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REPRESENTABILITY OF THE SPLIT EXTENSION FUNCTOR FOR CATEGORIES OF GENERALIZED LIE ALGEBRAS

by James Richard Andrew GRAY

Abstract

For an additive symmetric closed monoidal category $\mathbb{C}$ with equalizers, suppose M is a monoid defined with respect to the monoidal structure. In this setting we can define a *Lie algebra* with respect to M and the monoidal structure. For the category $\textbf{Lie}(M, \mathbb{C})$ of Lie algebras we show that the functor $\text{SplExt}(-, X) : \textbf{Lie}(M, \mathbb{C}) \to \textbf{Set}$ is representable by constructing a representation.

Pour une catégorie additive symétrique monoïdale fermée $\mathbb{C}$ avec égalisateurs, soit M un monoïde défini par rapport à la structure monoïdale. Dans ce contexte nous pouvons définir une *algèbre de Lie* par rapport à M et à la structure monoïdale. Pour la catégorie $\textbf{Lie}(M, \mathbb{C})$ d'algèbres de Lie nous montrons que le foncteur $\text{SplExt}(-, X) : \textbf{Lie}(M, \mathbb{C}) \to \textbf{Set}$ est représentable en construisant une représentation.

Introduction

We recall that for a Lie algebra X over a commutative ring R, a map $f : X \to X$ is called a derivation of X if f is linear and, for all x and y in X, $f(xy) = f(x)y + xf(y)$. The set $\text{Der}(X)$ of all derivations on X can be made into a Lie algebra with Lie multiplication $fg = f \circ g - g \circ f$ and all

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other operations defined pointwise. A diagram

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left, "p"] & G \arrow[l, shift left, "s"] \end{tikzcd}$$

where k is the kernel of p and $ps = 1_G$ is called a split extension of G with kernel X. Any morphism between split extensions, that is, a diagram

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, shift left, "p"] \arrow[d, "f"] & G \arrow[l, shift left, "s"] \arrow[d, equal] \\ X \arrow[r, "k'"] & A' \arrow[r, shift left, "p'"] & G \arrow[l, shift left, "s'"] \end{tikzcd}$$

where the top and bottom rows are split extensions, and $fk = k'$, $p = p'f$ and $fs = s'$, is invertible since the split short five lemma holds for Lie algebras. We define an equivalence relation on the set of split extensions of G with kernel X, by requiring that extensions are equivalent if and only if there is a morphism between them. The functor SplExt$(-, X)$: $\mathbf{Lie}_R \to \mathbf{Set}$ is defined on an object G as the set of equivalence classes of split extensions of G with kernel X and on a morphism $g : G' \to G$ by pulling back. A well-known classical result can be stated as: the functor SplExt$(-, X)$ is representable with Der(X) the object of the representation, that is, there is a natural isomorphism SplExt$(-, X) \cong \mathbf{Lie}_R(-, \text{Der}(X))$. This result can be extended to any category of internal Lie algebras defined in a cartesian closed category (see Theorem 5.2 in [1]). We will generalize this result in a different direction, namely to suitably define Lie algebras over a monoid M in an additive symmetric monoidal closed category. Introducing this concept requires some auxiliary observations:

Recall that a commutative monoid in a symmetric monoidal category $(\mathbb{C}, \otimes, I, \alpha, \rho, \lambda, \sigma)$ is an object M together with two morphisms

$$\mu : M \otimes M \to M, \qquad \eta : \mathbb{Z} \to M$$

such that the diagrams

$$\begin{tikzcd}
M \otimes (M \otimes M) \arrow[rr, "\alpha"] \arrow[d, "1\otimes\mu"] & & (M \otimes M) \otimes M \arrow[d, "\mu\otimes 1"] \\
M \otimes M \arrow[r, "\mu"] \arrow[d, "\sigma"] & M & M \otimes M \arrow[l, "\mu"'] \\
M \otimes M \arrow[ur, "\mu"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes M \arrow[r, "\eta\otimes 1"] \arrow[dr, "\lambda"'] & M \otimes M \arrow[d, "\mu"] & M \otimes I \arrow[l, "1\otimes\mu"'] \arrow[dl, "\rho"] \\
& M &
\end{tikzcd}$$

are commutative. Let us recall that when $(\mathbb{C}, \otimes, I, \alpha, \rho, \lambda, \sigma) = (\mathbf{Ab}, \otimes, \mathbb{Z}, \alpha, \rho, \lambda, \sigma)$ is the usual symmetric monoidal category of abelian groups, a commutative monoid in it is the same as a commutative ring. In this case the morphism $\mu : M \otimes M \to M$ corresponds, via the universal property of the tensor product, to a map $M \times M \to M$, call it multiplication, which is bilinear (distributive with respect to the addition of the abelian group M). The morphism $\eta : \mathbb{Z} \to M$ is determined by picking an element u in M, the image of 1. Furthermore, the commutativity of the first diagram means that multiplication is associative and commutative, while the commutativity of the second means that η makes u the identity element of M.

For an ordinary Lie algebra X over a commutative ring M, the scalar multiplication $M \times X \to X$ and the Lie multiplication $X \times X \to X$ are bilinear maps, and so by the universal property of the tensor product in **Ab** they can be described as morphisms $a : M \otimes X \to X$ and $b : X \otimes X \to X$ respectively. The commutativity of the diagrams

$$\begin{tikzcd}
M \otimes (M \otimes X) \arrow[r, "\alpha"] \arrow[d, "1\otimes a"] & (M \otimes M) \otimes X \arrow[d, "\mu\otimes 1"] \\
M \otimes X \arrow[dr, "a"] & M \otimes X \arrow[d, "a"'] \\
& X
\end{tikzcd}$$

$$\begin{tikzcd}
(M \otimes X) \otimes X \arrow[d, "a\otimes 1"] & M \otimes (X \otimes X) \arrow[l, "\alpha"'] \arrow[r, "\sigma\alpha(1\otimes\sigma)"] \arrow[d, "1\otimes b"] & X \otimes (M \otimes X) \arrow[d, "1\otimes a"] \\
X \otimes X \arrow[dr, "b"] & M \otimes X \arrow[d, "a"] & X \otimes X \arrow[dl, "b"'] \\
 & X &
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes X \arrow[r, "\sigma"] \arrow[d, "b"] & X \otimes X \arrow[dl, "-b"] \\
X &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes (X \otimes X) \arrow[r, "1+\sigma\alpha+\sigma\alpha\sigma\alpha"] \arrow[d, "0"] & X \otimes (X \otimes X) \arrow[d, "1\otimes b"] \\
X & X \otimes X \arrow[l, "b"]
\end{tikzcd}$$

state that

$$(mn)x = m(nx), \quad (mx)y = m(xy) = x(my), \quad xy = -yx,$$

$$x(yz) + z(xy) + y(zx) = 0$$

for all $m, n \in M$ and for all $x, y, z \in X$. These identities correspond to the axioms of a Lie algebra except that we have replaced the axiom $xx = 0$ ($x \in X$), with the axiom $xy = -yx$ ($x, y \in X$). Assuming the axiom $xx = 0$, the well known argument

$$xy = xx + xy + yx + yy - yx = (x + y)(x + y) - yx = -yx$$

shows that we have actually replaced an axiom with a formally weaker one. Assuming the axiom $xy = -yx$, the argument

$$2xx = xx + xx = xx - xx = 0$$

shows that when 2 has a multiplicative inverse in M, the two axioms are equivalent. When M is a field this corresponds to saying that M is not of characteristic 2. Since the axiom $xx = 0$ has a repeated variable in it, it is not possible to express it as the commutativity of a diagram involving tensor products. Therefore, in order to define a Lie algebra in an abstract symmetric monoidal category $(\mathbb{C}, \otimes, I, \alpha, \rho, \lambda, \sigma)$ we introduce an additional structure on $\mathbb{C}$. The structure we choose in this paper consists of a category $\mathbb{D}$, functors $U, V : \mathbb{C} \to \mathbb{D}$, and a natural transformation $\delta : U \to V(- \otimes -)$, satisfying suitable conditions (see Section 1). In Section 1 we define a generalized Lie algebra following the above as motivation. In Section 2 we define in this new setting the generalized Lie algebra of derivations and show, in Section 3, that the functor of split extensions from the category of these generalized Lie algebras to the category of sets is representable. We conclude Section 3 by remarking that the functor of split extensions of crossed modules of these generalized Lie algebras is representable.

1 Algebraic structures in monoidal categories

In this section we introduce the needed algebraic structures to define a generalized Lie algebra and construct in this context the functor which in the classical case takes associative algebras to Lie algebras. Throughout this paper we will assume that:

1. $\mathbb{C} = (\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is an additive symmetric monoidal category with all finite limits; in addition we assume it to be monoidal closed, although in this section we only use the fact that the tensor is distributive with respect to finite products;
2. $(M, \mu : M \otimes M \to M, \eta : I \to M)$ is a commutative monoid in $\mathbb{C}$;
3. $\mathbb{D}$ is a category in which hom-sets are abelian groups;
4. Composition of morphisms in $\mathbb{D}$ is distributive on the right with respect to addition of morphisms, that is, for any morphisms $f, g : B \to C$ and $h : A \to B$ we have $(f + g)h = fh + gh$;
5. U and V are functors from $\mathbb{C}$ to $\mathbb{D}$ and V restricted to hom-sets is an abelian group homomorphism;
6. δ is a natural transformation from U to $V(- \otimes -)$ such that:

   Condition 1.1. *For any $C \in \mathbb{C}$ the diagram*

$$\begin{tikzcd} UC \arrow[r, "\delta_C"] \arrow[dr, "\delta_C"'] & V(C \otimes C) \arrow[d, "V\sigma"] \\ & V(C \otimes C) \end{tikzcd}$$

   commutes.

Example 1.2. *$(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \sigma) = (\mathbf{Ab}, \otimes, \mathbb{Z}, \alpha, \rho, \lambda, \sigma)$ is the the usual symmetric monoidal category of abelian groups, $\mathbb{D} = \mathbf{ab}$ is the category with objects all abelian groups and morphisms all maps between their underlying sets, $U = V : \mathbf{Ab} \to \mathbf{ab}$ is the inclusion functor, and δ is defined by $\delta_C(c) = c \otimes c$ for all C in $\mathbf{Ab}$ and c in C. This example explains the main purpose of introducing $\mathbb{D}$, U, V, and δ: the axiom $xx = 0$ mentioned in the Introduction can now be expressed categorically as $V(b)\delta_X = 0$, where $b : X \otimes X \to X$ is a multiplication morphism on an object X (as in the Introduction).*

We recall: (i) an M-action is a pair (X, a), where X is an object in $\mathbb{C}$ and $a : M \otimes X \to X$ is a morphism in $\mathbb{C}$, such that the diagrams

$$\begin{tikzcd} M \otimes (M \otimes X) \arrow[rr, "\alpha"] \arrow[d, "1\otimes a"] & & (M \otimes M) \otimes X \arrow[d, "a\otimes 1"] \\ M \otimes X \arrow[r, "a"] & X & M \otimes X \arrow[l, "a"'] \end{tikzcd} \qquad \begin{tikzcd} I \otimes X \arrow[r, "\eta\otimes 1"] \arrow[dr, "\lambda"'] & M \otimes X \arrow[d, "a"] \\ & X \end{tikzcd}$$

commute; (ii) a magma defined with respect to the monoidal structure in $\mathbb{C}$ is a pair (X, b), where X is an object in $\mathbb{C}$ and $b : X \otimes X \to X$ is a morphism in $\mathbb{C}$.

Definition 1.3. *A triple $(X, a : M \otimes X \to X, b : X \otimes X \to X)$ is said to be an M-magma if (X, a) is an M-action for the monoid M. For M-magmas (X, a, b) and (X', a', b'), a morphism $f : X \to X'$ in $\mathbb{C}$ is an M-magma morphism if the diagrams*

$$\begin{tikzcd} M \otimes X \arrow[r, "1\otimes f"] \arrow[d, "a"] & M \otimes X' \arrow[d, "a'"] \\ X \arrow[r, "f"] & X' \end{tikzcd} \qquad \begin{tikzcd} X \otimes X \arrow[r, "f\otimes f"] \arrow[d, "b"] & X' \otimes X' \arrow[d, "b'"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

commute; that is, f must be a morphism of magmas and a morphism of M-actions at the same time. The category of M-magmas will be denoted M-$\mathbf{Mag}_0$.

For an M-magma (X, a, b) consider the following condition:

Condition 1.4. *(a) The diagram*

$$\begin{tikzcd} M \otimes (X \otimes X) \arrow[rr, "\alpha"] \arrow[d, "1\otimes b"] & & (M \otimes X) \otimes X \arrow[d, "a\otimes 1"] \\ M \otimes X \arrow[r, "a"] & X & X \otimes X \arrow[l, "b"'] \end{tikzcd}$$

commutes;

(b) The diagram

$$\begin{tikzcd} M \otimes (X \otimes X) \arrow[rr, "\sigma\alpha(1\otimes\sigma)"] \arrow[d, "1\otimes b"] & & X\otimes(M \otimes X) \arrow[d, "1\otimes a"] \\ M \otimes X \arrow[r, "a"] & X & X \otimes X \arrow[l, "b"'] \end{tikzcd}$$

commutes.

Let M-**Mag**$_1$ be the full subcategory of M-**Mag**$_0$ with objects all M-magmas satisfying Conditions 1.4(a) and 1.4(b). Let M-**Mag**$_2$ be the full subcategory of M-**Mag**$_1$ with objects all (X, a, b), in which the pair (X, b) is a semigroup, that is, the diagram

$$\begin{tikzcd} X \otimes (X \otimes X) \arrow[rr, "\alpha"] \arrow[d, "1\otimes b"] & & (X \otimes X) \otimes X \arrow[d, "b\otimes 1"] \\ X \otimes X \arrow[r, "b"] & X & X \otimes X \arrow[l, "b"'] \end{tikzcd}$$

commutes. In the situation of Example 1.2 the categories M-**Mag**$_1$ and M-**Mag**$_2$ are the categories of non-associative and associative M-algebras respectively.

For a magma (X, b) we are also going to use the following conditions:

Condition 1.5. $V(b)\delta_X = 0$.

Condition 1.6. *(a)* $b(1 + \sigma) = 0$ *(anticommutativity);*

(b) $b(1 \otimes b)(1 + \sigma\alpha + \sigma\alpha\sigma\alpha) = 0$ *(Jacobi identity).*

Remark 1.7. *When* $\mathbb{C} = \mathbb{D}$, $V = 1$, $U = (-\otimes -)$ *and* $\delta = 1 + \sigma$, *Condition 1.6(a) becomes an instance of Condition 1.5.*

Let $\mathbf{Lie}(M, \delta)$ be the full subcategory of M-$\mathbf{Mag}_1$ with objects all (X, a, b), in which the magma (X, b) satisfies Conditions 1.5, 1.6(a) and 1.6(b). In the situation of Example 1.2, as in fact mentioned in the Introduction, Conditions 1.5, 1.6(a) and 1.6(b) correspond to the identities

$$xx = 0, \qquad xy + yx = 0, \qquad x(yz) + z(xy) + y(zx) = 0$$

respectively, and recalling that the category M-$\mathbf{Mag}_1$ is the category of non-associative algebras we see that the category $\mathbf{Lie}(M, \delta)$ is the category of Lie algebras over the commutative ring M.

Remark 1.8. *If* $\mathbb{D} = \mathbb{C}$, $U = V = 1_{\mathbb{C}}$ *and* δ_C *is the zero morphism, then Condition 1.5 is trivially satisfied by any magma* (X, b). *If in addition, as in Example 1.2,* $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \sigma) = (\mathbf{Ab}, \otimes, \mathbb{Z}, \alpha, \lambda, \rho, \sigma)$ *is the usual symmetric monoidal category of abelian groups, the category* $\mathbf{Lie}(M, \delta)$ *has as objects Lie algebras, except that the axiom* $xx = 0$ *has been replaced by the axiom* $xy = -yx$.

If $(X, a : R \times X \to X, b : X \times X \to X)$ is an associative algebra over a ring R and if we define $\tilde{b} : X \times X \to X$ as

$$\tilde{b}(x, y) = b(x, y) - b(y, x)$$

for all $x, y \in X$, then the triple $(X, a, \tilde{b})$ is a Lie algebra defined with respect to the ring R. This correspondence of associative algebras and Lie algebras is functorial and can been extended to our setting.

Theorem 1.9. *If* $(X, a, b) \in M$-$\mathbf{Mag}_2$, *then* $(X, a, b(1 - \sigma)) \in \mathbf{Lie}(M, \delta)$ *and the assignment* $(X, a, b) \longmapsto (X, a, b(1 - \sigma))$ *defines a functor* $L : M$-$\mathbf{Mag}_2 \to \mathbf{Lie}(M, \delta)$ *which is identity on morphisms.*

Proof. Let $\tilde{b} = b(1 - \sigma)$. It is clear that $(X, a, \tilde{b})$ is an M-magma. Condition 1.4(a) holds for $(X, a, \tilde{b})$ since

$$\begin{aligned} a(1 \otimes \tilde{b}) &= a(1 \otimes (b(1 - \sigma))) = a(1 \otimes b) - a(1 \otimes b)(1 \otimes \sigma) \\ &= b(a \otimes 1)\alpha - b(1 \otimes a)\sigma\alpha(1 \otimes \sigma)(1 \otimes \sigma) \\ &= b(1 - \sigma)(a \otimes 1)\alpha = \tilde{b}(a \otimes 1)\alpha, \end{aligned}$$

where the third equality follows by Conditions 1.4(a) and 1.4(b) for (X,a,b). Similarly, it can easily be seen that Condition 1.4(b) holds for $(X,a,\tilde{b})$. To show that the Jacobi identity, Condition 1.6(b), holds for $(X,a,\tilde{b})$, consider the equation:

$$\begin{aligned}
&\tilde{b}(1\otimes\tilde{b})(1+\sigma\alpha+\sigma\alpha\sigma\alpha)\\
&\quad= b(1-\sigma)(1\otimes b)(1-1\otimes\sigma)(1+\sigma\alpha+\sigma\alpha\sigma\alpha)\\
&\quad= b((1\otimes b)(1-1\otimes\sigma)-\sigma(1\otimes b)(1-1\otimes\sigma))(1+\sigma\alpha+\sigma\alpha\sigma\alpha)\\
&\quad= b(1\otimes b)^{(1)}+b(1\otimes b)\sigma\alpha^{(2)}+b(1\otimes b)\sigma\alpha\sigma\alpha^{(3)}\\
&\qquad -b(1\otimes b)(1\otimes\sigma)^{(4)}-b(1\otimes b)(1\otimes\sigma)\sigma\alpha^{(5)}-b(1\otimes b)(1\otimes\sigma)\sigma\alpha\sigma\alpha^{(6)}\\
&\qquad -b(b\otimes 1)\sigma^{(3)}-b(b\otimes 1)\alpha^{(1)}-b(b\otimes 1)\alpha\sigma\alpha^{(2)}\\
&\qquad +b(b\otimes 1)(\sigma\otimes 1)\sigma^{(5)}+b(b\otimes 1)(\sigma\otimes 1)\alpha^{(6)}+b(b\otimes 1)(\sigma\otimes 1)\alpha\sigma\alpha^{(4)}\\
&\quad= 0,
\end{aligned}$$

where composites labelled with the same superscript are equal. For, we only need to observe that $b(1\otimes b)=b(b\otimes 1)\alpha$ since (X,b) is a semigroup, and use that directly for (1) and (2), or together with $\alpha\sigma\alpha\sigma\alpha=\sigma$ for (3), or together with $\alpha(1\otimes\sigma)=(\sigma\otimes 1)\alpha\sigma\alpha$ for (4), or together with $\alpha(1\otimes\sigma)\sigma\alpha=(\sigma\otimes 1)\sigma$ for (5), or together with $\alpha(1\otimes\sigma)\sigma\alpha\sigma\alpha=(\sigma\otimes 1)\sigma$ for (6). From Condition 1.1 and the definition of $\tilde{b}$ it follows that Conditions 1.5 and 1.6(a) hold for $(X,\tilde{b})$. For a morphism

$$(X,a,b)\xrightarrow{f}(X',a',b')$$

let $\tilde{b}'=b'(1-\sigma)$. By calculating

$$\begin{aligned}
\tilde{b}'(f\otimes f) &= b'(1-\sigma)(f\otimes f)\\
&= b'(f\otimes f-\sigma(f\otimes f))\\
&= b'(f\otimes f-(f\otimes f)\sigma)\\
&= b'(f\otimes f)(1-\sigma)\\
&= fb(1-\sigma)\\
&= f\tilde{b},
\end{aligned}$$

we see that f is a morphism in **Lie**(M,δ). □

2 Construction of derivations

In this section we construct, for an object (X, a, b) in **Lie**(M, δ), the object Der(X), which will be shown in Section 3 to be the representing object for the functor SplExt$(-, X)$: **Lie**$(M, \delta) \to$ **Set**.

Recall that, for a Lie algebra X over a commutative ring M, the Lie algebra of derivations, Der(X), can be constructed as follows. For abelian groups A and B, let Hom(A, B) be the abelian group of homomorphisms from A to B. Defining multiplication by composition and scalar multiplication pointwise, it is easily seen that Hom(X, X) satisfies the axioms of a ring as well as those of an M-module and, moreover has scalar multiplication with the property

$$m(h_1 \circ h_2) = (mh_1) \circ h_2$$

for all $m \in M$ and $h_1, h_2 \in$ Hom(X, X). The abelian group $E(X)$ of M-module morphisms from X to X can be constructed as the equalizer of the diagram

$$\mathrm{Hom}(X, X) \underset{f_2}{\overset{f_1}{\rightrightarrows}} \mathrm{Hom}(M \times X, X)$$

where f_1 and f_2 are defined by

$$f_1(h)(m, x) = mh(x), \qquad f_2(h)(m, x) = h(mx)$$

for all $h \in$ Hom(X, X), $m \in M$ and $x \in X$. It is easily seen that $E(X)$ is closed under the operations defined for Hom(X, X) and has the property

$$m(h_1 \circ h_2) = h_1 \circ (mh_2)$$

for all $m \in M$ and $h_1, h_2 \in E(X)$, i.e. $E(X)$ is an associative M-algebra. As described before, any associative M-algebra $E(X)$ becomes a Lie algebra with Lie multiplication defined by

$$h_1 h_2 = h_1 \circ h_2 - h_2 \circ h_1$$

for all $h_1, h_2 \in E(L)$. Finally, the Lie algebra of derivations Der(X), can be constructed as the equalizer of the diagram

$$E(X) \underset{g_2 e}{\overset{g_1 e}{\rightrightarrows}} \mathrm{Hom}(X \times X, X)$$

where $e : E(X) \to \mathrm{Hom}(X, X)$ is the equalizer of f_1 and f_2, and g_1 and g_2 are defined by

$$g_1(h)(x_1, x_2) = h(x_1 x_2), \qquad g_2(h)(x_1, x_2) = h(x_1)x_2 + x_1 h(x_2)$$

for all $h \in \mathrm{Hom}(X, X)$ and $x_1, x_2 \in L$. $\mathrm{Der}(X)$ can be seen to be closed under the operations defined for $E(X)$ and hence is a Lie algebra.

