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ON QUASI-EQUATIONS IN LOCALLY PRESENTABLE CATEGORIES II: A LOGIC

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Dedicated to Francis Borceux on the occasion of his sixtieth birthday

Résumé

Les quasi-équations, données par des paires parallèles de morphismes finitaires, représentent des propriétés des objets: un objet satisfait la propriété si son foncteur hom contravariant fusionne les morphismes de la paire. Récemment Adámek et Hébert ont caractérisé les sous-catégories des catégories localement de présentation finie spécifiées par des quasi-équations. Nous présentons ici une logique de quasi-équations proche de la logique classique équationnelle de Birkhoff. Nous prouvons qu'elle est consistante et complète dans toute catégorie localement présentation finie avec relations d'équivalence effectives.

Abstract

Quasi-equations, given by parallel pairs of finitary morphisms, represent properties of objects: an object satisfies the property if its contravariant hom-functor merges the parallel pair. Recently Adámek and Hébert characterized subcategories of locally finitely presentable categories specified by quasi-equations. We now present a logic of quasi-equations close to Birkhoff's classical equational logic. We prove that it is sound and complete in all locally finitely presentable categories with effective equivalence relations.

Mathematics Subject Classification: 18A15, 08B05, 18C35, 18E10.

Key words: quasi-equations, finitary morphisms, locally finitely presentable categories, exact categories, equational logic, logic of quasi-equations.

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1 Introduction

It was Bill Hatcher who first considered a representation of properties of objects via a parallel pair $u, v : R \to X$ of morphisms in the sense that an object A has the property iff every morphism $f : X \to A$ fulfils $f \cdot u = f \cdot v$, see [11]. Later Bernhard Banaschewski and Horst Herrlich [5] considered the related concept of injectivity w.r.t. a regular epimorphism $c : X \to Y$: this is just the step from parallel pairs to their coequalizers. For regular epimorphisms which are finitary, that is, have finitely presentable domain and codomain, Banaschewski and Herrlich [5] characterized full subcategories of "suitable" categories which can be specified by such injectivity: they are precisely the subcategories closed under products, subobjects, and filtered colimits. Recently the same result was proved for all locally finitely presentable categories, see [2], where parallel pairs of morphisms u, v with finitely presentable domain and codomain are called *quasi-equations*. Notation: $u \equiv v$.

In the present paper we introduce a logic of quasi-equations: for every set Q of quasi-equations we characterize its *consequences*, that is, quasi-equations $u \equiv v$ which hold in every object satisfying every quasi-equation in Q. In fact, we introduce two logics. The first one is sound and complete in every locally finitely presentable category. Moreover, this logic is extremely simple: it states that (1) $u \equiv u$ always holds, (2) if $u \equiv v$ holds, then also $q \cdot u \equiv q \cdot v$ holds, and (3) if $u \equiv v$ holds and c is a coequalizer of u and v

$$\begin{tikzcd} & {} \arrow[d, dashed, shift right, "u'"'] \arrow[d, dashed, shift left, "v'"] & \\ {} \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & {} \arrow[r, "c"] & {} \end{tikzcd}$$

then for all pairs with $c \cdot u' = c \cdot v'$ we have that $u' \equiv v'$ holds. However this last rule makes the logic disputable in applications. Think of Birkhoff's Equational Logic in the category $\mathbf{Alg}\,\Sigma$: its aim is to describe the fully invariant congruence generated by (u, v), whereas the coequalizer rule takes the congruence that (u, v) generates for granted.

We therefore present our main logic, called the Quasi-Equational Logic, without the coequalizer rule. Instead, we work with the parallel pairs alone. This logic is a bit more involved than (1)-(3) above, but is much nearer to

Birkhoff's classical result [7]. We prove its completeness in

(i) every locally finitely presentable category with effective equivalence relations

and

(ii) in $\mathbf{Mod}\,\Sigma$, the category of Σ-structures for every (many-sorted) first-order signature.

However, we also present an example of a regular, locally finitely presentable category in which the Quasi-Equational Logic is not complete.

Related Work Satisfaction of a quasi-equation $u \equiv v$ is equivalent to injectivity w.r.t. the coequalizer of u and v. Our simple logic is just a translation of the injectivity logic w.r.t. epimorphisms presented in [4]. The full logic we introduce below is based on a description of the kernel pairs which for regular, locally finitely presentable categories was presented by Pierre Grillet [10], and the generalization to all locally finitely presentable categories we use stems from [1].

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2 The Coequalizer Logic

Here we present a (surprisingly simple) deduction system for quasi-equations which is sound and complete in all locally finitely presentable categories. Its only disadvantage is that it uses the concept of coequalizer, and this makes the usufelness in applications a bit questionable.

Throughout the paper we assume that a locally finitely presentable category is given, see [9] or [3].

2.1. Definition A *finitary morphism* is one whose domain and codomain are finitely presentable objects. A *quasi-equation* is a parallel pair of finitary morphisms $u, v : R \to X$. We use the notation $u \equiv v$. An object A *satisfies* $u \equiv v$ if $f \cdot u \equiv f \cdot v$ holds for all $f : X \to A$. A quasi-equation $u \equiv v$ is said to be a *consequence* of a set Q of quasi-equations, written $Q \models u \equiv v$, if every object satisfying all members of Q also satisfies $u \equiv v$.

2.2. Observation Let the diagram

$$\begin{tikzcd} & R' \arrow[d, shift right, "u'"'] \arrow[d, shift left, "v'"] & \\ R \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & X \arrow[r, "c"] & C \end{tikzcd}$$

be such that we have

$$c \cdot u' = c \cdot v' \quad \text{and} \;\; c = \text{coeq}(u, v).$$

Then the quasi-equation $u' \equiv v'$ is a consequence of $u \equiv v$. In fact, if A satisfies $u \equiv v$ then for every $f : X \to A$ we see that f factors through c, consequently, $f \cdot u' \equiv f \cdot v'$.

This suggests the following

2.3. Definition The *Coequalizer Logic* uses the following deduction rules:

Reflexivity: $$\frac{}{u \equiv u}$$

Left Composition: $$\frac{u \equiv v}{q \cdot u \equiv q \cdot v}$$ given $\begin{tikzcd} \cdot \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] & \cdot \arrow[r, "q"] & \cdot \end{tikzcd}$

Coequalizer: $$\frac{u \equiv v \quad c \cdot u' = c \cdot v'}{u' \equiv v'}$$ for $c = \text{coeq}(u, v)$

2.4. Remark (i) The Coequalizer Deduction System is obviously sound: whenever we can prove a quasi-equation $u \equiv v$ from a given set Q by using the above three deduction rules, it follows that $u \equiv v$ is a consequence of Q.

(ii) We will prove the completeness of the above deduction system by reducing it to the completeness of the logic presented by Manuela Sobral and the authors in [4]. That logic concerned injectivity w.r.t. finitary epimorphisms $e : X \to Y$. Recall that an object A is injective w.r.t. e if every morphism from X to A factors through e. We say that e is an *injectivity consequence* of a set $\mathcal{E}$ of finitary epimorphisms provided that every object injective w.r.t. members of $\mathcal{E}$ is also injective w.r.t. e. We formulated the following logic of injectivity consisting of one axiom and three deduction rules (where e and e' are finitary epimorphisms):

(A) $\dfrac{}{\mathrm{id}_X}$ for finitely presentable objects X

(P) $\dfrac{e}{e'}$ for every pushout
$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, "e'"'] & \cdot \end{tikzcd}$$

(C) $\dfrac{e \quad e'}{e \cdot e'}$ given $\xrightarrow{e'} \xrightarrow{e}$

(L) $\dfrac{e \cdot e'}{e'}$

And we proved that this represents a sound and complete injectivity logic in every locally finitely presentable category. That is, given a set Q of finitary epimorphisms, then the injectivity consequences e of Q are precisely those which have a (finite) proof applying the above axiom and deduction rules to members of Q.

(iii) Before proceeding with our logic of quasi-equations, we observe an unexpected property of proofs based on the rules above: Let Q be a set of finitary epimorphisms containing all finitary identity morphisms. Then for every injectivity consequence e of Q there exists a proof of the following form

$$\left\{ \begin{array}{l} e_1 \\ \vdots \\ e_{k_1} \end{array} \right.$$

$$\text{(P)} \left\{ \begin{array}{l} e_{k_1+1} \\ \vdots \\ e_{k_2} \end{array} \right.$$

$$\text{(C)}\begin{cases} e_{k_2+1} \\ \vdots \\ e_{k_3} \end{cases}$$

$$\text{(L)}\begin{cases} e_{k_3+1} \\ \vdots \\ e_{k_4} = e \end{cases}$$

whose first part consists of elements of Q, the second part uses only (P), the third one only (C), and the last one only (L). This follows from the next lemma in which we put

$Q_C = \{e_1 \cdot e_2 \ldots e_k;\ e_i \in Q\}$ (the closure under (C))
$Q_L = \{e';\ e \cdot e' \in Q \text{ for some } e\}$ (the closure under (L))

and

$Q_P = \{e;\ e \text{ finitary and opposite to a member of } Q \text{ in a pushout}\}$
(the closure under (P))

2.5. Lemma *Let Q be a set of finitary epimorphisms containing all id_X, X finitely presentable. Then $((Q_P)_C)_L$ is closed under pushout, composition and left cancellation.*

Proof Observe that $(Q_P)_C$ is closed under pushout (and composition) since a pushout of a composite is the composite of pushouts.

To prove the statement, let us first prove that $((Q_P)_C)_L$ is closed under pushout: Given $e' \in ((Q_P)_C)_L$, there exists e finitary such that $ee' \in (Q_P)_C$. Consider the pushout e'' of e' along u

$$\begin{tikzcd} {} \arrow[r, "e'"] \arrow[d, "u"'] & {} \arrow[r, "e"] \arrow[d, "v"'] & {} \arrow[d, dotted, "w"] \\ {} \arrow[r, "e''"'] & {} \arrow[r, dotted, "f"'] & P \end{tikzcd}$$

and form a pushout P of e along v to get, by the above, $f \cdot e'' \in (Q_P)_C$, thus, $e'' \in ((Q_P)_C)_L$. Next we prove that $((Q_P)_C)_L$ is closed under composition:

Consider a composite $f' \cdot e'$

$$
\begin{tikzcd}
{} \arrow[r, "e'"] & {} \arrow[rr, "f'"] \arrow[d, "e"] & & {} \arrow[d, "f"] \\
& {} \arrow[dr, dotted, "v"'] & & {} \arrow[dl, dotted, "w"] \\
& & P &
\end{tikzcd}
$$

where $e \cdot e' \in (Q_P)_C$ and $f \cdot f' \in (Q_P)_C$. Form the pushout P of e and $f \cdot f'$ to get $v \in (Q_P)_C$, thus $v \cdot e \cdot e' = w \cdot f \cdot f' \cdot e' \in (Q_P)_C$. This proves $f' \cdot e' \in ((Q_P)_C)_L$. $\square$

2.6. Theorem *The Coequalizer Deduction System is sound and complete in every locally finitely presentable category. That is, a quasi-equation is a consequence of a set Q of quasi-equations iff it can be deduced from Q.*

Proof We apply the result of [4] mentioned in 2.4: given a set $\mathcal{H}$ of finitary epimorphisms containing all finitary identity morphisms, then the injectivity consequences of e form the closure of $\mathcal{H}$ under composition, pushout, and left cancellation.

Denote by $\mathcal{A}_{fp}$ the full subcategory of all finitely presentable objects in the category $\mathcal{A}$ and by

$$K : \mathcal{A}_{fp}^{\rightrightarrows} \longrightarrow \mathcal{A}_{fp}^{\rightarrow}$$

the functor assigning to every quasi-equation its coequalizer. We have

$$Q \models u \equiv v \quad \text{iff} \quad K(u,v) \text{ is an injectivity consequence of } K[Q].$$

Assume, without loss of generality, that Q contains all pairs $u \equiv u$. Then the above result together with Lemma 2.5 tells us that

$$Q \models u \equiv v \quad \text{iff} \quad K(u,v) \in ((K[Q]_P)_C)_L.$$

Thus, all we need to do is to present a proof of $u \equiv v$ from Q given that the coequalizer $c = K(u,v)$ lies in the left-cancellation hull of $(K[Q]_P)_C$, i.e., it has the form

$$\begin{tikzcd}
 & & & & \cdot \\
 & & & \cdots \arrow[ur, "c_n"] & \\
 & & \cdot \arrow[ur, "c_2"] & & \\
 & \cdot \arrow[ur, "c_1"] \arrow[rrr, "c"] & & & \cdot \arrow[uuu, "f"'] \\
 \cdot \arrow[ur, phantom] & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\cdot \arrow[r, dotted, shift left, "u"] \arrow[r, dotted, shift right, "v"'] & \cdot \arrow[r, "c_1"] \arrow[rr, bend right, "c"'] & \cdots \arrow[r, "c_n"] & \cdot
\end{tikzcd}$$

and for every i we have a pushout

$$\begin{tikzcd}
\cdot \arrow[r, shift left, "u_i"] \arrow[r, shift right, "v_i"'] & \cdot \arrow[r, "k_i"] \arrow[d, "g_i"'] & \cdot \arrow[d, "h_i"] \\
 & \cdot \arrow[r, "c_i"'] & \cdot
\end{tikzcd}$$

for some $u_i \equiv v_i$ in Q and $k_i = K(u_i, v_i)$. Observe first that c_i is a coequalizer of $u'_i = g_i \cdot u_i$ and $v'_i = g_i \cdot v_i$ and we have

$$\frac{u_i \equiv v_i}{u'_i \equiv v'_i}$$

due to Left Composition. The Coequalizer Rule then yields

$$\cfrac{\cfrac{c_n c_{n-1} \dots c_1 u = c_n c_{n-1} \dots c_1 v \qquad u'_n \equiv v'_n}{c_{n-1} \dots c_1 u = c_{n-1} \dots c_1 v \qquad u'_{n-1} \equiv v'_{n-1}}}{\vdots}$$

$$\cfrac{\cfrac{}{c_1 u = c_1 v \qquad u_1 \equiv v_1}}{u \equiv v}$$

□

3 The Quasi-Equational Logic in Exact Categories

In the present section we introduce the logic of quasi-equations that only works with parallel pairs (and does not use coequalizers). This logic is sound in all locally finitely presentable categories, and we prove here that it is complete whenever the category is *exact*, see [6] or [8], which means that

(a) it is *regular* in the sense of Michael Barr (that is, it has regular factorizations, meaning regular epimorphism followed by a monomorphism, and regular epimorphisms are closed under pullback)

and

(b) it has effective equivalence relations (see 3.5 for details).

We present also important examples (graphs, posets, first-order structures) of categories in which our logic is complete, although they are not exact. However, a counter-example demonstrates that the logic is **not** complete in every regular, locally finitely presentable category.

3.1. Definition The *Quasi-Equational Logic* uses the following deduction rules

Reflexivity: $$\frac{}{u \equiv u}$$

Symmetry: $$\frac{u \equiv v}{v \equiv u}$$

Transitivity: $$\frac{u \equiv v \qquad v \equiv w}{u \equiv w}$$

Union: $$\frac{u \equiv v \qquad u' \equiv v'}{u + u' \equiv v + v'}$$

Composition: $$\frac{u \equiv v}{q \cdot u \cdot p \equiv q \cdot v \cdot p}$$ given $\xrightarrow{p} \underset{v}{\overset{u}{\rightrightarrows}} \xrightarrow{q}$

Epi-Cancellation: $$\frac{u \cdot p \equiv v \cdot p}{u \equiv v}$$ for epimorphisms p

We say that a quasi-equation $u \equiv v$ is *deducible* from a set Q of quasi-equations, in symbols

$$Q \vdash u \equiv v$$

if there exists a (finite) proof of $u \equiv v$ applying the above deduction rules to members of Q.

3.2. Remark The Quasi-Equational Logic is obviously sound: whenever $Q \vdash u \equiv v$, then the quasi-equation $u \equiv v$ is a consequence of Q. That is, every object satisfying all quasi-equations in Q satisfies $u \equiv v$ too.

We will discuss the completeness in this and the next section.

3.3. Remark Every proof in Birkhoff's Equational Logic has an easy translation into the Quasi-Equational Logic: Recall that that logic for a given signature Σ consists of Reflexivity, Symmetry, Transitivity, and the following rules:

Invariance: $$\frac{u \equiv v}{\sigma(u) \equiv \sigma(v)} \quad \text{for all substitutions } \sigma$$

Congruence: $$\frac{u_1 \equiv v_1,\ u_2 \equiv v_2,\ \ldots,\ u_n \equiv v_n}{h(u_1, u_2, \ldots, u_n) \equiv h(v_1, v_2, \ldots, v_n)} \quad \text{for all } n\text{-ary symbols } h \text{ in } \Sigma$$

Let $F : \mathbf{Set} \to \mathbf{Alg}\,\Sigma$ be the left adjoint of the forgetful functor of $\mathbf{Alg}\,\Sigma$. A (finitary) equation $u \equiv v$ (where $u, v : 1 \to FX$ are Σ-terms for some finite set X of variables) may be regarded as a pair of morphisms of $\mathbf{Alg}\,\Sigma$

$$F1 \overset{\overline{v}}{\underset{\overline{u}}{\rightrightarrows}} FX$$

extending u and v. This replacement of equations by quasi-equations, together with a convenient translation of the deduction rules, transforms every formal proof in Birkhoff's equational logic into one in the Quasi-Equational Logic. The Invariance Rule is a special case of Left Composition (recall that a substitution is nothing else than an endomorphism $\sigma : FX \to FX$):

$$\frac{u \equiv v}{\sigma \cdot u \equiv \sigma \cdot v}$$

For the Congruence Rule, consider the homomorphism $\overline{h} : F1 \to Fn$ taking the generator of $F1$ to the term $h(0, \ldots, n-1)$ in Fn. By applying Union we obtain

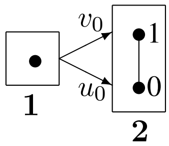
$$\overline{u}_0 + \overline{u}_1 + \cdots + \overline{u}_{n-1} \equiv \overline{v}_0 + \overline{v}_1 + \cdots + \overline{v}_{n-1} : Fn \to FX$$

and then we just compose with $\overline{h}$ from the right and the codiagonal from the left:

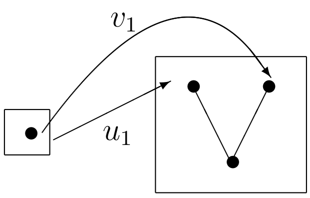
$$F1 \xrightarrow{\overline{h}} Fn \underset{\overline{v}_0+\overline{v}_1+\cdots+\overline{v}_{n-1}}{\overset{\overline{u}_0+\cdots+\overline{u}_{n-1}}{\rightrightarrows}} FX + \cdots + FX \xrightarrow{\nabla} FX$$

3.4. Example In the category of posets deduction of quasi-equations is rather trivial:

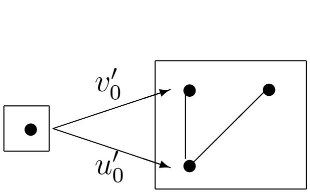
(i) Consider the following quasi-equation



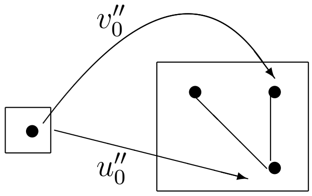
From $u_0 \equiv v_0$ we can deduce the following quasi-equation $u_1 \equiv v_1$:



In fact, by using Composition we deduce from $u_0 \equiv v_0$ the following



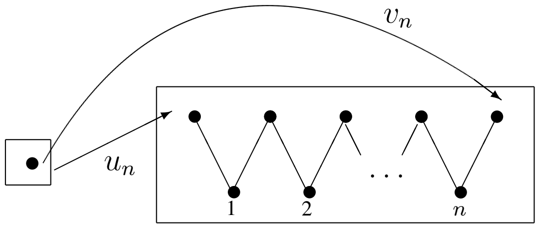
and



Symmetry yields $v_0' \equiv u_0'$ and, since $u_0' = u_0''$, Transitivity yields

$$u_0 \equiv v_0 \vdash u_1 \equiv v_1 \, .$$

(ii) Analogously we deduce from $u_0 \equiv v_0$ the following quasi-equations



(iii) More generally, we will show that the consequences of $u_0 \equiv v_0$ are all quasi-equations $u,\ v : A \to B$ such that

(*) $\qquad u(a)$ and $v(a)$ lie in the same component of B for all $a \in A$.

Given a quasi-equation $u \equiv v$ satisfying (*) then

$$u_0 \equiv v_0 \ \vdash \ u \equiv v\,.$$

This is clear from (ii) in case $A = \mathbf{1} = \{0\}$ is the terminal object: since $u(0)$ and $v(0)$ lie in the same component they are connected by a zig-zag. By using Union and Composition (with the codiagonal as q and $p = \mathrm{id}$) we conclude that the statement holds for all $u, v : A \to B$ with $A = \mathbf{1} + \cdots + \mathbf{1}$. And if A is arbitrary use the epimorphism $e : \mathbf{1} + \cdots + \mathbf{1} \to A$ carried by the identity map: since $u_0 \equiv v_0 \ \vdash \ u \cdot e \equiv v \cdot e$, Epi-Cancellation yields $u_0 \equiv v_0 \vdash u \equiv v$.

(iv) Conversely, every quasi-equation $u \equiv v$ where $u, v : A \to B$ are distinct implies $u_0 \equiv v_0$. In fact, choose $p \in A$ with $u(p) \neq v(p)$; say, $u(p) \not\geq v(p)$. Then we have an isotone map $q : B \to \mathbf{2} = \{0, 1\}$ where $q(u(p)) = 0$ and $q(v(p)) = 1$. Consequently, $u \equiv v \vdash u_0 \equiv v_0$ by Composition:

$$\begin{tikzcd}
1 \arrow[r, shift left, "u_0"] \arrow[r, shift right, "v_0"'] \arrow[d, "p"'] & 2 \\
A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \arrow[u, "q"']
\end{tikzcd}$$

(v) Given $u, v : A \to B$ such that (*) does not hold, then $u \equiv v$ implies the quasi-equation $l \equiv r$ for the coproduct injections $l, r : \mathbf{1} \to \mathbf{1} + \mathbf{1}$: use Composition picking $p : \mathbf{1} \to A$ such that $u \cdot p$ and $v \cdot p$ lie in different components and $q : B \to \mathbf{1} + \mathbf{1}$ which maps one of the components to l and the rest to r.

(vi) Conversely, $l \equiv r$ implies every quasi-equation. In fact, by Composition we clearly derive quasi-equations $u, v : \mathbf{1} \to B$. Using Union and Composition this yields all $u, v : \mathbf{1} + \mathbf{1} \cdots + \mathbf{1} \to B$. Finally, use $e : A' \to A$ as in (ii) above.

3.5. Remark Recall from [6] or [10] that in a regular, locally finitely presentable category:

(i) By a *relation* R on an object X is meant a subobject of $X \times X$. We can represent it by a collectively monic pair $u, v : R \to X$.

(ii) The *inverse relation* R^{-1} is represented by $v, u : R \to X$.

(iii) The *relation composite* $R \cdot R'$ of relations represented by collectively monic pairs $u, v : R \to X$ and $u', v' : R' \to X$ is obtained from the pullback P of v and u' via a factorization of $u \cdot p,\ v' \cdot p' : P \to X$:

$$
\begin{tikzcd}
& & P \arrow[dl, "p"'] \arrow[dr, "p'"] \arrow[ddd, two heads, bend right=90, "e"'] & & \\
& R \arrow[dl, "u"'] \arrow[dr, "v"] & & R' \arrow[dl, "u'"'] \arrow[dr, "v'"] & \\
X & & X & & X \\
& & \overline{R} = R \cdot R' \arrow[ull, "\overline{u}"] \arrow[urr, "\overline{v}"'] & &
\end{tikzcd}
$$

as a regular epimorphism $e : P \to R \cdot R'$ followed by a collectively monic pair $\overline{u}, \overline{v} : R \cdot R' \to X$. This composition is associative.

(iv) An *equivalence relation* is a relation R which is

a. reflexive, i.e., $\Delta_X \subseteq R$

b. symmetric, i.e., $R = R^{-1}$, and

c. transitive, i.e., $R = R \cdot R$.

Example: every kernel pair is an equivalence relation.

