as)t = \bot t = \bot = a(st)$ as required.

Also, since $\bot, \top$ are fixed, $x \leq y$ gives that $xs \leq ys$, for every $x, y \in A, s \in S$. Finally, let $s, t \in S$ and $s \leq t$. It suffices to verify that $as \leq at$. By the assumption, we get that $s, t \in T$ or $s, t \in I$ or $s \in I, t \in T$. If $s, t \in T$, $as = a = at$. If $s, t \in I, as = \bot = at$. If $s \in I, t \in T, as = \bot \leq a = at$. Since $(a \vee b)i = \top i = \top$, and $ai \vee bi = \bot \vee b = b$ for every $i \in I$, we conclude that A is a complete S-poset which is not continuously complete.

Case 2: $\forall t \in T, \forall i \in I, i \nleq t$. Consider the poset $B = \{\bot, a, b, c, \top\}$ as in the proof of Proposition 2.2, and the actions given by: $cs = c$ if $s \in T, cs = \top$ if $s \in I$; and $\bot, a, b, \top$ are fixed. Then B is an S-poset. To see this, let $s, t \in S$. It suffices to show that $(cs)t = c(st)$. There are 4 possible cases:

(i) Let $s, t \in T$. Then $st \in T$ and so $(cs)t = ct = c = c(st)$.

(ii) Let $s, t \in I$. Then $st \in I$ and so $(cs)t = \top t = \top = c(st)$.

(iii) Let $s \in T,\ t \in I$. Then $st \in I$ and so $(cs)t = ct = \top = c(st)$.

(iv) Let $s \in I,\ t \in T$. Then $st \in I$ and so $(cs)t = \top t = \top = c(st)$ as required.

Also, since $\bot, a, b, \top$ are fixed, $x \leq y$ gives that $xs \leq ys$, for every $x, y \in B, s \in S$. Finally, let $s, t \in S$ and $s \leq t$. It suffices to verify that $cs \leq ct$. By the assumption, we get that $s, t \in T$ or $s, t \in I$ or $s \in T, t \in I$. If $s, t \in T$, $cs = c = ct$. If $s, t \in I, cs = \top = ct$. If $s \in T, t \in I, cs = c \leq \top = ct$. Since $(a \vee b)i = ci = \top$, and $ai \vee bi = a \vee b = c$ for every $i \in I$, we conclude that B is a complete S-poset which is not continuously complete. □

Corollary 2.7. *Let (S,=) be a pomonoid. If all complete S-posets are continuously complete, then S has no prime ideal.*

Theorem 2.8. *Let S be a nontrivial pomonoid. If all complete S-posets are continuously complete, then either the identity of S is not externally adjoined (there exist $s, t \in S \setminus \{1\}$ with $st = 1$) or $\downarrow 1 \neq \{1\} \neq \uparrow 1$ (there exist $u, v \in S$ such that $u < 1 < v$).*

Proof. Let the identity of S be externally adjoined and $\downarrow 1 = \{1\}$ or $\uparrow 1 = \{1\}$. Then $P = S \backslash \{1\}$ is a prime ideal of S such that $\forall p \in P, p \nleq 1$ or $\forall p \in P, 1 \nleq p$. Thus S is strongly decomposable, contradicting Theorem 2.6. □

Corollary 2.9. *Consider the nontrivial pomonoid S with equality as its order. If all complete S-posets are continuously complete, then the identity of S is not externally adjoined.*

Corollary 2.10. *Let S be a nontrivial pomonoid. If all complete S-posets are continuously complete, then the identity of S is neither the bottom nor the top element of S.*

Proof. Let 1 be the bottom element of S. We apply Theorem 2.8. If $st = 1$ for some $s, t \neq 1$ in S, then from $1 \leq s$ we get that $1 \leq t \leq st = 1$, and so $t = 1$ which is a contradiction. Also for $u, v \in S$, the case $u < 1 < v$ can not happen because 1 is the bottom element. Similarly, 1 is not the top element. □

Corollary 2.11. *Let S be a nontrivial pomonoid. If all complete S-posets are continuously complete, then the zero of S (if exists) is not externally adjoined.*

Proof. Let S have the zero element 0 which is externally adjoined. Then, $P = \{0\}$ is a prime ideal of S. Since for all $s, t \neq 0$, $st \neq 0$, there exist no nonzero elements $s, t \in S$ with $s \leq 0 \leq t$. In fact, otherwise $st \leq 0t = 0 = s0 \leq st$ and so $st = 0$ which is a contradiction. This clearly implies that for every $s \in S \backslash P, s \nleq 0$ or for every $s \in S \backslash P, 0 \nleq s$. Consequently, S is strongly decomposable, which contradicts Theorem 2.6. □

The following example shows that the converses of Corollaries 2.9, 2.10 and 2.11 are not true in general.

Example 2.12. Consider the pomonoid $S = \{0, 1, s, t\}$ with the equality order and the operation defined as $s^2 = 1$, $t^2 = 0$, $st = ts = t$. Then $P = \{0, t\}$ is a prime ideal. So, by Corollary 2.7, there exists a complete S-poset which is not continuously complete.

Recall the following definition and lemma from [1].

Definition 2.13. [1] A *divisibility monoid* is a pomonoid S in which $s \leq t$ is equivalent to $t \in Ss$, and also to $t \in sS$.

Lemma 2.14. [1] *In any divisibility monoid, $st = 1$ implies $s = t = 1$.*

Now, in view of Theorem 2.8 and Lemma 2.14, the following result is immediate.

Proposition 2.15. *Let S be a nontrivial pomonoid. If all complete S-posets are continuously complete, then either S is not a divisibility monoid or $\downarrow 1 \neq \{1\} \neq \uparrow 1$.*

Remark 2.16. For a nontrivial left simple, or right simple, or commutative, or idempotent pomonoid S, not all complete S-posets are continuously complete. To see this, consider the pomonoid $S = \{1, s\}$ with $s^2 = s$ and $1 \leq s$. Then S satisfies all the mentioned properties. Also S is strongly decomposable because $P = \{s\}$ is a prime ideal of S and $s \nleq 1$. So, by Theorem 2.6, not all complete S-posets are continuously complete.

3. Characterizing S by continuous completeness

In this section, we give some sufficient conditions on S under which continuous completeness coincides with completeness.

First, notice that by Lemma 4.1 of [8] we get

Lemma 3.1. *For a pogroup S, all complete S-posets are continuously complete.*

The converse of the above lemma is true if we have the additional condition that "for every $s, t \in S$, $st \leq 1$ if and only if $1 \leq ts$". To see this, first recall from [8] that a pomonoid S which has no proper non-empty left (right) poideal is said to be *left* (*right*) *simple*.

Lemma 3.2. *A pomonoid S is left* (*right*) *simple if and only if for all $s \in S$ there exists $x \in S$ such that $1 \leq xs$ $(1 \leq sx)$.*

Also, we see that:

Lemma 3.3. *Let S be a left* (*right*) *simple pomonoid with the property that for every $s, t \in S$, $st \leq 1$ if and only if $1 \leq ts$. Then S is a pogroup.*

Proof. Let S be left simple and $s \in S$. Then there exist $t, u \in S$ such that $1 \leq ts$ and $1 \leq u(ts) = (ut)s$, by Lemma 3.2, and hence $s(ut) \leq 1$ by the hypothesis. On the other hand, again using the hypothesis, we have $1 \leq u(ts)$ implies $t(su) = (ts)u \leq 1$ and hence $1 \leq (su)t = s(ut)$. Consequently, $s(ut) = 1$ which implies that S is a pogroup. A similar argument can be applied for the case where S is a right simple pomonoid. □

Theorem 3.4. *Let S be a pomonoid with the property that for every $s, t \in S$, $st \leq 1$ if and only if $1 \leq ts$. Then the following are equivalent:*

(i) *All complete S-posets are continuously complete.*
(ii) *S is left simple.*
(iii) *S is right simple.*
(iv) *S is a pogroup.*

Proof. The equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) follow from Lemma 3.3. Also, Lemma 3.1 gives (iv) $\Rightarrow$ (i). It remains to prove (i) $\Rightarrow$ (ii). Suppose that S is not left simple. Then, by Lemma 3.2, there exists $s_0 \in S$ such that for all $t \in S$, $1 \nleq ts_0$ and hence $s_0 t \nleq 1$ by the hypothesis. Now, applying Theorem 2.6, we claim that S is strongly decomposable. Take $P = \{s \in S \mid \forall t \in S, st \nleq 1\}$. Since $s_0 \in P$, P is a nonempty, also it is clearly a proper subset of S. To show that P is a prime ideal of S, take $p \in P, s \in S$. Then $sp, ps \in P$, since :

(a) if $sp \in S \backslash P$, then $(sp)t \leq 1$ for some $t \in S$. Using the assumption, we get that $1 \leq t(sp) = (ts)p$ and then $p(ts) \leq 1$. This means that $p \in S \backslash P$ which is a contradiction, and

(b) if $ps \in S\backslash P$, then $p(st) = (ps)t \leq 1$ for some $t \in S$ and so $p \in S\backslash P$, which is a contradiction.

Also, let $st \in P$ for some $s, t \in S$. If $s, t \in S\backslash P$, then $su \leq 1$ and $tv \leq 1$ for some $u, v \in S$. This implies $(st)(vu) = s(tv)u \leq s(1)u \leq 1$ which means $st \in S\backslash P$, a contradiction. Finally, let $p \leq s$ for some $p \in P, s \in S\backslash P$. Then, $st \leq 1$ for some $t \in S$. Since $p \leq s$, $pt \leq st \leq 1$. This implies that $p \in S\backslash P$ which is a contradiction. So, $\forall p \in P, \forall s \in S\backslash P, p \nleq s$. Consequently, the pair $(S\backslash P, P)$ forms a strong decomposition for S, as claimed. □

Corollary 3.5. *Let $(S,=)$ be a pomonoid whose right invertible elements are invertible. Then, all complete S-posets are continuously complete if and only if S is a pogroup.*

Now, using the notion of *residuation*, we find a sufficient condition for the equivalence of completeness and continuous completeness. First, recall the following definition from [6].

Definition 3.6. Let A, B be posets. A monotone mapping $f : A \to B$ is called *residuated* if there exists a (necessarily unique) monotone map $f^+ : B \to A$, called the *residual* of f, such that $ff^+ \leq id_B$ and $id_A \leq f^+f$. In fact, the map f^+ is defined as

$$f^+(b) = max\{a \in A \mid f(a) \leq b\}, \text{ for } b \in B.$$

Definition 3.7. A pomonoid S is called *strongly right residuated* if for every $s \in S$, the right translation mapping $\rho_s : S \to S$, $\rho_s(x) = xs$, is residuated with the residual $\rho_s^+ = \rho_t$, for some $t \in S$.

By an easy verification, we have

Lemma 3.8. *A pomonoid S is strongly right residuated if and only if for each $s \in S$ there exists a unique $t \in S$ such that $ts \leq 1 \leq st$. It then follows that $sts = s$ and $tst = t$.*

Theorem 3.9. *Let S be a strongly right residuated pomonoid. Then all complete S-posets are continuously complete.*

Proof. Let A be a complete S-poset, $X \subseteq A$ and $s \in S$. The inequality $\bigvee(Xs) \leq (\bigvee X)s$ always holds. For the converse, since S is strongly right residuated, by Lemma 3.8, $ts \leq 1 \leq st$, for some $t \in S$. Now $1 \leq st$ implies $x \leq x(st) = (xs)t \leq (\bigvee(Xs))t$, for every $x \in X$. This implies that $\bigvee X \leq (\bigvee(Xs))t$ and so $(\bigvee X)s \leq (\bigvee(Xs))ts$. On the other hand, since $ts \leq 1$, $(\bigvee(Xs))ts \leq \bigvee(Xs)$. So, the equality holds and A is continuously complete. □

Theorem 3.10. *Let S be a pomonoid with the property that for every $s, t \in S$, $st \leq 1$ if and only if $1 \leq ts$. Then all complete S-posets are continuously complete if and only if S is strongly right residuated.*

Proof. First notice that under the given condition, S is strongly right residuated if and only if S is right simple. Now the result follows from Theorem 3.4. □

Finally, we give some more necessary and sufficient conditions on S under which continuous completeness and completeness coincide.

Corollary 3.11. *Let $(S,=)$ be a commutative or a finite pomonoid. Then, all complete S-posets are continuously complete if and only if S is a pogroup.*

Proof. Notice that in a commutative monoid, every right invertible element is invertible. Also, the same note is true in a finite monoid (since every map on a finite set is onto if and only if it is one-one, and every monoid S is isomorphic to a submonoid of the monoid of all maps on S). So, by Corollary 3.5, the proof is complete. □

Definition 3.12. Let S be a pomonoid. An S-poset A is called *residuated* if for all $s \in S$, $\rho_s : A \to A$, $\rho_s(a) = as$ for any $a \in A$, is residuated.