We show that this construction extends to our general context. We begin by showing that for $(X, a, b) \in \mathbf{Lie}(M, \delta)$ the internal hom-object X^X can be given a semigroup structure as well as an M-magma structure that satisfies Condition 1.4(a). We then construct the semigroup $E(X)$ as a regular sub-M-magma of the internal hom-object X^X and show that it satisfies Condition 1.4(b). We then apply the functor $L : M\text{-}\mathbf{Mag}_2 \to \mathbf{Lie}(M, \delta)$ to $E(X)$ and construct $\mathrm{Der}(X)$ as a regular subobject of $L(E(X))$.

For each object B in $\mathbb{C}$, we will denote the chosen right adjoint to the functor $- \otimes B$ by $-^B$ and denote the chosen counit of the associated adjunction by ϵ^B. For functors $F : \mathbb{X} \to \mathbb{A}$ and $G : \mathbb{A} \to \mathbb{X}$, where G is the right adjoint of F, given a morphism $h : FX \to A$, the corresponding morphism $X \to GA$ will be called the right adjunct of h (as in [6]). Similarly, given a morphism $g : X \to GA$, the corresponding morphism $FX \to A$ will be called the left adjunct of g. That is, for $g : A \to C^B$, the left adjunct of g is $\epsilon^B_C(g \otimes 1) : A \otimes B \to C$.

For a pair $(X, a_X : M \otimes X \to X)$ where $M = (M, \mu, \eta)$ is a monoid in $\mathbb{C}$ as above, consider the following condition, which is part of the definition of an action for a monoid:

Condition 2.1. *The diagram*

$$\begin{tikzcd} I\otimes X \arrow[r, "\eta\otimes 1"] \arrow[dr, "\lambda"'] & M\otimes X \arrow[d, "a_X"] \\ & X \end{tikzcd}$$

commutes.

Proposition 2.2. *If (X, a_X) satisfies Condition 2.1 and if $a_{X^X} : M \otimes X^X \to X^X$ is the right adjunct of $a_X(1 \otimes \epsilon^X_X)\alpha^{-1} : (M \otimes X^X) \otimes X \to X$ then (X^X, a_{X^X}) satisfies Condition 2.1.*

Proof. In the diagram

$$
\begin{tikzcd}
(I\otimes X^X)\otimes X \arrow[rrrr, "(\eta\otimes 1)\otimes 1"] \arrow[dr, "\alpha^{-1}"'] \arrow[ddddr, bend right, "\lambda\otimes 1"'] \arrow[ddr, phantom, "\boxed{2}"] & & & & (M\otimes X^X)\otimes X \arrow[dl, "\alpha^{-1}"] \arrow[ddddlll, bend left, "a_{X^X}\otimes 1"] \\
& I\otimes(X^X\otimes X) \arrow[rr, "\eta\otimes 1"] \arrow[d, "1\otimes\epsilon_X^X"] \arrow[dddd, bend right, "\lambda"'] \arrow[ur, phantom, "\boxed{1}"] \arrow[drr, phantom, "\boxed{3}"] & & M\otimes(X^X\otimes X) \arrow[d, "1\otimes\epsilon_X^X"] & \\
& I\otimes X \arrow[rr, "\eta\otimes 1"] \arrow[d, "\lambda", "\boxed{4}"'] & & M\otimes X \arrow[dll, "a_X", "\boxed{6}"'] \arrow[ddl, phantom, "\boxed{5}"] & \\
& X & & & \\
& X^X\otimes X \arrow[u, "\epsilon_X^X"'] & & &
\end{tikzcd}
$$

$\boxed{1}$ commutes since α is a natural transformation; $\boxed{2}$ commutes as an immediate consequence of the axioms of a monoidal category; $\boxed{3}$ commutes since $\otimes$ is a bifunctor; $\boxed{4}$ commutes since λ is a natural transformation; $\boxed{5}$ commutes by definition of $a_{X^X} : M \otimes X^X \to X^X$; $\boxed{6}$ commutes by assumption on (X, a_X). That is, $\lambda\otimes 1 = (a_{X^X}\otimes 1)((\eta\otimes 1)\otimes 1) = (a_{X^X}(\eta\otimes 1))\otimes 1$, which tells us that the left adjuncts of the morphims $\lambda, a_{X^X}(\eta\otimes 1) : I\otimes X^X \to X^X$ are equal to each other. Therefore these two morphisms are equal to each other themselves, as desired. $\square$

For a sextuple $(P, Q, X, u : P\otimes Q \to Q, p : P\otimes X \to X, q : Q\otimes X \to X)$ we consider the following condition:

Condition 2.3. *The diagram*

$$
\begin{tikzcd}
P\otimes(Q\otimes X) \arrow[rr, "\alpha"] \arrow[d, "1\otimes q"] & & (P\otimes Q)\otimes X \arrow[d, "u\otimes 1"] \\
P\otimes X \arrow[r, "p"] & X & Q\otimes X \arrow[l, "q"']
\end{tikzcd}
$$

commutes.

Lemma 2.4. *Suppose $(P, Q, X, u : P\otimes Q \to Q, p : P\otimes X \to X, q : Q\otimes X \to X)$ satisfies Condition 2.3, $p' : P\otimes X^X \to X^X$ is the right adjunct of $p(1\otimes\epsilon_X^X)\alpha^{-1} : (P\otimes X^X)\otimes X \to X$ and $q' : Q\otimes X^X \to X^X$ is the right* *adjunct of* $q(1\otimes\epsilon^X_X)\alpha^{-1} : (Q\otimes X^X)\otimes X \to X$ *then* (P,Q,X^X,u,p',q') *satisfies Condition 2.3.*

Proof. In the diagram

$$\begin{tikzcd}
(P\otimes(Q\otimes X^X))\otimes X \arrow[rrr, "\alpha\otimes1"] \arrow[d, "\alpha^{-1}"] \arrow[drrr, phantom, "\boxed{1}"] \arrow[ddddd, bend right=40, "(1\otimes q')\otimes1"'] \arrow[ddddd, phantom, bend right=25, "\boxed{2}"] & & & ((P\otimes Q)\otimes X^X)\otimes X \arrow[d, "\alpha^{-1}"] \arrow[ddddd, bend left=40, "(u\otimes1)\otimes1"] \arrow[ddddd, phantom, bend left=25, "\boxed{3}"] \\
P\otimes((Q\otimes X^X)\otimes X) \arrow[r, "1\otimes\alpha^{-1}"] \arrow[dd, "1\otimes(q'\otimes1)"] \arrow[ddr, phantom, "\boxed{4}"] & P\otimes(Q\otimes(X^X\otimes X)) \arrow[rr, "\alpha"'] \arrow[d, "1\otimes(1\otimes\epsilon^X_X)"] \arrow[dr, phantom, "\boxed{5}"] & & (P\otimes Q)\otimes(X^X\otimes X) \arrow[dl, "1\otimes\epsilon^X_X"] \arrow[ddd, "u\otimes1"'] \\
& P\otimes(Q\otimes X) \arrow[r, "\alpha"] \arrow[d, "1\otimes q"] \arrow[ddr, phantom, "\boxed{6}"] & (P\otimes Q)\otimes X \arrow[dd, "u\otimes1"] \arrow[ddr, phantom, "\boxed{7}"] & \\
P\otimes(X^X\otimes X) \arrow[r, "1\otimes\epsilon^X_X"] \arrow[ddr, phantom, "\boxed{8}"] & P\otimes X \arrow[d, "p"] & & \\
& X & Q\otimes X \arrow[l, "q"] \arrow[d, phantom, "\boxed{9}"] & Q\otimes(X^X\otimes X) \arrow[l, "1\otimes\epsilon^X_X"'] \\
(P\otimes X^X)\otimes X \arrow[r, "p'\otimes1"] \arrow[uu, "\alpha^{-1}"] & X^X\otimes X \arrow[u, "\epsilon^X_X"] & & (Q\otimes X^X)\otimes X \arrow[ll, "q'\otimes1"'] \arrow[u, "\alpha^{-1}"']
\end{tikzcd}$$

$\boxed{1}$ commutes by the axioms of a monoidal category; $\boxed{2}$, $\boxed{3}$ and $\boxed{5}$ commute since α is natural transformation; $\boxed{4}$ and $\boxed{9}$ commute from the definition of q'; $\boxed{8}$ commutes by the definition of p'; $\boxed{7}$ commutes since $\otimes$ is a bifunctor; $\boxed{6}$ commutes by assumption on u, p and q (Condition 2.3). That is, $(q'\otimes 1)((u\otimes1)\otimes1)(\alpha\otimes1) = (p'\otimes1)((1\otimes q')\otimes1)$, or, equivalently, $(q'(u\otimes 1)\alpha)\otimes 1 = (p'(1\otimes q'))\otimes 1$ – which means that the left adjuncts of the morphisms $p'(1\otimes q'), q'(u\otimes1)\alpha : P\otimes(Q\otimes X^X)\to X^X$ are equal to each other. Therefore these two morphisms are equal to each other themselves, as desired. $\square$

Proposition 2.5. *Let* (X,a_X) *be an* M*-action and, let* $a_{X^X} : M\otimes X^X \to X^X$ *and* $b_{X^X} : X^X\otimes X^X\to X^X$ *be the right adjuncts of* $a(1\otimes\epsilon^X_X)\alpha^{-1} : (M\otimes X^X)\otimes X \to X$ *and* $\epsilon^X_X(1\otimes\epsilon^X_X)\alpha^{-1} : (X^X\otimes X^X)\otimes X\to X$ *respectively. Then* (X^X,a_{X^X}) *is an* M*-action,* (X^X,b_{X^X}) *is a semigroup, and Condition 1.4(a) is satisfied.*

Proof. It is clear that since (X, a_X) is an M-action, the sextuple (M, M, X, μ, a_X, a_X) satisfies Condition 2.3. From Lemma 2.4 it follows that $(M, M, X^X, \mu, a_{X^X}, a_{X^X})$ satisfies Condition 2.3. This together with Proposition 2.2 applied to (X, a_X) shows that (X^X, a_{X^X}) is an M-action. From the definition of b_{X^X} we see that $(X^X, X^X, X, b_{X^X}, \epsilon_X^X, \epsilon_X^X)$ satisfies Condition 2.3 and by Lemma 2.4 $(X^X, X^X, X^X, b_{X^X}, b_{X^X}, b_{X^X})$ satisfies Condition 2.3 and therefore (X^X, b_{X^X}) is a semigroup. From the definition of a_{X^X} the sextuple $(M, X^X, X, a_{X^X}, a_X, \epsilon_X^X)$ satisfies Condition 2.3 and by Lemma 2.4 the sextuple $(M, X^X, X^X, a_{X^X}, a_{X^X}, b_{X^X})$ satisfies Condition 2.3 and therefore (X^X, a_{X^X}, b_{X^X}) satisfies Condition 1.4(a). $\square$

Let $f_1 : X^X \to X^{M\otimes X}$ and $f_2 : X^X \to X^{M\otimes X}$ be the right adjuncts of $\epsilon_X^X(1 \otimes a_X) : X^X \otimes (M \otimes X) \to X$ and $a_X(1 \otimes \epsilon_X^X)\sigma\alpha(1 \otimes \sigma) : X^X \otimes (M \otimes X) \to X$ respectively, and let $e : E(X) \to X^X$ be the equalizer of f_1 and f_2.

Proposition 2.6. *For the object $E(X)$ there exist unique morphisms $b_{E(X)} : E(X) \otimes E(X) \to E(X)$ and $a_{E(X)} : M \otimes E(X) \to E(X)$ for which e becomes an M-magma morphism and $(E(X), a_{E(X)}, b_{E(X)})$ is in M-**Mag**$_2$.*

Proof. In the diagram

$$\begin{tikzcd}
E(X) \otimes E(X) \arrow[r, "e\otimes e"] \arrow[d, dashed, "b_{E(X)}"'] & X^X \otimes X^X \arrow[d, "b_{X^X}"'] & \\
E(X) \arrow[r, "e"] & X^X \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & X^{M\otimes X} \\
M \otimes E(X) \arrow[r, "1\otimes e"] \arrow[u, dashed, "a_{E(X)}"] & M \otimes X^X \arrow[u, "a_{X^X}"] &
\end{tikzcd}$$

it can be seen, by considering the left adjuncts of $f_1 b_{X^X}(e\otimes e)$ and $f_2 b_{X^X}(e\otimes e)$ and the left adjuncts of $f_1 a_{X^X}(1 \otimes e)$ and $f_1 a_{X^X}(1 \otimes e)$, that the arrows $b_{X^X}(e \otimes e)$ and $a_{X^X}(1 \otimes e)$ equalize f_1 and f_2 and so, by the universal property of the equalizer e, there exist unique arrows $b_{E(X)}$ and $a_{E(X)}$ making the diagram commute. The left adjuncts of the morphisms $ea_{E(X)}(1 \otimes b_{E(X)})$ and $eb_{E(X)}(1 \otimes a_{E(X)})\sigma\alpha(1 \otimes \sigma)$ can been seen to be equal and since e is a monomorphism this shows that $(E(X), a_E(X), b_E(X))$ satisfies Condition 1.4(b). On the other hand, according to our construction of

a_{X^X} and b_{X^X}, the monomorphism e becomes an M-magma morphism from $(E(X), a_E(X), b_E(X))$ to (X^X, a_{X^X}, b_{X^X}), which implies that $(E(X), a_E(X), b_E(X))$ satisfies Condition 1.4(a) and that $(E(X), b_E(X))$ is a semigroup. This completes the proof. $\square$

By Theorem 1.9 we have that $L(E(X), b_{E(X)}, a_{E(X)}) = (E(X), \tilde{b}_{E(X)} = b_{E(X)}(1-\sigma), a_{E(X)})$ is in **Lie**(M, δ). For $(X, a_X, b_X) \in$ **Lie**(M, δ) let $g_1 : X^X \to X^{X \otimes X}$ be the right adjunct of $\epsilon_X^X(1 \otimes b_X) : X^X \otimes (X \otimes X) \to X$, let $g_2 : X^X \to X^{X \otimes X}$ be the right adjunct of the sum of the morphisms $b_X(\epsilon_X^X \otimes 1)\alpha : X^X \otimes (X \otimes X) \to X$ and $b_X(1 \otimes \epsilon_X^X)\sigma\alpha(1 \otimes \sigma) : X^X \otimes (X \otimes X) \to X$, and let $d : D(X) \to E(X)$ be the equalizer of $g_1 e$ and $g_2 e$.

Proposition 2.7. *For the object $D(X)$ there exist unique morphisms $b_{D(X)} : D(X) \otimes D(X) \to D(X)$ and $a_{D(X)} : M \otimes D(X) \to D(X)$ for which d is an M-magma morphism from $(D(X), a_{D(X)}, b_{D(X)})$ to $L(E(X), a_E(X), b_E(X))$ and $(D(X), a_{D(X)}, b_{D(X)})$ is in* **Lie**(M, δ).

Proof. In the diagram

$$\begin{tikzcd}
D(X) \otimes D(X) \arrow[r, "d \otimes d"] \arrow[dd, dashed, "b_{D(X)}"] & E(X) \otimes E(X) \arrow[d, "(1-\sigma)"] & & \\
& E(X) \otimes E(X) \arrow[r, "e \otimes e"] \arrow[d, "b_{E(X)}"] & X^X \otimes X^X \arrow[d, "b_{X^X}"] & \\
D(X) \arrow[r, "d"] & E(X) \arrow[r, "e"] & X^X \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & X^{X \otimes X} \\
M \otimes D(X) \arrow[r, "1 \otimes d"] \arrow[u, dashed, "a_{D(X)}"] & M \otimes E(X) \arrow[r, "1 \otimes e"] \arrow[u, "a_{E(X)}"] & M \otimes X^X \arrow[u, "a_{X^X}"] &
\end{tikzcd}$$

it can be seen, by considering the left adjuncts of $g_1 b_{X^X}(e \otimes e)(1-\sigma)(d \otimes d)$ and $g_2 b_{X^X}(e \otimes e)(1-\sigma)(d \otimes d)$ and the left adjuncts of $g_1 a_{X^X}(1 \otimes e)(1 \otimes d)$ and $g_2 a_{X^X}(1 \otimes e)(1 \otimes d)$, that the morphisms $b_{X^X}(e \otimes e)(1-\sigma)(d \otimes d)$ and $a_{X^X}(1 \otimes e)(1 \otimes d)$ equalize g_1 and g_2 and so, by the universal property of the equalizer d, there exist unique arrows $b_{D(X)}$ and $a_{D(X)}$ making the diagram commute. Since d is a monomorphism we see that $(D(X), a_{D(X)}, b_{D(X)})$ is in **Lie**(M, δ). $\square$

We now define the object Der(X) of a derivation of $X = (X, a_X, b_X)$ as Der$(X) = D(X) = (D(X), a_D(X), b_D(X))$.

3 Representability of split extension functor for the category Lie($\boldsymbol{M}, \delta$)

In this section we show that the functor SplExt$(-, X)$ can be defined for the category **Lie**(M, δ) and prove that it is representable by showing that Der$(X) = D(X)$ is the representing object.

To define the functor SplExt$(-, X)$ it is sufficient to show that the split short five lemma holds for **Lie**(M, δ) and that the category **Lie**(M, δ) has pullbacks of all split epimorphisms along arbitrary morphisms.

It is easily seen that the category **Lie**(M, δ) is pointed and finitely complete. Since $\mathbb{C}$ is additive the split short five lemma holds in $\mathbb{C}$ and since the forgetful functor W : **Lie**$(M, \delta) \to \mathbb{C}$ preserves limits and reflects isomorphisms, the split short five lemma holds also in **Lie**(M, δ).

Consider the diagram

$$\begin{tikzcd} X \arrow[r, shift right, "k"'] \arrow[d, equal] & A \arrow[l, shift right, "l"'] \arrow[r, shift left, "p"] \arrow[d, "f"] & G \arrow[l, shift left, "s"] \arrow[d, equal] \\ X \arrow[r, shift right, "k'"'] & A' \arrow[l, shift right, "l'"'] \arrow[r, shift left, "p'"] & G \arrow[l, shift left, "s'"] \end{tikzcd}$$

where f is a morphism (hence an isomorphism) of split extensions in **Lie**(M, δ), and l and l' are the unique M-action morphisms with $kl = 1_A - sp$ and $k'l' = 1_{A'} - s'p'$; we shall write $A = (A, a, b)$ and $A' = (A', a', b')$. Since k' is a monomorphism and $k'l'f = (1_{A'} - s'p')f = f - s'p'f = f - fsp = f(1_A - sp) = fkl = k'l$ we have $l'f = l$; therefore

$$lb(s \otimes k) = l'fb(s \otimes k) = l'b'(f \otimes f)(s \otimes k) = l'b'(s' \otimes k').$$

Consequently, if we define $h : G \to X^X$ as the right adjunct of the composite $lb(s \otimes k)$, we see that h depends only on the isomorphism class of the split extensions.

In the diagram

$$
\begin{tikzcd}
G \arrow[dr, dashed, "i"] \arrow[drr, "h"] \arrow[dd, dashed, "j"'] & & & X^{R\otimes X} \\
& E(X) \arrow[r, "e"'] & X^X \arrow[ur, shift left, "f_1"] \arrow[ur, shift right, "f_2"'] \arrow[dr, shift left, "g_1"] \arrow[dr, shift right, "g_2"'] & \\
D(X) \arrow[ur, "d"] & & & X^{X\otimes X}
\end{tikzcd}
$$

where the solid arrows are defined as before, it can be seen, by considering the left adjuncts of f_1h and f_2h and the left adjuncts of g_1h and g_2h, that h equalizes f_1 and f_2 as well as g_1 and g_2, and so by the universal properties of the equalizers e and d, there exist arrows i and j making the diagram commute.

Proposition 3.1. *The morphism $j : G \to D(X)$ is a morphism in* $\mathbf{Lie}(M,\delta)$.

Proof. Consider the diagrams

$$
\begin{tikzcd}
M\otimes G \arrow[r, "1\otimes j"'] \arrow[rr, bend left, "1\otimes h"] \arrow[d, "a_G"] & M\otimes D(X) \arrow[r, "1\otimes ed"'] \arrow[d, "a_{D(X)}"] & M\otimes X^X \arrow[d, "a_{X^X}"] \\
G \arrow[r, "j"] \arrow[rr, bend right, "h"'] & D(X) \arrow[r, "ed"] & X^X
\end{tikzcd}
$$

$$
\begin{tikzcd}
G\otimes G \arrow[r, "j\otimes j"'] \arrow[rr, bend left, "h\otimes h"] \arrow[dd, "b_G"] & D(X)\otimes D(X) \arrow[r, "ed\otimes ed"'] \arrow[dd, "b_{D(X)}"] & X^X\otimes X^X \arrow[d, "1-\sigma"] \\
& & X^X\otimes X^X \arrow[d, "b_{X^X}"] \\
G \arrow[r, "j"] \arrow[rr, bend right, "h"'] & D(X) \arrow[r, "ed"] & X^X
\end{tikzcd}
$$

where $G = (G, a_G, b_G)$. Considering the left adjuncts of $a_{X^X}(1\otimes h)$ and ha_G (in the first diagram), and considering the left adjuncts of $b_{X^X}(1-\sigma)(h\otimes h)$

and hb_G (in the second diagram), the diagram formed by the outer arrows can be seen to commute. Therefore, since e and d are monomorphisms and the right hand square in each diagram commutes, the left hand squares also commute. $\square$

For each G in **Lie**(M,δ), using the above construction we define the map τ_G : SplExt$(G,X) \to$ **Lie**$(M,\delta)(G, \mathrm{Der}(X))$ as follows:

$$\tau_G([\ X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} G\ ]) = j.$$

Proposition 3.2. *The maps τ_G form a natural transformation.*

Proof. Let $A = (A,a,b)$ and $A' = (A',a',b')$ be objects in **Lie**(M,δ) and let $f : G' \to G$ be any morphism in **Lie**(M,δ), such that in the diagram

$$\begin{tikzcd}
X \arrow[r, shift right, "k'"'] \arrow[d, "1_X"] & A' \arrow[l, shift right, "l'"'] \arrow[r, shift left, "p'"] \arrow[d, "f'"] & G' \arrow[l, shift left, "s'"] \arrow[d, "f"] \\
X \arrow[r, shift right, "k"'] & A \arrow[l, shift right, "l"'] \arrow[r, shift left, "p"] & G \arrow[l, shift left, "s"]
\end{tikzcd}$$

(A',f',p') is the pullback of f and p in **Lie**(M,δ), l and l' are the unique M-action morphisms with $kl = 1_A - sp$ and $k'l' = 1'_A - s'p'$, and the top and bottom rows excluding l and l' are split extensions. Let h' be the right adjunct of $l'b'(s' \otimes k')$ and j' be the unique morphism with $edj' = h'$, that is,

$$\tau'_G([\ X \xrightarrow{k'} A' \underset{s'}{\overset{p'}{\rightleftarrows}} G'\ ]) = j'.$$

Since $lb(s \otimes k)(f \otimes 1) = lb(sf \otimes k) = lb(f's' \otimes f'k') = lfb'(s' \otimes k') = l'b'(s' \otimes k')$ and h and h' are the right adjuncts of $lb(s \otimes k)$ and $l'b'(s' \otimes k')$ respectively, it follows that $hf = h'$. Therefore we have $edjf = hf = h' = edj'$ and since ed is monomorphism we conclude that $jf = j'$ and that the diagram

$$\begin{tikzcd}
\mathrm{SplExt}(G,X) \arrow[r, "\tau_G"] \arrow[d, "\mathrm{SplExt}(f,X)"] & \mathbf{Lie}(M,\delta)(G,\mathrm{Der}(X)) \arrow[d, "\mathbf{Lie}(M,\delta)(f,\mathrm{Der}(X))"] \\
\mathrm{SplExt}(G',X) \arrow[r, "\tau_{G'}"] & \mathbf{Lie}(M,\delta)(G',\mathrm{Der}(X))
\end{tikzcd}$$

commutes. $\square$

Theorem 3.3. *The functor* $\mathrm{SplExt}(-, X) : \mathbf{Lie}(M, \delta) \to \mathbf{Set}$ *is representable with representation* $(\tau, \mathrm{Der}(X))$.