(v) A regular category has *effective equivalence relations* if every equivalence relation $u, v : R \to X$ is a kernel pair (of some morphism – it follows that it is the kernel pair of coeq(u, v)).

(vi) Let R be a reflexive and symmetric relation. Then the smallest equivalence relation containing R is

$$\widehat{R} = R \cup (R \cdot R) \cup (R \cdot R \cdot R) \cup \dots$$

see [10], 1.6.8. That is, we form the chain $R^1 \subseteq R^2 \subseteq R^3 \subseteq \dots$ of subobjects of $X \times X$ defined by $R^1 = R$ and $R^{n+1} = R \cdot R^n$, and the union of this chain (a) is an equivalence relation and (b) is contained in every equivalence relation containing R.

3.6. Examples (i) Sets, presheaves, Σ-algebras (for every finitary, possibly many-sorted signature Σ) and their varieties all form exact, locally finitely presentable categories.

(ii) Every coherent Grothendieck topos is an exact, locally finitely presentable category.

(iii) The category

$$\mathbf{Mod}\,\Sigma$$

of models of a (possibly many-sorted) first-order signature is a regular, locally finitely presentable category. Recall that Σ is given by a set Σ_f of function symbols with prescribed arities $\sigma : s_1 \dots s_n \to s$ (for $s_1 \dots s_n \in S^*$ and $s \in S$) and a set Σ_r of relation symbols with prescribed arities $s_1 \dots s_n$ in S^*. A model of Σ is an S-sorted set $A = (A_s)_{s \in S}$ together with functions $\sigma^A : A_{s_1} \times \dots \times A_{s_n} \to A_s$ for all $\sigma : s_1 \dots s_n \to s$ in Σ_f and relations $\rho^A \subseteq A_{s_1} \times \dots \times A_{s_n}$ for all ρ in Σ_r of arity $s_1 \dots s_n$.

The regularity of $\mathbf{Mod}\,\Sigma$ is due to the fact that a homomorphism $h : A \to B$ is a regular epimorphism iff every sort $h_s : A_s \to B_s\,(s \in S)$ is an epimorphism in $\mathbf{Set}$ and for every relation ρ of arity $s_1 \dots s_n$ the derived function from ρ^A to ρ^B (restricting $h_{s_1} \times \dots \times h_{s_n}$) is an epimorphism in $\mathbf{Set}$.

These categories are not exact in general. A simple example in the category of directed graphs (Σ given by one binary relation): let $u, v : \mathbf{2} \times \mathbf{2} \to \mathbf{2}$ (where $\mathbf{2}$ is the chain $0 < 1$) be the kernel pair of the morphism $\mathbf{2} \to \mathbf{1}$. If R is the subobject of $\mathbf{2} \times \mathbf{2}$ with the same underlying set which has $(0,0) < (1,1)$ as the only strict relation, then $u, v : R \to \mathbf{2}$ is an equivalence relation that is not a kernel pair.

(iv) The category of posets (and monotone maps) is not regular. In fact, let A be a coproduct of two 2-chains $a < a'$ and $b < b'$, and let $e : A \to B$

be the surjection which merges a' with b to get the 3-chain $a < a' < b'$. The map $e : A \to B$ is a regular epimorphism, but its pullback along the embedding of the 2-chain $a < b'$ into B is not: the pullback is the map from the discrete two-point set into a 2-chain.

3.7. Notation Given a parallel pair $u, v : R \to X$ we denote by

$$u_0, v_0 : R_0 \to X$$

the reflexive and symmetric relation it generates in the following sense: factorize the pair

$$[u, v, \mathrm{id}], [v, u, \mathrm{id}] : R + R + X \to X$$

as a regular epimorphism $e_0 : R + R + X \twoheadrightarrow R_0$ followed by a collectively monic pair (u_0, v_0). Then we denote by

$$R_0^n \rightarrowtail \widehat{R}$$

the inclusion of the n-subobject in the union of 3.5(vi), represented by

$$u_n, v_n : R_0^n \to X.$$

3.8. Remark For further use let us recall here that in a locally finitely presentable category every directed union $R = \bigcup_{i \in I} R_i$ of subobjects is the colimit $R = \operatorname{colim} R_i$ of the corresponding diagram of inclusion maps, see [3], 1.62.

3.9. Theorem *The Quasi-Equational Logic is sound and complete in every exact, locally finitely presentable category. That is, for every set Q of quasi-equations and every quasi-equation $u \equiv v$, $Q \models u \equiv v$ iff $Q \vdash u \equiv v$.*

Proof (1) We prove first that for every quasi-equation $u \equiv v$ the relations $u_n, v_n : R_0^n \to X$ of 3.7 have the following property:

(*) $u \equiv v \vdash u_n \cdot s \equiv v_n \cdot s$ for every $s : S \to R_0^n$ with S finitely presentable.

The proof is by induction in n.

Case $n = 0$: Given $s : S \to R_0$:

$$\begin{tikzcd}
Q_i \arrow[r, "q_i"] \arrow[d, two heads, "\overline{q}_i"'] & Q \arrow[r, "\overline{s}"] \arrow[d, two heads, "\overline{e}_0"'] & R+R+X \arrow[r, shift left, "{[u,v,\mathrm{id}]}"] \arrow[r, shift right, "{[v,u,\mathrm{id}]}"'] \arrow[d, two heads, "e_0"] & X \\
S_i \arrow[r, tail, "s_i"'] & S \arrow[r, "s"] & R_0 \arrow[ur, shift left, "u_0"] \arrow[ur, shift right, "v_0"'] &
\end{tikzcd}$$

we form the pullback Q of s along e_0 and express Q as a filtered colimit of finitely presentable objects with the colimit cocone $q_i : Q_i \to Q\,(i \in I)$. Then we form the regular factorization of $\overline{e}_0 \cdot q_i$ as indicated in the diagram above. The object S is the union of the subobjects $s_i : S_i \to S\,(i \in I)$ because $[s_i] : \coprod_{i \in I} S_i \to S$ is a regular epimorphism. In fact, $[s_i] \cdot \coprod \overline{q}_i = \overline{e}_0 \cdot [q_i]$ obviously is a regular epimorphism (since in the regular category $\overline{e}_0$ is a regular epimorphism), thus, so is $[s_i]$. By 3.8 we have $S = \operatorname{colim} S_i$, therefore, the fact that S is finitely presentable implies that s_j is an isomorphism for some $j \in I$. We now have a derivation of $u_0 \cdot s \equiv v_0 \cdot s$ as follows:

$$\begin{array}{c}
u \equiv v \\
\hline
u \equiv v \quad v \equiv u \quad \mathrm{id} \equiv \mathrm{id} \\
\hline
\\
[u, v, \mathrm{id}] \equiv [v, u, \mathrm{id}] \\
\hline
u_0 \cdot s \cdot s_j \quad \overline{q}_j \equiv v_0 \cdot s \cdot s_j \cdot \overline{q}_j \\
\hline
u_0 \cdot s \equiv v_0 \cdot s
\end{array}$$

by Symmetry and Reflexivity

by Union and Composition (with $p = \mathrm{id}$, $q = \nabla : X + X + X \to X$)

by Composition ($p = \overline{s} \cdot q_j$, $q = \mathrm{id}$)

by Epi-Cancellation

Induction Case: Suppose (*) holds and $s : S \to R_0^{n+1}$ with S finitely presentable is given.

$$
\begin{tikzcd}
Q_i \arrow[r, "q_i"] \arrow[ddd, two heads, "\overline{q}_i"'] & Q \arrow[rrrr, "\overline{s}"] \arrow[ddd, two heads, "\overline{e}_{n+1}"'] & & & & P_n \arrow[dl, "p_n"'] \arrow[dr, "p'_n"] \arrow[ddd, bend left=90, two heads, "e_{n+1}"] & & & \\
& & & & R_0^n \arrow[dl, "u_n"'] \arrow[dr, "v_n"] & & R_0 \arrow[dl, "u_0"'] \arrow[dr, "v_0"] & & \\
& & & X & & X & & X & \\
S_i \arrow[r, tail, "s_i"'] & S \arrow[rrrr, "s"'] & & & & R_0^{n+1} \arrow[ull, "u_{n+1}"] \arrow[urr, "v_{n+1}"'] & & &
\end{tikzcd}
$$

Analogously to the above case we form the pullback Q of s and e_n and express Q as a filtered colimit of finitely presentable objects Q_i with the colimit cocone $q_i : Q_i \to Q\,(i \in I)$. We then form regular factorizations of $\overline{e}_{n+1} \cdot q_i$ as indicated, and by the above argument we conclude that s_j is an isomorphism for some $j \in I$. Therefore, by induction hypothesis, from $u \equiv v$, we can deduce

$$u_0 \cdot p'_n \cdot \overline{s} \cdot q_j \equiv v_0 \cdot p'_n \cdot \overline{s} \cdot q_j \qquad \text{and} \qquad u_n \cdot p_n \cdot \overline{s} \cdot q_j \equiv u_0 \cdot p'_n \cdot \overline{s} \cdot q_j \quad (3.1)$$

since $v_n \cdot p_n = u_0 \cdot p'_n$. Hence, by Transitivity,

$$u_n \cdot p_n \cdot \overline{s} \cdot q_j \equiv v_0 \cdot p'_n \cdot \overline{s} \cdot q_j$$

that is,

$$u_{n+1} \cdot s \cdot s_j \cdot \overline{q}_j \equiv v_{n+1} \cdot s \cdot s_j \cdot \overline{q}_j \,.$$

Now, by Epi-Cancellation, we conclude

$$u_{n+1} \cdot s \equiv v_{n+1} \cdot s \,.$$

(2) We are ready to prove the completeness of the Quasi-Equational Logic. Since the Coequalizer Deduction System is complete, and the only

deduction rule not contained in 3.1 is the Coequalizer rule, it is sufficient to find a translation of that rule:

$$\begin{tikzcd} & R' \arrow[d, shift right, "u'"'] \arrow[d, shift left, "v'"] & \\ R \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & X \arrow[r, "c"] & Y \end{tikzcd}$$

Suppose $u \equiv v$ and $u' \equiv v'$ are quasi-equations such that the coequalizer c of u, v fulfils $c \cdot u' = c \cdot v'$. Then we will find a derivation of $u' \equiv v'$ from $u \equiv v$ in the deduction system of 3.1. Let $\hat{u}, \hat{v} : \widehat{R} \to X$ be the kernel pair of c. Then $\widehat{R}$, being an equivalence relation, is the smallest equivalence relation containing R_0 in 3.7, consequently $\widehat{R} = \bigcup_{n \in \mathbb{N}} R_0^n$ by 3.5(vi). Then the pair u', v' factorizes through it via a morphism $t : R' \to \hat{R}$. Now $\widehat{R}$ is a chain colimit by 3.8, and R' is finitely presentable, thus, t factors through one of the colimit morphisms $r_n = [u_n, v_n] : R_0^n \rightarrowtail \widehat{R}$:

$$\begin{tikzcd} & & R' \arrow[dll, "\bar{t}"'] \arrow[dl, "t"] \arrow[d, shift right, "u'"'] \arrow[d, shift left, "v'"] \\ R_0^n \arrow[r, tail, "r_n"] \arrow[rr, bend left, "u_n"] \arrow[rr, bend right, "v_n"'] & \widehat{R} \arrow[r, shift left, "\hat{u}"] \arrow[r, shift right, "\hat{v}"'] & X \end{tikzcd}$$

That is, we have $\bar{t} : R' \to R_0^n$ such that $u_n \cdot \bar{t} = u'$ and $v_n \cdot \bar{t} = v'$. Thus, we can derive $u' \equiv v'$ from $u \equiv v$, see (1). $\square$

3.10. Remark (i) Observe that the effectivity of equivalence relations was not used in the first part of the proof.

(ii) Observe also that Epi-Cancellation was only used for regular epimorphisms in the above proof. We will use it more generally in 3.12 below.

3.11. Remark The above theorem implies that in categories

$$\mathbf{Alg}\,\Sigma$$

of algebras of an arbitrary finitary S-sorted (algebraic) signature Σ the Quasi-Equational Logic is complete: in fact, $\mathbf{Alg}\,\Sigma$ is an exact, locally finitely presentable category. We want to extend this result to categories $\mathbf{Mod}\,\Sigma$ of 3.6(iii). Although $\mathbf{Mod}\,\Sigma$ does not have effective equivalence relations, we have the following:

3.12. Proposition *The Quasi-Equational Logic is complete in* $\mathbf{Mod}\,\Sigma$.

Proof Consider the adjoint situation

$$\begin{tikzcd} \mathbf{Mod}\,\Sigma \arrow[r, bend left, "W"] \arrow[r, phantom, "\top"] & \mathbf{Alg}\,\Sigma_f \arrow[l, bend left, "D"] \end{tikzcd}$$

where W forgets the relations and D defines them to be empty. Both W and D preserve limits, colimits and finitely presentable objects. Consequently, they preserve regular factorizations and composition of relations.

As in the previous proof, we just need to translate the Coequalizer rule: given quasi-equations in $\mathbf{Mod}\,\Sigma$:

$$\begin{tikzcd} & R' \arrow[d, shift right, "u'"'] \arrow[d, shift left, "v'"] & \\ R \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & X \arrow[r, "c"] & Y \end{tikzcd}$$

with $c \cdot u' = c \cdot v'$ for $c = \text{coeq}(u, v)$, we will prove that

$$u \equiv v \vdash u' \equiv v'.$$

From the proof of 3.9 and 3.10 we have that $u \equiv v \vdash u_n \cdot s \equiv v_n \cdot s$ for all $s : S \to R_0^n$ with S finitely presentable. Further, since Wc is the coequalizer of Wu, Wv and the kernel pair of Wc is represented by the relation

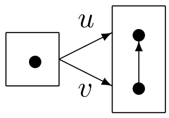
$$W\widehat{R} = \bigcup WR_0^n = \bigcup (WR)_0^n$$

we see that the pair Wu', Wv' factorizes through some $Wu_n, Wv_n : WR_0^n \to WX$ via a morphism $\bar{t} : WR' \to WR_0^n$. In case $R' = DWR'$ we have a morphism $s : R' \to R_0^n$ with $\bar{t} = Ws$, and then $u \equiv v \vdash u' \equiv v'$ because $u' = u_n \cdot s$ and $v' \equiv v_n \cdot s$. In general, the counit of $D \dashv W$ gives an epimorphism $e : DWR' \to R'$ (carried by the identity map) and the above consideration yields $u \equiv v \vdash u' \cdot e \equiv v' \cdot e$. Using Epi-Cancellation, we derive $u \equiv v \vdash u' \equiv v'$. $\square$

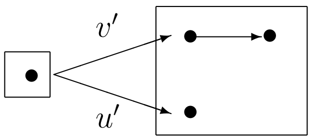
3.13. Example The Quasi-Equational Logic is complete in the category of posets. This follows easily from Example 3.4: If $u \equiv v$ is a consequence of a set Q of quasi-equations, and if some member of Q does not satisfy (*), then $Q \vdash l \equiv r$, and from that $Q \vdash u \equiv v$ follows. If all members of Q satisfy (*) then also $u \equiv v$ does (it is easy to see that the set of all quasi-equations satisfying (*) is closed under the deduction rules of 3.1). Thus, either Q contains a nontrivial quasi-equation, in which case we deduce $u_0 \equiv v_0$ from Q and we also deduce $u \equiv v$ from $u_0 \equiv v_0$. Or Q contains only quasi-equations $w \equiv w$, but then $u = v$.

3.14. Example of incompleteness of the Quasi-Equational Logic. For the language Σ_2 of one binary relation the category $\mathbf{Mod}\,\Sigma_2$ (of directed graphs and homomorphisms) has complete Quasi-Equational Logic by 3.12. Let $\mathcal{A}$ be the full subcategory of all graphs (X, R) which are antireflexive ($R \cap \Delta_X = \emptyset$) with the terminal object added. $\mathcal{A}$ is closed under limits, filtered colimits and regular factorizations in $\mathbf{Mod}\,\Sigma_2$, thus, it is a regular, locally finitely presentable subcategory.

The quasi-equation



is satisfied by precisely those graphs in $\mathcal{A}$ that are discrete or terminal. Therefore, it has as a consequence the quasi-equation



However, we cannot derive $u' \equiv v'$ from $u \equiv v$. In fact, all quasi-equations $\overline{u} \equiv \overline{v}$ that can be deduced from $u \equiv v$ have the property (*) in 3.4, since the quasi-equation $u \equiv v$ fulfils it and the set of all quasi-equations $\overline{u} \equiv \overline{v}$ fulfilling it is closed under all deduction rules. Since $u' \equiv v'$ does not, the proof is concluded.

4 The Quasi-Equational Logic in Non-Exact Categories

In the present section we work in a locally finitely presentable category with effective equivalence relations – but we do not assume regularity. We prove, again, that the Quasi-Equational Logic is complete. However, we need to extend slightly the concept of quasi-equation: we will consider all parallel pairs $u, v : R \to X$ where X is finitely presentable but R only finitely generated. Since finitely generated objects are precisely the strong quotients $e : \overline{R} \twoheadrightarrow R$ of finitely presentable objects $\overline{R}$, the difference is just a small technicality: for the quasi-equations (in the sense of preceding sections) $u' \equiv v'$ where $u' = u \cdot e$, $v' = v \cdot e$ we have $u \equiv v \vdash u' \equiv v'$ by Composition and, conversely, $u' \equiv v' \vdash u \equiv v$ by Epi-Cancellation.

4.1. Definition A *weak quasi-equation* is a parallel pair of morphisms (u, v) whose domain is finitely generated and codomain is finitely presentable. An object A satisfies $u \equiv v$ if $\mathcal{A}(-, A)$ merges u and v.

4.2. Theorem *The Quasi-Equational Logic is complete and sound in every locally finitely presentable category with effective equivalence relations. That is, given a set Q of weak quasi-equations, then a weak quasi-equation $u \equiv v$ is a consequence of Q iff it can be deduced from Q.*

4.3. Remark Before we prove this theorem, we need to modify Remark 3.5. Every locally finitely presentable category has the factorization system (strong epi, mono), see [3], 1.61. By a relation we again understand a subobject of $X \times X$. In the definition of composite, see 3.5 (iii), we just use the (strong epi, mono)-factorization of $u \cdot p$, $v' \cdot p'$. Then the concept of equivalence relation and having effective equivalence relations as in 3.5. However, relation composition is not associative in general.

Let R be a reflexive and symmetric relation. Then the smallest equivalence relation containing R is

$$\widehat{R} = R \cup (R \cdot R) \cup (R \cdot (R \cdot R)) \cup ((R \cdot R) \cdot R) \cup \dots$$

that is, the union

$$\widehat{R} = \bigcup_{i \in I} R_i$$

of the smallest set $R_i\,(i \in I)$ of relations containing R and closed under composition. This is essentially proved in [1]. For the sake of easy reference here is a proof:

(a) $\widehat{R}$ is reflexive since R is (so that R_i is reflexive for every i since a composite of reflexive relations is reflexive).

(b) $\widehat{R}$ is symmetric since R is: the formula

$$(R_j \cdot R_i)^{-1} = R_i^{-1} \cdot R_j^{-1}$$

implies that the set $\{R_i\}_{i \in I}$ is closed under the formation of inverses.

(c) $\widehat{R}$ is transitive because by 3.8

$$\widehat{R} = \operatorname*{colim}_{i \in I} R_i$$

and in locally finitely presentable categories pullbacks commute with filtered colimits. Indeed, let $u_i, v_i : R_i \to X$ be the pair representing r_i and $\hat{u}, \hat{v} : \widehat{R} \to X$ that representing $\hat{r}$. Form the pullback

$$\begin{tikzcd}
 & & \widehat{P} \arrow[dl, "p"'] \arrow[dr, "p'"] & & \\
 & \widehat{R} \arrow[dl, "\hat{u}"'] \arrow[dr, "\hat{v}"] & & \widehat{R} \arrow[dl, "\hat{u}"'] \arrow[dr, "\hat{v}"] & \\
X & & X & & X
\end{tikzcd}$$

Transitivity of $\widehat{R}$ means that the pair $\hat{u} \cdot p$, $\hat{v} \cdot p' : \widehat{P} \to X$ factors through $\hat{u}, \hat{v}$. The above pullback is a colimit of the pullbacks

$$\begin{tikzcd}
 & & P_i \arrow[dl, "p_i"'] \arrow[dr, "p_i'"] & & \\
 & R_i \arrow[dl, "u_i"'] \arrow[dr, "v_i"] & & R_i \arrow[dl, "u_i"'] \arrow[dr, "v_i"] & \\
X & & X & & X
\end{tikzcd}$$

and for each $i \in I$ we have $j \in J$ with $R_j = R_i \cdot R_i$, therefore, the pair $u_i \cdot p_i, v_i \cdot p_i' : P_i \to X$ factors through u_j, v_j. From $p = \operatorname{colim} p_i$ and

$p' = \operatorname{colim} p'_i$ we conclude that the pair $\hat{u} \cdot p$, $\hat{v} \cdot p'$ factors through $\hat{u}, \hat{v}$, as requested.

(d) It is obvious that an equivalence relation S containing R contains each R_i, thus, $\widehat{R} \subseteq S$. Moreover, it is easy to see that for every morphism $c : X \to Y$ we have

$$c \cdot u = c \cdot v \quad \text{iff} \quad c \cdot \hat{u} = c \cdot \hat{v}$$

(since $c \cdot u = c \cdot v$ implies that the set of all relations u', v' with $c \cdot u' = c \cdot v'$ is closed under inverse and relation composite – thus, $c \cdot u_i = c \cdot v_i$ for all $i \in I$.)

4.4. Notation For a weak quasi-equation $u, v : R \to X$ we denote by $u_0, v_0 : R_0 \to X$ the reflexive-and-symmetric hull given by a factorization of $[u, v, \mathrm{id}]$, $[v, u, \mathrm{id}] : R + R + X \to X$ as a strong epimorphism followed by a collectively monic pair (u_0, v_0). Then we have the above subobjects

$$r_i : R_i \to \widehat{R} \;\; (i \in I)$$

forming the least equivalence relation $\widehat{R} = \bigcup_{i \in I} R_i$ containing R_0 represented by pairs $u_i, v_i : R_i \to X$. If the pair $\hat{u}, \hat{v} : \widehat{R} \to X$ represents the equivalence relation $\widehat{R}$, then $u_i = \hat{u} \cdot r_i$ and $v_i = \hat{v} \cdot r_i$.

4.5. Proof of Theorem 4.2 Let $u, v : R \to X$ be a weak quasi-equation which is a consequence of a set Q of weak quasi-equations. We prove $Q \vdash u \equiv v$.

(1) We first prove that for every weak quasi-equation $u \equiv v$ we have

$$u \equiv v \vdash u_i \cdot s \equiv v_i \cdot s \quad \text{for every } s : S \to R_i \text{ with } S \text{ finitely generated}$$

by structural induction on $i \in I$: we verify first the case $s : S \to R_0$ for the reflexive-and-symmetric hull R_0 of R, and then show that if the above holds for R_i and R_j, then it holds for $R_i \cdot R_j$.

Base case: As in 3.9 derive $[u, v, \mathrm{id}] \equiv [v, u, \mathrm{id}]$ from $u \equiv v$, then use Epi-Cancellation to get $u_0 \equiv v_0$. Using Composition $u \equiv v \vdash u_0 \cdot s \equiv v_0 \cdot s$.