Theorem 3.13. *Let S be a pomonoid and A be a complete S-poset. Then A is continuously complete if and only if it is residuated.*

Proof. Let A be continuously complete and $s \in S$. We must prove that the mapping ρ_s on A is residuated. It suffices to show that for every $y \in A$ the set $X = \{x \in A \mid xs = \rho_s(x) \leq y\}$ is non-empty and has the top element. Since A is (continuously) complete, it has the bottom element $\perp$ and $\perp s = (\bigvee \emptyset)s = \bigvee \emptyset s = \bigvee \emptyset = \perp \leq y$ which implies that $\perp \in X$ and so $X \neq \emptyset$. Also, $\bigvee X$ exists in A. We see that $\bigvee X \in X$, because $(\bigvee X)s = \bigvee(Xs) = \bigvee\{xs \mid x \in X\} \leq y$.

For the converse, let A be residuated, $X \subseteq A$, and $s \in S$. It suffices to prove the nontrivial equality $(\bigvee X)s \leq \bigvee(Xs)$. By the hypothesis, ρ_s is residuated and so $\rho_s\rho_s^+ \leq id \leq \rho_s^+\rho_s$. Now, for every $x \in X$, we have

$$x \leq \rho_s^+(\rho_s(x)) \leq \rho_s^+(\bigvee \rho_s(X))$$

and so $\bigvee X \leq \rho_s^+(\bigvee \rho_s(X))$. This implies

$$(\bigvee X)s = \rho_s(\bigvee X) \leq \rho_s\rho_s^+(\bigvee \rho_s(X)) \leq \bigvee \rho_s(X) = \bigvee(Xs)$$

as required. □

By Theorems 3.4, 3.10, and 3.13, we get the following result.

Proposition 3.14. *Let S be a pomonoid with the property that for every $s, t \in S$, $st \leq 1$ if and only if $1 \leq ts$. Then, the following are equivalent:*

(i) *All complete S-posets are continuously complete.*
(ii) *All complete S-posets are residuated.*
(iii) *S is strongly right residuated.*

(iv) *S is left simple.*
(v) *S is right simple.*
(vi) *S is a pogroup.*

Corollary 3.15. *Let $(S, =)$ be a pomonoid whose right invertible elements are invertible (such as commutative and finite monoids). Then, the following are equivalent:*
(i) *All complete S-posets are continuously complete.*
(ii) *All complete S-posets are residuated.*
(iii) *S is a pogroup.*

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ON CONNECTIVITY SPACES

by Stéphane DUGOWSON

Résumé. Cet article présente les bases d'une théorie des espaces connectifs. Il étudie notamment l'engendrement des structures, l'existence des (co)limites dans les catégories concernées, le produit tensoriel et la structure de catégorie monoïdale fermée associée. On y définit une notion d'homotopie ainsi que le *smash product* des espaces connectifs intègres pointés et la structure de catégorie monoïdale fermée associée. On étudie ensuite les espaces connectifs finis et l'on introduit un nouvel invariant numérique pour les entrelacs : l'ordre connectif. On présente enfin le théorème peu connu de Brunn-Debrunner-Kanenobu, qui affirme que tout espace connectif fini intègre peut être représenté par un entrelacs.

Abstract. This paper presents some basic facts about connectivity spaces. In particular, it explains how to generate connectivity structures, the existence of limits and colimits in the main categories of connectivity spaces, the closed monoidal category structure given by the tensor product of integral connectivity spaces; it defines homotopy for connectivity spaces and mentions briefly some related difficulties; it defines the smash product of pointed integral connectivity spaces and shows that this operation results in a closed monoidal category with such spaces as objects. Then, it studies finite connectivity spaces, associating a directed acyclic graph with each such space and then defining a new numerical invariant for links: the connectivity order. Finally, it mentions the not very well-known Brunn-Debrunner-Kanenobu theorem which asserts that every finite integral connectivity space can be represented by a link.

Keywords: Connectivity. Closed Monoidal Categories. Links. Borromean. Brunnian.

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Connectivity spaces are topological objects which have not yet received much attention. This paper presents results we have recently obtained relating to them. In the first section we recall their definition. The second section explains how to generate a connectivity structure from a given family of subsets to be regarded as connected. The third section is about categorical constructions in the main categories of connectivity spaces, by seeing them as particular cases of "categories with lattices of structures". The fourth section studies the closed monoidal category structure given by the tensor product of integral connectivity spaces. The fifth section defines homotopy for connectivity spaces and briefly mentions some difficulties related to this notion. The sixth section is devoted to pointed integral connectivity spaces and to the smash product of such spaces. In the last section we study finite connectivity spaces, associating a directed acyclic graph with each such space, and then defining a new numerical invariant for links: the connectivity index. Finally, we discuss the not very well-known Brunn-Debrunner-Kanenobu theorem, which asserts that every finite integral connectivity space can be represented by a link in the space $\mathbf{R}^3$ (or in $\mathbf{S}^3$).

Notations

If X is a set, the set of subsets of X is denoted by $\mathcal{P}(X)$ or $\mathcal{P}_X$, and the set $\mathcal{P}(\mathcal{P}_X)$ by $\mathcal{Q}_X$. For any $\mathcal{A} \in \mathcal{Q}_X$, $\mathcal{A}^\bullet$ denotes the set $\{A \in \mathcal{A}, \mathrm{card}(A) \geq 2\}$. If $\sim$ is an equivalence relation on X, the equivalence class of $x \in X$ is denoted by $\tilde{x}$. If Y is a subset of X, $\sim_Y$ denotes the equivalence relation defined on X by $a \sim_Y b$ if and only if $a = b$ or $(a, b) \in Y^2$, and X/Y denotes the quotient $X/\sim_Y$.

1 Definitions, Examples

Let us recall the definition of connectivity spaces and connectivity morphisms [2, 7].

Definition 1 (Connectivity spaces)**.** *A* connectivity space *is a pair* $(X, \mathcal{K})$ *where* X *is a set and* $\mathcal{K}$ *is a set of subsets of* X *such that* $\emptyset \in \mathcal{K}$ *and*

$$\forall \mathcal{I} \in \mathcal{P}(\mathcal{K}), \bigcap_{K \in \mathcal{I}} K \neq \emptyset \Longrightarrow \bigcup_{K \in \mathcal{I}} K \in \mathcal{K}.$$

The set X *is called the* carrier *of the space* $(X, \mathcal{K})$*, the set* $\mathcal{K}$ *is its* connectivity structure*. The elements of* $\mathcal{K}$ *are called the* connected subsets *of the space. The morphisms between two connectivity spaces are the functions which transform connected subsets into connected subsets. They are called the* connectivity morphisms*, or the* connecting maps[1]*. A connectivity space is called* integral *if every singleton subset is connected. The connected subsets with cardinality greater than one will be called the non-trivial connected subsets. A connectivity space is called* finite *if its carrier is a finite set.*

If X is a connectivity space, $|X|$ will denote its carrier, and $\kappa(X)$ its connectivity structure, so $X = (|X|, \kappa(X))$.

Remark 1. Instead of supposing that the empty set is always a member of connectivity structures, we could suppose without any substantial change that it is never such a member. But it seems preferable to choose one or the other of those two assumptions, to avoid "doubling" the involved categories.

Remark 2. Each point of an integral connectivity space belongs to a maximal connected subset. Those subsets are the connected components of the space; they constitute a partition of it.

In [2], Börger denotes *Zus* the category of integral connectivity spaces, because of the German word *Zusammenhangsräume*. We propose here to use rather **Cnc** to denote the category of connectivity spaces, **Cnct** to denote the category of integral connectivity spaces and **fCnct** to denote the category of finite integral connectivity spaces.

[1]Though *non-disconnecting maps* would be more accurate.

Example 1. Let $U_T : \mathbf{Top} \to \mathbf{Cnct}$ be the functor whose value is defined on each topological space (X, τ) as the connectivity space $(X, \mathcal{K})$ with $\mathcal{K}$ the set of connected subsets (in ordinary topological sense) of (X, τ). Then U_T is not full and not surjective (up to isomorphism) on objects ; it is faithful but is neither strictly injective nor injective up to isomorphism on objects : for example, if $X = \{a, b\}$, $\tau_1 = \{\emptyset, \{a\}, X\}$ and $\tau_2 = \{\emptyset, X\}$, then (X, τ_1) and (X, τ_2) are not isomorphic but $U_T(X, \tau_1) = U_T(X, \tau_2)$.

Example 2. Let **Grf** be the topological construct[2] whose objects are the simple undirected graphs and whose morphisms are the functions which send edges to edges or singletons. More precisely, such a graph can be defined as a pair $(X, \mathcal{G})$ with $\mathcal{G} \in \mathcal{Q}_X$ such that

$$\{A \in \mathcal{P}_X, \mathrm{card} A = 1\} \subseteq \mathcal{G} \subseteq \{A \in \mathcal{P}_X, \mathrm{card} A = 2\},$$

and morphisms $f : (X, \mathcal{G}) \to (Y, \mathcal{H})$ are functions $f : X \to Y$ such that $\forall A \in \mathcal{G}, f(A) \in \mathcal{H}$. A subset K of such a graph $(X, \mathcal{G})$ is said to be connected if for every pair (x, x') of elements of K, there exists a finite path $x = x_0, x_1, \cdots, x_n = x'$ such that each x_i is in K and each $\{x_i, x_{i+1}\}$ is in $\mathcal{G}$. The forgetful functor $U_G : \mathbf{Grf} \to \mathbf{Cnct}$, whose value is defined for each simple undirected graph $(X, \mathcal{G})$ as $(X, \mathcal{K})$ with $\mathcal{K}$ the set of connected subsets of X, is a full embedding.

Example 3. With each tame link[3] L in $\mathbf{R}^3$ or $\mathbf{S}^3$, we associate an integral connectivity space S_L taking the components of the link L as points of S_L, the connected subsets of it being defined by the nonsplittable sublinks of L. The connectivity structure $\kappa(S_L)$ will be called the *splittability structure* of L.

---

[2] Following [1], §5.1, p. 61, a category of structured sets and structure preserving functions between them is called a *construct*. More precisely, a construct is a concrete category over the category **Set** of sets, that is a pair $(\mathbf{A}, U)$ where $\mathbf{A}$ is a category and $U : \mathbf{A} \to \mathbf{Set}$ is a faithful functor (forgetful functor). A *topological* construct is then a construct $(\mathbf{A}, U)$ such that the functor U is topological, *i.e.* such that every U-structured source $(f_i : E \to UA_i)_I$ has a unique U-initial lift $(\bar{f}_i : A \to A_i)_I$(see [1], 10.57, p. 182 and §21.1, p. 359, and *infra*, the section 3.1 of the present article).

[3] A link is called *tame* if it is not *wild*, that is if it is (ambient) isotopic to a polygonal link (or to a smooth link, see [4]).

Example 4. The simplest integral connectivity space which is neither in $U_T(\mathbf{Top})$ nor in $U_G(\mathbf{Grf})$ is the *Borromean* space $\mathbf{B}_3$, defined by $|\mathbf{B}_3| = 3 = \{0, 1, 2\}$ and $\kappa(\mathbf{B}_3) = \mathcal{B}_3$ such that $\mathcal{B}_3^\bullet = \{|\mathbf{B}_3|\}$. More generally, for each integer $n \in \mathbf{N}$, the n-points *Brunnian* space $\mathbf{B}_n$ is the integral connectivity space defined by $|\mathbf{B}_n| = n$ and $\kappa(\mathbf{B}_n) = \mathcal{B}_n$ such that $\mathcal{B}_n^\bullet = \{|\mathbf{B}_n|\}$. The names *Borromean* and *Brunnian* are justified by the fact that the corresponding spaces are the ones associated with the links with the same names.

Example 5. More generally, for each set X and each cardinal ν, there is a unique integral connectivity space whose non-trivial connected subsets are those with cardinal greater than ν.

Example 6. Let p be an integer. The *hyperbrunnian space* $\mathbf{HB}_p$ is the integral connectivity space such that $|\mathbf{HB}_p| = \{0, 1, \cdots, p-1\}^{\mathbf{N}}$ and with non-trivial connected subsets all the $K \subseteq |\mathbf{HB}_p|$ for which there exist $k \in \mathbf{N}$ and $a \in |\mathbf{HB}_p|$ such that K be of the form

$$K = \{x \in |\mathbf{HB}_p|, \forall n < k, x_n = a_n\}.$$

The space $\mathbf{HB}_3$ will be called the *hyperborromean* space. For each $k \in \mathbf{N}$, the function $\phi_k : \mathbf{HB}_p \to \mathbf{B}_p$ defined by $f(x) = x_k$ is a connectivity morphism. If $p \geq 2$, the function $f : \mathbf{HB}_p \to \mathbf{I}$ defined by

$$f(x) = \sum_{n=0}^{n=\infty} \frac{x_n}{p^{n+1}}$$

is a surjective connectivity morphism onto $\mathbf{I} = [0, 1]$, the connectivity space associated with the usual topological interval $[0, 1]$.