Proof. We show that the natural transformation $\tau : \mathrm{SplExt}(-, X) \to \mathbf{Lie}(M, \delta)(-, \mathrm{Der}(X))$ is a natural isomorphism. For an arrow $z : G \to \mathrm{Der}(X)$ in $\mathbf{Lie}(M, \delta)$ let $r : G \otimes X \to X$ be the left adjunct of edz, and let $X \rtimes_z G = (X \oplus G, a, b)$, where

$$a = \iota_1 a_X(1 \otimes \pi_1) + \iota_2 a_G(1 \otimes \pi_2)$$

and

$$b = \iota_1(b_X(\pi_1 \otimes \pi_1) + r(\pi_2 \otimes \pi_1)(1 - \sigma)) + \iota_2 b_G(\pi_2 \otimes \pi_2),$$

in obvious notation. It can been seen that $X \rtimes_z G$ is in $\mathbf{Lie}(M, \delta)$ and that the diagram

$$X \xrightarrow{\iota_1} X \rtimes_r G \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} G$$

is a split extension in $\mathbf{Lie}(M, \delta)$. Let $\hat{\tau}_G : \mathbf{Lie}(M, \delta)(G, \mathrm{Der}(X)) \to \mathrm{SplExt}(G, X)$ be the map defined as follows:

$$\hat{\tau}_G(z) = [\ X \xrightarrow{\iota_1} X \rtimes_r G \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} G\ ]$$

It can be seen that $\hat{\tau_G} = {\tau_G}^{-1}$ and hence $(\tau, \mathrm{Der}(X))$ is a representation of $\mathrm{SpltExt}(-, X)$. $\square$

Remark 3.4. *Since the category* $\mathbf{Cat}(\mathbf{Lie}(M, \delta))$ *of internal categories in* $\mathbf{Lie}(M, \delta)$ *can be presented as* $\mathbf{Lie}(M', \delta')$ *for suitable* M' *and* δ' *(it essentially follows from the results of [5]), by Theorem 3.3 the functor* SplExt: $\mathbf{Cat}(\mathbf{Lie}(M, \delta)) \to \mathbf{Set}$ *is representable.*

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REPRESENTATION OF METRIC JETS

by Elisabeth BURRONI and Jacques PENON

We dedicate this article to Francis Borceux

Abstract

Guided by the heuristic example of the tangential $\mathrm{T}f_a$ of a map f differentiable at a which can be canonically represented by the unique continuous affine map it contains, we extend, in this article, into a specific metric context, this property of representation of a metric jet. This yields a lot of relevant examples of such representations.

L'application affine continue qui est tangente, en un point a fixé, à une application f différentiable en ce point, peut être très naturellement considérée comme un représentant de la tangentielle Tf_a de f en a. Cet exemple sera notre guide heuristique pour trouver un context métrique spécifique dans lequel cette propriété de représentation d'un jet métrique soit possible. Au passage, on fournit de nombreux exemples pertinents de telles représentations.

Key words : differential calculus, Gateaux differentials, fractal maps, jets, metric spaces, categories

AMS classification : 58C25, 58C20, 28A80, 58A20, 54E35, 18D20

INTRODUCTION

This article is the sequel of a paper published in TAC [6] ; most of the proofs of the statements given here can be found in the second chapter of a paper published in arXiv [5].

We recall that maps $f, g : M \longrightarrow M'$ (where M,M' are metric spaces) are tangent at a (not isolated in M), what we denote $f \succ\!\!\prec_a g$, if $f(a) = g(a)$ and $\lim_{a \neq x \to a} \frac{d(f(x),g(x))}{d(x,a)} = 0$; a metric jet (in short, a jet) is an equivalence class for this relation $\succ\!\!\prec_a$, restricted to the set of the maps which are locally lipschitzian at a (in short, LL_a). We say that a map f is tangentiable at a (in short, $Tang_a$) if it possesses a tangent at a which is LL_a ; then the jet composed of all the LL_a maps which are tangent to f at a is called the tangential of f at a and denoted $\mathrm{T}f_a$.

$\mathbb{LL}$ and $\mathbb{J}$et are the cartesian categories whose objects are pointed metric spaces (for both) and morphisms $(M, a) \longrightarrow (M', a')$, the LL_a maps f verifying $f(a) = a'$ (for $\mathbb{LL}$), and the jets φ whose elements are forequoted maps which are tangent altogeter at a (for $\mathbb{J}$et). We denote $q : \mathbb{LL} \longrightarrow \mathbb{J}$et the canonical surjection which associates its tangential $\mathrm{T}f_a$ to a LL_a map $f : (M, a) \longrightarrow (M', a')$.

This paper takes up and develops two previous talks ([2] and [3]). Here, we propose a frame in which each jet can be canonically represented by one of its elements. This frame is the algebraico-metric structure of "Σ-contracting" metric space (equipped with a "central point" denoted ω) ; the morphisms between such spaces, called "Σ-homogeneous" maps, have a fundamental "Σ-uniqueness property" : two Σ-homogeneous maps which are tangent at ω are equal.

Carrying on the analogy with the classical differential calculus, we are interested (in the Σ-contracting metric world) in maps f which are tangentiable at ω and whose tangential $\mathrm{T}f_\omega$ possesses a Σ-homogeneous element which can represent it ; such a map is said "Σ-contactable" at ω, the unique Σ-homogeneous LL_ω element tangent to it at ω being its "Σ-contact" at ω. In many respects, the properties of "Σ-contactibility" are similar to those of differentiability, as, for instance, the search of extrema for a map taking its values in $\mathbb{R}$ (section 5).

We will mainly be interested in two special cases, in the normed vector space (in short n.v.s.), which will provide many examples. The first one with $\Sigma = \mathbb{R}_+$ (section 3), brings back the "old" interesting notion of maps which are differentiable in the sense of Gateaux [7]. The second one (section 4) immerses ourselves in the fractal universe. We finally notice that the notion of contactibility does not entirely exhaust the one of tangentiability, since there exist maps which are tangentiable (at a central point) and not contactable (at this point).

For general definitions in category theory (for instance cartesian or enriched categories), see [1].

Acknowledgements : It is a talk about Ehresmann's jets, given by Francis Borceux at the conference organised in Amiens in 2002 in honour of Andrée and Charles Ehresmann which has initiated our work. Since at that epoch we where interested, in our teaching, in what could be described uniquely with metric tools ... hence the idea of the metric jets !

1 Valued monoids, contracting spaces

Our aim, in this section, is to define the algebraico-metric notion of contracting spaces and prove they possess the above mentionned uniqueness property.

Definition 1.1 *a* valued monoid Σ *is a monoid (its law of composition is denoted here multiplicatively) equipped with a particular element denoted 0, and with a monoid homomorphism* $v : \Sigma \longrightarrow \mathbb{R}_+$ *(where* $\mathbb{R}_+ = [0, +\infty[$*), called the* valuation *of* Σ*, verifying the two conditions :*

(1) $\forall t \in \Sigma \quad (v(t) = 0 \Longleftrightarrow t = 0)$,

(2) $\exists t \in \Sigma \quad (0 < v(t) < 1)$.

Thanks to (1), we have that 0 is an absorbing element in Σ.

Examples 1.2

1) $\mathbb{R}$ and its multiplicative submonoids $\mathbb{R}_+$ and $[0, 1]$ are valued monoids, where 0 is their absorbing element (their valuation being the absolute value).

2) If r is a real number verifying $0 < r < 1$, we denote $\mathbb{N}'_r$ the additive monoid $\mathbb{N}'_r = \mathbb{N} \cup \{\infty\}$, where ∞ is its absorbing element, and whose valuation $v_r : \mathbb{N}'_r \longrightarrow \mathbb{R}_+$ is given by $v_r(n) = r^n$ if $n \in \mathbb{N}$ and $v_r(\infty) = 0$.

Definition 1.3

1) A morphism of valued monoids $\sigma : \Sigma \longrightarrow \Sigma'$ *is a monoid homomorphism verifying :* $\forall t \in \Sigma \quad (v(\sigma(t)) = v(t))$.

2) A valued submonoid *of* Σ' *is a valued monoid* Σ *verifying* $\Sigma \subset \Sigma'$ *such that the canonical injection* $j : \Sigma \hookrightarrow \Sigma'$ *is a morphism of valued monoids.*

Remark 1.4

A valuation $v : \Sigma \longrightarrow \mathbb{R}_+$ is itself a morphism of valued monoids.

Definition 1.5 Σ *being a valued monoid, a* Σ-contracting metric space *(in short a* Σ*-contracting space) is a metric space* M *pointed by* ω *(said to be* central*), equipped with an external operation* $\Sigma \times M \longrightarrow M$: $(t, x) \mapsto t \star x$ *which, in addition of the usual properties :*

(1) $\forall x \in M \;\; (1 \star x = x)$,

(2) $\forall t, t' \in \Sigma \; \forall x \in M \;\; (t' \star (t \star x) = (t'.t) \star x)$,

verifies also the following conditions :

(3) $0 \star \omega = \omega$,

(4) $\forall t \in \Sigma \;\; \forall x, y \in M \;\; (d(t \star x, t \star y) = v(t)d(x, y))$

The central point of Σ*-contracting spaces will usually be denoted* ω.

Remark 1.6 A Σ-contracting space M verifies the following properties : (1) $\forall t \in \Sigma \;\; (t \star \omega = \omega)$, (2) $\forall x \in M \;\; (0 \star x = \omega)$.

Examples 1.7

1) Let E be a n.v.s. ; fixing a point $a \in E$, the pointed n.v.s. (E, a) is denoted E_a. We make this E_a a $\mathbb{R}$-contracting space (with central point a), setting, for $t \in \mathbb{R}$ and $x \in E$, $t \star x = a + t(x - a)$. This external operation on E_a is said to be "standard".

2) When Σ is a valued monoid whose valuation $v : \Sigma \longrightarrow \mathbb{R}_+$ is injective, then Σ becomes itself a Σ-contracting space setting $\omega = 0$ and, for $s, t \in \Sigma$, $t \star s = t.s$ and $d(s, t) = |v(t) - v(s)|$.

3) If M, M' are Σ-contacting spaces, then so is $M \times M'$.

4) Let $\sigma : \Sigma \longrightarrow \Sigma'$ be a morphism of valued monoids ; then every Σ'-contracting space can be canonically equipped with a structure of Σ-contracting space, the new Σ-operation being : $(t, x) \mapsto \sigma(t) \star x$.

Remarks 1.8

1) If Σ is a valued monoid, every $\mathbb{R}_+$-contracting space M has also a "canonical" structure of Σ-contracting space (its external operation being : $(t,x) \mapsto v(t) \star x$ for every $t \in \Sigma$ and $x \in M$). In particular, a pointed n.v.s. E_a has also a canonical structure of Σ-contracting space, its external operation being, for $t \in \Sigma$ and $x \in E$, $t \star x = a + v(t)(x-a)$.

2) We notice that we have two structures of $\mathbb{R}$-contracting space on E_a : the standard one of 1.7, and the above canonical one, whose external operation is $t \star x = a + |t|(x-a)$.

3) In a Σ-contracting space $M \neq \{\omega\}$, ω is never isolated in M.

Definition 1.9

A Σ-contracting space is revertible *if, for every $t \in \Sigma$ verifying $t \neq 0$, the map $t \star (-) : M \longrightarrow M$ is bijective. In this case, we set $t \overset{-1}{\star} x = (t \star (-))^{-1}(x)$ for $x \in M$.*

Remark 1.10

If $\sigma : \Sigma \longrightarrow \Sigma'$ is a morphism of valued monoids and M a revertible Σ'-contracting space, then M is a revertible Σ-contracting space.

Examples and counter-examples 1.11

1) The pointed n.v.s. E_a is a revertible $\mathbb{R}_+$(or $\mathbb{R}$)-contracting space for the standard structure; with $t \overset{-1}{\star} x = t^{-1} \star x = a + t^{-1}(x-a)$ for every $t \in \Sigma$ $(t \neq 0)$ and $x \in E$. Actually, for each valued monoid Σ, E_a is a revertible Σ-contracting space for the canonical structure; with $t \overset{-1}{\star} x = t^{-1} \star x = a + v(t)^{-1}(x-a)$ for every $t \in \Sigma$ $(t \neq 0)$ and $x \in E$.

2) $[0,1]$ and $\mathbb{N}'_r$ are Σ-contracting spaces with, respectively, $\Sigma = [0,1]$, $\omega = 0$, and $\Sigma = \mathbb{N}'_r$, $\omega = \infty$. But none of them is revertible.

Definition 1.12 *Let Σ be a valued monoid.*

1) A map $h : M \longrightarrow M'$ is Σ-homogeneous *if M, M' are Σ-contracting spaces and h verifies : $\forall t \in \Sigma\ \ \forall x \in M \quad (h(t \star x) = t \star h(x))$.*

2) M' being a Σ-contracting space and M a metric subspace of M', then M is a Σ-contracting subspace *of M' if $\omega \in M$ and $t \star x \in M$*

for all $t \in \Sigma$ and $x \in M$; then the canonical injection $M \hookrightarrow M'$ is Σ-homogeneous.

Remarks 1.13

1) A Σ-homogeneous map $h : M \longrightarrow M'$ verifies $h(\omega) = \omega$.

2) Let $\sigma : \Sigma \longrightarrow \Sigma'$ be a morphism of valued monoids, M and M' two Σ'-contracting spaces. Then, every Σ'-homogeneous map $M \longrightarrow M'$ is Σ-homogeneous ; the inverse is true when σ is surjective.

Proposition 1.14 *For every Σ-homogeneous map $h : M \longrightarrow M'$, we have the equivalence :* $\quad h$ *k-lipschitzian* $\iff$ h *k-LL_ω*

Proof : $r > 0$ being such that $h|_{B(\omega,r)}$ is k-lipschitzian, we choose $t \in \Sigma$ such that $0 < v(t) < 1$; then, for each $x, y \in M$, there exists $n \in \mathbb{N}$ verifying $t^n \star x,\ t^n \star y \in B(\omega, r)$. $\square$

Theorem 1.15 (of Σ-uniqueness)
Let $h_1, h_2 : M \longrightarrow M'$ be two Σ-homogeneous maps verifying $h_1 \asymp_\omega h_2$; then $h_1 = h_2$.

Proof : Let us take $t \in \Sigma$ verifying $0 < v(t) < 1$ and fix $x \in M$. We can assume that $x \neq \omega$. Let us set $x_n = t^n \star x$ for each $n \in \mathbb{N}$. Then, we have $x_n \neq \omega$ and $\lim_n x_n = \omega$, so that we can write : $\frac{d(h_1(x_n),h_2(x_n))}{d(x_n,\omega)} = \frac{d(h_1(t^n \star x),h_2(t^n \star x))}{d(t^n \star x,t^n \star \omega)} = \frac{d(t^n \star h_1(x),t^n \star h_2(x))}{d(t^n \star x,t^n \star \omega)} = \frac{v(t^n)d(h_1(x),h_2(x))}{v(t^n)d(x,\omega)} = \frac{d(h_1(x),h_2(x))}{d(x,\omega)}$. But, as $\lim_n x_n = \omega$ and $h_1 \asymp_\omega h_2$, this provides $0 = \lim_n \frac{d(h_1(x_n),h_2(x_n))}{d(x_n,\omega)} = \frac{d(h_1(x),h_2(x))}{d(x,\omega)}$ which implies $d(h_1(x), h_2(x)) = 0$, i.e $h_1(x) = h_2(x)$. $\square$

Theorem 1.16 *Let M, M' be Σ-contracting spaces with M' revertible, V a neighborhood of ω in M, and maps $f : V \longrightarrow M'$, $h : M \longrightarrow M'$ such that h is Σ-homogeneous and verifies $f \asymp_\omega h|_V$. Then, for all $x \in M$, we have $h(x) = \lim_{0 \neq v(t) \to 0} t \overset{-1}{\star} f(t \star x)$.*

Proof : The above equality is clearly true for $x = \omega$. If $x \neq \omega$, we have $\omega = \lim_{v(t)\to 0} t \star x$ (since $d(t \star x, \omega) = d(t \star x, t \star \omega) = v(t)d(x, \omega)$), which insures that there exists $\varepsilon > 0$ such that, for all

$t \in \Sigma$ verifying $0 < v(t) < \varepsilon$, we have $t \star x \in V - \{\omega\}$. Thus, for all these t, we can write : $\frac{d(f(t\star x),h(t\star x))}{d(t\star x,\omega)} = \frac{d(f(t\star x),t\star h(x))}{d(t\star x,t\star\omega)} = \frac{d(t\star(t\overset{-1}{\star} f(t\star x)),t\star h(x))}{v(t)d(x,\omega)} = \frac{v(t)d(t\overset{-1}{\star} f(t\star x),h(x))}{v(t)d(x,\omega)} = \frac{d(t\overset{-1}{\star} f(t\star x),h(x))}{d(x,\omega)}$. Since $f \asymp_\omega h|_V$, we have $\lim_{0\neq v(t)\to 0} \frac{d(t\overset{-1}{\star} f(t\star x),h(x))}{d(x,\omega)} = \lim_{0\neq v(t)\to 0} \frac{d(f(t\star x),h(t\star x))}{d(t\star x,\omega)} = 0$, which finally means that $\lim_{0\neq v(t)\to 0} t \overset{-1}{\star} f(t \star x) = h(x)$. $\square$

Theorem 1.17 *Let M, M' be Σ-contracting spaces with M' revertible, V a neighborhood of ω in M, $g : V \longrightarrow M'$ a k-LL_ω map, and $h : M \longrightarrow M'$ a Σ-homogeneous map verifying $g \asymp_\omega h|_V$. Then, h is k-lipschitzian.*

<u>Proof</u> : Let W be a neighborhood of ω in V such that $g|_W$ is k-lipschitzian, $x, y \in M$, and $t \in \Sigma$ such that $0 < v(t) < 1$; then, there exists $N \in \mathbb{N}$ such that $t^n \star x, t^n \star y \in W$ for all $n \geq N$; so that, for all these n, we have $d(g(t^n \star x), g(t^n \star y)) \leq kd(t^n \star x, t^n \star y)$ $= kv(t^n)d(x, y)$, which provides $d(t^n \overset{-1}{\star} g(t^n \star x), t^n \overset{-1}{\star} g(t^n \star y)) = (v(t^n))^{-1}d(g(t^n \star x), g(t^n \star y)) \leq kd(x, y)$. Now, d being continuous, we obtain (doing $n \to +\infty$) : $d(h(x), h(y)) \leq kd(x, y)$. $\square$

Corollary 1.18 *M, M' being as in 1.17, and $h : M \longrightarrow M'$ being a Σ-homogeneous map, we have the equivalences :*

$$h \textit{ lipschitzian} \iff h\ LL_\omega \iff h\ Tang_\omega$$

Counter-example 1.19 We give here an example of function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ which is $\mathbb{R}$-homogeneous and continuous (since it is LSL : see 1.20 below), but not lipschitzian (thus not $Tang_0$). Consider the function $f(x, y) = x \sin \frac{y}{x}$ if $x \neq 0$ and $f(0, y) = 0$ (see Figure 1 in [5]). This f is clearly $\mathbb{R}$-homogeneous $\mathbb{R}^2_0 \longrightarrow \mathbb{R}_0$. We also notice that f is LSL (at every point) since it is differentiable on $\mathbb{R}^* \times \mathbb{R}$; and it is clear at each point $(0, a)$. Now, if $x \neq 0$, we have $\frac{\partial f}{\partial x}(x, y) = \sin \frac{y}{x} - \frac{y}{x} \cos \frac{y}{x}$ and thus, putting $x_n = \frac{1}{n^2}$ and $y_n = \frac{2\pi}{n}$, we obtain $\lim_n \frac{\partial f}{\partial x}(x_n, y_n) = -\infty$, where $\lim_n(x_n, y_n) = (0, 0)$. Thus, this function f cannot be lipschitzian !

Remarks 1.20

1) For a map $f : M \longrightarrow M'$ between metric spaces; LSL means "locally semi-lipschitzian" at every point of M; and "semi-lipschitzian" at $a \in M$ means that, there exists a real $k > 0$ such that, for all $x \in M$, we have $d(f(x), f(a)) \leq kd(x, a)$.

2) By the way, in 1.19, we have prove that : $LSL_a \not\Rightarrow Tang_a$. Thus, we cannot complete our equivalences of 1.18, adding the properties of being LSL_ω or continuous at ω! Even though, for linear maps, all these properties are equivalent.

2 Representability and Contactibility

The Σ-uniqueness property allows to choose at most one canonical representative element in each jet between Σ-contracting spaces (pointed by their central point ω); hence the term of "Σ-representable" jets. The maps f which are tangentiable at ω and whose tangential $\mathrm{T}f_\omega$ is Σ-representable are called "Σ-contactable" at ω.

Remarks 2.1

1) Let Σ be a valued monoid; a map $h : M \longrightarrow M'$ which is Σ-homogeneous and lipschitzian will be called Σ-Lhomogeneous.

2) A Σ-Lhomogeneous map is a Σ-homogeneous which is LL_ω.

