cahiers de topologie et géométrie différentielle catégoriques

créés par CHARLES EHRESMANN en 1958
dirigés par Andrée CHARLES EHRESMANN

VOLUME LI-2, 2e trimestre 2010

SOMMAIRE
4th Volume dedicated to F. BORCEUX

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES *Vol. LI-2 (2010)*

BIPULLBACKS AND CALCULUS OF FRACTIONS

by Enrico M. VITALE

Dedicated to Francis Borceux on the occasion of his 60th birthday

RÉSUMÉ. Nous démontrons que la classe des équivalences faibles entre groupoïdes internes dans une catégorie régulière protomodulaire est une "congruence à biproduits fibrés" et, par conséquent, elle admet un calcul à droite des fractions. Comme application, nous montrons que les foncteurs monoïdaux entre groupoïdes internes dans les groupes et les homomorphismes entre 2-algèbres de Lie strictes sont les fractions des foncteurs internes par rapport aux équivalences faibles.

RÉSUMÉ. We prove that the class of weak equivalences between internal groupoids in a regular protomodular category is a bipullback congrence and, therefore, has a right calculus of fractions. As an application, we show that monoidal functors between internal groupoids in groups and homomorphisms of strict Lie 2-algebras are fractions of internal functors with respect to weak equivalences.

1. Introduction

It is well known that any monoidal category is monoidally equivalent to a strict one. This is not true for strong monoidal functors: not every strong monoidal functor is naturally isomorphic to a strict one (i.e., to a functor F such that the structural isomorphisms $FA \otimes FB \to F(A \otimes B)$ and $I \to FI$ are identities). An important example of this fact is given by Schreier theory of group extensions. In fact, let A and B be groups and write $D(A)$ for A seen as a discrete internal groupoid in the category *Grp* of groups, and $OUT(B)$ for the internal groupoid in *Grp* corresponding to the crossed module $B \to Aut(B)$ of inner automorphisms.

Financial support: FNRS grant 1.5.276.09 is gratefully acknowledged.
2000 Mathematics Subject Classification: 18D05,18B40, 18D10, 18D35.
Key words and phrases: bipullback, bicategory of fractions, monoidal functor, weak equivalence.

Then internal (= strict) functors from $D(A)$ to $OUT(B)$ correspond to split extensions of A through B, whereas monoidal functors from $D(A)$ to $OUT(B)$ correspond to arbitrary extensions of A by B.

The previous example leads to the following question: what is the precise relation between the 2-category of internal groupoids and internal functors in *Grp* and the 2-category of internal groupoids in *Grp* and monoidal functors? The same question can be asked working internally to the category *Lie* of Lie K-algebras (for K a fixed field), replacing monoidal functors by homomorphisms of strict Lie 2-algebras (precise definitions are in Section 7).

A possible answer to the previous questions is suggested by the fact that if $F\colon \mathbb{C} \to \mathbb{D}$ is an internal functor in *Grp* which is a weak equivalence (i.e., full, faithful and essentially surjective on objects) then the quasi-inverse functor $F^{-1}\colon \mathbb{D} \to \mathbb{C}$ is no longer an internal functor, but it is still a monoidal one. More precisely, we prove that:

1. The 2-category of internal groupoids in *Grp* and monoidal functors is the 2-category of fractions of the 2-category of internal groupoids and internal functors in *Grp* with respect to weak equivalences.

2. The 2-category of internal groupoids in *Lie* and homomorphisms is the 2-category of fractions of the 2-category of internal groupoids and internal functors in *Lie* with respect to weak equivalences.

The paper is organized as follows:

- In Section 2 we recall some basic facts on bicategories of fractions established by D. Pronk in [16]. We then revisit the right calculus of fractions for classes of 1-cells using bipullbacks.

- In Section 3 we show that, for a category $\mathcal{C}$ with finite limits, the 2-category *Grpd*($\mathcal{C}$) of internal groupoids and internal functors has bipullbacks. More precisely, we show that the standard homotopy pullback in *Grpd*($\mathcal{C}$) also satisfies the universal property of a bipullback.

- Using bipullbacks, we show in Section 4 that if $\mathcal{C}$ is regular, then the class of weak equivalences in *Grpd*($\mathcal{C}$) has a right calculus of fractions.

- In Section 5 we refine the previous result showing that if $\mathcal{C}$ is regular and protomodular, then weak equivalences satisfy the "$2 \Rightarrow 3$" property and therefore they are a bipullback congruence, a notion inspired by Bénabou's approach to categories of fractions (see [4]).
- In the last two sections we choose as base category $\mathcal{C}$ the category of groups (Section 6) and the category of Lie K-algebras (Section 7) and we prove the results announced above.

Since *Grp* and *Lie* are Mal'cev categories, internal categories coincide with internal groupoids (see [11]). This is the reason why we restrict our attention to internal groupoids.

Let me finish with some comments. The result established in Section 6 is not at all a surprise. In fact, if we work with isomorphism classes of internal functors, then Proposition 6.4 becomes a result on categories of fractions (not on 2-categories of fractions) quite easy to prove directly and also easy to deduce using the Quillen model structures studied in [13] and in [15]. So, in my opinion, what is interesting is not the result *per se* but the fact that the 2-categorical nature of its proof requires the use of bipullbacks, whereas other kinds of 2-dimensional limits (like homotopy pullbacks) are not convenient in this context (see the Introduction in [4] for some comments on bilimits). Concerning the analogous result for Lie algebras stated in Section 7, I think it is interesting for a completely different reason. The notion of monoidal functor is a well-established one, whereas the notion of homomorphism of Lie 2-algebras is much more recent, so Proposition 7.4 could help to understand the 2-dimensional theory of Lie algebras.

Notation: the composite of $f: A \to B$ and $g: B \to C$ is written $f \cdot g$ or fg.

Terminology: bicategory means bicategory with invertible 2-cells.

2. Bicategories of fractions

2.1 Categories of fractions have been introduced by P. Gabriel and M. Zisman in [14] (see also Ch. 5 in [5]). If $\mathcal{C}$ is a category and Σ a class of arrows in $\mathcal{C}$, the category of fractions of $\mathcal{C}$ with respect to Σ is a functor

$$P_\Sigma : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$$

universal among all functors $\mathcal{F}\colon \mathcal{C} \to \mathcal{A}$ such that $\mathcal{F}(s)$ is an isomorphism for all $s \in \Sigma$. This can be restated saying that for every category $\mathcal{A}$

$$P_\Sigma \cdot - \,:\, \mathit{Funct}(\mathcal{C}[\Sigma^{-1}], \mathcal{A}) \to \mathit{Funct}_\Sigma(\mathcal{C}, \mathcal{A})$$

is an equivalence of categories, where $\mathit{Funct}_\Sigma(\mathcal{C}, \mathcal{A})$ is the category of functors making the elements of Σ invertible. If the class Σ has a right calculus of fractions, then $\mathcal{C}[\Sigma^{-1}]$ has a quite simple description:

Proposition 2.2 *(Gabriel-Zisman) Assume that Σ satisfies the following conditions:*

CF1. Σ contains all identities;

CF2. Σ is closed under composition;

CF3. For every pair $f\colon A \to B \leftarrow C\colon g$ with $g \in \Sigma$ there exist $g'\colon P \to A$ and $f'\colon P \to C$ such that $g' \cdot f = f' \cdot g$ and $g' \in \Sigma$;

CF4. If a pair of parallel arrows is coequalized by an element of Σ, then it is also equalized by an element of Σ.

Then the objects of $\mathcal{C}[\Sigma^{-1}]$ are those of $\mathcal{C}$ and an arrow from A to B is $\mathcal{C}[\Sigma^{-1}]$ is a class of spans

$$A \xleftarrow{\;s\;} I \xrightarrow{\;f\;} B$$

with $s \in \Sigma$. Two spans (s, I, f) and (s', I', f') are equivalent if there exist arrows x, x' in $\mathcal{C}$ such that $x \cdot s = x' \cdot s' \in \Sigma$ and $x \cdot f = x' \cdot f'$.

The analogous problem for bicategories has been solved by D. Pronk in [16]. For an introduction to bicategories see [3] or Ch. 7 in [5] where 2-categories are also discussed.

Definition 2.3 *(Pronk)* Let $\mathcal{B}$ be a bicategory and Σ a class of 1-cells in $\mathcal{B}$. The bicategory of fractions of $\mathcal{B}$ with respect to Σ is a homomorphism of bicategories

$$P_\Sigma\colon \mathcal{B} \to \mathcal{B}[\Sigma^{-1}]$$

universal among all homomorphisms $\mathcal{F}\colon \mathcal{B} \to \mathcal{A}$ such that $\mathcal{F}(S)$ is an equivalence for all $S \in \Sigma$. This can be restated saying that for every bicategory $\mathcal{A}$

$$P_\Sigma \cdot - \,:\, Hom(\mathcal{B}[\Sigma^{-1}], \mathcal{A}) \to Hom_\Sigma(\mathcal{B}, \mathcal{A})$$

is a biequivalence of bicategories, where $Hom_\Sigma(\mathcal{B}, \mathcal{A})$ is the bicategory of those homomorphisms $\mathcal{F}$ such that $\mathcal{F}(S)$ is an equivalence for all $S \in \Sigma$.

Definition 2.4 *(Pronk)* Let $\mathcal{B}$ be a bicategory and Σ a class of 1-cells in $\mathcal{B}$. The class Σ has a right calculus of fractions if the following conditions hold:

BF1. Σ contains all equivalences;

BF2. Σ is closed under composition;

BF3. For every pair $F\colon \mathbb{A} \to \mathbb{B} \leftarrow \mathbb{C}\colon G$ with $G \in \Sigma$ there exist $G'\colon \mathbb{P} \to \mathbb{A}$, $F'\colon \mathbb{P} \to \mathbb{C}$ and $\varphi\colon G' \cdot F \Rightarrow F' \cdot G$ with $G' \in \Sigma$;

BF4. For every $\alpha\colon F \cdot W \Rightarrow G \cdot W$ with $W \in \Sigma$ there exist $V \in \Sigma$ and $\beta\colon V \cdot F \Rightarrow V \cdot G$ such that $V \cdot \alpha = \beta \cdot W$, and for any other $V' \in \Sigma$ and $\beta'\colon V' \cdot F \Rightarrow V' \cdot G$ such that $V' \cdot \alpha = \beta' \cdot W$ there exist U, U' and $\varepsilon\colon U \cdot V \Rightarrow U' \cdot V'$ such that $U \cdot V \in \Sigma$ and

$$\begin{array}{ccc} U \cdot V \cdot F & \xrightarrow{U \cdot \beta} & U \cdot V \cdot G \\ {\scriptstyle \varepsilon \cdot F}\downarrow & & \downarrow{\scriptstyle \varepsilon \cdot G} \\ U' \cdot V' \cdot F & \xrightarrow[U' \cdot \beta']{} & U' \cdot V' \cdot G \end{array}$$

commutes;

BF5. If $\alpha\colon F \Rightarrow G$ is a 2-cell, then $F \in \Sigma$ if and only if $G \in \Sigma$.

If the class Σ has a right calculus of fractions, the bicategory $\mathcal{B}[\Sigma^{-1}]$ can be described in a way similar to that recalled in Proposition 2.2. Here we do not give full details because what we will use in Sections 6 and 7 is the following useful result:

Proposition 2.5 *(Pronk) Let $\mathcal{B}$ be a bicategory and Σ a class of 1-cells in $\mathcal{B}$ which has a right calculus of fractions. Consider a bifunctor $\mathcal{F}\colon \mathcal{B} \to \mathcal{A}$ such that $\mathcal{F}(S)$ is an equivalence for all $S \in \Sigma$ and let $\widehat{\mathcal{F}}\colon \mathcal{B}[\Sigma^{-1}] \to \mathcal{A}$ be its extension. Then $\widehat{\mathcal{F}}$ is a biequivalence provided that $\mathcal{F}$ satisfies the following conditions:*

EF1. $\mathcal{F}$ is surjective up to equivalence on objects;

EF2. $\mathcal{F}$ is full and faithful on 2-cells;

EF3. For every 1-cell F in $\mathcal{A}$ there exist 1-cells G and W in $\mathcal{B}$ with W in Σ and a 2-cell $\mathcal{F}(G) \Rightarrow \mathcal{F}(W) \cdot F$.

(In [16] it is stated that conditions EF1-EF3 are also necessary for $\widehat{\mathcal{F}}$ being a biequivalence. This is not true, as proved by M. Dupont in [12].)

2.6 Recall that a diagram

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\ {\scriptstyle G'}\downarrow & \overset{\varphi}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle G} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

in a bicategory $\mathcal{B}$ is a bipullback of F and G if for any other diagram

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{K} & \mathbb{C} \\ {\scriptstyle H}\downarrow & \overset{\psi}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle G} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

there exists a fill-in, that is a triple ($L\colon \mathbb{X} \to \mathbb{P}$, $\alpha\colon L \cdot G' \Rightarrow H$, $\beta\colon L \cdot F' \Rightarrow K$) such that

$$\begin{array}{ccc} L \cdot G' \cdot F & \xrightarrow{L\cdot\varphi} & L \cdot F' \cdot G \\ {\scriptstyle \alpha\cdot F}\downarrow & & \downarrow{\scriptstyle \beta\cdot G} \\ H \cdot F & \xrightarrow[\psi]{} & K \cdot G \end{array}$$

commutes, and for any other fill-in (L', α', β') there exists a unique $\lambda\colon L' \Rightarrow L$ such that

$$\begin{array}{ccccccc} L'\cdot G' & \xrightarrow{\lambda\cdot G'} & L\cdot G' & \qquad & L'\cdot F' & \xrightarrow{\lambda\cdot F'} & L\cdot F' \\ & {\scriptstyle\alpha'}\searrow \quad \swarrow{\scriptstyle\alpha} & & & & {\scriptstyle\beta'}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & H & & & & K & \end{array}$$

commute.

Remark 2.7 1. Bipullbacks are determined uniquely up to equivalence.

2. A 1-cell $W\colon \mathbb{B} \to \mathbb{A}$ is called full and faithful if for every $\mathbb{X}$ the hom-functor

$$\mathcal{B}(\mathbb{X}, W)\colon \mathcal{B}(\mathbb{X}, \mathbb{B}) \to \mathcal{B}(\mathbb{X}, \mathbb{A})$$

is full and faithful in the usual sense. Consider now the following diagrams, the first one being a bipullback,

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{W_2} & \mathbb{B} \\ {\scriptstyle W_1}\downarrow & \overset{w}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle W} \\ \mathbb{B} & \xrightarrow[W]{} & \mathbb{A} \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{B} & \xrightarrow{id} & \mathbb{B} \\ {\scriptstyle id}\downarrow & \overset{W}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle W} \\ \mathbb{B} & \xrightarrow[W]{} & \mathbb{A} \end{array}$$

Let $(D_W\colon \mathbb{B} \to \mathbb{K}, \delta_1\colon D_W \cdot W_1 \Rightarrow id, \delta_2\colon D_W \cdot W_2 \Rightarrow id)$ be the fill-in of the second diagram through the first one. Then W is full and faithful iff the second diagram is a bipullback iff the diagonal D_W is an equivalence.

Proposition 2.8 *Let $\mathcal{B}$ be a bicategory with bipullbacks and Σ a class of 1-cells in $\mathcal{B}$. Assume that Σ satisfies the following conditions:*

BP1. Σ contains all equivalences;

BP2. Σ is closed under composition;

BP3. Σ is stable under bipullbacks;

BP4. If W is in Σ, then the diagonal D_W is in Σ;

BP5. If $\alpha\colon F \Rightarrow G$ is a 2-cell, then $F \in \Sigma$ if and only if $G \in \Sigma$.

Then Σ has a right calculus of fractions.

Proof. Clearly BP3 implies BF3. We have to show that BF4 holds. Consider the following diagrams, the first one being a bipullback,

$$
\begin{array}{ccc}
\mathbb{K} & \xrightarrow{W_2} & \mathbb{B} \\
{\scriptstyle W_1}\downarrow & \overset{w}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle W} \\
\mathbb{B} & \xrightarrow[W]{} & \mathbb{A}
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{B} & \xrightarrow{id} & \mathbb{B} \\
{\scriptstyle id}\downarrow & \overset{W}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle W} \\
\mathbb{B} & \xrightarrow[W]{} & \mathbb{A}
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{G} & \mathbb{B} \\
{\scriptstyle F}\downarrow & \overset{\alpha}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle W} \\
\mathbb{B} & \xrightarrow[W]{} & \mathbb{A}
\end{array}
$$

Let $(D_W \colon \mathbb{B} \to \mathbb{K}, \delta_1, \delta_2)$ be the fill-in of the second diagram through the first one, and $(H \colon \mathbb{C} \to \mathbb{K}, \alpha_1, \alpha_2)$ the fill-in of the third diagram through the first one. Consider also the bipullback

$$
\begin{array}{ccc}
\mathbb{D} & \xrightarrow{V} & \mathbb{C} \\
{\scriptstyle L}\downarrow & \overset{\varphi}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle H} \\
\mathbb{B} & \xrightarrow[D_W]{} & \mathbb{K}
\end{array}
$$

and define $\beta \colon V \cdot F \Rightarrow V \cdot G$ as follows

$$VF \xrightarrow{V\alpha_1^{-1}} VHW_1 \xrightarrow{\varphi^{-1}W_1} LD_W W_1 \xrightarrow{L\delta_1} L \xrightarrow{L\delta_2^{-1}} LD_W W_2 \xrightarrow{\varphi W_2} VHW_2 \xrightarrow{V\alpha_2} VG$$

Observe that since $W \in \Sigma$, then $D_W \in \Sigma$ by BP4, and then $V \in \Sigma$ by BP3. Moreover, the condition $V \cdot \alpha = \beta \cdot W$ follows from the fill-in condition on $(D_W, \delta_1, \delta_2)$ and (H, α_1, α_2).