Example 7. More generally, if X is a set and $(T, \leq)$ is a totally ordered set, we define the integral connectivity space $\mathbf{B}_T(X)$ by $|\mathbf{B}_T(X)| = X^T$ and $\kappa(\mathbf{B}_T(X))^\bullet = \{K_{f,t}, (f, t) \in X^T \times T\}$ where $K_{f,t} = \{g \in X^T, \forall s \in T, s < t \Rightarrow g(s) = f(s)\}$. Then $\mathbf{B}_p = \mathbf{B}_{\{*\}}(p)$, and $\mathbf{HB}_p = \mathbf{B}_{\mathbf{N}}(p)$. If $\mathrm{card}(X) \geq 2$, then $\mathbf{B}_T(X)$ is a connected space iff T has a least element.

Example 8. Let $(X, \leq)$ be a totally ordered set. The set of all intervals (of any form) of X constitutes an integral connectivity structure on X, called the *order connectivity structure*. In particular, ordinal numbers define connectivity spaces, called the *ordinal connectivity spaces*.

2 Generating Connectivity Structures

2.1 The Theorem of Generation

Proposition 1. *Let X be a set, and Cnc_X (resp. $Cnct_X$) the set of connectivity structures on X (resp. the set of integral connectivity structures on X). For the order defined by*

$$\mathcal{X}_1 \leq \mathcal{X}_2 \Leftrightarrow \mathcal{X}_1 \subseteq \mathcal{X}_2,$$

$(Cnc_X, \leq)$ *and* $(Cnct_X, \leq)$ *are complete lattices.*

Proof. These ordered sets have $\mathcal{P}_X$ as a maximal element, and for each nonempty family $(\mathcal{X}_i)_{i\in I}$ of (integral) connectivity structures on X, $\bigcap_i \mathcal{X}_i$ is again an (integral) connectivity structure on X.

□

If $\mathcal{X}_1 \leq \mathcal{X}_2$, we say that $\mathcal{X}_1$ is *finer* than $\mathcal{X}_2$, or that $\mathcal{X}_2$ is *coarser* than $\mathcal{X}_1$. $\mathcal{P}_X$, the coarsest structure on X, is called the *indiscrete* structure on X. The finest connectivity structure contains only the empty set; it is called the *discrete* connectivity structure. The finest integral connectivity structure contains only the empty set and the singletons; it is called the *discrete* integral connectivity structure, or simply the discrete structure.

Remark 3. The lattices Cnc_X and $Cnct_X$ are not distributive, unless X has no more than two points. For example, if $X = \{1, 2, 3\}$ and, for each $i \in X$, $\mathcal{X}_i$ is the integral connectivity structure on X with $(X \setminus \{i\})$ as the only non trivial connected set, then $\bigvee_i(\mathcal{X}_i) = \mathcal{P}_X$, so $\mathcal{B}_3 \wedge (\bigvee_i(\mathcal{X}_i)) = \mathcal{B}_3$, while $\bigvee_i(\mathcal{B}_3 \wedge \mathcal{X}_i)$ is the discrete integral connectivity structure on X.

Definition 2. *Let X be a set, and $\mathcal{A} \in \mathcal{Q}_X$ a set of subsets of X. The finest connectivity structure (resp. integral connectivity structure) on X which contains $\mathcal{A}$ is called the* connectivity structure *(resp.* integral connectivity structure*)* generated *by $\mathcal{A}$ and is denoted by $[\mathcal{A}]_0$ (resp. $[\mathcal{A}]$).*

Thus, $[\mathcal{A}]_0 = \bigwedge\{\mathcal{X} \in Cnc_X, \mathcal{A} \subseteq \mathcal{X}\}$ and $[\mathcal{A}] = \bigwedge\{\mathcal{X} \in Cnct_X, \mathcal{A} \subseteq \mathcal{X}\}$.

Proposition 2. *Let X be a set, $\mathcal{A}$ a set of subsets of X, $(Y, \mathcal{Y})$ a connectivity space (resp. integral connectivity space) and $f : X \to Y$ a function. Then f is a connectivity morphism from $(X, [\mathcal{A}]_0)$ (resp. $(X, [\mathcal{A}])$) to $(Y, \mathcal{Y})$ if and only if $f(A) \in \mathcal{Y}$ for all $A \in \mathcal{A}$.*

Proof. $\{A \in \mathcal{P}_X, f(A) \in \mathcal{Y}\}$ is a connectivity structure on X containing $\mathcal{A}$ and then containing $[\mathcal{A}]_0$ (resp. $[\mathcal{A}]$).

□

The expression "*generated structure*" is justified by the next theorem, in which ω_0 denotes the smallest infinite ordinal.

Theorem 3 (Generation of connectivity structures)**.** *Let X be a set and $\mathcal{A} \in \mathcal{Q}_X$ a set of subsets of X. Then there exists an ordinal $\alpha_0 \leq \omega_0 + 1$ such that*

$$[\mathcal{A}]_0 = \Phi^\alpha(\mathcal{A}) \text{ for all } \alpha \geq \alpha_0,$$

where the Φ^α are the operators $\mathcal{Q}_X \to \mathcal{Q}_X$ defined by induction for every ordinal α by

- $\Phi^0 = id_{\mathcal{Q}_X}$,
- *if there is an ordinal β such that $\alpha = \beta + 1$, then $\Phi^\alpha = \Phi \circ \Phi^\beta$*
- *otherwise, for all $\mathcal{U} \in \mathcal{Q}_X$, $\Phi^\alpha(\mathcal{U}) = \bigcup_{\beta<\alpha} \Phi^\beta(\mathcal{U})$,*

and with Φ the operator defined for all $\mathcal{U} \in \mathcal{Q}_X$ by

$$\Phi(\mathcal{U}) = \{\emptyset\} \cup \{\bigcup_{A \in \mathcal{E}} A, \mathcal{E} \in \mathcal{L}_\mathcal{U}\},$$

where $\mathcal{L}_\mathcal{U} = \{\mathcal{E} \in \mathcal{P}(\mathcal{U}), \bigcap_{A \in \mathcal{E}} A \neq \emptyset\}$.

The integral connectivity structure $[\mathcal{A}]$ generated by $\mathcal{A}$ is obtained in the same way, adding the singletons of X at any stage of the process.

Proof. We only have to prove the part of the theorem concerning the generation of connectivity structures, the last claim about *integral* connectivity structure being then obvious.

For every $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{Q}_X$, the following three properties are easy to check:

- $\mathcal{U} \subseteq \Phi(\mathcal{U})$,
- $\mathcal{U} \subseteq \mathcal{V} \Rightarrow \Phi(\mathcal{U}) \subseteq \Phi(\mathcal{V})$,
- $\mathcal{U} \in Cnc_X \Leftrightarrow \Phi(\mathcal{U}) = \mathcal{U}$.

The first two properties then imply by induction that for all ordinal numbers α and β with $\alpha \leq \beta$, one has $\Phi^\alpha(\mathcal{U}) \subseteq \Phi^\beta(\mathcal{U})$, and the last two properties imply $\Phi(\mathcal{U}) \subseteq [\mathcal{U}]_0$ and, by induction, $\Phi^\alpha(\mathcal{U}) \subseteq [\mathcal{U}]_0$ for all ordinal numbers α. Then, if for an ordinal number α_0 the set $\Phi^{\alpha_0}(\mathcal{A})$ is a connectivity structure on X, it coincides with $[\mathcal{A}]_0$. So, to complete the proof, it suffices to verify that the set $\mathcal{C} = \Phi^{\omega_0+1}(\mathcal{A})$ is such a structure, *i.e.* $\Phi(\mathcal{C}) = \mathcal{C}$. For this, let $\mathcal{W}$ be the set $\Phi^{\omega_0}(\mathcal{A})$, so that $\mathcal{C} = \Phi(\mathcal{W})$. Then $\mathcal{W}$ is stable by union of *finite* families with nonempty intersections since $\Phi^{\omega_0}(\mathcal{A}) = \bigcup_{n\in\mathbf{N}} \Phi^n(\mathcal{A})$ so every such family is included in $\Phi^n(\mathcal{A})$ for some integer n, and its union is again in $\mathcal{W}$. Now, let $(S_u)_{u\in U}$ be any family of subsets of X belonging to $\mathcal{C}$ and such that $\bigcap_{u\in U} S_u \neq \emptyset$. We want to verify that $\bigcup_{u\in U} S_u \in \mathcal{C}$. For each $u \in U$, $S_u \in \mathcal{C}$ implies that there exists a family $(S_{u,i})_{i\in I_u}$ of subsets of X belonging to $\mathcal{W}$ such that $\bigcap_{i\in I_u} S_{u,i} \neq \emptyset$ and $\bigcup_{i\in I_u} S_{u,i} = S_u$. Let x be an element of $\bigcap_{u\in U} S_u$. For each $u \in U$, there exists an index $i_u \in I_u$ such that $x \in S_{u,i_u}$. For all $u \in U$ and $i \in I_u$, let $T_{u,i}$ be the set $S_{u,i} \cup S_{u,i_u}$. We have $S_{u,i} \in \mathcal{W}$, $S_{u,i_u} \in \mathcal{W}$ and $S_{u,i} \cap S_{u,i_u} \neq \emptyset$ (since $\bigcap_{i\in I_u} S_{u,i} \neq \emptyset$) so $T_{u,i} \in \mathcal{W}$ by the property of $\mathcal{W}$ we emphasized. Then $\bigcap_{u\in U, i\in I_U} T_{u,i} \neq \emptyset$, so $\bigcup_{u\in U, i\in I_U} T_{u,i} \in \Phi(\mathcal{W})$, that is $\bigcup_{u\in U} S_u \in \mathcal{C}$.

□

Remark 4. In the above proof, the existence of the families $(I_u)_{u\in U}$, $((S_{u,i})_{i\in I_u})_{u\in U}$ and $(i_u)_{u\in U}$ depends on the axiom of choice.

Example 9. Let X be the connectivity space such that $|X| = \mathbf{R}^2 \simeq \mathbf{C}$ and $\kappa(X) = [\mathcal{D}]_0$, where $\mathcal{D}$ is the set of open disks of the Euclidean plane $\mathbf{R}^2$. For $k \in \{1, 2, 3\}$, let $r_k = (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)^k$ be the cubic roots of unity. For each $(x_0, y_0) = z_0 \in \mathbf{C}$, let (z_n) be the sequence of complex numbers defined by the Newton's method for the equation $z^3 - 1 = 0$ and with first term z_0. If the sequence $(z_n)_{n\in\mathbf{N}}$ converges to r_k, we put $f(z_0) = k$, otherwise — in particular if the sequence (z_n) is defined only for a finite number of terms — we put $f(z_0) = 0$. Then the function $f : X \to \mathbf{B}_4$ defined in this way is a connectivity epimorphism. Indeed, the three basins of attraction $W_k = f^{-1}(k)$, $k \in \{1, 2, 3\}$, have the Wada property : their common boundary is the Julia set $W_0 = f^{-1}(0)$ (see [9]). If K is a nonempty element of the connectivity structure $[\mathcal{D}]_0$, it is open and connected for the usual topology of the plane and then either $K \subset W_k$ for a $k \in \{1, 2, 3\}$ and $f(K) = \{k\} \in \kappa(\mathbf{B}_4)$, or K intersects W_0 and then $f(K) = |\mathbf{B}_4|$ which is again in $\kappa(\mathbf{B}_4)$. Note that if we replace $\mathbf{B}_4$ by $(|\mathbf{B}_4|, \kappa(\mathbf{B}_4) \setminus \{\{0\}\})$, the function f is still a connectivity morphism. Moreover, it is easy to use this function f to define other surjective connectivity morphisms from the same connectivity plane X to the borromean space $\mathbf{B}_3$.

Example 10. There are several general ways to associate a connectivity space with each (partially) ordered set. We can for example define closed intervals of such a set exactly like in the totally ordered case, and then associate with each ordered set $(S, \leq)$ the connectivity space $(S, [\mathcal{J}])$ with $\mathcal{J}$ the set of closed intervals of S. In particular, for each topological construct and each set X, we obtain a connectivity space whose points are the structures on X.