We denote Σ-$\mathbb{C}$ontr, the category whose objects are the Σ-contracting spaces and whose morphisms are the Σ-Lhomogeneous maps (it is a suitable world for guarantying the Σ-uniqueness property). When $\sigma : \Sigma \longrightarrow \Sigma'$ is a morphism of valued monoids, there exists a canonical functor $\widehat{\sigma} : \Sigma'$-$\mathbb{C}$ontr $\longrightarrow \Sigma$-$\mathbb{C}$ontr. For every valued monoid Σ, we also have another canonical functor $U : \Sigma$-$\mathbb{C}$ontr $\longrightarrow \mathbb{LL}$ defined by $U(M) = (M, \omega)$ and $U(h) = h$; then we call J the following composite : Σ-$\mathbb{C}$ontr $\xrightarrow{U} \mathbb{LL} \xrightarrow{q} \mathbb{J}$et (refer to the introduction for $q : \mathbb{LL} \longrightarrow \mathbb{J}$et).

Proposition 2.2 *Σ-$\mathbb{C}$ontr is a cartesian category and the previous functors U, and then J, are strict morphisms of cartesian categories.*

Now, since for each $M, M' \in |\Sigma\text{-}\mathbb{C}\mathrm{ontr}|$, the canonical map : $\Sigma\text{-}\mathbb{C}\mathrm{ontr}(M, M') \xrightarrow{can} \mathbb{J}\mathrm{et}(JM, JM')$, defined by $can(h) = J(h) = \mathrm{T}h_\omega$, is injective (thanks to the Σ-uniqueness property), we can equip $\Sigma\text{-}\mathbb{C}\mathrm{ontr}(M, M')$ with the distance $d(h, h') = d(J(h), J(h'))$ (we recall (see [6]) that the category $\mathbb{J}\mathrm{et}$ is enriched in $\mathbb{M}\mathrm{et}$, the cartesian category whose objects are the metric spaces and morphisms, the LSL maps).

Proposition 2.3

1) The cartesian category $\Sigma\text{-}\mathbb{C}ontr$ is enriched in $\mathbb{M}et$.

2) For each $M, M_0, M_1 \in |\Sigma-\mathbb{C}\mathrm{ontr}|$, the following canonical map : $\Sigma-\mathbb{C}\mathrm{ontr}(M, M_0 \times M_1) \xrightarrow{can} \Sigma-\mathbb{C}\mathrm{ontr}(M, M_0) \times \Sigma-\mathbb{C}\mathrm{ontr}(M, M_1)$ is an isometry.

Proposition 2.4 *Let $h, h' \in \Sigma\text{-}\mathbb{C}\mathrm{ontr}(M, M')$; then, we have $d(h, h') = \sup\{C(x) \,|\, x \in B'(\omega, 1)\}$, where $C(x) = \frac{d(h(x),h'(x))}{d(x,\omega)}$ if $x \neq \omega$, $C(\omega) = 0$.*

Proposition 2.5 *Let M, M' be Σ-contracting spaces where M' is revertible, and $h : M \longrightarrow M'$ a Σ-Lhomogeneous map. Let us set $k = \sup\{\frac{d(h(x),h(y))}{d(x,y)} \,|\, x, y \in B'(\omega, 1); x \neq y\}$. Then :*

1) h is k-lipschitzian,

2) $\rho(\mathrm{T}h_\omega) = k$ (if φ is a jet, its lipschitzian ratio $\rho(\varphi) = \inf K(\varphi)$ where $K(\varphi) = \{k > 0 \mid \exists f \in \varphi,\ f \text{ is } k - LL_a\}$; see [6]).

<u>*Proof*</u> : Come from 1.17. □

Definition 2.6 *Consider two Σ-contracting spaces M, M' and a jet $\varphi : (M, \omega) \longrightarrow (M', \omega)$. We say that : φ is* Σ-representable *if there exists a Σ-Lhomogeneous element $h : M \longrightarrow M'$ verifying $J(h) = \varphi$ (i.e. $\mathrm{T}h_\omega = \varphi$). Thanks to the uniqueness theorem, such a h is unique, and may thus be called the* Σ-representative element *of the jet φ.*

Remark 2.7 The Σ-representable jets are stable under composition, and pairs (and thus products).

We now call Σ-*contracting domain* a pair (M, V) where M is a Σ-contracting space and V a neighborhood of ω in M. Besides, $f : (M, V) \longrightarrow (M', V')$ is said to be a centred map if (M, V) and (M', V') are Σ-contracting domains, and $f : V \longrightarrow V'$ verifies $f(\omega) = \omega$.

Definition 2.8 *Let $f : (M, V) \longrightarrow (M', V')$ be a centred map. We say that f is* Σ-contactable *if $f : V \longrightarrow V'$ is tangentiable at ω and if the following composite jet is Σ-representable :*

$$(M, \omega) \xrightarrow{\quad\sim\quad} (V, \omega) \xrightarrow{\quad \mathrm{T}f_\omega \quad} (V', \omega) \xrightarrow{\quad\sim\quad} (M', \omega)$$

We denote $\mathrm{K}_\Sigma f : M \longrightarrow M'$ (or merely $\mathrm{K}f$ if none ambiguity about Σ) the unique representative element of the above composite jet ; and we call it the Σ-contact *of f.*

Remarks 2.9

1) In other words, f is Σ-contactable if there exists a Σ-Lhomogeneous $h : M \longrightarrow M'$ such that $f \asymp_\omega h|_V$ (where, here, f is seen as a map $V \longrightarrow M'$). In that case, $h = \mathrm{K}_\Sigma f$; and $\mathrm{K}_\Sigma f \in \Sigma\text{-}\mathbb{C}\mathrm{ontr}(M, M')$.

2) Let $\sigma : \Sigma \longrightarrow \Sigma'$ be a morphism of valued monoids. If f is a Σ'-contactable centred map, f is also Σ-contactable, with $\mathrm{K}_\Sigma f = \mathrm{K}_{\Sigma'} f$.

3) Let E, E' be n.v.s., U an open subset of E, $a \in U$, $f : U \longrightarrow E'$ a map. If f is differentiable at a, then $f : (E_a, U) \longrightarrow (E'_{f(a)}, E')$ is standard $\mathbb{R}$-contactable with $\mathrm{K}_\mathbb{R} f(x) = f(a) + \mathrm{d}f_a(x-a))$, its continuous affine tangent at a ; and, for every valued monoid Σ, it is even canonically Σ-contactable with $\mathrm{K}_\Sigma f = \mathrm{K}_\mathbb{R} f$ written as above.

Proposition 2.10 *Let $f : (M, V) \longrightarrow (M', V')$ be a centred Σ-contactable map. Then, for all $x \in M$, $\mathrm{K}f(x) = \lim_{0 \neq v(t) \to 0} t \overset{-1}{\star} f(t \star x)$.*

Proposition 2.11 *Here, for lightening, we omit the surrounding Σ-contracting spaces of the several neigborhoods of ω.*

1) Let $f : V \longrightarrow V'$, $g : V' \longrightarrow V''$ be two centred maps. If f and g are Σ-contactable, so is $g.f$, with $\mathrm{K}(g.f) = \mathrm{K}g.\mathrm{K}f$.

2) Let $f_0 : V \longrightarrow V_0$, $f_1 : V \longrightarrow V_1$ be two centred maps. If f_0 and f_1 are Σ-contactable, so is the pair $(f_0, f_1) : V \longrightarrow V_0 \times V_1$, with $\mathrm{K}(f_0, f_1) = (\mathrm{K}f_0, \mathrm{K}f_1)$.

3) Let $f_0 : V_0 \longrightarrow V'_0$, $f_1 : V_1 \longrightarrow V'_1$ be two centred maps. If f_0 and f_1 are Σ-contactable, so is the product $f_0 \times f_1 : V_0 \times V_1 \longrightarrow V'_0 \times V'_1$, with $\mathrm{K}(f_0 \times f_1) = \mathrm{K}f_0 \times \mathrm{K}f_1$.

In the n.v.s. frame, we can define, with the help of an isometric translation, the analog $\mathrm{k}f_a$ of the differential at a $\mathrm{d}f_a$, for the contact $\mathrm{K}f$ (this one generalizing the continuous affine tangent to f at a). More precisely, we recall that, for every n.v.s. E and $a \in E$, the pointed n.v.s. E_a is a Σ-contracting space for any valued monoid Σ. Now, if $f : (E_a, U) \longrightarrow (E'_{f(a)}, U')$ is a Σ-contactable map (in the sens of 2.8 and 2.9), then we say that f is Σ-contactable at a, its Σ-contact at a being $\mathrm{k}_\Sigma f_a = \Theta^{-1}_{af(a)}(\mathrm{K}f_\Sigma)$, in short $\mathrm{k}f_a = \Theta^{-1}_{af(a)}(\mathrm{K}f)$; where $\Theta_{aa'} : \Sigma\text{-}\mathbb{C}\mathrm{ontr}(E_0, E'_0) \longrightarrow \Sigma\text{-}\mathbb{C}\mathrm{ontr}(E_a, E'_{a'})$ is the isometric translation defined by $\Theta_{aa'}(h)(x) = a' + h(x - a)$. The Σ-Lhomogeneous map $\mathrm{k}f_a : E_0 \longrightarrow E'_0$ is thus the translate at 0 of the Σ-Lhomogeneous map $\mathrm{K}f : E_a \longrightarrow E'_{f(a)}$, the Σ-contact of f.

The formulas of 2.11 can then be rewritten in this context for the $\mathrm{k}f_a$, absolutely similar to those of $\mathrm{d}f_a$ (the $\mathrm{k}f_a$ are stable under composition, pairs and products, with : $\mathrm{k}(g.f)_a = \mathrm{k}g_{f(a)}\mathrm{k}f_a$, $\mathrm{k}(f_1, f_2)_a = (\mathrm{k}f_{1a}, \mathrm{k}f_{2a})$ and $\mathrm{k}(f_1 \times f_2)_{(a_1,a_2)} = \mathrm{k}f_{1a_1} \times \mathrm{k}f_{2a_2}$).

Remarks 2.12

1) Let E, E' two n.v.s., U an open subset of E, $a \in U$; then $f : U \longrightarrow E'$ differentiable at $a \implies f : (E_a, U) \longrightarrow (E'_{f(a)}, E')$ Σ-contactable at a for any Σ ; with $\mathrm{k}_\Sigma f_a = \mathrm{d}f_a$. More precisely, if f is Σ-contactable at a, then f is differentiable at a iff $\mathrm{k}_\Sigma f_a$ is linear.

2) $f : (\mathbb{R}_a, U) \longrightarrow (E'_{f(a)}, U')$ is standard $\mathbb{R}$-contactable at a iff $f : U \longrightarrow U'$ is differentiable at a with $\mathrm{k}_\mathbb{R} f_a = \mathrm{d}f_a$. It is not always the case : see 2)below.

3) We prove here that standard $\mathbb{R}$-contactable at a $\not\Longrightarrow$ differentiable at a : consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(0,0) = 0$ and $f(x,y) = \frac{xy^2}{x^2+y^2}$ if $(x,y) \neq (0,0)$. This f is differentiable on $\mathbb{R}^2 - \{(0,0)\}$, and, since for $(x,y) \neq (0,0)$, $\|df_{(x,y)}\|$ is bounded, f is lipschitzian on $\mathbb{R}^2$. Besides, it is obviously $\mathbb{R}$-homogeneous, so that f is standard $\mathbb{R}$-contactable at 0, with $\mathrm{k}_\mathbb{R} f_0 = f$. However, since f is not linear, it cannot be differentiable at 0 (see Figure 5 in [5]) !

Proposition 2.13 *If $f : (E_a, U) \longrightarrow (E'_{f(a)}, U')$ is Σ-contactable at a, we have :*

1) $\mathrm{k}f_a(x) = \lim_{0 \neq v(t) \to 0} \frac{f(a+v(t)x) - f(a)}{v(t)}$ *for all $x \in E$,* *2)* $\rho(\mathrm{T}f_a) = \sup\{\frac{\|h(x)-h(y)\|}{\|x-y\|} \mid x, y \in B'(0,1), x \neq y\}$*, where* $h = \mathrm{k}f_a$ *is* $\rho(\mathrm{T}f_a)$*-lipschitzian (see 2.5).*

3 G-differentiability

Here $\Sigma = \mathbb{R}_+$. René Gateaux [1], defined maps (said "differentiable in the sense of Gateaux") which are very close to our $\mathbb{R}_+$-contactable maps : the main difference being the fact that we replace his continuous maps (see Bouligand in [7]) by lipschitzian maps. In homage to Gateaux, we choose "G-differentiable" for "$\mathbb{R}_+$- contactable" [2].

E and E' being two n.v.s., we merely write $\mathbb{H}\mathrm{om}_+(E, E')$ for the n.v.s. that we should denote $\mathbb{R}_+$-$\mathbb{C}\mathrm{ontr}(E_0, E'_0)$; we thus recall that its elements are the $\mathbb{R}_+$-Lhomogeneous maps $h : E_0 \longrightarrow E'_0$, i.e the maps $h : E \longrightarrow E'$ which are lipschitzian and verify $h(tx) = th(x)$ for all $t \in \mathbb{R}_+$ and $x \in E$.

Examples 3.1

1) Standard $\mathbb{R}$-Lhomogeneous implies $\mathbb{R}_+$-Lhomogeneous ; in particular, the continuous linear maps from E to E' are in $\mathbb{H}\mathrm{om}_+(E, E')$.

2) Let E be a n.v.s. ; then every norm N on E, which is equivalent to the given norm $\| \ \|$ on E, is in $\mathbb{H}\mathrm{om}_+(E, \mathbb{R})$.

3) The maps $\mathrm{Max}, \mathrm{Min} : \mathbb{R}^n \longrightarrow \mathbb{R}$ are in $\mathbb{H}\mathrm{om}_+(\mathbb{R}^n, \mathbb{R})$, since they are 1-lipschitzian.

Proposition 3.2

1) Let $h \in \mathbb{H}\mathrm{om}_+(E, E')$*, where* $E \neq \{0\}$ *; we have* $\|h\| = \sup\{\|h(x)\| \mid \|x\| = 1\}$.

2) Let $h \in \mathbb{H}\mathrm{om}_+(E, E')$*, then, for all* $\varepsilon > 0$*, we have* $\rho(\mathrm{T}h_0) = \sup\{\frac{\|h(x)-h(y)\|}{\|x-y\|} \mid x \neq y;\ x, y \in C(\varepsilon)\}$*, where* $C(\varepsilon) = \{x \in E \mid 1 - \varepsilon < \|x\| < 1 + \varepsilon\}$.

1. The young French René Gateaux was one of the first victims of the first world war, he was twenty five years old when he died on the third of october 1914.

2. it is shorter than "lipschitzian Gateaux-differentiable" which would be more convenient.

<u>*Proof*</u> : 2) Let us set $r(\varepsilon) = \sup\{\frac{\|h(x)-h(y)\|}{\|x-y\|} \mid x \neq y;\ x, y \in C(\varepsilon)\}$. It is clear that $r(\varepsilon) \leq \rho(\mathrm{T}h_0)$ since h is $\rho(\mathrm{T}h_0)$-lipschitzian (by 2.12). Conversely, take $a \in E$, $a \neq 0$ and let us set $\varepsilon' = \|a\|\varepsilon$. Then, for every $x \in B(a, \varepsilon')$, we have $\frac{x}{\|a\|}, \frac{a}{\|a\|} \in C(\varepsilon)$. Then $r(\varepsilon) \geq \frac{\|h(\frac{x}{\|a\|})-h(\frac{a}{\|a\|})\|}{\|\frac{x}{\|a\|}-\frac{a}{\|a\|}\|} = \frac{\|h(x)-h(a)\|}{\|x-a\|}$ if $x \neq a$; so that h is $r(\varepsilon)$-LSL_a. In particular, for every $a \in B'(0,1) - \{0\}$, h is $r(\varepsilon)$- LSL_a, with $B'(0,1)$ convex. Thus, $h|_{B'(0,1)}$ is $r(\varepsilon)$-lipschitzian (see section 1 in [6]), so that $\rho(\mathrm{T}h_0) \leq r(\varepsilon)$. $\square$

Proposition 3.3 *Let E be a n.v.s. ; then the following map is a linear isometry :* $\quad can : \mathbb{H}\mathrm{om}_+(\mathbb{R}, E) \longrightarrow E \times E : h \mapsto (-h(-1), h(1))$.

Definition 3.4 *A centred map $f : (E_a, U) \longrightarrow (E'_{f(a)}, U')$ is said to be* G-differentiable *at a if it is $\mathbb{R}_+$-contactable at a ; in this case,* $\mathrm{K}_+ f$ *and* $\mathrm{k}_+ f_a$ *will respectively merely denote* $\mathrm{K}_{\mathbb{R}_+} f$ *and* $\mathrm{k}_{\mathbb{R}_+} f_a$.

Remarks 3.5 In the following cases, E and E' are n.v.s., $f : U \longrightarrow E'$ with U an open subset of E, and $a \in U$.

1) If f is differentiable at a, then f is G-differentiable at a with $\mathrm{k}_+ f_a = \mathrm{d} f_a$. More precisely, if f is G-differentiable at a, then f is differentiable at a iff $\mathrm{k}_+ f_a$ is linear.

2) If f is G-differentiable at a, then f is standard $\mathbb{R}$-contactable at a iff $\mathrm{k}_+ f_a$ is standard $\mathbb{R}$-homogeneous with $\mathrm{k}_+ f_a = \mathrm{k}_{\mathbb{R}} f_a$.

3) If f is G-differentiable at a, then f is Σ-contactable at a for all valued monoid Σ, with $\mathrm{k}_\Sigma f_a = \mathrm{k}_+ f_a$.

Examples 3.6 The following examples are all G-differentiable :

1) The norm function $\vartheta : \mathbb{R} \longrightarrow \mathbb{R} : x \mapsto |x|$, which verifies $\mathrm{k}_+ \vartheta_0 = \vartheta$ and, for $a \neq 0$, $\mathrm{k}_+ \vartheta_a = \mathrm{d}\vartheta_a = sign(a) Id_{\mathbb{R}}$ where $sign(a) = \frac{a}{|a|}$.

2) The euclidian norm function $N^2 : \mathbb{R}^n \longrightarrow \mathbb{R} : x \mapsto \|x\|_2$, which verifies $\mathrm{k}_+ N^2_0 = N^2$ and, for $a \neq 0$, $\mathrm{k}_+ N^2_a = \mathrm{d} N^2_a$.

3) The functions Max, Min : $\mathbb{R}^n \longrightarrow \mathbb{R}$, which verify :
$\mathrm{k}_+\mathrm{Max}_a(x) = \mathrm{Max}_{i \in \overline{\mathcal{I}}(a)}(x_i)$ and $\mathrm{k}_+\mathrm{Min}_a(x) = \mathrm{Min}_{i \in \underline{\mathcal{I}}(a)}(x_i)$, where $\overline{\mathcal{I}}(a) = \{i \in \{1, \ldots, n\} \mid a_i = \mathrm{Max}(a)\}$ and $\underline{\mathcal{I}}(a) = \{i \in \{1, \ldots, n\} \mid a_i = \mathrm{Min}(a)\}$; which provides $\mathrm{k}_+\mathrm{Max}_0 = \mathrm{Max}$ and $\mathrm{k}_+\mathrm{Min}_0 = \mathrm{Min}$.

4) The product norm function $N^\infty : \mathbb{R}^n \longrightarrow \mathbb{R} : x \mapsto \|x\|_\infty$, which verifies $\mathrm{k}_+N^\infty_0 = N^\infty$ and, for $a \neq 0$, $\mathrm{k}_+N^\infty_a(x) = \mathrm{Max}_{i\in\overline{\mathcal{I}}(|a|)}(sign(a_i)x_i)$, where $|a| = (|a_1|, \ldots, |a_n|)$.

5) The norm function $N^1 : \mathbb{R}^n \longrightarrow \mathbb{R} : x \mapsto \|x\|_1$, which verifies $\mathrm{k}_+N^1_a(x) = \sum_{i\in\mathcal{I}_0(a)} |x_i| + \sum_{i\notin\mathcal{I}_0(a)} sign(a_i)x_i$, where $\mathcal{I}_0(a) = \{i \in \{1,\ldots,n\} \mid a_i = 0\}$.

Proof : First, all the above functions h being $\mathbb{R}_+$-Lhomogeneous, they are all G-differentiable at 0 with $\mathrm{k}_+h_0 = h$.

3) It comes from the equality $\mathrm{Max}(a + x) = \mathrm{Max}(a) + \mathrm{Max}_{i\in\overline{\mathcal{I}}(a)}x_i$, which is true for all a, locally in a neighborhood of 0. Indeed :

- If $\overline{\mathcal{I}}(a) = \{1, \cdots, n\}$ (i.e, if a is a constant n-uplet), then, for all $x \in \mathbb{R}^n$, we have $\mathrm{Max}(a + x) = \mathrm{Max}(a) + \mathrm{Max}(x) = \mathrm{Max}(a) + \mathrm{Max}_{i\in\overline{\mathcal{I}}(a)}x_i$.

- If $\overline{\mathcal{I}}(a) \neq \{1, \cdots, n\}$. Let $j \in \overline{\mathcal{I}}(a)$ such that $\mathrm{Max}_{i\in\overline{\mathcal{I}}(a)}(x_i) = x_j$; we can write : $\mathrm{Max}(a) + \mathrm{Max}_{i\in\overline{\mathcal{I}}(a)}(x_i) = a_j + x_j \leq \mathrm{Max}(a + x)$. Conversely, we set $r = \frac{1}{2}\mathrm{Min}_{i\notin\overline{\mathcal{I}}(a)}(\mathrm{Max}(a) - a_i)$; then $r > 0$. Let V be the open ball $B_\infty(0, r)$ (for the product norm $\| \ \|_\infty$) ; then, for $x \in V$, we have :

- if $j \notin \overline{\mathcal{I}}(a)$, $x_j - \mathrm{Max}_{i\in\overline{\mathcal{I}}(a)}(x_i) \leq |x_j| + |\mathrm{Max}_{i\in\overline{\mathcal{I}}(a)}(x_i)| \leq 2\|x\|_\infty < 2r \leq \mathrm{Max}(a) - a_j$, and thus again $a_j + x_j \leq \mathrm{Max}(a) + \mathrm{Max}_{i\in\overline{\mathcal{I}}(a)}(x_i)$,

- if $j \in \overline{\mathcal{I}}(a)$, $a_j + x_j = \mathrm{Max}(a) + x_j \leq \mathrm{Max}(a) + \mathrm{Max}_{i\in\overline{\mathcal{I}}(a)}(x_i)$; finaly, for all $x \in V$, we have $\mathrm{Max}(a + x) \leq \mathrm{Max}(a) + \mathrm{Max}_{i\in\overline{\mathcal{I}}(a)}(x_i)$; hence the result. Same for Min.