Let $\beta' \colon V' \cdot F \Rightarrow V' \cdot G$ be such that $V' \in \Sigma$ and $V' \cdot \alpha = \beta' \cdot W$. We obtain two fill-in of

$$
\begin{array}{ccc}
\mathbb{D}' & \xrightarrow{V' \cdot F} & \mathbb{B} \\
{\scriptstyle V' \cdot F}\downarrow & \overset{V'FW}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle W} \\
\mathbb{B} & \xrightarrow[W]{} & \mathbb{A}
\end{array}
$$

through the bipullback $(\mathbb{K}, W_1, W_2, w)$: the first one is

$$(\,\mathbb{D}' \xrightarrow{V'} \mathbb{C} \xrightarrow{F} \mathbb{B} \xrightarrow{D_W} \mathbb{K}\,, V' \cdot F \cdot \delta_1, V' \cdot F \cdot \delta_2)$$

and the second one is

$$(\,\mathbb{D}' \xrightarrow{V'} \mathbb{C} \xrightarrow{H} \mathbb{K}\,, V' \cdot \alpha_1,\ V'HW_2 \xrightarrow{V'\alpha_2} V'G \xrightarrow{(\beta')^{-1}} V'F\,)$$

By the universal property of $(\mathbb{K}, W_1, W_1, w)$, there exists a unique $\beta^* \colon V' \cdot F \cdot D_W \Rightarrow V' \cdot H$ such that

$$\begin{array}{ccc} V' \cdot F \cdot D_W \cdot W_1 & \xrightarrow{\beta^* \cdot W_1} & V' \cdot H \cdot W_1 \\ & {\scriptstyle V' \cdot F \cdot \delta_1} \searrow \quad \swarrow {\scriptstyle V' \cdot \alpha_1} & \\ & V' \cdot F & \end{array}$$

and

$$\begin{array}{ccc} V' \cdot F \cdot D_W \cdot W_2 & \xrightarrow{\beta^* \cdot W_2} & V' \cdot H \cdot W_2 \\ {\scriptstyle V' \cdot F \cdot \delta_2} \downarrow & & \downarrow {\scriptstyle V' \cdot \alpha_2} \\ V' \cdot F & \xrightarrow[\beta']{} & V' \cdot G \end{array}$$

commute. Let $(U \colon \mathbb{D}' \to \mathbb{D}, \eta \colon U \cdot L \Rightarrow V' \cdot F, \varepsilon \colon U \cdot V \Rightarrow V')$ be the fill-in of

$$\begin{array}{ccc} \mathbb{D}' & \xrightarrow{V'} & \mathbb{C} \\ {\scriptstyle V' \cdot F} \downarrow & \overset{\beta^*}{\nearrow\!\!\!\!\nearrow} & \downarrow {\scriptstyle H} \\ \mathbb{B} & \xrightarrow[D_W]{} & \mathbb{K} \end{array}$$

through the bipullback $(\mathbb{D}, L, V, \varphi)$. If we choose $U' = id$, we have $\varepsilon \colon U \cdot V \Rightarrow U' \cdot V'$. Since $V' \in \Sigma$, then also $U' \cdot V'$ and $U \cdot V$ are in Σ because of BP1, BP2 and BP5. It remains to check the compatibility of ε, β and β' as in BF4, but this is just a diagram chasing. ■

3. Bipullbacks in *Grpd*($\mathcal{C}$)

The aim of this section is to prove the following result:

Proposition 3.1 *Let $\mathcal{C}$ be a category with finite limits, and let Grpd($\mathcal{C}$) be the 2-category of internal groupoids, internal functors and internal natural transformations in $\mathcal{C}$. The 2-category Grpd($\mathcal{C}$) has bipullbacks.*

3.2 Let us fix notation (details can be found in Ch. 7 of [5] or in Appendix 3 of [7]):

- An internal groupoid $\mathbb{C}$ is represented by

$$C_1 \times_{c,d} C_1 \xrightarrow{m} C_1 \underset{c}{\overset{d}{\rightrightarrows}} C_0 \;(\text{with } e\colon C_0 \to C_1) \qquad C_1 \xrightarrow{i} C_1$$

  where the following diagram is a pullback

$$\begin{array}{ccc} C_1 \times_{c,d} C_1 & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle d} \\ C_1 & \xrightarrow[c]{} & C_0 \end{array}$$

- An internal functor $F\colon \mathbb{C} \to \mathbb{D}$ is represented by

$$\begin{array}{ccc} C_1 & \xrightarrow{F_1} & D_1 \\ {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ C_0 & \xrightarrow[F_0]{} & D_0 \end{array}$$

- An internal natural transformation $\alpha\colon F \Rightarrow G\colon \mathbb{C} \to \mathbb{D}$ is represented by

$$\begin{array}{ccc} C_1 & \underset{G_1}{\overset{F_1}{\rightrightarrows}} & D_1 \\ {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} & \nearrow^{\alpha} & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ C_0 & \underset{G_0}{\overset{F_0}{\rightrightarrows}} & D_0 \end{array}$$

3.3 It is helpful to start recalling that in *Grpd*(*Set*) bipullbacks are comma-squares. With the notations of 2.6:

- an object in $\mathbb{P}$ is a triple $(a_0 \in A_0, b_1\colon F_0(a_0) \to G_0(c_0), c_0 \in C_0)$,

- an arrow from (a_0, b_1, c_0) to (a'_0, b'_1, c'_0) is a pair of arrows $(a_1\colon a_0 \to a'_0, c_1\colon c_0 \to c'_0)$ such that $F_1(a_1) \cdot b'_1 = b_1 \cdot G_1(c_1)$,

- $G'\colon \mathbb{P} \to \mathbb{A}$ and $F'\colon \mathbb{P} \to \mathbb{C}$ are the obvious projections, and $\varphi(a_0, b_1, c_0) = b_1$,

- $L_0(x_0) = (H_0(x_0), \psi(x_0), K_0(x_0))$, $L_1(x_1) = (H_1(x_1), K_1(x_1))$, $\alpha = id$ and $\beta = id$,
- $\lambda(x_0) = (\alpha'(x_0), \beta'(x_0))$.

3.4 The description of bipullbacks in *Grpd*(*Set*) recalled in 3.3 indicates that the first step to obtain bipullbacks in *Grpd*($\mathcal{C}$) is to construct from an internal groupoid $\mathbb{B}$ a new internal groupoid $\vec{\mathbb{B}}$ whose objects are arrows in $\mathbb{B}$ and whose arrows are commutative squares in $\mathbb{B}$. The construction of $\vec{\mathbb{B}}$ is quite standard:

$$\vec{\mathbb{B}} = \left(\vec{B}_1 \times_{\vec{c},\vec{d}} \vec{B}_1 \xrightarrow{\ \vec{m}\ } \vec{B}_1 \underset{\vec{c}}{\overset{\vec{d}}{\rightrightarrows}} B_1 \ (\vec{e} : B_1 \to \vec{B}_1) \qquad \vec{B}_1 \xrightarrow{\ \vec{i}\ } \vec{B}_1 \right)$$

- $\vec{B}_1$ is defined by the following pullback

$$\begin{array}{ccc} \vec{B}_1 & \xrightarrow{\ m_2\ } & B_1 \times_{c,d} B_1 \\ {\scriptstyle m_1}\downarrow & & \downarrow{\scriptstyle m} \\ B_1 \times_{c,d} B_1 & \xrightarrow[\ m\ ]{} & B_1 \end{array}$$

- $\vec{d} = m_1 \cdot \pi_1$ and $\vec{c} = m_2 \cdot \pi_2$,
- $\vec{e}$ is the unique factorization through $\vec{B}_1$ of the following commutative diagram

$$\begin{array}{ccccc} B_1 & \xrightarrow{\ <d,1>\ } & B_0 \times B_1 & \xrightarrow{\ e\times 1\ } & B_1 \times_{c,d} B_1 \\ {\scriptstyle <1,c>}\downarrow & & & & \downarrow{\scriptstyle m} \\ B_1 \times B_0 & \xrightarrow[\ 1\times e\ ]{} & B_1 \times_{c,d} B_1 & \xrightarrow[\ m\ ]{} & B_1 \end{array}$$

- we leave to the reader the task of describing $\vec{m}$ and $\vec{i}$.

3.5 The internal groupoid $\vec{\mathbb{B}}$ is equipped with two internal functors $\delta, \gamma \colon \vec{\mathbb{B}} \to \mathbb{B}$ specified by

$$\begin{array}{ccc} \vec{B}_1 & \xrightarrow{\ \delta_1 = m_2\cdot\pi_1\ } & B_1 \\ {\scriptstyle \vec{d}}\downarrow\downarrow{\scriptstyle \vec{c}} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ B_1 & \xrightarrow[\ \delta_0 = d\ ]{} & B_0 \end{array} \qquad \begin{array}{ccc} \vec{B}_1 & \xrightarrow{\ \gamma_1 = m_1\cdot\pi_2\ } & B_1 \\ {\scriptstyle \vec{d}}\downarrow\downarrow{\scriptstyle \vec{c}} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ B_1 & \xrightarrow[\ \gamma_0 = c\ ]{} & B_0 \end{array}$$

and it tourns out that to give an internal natural transformation $\alpha\colon F \Rightarrow G\colon \mathbb{A} \to \mathbb{B}$ is the same as giving an internal functor $\alpha\colon \mathbb{A} \to \vec{\mathbb{B}}$ such that $\alpha \cdot \delta = F$ and $\alpha \cdot \gamma = G$. Indeed, the internal functor α is specified by

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_1} & \vec{B}_1 \\ {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} & & {\scriptstyle \vec{d}}\downarrow\downarrow{\scriptstyle \vec{c}} \\ A_0 & \xrightarrow[\alpha]{} & B_1 \end{array}$$

where α_1 is the unique factorization through $\vec{B}_1$ of the following commutative diagram

$$\begin{array}{ccccc} A_1 & \xrightarrow{<1,c>} & A_1 \times A_0 & \xrightarrow{F_1 \times \alpha} & B_1 \times_{c,d} B_1 \\ {\scriptstyle <d,1>}\downarrow & & & & \downarrow{\scriptstyle m} \\ A_0 \times A_1 & \xrightarrow[\alpha \times G_1]{} & B_1 \times_{c,d} B_1 & \xrightarrow[m]{} & B_1 \end{array}$$

3.6 We are ready to prove Proposition 3.1. We use the notations of 2.6.

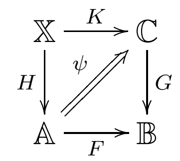
Proof. Given $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{C} \to \mathbb{B}$ in *Grpd*($\mathcal{C}$), a bipullback

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\ {\scriptstyle G'}\downarrow & \overset{\varphi}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle G} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

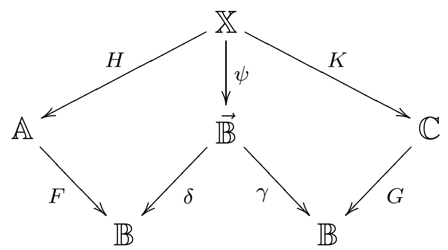
is given by the following limit in *Grpd*($\mathcal{C}$) (recall that *Grpd*($\mathcal{C}$) has limits computed componentwise in $\mathcal{C}$)

$$\begin{array}{ccccccc} & & & \mathbb{P} & & & \\ & & \overset{G'}{\swarrow} & \downarrow{\scriptstyle \varphi} & \overset{F'}{\searrow} & & \\ & \mathbb{A} & & \vec{\mathbb{B}} & & \mathbb{C} & \\ & & \underset{F}{\searrow} \; \underset{\delta}{\swarrow} & & \underset{\gamma}{\searrow} \; \underset{G}{\swarrow} & & \\ & & \mathbb{B} & & \mathbb{B} & & \end{array}$$

Indeed, any diagram

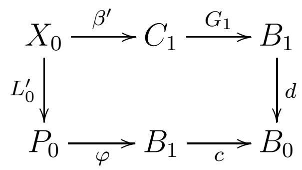


produces a commutative diagram

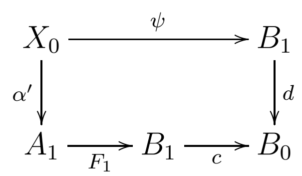


so that following the universal property of $\mathbb{P}$ as a limit there exists a unique $L\colon \mathbb{X} \to \mathbb{P}$ such that $L \cdot G' = H$, $L \cdot F' = K$ and $L \cdot \varphi = \psi$. (In other words, $(\mathbb{P}, G', F', \varphi)$ is the standard homotopy pullback of F and G.)
Clearly, $(L, \alpha = id, \beta = id)$ is a fill-in of $(\mathbb{X}, H, K, \psi)$ through $(\mathbb{P}, G', F', \varphi)$. Let (L', α', β') be another fill-in of $(\mathbb{X}, H, K, \psi)$ through $(\mathbb{P}, G', F', \varphi)$. We have to show that there exists a unique $\lambda\colon L' \Rightarrow L$ such that $\lambda \cdot G' = \alpha'$ and $\lambda \cdot F' = \beta'$. Define:

- τ_1 to be the unique factorization through $B_1 \times_{c,d} B_1$ of the following diagram



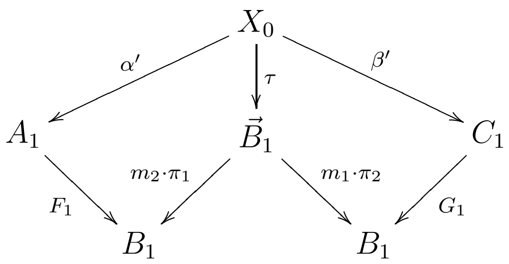
- τ_2 to be the unique factorization through $B_1 \times_{c,d} B_1$ of the following diagram



- τ to be the unique factorization through $\vec{B}_1$ of the following diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\tau_2} & B_1 \times_{c,d} B_1 \\ {\scriptstyle \tau_1}\downarrow & & \downarrow{\scriptstyle m} \\ B_1 \times_{c,d} B_1 & \xrightarrow[m]{} & B_1 \end{array}$$

Finally, λ is the unique factorization through P_1 of the following diagram



Clearly, $\lambda \cdot G' = \alpha'$ and $\lambda \cdot F' = \beta'$. To check that $\lambda \cdot d = L'_0$ and $\lambda \cdot c = L_0$, the naturality of λ, and its uniqueness is a diagram chasing using that $\{G'_1, \varphi_1, F'_1\}$, $\{m_1, m_2\}$ and $\{\pi_1, \pi_2\}$ are jointly monomorphic. ∎

4. Weak equivalences in *Grpd*($\mathcal{C}$)

Definition 4.1 *(Bunge-Paré)* Let $F\colon \mathbb{C} \to \mathbb{B}$ be in *Grpd*($\mathcal{C}$).

1. F is essentially surjective on objects if

$$C_0 \times_{F_0,d} D_1 \xrightarrow{t_2} D_1 \xrightarrow{c} D_0$$

   is a regular epimorphism, where t_2 is given by the following pullback

$$\begin{array}{ccc} C_0 \times_{F_0,d} D_1 & \xrightarrow{t_2} & D_1 \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle d} \\ C_0 & \xrightarrow[F_0]{} & D_0 \end{array}$$

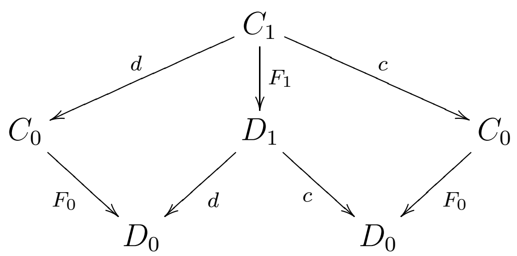
2. F is a weak equivalence if it is full and faithful (see 2.7) and essentially surjective on objects.

The previous definition is due to M. Bunge and R. Paré (see [10]). In [13] a more general notion of weak equivalence involving a Grothendieck topology on $\mathcal{C}$ has been considered. Since in Sections 6 and 7 the base category $\mathcal{C}$ is regular, I adopt for the moment the definition of Bunge and Paré. More on this point is contained in 5.10.

Next lemma is well-known and we only sketch the proof.

Lemma 4.2 *Let $F: \mathbb{C} \to \mathbb{D}$ be in $Grpd(\mathcal{C})$.*

1. *F is full and faithful if and only if the following is a limit diagram*

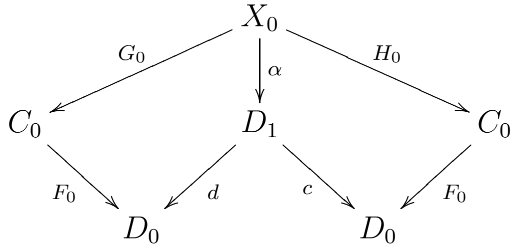


2. *F is an equivalence if and only if it is full and faithful and*

$$C_0 \times_{F_0,d} D_1 \xrightarrow{t_2} D_1 \xrightarrow{c} D_0$$

is a split epimorphism.

Proof. 1. If the diagram is a limit diagram and $\alpha: G{\cdot}F \Rightarrow H{\cdot}F: \mathbb{X} \to \mathbb{D}$ is an internal natural transformation, then $\alpha \cdot d = G_0 \cdot F_0$ and $\alpha \cdot c = H_0 \cdot F_0$. By the universal property of C_1 we get a unique $\beta: X_0 \to C_1$ such that $\beta \cdot d = G_0$, $\beta \cdot d = H_0$ and $\beta \cdot F_1 = \alpha$. So we have $\beta: G \Rightarrow H$ such that $\beta \cdot F = \alpha$. (The naturality of β follows from that of α.)
Conversely, any commutative diagram



gives rise to internal functors $G, H \colon \mathbb{X} \to \mathbb{C}$ with discrete domain

$$\begin{array}{ccc} X_0 & \overset{G_0 \cdot e}{\underset{H_0 \cdot e}{\rightrightarrows}} & C_1 \\ {\scriptstyle 1}\downarrow\downarrow{\scriptstyle 1} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ X_0 & \overset{G_0}{\underset{H_0}{\rightrightarrows}} & C_0 \end{array}$$

and to an internal natural transformation $\alpha \colon G \cdot F \Rightarrow H \cdot F$. To give an internal natural transformation $\beta \colon G \Rightarrow H$ such that $\beta \cdot F = \alpha$ means precisely to give a factorization $\beta \colon X_0 \to C_1$ of (G_0, α, H_0) through (d, F_1, c).
2. Let F be an equivalence and consider an internal natural transformation $\beta \colon G \cdot F \Rightarrow Id_{\mathcal{D}}$. Since $\beta \cdot d = G_0 \cdot F_0$, there exists a unique $j \colon D_0 \to C_0 \times_{F_0,d} D_1$ such that $j \cdot t_1 = G_0$ and $j \cdot t_2 = \beta$. Therefore $j \cdot t_2 \cdot c = \beta \cdot c = id$.
Conversely, if $j \colon D_0 \to C_0 \times_{F_0,d} D_1$ such that $j \cdot t_2 \cdot c = id$, we can construct a quasi-inverse internal functor $G \colon \mathbb{D} \to \mathbb{C}$ as follows: first define G_0 by

$$G_0 = j \cdot t_1 \colon D_0 \to C_0 \times_{F_0,d} D_1 \to C_0$$

Then, define $j_1 \colon D_1 \to D_1$ by

$$j_1 = < d \cdot j \cdot t_2, 1, c \cdot j \cdot t_2 \cdot i > \cdot (m \times 1) \cdot m \colon D_1 \to D_1 \times_{c,d} D_1 \times_{c,d} D_1 \to D_1$$

Finally, since F is full and faithful, by the first part of the lemma we get a unique arrow $G_1 \colon D_1 \to C_1$ such that $G_1 \cdot d = d \cdot G_0$, $G_1 \cdot F_1 = j_1$ and $G_1 \cdot c = c \cdot G_0$. ∎

Corollary 4.3 *Every equivalence in Grpd($\mathcal{C}$) is a weak equivalence. The converse is true provided that in $\mathcal{C}$ the axiom of choice holds (i.e., regular epimorphisms split).*

4.4 Regular categories have been introduced by M. Barr in [2] (see also Ch. 2 in [6]). In a regular category regular epimorphisms behave well: they are closed under composition and finite products, stable under pullbacks, and if a composite arrow $f \cdot g$ is a regular epimorphism, then g is a regular epimorphism. It follows that if $F \colon \mathbb{C} \to \mathbb{D}$ is in *Grpd*($\mathcal{C}$) with $\mathcal{C}$ regular and if F_0 is a regular epimorphism, then F is essentially surjective on objects.