Induction case: Let $R_k = R_i \cdot R_j$ and let

$$u \equiv v \;\vdash\; u_i \cdot s \equiv v_i \cdot s \quad \text{and} \quad u \equiv v \;\vdash\; u_j \cdot s \equiv v_j \cdot s$$

hold for all morphisms s with finitely generated domain and codomain such that the composites are defined. Given

$$s : S \to R_k\,,\ \ S \text{ finitely generated,}$$

we prove $u \equiv v \ \vdash\ u_k \cdot s \equiv v_k \cdot s$. Let us recall the definition of $R_k = R_i \cdot R_j$:

$$\begin{tikzcd}
Q_d \arrow[rrr, "q_d"] \arrow[dd, "\overline{q}_d"'] & & & P_k \arrow[dl, "p_i"'] \arrow[dr, "p_j"] \arrow[ddd, bend left=70, two heads, "e_k"] & & & \\
& & R_i \arrow[dl, "u_i"'] \arrow[dr, "v_i"] & & R_j \arrow[dl, "u_j"'] \arrow[dr, "v_j"] & & \\
\overline{Q}_d \arrow[drrr, tail, "m_d"'] & X & & X & & X & \\
S \arrow[u, dashed, "\overline{s}"] \arrow[rrr, "s"'] & & & R_k \arrow[ull, "u_k"'] \arrow[urr, "v_k"'] & & &
\end{tikzcd}$$

Express P_k as a filtered colimit of finitely presentable objects $Q_d\,(d \in D)$ with the colimit cocone $q_d : Q_d \to P_k\,(d \in D)$ and let the (strong epi, mono)-factorization of $e_k \cdot q_d$ be

$$e_k \cdot q_d = m_d \cdot \overline{q}_d \quad \text{for } \ m_d : \overline{Q}_d \rightarrowtail R_k\,.$$

Then $R_k = \bigcup_{d\in D} \overline{Q}_d$ because $[m_d] \cdot \coprod_{d\in D} \overline{q}_d = e_k \cdot [q_d]$ is a strong epimorphism, thus, so is $[m_d]$. By 3.8

$$R_k = \operatorname{colim} \overline{Q}_d$$

is a colimit of a directed diagram of monomorphisms. Since S is finitely generated, $\mathcal{A}(S, -)$ preserves this colimit, consequently, $s : S \to \operatorname{colim} \overline{Q}_d$ factors through some m_d:

$$s = m_d \cdot \overline{s} \quad \text{for some } d \in D \text{ and } \overline{s} : S \to \overline{Q}_d\,.$$

By induction hypothesis,

$$u \equiv v \ \vdash\ u_i \cdot p_i \cdot q_d \equiv v_i \cdot p_i \cdot q_d \quad \text{and} \quad u \equiv v \ \vdash\ u_j \cdot p_j \cdot q_d \equiv v_j \cdot p_j \cdot q_d$$

which by Transitivity and $v_i \cdot p_i = u_j \cdot p_j$ yields

$$u \equiv v \ \vdash \ u_i \cdot p_i \cdot q_d = v_j \cdot p_j \cdot q_d \,.$$

In other words,

$$u \equiv v \ \vdash \ u_k \cdot e_k \cdot q_d \equiv v_k \cdot e_k \cdot q_d \,.$$

Now from $e_k \cdot q_d = m_d \cdot \overline{q}_d$ we deduce, due to Epi-Cancellation,

$$u \equiv v \vdash u_k \cdot m_d \equiv v_k \cdot m_d$$

and using $s = m_d \cdot \overline{s}$ we get, via Composition,

$$u \equiv v \vdash u_k \cdot s \equiv v_k \cdot s$$

as desired.

(2) The rule Coequalizer (for finitary morphisms) is, due to (1), translated to the rules of 3.1 quite analogously as in the proof of 3.9, part (2).

(3) To prove the completeness, let $u, v : R \to X$ be a weak quasi-equation which is a consequence of the set Q. Since R is finitely generated, it is a strong quotient $e : R^* \twoheadrightarrow R$ of a finitely presentable object R^* and we consider the quasi-equation $u^* \equiv v^*$ obtained from $u \equiv v$ by composition with e. Analogously, for every member $\overline{u} \equiv \overline{v}$ of Q we form a quasi-equation $\overline{u}^* \equiv \overline{v}^*$ in the above manner and get a set Q^* of quasi-equations.

It is clear that $u \equiv v$ is a consequence of Q iff $u^* \equiv v^*$ is a consequence of Q^*: use the soundness of Epi-Cancellation and Composition. By Theorem 2.6, there is a formal proof of $u^* \equiv v^*$ from Q^* using the Coequalizer Deduction System. We see from (2) that this formal proof gives rise to a proof of $u^* \equiv v^*$ from Q^* using the deduction rules of 3.1. Now $Q \vdash u \equiv v$ follows from the fact that $Q \vdash Q^*$ and $u^* \equiv v^* \vdash u \equiv v$. □

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WHAT SEPARABLE FROBENIUS MONOIDAL FUNCTORS PRESERVE?

by Micah McCURDY & Ross STREET

Abstract

Separable Frobenius monoidal functors were defined and studied under that name by Kornél Szlachányi [14], [15], and by Brian Day and Craig Pastro[5]. They are a special case of the linearly distributive functors of Robin Cockett and Robert Seely [4]. Our purpose is to develop the theory of such functors in a very precise sense. We characterize geometrically which monoidal expressions are preserved by these functors (or rather, are stable under conjugation in an obvious sense). We show, by way of corollaries, that they preserve lax (meaning not necessarily invertible) Yang-Baxter operators, weak Yang-Baxter operators in the sense of [2], and (in the braided case) weak bimonoids in the sense of [12]. Actually, every weak Yang-Baxter operator is the image of a genuine Yang-Baxter operator under a separable Frobenius monoidal functor. Prebimonoidal functors are also defined and discussed.

Les foncteurs monoïdaux Frobenius séparables ont été définis et étudiés, sous ce nom, par Kornél Szlachányi [14], [15], et par Brian Day et Craig Pastro [5]. Ils sont un cas spécial des foncteurs linéaires entre catégories linéarement distributives, introduits par Robin Cockett et Robert Seely [4]. Notre objet est de développer la théorie de ces foncteurs en un sens très précis. Nous caractérisons géométriquement les expressions qui sont préservées par ces foncteurs (c'est-à-dire, sont stables sous conjugaison en un sens évident). Nous montrons sous forme de corollaire qu'ils préservent les opérateurs Yang-Baxter lax (non-nécessairement inversibles), les opérateurs Yang-Baxter faibles dans le sens de [2], et (dans le cas tressé) les bimonoïdes faibles dans le sens de [12]. En fait, chaque opérateur Yang-Baxter faible est une image d'un opérateur Yang-Baxter véritable par un foncteur Frobenius séparable. Les foncteurs prébimonoïdaux sont aussi définis et discutés.

Mathematics Subject Classification: 18D10

Keywords: Frobenius monoidal functor, monoidal category, weak bimonoid, Yang-Baxter operator, separable Frobenius algebra, weak distributive law.

Dedicated to Francis Borceux on the occasion of his 60$^{\text{th}}$ birthday.

1 Introduction

Frobenius monoidal functors $F : \mathcal{C} \to \mathcal{X}$ between monoidal categories were defined and studied under that name in [14], [15] and [5] and in a more general context in [4]. If the domain $\mathcal{C}$ is the terminal category 1, then F amounts to a Frobenius monoid in $\mathcal{X}$. It was shown in [5] that Frobenius monoidal functors compose, so that, by the last sentence, they take Frobenius monoids to Frobenius monoids. We concentrate here on separable Frobenius F and show that various kinds of Yang-Baxter operators and (in the braided case) weak bimonoids are preserved by F.

We introduce prebimonoidal functors $F : \mathcal{C} \to \mathcal{X}$ between monoidal categories which are, say, braided. If the domain $\mathcal{C}$ is the terminal category 1, then any (weak) bimonoid in $\mathcal{X}$ gives an example of such an F. We show that prebimonoidal functors compose and relate them to separable Frobenius functors.

2 Definitions

Justified by coherence theorems (see [8] for example), we write as if our monoidal categories were strict. A functor $F : \mathcal{C} \to \mathcal{X}$ between monoidal categories is *Frobenius* when it is equipped with a monoidal structure

$$\phi_{A,B} : FA \otimes FB \to F(A \otimes B) \qquad \phi_0 : I \to FI,$$

and an opmonoidal structure

$$\psi_{A,B} : F(A \otimes B) \to FA \otimes FB \qquad \psi_0 : FI \to I$$

such that

$$\begin{tikzcd}
F(A\otimes B)\otimes FC \arrow[r, "\phi_{A\otimes B,C}"] \arrow[d, "\psi_{A,B}\otimes 1"'] & F(A\otimes B\otimes C) \arrow[d, "\psi_{A,B\otimes C}"] \\
FA\otimes FB\otimes FC \arrow[r, "1\otimes\phi_{B,C}"'] & FA\otimes F(B\otimes C)
\end{tikzcd}$$

$$\begin{tikzcd}
FA\otimes F(B\otimes C) \arrow[r, "\phi_{A,B\otimes C}"] \arrow[d, "1\otimes\psi_{B,C}"'] & F(A\otimes B\otimes C) \arrow[d, "\psi_{A\otimes B,C}"] \\
FA\otimes FB\otimes FC \arrow[r, "\phi_{A,B}\otimes 1"'] & F(A\otimes B)\otimes FC
\end{tikzcd}$$

We shall call $F : \mathcal{C} \to \mathcal{X}$ *separable Frobenius monoidal* when it is Frobenius monoidal and each composite

$$F(A\otimes B) \xrightarrow{\psi_{A,B}} FA\otimes FB \xrightarrow{\phi_{A,B}} F(A\otimes B)$$

is the identity. We call $F : \mathcal{C} \to \mathcal{X}$ *strong monoidal* when it is separable Frobenius monoidal, $\phi_{A,B}$ is invertible, and ϕ_0 and ψ_0 are mutually inverse.

Suppose $F : \mathcal{C} \to \mathcal{X}$ is both monoidal and opmonoidal. By coherence, we have canonical morphisms

$$\phi_{A_1,\ldots,A_n} : FA_1 \otimes\cdots\otimes FA_n \longrightarrow F(A_1\otimes\cdots\otimes A_n)$$

and

$$\psi_{A_1,\ldots,A_n} : F(A_1\otimes\cdots\otimes A_n) \longrightarrow FA_1\otimes\cdots\otimes FA_n$$

defined by composites of instances of ϕ and ψ. If $n = 0$ then these reduce to ϕ_0 and ψ_0; if $n = 1$, they are identities.

The *F-conjugate* of a morphism

$$f : A_1\otimes\cdots\otimes A_n \longrightarrow B_1\otimes\cdots\otimes B_m$$

in $\mathcal{C}$ is the composite f^F :

$$\begin{tikzcd}[column sep=small]
FA_1 \otimes \cdots \otimes FA_n \arrow[dr, "\phi_{A_1,\ldots,A_n}"] & & & FB_1 \otimes \cdots \otimes FB_m \\
& F(A_1 \otimes \cdots \otimes A_n) \arrow[r, "Ff"] & F(B_1 \otimes \cdots \otimes B_m) \arrow[ur, "\psi_{B_1,\ldots,B_m}"] &
\end{tikzcd}$$

in $\mathcal{X}$. For $m = 1$, this really only requires F to be monoidal while, for $n = 1$, this really only requires F to be opmonoidal. If a structure in $\mathcal{C}$ is defined in terms of morphisms between multiple tensors, we can speak of the F-*conjugate* of the structure in $\mathcal{X}$. For example, we can easily see the well-known fact that the F-conjugate of a monoid, for F monoidal, is a monoid; dually, the F-conjugate of a comonoid, for F opmonoidal, is a comonoid. It was shown in [5] that the F-conjugate of a Frobenius monoid is a Frobenius monoid.

Notice that, for a separable Frobenius monoidal functor F, we have $\phi_n \circ \psi_n = 1$ for $n > 0$.

Suppose $\mathcal{C}$ and $\mathcal{X}$ are braided monoidal. We say that a separable Frobenius monoidal functor $F : \mathcal{C} \longrightarrow \mathcal{X}$ is *braided* when the F-conjugate of the braiding $c_{A,B} : A \otimes B \to B \otimes A$ in $\mathcal{C}$ is equal to $c_{FA,FB} : FA \otimes FB \to FB \otimes FA$ in $\mathcal{X}$. Because of separability, it follows that F is braided as both a monoidal and opmonoidal functor.

A *lax Yang-Baxter (YB) operator* on an object A of a monoidal category $\mathcal{C}$ is a morphism $y : A \otimes A \longrightarrow A \otimes A$ satisfying the condition

$$(y \otimes 1) \circ (1 \otimes y) \circ (y \otimes 1) = (1 \otimes y) \circ (y \otimes 1) \circ (1 \otimes y)$$

A *Yang-Baxter (YB) operator* is an invertible lax YB-operator.

Recall that the Cauchy (idempotent splitting) completion $\mathcal{QC}$ of a category $\mathcal{C}$ is the category whose objects are pairs (A, e) where $e : A \to A$ is an idempotent on A and whose morphisms $f : (A, e) \to (B, p)$ are morphisms $f : A \to B$ in $\mathcal{C}$ satisfying $pfe = f$ (or equivalently $pf = f$ and $fe = f$). Note emphatically that the identity morphism of (A, e) is $e : (A, e) \to (A, e)$; in particular, this means the forgetful $\mathcal{QC} \to \mathcal{C}$, $(A, e) \mapsto A$, is not a functor. If $\mathcal{C}$ is monoidal then so is $\mathcal{QC}$ with $(A, e) \otimes (A', e') = (A \otimes A', e \otimes e')$ and unit $(I, 1)$.

A *weak Yang-Baxter operator on A* (compare [2]) in $\mathcal{C}$ consists of an idempotent $\nabla : A\otimes A \longrightarrow A\otimes A$, and lax YB-operators $y : A\otimes A \longrightarrow A\otimes A$ and $y' : A\otimes A \longrightarrow A\otimes A$, subject to the following conditions:

$$\nabla\circ y = y = y\circ\nabla \tag{2.1}$$
$$\nabla\circ y' = y' = y'\circ\nabla \tag{2.2}$$
$$y\circ y' = \nabla = y'\circ y \tag{2.3}$$

$$(1\otimes\nabla)\circ(\nabla\otimes 1) = (\nabla\otimes 1)\circ(1\otimes\nabla) \tag{2.4}$$
$$(1\otimes y)\circ(\nabla\otimes 1) = (\nabla\otimes 1)\circ(1\otimes y)\,, \tag{2.5}$$
$$(1\otimes\nabla)\circ(y\otimes 1) = (y\otimes 1)\circ(1\otimes\nabla)\,. \tag{2.6}$$

Notice that Equations 2.1, 2.2 and 2.3 say that $y : (A\otimes A,\nabla) \longrightarrow (A\otimes A,\nabla)$ is a morphism with inverse y' in $\mathcal{QC}$.

Suppose $(A,\mu : A\otimes A \longrightarrow A,\eta : I \longrightarrow A)$ and $(B,\mu : B\otimes B \longrightarrow A,\eta : I \longrightarrow B)$ are monoids in the monoidal category $\mathcal{C}$. Let a morphism $\lambda : A\otimes B \longrightarrow B\otimes A$ be given. The following conditions imply that $A\otimes B$ becomes a monoid with multiplication $A\otimes B\otimes A\otimes B \xrightarrow{1\otimes\lambda\otimes 1} A\otimes A\otimes B\otimes B \xrightarrow{\mu\otimes\mu} A\otimes B$ and unit $I \xrightarrow{\eta\otimes\eta} A\otimes B$:

$$\lambda\circ(\mu\otimes 1_B) = (1_B\otimes\mu)\circ(\lambda\otimes 1_A)\circ(1_A\otimes\lambda)\,, \tag{2.7}$$
$$\lambda\circ(1_A\otimes\mu) = (\mu\otimes 1_A)\circ(1_B\otimes\lambda)\circ(\lambda\otimes 1_B)\,, \tag{2.8}$$

$$\lambda\circ(\eta\otimes 1_B) = 1_B\otimes\eta, \qquad \lambda\circ(1_A\otimes\eta) = \eta\otimes 1_A. \tag{2.9}$$

These are the conditions for λ to be a *distributive law* [3]. A *weak distributive law* [13] is the same except that Equations 2.9 are replaced by:

$$(1\otimes\mu)\circ(\lambda\otimes 1)\circ(\eta\otimes 1\otimes 1) = (\mu\otimes 1)\circ(1\otimes\lambda)\circ(1\otimes 1\otimes\eta)\,. \tag{2.10}$$

In the monoidal category $\mathcal{C}$, suppose A is equipped with a multiplication $\mu : A\otimes A \longrightarrow A$ and a "switch morphism" $\lambda : A\otimes A \longrightarrow A\otimes A$. Supply $A\otimes A$ with the multiplication $A\otimes A\otimes A\otimes A \xrightarrow{1\otimes\lambda\otimes 1} A\otimes A\otimes A\otimes A \xrightarrow{\mu\otimes\mu} A\otimes A$. Then a comultiplication $\delta : A \longrightarrow A\otimes A$ preserves multiplication when the following holds:

$$\delta\circ\mu = (\mu\otimes\mu)\circ(1\otimes\lambda\otimes 1)\circ(\delta\otimes\delta)\,. \tag{2.11}$$

Dually, if we start with δ and λ, define the comultiplication $A \otimes A \xrightarrow{\delta\otimes\delta} A \otimes A \otimes A \otimes A \xrightarrow{1\otimes\lambda\otimes 1} A \otimes A \otimes A \otimes A$ on $A \otimes A$, and ask for μ to preserve comultiplication, we are led to the same Equation 2.11

In a braided monoidal category $\mathcal{C}$, a *weak bimonoid* (see [12]) is an object A equipped with a monoid structure and a comonoid structure satisfying Equation 2.11 (with $\lambda = c_{A,A}$) and the "weak unit and counit" conditions:

$$\begin{aligned} \varepsilon \circ \mu \circ (1 \otimes \mu) &= (\varepsilon \otimes \varepsilon) \circ (\mu \otimes \mu) \circ (1 \otimes \delta \otimes 1) \\ &= (\varepsilon \otimes \varepsilon) \circ (\mu \otimes \mu) \circ \left(1 \otimes c_{A,A}^{-1} \otimes 1\right) \circ (1 \otimes \delta \otimes 1) \end{aligned} \tag{2.12}$$

$$\begin{aligned} (1 \otimes \delta) \circ \delta \circ \eta &= (1 \otimes \mu \otimes 1) \circ (\delta \otimes \delta) \circ (\eta \otimes \eta) \\ &= (1 \otimes \mu \otimes 1) \circ \left(1 \otimes c_{A,A}^{-1} \otimes 1\right) \circ (\delta \otimes \delta) \circ (\eta \otimes \eta) \end{aligned} \tag{2.13}$$

A *lax Yang-Baxter (YB) operator* on a functor $T : \mathcal{A} \to \mathcal{C}$ into a monoidal category $\mathcal{C}$ is a natural family of morphisms

$$y_{A,B} : TA \otimes TB \longrightarrow TB \otimes TA$$

satisfying the condition

$$\begin{tikzcd} & TB \otimes TA \otimes TC \arrow[r, "1\otimes y"] & TB \otimes TC \otimes TA \arrow[dr, "y\otimes 1"] & \\ TA \otimes TB \otimes TC \arrow[ur, "y\otimes 1"] \arrow[dr, "1\otimes y"'] & & & TC \otimes TB \otimes TA \\ & TA \otimes TC \otimes TB \arrow[r, "y\otimes 1"'] & TC \otimes TA \otimes TB \arrow[ur, "1\otimes y"'] & \end{tikzcd}$$

One special case is where $\mathcal{A} = 1$ so that T is an object of $\mathcal{C}$: then we obtain a lax YB-operator on the object T as above. Another case is where $\mathcal{A} = \mathcal{C}$ and T is the identity functor: each (lax) braiding c on $\mathcal{C}$ gives an example with $y_{A,B} = c_{A,B}$.

Suppose $T : \mathcal{A} \to \mathcal{C}$ is a functor and $F : \mathcal{C} \to \mathcal{X}$ is a functor between monoidal categories. Suppose lax YB-operators y on T and z on FT are given. We define F to be *prebimonoidal relative to y and z* when it is monoidal and opmonoidal, and satisfies

$$\begin{tikzcd}[column sep=small]
FTA\otimes FTB\otimes FTC\otimes FTD \arrow[rr, "1\otimes z\otimes 1"] & & FTA\otimes FTC\otimes FTB\otimes FTD \arrow[dr, "\phi\otimes\phi"] \\
F(TA\otimes TB)\otimes F(TC\otimes TD) \arrow[u, "\psi\otimes\psi"] \arrow[dr, "\phi"'] & & & F(TA\otimes TC)\otimes F(TB\otimes TD) \\
& F(TA\otimes TB\otimes TC\otimes TD) \arrow[r, "F(1\otimes y\otimes 1)"'] & F(TA\otimes TC\otimes TB\otimes TD) \arrow[ur, "\psi"'] &
\end{tikzcd}$$

When $\mathcal{C}$ and $\mathcal{X}$ are (lax) braided and T is the identity with $y_{A,B} = c_{A,B}$ and $z_{A,B} = c_{FA,FB}$, we merely say F is *prebimonoidal*. Such an F is bimonoidal when, furthermore, FI, with its natural monoid and comonoid structure, is a bimonoid. We were surprised not to find this concept in the literature, however, we have found that it was presented in preliminary versions of the forthcoming book [1], and in talks by the authors of the same.

3 Separable invariance and connectivity

We begin by reviewing some concepts from [9]. Progressive plane string diagrams are deformation classes of progressive plane graphs. Here we will draw them progressing from left to right (direction of the x-axis) rather than from down to up (direction of the y-axis). A tensor scheme is a combinatorial directed graph with vertices and edges such that the source and target of each edge is a word of vertices (rather than a single vertex). Progressive string diagrams Γ can be labelled (or can have valuations) in a tensor scheme $\mathcal{D}$: for a given labelling $v : \Gamma \to \mathcal{D}$, the labels on the edges (strings) γ of Γ are vertices $v(\gamma)$ of $\mathcal{D}$ while the labels on the vertices (nodes) x of Γ are edges $v(x) : v(\gamma_1)\cdots v(\gamma_m) \to v(\delta_1)\cdots v(\delta_n)$ of $\mathcal{D}$ where $\gamma_1, \cdots, \gamma_m$ are the input edges and $\delta_1, \cdots, \delta_n$ are the output edges of x read from top to bottom; see Figure 1 where $f = v(x)$, $A_1 = v(\gamma_1)$, $B_n = v(\delta_n)$, and so on. The free monoidal category $\mathcal{F}\mathcal{D}$ on a tensor scheme $\mathcal{D}$ has objects words of vertices and morphisms progressive plane string diagrams labelled in $\mathcal{D}$; composition progresses horizontally while tensoring is defined by stacking diagrams vertically.

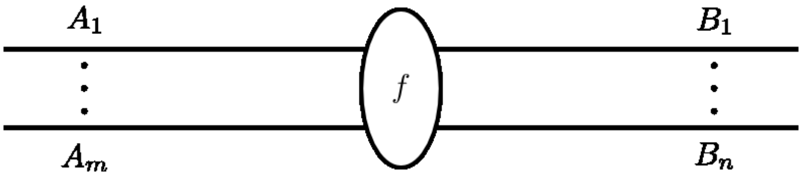


Figure 1:

Every monoidal category $\mathcal{C}$ has an underlying tensor scheme: the vertices are the objects of $\mathcal{C}$ and the edges from one word $A_1 \cdots A_m$ of objects to another $B_1 \cdots B_n$ is a morphism $f : A_1 \otimes \cdots \otimes A_m \to B_1 \otimes \cdots \otimes B_n$ in $\mathcal{C}$; see Figure 1. When we speak of a labelling of a string diagram in $\mathcal{C}$ we mean a labelling in the underlying tensor scheme; here we will simply call this a *string diagram in* $\mathcal{C}$. The value $v(\Gamma)$ of the string diagram $v : \Gamma \to \mathcal{C}$ is a morphism obtained by deforming Γ so that no two vertices of Γ are on the same vertical line then by horizontally composing strips of the form

$$1_{C_1} \otimes \cdots \otimes 1_{C_h} \otimes f \otimes 1_{D_1} \otimes \cdots \otimes 1_{D_k}.$$

Calculations in monoidal categories can be performed using string diagrams rather than the traditional diagrams of category theory. The value of Figure 1 is of course f. Figure 2 shows a string diagram $v : \Gamma \to \mathcal{C}$ whose value $v(\Gamma)$ is

$$\begin{array}{c} A_1 \otimes \cdots \otimes A_m \otimes D_1 \otimes \cdots \otimes D_q \\ \big\downarrow {\scriptstyle f \otimes 1} \\ B_1 \otimes \cdots \otimes B_n \otimes C_1 \otimes \cdots \otimes C_p \otimes D_1 \otimes \cdots \otimes D_q \\ \big\downarrow {\scriptstyle 1 \otimes g} \\ E_1 \otimes \cdots \otimes E_p \end{array}$$

Now we return to our study of separable Frobenius monoidal functors.