4) Since $N^\infty = \mathrm{Max}.\vartheta^n$, (where $\vartheta^n = \vartheta\times\cdots\times\vartheta$, n times), the function N^∞ is G-differentiable by composition (section 2).

5) Since $N^1 = \sigma.\vartheta^n$, where σ is the addition of $\mathbb{R}^n$, N^1 is G-differentiable still by composition. □

Remarks 3.7

1) We are giving here, for each G-differentiable function h studied in 3.6, the domain $D(h)$ on which h is differentiable.

a) $D(\vartheta) = \mathbb{R}^* = \mathbb{R} - \{0\}$.

b) $D(N^2) = (\mathbb{R}^*)^n$.

c) $D(\mathrm{Max}) = \{a \in \mathbb{R}^n \mid \exists! i \leq n \ (a_i = \mathrm{Max}(a))\}$. For instance, for $n = 2$, $\mathrm{D(Max)} = \Delta^c$ where Δ is the diagonal.

d) Here $D(N^\infty) = \{a \in \mathbb{R}^n \mid \exists! i \leq n \ (|a_i| = \|a\|_\infty)\}$. For $n = 2$, $D(N^\infty) = (\{a \in \mathbb{R}^2 \mid |a_1| = |a_2|\})^c$.

e) For N^1, we have $D(N^1) = \{a \in \mathbb{R}^n \mid \forall i \leq n \ \ a_i \neq 0\}$.

2) Of course, there exist G-differentiable maps which are not $\mathbb{R}_+$-homogeneous : 3.6 gives a lot of such examples. Indeed, a translate $g(x) = f(a+x)$ of a $\mathbb{R}_+$-homogeneous map f is not necessarily still $\mathbb{R}_+$-homogeneous, although, in our examples, such a translate remains G-differentiable (by composition).

Proposition 3.8 *Let $f : (\mathbb{R}_a, U) \longrightarrow (E'_{f(a)}, U')$ be a centred map. Then f is G-differentiable iff f admits left and right derivatives at a. In this case, referring to 3.3 for the linear isometry can, we have $\mathrm{k}_+ f_a = can^{-1}(f'_l(a), f'_r(a))$.*

Continuously G-differentiable maps

Our aim here is to prove that, in finite dimension, every *continuously G-differentiable* map f (i.e G-differentiable such that $\mathrm{k}_+ f$ is continuous) is in fact of class C^1. We need some preliminary results.

Proposition 3.9 *Let U be an open subset of $\mathbb{R}$ and $f : U \longrightarrow \mathbb{R}$ a continuous function admitting left and right derivatives at every point of U and such that the functions $f'_l, f'_r : U \longrightarrow \mathbb{R}$ are continuous at $a \in U$. Then $f'_l(a) = f'_r(a)$, so that f is derivable at a.*

<u>*Proof*</u> : We need the following well-known lemma :

Lemma 3.10

1) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function which admits a right derivative at every point of $]a, b[$, and $k \in \mathbb{R}$. Then,

a) If, for all $t \in]a, b[$ $f'_r(t) \leq k$, then $f(b) - f(a) \leq k(b - a)$,

b) If, for all $t \in]a, b[$ $f'_r(t) \geq k$, then $f(b) - f(a) \geq k(b - a)$.

2) Same statements with f'_l instead of f'_r.

We come back to the proof of 3.9 : Let $\varepsilon > 0$; let us prove that $f'_l(a) \leq f'_r(a) + \varepsilon$. Since f'_r is continuous at a, there exists $\eta > 0$ such that $f'_r(x) < f'_r(a) + \varepsilon$ for all $x \in]a - \eta, a + \eta[\subset U$. Let $k = f'_r(a) + \varepsilon$. Then, 3.10 provides that, for all $x \in \mathbb{R}$ verifying $a - \eta < x < a$, we have $f(a) - f(x) \leq k(a - x)$, i.e $\frac{f(a)-f(x)}{a-x} \leq k$, which gives $f'_l(a) \leq k$ (doing $x \to a-$) ; hence $f'_l(a) \leq f'_r(a)$, doing $\varepsilon \to 0$. Same for the reverse inequality. $\square$

Corollary 3.11 *Let $f : (E, U) \longrightarrow (\mathbb{R}, U')$ be a G-differentiable map, such that the map $\mathrm{k}_+ f : U \longrightarrow \mathbb{H}\mathrm{om}_+(E, \mathbb{R}) : x \mapsto \mathrm{k}_+ f_x$ is continuous at $a \in U$. Then, for all $v \in E$, the directionnal derivative at a $\frac{\partial f}{\partial v}(a) = \lim_{0 \neq t \to 0} \frac{f(a+tv)-f(a)}{t}$ exists in $\mathbb{R}$.*

Theorem 3.12 *We assume that E and E' are n.v.s. of finite dimensions and that $f : (E, U) \longrightarrow (E', U')$ is a centred map which is continuously G-differentiable. Then, f is of class C^1.*

Proof : Begin first with $E' = \mathbb{R}$. $\square$

4 Fractality and neo-fractality

Here $\Sigma = \mathbb{N}'_r$ (see section 1). The interest of this particular case is to speak of fractality.

As in section 3, we remain in the n.v.s. context. Let us fix a real number $0 < r < 1$. Now, E, E' being two n.v.s., we specify (referring to 1.2 and 1.11 for $\mathbb{N}'_r$) that the $\mathbb{N}'_r$-Lhomogeneous maps $h : E_0 \longrightarrow E'_0$ are the maps $h : E \longrightarrow E'$ which are lipschitzian and which satisfy the following fractality property : $h(rx) = rh(x)$ for all $x \in E$; such maps will be called "r-Lfractal". Thus, we merely write $\mathbb{F}\mathrm{rac}_r(E, E')$ for the n.v.s. which we should denote $\mathbb{N}'_r$-$\mathbb{C}\mathrm{ontr}(E_0, E'_0)$: see section 2.

Examples 4.1

1) The $\mathbb{R}_+$-Lhomogeneous maps $E_0 \longrightarrow E'_0$ are r-Lfractal.

2) Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(0) = 0$ and $f(x) = x \sin \log |x|$ for all $x \neq 0$. Then, f is r-Lfractal for $r = e^{-2\pi}$.

3) More generally, for $p \in \{1, 2, \infty\}$, the map $f^p : \mathbb{R}^n \longrightarrow \mathbb{R}^n : x \mapsto \lambda_p(x)x$, where $\lambda_p : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the function defined by $\lambda_p(0) = 0$ and $\lambda_p(x) = \sin \log \|x\|_p$ for $x \neq 0$. Then f^p is r-Lfractal for $r = e^{-2\pi}$.

4) $\mathbb{K}$ being the triadic Cantor set, let $K_\infty = \cup_{n\in\mathbb{N}} 3^n\mathbb{K}$ and $g : \mathbb{R} \longrightarrow \mathbb{R} : x \mapsto d(x, K_\infty)$. Then g is $\frac{1}{3}$-Lfractal.

<u>*Proof*</u> :

2) is a particular case of 3) ... see Fig 1.

Figure 1

3) The fact that f^p is lipschitzian, comes from the formulas $\mathrm{k}_+ f^p_x(y) = (\sin\log\|x\|_p)y + \frac{\cos\log\|x\|_p}{\|x\|_p}\,\mathrm{k}_+ N^p_x(y)x$ and $|\mathrm{k}_+ N^p_x(y)| \le \|y\|_p$ (referring to the examples of 4.6). Using now 3.11, we obtain that f^p is 2-lipschitzian.

4) g is 1-lipschitzian. Besides, as $\frac{1}{3}K_\infty = K_\infty$, we have $g(\frac{1}{3}x) = d(\frac{1}{3}x, K_\infty) = d(\frac{1}{3}x, \frac{1}{3}K_\infty) = \frac{1}{3}d(x, K_\infty) = \frac{1}{3}g(x)$. See Fig 2 [3]

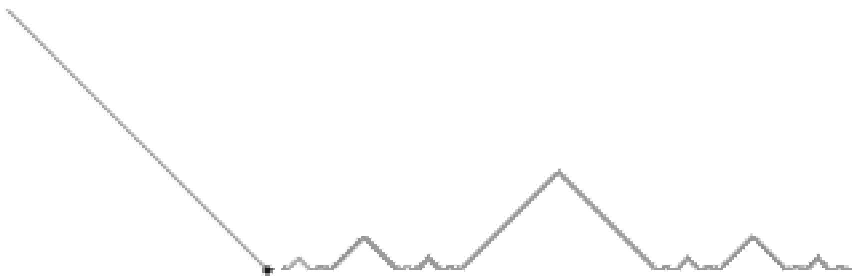

Figure 2

$\square$

Proposition 4.2 *Let $h \in \mathbb{F}\mathrm{rac}_r(E, E')$.*

1) If $E \neq \{0\}$, we have : $\|h\| = \sup\{\frac{\|h(x)\|}{\|x\|} \mid r < \|x\| \le 1\}$.

3. We could call g the "Giseh" function, if, as Napoleon, we gaze at the Giseh pyramides diminishing at the horizon !

2) For every $\varepsilon > 0$, we have $\rho(\mathrm{T}h_0) = \sup\{\frac{\|h(x)-h(y)\|}{\|x-y\|} \,|\, x \neq y,$ $x, y \in C(r,\varepsilon)\}$, with $C(r,\varepsilon) = \{x \in E \,|\, r < \|x\| < 1+\varepsilon\}$.

<u>*Proof*</u> : 2) Let us put $R(\varepsilon) = \sup\{\frac{\|h(x)-h(y)\|}{\|x-y\|} \,|\, x \neq y \; x, y \in C(r,\varepsilon)\}$. Clearly, , $R(\varepsilon) \leq \rho(\mathrm{T}h_0)$ since h is $\rho(\mathrm{T}h_0)$-lipschitzian (by 2.12). Let us now show that $\rho(\mathrm{T}h_0) \leq R(\varepsilon)$. Let $a \in E$ verifying $0 < \|a\| \leq 1$ and $n \in \mathbb{N}$ such that $r^{n+1} < \|a\| \leq r^n$, i.e $r < \|r^{-n}a\| \leq 1$. Let us put $\varepsilon' = r^n \mathrm{Min}\{\varepsilon, r^{-n}\|a\| - r\}$ and let $x \in B(a,\varepsilon')$. Then, $r^{-n}a$, $r^{-n}x \in C(r,\varepsilon)$; so that, if $x \neq a$, we have $R(\varepsilon) \geq \frac{\|h(r^{-n}x)-h(r^{-n}a)\|}{\|r^{-n}x-r^{-n}a\|} = \frac{\|h(x)-h(a)\|}{\|x-a\|}$, which proves that h is $R(\varepsilon)$-LSL_a. This being true for every $a \in B'(0,1)-\{0\}$, we deduce that the restriction $h|_{B'(0,1)}$ is $R(\varepsilon)$-lipschitzian (see section 1 in [5]) which finally implies that $\rho(\mathrm{T}h_0) \leq R(\varepsilon)$. $\square$

Definition 4.3 *A centred map $f(E_a, U) \longrightarrow (E'_{f(a)}, U')$ is said to be* r-neo-fractal *at $a \in U$ if f is $\mathbb{N}'_r$-contactable at a ; in this case, $\mathrm{k}_r f_a$ will merely denote $\mathrm{k}_{\mathbb{N}'_r} f_a$.*

Remarks 4.4 In each case, E and E' are n.v.s. and $f : U \longrightarrow E'$ where U is an open subset of E, and $a \in U$.

1) When f is G-differentiable at a, it is also r-neo-fractal for every $0 < r < 1$, and we have $\mathrm{k}_r f_a = \mathrm{k}_+ f_a$.

2) Every r-fractal map $h : E \longrightarrow E'$ is r-neo-fractal at 0 and we have $\mathrm{k}_r h_0 = h$.

3) Every r-neo-fractal map at a is tangentiable at a.

4) When f is r-neo-fractal at a, then f is G-differentiable at a iff $\mathrm{k}_r f_a \in \mathbb{H}\mathrm{om}_+(E,E')$; in this case, $\mathrm{k}_+ f_a = \mathrm{k}_r f_a$.

Examples 4.5

We consider successively the examples 2),3),4) already studied in 4.1 :

1) The function f is $e^{-2\pi}$-neo-fractal at 0, and differentiable on $\mathbb{R}^*$.

2) For $p \in \{1, 2, \infty\}$, the map f^p is $e^{-2\pi}$-neo-fractal at 0, and G-differentiable at every $x \neq 0$.

3) The Giseh function g is G-differentiable at every $x \notin K_\infty$. Furthermore, if we denote K_∞^+ and K_∞^- the subsets of K_∞ defined, for $x \in K_\infty$, by : $\quad x \in K_\infty^+ \iff \exists \varepsilon > 0 \;\; (]x-\varepsilon, x[\cap K_\infty = \emptyset)$,

$$x \in K_\infty^- \Longleftrightarrow \exists \varepsilon > 0 \ \ (]x, x+\varepsilon[\cap K_\infty = \emptyset),$$

then g is $\frac{1}{3}$-neo-fractal at every point of $K_\infty^+ \cup K_\infty^-$, and we have : for $a \in K_\infty^+$, $\mathrm{k}_{\frac{1}{3}} g_a = g$; and for $a \in K_\infty^-$, $\mathrm{k}_{\frac{1}{3}} g_a = g_-$, where $g_-(x) = g(-x)$.

Proof : 1), 2) Use 4.4.

3) For the Giseh function, we verify that, in the neighborhood of a, we have $g(x) = g(x-a)$ if $a \in K_\infty^+$, and $g(x) = g(a-x)$ if $a \in K_\infty^-$ ... for a detailed proof, see section 5 in [5]. □

Remarks 4.6 In the previous examples, we notice that :

1) a) Thanks to 4.4, we see that f is $e^{-2\pi}$-neo-fractal at 0, but not G-differentiable at 0. Same remark for the f^p where $p \in \{1, 2, \infty\}$.

b) g is not G-differentiable at all $x \in K_\infty^+ \cup K_\infty^-$, although it is $\frac{1}{3}$-neo-fractal at these points.

2) a) Of course, there exist neo-fractal maps which are not Lfractal : we have just, as in 3.7, to translate our previous examples at every point where they are neo-fractal.

b) As for the function f of 4.5 which remains differentiable at 0, although no more Lfractal, we obtain a convincing example considering the function $x^2 + f(x)$ ... So guided, we can find a lot of other good examples of neo-fractal maps which are not Lfractal.

Construction of fractal functions

Let s and T be strictly positive real numbers and $f : \mathbb{R} \longrightarrow \mathbb{R}$ a T-periodic and s-lipschitzian function which admits a right derivative at every point (we have $|f'_r(x)| \leq s$ for all $x \in \mathbb{R}$) ; in particular, f is bounded on $\mathbb{R}$. Then, we associate to f the function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\varphi(0) = 0$ and, for $x \neq 0$, $\varphi(x) = x f(\log |x|)$. Then φ admits a right derivative at every points of $]0, +\infty[$ which is bounded, thus lipschitzian on $\mathbb{R}$. Clearly, φ is r-Lfractal.

Let us consider now the set $\mathcal{P}_T$ of the T-periodic functions $\mathbb{R} \longrightarrow \mathbb{R}$ which are lipschitzian and which admit a right derivative at every point. Then, $\mathcal{P}_T$ has a structure of vectorial subspace of $\mathbb{R}^{\mathbb{R}}$. The previous construction provides a map $j : \mathcal{P}_T \longrightarrow \mathbb{F}\mathrm{rac}_r(\mathbb{R}, \mathbb{R})$, where $j(f)$ is the function φ associated to f as above. This map is clearly linear and injective. By composition, we have an injective linear map :

$$\mathcal{P}_T \xrightarrow{j} \mathbb{F}\mathrm{rac}_r(\mathbb{R},\mathbb{R}) \xrightarrow{J} \mathbb{J}\mathrm{et}((\mathbb{R},0),(\mathbb{R},0)) \xrightarrow{can} \mathbb{J}\mathrm{et}_{free}(\mathbb{R},\mathbb{R})$$

which provides the following statement (for the free jets, refer to [6]) :

Proposition 4.7 *The space* $\mathbb{J}\mathrm{et}_{free}(\mathbb{R},\mathbb{R})$ *is thus a vectorial space of infinite dimension.*

Summary

The tangentiable maps have made in evidence new classes of maps. We give a recapitulative diagram of the various implications mentionned all along this paper :

$$\begin{tikzcd}
C^1 \arrow[r, Rightarrow, "1"] \arrow[dr, Rightarrow, "5"'] & Diff_a \arrow[r, Rightarrow, "2"] & \mathbb{R}-cont_a \arrow[r, Rightarrow, "2'"] & G-diff_a \arrow[r, Rightarrow, "3"] & r-neofr_a \arrow[dl, Rightarrow, "4"'] & \\
& LL_a \arrow[rr, Rightarrow, "6"'] & & Tang_a \arrow[r, Rightarrow, "7"'] & LSL_a \arrow[r, Rightarrow, "8"'] & C^0_a
\end{tikzcd}$$

Where, here, $Diff_a$, $G\text{-}diff_a$ and $r\text{-}neofr_a$ and $\mathbb{R}-cont_a$ stand respectively for differentiable, G-differentiable , r-neo-fractal and standard $\mathbb{R}$-contactable at a for all of them (C^1 means "of class C^1", and C^0_a means continuous at a). In the above diagram, every inverse implication is false ; we give counter-examples below.

Counter-examples 4.8

In each case (except for 2), we denote $f : \mathbb{R} \longrightarrow \mathbb{R}$ the given counter-example (here $a = 0$ and $f(0) = 0$; the "number" i) corresponding to a counter-example to the i^{ieth} above implication).

1) $f(x) = x^2 \sin \frac{1}{x}$ (well-known),

2) see 2.12,

2') $f(x) = |x|$ (see 3.6 and 3.7),

3) $f(x) = x \sin(\log |x|)$ (see example 4.6 and 4.7),

4) $f(x) = x \sin(\log |\log |x||)$ if $x \neq 0$ [4].

5) $f(x) = |x|$ (lipschitzian but not differentiable at 0).

6) $f(x) = x^2 \sin \frac{1}{x^2}$ (see section 3 in [5]),

7) $f(x) = x \sin \frac{1}{x}$ (same as for 6)),

8) $f(x) = x^{\frac{1}{3}}$ (same as for 6)).

Remarks 4.9 We complete the above diagram of implications, adding the following diagram (where $\mathbb{R}$-Lhom, $\mathbb{R}_+$- Lhom and r-Lfrac stand for standard $\mathbb{R}$-Lhomogeneous, $\mathbb{R}_+$-Lhomogeneous and r-Lfractal) :

$$\begin{array}{ccccc} \mathbb{R}-Lhom & \Longrightarrow & \mathbb{R}_+-Lhom & \Longrightarrow & r-Lfrac \\ \Downarrow & & \Downarrow & & \Downarrow \\ \mathbb{R}-cont_0 & \Longrightarrow & G-diff_0 & \Longrightarrow & r-neofr_0 \end{array}$$

The inverse implications are false : refer to 2') and 3) in 4.8 for the horizontal non-implications, and to 3.7 and 4.6 for the vertical ones.

5 Local extrema

In this last section, we present nice generalisations of classical theorems about extrema of functions taking their values in $\mathbb{R}$. In particular, we give a sufficient condition for having an extremum which only needs hypotheses at order 1 ! In 5.1 and 5.2, Σ is a valued monoid.

Theorem 5.1 *Let $f : (M, U) \longrightarrow (\mathbb{R}_b, \mathbb{R})$ be a centred Σ-contactable function which admits a local minimum at $\omega \in U$; then $\mathrm{K}_\Sigma f$ admits a global minimum at ω.*

<u>*Proof*</u> : We recall (see 1.8 and 1.11) and that, $\mathbb{R}_b$ is a revertible Σ-contracting space for the canonical structure, with $t \star y = v(t)(y - b) + b$ (for every $t \in \Sigma$ and $y \in \mathbb{R}$), and $t \overset{-1}{\star} y = t^{-1} \star y = \frac{y-b}{v(t)} + b$ (if $t \neq 0$). If $h = \mathrm{K}_\Sigma f$, we have $h(x) = \lim_{0 \neq v(t) \to 0} t \overset{-1}{\star} f(t \star x) = \lim_{0 \neq v(t) \to 0} \frac{f(t \star x) - b}{v(t)} + b$, for every $x \in M$. Since f admits a local minimum at ω, we have $f(x) \geq f(\omega) = b$ on a neighborhood V of ω in U. Fixing $x \in M$, and since $\lim_{v(t) \to 0} t \star x = \omega$, there exists $\varepsilon > 0$ such that, for all $t \in \Sigma$, $0 < v(t) < \varepsilon \Longrightarrow t \star x \in V$. So, when $0 < v(t) < \varepsilon$, we have $f(t \star x) \geq b$, which implies $t \overset{-1}{\star} f(t \star x) = \frac{f(t \star x) - b}{v(t)} + b \geq b$. Doing $v(t) \to 0$, we obtain $h(x) \geq b = h(\omega)$. $\square$

Corollary 5.2 *Let $f : (E_a, U) \longrightarrow (\mathbb{R}_{f(a)}, \mathbb{R})$ a centred Σ-contactable function which admits a local minimum at $a \in U$; then $\mathrm{k}_\Sigma f_a$ admits a global minimum at 0.*

Proof : It, comes from the fact that, for all $x \in E$, we have $\mathrm{k}_\Sigma f_a(x) = \mathrm{K}_\Sigma f(x+a) - f(a) \geq 0 = \mathrm{k}_\Sigma f_a(0)$, where $\mathrm{K}_\Sigma f : E_a \longrightarrow \mathbb{R}_{f(a)}$. □

Remark 5.3 This gives back the well-known result of the differentiable case : " f admits a local minimum at $a \Longrightarrow a$ is a critical point of f" since there exists a unique continuous linear function $E \longrightarrow \mathbb{R}$ which admits a global minimum at 0 : the null function.