Proposition 4.5 *Let $\mathcal{C}$ be a regular category and let Σ be the class of weak equivalences in Grpd($\mathcal{C}$). Then Σ has a right calculus of fractions.*

Proof. Since by Proposition 3.1 *Grpd*($\mathcal{C}$) has bipullbacks, to prove that Σ has a right calculus of fractions we check conditions BP1–BP5 in Proposition 2.8.
BP1 is given by Corollary 4.3, BP4 follows from 2.7 and BP5 is an exercise for the reader.
BP2: full and faithful internal functors are closed under composition because so they are in *Grpd*(*Set*). Assume now that $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{B} \to \mathbb{C}$ are essentially surjective. Consider the following pullbacks

$$\begin{array}{ccc} A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle d} \\ A_0 & \xrightarrow[F_0]{} & B_0 \end{array} \qquad \begin{array}{ccc} B_0 \times_{G_0,d} C_1 & \xrightarrow{t_2} & C_1 \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle d} \\ B_0 & \xrightarrow[G_0]{} & C_0 \end{array} \qquad \begin{array}{ccc} A_0 \times_{F_0 G_0,d} C_1 & \xrightarrow{\tau_2} & C_1 \\ {\scriptstyle \tau_1}\downarrow & & \downarrow{\scriptstyle d} \\ A_0 & \xrightarrow[F_0 \cdot G_0]{} & C_0 \end{array}$$

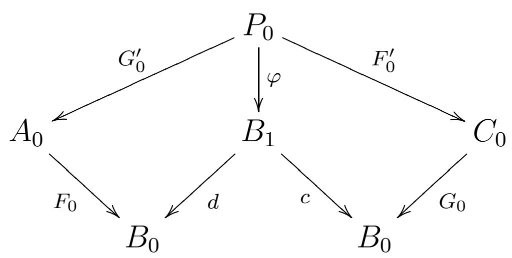
The essential surjectivity of $F \cdot G$ comes from the commutativity of the following diagram

$$\begin{array}{ccccc} A_0 \times_{F_0,d} B_1 \times_{G_1 c,d} C_1 & \xrightarrow{t_2 \times 1} & B_1 \times_{G_1 c,d} C_1 & \xrightarrow{c \times 1} & B_0 \times_{G_0,d} C_1 \\ {\scriptstyle 1 \times G_1 \times 1}\downarrow & & & & \downarrow{\scriptstyle t_2} \\ A_0 \times_{F_0 G_0,d} C_1 \times_{c,d} C_1 & & & & C_1 \\ {\scriptstyle 1 \times m}\downarrow & & & & \downarrow{\scriptstyle c} \\ A_0 \times_{F_0 G_0,d} C_1 & \xrightarrow[\tau_2]{} & C_1 & \xrightarrow[c]{} & C_0 \end{array}$$

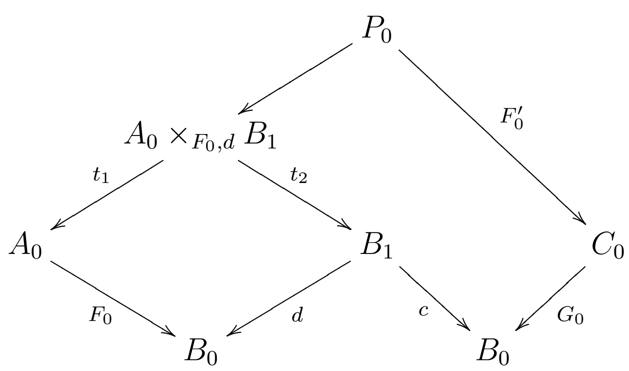
BP3: full and faithful internal functors are stable under bipullbacks because so they are in *Grpd*(*Set*) (use 3.3) and $Grpd(\mathcal{C})(\mathbb{X}, -)\colon Grpd(\mathcal{C}) \to Grpd(Set)$ preserves bipullbacks. Consider now a bipullback

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\ {\scriptstyle G'}\downarrow & \overset{\varphi}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle G} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

and assume that F is essentially surjective. Following the description of $\mathbb{P}$ given at the beginning of 3.6, we have a limit diagram in $\mathcal{C}$



But such a limit can be obtained performing two pullbacks as follows



Since by assumption $t_2 \cdot c \colon A_0 \times_{F_0,d} B_1 \to B_1 \to B_0$ is a regular epimorphism, F'_0 also is a regular epimorphism and then F' is essentially surjective (see 4.4). ∎

5. Bipullback congruences

Next definition is the direct bicategorical generalization of the notion of pullback congruence introduced by J. Bénabou in [4].

Definition 5.1 Let $\mathcal{B}$ be a bicategory with bipullbacks and Σ a class of 1-cells in $\mathcal{B}$. The class Σ is a bipullback congruence if the following conditions hold:

BC1. Σ contains all equivalences;

BC2. Σ satisfies the "$2 \Rightarrow 3$" property: let $F\colon \mathbb{C} \to \mathbb{D}$ and $G\colon \mathbb{D} \to \mathbb{E}$ be 1-cells in $\mathcal{B}$; if two of F, G and $F \cdot G$ are in Σ, then the third one is in Σ;

BC3. Σ is stable under bipullbacks;

BC4. If $\alpha\colon F \Rightarrow G$ is a 2-cell, then $F \in \Sigma$ if and only if $G \in \Sigma$.

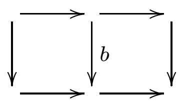
Proposition 5.2 *Let $\mathcal{B}$ be a bicategory with bipullbacks. Any bipullback congruence has a right calculus of fractions.*

Proof. It is enough to prove that a bipullback congruence Σ satisfies condition BP3 in Proposition 2.8. Let $W\colon \mathbb{B} \to \mathbb{A}$ be in Σ and let $(D_W\colon \mathbb{B} \to \mathbb{K}, \delta_1\colon D_W \cdot W_1 \Rightarrow id, \delta_2\colon D_W \cdot W_2 \Rightarrow id)$ be the diagonal fill-in as in 2.7. By BC1, $id \in \Sigma$, and then by BC4 $D_W \cdot W_1 \in \Sigma$. Since by BC3 $W_1 \in \Sigma$, we conclude by BC2 that $D_W \in \Sigma$. ■

5.3 Protomodular categories have been introduced by D. Bourn in [8] (see also [7]). Since we are concerned only with regular categories, we can consider the next lemma, proved in [9], as a definition of protomodular category. This lemma makes also evident the analogy between bipullback congruences and regular protomodular categories: in a regular protomodular category pullbacks satisfies the "$2 \Rightarrow 3$" property. This analogy will be made precise in Proposition 5.5.

Lemma 5.4 *(Bourn-Gran) Let $\mathcal{C}$ be a regular category. The following conditions are equivalent:*

1. *$\mathcal{C}$ is protomodular;*
2. *In any commutative diagram*



where b is a regular epimorphism, if the left hand square and the outer rectangle are pullbacks, then the right hand square is a pullback.

Proposition 5.5 *Let $\mathcal{C}$ be a regual protomodular category. The class of weak equivalences in Grpd$(\mathcal{C})$ is a bipullback congruence.*

Let $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{B} \to \mathbb{C}$ be in *Grpd*$(\mathcal{C})$. In order to prove Proposition 5.5 we need two lemmas on the shape of certains limits. The proof is routine.

Lemma 5.6 *Consider the pullbacks*

$$\begin{array}{ccc} A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 \\ {\scriptstyle t_1}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ A_0 & \xrightarrow[F_0]{} & B_0 \end{array} \qquad \begin{array}{ccc} B_1 \times_{c,F_0} A_0 & \xrightarrow{s_2} & A_0 \\ {\scriptstyle s_1}\big\downarrow & & \big\downarrow{\scriptstyle F_0} \\ B_1 & \xrightarrow[c]{} & B_0 \end{array}$$

and the commutative diagrams

$$\begin{array}{ccccc} A_1 & & \xrightarrow{\quad c \quad} & & A_0 \\ {\scriptstyle <d,F_1>}\big\downarrow & & (1) & & \big\downarrow{\scriptstyle F_0} \\ A_0 \times_{F_0,d} B_1 & \xrightarrow[t_2]{} & B_1 & \xrightarrow[c]{} & B_0 \end{array} \qquad \begin{array}{ccccc} A_1 & & \xrightarrow{\quad d \quad} & & A_0 \\ {\scriptstyle <F_1,c>}\big\downarrow & & (2) & & \big\downarrow{\scriptstyle F_0} \\ B_1 \times_{c,F_0} A_0 & \xrightarrow[s_1]{} & B_1 & \xrightarrow[d]{} & B_0 \end{array}$$

Then $F\colon \mathbb{A} \to \mathbb{B}$ is full and faithful iff (1) *is a pullback iff* (2) *is a pullback.*

Lemma 5.7 *Consider the pullbacks*

$$\begin{array}{ccc} A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 \\ {\scriptstyle t_1}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ A_0 & \xrightarrow[F_0]{} & B_0 \end{array} \qquad \begin{array}{ccc} C_1 \times_{c,G_0} B_0 & \xrightarrow{s_2} & B_0 \\ {\scriptstyle s_1}\big\downarrow & & \big\downarrow{\scriptstyle G_0} \\ C_1 & \xrightarrow[c]{} & C_0 \end{array} \qquad \begin{array}{ccc} A_0 \times_{F_0G_0,d} C_1 & \xrightarrow{\tau_2} & C_1 \\ {\scriptstyle \tau_1}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ A_0 & \xrightarrow[F_0 \cdot G_0]{} & C_0 \end{array}$$

and the commutative diagrams

$$\begin{array}{ccccc} A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 & \xrightarrow{<G_1,c>} & C_1 \times_{c,G_0} B_0 \\ \big\downarrow & & & & \big\downarrow{\scriptstyle s_1} \\ {\scriptstyle t_1}\big\downarrow & & (3) & & C_1 \\ \big\downarrow & & & & \big\downarrow{\scriptstyle d} \\ A_0 & \xrightarrow[F_0]{} & B_0 & \xrightarrow[G_0]{} & C_0 \end{array} \qquad \begin{array}{ccccc} A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 & \xrightarrow{c} & B_0 \\ {\scriptstyle 1\times G_1}\big\downarrow & & (4) & & \big\downarrow{\scriptstyle G_0} \\ A_0 \times_{F_0G_0,d} C_1 & \xrightarrow[\tau_2]{} & C_1 & \xrightarrow[c]{} & C_0 \end{array}$$

Then (3) *is a pullback iff* (4) *is a pullback.*

5.8 We are ready to prove Proposition 5.5.

Proof. Let Σ be the class of weak equivalences in *Grpd*($\mathcal{C}$). We have to show that condition BC2 holds, since the other conditions have been checked in the proof of Proposition 4.5. More precisely, given $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{B} \to \mathbb{C}$ in *Grpd*($\mathcal{C}$) such that $F \cdot G \in \Sigma$, we have to prove that $F \in \Sigma$ iff $G \in \Sigma$. There are two not obvious steps. (The protomodularity of $\mathcal{C}$ is needed only for the first step.)
1. If $F \cdot G$ is full and faithful and F is a weak equivalence, then G is full and faithful. Consider the following commutative diagram

$$
\begin{array}{ccccc}
A_1 & & \xrightarrow{\quad c \quad} & & A_0 \\
\downarrow{\scriptstyle <d,F_1>} & & & & \downarrow{\scriptstyle F_0} \\
A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 & \xrightarrow{c} & B_0 \\
\downarrow{\scriptstyle 1\times G_1} & & & & \downarrow{\scriptstyle G_0} \\
A_0 \times_{F_0G_0,d} C_1 & \xrightarrow[\tau_2]{} & C_1 & \xrightarrow[c]{} & C_0
\end{array}
$$

Since F is full and faithful, by Lemma 5.6 the top square is a pullback. Since $F \cdot G$ is full and faithful, by Lemma 5.6 the outer rectangle is a pullback. Since F is essentially surjective, the second row is a regular epimorphism. Following Lemma 5.4 the bottom square is a pullback. Therefore, by Lemma 5.7, the outer rectangle of the following commutative diagram is a pullback

$$
\begin{array}{ccccc}
A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 & \xrightarrow{<G_1,c>} & C_1 \times_{c,G_0} B_0 \\
 & & & & \downarrow{\scriptstyle s_1} \\
\downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle d} & & C_1 \\
 & & & & \downarrow{\scriptstyle d} \\
A_0 & \xrightarrow[F_0]{} & B_0 & \xrightarrow[G_0]{} & C_0
\end{array}
$$

Since the left hand square is a pullback by definition and the second column is a split epimorphism, by Lemma 5.4 the right hand square is a pullback. By Lemma 5.6 again we conclude that G is full and faithful.
2. If $F \cdot G$ is essentially surjective and G is full and faithful, then F is

essentially surjective. Consider the following pullback (notations as in Lemma 5.7)

$$\begin{array}{ccccc} Q & & \xrightarrow{\lambda_2} & & B_0 \\ \downarrow{\scriptstyle \lambda_1} & & & & \downarrow{\scriptstyle G_0} \\ A_0 \times_{F_0 G_0, d} C_1 & \xrightarrow[\tau_2]{} & C_1 & \xrightarrow[c]{} & C_0 \end{array}$$

By assumption $\tau_2 \cdot c$ is a regular epimorphism, so that λ_2 also is a regular epimorphism. Since G is full and faithful, there exists $\lambda \colon Q \to B_1$ such that $\lambda \cdot d = \lambda_1 \cdot \tau_1 \cdot F_0$, $\lambda \cdot G_1 = \lambda_1 \cdot \tau_2$ and $\lambda \cdot c = \lambda_2$. From the first equation on λ, we deduce the existence of $\mu \colon Q \to A_0 \times_{F_0,d} B_1$ such that $\mu \cdot t_1 = \lambda_1 \cdot \tau_1$ and $\mu \cdot t_2 = \lambda$. Finally, $\mu \cdot t_2 \cdot c = \lambda \cdot c = \lambda_2$, so that $t_2 \cdot c$ is a regular epimorphism. (Note that we need only the existence of λ, not its uniqueness. In other words we only use the "fullness" of G, and not its "faithfulness".) ■

5.9 Observe that, contrarily to Lemma 5.4, Proposition 5.5 is not a characterization of regular protomodular categories. Indeed, if $\mathcal{C}$ is *Set* (more generally, if in $\mathcal{C}$ the axiom of choice holds) then weak equivalences in *Grpd*($\mathcal{C}$) are the same that equivalences (see Corollary 4.3), and the class of equivalences obviously is a bipullback congruence.

5.10 G. Janelidze pointed out to me that condition 2 in Lemma 5.4 holds in any protomodular (not necessarily regular) category $\mathcal{C}$ provided that the arrow b is a pullback stable strong epimorphism. This fact has an interesting consequence. Indeed, Proposition 4.5 holds when $\mathcal{C}$ is any finitely complete category and Σ is the class of "weak $\mathcal{E}$-equivalences", where:

- $\mathcal{E}$ is any class of arrows that behaves well (in the sense explained in 4.4) and contains the split epimorphisms,
- an internal functor F is a weak $\mathcal{E}$-equivalence if it is full and faithful and essentially $\mathcal{E}$-surjective (that is, the arrow $t_2 \cdot c \colon C_0 \times_{F_0,d} D_1 \to D_1 \to D_0$ of Definition 4.1 is in $\mathcal{E}$).

Therefore, Proposition 5.5 holds for weak $\mathcal{E}$-equivalences in any protomodular category $\mathcal{C}$ provided that $\mathcal{E}$ behaves well, contains the split epimorphisms and is contained in the class of pullback stable strong epimorphisms. Examples are:

i. the class of pullback stable regular epimorphisms,

ii. the class of pullback stable regular epimorphisms that are effective descent morphisms.

6. Monoidal functors

All along this section we fix $\mathcal{C} = Grp$, the category of groups, which is a regular and protomodular category. I use additive notation for groups.

6.1 The aim of this section is to prove that the 2-category *MON* described hereunder is the bicategory of fractions of *Grpd*($\mathcal{C}$) with respect to weak equivalences.

1. Objects of *MON* are internal groupoids in *Grp*. Note that since the forgetful functor $Grp \to Set$ preserves finite limits, any object of *MON* is also a groupoid in the usual sense.

2. 1-cells $F\colon \mathbb{A} \to \mathbb{B}$ in *MON* are monoidal functors, that is, pairs (F, F_2) where F is a (not necessarily internal) functor and

$$F_2 = \{F_2^{a,b}\colon Fa + Fb \to F(a+b)\}_{a,b \in A_0}$$

is a natural family of arrows in $\mathbb{B}$ satisfying the cocycle condition

$$\begin{array}{ccc} Fa + Fb + Fc & \xrightarrow{1+F_2^{b,c}} & Fa + F(b+c) \\ {\scriptstyle F_2^{a,b}+1}\downarrow & & \downarrow{\scriptstyle F_2^{a,b+c}} \\ F(a+b) + Fc & \xrightarrow[F_2^{a+b,c}]{} & F(a+b+c) \end{array}$$

(and suitable $F_0\colon 0 \to F0$ is uniquely determined by F and F_2).