2.2 Irreducibility

Definition 3. *Let X be a connectivity space. A connected subset K of $|X|$ is called* reducible *if it belongs to the connectivity structure generated by the others, that is*

$$K \in [\kappa(X) \setminus \{K\}]_0.$$

A nonempty connected subset of $|X|$ is said to be irreducible *if it is not reducible. The space X is said to be* irreducible *if $|X|$ is an irreducible connected subset of itself. It is said to be* distinguished *if each of its nonempty connected subsets is irreducible.*

Remark 5. With the notation of the theorem 3 we have either $\Phi(\kappa(X) \setminus \{K\}) = \kappa(X) \setminus \{K\}$, and then K is irreducible, or $\Phi(\kappa(X) \setminus \{K\}) = \kappa(X)$. In any case, $[\kappa(X) \setminus \{K\}]_0 = \Phi(\kappa(X) \setminus \{K\})$, and K is reducible iff there is a family $\mathcal{E}$ of proper connected subsets $A \subsetneq K$ such that $\bigcap_{A \in \mathcal{E}} A \neq \emptyset$ and $K = \bigcup_{A \in \mathcal{E}} A$.

Remark 6. A connected singleton is necessarily irreducible.

Example 11. If X is a finite connectivity space, a subset K of $|X|$ is reducible iff there are two connected subsets $A \subsetneqq \kappa(X)$ and $B \subsetneqq \kappa(X)$ such that

$$K = A \cup B \text{ and } A \cap B \neq \emptyset.$$

Example 12. The only irreducible connected subsets of $\mathbf{R}$ are the trivial ones.

Example 13. Brunnian spaces and hyperbrunnian spaces are connected and distinguished spaces. Nevertheless, note that $\mathbf{B}_T(X)$ is not a distinguished space for every set X and every totally ordered set T. For example, $\mathbf{B}_{[0,1]}(\{a,b\})$ is not a distinguished space, since $\{f \in \{a,b\}^{[0,1]}, \exists \epsilon \in]0,1], t < \epsilon \Rightarrow f(t) = a\}$ is a connected subset which is reducible.

Definition 4. *Let X be a connectivity space. Its* Brunnian closure *is* $\overline{X} = (|X|, \kappa(X) \cup \{|X|\})$.

Example 14. $\mathbf{B}_n$ is the Brunnian closure of the n-points discrete integral space. $\mathbf{HB}_n$ is the Brunnian closure of the disjoint union (*cf. infra*, section 3.2) of n copies of itself.

The next proposition is obvious.

Proposition 4. *If X is a nonempty irreducible space, then $(|X|, \kappa(X) \setminus \{X\})$ is a connectivity space. If X is a non-connected connectivity space, then $\overline{X}$ is an irreducible connected space.*

Because of the next proposition, the notion of irreducibility will play a fundamental role in the case of finite connectivity spaces.

Proposition 5. *A connectivity structure on a given finite set is characterised by the set of the irreducible connected subsets, which is the minimal set of subsets which generates this structure.*

Proof. For any connectivity space X, let $\iota(X)$ denote the set of the irreducible connected subsets of X. Then, for any $\mathcal{A} \in \mathcal{Q}_X$ such that $[\mathcal{A}]_0 = \kappa(X)$, one has $\mathcal{A} \supseteq \iota(X)$ since, by construction, each set $C \in [\mathcal{A}]_0$ which is not in $\mathcal{A}$ is reducible. On the other hand, an easy induction shows that, for every integer k, every reducible connected subset of X with cardinal smaller than k is an element of $[\iota(X)]_0$. Thus, if X is finite, $\kappa(X) = [\iota(X)]_0$.

□

2.3 Connectivity Spaces and Hypergraphs

A hypergraph is a set of vertices endowed with a set of nonempty sets of vertices, these sets of vertices being considered as generalized edges, the so-called *hyperedges*. There is some similarity between hypergraphs and connectivity spaces — for example it is possible to consider Borromean structures in both cases — but

- the union of two hyperedges with a nonempty intersection is not necessarily an hyperedge, so hyperedges are not the same as connected subsets,
- the union of two hyperedges with a nonempty intersection can be an hyperedge, so hyperedges are not the same as irreducible connected subsets.

To clarify the relation between the two concepts, let us consider the category **HypG** of hypergraphs, that is the category whose objects are the pairs $(X, \mathcal{H})$ with X a set and $\mathcal{H} \in \mathcal{Q}_X$ a set whose elements are called hyperedges, and whose morphisms $f : (X, \mathcal{H}) \to (X', \mathcal{H}')$ are functions $X \to X'$ which preserve hyperedges : $H \in \mathcal{H} \Rightarrow f(H) \in \mathcal{H}'$. Then the proposition 2 implies

Corollary 6. *The category* **Cnc** *is concrete on* **HypG** *with a forgetful functor admiting as a left adjoint the functor* **HypG** $\to$ **Cnc** *which associates with each hypergraph* $(X, \mathcal{H})$ *the space whose connectivity structure is generated by* $\mathcal{H}$*, i.e.* $(X, [\mathcal{H}]_0)$*, and with each morphism itself as a connectivity morphism. Similarly, the generation of integral connectivity structures* $[\mathcal{H}]$ *from sets* $\mathcal{H} \in \mathcal{Q}_X$ *defines a left adjoint to the forgetful functor* **Cnct** $\to$ **HypG***, and the situation is the same between finite hypergraphs and finite connectivity spaces.*

3 Limits and Colimits

3.1 Categories with Lattices of Structures

Let **JCPos** be the category of complete (small) lattices and join-preserving maps. If S is a functor from a category $\mathbf{X}$ to **JCPos**, $S(X)$ or S_X will denote the lattice associated by S with an object X, and (while it is unambigous) $f_!$ the map between lattices associated by S with a morphism f. The elements of the lattice S_X will be called the S-structures on X.

Definition 5. *Let* $\mathbf{X}$ *be a category and* $S : \mathbf{X} \to \mathbf{JCPos}$ *a functor. The category* $\mathbf{X}_S$*, which we shall refer to as* the category with lattices of structures associated with S *or more briefly as* the category structured by S, *is defined as follows. Its objects are pairs* $(X, \mathcal{X})$ *with* X *an object of* $\mathbf{X}$ *and* $\mathcal{X} \in S_X$ *an* S*-structure. A morphism* $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ *is an* $\mathbf{X}$*-morphism* $f : X \to Y$ *such that* $f_!(\mathcal{X}) \leq \mathcal{Y}$ *in the lattice* S_Y*.*

In the category $\mathbf{X}_S$, spaces $(X, 1_{S_X})$ are called *indiscrete* spaces, and spaces $(X, 0_{S_X})$ are called *discrete* spaces. If, in the lattice S_X, we have $\mathcal{X} \leq \mathcal{X}'$, then the structure $\mathcal{X}$ is said to be *finer* than $\mathcal{X}'$ and the latter is said to be *coarser* than the former.

Remark 7. An equivalent definition is given by considering *contravariant* functors from the basis category $\mathbf{X}$ to the category **MCPos** of complete (small) lattices and meet-preserving maps: an object of the category defined by such a functor T is a pair $(X, \mathcal{X})$ with $\mathcal{X} \in T_X$, and a morphism $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ is a $\mathbf{X}$-morphism $f : X \to Y$ such that $\mathcal{X} \leq f^*(\mathcal{Y})$, where $f^* = T(f)$. Then for each covariant $S : \mathbf{X} \to \mathbf{JCPos}$, there is an *associated contravariant functor* T defining the category $\mathbf{X}_S$ in this way. This functor T is defined on objects X by $T_X = S_X$ and on $\mathbf{X}$-morphisms $f : X \to Y$ by $T(f) = f^*$ with, for each $\mathcal{Y} \in T_Y$,

$$f^*\mathcal{Y} = \bigvee \{\mathcal{X} \in T_X, f_!\mathcal{X} \leq \mathcal{Y}\}.$$

In the next proposition, we use the definition of a topological category given in [1] : a topological category on $\mathbf{X}$ is a concrete category $U : \mathbf{A} \to \mathbf{X}$ (that is, a faithful functor U), such that every U-source $(X \to UA_i)_{i \in I}$ in $\mathbf{X}$ has a unique U-initial lift $(A \to A_i)_{i \in I}$ in $\mathbf{A}$.

Proposition 7. *A category is a small-fibred topological one if and only if it is a category with lattices of structures. More precisely :*

- *For each functor $S : \mathbf{X} \to \mathbf{JCPos}$, the functor $U : \mathbf{X}_S \to \mathbf{X}$ defined by $U(X, \mathcal{X}) = X$ and $Uf = f$ is a small-fibred topological category.*
- *Each small-fibred topological category $U : \mathbf{A} \to \mathbf{X}$ is isomorphic to the category $\mathbf{X}_S$ with S the functor defined for each object X of $\mathbf{X}$ by the fibre $S_X = \{A \in \mathbf{A}, UA = X\}$ with the usual order (i.e. $A_1 \leq A_2$ iff id_X has a lift $A_1 \to A_2$), and for each arrow $f : X \to Y$ in $\mathbf{X}$ and each $A \in S_X$ by $f_!(A) = \wedge\{B \in S_Y, f$ has lift $A \to B\}$.*

Proof. Let $S : \mathbf{X} \to \mathbf{JCPos}$ be any functor. The functor $U : \mathbf{X}_S \to \mathbf{X}$ defined by $U(f : (X, \mathcal{X}) \to (Y, \mathcal{Y})) = (f : X \to Y)$ is trivially faithful, its fibres are the sets S_X, and it is topological : each U-source $(f_i : X \to UA_i)_{i \in I}$ has a unique U-initial lift, that is $(f_i : (X, \mathcal{X}_0) \to A_i)_{i \in I}$, where $\mathcal{X}_0$ is the coarsest S-structure on X such that all f_i be (have lifts as) $\mathbf{X}_S$-morphisms, that is $\mathcal{X}_0 = \wedge_i f_i^*(\mathcal{Y}_i)$ where $\mathcal{Y}_i$ is the S-structure of A_i and, for each $f : X \to Y$ and each $\mathcal{Y} \in S_Y$,$f^*(\mathcal{Y})$ is the coarsest S-structure $\mathcal{X}$ on X such that f is an $\mathbf{X}_S$-morphism $(X, \mathcal{X}) \to (Y, \mathcal{Y})$, that is $f^*(\mathcal{Y}) = \vee\{\mathcal{X} \in S_X, f_!(\mathcal{X}) \leq \mathcal{Y}\}$.

On the other hand, let now $U : \mathbf{A} \to \mathbf{X}$ be a topological category with small fibres. One knows (see [1]) that such fibres S_X are then complete lattices. We can remark also that, for a given $f : X \to Y$ in $\mathbf{X}$ and an object $A \in S_X$, the set $\{B \in S_Y, f$ has a lift $A \to B\}$ is nonempty, because Y has an indiscrete lift. Then $f_!$ is well-defined as a function. Now, if $(A_i)_{i \in I}$ is any family in the fibre S_X, and $B \in S_Y$ is such that $f : X \to Y$ has a lift $\vee_i A_i \to B$, then id_X has a lift $A_i \to \vee_i A_i$ for each i, so f has a lift $A_i \to B$ for each i. On the other hand, if $f : X \to UB$ has a lift $A_i \to B$ for each i, then $\forall i \in I, A_i \leq A$, where the U-initial lift of f is $A \to B$; but $\vee_i A_i \leq A$, so id_X has a lift $\vee_i A_i \to A$ and f has a lift $\vee_i A_i \to B$. Thus, for a given $f : X \to Y$ and a given family $(A_i)_{i \in I}$ in S_X, we have $\{B \in S_Y, f$ has a lift $\vee_i A_i \to B\} = \{B \in S_Y, \forall i \in I, f$ has a lift $A_i \to B\}$. Let $\beta_i = f_!(A_i) = \wedge\{B \in S_Y, f$ has a lift $A_i \to B\}$. Then

$$f_!(\vee_i A_i) = \wedge\{B \in S_Y, \forall i \in I, f \text{ has a lift } A_i \to B\}$$

$$= \wedge\{B \in S_Y, \forall i \in I, B \geq \beta_i\} = \vee_i \beta_i,$$

so $f_!(\vee_i A_i) = \vee_i f_!(A_i)$: $f_!$ is a **JCPos**-morphism, and the functor S is well-defined. It is then easy to verify that the functor $\mathbf{A} \to \mathbf{X}_S$ defined by

$$(f : A \to B) \mapsto (Uf : (UA, A) \to (UB, B))$$

is an isomorphism of categories, with inverse

$$(f : (X, \mathcal{X}) \to (Y, \mathcal{Y})) \mapsto (\tilde{f} : \mathcal{X} \to \mathcal{Y}),$$

where $\tilde{f}$ is the lift of f, which exists since $f_!(\mathcal{X}) \leq \mathcal{Y}$.