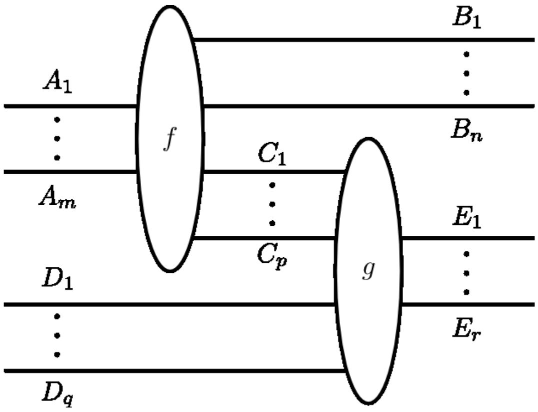


Figure 2:

Suppose $v : \Gamma \to \mathcal{C}$ is a string diagram in a monoidal category $\mathcal{C}$ and $F : \mathcal{C} \to \mathcal{X}$ is a monoidal and opmonoidal functor. We obtain a *conjugate string diagram* $v^F : \Gamma \to \mathcal{X}$ in $\mathcal{X}$ by defining

$$v^F(\gamma) = Fv(\gamma) \text{ and } v^F(x) = v(x)^F$$

for each edge γ and each node x of Γ. The conjugate of the string diagram in Figure 2 is shown in Figure 3. A (progressive plane) string diagram Γ is called *[separable] Frobenius invariant* when, for any labelling $v : \Gamma \to \mathcal{C}$ of Γ in any monoidal category $\mathcal{C}$ and any [separable] Frobenius monoidal functor $F : \mathcal{C} \to \mathcal{X}$, the value of the conjugate diagram v^F in $\mathcal{X}$ is equal to the conjugate of the value of v; that is,

$$v^F(\Gamma) = v(\Gamma)^F. \tag{3.1}$$

As mentioned before, for a separable Frobenius monoidal functor F, we have $\phi_n \circ \psi_n = 1$ for $n > 0$.

The following two theorems characterize which string diagrams are preserved by Frobenius and separable Frobenius monoidal functors in terms of connectedness and acyclicity. Robin Cockett pointed out to us that similar geometric conditions occur in the work of Girard [7] and Fleury and Retoré ([6], §3.1). There may be a relationship with our results but the precise nature is unclear.

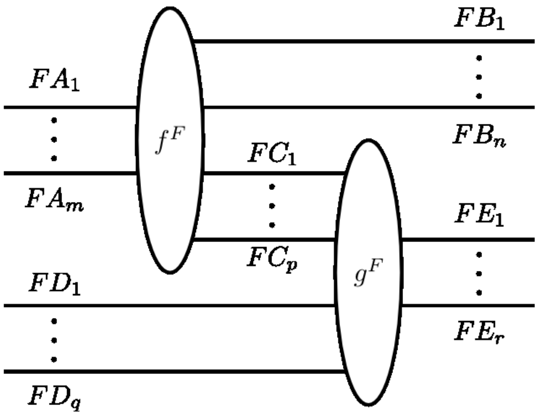


Figure 3:

Theorem 3.1. *A progressive plane string diagram is separable Frobenius invariant if and only if it is connected.*

Proof. In Figure 4, we show that Equation 3.1 holds for the string diagram $v : \Gamma \to \mathcal{C}$ as in Figure 2, provided $p > 0$ (as required for Γ to be connected). To simplify notation we write FA for $F(A_1 \otimes \cdots \otimes A_m)$ and write A^F for $FA_1 \otimes \cdots \otimes FA_m$. We also leave out some tensor symbols $\otimes$. The second equality in Figure 4 is where separability, and the fact that the length p of the word C is strictly positive, are used; the third is where a Frobenius property is used.

Similarly, an obvious (horizontal) dual diagram to Figure 2 (look through the back of the page!) can be shown separably invariant. Furthermore, it is simple to show that diagrams of the form shown in Figure 5 are separably invariant, as well as their diagrams of the dual form.

By a similar proof to the above, such diagrams and their duals are separably invariant. Every connected string diagram can be constructed by iterating these four processes, this proves “if”. For “only if”, we exploit the fact that every string diagram can be interpreted in the terminal monoidal category 1 and that separable Frobenius monoidal functors $F : 1 \to \mathcal{C}$ are precisely separable Frobenius algebras in $\mathcal{C}$.

Suppose for a contradiction that a disconnected string diagram Γ with n input wires and m output wires is invariant under conjugation by such a

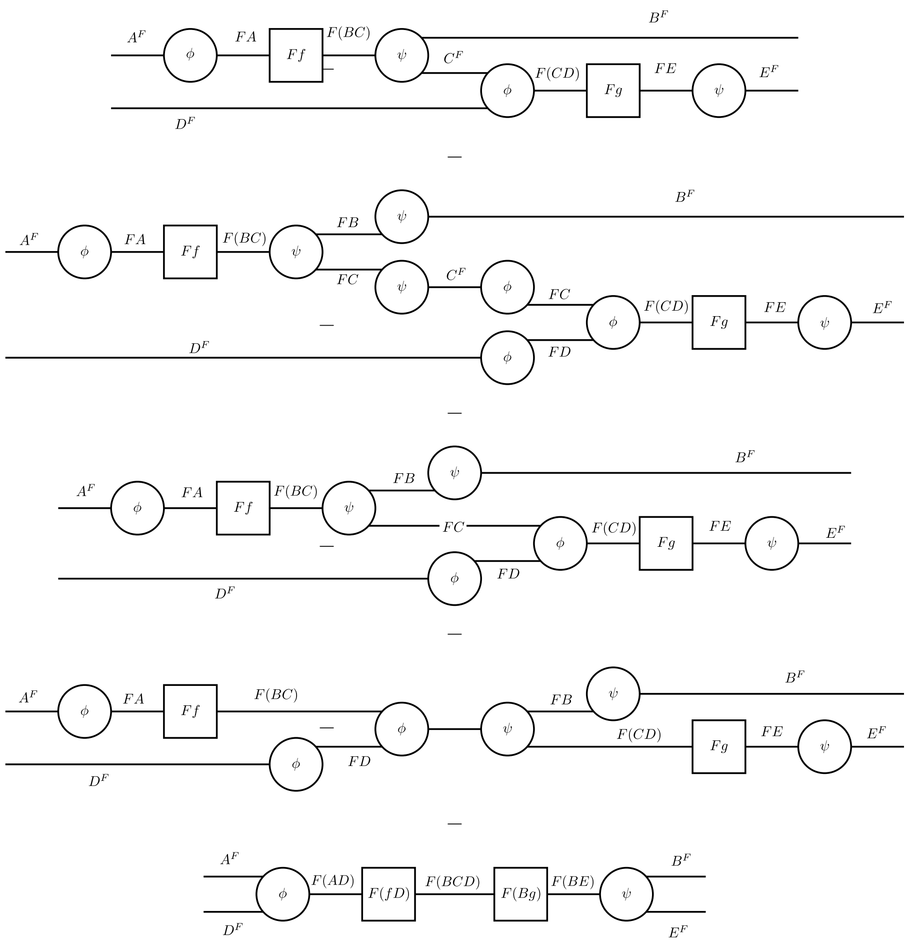


Figure 4:

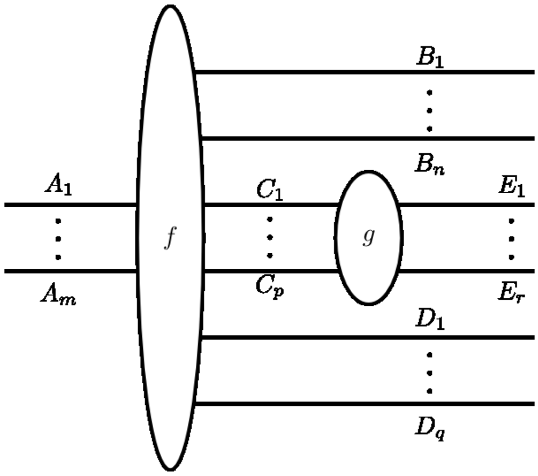


Figure 5:

separable Frobenius F, that is, a separable Frobenius algebra C. This asserts the equality of two morphisms $C^{\otimes n} \longrightarrow C^{\otimes m}$, the first (obtained by taking the (trivial) value of the labelling in 1 and then applying F) is the composite of n-fold multiplication followed by m-fold comultiplication; the second (obtained by applying F to the labelling and then taking the value in $\mathcal{C}$) is considerably more complicated, containing at least two connected components since Γ is assumed to be disconnected. By prepending n units and appending m counits, the first becomes the barbell of unit followed by counit; the latter becomes an endomap of the tensor unit of $\mathcal{C}$ with at least two connected components. If the tensor product of $\mathcal{C}$ is symmetric, this last simplifies to as many copies of the barbell as there are connected components of Γ; hence, it suffices to find a separable Frobenius algebra for which the barbell does not equal any non-trivial power of itself.

We give two examples of such separable Frobenius algebras, a simple algebraic example and a more complicated geometric example. First, consider the complex numbers as a Frobenius algebra over the reals. Kock ([10], Example 2.2.14) notes that $\mathbb{C} \longrightarrow \mathbb{R}$ given by $x + iy \mapsto ax + by$ is a Frobenius form on $\mathbb{C}$ for all a and b not both zero. Choosing $a = 2, b = 0$ gives a *separable* Frobenius structure, and the "barbell" $\mathbb{R} \longrightarrow \mathbb{C} \longrightarrow \mathbb{R}$ is multiplication by 2, which does not equal any non-trivial power of itself. This completes the proof of the converse of the theorem.

We sketch the construction of a more complicated but perhaps more pleasing, geometric example: consider the category **2Thick**, as described in [11], whose objects are finite disjoint unions of the interval (identified with the natural numbers), embedded in the plane, and whose morphisms are boundary-preserving-diffeomorphism classes smooth oriented surfaces embedded in the plane with boundary equal to the union of domain and codomain. For instance, Figure 6 shows a morphism in **2Thick** from 2 to 1. Lauda proves that **2Thick** is the free monoidal category containing a

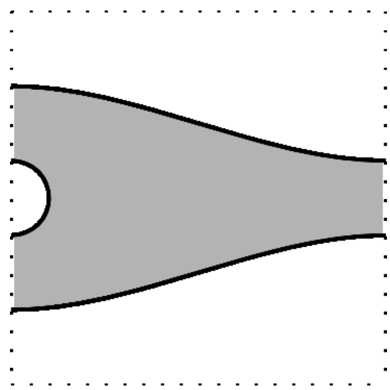


Figure 6: A morphism in **2Thick**

Frobenius algebra; the morphism from 2 to 1 shown in Figure 6 is the multiplication for this Frobenius algebra; the obvious similar map from 1 to 2 is the comultiplication. However, for this theorem, we require a *separable* Frobenius algebra, so we modify **2Thick** to obtain a category in which the equality in Figure 7 holds; in fact, we conjecture, to obtain the free monoidal category containing a separable Frobenius algebra. Specifically, instead of taking boundary-preserving diffeomorphism classes of morphisms, we say that two morphisms $k \longrightarrow l$ are equal if there is a suitable 3-manifold M with corners which can be embedded in the unit cube in such that the intersection with the top face is the first morphism and the intersection with the bottom face is the second morphism. Here "suitable" means that the 3-manifold must be trivial on the domains and codomains k and l, and, crucially, the only critical points of the boundary of M permitted are "cups" –

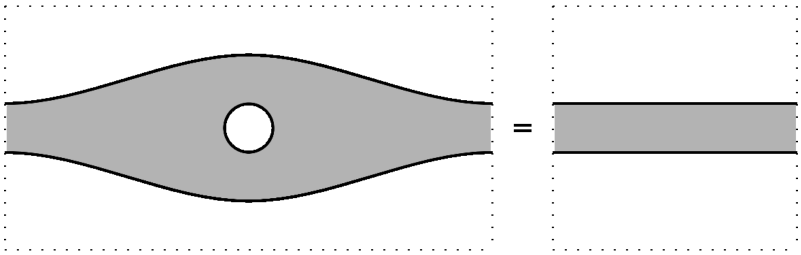


Figure 7:

that is, critical points which are not saddle points where the convex portion of the critical point lies *outside* the manifold M. There is an evident such "cup" which will witness the desired equality shown in Figure 7. Let us call this quotient of **2Thick** by the name **2Thick**$'$.

Most importantly, it is clear that no two morphisms with different numbers of connected components can be identified by this equivalence relation, so any disconnected string diagram will fail to be separably invariant with respect to the canonical separable Frobenius functor $1 \longrightarrow$ **2Thick**$'$. $\square$

What this implies is that separable Frobenius monoidal functors preserve equations in monoidal categories for which both sides of the equation are values of connected string diagrams. For example:

Corollary 3.2. *For $n > 1$, equations of the form:*

$$(a_n \otimes 1)\,(1 \otimes a_{n-1})\,(a_{n-2} \otimes 1) \cdots = (1 \otimes b_n)\,(b_{n-1} \otimes 1)\,(1 \otimes b_{n-2}) \cdots ,$$

involving morphisms

$$a_1, \ldots, a_n, b_1, \ldots, b_n : A \otimes A \longrightarrow A \otimes A,$$

are stable under F-conjugation. In fact, for $n = 2$, Frobenius F will do.

The proof of Theorem 3.1 can be slightly modified to give the analagous result for merely Frobenius monoidal functors instead of separable Frobenius monoidal functors.

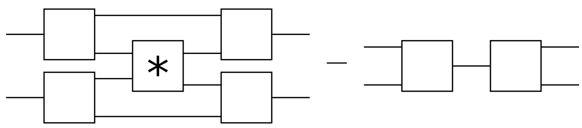
Theorem 3.3. *A progressive plane string diagram is Frobenius invariant if and only if it is connected and simply connected.*

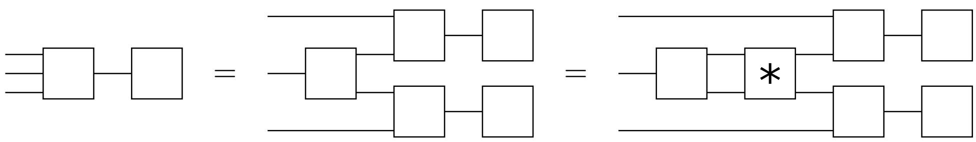
Proof. We have noted that all connected string diagrams can be obtained as iterations of the constructions shown in Figures 2 and 5 and their duals, with the restriction that $p > 0$. All connected and simply connected string diagrams can be obtained in this way with the restriction that $p = 1$. The only step of the proof in Figure 4 (and also the corresponding proof for the case shown in Figure 5) which requires separability is the cancellation of $FC \xrightarrow{\psi} C^F \xrightarrow{\phi} FC$ to obtain the identity on FC; since $p = 1$, we have $FC = C^F = FC_1$, and both of these maps are identities. Hence, the same proof will go through in this case, establishing "if".

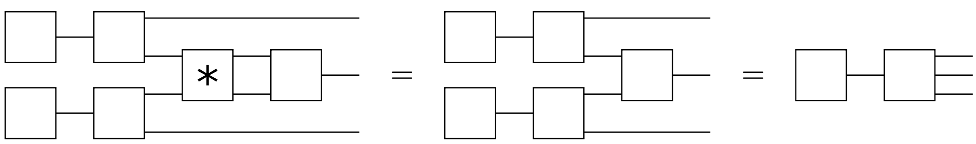
Conversely, suppose that Γ is a string diagram which is not connected and acyclic. By Theorem 3.1, we may assume that Γ is connected and therefore is not acyclic. Then for Γ to be invariant under the canonical Frobenius monoidal functor $1 \longrightarrow$ **2Thick** described in [11] and referred to already in Theorem 3.1 would imply that there is a diffeomorphism between two 2-manifolds of different genus; this is not the case. $\square$

Corollary 3.4. *Weak bimonoids are preserved by braided separable Frobenius functors.*

Proof. Weak bimonoids satisfy Equations 2.11, 2.12, and 2.13 these equations are labelled versions of the following string-diagrams:







These are clearly connected and hence preserved by separable Frobenius functors. The asterisks indicate labellings by braids or their inverses, these are preserved by braided Frobenius functors. □

However, genuine bimonoids are not preserved in general: the three unit and counit equations for a bimonoid involve non-connected string diagrams.

Corollary 3.5. *Weak distributive laws are preserved under F-conjugation.*

However, distributive laws are not preserved in general: the string diagrams for the right-hand sides of Equations 2.9 are not connected.

Corollary 3.6. *Lax YB-operators are preserved under F-conjugation.*

However, YB-operators are not preserved: invertibility involves an equation whose underlying diagram is a pair of disjoint strings and so is disconnected.

Corollary 3.7. *Weak YB-operators are preserved under F-conjugation. In particular, the F-conjugate of a YB-operator is a weak YB-operator.*

Proposition 3.8. *Every weak YB-operator in a monoidal category in which idempotents split is the conjugate of a YB-operator under some separable Frobenius monoidal functor.*

Proof. Let $\mathcal{C}$ be a such a monoidal category containing an object D and an idempotent $\nabla : D\otimes D \longrightarrow D\otimes D$ such that $(\nabla\otimes 1)(1\otimes\nabla) = (1\otimes\nabla)(\nabla\otimes 1)$. Then there is an idempotent $\nabla_n : D^{\otimes n} \longrightarrow D^{\otimes n}$ recursively defined by:

$$\begin{aligned}
&\nabla_0 = 1_I\\
&\nabla_1 = 1_D\\
&\nabla_2 = \nabla\\
&\nabla_n = (1\otimes\nabla_{n-1})\circ(\nabla\otimes 1) \qquad \text{for } n > 2
\end{aligned}$$

Let $\mathcal{C}(D)$ be the subcategory of $\mathcal{QC}$ whose objects are the pairs $(D^{\otimes n}, \nabla_n)$ and whose morphisms $f : (D^{\otimes n}, \nabla_n) \to (D^{\otimes m}, \nabla_m)$ are those in $\mathcal{QC}$ for which:

$$(1\otimes f)(\nabla_n\otimes 1) = (\nabla_m\otimes 1)(1\otimes f) \tag{3.2}$$

$$(f\otimes 1)(1\otimes\nabla_n) = (1\otimes\nabla_m)(f\otimes 1). \tag{3.3}$$

The category $\mathcal{C}(D)$ becomes monoidal via

$$(D^{\otimes n}, \nabla_n) \otimes (D^{\otimes m}, \nabla_m) = (D^{\otimes(n+m)}, \nabla_{n+m}).$$

Note that this is not the same as the usual tensor product on $\mathcal{QC}$ which is inherited from that of $\mathcal{C}$. A weak Yang-Baxter operator on D in $\mathcal{C}$ is a Yang-Baxter operator on D in $\mathcal{C}(D)$. Since idempotents split in $\mathcal{C}$ then we have a functor $\mathcal{C}(D) \to \mathcal{C}$ taking each idempotent to a splitting. Moreover, this functor $\mathcal{C}(D) \to \mathcal{C}$ is separable Frobenius (although not strong) and so each weak YB-operator is the image of a genuine YB-operator.

Proposition 3.9. *Prebimonoidal functors compose.*

Proof. Suppose that $F : \mathcal{C} \longrightarrow \mathcal{X}$ is prebimonoidal with respect to a YB-operator y on $T : \mathcal{A} \longrightarrow \mathcal{C}$ and a YB-operator z on FT, and suppose further that $G : \mathcal{X} \longrightarrow \mathcal{Y}$ is prebimonoidal with respect to z and a YB-operator a on GFT. Then the diagram in Figure 8 proves that GF is prebimonoidal with respect to y and a.
The diamonds commute by naturality of ϕ and ψ and the left and right pentagons commute by prebimonoidality of F and G, respectively. □

Proposition 3.10. *If F is separable Frobenius then it is prebimonoidal relative to y and $z = y^F$.*

Proof. The proof is contained in Figure 9.
The five diamonds commute since F is Frobenius, and the two right-hand triangles commute since F is separable. The rhombus commutes by definition of y^F, the parallelograms by naturality of ϕ and ψ, and the two irregular cells are trivial. □

Proposition 3.11. *A strong monoidal functor between braided monoidal categories is prebimonoidal if and only if it is braided.*

Proof. As noted above, strong monoidal functors are separable Frobenius, and strong monoidal functors are braided precisely when $c^F_{A,B} = c_{FA,FB}$, so Proposition 3.10 establishes "if". Conversely, suppose that F is prebimonoidal with respect to the two braidings, and consider the commutative diagram in Figure 10.

$$
\begin{tikzcd}
 & GF(Tu\otimes Tv)\otimes GF(Tw\otimes Tx) \arrow[dl, "\phi"'] \arrow[dr, "G\psi\otimes G\psi"] & \\
G(F(Tu\otimes Tv)\otimes F(Tw\otimes Tx)) \arrow[d, "G\phi"'] \arrow[ddr, bend left, "G(\psi\otimes\psi)"] & & G(FTu\otimes FTv)\otimes G(FTw\otimes FTx) \arrow[d, "\psi\otimes\psi"] \arrow[ddl, bend right, "\phi"'] \\
GF(Tu\otimes Tv\otimes Tw\otimes Tx) \arrow[ddd, "GF(1\otimes y\otimes 1)"'] & & GFTu\otimes GFTv\otimes GFTw\otimes GFTx \arrow[ddd, "1\otimes a\otimes 1"] \\
 & G(FTu\otimes FTv\otimes FTw\otimes FTx) \arrow[d, "G(1\otimes z\otimes 1)"] & \\
 & G(FTu\otimes FTw\otimes FTv\otimes FTx) \arrow[ddl, bend left, "G(\phi\otimes\phi)"'] \arrow[ddr, bend right, "\psi"] & \\
GF(Tu\otimes Tw\otimes Tv\otimes Tx) \arrow[d, "G\psi"'] & & GFTu\otimes GFTw\otimes GFTv\otimes GFTx \arrow[d, "\phi\otimes\phi"] \\
G(F(Tu\otimes Tw)\otimes F(Tv\otimes Tx)) \arrow[dr, "\psi"'] & & G(FTu\otimes FTw)\otimes G(FTv\otimes FTx) \arrow[dl, "G\phi\otimes G\phi"] \\
 & GFTu\otimes GFTw\otimes GFTv\otimes GFTx &
\end{tikzcd}
$$

Figure 8: Proof of Proposition 3.9

$$\begin{tikzcd}[column sep=tiny]
& & F(Tu\otimes Tv)\otimes F(Tw\otimes Tx) \arrow[dl, "\psi\otimes 1"'] \arrow[dr, "\phi"] \arrow[ddll, bend right, "\psi\otimes\psi"'] & \\
& FTu\otimes FTv\otimes F(Tw\otimes Tx) \arrow[dl, "1\otimes 1\otimes\psi"'] \arrow[dr, "1\otimes\phi"] & & F(Tu\otimes Tv\otimes Tw\otimes Tx) \arrow[dl, "\psi"'] \arrow[dd, equal] \\
FTu\otimes FTv\otimes FTw\otimes FTx \arrow[dr, "1\otimes\phi\otimes 1"] \arrow[dddd, "1\otimes y^F\otimes 1"'] & & FTu\otimes F(Tv\otimes Tw\otimes Tx) \arrow[dl, "1\otimes\psi"'] \arrow[dr, "\phi"] & \\
& FTu\otimes F(Tv\otimes Tw)\otimes FTx \arrow[dr, "\phi\otimes 1"] \arrow[dd, "1\otimes Fy\otimes 1"'] & & F(Tu\otimes Tv\otimes Tw\otimes Tx) \arrow[dl, "\psi"'] \arrow[dd, "F(1\otimes y\otimes 1)"] \\
& & F(Tu\otimes Tv\otimes Tw)\otimes FTx \arrow[dd, "F(1\otimes y)\otimes 1"] & \\
& FTu\otimes F(Tw\otimes Tv)\otimes FTx \arrow[dl, "1\otimes\psi\otimes 1"'] \arrow[dr, "\phi\otimes 1"] & & F(Tu\otimes Tw\otimes Tv\otimes Tx) \arrow[dl, "\psi"'] \arrow[dd, equal] \\
FTu\otimes FTw\otimes FTv\otimes FTx \arrow[dr, "\phi\otimes 1\otimes 1"] \arrow[ddrr, bend right, "\phi\otimes\phi"'] & & F(Tu\otimes Tw\otimes Tv)\otimes FTx \arrow[dl, "\psi\otimes 1"'] \arrow[dr, "\phi"] & \\
& F(Tu\otimes Tw)\otimes FTv\otimes FTx \arrow[dr, "1\otimes\phi"] & & F(Tu\otimes Tw\otimes Tv\otimes Tx) \arrow[dl, "\psi"] \\
& & F(Tu\otimes Tw)\otimes F(Tv\otimes Tx) &
\end{tikzcd}$$