3. 2-cells $\lambda\colon F \Rightarrow G$ in *MON* are monoidal natural transformations, that is, natural transformations such that the following diagram

commutes

$$\begin{array}{ccc} Fa + Fb & \xrightarrow{F_2^{a,b}} & F(a+b) \\ {\scriptstyle \lambda_a+\lambda_b}\downarrow & & \downarrow{\scriptstyle \lambda_{a+b}} \\ Ga + Gb & \xrightarrow[G_2^{a,b}]{} & G(a+b) \end{array}$$

Remark 6.2 1. The 2-category *Grpd*($\mathcal{C}$) embeds into the 2-category *MON*: internal functors $F\colon \mathbb{A} \to \mathbb{B}$ are precisely those monoidal functors for which all the $F_2^{a,b}$ are identites. Indeed, in this case the naturality of F_2 corresponds to the fact that $F_1\colon A_1 \to B_1$ is a group homomorphism, and the cocycle condition is verified because $e\colon B_0 \to B_1$ is a group homomorphism.

2. The embedding $\mathcal{F}\colon$ *Grpd*($\mathcal{C}$) $\to$ *MON* is full and faithful on 2-cells. Indeed, if $F_2^{a,b} = id = G_2^{a,b}$, then the fact that λ is monoidal corresponds to the fact that $\lambda\colon A_0 \to B_1$ is a group homomorphism.

3. The embedding $\mathcal{F}\colon$ *Grpd*($\mathcal{C}$) $\to$ *MON* preserves weak equivalences. In fact, the forgetful functor *Grp* $\to$ *Set* preserves and reflects finite limits and regular epimorphisms (this is because *Grp* is an algebraic category, see Ch. 3 in [6]), so that weak equivalences in *Grpd*($\mathcal{C}$) and in *MON* are 1-cells which are full, faithful and essentially surjective in the usual sense.

4. In *MON* weak equivalences coincide with equivalences. Indeed, if $F\colon \mathbb{A} \to \mathbb{B}$ is a weak equivalence, any quasi-inverse $G\colon \mathbb{B} \to \mathbb{A}$ can be equipped with a monoidal structure as follows: choose, for each $x \in B_0$, an arrow $\beta_x\colon F(Gx) \to x$ so to have a natural transformation $\beta\colon G \cdot F \Rightarrow Id$. Then define

$$G_2^{x,y}\colon Gx + Gy \to G(x+y)$$

to be the unique arrow making the following diagram commutative

$$\begin{array}{ccc} F(Gx+Gy) & \xrightarrow{F(G_2^{x,y})} & F(G(x+y)) \\ {\scriptstyle F_2^{Gx,Gy}}\uparrow & & \downarrow{\scriptstyle \beta_{x+y}} \\ F(Gx)+F(Gy) & \xrightarrow[\beta_x+\beta_y]{} & x+y \end{array}$$

It is straightforward to check naturality and cocycle condition for G_2 and that β is monoidal. Moreover, we get a monoidal natural transformation $\alpha\colon F \cdot G \Rightarrow Id$ via the equation $F(\alpha_a) = \beta_{Fa}$.

5. The above construction of G_2 makes clear that even if F is a weak equivalence in *Grpd*($\mathcal{C}$) in general G is in *MON* but not in *Grpd*($\mathcal{C}$).

Lemma 6.3 *The 2-category MON has bipullbacks. Moreover, given 1-cells* $F\colon \mathbb{A} \to \mathbb{B}$ *and* $G\colon \mathbb{C} \to \mathbb{B}$, *it is possible to choose a bipullback of* F *and* G

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\ {\scriptstyle G'}\downarrow & \nearrow\!\!\!\nearrow {\scriptstyle \varphi} & \downarrow{\scriptstyle G} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

in such a way that F' *and* G' *are internal functors in Grp.*

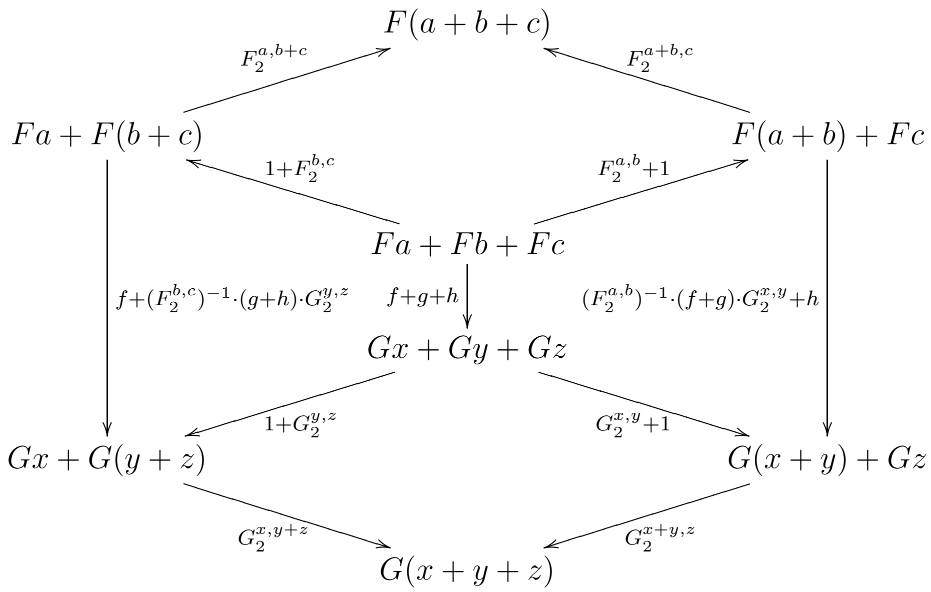
Proof. The construction of the pullback $\mathbb{P}$ is as in 3.3. The interesting point is that, even if F and G are monoidal (not necessarily internal) functors, $\mathbb{P}$ is an internal groupoid in *Grp* and not just a monoidal category. Indeed, if

$$(a, f\colon Fa \to Gx, x) \ \text{ and } \ (b, g\colon Fb \to Gy, y)$$

are objects in $\mathbb{P}$, their tensor product $(a, f\colon Fa \to Gx, x) + (b, g\colon Fb \to Gy, y)$ is given by

$$(a+b,\ F(a+b) \xrightarrow{(F_2^{a,b})^{-1}} Fa + Fb \xrightarrow{f+g} Gx + Gy \xrightarrow{G_2^{x,y}} G(x+y)\ , x+y)$$

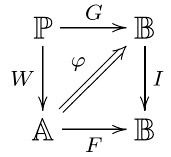
If $(c, h\colon Fc \to Gz, z)$ is a third object in $\mathbb{P}$, to check that the above tensor product is strictly associative easily reduces to the commutativity of the following diagram



that is, to the cocycle condition on F_2 and G_2.
The fact that F' and G' are internal functors is obvious. ■

Proposition 6.4 *The embedding $\mathcal{F}\colon$ Grpd($\mathcal{C}$) $\to$ MON is the bicategory of fractions of Grpd($\mathcal{C}$) with respect to the class of weak equivalences.*

Proof. Let Σ be the class of weak equivalences in *Grpd*($\mathcal{C}$). From Proposition 4.5 we know that Σ has a right calculus of fractions. Moreover, by 6.2.3 and 6.2.4, $\mathcal{F}(W)$ is an equivalence for every $W \in \Sigma$. It remains to check conditions EF1–EF3 in Proposition 2.5: EF1 is obvious and EF2 is precisely 6.2.2. As far as EF3 is concerned, consider a 1-cell $F\colon \mathbb{A} \to \mathbb{B}$ in *MON* and perform the bipullback of F along the identity 1-cell I as in Lemma 6.3



so that both W and G are internal functors. Since equivalences are stable under bipullbacks, W is an equivalence in *MON* and therefore it is a weak equivalence in *Grpd*($\mathcal{C}$). Finally, $\varphi\colon \mathcal{F}(W)\cdot F \Rightarrow \mathcal{F}(G)$ is the 2-cell needed in EF3. Following Proposition 2.5, $\mathcal{F}\colon$ *Grpd*($\mathcal{C}$) $\to$ *MON* is the bicategory of fractions with respect to Σ. ■

Remark 6.5 Observe that we cannot expect to describe a class larger than the class of monoidal functors as fractions of internal functors with respect to weak equivalences. Indeed, the existence of a 2-cell $\mathcal{F}(W)\cdot F \Rightarrow \mathcal{F}(G)$ as in condition EF3 implies that F is monoidal.

7. Homomorphisms of strict Lie 2-algebras

In this section the base category $\mathcal{C}$ is the category *Lie* of Lie algebras over a fixed field K, which is a regular and protomodular category. The situation is completely analogous to the situation described in Section 6 for groups. The reason is that the forgetful functors *Lie* $\to$ *Vect* (where *Vect* is the category of vector spaces over K) and *Vect* $\to$ *Set* preserve and reflect finite limits and regular epimorphisms (because *Lie* and *Vect* are algebraic categories) and moreover in *Vect* the axiom of choice holds (because every vector space is free and therefore regular projective).

7.1 The aim of this section is to prove that the 2-category *LIE* described hereunder is the bicategory of fractions of *Grpd*($\mathcal{C}$) with respect to weak equivalences.

1. Objects of *LIE* are internal groupoids in *Lie*, also called strict Lie 2-algebras in [1].

2. 1-cells $F\colon \mathbb{A} \to \mathbb{B}$ in *LIE* are internal functors in *Vect* equipped with a family of arrows in $\mathbb{B}$

$$F_2 = \{F_2^{a,b}\colon [Fa, Fb] \to F[a,b]\}_{a,b\in A_0}$$

which is natural, bilinear, antisymmetric, and satisfies the following Jacobi condition

$$\begin{array}{ccc}
[Fa,[Fb,Fc]] & = & [[Fa,Fb],Fc]+[Fb,[Fa,Fc]] \\
\downarrow {\scriptstyle [1,F_2^{b,c}]} & & \downarrow {\scriptstyle [F_2^{a,b},1]+[1,F_2^{a,c}]} \\
[Fa,F[b,c]] & & [F[a,b],Fc]+[Fb,F[a,c]] \\
\downarrow {\scriptstyle F_2^{a,[b,c]}} & & \downarrow {\scriptstyle F_2^{[a,b],c}+F_2^{b,[a,c]}} \\
F[a,[b,c]] & = & F[[a,b],c]+F[b,[a,c]]
\end{array}$$

These 1-cells are simply called homomorphisms in [1], where in fact they are defined for more general semi-strict Lie 2-algebras.

3. 2-cells $\lambda\colon F \Rightarrow G$ in *LIE* are internal natural transformations in *Vect* such that the following diagram commutes

$$\begin{array}{ccc}
[Fa,Fb] & \xrightarrow{F_2^{a,b}} & F[a,b] \\
\downarrow {\scriptstyle [\lambda_a,\lambda_b]} & & \downarrow {\scriptstyle \lambda_{[a,b]}} \\
[Ga,Gb] & \xrightarrow[G_2^{a,b}]{} & G[a,b]
\end{array}$$

Remark 7.2 1. The 2-category *Grpd*($\mathcal{C}$) embeds into the 2-category *LIE*: internal functors $F\colon \mathbb{A} \to \mathbb{B}$ are precisely those homomorphisms for which all the $F_2^{a,b}$ are identites. The embedding $\mathcal{F}\colon$ *Grpd*($\mathcal{C}$) $\to$ *LIE* is full and faithful on 2-cells, and preserves weak equivalences.

2. In *LIE* weak equivalences coincide with equivalences. Indeed, let $F\colon \mathbb{A} \to \mathbb{B}$ be a weak equivalence in *LIE*. Then F is also a weak equivalence in the 2-category of internal groupoids and internal functors in *Vect*. Since in *Vect* the axiom of choice holds, F has a quasi-inverse $G\colon \mathbb{B} \to \mathbb{A}$ which is an internal functor in *Vect* (see Corollary 4.3). Now G can be equipped with a structure of homomorphism as follows: consider the internal (in *Vect*) natural transformation $\beta\colon G \cdot F \Rightarrow Id$ and define

$$G_2^{x,y}\colon [Gx,Gy] \to G[x,y]$$

to be the unique arrow making the following diagram commutative

$$\begin{array}{ccc} F[Gx,Gy] & \xrightarrow{F(G_2^{x,y})} & F(G[x,y]) \\ {\scriptstyle F_2^{Gx,Gy}}\uparrow & & \downarrow{\scriptstyle \beta_{[x,y]}} \\ [F(Gx),F(Gy)] & \xrightarrow[{[\beta_x,\beta_y]}]{} & [x,y] \end{array}$$

Lemma 7.3 *The 2-category LIE has bipullbacks. Moreover, given 1-cells $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{C} \to \mathbb{B}$, it is possible to choose a bipullback of F and G*

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\ {\scriptstyle G'}\downarrow & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle G} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

in such a way that F' and G' are internal functors in Lie.

Proof. Once again the point is that, even if F and G are homomorphisms, the bipullback $\mathbb{P}$ constructed as in 3.3 is an internal groupoid in *Lie* and not just a semi-strict Lie 2-algebra. Indeed, the Lie operation in $\mathbb{P}$ is defined by

$$([a,b],\ F[a,b] \xrightarrow{(F_2^{a,b})^{-1}} [Fa,Fb] \xrightarrow{[f,g]} [Gx,Gy] \xrightarrow{G_2^{x,y}} G[x,y]\ , [x,y])$$

and the Jacobi identity is strictly verified thanks to the Jacobi condition on F_2 and G_2. ■

Proposition 7.4 *The embedding $\mathcal{F}\colon$ Grpd($\mathcal{C}$) $\to$ LIE is the bicategory of fractions of Grpd($\mathcal{C}$) with respect to the class of weak equivalences.*

Proof. The proof is analogous to that of Proposition 6.4 and we omit details. ■

References

[1] J.C. Baez and A.S. Crans, Higher-dimensional algebra VI: Lie 2-algebras, *Theory and Applications of Categories* **12** (2004) 492–538.

[2] M. Barr, Exact categories, *Springer LNM* **236** (1971) 1–120.

[3] J. Bénabou, Introduction to bicategories, *Springer LNM* **40** (1967) 1–77.

[4] J. Bénabou, Some remarks on 2-categorical agebra, *Bulletin de la Société Mathématique de Belgique* **41** (1989) 127–194.

[5] F. Borceux, Handbook of Categorical Algebra 1, *Cambridge University Press* (1994).

[6] F. Borceux, Handbook of Categorical Algebra 2, *Cambridge University Press* (1994).

[7] F. Borceux and D. Bourn, Mal'cev, Protomodular, Homological and Semi-abelian Categories, *Kluwer Academic Publishers* (2004).

[8] D. Bourn, Normalization equivalence, kernel equivalence and affine categories, *Springer LNM* **1488** (1991) 43–62.

[9] D. Bourn and M. Gran, Regular, protomodular and abelian categories. In: Categorical Foundations, M.C. Pedicchio and W. Tholen Editors, *Cambridge University Press* (2004) 165–211.

[10] M. Bunge and R. Paré, Stacks and equivalence of indexed categories, *Cahiers de Topologie et Géométrie Différentielle Catégorique* **20** (1979) 373–399.

[11] A. Carboni, M.C. Pedicchio and N. Pirovano, Internal graphs and internal groupoids in Mal'cev categories, *CMS Conference Proceedings* **13** (1992) 97–109.

[12] M. Dupont, Abelian categories in dimension 2, arXiv: 0809.1760 (2008).

[13] T. Everaert, R.W. Kieboom and T. Van der Linden, Model structures for homotopy of internal categories, *Theory and Applications of Categories* **15** (2005) 66–94.

[14] P. Gabriel and M. Zisman, Calculus of Fractions and Homotopy Theory, *Springer* (1967).

[15] B. Noohi, Notes on 2-groupoids, 2-groups and crossed modules, *Homology, Homotopy and Applications* **9** (2007) 75–106.

[16] D. Pronk, Etendues and stacks as bicategories of fractions, *Compositio Mathematica* **102** (1996) 243–303.

Institut de Recherche en Mathématique et Physique
Université catholique de Louvain
Chemin du Cyclotron 2
B 1348 Louvain-la-Neuve, Belgique
Email: enrico.vitale@uclouvain.be

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES *Vol. LI-2 (2010)*

IDEAL DETERMINED CATEGORIES

by G. JANELIDZE, L. MARKI**, W. THOLEN° and A. URSINI*

Dedicated to Francis Borceux at the occasion of his sixtieth birthday

Résumé. Nous clarifions le rôle de l'axiome de Hofmann dans la définition "à l'ancienne mode" de catégorie semi-abélienne. En enlevant cet axiome nous obtenons la contrepartie catégorique de la notion de variété avec détermination des idéaux ("ideal determined") d'algèbres universelles – que nous appelons alors catégorie idéal déterminée. En utilisant des contre-exemples provenant de l'algèbre universelle nous pouvons conclure qu'il y a des catégories idéal déterminées qui ne sont pas des catégories de Mal'tsev. Nous montrons aussi qu'il existe des catégories de Mal'tsev idéal déterminées qui ne sont pas semi-abéliennes.

Abstract. We clarify the role of Hofmann's Axiom in the old-style definition of a semi-abelian category. By removing this axiom we obtain the categorical counterpart of the notion of an ideal determined variety of universal algebras – which we therefore call an ideal determined category. Using known counter-examples from

* Partially supported by the South African NRF.
**Research partially supported by the Hungarian National Foundation for Scientific Research grant K61007 and NK72523.
° Research partially supported by the Natural Sciences and Engineering Research Council of Canada

universal algebra we conclude that there are ideal determined categories which fail to be Mal'tsev. we also show that there are ideal determined Mal'tsev categories which fail to be semi-abelian.

Keywords: semi-abelian category, ideal determined category, normal subobject, ideal
MSC: 18A32, 08A30, 08C05, 18C99

1. Introduction

In modern terms, a pointed category C with finite limits and finite colimits is semi-abelian if it is Barr exact and Bourn protomodular. As shown in [JMT], these two conditions may be equivalently replaced by the following older-style axioms:

(A) Every morphism admits a pullback stable (normal epi, mono)-factorization (where "normal epimorphism" means "cokernel of some morphism").

(B) For every commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{q} & C \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v} \\ E & \xrightarrow[p]{} & B \end{array} \qquad (1.1)$$

with normal epimorphisms *p*, *q* and monomorphisms *v*, *w*, one has

(B1) if *w* is normal, then so is *v*;

(B2) (*"Hofmann's Axiom"*) if v is normal and ker(*p*) ≤ *w* as subobjects of *E*, then *w* is also normal.