□

By proposition 21.15, theorem 21.16 and corollary 21.17 of [1], we have then

Corollary 8. *If* $\mathbf{X}$ *denotes the category* **Set** *of sets (resp. the category* **fSet** *of finite sets),* $S : \mathbf{X} \to \mathbf{JCPos}$ *any functor,* $T : \mathbf{X}^{op} \to \mathbf{MCPos}$ *the contravariant functor associated with* S *and* $U : \mathbf{A} = \mathbf{X}_S \to \mathbf{X}$ *the construct*[4] *(resp. "finite" construct) defined by* S*, then the following hold*

1. $\mathbf{A}$ *is (co)complete (resp. finitely (co)complete),*

2. U *has a left adjoint* O *(the discrete structure) and a right adjoint* I *(the indiscrete structure) :* $O \dashv U \dashv I$*, so* U *preserves (co)limits,*

3. *the limit* $(l_i : L \to D_i)_{i \in \mathbf{I}}$ *of a small (resp. finite) diagram* $D : \mathbf{I} \to \mathbf{A}$ *is the initial lift of the underlying limit in* $\mathbf{X}$*, that is: if* $(l_i : |L| \to UD_i)_{i \in \mathbf{I}}$ *is the limit of* UD*, then* $L = (|L|, \bigwedge_{i \in \mathbf{I}} l_i^*(\mathcal{X}_i))$*, where* $D_i = (X_i, \mathcal{X}_i)$ *and* $l_i^* = T(l_i)$*,*

4. *colimits are given in the same way, as final lifts: if* $(c_i : |C| \leftarrow UD_i)_{i \in \mathbf{I}}$ *is the colimit of* UD*, then the colimit of* D *in* $\mathbf{A}$ *is* $(c_i : C \leftarrow D_i)_{i \in \mathbf{I}}$ *with* $C = (|C|, \bigvee_{i \in \mathbf{I}} c_{i!}(\mathcal{X}_i))$*, where* $D_i = (X_i, \mathcal{X}_i)$ *and* $c_{i!} = S(c_i)$*,*

[4] See *supra* the note 2.

5. **A** *is wellpowered and cowellpowered,*

6. **A** *is an* $(Epi, ExtremalMonoSource)$*-category,*

7. **A** *has regular factorizations, i.e. is an* $(RegEpi, MonoSource)$*-category (and thus is, in particular, a* $(RegEpi, Mono)$*-category),*

8. *in* **A***, the classes of embeddings (i.e. initial monomorphisms), of extremal monomorphisms and of regular monomorphisms coincide,*

9. *in* **A***, the classes of quotient morphisms (i.e. final epimorphisms), of extremal epimorphisms and of regular epimorphisms coincide,*

10. **A** *has separators and coseparators.*

Example 15. Let $\mathcal{P} : \mathbf{Set} \to \mathbf{JCPos}$ be the (covariant) functor which associates with each set the complete lattice of its subsets. For any functor $T : \mathbf{X} \to \mathbf{Set}$, the category $\mathbf{X}_{\mathcal{P}T}$ structured by the functor $\mathcal{P} \circ T : \mathbf{X} \to \mathbf{JCPos}$ coincides with the topological category $\mathbf{Spa}(T)$ of T-spaces on $\mathbf{X}$ ([1], p. 76). Thus, the "functor-structured categories" $\mathbf{Spa}(T)$ are special cases of the categories structured by functors $\mathbf{X} \to \mathbf{JCPos}$. In particular, for $T = \mathcal{P}$, we obtain $\mathbf{Spa}(\mathcal{P}) = \mathbf{Set}_{\mathcal{Q}} = \mathbf{HypG}$.

3.2 (Co)limits in the Categories of Connectivity spaces

In [2], Börger showed that

Proposition 9. Cnct *is a topological category. It is not cartesian closed.*

It is easy to check that, as a category with lattice of structures, **Cnct** is defined by the covariant functor $Cnct : \mathbf{Set} \to \mathbf{JCPos}$ such that $Cnct_X$ is the lattice of all integral connectivity structures on X and, for every $f : X \to X'$, $Cnct(f) = f_!$ is the **JCPos**-morphism $Cnct_X \to Cnct_{X'}$ such that, for all $\mathcal{K} \in Cnct_X$,

$$f_!(\mathcal{K}) = [\{f(K), K \in \mathcal{K}\}]. \quad (1)$$

Equivalently, the contravariant definition of **Cnct** is given, for all $\mathcal{K}' \in Cnct_{X'}$, by

$$f^*(\mathcal{K}') = \{K \in \mathcal{P}_X, f(K) \in \mathcal{K}'\}. \tag{2}$$

The same formulas hold on **fSet**, defining a functor $fCnct$ such that $\mathbf{fCnct} = \mathbf{fSet}_{fCnct}$, which is thus a topological category on **fSet**. For **Cnc**, it suffices to use $[\{f(K), K \in \mathcal{K}\}]_0$ instead of $[\{f(K), K \in \mathcal{K}\}]$ in the expression of $f_!$ to define a functor Cnc such that $\mathbf{Cnc} = \mathbf{Set}_{Cnc}$, which is thus a topological construct[5].

From the formula (1) and the corollary 8, we deduce that the connectivity structure $\kappa(C)$ of the colimit C of a small diagram $D : \mathbf{I} \to \mathbf{Cnct}$ is given by $\kappa(C) = \bigvee_{i\in\mathbf{I}}[\{c_i(K), K \in \kappa(D_i)\}]$ and then

$$\kappa(C) = [\{c_i(K), i \in \mathbf{I}, K \in \kappa(D_i)\}], \tag{3}$$

where the $c_i : |D_i| \to |C|$ are the coprojections. The same formula holds for colimits of finite diagrams in **fCnct**, and, using $[-]_0$ instead of $[-]$, for small diagrams in **Cnc**.

From the formula (2), one likewise deduces the connectivity structure $\kappa(L)$ of the limit L of a small diagram $D : \mathbf{I} \to \mathbf{Cnct}$,

$$\kappa(L) = \bigcap_{i\in\mathbf{I}}\{K \in \mathcal{P}_{|L|}, l_i(K) \in \kappa(D_i)\}, \tag{4}$$

where the $l_i : |L| \to |D_i|$ are the projections. The same formula holds for limits of small diagrams in **Cnc** and of finite diagrams in **fCnct**.

For example, the cartesian product $C_1 \times C_2$ of two connectivity spaces is characterised by $|C_1 \times C_2| = |C_1| \times |C_2|$ and

$$\kappa(C_1 \times C_2) = \{A \in \mathcal{P}(|C_1| \times |C_2|), \pi_i(A) \in \kappa(C_i) \text{ for } i \in \{1, 2\}\},$$

where the π_i are the projections, whereas the coproduct, or disjoint union, satisfies $|C_1 \amalg C_2| = |C_1| \amalg |C_2|$ and $\kappa(C_1 \amalg C_2) = \kappa(C_1) \amalg \kappa(C_2)$.

With those formulas, it is easy to check that none of the three categories considered here is cartesian closed. It suffices to exhibit a colimit

[5] **Cnc** is not *well-fibred*, so it is not a topological category according to the definition given in 1983 by Herrlich [10], but, as we said, we use here the less restrictive definition finally retained by Herrlich, Adámek and Strecker in [1].

which is not preserved by a product, and this can be done simultaneously in the three categories. For example, let $\{a, *, b\}$ be a set with three distinct elements, A_u be the indiscrete connectivity space defined for each $u \in \{a, b\}$ by its carrier $|A_u| = \{*, u\}$, and B the space with carrier $\{1, 2, 3\}$ and with structure $[\{\{1, 2\}, \{2, 3\}\}]$. Then, in each of the categories concerned, the colimit C of the diagram $A_a \hookleftarrow \{*\} \hookrightarrow A_b$ (with arrows the inclusions) is $C = (\{a, *, b\}, [\{\{a, *\}, \{*, b\}\}])$, its product $C \times B$ with B is the cartesian product $\{a, *, b\} \times \{1, 2, 3\}$ endowed with the integral connectivity structure including all subsets having their two projections connected. For example, the set $\{(a, 1), (*, 3), (b, 2)\}$ is connected in $C \times B$; but it is easy to verify that the same set is not connected in the colimit of the diagram $A_a \times B \hookleftarrow \{*\} \times B \hookrightarrow A_b \times B$. Thus, in each of the categories considered, the endofunctor $- \times B$ does not preserve colimits. We thus proved

Proposition 10. **Cnc** *and* **fCnct** *are topological categories; they are not cartesian closed.*

3.3 Quotients and Embeddings

This section gives trivial but useful consequences of the corollary 8 and of the formulas (1) and (2).

Proposition 11. *In* **Cnct** *and* **fCnct** *(resp.* **Cnc***), a morphism* $f : A \to B$ *is a* regular epimorphism *iff* $|f|$ *is surjective and* $\kappa(B) = [f(\kappa(A))]$ *(resp.* $\kappa(B) = [f(\kappa(A))]_0$*). In* **fCnct**, **fCnct** *and* **Cnc**, *a morphism* $f : A \to B$ *is a* regular monomorphism *iff* $|f|$ *is injective and* $\kappa(A) = \{K \in \mathcal{P}_{|A|}, f(K) \in \kappa(B)\}$.

Now, in every topological construct, a regular epimorphism, *i.e.* a coequalizer, is the same as a quotient morphism, *i.e.* a final morphism which is surjective as a function, and can also be viewed as (the unique final lift of) the canonical map associated with an equivalence relation. This remark results in the definition of the quotient of a connectivity space by an equivalence relation.

Definition 6 (Quotient by an equivalence relation)**.** *If C is a connectivity space and $\sim$ is an equivalence relation on $|C|$, the* quotient space

$C/\sim$ is defined by $|C/\sim| = |C|/\sim$ and

$$\kappa(C/\sim) = s_!(\kappa(C)) = [s(\kappa(C))]_0 \tag{5}$$

where s is the canonical map $s : |C| \twoheadrightarrow |C|/\sim$. In particular, if T is a subset of $|C|$, C/T denotes the space $C/\sim_T$.

Remark 8. Note that if C is an integral connectivity space, then for any surjective map $s : |C| \twoheadrightarrow Y$ we have $[s(\kappa(C))]_0 = [s(\kappa(C))]$.

Likewise, in every topological construct, a regular monomorphism, *i.e.* an equalizer, is the same as an embedding, *i.e.* an initial morphism which is injective as a function, and can also be viewed as (the unique initial lift of) the inclusion map of a subspace. This leads to the definition of the connectivity structure induced by a connectivity space on a subset of its carrier.

Definition 7 (Structure induced on a subset)**.** *If C is a connectivity space and S is a subset of $|C|$, the* connectivity space induced *on S by C is the space $C_{|S}$ defined by $|C_{|S}| = S$ and*

$$\kappa(C_{|S}) = i^*(\kappa(C)) = \mathcal{P}_S \cap \kappa(C) \tag{6}$$

where i is the inclusion map $i : S \hookrightarrow |C|$.

4 Tensor Product of Connectivity Spaces

The formula (4) suggests that the cartesian product of connectivity spaces is in some way "too coarse" to be really useful in algebra. For example, let $\mathbf{N}$ be the set of natural numbers with the integral connectivity structure generated by the subsets $\{n, n+1\}$; it is easy to check that the addition $+ : \mathbf{N}^2 \to \mathbf{N}$ is not a connectivity morphism (when $\mathbf{N}^2$ is endowed with the cartesian square structure of $\mathbf{N}$). Likewise for the addition of real numbers. This section presents a more interesting connectivity product than the cartesian one for algebraic structures.

Let X_i $(i = 1, 2)$ and Y be connectivity spaces. For each $x_1 \in |X_1|$ (resp. $x_2 \in |X_2|$), we denote by $f(x_1, -)$ (resp. $f(-, x_2)$) the partial function associated with a given function $f : |X_1| \times |X_2| \to |Y|$.

Definition 8. *A function* $f : |X_1| \times |X_2| \to |Y|$ *is said to be* partially connecting *from* $X_1 \times X_2$ *to* Y *if* $f(x_1, -) : X_2 \to Y$ *and* $f(-, x_2) : X_1 \to Y$ *are connectivity morphisms for all* $x_1 \in |X_1|$ *and all* $x_2 \in |X_2|$.

Definition 9. *The* connectivity tensor product $X_1 \boxtimes X_2$ *of two connectivity spaces* X_i $(i = 1, 2)$ *is the space with carrier* $|X_1 \boxtimes X_2| = |X_1| \times |X_2|$ *and with connectivity structure* $\kappa(X_1 \boxtimes X_2) = [\{K_1 \times K_2, (K_1, K_2) \in \kappa(X_1) \times \kappa(X_2)\}]_0$.