Figure 9: Proof of Proposition 3.10

$$\begin{tikzcd}
 & & FA \otimes FB \arrow[dll, equal] \arrow[d, equal] \arrow[drr, "\phi"] & & \\
I \otimes FA \otimes FB \otimes I \arrow[ddd, "1\otimes c\otimes 1"'] & & F(I\otimes A)\otimes F(B\otimes I) \arrow[dl, "\psi\otimes\psi"'] \arrow[dr, "\phi"] & & F(A\otimes B) \arrow[dl, equal] \arrow[ddd, "Fc"] \\
 & FI\otimes FA\otimes FB\otimes FI \arrow[ul, "\psi_0\otimes 1\otimes 1\otimes\psi_0"] \arrow[d, "1\otimes c\otimes 1"] & & F(I\otimes A\otimes B\otimes I) \arrow[d, "F(1\otimes c\otimes 1)"] & \\
 & FI\otimes FB\otimes FA\otimes FI \arrow[dr, "\phi\otimes\phi"] & & F(I\otimes B\otimes A\otimes I) \arrow[dl, "\psi"'] \arrow[dr, equal] & \\
I\otimes FB\otimes FA\otimes I \arrow[ur, "\phi_0\otimes 1\otimes 1\otimes\phi_0"] \arrow[drr, equal] & & F(I\otimes B)\otimes F(A\otimes I) \arrow[d, equal] & & F(B\otimes A) \arrow[dll, "\psi"] \\
 & & FB\otimes FA & &
\end{tikzcd}$$

Figure 10: Proof of Proposition 3.11

The middle cell commutes since F is prebimonoidal, the bottom left since ϕ is monoidal, and the top left since ψ is opmonoidal. The three right-hand cells commute by definition, and, noting that ϕ_0 and ψ_0 are both natural and mutually inverse, the left-hand cell does so also. Hence, the full diagram shows that $c^F_{A,B} = c_{FA,FB}$, as desired. $\square$

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"HAUSDORFF DISTANCE" VIA CONICAL COCOMPLETION

*by Isar STUBBE**

À Francis Borceux, qui m'a tant appris et qui m'apprend toujours

Résumé. Dans le contexte des catégories enrichies dans un quantaloïde, nous expliquons comment toute classe de poids saturée définit, et est définie par, une unique sous-KZ-doctrine pleine de la doctrine pour la cocomplétion libre. Les KZ-doctrines qui sont des sous-KZ-doctrines pleines de la doctrine pour la cocomplétion libre, sont caractérisées par deux conditions simples de "pleine fidélité". Les poids coniques forment une classe saturée, et la KZ-doctrine correspondante est exactement (la généralisation aux catégories enrichies dans un quantaloïde de) la doctrine de Hausdorff de [Akhvlediani *et al.*, 2009].
Abstract. In the context of quantaloid-enriched categories, we explain how each saturated class of weights defines, and is defined by, an essentially unique full sub-KZ-doctrine of the free cocompletion KZ-doctrine. The KZ-doctrines which arise as full sub-KZ-doctrines of the free cocompletion, are characterised by two simple "fully faithfulness" conditions. Conical weights form a saturated class, and the corresponding KZ-doctrine is precisely (the generalisation to quantaloid-enriched categories of) the Hausdorff doctrine of [Akhvlediani *et al.*, 2009].
Keywords. Enriched category, cocompletion, KZ-doctrine, Hausdorff distance
Mathematics Subject Classification (2010). 18D20, 18A35, 18C20

1. Introduction

At the meeting on "Categories in Algebra, Geometry and Logic" honouring Francis Borceux and Dominique Bourn in Brussels on 10–11 October 2008, Walter Tholen gave a talk entitled "On the categorical meaning of Hausdorff and Gromov distances", reporting on joint work with Andrei Akhvlediani and Maria Manuel Clementino [2009]. The term 'Hausdorff distance' in his title refers to the following construction: if (X, d) is a metric space and $S, T \subseteq X$, then

$$\delta(S,T) := \bigvee_{s\in S} \bigwedge_{t\in T} d(s,t)$$

defines a (generalised) metric on the set of subsets of X. But Bill Lawvere [1973] showed that metric spaces are examples of enriched categories, so one can aim at

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suitably generalising this 'Hausdorff distance'. Tholen and his co-workers achieved this for *categories enriched in a commutative quantale $\mathcal{V}$*. In particular they devise a KZ-doctrine on the category of $\mathcal{V}$-categories, whose algebras – in the case of metric spaces – are exactly the sets of subsets of metric spaces, equipped with the Hausdorff distance.

We shall argue that the notion of Hausdorff distance can be developed for *quantaloid-enriched categories* too, using *enriched colimits* as main tool. In fact, very much in line with the work of [Albert and Kelly, 1988; Kelly and Schmitt, 2005; Schmitt, 2006] on cocompletions of categories enriched in a symmetric monoidal category and the work of [Kock, 1995] on the abstraction of cocompletion processes, we shall see that, for quantaloid-enriched categories, each saturated class of weights defines, and is defined by, an essentially unique KZ-doctrine. The KZ-doctrines that arise in this manner are the full sub-KZ-doctrines of the free cocompletion KZ-doctrine, and they can be characterised with two simple "fully faithfulness" conditions. As an application, we find that the conical weights form a saturated class and the corresponding KZ-doctrine is precisely (the generalisation to quantaloid-enriched categories of) the Hausdorff doctrine of [Akhvlediani *et al.*, 2009].

In this paper we do not speak of 'Gromov distances', that other metric notion that Akhvlediani, Clementino and Tholen [2009] refer to. As they analyse, Gromov distance is necessarily built up from *symmetrised* Hausdorff distance; and because their base quantale $\mathcal{V}$ is commutative, they can indeed extend this notion too to $\mathcal{V}$-enriched categories. More generally however, symmetrisation for quantaloid-enriched categories makes sense when that quantaloid is involutive. Preliminary computations indicate that 'Gromov distance' ought to exist on this level of generality, but quickly got too long to include them in this paper: so we intend to work this out in a sequel.

2. Preliminaries

2.1 Quantaloids

A **quantaloid** is a category enriched in the monoidal category Sup of complete lattices (also called sup-lattices) and supremum preserving functions (sup-morphisms). A quantaloid with one object, i.e. a monoid in Sup, is a **quantale**. Standard references include [Rosenthal, 1996; Paseka and Rosicky, 2000] .

Viewing $\mathcal{Q}$ as a locally ordered category, the 2-categorical notion of *adjunction in $\mathcal{Q}$* refers to a pair of arrows, say $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow A$, such that $1_A \leq g \circ f$ and $f \circ g \leq 1_B$ (in which case f is left adjoint to g, and g is right adjoint to f, denoted $f \dashv g$).

Given arrows

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "h"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

in a quantaloid $\mathcal{Q}$, there are adjunctions between sup-lattices as follows:

$$\begin{tikzcd} \mathcal{Q}(B,C) \arrow[r, bend left, "- \circ f"] \arrow[r, phantom, "\perp"] & \mathcal{Q}(A,C), \arrow[l, bend left, "\{f,-\}"] \end{tikzcd} \quad \begin{tikzcd} \mathcal{Q}(A,B) \arrow[r, bend left, "g \circ -"] \arrow[r, phantom, "\perp"] & \mathcal{Q}(A,C), \arrow[l, bend left, "{[g,-]}"] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{Q}(A,B) \arrow[r, bend left, "\{-,h\}"] \arrow[r, phantom, "\perp"] & \mathcal{Q}(B,C)^{\mathsf{op}}. \arrow[l, bend left, "{[-,h]}"] \end{tikzcd}$$

The arrow $[g,h]$ is called the **lifting** of h trough g, whereas $\{f,h\}$ is the **extension** of h through f. Of course, every left adjoint preserves suprema, and every right adjoint preserves infima. For later reference, we record some straightforward facts:

Lemma 2.1 *If $g\colon B \longrightarrow C$ in a quantaloid has a right adjoint g^*, then $[g,h] = g^* \circ h$ and therefore $[g,-]$ also preserves suprema. Similarly, if $f\colon A \longrightarrow B$ has a left adjoint $f_!$ then $\{f,h\} = h \circ f_!$ and thus $\{f,-\}$ preserves suprema.*

Lemma 2.2 *For any commutative diagram*

$$\begin{tikzcd} A \arrow[dr, "f"'] & & B \arrow[dl, "g"'] \arrow[dr, "h"] & & C \arrow[dl, "i"] \\ & D \arrow[rr, "j"'] & & E & \end{tikzcd}$$

in a quantaloid, we have that $[i,h] \circ [g,f] \leq [i, j \circ f]$. If all these arrows are left adjoints, and g moreover satisfies $g \circ g^* = 1_D$, then $[i,h] \circ [g,f] = [i, j \circ f]$.

Lemma 2.3 *If $f\colon A \longrightarrow B$ in a quantaloid has a right adjoint f^* such that moreover $f^* \circ f = 1_A$, then $[f \circ x, f \circ y] = [x,y]$ for any $x, y\colon X \rightrightarrows A$.*

2.2 Quantaloid-enriched categories

From now on $\mathcal{Q}$ denotes a *small* quantaloid. Viewing $\mathcal{Q}$ as a (locally ordered) bicategory, it makes perfect sense to consider categories enriched in $\mathcal{Q}$. Bicategory-enriched categories were invented at the same time as bicategories by Jean Bénabou [1967], and further developed by Ross Street [1981, 1983]. Bob Walters [1981] particularly used quantaloid-enriched categories in connection with sheaf theory. Here we shall stick to the notational conventions of [Stubbe, 2005], and refer to that paper for additional details, examples and references.

A **$\mathcal{Q}$-category** $\mathbb{A}$ consists of a set of objects $\mathbb{A}_0$, a type function $t\colon \mathbb{A}_0 \longrightarrow \mathcal{Q}_0$, and $\mathcal{Q}$-arrows $\mathbb{A}(a',a)\colon ta \longrightarrow ta'$; these must satisfy identity and composition axioms, namely:

$$1_{ta} \leq \mathbb{A}(a,a) \ \text{ and } \ \mathbb{A}(a'',a') \circ \mathbb{A}(a',a) \leq \mathbb{A}(a'',a).$$

A **$\mathcal{Q}$-functor** $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is a type-preserving object map $a \mapsto Fa$ satisfying the functoriality axiom:

$$\mathbb{A}(a',a) \leq \mathbb{B}(Fa',Fa).$$

And a **$\mathcal{Q}$-distributor** $\Phi\colon \mathbb{A} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{B}$ is a matrix of $\mathcal{Q}$-arrows $\Phi(b,a)\colon ta \longrightarrow tb$, indexed by all couples of objects of $\mathbb{A}$ and $\mathbb{B}$, satisfying two action axioms:

$$\Phi(b,a') \circ \mathbb{A}(a',a) \leq \Phi(b,a) \ \text{ and } \ \mathbb{B}(b,b') \circ \Phi(b',b) \leq \Phi(b,a).$$

Composition of functors is obvious; that of distributors is done with a "matrix" multiplication: the composite $\Psi \otimes \Phi\colon \mathbb{A} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{C}$ of $\Phi\colon \mathbb{A} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{B}$ and $\Psi\colon \mathbb{B} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{C}$ has as elements

$$(\Psi \otimes \Phi)(c,a) = \bigvee_{b \in \mathbb{B}_0} \Psi(c,b) \circ \Phi(b,a).$$

Moreover, the elementwise supremum of parallel distributors $(\Phi_i\colon \mathbb{A} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{B})_{i\in I}$ gives a distributor $\bigvee_i \Phi_i\colon \mathbb{A} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{B}$, and it is easily checked that we obtain a (large) quantaloid $\mathsf{Dist}(\mathcal{Q})$ of $\mathcal{Q}$-categories and distributors. Now $\mathsf{Dist}(\mathcal{Q})$ is a 2-category, so we can speak of adjoint distributors. In fact, any functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ determines an adjoint pair of distributors:

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "\mathbb{B}(-,F-)"] \arrow[r, phantom, "\perp"] & \mathbb{B}. \arrow[l, bend left, "\mathbb{B}(F-,-)"] \end{tikzcd} \qquad (1)$$

Therefore we can sensibly order parallel functors $F,G\colon \mathbb{A} \rightrightarrows \mathbb{B}$ by putting $F \leq G$ whenever $\mathbb{B}(-,F-) \leq \mathbb{B}(-,G-)$ (or equivalently, $\mathbb{B}(G-,-) \leq \mathbb{B}(F-,-)$) in $\mathsf{Dist}(\mathcal{Q})$. Doing so, we get a locally ordered category $\mathsf{Cat}(\mathcal{Q})$ of $\mathcal{Q}$-categories and functors, together with a 2-functor

$$i\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})\colon \Big(F\colon \mathbb{A} \longrightarrow \mathbb{B}\Big) \mapsto \Big(\mathbb{B}(-,F-)\colon \mathbb{A} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{B}\Big). \qquad (2)$$

(The local order in $\mathsf{Cat}(\mathcal{Q})$ need not be anti-symmetric, i.e. it is not a partial order but rather a preorder, which we prefer to call simply an order.)

This is the starting point for the theory of quantaloid-enriched categories, including such notions as:

- **fully faithful functor**: an $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ for which $\mathbb{A}(a', a) = \mathbb{B}(Fa', Fa)$, or alternatively, for which the unit of the adjunction in (1) is an equality,
- **adjoint pair**: a pair $F\colon \mathbb{A} \longrightarrow \mathbb{B}$, $G\colon \mathbb{B} \longrightarrow \mathbb{A}$ for which $1_{\mathbb{A}} \leq G \circ F$ and also $F \circ G \leq 1_{\mathbb{B}}$, or alternatively, for which $\mathbb{B}(F-, -) = \mathbb{A}(-, G-)$,
- **equivalence**: an $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ which are fully faithful and essentially surjective on objects, or alternatively, for which there exists a $G\colon \mathbb{B} \longrightarrow \mathbb{A}$ such that $1_{\mathbb{A}} \cong G \circ F$ and $F \circ G \cong 1_{\mathbb{B}}$,
- **left Kan extension**: given $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ and $G\colon \mathbb{A} \longrightarrow \mathbb{C}$, the left Kan extension of F through G, written $\langle F, G\rangle\colon \mathbb{C} \longrightarrow \mathbb{B}$, is the smallest such functor satisfying $F \leq \langle F, G\rangle \circ G$,

and so on. In the next subsection we shall recall the more elaborate notions of presheaves, weighted colimits and cocompletions.

2.3 Presheaves and free cocompletion

If X is an object of $\mathcal{Q}$, then we write $*_X$ for the one-object $\mathcal{Q}$-category, whose single object $*$ is of type X, and whose single hom-arrow is 1_X.

Given a $\mathcal{Q}$-category $\mathbb{A}$, we now define a new $\mathcal{Q}$-category $\mathcal{P}(\mathbb{A})$ as follows:

- objects: $(\mathcal{P}(\mathbb{A}))_0 = \{\phi\colon *_X \looparrowright \mathbb{A} \mid X \in \mathcal{Q}_0\}$,
- types: $t(\phi) = X$ for $\phi\colon *_X \looparrowright \mathbb{A}$,
- hom-arrows: $\mathcal{P}(\mathbb{A})(\psi, \phi) =$ (single element of) the lifting $[\psi, \phi]$ in $\mathsf{Dist}(\mathcal{Q})$.

Its objects are **(contravariant) presheaves** on $\mathbb{A}$, and $\mathcal{P}(\mathbb{A})$ itself is the **presheaf category** on $\mathbb{A}$.

The presheaf category $\mathcal{P}(\mathbb{A})$ **classifies distributors** with codomain $\mathbb{A}$: for any $\mathbb{B}$ there is a bijection between $\mathsf{Dist}(\mathcal{Q})(\mathbb{B}, \mathbb{A})$ and $\mathsf{Cat}(\mathcal{Q})(\mathbb{B}, \mathcal{P}(\mathbb{A}))$, which associates to any distributor $\Phi\colon \mathbb{B} \looparrowright \mathbb{A}$ the functor $Y_\Phi\colon \mathbb{B} \longrightarrow \mathcal{P}(\mathbb{A})\colon b \mapsto \Phi(-, b)$, and conversely associates to any functor $F\colon \mathbb{B} \longrightarrow \mathcal{P}(\mathbb{A})$ the distributor $\Phi_F\colon \mathbb{B} \looparrowright \mathbb{A}$ with elements $\Phi_F(a, b) = (Fb)(a)$. In particular is there a functor, $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{P}(\mathbb{A})$, that corresponds with the identity distributor $\mathbb{A}\colon \mathbb{A} \looparrowright \mathbb{A}$: the elements in the image of $Y_{\mathbb{A}}$ are the **representable presheaves on** $\mathbb{A}$, that is to say, for each $a \in \mathbb{A}$ we have $\mathbb{A}(-, a)\colon *_{ta} \looparrowright \mathbb{A}$. Because such a representable presheaf is a left adjoint in $\mathsf{Dist}(\mathcal{Q})$, with right adjoint $\mathbb{A}(a, -)$, we can verify that

$$\mathcal{P}(\mathbb{A})(Y_{\mathbb{A}}(a), \phi) = [\mathbb{A}(-, a), \phi] = \mathbb{A}(a, -) \otimes \phi = \phi(a).$$

This result is known as **Yoneda's Lemma**, and implies that $Y_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathcal{P}(\mathbb{A})$ is a fully faithful functor, called the **Yoneda embedding** of $\mathbb{A}$ into $\mathcal{P}(\mathbb{A})$.

By construction there is a 2-functor

$$\mathcal{P}_0\colon \mathsf{Dist}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})\colon (\Phi\colon\mathbb{A}\,{\circ\!\!\!\longrightarrow}\,\mathbb{B})\mapsto(\Phi\otimes -\colon\mathcal{P}(\mathbb{A})\longrightarrow\mathcal{P}(\mathbb{B})),$$

which is easily seen to preserve local suprema. Composing this with the one in (2) we define two more 2-functors:

$$\begin{tikzcd} \mathsf{Dist}(\mathcal{Q}) \arrow[r, dotted, "\mathcal{P}_1"] \arrow[dr, "\mathcal{P}_0"] & \mathsf{Dist}(\mathcal{Q}) \\ \mathsf{Cat}(\mathcal{Q}) \arrow[u, "i"] \arrow[r, dotted, "\mathcal{P}"'] & \mathsf{Cat}(\mathcal{Q}) \arrow[u, "i"'] \end{tikzcd} \qquad (3)$$

In fact, $\mathcal{P}_1$ is a Sup-functor (a.k.a. a homomorphism of quantaloids). Later on we shall encounter these functors again.

For a distributor $\Phi\colon\mathbb{A}\,{\circ\!\!\!\longrightarrow}\,\mathbb{B}$ and a functor $F\colon\mathbb{B}\longrightarrow\mathbb{C}$ between $\mathcal{Q}$-categories, the **Φ-weighted colimit of** F is a functor $K\colon\mathbb{A}\longrightarrow\mathbb{C}$ such that $[\Phi,\mathbb{B}(F-,-)]=\mathbb{C}(K-,-)$. Whenever a colimit exists, it is essentially unique; therefore the notation $\mathrm{colim}(\Phi,F)\colon\mathbb{A}\longrightarrow\mathbb{C}$ makes sense. These diagrams picture the situation:

$$\begin{tikzcd} \mathbb{B} \arrow[r, "F"] & \mathbb{C} \\ \mathbb{A} \arrow[u, "\Phi", "\circ" marking] \arrow[ur, dotted, "\mathrm{colim}(\Phi,F)"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{B} & \mathbb{C} \arrow[l, "\mathbb{C}(F-,-)"', "\circ" marking] \arrow[dl, dotted, "\circ" marking, "{[\Phi,\mathbb{C}(F-,-)]=\mathbb{C}(\mathrm{colim}(\Phi,F)-,-)}"] \\ \mathbb{A} \arrow[u, "\Phi", "\circ" marking] & \end{tikzcd}$$

A functor $G\colon\mathbb{C}\longrightarrow\mathbb{C}'$ is said to **preserve** $\mathrm{colim}(\Phi,F)$ if $G\circ\mathrm{colim}(\Phi,F)$ is the Φ-weighted colimit of $G\circ F$. A $\mathcal{Q}$-category admitting all possible colimits, is **cocomplete**, and a functor which preserves all colimits which exist in its domain, is **cocontinuous**. (There are, of course, the dual notions of limit, completeness and continuity. We shall only use colimits in this paper, but it is a matter of fact that a $\mathcal{Q}$-category is complete if and only if it is cocomplete [Stubbe, 2005, Proposition 5.10].)

For two functors $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ and $G\colon\mathbb{A}\longrightarrow\mathbb{C}$, we can consider the $\mathbb{C}(G-,-)$-weighted colimit of F. Whenever it exists, it is $\langle F,G\rangle\colon\mathbb{C}\longrightarrow\mathbb{B}$, the left Kan extension of F through G; but not every left Kan extension need to be such a colimit. Therefore we speak of a **pointwise left Kan extension** in this case.

Any presheaf category $\mathcal{P}(\mathbb{C})$ is cocomplete, as follows from its classifying property: given a distributor $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ and a functor $F\colon \mathbb{B} \longrightarrow \mathcal{P}(\mathbb{C})$, consider the unique distributor $\Phi_F\colon \mathbb{B} \longrightarrow \mathbb{C}$ corresponding with F; now in turn the composition $\Phi_F \otimes \Phi\colon \mathbb{A} \multimap \mathbb{C}$ corresponds with a unique functor $Y_{\Phi_F \otimes \Phi}\colon \mathbb{A} \longrightarrow \mathcal{P}(\mathbb{C})$; the latter is $\mathrm{colim}(\Phi, F)$.

In fact, the 2-functor

$$\mathcal{P}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$$

is the **Kock-Zöberlein-doctrine**[1] **for free cocompletion**; the components of its multiplication $M\colon \mathcal{P} \circ \mathcal{P} \Longrightarrow \mathcal{P}$ and its unit $Y\colon 1_{\mathsf{Cat}(\mathcal{Q})} \Longrightarrow \mathcal{P}$ are

$$\mathrm{colim}(-, 1_{\mathcal{P}(\mathbb{C})})\colon \mathcal{P}(\mathcal{P}(\mathbb{C})) \longrightarrow \mathcal{P}(\mathbb{C}) \text{ and } Y_{\mathbb{C}}\colon \mathbb{C} \longrightarrow \mathcal{P}(\mathbb{C}).$$

This means in particular that $(\mathcal{P}, M, Y)$ is a monad on $\mathsf{Cat}(\mathcal{Q})$, and a $\mathcal{Q}$-category $\mathbb{C}$ is cocomplete if and only if it is a $\mathcal{P}$-algebra, if and only if $Y_{\mathbb{C}}\colon \mathbb{C} \longrightarrow \mathcal{P}(\mathbb{C})$ admits a left adjoint in $\mathsf{Cat}(\mathcal{Q})$.

2.4 Full sub-KZ-doctrines of the free cocompletion doctrine

The following observation will be useful in a later subsection.

Proposition 2.4 *Suppose that $\mathcal{T}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ is a 2-functor and that*

$$\begin{tikzcd} \mathsf{Cat}(\mathcal{Q}) \arrow[r, bend left, "\mathcal{P}"] \arrow[r, bend right, "\mathcal{T}"'] \arrow[r, phantom, "\Uparrow\varepsilon"] & \mathsf{Cat}(\mathcal{Q}) \end{tikzcd}$$

is a 2-natural transformation, with all components $\varepsilon_{\mathbb{A}}\colon \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$ fully faithful functors, such that there are (necessarily essentially unique) factorisations

$$\begin{tikzcd} \mathcal{P} \circ \mathcal{P} \arrow[r, Rightarrow, "M"] & \mathcal{P} & \\ & & 1_{\mathsf{Cat}(\mathcal{Q})} \arrow[ul, Rightarrow, "Y"'] \arrow[dl, Rightarrow, dotted, "\eta"] \\ \mathcal{T} \circ \mathcal{T} \arrow[uu, Rightarrow, "\varepsilon * \varepsilon"] \arrow[r, Rightarrow, dotted, "\mu"'] & \mathcal{T} \arrow[uu, Rightarrow, "\varepsilon"'] & \end{tikzcd}$$

[1] A Kock-Zöberlein-doctrine (or KZ-doctrine, for short) $\mathcal{T}$ on a locally ordered category $\mathcal{K}$ is a 2-functor $\mathcal{T}\colon \mathcal{K} \longrightarrow \mathcal{K}$ for which there are a multiplication $\mu\colon \mathcal{T} \circ \mathcal{T} \Longrightarrow \mathcal{T}$ and a unit $\eta\colon 1_{\mathcal{K}} \Longrightarrow \mathcal{T}$ making $(\mathcal{T}, \mu, \eta)$ a 2-monad, and satisfying moreover the "KZ-inequation": $\mathcal{T}(\eta_K) \leq \eta_{\mathcal{T}(K)}$ for all objects K of $\mathcal{K}$. The notion was invented independently by Volker Zöberlein [1976] and Anders Kock [1972] in the more general setting of 2-categories. We refer to [Kock, 1995] for all details.