While the equivalence proof for the new-versus-old-style definitions given in [JMT] went a long way towards Mac Lane's [M] original quest for an appropriate categorical setting that would allow for a generalization of various classical group-theoretic constructions and results (see in particular [BB]), the following rather obvious question remained unanswered:

Question 1.1. Is Hofmann's Axiom redundant in the list of old-style axioms (i.e., does (B2) follow from (A), (B1) for pointed finitely complete and finitely cocomplete categories)?

This question draws particular relevance from the fact that some authors worked in settings that do not include Hofmann's Axiom, especially those working in Kurosh-Amitsur radical theory.

By exploiting known results and counterexamples from universal algebra, in this paper we provide the expected negative answer to Question 1.1. In fact, we will show that a pointed finitely complete and finitely cocomplete Barr exact category satisfying conditions (A), (B1)

- may fail to be Mal'tsev (which is a necessary condition for protomodularity in this context) and
- may fail to be protomodular even when it is Mal'tsev.

The pivotal step for this exploitation is the surprising realization that pointed varieties of universal algebras satisfying (A), (B1) were studied already in the 1970s and 80s under different names: they were called *BIT* (*"buona teoria degli ideali"*) in [U1] and *ideal determined* in [GU], and this fact leads us not just to a single counterexample but to an interesting class of them. We use the latter term to introduce the categorical notion given in the title of this paper and use results from [JMU1] and [JMU2] to demonstrate its relevance beyond the resolution of Question 1.1. We conclude the paper with some open questions that should form the basis for future work in this context, work that should also clarify more comprehensively the status of the notion of ideal determined category vis-a-vis Z. Janelidze's subtractive categories [J1].

2. Ideal determined categories

Let us recall some notions from universal algebra used in this paper:

Definition 2.1. A pointed variety **C** of universal algebras is said to be *BIT* in the sense of [U1], or, equivalently, *ideal determined* in the sense of [GU], if its congruences are determined by its ideals, i.e., if the following two conditions hold:

(a) every congruence on any algebra in **C** is generated by its 0-class (i.e., no smaller congruence has the same 0-class);

(b) every ideal in every algebra in **C** is normal, i.e., it is the 0-class of a congruence.

Varieties of universal algebras satisfying condition 2.1(a) are called 0-regular. On the other hand, as mentioned in [JMU1], in the language of categorical algebra condition 2.1(a) simply says that every regular epimorphism in **C** is normal. Since every variety of universal algebras admits a pullback stable (regular epi, mono)-factorization, condition 2.1(a) is nothing but the algebraic version of condition (A) of the Introduction.

As shown in [JMU2], a subalgebra S of an algebra A in a pointed variety **C** is an ideal if, and only if, there exist a surjective homomorphism $f : A' \to A$ in **C** and a normal subalgebra N in A' for which $f(N) = A'$. Therefore, under condition (A), condition 2.1(b) is nothing but the algebraic version of condition (B1) of the Introduction.

Accordingly we introduce:

Definition 2.2. A pointed finitely complete and finitely cocomplete category **C** is said to be *ideal determined* if it satisfies conditions (A) and ((B1).

We obtain immediately:

Proposition 2.3. A pointed variety of universal algebras is ideal determined as a category if and only if it is ideal determined in the sense of universal algebra. □

Furthermore, the universal-algebraic motivation for “ideal determined” can be reformulated categorically as follows:

According to [JMU2], a monomorphism $v : C \to B$ in a pointed category **C** with finite limits and colimits satisfying condition (A) should be called an *ideal* if there exists a commutative diagram of the form (1.1) in **C**, in which p and q are normal epimorphisms and w is a normal monomorphism. Hence, with this terminology **C** is ideal determined if, and only if, its ideals are normal monomorphisms. On the other hand, condition (A) simply says that **C** is a regular category in which every regular epimorphism is normal. Hence, in terms of the correspondence between normal monomorphisms and normal epimorphisms we may briefly say that *ideal determined categories are regular categories in which regular epimorphisms are determined by ideals.*

Let us now recall when a variety of universal algebras is semi-abelian, combining past work from both category theory and universal algebra. The universal-algebraic side of the story was discovered in [JMU1] (see Theorems 1.3 and 1.4 in [JMU1]), with another crucial remark made in [JMU2]. The equivalence (a)⇔(c) in the following theorem follows also from the main result of [BJ], while (b)⇔(c) had been proved originally in [Be]:

Theorem 2.4. The following conditions on a pointed variety **C** of universal algebras are equivalent:

(a) **C** is a semi-abelian category;

(b) **C** satisfies the Split Short Five Lemma (see [JMT]);

(c) **C** is 0-coherent in the sense of E. Beutler [Be], i.e., for every A in **C**, every subalgebra A' in A, and every congruence R on A, one has:

$\{a \in A \mid (0,a) \in R\} \subseteq A'$ implies $\{a \in A \mid (a',a) \in R\} \subseteq A'$ for all a' in A.

(d) **C** is *BIT speciale* in the sense of [U2] (=*classically ideal determined* in the sense of [U3]), i.e., there are binary terms t_1, ..., t_n, and an $(n+1)$-ary term t satisfying the identities $t(x,t_1(x,y),...,t_n(x,y)) = y$ and $t_i(x,x) = 0$ for each $i = 1, ..., n$. □

This theorem shows that various semi-abelian categorical constructions are closely related to the universal-algebraic theory of Magari ideals as developed by Ursini and his collaborators in [U1], [U2], and by the authors of various subsequent papers.

Remark 2.5. Already from [U2] it is well known that not every ideal determined (=*BIT*) variety of universal algebras is classically ideal determined (=*BIT speciale*). Hence, not every ideal determined category is semi-abelian, but is it always a Mal'tsev category? The negative answer is again provided by universal algebra. The first of a string of counter-examples was provided in [GU] ("implication algebras"), which eventually led to the proof of the following much stronger result by G. D. Barbour and J.G. Raftery [BR]: For every natural number $n \geq 2$ there is a pointed ideal determined variety of universal algebras which has (n+1)-permutable congruences but not n-permutable congruences.

3. Not every ideal determined Mal'tsev category/variety is semi-abelian

Throughout this section **C** denotes a pointed variety of universal algebras. We shall write M(**C**) for the (pointed) variety obtained from **C** by adding a ternary operation p satisfying the Mal'tsev identities:

$$p(x,y,y) = x = p(y,y,x). \qquad (3.1)$$

Given a morphism $\alpha : A \to B$ in **C** with B in M(**C**), we can always make A an object in M(**C**), by choosing any map (not necessarily a homomorphism) $\beta : \alpha(A) \to A$ with
$\beta(0) = 0$ and $\alpha\beta(b) = b$ for each $b \in \alpha(A)$, and then by defining p on A by

$$p(x,y,z) = \begin{cases} x \text{ if } y = z, \\ z \text{ if } x = y, \\ \beta(p(\alpha(x),\alpha(y),\alpha(z))) \text{ if } x \neq y \neq z; \end{cases}$$

we will denote that object by $A[\alpha,\beta]$. The morphism α determines a morphism
$A[\alpha,\beta] \to B$ in M(**C**), and if β is a morphism in **C,** it actually determines a morphism $B \to A[\alpha,\beta]$ in M(**C**).

Now, consider the diagram

$$\begin{array}{ccccc} K & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B \\ \| & & {\scriptstyle\iota}\downarrow & & \| \\ K & \xrightarrow[\kappa]{} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array} \qquad (3.2)$$

in **C** constructed as follows:

- α and β are arbitrary morphisms in **C** with $\alpha\beta = 1_B$;
- $K = \alpha^{-1}(0)$ is the kernel of α, and A' is a subalgebra in A containing K and $\beta(B)$;
- ι, κ, and κ' are the inclusion maps, and α' and β' the induced maps determined by $\alpha' = \alpha\iota$ and $\iota\beta' = \beta$, respectively.

There are many ways of making B an object in M(**C**); let us put

$$p(x,y,z) = \begin{cases} x \text{ if } y = z, \\ z \text{ if } x = y, \\ 0 \text{ if } x \neq y \neq z \end{cases}$$

in B and denote this object by B_0. After that we can form $A[\alpha,\beta]$, and, since A' contains $\beta(B)$, it determines a subalgebra in $A[\alpha,\beta]$; moreover, that subalgebra is nothing but $A'[\alpha',\beta']$, and the diagram (3.2) determines a similar diagram in M(**C**), namely,

$$\begin{array}{ccccc} K_0 & \xrightarrow{\kappa'} & A'[\alpha',\beta'] & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B_0 \\ \Vert & & \downarrow \iota & & \Vert \\ K_0 & \xrightarrow[\kappa]{} & A[\alpha,\beta] & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B_0, \end{array} \qquad (3.3)$$

where K_0 is constructed similarly to B_0. This proves:

Theorem 3.1. M(**C**) is semi-abelian if and only if so is **C**. □

In particular, since we know that not every (pointed) ideal determined variety is semi-abelian, we immediately conclude that not every ideal determined Mal'tsev variety is semi-abelian. Therefore there exist Barr exact, Mal'tsev and ideal determined categories that are not semi-abelian.

Remark 3.2. (a) The arguments used for obtaining (3.3) from (3.2) and then deducing Theorem 3.1, apply obviously not just for (3.1) but also for similar conditions involving equalities of "one-step" terms.

(b) One could use similar arguments with β in (3.2) being merely a map with
$\beta(0) = 0$ and $\alpha\beta(b) = b$ for every $b \in \alpha(A)$, not a homomorphism, as we originally required in the construction of $A[\alpha,\beta]$.

4. Four open questions

Question 4.1. Is every ideal determined category Barr exact?

An obvious candidate for a counter-example would be a quasi-variety of universal algebras that generates a familiar ideal determined variety. However, this does not work: in fact, it is easy to show that if a quasi-variety generates an ideal determined variety then it is a variety. This indicates that one should begin by studying exact completions of ideal determined categories.

Question 4.2. Is a pointed finitely complete and finitely cocomplete Barr exact category ideal determined if, and only if, it satisfies condition (A) and is subtractive in the sense of Z. Janelidze [J1]?

Since the subtractive categories that are varieties are the same as subtractive varieties, in the "varietal" case the affirmative answer is well known [GU], and this is in fact the reason why we are interested in this question. Barr exactness is essential here, and the reason for that is clear not only by the comment to Question 4.3 given below, but also by the fact that, say, the category of torsion-free abelian groups is subtractive and satisfies condition (A) but fails to be ideal determined.

Question 4.3. Is a category **C** abelian whenever both **C** and $\mathbf{C}^{op}$ are ideal determined?

This question is closely related to the previous ones since, as shown by Z. Janelidze [J2], under mild additional conditions (which are much weaker than the conjunction of (A) and our standard assumption of being pointed and finitely complete and finitely cocomplete), **C** is additive whenever both **C** and $\mathbf{C}^{op}$ are subtractive. And together with Barr exactness additivity would imply abelianness.

Our fourth question is rather vague:

Question 4.4. What is the role of finite cocompleteness in this work?

Finite cocompleteness is not very often used, but it holds in all algebraic examples. It is not clear to us to what extent it would be interesting to study the classes of categories considered above without this assumption.

References

[BR] G. D. Barbour and J.G. Raftery, Ideal determined varieties have unbounded degrees
of permutability, Quaest. Math. 20, 1997, 563-568

[Be] E. Beutler, An idealtheoretic characterization of varieties of abelian Ω-groups, Algebra Universalis 8, 1978, 91-100

[BB] F. Borceux and D. Bourn, Mal'cev, Protomodular, Homological and Semi-Abelian Categories, Mathematics and Its Applications, Kluwer, 2004

[B] D. Bourn, Normalization equivalence, kernel equivalence and affine categories. In: Category Theory (Proc. Conf. Como, 1990), pp. 43-62. Lecture Notes in Mathematics, 1488, Springer, 1991

[BJ] D. Bourn and G. Janelidze, Characterization of protomodular varieties of universal algebras, Theory Appl. Categ. 11, 2003, 143-147

[GU] H. P. Gumm and A. Ursini, Ideals in universal algebras, Algebra Universalis 19, 1984, 45-54

[JMT] G. Janelidze, L. Márki, and W. Tholen, Semi-abelian categories, J. Pure Appl. Algebra 168 (2002), 367-378

[JMU1] G. Janelidze, L. Márki, and A. Ursini, Ideals and clots in universal algebra and in semi-abelian categories, J. Algebra 307 (2007), 191-208

[JMU2] G. Janelidze, L. Márki, and A. Ursini, Ideals and clots in pointed regular categories, Appl. Categ. Struct. 17 (2009), 345-350

[J1] Z. Janelidze, Subtractive categories, Appl. Categ. Struct. 13, 2005, 343-350

[J2] Z. Janelidze, Closedness properties of internal relations IV: Expressing additivity of a category via subtractivity, J. Homotopy Rel. Struct. 1, 2006, 219-227

[M] S. Mac Lane, Duality for groups, Bull. Amer. Math. Soc. 56, 1950, 485-516

[U1] A. Ursini, Sulle varietà di algebre con una buona teoria degli ideali, Boll. Unione Mat. Ital. (4) 7, 1972, 90-95

[U2] A. Ursini, Osservazioni sulle varietà BIT, Boll. Unione Mat. Ital. (4) 7, 1973, 205-211

[U3] A. Ursini, On subtractive varieties I, Algebra Universalis 31, 1994, 204-222

G. JANELIDZE: Department of Mathematics and Applied Mathematics,
University of Cape Town, Rondebosch 7701, South Africa

L. MÁRKI: A. Rényi Institute of Mathematics, Hungarian Academy of Sciences,
H-1364 Budapest, Pf. 127, Hungary

W. THOLEN: Department of Mathematics and Statistics, York University,
4700 Keele St., Toronto, Ontario M3J 1P3, Canada

A. URSINI: Dipartimento di Scienze Matematiche e Informatiche "Roberto Magari",
Università di Siena, Pian dei Mantellini 44, 53100 Siena, Italy

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES Vol. LI-2 (2010)

ON REGULAR AND HOMOLOGICAL CLOSURE OPERATORS

by Maria Manuel CLEMENTINO and Gonçalo GUTIERRES

Dedicated to Francis Borceux on the occasion of his sixtieth birthday

Résumé

En ayant remarqué que la propriété d'hérédité faible des opérateurs réguliers de fermeture dans Top et des opérateurs de fermeture homologiques dans les catégories homologiques permet d'identifier les théories de torsion, nous étudions ces opérateurs de fermeture en parallèle, en montrant que les opérateurs réguliers de fermeture jouent en topologie le même rôle que les opérateurs de fermeture homologiques jouent en algèbre.

Abstract

Observing that weak heredity of regular closure operators in Top and of homological closure operators in homological categories identifies torsion theories, we study these closure operators in parallel, showing that regular closure operators play the same role in topology as homological closure operators do algebraically.

Mathematics Subject Classification: 18E10, 18E40, 54B30, 18B30, 18A32.

Key words: regular closure operator, homological closure operator, maximal closure operator, torsion theory.

The authors acknowledge partial financial support from the Center of Mathematics of the University of Coimbra/FCT.

Introduction

Homological categories were introduced by Borceux and Bourn [2], and have since then been studied by several authors, as the right non-abelian setting to study homology. As shown by Bourn and Gran [6], these categories provide also a suitable setting to study torsion theories. In [6] the authors introduce torsion theories in homological categories and show that they are identifiable by weak heredity of their homological closure operators. This result resembles the characterization of disconnectednesses of topological spaces via weak heredity of their regular closure operators, and encompasses the characterization of torsion-free subcategories of abelian categories via weak heredity of their regular closure operators obtained in [7] (see also [12]). Having as starting point this common property, we establish parallel properties of regular and homological closure operators, in topological spaces and in homological categories, respectively. Since in abelian categories regular closure operators are exactly the homological ones, this study raises the question of finding in which cases these closure operators coincide in homological categories. We show that it is necessary that they are induced by a subcategory of abelian objects. Moreover, in semi-abelian categories regular and homological closures coincide exactly when they are induced by a regular-epireflective subcategory of abelian objects.

In Section 1 we describe briefly disconnectednesses of topological spaces and torsion theories in homological categories. In Section 2 we introduce regular and homological closure operators, showing that the latter ones can be described as maximal closure operators. In Section 3 we establish parallel results for regular and homological closures, based on the results obtained in [6]. In Theorem 3.1.4 we show the validity of the corresponding topological version of the characterization of hereditary torsion theories via hereditary homological closure operators. Next we investigate openness and closedness of regular epimorphisms, with respect to the regular closure, showing that these properties are unlikely topological; see Propositions 3.2.2 and 3.3.2. Finally, in Corollary 3.4.2, we characterise the regular-epireflective subcategories of semi-abelian categories for which the regular and the homological closures coincide, generalising the result obtained in [13] for abelian categories.

1 (Dis)connectednesses and Torsion Theories

1.1 (Dis)connectednesses in Topology

Given a subcategory **A** of the category **Top** of topological spaces and continuous maps, we define the full subcategories

$$l\mathbf{A} := \{X \in \mathbf{Top} \mid \text{ if } f : X \to A \text{ and } A \in \mathbf{A}, \text{ then } f \text{ is constant}\},$$

$$r\mathbf{A} := \{X \in \mathbf{Top} \mid \text{ if } f : A \to X \text{ and } A \in \mathbf{A}, \text{ then } f \text{ is constant}\}.$$

A subcategory of the form $l\mathbf{A}$ for some **A** is said to be a *connectedness*, while a subcategory of the form $r\mathbf{A}$ is said to be a *disconnectedness*. Connectednesses and disconnectednesses of **Top** were thoroughly studied by Arhangel'skiĭ and Wiegandt in [1]. We list here some properties of these subcategories we will need throughout.

1.1.1 Proposition

(1) *Every disconnectedness is a regular-epireflective subcategory of* **Top**.

(2) **Top***, the subcategory of T_0-spaces* $\mathbf{Top}_0$*, the subcategory of T_1-spaces* $\mathbf{Top}_1$ *and the subcategory* **Sgl** *consisting of the empty and the singleton spaces are disconnectednesses.*

(3) *Let* **A** *be a disconnectedness. If* **A** *is different from* **Top** *and from* $\mathbf{Top}_0$*, then $A \subseteq \mathbf{Top}_1$. If* **A** *is different from* **Sgl***, then* **A** *contains the subcategory* **TDisc** *of totally disconnected spaces.*

(4) **Sgl***, the subcategory* **Ind** *of indiscrete spaces, and* **Top** *are connectednesses. These are the only connectednesses closed under subspaces.*

1.2 Torsion theories in homological categories

A pointed category **C** is *homological* if it is

(1) *(Barr-)regular*, that is if it is finitely complete and (regular epimorphisms, monomorphisms) is a pullback-stable factorization system in **C**, and

(2) *protomodular*, that is given a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & 1 & \vdots & 2 & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array}$$

where the dotted vertical arrow is a regular epimorphism, if $\boxed{1}$ and the whole rectangle are pullbacks, then $\boxed{2}$ is a pullback as well.