For every connectivity space X_i, $\kappa(X_1 \boxtimes X_2)$ is a finer connectivity structure on the set $|X_1| \times |X_2|$ than the one given by the connectivity cartesian product, since $K_1 \times K_2 \in \kappa(X_1 \times X_2)$ for each connected subsets K_1 and K_2. Thus, $id : X_1 \boxtimes X_2 \to X_1 \times X_2$ is a bijective connectivity morphism (but it is of course not an isomorphism in general). If X_1 and X_2 are *integral* connectivity spaces, then its inverse function, that is the function from $X_1 \times X_2$ to $X_1 \boxtimes X_2$ defined by $\tau(x_1, x_2) = (x_1, x_2)$, is a partially connecting function.

Theorem 12. *Let* X_1 *and* X_2 *be integral connectivity spaces,* Y *a connectivity space, and* $f : |X_1| \times |X_2| \to |Y|$ *a function. Then* f *is a partially connecting function from* $X_1 \times X_2$ *to* Y *if and only if it is a connectivity morphism from* $X_1 \boxtimes X_2$ *to* Y*,* i.e. *there exists a unique connectivity morphism* $\tilde{f} : X_1 \boxtimes X_2 \to Y$ *such that* $\tilde{f} \circ \tau = f$.

Proof. If $\tilde{f}$ is a connectivity morphism, then $\tilde{f} \circ \tau = f$ is a partially connecting function since τ is such a function. On the other hand, let f be a partially connecting function from $X_1 \times X_2$ to Y. Unicity of $\tilde{f}$ being obvious, since necessarily $\tilde{f}(x_1, x_2) = f(x_1, x_2)$, it suffices to check that this function is a connectivity morphism on $X_1 \boxtimes X_2$. Then, according to the proposition 2, it suffices to check that for every $K_i \in \kappa(X_i)$, $f(K_1 \times K_2) \in \kappa(Y)$. Let $K_1 \times K_2$ be such nonempty subset of $|X_1| \times |X_2|$, and let $x_1^0 \in K_1$. f being partially connecting, the sets $V = \{f(x_1^0, x_2), x_2 \in K_2\}$ and $H_{x_2} = \{f(x_1, x_2), x_1 \in K_1\}$ are, for all $x_2 \in K_2$, in $\kappa(Y)$. So are the sets $V \cup H_{x_2}$ (as $V \cap H_{x_2} \neq \emptyset$), and $\bigcup_{x_2 \in K_2}(V \cup H_{x_2})$; that is: $\tilde{f}(K_1 \times K_2) \in \kappa(Y)$.

□

Example 16. Let $f : \mathbf{R}_+^2 \to \mathbf{R}$ defined by

- $f(0,0) = 0$,
- for all x and y, $f(x,y) = f(y,x)$,
- $\forall x > 0, \forall y \in [0,x], f(x,y) = y/x$.

Then f is a partially connecting map since it is "partially continuous", but it is not continuous, and neither $\Delta = \{(x,x), x \geq 0\}$ nor $f(\Delta) = \{0,1\}$ are connected subsets of, respectively, $\mathbf{R}_+ \boxtimes \mathbf{R}_+$ and $\mathbf{R}$.

Note that for each integral connectivity space X, one has an endofunctor $X \boxtimes - : \mathbf{Cnct} \to \mathbf{Cnct}$ defined for each integral connectivity space Y by $X \boxtimes Y$ and for each connectivity morphism $g : Y_1 \to Y_2$ between integral connectivity spaces by $(X \boxtimes g)(x, y_1) = (x, g(y_1))$.

Now, let us define another endofunctor on **Cnct**. For every subset M of the set $Hom(X,Y)$ of connectivity morphisms from a connectivity space X to a connectivity space Y, and for every subset A of the set $|X|$, let $\langle M, A\rangle$ denotes $\bigcup_{f\in M} f(A)$. Then, for each integral connectivity space X, there is an endofunctor $\mathbf{Cnct}(X,-) : \mathbf{Cnct} \to \mathbf{Cnct}$ defined for every integral connectivity space Y by

- $|\mathbf{Cnct}(X,Y)| = Hom(X,Y)$,
- $\kappa(\mathbf{Cnct}(X,Y)) = \{M \in \mathcal{P}(Hom(X,Y)), \forall K \in \kappa(X), \langle M, K\rangle \in \kappa(Y)\}$,

and for every connectivity morphism $g : Y_1 \to Y_2$ by $\mathbf{Cnct}(X,g) = g_*$ such that

$$\forall \varphi \in \mathbf{Cnct}(X, Y_1), g_*(\varphi) = g \circ \varphi.$$

Remark 9. A set M of connectivity morphisms between two integral connectivity spaces X and Y is connected, that is belongs to $\kappa(\mathbf{Cnct}(X,Y))$, if (and only if) for all $x \in X$, $\langle M, \{x\}\rangle \in \kappa(Y)$. Indeed, if this condition is satisfied, then for every nonempty connected subset K of X and any $x \in K$, one has $\langle M, K\rangle = \bigcup_{f\in M}(f(K) \cup \langle M, \{x\}\rangle) \in \kappa(Y)$.

Theorem 13. *For every integral connectivity space X, the endofunctor $X \boxtimes -$ is left adjoint to the endofunctor* $\mathbf{Cnct}(X,-)$*. Thus,* $(\mathbf{Cnct}, \boxtimes)$ *is a closed symmetric monoidal category.*

Proof. The product $\boxtimes$ is obviously symmetric. Let X, Y and Z be integral connectivity spaces. For every connectivity morphism $\psi : X \boxtimes Y \to Z$, one has a morphism $\rho(\psi) : Y \to \mathbf{Cnct}(X, Z)$ defined for all $y \in Y$ by $\rho(\psi)(y) = \psi(-, y)$. Then ρ is clearly a bijection between the sets $Hom(X \boxtimes Y, Z)$ and $Hom(Y, \mathbf{Cnct}(X, Z))$, and it is natural since for all integral connectivity spaces Y, Y', Z and Z' and for all connectivity morphisms $u : Y \to Y'$, $v : Z \to Z'$ and $\psi : X \boxtimes Y' \to Z$, one has $\rho(v \circ \psi \circ (X \boxtimes u)) = \rho((x, y) \mapsto v(\psi(x, u(y)))) = (y \mapsto v \circ \psi(-, u(y)) = \mathbf{Cnct}(X, v) \circ \rho(\psi) \circ u$.

□

5 Homotopy

Let $\overrightarrow{I}$ be a triple $(I, 0, 1)$ with I a nonempty integral connectivity space, and 0 and 1 some elements of $|I|$. In particular, let $\mathbf{I}$ be the connectivity space associated with the usual topological space $[0, 1]$, and $\overrightarrow{\mathbf{I}} = (\mathbf{I}, 0, 1)$.

Definition 10 (Homotopy)**.** *Let X and Y be integral connectivity spaces, and $f, g : X \to Y$ some connectivity morphisms. The function g is said to be $\overrightarrow{I}$-homotopic to f provided there exists a connectivity morphism*

$$h : I \to \mathbf{Cnct}(X, Y)$$

such that $h(0) = f$ and $h(1) = g$. In particular, in the case of $\overrightarrow{I} = \overrightarrow{\mathbf{I}}$, g is simply said to be homotopic to f.

We denote by $f \sim g$ the homotopy relation between connectivity morphisms. Like in the topological case, it is obviously an equivalence relation. The adjoint situation $(X \boxtimes -) \dashv \mathbf{Cnct}(X, -)$ leads to an alternative definition of homotopy for connectivity morphisms.

Definition 11 (Alternative definition of homotopy)**.** *Let X and Y be integral connectivity spaces. A function $g : X \to Y$ is $\overrightarrow{I}$-homotopic to $f : X \to Y$ provided there exists a connectivity morphism $h : I \boxtimes X \to Y$ such that $h(0, -) = f$ and $h(1, -) = g$, that is a function $h : I \times X \to Y$ such that*

- $h(0, -) = f$ *and* $h(1, -) = g$,
- $\forall t \in I, \forall K \in \kappa(X), h(t, K) \in \kappa(Y)$,
- $\forall D \in \kappa(I), \forall x \in X, h(D, x) \in \kappa(Y)$.

Definition 12 (Contractibility)**.** *An integral connectivity space X is said to be* contractible *provided the identity map $id : X \to X$ of the space be homotopic to a constant map $c : X \to X$.*

Examples. The connectivity space associated with the usual topological circle $S^1 = \{e^{i\theta}, \theta \in [0, 2\pi]\} \subset \mathbf{C}$ is contractible. Indeed, the function $h : \mathbf{I} \times S^1 \to S^1$ defined by

- for $t \in [0, 1[$ and $z \in S^1$, $h(t, z) = z.e^{i\frac{t}{1-t}}$,
- $\forall z \in S^1, h(1, z) = 1$,

realizes an homotopy between the identity of the circle and the constant function $z \mapsto 1 \in S^1$.

More generally, the same kind of argument shows that every n-sphere is contractible. On the other hand, there exist a connected connectivity space X such that no two distinct connectivity endomorphisms $X \to X$ are homotopic. For example, if $X = \mathcal{P}(\mathbf{R})$ is endowed with the integral connectivity structure for which non trivial connected subsets are subsets with a cardinal greater than the one of $\mathbf{R}$, then non-trivial connected subsets of $\mathbf{Cnct}(X, X)$ also have such a cardinal, and then every connectivity morphism from $\mathbf{I}$ to $\mathbf{Cnct}(X, X)$ is a constant function.

Those examples show that any theory of homotopy in the connectivity framework should be very different from the topological one. In particular, it could be interesting to use different kinds of discrete times instead of $\mathbf{I}$.

6 Pointed Connectivity Spaces

6.1 Pointed Sets

The category **pSet** of pointed sets and based maps is a concrete category on **Set**. The forgetful functor **pSet** $\to$ **Set** will be denoted by $|-|$, and the base-point of a pointed set P by $\beta(P)$, so $P = (|P|, \beta(P))$.

pSet has a zero object, $(\{*\}, *)$, it is complete and cocomplete. In particular, the cartesian product of two pointed sets P_1 and P_2 is defined by $|P_1 \times P_2| = |P_1| \times |P_2|$ and $\beta(P_1 \times P_2) = (\beta(P_1), \beta(P_2))$. The class of coequalizers coincides with the class of all epimorphisms, *i.e.* surjective based maps, and with the class of quotient morphisms (in **pSet** every morphism is final). If $\sim$ is an equivalence relation on $|P|$, the quotient pointed set $P/\sim$ is defined by $|P/\sim| = |P|/\sim$ and $\beta(P/\sim) = \widetilde{\beta(P)}$. In particular, if T is a subset of $|P|$, P/T denotes the pointed set $P/\sim_T$. The coproduct of P_1 and P_2 is denoted by $P_1 \vee P_2$. It can be defined either as the quotient of the set $|P_1| \amalg |P_2|$ by the equivalence relation which identifies $\beta(P_1)$ and $\beta(P_2)$ or alternatively by the formulas

$$|P_1 \vee P_2| = (|P_1| \times \{\beta(P_2)\}) \cup (\{\beta(P_1)\} \times |P_2|) \tag{7}$$

and

$$\beta(P_1 \vee P_2) = (\beta(P_1), \beta(P_2)).$$

The category **pSet** is not cartesian closed since, for example, if P is a pointed set with two elements and Q is the zero object, then $P \times (Q \vee Q) \simeq P$ whereas $(P \times Q) \vee (P \times Q)$ has three elements. Nevertheless, the set of based maps from a pointed set P to a pointed set Q has a "natural" special point, that is the constant map $x \mapsto \beta(Q)$, so there is a "natural" object in **pSet** representing $Hom(P, Q)$. Let

$$\mathbf{pSet}(P, Q) = (Hom(P, Q), x \mapsto \beta(Q))$$

denotes this object. For each pointed set P, we then have an endofunctor $\mathbf{pSet}(P, -)$ on **pSet**, with $\mathbf{pSet}(P, f) = f \circ -$. One knows that this functor has a left adjoint $P \wedge -$, the so-called *smash product*, defined on objects by

$$P \wedge Q = (P \times Q)/|P \vee Q|,$$

where the set $|P \vee Q|$ is defined by the formula (7), and on based maps $f : Q \to R$ by

$$\forall (p, q) \in |P| \times |Q|, (P \wedge f)(\widetilde{(p, q)}) = (\widetilde{p, f(q)}). \tag{8}$$

Then, endowed with the smash product, **pSet** is a closed symmetric monoidal category. Note that there are no projections associated with the smash product, and that the two-elements pointed set is a unit for it.