Theorem 5.4 *Let $f : (M, U) \longrightarrow (\mathbb{R}_b, \mathbb{R})$ a $\mathbb{R}_+$-contactable centred function. If M is a Daniel space (i.e a metric space in which every closed and bounded subset is compact) and if $\mathrm{K}_\Sigma f : M \longrightarrow \mathbb{R}_b$ admits a strict global minimum at ω, then f admits a strict local minimum at ω.*

Proof : We can suppose $M \neq \{\omega\}$ and set $S = \{x \in M \mid d(x, \omega) = 1\}$. Then S is a non empty compact (if $x \in M - \{\omega\}$, we have $\frac{1}{d(x,\omega)} \star x \in S$). Since $h = \mathrm{K}_\Sigma f$ is continuous, h reaches its inferior bound at $x_0 \in S$, so that $h(x) \geq h(x_0) > b$ for all $x \in S$. Consider $\varepsilon = h(x_0) - b > 0$. Since $f \rightarrowtail_\omega h|_U$, there exists $\eta > 0$ such that $B(\omega, \eta) \subset U$ and verifying the implication : $0 < d(x, \omega) < \eta \Longrightarrow |f(x) - h(x)| < \varepsilon d(x, \omega)$ for all $x \in M$. Let us fix $x \in B(\omega, \eta) - \{\omega\}$; it verifies $f(x) > h(x) - \varepsilon d(x, \omega)$. If $y = \frac{1}{d(x,\omega)} \star x$, we have $y \in S$, so that $h(y) \geq h(x_0)$ which implies $h(y) - b - \varepsilon \geq h(x_0) - b - \varepsilon = 0$. Hence (since $h : M \longrightarrow \mathbb{R}_b$ is $\mathbb{R}_+$-homogeneous, where $\mathbb{R}_+$ is a quasi-group) $h(x) - \varepsilon d(x, \omega) = h(d(x, \omega) \star y) - \varepsilon d(x, \omega) = d(x, \omega)(h(y) - b) + b - \varepsilon d(x, \omega) = d(x, \omega)(h(y) - b - \varepsilon) + b \geq b$. Thus, for all $x \in B(\omega, \eta) - \{\omega\}$, we have $f(x) > h(x) - \varepsilon d(x, \omega) \geq b = f(\omega)$. □

Corollary 5.5 *Let $f : (E_a, U) \longrightarrow (\mathbb{R}_{f(a)}, \mathbb{R})$ a centred G-differentiable map (where E is a n.v.s. of finite dimension), such that $\mathrm{k}_+ f_a > 0$ (i.e verifying $\mathrm{k}_+ f_a(x) > 0$ for every $x \in E - \{a\}$). Then, f admits a strict local minimum at a.*

Proof : For all $x \neq a$, we have $\mathrm{K}_+ f(x) = f(a) + \mathrm{k}_+ f_a(x - a) > f(a)$.

Remark 5.6 This theorem has not its equivalent, at order 1, in differential calculus, since a linear function cannot have a strict minimum. It is rather inspired by theorems giving sufficient conditions, at order 2, for the existence of extrema.

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A NOTE ON THE PENON DEFINITION OF *n*-CATEGORY

by Eugenia CHENG and Michael MAKKAI

Abstract

Nous étudions les tricatégories de Penon, et nous démontrons que, dans le cas des tricatégories deux fois dégénérées, on obtient les catégories monoïdales symétriques et non les catégories monoïdales tressées. Nous prouvons que les tricatégories de Penon ne peuvent pas donner toutes les tricatégories. Pour corriger cette situation, nous proposons une petite modification de la d'efinition, utilisant les ensembles globulaires non-réflexifs à la place des ensembles globulaires réflexifs, et nous démontrons qu'ainsi le problème précédent relatif aux tricatégories deux fois dégénérées n'apparaît plus.

We show that doubly degenerate Penon tricategories give symmetric rather than braided monoidal categories. We prove that Penon tricategories cannot give all tricategories, but we show how to modify the definition slightly in order to rectify the situation. We give the modified definition, using non-reflexive rather than reflexive globular sets, and show that the problem with doubly degenerate tricategories does not arise.

Keywords: tricategory, degenerate tricategory, braided monoidal category, symmetric monoidal category, globular set, reflexive, non-reflexive.

MSC2000: 18A05, 18D05, 18D10

Introduction

Many different definitions of weak n-category have been proposed but as yet the relationship between them and their validity have not been well understood. One preliminary check that can be applied to any proposed definition is that it "agrees", in some suitable sense, with well-established low-dimensional examples. Thus, one might begin by checking the definition of 1-category against the usual definition of category (and this is not always trivial) and then the definition of 2-category against the classical definition of bicategory [1].

After this point things become more difficult; the definition of tricategory [5] has been accepted but a completely algebraic version of the definition has since been proposed [7], and questions remain about what should be "the" definition of tricategory, if indeed a unique definition should be sought at all. However, there is a degenerate form of tricategory which is much better understood – that is, braided monoidal categories.

Corollary 8.7 of [5] states that "one-object, one-arrow tricategories are precisely braided monoidal categories". However, the results of [3] show that the correspondence is not straightforward, and one-object one-arrow tricategories ("doubly degenerate tricategories") in fact give rise to braided monoidal categories with various extra pieces of structure. However, it is shown that braided monoidal categories should at least *arise among* the totality of tricategories; by focussing on this in the present work we avoid the intricate questions involved in the above Corollary.

The main aim of this paper is to show that Penon's definition of n-category in its original form is not as general as it might be, as it gives symmetric rather than braided monoidal categories in the "doubly degenerate" case. This should not be seen as a serious problem – the situation is quite easily rectified by starting with globular sets instead of reflexive globular sets in Penon's definition. A reflexive globular set is one in which putative identities are already picked out. This can be thought of as being analogous to degeneracies being part of the structure of a simplicial set, but then the analogy breaks down. For simplicial sets the presence of degeneracies is a rich and crucial part of the structure, but in Penon's definition of n-category, it is precisely these degeneracies that cause the resulting definition to be slightly too strict, yielding a symmetry instead of a braiding in the one-object, one-arrow situation.

The problem in the reflexive case is that, since the identities are picked out in the underlying globular set, when forming the free 3-cateogry on such a structure it is possible to have non-identity cells whose source and target are the identity. In the non-reflexive case, the identities are added in *freely* in the free 3-category, so the only cells whose source or target are identities are themselves identities. It is a general principle that having non-trivial cells with identity source and target causes problems, as in the following situations:

- Strict computads do not form a presheaf category but strict many-to-one computads do; the problem is caused by the possibility of cells with source and target the identity [9], so this is avoided by insisting that the target is 1-ary (thus disallowing identities since they are "nullary").

- Coherence for tricategories [6] says that all diagrams of constraints in a *free* tricategory commute, but not all diagrams of constraints in a general tricategory commute; in a general tricategory a diagram of constraints commutes if it involves no non-identity cells with the identity in the source or target. For example, the diagram asserting that a braiding is in fact a symmetry does not necessarily commute; it involves cells whose source and target are the identity.

We present the result in two different ways. The first, more intuitively clear but less precise, says that "A degenerate Penon tricategory is a symmetric monoidal category". This is the subject of Section 2. We simply examine a degenerate Penon tricategory and express it as a braided monoidal category in the expected way; we then see that in fact the braiding is forced to be a symmetry. However, this is not a precise mathematical statement – all it says is that the generally expected method of producing a braided monoidal category from a doubly degenerate tricategory does not gives us *all* braided monoidal categories, only the symmetric ones. However this may be considered to be the heart of the problem, and was pointed out by the second author during the Workshop on n-categories at the IMA in June 2004.

In Section 3 we "go backwards" in order to make a precise statement. First we exhibit a monoidal category which can be equipped with a braiding but *cannot* be equipped with a symmetry. We then express this as a tricategory and show that it does not satisfy the axioms for a Penon tricategory. Thus we conclude that Penon's original notion of tricategory does not include all the examples we would like.

In Section 4 we give the non-reflexive version of Penon's definition, and in Section 5 we show that these problems do not arise in this case.

We begin in Section 1 by reviewing the basic definitions. Note that we will often use the term "n-category" even when n might be ω.

1 Basic definitions

In this section we recall the definition of n-category proposed by Penon [10]. According to this definition, an ω-category is an algebra for a certain monad P on the category of reflexive globular sets. Our statement of the definition is more similar to that of Leinster [8]; for more explanation we also refer the reader to [4]. The definition starts with the underlying data given by a *reflexive globular set*, then imposing the structure of a *magma* (for composition) and *contraction* (for coherence). For finite n a simple truncation is applied to the underlying data, while some care must be taken over the n-cells when defining contractions in this case.

1.1 Reflexive globular sets

We write **RGSet** for the category of reflexive globular sets. **RGSet** is the category of presheaves $[\mathbb{R}^{\text{op}}, \mathbf{Set}]$ where $\mathbb{R}$ is the category whose objects are the natural numbers and whose morphisms are as depicted below:

$$\begin{tikzcd} \cdots & 3 \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] & 2 \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \arrow[l, bend right=60, "i"', leftarrow] & 1 \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \arrow[l, bend right=60, "i"', leftarrow] & 0 \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \arrow[l, bend right=60, "i"', leftarrow] \end{tikzcd}$$

satisfying globularity and reflexivity conditions. However we will write a reflexive globular set explicitly as a diagram of sets

$$\begin{tikzcd} \cdots \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A(3) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A(2) \arrow[l, bend left, "i"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A(1) \arrow[l, bend left, "i"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A(0) \arrow[l, bend left, "i"] \end{tikzcd}$$

satisfying the globularity conditions $ss = st$ and $ts = tt$, and the reflexivity condition $si = ti = 1$. For the finite n-dimensional case we truncate the diagram to get the category $\mathbf{RGSet}_n$ of n-dimensional reflexive globular sets as below

$$\begin{tikzcd} A(n) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A(n-1) \arrow[l, bend left, "i"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \cdots \arrow[l, bend left, "i"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A(1) \arrow[l, bend left, "i"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A(0) \arrow[l, bend left, "i"] \end{tikzcd}$$

We call the elements of $A(k)$ the k-cells of A. The maps s and t give the source and target of each k-cell and the map i picks out the putative identity for each k-cell. Part of the structure of the monad P will be to ensure that these really do act as (weak) identities in the n-category structure.

A map of reflexive globular sets is a map of these diagrams making all the obvious squares commute.

Note Every strict n-category has an underlying n-dimensional reflexive globular set.

1.2 Magmas

A magma is a reflexive globular set equipped with binary composition at all levels. That is, for all $m \geq 1$ we can compose along bounding k-cells for any $0 \leq k \leq m-1$. So given $\alpha, \beta \in A(m)$ with

$$t^{m-k}\alpha = s^{m-k}\beta$$

we have composite $\beta \circ_k \alpha \in A(m)$ with source and target given by

$$s(\beta \circ_k \alpha) = \begin{cases} s(\beta) \circ_k s(\alpha) & \text{if } k < m-1 \\ s(\alpha) & \text{if } k = m-1 \end{cases}$$

$$t(\beta \circ_k \alpha) = \begin{cases} t(\beta) \circ_k t(\alpha) & \text{if } \ k < m-1 \\ t(\beta) & \text{if } \ k = m-1 \end{cases}$$

Note that the composites on the right hand side make sense because of the globularity conditions. For examples and diagrams illustrating these composites see [4].

A map of magmas is a map of the underlying reflexive globular sets preserving composition.

An n-dimensional magma is one whose underlying reflexive globular set is n-dimensional.

Note In a magma only binary composites are given (i.e. in the language of Leinster it is "biased") and no axioms are required to be satisfied. In particular, the putative identities are still not required to act as identities with respect to the composition. Further, note that any strict n-category has an underlying n-dimensional magma.

1.3 Contractions

Composition in a magma is not required to be in any way coherent; we achieve coherence for n-categories by way of a "contraction". A contraction is a piece of structure that can be defined on any map $A \xrightarrow{f} B$ of reflexive globular sets. The idea of a contraction is similar to contractibility of topological spaces, in that it measures holes, or rather lack thereof. A contraction on a map $A \xrightarrow{f} B$ essentially ensures that A has "no more holes up to homotopy" than B; the contraction cells witness the contraction of A onto B.

First we need a notion of parallel k-cells.

Definition 1.1. *A pair of k-cells α, β are called* parallel *if*

- $k = 0$*, or*
- $k \geq 1$ *and* $s\alpha = s\beta, t\alpha = t\beta$.

Definition 1.2. *A* contraction *[,] on a map $A \xrightarrow{f} B$ of reflexive globular sets gives, for any pair of parallel k-cells α and β such that $f\alpha = f\beta$, a $(k+1)$-cell*

$$[\alpha, \beta] : \alpha \longrightarrow \beta$$

such that

1. $f[\alpha, \beta] = i(f(\alpha))$
2. $[\alpha, \alpha] = i\alpha$.

For the n-dimensional case a contraction gives the above for $k < n$; given α and β as above for $k = n$ we must have $\alpha = \beta$.

The cells $[\alpha, \beta]$ are referred to generally as "contraction cells". This definition can also be thought of informally as saying that "any disc in B with a lift of its boundary to A gives a lift of the disc as well".

1.4 The crucial category $\mathcal{Q}$

We construct the monad P from an adjunction

$$\begin{tikzcd} \mathcal{Q} \arrow[d, shift left, "U"] \arrow[d, phantom, "\dashv"] \\ \mathbf{RGSet} \arrow[u, shift left, "F"] \end{tikzcd}$$

The category $\mathcal{Q}$ has objects of the form

$$\begin{tikzcd} A \arrow[d, "{f,\ [\ ,\ ]}"] \\ B \end{tikzcd}$$

where

- A is a magma
- B is a strict n-category
- f is a map of magmas
- $[\ ,\ ]$ is a contraction on f

The idea is that the contraction ensures that A has "no more holes up to homotopy" than the strict n-category B, so although it need not be a strict n-category itself, it cannot be too incoherent.

A morphism of such objects is a square

$$\begin{tikzcd} A \arrow[r, "g_1"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r, "g_2"'] & B' \end{tikzcd}$$

where g_1 is a map of magmas, g_2 is a map of strict n-categories, and the square of underlying magma maps commutes; furthermore the maps must preserve contraction cells, that is, for every contraction cell $[\alpha, \beta]$ in A, we must have

$$g_1[\alpha, \beta] = [g_1\alpha, g_1\beta].$$

1.5 The adjunction

There is a forgetful functor

$$\mathcal{Q} \xrightarrow{U} \mathbf{RGSet}$$

sending an object as above to the underlying reflexive globular set of A. This functor has a left adjoint F; we define P to be the monad induced by this adjunction. For the n-dimensional case we write P_n for the induced monad on $\mathbf{RGSet}_n$

The existence of this adjoint can be deduced from standard Adjoint Functor Theorems; it is proved by Penon in [10, Section 4], and it is also quite straightforward to construct using results of [2]. Given a reflexive globular set A we can express FA as

$$\begin{tikzcd} PA \arrow[d, "{\phi,\ [\ ,\ ]}"] \\ T_R A \end{tikzcd}$$

where T_R is the free strict ω-category monad on reflexive globular sets.

Note This is quite different from a free strict ω-category on a *non-reflexive* globular set; for example $T_R 1 = 1$ as reflexive globular sets (as the unique cell at each dimension must be the identity) which is certainly not true of non-reflexive globular sets. This difference may be thought of as being the heart of the problem considered in this paper.

The idea is to combine two types of structure: contraction and magma. We proceed dimension by dimension – at each level we first add in the required contraction cells freely, and then binary composites freely. ϕ then acts by sending all contraction cells to the identity in $T_R A$, and forgetting the parentheses in all composites.

For the finite n-dimensional case the final stage of the construction consists of *identifying* any n-dimensional composites that lie over the same cell in $T_R A$.

Definition 1.3. *An* ω-category *is defined to be a P-algebra. An* n-category *is defined to be a P_n-algebra.*

2 Doubly degenerate 3-categories as symmetric monoidal categories

In this section we show how a doubly degenerate Penon 3-category gives rise to a braided monoidal category, and that the braiding given in this way is in fact necessarily a symmetry. Since the main aim of this section is to show why the braiding must be a symmetry, we do not go through the details of checking all the axioms for a braided monoidal category.

A doubly degenerate tricategory is one that has only one 0-cell and one 1-cell. The general idea is as follows. We obtain a category from it by regarding the old 2-cells as objects and the old 3-cells as morphisms, that is, we take the unique hom-category on the unique 1-cell. We obtain a monoidal structure by taking the tensor product to be given by vertical composition of 2-cells. Finally we use an "up to isomorphism" Eckmann-Hilton argument to show that this tensor is "commutative up to isomorphism" – that is, it is a braiding.

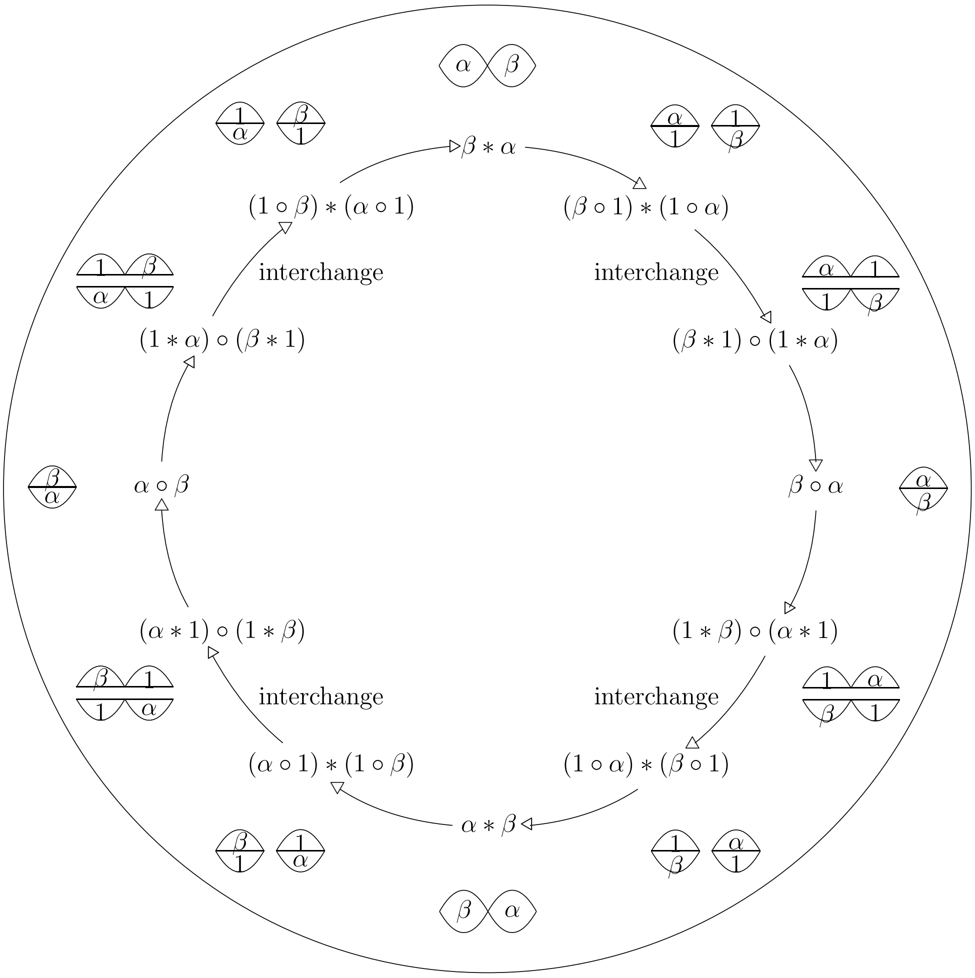
We now state this in the framework of Penon's definition. For convenience we now write $P = P_3$ for the "free Penon 3-category" monad on reflexive 3-globular sets. Let $\begin{pmatrix} PA \\ \downarrow\theta \\ A \end{pmatrix}$ be a P-algebra where A is a doubly degenerate reflexive 3-globular set i.e. it has only one 0-cell and only one 1-cell. We construct a braided monoidal category from it as follows:

- the objects are given by $A(2)$
- the morphisms are given by $A(3)$
- the tensor product is given by $\alpha \otimes \beta = \alpha \circ \beta$ as 2-cells of A
- the braiding $\gamma_{\alpha,\beta} : \alpha \otimes \beta \longrightarrow \beta \otimes \alpha$ is given by the contraction cell $[\alpha \circ \beta, \beta \circ \alpha]$.

To see that this contraction cell exists we need to show that

$$\phi(\alpha \circ \beta) = \phi(\beta \circ \alpha) \in T_R A$$

where ϕ is the map $PA \longrightarrow T_R A$. So we need to show that $\alpha \circ \beta = \beta \circ \alpha \in T_R A$. This is proved by an Eckmann-Hilton type argument using the fact that the source and target 1-cells of α and β are the identity in $T_R A$. We find it helpful to place the various stages of the argument on the following "clock face":



Since all these composites are equal in the strict 3-category $T_R A$, we have in particular a contraction cell in PA

$$[\alpha \circ \beta, \beta \circ \alpha] : \alpha \circ \beta \longrightarrow \beta \circ \alpha$$

It is routine to check the axioms for a braided monoidal category using the contraction conditions at the top dimension; we show further that the symmetry axiom must hold, that is:

$$\gamma_{\beta,\alpha} \circ \gamma_{\alpha,\beta} = 1$$

i.e.

$$[\beta \circ \alpha, \alpha \circ \beta] \circ [\alpha \circ \beta, \beta \circ \alpha] = 1.$$

This is also true by contraction; in fact for any contraction 3-cell $[x, y]$ we have

$$[x, y] \circ [y, x] = 1$$

since

$$\phi([x, y] \circ [y, x]) = \phi[x, y] \circ \phi[y, x] = 1 = \phi(1).$$

Thus we see that a doubly degenerate 3-category is forced to be a symmetric monoidal category, not just a braided monoidal category as originally expected.

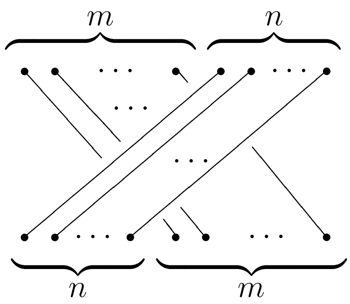
3 Comparison with braided monoidal categories

In this section we give a precise sense in which Penon 3-categories are not the same as classical tricategories. We exhibit a tricategory which does not arise as a Penon 3-category. We will later show that this problem does not arise in the non-reflexive version.

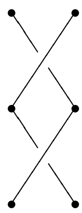
The tricategory we examine is a doubly degenerate one: the free braided (strict) monoidal category on one object. We show that its underlying monoidal category cannot be equipped with a symmetry and thus that it cannot be expressed as a doubly degenerate Penon 3-category.

Let $\mathcal{B}$ denote the free braided (strict) monoidal category on one object. This has

- objects the natural numbers
- homsets $\mathcal{B}(n, m) = \begin{cases} n\text{th braid group} & \text{if } m = n \\ \emptyset & \text{otherwise} \end{cases}$
- tensor product on objects addition, on morphisms juxtaposition of braids
- unit object 0
- braiding $\gamma_{m,n} : m + n \longrightarrow n + m$ depicted by



Now we observe that γ is not a symmetry: for example in the case $m = n = 1$ the composite $\gamma_{n,m} \circ \gamma_{m,n}$ is depicted by



which is not equal to the identity braid on 2.