C is said to be *semi-abelian* if it is pointed, exact and protomodular. That is, in addition to (1) and (2) the pointed category **C** also satisfies

(3) every equivalence relation is effective, i.e. a kernel pair relation.

A *torsion theory* in a homological category is a pair $(\mathbf{T}, \mathbf{F})$ of full and replete subcategories of **C** such that:

1. If $T \in \mathbf{T}$ and $F \in \mathbf{F}$, then the only morphism $T \to F$ is the zero morphism.
2. For each $X \in \mathbf{C}$ there is a short exact sequence

$$0 \longrightarrow T \longrightarrow X \xrightarrow{\rho_X} F \longrightarrow 0$$

with $T \in \mathbf{T}$ and $F \in \mathbf{F}$.

If $(\mathbf{T}, \mathbf{F})$ is a torsion theory, the subcategory **T** is called the *torsion subcategory*, and **F** is called the *torsion-free subcategory*. Every torsion-free subcategory **F** is regular-epireflective, with the **F**-reflection of X given by ρ_X as above.

Torsion theories in homological categories were introduced by Bourn and Gran in [6], encompassing the properties of Dickson's torsion theories in abelian categories [10].

The notion of *abelian object* has been studied in non-abelian settings (see [2]). In homological categories they can be defined as those objects which have an internal abelian group structure. As shown by Bourn in [5]:

1.2.1 Proposition

(1) *In a homological category, the following conditions are equivalent for an object X:*

   (i) *X has an internal abelian group structure;*

   (ii) *the diagonal $\delta_X : X \to X \times X$ is an equaliser.*

(2) *In a semi-abelian category, the following conditions are equivalent for an object X:*

   (i) *X has an internal abelian group structure;*

   (ii) *the diagonal $\delta_X : X \to X \times X$ is a kernel.*

2 Regular and homological closure operators

2.1 Closure operators

Throughout $\mathbf{C}$ is a finitely complete category with cokernel pairs and $\mathcal{M}$ is a pullback-stable class of monomorphisms of $\mathbf{C}$. This means that $\mathbf{C}$ *has inverse* $\mathcal{M}$*-images*, that is for each morphism $f : X \to Y$ there is a change-of-base functor

$$f^{-1}(\ ) : \mathcal{M}/Y \to \mathcal{M}/X$$

where $\mathcal{M}/X$ is the (preordered) category of $\mathcal{M}$*-subobjects of* X, that is of morphisms in $\mathcal{M}$ with codomain X. When, for each morphism $f : X \to Y$, the functor $f^{-1}(\ ) : \mathcal{M}/Y \to \mathcal{M}/X$ has a left adjoint $f(\ ) : \mathcal{M}/X \to \mathcal{M}/Y$, we say that $\mathbf{C}$ *has direct* $\mathcal{M}$*-images*.

A *closure operator* c on $\mathbf{C}$ with respect to $\mathcal{M}$ assigns to each $m : M \to X$ in $\mathcal{M}$ a morphism $c_X(m) : c_X(M) \to X$ in $\mathcal{M}$ such that, for every object X,

(C1) c_X is *extensive*: $m \le c_X(m)$ for every $m : M \to X$ in $\mathcal{M}$;

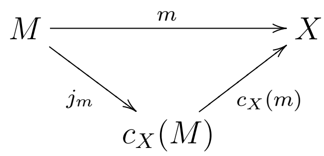
(C2) c_X is *monotone*: $m \le m' \;\Rightarrow\; c_X(m) \le c_X(m')$, for every $m : M \to X$, $m' : M' \to X$ in $\mathcal{M}$;

(C3) morphisms are *c-continuous*: $c_X(f^{-1}(n)) \le f^{-1}(c_Y(n))$ for every morphism $f : X \to Y$ and every $n : N \to Y$ in $\mathcal{M}$.

When $\mathbf{C}$ has direct $\mathcal{M}$-images, condition (C3) can be equivalently expressed by

(C3′) $f(c_X(m)) \leq c_Y(f(m))$, for every $\mathcal{M}$-subobject m of X.

Extensivity of c says that every $m : M \to X \in \mathcal{M}$ factors as



The morphism $m : M \to X$ is c-*closed* if $c_X(m) \cong m$, and c-*dense* if $c_X(m) \cong 1_X$.

A closure operator c is said to be

- *idempotent* if $c_X(m)$ is c-closed for every $m : M \to X \in \mathcal{M}$;
- *weakly hereditary* if j_m is c-dense for every $m \in \mathcal{M}$;
- *hereditary* if, for $m : M \to X$, $l : X \to Y$ and $l \cdot m$ in $\mathcal{M}$,

$$c_X(m) \cong l^{-1}(c_Y(l \cdot m)).$$

It is immediate that hereditary closure operators are in particular weakly hereditary.

Closure operators with respect to $\mathcal{M}$ can be preordered by

$$c \leq d \;:\Leftrightarrow\; \forall m : M \to X \in \mathcal{M}\; c_X(m) \leq d_X(m).$$

2.2 Regular versus homological closure operators

For any such class $\mathcal{M}$ of monomorphisms containing the regular monomorphisms, every reflective subcategory $\mathbf{A}$ of $\mathbf{C}$ induces a *regular closure operator* $\mathrm{reg}^{\mathbf{A}}$ on $\mathbf{C}$ with respect to $\mathcal{M}$, assigning to each $m : M \to X$ in $\mathcal{M}$ the equaliser of the following diagram

$$X \underset{v}{\overset{u}{\rightrightarrows}} Y \xrightarrow{\rho_Y} RY \;,$$

where (u,v) is the cokernel pair of m and ρ_Y is the **A**-reflection of Y; that is,

$$\mathrm{reg}^{\mathbf{A}}_X(m) = \mathrm{eq}(\rho_Y \cdot u, \rho_Y \cdot v).$$

Regular closure operators are idempotent but not weakly hereditary in general.

When the category **C** is pointed, replacing equalisers by kernels in the construction above gives rise to another interesting closure operator. Let $\mathcal{M}$ be a pullback-stable class of monomorphisms containing the kernels, and let **A** be a reflective subcategory of **C**. The *homological closure operator* $\mathrm{h}^{\mathbf{A}}$ *induced by* **A** in $\mathcal{M}$ assigns to each $m : M \to X$ the kernel of the following composition of morphisms

$$X \xrightarrow{\pi_M} Y \xrightarrow{\rho_Y} RY\ ,$$

where π_M is the cokernel of m and ρ_Y is the **A**-reflection of Y; that is,

$$\mathrm{h}^{\mathbf{A}}_X(m) = \ker(\rho_Y \cdot \pi_M).$$

Homological closure operators are idempotent but not weakly hereditary in general.

If **C** has direct $\mathcal{M}$-images, then $\mathrm{reg}^{\mathbf{A}}$ is completely determined by its restriction to **A**, via the formula

$$\mathrm{reg}^{\mathbf{A}}_X(m) \cong \rho_X^{-1}(\mathrm{reg}^{\mathbf{A}}_{RX}(\rho_X(m))), \tag{$\star$}$$

for any $m : M \to X$ in $\mathcal{M}$, with $\rho_X : X \to RX$ the **A**-reflection of X.

There is an alternative way of replacing equalisers by kernels in the definition of regular closure operator. Indeed, $\mathrm{reg}^{\mathbf{A}}$ is the maximal closure such that every equaliser in **A** is closed. In particular:

2.2.1 Lemma *If* **A** *is a reflective subcategory of* **Top** *and* X *is an object of* **A***, then:*

(1) *the diagonal* $\delta_X : X \to X \times X$ *is* $\mathrm{reg}^{\mathbf{A}}$*-closed;*

(2) *For every* $x \in X$*, the inclusion* $\{x\} \to X$ *is* $\mathrm{reg}^{\mathbf{A}}$*-closed.*

In a pointed finitely-complete category $\mathbf{C}$, given a pullback-stable class of monomorphisms $\mathcal{M}$ containing the zero-subobjects and a reflective subcategory $\mathbf{A}$, one calls *maximal closure operator induced by* $\mathbf{A}$, the maximal closure operator $\max^{\mathbf{A}}$ with $0_A : 0 \to A$ closed, for every $A \in \mathbf{A}$ (or, equivalently, with kernels of $\mathbf{A}$-morphisms closed). It is easily verified that:

2.2.2 Proposition *If* $\mathbf{A}$ *is a reflective subcategory of a pointed and finitely-complete category* $\mathbf{C}$ *with cokernels, then* $\mathrm{h}^{\mathbf{A}} = \max^{\mathbf{A}}$.

While regular closure operators were introduced by Salbany [15] more than 30 years ago, and widely studied since then, homological closure operators were introduced more recently by Bourn and Gran [6] in the context of homological categories.

For comprehensive accounts on closure operators and homological categories we refer the reader to [12] and [2, 14] respectively.

3 How close are regular and homological closure operators

3.1 (Weak) heredity

The study of weak heredity of regular closure operators presented in [7] encompasses the following topological and algebraic results.

3.1.1 Theorem

(1) *For a regular-epireflective subcategory* $\mathbf{A}$ *of* **Top***, the following assertions are equivalent:*

  (i) $\mathrm{reg}^{\mathbf{A}}$ *is weakly hereditary;*

  (ii) $\mathbf{A}$ *is a disconnectedness.*

(2) *For a (regular-)epireflective subcategory* $\mathbf{A}$ *of an abelian category* $\mathbf{C}$*, the following conditions are equivalent:*

  (i) $\mathrm{reg}^{\mathbf{A}}$ *is weakly hereditary;*

  (ii) $\mathbf{A}$ *is a torsion-free subcategory.*

Disconnectedness in topological spaces and torsion-free subcategories in abelian categories are particular cases of right-constant subcategories (see [9] for details), hence the two theorems above are instances of a more general result. Moreover, as shown in [13], if **C** is an abelian category, then the regular closure operator induced by an epireflective subcategory **A** coincides with the maximal closure operator induced by **A**. This shows, moreover, that Theorem 3.1.1.2 is a particular case of the following result, due to Bourn and Gran [6].

3.1.2 Theorem *For a regular-epireflective subcategory* **A** *of a homological category* **C***, the following conditions are equivalent:*

(i) $\max^{\mathbf{A}}$ *is weakly hereditary;*

(ii) **A** *is a torsion-free subcategory.*

In [6] Bourn and Gran show also that heredity of $\max^{\mathbf{A}}$ identifies hereditary torsion theories, that is those torsion theories with hereditary torsion part.

3.1.3 Theorem *For a regular-epireflective subcategory* **A** *of a homological category* **C***, the following conditions are equivalent:*

(i) $\max^{\mathbf{A}}$ *is hereditary;*

(ii) **A** *is a hereditary torsion-free subcategory.*

As for weak heredity there is a corresponding result in topology.

3.1.4 Theorem *For a regular-epireflective subcategory* **A** *of* **Top***, the following conditions are equivalent:*

(i) $\mathrm{reg}^{\mathbf{A}}$ *is hereditary;*

(ii) **A** *is an hereditary disconnectedness (that is, its connectedness counterpart* $\mathrm{l}(\mathbf{A})$ *is hereditary);*

(iii) **A** *is either* **Top** *or the category* $\mathbf{Top}_0$ *of* T_0*-spaces or the category* **Sgl** *consisting of singletons and the empty set.*

Proof. First we remark that (ii)⇔(iii) follows from Proposition 1.1.1(4).

(iii)⇔(i): If $\mathbf{A} = \mathbf{Top}$, then $\mathrm{reg}^{\mathbf{A}}$ is the discrete closure, which is trivially hereditary. If $\mathbf{A} = \mathbf{Top}_0$, then $\mathrm{reg}^{\mathbf{A}}$ is the b-closure, with, for $A \subseteq X$,

$$\mathrm{b}_X(A) = \{x \in X \mid \text{for every neighbourhood } U \text{ of } x, \overline{\{x\}} \cap U \cap A \neq \emptyset\},$$

which is known to be hereditary (see for instance [12]). If $\mathbf{A} = \mathbf{Sgl}$, then $\mathrm{reg}^{\mathbf{A}}$ is the indiscrete closure, that is

$$\mathrm{reg}_X^{\mathbf{Sgl}}(A) = X \text{ for every } \emptyset \neq A \subseteq X \text{ and } \mathrm{reg}_X^{\mathbf{Sgl}}(\emptyset) = \emptyset,$$

which is hereditary. Conversely, assume that $\mathbf{A}$ is none of these three subcategories. By Proposition 1.1.1(3), $\mathbf{TDisc} \subseteq \mathbf{A} \subseteq \mathbf{Top}_1$. Consider the Sierpinski space $S = \{0, 1\}$, with $\{0\}$ the only non-trivial open subset, and its product $S \times S$. The two-point discrete space $D = \{(0, 1), (1, 0)\}$ is a subspace of $S \times S$; the $\mathbf{A}$-reflection of $S \times S$ is a singleton, while $D \in \mathbf{A}$. Hence $\mathrm{reg}_D^{\mathbf{A}}(0, 1) = (0, 1)$ while $\mathrm{reg}_{S\times S}^{\mathbf{A}}(0, 1) = S \times S$, and therefore $\mathrm{reg}^{\mathbf{A}}$ is not hereditary.

□

3.2 Openness of regular epimorphisms

Another interesting feature of homological closure operators pointed out by Bourn and Gran [6] is to make regular epimorphisms open. Recall that, given a closure operator c, a morphism $f : X \to Y$ is *c-open* if, for every $n : N \to Y \in \mathcal{M}$,

$$c_X(f^{-1}(n)) \cong f^{-1}(c_Y(n));$$

that is, the inequality in the c-continuity condition (C3) becomes an isomorphism. It was shown in [8] that:

3.2.1 Proposition *For an idempotent closure operator c in a homological category $\mathbf{C}$ the following conditions are equivalent:*

(i) $c = \max^{\mathbf{A}}$ *for some regular-epireflective subcategory* $\mathbf{A}$*;*

(ii) *regular epimorphisms in* $\mathbf{C}$ *are c-open.*

It is easy to check that in general this is not a common property of regular closure operators in **Top**.

3.2.2 Proposition *For a closure operator c in* **Top** *the following conditions are equivalent:*

(i) *c is regular, and every regular epimorphism is c-open;*

(ii) *c is either the discrete or the indiscrete closure operator.*

Proof. (ii)⇒(i) is trivial.

(i)⇒(ii): Let c be a regular closure operator induced by a regular-epireflective subcategory $\mathbf{A}$ different from **Top**. Then either $\mathbf{A} = \mathbf{Top}_0$ or $\mathbf{A} \subseteq \mathbf{Top}_1$. If $\mathbf{A} = \mathbf{Top}_0$, then $\mathrm{reg}^{\mathbf{A}}$ is the b-closure, which does not satisfy (i): take $X = \{0,1,2,3\} \to Y = \{0,1,2\}$ with $f(i) = i$ if $i \leq 2$ and $f(3) = 2$, where the only non-trivial open subset of X is $\{1,2\}$, hence the quotient topology is indiscrete; then $f^{-1}(\mathrm{b}(0)) = X$ and $\mathrm{b}(f^{-1}(0)) = \{0,3\}$. If $A \subseteq \mathbf{Top}_1$, then $\mathrm{reg}^{\mathbf{A}}$ is indiscrete in the Sierpinski space. Hence, for every closed, non-open, subset C of a space Z, since $\chi_C : Z \to S$ is a quotient map, hence $\mathrm{reg}^{\mathbf{A}}$-open, one has $\mathrm{reg}^{\mathbf{A}}_Z(C) = \chi_C^{-1}(\mathrm{reg}^{\mathbf{A}}_S(1)) = Z$. Therefore, if Z is T_1 and non-discrete, it has a non-open point z, and so $\mathrm{reg}^{\mathbf{A}}_Z(z) = Z$, which implies that $Z \notin \mathbf{A}$. This means then that $\mathbf{A}$ has only discrete spaces, and then $\mathbf{A} \subset \mathbf{TDisc}$, which implies $\mathbf{A} = \mathbf{Sgl}$ by Proposition 1.1.1. □

3.3 Closedness of regular epimorphisms

Closed morphisms with respect to a closure operator are defined analogously to open morphisms, replacing inverse images by direct images. When $\mathbf{C}$ has direct $\mathcal{M}$-images, a morphism $f : X \to Y$ is said to be *c-closed* if, for every $m \in \mathcal{M}/X$,

$$f(c_X(m)) \cong c_Y(f(m)).$$

(As said before, the inequality $f(c_X(m)) \leq c_Y(f(m))$ is equivalent to c-continuity of f.)

We recall that an epireflective subcategory is said to be *Birkhoff* if it is closed under regular epimorphisms.

Next we analyse the topological counterpart of the following result.

3.3.1 Proposition [6] *If* $\mathbf{A}$ *is a regular-epireflective subcategory of a semi-abelian category* $\mathbf{C}$*, the following assertions are equivalent:*

(i) *regular epimorphisms are* $\max^{\mathbf{A}}$*-closed;*

(ii) $\mathbf{A}$ *is a Birkhoff subcategory.*

3.3.2 Proposition *For a regular-epireflective subcategory* $\mathbf{A}$ *of* $\mathbf{Top}$ *the following conditions are equivalent:*

(i) *regular epimorphisms are* $\mathrm{reg}^{\mathbf{A}}$*-closed;*

(ii) $\mathbf{A}$ *is a Birkhoff subcategory;*

(iii) $\mathbf{A} = \mathbf{Top}$ *or* $\mathbf{A} = \mathbf{Sgl}$.

Proof. Trivially (iii)⇒(ii). To show that (ii)⇒(iii), first note that $\mathbf{Top}_0$ is not closed under quotients, hence it is not a Birkhoff subcategory. Now, if $\mathbf{A} \subseteq \mathbf{Top}_1$ and $\mathbf{A}$ contains a non-discrete space Z, hence with a closed non-open subset C, then $\chi_C : Z \to S$ is a quotient map although the Sierpinski space S does not belong to $\mathbf{A}$. Hence every object of $\mathbf{A}$ is discrete, which implies that $\mathbf{A} = \mathbf{Sgl}$.