6.2 Pointed Integral Connectivity Spaces

Definition 13. *A* pointed integral connectivity space X *is a triple* $(S, \mathcal{K}, b)$*, where* $(S, \mathcal{K})$ *is an integral connectivity space and* b *a point of* S*, called the base-point of* X*.*

For every pointed connectivity space X, we will denote $|X|$ its underlying carrier set, $\kappa(X)$ its connectivity structure and $\beta(X)$ its base-point, so $X = (|X|, \kappa(X), \beta(X))$.

The category whose objects are the pointed integral connectivity spaces and whose morphisms are connectivity morphisms preserving base-points will be denoted by **pCnct**. It can be viewed as a category with lattices of structures on the base category **pSet** of pointed sets. Indeed, the choice of a base-point does not have any effect on the lattice of (integral) connectivity structures on a given set, and connectivity morphisms between pointed spaces are just based maps between underlying pointed sets which preserve connected subsets, so $\mathbf{pCnct} = \mathbf{pSet}_{pCnct}$ with $pCnct = Cnct \circ |-| : \mathbf{pSet} \to \mathbf{JCPos}$. Thus,

Proposition 14. **pCnct** *is a topological category on* **pSet**. *It is thus complete and cocomplete.*

The topological forgetful functor $\mathbf{pCnct} \to \mathbf{pSet}$ will be denoted $|-|_p$, so that $|X|_p = (|X|, \beta(X))$. The category **pCnct** can also be viewed as a concrete category on **Cnct**, and we will denote $|-|_\kappa$ the corresponding forgetful functor, so that $|X|_\kappa = (|X|, \kappa(X))$. Then, the product of two pointed integral connectivity spaces X_1 and X_2 is characterised by $|X_1 \times X_2|_p = |X_1|_p \times |X_2|_p$ and $|X_1 \times X_2|_\kappa = |X_1|_\kappa \times |X_2|_\kappa$. If $\sim$ is an equivalence relation on $|X|$, the quotient pointed space $X/_\sim$ is likewise characterised by $|X/_\sim|_p = |X|_p/_\sim$ and $|X/_\sim|_\kappa = |X|_\kappa/_\sim$. This gives in particular the definition of X/T with $T \subseteq |X|$. The coproduct satisfies $|X_1 \vee X_2|_p = |X_1|_p \vee |X_2|_p$, and its connectivity part $|X_1 \vee X_2|_\kappa$ can be defined either as the quotient of $|X_1|_\kappa \amalg |X_2|_\kappa$ by the relation $\beta(X_1) \sim \beta(X_2)$, or as induced by the space $|X_1|_\kappa \boxtimes |X_2|_\kappa$ on $|X_1 \vee X_2|$ seen as a subset of $|X_1| \times |X_2|$ according to the formula (7), the X_i replacing there the P_i. In the sequel, the expression $|X_1 \vee X_2|$ will keep this last meaning. Now, the same argument as for **pSet** shows that **pCnct** is not cartesian closed.

6.3 The Smash Product

Definition 14. *Let X_1 and X_2 be pointed integral connectivity spaces. Then,*

- *the* tensor product $X_1 \boxtimes X_2$ *is defined by the relations*
  1. $|X_1 \boxtimes X_2|_p = |X_1|_p \times |X_2|_p$,
  2. $|X_1 \boxtimes X_2|_\kappa = |X_1|_\kappa \boxtimes |X_2|_\kappa$,
- *the* smash product *is defined by* $X_1 \wedge X_2 = (X_1 \boxtimes X_2)/|X_1 \vee X_2|$,
- $\mathbf{pCnct}(X_1, X_2)$*, the* pointed connectivity space of connecting based maps *from X_1 to X_2, is defined by*
  1. $|\mathbf{pCnct}(X_1, X_2)| = |\mathbf{Cnct}(|X_1|_\kappa, |X_2|_\kappa)| \cap |\mathbf{pSet}(|X_1|_p, |X_2|_p)|$,
  2. $\kappa(\mathbf{pCnct}(X_1, X_2)) = i^*(\kappa(\mathbf{Cnct}(|X_1|_\kappa, |X_2|_\kappa)))$*, where i is the inclusion map* $i : |\mathbf{pCnct}(X_1, X_2)| \hookrightarrow |\mathbf{Cnct}(|X_1|_\kappa, |X_2|_\kappa)|$,
  3. $\beta(\mathbf{pCnct}(X_1, X_2))$ *is the constant map* $x \mapsto \beta(X_2)$.

Now, with those objects we can define, for every pointed integral connectivity space X, the endofunctors $\mathbf{pCnct}(X, -)$ and $X \wedge -$ on the category **pCnct**. In fact, for every morphism f, the morphisms $X \wedge f$ and $\mathbf{pCnct}(X, f)$ are given by the same formulas as for the corresponding endofunctors on **pSet**.

Theorem 15. *For every pointed integral connectivity space X, the endofunctor $(X \wedge -)$ on* **pCnct** *is left adjoint to the endofunctor* $\mathbf{pCnct}(X, -)$.

Proof. Let X, Y and Z be pointed integral connectivity spaces. For every based connecting map $\psi : X \wedge Y \to Z$, one has a based connecting map $\rho(\psi) : Y \to \mathbf{pCnct}(X, Z)$ defined for all $y \in Y$ by

$$\rho(\psi)(y) = \psi(\widetilde{(-, y)}).$$

Indeed, for every $y \in Y$, $\psi(\widetilde{(-, y)}) \in \mathbf{pCnct}(X, Z)$ since

- ψ is defined on classes $\widetilde{(x, y)}$, so $\psi(\widetilde{(-, y)})$ is a function from $|X|$ to $|Z|$,

- $\psi(\widetilde{(-,y)})(\beta(X)) = \psi(\beta(X \wedge Y)) = \beta(Z)$,
- for every $K \in \kappa(X)$, $s(K \times \{y\}) \in \kappa(X \wedge Y)$ so $\psi(\widetilde{(-,y)}(K) \in \kappa(Z)$,

where $s : X \boxtimes Y \twoheadrightarrow X \wedge Y$ denotes the canonical map. And the function $y \mapsto \psi(\widetilde{(-,y)})$ is a based connecting map from Y to $\mathbf{pCnct}(X,Z)$, since

- $\psi(\widetilde{(-,\beta(Y))}) = (x \mapsto \beta(Z)) = \beta(\mathbf{pCnct}(X,Z))$,
- for every $L \in \kappa(Y)$, $\{\psi(\widetilde{(-,y)}), y \in L\} \in \kappa(\mathbf{pCnct}(X,Z))$, since for every $x \in |X|$ one has $< \{\psi(\widetilde{(-,y)}), y \in L\}, x >= \psi(\widetilde{(x,L)}) \in \kappa(Z)$.

Now, one verifies as well that the formula

$$\theta(\varphi)(\widetilde{(x,y)}) = \varphi(y)(x)$$

defines a map θ from $Hom(Y, \mathbf{pCnct}(X,Z))$ to $Hom(X \wedge Y, Z)$, and that θ and ρ are inverses of each other. Finally, ρ is natural since for all pointed integral connectivity spaces Y, Y', Z and Z' and for all based connecting maps $u : Y \to Y'$, $v : Z \to Z'$ and $\psi : X \wedge Y' \to Z$, one has $\rho(v \circ \psi \circ (X \wedge u)) = (y \mapsto v \circ \psi(\widetilde{(-,u(y))})) = \mathbf{pCnct}(X,v) \circ \rho(\psi) \circ u$.

□

7 Finite Integral Connectivity Spaces

7.1 Generic Graphs

Definition 15. *Let X be a finite integral connectivity space. A* generic point *of X is a non-empty irreducible connected subset of X. The* generic graph *G_X of X is the directed graph whose vertices are the generic points of X and such that $g \to h$ is a directed edge of G_X if and only if $g \supsetneq h$ and there is no generic point k such that $g \supsetneq k \supsetneq h$.*

Associated with a partial order, the directed graph G_X is a so-called *directed acyclic graph*, that is a directed graph with no *directed* cycle; note that cycles are allowed in the undirected graph obtained by forgetting orientation of the edges. On the other hand, not every finite acyclic directed graph is a G_X for some finite integral connectivity space X. For example, the directed acyclic graph $a \to b$ is not such a G_X.

Notation. For the sake of simplicity, if G is a directed graph, $a \in G$ will express that a is a vertex of G and $(a \to b) \in G$ will express that $a \to b = (a, b)$ is a directed edge of this graph.

Proposition 16. *A finite integral connectivity space X is characterised, up to isomorphism, by its generic graph G_X (defined up to isomorphism).*

Proof. The space X being integral, every singleton is an irreducible connected subset, and appears in G_X as a sink, *i.e.* a vertex with no outgoing edges. Thus, the carrier $|X|$ of the space is given, up to bijection, by the set of sinks of G_X. Now, the connectivity structure is given by G_X as a consequence of the proposition 5.

□

Proposition 17. *If X is a non-empty finite integral connectivity space, then*

1. *X is connected iff G_X is connected,*
2. *there is a bijection between connected components of X and those of G_X,*
3. *X is irreducible iff G_X has exactly one source, i.e. a vertex with no incoming edges,*
4. *X is distinguished iff there is no triple (a, b, c) of distinct vertices in G_X such that $(a \to b)$ and $(b \leftarrow c)$ are in G_X.*
5. *X is connected and distinguished iff G_X is a directed tree.*

Proof.

1. If there is an arrow $(a \to b)$ in G_X then a and b, as subsets of $|X|$, are contained in the same connected component of X; thus, if G_X is connected then X is also connected. On the other hand, let (C_i) be the family of G_X connected components and, for each i, let $\sigma(C_i)$ be the union of sinks belonging to C_i; then, every connected subset produced at any step of the process described in theorem 3 stays in one of the $\sigma(C_i)$, otherwise there should be two irreducible connected subsets of X contained respectively in two distinct $\sigma(C_i)$ and with a non-empty intersection, which is not possible. Thus, if G_X is not connected, neither is X.

2. The generic graph G_X of the disjoint union X of any finite family of finite spaces X_i is clearly the disjoint union of the G_{X_i}, thus the connected components of any finite space X are the $\sigma(C_i)$ associated with the connected components C_i of G_X.

3. If X is irreducible then $|X|$ is a generic point which contains all other generic points so it is the only source in G_X.

   If G_X has only one source, then each irreducible connected proper subset of X is contained in a larger irreducible subset, so, X being finite and the set of irreducible connected sets being nonempty, $|X|$ is itself an irreducible connected subset.

4. If there is a triple (a, b, c) with $a \neq c$ and $a \to b \leftarrow c$ in G_X, then $a \cup c$ is a reducible connected subset of X which is thus not distinguished.

   If two irreducible connected subsets of X not included one in the other have a common point, then there must exist in G_X a triple of distinct points (a, b, c) with $a \to b \leftarrow c$ in G_X; thus, if G_X does not admit such a triple, then the inductive generation of connected subsets from irreducible ones (theorem 3) will not produce any new connected subsets.

5. The last affirmation is a direct consequence of the others.

□

Definition 16. *Let X be a non-empty finite integral connectivity space. The* order *of any irreducible subset of X is its height as a vertex of the directed acyclic graph G_X (i.e. the length of the longest path from that vertex to a sink of G_X). The order $\omega(X)$ of X is the maximum of orders of its irreducible connected subsets, that is the length of G_X.*

Example 17. A finite space of order 0 is totally disconnected, *i.e.* its structure is the discrete one.

Example 18. One has $\omega(U_G(S)) \leq 1$ for any finite simple undirected graph S.

The definition of the order of a finite integral connectivity space results in the definition of a new numerical invariant for links:

Definition 17. *The connectivity order of a tame link L in $\mathbf{R}^3$ (or $\mathbf{S}^3$) is $\omega(L) = \omega(S_L)$.*

Example 19. The connectivity order of the Borromean link or, more generally, of any Brunnian link, is $\omega(\mathbf{B}_n) = 1$.

Remark 10. The connectivity order is not a Vassiliev finite type invariant for links. For example, it is easy to check that the connectivity order of the *singular* link with two components, a circle and another component crossing this circle at $2n$ double-points, is greater than 2^n.