Proposition 3.1. *The underlying monoidal category of* $\mathcal{B}$ *cannot be equipped with a symmetry.*

Proof. We seek a symmetry

$$\sigma_{AB} : A \otimes B \longrightarrow B \otimes A$$

natural in A and B. Put $A = B = 1$. In particular we need a morphism

$$\sigma_{1,1} : 2 \longrightarrow 2.$$

The only such maps are given by $\gamma_{1,1}^k$ for all $k \in \mathbb{Z}$. We have seen above that $\gamma_{1,1}$ is not a symmetry, and similarly $\gamma_{1,1}^k$ is not a braiding for any $k \geq 0$. For $k = 0$ we have the identity, but if $\sigma_{1,1} = id$ then the braid axioms force $\sigma_{m,n} = id$ for all m, n which does not satisfy naturality.

Since there are no other morphisms $2 \longrightarrow 2 \in \mathcal{B}$ we conclude that there is no symmetry on this monoidal category. $\Box$

We can now realise this braided monoidal category as a doubly degenerate tricategory whose 2-cells are the natural numbers and whose 3-cells $n \longrightarrow m$ are given by $\mathcal{B}(n,m)$ as above. Composition along both 0-cells and 1-cells is given by $\otimes$ and the interchange constraint is derived from the braiding in the obvious way, with the homomorphism axiom following from the Yang-Baxter equation.

Note that this is in fact a Gray-category since everything is strict except interchange. So we do indeed have a tricategory, and it does not satisfy the axioms for a Penon 3-category.

Remark

We might ask if every tricategory is *equivalent* to a Penon 3-category but this cannot be true. The above braided monoidal category cannot be equivalent to a symmetric monoidal category since we know that a braided monoidal category is equivalent to a symmetric monoidal category if and only if it is itself symmetric [5].

4 The non-reflexive case

In this section we give a non-reflexive version of Penon's definition; we will then show that the problems encountered in the previous sections no longer arise.

An ω-category will now be defined as an algebra for a monad N on ordinary (non-reflexive) globular sets. We write a globular set A as a diagram of sets as below.

$$\cdots \underset{t}{\overset{s}{\rightrightarrows}} A(3) \underset{t}{\overset{s}{\rightrightarrows}} A(2) \underset{t}{\overset{s}{\rightrightarrows}} A(1) \underset{t}{\overset{s}{\rightrightarrows}} A(0)$$

Now we have a forgetful functor $\mathcal{Q} \longrightarrow \mathbf{GSet}$ given by composing the old forgetful functor $\mathcal{Q} \longrightarrow \mathbf{RGSet}$ with the forgetful functor $\mathbf{RGSet} \longrightarrow \mathbf{GSet}$. As before, this has a left adjoint

$$\begin{tikzcd} \mathcal{Q} \arrow[d, shift left=1ex] \arrow[d, phantom, "\dashv"] \\ \mathbf{GSet} \arrow[u, shift left=1ex] \end{tikzcd}$$

inducing a monad which we will call N. There is also an n-dimensional version as before, which we will call N_n.

Definition 4.1. *An ω-category is an N-algebra. An n-category is an N_n-algebra.*

Note that we thus have a commuting triangle of adjunctions

$$\begin{tikzcd} & \mathcal{Q} \arrow[dl, shift left=1ex, "G_2"] \arrow[dr, shift left=1ex, "G_3"] & \\ \mathbf{RGSet} \arrow[ur, shift left=1ex, "F_2"] \arrow[ur, phantom, "\dashv"] \arrow[rr, shift left=1ex, "G_1"] \arrow[rr, phantom, "\top"] & & \mathbf{GSet} \arrow[ll, shift left=1ex, "F_1"] \arrow[ul, shift left=1ex, "F_3"] \arrow[ul, phantom, "\top"] \end{tikzcd}$$

(where the bottom is monadic but the other two sides are not). Thus we immediately have a construction of NA – we first add in putative identities freely and then proceed as in the reflexive case.

We may write F_3A as

$$\left(\begin{array}{c} NA \\ \downarrow \\ T_R F_1 A \end{array} \right) = \left(\begin{array}{c} NA \\ \downarrow \\ TA \end{array} \right)$$

where T is the free strict ω-category monad on non-reflexive globular sets, and we observe the crucial difference between the reflexive and non-reflexive versions of the theory — in NA the identities are freely added, so there are no non-identity cells which have identities as their source or target, whereas in PA there may exist such cells.

5 Degenerate 3-categories in the non-reflexive version

In this section we briefly examine degenerate 3-categories in the non-reflexive case and show that the previous problem of braidings being forced to be symmetries does not now arise. We consider a doubly degenerate 3-category $\begin{pmatrix} PA \\ \downarrow\theta \\ A \end{pmatrix}$. As before, we construct a monoidal category from it with

- objects given by $A(2)$
- morphisms given by $A(3)$
- tensor product given by $\alpha \otimes \beta = \alpha \circ \beta$ as 2-cells of A.

However, we cannot copy the previous construction of a braiding as we no longer have $\alpha \circ \beta = \beta \circ \alpha$ in the strict 3-category TA. This is because TA is now the free strict 3-category on the non-reflexive globular set A, so $TA(1) \neq 1$ although $A(1) = 1$.

In the reflexive version, the unique 1-cell of A becomes the identity 1-cell of $T_R A$, so all the composites on the Eckmann-Hilton "clockface" are equal. In the non-reflexive version, the unique 1-cell of A generates the 1-cells of TA but there is a new (formal) 1-cell identity. So the composites on the Eckmann-Hilton clockface do not even have the same source and target, and are certainly not equal.

This shows that the previous problem no longer arises; it remains to see how to construct a braiding at all. We sketch a proposed argument here, but checking the axioms is not straightforward and we defer this to a future work.

Examining the Eckmann-Hilton clockface again we see that, apart from $\alpha \circ \beta$ and $\beta \circ \alpha$, the clock splits in two: the top half is all equal to $\beta * \alpha$ and the bottom half to $\alpha * \beta$. So in NA we do have a contraction cell

$$\chi = [(1 * \alpha) \circ (\beta * 1),\ (\beta * 1) \circ (1 * \alpha)]$$

("10 o'clock to 2 o'clock"), so we seek to extend this to a braiding

$$\alpha \circ \beta \longrightarrow \beta \circ \alpha.$$

In the following argument we write $\circ$ and $*$ for the formal composition in NA, and evaluate these composites in A by means of the algebra map θ. We write the unique 1-cell in A as e and the unit 1-cell in NA as I. Since this is only a sketch, we also ignore associativity issues with the understanding that for a precise construction these would need to be dealt with using further contraction cells.

We proceed in the following steps.

1. We have contraction 2-cells in NA $\mathcal{l}_e = [I * e, e]$, $\mathcal{r}_e = [e * I, e]$ and also $[I * I, I]$. We know that $\theta(I) = \theta(e) = e$, so by algebra associativity we have

$$\theta(\mathcal{l}_e) = \theta([I * I, I]) = \theta(\mathcal{r}_e)$$

in A. (This is the familiar result $\mathcal{l}_I = \mathcal{r}_I$ in any bicategory.) By contraction, we also have pseudo-inverses for these cells, which we will denote $(\ )^*$.

2. We have contraction 3-cells in NA

$$\lambda_\alpha = [\mathcal{l}_e \circ (1_I * \alpha) \circ \mathcal{l}_e^*, \alpha]$$

$$\rho_\beta = [\mathcal{r}_e \circ (\beta * 1_I) \circ \mathcal{r}_e^*, \beta].$$

3. We now form $\rho \circ_1 \lambda$, composing these 3-cells along the 1-cell boundary, and apply θ. Now,

$$s(\theta(\rho \circ_1 \lambda)) = \theta\left(\theta(\mathcal{r}_e) \circ \theta(\beta * 1_I) \circ \underbrace{\theta(\mathcal{r}_e^*) \circ \theta(\mathcal{l}_e)} \circ \theta(1_I * \alpha) \circ \theta(\mathcal{l}_e^*)\right)$$

so we can precompose by contraction cells at the middle factor (indicated), giving a composite 3-cell

$$\xi : \theta\left(\theta(\mathcal{r}_e) \circ \theta(\beta * 1_I) \circ \theta(1_I * \alpha) \circ \theta(\mathcal{l}_e^*)\right) \Longrightarrow \theta(\beta \circ \alpha)$$

4. Recall we have a contraction cell

$$\chi : \theta\left((1_e * \alpha) \circ (\beta * 1_e)\right) \Longrightarrow \theta\left((\beta * 1_e) \circ (1_e * \alpha)\right).$$

   Now, using algebra axioms we can rewrite this as

$$\theta\left(\theta(1_e * \alpha) \circ \theta(\beta * 1_e)\right) \Longrightarrow \theta\left(\theta(\beta * 1_e) \circ \theta(1_e * \alpha)\right)$$

   and thus, to make it composable with ξ it remains to compose it vertically with the identity 3-cells on r_e and l_e^*; we have then bridged the "gap" into 3 o'clock.

5. A similar argument then takes us from 9 o'clock to 10 o'clock.

Evidently the above arguments are not ideal and we hope to find a more efficient method for calculating in this framework. It remains to prove that this is in fact a braiding, but it is clear that the argument previously used to show that the braiding was a symmetry is no longer applicable.

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ABSTRACT PROJECTIVE LINES
by Anders KOCK

Abstract. We describe a notion of projective line (over a fixed field k): a groupoid with a certain structure. A *morphism* of projective lines is then a functor preserving the structure. We prove a structure theorem: such projective lines are isomorphic to the coordinate projective line (= set of 1-dimensional subspaces of k^2).

Résumé. Nous décrivons une notion de droite projective (sur un corps fixe k): un groupoïde avec une certaine structure. Un *morphisme* de droites projectives est alors un foncteur préservant la structure. Nous prouvons un théorème de structure: telle droite projective est isomorphe à la droite projective des coordonnées (= l'ensemble des sous-espaces linéaires de dimension 1 dans k^2).

Keywords: Projective line, groupoid, cross ratio.
MSC 2010: 14A25, 51A05.

Introduction

For V a vector space over a field k, one has the Grassmannian manifold $P(V)$ consisting of 1-dimensional linear subspaces of V. If V is $n+1$-dimensional, $P(V)$ is a copy of n-dimensional projective space. For $n \geq 2$, $P(V)$ has a rich combinatorial structure, in terms of incidence relations (essentially: the lattice of linear subspaces), in fact, this structure is so rich that one can essentially reconstruct V from the combinatorial structure.

But for $n = 1$, this combinatorial structure (in the form of a lattice), is trivial; as expressed by R. Baer, "*A line ... has no geometrical structure, if considered as an isolated or absolute phenomenon, since then it is nothing but a set of points with the number of points on the line as the only invariant...*", [1] p. 71.

However, it is our contention that a projective line has another kind of structure, making it possible to talk about a projective line as a set equipped with a certain structure, in such a way that isomorphisms (projectivities) between projective lines are bijective maps which preserve this structure.

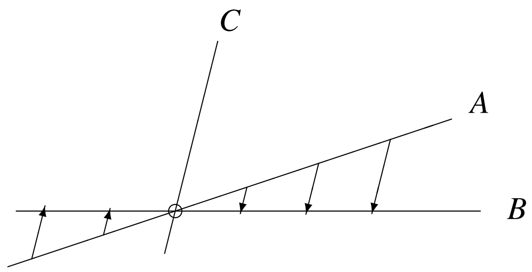
The structure we describe (Section 2) is that of a groupoid (i.e. a category where all arrows are invertible), and with certain properties. The fact that the coordinate projective line $P(k^2)$, more generally, a projective space of the form $P(V)$ (and also the projective plane in the classical synthetic sense) has such groupoid structure, was observed in [3], and further elaborated on in [2]; we shall recall the relevant notions and constructions from [3] in Section 1, and a crucial observation from [2] in Section 3. The present note may be seen as a completion of some of the efforts of these two papers.

1 Groupoid structure on $P(V)$

Let k be a field and let V a 2-dimensional vector space over k. We have a groupoid $\mathbf{L}(V)$, whose set of objects is the set $P(V)$ of 1-dimensional linear subspaces of V, and whose arrows are the linear isomorphisms between these. For $A \in P(V)$, the linear isomorphisms $A \to A$ are in canonical bijective correspondence with the invertible scalars,

$$\mathbf{L}(V)(A,A) = k^*;$$

on the other hand, if A and B are *distinct* 1-dimensional subspaces, then the linear isomorphisms $A \to B$ are all of the form "projection from A to B in a certain unique *direction* C", with $C \in P(V)$ and C distinct from A and B. (This also works in higher dimensions, cf. [3] and [2]; one just has to require that C belongs to the 2-dimensional subspace spanned by A and B.) This is in fact a bijective correspondence, so $\mathbf{L}(V)(A,B)$ is canonically identified with the set $P(V) \setminus \{A,B\}$. Here is a picture (essentially) from [3]:



The linear isomorphism $A \to B$ thus described, we shall denote $(C : A \to B)$. It is clear that the composite of $(C : A \to B)$ with $(C : B \to A)$ gives the identity map of A (projecting forth and back in the same direction). Also it is clear that $(C : A \to B)$ composes with $(C : B \to D)$ to give $(C : A \to D)$. These equations will appear in the axiomatics for abstract projecive lines as the "idempotency laws", (2) and (3) below.

Also, it is clear that two linear isomorphisms from A to B differ by a scalar $\in k^*$; thus, for A and B distinct, and $(C : A \to B)$ and $(D : A \to B)$, there is a unique scalar $\mu \in k^*$ such that

$$(C : A \to B) = \mu \cdot (D : A \to B) = (D : A \to B) \cdot \mu. \tag{1}$$

(We compose from left to right.) This scalar μ is (for A,B,C,D mutually distinct) the classical cross-ratio $(A,B;C,D)$, cf. [3] (3) and [2] Theorem 1.5.3. (For A,B,C distinct, and $D = C$, we have $(A,B;C,C) = 1$.) Permuting the four entries (assumed distinct) will change the cross ratio according to well known formulae (see e.g. [6], [5]) which we shall make explicit and take as axioms.

Thus, the groupoid $\mathbf{L}(V)$, which we in this way have associated to a 2-dimensional vector space V over k, will be an example of an abstract projective line $\mathbf{L}$, in the sense of the next Section.

2 Abstract projective lines: axiomatics

Let k be a field. By a *k-groupoid*, we understand a groupoid $\mathbf{L}$ which is transitive (i.e. the hom set $\mathbf{L}(A,B)$ is non-empty, for any pair of objects A,B in $\mathbf{L}$), and such that all vertex groups $\mathbf{L}(A,A)$ are identified with the (commutative, multiplicative) group k^* of non-zero elements of the field k. We assume that k^* is *central* in $\mathbf{L}$ in the sense that for all $f : A \to B$ and $\lambda \in k^* = \mathbf{L}(A,A) = \mathbf{L}(B,B)$, $\lambda \cdot f = f \cdot \lambda$.

A k-functor between k-groupoids is a functor which preserves k^* in the evident sense.

We now define the notion of *abstract projective line* over k; it is to be a k-groupoid $\mathbf{L}$, equipped with the following kind of structure (L denotes the set of objects of $\mathbf{L}$):
for any two different objects $A,B \in L$, there is given a bijection between the set $\mathbf{L}(A,B)$ and the set $L \setminus \{A,B\}$,

and these bijections should satisfy some equational axioms: the *idempotence* laws (2) and (3), and the *permutation* laws (4),...,(7). To state these laws, we use, as in Section 1, the notation:

if $C \in L \setminus \{A, B\}$, then the arrow $A \to B$ corresponding to it (under the assumed bijection) is denoted by $(C : A \to B)$, or just by C, if A and B are clear from the context (say, from a diagram).

Here is the first set of equations that we assume (the “idempotence equations”): Let A, B, F be mutually distinct, then

$$(F : A \to B) \cdot (F : B \to A) = 1 \in k^* \tag{2}$$

and for A, B, C, F mutually distinct

$$(F : A \to B) \cdot (F : B \to C) = (F : A \to C). \tag{3}$$

The permutation laws which we state next are concerned with the crucial notion of *cross ratio*: If A, B, C, D are four distinct elements of L, we let $(A, B; C, D)$ be the unique scalar (element of k^*) such that

$$\begin{tikzcd} A \arrow[r, "C"] \arrow[dr, "{(A,B;C,D)}"'] & B \arrow[d, "D"] \\ & A \end{tikzcd}$$

commutes; also, $(A, B; C, D)$ makes sense if $C = D$, and in this case equals $1 \in k^*$, by (2). This scalar is called the *cross ratio* of the 4-tuple A, B, C, D. [1]

Since the elements of L both appear as objects of $\mathbf{L}$ and as labels of arrows of $\mathbf{L}$, the four entries (assumed distinct) in a cross ratio expression can be permuted freely by the 24 possible permutations of four letters. We assume the standard formulas for these permutation instances of a given cross ratio $\mu = (A, B; C, D)$; they give six values,

$$\mu, \mu^{-1}, 1 - \mu, (1 - \mu)^{-1}, 1 - \mu^{-1}, (1 - \mu^{-1})^{-1},$$

[1] Convenience, as well as continuity, prompts us to define $(A, B; C, B) = 0$; this is consistent with determinant formulas for cross ratios in $P(k^2)$ to be given later. In fact, one may consistently define $(A, B; C, D)$ whenever $A \neq D$ and $B \neq C$; $(A, A; C, D) = (A, B; C, C) = 1$, and $(A, B; A, D) = (A, B; C, B) = 0$.

see e.g. [6] p. 8 or [5] 0.2. The equations are

$$(A,B;C,D) = (B,A;D,C) = (C,D;A,B) = (D,C;B,A), \tag{4}$$

and the following equations, where μ denotes $(A,B;C,D)$,

$$(A,B;C,D) = \mu; \qquad (A,B;D,C) = \mu^{-1}; \tag{5}$$

$$(A,C;B,D) = 1-\mu; \qquad (A,C;D,B) = (1-\mu)^{-1}; \tag{6}$$

$$(A,D;B,C) = 1-\mu^{-1}; \qquad (A,D;C,B) = (1-\mu^{-1})^{-1}. \tag{7}$$

(This set of equations is not independent.) We had not needed to be so specific about these "permutation equations", since we shall only need the following consequence: if a map $\Phi : L \to L'$ preserves a cross ratio $(A,B;C,D)$ for some distinct A,B,C,D, then it also preserves any other cross ratio in which the entries are A,B,C,D in some other order.

We have now stated what we mean by an abstract projective line $\mathbf{L}$. For (iso-)*morphisms* ("projectivities") between such: Let $\mathbf{L}$ and $\mathbf{L}'$ be abstract projective lines with object sets (underlying sets) L and L', respectively. By an *isomorphism* $\mathbf{L} \to \mathbf{L}'$ of projective lines, we understand a bijective map $\phi : L \to L'$ with the property that if we put

$$\overline{\phi}(F : A \to B) := (\phi(F) : \phi(A) \to \phi(B)), \tag{8}$$

(and $\overline{\phi}(\lambda) = \lambda$ for any scalar $\lambda \in k^*$), then $\overline{\phi}$ commutes with composition, i.e. it defines a *functor* $\mathbf{L} \to \mathbf{L}'$ (preserving scalars, i.e. it defines a k-functor). The noticeable aspect of the category $\mathscr{L}$ of abstract projective lines, with (iso)morphisms as just defined, is that the "underlying" functor $\mathbf{L} \mapsto L$ (from $\mathscr{L}$ to the category of sets) is a *faithful* functor, so that it makes sense to say whether a given function $L \to L'$ is a morphism (projectivity) or not.

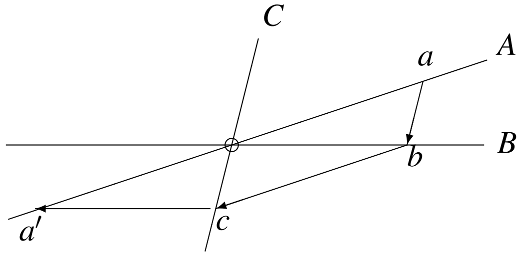
As always in such situations, it is convenient to use the same notation for the object itself, and its underlying set; so we henceforth do not have to distinguish notationally between $\mathbf{L}$ and L.

Cross ratio was defined as a special case of composition; projectivities, in the sense defined here, commute with composition, since projectivities are functors. Hence it is clear that projectivities preserve cross ratios.

In an (abstract) projective line $\mathbf{L}$, one may draw some diagrams that are meaningless in more general categories, like the following square (whose commutativity actually can be *proved* on basis of the axiomatics):

$$\begin{tikzcd} A \arrow[r, "C"] \arrow[d, "-1"'] & B \arrow[d, "A"] \\ A & C \arrow[l, "B"] \end{tikzcd} \tag{9}$$

(where A, B, C are three distinct points in $\mathbf{L}$). The commutativity of this diagram, for $\mathbf{L} = P(V)$, expresses an evident geometric fact that one sees by contemplating the figure (essentially from [3], p. 3):



The existence of this diagram (9) shows that "cross ratios do not immediately encode all the geometry" of projective lines; for, no cross ratio (except 1) can be concocted out of just three distinct points; four are needed.

3 Three-transitivity

The "Fundamental Theorem" for projective lines $P(V)$ coming from 2-dimensional vector spaces V is: for any two lists of three distinct points, there is a unique projectivity taking the points of the first list to the points of the second. This theorem, we shall prove holds for abstract projective lines.

Let $\mathbf{L}$ and $\mathbf{L}'$ be abstract projective lines over the field k.

Theorem 1 (Fundamental Theorem) *Given three distinct points A, B, C in $\mathbf{L}$, and given similarly A', B', C' three distinct distinct points in $\mathbf{L}'$. Then there is a unique projectivity $\phi : \mathbf{L} \to \mathbf{L}'$ taking A to A', B to B' and C to C'.*

Proof. For D distinct from A,B,C, we put $\phi(D) := D'$, where D' is the unique element in $\mathbf{L}'$ with $(A',B';C',D') = (A,B;C,D)$; equivalently D' is determined by the equation

$$(C' : A' \to B') \cdot (D' : B' \to A') = (A,B;C,D).$$

By construction and the permutation equations, ϕ preserves cross ratios of any distinct 4-tuple, three of whose entries are A,B,C. Next, by the idempotence equations (2) and (3),

$$(A,B;D,E) = (A,B;D,C) \cdot (A,B;C,E),$$

and similarly for the $A',\ldots,E'$. Each of the cross ratios on the right have three entries from the original set A,B,C, and so are preserved, hence so is the cross ratio on the left hand side, $(A,B;D,E)$. So we conclude that any cross ratio, two of whose entries are A and B, is preserved. Next,

$$(A,D;E,F) = (A,D;E,B) \cdot (A,D;B,F),$$

and similarly for the $A',\ldots,F'$, so we conclude that any cross ratio with A as one of its entries is preserved. Finally,

$$(D,E;F,G) = (D,E;F,A) \cdot (D,E;A,G),$$

and similarly for the $A',\ldots,G'$, so we conclude that all cross ratios are preserved.