Then $(\mathcal{T},\mu,\eta)$ is a sub-2-monad of $(\mathcal{P},M,Y)$, and is a KZ-doctrine. We call the pair $(\mathcal{T},\varepsilon)$ a **full sub-KZ-doctrine** *of $\mathcal{P}$.*

Proof : First note that, because each $\varepsilon_{\mathbb{A}}\colon\mathcal{T}(\mathbb{A})\longrightarrow\mathcal{P}(\mathbb{A})$ is fully faithful, for each $F,G\colon\mathbb{C}\longrightarrow\mathcal{T}(\mathbb{A})$,

$$\varepsilon_{\mathbb{A}}\circ F\leq\varepsilon_{\mathbb{A}}\circ G\Longrightarrow F\leq G,$$

thus in particular $\varepsilon_{\mathbb{A}}$ is (essentially) a monomorphism in $\mathsf{Cat}(\mathcal{Q})$: if $\varepsilon_{\mathbb{A}}\circ F\cong\varepsilon_{\mathbb{A}}\circ G$ then $F\cong G$. Therefore we can regard $\varepsilon\colon\mathcal{T}\Longrightarrow\mathcal{P}$ as a subobject of the monoid $(\mathcal{P},M,Y)$ in the monoidal category of endo-2-functors on $\mathsf{Cat}(\mathcal{Q})$. The factorisations of M and Y then say precisely that $(\mathcal{T},\mu,\eta)$ is a submonoid, i.e. a 2-monad on $\mathsf{Cat}(\mathcal{Q})$ too.

But $\mathcal{P}\colon\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})$ maps fully faithful functors to fully faithful functors, as can be seen by applying Lemma 2.3 to the left adjoint $\mathbb{B}(-,F-)\colon\mathbb{A}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{B}$ in $\mathsf{Dist}(\mathcal{Q})$, for any given fully faithful $F\colon\mathbb{A}\longrightarrow\mathbb{B}$. Therefore each

$$(\varepsilon*\varepsilon)_{\mathbb{A}}\colon\mathcal{T}(\mathcal{T}(\mathbb{A}))\longrightarrow\mathcal{P}(\mathcal{P}(\mathbb{A}))$$

is fully faithful: for $(\varepsilon*\varepsilon)_{\mathbb{A}}=\mathcal{P}(\varepsilon_{\mathbb{A}})\circ\varepsilon_{\mathcal{T}\mathbb{A}}$ and by hypothesis both $\varepsilon_{\mathbb{A}}$ and $\varepsilon_{\mathcal{T}\mathbb{A}}$ are fully faithful. The commutative diagrams

$$\begin{tikzcd}
\mathcal{P}(\mathbb{A}) \arrow[r, "\mathcal{P}(Y_{\mathbb{A}})"] & \mathcal{P}(\mathcal{P}(\mathbb{A})) \\
 & \mathcal{T}(\mathcal{P}(\mathbb{A})) \arrow[u, "Y_{\mathcal{T}(\mathbb{A})}"'] \\
\mathcal{T}(\mathbb{A}) \arrow[uu, "\varepsilon_{\mathbb{A}}"] \arrow[ur, "\mathcal{T}(Y_{\mathbb{A}})"] \arrow[r, "\mathcal{T}(\eta_{\mathbb{A}})"'] & \mathcal{T}(\mathcal{T}(\mathbb{A})) \arrow[u, "\mathcal{T}(\varepsilon_{\mathbb{A}})"'] \arrow[uu, bend right, "(\varepsilon*\varepsilon)_{\mathbb{A}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(\mathbb{A}) \arrow[r, "Y_{\mathcal{P}(\mathbb{A})}"] \arrow[dr, "\eta_{\mathcal{P}\mathbb{A}}"'] & \mathcal{P}(\mathcal{P}(\mathbb{A})) \\
 & \mathcal{T}(\mathcal{P}(\mathbb{A})) \arrow[u, "Y_{\mathcal{T}(\mathbb{A})}"'] \\
\mathcal{T}(\mathbb{A}) \arrow[uu, "\varepsilon_{\mathbb{A}}"] \arrow[r, "\eta_{\mathcal{T}(\mathbb{A})}"'] & \mathcal{T}(\mathcal{T}(\mathbb{A})) \arrow[u, "\mathcal{T}(\varepsilon_{\mathbb{A}})"'] \arrow[uu, bend right, "(\varepsilon*\varepsilon)_{\mathbb{A}}"']
\end{tikzcd}$$

thus imply, together with the KZ-inequation for $\mathcal{P}$, the KZ-inequation for $\mathcal{T}$. $\square$

Some remarks can be made about the previous Proposition. Firstly, about the fully faithfulness of the components of $\varepsilon\colon\mathcal{T}\Longrightarrow\mathcal{P}$. In any locally ordered category $\mathcal{K}$ one defines an arrow $f\colon A\longrightarrow B$ to be *representably fully faithful* when, for any object X of $\mathcal{K}$, the order-preserving function

$$\mathcal{K}(f,-)\colon\mathcal{K}(X,A)\longrightarrow\mathcal{K}(X,B)\colon x\mapsto f\circ x$$

is order-reflecting – that is to say, $\mathcal{K}(f,-)$ is a fully faithful functor between ordered sets viewed as categories – and therefore f is also essentially a monomorphism in $\mathcal{K}$. But the converse need not hold, and indeed does not hold in $\mathcal{K} = \mathsf{Cat}(\mathcal{Q})$: not every monomorphism in $\mathsf{Cat}(\mathcal{Q})$ is representably fully faithful, and not every representably fully faithful functor is fully faithful. Because the 2-functor $\mathcal{P}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ preserves representable fully faithfulness as well, the above Proposition still holds (with the same proof) when the components of $\varepsilon\colon \mathcal{T} \Longrightarrow \mathcal{P}$ are merely *representably* fully faithful; and in that case it might be natural to say that $\mathcal{T}$ is a "sub-KZ-doctrine" of $\mathcal{P}$. But for our purposes later on, the interesting notion is that of *full* sub-KZ-doctrine, thus with the components of $\varepsilon\colon \mathcal{T} \Longrightarrow \mathcal{P}$ being fully faithful.

A second remark: in the situation of Proposition 2.4, the components of the transformation $\varepsilon\colon \mathcal{T} \Longrightarrow \mathcal{P}$ are necessarily given by pointwise left Kan extensions. More precisely, $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle\colon \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$ is the $\mathcal{T}(\mathbb{A})(\eta_{\mathbb{A}}-,-)$-weighted colimit of $Y_{\mathbb{A}}$ (which exists because $\mathcal{P}(\mathbb{A})$ is cocomplete), and can thus be computed as

$$\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle\colon \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})\colon t \mapsto \mathcal{T}(\mathbb{A})(\eta_{\mathbb{A}}-,t).$$

By fully faithfulness of $\varepsilon_{\mathbb{A}}\colon \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$ and the Yoneda Lemma, we can compute that

$$\mathcal{T}(\mathbb{A})(\eta_{\mathbb{A}}-,t) = \mathcal{P}(\mathbb{A})(\varepsilon_{\mathbb{A}} \circ \eta_{\mathbb{A}}-, \varepsilon_{\mathbb{A}}(t)) = \mathcal{P}(\mathbb{A})(Y_{\mathbb{A}}-, \varepsilon_{\mathbb{A}}(t)) = \varepsilon_{\mathbb{A}}(t).$$

Hence the component of $\varepsilon\colon \mathcal{T} \Longrightarrow \mathcal{P}$ at $\mathbb{A} \in \mathsf{Cat}(\mathcal{Q})$ is necessarily the Kan extension $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle$. We can push this argument a little further to obtain a characterisation of those KZ-doctrines which occur as full sub-KZ-doctrines of $\mathcal{P}$:

Corollary 2.5 *A KZ-doctrine $(\mathcal{T},\mu,\eta)$ on $\mathsf{Cat}(\mathcal{Q})$ is a full sub-KZ-doctrine of $\mathcal{P}$ if and only if all $\eta_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{T}(\mathbb{A})$ and all left Kan extensions $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle\colon \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$ are fully faithful.*

Proof : If $\mathcal{T}$ is a full sub-KZ-doctrine of $\mathcal{P}$, then we have just remarked that $\varepsilon_{\mathbb{A}} = \langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle$, and thus these Kan extensions are fully faithful. Moreover – because $\varepsilon_{\mathbb{A}} \circ \eta_{\mathbb{A}} = Y_{\mathbb{A}}$ with both $\varepsilon_{\mathbb{A}}$ and $Y_{\mathbb{A}}$ fully faithful – also $\eta_{\mathbb{A}}$ must be fully faithful.

Conversely, if $(\mathcal{T},\mu,\eta)$ is a KZ-doctrine with each $\eta_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{T}(\mathbb{A})$ fully faithful, then – e.g. by [Stubbe, 2005, Proposition 6.7] – the left Kan extensions $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle$ (exist and) satisfy $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle \circ \eta_{\mathbb{A}} \cong Y_{\mathbb{A}}$. By assumption each of these Kan extensions is fully faithful, so we must now prove that they are the components of a natural transformation and that this natural transformation commutes with the multiplications of $\mathcal{T}$ and $\mathcal{P}$. We do this in four steps:

(i) For any $\mathbb{A} \in \mathsf{Cat}(\mathcal{Q})$, there is the free $\mathcal{T}$-algebra $\mu_{\mathbb{A}}\colon \mathcal{T}(\mathcal{T}(\mathbb{A})) \longrightarrow \mathcal{T}(\mathbb{A})$. But the free $\mathcal{P}$-algebra $M_{\mathbb{A}}\colon \mathcal{P}(\mathcal{P}(\mathbb{A})) \longrightarrow \mathcal{P}(\mathbb{A})$ on $\mathcal{P}(\mathbb{A})$ also induces a $\mathcal{T}$-algebra on $\mathcal{P}(\mathbb{A})$: namely, $M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle : \mathcal{T}(\mathcal{P}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$. To see this, it suffices to prove the adjunction $M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle \dashv \eta_{\mathcal{P}(\mathbb{A})}$. The counit is easily checked:

$$M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle \circ \eta_{\mathcal{P}(\mathbb{A})} = M_{\mathbb{A}} \circ Y_{\mathcal{P}(\mathbb{A})} = 1_{\mathcal{P}(\mathbb{A})},$$

using first the factorisation property of the Kan extension and then the split adjunction $M_{\mathbb{A}} \dashv Y_{\mathcal{P}(\mathbb{A})}$. As for the unit of the adjunction, we compute that

$$\begin{aligned}
\eta_{\mathcal{P}(\mathbb{A})} \circ M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle &= \mathcal{T}(M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle) \circ \eta_{\mathcal{T}(\mathcal{P}(\mathbb{A}))} \\
&\geq \mathcal{T}(M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle) \circ \mathcal{T}(\eta_{\mathcal{P}(\mathbb{A})}) \\
&= \mathcal{T}(M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle \circ \eta_{\mathcal{P}(\mathbb{A})}) \\
&= \mathcal{T}(1_{\mathcal{P}(\mathbb{A})}) \\
&= 1_{\mathcal{T}(\mathcal{P}(\mathbb{A}))},
\end{aligned}$$

using naturality of η and the KZ inequality for $\mathcal{T}$, and recycling the computation we made for the counit.

(ii) Next we prove, for each $\mathcal{Q}$-category $\mathbb{A}$, that $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle : \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$ is a $\mathcal{T}$-algebra homomorphism, for the algebra structures explained in the previous step. This is the case if and only if $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle = (M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle) \circ \mathcal{T}(\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle) \circ \eta_{\mathcal{T}(\mathbb{A})}$ (because the domain of $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle$ is a free $\mathcal{T}$-algebra), and indeed:

$$\begin{aligned}
& M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle \circ \mathcal{T}(\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle) \circ \eta_{\mathcal{T}(\mathbb{A})} \\
&= M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle \circ \eta_{\mathcal{P}(\mathbb{A})} \circ \langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle \\
&= M_{\mathbb{A}} \circ Y_{\mathcal{P}(\mathbb{A})} \circ \langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle \\
&= 1_{\mathcal{P}(\mathbb{A})} \circ \langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle \\
&= \langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle.
\end{aligned}$$

(iii) To check that the left Kan extensions are the components of a natural transformation we must verify, for any $F: \mathbb{A} \longrightarrow \mathbb{B}$ in $\mathsf{Cat}(\mathcal{Q})$, that $\mathcal{P}(F) \circ \langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle = \langle Y_{\mathbb{B}}, \eta_{\mathbb{B}}\rangle \circ \mathcal{T}(F)$. Since this is an equation of $\mathcal{T}$-algebra homomorphisms for the $\mathcal{T}$-algebra structures discussed in step (i) – concerning $\mathcal{P}(F)$, it is easily seen to be a left adjoint and therefore also a $\mathcal{T}$-algebra homomorphism [Kock, 1995, Proposition 2.5] – it suffices to show that $\mathcal{P}(F) \circ \langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle \circ \eta_{\mathbb{A}} = \langle Y_{\mathbb{B}}, \eta_{\mathbb{B}}\rangle \circ \mathcal{T}(F) \circ \eta_{\mathbb{A}}$. This is straightforward from the factorisation property of the Kan extension and the naturality of $Y_{\mathbb{A}}$ and $\eta_{\mathbb{A}}$.

(iv) Finally, the very fact that $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle\colon \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$ is a $\mathcal{T}$-algebra homomorphism as in step (ii), means that

$$\begin{tikzcd}
\mathcal{T}(\mathcal{T}(\mathbb{A})) \arrow[r, "\mathcal{T}(\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle)"] \arrow[d, "\mu_{\mathbb{A}}"'] & \mathcal{T}(\mathcal{P}(\mathbb{A})) \arrow[r, "\langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle"] & \mathcal{P}(\mathcal{P}(\mathbb{A})) \arrow[d, "M_{\mathbb{A}}"] \\
\mathcal{T}(\mathbb{A}) \arrow[rr, "\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle"'] & & \mathcal{P}(\mathbb{A})
\end{tikzcd}$$

commutes: it expresses precisely the compatibility of the natural transformation whose components are the Kan extensions, with the multiplications of, respectively, $\mathcal{T}$ and $\mathcal{P}$. $\square$

3. Interlude: classifying cotabulations

In this section it is Proposition 3.3 which is of most interest: it explains in particular how the 2-functors on $\mathsf{Cat}(\mathcal{Q})$ of Proposition 2.4 can be extended to $\mathsf{Dist}(\mathcal{Q})$. It could easily be proved with a direct proof, but it seemed more appropriate to include first some material on classifying cotabulations, then use this to give a somewhat more conceptual proof of (the quantaloidal generalisation of) Akhvlediani *et al.*'s 'Extension Theorem' [2009, Theorem 1] in our Proposition 3.2, and finally derive Proposition 3.3 as a particular case.

A **cotabulation** of a distributor $\Phi\colon \mathbb{A} \xrightarrow{\circ} \mathbb{B}$ between $\mathcal{Q}$-categories is a pair of functors, say $S\colon \mathbb{A} \longrightarrow \mathbb{C}$ and $T\colon \mathbb{B} \longrightarrow \mathbb{C}$, such that $\Phi = \mathbb{C}(T-, S-)$. If $F\colon \mathbb{C} \longrightarrow \mathbb{C}'$ is a fully faithful functor then also $F \circ S\colon \mathbb{A} \longrightarrow \mathbb{C}'$ and $F \circ T\colon \mathbb{B} \longrightarrow \mathbb{C}'$ cotabulate Φ; so a distributor admits many different cotabulations. But the classifying property of $\mathcal{P}(\mathbb{B})$ suggests a particular one:

Proposition 3.1 *Any distributor $\Phi\colon \mathbb{A} \xrightarrow{\circ} \mathbb{B}$ is cotabulated by $Y_\Phi\colon \mathbb{A} \longrightarrow \mathcal{P}(\mathbb{B})$ and $Y_{\mathbb{B}}\colon \mathbb{B} \longrightarrow \mathcal{P}(\mathbb{B})$. We call this pair the* **classifying cotabulation** *of $\Phi\colon \mathbb{A} \xrightarrow{\circ} \mathbb{B}$.*

Proof: We compute for $a \in \mathbb{A}$ and $b \in \mathbb{B}$ that $\mathcal{P}(\mathbb{B})(Y_{\mathbb{B}}(b), Y_\Phi(a)) = Y_\Phi(a)(b) = \Phi(b, a)$ by using the Yoneda Lemma. $\square$

For two distributors $\Phi\colon \mathbb{A} \xrightarrow{\circ} \mathbb{B}$ and $\Psi\colon \mathbb{B} \xrightarrow{\circ} \mathbb{C}$ it is easily seen that $Y_{\Psi \otimes \Phi} = \mathcal{P}_0(\Psi) \circ Y_\Phi$, so the classifying cotabulation of the composite $\Psi \otimes \Phi$ relates to those of Φ and Ψ as

$$\Psi \otimes \Phi = \mathcal{P}(\mathbb{C})(\mathcal{P}_0(\Psi) \circ Y_\Phi -, Y_{\mathbb{C}} -). \tag{4}$$

For a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ it is straightforward that $Y_{\mathbb{B}(-,F-)} = Y_{\mathbb{B}} \circ F$, so

$$\mathbb{B}(-,F-) = \mathcal{P}(\mathbb{B})(Y_{\mathbb{B}}-, Y_{\mathbb{B}} \circ F-). \tag{5}$$

In particular, the identity distributor $\mathbb{A}\colon \mathbb{A} \multimap \mathbb{A}$ has the classifying cotabulation

$$\mathbb{A} = \mathcal{P}(\mathbb{A})(Y_{\mathbb{A}}-, Y_{\mathbb{A}}-). \tag{6}$$

Given that classifying cotabulations are thus perfectly capable of encoding composition and identities, it is natural to extend a given endo-functor on $\mathsf{Cat}(\mathcal{Q})$ to an endo-functor on $\mathsf{Dist}(\mathcal{Q})$ by applying it to classifying cotabulations. Now follows a statement of the 'Extension Theorem' of [Akhvlediani *et al.*, 2009] in the generality of quantaloid-enriched category theory, and a proof based on the calculus of classifying cotabulations.

Proposition 3.2 (Akhvlediani et al., 2009) *Any 2-functor $\mathcal{T}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ extends to a lax 2-functor $\mathcal{T}'\colon \mathsf{Dist}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})$, which is defined to send a distributor $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ to the distributor cotabulated by $\mathcal{T}(Y_\Phi)\colon \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{T}(\mathcal{P}(\mathbb{B}))$ and $\mathcal{T}(Y_{\mathbb{B}})\colon \mathcal{T}(\mathbb{B}) \longrightarrow \mathcal{T}(\mathcal{P}(\mathbb{B}))$. This comes with a lax transformation*

$$\begin{tikzcd} \mathsf{Dist}(\mathcal{Q}) \arrow[r, "\mathcal{T}'"] & \mathsf{Dist}(\mathcal{Q}) \\ \mathsf{Cat}(\mathcal{Q}) \arrow[u, "i"] \arrow[r, "\mathcal{T}"'] \arrow[ur, phantom, "\Nwarrow"] & \mathsf{Cat}(\mathcal{Q}) \arrow[u, "i"'] \end{tikzcd} \tag{7}$$

all of whose components are identities. This lax transformation is a (strict) 2-natural transformation (i.e. this diagram is commutative) if and only if $\mathcal{T}'$ is normal, if and only if each $\mathcal{T}(Y_{\mathbb{A}})\colon \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{T}(\mathcal{P}(\mathbb{A}))$ is fully faithful.

Proof : If $\Phi \leq \Psi$ holds in $\mathsf{Dist}(\mathcal{Q})(\mathbb{A},\mathbb{B})$ then (and only then) $Y_\Phi \leq Y_\Psi$ holds in $\mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathcal{P}(\mathbb{B}))$. By 2-functoriality of $\mathcal{T}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ we find that $\mathcal{T}(Y_\Phi) \leq \mathcal{T}(Y_\Psi)$, and thus $\mathcal{T}'(\Phi) \leq \mathcal{T}'(\Psi)$.

Now suppose that $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ and $\Psi\colon \mathbb{B} \multimap \mathbb{C}$ are given. Applying $\mathcal{T}$ to the commutative diagram

$$\begin{tikzcd} \mathbb{A} \arrow[dr, "Y_\Phi"'] & & \mathbb{B} \arrow[dl, "Y_{\mathbb{B}}"'] \arrow[dr, "Y_\Psi"] & & \mathbb{C} \arrow[dl, "Y_{\mathbb{C}}"'] \\ & \mathcal{P}(\mathbb{B}) \arrow[rr, "\mathcal{P}_0(\Psi)"'] & & \mathcal{P}(\mathbb{C}) & \end{tikzcd}$$

gives a commutative diagram in $\mathsf{Cat}(\mathcal{Q})$, which embeds as a commutative diagram of left adjoints in the quantaloid $\mathsf{Dist}(\mathcal{Q})$ by application of $i\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})$. Lemma 2.2, the formula in (4) and the definition of $\mathcal{T}'$ allow us to conclude that $\mathcal{T}'(\Psi) \otimes \mathcal{T}'(\Phi) \leq \mathcal{T}'(\Psi \otimes \Phi)$.

Similarly, given $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ in $\mathsf{Cat}(\mathcal{Q})$, applying $\mathcal{T}$ to the commutative diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[dr, "F"'] & & \mathbb{B} \arrow[dl, "1_{\mathbb{B}}"'] \arrow[dr, "Y_{\mathbb{B}}"] & & \mathbb{B} \arrow[dl, "Y_{\mathbb{B}}"] \\
& \mathbb{B} \arrow[rr, "Y_{\mathbb{B}}"'] & & \mathcal{P}(\mathbb{B}) &
\end{tikzcd}$$

gives a commutative diagram in $\mathsf{Cat}(\mathcal{Q})$. This again embeds as a diagram of left adjoints in $\mathsf{Dist}(\mathcal{Q})$ via $i\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})$. Lemma 2.2, the formula in (5) and the definition of $\mathcal{T}'$ then straightforwardly imply that

$$\begin{array}{rcl}
\mathcal{T}'(\mathbb{B}(-, F-)) & = & \mathcal{T}(\mathcal{P}(\mathbb{B}))(\mathcal{T}(Y_{\mathbb{B}})-, \mathcal{T}(Y_{\mathbb{B}})-) \otimes \mathcal{T}(\mathbb{B})(-, \mathcal{T}(F)-) \\
& \geq & \mathcal{T}(\mathbb{B})(-, \mathcal{T}(F)-),
\end{array}$$

accounting for the lax transformation in (7).

It further follows from this inequation, by applying it to identity functors, that $\mathcal{T}'$ is in general lax on identity distributors. But Lemma 2.2 also says: (i) if each $\mathcal{T}(Y_{\mathbb{B}})\colon \mathcal{T}(\mathbb{B}) \longrightarrow \mathcal{T}(\mathcal{P}(\mathbb{B}))$ is fully faithful (equivalently, if $\mathcal{T}'$ is normal), then necessarily $(i \circ \mathcal{T})(F) \cong (\mathcal{T}' \circ i)(F)$ for all $F : \mathbb{A} \longrightarrow \mathbb{B}$ in $\mathsf{Cat}(\mathcal{Q})$, asserting that the diagram in (7) commutes; (ii) and conversely, if that diagram commutes, then chasing the identities in $\mathsf{Cat}(\mathcal{Q})$ shows that $\mathcal{T}'$ is normal. $\square$

We shall be interested in extending full sub-KZ-doctrines of the free cocompletion doctrine $\mathcal{P}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ to $\mathsf{Dist}(\mathcal{Q})$; for this we make use of the functor $\mathcal{P}_1\colon \mathsf{Dist}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})$ defined in the diagram in (3).