(iii)⇒(i) is clear, since $\mathrm{reg}^{\mathbf{Top}}$ is the discrete closure and $\mathrm{reg}^{\mathbf{Sgl}}$ is the indiscrete closure, both making regular epimorphisms c-closed.

(i)⇒(iii): If $\mathbf{A} = \mathbf{Top}_0$, $\mathrm{reg}^{\mathbf{A}}$ is the b-closure. The quotient map $X \to Y$ used in the proof of Proposition 3.2.2 is not b-closed since

$$f(\mathrm{b}(0)) = f(\{0,3\}) = \{0,2\} \text{ and } \mathrm{b}(f(0)) = \mathrm{b}(0) = \{0,1,2\}.$$

If $\mathbf{A} \subseteq \mathbf{Top}_1$ and C is a closed, non-open, subset of $Z \in \mathbf{A}$, then $\chi_C : Z \to S$ is a quotient map. Moreover, $\mathrm{reg}^{\mathbf{A}}$ is indiscrete in S, since the $\mathbf{A}$-reflection of S is a singleton, and every point in Z is $\mathrm{reg}^{\mathbf{A}}$-closed, because $Z \in \mathbf{A}$ (see Lemma 2.2.1). For any $z \in C$ one has

$$\chi_C(\mathrm{reg}^{\mathbf{A}}_Z(z)) = \chi_C(z) = 1 \neq \mathrm{reg}^{\mathbf{A}}_S(\chi_C(z)) = \mathrm{reg}^{\mathbf{A}}_S(1) = S.$$

Therefore every object of $\mathbf{A}$ is discrete, and so $\mathbf{A} = \mathbf{Sgl}$. □

3.4 When regular and homological closures coincide

Finally, it is natural to ask in which pointed regular categories regular and maximal closure operators coincide. Until the end of this section, we will assume that these closure operators are defined in the class of monomorphisms of $\mathbf{C}$.

3.4.1 Theorem *Let* $\mathbf{A}$ *be a regular-epireflective subcategory of a pointed regular category with cokernels. The following assertions are equivalent:*

(i) *when restricted to* $\mathbf{A}$*,* $\mathrm{reg}^{\mathbf{A}}$ *and* $\max^{\mathbf{A}}$ *coincide;*

(ii) $\mathrm{reg}^{\mathbf{A}} = \max^{\mathbf{A}}$*;*

(iii) *in* $\mathbf{A}$ *every equaliser is a kernel;*

(iv) *for every object* A *of* $\mathbf{A}$*, the diagonal* δ_A *is a kernel in* $\mathbf{A}$*.*

Proof. (i)⇒(ii): On one hand, since the maximal closure $\max^{\mathbf{A}}$ is the largest closure c with $0_A : 0 \to A$ c-closed for any $A \in \mathbf{A}$ and $\mathrm{reg}^{\mathbf{A}}$ and $\max^{\mathbf{A}}$ coincide in $\mathbf{A}$, $\mathrm{reg}^{\mathbf{A}} \leq \max^{\mathbf{A}}$.

On the other hand, denoting by ρ the $\mathbf{A}$-reflection, by $(\star)$ of Section 2 we have that $\mathrm{reg}^{\mathbf{A}}_X(m) \cong \rho_X^{-1}(\mathrm{reg}^{\mathbf{A}}_{RX}(\rho_X(m)))$ is $\max^{\mathbf{A}}$-closed since, by (i), $\mathrm{reg}^{\mathbf{A}}_{RX}(\rho_X(m)) \cong \max^{\mathbf{A}}_{RX}(\rho_X(m))$, hence $\mathrm{reg}^{\mathbf{A}}_X \geq \max^{\mathbf{A}}_X$.

(ii)⇒(iii): Since every equaliser $m : M \to A$ in $\mathbf{A}$ is $\mathrm{reg}^{\mathbf{A}}$-closed, hence $\max^{\mathbf{A}}$-closed by (ii), and the $\max^{\mathbf{A}}$-closure of m in $\mathbf{A}$ is the kernel of

$$A \xrightarrow{\pi_M} Y \xrightarrow{\rho_Y} RY \ \in \mathbf{A},$$

$m \cong \max^{\mathbf{A}}_A(m) \cong \ker(\rho_Y \cdot \pi_M)$ is a kernel in $\mathbf{A}$ as claimed.

(iii)⇒(iv) is obvious, while (iv)⇒(iii) follows from the fact that the equaliser of $f, g : A \to B$ is the pullback of $\delta_B : B \to B \times B$ along $< f, g >: A \to B \times B$.

(iii)⇒(i): A monomorphism in $\mathbf{A}$ is $\mathrm{reg}^{\mathbf{A}}$-closed (resp. $\max^{\mathbf{A}}$-closed) if, and only if, it is an equaliser in $\mathbf{A}$ (resp. a kernel in $\mathbf{A}$). If equalisers are kernels, then, as idempotent closure operators, necessarily $\mathrm{reg}^{\mathbf{A}}$ and $\max^{\mathbf{A}}$ coincide in $\mathbf{A}$. □

If $\mathbf{A}$ is a regular-epireflective subcategory of a homological category, then $\mathbf{A}$ is homological as well (see [4]), and so in $\mathbf{A}$ every coequaliser is a cokernel. In the theorem above the dual property is required for **A** so that its homological and regular closure operators coincide. Indeed this condition leads us again to an abelian-like condition, as we show next.

3.4.2 Corollary

(1) *If* **A** *is a regular-epireflective subcategory of a homological category* **C** *with* $\mathrm{reg}^{\mathbf{A}} = \max^{\mathbf{A}}$*, then* **A** *consists of abelian objects.*

(2) *If* **A** *is a regular-epireflective subcategory of a semi-abelian category* **C***, then the following conditions are equivalent:*

  (i) $\mathrm{reg}^{\mathbf{A}} = \max^{\mathbf{A}}$*;*

  (ii) *every object in* **A** *is abelian.*

Proof. First we remark that both **C** and **A** are homological (semi-abelian resp.), and so the result follows from Proposition 1.2.1 since:

$$X \text{ is abelian } :\Leftrightarrow X \text{ has an internal abelian group structure}$$

If $\mathrm{reg}^{\mathbf{A}} = \max^{\mathbf{A}}$, then $\delta_A : A \to A \times A$ is a kernel, for every $A \in \mathbf{A}$. Hence, A is abelian. Conversely, if A is abelian then it has an internal abelian group structure in **C**, hence also in **A**, and so δ_A must be a kernel in **A** in case **A** is semi-abelian. □

We point out that there are non (semi-)abelian homological categories where every equaliser is a kernel. In fact such categories are necessarily additive but may fail to be exact. (We recall that an exact and additive category is abelian: see [14].) This is the case, for instance, for the category of topological abelian groups, which is regular and protomodular but not exact (see [3] for details.)

Acknowledgments. We thank Marino Gran for valuable comments on the subject of this paper.

References

[1] A. Arhangel'skiĭ and R. Wiegandt, Connectednesses and disconnectednesses in Topology, *Gen. Topology and Appl.* **5** (1975), 9–33.

[2] F. Borceux and D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Mathematics and its Applications, 566. Kluwer Academic Publishers, Dordrecht, 2004.

[3] F. Borceux and M. M. Clementino, Topological semi-abelian algebras, *Adv. Math.* **190** (2005), no. 2, 425–453.

[4] F. Borceux, M. M. Clementino, M. Gran and L. Sousa, Protolocalisations of homological categories, *J. Pure Appl. Algebra* **212** (2008), 1898-1927.

[5] D. Bourn, Normal subobjects and abelian objects in protomodular categories, *J. Algebra* **228** (2000), 143–164.

[6] D. Bourn and M. Gran, Torsion theories and closure operators, *J. Algebra* **305** (2006), 18–47.

[7] M. M. Clementino, Weakly hereditary regular closure operators, *Topology Appl.* **49** (1993), 129–139.

[8] M. M. Clementino, D. Dikranjan and W. Tholen, Torsion theories and radicals in normal categories, *J. Algebra* **305** (2006), 98–129.

[9] M. M. Clementino and W. Tholen, Separation versus connectedness, *Topology Appl.* **75** (1997), no. 2, 143–181.

[10] S. Dickson, A torsion theory for abelian categories, *Trans. Amer. Math. Soc.* **121** (1966), 223–235.

[11] D. Dikranjan, E. Giuli and W. Tholen, *Closure Operators II*, in: Proc. Int. Conf. on Categorical Topology, Prague 1988 (World Scientific, Singapore 1989), pp. 297–335.

[12] D. Dikranjan and W. Tholen, *Categorical structure of closure operators. With applications to topology, algebra and discrete mathematics*, Mathematics and its Applications, **346**, Kluwer, Dordrecht, 1995.

[13] G. Gutierres, *On coregular closure operators*, Preprint 99-21, Department of Mathematics, University of Coimbra (1999), http://www.mat.uc.pt/preprints/1999.html.

[14] G. Janelidze, L. Márki and W. Tholen, Semi-abelian categories, *J. Pure Appl. Algebra* **168** (2002), no. 2–3, 367–386.

[15] S. Salbany, *Reflective subcategories and closure operators*, in: Lecture Notes in Math. (Springer 1976), pp. 548–565.

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Vol. LI-2 (2010)

A NOTE ON DOUBLE CENTRAL EXTENSIONS IN EXACT MAL'TSEV CATEGORIES

by Tomas EVERAERT and Tim VAN DER LINDEN

Dedicated to Francis Borceux on the occasion of his sixtieth birthday

Abstract

La caractérisation des extensions centrales doubles en termes de commutateurs de Janelidze (dans le cas des groupes) et de Gran et Rossi (dans le cas des variétés de Mal'tsev) est montrée d'être toujours valide dans le contexte des catégories exactes de Mal'tsev avec coégalisateurs.

The characterisation of double central extensions in terms of commutators due to Janelidze (in the case of groups) and Gran and Rossi (in the case of Mal'tsev varieties) is shown to be still valid in the context of exact Mal'tsev categories with coequalisers.

In his article [10], George Janelidze gave a characterisation of the double central extensions of groups in terms of commutators. Not only did he thus relate Galois theory to commutator theory, but he also sowed the seeds for a new approach to homological algebra, where higher-dimensional (central) extensions are used as a basic tool—see, for instance, [5, 6, 11, 16].

Expressed in terms of commutators of equivalence relations [15, 17], his result amounts to the following: a double extension

$$\begin{array}{ccc} X & \xrightarrow{c} & C \\ {\scriptstyle d}\downarrow & & \downarrow \\ D & \xrightarrow[f]{} & Z \end{array} \qquad \textbf{(A)}$$

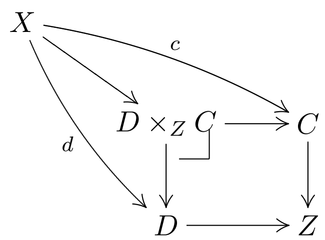
is central if and only if $[R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X]$. Here $R[d]$ and $R[c]$ denote the kernel pairs of d and c, and Δ_X and ∇_X are the smallest and the largest

Research supported by Centro de Matemática da Universidade de Coimbra, by Fundação para a Ciência e a Tecnologia (under grant number SFRH/BPD/38797/2007) and by Fonds voor Wetenschappelijk Onderzoek (FWO-Vlaanderen).

equivalence relation on X. This characterisation was generalised to the context of Mal'tsev varieties by Marino Gran and Valentina Rossi [9]. Although one of the implications (the "only if"-part) of the proof given in [9] is entirely categorical and easily seen to be valid in any Barr exact Mal'tsev category with coequalisers, the other implication is not, and makes heavy use of universal-algebraic machinery. The aim of this note is to provide a proof of the other implication which is valid in any exact Mal'tsev category with coequalisers.

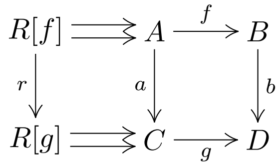
In our proof, we shall not only consider double extensions, but also three-fold and four-fold extensions. Therefore, we begin this note with a few introductory words on higher-dimensional extensions. For an in-depth discussion on this subject in the context of semi-abelian categories we refer the reader to [6] and [5].

Let $\mathcal{A}$ be a regular category, i.e., a finitely complete category with pullback-stable regular epi-mono factorisations. Given $n \geq 0$, denote by $\mathsf{Arr}^n\mathcal{A}$ the category of n-dimensional arrows in $\mathcal{A}$. (A zero-dimensional arrow is an object of $\mathcal{A}$.) n-fold extensions are defined inductively as follows. A **(one-fold) extension** is a regular epimorphism in $\mathcal{A}$. For $n \geq 1$, an $(n+1)$**-fold extension** is a commutative square **A** in $\mathsf{Arr}^{n-1}\mathcal{A}$ (an arrow in $\mathsf{Arr}^n\mathcal{A}$) such that in the induced commutative diagram



every arrow is an n-fold extension. Thus for $n = 2$ we regain the notion of double extension. Note that, since in the regular category $\mathsf{Arr}^{n-1}\mathcal{A}$ a pullback of regular epimorphisms is always a pushout, it follows that an $(n+1)$-fold extension is necessarily a pushout in $\mathsf{Arr}^{n-1}\mathcal{A}$, for any $n \geq 1$.

Suppose from now on that $\mathcal{A}$ is, moreover, Mal'tsev [4, 3], i.e., every (internal) reflexive relation in $\mathcal{A}$ is an equivalence relation. It was shown in [1] that, for a regular category $\mathcal{A}$, the Mal'tsev condition is equivalent to the following property: if, in a commutative diagram



f, g, a and b are extensions, then the right hand square is a double extension if and only if its kernel pair in $\mathsf{Arr}\mathcal{A}$—the morphism r in the diagram—is an extension. Since the concept of double extension is symmetric, this has the following consequences:

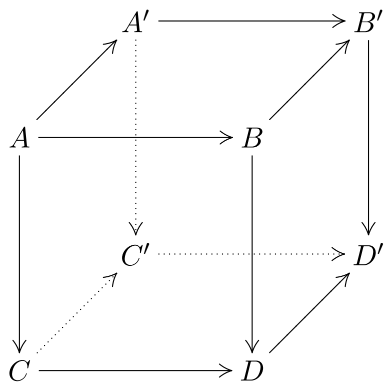
- double extensions are stable under composition;
- if a composite $g \circ f \colon A \to B \to C$ of arrows in $\mathsf{Arr}\mathcal{A}$ is a double extension and B is an extension, then $g \colon B \to C$ is a double extension;
- any split epimorphism of extensions is a double extension.

And then also the following is straightforward to prove:

- the pullback in $\mathsf{Arr}\mathcal{A}$ of a double extension $A \to B$ along a double extension $C \to B$ is a double extension.

In fact, for *any* $n \geq 2$, a commutative square in $\mathsf{Arr}^{n-2}\mathcal{A}$ consisting of $(n-1)$-fold extensions is an n-fold extension if and only if its kernel pair in $\mathsf{Arr}^{n-1}\mathcal{A}$ is an $(n-1)$-fold extension, and thus for all of the above listed properties one obtains higher dimensional versions as well. This is easily shown by induction, if one takes into account that the notion of n-fold extension (for $n \geq 3$) is symmetric in the following sense: any commutative cube in $\mathsf{Arr}^{n-3}\mathcal{A}$ can be considered in three ways as a commutative square in $\mathsf{Arr}^{n-2}\mathcal{A}$; if any of the three squares is an n-fold extension, then the same is true for the other two.

Lemma. *Let $n \geq 3$, and suppose that the following commutative cube in $\mathsf{Arr}^{n-3}\mathcal{A}$ is an n-fold extension.*



If the top square is a pullback, then so is the bottom square.

Proof. Taking pullbacks in the top and bottom squares of the cube, we obtain the comparison square

$$\begin{array}{ccc} A & \longrightarrow & A' \times_{B'} B \\ \downarrow & & \downarrow \\ C & \longrightarrow & C' \times_{D'} D. \end{array}$$

Since the cube is an n-fold extension, this square is an $(n-1)$-fold extension. In particular, it is a pushout in $\mathsf{Arr}^{n-3}\mathcal{A}$, and it follows that the lower comparison morphism is an isomorphism as soon as the upper one is so. $\square$

From now on, we assume that the regular Mal'tsev category $\mathcal{A}$ is, moreover, (Barr) exact (every equivalence relation in $\mathcal{A}$ is effective) and that $\mathcal{A}$ admits coequalisers. This allows us to consider the commutator of equivalence relations defined in [15] (see also [14]), which is a generalisation of Jonathan Smith's definition in the context of Mal'tsev varieties [17]. Following [7] we call an object $A \in \mathcal{A}$ **abelian** if $[\nabla_A, \nabla_A] = \Delta_A$, and we write $\mathsf{Ab}\mathcal{A}$ for the full subcategory of $\mathcal{A}$ determined by all abelian objects. Then $\mathsf{Ab}\mathcal{A}$ is a reflective subcategory of $\mathcal{A}$ and the abelianisation of an object $A \in \mathcal{A}$ is given the quotient $\mathsf{ab}A = A/[\nabla_A, \nabla_A]$. It was shown in [7] that $\mathsf{Ab}\mathcal{A}$ is a **Birkhoff subcategory** of $\mathcal{A}$, which means that it is, moreover, closed in $\mathcal{A}$ under subobjects and regular quotients. Recall from [12] that the Birkhoff condition is equivalent to the following one: for any extension $f\colon A \to B$ in $\mathcal{A}$ the commutative square canonically induced by the unit η

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathsf{ab}A \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \xrightarrow[\eta_B]{} & \mathsf{ab}B \end{array} \qquad (\mathbf{B})$$

is a double extension. Note that this condition, together with the lemma above for $n = 3$, implies that

- the abelianisation functor $\mathsf{ab}\colon \mathcal{A} \to \mathsf{Ab}\mathcal{A}$ preserves pullbacks of split epimorphisms along extensions.

(To see this, keep in mind that a split epimorphism of double extensions is always a three-fold extension.) This important property was first discovered by Marino Gran in [8], and we shall need it in the proof of our theorem.