Proposition 18. *One has $\omega(X) \leq \mathrm{card}(X) - 1$ for every finite integral space X; and the integral connectivity space $\mathbf{V}_n$ defined by $|\mathbf{V}_n| = n$ and $\kappa(\mathbf{V}_n)^\bullet = \{2, 3, \cdots, n\}$ is, up to isomorphism, the only integral connectivity space such that $\mathrm{card}(\mathbf{V}_n) = n$ and $\omega(\mathbf{V}_n) = n - 1$.*

Proof. A trivial induction results in the first claim. The second one is obvious if $n = 1$. Suppose that it is true for an integer n, and let X be an integral connectivity space with $n+1$ points and with order n. Then there must exist an irreducible connected subset K of X with order $n-1$, and one has necessarily $\mathrm{card}(K) \geq n$, so $\mathrm{card}(K) = n$. By induction, $K \backsimeq \mathbf{V}_n$. Let x be the unique element of $X \setminus K$. $|X|$ is necessarily the only non-trivial connected subset which contains x, otherwise X would be of order smaller than n, then $\kappa(X) = \{\{x\}\} \cup \kappa(K) \cup \{|X|\}$, and thus $X \simeq \mathbf{V}_{n+1}$.

□

Let us now describe two ways to produce finite spaces from two given non-empty finite integral connectivity spaces X and Y, Y being supposed irreducible.

1. Let x be a point of $|X|$. We denote by $X \rhd_x Y$ the connectivity space whose generic graph is obtained by replacing in G_X the sink $\{x\}$ by (a copy of) G_Y, arrows to x in G_X being replaced by arrows to the unique source of (the copy of) G_Y. In other words, $X \rhd_x Y$ is the integral space such that $|X \rhd_x Y| = |X| \setminus \{x\} \cup |Y'|$ and the set $\kappa_0(X \rhd_x Y)$ of irreducible connected sets is given by

$$\{K \in \kappa_0(X), x \notin K\} \cup \kappa_0(Y') \cup \{K \cup |Y'|, x \in K \in \kappa_0(X)\},$$

where Y' is a copy of Y such that $|X| \cap |Y'| = \emptyset$.

2. We can replace simultaneously every sink of G_X by (a copy of) G_Y to produce a space denoted by $X \rhd Y$. That is, $X \rhd Y$ is the connectivity space such that $|X \rhd Y| = |X| \times |Y|$ and the set $\kappa_0(X \rhd Y)$ of irreducible connected sets is given by

$$\kappa_0(X \rhd Y) = \{\{x\} \times L, x \in |X|, L \in \kappa_0(Y)\} \cup \{K \times |Y|, K \in \kappa_0(X)\}.$$

Example 20. $\mathbf{B}_2 \rhd_x \mathbf{V}_n \simeq \mathbf{V}_{n+1}$, where x is any of the two points of $\mathbf{B}_2$.

Proposition 19. *For any non-empty finite integral connectivity space X and any non-empty irreducible finite integral connectivity space Y, one has $\omega(X \rhd Y) = \omega(X) + \omega(Y)$.*

Proof. By construction, $G_{X \rhd Y}$ is obtained by replacing each sink of G_X by a copy of G_Y, so its length is $\omega(X) + \omega(Y)$.

□

Example 21. The link depicted on figure 1 is a Borromean assembly of three Borromean links. Its generic graph is (isomorphic to) $\mathbf{B}_3 \rhd \mathbf{B}_3$, and its connectivity order is 2.

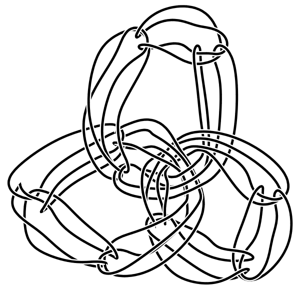

Figure 1: A Borromean ring of borromean rings.

7.2 Representation by Links

In [7, 6], I asked whether every finite connectivity space can be represented by a link, *i.e.* whether there exists a link whose connectivity structure is (isomorphic to) the one given. It turns out that in 1892, Brunn [3] first asked this question, without clearly bringing out the notion of a connectivity space. His answer was positive, and he gave the idea of a proof based on a construction using some of the links now called "Brunnian". In 1964, Debrunner [5], rejecting Brunn's "proof", gave another construction, but proving it only for n-dimensional links with $n \geq 2$. In 1985, Kanenobu [11, 12] seems to be the first to give a proof of the possibility of representing every finite connectivity structure by a classical link, a result which is still little known at this date. The key idea of those different constructions is already in Brunn's original article; it consists in using some Brunnian structures to successively link the sets of components which are desired to become unsplittable.

Thus already from Brunn's point of view, the links we now call "Brunnian links" are not so interesting in and of themselves, but rather because they allow one to construct *all* finite connectivity structures from links.

Theorem 20 (Brunn-Debrunner-Kanenobu)**.** *Every finite connectivity structure is the splittability structure of at least one link in* $\mathbf{R}^3$.

Remark 11. Note that the structure of the links used by Brunn is well described by the so-called *Brunnian groups* constituted by the *Brun-*

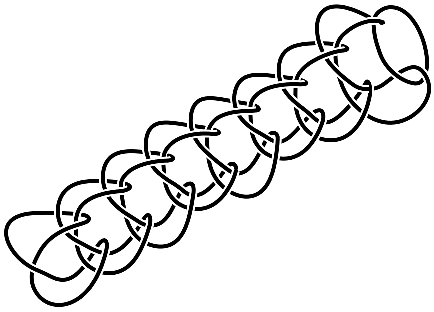

Figure 2: A link with a connectivity order 8.

nian braids introduced as *decomposable braids* by Levinson [13, 14] (see also [16] and [15]) and by the *Brunnian words* studied by Gartside and Greenwood [8].

Example 22. The structure of the connectivity space $\mathbf{V}_9$ with 9 points and maximal connectivity order 8 is the splittability structure of the link depicted on figure 2.

Acknowledgments. Thanks to Jean Bénabou, who introduced me to categories with lattices of structures, to Sergei Soloviev, who asked me about smash products of connectivity spaces, to David C. Ullrich who, on the forum `sci.math`, gave the upper bound ω_0 for the construction I give in the theorem 3. To René Guitart, Mark Weber, Albert Burroni and Quentin Donner for various talks and to Christopher-David Booth, Behrouz Roumizadeh and Anne Richards who helped me to correct my English.

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RESUMES DES ARTICLES PUBLIES dans le Volume LI (2010)

ADAMEK & SOUSA. On quasi-equations on locally presentable categories II: A logic, 3-28.
Quasi-equations, given by parallel pairs of finitary morphisms, represent properties of objects: an object satisfies the property if its contravariant hom-functor merges the parallel pair. Recently Adamek and Hébert characterized subcategories of locally finitely presentable categories specified by quasi-equations. The authors now present a logic of quasi-equations close to Birkhoff's classical equational logic. They prove that it is sound and complete in all locally finitely presentable categories with effective equivalence relations.

MCCURDY & STREET, What separable Frobenius monoidal functors preserve? 29-50.
Separable Frobenius monoidal functors were defined and studied under that name by Kornél Szlachányi and by Brian Day and Craig Pastro. They are a special case of the linearly distributive functors of Robin Cockett and Robert Seely. The purpose of the authors is to develop the theory of such functors in a very precise sense. They characterize geometrically which monoidal expressions are preserved by these functors (or rather, are stable under conjugation in an obvious sense). They show, by way of corollaries, that they preserve lax (meaning not necessarily invertible) Yang-Baxter operators, weak Yang-Baxter operators (in the sense of Alvarez, Vilaboa and González Rodríguez), and (in the braided case) weak bimonoids in the sense of Pastro and Street. Actually every weak Yang-Baxter operator is the image of a genuine Yang-Baxter operator under a separable Frobenius monoidal functor. Prebimonoidal functors are also defined and discussed.

I. STUBBE, "Hausdorff distance" via conical cocompletion, 51-76.
In the context of quantaloid-enriched categories, the author explains how each saturated class of weights defines, and is defined by, an essentially unique full sub-KZ-doctrine of the free cocompletion KZ-doctrine. The KZ-doctrines which arise as full sub-KZ-doctrines of the free cocompletion, are characterised by two simple "fully faithfulness" conditions. Conical weights form a saturated class, and the corresponding KZ-doctrine is precisely (the generalisation to quantaloid-enriched categories of) the Hausdorff doctrine of Akhvlediani et al (2009).

E. VITALE, Bipullbacks and calculus of fractions, 83-113.
In this article it is proved that the class of weak equivalences between internal groupoids in a regular protomodular category is a bipullback congruence and, therefore, has a right calculus of fractions. As an application, it is shown that monoidal functors between internal groupoids in groups and homomorphisms of strict Lie 2-algebras are fractions of internal functors with respect to weak equivalences.

JANELIDZE, MARKI, THOLEN & URSINI, Ideal determined categories, 115-125.
The authors clarify the role of Hofmann's Axiom in the old-style definition of a semi-abelian category. By removing this axiom they obtain the categorical counterpart of the notion of an ideal determined variety of universal algebras – which they therefore call an ideal determined category. Using known counter-examples from universal algebra they conclude that there are ideal determined categories which fail to be Mal'tsev. They also show that there are ideal determined Mal'tsev categories which fail to be semi-abelian.

M.M. CLEMENTINO & GUTIERRES, On regular and homological closure operators, 127-142.
Observing that weak heredity of regular closure operators in Top and of homological closure operators in homological categories identifies torsion theories, the authors study these closure operators in parallel, showing that regular closure operators play the same role in topology as homological closure operators do algebraically.

EVERAERT & VAN DER LINDEN, A note on double central extensions in exact Mal'tsev categories, 143-153.
The characterisation of double central extensions in terms of commutators due to Janelidze (in the case of groups) and Gran and Rossi (in the case of Mal'tsev varieties) is shown to be still valid in the context of exact Mal'tsev categories with coequalisers.

J.R.A. GRAY, Representability of the strict extension functor for categories of generalized Lie algebras, 162-181.
For an additive symmetric closed monoidal category C with equalizers, suppose M is a monoid defined with respect to the monoidal structure. In this setting we can define a Lie algebra with respect to M and the monoidal structure. For the category **Lie**(M; C) of Lie algebras the author shows that the functor SplExt(-, X) from **Lie**(M, C) to **Set** is representable by constructing a representation.

E. BURRONI & PENON, Representation of metric jets, 182-204.
Guided by the heuristic example of the tangential Tf_a of a map *f* differentiable at *a* which can be canonically represented by the unique continuous affine map it contains, the authors extend this property of representation of a metric jet, into a specific metric context. This yields a lot of relevant examples of such representations.

CHENG & MAKKAI, A Note on the Penon definition of *n*-category, 205-223.
The authors show that doubly degenerate Penon tricategories give symmetric rather than braided monoidal categories. They prove that Penon tricategories cannot give all tricategories, but they show how to modify the definition slightly in order to rectify the situation. They give the modified definition, using non-reflexive rather than reflexive globular sets, and show that the problem with doubly degenerate tricategories does not arise.

A. KOCK, Abstract projective lines, 224-240.
The article describes a notion of projective line (over a fixed field *k*): a groupoid with a certain structure. A morphism of projective lines is then a functor preserving the structure. The author proves a structure theorem, namely: such projective lines are isomorphic to the coordinate projective line (= set of 1-dimensional subspaces of k^2).

W. RUMP, Objective categories and schemes, 243-271.
Quasi-coherent sheaves over a scheme are regarded as modules over an objective category. The category Obj of objective categories is shown to be dual to the category of schemes. The author exhibits Obj as a reflective full subcategory of a category POb (pre-objective categories) whose objects are contravariant functors from a poset to the category of commutative rings while the morphisms of POb take care of the structure responsible for the generation of schemes. In this context, morphisms of schemes just turn into functors between objective categories preserving the relevant structure. The main result gives a more explicit version of Rosenberg's reconstruction of schemes (1998).

EBRAHIMI, MAHMOUDI & RASOULI, Characterizing pomonoids S by complete S-posets, 272-281.
A poset with an action of a pomonoid S on it is called an S-poset. There are two different notions of completeness for an S-poset. One just as a poset, and the other as a poset taking also account of the actions which are distributive over the joins. In this paper, comparing these two notions with each other, the authors find characterizations for some pomonoids.

S. DUGOWSON, On connectivity spaces, 282-315.
This paper presents some basic facts about connectivity spaces. In particular, it explains how to generate connectivity structures, the existence of limits and colimits in the main categories of connectivity spaces, the closed monoidal category structure given by the tensor product of integral connectivity spaces; it defines homotopy for connectivity spaces (mentioning some related difficulties) and the smash product of pointed integral connectivity spaces, showing that this operation results in a closed monoidal category with such spaces as objects. Then, it studies finite connectivity spaces, associating a directed acyclic graph with each such space and then defining a new numerical invariant for links: the connectivity order. Finally, it mentions the not very well-known Brunn-Debrunner-Kanenobu Theorem which asserts that every finite integral connectivity space can be represented by a link.

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Fascicule 2: 4th issue dedicated to Francis Borceux

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