Proposition 3.3 *Let $(\mathcal{T}, \varepsilon)$ be a full sub-KZ-doctrine of $\mathcal{P}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$. The lax extension $\mathcal{T}'\colon \mathsf{Dist}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})$ of $\mathcal{T}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ (as in Proposition 3.2) can then be computed as follows: for $\Phi\colon \mathbb{A} \multimap \mathbb{B}$,*

$$\mathcal{T}'(\Phi) = \mathcal{P}(\mathbb{B})(\varepsilon_{\mathbb{B}}-, -) \otimes \mathcal{P}_1(\Phi) \otimes \mathcal{P}(\mathbb{A})(-, \varepsilon_{\mathbb{A}}-). \tag{8}$$

Moreover, $\mathcal{T}'$ is always a normal lax Sup-functor, thus the diagram in (7) commutes.

Proof : Let $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ be a distributor. Proposition 3.2 defines $\mathcal{T}'(\Phi)$ to be the distributor cotabulated by $\mathcal{T}(Y_\Phi)$ and $\mathcal{T}(Y_{\mathbb{B}})$; but by fully faithfulness of the components of $\varepsilon\colon \mathcal{T} \Longrightarrow \mathcal{P}$, and its naturality, we can compute that

$$\mathcal{T}(\mathcal{P}(\mathbb{B}))(\mathcal{T}(Y_{\mathbb{B}})-, \mathcal{T}(Y_\Phi)-)$$

$$
\begin{aligned}
&= \mathcal{P}(\mathcal{P}(\mathbb{B}))((\varepsilon_{\mathcal{P}(\mathbb{B})} \circ \mathcal{T}(Y_{\mathbb{B}}))-, (\varepsilon_{\mathcal{P}(\mathbb{B})} \circ \mathcal{T}(Y_{\Phi}))-) \\
&= \mathcal{P}(\mathcal{P}(\mathbb{B}))((\mathcal{P}(Y_{\mathbb{B}}) \circ \varepsilon_{\mathbb{B}})-, (\mathcal{P}(Y_{\Phi}) \circ \varepsilon_{\mathbb{A}})-) \\
&= \mathcal{P}(\mathbb{B})(\varepsilon_{\mathbb{B}}-, -) \otimes \mathcal{P}(\mathcal{P}(\mathbb{B}))(\mathcal{P}(Y_{\mathbb{B}})-, \mathcal{P}(Y_{\Phi})-) \otimes \mathcal{P}(\mathbb{A})(-, \varepsilon_{\mathbb{A}}-).
\end{aligned}
$$

The middle term in this last expression can be reduced:

$$
\begin{aligned}
\mathcal{P}(\mathcal{P}(\mathbb{B}))(\mathcal{P}(Y_{\mathbb{B}})-, \mathcal{P}(Y_{\Phi})-) &= [\mathcal{P}(\mathbb{B})(-, Y_{\mathbb{B}}-) \otimes -,\ \mathcal{P}(\mathbb{B})(-, Y_{\Phi}-) \otimes -] \\
&= [-, \mathcal{P}(\mathbb{B})(Y_{\mathbb{B}}-, -) \otimes \mathcal{P}(\mathbb{B})(-, Y_{\Phi}-) \otimes -] \\
&= [-, \mathcal{P}(\mathbb{B})(Y_{\mathbb{B}}-, Y_{\Phi}-) \otimes -] \\
&= [-, \Phi \otimes -] \\
&= \mathcal{P}(\mathbb{B})(-, \mathcal{P}_0(\Phi)-) \\
&= (i \circ \mathcal{P}_0)(\Phi)(-, -) \\
&= \mathcal{P}_1(\Phi)(-, -).
\end{aligned}
$$

Thus we arrive at (8). Because $\mathcal{P}_1$ is a (strict) functor and because each $\varepsilon_{\mathbb{A}}$ is fully faithful, it follows from (8) that $\mathcal{T}'$ is normal. Similarly, because $\mathcal{P}_1$ is a Sup-functor, $\mathcal{T}'$ preserves local suprema too. $\square$

If we apply Proposition 3.2 to the 2-functor $\mathcal{P}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ itself, then we find that $\mathcal{P}' = \mathcal{P}_1$ (and thus it is strictly functorial, not merely normal lax). In general however, $\mathcal{T}'$ does *not* preserve composition.

4. Cocompletion: saturated classes of weights vs. KZ-doctrines

The Φ-weighted colimit of a functor F exists if and only if, for every $a \in \mathbb{A}_0$, $\mathrm{colim}(\Phi(-,a), F)$ exists:

$$
\begin{tikzcd}
\mathbb{B} \arrow[r, "F"] & \mathbb{C} \\
\mathbb{A} \arrow[u, "\Phi"'] \arrow[ur, dotted, "\mathrm{colim}(\Phi, F)"] & \\
*_{ta} \arrow[u, "\mathbb{A}(-,a)"'] \arrow[uu, bend left, "\Phi(-,a) = \Phi \otimes \mathbb{A}(-,a)"] \arrow[uur, dotted, "\mathrm{colim}(\Phi(-,a), F)"'] &
\end{tikzcd}
$$

Indeed, $\mathrm{colim}(\Phi, F)(a) = \mathrm{colim}(\Phi(-,a), F)(*)$. But now $\Phi(-,a)\colon *_{ta} \longrightarrow \mathbb{B}$ is a presheaf on $\mathbb{B}$. As a consequence, a $\mathcal{Q}$-category $\mathbb{C}$ is cocomplete if and only if it admits all colimits weighted by presheaves.

It therefore makes perfect sense to fix a class $\mathcal{C}$ of presheaves and study those $\mathcal{Q}$-categories that admit all colimits weighted by elements of $\mathcal{C}$: by definition these are the **$\mathcal{C}$-cocomplete categories**. Similarly, a functor $G\colon \mathbb{C} \longrightarrow \mathbb{C}'$ is **$\mathcal{C}$-cocontinuous** if it preserves all colimits weighted by elements of $\mathcal{C}$.

As [Albert and Kelly, 1988; Kelly and Schmitt, 2005] demonstrated in the case of $\mathcal{V}$-categories (for $\mathcal{V}$ a symmetric monoidal closed category with locally small, complete and cocomplete underlying category $\mathcal{V}_0$), and as we shall argue here for $\mathcal{Q}$-categories too, it is convenient to work with classes of presheaves that "behave nicely":

Definition 4.1 *A class $\mathcal{C}$ of presheaves on $\mathcal{Q}$-categories is* **saturated** *if:*

i. $\mathcal{C}$ contains all representable presheaves,

ii. for each $\phi\colon *_X \circ\!\!\!\rightarrow \mathbb{A}$ in $\mathcal{C}$ and each functor $G\colon \mathbb{A} \longrightarrow \mathcal{P}(\mathbb{B})$ for which each $G(a)$ is in $\mathcal{C}$, $\mathrm{colim}(\phi, G)$ is in $\mathcal{C}$ too.

There is another way of putting this. Observe first that any class $\mathcal{C}$ of presheaves on $\mathcal{Q}$-categories defines a sub-2-graph $k\colon \mathsf{Dist}_{\mathcal{C}}(\mathcal{Q}) \hookrightarrow \mathsf{Dist}(\mathcal{Q})$ by

$$\Phi\colon \mathbb{A} \circ\!\!\!\rightarrow \mathbb{B} \text{ is in } \mathsf{Dist}_{\mathcal{C}}(\mathcal{Q}) \overset{\text{def.}}{\iff} \text{ for all } a \in \mathbb{A}_0\text{: } \Phi(-,a) \in \mathcal{C}. \tag{9}$$

Then in fact we have:

Proposition 4.2 *A class $\mathcal{C}$ of presheaves on $\mathcal{Q}$-categories is saturated if and only if $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$ is a sub-2-category of $\mathsf{Dist}(\mathcal{Q})$ containing (all objects and) all identities. In this case there is an obvious factorisation*

$$\begin{tikzcd}
\mathsf{Cat}(\mathcal{Q}) \arrow[rr, "i"] \arrow[dr, dotted, "j"'] & & \mathsf{Dist}(\mathcal{Q}) \\
& \mathsf{Dist}_{\mathcal{C}}(\mathcal{Q}) \arrow[ur, hook, "k"'] &
\end{tikzcd}$$

Proof : With (9) it is trivial that $\mathcal{C}$ contains all representable presheaves if and only if $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$ contains all objects and all identities.

Next, assume that $\mathcal{C}$ is a saturated class of presheaves, and let $\Phi\colon \mathbb{A} \circ\!\!\!\rightarrow \mathbb{B}$ and $\Psi\colon \mathbb{B} \circ\!\!\!\rightarrow \mathbb{C}$ be arrows in $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$. Invoking the classifying property of $\mathcal{P}(\mathbb{C})$ and the computation of colimits in $\mathcal{P}(\mathbb{C})$, we find $\mathrm{colim}(\Phi(-,a), Y_\Psi) = \Psi \otimes \Phi(-,a)$ for each $a \in \mathbb{A}_0$. But because $\Phi(-,a) \in \mathcal{C}$ and for each $b \in \mathbb{B}_0$ also $Y_\Psi(b) = \Psi(-,b) \in \mathcal{C}$, this colimit, i.e. $\Psi \otimes \Phi(-,a)$, is an element of $\mathcal{C}$. This holds for all $a \in \mathbb{A}_0$, thus the composition $\Psi \otimes \Phi\colon \mathbb{A} \circ\!\!\!\rightarrow \mathbb{C}$ is an arrow in $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$.

Conversely, assuming $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$ is a sub-2-category of $\mathsf{Dist}(\mathcal{Q})$, let $\phi\colon *_A \to \mathbb{B}$ be in $\mathcal{C}$ and let $F\colon \mathbb{B} \longrightarrow \mathcal{P}(\mathbb{C})$ be a functor such that, for each $b \in \mathbb{B}$, $F(b)$ is in $\mathcal{C}$. By the classifying property of $\mathcal{P}(\mathbb{C})$ we can equate the functor $F\colon \mathbb{B} \longrightarrow \mathcal{P}(\mathbb{C})$ with a distributor $\Phi_F\colon \mathbb{B} \to \mathbb{C}$ and by the computation of colimits in $\mathcal{P}(\mathbb{C})$ we know that $\mathrm{colim}(\phi, F) = \Phi_F \otimes \phi$. Now $\Phi_F(-,b) = F(b)$ by definition, so $\Phi_F\colon \mathbb{B} \to \mathbb{C}$ is in $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$; but also $\phi : *_A \to \mathbb{B}$ is in $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$, and therefore their composite is in $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$, i.e. $\mathrm{colim}(\phi, F)$ is in $\mathcal{C}$, as wanted.

Finally, if $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is any functor, then for each $a \in \mathbb{A}$ the representable $\mathbb{B}(-, Fa)\colon *_{ta} \to \mathbb{B}$ is in the saturated class $\mathcal{C}$, and therefore $\mathbb{B}(-, F-)\colon \mathbb{A} \to \mathbb{B}$ is in $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$. This accounts for the factorisation of $\mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})$ over $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q}) \hookrightarrow \mathsf{Dist}(\mathcal{Q})$. $\square$

We shall now characterise saturated classes of presheaves on $\mathcal{Q}$-categories in terms of KZ-doctrines on $\mathsf{Cat}(\mathcal{Q})$. (We shall indeed always deal with a saturated class of presheaves, even though certain results hold under weaker hypotheses.) We begin by pointing out a classifying property:

Proposition 4.3 *Let $\mathcal{C}$ be a saturated class of presheaves and, for a $\mathcal{Q}$-category $\mathbb{A}$, write $J_{\mathbb{A}}\colon \mathcal{C}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$ for the full subcategory of $\mathcal{P}(\mathbb{A})$ determined by those presheaves on $\mathbb{A}$ which are elements of $\mathcal{C}$. A distributor $\Phi\colon \mathbb{A} \to \mathbb{B}$ belongs to $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$ if and only if there exists a (necessarily unique) factorisation*

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "Y_\Phi"] \arrow[dr, dotted, "I_\Phi"'] & & \mathcal{P}(\mathbb{B}) \\ & \mathcal{C}(\mathbb{B}) \arrow[ur, "J_{\mathbb{B}}"'] & \end{tikzcd} \tag{10}$$

in which case Φ is cotabulated by $I_\Phi\colon \mathbb{A} \longrightarrow \mathcal{C}(\mathbb{B})$ and $I_{\mathbb{B}}\colon \mathbb{B} \longrightarrow \mathcal{C}(\mathbb{B})$ (the latter being the factorisation of $Y_{\mathbb{B}}$ through $J_{\mathbb{B}}$).

Proof : The factorisation property in (10) literally says that, for any $a \in \mathbb{A}$, the presheaf $Y_\Phi(a)$ on $\mathbb{B}$ must be an element of the class $\mathcal{C}$. But $Y_\Phi(b) = \Phi(-, b)$ hence this is trivially equivalent to the statement in (9), defining those distributors that belong to $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$. In particular, if $\mathcal{C}$ is saturated then $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$ contains all identities, hence we have factorisations $Y_{\mathbb{B}} = J_{\mathbb{B}} \circ I_{\mathbb{B}}$ of the Yoneda embeddings. Hence, whenever a factorisation as in (10) exists, we can use the fully faithful $J_{\mathbb{B}}\colon \mathcal{C}(\mathbb{B}) \longrightarrow \mathcal{P}(\mathbb{B})$ to compute, starting from the classifying cotabulation of Φ, that

$$\Phi = \mathcal{P}(\mathbb{B})(Y_{\mathbb{B}}-, Y_\Phi-) = \mathcal{P}(\mathbb{B})(J_{\mathbb{B}}(I_{\mathbb{B}}(-)), J_{\mathbb{B}}(I_\Phi(-))) = \mathcal{C}(\mathbb{B})(I_{\mathbb{B}}-, I_\Phi-),$$

confirming the cotabulation of Φ by I_Φ and $I_{\mathbb{B}}$. $\square$

Any saturated class $\mathcal{C}$ thus automatically comes with the 2-functor

$$\mathcal{C}_0\colon \mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})\colon \Big(\Phi\colon\mathbb{A}\,-\!\!\!\circ\!\!\!\rightarrow\mathbb{B}\Big)\mapsto\Big(\Phi\otimes -\colon\mathcal{C}(\mathbb{A})\longrightarrow\mathcal{C}(\mathbb{B})\Big)$$

and the full embeddings $J_{\mathbb{A}}\colon\mathcal{C}(\mathbb{A})\longrightarrow\mathcal{P}(\mathbb{A})$ are the components of a 2-natural transformation

$$\begin{tikzcd} & \mathsf{Dist}(\mathcal{Q}) \arrow[dr, bend left, "\mathcal{P}_0"] \arrow[d, phantom, "\Uparrow J"] & \\ \mathsf{Dist}_{\mathcal{C}}(\mathcal{Q}) \arrow[ur, bend left, "k"] \arrow[rr, "\mathcal{C}_0"'] & {} & \mathsf{Cat}(\mathcal{Q}) \end{tikzcd}$$

Composing $\mathcal{C}_0$ with $j\colon\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$ it is natural to define

$$\mathcal{C}\colon\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})\colon\Big(F\colon\mathbb{A}\longrightarrow\mathbb{B}\Big)\mapsto\Big(\mathbb{B}(-,F-)\otimes -\colon\mathcal{C}(\mathbb{A})\longrightarrow\mathcal{C}(\mathbb{B})\Big)$$

together with

$$\begin{tikzcd} \mathsf{Cat}(\mathcal{Q}) \arrow[r, bend left, "\mathcal{P}"] \arrow[r, bend right, "\mathcal{C}"'] \arrow[r, phantom, "\Uparrow J"] & \mathsf{Cat}(\mathcal{Q}) \end{tikzcd}$$

(slightly abusing notation). We apply previous results, particularly Proposition 2.4:

Proposition 4.4 *If $\mathcal{C}$ is a saturated class of presheaves on $\mathcal{Q}$-categories then the 2-functor $\mathcal{C}\colon\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})$ together with the transformation $J\colon\mathcal{C}\Longrightarrow\mathcal{P}$ forms a full sub-KZ-doctrine of $\mathcal{P}$. Moreover, the $\mathcal{C}$-cocomplete $\mathcal{Q}$-categories are precisely the $\mathcal{C}$-algebras, and the $\mathcal{C}$-cocontinuous functors between $\mathcal{C}$-cocomplete $\mathcal{Q}$-categories are precisely the $\mathcal{C}$-algebra homomorphisms.*

Proof : To fulfill the hypotheses in Proposition 2.4, we only need to check the factorisation of the multiplication: if we prove, for any $\mathcal{Q}$-category $\mathbb{A}$ and each $\phi\in\mathcal{C}(\mathcal{C}(\mathbb{A}))$, that the $(J*J)_{\mathbb{A}}(\phi)$-weighted colimit of $1_{\mathcal{P}(\mathbb{A})}$ is in $\mathcal{C}(\mathbb{A})$, then we obtain the required commutative diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(\mathbb{A})) \arrow[r, "\mathrm{colim}(-,1_{\mathcal{P}(\mathbb{A})})"] & \mathcal{P}(\mathbb{A}) \\ \mathcal{C}(\mathcal{C}(\mathbb{A})) \arrow[u, "(J*J)_{\mathbb{A}}"] \arrow[r, dotted] & \mathcal{C}(\mathbb{A}) \arrow[u, "J_{\mathbb{A}}"'] \end{tikzcd}$$

But because $(J*J)_{\mathbb{A}}=\mathcal{P}(J_{\mathbb{A}})\circ J_{\mathcal{C}(\mathbb{A})}$ we can compute that

$$\mathrm{colim}((J*J)_{\mathbb{A}}(\phi),1_{\mathcal{P}(\mathbb{A})})=\mathrm{colim}(\mathcal{P}(\mathbb{A})(-,J_{\mathbb{A}}-)\otimes\phi,1_{\mathcal{P}(\mathbb{A})})=\mathrm{colim}(\phi,J_{\mathbb{A}})$$

and this colimit indeed belongs to the saturated class $\mathcal{C}$, because both ϕ and (the objects in) the image of $J_{\mathbb{A}}$ are in $\mathcal{C}$.

A $\mathcal{Q}$-category $\mathbb{B}$ is a $\mathcal{C}$-algebra if and only if $I_{\mathbb{B}}\colon\mathbb{B}\longrightarrow\mathcal{C}(\mathbb{B})$ admits a left adjoint in $\mathsf{Cat}(\mathcal{Q})$ (because $\mathcal{C}$ is a KZ-doctrine). Suppose that $\mathbb{B}$ is indeed a $\mathcal{C}$-algebra, and write the left adjoint as $L_{\mathbb{B}}\colon\mathcal{C}(\mathbb{B})\longrightarrow\mathbb{B}$. If $\phi\colon *_X \multimapdotinv \mathbb{A}$ is a presheaf in $\mathcal{C}$ and $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ is any functor, then $\mathcal{C}(F)(\phi)$ is an object of $\mathcal{C}(\mathbb{B})$, thus we can consider the object $L_{\mathbb{B}}(\mathcal{C}(F)(\phi))$ of $\mathbb{B}$. This is precisely the ϕ-weighted colimit of F, for indeed its universal property holds: for any $b\in\mathbb{B}$,

$$\begin{aligned}
\mathbb{B}(L_{\mathbb{B}}(\mathcal{C}(F)(\phi)),b) &= \mathcal{C}(\mathbb{B})(\mathcal{C}(F)(\phi),I_{\mathbb{B}}(b))\\
&= \mathcal{P}(\mathbb{B})(J_{\mathbb{B}}(\mathcal{C}(F)(\phi)),J_{\mathbb{B}}(I_{\mathbb{B}}(b)))\\
&= [\mathcal{P}(F)(J_{\mathbb{B}}(\phi)),Y_{\mathbb{B}}(b)]\\
&= [\mathbb{B}(-,F-)\otimes J_{\mathbb{B}}(\phi),\mathbb{B}(-,b)]\\
&= [J_{\mathbb{B}}(\phi),\mathbb{B}(F-,-)\otimes\mathbb{B}(-,b)]\\
&= [\phi,\mathbb{B}(F-,b)].
\end{aligned}$$

(Apart from the adjunction $L_{\mathbb{B}}\dashv I_{\mathbb{B}}$ we used the fully faithfulness of $J_{\mathbb{B}}$ and its naturality, and then made some computations with liftings and adjoints in $\mathsf{Dist}(\mathcal{Q})$.)

Conversely, suppose that $\mathbb{B}$ admits all $\mathcal{C}$-weighted colimits. In particular can we then compute, for any $\phi\in\mathcal{C}(\mathbb{B})$, the ϕ-weighted colimit of $1_{\mathbb{B}}$, and doing so gives a function $f\colon\mathcal{C}(\mathbb{B})\longrightarrow\mathbb{B}\colon\phi\mapsto\mathrm{colim}(\phi,1_{\mathbb{B}})$. But for any $\phi\in\mathcal{C}(\mathbb{B})$ and any $b\in\mathbb{B}$ it is easy to compute, from the universal property of colimits and using the fully faithfulness of $J_{\mathbb{B}}$, that

$$\begin{aligned}
\mathbb{B}(f(\phi),b) &= [\phi,\mathbb{B}(1_{\mathbb{B}}-,b)] = \mathcal{P}(\mathbb{B})(\phi,Y_{\mathbb{B}}(b))\\
&= \mathcal{P}(\mathbb{B})(J_{\mathbb{B}}(\phi),J_{\mathbb{B}}(I_{\mathbb{B}}(b))) = \mathcal{C}(\mathbb{B})(\phi,I_{\mathbb{B}}(b)).
\end{aligned}$$

This straightforwardly implies that $\phi\mapsto f(\phi)$ is in fact a functor (and not merely a function), and that it is left adjoint to $I_{\mathbb{B}}$; thus $\mathbb{B}$ is a $\mathcal{C}$-algebra.

Finally, let $G\colon\mathbb{B}\longrightarrow\mathbb{C}$ be a functor between $\mathcal{C}$-cocomplete $\mathcal{Q}$-categories. Supposing that G is $\mathcal{C}$-cocontinuous, we can compute any $\psi\in\mathcal{C}(\mathbb{B})$ that

$$G(L_{\mathbb{B}}(\psi))=G(\mathrm{colim}(\psi),1_{\mathbb{B}})=\mathrm{colim}(\psi),G)=L_{\mathbb{C}}(\mathcal{C}(G)(\psi)),$$

proving that G is a homomorphism between the $\mathcal{C}$-algebras $(\mathbb{B},L_{\mathbb{B}})$ and $(\mathbb{C},L_{\mathbb{C}})$. Conversely, supposing now that G is a homomorphism, we can compute for any presheaf $\phi\colon *_X \multimapdotinv \mathbb{A}$ in $\mathcal{C}$ and any functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ that

$$G(\mathrm{colim}(\phi,F))=G(L_{\mathbb{B}}(\mathcal{C}(F)(\phi)))=L_{\mathbb{C}}(\mathcal{C}(G)(\mathcal{C}(F)(\phi)))=\mathrm{colim}(\phi,G\circ F),$$

proving that G is $\mathcal{C}$-cocontinuous. $\square$

Also the converse of the previous Proposition is true:

Proposition 4.5 *If $(\mathcal{T},\varepsilon)$ is a full sub-KZ-doctrine of $\mathcal{P}\colon \mathsf{Cat}(\mathcal{Q})\longrightarrow \mathsf{Cat}(\mathcal{Q})$ then*

$$\mathcal{C}_{\mathcal{T}} := \{\varepsilon_{\mathbb{A}}(t) \mid \mathbb{A}\in \mathsf{Cat}(\mathcal{Q}),\ t\in\mathcal{T}(\mathbb{A})\} \tag{11}$$

is a saturated class of presheaves on $\mathcal{Q}$-categories. Moreover, the $\mathcal{T}$-algebras are precisely the $\mathcal{C}_{\mathcal{T}}$-cocomplete categories, and the $\mathcal{T}$-algebra homomorphisms are precisely the $\mathcal{C}_{\mathcal{T}}$-cocontinuous functors between the $\mathcal{C}_{\mathcal{T}}$-cocomplete categories.