We have now described the bijection $\phi : \mathbf{L} \to \mathbf{L}'$, and proved that it preserves cross ratio of any four distinct points. To prove that it is a projectivity, in the sense defined, we need to argue that the corresponding $\overline{\phi}$ (as described by (8)) preserves composition of arrows. This is essentially an argument from [2] 2-4-4, which we make explicit:

Proposition 2 *If a bijection $\phi : \mathbf{L} \to \mathbf{L}'$ preserves cross ratio formation, then $\overline{\phi}$ preserves composition.*

It suffices to prove that commutative triangles go to commutative triangles. If the three vertices of the triangle agree, these arrows are scalars $\in k^*$, and $\overline{\phi}$ preserves scalars. If two, but not all three, vertices agree, one arrow is a scalar, and commutativity of the triangle expresses that this scalar is the cross ratio (or its inverse) of the four points that appear as the two vertices and those two labels (likewise points in $\mathbf{L}$) that appear on the non-scalar arrows in the triangle; this is a an immediate consequence of the definition (1), possibly combined with the idempotence law (2). We conclude that composites of this form are likewise preserved by $\overline{\phi}$. Finally, we consider the case where the three vertices of the triangle are distinct, so the three arrows in the triangle are of the form $(E : A \to B)$, $(F : B \to C)$, and $(G : A \to C)$ with A, B, C distinct. Consider $(E : A \to B) \cdot (F : B \to C) \cdot (G : C \to A)$, displayed as the top composite in the diagram

$$\begin{tikzcd}
A \arrow[r, "E"] \arrow[d, "1"'] & B \arrow[r, "F"] \arrow[d, "1"] & C \arrow[r, "G"] \arrow[d, "1"] & A \arrow[d, "{(A,C;G,F)}"] \\
A \arrow[r, "E"] \arrow[d, "1"'] & B \arrow[r, "F"] \arrow[d, "1"'] & C \arrow[r, "F"] & A \arrow[d, "{(A,B;F,E)}"] \\
A \arrow[r, "E"'] & B \arrow[rr, "E"'] & & A.
\end{tikzcd}$$

All squares commute; the lower right hand rectangle commutes because of the idempotence law (3) (the two Fs combine into one). The lower composite is 1, because of an idempotence law (2). So we conclude:

$$(A,C;G,F) \cdot (A,B;F,E) = 1 \text{ iff } (E : A \to B) \cdot (F : B \to C). \cdot (G : C \to A) = 1.$$

Multiplying on the right by $G : A \to C$ (which is inverse to $G : C \to A$), we conclude

$$(A,C;G,F) \cdot (A,B;F,E) = 1 \text{ iff } (E : A \to B) \cdot (F : B \to C) = (G : A \to C).$$

Thus commutativity of diagrams can be expressed in terms of cross ratio. Hence since cross ratio are preserved, the composite of $(E : A \to B)$ and $(F : B \to C)$ is preserved by $\overline{\phi}$. This proves the Proposition, and therefore also the existence assertion of the Theorem. The uniqueness is clear, since a projectivity preserves cross ratios, so that we are forced to define $\phi(D)$ as the $D' \in \mathbf{L}'$ with $(A',B';C',D') = (A,B;C,D)$.

4 $\mathbf{L} = P(k^2)$ as an abstract projective line

The content of the present Section is mostly classical, but it emphasizes the category aspects of $P(k^2)$. Non-zero vectors in k^2 are denoted $a = (a_1,a_2)$, $b = (b_1,b_2)$ etc., and the 1-dimensional linear subspace of k^2 spanned by a is denoted A; similarly, b spans B, etc; A, B, $\ldots$, are the points of the set $\mathbf{L} = P(k^2)$. We now have available the precious tool of *determinants* of 2×2 matrices. We denote the determinant whose rows (or columns) are a,b by the symbol $|a,b|$.

Given distinct A, B, and C, spanned by a, b, and c, respectively. We describe the linear map "projection from A to B in the direction of C" by describing its value on $a \in A$; this value, being in B, is of the form $\lambda \cdot b$ for some unique scalar $\lambda \in k^*$, and an elementary calculation with linear equation systems (say, using Cramer's rule) gives that $\lambda = |c,a|/|c,b]$. Thus

$$(C : A \to B)(a) = \frac{|c,a|}{|c,b|} \cdot b \tag{10}$$

is the basic formula. We can calculate the value of the composite $(C : A \to B) \cdot (D : B \to E)$ on $a \in A$; it takes $a \in A$ into

$$\frac{|c,a|}{|c,b|} \cdot \frac{|d,b|}{|d,e|} \cdot e \in E. \tag{11}$$

In particular, if $E = A$, $a \in A$ goes into $(A,B;C,D)\cdot a$, where

$$(A,B;C,D) := \frac{|c,a|}{|c,b|} \cdot \frac{|d,b|}{|d,a|}$$

$$= \frac{|a,c|\cdot|b,d|}{|a,d|\cdot|b,c|}$$

(using $|c,a| = -|a,c|$, and similarly for the other factors). This is the standard cross ratio $(A,B;C,D)$, and the standard permutation rules follow by known determinant calculations, as do the idempotency laws. So $P(k^2)$ is indeed an abstract projective line, in our sense.

In $\mathbf{L} = P(k^2)$, we describe the points $A \in \mathbf{L}$ by homogeneous coordinates $[a_1 : a_2]$, where a is any vector spanning A. It is convenient to select three particular points in $P(k^2)$, called V, H, and D (for "vertical", "horizontal", and "diagonal", respectively):

$$V = [0:1], \quad H = [1:0], \quad D = [1:1].$$

For any point X distinct from V, there exists a unique $x \in k$ so that $X = [1:x]$. Thus, the $x \in k$ corresponding to H and D are 0 and 1, respectively. For X distinct from V, the corresponding $x \in k$ may be calculated in terms of a cross ratio,

$$x = (V, H; D, X),$$

again by an easy calculation with determinants. Thus $\mathbf{L} \backslash \{V\}$ has, by the chosen conventions, been put in 1-1 correspondence with the affine line k, so

$$\mathbf{L} = \{V\} + k;$$

V is the "point at infinity" of the ("vertical") copy $\{(1,x) \mid x \in k\}$ of the affine line k inside k^2.

The Fundamental Theorem then has the following

Corollary 3 *For every abstract projective line* $\mathbf{L}$ *over k, there exists an isomorphism (= "projective equivalence") with the projective line* $P(k^2)$.

Proof. Pick three distinct points A, B, C in $\mathbf{L}$, and let $\phi = \phi_{A,B,C}$ be the unique projectivity (as asserted by the Theorem) $\mathbf{L} \to P(k^2)$ sending A to $[1:0]$, B to $[0:1]$, and $[C]$ to $[1:1]$.

The isomorphism ϕ described is not canonical, since it depends on choice of three distinct points A, B, C. However, it gives rise to certain canonical *bundles*; this will be exploited in Section 6.

The isomorphism/projectivity ϕ described in this Corollary, although not canonical, allows us to perform calculations in $\mathbf{L}$ using coordinates, in the form of such projective equivalence $\mathbf{L} \cong P(k^2)$.

Let us for instance prove commutativity of (9). It suffices to prove that it holds in $\mathbf{L} = P(k^2)$. For, then it follows from the Fundamental Theorem that it also holds for three distinct points in an abstract projective line $\mathbf{L}$.

So consider points A, B, C in $P(k^2)$, and pick non-zero vectors $a \in A$, $b \in B$, and $c \in C$. Using (11), we see that the composite $(C : A \to B) \cdot (A : B \to C)$ takes $a \in A$ into

$$\frac{|c,a|}{|c,b|} \cdot \frac{|a,b|}{|a,c|} \cdot c,$$

and since $|c,a| = -|a,c|$, two factors cancel except for the sign, and we are left with

$$-\frac{|a,b|}{|c,b|} \cdot c = -\frac{|b,a|}{|b,c|} \cdot c;$$

an easy calculation shows that this by $B : C \to A$ goes to $-a$. (See [2] 1-4-2 for a more coordinate free proof.)

To complete the comparison with the classical "coordinate-" projective line $P(k^2)$, we need to compare projectivities in our sense (functors) with classical projectivities, meaning maps $P(k^2) \to P(k^2)$ that are "tracked" by linear automorphisms $k^2 \to k^2$.

Let $f : k^2 \to k^2$ be such linear automorphism. Then it defines a map $P(f) : P(k^2) \to P(k^2)$ by $[a_1 : a_2] \mapsto [f(a_1) : f(a_2)]$. We shall see that this map preserves composition of arrows, hence is a functor; for, by (10), $f(C : A \to B))$ takes $f(a) \in P(f)(A)$ to

$$\frac{|f(c), f(a)|}{|f(c), f(b)|} \cdot f(b) = \frac{|c,a|}{|c,b|} \cdot f(b) \in P(f)(B)$$

(using the product rule for determinants and then cancelling the occurrences of the determinant of f that appear). The fact that composition is preserved is then a consequence of the formula (11).

On the other hand, every projectivity $\phi : P(k^2) \to P(k^2)$ (in our sense) is of the form $P(f)$ for some linear automorphism $f : k^2 \to k^2$ (which is in fact unique modulo k^*). Let $\phi(H) = A$, $\phi(V) = B$ and $\phi(D) = C$. Pick non-zero vectors $a \in A$, $b \in B$ and $c \in C$. Any linear automorphism $f : k^2 \to k^2$ with matrix

$$f = \begin{bmatrix} a_1 & \lambda b_1 \\ a_2 & \lambda b_2 \end{bmatrix},$$

with $\lambda \neq 0$ has the property that it takes $(1,0)$ to a and $(0,1)$ to λb, hence $P(f)$ takes H to A and V to B. Also f takes $(1,1)$ to $a+\lambda b$; so to ensure $P(f)(D) = C$, we must ensure $a+\lambda b \in C$, i.e. we must ensure linear dependence of the pair consisting of $a+\lambda b$ and c. This means that we should pick λ so that the determinant $|a+\lambda b, c|$ is 0; there is a unique λ solving this, namely $-|a,c|/|b,c|$. With this λ, the maps ϕ and $P(f)$ agree on H, V, and D, and since they both are projectivities, they agree everywhere, by the Fundamental Theorem. This proves that every projectivity $\phi : P(k^2) \to P(k^2)$ (functor) is indeed tracked by a linear automorphism $k^2 \to k^2$.

Remark. The projectivity $\phi : P(k^2) \to P(k^2)$ tracked by a linear automorphism $f : k^2 \to k^2$ with matrix $[\alpha_{ij}]$ is also classically described as the *fractional linear transformation*

$$x \mapsto \frac{\alpha_{21} + \alpha_{22}x}{\alpha_{11} + \alpha_{12}x}.$$

This refers to the identification of $x \in k$ with $[1 : x] \in P(k^2)$.

5 Structures on punctured projective lines

We identify k with the subset $P(k^2)\backslash\{V\} \subseteq P(k^2)$ via $x \mapsto [1 : x]$ (recall that V denotes $[0 : 1]$). The group $PGL(2,k)$ of auto-projectivities of $P(k^2)$ contains a subgroup of those auto-projectivities which are tracked by matrices of the form $\begin{bmatrix} 1 & 0 \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$ (with $\alpha_{22} \neq 0$); they are those $\phi \in PGL(2,k)$ which satisfy $\phi(V) = V$, or equivalently which map $k \subseteq P(k^2)$ to itself. Such ϕ map k to itself by an *affine* bijection, $x \mapsto \alpha_{21} + \alpha_{22}x$. Thus, the subgroup of auto-projectivities of $P(k^2)$ which fix V, is identified with the group $\mathrm{Aff}(k)$ of affine bijections $k \to k$.

We shall prove

Proposition 4 *Given an abstract projective line* $\mathbf{L}$*. 1) For any point* A *in* $\mathbf{L}$*, the set* $\mathbf{L}\backslash\{A\}$ *carries a canonical structure of affine line. 2) For any two distinct points* A, B *in* $\mathbf{L}$*, the set* $\mathbf{L}\backslash\{A\}$ *carries a canonical structure of vector line, with* B *as* 0*. 3) For any three distinct points* A, B, C *in* $\mathbf{L}$*, the set* $\mathbf{L}\backslash\{A\}$ *carries a canonical structure of vector line with a chosen basis, with* B *as* 0 *and* C *as the chosen basis vector.*

Proof. For 1): Given $A \in \mathbf{L}$. Pick distinct B and C in $\mathbf{L}\setminus\{A\}$. Consider the unique projectivity $\phi_{A,B,C} : \mathbf{L} \to P(k^2)$ with $A \mapsto V$, $B \mapsto H$ and $C \mapsto D$, as in the Fundamental Theorem. Then since $\phi_{A,B,C}$ maps A to V, it maps $\mathbf{L}\setminus\{A\}$ bijectively to $P(k^2)\setminus\{V\} = k$. We import the affine structure which k has back to $\mathbf{L}\setminus\{A\}$ via this bijection. This affine structure on $\mathbf{L}\setminus\{A\}$ does not depend on the choice of B and C; for, $\phi_{A,B,C}$ and $\phi_{A,B',C'}$ (same A!) will differ by a projectivity $\phi_{A,B,C} \circ \phi^{-1}_{A,B',C'} : P(k^2) \to P(k^2)$ which fixes V, and such projectivity preserves the affine structure on $k \subseteq P(k^2)$, as we observed. This proves 1). For 2) and 3): It is a well known fact that an affine line with a chosen point B carries a canonical structure of vector line with B as 0. Also, a vector line with a chosen point $C \neq 0$ carries a canonical structure of vector line with a chosen basis vector C, i.e. it is *canonically* isomorphic to the vector space k. Then it is clear that 2) and 3) are consequences of 1).

Remark. Instead of deriving 2) from 1) using the "well known fact", we might prove 2) directly along the same lines as used for proving 1), namely by observing that a projectivity $P(k^2) \to P(k^2)$, which fixes V as well as H, is tracked by a diagonal matrix, and therefore restricts along $k \subseteq P(k^2)$ to a linear $k \to k$. Also, 3) may be seen as an immediate consequence of the Fundamental Theorem.

The affine, resp. vector line, structures described in this Proposition depend on the choice of the point A, resp. on the choice of the points A, B. We can record the dependence using some notions from fibre bundle theory. This is the content of the following Section.

6 The canonical fibre bundles

We consider a pull-back diagram of sets, having the following form

$$\begin{tikzcd} U \times F \arrow[r, "h"] \arrow[d, "proj"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "\pi"] \\ U \arrow[r, "\gamma"'] & B \end{tikzcd}$$

with γ a surjection. Thus for each $u \in U$, h provides a bijection $h(u,-)$ from F to the fibre $E_{\gamma(u)}$. One may say that such pull-back diagram provides $E \to B$ with structure of *fibre* bundle with *fibre* F (or *modelled on* F). The situation gives rise to a "cocycle" $\overline{h} : U \times_B U \to Aut(F)$, where $Aut(F)$ is the group of all bijections $F \to F$, namely

$$\overline{h}(u_1,u_2) := h(u_1,-)^{-1} \circ h(u_2,-)$$

(composing from right to left).

If F carries some structure T, say structure of vector space, affine space, or projective line, the structure defines a subgroup $Aut_T(F)$ of $Aut(F)$, namely the subgroup of bijections $F \to F$ which preseve the structure in question. For vector space structure T, the customary notation for $Aut_T(F)$ is $GL(F)$, and similarly $\mathrm{Aff}(F)$ for affine space structure.

If now F carries T-structure, and if the cocycle $\overline{h}$ factors through $Aut_T(F)$, the fibres E_x can canonically be provided with T-structure as well. Namely, pick a $u \in U$ with $\gamma(u) = x$, and transport the T-structure from F to E_x via the bijection $h(u,-) : F \to E_x$. The structure thus defined does not depend on the choice of u; any two choices u_1 and u_2 will give the same T-structure on E_x since $h(u_1,-)^{-1} \circ h(u_2,-)$ was assumed to be a T-automorphism.

Thus if F is a vector space, and if the cocycle $\overline{h}$ takes values in $GL(F)$, $E \to X$ acquires structure of a vector bundle; similarly for affine-space bundles or projective-line bundles.

We consider now a fixed (abstract) projective line $\mathbf{L}$. The isomorphisms $\mathbf{L} \cong P(k^2)$, as given by the Fundamental Theorem, are not canonical, but depend on the choice of three distinct points in $\mathbf{L}$. Out of these isomorphisms grow, however, certain canonical fibre bundles:

Let $\mathbf{L}^{(2)}$, resp. $\mathbf{L}^{(3)}$, denote the set of pairs, resp. triples, of distinct points in $\mathbf{L}$. We also have the set $\mathbf{L}^{(2)} \times_{\mathbf{L}} \mathbf{L}^{(2)}$ of quadruples $((A,B),(A,B'))$ with A distinct from B and from B'. These sets appear in pull-back diagrams

$$\begin{tikzcd} \mathbf{L}^{(3)} \times k \arrow[r, "h_a"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbf{L}^{(2)} \arrow[d] \\ \mathbf{L}^{(3)} \arrow[r] & \mathbf{L} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{L}^{(3)} \times k \arrow[r, "h_l"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbf{L}^{(2)} \times_{\mathbf{L}} \mathbf{L}^{(2)} \arrow[d] \\ \mathbf{L}^{(3)} \arrow[r] & \mathbf{L}^{(2)} \end{tikzcd} \tag{12}$$

where the displayed vertical maps are projection onto the first factor, or factors, and likewise for the horizontal maps in the bottom row. The maps in the top row are essentially given by the ϕs of the Fundamental Theorem; thus $h_a(A,B,C,\mu)$ is (A,X), where $X \in \mathbf{L}\backslash\{A\}$ is the unique point with $\phi_{A,B,C}(X) = \mu$; and $\phi_l(A,B,C,\mu)$ is $((A,B),(A,X))$ where X, as before, is the unique point with $\phi_{A,B,C}(X) = \mu$.

To the left of these pull-back diagrams, we can, if we want, adjoin yet another one, namely the following pull-back over $\mathbf{L}^{(0)} = 1$ (so a cartesian product diagram), i.e.

$$\begin{tikzcd} \mathbf{L}^{(3)} \times P(k^2) \arrow[r, "h_p"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{L} \arrow[d] \\ \mathbf{L}^{(3)} \arrow[r] & 1 \end{tikzcd} \tag{13}$$

with $h_p(A,B,C,\mu)$ the unique X such that $\phi_{A,B,C}(X) = \mu$. Here $\phi_{A,B,C} : \mathbf{L} \to P(k^2)$ is the isomorphism of projective lines provided by the Fundamental Theorem, cf. Corollary 3 (and its proof). The cocycle $\mathbf{L}^{(3)} \times \mathbf{L}^{(3)} \to Aut(P(k^2))$ takes values in the subgroup of projectivities of $P(k^2)$, i.e. in $PGL(2,k)$. – Also, to the right of the pull-back diagrams in (12), we may adjoin another "extreme" one (which we shall not use), likewise deriving from the Fundamental Theorem.

$$\begin{tikzcd} \mathbf{L}^{(3)} \times P(k^2) \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{L}^{(3)} \times \mathbf{L} \arrow[d] \\ \mathbf{L}^{(3)} \arrow[r, "="'] & \mathbf{L}^{(3)}. \end{tikzcd}$$

The cocycles associated to the two pull-back diagrams in (12) take values in the subgroup $\mathrm{Aff}(k) \subseteq PGL(2,k)$, resp. in $GL(k) = k^* \subseteq PGL(2,k)$. For the first of these diagrams, note that the right hand vertical map $\mathbf{L}^{(2)} \to \mathbf{L}$ has for its fibre over $A \in \mathbf{L}$ (a set which may canonically be identified with) $\mathbf{L}\backslash\{A\}$; and for given $(A,B,C) \in \mathbf{L}^{(3)}$, the map h_a maps (A,B,C,μ)

according to the recipe in terms of $\phi_{A,B,C}$, given in the proof of Proposition 4; and, as we observed there, for $((A,B,C),(A,B',C')) \in \mathbf{L}^{(3)} \times_{\mathbf{L}} \mathbf{L}^{(3)}$, (same A!) the value of the "difference" (cocycle) $\phi_{A,B,C} \circ \phi^{-1}_{A,B',C'}$ belongs to $\mathrm{Aff}(k)$.

For the second of the diagrams, the right hand vertical map $\mathbf{L}^{(2)} \times_{\mathbf{L}} \mathbf{L}^{(2)} \to \mathbf{L}^{(2)}$ has for its fibre over $(A,B) \in \mathbf{L}^{(2)}$ a set which again may be canonically identified with $\mathbf{L}\backslash\{A\}$ (by identifying $((A,B),(A,C))$ with C). And for given $(A,B,C) \in \mathbf{L}^{(3)}$, the map h_l maps (A,B,C,μ) according to the same recipe as the one for h_a. The cocycle now takes for its input tuples $((A,B,C),(A,B,C'))$ (same A and B!), and, as we observed in the Remark, $\phi_{A,B,C} \circ \phi^{-1}_{A,B,C'}$ takes values in k^*.

Thus, the first diagram in (12) exhibits the $\mathbf{L}\backslash\{A\}$s as the fibres of an affine line bundle over $\mathbf{L}$; and the second diagram in (12) exhibits the $\mathbf{L}\backslash\{A\}$s as the fibres of a vector line bundle over $\mathbf{L}^{(2)}$, both these bundles modelled on the fibre k, viewed as, respectively, an affine line or a vector line.

One may similarly consider the diagram (13) as exhibiting the (abstract) projective line $\mathbf{L}$ as a projective line bundle over 1, modelled on the coordinate projective line $P(k^2)$.

Stacks of projective lines

The notion of projective line, and of morphism (= isomorphism = projectivity) between such, as described here, is a (1-sorted) first order theory. This immediately implies that the notion of a *bundle* of projective lines over a space M makes sense, and in fact, such bundles pull back along maps, and descend along surjections, so projective line bundles form canonically a stack over the base category of sets, or, with suitable modifications, over the base category of spaces, say. Continuity, or other forms of cohesion, will usually follow by the the fact that the constructions employed are canonical, as in [4], Section A.5. The study of bundles of projective lines in the category of schemes, from [5], was the input challenge for the present work, and I hope to push further into loc. cit. using the abstract-projective-line concepts.

Example. Let k denote the field of three elements $\mathbb{Z}_3$. Every 4-element set carries a *unique* structure of abstract projective line over this k. We invite the reader to construct this structure (a groupoid with 4 objects, and each hom-set a 2-element set); the composition laws follow from the idempotence equations; the cross ratio of the four distinct points (in any order) is -1.

(Another argument: the group $PGL(2;\mathbb{Z}_3)$ has 24 elements, which is also the number of permutations of a 4-element set, hence every permutation of a 4-element set is a projectivity.)

It follows that for any space M, and for any 4-fold covering $E \to M$, the bundle $E \to M$ is uniquely a bundle of projective lines over k. Clearly, such $E \to M$ need not have a section $M \to E$, so does not come about from a bundle of affine lines over M, by completing the fibres by points at infinity (the fibrewise infinity points would provide a cross section).

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