Recall from [12] that an extension $f\colon A \to B$ is **trivial** (with respect to the Birkhoff subcategory $\mathsf{Ab}\mathcal{A}$) if the induced square **B** is a pullback; it is **central** if there exists an extension $p\colon E \to B$ such that the pullback $p^*(f)\colon E \times_B A \to E$ of f along

p is a trivial extension; it is **normal** when the projections of its kernel pair $R[f]$ are trivial. Let us denote by $\mathsf{Ext}\mathcal{A}$ and $\mathsf{CExt}\mathcal{A}$ the full subcategories of $\mathsf{Arr}\mathcal{A}$ determined by all extensions and all central extensions, respectively. It was shown in [8] (see also [2, 13]) that the central extensions (with respect to $\mathsf{Ab}\mathcal{A}$) are precisely those extensions $f\colon A \to B$ with $[R[f], \nabla_A] = \Delta_A$. As explained in [13] (in the case of Mal'tsev varieties—but the argument remains valid), this implies in particular that the category $\mathsf{CExt}\mathcal{A}$ is reflective in $\mathsf{Ext}\mathcal{A}$ and that the centralisation of an extension $f\colon A \to B$ is given by the induced quotient $\mathsf{centr}f = A/[R[f], \nabla_A] \to B$. The centralisation functor $\mathsf{centr}\colon \mathsf{Ext}\mathcal{A} \to \mathsf{CExt}\mathcal{A}$ has the following property, which is a consequence of the fact that the commutator of equivalence relations is preserved by regular images [15]: for any double extension $f\colon A \to B$, the square in $\mathsf{Arr}\mathcal{A}$ canonically induced by the unit η^1

$$\begin{array}{ccc} A & \xrightarrow{\eta^1_A} & \mathsf{centr}A \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \xrightarrow[\eta^1_B]{} & \mathsf{centr}B \end{array} \qquad (\mathbf{C})$$

is a three-fold extension. Using the terminology of [5, 6] this means that $\mathsf{CExt}\mathcal{A}$ is a strongly $\mathcal{E}^1$-Birkhoff subcategory of $\mathsf{Ext}\mathcal{A}$, where $\mathcal{E}^1$ denotes the class of all double extensions. Applying the lemma for $n = 4$, it follows that

- the centralisation functor $\mathsf{centr}\colon \mathsf{Ext}\mathcal{A} \to \mathsf{CExt}\mathcal{A}$ preserves pullbacks of split epimorphisms of extensions along double extensions.

(To see this, keep in mind that a split epimorphism of three-fold extensions is always a four-fold extension.) Taking this into account, one is then able to prove also the following consequences of the strong $\mathcal{E}^1$-Birkhoff property of $\mathsf{CExt}\mathcal{A}$, all of which are well-known in the case of one-fold extensions [12]. Analogous to the one-dimensional case, a *double* extension $f\colon A \to B$ is **trivial** when the induced square **C** is a pullback; it is **central** if there exists a double extension $p\colon E \to B$ such that the pullback $p^*(f)\colon E \times_B A \to E$ of f along p is a trivial double extension; it is **normal** when the projections of its kernel pair $R[f]$ are trivial.

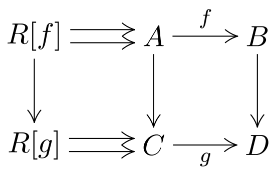
- The pullback in $\mathsf{Arr}\mathcal{A}$ of a trivial double extension along a double extension is a trivial double extension;
- the pullback in $\mathsf{Arr}\mathcal{A}$ of a double central extension along a double extension is a double central extension;

- a double central extension that is a split epimorphism in $\mathsf{Arr}\mathcal{A}$ is necessarily trivial.

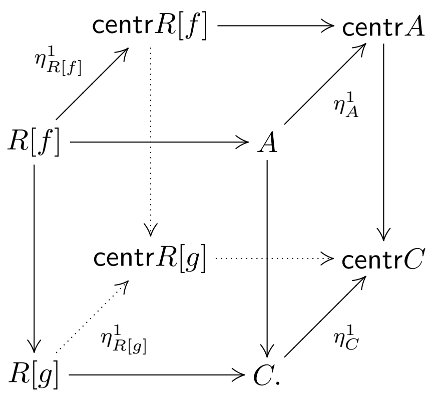
And it follows that

- the concepts of central and normal double extension coincide.

We need one last consequence of the strong $\mathcal{E}^1$-Birkhoff property of $\mathsf{CExt}\mathcal{A}$. For this, consider a three-fold extension, pictured as the right hand square in the following diagram in $\mathsf{Arr}\mathcal{A}$.



By applying the centralisation functor $\mathsf{centr}\colon \mathsf{Ext}\mathcal{A} \to \mathsf{CExt}\mathcal{A}$ to the left hand commutative square of (say) first projections, we obtain a commutative cube in $\mathsf{Arr}\mathcal{A}$ which is a four-fold extension as a split epimorphism of three-fold extensions:



It follows from the lemma that the bottom square in this cube is a pullback as soon as the top square is a pullback, i.e., if f is a normal extension, then so is g. Since the concepts of central and normal double extension coincide, it follows that

- a quotient of a double central extension by a three-fold extension is again a double central extension.

We are now in a position to prove the characterisation of double central extensions. As mentioned before, we only need to consider one implication: for the other, we refer the reader to [9].

Let **A** be a double extension such that $[R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X]$. The first condition $[R[d], R[c]] = \Delta_X$ says that there exists a partial Mal'tsev operation $p\colon R[c] \times_X R[d] \to X$, i.e., a morphism p that satisfies the conditions $p(\alpha, \gamma, \gamma) = \alpha$ and $p(\alpha, \alpha, \gamma) = \gamma$. Recall from [4] that such a p, in a regular Mal'tsev category, necessarily satisfies the conditions $dp(\alpha, \beta, \gamma) = d(\gamma)$ and $cp(\alpha, \beta, \gamma) = c(\alpha)$. We use the notation $R[d]\square R[c]$ for the largest double equivalence relation on $R[d]$ and $R[c]$, which "consists" of all quadruples $(\alpha, \beta, \delta, \gamma)$ of "elements" of X that satisfy $c(\alpha) = c(\beta)$, $c(\delta) = c(\gamma)$, $d(\alpha) = d(\delta)$ and $d(\beta) = d(\gamma)$. Such a quadruple may be pictured as

$$\begin{pmatrix} \alpha & c & \beta \\ d & & d \\ \delta & c & \gamma \end{pmatrix}. \tag{D}$$

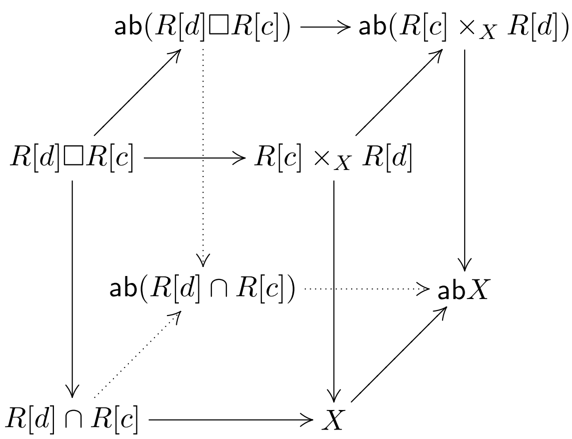
Writing

$$\pi\colon R[d]\square R[c] \to R[c] \times_X R[d]$$

for the canonical comparison map (π sends a quadruple **D** in $R[d]\square R[c]$ to the triple (α, β, γ)) and $q\colon R[d]\square R[c] \to R[d] \cap R[c]$ for the map which sends a quadruple **D** to the couple $(p(\alpha, \beta, \gamma), \delta)$ in $R[d]\cap R[c]$, we obtain the pullback of split epimorphisms

$$\begin{array}{ccc} R[d]\square R[c] & \xrightarrow{\ \pi\ } & R[c] \times_X R[d] \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ R[d] \cap R[c] & \xrightarrow[\ p_1\ ]{} & X. \end{array}$$

Applying the abelianisation functor gives us the following commutative cube, in which the slanted arrows are components of the unit η.

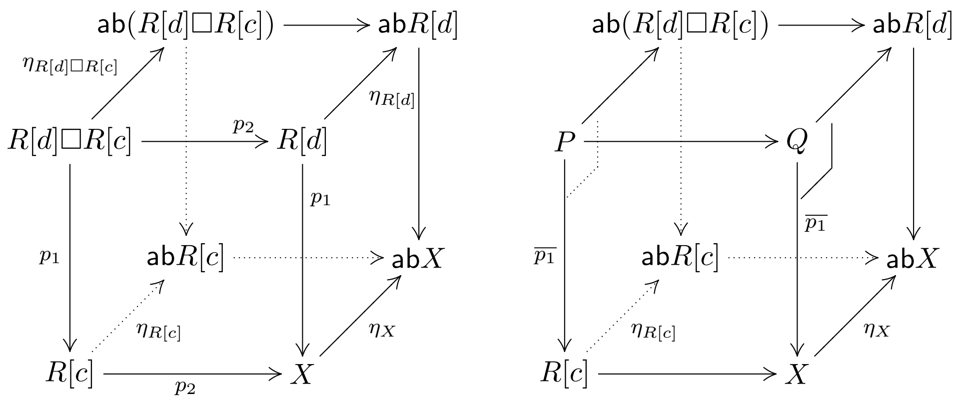


Since the reflector ab preserves pullbacks of extensions along split epimorphisms, the back square of this cube is a pullback.

The second condition $[R[d] \cap R[c], \nabla_X] = \Delta_X$ tells us that the extension $(d, c)\colon X \to D \times_X C$ is central. This is equivalent to the kernel pair projection $p_1\colon R[d] \cap R[c] \to X$ being a trivial extension, which is another way to say that the bottom square in the above cube is a pullback. Hence the two conditions together imply that so is its top square

$$\begin{array}{ccc} R[d]\Box R[c] & \xrightarrow{\pi} & R[c] \times_X R[d] \\ {\scriptstyle \eta_{R[d]\Box R[c]}}\downarrow & & \downarrow{\scriptstyle \eta_{R[c]\times_X R[d]}} \\ \mathsf{ab}(R[d]\Box R[c]) & \xrightarrow[\mathsf{ab}\pi]{} & \mathsf{ab}(R[c] \times_X R[d]). \end{array}$$

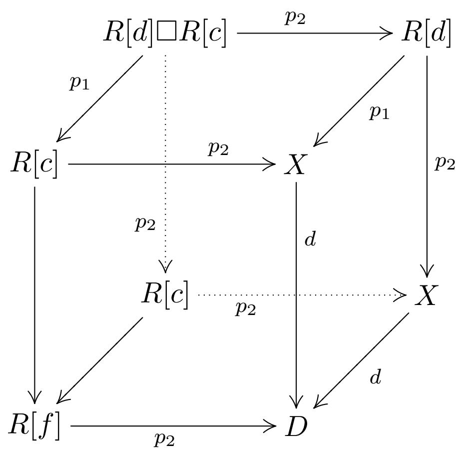
Now consider the left hand side cube and the induced right hand side cube of pullbacks.



Taking into account that, since $R[c] \times_X R[d]$ is a pullback of a split epimorphism along a split epimorphism, $\mathsf{ab}(R[c] \times_X R[d]) = \mathsf{ab}R[c] \times_{\mathsf{ab}X} \mathsf{ab}R[d]$, the foregoing results imply that the left hand side cube is a limit diagram. Hence the comparison square

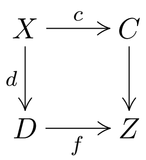
$$\begin{array}{ccc} R[d]\Box R[c] & \longrightarrow & R[d] \\ \downarrow & & \downarrow \\ P & \longrightarrow & Q \end{array}$$

between the two cubes is a pullback, which means that the front square (considered as a horizontal arrow) of the left hand side cube is a trivial double extension. (The vertical arrows p_1 in this double extension are split epimorphisms, so their centralisation is their trivialisation—the two arrows $\overline{p_1}$ on the right hand side.) A fortiori, it is a double central extension. Now consider the commutative cube below. Considered as a horizontal arrow, it is a split epimorphism between pullbacks of regular epimorphisms; consequently it is a three-fold extension.



We have just seen that this cube's top square, considered as a horizontal arrow, is a double central extension. It follows that the bottom square, also considered as a horizontal arrow, is a double central extension as well, being a quotient of a double extension along a three-fold extension. But this bottom square is one of the projections of the kernel pair of the double extension **A**, so that also **A** is central, and we obtain:

Theorem. *In a Barr exact Mal'tsev category with finite colimits, a double extension*



is central if and only if $[R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X]$. □

References

[1] D. Bourn, *The denormalized* 3×3 *lemma*, J. Pure Appl. Algebra **177** (2003), 113–129.

[2] D. Bourn and M. Gran, *Central extensions in semi-abelian categories*, J. Pure Appl. Algebra **175** (2002), 31–44.

[3] A. Carboni, G. M. Kelly, and M. C. Pedicchio, *Some remarks on Maltsev and Goursat categories*, Appl. Categ. Struct. **1** (1993), 385–421.

[4] A. Carboni, M. C. Pedicchio, and N. Pirovano, *Internal graphs and internal groupoids in Mal'cev categories*, Proceedings of Conf. Category Theory 1991, Montreal, Am. Math. Soc. for the Canad. Math. Soc., Providence, 1992, pp. 97–109.

[5] T. Everaert, *Higher central extensions and Hopf formulae*, J. Algebra, to appear, 2008.

[6] T. Everaert, M. Gran, and T. Van der Linden, *Higher Hopf formulae for homology via Galois Theory*, Adv. Math. **217** (2008), 2231–2267.

[7] M. Gran, *Central extensions and internal groupoids in Maltsev categories*, J. Pure Appl. Algebra **155** (2001), 139–166.

[8] M. Gran, *Applications of categorical Galois theory in universal algebra*, Galois Theory, Hopf Algebras, and Semiabelian Categories (G. Janelidze, B. Pareigis, and W. Tholen, eds.), Fields Institute Communications Series, vol. 43, American Mathematical Society, 2004, pp. 243–280.

[9] M. Gran and V. Rossi, *Galois theory and double central extensions*, Homology, Homotopy and Appl. **6** (2004), no. 1, 283–298.

[10] G. Janelidze, *What is a double central extension? (The question was asked by Ronald Brown)*, Cah. Top. Géom. Diff. Catég. **XXXII** (1991), no. 3, 191–201.

[11] G. Janelidze, *Galois groups, abstract commutators and Hopf formula*, Appl. Categ. Struct. **16** (2008), 653–668.

[12] G. Janelidze and G. M. Kelly, *Galois theory and a general notion of central extension*, J. Pure Appl. Algebra **97** (1994), 135–161.

[13] G. Janelidze and G. M. Kelly, *Central extensions in Mal'tsev varieties*, Theory Appl. Categ. **7** (2000), no. 10, 219–226.

[14] G. Janelidze and M. C. Pedicchio, *Pseudogroupoids and commutators*, Theory Appl. Categ. **8** (2001), no. 15, 408–456.

[15] M. C. Pedicchio, *A categorical approach to commutator theory*, J. Algebra **177** (1995), 647–657.

[16] D. Rodelo and T. Van der Linden, *The third cohomology group classifies double central extensions*, Theory Appl. Categ. **23** (2010), no. 8, 150–169.

[17] J. D. H. Smith, *Mal'cev varieties*, Lecture notes in mathematics, vol. 554, Springer, 1976.

Tomas Everaert
Vakgroep Wiskunde
Vrije Universiteit Brussel
Pleinlaan 2
1050 Brussel
Belgium
teveraer@vub.ac.be

Tim Van der Linden
Centro de Matemática da Universidade de Coimbra
3001-454 Coimbra
Portugal
tvdlinde@vub.ac.be

CONTENTS OF THE VOLUME DEDICATED TO FRANCIS BORCEUX

The Volume dedicated to Francis Borceux on the occasion of his sixtieth birthday extends on four and a half issues of the "Cahiers", as indicated below.

Volume L-3

Volume L- 4

Volume LI-1

Volume LI-2

Volume LI-3 (to be published)

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES Vol. LI-2 (2010)

THE "CAHIERS" ON INTERNET

by Andrée C. EHRESMANN

We have created a new Internet site for the "Cahiers":

http://pagesperso-orange.fr/ehres/Cahiers/Ctgdc.htm

It extends and will replace the former site

http://pagesperso-orange.fr/vbm-ehr/CT/

It gives a general presentation of the journal, an Index of all the papers published since Volume I and abstracts (in English) of the papers published since 1999. It also contains pdf files of some recent issues of the "Cahiers" (before their posting on NUMDAM), and the list of contents of the volumes of the pre-publication

"Esquisses Mathématiques"

which we had published from 1970 to 1981.

We recall that all the former volumes of the "Cahiers" are freely accessible on the site of the NUMDAM cell of the Centre National de la Recherche Scientifique. The papers can be downloaded as pdf or dvi files, and many hyperlinks have been added to refer to other papers accessible on NUMDAM. The first 8 volumes of the "Cahiers" are classified under the heading "Séminaire Ehresmann" (which was their sub-title at this time) on the site

http://www.numdam.org/numdam-bin/feuilleter?j=SE

For the following volumes, the site is

http://www.numdam.org/numdam-bin/feuilleter?j=CTGDC

The new volumes become freely accessible to all on this site about two years after the publication of the printed version.

We also recall that, since many years, the Table of Contents of the volumes of the "Cahiers" are also posted on the Internet site

htp://www.tac.mta.ca/tac/

of the electronic journal TAC. And, in exchange, each year the Table of Contents of the present volumes of this journal are printed in one of the issues of the "Cahiers".

GUIDE TO AUTHORS

The papers accepted for the '*Cahiers*' are reproduced directly from the pdf files provided by the Authors. So Authors are required to send 1 *hard copy,* prepared with T_EX and laser printed, as well as the corresponding pdf file, satisfying the following format A4 requirements.

Left margin: 4,1 cm.

Width of the page: 12,8 cm between the margins.

Top margin: 6 cm before the first line of the text.

Length from the first to the last line of the text: 18,5 cm (without headlines nor page numbers).

Type: Times New Roman, 12 points.

Interline: 0,49 cm (with larger interlines before and after centered formulas and between Sections).

Section titles: in bold face, as well as the words 'Proposition', 'Theorem', 'Definition'..., with the statements of these in italics.

Diagrams and figures: drawn very carefully and presented in the text, as they should appear after printing.

References: numbered and listed at the end of the paper in the alphabetic order of their authors, and with the standard abbreviations.

Address of the author(s): at the end of the paper.

First page: The title and the name of the author(s) (and the headlines and page numbers) will be added by the Editors, so that the length of this page from the first to the last line should be only 15 cm. It should begin by an Abstract in English and an abstract in French (from 5 to 15 lines according to the length of the paper), presented with a left margin of 1,5 cm more than the current margin, followed by the key words and MS classifications. (Authors may ask the editor for a French translation of their abstract.)

For more information, please contact:

Mme Ehresmann, Faculté des Sciences, Math.
33 rue Saint-Leu, F-80039 Amiens. France.
E-mail: ehres@u-picardie.fr
http://pagesperso-orange.fr/ehres