Proof : We shall write $\mathsf{Dist}_{\mathcal{T}}(\mathcal{Q})$ for the sub-2-graph of $\mathsf{Dist}(\mathcal{Q})$ determined – as prescribed in (9) – by the class $\mathcal{C}_{\mathcal{T}}$, and we shall show that it is a sub-2-category containing all (objects and) identities of $\mathsf{Dist}(\mathcal{Q})$. But a distributor $\Phi\colon\mathbb{A}\xrightarrow{\circ}\mathbb{B}$ belongs to $\mathsf{Dist}_{\mathcal{T}}(\mathcal{Q})$ if and only if the classifying functor $Y_\Phi\colon\mathbb{A}\longrightarrow\mathcal{P}(\mathbb{B})$ factors (necessarily essentially uniquely) through the fully faithful $\varepsilon_{\mathbb{B}}\colon\mathcal{T}(\mathbb{B})\longrightarrow\mathcal{P}(\mathbb{B})$.

By hypothesis there is a factorisation $Y_{\mathbb{A}}=\varepsilon_{\mathbb{A}}\circ\eta_{\mathbb{A}}$ for any $\mathbb{A}\in\mathsf{Cat}(\mathcal{Q})$, so $\mathsf{Dist}_{\mathcal{T}}(\mathcal{Q})$ contains all identities. Secondly, suppose that $\Phi\colon\mathbb{A}\xrightarrow{\circ}\mathbb{B}$ and $\Psi\colon\mathbb{B}\xrightarrow{\circ}\mathbb{C}$ are in $\mathsf{Dist}_{\mathcal{T}}(\mathcal{Q})$, meaning that there exist factorisations

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "Y_\Phi"] \arrow[dr, dotted, "I_\Phi"'] & & \mathcal{P}(\mathbb{B}) \\
& \mathcal{T}(\mathbb{B}) \arrow[ur, "\varepsilon_{\mathbb{B}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{B} \arrow[rr, "Y_\Psi"] \arrow[dr, dotted, "I_\Psi"'] & & \mathcal{P}(\mathbb{C}) \\
& \mathcal{T}(\mathbb{C}) \arrow[ur, "\varepsilon_{\mathbb{C}}"'] &
\end{tikzcd}$$

The following diagram is then easily seen to commute:

$$\begin{tikzcd}
& \mathcal{T}(\mathbb{B}) \arrow[r, "\varepsilon_{\mathbb{B}}"] \arrow[d, "\mathcal{T}(I_\Psi)"'] & \mathcal{P}(\mathbb{B}) \arrow[d, "\mathcal{P}(I_\Psi)"] \arrow[dd, bend left, "\mathcal{P}(Y_\Psi)"] \\
& \mathcal{T}(\mathcal{T}(\mathbb{C})) \arrow[r, "\varepsilon_{\mathcal{T}(\mathbb{C})}"] \arrow[dl, "\mu_{\mathbb{C}}"'] \arrow[d, "\mathcal{T}(\varepsilon_{\mathbb{C}})"'] \arrow[dr, dotted, "(\varepsilon*\varepsilon)_{\mathbb{C}}"] & \mathcal{P}(\mathcal{T}(\mathbb{C})) \arrow[d, "\mathcal{P}(\varepsilon_{\mathbb{C}})"] \\
\mathcal{T}(\mathbb{C}) \arrow[dr, "\varepsilon_{\mathbb{C}}"'] & \mathcal{T}(\mathcal{P}(\mathbb{C})) \arrow[r, "\varepsilon_{\mathcal{P}(\mathbb{C})}"'] & \mathcal{P}(\mathcal{P}(\mathbb{C})) \arrow[dl, "M_{\mathbb{C}}"] \\
& \mathcal{P}(\mathbb{C}) &
\end{tikzcd}$$

But we can compute, for any $\phi\in\mathcal{P}(\mathbb{B})$, that

$$(M_{\mathbb{C}}\circ\mathcal{P}(Y_\Psi))(\phi) \quad = \quad \mathrm{colim}(\mathcal{P}(\mathbb{A})(-,Y_\Psi-)\otimes\phi, 1_{\mathcal{P}(\mathbb{A})})$$

$$\begin{aligned} &= \mathrm{colim}(\phi, Y_\Psi) \\ &= \Psi \otimes \phi \\ &= \mathcal{P}_0(\Psi)(\phi) \end{aligned}$$

and therefore $Y_{\Psi\otimes\Psi} = \mathcal{P}_0(\Psi) \circ Y_\Phi = M_{\mathbb{C}} \circ \mathcal{P}(Y_\Psi) \circ \varepsilon_{\mathbb{B}} \circ I_\Phi = \varepsilon_{\mathbb{C}} \circ \mu_{\mathbb{C}} \circ \mathcal{T}(I_\Psi) \circ I_\Phi$, giving a factorisation of $Y_{\Psi\otimes\Phi}$ through $\varepsilon_{\mathbb{C}}$, as wanted.

The arguments to prove that a $\mathcal{Q}$-category $\mathbb{B}$ is a $\mathcal{T}$-algebra if and only if it is $\mathcal{C}_\mathcal{T}$-cocomplete, and that a $\mathcal{T}$-algebra homomorphism is precisely a $\mathcal{C}_\mathcal{T}$-cocontinuous functor between $\mathcal{C}_\mathcal{T}$-cocomplete $\mathcal{Q}$-categories, are much like those in the proof of Proposition 4.4. Omitting the calculations, let us just indicate that for a $\mathcal{T}$-algebra $\mathbb{B}$, thus with a left adjoint $L_{\mathbb{B}}: \mathcal{T}(\mathbb{B}) \longrightarrow \mathbb{B}$ to $\eta_{\mathbb{B}}$, for any weight $\phi: *_X \looparrowright \mathbb{A}$ in $\mathcal{C}_\mathcal{T}$ – i.e. $\phi = \varepsilon_{\mathbb{A}}(t)$ for some $t \in \mathcal{T}(\mathbb{A})$ – and any functor $F: \mathbb{A} \longrightarrow \mathbb{B}$, the object $L_{\mathbb{B}}(\mathcal{T}(F))(t)$ is the ϕ-weighted colimit of F. And conversely, if $\mathbb{B}$ is a $\mathcal{C}_\mathcal{T}$-cocomplete $\mathcal{Q}$-category, then $\mathcal{T}(\mathbb{B}) \longrightarrow \mathbb{B}: t \mapsto \mathrm{colim}(\varepsilon_{\mathbb{B}}(t), 1_{\mathbb{B}})$ is the left adjoint to $\eta_{\mathbb{B}}$, making $\mathbb{B}$ a $\mathcal{T}$-algebra. $\square$

If $\mathcal{C}$ is a saturated class of presheaves and we apply Proposition 4.4 to obtain a full sub-KZ-doctrine $(\mathcal{C}, J)$ of $\mathcal{P}: \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$, then the application of Proposition 4.5 gives us back precisely that same class $\mathcal{C}$ that we started from. The other way round is slightly more subtle: if $(\mathcal{T}, \varepsilon)$ is a full sub-KZ-doctrine of $\mathcal{P}$ then Proposition 4.5 gives us a saturated class $\mathcal{C}_\mathcal{T}$ of presheaves, and this class in turn determines by Proposition 4.4 a full KZ-doctrine of $\mathcal{P}$, let us write it as $(\mathcal{T}', \varepsilon')$, which is *equivalent* to $\mathcal{T}$. More exactly, each (fully faithful) $\varepsilon_{\mathbb{A}}: \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$ factors over the fully faithful and injective $\varepsilon'_{\mathbb{A}}: \mathcal{T}'(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$, and this factorisation is fully faithful and surjective, thus an equivalence. These equivalences are the components of a 2-natural transformation $\delta: \mathcal{T} \Longrightarrow \mathcal{T}'$ which commutes with ε and ε'.

We summarise all the above in the following:

Theorem 4.6 *Propositions 4.4 and 4.5 determine an essentially bijective correspondence between, on the one hand, saturated classes $\mathcal{C}$ of presheaves on $\mathcal{Q}$-categories, and on the other hand, full sub-KZ-doctrines $(\mathcal{T}, \varepsilon)$ of the free cocompletion KZ-doctrine $\mathcal{P}: \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$; a class $\mathcal{C}$ and a doctrine $\mathcal{T}$ correspond with each other if and only if the $\mathcal{T}$-algebras and their homomorphisms are precisely the $\mathcal{C}$-cocomplete $\mathcal{Q}$-categories and the $\mathcal{C}$-cocontinuous functors between them. Proposition 3.3 implies that, in this case, there is a normal lax* Sup-*functor $\mathcal{T}': \mathsf{Dist}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})$, sending a distributor $\Phi: \mathbb{A} \looparrowright \mathbb{B}$ to the distributor $\mathcal{T}'(\Phi): \mathcal{T}(\mathbb{A}) \looparrowright \mathcal{T}(\mathbb{B})$ with elements*

$$\mathcal{T}'(\Phi)(t, s) = \mathcal{P}(\mathbb{B})(\varepsilon_{\mathbb{B}}(t), \Phi \otimes \varepsilon_{\mathbb{A}}(s)), \quad \textit{for } s \in \mathcal{T}(\mathbb{A}),\ t \in \mathcal{T}(\mathbb{B}),$$

which makes the following diagram commute:

$$\begin{tikzcd} \mathsf{Dist}(\mathcal{Q}) \arrow[r, "\mathcal{T}'"] & \mathsf{Dist}(\mathcal{Q}) \\ \mathsf{Cat}(\mathcal{Q}) \arrow[u, "i"] \arrow[r, "\mathcal{T}"'] & \mathsf{Cat}(\mathcal{Q}) \arrow[u, "i"'] \end{tikzcd}$$

5. Conical cocompletion and the Hausdorff doctrine

5.1 Conical colimits

Let $\mathbb{A}$ be a $\mathcal{Q}$-category. Putting, for any $a, a' \in \mathbb{A}$,

$$a \leq a' \overset{\text{def.}}{\iff} ta = ta' \text{ and } 1_{ta} \leq \mathbb{A}(a, a')$$

defines an order relation on the objects of $\mathbb{A}$. (There are equivalent conditions in terms of representable presheaves.) For a given $\mathcal{Q}$-category $\mathbb{A}$ and a given object $X \in \mathcal{Q}_0$, we shall write $(\mathbb{A}_X, \leq_X)$ for the ordered set of objects of $\mathbb{A}$ of type X. Because elements of different type in $\mathbb{A}$ can never have a supremum in $(\mathbb{A}_0, \leq)$, it would be very restrictive to require this order to admit arbitrary suprema; instead, experience shows that it makes good sense to require each $(\mathbb{A}_X, \leq_X)$ to be a sup-lattice: we then say that $\mathbb{A}$ is **order-cocomplete** [Stubbe, 2006]. As spelled out in that reference, we have:

Proposition 5.1 *For a family $(a_i)_{i\in I}$ in $\mathbb{A}_X$, the following are equivalent:*

i. $\bigvee_i a_i$ *exists in* $\mathbb{A}_X$ *and* $\mathbb{A}(\bigvee_i a_i, -) = \bigwedge_i \mathbb{A}(a_i, -)$ *holds in* $\mathsf{Dist}(\mathcal{Q})(\mathbb{A}, *_X)$,

ii. $\bigvee_i a_i$ *exists in* $\mathbb{A}_X$ *and* $\mathbb{A}(-, \bigvee_i a_i) = \bigvee_i \mathbb{A}(-, a_i)$ *holds in* $\mathsf{Dist}(\mathcal{Q})(*_X, \mathbb{A})$,

iii. if we write $(I, \leq)$ for the ordered set in which $i \leq j$ precisely when $a_i \leq_X a_j$ and $\mathbb{I}$ for the free $\mathcal{Q}(X,X)$-category on the poset $(I, \leq)$, $F\colon \mathbb{I} \longrightarrow \mathbb{A}$ for the functor $i \mapsto a_i$ and $\gamma\colon *_X \,-\!\!\!\circ\!\!\!\rightarrow \mathbb{I}$ for the presheaf with values $\gamma(i) = 1_X$ for all $i \in \mathbb{I}$, then the γ-weighted colimit of F exists.

In this case, $\mathrm{colim}(\gamma, F) = \bigvee_i a_i$ *and it is the* **conical colimit** *of* $(a_i)_{i\in I}$ *in* $\mathbb{A}$.

It is important to realise that such conical colimits – which are enriched colimits! – can be characterised by a property of weights:

Proposition 5.2 *For a presheaf $\phi\colon *_X \,-\!\!\!\circ\!\!\!\rightarrow \mathbb{A}$, the following conditions are equivalent:*

i. there exists a family $(a_i)_{i\in I}$ in $\mathbb{A}_X$ such that for any functor $G\colon\mathbb{A}\longrightarrow\mathbb{B}$, if the ϕ-weighted colimit of G exists, then it is the conical colimit of the family $(G(a_i))_i$,

ii. there exists a family $(a_i)_{i\in I}$ in $\mathbb{A}_X$ for which $\phi = \bigvee_i \mathbb{A}(-,a_i)$ holds in $\mathsf{Dist}(\mathcal{Q})(*_X,\mathbb{A})$,

iii. there exist an ordered set $(I,\leq)$ and a functor $F\colon\mathbb{I}\longrightarrow\mathbb{A}$ with domain the free $\mathcal{Q}(X,X)$-category on $(I,\leq)$ such that, if we write $\gamma\colon *_X \multimap \mathbb{I}$ for the presheaf with values $\gamma(i)=1_X$ for all $i\in\mathbb{I}$, then $\phi=\mathbb{A}(-,F-)\otimes\gamma$.

In this case, we call ϕ a **conical presheaf***.*

Proof : (i$\Rightarrow$ii) Applying the hypothesis to the functor $Y_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathcal{P}(\mathbb{A})$ – indeed $\mathrm{colim}(\phi,Y_{\mathbb{A}})$ exists, and is equal to ϕ by the Yoneda Lemma – we find a family $(a_i)_{i\in I}$ such that ϕ is the conical colimit in $\mathcal{P}(\mathbb{A})$ of the family $(Y_{\mathbb{A}}(a))_i$. This implies in particular that $\phi=\bigvee_i\mathbb{A}(-,a_i)$.

(ii$\Rightarrow$iii) For $\phi=\bigvee_i\mathbb{A}(-,a_i)$ it is always the case that $[\bigvee_i\mathbb{A}(-,a_i),-]=\bigwedge_i[\mathbb{A}(-,a_i),-]$, i.e. $\mathcal{P}(\mathbb{A})(\bigvee_i\mathbb{A}(-,a_i),-)=\bigwedge_i\mathcal{P}(\mathbb{A})(\mathbb{A}(-,a_i),-)$. Thus ϕ is the conical colimit in $\mathcal{P}(\mathbb{A})$ of the family $(\mathbb{A}(-,a_i))_i$, and Proposition 5.1 allows for the conclusion.

(iii$\Rightarrow$i) If, for some functor $G\colon\mathbb{A}\longrightarrow\mathbb{B}$, $\mathrm{colim}(\phi,G)$ exists, then, by the hypothesis that $\phi=\mathbb{A}(-,F-)\otimes\gamma$, it is equal to $\mathrm{colim}(\mathbb{A}(-,F-)\otimes\gamma,G)=\mathrm{colim}(\gamma,G\circ F)$. The latter is the conical colimit of the family $(G(F(i)))_{i\in I}$; thus the family $(F(i))_i$ fulfills the requirement. □

A warning is in order. Proposition 5.2 attests that the conical presheaves on a $\mathcal{Q}$-category $\mathbb{A}$ are those which are a supremum of some family of representable presheaves on $\mathbb{A}$. Of course, neither that family of representables, nor the family of representing objects in $\mathbb{A}$, need to be unique.

Now comes the most important observation concerning conical presheaves.

Proposition 5.3 *The class of conical presheaves is saturated.*

Proof : We shall check both conditions in Proposition 4.1. All representable presheaves are clearly conical, so the first condition is fulfilled. As for the second condition, consider a conical presheaf $\phi\colon *_X\multimap\mathbb{A}$ and a functor $G\colon\mathbb{A}\longrightarrow\mathcal{P}(\mathbb{B})$ such that each $G(a)\colon *_{ta}\multimap\mathbb{B}$ is a conical presheaf too. The ϕ-weighted colimit of G certainly exists, hence the first statement in Proposition 5.2 applies: it says that $\mathrm{colim}(\phi,G)$ is the conical colimit of a family of conical presheaves. In other words, $\mathrm{colim}(\phi,G)$ is a supremum of a family of suprema of representables, and is therefore a supremum of representables too, hence a conical presheaf. □

5.2 The Hausdorff doctrine

Applying Theorem 4.6 to the class of conical presheaves we get:

Definition 5.4 *We write $\mathcal{H}\colon \mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})$ for the KZ-doctrine associated with the class of conical presheaves. We call it the* **Hausdorff docrine on** $\mathsf{Cat}(\mathcal{Q})$*, and we say that $\mathcal{H}(\mathbb{A})$ is the* **Hausdorff $\mathcal{Q}$-category** *associated to a $\mathcal{Q}$-category $\mathbb{A}$. We write $\mathcal{H}'\colon \mathsf{Dist}(\mathcal{Q})\longrightarrow\mathsf{Dist}(\mathcal{Q})$ for the normal lax* Sup*-functor which extends $\mathcal{H}$ from* $\mathsf{Cat}(\mathcal{Q})$ *to* $\mathsf{Dist}(\mathcal{Q})$.

To justify this terminology, and underline the concordance with [Akhvlediani *et al.*, 2009], we shall make this more explicit. According to Proposition 4.4, $\mathcal{H}(\mathbb{A})$ is the full subcategory of $\mathcal{P}(\mathbb{A})$ determined by the conical presheaves on $\mathbb{A}$. By Proposition 5.2 however, the objects of $\mathcal{H}(\mathbb{A})$ can be equated with suprema of representables; so suppose that

$$\phi=\bigvee_{a\in A}\mathbb{A}(-,a)\quad\text{and}\quad\phi'=\bigvee_{a'\in A'}\mathbb{A}(-,a')$$

for subsets $A\subseteq\mathbb{A}_X$ and $A'\subseteq\mathbb{A}_Y$. Then we can compute that

$$\begin{aligned}
\mathcal{H}(\mathbb{A})(\phi',\phi) &= \mathcal{P}(\mathbb{A})(\phi',\phi)\\
&= [\phi',\phi]\\
&= [\bigvee_{a'}\mathbb{A}(-,a'),\bigvee_{a}\mathbb{A}(-,a]\\
&= \bigwedge_{a'}[\mathbb{A}(-,a'),\bigvee_{a}\mathbb{A}(-,a)]\\
&= \bigwedge_{a'}\bigvee_{a}[\mathbb{A}(-,a'),\mathbb{A}(-,a)]\\
&= \bigwedge_{a'}\bigvee_{a}\mathbb{A}(a',a).
\end{aligned}$$

(The penultimate equality is due to the fact that each $\mathbb{A}(-,a')\colon *_Y \multimap \mathbb{A}$ is a left adjoint in the quantaloid $\mathsf{Dist}(\mathcal{Q})$, and the last equality is due to the Yoneda lemma.) This is precisely the expected formula for the "Hausdorff distance between (the conical presheaves determined by) the subsets A and A' of $\mathbb{A}$". It must be noted that [Schmitt, 2006, Proposition 3.42] describes a very similar situation particularly for symmetric categories enriched in the commutative quantale of postive real numbers.

Similarly for functors: given a functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ between $\mathcal{Q}$-categories, the functor $\mathcal{H}(F)\colon\mathcal{H}(\mathbb{A})\longrightarrow\mathcal{H}(\mathbb{B})$ sends a conical presheaf ϕ on $\mathbb{A}$ to the conical presheaf $\mathbb{B}(-,F-)\otimes\phi$ on $\mathbb{B}$. Supposing that $\phi=\bigvee_{a\in A}\mathbb{A}(-,a)$ for some $A\subseteq\mathbb{A}_X$, it is straightforward to check that

$$\begin{aligned}\mathbb{B}(-,F-)\otimes\phi &= \bigvee_{x\in\mathbb{A}}\left(\mathbb{B}(-,Fx)\circ\bigvee_{a\in A}\mathbb{A}(x,a)\right)\\ &= \bigvee_{a\in A}\left(\bigvee_{x\in\mathbb{A}}\mathbb{B}(-,Fx)\circ\mathbb{A}(x,a)\right)\\ &= \bigvee_{a\in A}\mathbb{B}(-,Fa).\end{aligned}$$

That is to say, "$\mathcal{H}(F)$ sends (the conical presheaf determined by) $A\subseteq\mathbb{A}$ to (the conical presheaf determined by) $F(A)\subseteq\mathbb{B}$".

Finally, by Proposition 3.3, the action of $\mathcal{H}'$ on a distributor $\Phi\colon\mathbb{A}\multimap\mathbb{B}$ gives a distributor $\mathcal{H}'(\Phi)\colon\mathcal{H}(\mathbb{A})\multimap\mathcal{H}'(\mathbb{B})$ whose value in $\phi\in\mathcal{H}(\mathbb{A})$ and $\psi\in\mathcal{H}(\mathbb{B})$ is $\mathcal{P}(\mathbb{B})(\psi,\Phi\otimes\phi)$. Assuming that

$$\phi=\bigvee_{a\in A}\mathbb{A}(-,a)\text{ and }\psi=\bigvee_{b\in B}\mathbb{B}(-,b)$$

for some $A\subseteq\mathbb{A}_X$ and $B\subseteq\mathbb{B}_Y$, a similar computation as above shows that

$$\mathcal{H}'(\Phi)(\psi,\phi)=\bigwedge_{b\in B}\bigvee_{a\in A}\Phi(b,a).$$

This is the expected generalisation of the previous formula, to measure the "Hausdorff distance between (the conical presheaves determined by) $A\subseteq\mathbb{A}$ and $B\subseteq\mathbb{B}$ through $\Phi\colon\mathbb{A}\multimap\mathbb{B}$".

5.3 Other examples

The following examples of saturated classes of presheaves have been considered by [Kelly and Schmitt, 2005] in the case of categories enriched in symmetric monoidal categories.

Example 5.5 (Minimal and maximal class) The smallest saturated class of presheaves on $\mathcal{Q}$-categories is, of course, that containing only representable presheaves. It is straightforward that the KZ-doctrine on $\mathsf{Cat}(\mathcal{Q})$ corresponding with this class is the identity functor. On the other hand, the class of all presheaves on $\mathcal{Q}$-categories corresponds with the free cocompletion KZ-doctrine on $\mathsf{Cat}(\mathcal{Q})$.

Example 5.6 (Cauchy completion) The class of all left adjoint presheaves, also known as **Cauchy presheaves**, on $\mathcal{Q}$-categories is saturated. Indeed, all representable presheaves are left adjoints. And suppose that $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ and $\Psi\colon \mathbb{B} \multimap \mathbb{C}$ are distributors such that, for all $a \in \mathbb{A}$ and all $b \in \mathbb{B}$, $\Phi(-,a)\colon *_{ta} \multimap \mathbb{B}$ and $\Psi(-,b)\colon *_{tb} \multimap \mathbb{C}$ are left adjoints. Writing $\rho_b\colon \mathbb{C} \multimap *_{tb}$ for the right adjoint to $\Psi(-,b)$, it is easily verified that Ψ is left adjoint to $\bigvee_{b\in\mathbb{B}} \mathbb{B}(-,b) \otimes \rho_b$. This makes sure that $(\Psi \otimes \Phi)(-,a) = \Psi \otimes \Phi(-,a)$ is a left adjoint too, and by Proposition 4.2 we can conclude that the class of Cauchy presheaves is saturated. The KZ-doctrine on $\mathsf{Cat}(\mathcal{Q})$ which corresponds to this saturated class of presheaves, sends a $\mathcal{Q}$-category $\mathbb{A}$ to its **Cauchy completion** [Lawvere, 1973; Walters, 1981; Street, 1983].

Inspired by the examples in [Lawvere, 1973] and the general theory in [Kelly and Schmitt, 2005], Vincent Schmitt [2006] has studied several other classes of presheaves for ordered sets (viewed as categories enriched in the 2-element Boolean algebra) and for generalised metric spaces (viewed as categories enriched in the quantale of positive real numbers). He constructs *saturated* classes of presheaves by requiring that each element of the class "commutes" (in a suitable way) with all elements of a given (*not-necessarily saturated*) class of presheaves. These interesting examples do not seem to generalise straightforwardly to general quantaloid-enriched categories, so we shall not survey them here, but refer instead to [Schmitt, 2006] for more details.

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